Homework Assignment #6

Note: You must provide sufficient detail in your derivations or proofs to earn full credit. No late homework will be graded.

1. Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$ where $\mu \geq 0$ and $\sigma \geq 1$. If $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $S^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2$, are $\hat{\mu} = \max\{\bar{X}_n, 0\}$ and $\hat{\sigma}^2 = \max\{S^2, 1\}$ the maximum likelihood estimators of μ and σ^2?

2. Let X be the number of successes from n independent Bernoulli trials with success probability p.

(a) If we impose a prior distribution on p as the uniform distribution on (0,1), find the Bayes estimator of p.

(b) If the prior distribution on $\theta = p^2$ is uniform on (0,1), what is the posterior mean of p?

(c) The uniform prior is sometimes viewed as a prior that carries no prior information about the parameter. From this exercise, what do you learn about the uniform prior?

3. Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from a bivariate normal distribution, where X_i is distributed as $N(0, 1)$, and $Y_i - \beta X_i$ is independent of X_i and distributed as $N(0, 1)$. Find the best unbiased estimator of β for any given values of $X_1, \cdots, X_n$.

4. Suppose that $X = (X_1, \ldots, X_k)$ is Multinomial with $X \mid \theta \sim M_k(n; \theta_1, \theta_2, \ldots, \theta_k)$, where $\theta = (\theta_1, \cdots, \theta_k)$ and $\sum_{i=1}^k \theta_i = 1$. Now assume that θ has a prior distribution with the density

$$\pi(\theta) \propto \theta_1^{\alpha_1 - 1}\theta_2^{\alpha_2 - 1} \cdots \theta_k^{\alpha_k - 1},$$

with the (prior) expectation $E[\theta_i] = \frac{\alpha_i}{\alpha_1 + \cdots + \alpha_k}$. This prior distribution is often called the Dirichlet Distribution. Given a vector observation $\mathbf{X} = \mathbf{x}$,

(a) calculate the posterior distribution of θ.

(b) What is the Bayes estimator for θ_i $(i = 1, \cdots, k)$ under the squared error loss?