Contents

1 The Continuum

1.1 $\sqrt{2}$

Until now, we have looked at *discrete* numbers, that is numbers that are nicely separated from each other so that we can write them down unmistakebly and always know of which number we are currently talking. Now we enter a completely different universe. The universe of continuous numbers that cannot be written down in a finite number of steps. The representation of these numbers consists of infinitely many elements and, therefore, we will never be able to write the number down completely with all its elements. We may give a finite formula that describes how to compute the specific number, but we will never see the whole number written down. Those numbers are known since antiquity and, apparently, their existence came as a great surprise to Greek mathematicians.

The first step of our investigations into this kind of numbers, is to show that they *exist*, *i.e.*, there are contexts where they arise naturally. To start, we will assume that they are not necessary. We assume that all numbers are either natural, integral or fractional. Indeed, any of the fundamental arithmetic operations, $+, -, \times$ and $/$, applied on two rational numbers results always in a rational number, *i.e.* an integer or a fraction. We could therefore suspect that the result of any operation is a rational number, *i.e.* an integer or a fraction.

What about $\sqrt{2}$, the square root of 2? Let us assume that $\sqrt{2}$ is as well a fraction. Then we have two integers n and d, coprime to each other, such that

$$\sqrt{2} = \frac{n}{d} \tag{1.1}$$

and

$$2 = \left(\frac{n}{d}\right)^2 = \frac{n^2}{d^2}. \tag{1.2}$$

Any number can be represented as a product of primes. If $p_1 p_2 \dots p_n$ is the prime factorsiation of n and $q_1 q_2 \dots q_d$ is that of d, we can write:

$$\sqrt{2} = \frac{p_1 p_2 \dots p_n}{q_1 q_2 \dots q_d}. \tag{1.3}$$

It follows that

$$2 = \frac{p_1^2 p_2^2 \dots p_n^2}{q_1^2 q_2^2 \dots q_d^2}. \tag{1.4}$$

As we know from the previous chapter, two different prime numbers p and q squared (or raised to any integer) do not result in two numbers that share factors. The factorisation of p^n is just p^n and that of q^n is just q^n. They are coprime to each other. The fraction in equation 1.4, thus, cannot represent an integer, such as 2. There is only one way for such a fraction to result in an integer, *viz.*, when the numerator is an integer and the denominator is 1, which is obviously not the case for $\sqrt{2}$. It follows that, if the root of an integer is not an integer itself, it is not a rational number either.

But, if $\sqrt{2}$ is not a rational number, a number that can be represented as the fraction of two integers, what the heck is it then?

There are several methods to approximate the number $\sqrt{n}$. The simplest and oldest is the *Babylonian* method, also called *Heron's* method for Heron of Alexandria, a Greek mathematician of the first century who lived in Alexandria.

The idea of Heron's method, basically, is to iteratively approximate the real value starting with a guess. We can start with some arbitrary value. If the first guess, say g, does not equal $\sqrt{n}$, *i.e.* $gg \neq n$, then g is either slightly too big or too small. We either have $gg > n$ or $gg < n$. So, on each step, we improve a bit on the value by taking the average of g and its counterpart n/g. If $gg > n$, then clearly $n/g < g$ and, if $gg < n$, then $n/g > g$. The average of g and a/g is calculated as $(g + a/g)/2$. The result is used as input for the next round. The more iterations of this kind we do, the better is the approximation. In Haskell:

```
heron :: Natural → Natural → Double
heron n s = let a = fromIntegral n in go s a (a / 2)
  where go 0 _ x = x
        go i a x | xx = a     = x
                 | otherwise = go (i − 1) a ((x + a / x) / 2)
```

The function takes two arguments. The first is the number whose square root we want to calculate and the second is the number of iterations we want to do. We then call *go* with s, the number of iterations, a, the *Double* representation of n, and our first guess $a/2$.

In *go*, if we have reached $i = 0$, we yield the result x. Otherwise, we call *go* again with $i - 1$, a and the average of x and a/x.

