Stat511(Section001): Homework #6

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Problem 1

The log-likelihood function of normal distribution is:

$$l(x_1,...,x_n) = -\frac{n}{2}ln(2\pi) - \frac{n}{2}ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2, \mu \geq 0, \sigma^2 \geq 1$$

We can get:

$$\frac{\partial l(x_1,...,x_n)}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)$$

$$\frac{\partial l(x_1,...,x_n)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2$$

Then if $\bar{X} \geq 0$, let the above formula to be zero, we have:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}x_i = \bar{X}$$

Now let us consider $\bar{X} < 0$:

$$\frac{\partial l(x_1,...,x_n)}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = \frac{1}{\sigma^2}(\sum_{i=1}^{n}x_i - n\mu) < 0$$

Hence, it is a decreasing function when $\bar{X} < 0$ and $\mu > 0$, Then to make $l(x_1,...,x_n)$ as larger as possible, μ should be zero. That is to say $\hat{\mu} = max(\bar{X},0)$ is MLE of μ.However, when we consider σ^2 , we have:

$$\frac{\partial l(x_1,...,x_n)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\bar{X}+\bar{X}-\mu)^2$$

For $(x_i-\bar{X}+\bar{X}-\mu)^2$ we have:

$$(x_i-\bar{X}+\bar{X}-\mu)^2 = (x_i-\bar{X})^2 + 2(x_i-\bar{X})(\bar{X}-\mu) + (\bar{X}-\mu)^2$$

We know:

$$\sum_{i=1}^{n}(x_i-\bar{X}) = \sum_{i=1}^{n}x_i - n\bar{X} = 0$$

Thus:

$$\sum_{i=1}^{n}2(x_i-\bar{X})(\bar{X}-\mu) = 0$$

Hence plug back in we can get:

$$\begin{aligned}\frac{\partial l(x_1,...,x_n)}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}[\sum_{i=1}^{n}(x_i-\bar{X})^2 + n(\bar{X}-\mu)^2] \\ &= -\frac{n}{2\sigma^2} + \frac{n}{2\sigma^4}(S^2 + (\bar{X}-\mu)^2) \\ &= \frac{n}{2\sigma^4}(S^2 + (\bar{X}-\mu)^2 - \sigma^2)\end{aligned}$$

Then let it to be zero and assumed $\bar{X} < 0, S^2 > 1$,we have:

$$\hat{\sigma^2} = S^2 + (\bar{X}-\mu)^2 \neq S^2$$

Since $\bar{X} < 0, \hat{\mu} = 0$ and $(\bar{X}-\mu)^2 \neq 0$. Thus $\hat{\sigma^2} \neq max(S^2,1)$. For σ^2 it is not a MLE estimator. To sum up, $\hat{\mu} = max(\bar{X},0)$ is the MLE estimator of μ where $\hat{\sigma^2} = max(S^2,1)$ is not a MLE estimator of σ^2.

Problem 2

(a)
We know:

$$f(\theta|x) \propto f(x|\theta)f(\theta)$$

Thus we have:

$$f(\theta|x) \propto p^x(1-\theta)^{n-x}I(0<p<1)$$

We know p is always staisfy $0<p<1$. Thus:

$$f(\theta|x) \propto p^x(1-\theta)^{n-x}$$

Then, it must be a Beta distribution. Thus we can have;

$$f(\theta|x) \sim Beta(x+1, n-x+1)$$

Thus, we get Bayes estimator of p is $\frac{x+1}{n+2}$.
(b)
We have:

$$P(p \leq y) = P(p^2 \leq y^2) = y^2$$

Thus, we have:

$$f_p(y) = 2y$$

Hence, we can get:

$$f(\theta|x) \propto f(x|\theta)f(\theta) = p^{x+1}(1-\theta)^{n-x}$$

Agian, this beta distribution and $f(p|x) \sim Beta(x+2, n-x+1)$. Thus the posterior mean of p is $\frac{x+2}{n+3}$
(c)
We can see that the uniform in p^2 does not mean the uniform in p. Hence they will get different Bayes estimator. Hence, we can say even uniform distribution carry the information.

Problem 3

To seek the best unbiased estimator, we need to find a sufficient statistic for β. First we know $Y_i - \beta X_i$ is independent with X_i. Then it is independent with βX_i. We know that when X_i is given, then βX_i is a scalar. Hence, we consider $Y_i|X_i$, We have:

$$Y_i|X_i = Y_i - \beta X_i + \beta X_i \sim N(\beta X_i, 1)$$

Then we have :

$$f(\beta; Y|X) = \frac{1}{(2\pi)^{\frac{n}{2}}} exp(-\frac{1}{2}\sum_{i=1}^{n}(Y_i - \beta X_i)^2) = \frac{1}{(2\pi)^{\frac{n}{2}}} exp(-\sum_{i=1}^{n}(0.5Y_i^2 - \beta Y_i X_i + 0.5\beta^2 X_i^2))$$

It is a expontional family of distribution. Thus its minimal sufficient statistic is $\sum_{i=1}^n Y_i X_i = T$. And let us find a unbiased estimator of β. We have:

$$E(T) = nE(Y_iX_i) = nE_X[E(Y_iX_i|X)] = nE[X_iE(Y_i|X_i)] = \beta nE(X_i^2)$$

Thus:

$$\beta = \frac{nE(Y_iX_i)}{nE(X_i^2)}$$

Hence:

$$\hat{\beta|T} = \frac{\sum_{i=1}^n Y_iX_i}{\sum_{i=1}^n X_i^2} = \frac{T}{\sum_{i=1}^n X_i^2}$$

is the best unbiased estimator of β for any given value of $X_1, .., X_n$ by Rao-Blackwell theorem.

Problem 4

(a)
we have:

$$f(\theta|x) \propto f(x|\theta)f(\theta) = \theta_1^{\alpha_1+x_1-1}\theta_2^{\alpha_2+x_2-1}...\theta_n^{\alpha_n+x_n-1}$$

Thus the posterior distribution of θ is also a Dirichlet Distribution.
(b)
We know under MSE, the bayes estimator is just posterior mean. Thus we can get:

$$\hat{\theta}_i = \frac{\alpha_i + x_i}{\sum_{j=1}^{k} \alpha_j + x_j}$$