Stat511(Section001): Homework #2

Due on Jan. 19, 2021 at 5:00pm

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Problem 1

We can know the pdf of x is:

$$f(x) = b^2xe^{-bx}, x > 0$$

Hence the cdf of x is:

$$F(x) = \int_0^x b^2xe^{-bx}dx$$

let $t = bx$, we have:

$$F(x) = \int_0^{bx} te^{-t}dt$$

This is incomplete gamma function, we will use γ to present the function. And we can get the $F(x)$ is $\gamma(2, bx)$. for quantile function $Q_x(p)$, when p is 0, it is easy to see that,$Q_x(P) = -\infty$. if $p > 0$,we have:

$$p = \gamma(2, bx)$$

$$\gamma^{-1}(2, p) = bx$$

$$x = \frac{\gamma^{-1}(2, p)}{b}$$

Hence,

$$Q_x(p) = \frac{\gamma^{-1}(2, p)}{b}, 0 < p \leq 1$$

The quatiles are:

$$Q_x(0.25) = \frac{\gamma^{-1}(2, 0.25)}{b}, Q_x(0.75) = \frac{\gamma^{-1}(2, 0.75)}{b}$$

Problem 2

(1) Since we know $p < 0.5$, then we have $g(p) = \sqrt{p} + \sqrt{1-p}$ is one to one function when $p \in (0, 0.5)$. And we have:

$$g'(p) = 0.5(\frac{\sqrt{p}}{p} - \frac{\sqrt{1-p}}{1-p})$$

$$(g'(p))^2 = \frac{1 - \sqrt{p(1-p)}}{4p(1-p)}$$

Hence, we can get:

$$\sqrt{n}(g(\bar{X_n}) - g(p)) \approx N(0, \frac{1 - 2\sqrt{p(1-p)}}{4})$$

Hence:

$$g(\bar{X_n}) \approx N(\sqrt{p} + \sqrt{1-p}, \frac{1 - 2\sqrt{p(1-p)}}{4n}))$$

(2) when p is 0.5, $g'(p) = 0$, Hence we can not use δ-method.We should consider second order. We have:

$$g''(p) = -\frac{0.25}{(1-p)^{1.5}} - \frac{0.25}{p^{1.5}}$$

We can see $g''(0.5) = -\sqrt{2} \neq 0$.Hence we can know:

$$n(g(\bar{X_n}) - g(p)) \approx -\frac{\sqrt{2}}{8}\chi_1^2$$

$$g(\bar{X_n}) \approx -\frac{\sqrt{2}}{8n}\chi_1^2 + \sqrt{2}$$

Problem 3

(1) We have:

$$P(X_{(n)} \leq x) = 1 - P(X_{(n)} > x) = 1 - P(X_1 > x, ..., X_n > x) = 1 - \prod_{i=1}^{n} P(X_i > x)$$

Hence, we can get:

$$P(X_{(n)} \leq x) = 1 - e^{-nx}, x \geq 0$$

It is easy for us to get the pdf:

$$f_{(n)}(x) = ne^{-nx}, x \geq 0$$

(2) Since all X_i are iid, the probability to be $X_{(n)}$ should be same for every X_i.Hence

$$P(X_{(n)} = X_1) = \frac{1}{n}$$

Problem 4

Since X_i are independent, their joint pdf should be mulitvariant normal distribution, and their linear combination is a normal distribution. And we know that $X_i - X_j \sim N(0, 2)$ for $i \neq j$. Hence, if $a(X_i - X_j)^2 \sim \chi_1^2$, Then $\sqrt{a}(X_i - X_j) \sim N(0, 1)$ and we can get $a = \frac{1}{2}$.
Now we consider T_n. T_n can be chi-square distribution with degree no greater than 3. And we can list all the possible situation:

1. T_n is a chi-square distribution with degree 1.
   For one a_i is $\frac{1}{2}$, and other are 0.

2. T_n is a chi-square distribution with degree 2.
   For one a_i is 0, and the other are 0.5.

3. T_n is a chi-square distribution with degree 3.
   All a_i is 0.5

Hence we can know all possible values of $a_1 + a_2 + a_3$ are 0.5,1 and 1.5.