Let us compute $\sqrt{2}$ following this approach for, say, five iterations. We first have

go 5 2 1 = *go* 4 2 ((1 + 2) / 2)
go 4 2 1.5 = *go* 3 2 ((1.5 + 2 / 1.5) / 2)
go 3 2 1.416666 = *go* 2 2 ((1.416666 + 2 / 1.416666) / 2)
go 2 2 1.414215 = *go* 1 2 ((1.414215 + 2 / 1.414215) / 2)
go 1 2 1.414213 = *go* 0 2 ((1.414213 + 2 / 1.414213) / 2)
go 1 2 1.414213 = 1.414213.

Note that we do not show the complete *Double* value, but only the first six digits. The results of the last two steps, therefore, are identical. They differ, in fact, at the twelfth digit: 1.4142135623746899 (*heron* 2 4) versus 1.414213562373095 (*heron* 2 5). Note that the result of five iterations has one digit less than that of four iterations. This is because that after the last digit, 5, the digit 0 follows and then the *precision* of the Double number type is exhausted. *Irrational* numbers, this is the designation of the type of numbers we are talking about, consist of infinitely many digits. Therefore, the last digit in the number presented above is in fact not the last digit of the number. With slightly higher precision, we would see that the number continues like 03 . . .

The result of (*heron* 2 5) ↑ 2 is fairly close to 2: 1.9999999999999996. It will not get any closer using *Double* representation.

1.2 Φ

Another interesting irrational number is τ or Φ, known as the *divine proportion* or *golden ratio*. The golden ratio is the relation of two quantities, such that the greater relates to the lesser as the sum of both to the greater. This may sound confusing, so here are two symbolic representations:

$$\frac{a}{b} = \frac{a+b}{a}. \tag{1.5}$$

In this equation, a is the greater number and b the lesser. The equation states that the relation of a to b equals the relation of $a + b$ to a. Alternatively we can say

$$\frac{b}{a} = \frac{a}{b-a}. \tag{1.6}$$

Here b is the sum of the two quantities and a is the greater one. The equation states that the relation of b to a is the same as the relation of a to $b - a$, which, of course, is then the smaller quantity.

The golden ratio is known since antiquity. The symbol Φ is an homage to ancient Greek sculptor and painter Phidias (480 – 430 BC). Its first heyday after antiquity was during Renaissance and then it was again extremely popular in the 18^{th} and 19^{th} centuries. Up

to our times, artists, writers and mathematicians have repeatedly called the golden ratio especially pleasing.

From equation 1.6 we can derive the numerical value of Φ in the form $\frac{\Phi}{1}$. We can then calculate b, for any value a, as $a\Phi$. The derivation uses some algebraic methods, which we will study more closely in the next part, especially *completing the square*. As such this section is also an algebra teaser.

We start by setting $\Phi = \frac{b}{a}$ with $a = 1$. We then have on the left side of the equation $\Phi = \frac{b}{1} = b$. On the right-hand side, we have $\frac{a=1}{\Phi-1}$:

$$\Phi = \frac{1}{\Phi - 1}. \tag{1.7}$$

Multiplying both sides by $\Phi - 1$, we get

$$\Phi^2 - \Phi = 1. \tag{1.8}$$

This is a quadratic equation and, with some phantasy, we even recognise the fragment of a binomial formula on the left side. A complete binomial formula would be

$$(a - b)^2 = a^2 - 2ab + b^2. \tag{1.9}$$

We try to pattern match the fragment above such that Φ^2 is a^2 and $-\Phi$ is $-2ab$. That would mean that the last term, b^2 would correspond to the square half of the number that is multiplied by Φ to yield $-\Phi$. $-\Phi$ can be seen as $-1 \times \Phi$. Half of that number is $-\frac{1}{2}$. That squared is $\frac{1}{4}$. So, if we add $\frac{1}{4}$ to both sides of the equation, we would end up with a complete binomial formula on the left side:

$$\Phi^2 - \Phi + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}. \tag{1.10}$$

