Uniform Consistency of the Highly Adaptive Lasso Estimator of Infinite Dimensional Parameters

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Abstract

Consider the case that we observe n independent and identically distributed copies of a random variable with a probability distribution known to be an element of a specified statistical model. We are interested in estimating an infinite dimensional target parameter that minimizes the expectation of a specified loss function. In van der Laan (2015) we defined an estimator that minimizes the empirical risk over all multivariate real valued cadlag functions with variation norm bounded by some constant M in the parameter space, and selects M with cross-validation. We referred to this estimator as the Highly-Adaptive-Lasso estimator due to the fact that the constrained can be formulated as a bound M on the sum of the coefficients a linear combination of a very large number of basis functions. Specifically, in the case that the target parameter is a conditional mean, then it can be implemented with the standard LASSO regression estimator. In van der Laan (2015) we proved that the HAL-estimator is consistent w.r.t. the (quadratic) loss-based dissimilarity at a rate faster than $n^{-1/2}$ (i.e., faster than $n^{-1/4}$ w.r.t. a norm), even when the parameter space is completely nonparametric. The only assumption required for this rate is that the true parameter function has a finite variation norm. The loss-based dissimilarity is often equivalent with the square of an $L^2(P_0)$-type norm. In this article, we establish that under some weak continuity condition, the HAL-estimator is also uniformly consistent.

Keywords: Cadlag, cross-validation, empirical risk, Highly-Adaptive-Lasso estimator, loss-function, oracle inequality, variation norm.

1 Introduction

Let $O \sim P_0 \in \mathcal{M}$ and $\Psi : \mathcal{M} \rightarrow \mathbf{\Psi}$ be an infinite dimensional target parameter of interest, where $\mathbf{\Psi} = \{\Psi(P) : P \in \mathcal{M}\}$ is the parameter space of Ψ. The estimand is

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thus given by $\psi_0 = \Psi(P_0)$. We observe n i.i.d. copies of O. We assume there exists a loss function $L(\psi)(O)$ such that $P_0 L(\psi_0) = \min_{\psi \in \mathbf{\Psi}} P_0 L(\psi)$. We assume that the loss function is uniformly bounded:

$$\sup_{\psi \in \mathbf{\Psi}} \sup_{o} \mid L(\psi)(o) \mid < \infty. \tag{1}$$

In the case that the loss-based dissimilarity $d_0(\psi, \psi_0) \equiv P_0 L(\psi) - P_0 L(\psi_0)$ is quadratic, we often also assume

$$\sup_{\psi \in \mathbf{\Psi}} \frac{P_0 (L(\psi) - L(\psi_0))^2}{P_0 L(\psi) - P_0 L(\psi_0)} < \infty. \tag{2}$$

We assume that the parameter space $\mathbf{\Psi}$ is a subset of d-variate real valued cadlag functions $D[0, \tau]$ on a cube $[0, \tau] \subset \mathbb{R}^d_{\geq 0}$. A function in $D[0, \tau]$ is right-continuous with left-hand limits, and we also assume that it is left-continuous at any point on the right-edge of $[0, \tau]$: so if $x_j = \tau_j$ for some $j \in \{1, \ldots, d\}$, then we assume that ψ is continuous at such an x. We also assume that each function ψ in the parameter space $\mathbf{\Psi}$ has a uniform sectional variation norm bounded by some universal $M < \infty$, but one can also select M with cross-validation to avoid this assumption $\sup_{\psi \in \mathbf{\Psi}} \parallel \psi \parallel_v < \infty$ (see van der Laan (2015)), in which case we only need to assume that the variation norm of each single ψ is finite. We define the uniform sectional variation norm of a multivariate real valued cadlag function ψ as

$$\parallel \psi \parallel_v = \psi(0) + \sum_{s \subset \{1, \ldots, d\}} \int_{0_s}^{\tau_s} \mid \psi_s(du_s) \mid,$$

where the sum is over all subsets s of $\{1, \ldots, d\}$; for a given subset s, we define $u_s = (u_j : j \in s)$, $u_{-s} = (u_j : j \notin s)$; and we define the section $\psi_s(u_s) \equiv \psi(u_s, 0_{-s})$ that sets the components in the complement of s equal to zero. Any cadlag function that has a bounded variation norm generates a finite measure so that integrals w.r..t this function are well defined. We also assume that for each $\psi \in \mathbf{\Psi}$, $O \rightarrow L(\psi)(O)$ is a d_1-variate cadlag function on a compact support $[0, \tau_1] \subset \mathbb{R}^{d_1}_{\geq 0}$ with universally bounded variation norm:

$$\sup_{\psi \in \mathbf{\Psi}} \parallel \psi \parallel_v < \infty. \tag{3}$$

If $O = (B, O_1)$ for a discrete variable $B \in \{1, \ldots, K\}$ and continuous component O_1, then one only needs to assume this for $O_1 \rightarrow L(\psi)(b, O_1)$ for each b.

