Stat511(Section001): Homework #9

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Problem 1

We know:

$$Y_i|X_i \sim N(\beta X_i, \sigma^2), f(Y_i|X_i) = \frac{1}{\sqrt{2\pi}\sigma} exp(-\frac{1}{2\sigma^2}(y_i - \beta x_i)^2)$$

First we consider the LRT of simple case, which means $H_0 : \beta = 0, H_1 : \beta = \beta_1 > 0$. We can get:

$$L = \frac{L_{H_0}(\beta|X_i)}{L_{H_1}(\beta|X_i)} = exp(\frac{1}{2\sigma^2}\sum_{i=1}^{N}(y_i - \beta_1 x_i)^2 - y_i^2) = exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{N} 2\beta_1 y_i x_i - \beta_1^2 x_i^2)$$

Then we use:

$$T(y|x) = \sum_{i=1}^{N} y_i x_i$$

as test statistic. Now we need to find a k so that $P(L < k) = \alpha$ and α is level of the test. And we know:

$$L < k \to T(y|x) > k_1$$

And we know under H_0, we have:

$$x_i y_i | x_i \sim N(0, x_i^2\sigma^2)$$

Thus we have:

$$T(y|x) \sim N(0, \sigma^2 \sum_{i=1}^{N} x_i^2)$$

Thus we can get:

$$P(T(y|x) > k_1) = \alpha \to k_1 = \sqrt{\sigma^2 \sum_{i=1}^{N} x_i^2} Z_{1-\alpha}$$

where $Z_{1-\alpha}$ is the $1 - \alpha$ precentile of standard normal distribution. And this works for every $\beta_1 > 0$. And we know LRT is the most powerful test. Thus, it is the UMP test.

Problem 2

(a)

We know:

$$f(X_i) = \frac{1}{\sqrt{2\pi}\sigma} exp(-\frac{1}{2\sigma^2}(x_i - \mu_1)^2)$$

$$f(Y_i) = \frac{1}{\sqrt{2\pi}2\sigma} exp(-\frac{1}{8\sigma^2}(y_i - \mu_2)^2)$$

And under H_0, $\mu_1 = \mu_2$. And we can get the log-likelihood function of X,Y is:

$$logL(\mu_1, \sigma^2) = -\frac{N}{2}log(\sigma^2) - \frac{N}{2}log(4\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i - \mu_1)^2 - \frac{1}{8\sigma^2}\sum_{i=1}^{N}(y_1 - \mu_1)^2$$

Then we can get:

$$\frac{\partial logL}{\partial \mu_1} = \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i - \mu_1) + \frac{1}{4\sigma^2}\sum_{i=1}^{N}(y_i - \mu_1) = 0 \to \hat{\mu_1} = \frac{4\bar{X} + \bar{Y}}{5}$$

$$\frac{\partial logL}{\partial \sigma^2} = -\frac{N}{2\sigma^2} - \frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{N}(x_i - \mu_1)^2 + \frac{1}{8\sigma^4}\sum_{i=1}^{N}(y_i - \mu_1)^2 = 0 \to \hat{\sigma^2} = \frac{1}{2N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2 + \frac{1}{8N}\sum_{i=1}^{N}(y_i - \hat{\mu})^2$$

Problem 2 continued on next page. . .

Thus:

$$l_0 = -\frac{N}{2}log(\hat{\sigma^2}) - \frac{N}{2}log(4\hat{\sigma^2}) - n = -Nlog(\hat{\sigma^2}) - n + c$$

where c is a constant, We can get $c = -\frac{N}{2}log4$.

(b)

Under H_1, we know $\mu_1 \neq \mu_2$. And we can get the log-likelihood function is:

$$logL(\mu_1, \mu_2, \sigma^2) = -\frac{N}{2}log(\sigma^2) - \frac{N}{2}log(4\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i - \mu_1)^2 - \frac{1}{8\sigma^2}\sum_{i=1}^{N}(y_1 - \mu_2)^2$$

And we can get the MLE of them is:

$$\hat{\mu_1} = \bar{X}, \hat{\mu_2} = \bar{Y}, \hat{\sigma^2} = \frac{1}{2N}\sum_{i=1}^{N}(x_i - \bar{X})^2 + \frac{1}{8N}\sum_{i=1}^{N}(y_i - \bar{Y})^2$$

To distinguish with part(a), we use $\hat{\sigma_1^2}$ to present this part $\hat{\sigma^2}$. Then we have:

$$l_1 = -\frac{N}{2}log(\hat{\sigma_1^2}) - \frac{N}{2}log(4\hat{\sigma_1^2}) - n = -Nlog(\hat{\sigma_1^2}) - n + c$$

Where again $c = -\frac{N}{2}log4$.