We can apply the binomial theorem on the left side and get

$$\left(\Phi - \frac{1}{2}\right)^2 = \frac{5}{4}. \tag{1.11}$$

Now we take the square root:

$$\Phi - \frac{1}{2} = \frac{\pm\sqrt{5}}{2}. \tag{1.12}$$

Note that the $\pm$ in front of $\sqrt{5}$ reflects the fact that the square root of 5 (or any other number) may be positive or negative. Both solutions are possible. However, since we are looking for a positive relation, we only consider the positive solution and ignore the other one. We can therefore simplify to

$$\Phi - \frac{1}{2} = \frac{\sqrt{5}}{2}. \tag{1.13}$$

Finally, we add $\frac{1}{2}$ to both sides:

$$\Phi = \frac{1+\sqrt{5}}{2} \tag{1.14}$$

and voilá that is Φ. The numerical value is approximately 1.618 033 988 749 895. We can define it as a constant in Haskell as

```
phi :: Double
phi = 0.5 * (1 + sqrt 5)
```

We can now define a function that, for any given a, yields b:

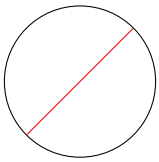
```
golden :: Double → Double
golden a = a * phi
```

golden 1, of course, is just Φ, *i.e.* 1.618 033 988 749 895. *golden* 2 is twice that value, namely 3.236 067 977 499 79. Furthermore, we can state that 2 / (*golden* 2 − 2), which is $\frac{a}{b-a}$, is again an approximation of Φ.

That is very nice. But there is more to come. Φ, in fact, is intimately connected to the Fibonacci sequence, as we will se in the next chapter.

1.3 π

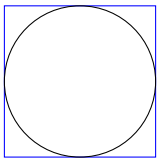
π is probably the most famous irrational number. It emerged in antique mathematics in studying the circle where it expresses the relation between the diameter (depicted in red in the image below) and the circumference (depicted in black):



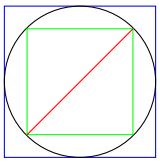
Since, often, the radius, which is half the diameter, is much more important in mathematics, it has been proposed to use $\tau = 2\pi$ where π is used today. But π has survived

through history and, even though slightly suboptimal in some situations, it is still in use today.

The reason why the perimeter instead of the radius was used to define the circle constant is probably because classic approaches to approximate π take the perimeter as basis. They start by drawing a square with side length 1 and inscribe a circle into the square with perimeter 1:



Since the square has side length 1, its perimeter, the sum of all its sides is $1+1+1+1 = 4$ and, as we can see clearly in the picture above, this perimeter is greater than that of the circle. 4, hence, is an upper bound for the circumference of the circle with perimeter 1. A lower bound would then be given by a square inscribed in the circle, such that the distance between its opposing corners (red) is 1, the perimeter of the circle:



We see two right triangles with two green sides on a red basis. The basis is the perimeter of the circle, of which we know that its length is 1. You certainly know the Pythagorean theorem, probably the most famous or notorious theorem of all mathematics, which states that, in a right triangle, one with a right angle, an angle of 90°, the sum of the squares of the sides to the left and right of that angle (the green sides) equals the square of the hypothenuse, the red side, which is opposite to the right angle. This can be stated as:

$$a^2 = b^2 + c^2, \tag{1.15}$$

where a is the red side, whose length we know, namely 1. We further know that the green sides are equal. We hence have:

$$1^2 = 2b^2. \tag{1.16}$$

and further derive

$$1 = \sqrt{2b^2}, \tag{1.17}$$

which is

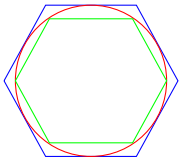
$$1 = \sqrt{2}b. \tag{1.18}$$

Dividing both sides by $\sqrt{2}$, we get

$$b = \frac{1}{\sqrt{2}}, \tag{1.19}$$

the side length of the green square, which is approximately 0.707. The perimeter of the inner square is thus 4×0.707, which is approximately 2.828. Thus π is some value between 2.828 and 4.

That result is not very satifactory, of course. There is room for a lot of numbers between 2.828 and 4. The method was therefore extended by choosing polygons with more than four sides to come closer to the real value of π. The ancient record holder for approximating π is Archimedes who started off with a hexagon, which is easy to construct with compass and ruler:



Then he subsequently doubled the number of sides of the polygon, so that he obtained polygons with 12, 24, 48 and, finally, 96 sides. With this approach he concluded that $\frac{223}{71} < \pi < \frac{22}{7}$, which translates to a number between 3.1408 and 3.1428 and is pretty close to the approximated value 3.14159.