Consider the following estimator $\hat{\Psi} : \mathcal{M}_{np} \rightarrow \mathbf{\Psi}$ defined by

$$\hat{\Psi}(P_n) = \arg\min_{\psi \in \mathbf{\Psi}} P_n L(\psi). \tag{4}$$

In van der Laan (2015) we proved that this estimator converges in loss-based dissimilarity at a rate faster than $n^{-1/2}$ to its true counterpart:

$$d_0(\psi_n, \psi_0) = P_0 L(\psi_n) - P_0 L(\psi_0) = O_P(n^{-1/2 - \alpha(d)}), \tag{5}$$

where $\alpha(d) > 0$ is a specified number that behaves in the worst case as $1/d$. The worst case corresponds with $\mathbf{\Psi} = \mathbf{\Psi}_{NP} \equiv \{\psi \in D[0,\tau] : \parallel \psi \parallel_v < M\}$ being equal to the set of cadlag functions with variation norm bounded by M, while this rate will be better for smaller parameter spaces $\mathbf{\Psi}$ and can be expressed in terms of the entropy of $\mathbf{\Psi}$. For the case that the parameter space equals the nonparametric parameter space $\mathbf{\Psi}_{NP}$, this estimator can be defined as the minimizer of the empirical risk $P_n L(\psi)$ over a linear combination of around $n2^{d-1}$ indicator basis functions under the constrained that the sum of the absolute value of its coefficients is bounded by M. This is shown by using the following representation of a function $\psi \in D[0,\tau]$ with $\parallel \psi \parallel_v < \infty$:

$$\psi(x) = \psi(0) + \sum_{s \subset \{1,\ldots,d\}} \int_{0_s}^{x_s} d\psi_s(u_s).$$

This representation shows that ψ can be represented as an infinite linear combination of indicators $x_s \rightarrow I(u_s \leq x_s)$ indexed by a cut-off u_s and subset s, where the sum of the absolute values of the "coefficients" $d\psi_s(u_s)$ equals $\parallel \psi \parallel_v$. This motivated us to name it the Highly Adaptive Lasso (HAL) estimator, and indeed in the case of a squared error or log-likelihood loss for binary outcomes it reduces to the standard Lasso regression estimator as implemented in standard software packages, but where one runs it with a possibly enormous amount of basis functions.

For example, for the squared error loss and $\psi_0 = E_{P_0}(Y \mid W)$ being a regression function, $d_0(\psi, \psi_0) = P_0(\psi - \psi_0)^2$ is the square of the $L^2(P_0)$-norm. Thus, our general convergence result will typically imply convergence in an $L^2(P_0)$ or Kullback-Leibler norm. In this article we are concerned with showing that this general HAL-estimator is also uniformly consistent under certain additional smoothness conditions. Let $\parallel \psi \parallel_\infty = \sup_{x \in [0,\tau]} \mid \psi(x) \mid$ be the supremum norm. We want to prove that

