probability space (Ω, F, P)

Ω is collection of elementary event

F is collection of subsets of Ω, called random events

P is function on F with values in $[0,1]$

$\Omega \in F$

if $A \in F$, then $\Omega - A \in F$

If $A_1, A_2 \cdots A_n$, then $\bigcup_{n=1}^{\infty} A_n \in F$

$A \in F$, $B \in F$ $\Rightarrow A \cap B \in F$

$\Rightarrow A - B \in F$

F is called a σ-alphy

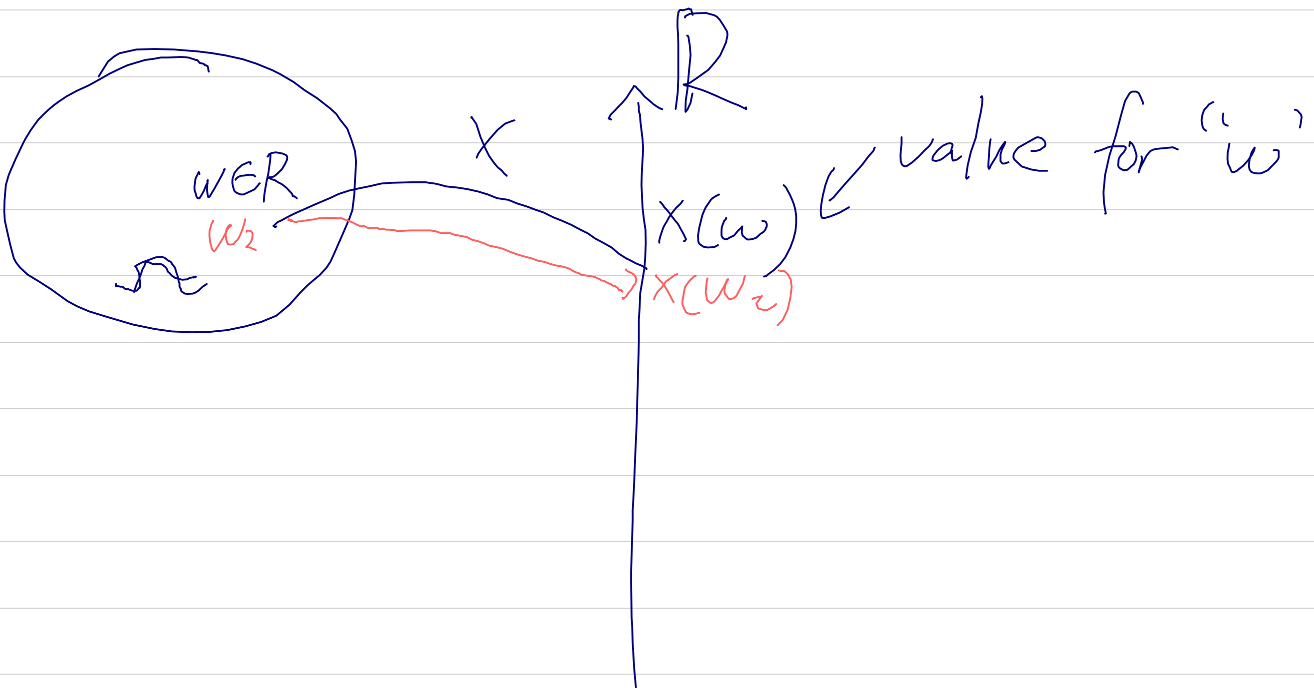
If $A_1, A_2, \cdots, A_n \cdots \in F$ and $A_i \cap A_j = \emptyset$ for all i and j

$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ (没有交集，并=加)