In modern times, mathematicians started to search for approximations by other means than geometry, in particular by infinite series. One of the first series was discovered by Indian mathematician Nilakantha Somayaji (1444 – 1544). It goes like

$$\pi = 3 + \frac{4}{2 \times 3 \times 4} - \frac{4}{4 \times 5 \times 6} + \frac{4}{6 \times 7 \times 8} - \ldots \tag{1.20}$$

We can implement this in Haskell as

```
nilak :: Int → Double
nilak i     | even i     = nilak (i + 1)
            | otherwise = go i 2 3 4
  where go 0 _ _ _ = 3
        go n a b c = let k  | even n = −4
                            | otherwise = 4
                     in (k / (a ∗ b ∗ c)) + go (n − 1) c (c + 1) (c + 2)
```

Here we use a negative term, whenever n, the counter for the step we are performing, is even. Since, with this approach, an even number of steps would produce a bad approximation, we perform, for i even, $i + 1$ and hence an odd number of steps. This way, the series converges to 3.14159 after about 35 steps, *i.e. nilak* 35 is some number that starts with 3.14159.

An even faster convergence is obtained by the beautiful series discovered by French mathematician François Viète (1540 – 1603) in 1593:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times \ldots \tag{1.21}$$

In Haskell this gives rise to a very nice recursive function:

```
vietep :: Int → Double
vietep i = 2 / (go 0 (sqrt 2))
  where go n t | n ≡ i       = 1
               | otherwise = (t / 2) ∗ go (n + 1) (sqrt (2 + t))
```

The approximation 3.14159 is reached with *vietep* 10.

There are many other series, some focusing on early convergence, others on beauty. An exceptionally beautiful series is that of German polymath Gottfried Wilhelm Leibniz (1646 – 1716), who we will get to know more closely later on:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \tag{1.22}$$

In Haskell this is, for instance:

```
leipi :: Int → Double
leipi i = 4 ∗ go 0 1
  where go n d | n ≡ i       = 0
               | otherwise = let x | even n = 1
                                   | otherwise = −1
                             in x / d + go (n + 1) (d + 2)
```

This series converges really slowly. We reach 3.14159 only after about 400 000 steps.

π appears quite often in mathematics, particularly in geometry. But there are also some unexpected entries of this number. The inevitable Leonhard Euler solved a function, which today is called *Riemann zeta function*, for the special case $s = 2$:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.23}$$

Euler showed that, for the special case $s = 2$, $\zeta(s)$ converges to $\frac{\pi^2}{6}$; in fact, for any n, n a multiple of 2, $\zeta(n)$ converges to some fraction of a power of π, *e.g.* $\zeta(4)$ approaches $\frac{\pi^4}{90}$, $\zeta(6)$ approaches $\frac{\pi^6}{945}$ and so on.

This is surprising, because the zeta function is not related to circles, but to number theory. It appears for example, when calculating the probability of two numbers being coprime to each other. Two numbers are coprime if they do not share prime factors. The probability of a number being divisible by a given prime p is $\frac{1}{p}$, since every p^{th} number is divisible by p. For two independently chosen numbers, the probability that both are divisible by prime p is therefore $\frac{1}{p} \times \frac{1}{p} = \frac{1}{p^2}$. The reverse probability that both are not divisible by that prime, hence, is $1 - \frac{1}{p^2}$. The probablitiy that there is no prime at all that divides both is then

$$\prod_{p}^{\infty} 1 - \frac{1}{p^2}. \tag{1.24}$$

To cut a long story short, this equation can be transformed into the equation

$$\frac{1}{1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots} = \frac{1}{\zeta(2)} = \frac{1}{\frac{\pi^2}{6}} = \frac{6}{\pi^2} = 0.607 \approx 61\% \tag{1.25}$$

and with this π appears as a constant in number theory expressing the probability of two randomly chosen numbers being coprime to each other.