(c)

We have:

$$\begin{aligned}
T &= 2(l_1 - l_0) \\
&= 2(-Nlog(\hat{\sigma_1^2}) - n + c + Nlog(\hat{\sigma^2}) + n - c) \\
&= 2Nlog\frac{\hat{\sigma^2}}{\hat{\sigma_1^2}}
\end{aligned}$$

$$\begin{aligned}
\hat{\sigma^2} &= \frac{1}{2N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2 + \frac{1}{8N}\sum_{i=1}^{N}(y_i - \hat{\mu})^2 \\
&= \frac{1}{2N}\sum_{i=1}^{N}(x_i - \bar{X} + \frac{1}{5}(\bar{X} - \bar{Y}))^2 + \frac{1}{8N}\sum_{i=1}^{N}(y_i - \bar{Y} - \frac{4}{5}(\bar{X} - \bar{Y}))^2 \\
&= \hat{\sigma_1^2} + \frac{1}{2}(\frac{1}{5}(\bar{X} - \bar{Y}))^2 + \frac{1}{8}(\frac{4}{5}(\bar{X} - \bar{Y}))^2 \\
&= \hat{\sigma_1^2} + \frac{1}{10}(\bar{X} - \bar{Y})^2
\end{aligned}$$

We can plug this in T to get:

$$\begin{aligned}
T &= 2Nlog\frac{\hat{\sigma^2}}{\hat{\sigma_1^2}} \\
&= 2Nlog(1 + \frac{\frac{1}{10}(\bar{X} - \bar{Y})^2}{\frac{1}{2N}\sum_{i=1}^{N}(x_i - \bar{X})^2 + \frac{1}{8N}\sum_{i=1}^{N}(y_i - \bar{Y})^2}) \\
&= 2Nlog(1 + \frac{8N}{10(N-1)}\frac{(\bar{X} - \bar{Y})^2}{4S_x^2 + S_y^2}) \\
&= 2Nlog(1 + \frac{8N}{10(N-1)}T^*)
\end{aligned}$$

We can see that T is a monotonous increasing function of T^*. And they have one-to-one relationship. We will reject H_0 when $T > k_0$. Thus, it is equivalent $T^* > k$ where k is a constant related to k_0.

Problem 3

(a)
We know under H_0:

$$\bar{X} \sim N(\mu_1, \sigma^2/N), \bar{Y} \sim N(\mu_1, 4\sigma^2/N)$$

Thus, we can get:

$$\bar{X} - \bar{Y} \sim N(0, \frac{5\sigma^2}{N})$$

Thus:

$$\frac{N}{5\sigma^2}(\bar{X} - \bar{Y})^2 \sim \chi^2(1)$$

For sample variance, we know:

$$\frac{(N-1)S_x^2}{\sigma^2} \sim \chi(N-1), \frac{(N-1)S_y^2}{4\sigma^2} \sim \chi^2(N-1)$$

$$\frac{(N-1)S_x^2}{\sigma^2} + \frac{(N-1)S_y^2}{4\sigma^2} = \frac{N-1}{4\sigma^2}(4S_x^2 + S_y^2) \sim \chi^2(2N-2)$$

Now consider T^*, we have:

$$T^* = \frac{(\bar{X} - \bar{Y})^2}{4S_x^2 + S_y^2} = \frac{\frac{N}{5\sigma^2}(\bar{X} - \bar{Y})^2}{\frac{N-1}{4\sigma^2(2N-2)}(4S_x^2 + S_y^2)} \frac{5}{8N} \sim \frac{5}{8N}F(1, 2N-2)$$

Thus we can get $k = \frac{5}{8N}F_\alpha(1, 2N-2)$. where $F_{1-\alpha}(1, 2N-2)$ is the $1-\alpha$ precentile of $F(1, 2N-2)$. When $N = 10, \alpha = 0.05$, we have: $k = \frac{5}{80}F_{0.95}(1, 18) = 0.275867$.

(b)
By wilks' theorem, we know $2(l_1 - l_0) = T \xrightarrow{N\to\infty} \chi^2(1)$. Then we have $k_0 = \chi^2_{0.95}(1) = 3.841$. Then the reject region is $T > 3.841$. To compare two region, we plug $T^* = k = 0.275867$ to $T = 2Nlog(1 + \frac{8N}{10(N-1)}T^*)$ and get the reject region for part(a) is $T > 4.3861659$. They are close, but not that close. I think that is because 10 is a small number for N, which makes the T is not close to its asymptotic distribution.

Problem 4

(a)
By fisher's method, we can get:

$$T = -2\sum_{i=1}^{3} log(p_i) \sim \chi^2(6)$$

And we can get $T = 13.004$. And we can get $\chi^2_{0.95}(6) = 12.591$. Thus we can get $T > 12.591$. And we can reject null hypothesis at significance level of 0.05.

(b)
Consider a extreme situation where $\rho(p_i, p_j) = 1$ for all $i, j \leq 3$. That is to say $p_i = a \times p_j + b$ for every i, j. And this means these 3 tests are almost equivalent. And we can assume $p_1 = 10p_3 + 0.5, p_2 = 5p_3 + 0.2$. And we can find that if we only consider p_1, p_2, we will always not reject null hypothesis, And if we only consider p_3, we will always reject null hypothesis(since $10p_3 + 0.5 \leq 1, p_3 \leq 0.05$). Then we cannot find out which test are correct and which test are not without further information. Then we cannot reach the same conclusion.