many random event

Get Expected value

Random variable $X : \Omega \to \mathbb{R}$



X is F measuriable function

For all $a \in b$

$\{\omega ; a < X(\omega) \le b\}$

is element of F

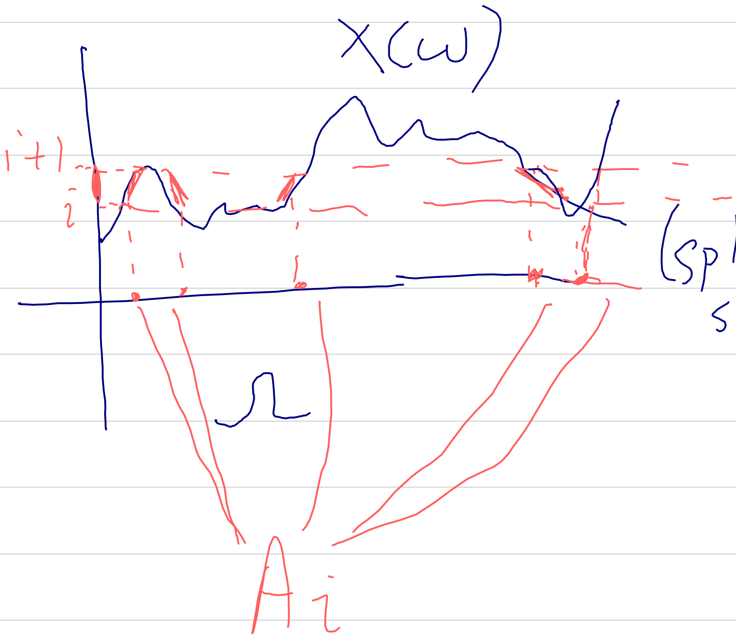
Expected value of X_i

$$E(X) \triangleq \int_\Omega X(\omega) P(d\omega)$$ Lebesgue Integral

Step 1: Consider $X \geq 0$

$$A_i = \left\{ \omega \in \Omega \quad \frac{i}{2^n} < X(\omega) \leq \frac{i+1}{2^n} \right\}$$

$i = 0, 1, \cdots, n2^n - 1$



(split into small pieces)
split y-axis)

$$A_{n \cdot 2^n} = \{ \omega \in \Omega ; X(\omega) \geq n \}$$

$$X^{(n)}_{(i)} = \begin{cases} \frac{i+1}{2^n} & \text{if } \omega \in A_i \\ n & \text{if } \omega \in A_{n \cdot 2^n} \end{cases}$$

$n \to \infty$

$$E(X^{(n)}) = \sum_{i=0}^{n \cdot 2^n - 1} \frac{i+1}{2^n} P[A_i] + \cdots + n P[A_{n \cdot 2^n}]$$

$$\begin{cases} X_+ = \max(0, X) \geq 0 \\ \quad \text{if } X \text{ negitive} \rightarrow \text{input } 0 \\ X_- = \max(0, -X) \geq 0 \end{cases}$$

$$X = X_+ - X_-$$

$$E[X] = E[X_+] - E[X_-]$$

Distribution Function X:

$$F(a) = P[\underbrace{w \in \Omega, \overset{\text{random select}}{\quad} X(w) \leq a}_{A}]$$

$a \leq b$

$$F(b) = P[\underbrace{w \in \Omega, \ X(w) \leq b}_{B}] \geq F(a)$$

$i = 0, 1, \ldots n2^n - 1$

$$A_i = \{w \in \Omega, \ \frac{i}{2^n} \leq X(w) \leq \frac{i+1}{2^n}\}$$

$$E[X^{(n)}] = \sum_{i=0}^{n2^n-1} \frac{i+1}{2^n} P[A_i] + \cdots + nP(A_{n2^n})$$

