1 Real Numbers

1.1 $\sqrt{2}$

Until now, we have looked at *discrete* numbers, that is numbers that are nicely separated from each other so that we can write them down unmistakebly and always know of which number we are currently talking. Now we enter a completely different universe. The universe of continuous numbers that cannot be written down in a finite number of steps. The representation of these numbers consists of infinitely many elements and, therefore, we will never be able to write the number down completely with all its elements. We may give a finite formula that describes how to compute the specific number, but we will never see the whole number written down. Those numbers are known since antiquity and, apparently, their existence came as a great surprise to Greek mathematicians.

The first step of our investigations into this kind of numbers, is to show that they *exists*, *i.e.* there are contexts where they arise naturally. To start, we will assume that they are not necessary. We assume that all numbers are either natural, integral or fractional. Indeed, any of the fundamental arithmetic operations, $+, -, \times$ and $/$, applied on two natural numbers, results either in another natural number, a negative number or a fraction. We could therefore suspect that the result of any operation is a rational number, *i.e.* an integeger or a fraction.

What about $\sqrt{2}$, the square root of 2? Let us assume that $\sqrt{2}$ is as well a fraction. Then we have two integers n and d, coprime to each other, such that

$$\sqrt{2} = \frac{n}{d} \tag{1.1}$$

and

$$2 = \left(\frac{n}{d}\right)^2 = \frac{n^2}{d^2}. \tag{1.2}$$

Any number can be represented as a product of primes. If $p_1p_2 \dots p_n$ is the prime factorsiation of n and $q_1q_2 \dots q_d$ is that of d, we can write:

$$\sqrt{2} = \frac{p_1p_2 \dots p_n}{q_1q_2 \dots q_d}. \tag{1.3}$$

It follows that

$$2 = \frac{p_1^2 p_2^2 \dots p_n^2}{q_1^2 q_2^2 \dots q_d^2}. \tag{1.4}$$

As we know from the previous chapter, two different prime numbers p and q squared (or raised to any integer) do not result in two numbers that share factors. The factorisation of p^n is just p^n and that of q^n is just q^n. They are coprime to each other. The fraction in equation 1.4, thus, cannot be an integer. There is only one way for such a fraction to result in an integer, *viz.* when it is an integer itself. From this follows that, if the root of an integer is not an integer itself, it is not a rational number either.

But, if $\sqrt{2}$ is not a rational number, a number that can be represented as the fraction of two integers, what the heck is it then?

There are several methods approximate that number. The simplest and oldest is the *Babylonian* method, also called *Heron's* method for Heron of Alexandria, a Greek mathematician of the first century who lived in Alexandria.

The idea of Heron's method, basically, is to iteratively approximate the real value starting with a guess. We can use the root algorithm defined in the first chapter to generate a guess or start with some arbitrary value. This value, x, if it is not an integer, is either slightly too big or too small, *i.e.* we either have $xx > a$ or $xx < a$. So, on each step, we improve a bit on the value by taking the average of x and its counterpart a/x. If $xx > a$, then $a/x < x$ and, if $xx < a$, then $a/x > x$. The average of x and a/x is calculated as $(x + a/x)/2$. The result is used as guess for the next step. The more iterations of this kind we do, the better is the approximation. In Haskell:

```
heron :: Natural -> Natural -> Double
heron n s = let a = fromIntegral n in go s a (a / 2)
  where go 0 _ x = x
        go i a x = go (i - 1) a ((x + a / x) / 2)
```

The function takes two arguments. The first is the number whose square root we want to calculate and the second is the number of iterations. We then call *go* with s, the number of iterations, a, the *Double* representation of n and our first guess $a/2$.

In *go*, if we have reached $i = 0$, we yield the result x. Otherwise, we call *go* again with $i - 1$, a and the average of x and a/x.

Let us compute $\sqrt{2}$ following this approach for, say, five iterations. We first have

$go\ 5\ 2\ 1 = go\ 4\ 2\ ((1 + 2)\ /\ 2)$
$go\ 4\ 2\ 1.5 = go\ 3\ 2\ ((1.5 + 2\ /\ 1.5)\ /\ 2)$
$go\ 3\ 2\ 1.416666 = go\ 2\ 2\ ((1.416666 + 2\ /\ 1.416666)\ /\ 2)$
$go\ 2\ 2\ 1.414215 = go\ 1\ 2\ ((1.414215 + 2\ /\ 1.414215)\ /\ 2)$
$go\ 1\ 2\ 1.414213 = go\ 0\ 2\ ((1.414213 + 2\ /\ 1.414213)\ /\ 2)$
$go\ 1\ 2\ 1.414213 = 1.414213.$

Note that we do not show the complete *Double* value, but only the first six digits. The results of the last two steps, therefore, are identical. They differ, in fact, at the twelfth digit: 1.4142135623746899 (*heron* 2 4) versus 1.414213562373095 (*heron* 2 5). The result of (*heron* 2 5)↑2 is fairly close to 2: 1.9999999999999996. It will not get any closer using *Double* representation.

1.2 Φ

1.3 π

1.4 e

1.5 γ

1.6 Arithmetic with Real Numbers

1.7 Representation of Real Numbers

1.8 $\mathbb{R}$

1.9 Real Factorials

1.10 Real Binomial Coefficients

1.11 The Continuum

1.12 Review of the Number Zoo