2 - 5

asset $1, 2, \cdots n$ Current Prices $S_1, S_2, \cdots S_n$

Scenarios $1, 2, \cdots m$

Matrix of future prices (after 1 period)

$$A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{\text{assets}} \text{ scenarios}$$

a_{ij} - price of asset j in scenario i

x_j — # of units of asset j "bought"

$j = 1, 2 \cdots n$ $x_j > 0$ long

$x_j < 0$ short

$\langle S, x \rangle = S^T x$

Cost today: $S^1 x_1 + S^2 x_2 + \cdots + S^n x_n$ (Price $\times$ number of asset)

Value in Scenario i: $a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n$, $i = 1, \cdots n$

if buy negative value asset $\longrightarrow$ T long, $\rightarrow$ get non-negative value

NO arbitrage:

for scenario ≥ 0, $Ax \geq 0$, then $S^T x \geq 0$

$$a_{11}x_1 + \cdots a_{1n}x_n \geq 0$$
$$\vdots \qquad \qquad \vdots$$
$$a_{m1}x_1 + \cdots a_{mn}x_n \geq 0$$

$$q_1 \geq 0$$
$$\vdots$$
$$q_m \geq 0$$

$$s^1 x_1 + \cdots + s^n x_n \geq 0$$

FARKAS LEMAS : $q_1 \geq 0, \cdots q_m \geq 0$ exist,

then $s^1 = q_1 a_{11} + q_2 a_{21} + \cdots + q_m a_{m1}$

$s^2 = q_1 a_{12} + q_2 a_{22} + \cdots + q_m a_{m2}$

$\vdots$

$s^n = q_1 a_{1n} + q_2 a_{2n} + \cdots + q_m a_{mn}$

if $s^1 = 1$, $\longrightarrow 1 = q_1 + q_2 + \cdots + q_m$

$$A = \begin{bmatrix} a_{11} & - & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{m1} & - & - & a_{mn} \end{bmatrix}$$

all 1

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if no arbitrage, S_1 is riskless

$S = E^Q[a]$ is the Expected Future value of a

Two assets

Stock

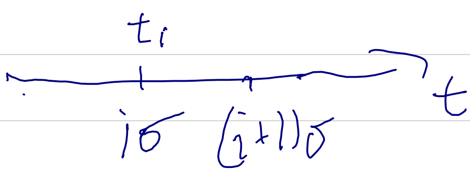
price at time i: S_i

Scenior of tim $i+1$ $\begin{cases} \nearrow S_{i+1}(u) \\ \searrow S_{i+1}(d) \end{cases}$

Bond

value at i: $B_i = 1$

Senarios at $i+1$ $\begin{cases} B_i e^{r\delta} \\ B_i e^{r\delta} \end{cases}$



$r > 0$ int-rate

$\delta > 0$ time measured from i to $i+1$

Discount Price

$$\tilde{S}_{i+1} = e^{-r\delta} S_{i+1} \begin{cases} e^{-r\delta} S_{i+1}(u) \\ e^{-r\delta} S_{i+1}(d) \end{cases}$$

$$\tilde{B}_{i+1} = e^{-r\delta} B_{i+1} = 1$$

$$A = \boxed{\begin{matrix} 1 & e^{-r\delta} S_{i+1}(u) \\ 1 & e^{-r\delta} S_{i+1}(d) \end{matrix}}$$