$$\| $$

$$\left[F\left(\frac{i+1}{2^n}\right) - F\left(\frac{i}{2^n}\right)\right]$$

$dF(x)$

$A \subset B$

$B = A \cup (B - A)$

$P[B] = P[A] + P[B-A]$

$\Rightarrow P(B) \geq P[A]$

$i \leq n2^n$

$$E[X] = \int_0^\infty X\,dF(x)$$

Stieltjies integral

If $f(x) = \frac{df(x)}{f(x)}$ exists

$$E(x) = \int x f(x)\,dx$$

Possibality density Function

$$E(x) = \int_\Omega X(\omega)\,P(d\omega)$$

$$E(x+y) = E(x) + E(y)$$

$$E(\alpha X) = \alpha E(x)$$

Conditional Expected Value

(Ω, F, P) — probality space $X : \Omega \to R$

$E(x)$ exists

$\mathcal{G}$ — σ-alphra $\mathcal{G} \in F$

If $A \in \mathcal{G}$, then I know whether $\Omega \in A$ or not

If $B \in F - \mathcal{G}$ then I do not know whether $\omega \in B$

$E(X|\mathcal{G})$: condition expected values $\omega' \in \mathcal{G}$

Y is a random variable

it is measurable with respect to $\mathcal{G}$

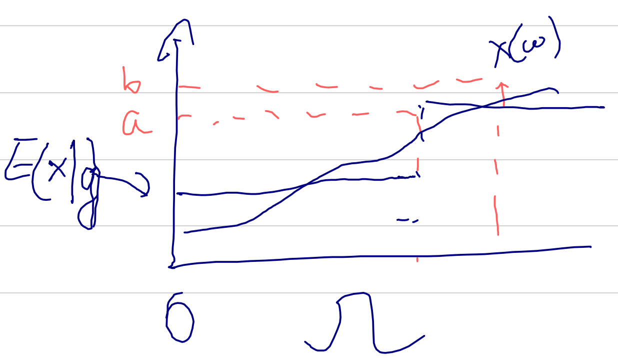
$\{\omega \in \Omega ;\ a \le Y(\omega) \le b\} \in \mathcal{G}$ for all $a \le b$

$$\int_A Y(\omega) P(d\omega) = \int_A X(\omega) P(d\omega) \quad \text{for all } A \in \mathcal{G}$$

Simple case : $\mathcal{G} = \{\phi, \Omega\} \Rightarrow Y$ is a constant

$$\int_\Omega \underset{\text{(constant)}}{Y} P(d\omega) = \int_\Omega X(\omega) P(d\omega)$$

$Y = E(X)$



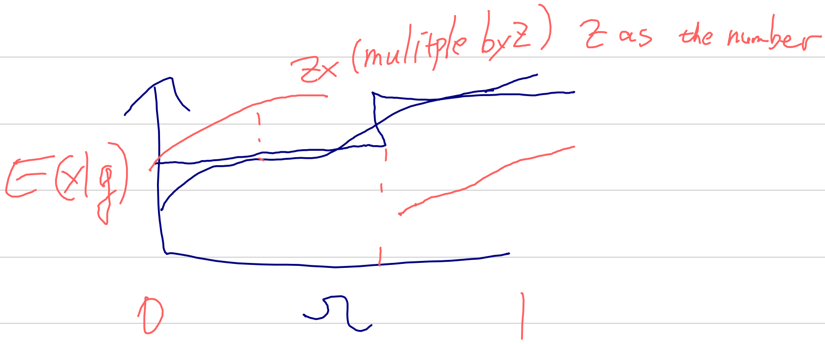
F contains all intervals $[a,b]$ and this conditional unions/mixation

$\mathcal{G} = \{\phi, \Omega, [0, \frac{1}{2}), [\frac{1}{2}, 1]\}$

$E[\alpha X + \beta Z | \mathcal{G}] = \alpha E[X|\mathcal{G}] + \beta E[Z|\mathcal{G}]$

If Z is $\mathcal{G}$ measurable

$E[ZX|\mathcal{G}] = Z E(X|\mathcal{G})$, we take Z as number



If $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$

$E[X|\mathcal{G}_1] = E\{E[X|\mathcal{G}_2] | \mathcal{G}_1\}$ tower property

Example : $\mathcal{G}_1 = \{\varphi, \Omega\}$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\mathcal{G} \subset \mathcal{F}$

event

$P(A|\mathcal{G})$ for all $A \in \mathcal{F}$

$X_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$ $P[A|\mathcal{G}] \sim E(X_A|\mathcal{G})$

Suppose $A \in \mathcal{G}$ A is in $\mathcal{G}$ $\chi_A = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$

$$P[A|\mathcal{G}] \triangleq E[\chi_A|\mathcal{G}]$$

$P(A|\mathcal{G}](\omega)$

χ_A is measurable of $\mathcal{G}$

in this case (A is in $\mathcal{G}$) $P[A|\mathcal{G}] = \chi_A$

A is known

If A is not in $\mathcal{G}$

$A \notin \mathcal{G}$

$P(A|\mathcal{G}](\omega)$

$A \in \mathcal{F}$ is probability measurement

$(\Omega, \mathcal{F}, P)$ $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$ Filtration

$X_1, X_2, \ldots, X_n, \ldots$ random variables

each X_n is $\mathcal{F}_n$ measurable

adopted to the filtroable

$X_n \to X$ almost surely (a.s) if

$P(\omega \in \Omega, X_n(\omega) \to X(\omega)) = 1$

Convergence in r/e with mean

$X_n \to X$ in the with mean

$$\lim_{n \to \infty} E[|X_n - X|^r] = 0$$

Lebesgue Process

If $|X_n| \le y$ and $E(y) < \infty$

then $\lim_{n \to \infty} E(X_n) = E(\lim_{n \to \infty} X_n)$ if (a.s.) (exchange)

X

Convert to