1.4 e

The Bernoullis were a family of Huguenots from Antwerp in the Spanish Netherlands from where they fled the repression by the Catholic Spanish authorities, first to Frankfurt am Main, later to Basel in Switzerland. Among the Bernoullis, there is a remarkable number of famous mathematicians who worked in calculus, probability theory, number theory and many areas of applied mathematics. One of the Basel Bernoullis was Johann Bernoulli (1667 – 1748) who worked mainly in calculus and tutored famous mathematicians like Guillaume L'Hôpital, but whose greatest contribution to the history of math

was perhaps to recognise the enormous talent of another of his pupils whose name was Leonhard Euler.

His brother Jacob Bernoulli (1655 – 1705), who worked, as his brother, in calculus, but most prominently in probability theory, is much better known today, partly perhaps because many of Johann's achievements were published under the name of L'Hôpital. Unfortunately, early modern mathematics and science in general was plagued with disputes over priorities in the authorship of contributions, a calamity that authors and authorities later tried to solve by introducing the *droite d'auteur*, better known in the English speaking world as *copyright*.

Among the many problems Jacob studied was the calculation of interests. He started off with a very simple problem. Suppose we have a certain amount of money and a certain interest credited after a given amount of time. To keep it simple, let the amount equal 1 (of any currency of your liking – currencies in Jacob's lifetime were extremely complicated, so we better ignore that detail). After one year 100% interest is paid. After that year, we hence have $1 + \frac{1\times 100}{100} = 2$ in our account. That is trivial. But what, if the interest is paid in shorter periods during the year? For instance, if the interest is paid twice a year, then the interest for that period would be 50%. After six months we would have $1 + \frac{1\times 50}{100} = 1.5$ in our account. After one year, the account would then be $1.5 + \frac{1.5\times 50}{100} = 1.5 + \frac{75}{100} = 1.5 + 0.75 = 2.25$.

Another way to see this is that the initial value is multiplied by 1.5 (the initial value plus the interest) twice: $1 \times 1.5 \times 1.5 = 1 \times 1.5^2 = 2.25$. When we reduce the period even further, say, to three months, then we had $1.25^4 \approx 2.4414$. On a monthly base, we would get $\left(1 + \frac{1}{12}\right)^{12} \approx 2.613$. On a daily basis, we would have $\left(1 + \frac{1}{365}\right)^{365} \approx 2.7145$. With hourly interests and the assumption that one year has $24 \times 365 = 8760$ hours, we would get $\left(1 + \frac{1}{8760}\right)^{8760} \approx 2.71812$. With interest paid per minute we would get $\left(1 + \frac{1}{525600}\right)^{525600} \approx 2.71827$ and on interest paid per second, we would get $\left(1 + \frac{1}{3156000}\right)^{3156000} \approx 2.71828$. In general, for interest on period n, we get:

$$\left(1 + \frac{1}{n}\right)^n.$$

You may have noticed in the examples above that this formula converges with greater and greater ns. For n approaching ∞, it converges to 2.71828, a number that is so beautiful that we should look at more than just the first 5 digits:

$$2.7\ 1828\ 1828\ 4590\ 4523\ldots$$

This is e. It is called Euler's number or, for the first written appearance of concepts related to it in 1618, Napier's number. It is a pity that its first mentioning was not in the year 1828. But who knows – perhaps in some rare Maya calendar the year 1618 actually is the year 1828.

An alternative way to approach e that converges much faster than the closed form above is the following:

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$$

or, in other words:

$$e = \sum_{n=1}^{\infty} \frac{1}{n!}. \tag{1.26}$$

We can implement this equation in Haskell as

```
e_ :: Integer → Double
e_ p = 1 + sum [1 / (dfac n) | n ← [1 .. p]]
  where dfac = fromInteger ∘ fac
```

After some experiments with this function, we see that it converges already after 17 recursions to a value that does not change with greater arguments at *Double* precision, such that $e\_\ 17 \equiv e\_\ 18 \equiv e\_\ 19 \equiv \dots$ We could then implement e as

```
e :: Double
e = e_ 17
```

The fact that e is related to the factorial may lead to the suspicion that it also appears directly in a formula dealing with factorials. There, indeed, is a formula derived by James Stirling who we already know for the Stirling numbers. This formula approximates the value of $n!$ without the need to go through all the steps of its recursive definition. Stirling's formula is as follows:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \tag{1.27}$$

This equation is nice already because of the fact that e and π appear together to compute the result of an important function. But how precise is the approximation? To answer this question, we first implement Stirling's formula:

```
stirfac :: Integer → Integer
stirfac i = ceiling $ (sqrt (2 * pi * n)) * (n / e) ↑ i
  where n = fromIntegral i
```

Note that we *ceil* the value, instead of rounding it just to the next integer value.