$\quad 1 \qquad S_i$

$q_1 > 0 \qquad q_1 + q_2 = 1$

$q_2 > 0$

$$S_i = q_1 e^{-r\delta} S_{i+1}(u) + q_2 e^{-r\delta} S_{i+1}(d)$$

$$= q e^{-r\delta} S_{i+1}(u) + (1-q) e^{-r\delta} S_{i+1}(d)$$

q : Prob of upward

$U > 1$, multiplier

$S_{i+1}(u) = u \cdot S_i$

$S_{i+1}(d) = \frac{1}{U} S_i$

$S_i = e^{-r\delta}\left[q \cdot u S_i + (1-q)\frac{1}{U} S_i\right]$

$1 = e^{-rd}\left[q \cdot n + (1-q)\frac{1}{U}\right]$

$e^{r\delta} = q\left(u - \frac{1}{U}\right) + \frac{1}{U}$

$q = \dfrac{e^{r\delta} - \frac{1}{U}}{U - \frac{1}{U}}$

$\boxed{\dfrac{e^{\sigma r} - d}{U - d} \qquad d = \frac{1}{U}}$

Binary tree

$e^{r\delta} \le U$

Another tradatible asset

$V_i \to V_{i+1}(u)$, $V_i \to V_{i+1}(d)$

discount ns

$\tilde{V}_{i+1}(u) = e^{-r\delta} V_{i+1}(u)$

$\tilde{V}_{i+1}(d) = e^{-r\delta} V_{i+1}(d)$

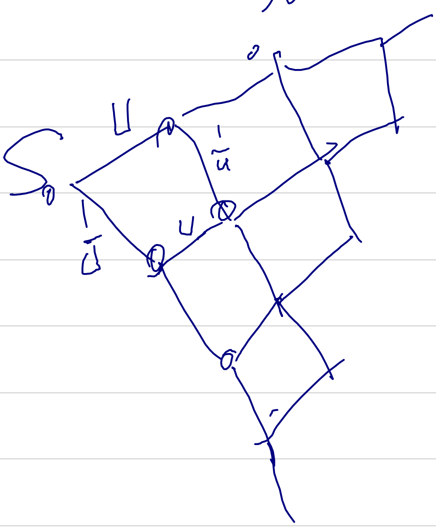
From Farkas Lamma :

$V_i = q\,\tilde{V}_{i+1}(u) + (1-q)\,\tilde{V}_{i+1}(d)$

Martingale Model

$$S_{i+1} = \begin{cases} S_i \cdot u \\ S_i \frac{1}{u} \end{cases}$$

$$B_{i+1} = B_i e^{r\delta} \qquad i = 0, 1, \dots, N-1$$



stock price $\quad \tilde{S}_i = S_i e^{-ir\delta} \qquad \tilde{B}_i = e^{-ri\delta} B_i$

$$S_{i+1} = \begin{cases} \tilde{S}_i e^{-r\delta} \cdot u \\ \tilde{S}_i e^{-r\delta} \cdot \frac{1}{u} \end{cases}$$

$$\tilde{S}_i = q \cdot \overbrace{\tilde{S}_i e^{-r\delta} u}^{\tilde{S}_{i+1}(u)} + (1-q) \overbrace{\hat{S}_i e^{-r\delta} \frac{1}{u}}^{\tilde{S}_{i+1}(d)}$$

at every node at time i

q : Prob

$\tilde{B}_i = 1$

$\mathcal{F}_i$ - all events happened at $t=0, \cdots i$

$$\underbrace{\tilde{S}_i = E^Q[S_{i+1} | F_i]}_{\text{Martingle with respect to } Q} \quad i = 0, 1, 2 \cdots, n-1$$

Dynamic Portfolio: ϕ_{i+1} stocks from i to $i+1$

$\underline{\psi_{i+1} \text{ bonds from } i \text{ to } i+1}$

must be determined of time i

W_0 - initial value

Nominal value of i: $\phi_{i+1} S_i + \psi_{i+1} B_i = W_i$

- - - - $i+1$: $\left\{ \begin{array}{l} \phi_{i+1} S_{i+1} + \psi_{i+1} B_{i+1} = W_{i+1} \\ \phi_{i+2} S_{i+1} + \psi_{i+2} B_{i+1} = W_{i+1} \end{array} \right.$ trade

New Prob of $i+1$:

$\phi_{i+1} S_{i+1} + \psi_{i+1} B_{i+1} = \phi_{i+2} S_{i+1} + \psi_{i+2} B_{i+1}$ - $\boxed{\text{self-financing}}$

$\tilde{w}_i = e^{-r\delta} w_i$ — discounted portfolio value

$\tilde{w}_i = \varphi_{i+1} \tilde{S}_i + \psi_{i+1}$

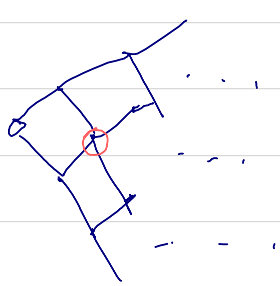
$\tilde{w}_{i+1} = \varphi_{i+1} \tilde{S}_{i+1} + \psi_{i+1}$

$\tilde{w}_{i+1} - \tilde{w}_i = \varphi_{i+1} [\tilde{S}_{i+1} - \tilde{S}_i]$

F_i measurable

$\Downarrow$

$\{\tilde{w}_i\}_{i=1,\cdots,n}$ is a Q-martingale



$\{V_i\}_{i=0,1,\cdots,n}$: nominal values of a tradable asset

$\tilde{V}_i = e^{-ird} V_i$: discounted values

No arbitrage $:\Longrightarrow$ $\tilde{V}_i = E^Q[\tilde{V}_{i+1} | F_i]$, $i=0,1,\cdots,n-1$

Q-martingale

Create a replicating portfolio? $\tilde{V}_i = \tilde{w}_i$, $i = 0,1,\cdots,n-1$