$$\parallel \psi_n - \psi_0 \parallel_\infty \rightarrow_p 0. \tag{6}$$

The case that the observed data has a discrete and continuous component Before we proceed we demonstrate how one can apply our results to a setting in which ψ_0 is a function of a purely discrete component B and continuous component. Suppose that $O = (B, O_1)$, where B is discrete with finite number of values $\{1, \ldots, K\}$, and $\psi = (\psi_b : b = 1, \ldots, K)$, where the components ψ_b are variation independent so that $\mathbf{\Psi} = \prod_{b=1}^K \mathbf{\Psi}_b$ with $\mathbf{\Psi}_b$ being the parameter space of $\Psi_b : \mathcal{M} \rightarrow \mathbf{\Psi}_B$. One now assumes that for each b $\mathbf{\Psi}_b$ is a subset of d_b-dimensional cadlag functions with variation norm smaller than some $M_b < \infty$. We have $d_0(\psi_n, \psi_0) = \sum_{b=1}^K \int \{L(\psi_n)(b, o_1) - L(\psi_0)(b, o_1)\} dP_0(b, o_1)$. Suppose that $L(\psi)(b, o_1)$ only depends on ψ through a ψ_b and suppose that $\psi = (\psi_b : b = 1, \ldots, K)$ is a variation independent parameterization. Then, $\psi_{0,b}$ is the minimizer of $\psi \rightarrow P_0 L_b(\psi)$ where $L_b(\psi)(O_1) = I(B = b) L(\psi)(b, O_1)$, and $\psi_{n,b} = \arg\min_{\mathbf{\Psi}_b} P_0 L_b(\psi)$. In addition, $d_0(\psi_n, \psi_0) = \sum_b d_{0,b}(\psi_{n,b}, \psi_{0,b})$, where $d_{0,b}(\psi_b, \psi_{0,b}) = P_0 L_b(\psi_b) - P_0 L_b(\psi_{0,b})$. Thus the estimator ψ_n above can then be analyzed separately as an estimator $\psi_{n,b}$ for

$\psi_{0,b}$ for each b. In particular, the rate of convergence result above now applies to each $\psi_{n,b}$ with dimension d replaced by d_b and loss function $L_b(\psi)$. Our goal is then reduced to establishing that $\psi_{n,b} - \psi_{0,b}$ converges uniformly to zero in probability. In the sequel we suppress this index b, but the reader needs to know that in such applications we simply apply our results to $\psi_{0,b}$ and $\psi_{n,b}$ with loss function $L_b(\psi_b)$, for each b separately. In order to establish our uniform consistency result, we will assume that each $P_0(B = b, \cdot)$ is a continuous measure for O_1, which corresponds with the stated assumption **A2** below that P_0 is continuous on the support of L_b.

To establish the uniform consistency we will make the following assumptions:

A0 : $d_0(\psi_n, \psi_0) = o_P(1)$ and the loss function is uniformly bounded (1).

A1 : $d_0(\psi, \psi_0) = 0$ implies $\parallel \psi - \psi_0 \parallel_{P_0} = 0$.

A2 : ψ_0 is continuous on $[0, \tau]$, and P_0 is continuous measure on the set of o-values for which $\sup_\psi \mid L(\psi)(o) \mid > 0$.

A3 : If ψ_n converges pointwise to $\psi_\infty \in \mathbf{\Psi}$ on $[0, \tau]$ at each continuity point of $\psi_\infty \in \mathbf{\Psi}$, then $L(\psi_n)$ converges pointwise to $L(\psi_\infty)$ on a support of P_0.

Regarding assumption **A0**, above we provided sufficient assumptions that even guarantee $d_0(\psi_n, \psi_0) = O_P(n^{-1/2-\alpha(d)})$, which could thus easily be weakened, as long as we keep assuming that the loss function is uniformly bounded. Assumption **A1** is a very weak assumption. Regarding assumption **A3**, since P_0 is continuous by **A2**, one only needs to show that $L(\psi_n)$ converges to $L(\psi_\infty)$ on a set that can exclude any finite or countable set. Since the number of discontinuity points of ψ_∞ is finite or countable, the lack of convergence of ψ_n at these points should not be an issue.

We have the following theorem.

Theorem 1 *Let ψ_n be the HAL-estimator defined by (4). Assume **A0, A1, A2** and **A3**. Then,* $\sup_{x \in [0,\tau]} \mid \psi_n(x) - \psi_0(x) \mid \to 0$ *in probability as $n \to \infty$.*

2 Proof of Theorem 1

Using that $\sup_{\psi \in \mathbf{\Psi}} \sup_o \mid L(\psi)(o) \mid < \infty$, the dominated convergence theorem combined with **A3** proves the following lemma.

Lemma 1 *Assume **A0** and **A3**. If ψ_n converges pointwise to $\psi_\infty \in \mathbf{\Psi}$ on $[0, \tau]$ at each continuity point of ψ_∞, then $P_0 L(\psi_n) - P_0 L(\psi_\infty) \to 0$.*

The following lemma proves that if $d_0(\psi, \psi_0) = 0$, then ψ equals ψ_0 pointwise as well.

