Homework Assignment #7

Note: You must provide sufficient detail in your derivations or proofs to earn full credit. No late homework will be graded.

1. Let $X_1, \cdots, X_n$ be a random sample from the Bernoulli distribution with success probability p.

(a) Show that the variance of the maximum likelihood estimator of p attains the Cramér-Rao lower bound.

(b) For $n > 4$, show that the product $X_1X_2X_3X_4$ is an unbuased estimator of p^4, and use this fact to find the best unbiased estimator of p^4.

2. Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where both μ and $\sigma > 0$ are unknown. Let $\theta = \sigma^p$ for some $p > 0$.

(a) Find the Fisher information about θ.

(b) Find the Cramér-Rao lower bound for the variance of any unbiased estimator for θ.

3. Let $X_1, \cdots, X_n$ be a random sample from the uniform distribution on $[0, \theta]$.

(a) Calculate the variance of the maximum likelihood estimator of θ. Does the variance decrease at the rate of $1/n$?

(c) Does the Cramér-Rao information bound hold in this case? Why?

4. Let $(X_1, Y_i), \cdots, (X_n, Y_n)$ be a random sample that follows the regression model

$$Y_i = \beta X_i + e_i, \quad i = 1, \cdots, n,$$

where X_i follow a continuous distribution, e_i are independent of X_i and are normally distributed with mean zero and unknown variance σ^2.

(a) Can you use the Rao-Blackwell Theorem to find the best unbiased estimator of β? Why? What if σ were known? (Hint: If you cannot find the complete sufficient statistic, then you cannot use the Rao-Blackwell approach to find the BUE.)

(b) Can you use the Cramér-Rao information bound to find the best unbiased estimator of β? If so, how? If not, why?