Stat511(Section001): Homework #1

Due on Jan. 12, 2021 at 11:59pm

Instructor:Xuming He

Tiejin Chen **tiejin@umich.edu**

Problem 1

Consider $X \sim Uni(-1,1), Y = X^2$. It is obvious that they are dependent. However, we can heve:

$$Cov(X,Y) = E(XY) - E(X)E(Y) = E(X^3) - E(X)E(X^2) = 0 - 0 = 0$$

That is to say, X and Y are dependent but uncorrelated. [2]

Problem 2

We should use Scheme2. Because, for Scheme1, we obtain w_A,w_B respectively, for each w, we have standard deviation is σ. And we consider Scheme 2, we can get:

$$w_A = \frac{(w_{A+B} + w_{A-B})}{2}, w_B = \frac{(w_{A+B} - w_{A-B})}{2}$$

Since two measurement should be independent, we can get:

$$Var(w_A) = \frac{Var(w_{A+B}) + Var(w_{A-B})}{4} = Var(w_B) = \frac{\sigma^2}{2}$$

$$Sd(w_A) = Sd(w_B) = \sqrt{Var(w_A)} = \frac{\sqrt{2}}{2}\sigma$$

which is smaller than the standard deviation in Scheme1. Hence, we think Scheme2 is more precise and we prefer Scheme2.

Problem 3

Since (X,Y) is a joint normal distribution, we can know that $X - Y$ is also a normal distribution, with expectation equal to 0 and variance equal to $Var(X) + Var(Y) - 2Cov(X,Y)$. And we have:

$$Cov(X,Y) = \rho Sd(X)Sd(Y) = \rho$$

Hence, we can know $X - Y \sim N(0, 2-2\rho)$.Let $Z = X - Y$,Hence:

$$P(X_i > Y_i) = P(X_i - Y_i > 0) = \frac{1}{\sqrt{2\pi(2-2p)}}\int_{z=0}^{\infty} e^{-\frac{z^2}{2(2-2\rho)}}dz = \frac{1}{2}$$

Now we can see that for one $i, I(X_i > Y_i)$ is a Bernouli distribution with $p = \frac{1}{2}$. T is sum of n Bernouli distributio. Hence T is a binomial distribution, and $T \sim Bin(n, \frac{1}{2})$

Problem 4

First, let us compare the mean. for $\frac{T}{n-1}$, we have:

$$E(\frac{T}{n-1}) = E(\frac{1}{n-1}(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2)) = \frac{1}{n-1}(nE(X_1^2) - nE(\bar{X}^2)) = \frac{1}{n-1}(n(\sigma^2 + \mu^2) - n(\frac{\sigma^2}{n} + \mu^2)) = \sigma^2$$

$$E(\frac{T}{n}) = E(\frac{n-1}{n}\frac{T}{n-1}) = \frac{n-1}{n}E(\frac{T}{n-1}) = \frac{n-1}{n}\sigma^2$$

We can get the conclusion that $n^{-1}T$ is a bias estimator while $(n-1)^{-1}T$ is an unbiasd estimator. $(n-1)^{-1}T$ seems a better estimator under this circumstance.

Problem 4 continued on next page...

Now, let us consider the variance.For this part, we will use a conclusion that $\frac{T}{\sigma^2} \sim \chi^2(n-1)$. [1]
We can know:

$$Var(\chi^2(n-1)) = 2(n-1)$$

$$Var(\frac{T}{n-1}) = \frac{1}{(n-1)^2}Var(\sigma^2\chi^2(n-1)) = \frac{2}{n-1}\sigma^4$$

$$Var(\frac{T}{n}) = \frac{2(n-1)}{n^2}\sigma^4$$

It is easy to see that $n^{-1}T$ has smaller variance than $(n-1)^{-1}T$. Hence $n^{-1}T$ is a better estimator when we only consider variance. In fact, $n^{-1}T$ also has smaller mean square error than $(n-1)^{-1}T$.

References

[1] William G Cochran. The distribution of quadratic forms in a normal system, with applications to the analysis of covariance. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 30, pages 178–191. Cambridge University Press, 1934.

[2] Clare Brown (https://stats.stackexchange.com/users/38489/clare brown). Simple examples of uncorrelated but not independent x and y. Cross Validated. URL:https://stats.stackexchange.com/q/85363 (version: 2016-03-05).