Lemma 2 *Assume **A1**. If $d_0(\psi, \psi_0) = 0$ for a $\psi, \psi_0 \in D[0, \tau]$, then $\parallel \psi - \psi_0 \parallel_\infty = 0$.*

Proof: Assume $d_0(\psi,\psi_0)=0$. Suppose that $\psi-\psi_0>0$ (same for <0) at a point $x\in[0,\tau)$, then it will also be larger than 0 at a small neighborhood $[x,x+\delta)$ for some $\delta>0$ due to the right-continuity of $\psi-\psi_0$. As a consequence, if $\psi-\psi_0>0$ at a point x, then $\parallel \psi-\psi_0 \parallel_{P_0}>0$. By assumption **A1** this implies that $d_0(\psi,\psi_0)>0$. Finally, if $x\in[0,\tau)^c\subset[0,\tau]$, then we assumed that ψ,ψ_0 are left-continuous, so that the same argument applies if we assume that $\psi-\psi_0>0$ at an x on the right-edge of $[0,\tau]$. This proves that $d_0(\psi,\psi_0)=0$ implies that $\psi-\psi_0=0$ on $[0,\tau]$. $\Box$

The following lemma establishes that our parameter space ψ is weakly compact so that each sequence has a weakly converging (i.e., poinwise) subsequence. In addition, if we also assume that the sequence is consistent for ψ_0, then the limit of this weakly converging subsequence has to equal ψ_0 as well.

Lemma 3 *Assume **A0**, **A1**, **A2**, and **A3**. Any sequence $(\psi_n:n=1,\ldots)$ in $\mathbf{\Psi}$ has a subsequence $(\psi_{n(k)}:k=1,\ldots)$ so that there exists a $\psi_\infty\in\mathbf{\Psi}$ and $\psi_{n(k)}$ converges pointwise to ψ_∞ at each continuity point of ψ_∞.*

If we also know that $d_0(\psi_n,\psi_0)\rightarrow 0$, then we have that $\parallel \psi_\infty-\psi_0 \parallel_\infty=0$.

Proof: By Hildebrandt (1963) (see also lemma 1.2 in van der Laan (1993)), any cadlag function of bounded variation can be represented as a difference of two monotone cadlag functions generating positive finite measures, i.e. the analogue of cumulative distributions functions but not bounded by $[0,1]$. Thus $\psi_n=F_n-G_n$ for monotone increasing functions $F_n,G_n\in D[0,\tau]$. Any sequence $(F_n:n)$ of cumulative distribution functions has a subsequence that converges weakly to a limit F_∞, and similarly, any sequence $(G_n:n)$ has a subsequence that converges weakly to a limit G_∞, where weak convergence is equivalent with pointwise convergence at each continuity point of the limit. This shows that we can find a subsequence $(F_{n(k)}-G_{n(k)}:k)$ of $(F_n-G_n:n)$ and limit $\psi_\infty=F_\infty-G_\infty$ so that $F_{n(k)}-G_{n(k)}$ converges pointwise to $F_\infty-G_\infty$ at each point in which both F_∞ and G_∞ are continuous. We now want to show that the points at which ψ_∞ are continuous are equal to the point at which both F_∞ and G_∞ are continuous. By the Hahn decomposition theorem both F_∞ and G_∞ are the sum of a continuous measure and purely discrete measure. The continuous measure corresponds with a continuous function. The discrete support of F_∞ and G_∞ has to be disjoint since if a measure assigns at a point both a negative and positive mass then we can replace that by just assigning a single mass that is either positive or negative. Thus we have shown that $(\psi_{n(k)}:k)$ converges pointwise to ψ_∞ at each continuity point of ψ_∞.

Consider now the second statement in the lemma. Suppose now that we also know that $d_0(\psi_n,\psi_0)\rightarrow_p 0$. Then we also have $d_0(\psi_{n(k)},\psi_0)\rightarrow 0$. By Lemma 1, the fact that $\psi_{n(k)}$ converges pointwise to ψ_∞ at each continuity point of ψ_∞ implies that $P_0L(\psi_{n(k)})-P_0L(\psi_\infty)\rightarrow 0$. Now use that $d_0(\psi_{n(k)},\psi_0)=P_0L(\psi_{n(k)})-P_0L(\psi_\infty)+d_0(\psi_\infty,\psi_0)$. Since the left-hand side converge to zero, and the first term on the right-hand side converges to zero as well, this implies that $d_0(\psi_\infty,\psi_0)=0$.

By Lemma 2, this implies that $\| \psi_\infty - \psi_0 \|_\infty = 0$. This completes the proof of the lemma. □

Consider our HAL-estimator ψ_n. Given $d_0(\psi_n, psi_0) \rightarrow_p 0$, Lemma 3 proves that ψ_n converges pointwise to ψ_0 at each point in $[0,\tau]$, where ψ_0 is continuous. Thus, we have translated the consistency of ψ_n w.r.t. loss-based dissimilarity into pointwise convergence.

