Chapter VII

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Summary

1 Ex. 7.3

Let $\hat{\mathbf{f}} = \mathbf{S}\mathbf{y}$ be a linear smoothing of $\mathbf{y}$.

(a) If S_{ii} is the ith diagonal element of $\mathbf{S}$, showing that for $\mathbf{S}$ arising from least squares projections and cubic smoothing splines, the cross-validated residual can be written by as

$$y_i - \hat{f}^{-i}(x_i) = \frac{y_i - \hat{f}(x_i)}{1 - S_{ii}} \tag{1}$$

(b) Use this result to show that $|y - \hat{f}^{-i}(x_i)| \geq |y_i - \hat{f}(x_i)|$.

(c) Find general conditions on any smoother $\mathbf{S}$ to make result (7.64) hold.

Proof: **(a)** Denote $A = X^T X$; for the least square with the i^{th} data points omitted:

$$\hat{f}^{-i}(x_i) = x_i^T((X^{-i})^T X^{-i})^{-1}(X^{-i})^T y^{-i} \tag{2}$$

It is true that $X^T X = \Sigma_{j=1}^{j=n} x_i x_i^T$, and $(X^{-i})^T X^{-i} = X^T X - x_i x$. And it is obvious that $(X^{-i})^T y^{-i} = X^T y - x_i y_i$. Finally, (2) can be simplified as following:

$$\hat{f}^{-i}(x_i) = x_i^T(A - x_i x_i^T)^{-1}(X^T y - x_i y_i) \tag{3}$$

There exists a tricks when simplifing (3). In mathematics, the *Sherman-Morrison formula* (see `https://en.wikipedia.org/wiki/Sherman-Morrison_formula`) computes the inverse of the sun of an invertible matrix A and the outer product, uv^T:

$$(A - x_i x_i^T)^{-1} = A^{-1} + \frac{A^{-1} x_i x_i^T A^{-1}}{1 - x_i^T A^{-1} x_i} \tag{4}$$

Since $X^T \in \Re^{pn}$ equals to $[x_1, ..., x_i, ..., x_n]$, we can write x_i^T as $\delta_i^T X (\delta_i$ is a vector $\delta \in \Re^n$ where all elements expect the i^{th} component valued as 1 is zero, timely $\delta_i^T = [0, ..., 1^i, ..., 0])$. We denote:

$$\begin{aligned} x_i^T A^{-1} x_i &= \delta_i^T X A^{-1} X^T \delta_i && (5)\\ &= \delta_i^T S \delta_i && (6)\\ &= S_{ii} && (7) \end{aligned}$$

where $S = X(X^T X)^{-1} X^T$. To simplify (3), we substitute (4) (7) into this equation and we get:

$$\begin{aligned} \hat{f}^{-i}(x_i) &= x_i^T [A^{-1} + \frac{A^-1 x_i x_i^T A^{-1}}{1 - S_{ii}}] && (8)\\ &= x_i^T A^{-1} X^T y - x_i^T A^{-1} x_i y_i + \frac{x_i^T A^{-1} x_i (x_i^T A^{-1} X^T y) - (x_i^T A^{-1} x_i)^2 y_i}{1 - S_{ii}} && (9)\\ &= \hat{f}(x_i) - S_{ii} y_i + \frac{S_{ii} \hat{f}(x_i) - S_{ii}^2 y_i}{1 - S_{ii}} && (10)\\ &= \frac{\hat{f}(x_i) - y_i S_{ii} - \hat{f}(x_i) S_{ii} + y_i S_{ii}^2 + \hat{f}(x_i) S_{ii} - y_i S_{ii}^2}{1 - S_{ii}} && (11)\\ &= \frac{\hat{f}(x_i) - y_i S_{ii}}{1 - S_{ii}} && (12)\\ &= \frac{\hat{f}(x_i) - y_i + y_i(1 - S_{ii})}{1 - S_{ii}} && (13)\\ &= y_i - \frac{y_i - \hat{f}(x_i)}{1 - S_{ii}} && (14) \end{aligned}$$

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List of Listings

Adding the comma-separated parameter caption=Python example inside the brackets, enables the caption. This caption can be later used in the list of Listings.

Listings