Stat511(Section001): Homework #10

Due on Apr. 13, 2022 at 5:00pm

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Problem 1

(a)

We know under H_0 or H_1, $\hat{\mu} = \bar{X}$. And under H_1, we also know the the MLE of σ^2 is:

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Then we can get the log LR for this test is:

$$logL = maxL_{H_0}(x) - maxLH_1(x) = -\frac{n}{2}log(\sigma_0^2) + \frac{n}{2}log(\hat{\sigma^2}) - \frac{n}{2}\frac{\hat{\sigma^2}}{\sigma_0^2} + \frac{n}{2} = \frac{n}{2}log(\frac{\hat{\sigma^2}}{\sigma_0^2}) - \frac{n}{2}\frac{\hat{\sigma^2}}{\sigma_0^2} + \frac{n}{2}$$

We also know that, under H_0:

$$\frac{n\hat{\sigma^2}}{\sigma_0^2} \sim \chi^2(n-1)$$

Then choose $\frac{n\hat{\sigma^2}}{\sigma_0^2} = T$, we have:

$$logL = \frac{n}{2}log(\frac{T}{n}) - \frac{1}{2}T + \frac{n}{2}$$

We will reject H_0, when $logL < k$. And we can see that:

$$\frac{dlogL}{dT} = \frac{n}{2T} - \frac{1}{2}$$

Thus, when $T < n$, log L is a increasing function of T, and when $T > n$ it is a decreasing function of T. Hence:

$$logL < k \to T > k_1 \ or \ T < k_2$$

Then we can know:

$$k_1 = \chi^2(n-1)_{1-\alpha/2}, k_2 = \chi^2(n-1)_{\alpha/2}$$

where $\chi^2(n-1)_{\alpha/2}$ is the $\frac{\alpha}{2}$ percentile of $\chi^2(n-1)$.

(b)

We know the $1-\alpha$ confidence interval for T is:

$$k_2 < T < k_1 \to k_2 < \frac{n\hat{\sigma^2}}{\sigma^2} < k_1$$

Thus we can get:

$$\frac{n\hat{\sigma^2}}{k_1} < \sigma^2 < \frac{n\hat{\sigma^2}}{k_2} \to \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{k_1} < \sigma^2 < \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{k_2}$$

$[\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{k_1}, \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{k_2}]$ is the $1-\alpha$ confidence interval of σ^2,where $k_1 = \chi^2(n-1)_{1-\alpha/2}, k_2 = \chi^2(n-1)_{\alpha/2}$.

Problem 2

(a)

We know:

$$E(X_i) = \lambda$$

$$Var(X_i) = \lambda$$

Thus, by central limit theorem, we have:

$$\sqrt{n}(\bar{X} - \lambda) \sim N(0, \lambda) \to \frac{T - n\lambda}{\sqrt{n\lambda}} \sim N(0,1)$$

Problem 2 continued on next page. . .

And we can get:

$$\frac{T-n\lambda}{\sqrt{2n\lambda}} \sim N(0,\frac{1}{2})$$

Thus it is asymptotically pivotal.

(b)

From (a), we know the $1-\alpha$ confidence level of $\frac{T-n\lambda}{\sqrt{n\lambda}}$ is:

$$z_{\alpha/2} < \frac{T-n\lambda}{\sqrt{n\lambda}} < z_{1-\alpha/2}$$

where $z_{\alpha/2}$ is $\frac{\alpha}{2}$ percentile of standard nomral. $z_{1-\alpha/2}$ is $1-\frac{\alpha}{2}$ precentile of standard normal. Then we have:

$$z_{\alpha/2} < \frac{T-n\lambda}{\sqrt{n\lambda}} < -z_{\alpha/2}$$

Solving to get:

$$\frac{2T+z_{\alpha/2}^2-\sqrt{4Tz_{\alpha/2}^2+z_{\alpha/2}^4}}{2n} < \lambda < \frac{2T+z_{\alpha/2}^2+\sqrt{4Tz_{\alpha/2}^2+z_{\alpha/2}^4}}{2n}$$

And that is the confidence interval for λ. **(c)**

By (b), we can get the 95% interval is $[1.094, 2.888]$

Problem 3

It easy to know if X is the number of success in 5 independent missile trials. Then we know $X \sim Bin(5,p)$. Now to seek lower confidence bound. We consider $H_0: p=p_0, H_1: p>p_0$ with $\alpha = 0.1$. Then under H_0, we have:

$$P(X\geq 4) = 5p_0^4(1-p_0)+p_0^5 = 5p_0^4-4p_0^5 > 0.1$$

Then we can have $p_0 = 0.416$. And we can get 90% lower confidence bound is 0.416.

Problem 4

(a)

We know the likelihood function for one X_i can been writeen as:

$$f(X_i) = (1-2\theta)^{I(X_i=0)}\theta^{I(X_i=1)+I(X_i=-1)} = (1-2\theta)^{I(X_i=0)}\theta^{1-I(X_i=0)}$$

Thus, take $T = sum_{i=1}^n I(X_i=0)$, then we can get the likelihood function of $X_1,...,X_n$ is:

$$L(\theta) = \theta^{n-T}(1-2\theta)^T$$

$$logL(\theta) = (n-T)log(\theta)+Tlog(1-2\theta)$$

Then, we have the MLE of θ is:

$$hat\theta = \frac{n-T}{2n}$$

Hence, under H_1 , if $\frac{n-T}{2n} > \theta_0$ $\theta_{H_1} = \frac{n-T}{2n}$. else we get $\theta_{H_1} = \theta_0$. Thus we can get the Likelihood Ratio is:

$$L = \begin{cases} 1 & \theta_0 > \dfrac{n-T}{2n} \\ \dfrac{\theta_0^{n-T}(1-2\theta_0)^T}{(\frac{n-T}{2n})^{n-T}(\frac{T}{n})^T} & else \end{cases}$$

Problem 4 continued on next page...

This is a function a T. We will reject H_0 when this function is less than k_1. Then we have:

$$\frac{dlogL}{dT} = log(\frac{n-T}{T}) + log(\frac{1-2\theta_0}{\theta_0}) - log2 \geq 0$$

Thus $L < k_1$ is equvalent to $T < k$. That is end of our proof.

(b)

We have the reject region $T < k(\alpha)$. Thus $T \geq k(\alpha)$ is the acceptance region. Then we can get $1-\alpha$ lower confidence bound for θ is just $k(\alpha)$.

(c)

We know under H_0 $T \sim Bin(10, \frac{1}{3})$. Then we need to find to biggest k. $P(T < k) \leq 0.05$. And By python, we get $k = 0$.