Lemma 4 *Let ψ_n be the HAL-estimator defined by (4). Assume **A0**, **A1**, **A2** and **A3**. Then, $\psi_n(x) - \psi_0(x) \rightarrow_p 0$ at each $x \in [0,\tau]$. More generally, we have $\psi_n = F_n - G_n$ for F_n, G_n that generate positive uniformly finite measures, $\psi_0 = F_0 - G_0$ for F_0, G_0 that generates finite positive measures, and $F_n(x) - F_0(x) \rightarrow_p 0$ and $G_n(x) - G_0(x) \rightarrow 0$ for each $x \in [0,\tau]$.*

So we have shown $\psi_n = F_n - G_n$, $\psi_0 = F_0 - G_0$, where F_n, G_n converge pointwise to F_0, G_0 at each point in $[0,\tau]$. Finally, we establish that the pointwise convergence of F_n (G_n) to a continuous F_0 (G_0) implies uniform convergence, thereby showing that ψ_n converges uniformly to ψ_0 as well.

Lemma 5 *If F_n is a sequence of cadlag functions that generate a positive measure on $[0,\tau]$, $F_n(x) \rightarrow F_0(x)$ for each $x \in [0,\tau]$, and F_0 is continuous on $[0,\tau]$, then $\| F_n - F_0 \|_\infty \rightarrow 0$.*

Proof: Let $\epsilon > 0$.

By Heine's theorem, since F_0 is continuous on the compact set $[0,\tau]$, it is uniformly continuous on $[0,\tau]$.

By uniform continuity of F_0, there exists $\eta > 0$ such that for any $x, y \in [0,\tau]$, $\|x - y\| < \eta$ implies $|F_0(x) - F_0(y)| < \epsilon$. Consider a grid on $[0,\tau]$ with grid points $x_{\mathbf{i}} \equiv (i_1\eta, ..., i_d\eta)$.

Consider an arbitrary $x \in [0,\tau]$. For a certain $\mathbf{i} \in \mathbb{N}^d$, x falls in the hypercube $[x_{\mathbf{i}}, x_{\mathbf{i}\,+\,\mathbf{1}}]$, where $\mathbf{1} \equiv (1, ..., 1)$.

Since F_0 and F_n generate positive measures,

$$F_n(x_{\mathbf{i}}) - F_0(x_{\mathbf{i+1}}) \leq F_n(x) - F_0(x) \leq F_n(x_{\mathbf{i+1}}) - F_0(x_{\mathbf{i}}). \tag{7}$$

Observe that

$$F_n(x_{\mathbf{i+1}}) - F_0(x_{\mathbf{i}}) = (F_n(x_{\mathbf{i+1}}) - F_0(x_{\mathbf{i+1}})) + F_0(x_{\mathbf{i+1}}) - F_0(x_{\mathbf{i+1}})). \tag{8}$$

Since $F_n(x_{\mathbf{j}}) - F_0(x_{\mathbf{j}})$ converges to zero for all the $x_{\mathbf{j}}$'s in $[0,\tau]$, and since there are a finite number of such $x_{\mathbf{j}}$'s, there exists $n_0 > 0$ such that for all $x_{\mathbf{j}} \in [0,\tau]$, $n > n_0$, $|F_n(x_{\mathbf{j}}) - F_0(x_{\mathbf{j}})| < \frac{\epsilon}{2}$.

Therefore, going back to (8) and using this latter fact and the uniform continuity, we have $F_n(x_{\mathbf{i+1}}) - F_0(x_{\mathbf{i}}) \leq \frac{\epsilon}{2}$ for any $n > n_0$.

Since we can apply the exact same arguments to the lower bound in (7), we have that for $n > n_0$,

$$-\frac{\epsilon}{2} \leq F_n(x) - F_0(x) \leq \frac{\epsilon}{2}. \tag{9}$$

Since n_0 does not depend on x, we have proved uniform convergence of F_n to F_0 over $[0, \tau]$. $\square$

References

Theophil Henry Hildebrandt. *Introduction to the theory of integration.* Academic Press, 1963.

Mark J. van der Laan. *Efficient and Inefficient Estimation in Semiparametric Models.* PhD thesis, 1993.

Mark J. van der Laan. A generally efficient targeted minimum loss based estimator. *U.C. Berkeley Division of Biostatistics Working Paper Series*, 2015.