Fixed a node on the lattice of time i

$$\begin{tikzcd}[row sep=small] & \tilde{V}_{i+1}(u) \\ \tilde{V}_i \arrow[ur, no head] \arrow[dr, no head] & \\ & \tilde{V}_{i+1}(d) \end{tikzcd}$$

$$\tilde{V}_i = q\,\tilde{V}_{m}(u) + (1-q)\,\tilde{V}_{i+1}(d)$$

$$\tilde{S}_i = q\,\tilde{S}_{i+1}(u) + (1-q)\,\tilde{S}_{i+1}(d)$$

Suppose $\tilde{W}_i - \tilde{V}_i$ at this node

$$\tilde{W}_{i+1}(u) - \tilde{W}_i = \varphi_{i+1}\left[\tilde{S}_{i+1}(u) - \tilde{S}_i\right]$$

$$\underline{\underline{\tilde{V}_{i+1}(u) - \tilde{V}_i}} = \varphi_{i+1}\left[\tilde{S}_i e^{-rd} u - \tilde{S}_i\right]$$

To match the up-move

$$\varphi_{i+1} = \frac{\tilde{V}_{i+1}(u) - \tilde{V}_i}{\tilde{S}_i\left[e^{-rd}u - 1\right]}$$

For the down-move

$$\tilde{V}_{i+1}(d) - \tilde{V}_i = \varphi_{i+1}\left[\tilde{S}_i e^{-r\delta}\frac{1}{u} - \tilde{S}_i\right]$$

Perhaps

$$\varphi'_{i+1} = \frac{\tilde{V}_{i+1}(d) - \tilde{V}_i}{\tilde{S}_i\left[e^{-r\delta}\cdot\frac{1}{u} - 1\right]} = \frac{\tilde{V}_{i+1}(d) - \tilde{V}_i}{\tilde{S}_{i+1}(d) - \tilde{S}_i}$$

$$= \frac{\tilde{V}_{i+1}(d) - \left[q\,\tilde{V}_{i+1}(u) + (1-q)\,\tilde{V}_{i+1}(d)\right]}{\tilde{S}_i\left[e^{-rd}\frac{1}{u} - 1\right)} = \frac{q\left[\tilde{V}_{i+1}(d) - \tilde{V}_{i+1}(u)\right]}{q\left[\tilde{S}_{i+1}(d) - \tilde{S}_{i+1}(u)\right]}$$

$$= \tilde{S}_{i+1}(d) - \left[q\,\tilde{S}_{i+1}(u) + (1-q)\,\tilde{S}_{i+1}(d)\right)$$

$$\phi_{i+1}^{up} = \frac{\tilde{V}_{i+1}(u) - [q\tilde{V}_{i+1}(u) + (1-q)\tilde{V}_{i+1}(d)]}{\tilde{S}_{i+1}(u) - [q\tilde{S}_{i+1}(u) + (1-q)\tilde{S}_{i+1}(d)]}$$

$$= \frac{(1-q)[\tilde{V}_{i+1}(u) - \tilde{V}_{i+1}(d)]}{(1-q)[\tilde{S}_{i+1}(u) - \tilde{S}_{i+1}(d)]}$$

$i = 0, 1, \dots, N$ $\qquad t_i = i\delta \qquad \delta = \frac{T}{N}$

Put option at time T, for stock price K

Call - - — ——

V_i: value of the option at time $t_i = i\delta$

(Call) $V_N = \begin{cases} S_N - K & \text{if } S_N \geq K \\ 0 & \text{otherwise} \end{cases}$

Discount option price:

$\tilde{V}_i = e^{-ir\delta} V_i$

$i = 0, 1, \dots, N-1$

they must be a Q-martingale

$\tilde{V}_i = E^Q[\tilde{V}_{i+1} | F_i]$

$\to E^Q[\tilde{V}_{i+2} | F_{i+1}]$

$\to E[V_{i+3} | F_{i+2}]$

$$\tilde{V}_i = E^Q\left\{E^Q\left[E^Q\left(\cdots E^Q[\tilde{V}_N \mid F_{N-1}]\cdots \mid F_{i+1}\right)\right] \Big| F_i\right\}$$

first-martingale: $\tilde{V}_i = E^Q[\tilde{V}_N \mid F_i]$ for all $i = 0, 1, \cdots, N$

$$\tilde{V}_N = e^{-Nr\delta} \max(0, S_N - K)$$

$$\tilde{V}_i = E^Q[e^{-Nr\delta} \max(0, S_N - K) \mid F_i]$$

$$e^{-ir\delta} V_i = E^Q[e^{-Nr\delta} \max(0, S_N - K) \mid F_i]$$

nominal price of the option

$$V_i = E^Q[e^{-(N-i)r\delta} \max(0, S_N - K) \mid F_i]$$

Creat a replicated portfolio $\{\tilde{W}_i\} = \tilde{V}_i$

martingale

Suppose option price

$\tilde{P}_i \neq \tilde{V}_i$ — martingale

at same node

case 1: $\tilde{P}_i < \tilde{W}_i$ at the node:

buy option, sell portfolio

Case 2: $\tilde{P}_i > \tilde{W}_i$