Stat511(Section001): Homework #8

Due on Mar. 25, 2022 at 5:00pm

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Problem 1

We can see the hypothesis testing problem as:

$$H_0 : \lambda = 1000, H_1 : \lambda \neq 1000$$

Where λ is the parameter of exponential distribution. And let X_i be random variable of the i-th bulb's life time. We know under $H_0, X_i \sim exp(1000) = Gamma(1, 1000)$. Thus under H_0, we have:

$$\sum_{i=1}^{4} X_i \sim Gamma(4, 1000)$$

Then we use P-value to get the test, Since $\alpha = 0.05$, we have:

$$p(\sum_{i=1}^{4} X_i > c_1) = 0.05/2 = 0.025$$

$$p(\sum_{i=1}^{4} X_i < c_2) = 0.05/2 = 0.025$$

Then we need to find c_1, c_2. It is easy to see that c_1 is 97.5-th percentile of $Gamma(4, 1000)$ and c_2 is 2.5th percentile of the distribution. Here, we can adjust Gamma distribution to get a chi-square distribution. However, since we can use computer here, we can direct get the c_1 is 8767.273,c_2 is 1089.865 by python and scipy. And we can get:

$$\sum_{i=1}^{4} X_i = 4 \times \bar{X} = 4 \times 900 = 3600, 1089.865 < 3600 < 8767.273$$

Thus we fail to reject H_0, which means we do not have significant evidence against the GE's claim. And this is our code for this part:

```
from scipy.stats import gamma
print("c1:",gamma.ppf(0.025,4,scale=1000))
print("c2:",gamma.ppf(0.975,4,scale=1000))
```

Problem 2

Now we have:

$$H_0 : \lambda = 1000, H_1 : \lambda = 900$$

Under H_1, we have:

$$E_{\theta_1}(\phi(x)) = p_{\lambda=900}(\sum_{i=1}^{4} X_i > 8767.273) + p_{\lambda=900}(\sum_{i=1}^{4} X_i < 1089.865)$$

$$\sum_{i=1}^{4} X_i \sim Gamma(4, 900)$$

By python and scipy, we can get the power of our test is 0.01248+0.03472 =0.04720. And our code for this part is:

```
from scipy.stats import gamma
print("p1:",1-gamma.cdf(8767.273,4,scale=900))
print("p2:",gamma.cdf(1089.865,4,scale=900))
```

Problem 3

(a)
we have $X_i \sim exp(\mu_x)$, $Y_i \sim exp(\mu_y)$. Then, Under H_0, we have:

$$L_{H_0}(\mu_x|X,Y) = \mu_x^{-(n+m)} exp(-\frac{1}{\mu_x}(\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i))$$

since under H_0, Y_i, X_i has same distribution. And we can see Y_i as X_{n+1}. Then we can get the MLE of them is $\hat{\mu_x} = \frac{1}{n+m}(\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i)$. Then we have:

$$maxL_{H_0}(\mu_x|X,Y) = (\frac{n+m}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i})^{n+m} exp(-(n+m))$$

Similarly, we have:

$$maxL_{H_1}(\mu_x, \mu_y|X,Y) = \mu_x^{-n}\mu_y^{-m} exp(-\frac{1}{\mu_x}\sum_{i=1}^{n} X_i) exp(\frac{1}{\mu_y}\sum_{i=1}^{m} Y_i) = (\frac{n}{\sum_{i=1}^n X_i})^n (\frac{m}{\sum_{i=1}^m Y_i})^m exp(-(n+m))$$

Then we have LRT statistic:

$$T(X,Y) = \frac{(\frac{n+m}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i})^{n+m}}{(\frac{n}{\sum_{i=1}^n X_i})^n (\frac{m}{\sum_{i=1}^m Y_i})^m} = (\frac{(n+m)\sum_{i=1}^n X_i}{n(\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i)})^n (\frac{(n+m)\sum_{i=1}^m Y_i}{m(\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i)})^m$$

(b)
when $T = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i}$, $1 - T = \frac{\sum_{i=1}^m Y_i}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i}$. Then we have:

$$T(X,Y) = (\frac{n+m}{n}T)^n (\frac{n+m}{m}(1-T))^m$$

(c)
Under H_0, we have:

$$\sum_{i=1}^{n} X_i \sim Gamma(n, \mu_x) \to \frac{1}{\mu_x}\sum_{i=1}^{n} X_i \sim Gamma(n,1)$$

$$\frac{1}{\mu_x}(\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i) \sim Gamma(n+m, 1)$$

Thus, we have:

$$T(X,Y) = \frac{\frac{1}{\mu_x}\sum_{i=1}^n X_i}{\frac{1}{\mu_x}(\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i)} \sim Beta(n,m)$$

Then we know $\frac{L_0}{L_1}$ need to be less than some $k \in (0,1)$ to reject H_0. Hence we have:

$$P(T(X,Y) < k) = \alpha$$

Thus we can get k is the α-th percentile of $Beta(n,m)$. We will reject H_0 when the LRT statistic is less than k.

Problem 4

(a)
We have:

$$maxL_{H_0} = \begin{cases} 0 & \theta_0 < X_{(n)} \\ (\frac{1}{X_{(n)}})^n & X_{(n)} \leq \theta_0 \end{cases}$$

Problem 4 continued on next page. . .

Where $X_{(n)}$ the highest order statistic. For H_1, we have:

$$L_{H_1} = maxL_{H_0 \cup H_1} = (\frac{1}{X_{(n)}})^n$$

Thus we can get:

$$T(X) = \frac{L_{H_0}}{L_{H_1}} = \begin{cases} 0 & \theta_0 < X_{(n)} \\ 1 & X_{(n)} \leq \theta_0 \end{cases}$$

We will only reject H_0 when $\theta_0 < X_{(n)}$ for any level α.

(b)

we have $X_{(n)} = 22$ and $\theta_0 = 25$. Thus we have $\theta_0 > X_{(n)}$. And from the result in first part, we say we cannot reject the null hypothesis under level $\alpha = 0.05$.