Then we define a function to compute the difference $difac\ n = fac\ n - stirfac\ n$. The result for the first 15 numbers is

$$0, 0, 0, 0, 1, 9, 59, 417, 3343, 30104, 301174, 3314113, 39781324, 517289459, 7243645800.$$

For the first numbers, the difference is 0. Indeed:

$$\begin{array}{rcrcr} 1! & = & stirfac(1) & = & 1 \\ 2! & = & stirfac(2) & = & 2 \\ 3! & = & stirfac(3) & = & 6 \\ 4! & = & stirfac(4) & = & 24 \end{array}$$

Then, the functions start to disagree, for instance $5! = 120 \neq stirfac(5) = 119$. The difference grows rapidly and reaches more than 3 million with 12!. But what is the deviation in relation to the real value? We define the function $100 * (\mathit{fromIntegral}\ \$\ \mathit{difac}\ n)\ /\ (\mathit{fromIntegral}\ \$\ \mathit{fac}\ n)$ to obtain the difference in terms of a percentage of the real value. We see starting from 5 (where the first difference occurs):

$$0.8333, 1.25, 1.1706, 1.0342, 0.9212, 0.8295, 0.7545, 0.6918, 0.6388, 0.5933, 0.5539, \ldots$$

For 5, the value jumps up from 0 to 0.8333%, climbs even higher to 1.25% and then starts to descrease slowly. At 42 the deviation falls below 0.2%. At 84, it falls below 0.1% and keeps falling. Even though the difference appears big in terms of absolute numbers, the percentage quickly shrinks and, for some problems, may even be neglible.

A completely different way to approximate e is by *continued fractions*. Continued fractions are infinite fractions, where each denominator is again a fraction. For instance:

$$e = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}}} \tag{1.28}$$

A more readable representation of continued fractions is by sequences of the denominator like:

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots] \tag{1.29}$$

where the first number is separated by a semicolon to highlight the fact that it is not a denominator, but an integral number added to the fraction that follows. We can capture this very nicely in Haskell, using just a list of integers. However, in some cases we might

have a fraction with numerators other than 1. An elegant way to represent this case is by using fractions instead of integers. We would then represent $\frac{2}{a+...}$ as $\frac{1}{\frac{1}{2}a+...}$. Here is an implementation:

```
contfrac :: [Quoz] → Double
contfrac [] = 1
contfrac [i] = fromQuoz i
contfrac (i : is) = n + 1 / (contfrac is)
  where n = fromQuoz i

fromQuoz :: Quoz → Double
fromQuoz i = case i of
  (Pos (Q nu d)) → fromIntegral nu / fromIntegral d
  (Neg (Q nu d)) → negate (fromIntegral nu / fromIntegral d)
```

For *contfrac* $[2, 1, 2, 1, 1, 4, 1, 1, 6]$ we get 2.7183, which is not bad, but not yet too close to e. With $[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10]$ we get 2.718 281 828, which is pretty close.

Examining the sequence a bit further, we see that is has a regular structure. We can generate it by means of the *Engel expansion* named for Friedrich Engel (1861 – 1941), a German mathematician who worked close with the great Norwegian algebraist Sophus Lie (1842 – 1899). The Engel expansion can be implemented as follows:

```
engelexp :: [Integer]
engelexp = 2 : 1 : go 1
  where go n = (2 * n) : 1 : 1 : go (n + 1)
```

The following fraction, however, conoverges must faster than the Engel expansion: $[1, \frac{1}{2}, 12, 5, 28, 9, 44, 13]$. Note that we take advantage of the datatype *Quoz* to represent a numerator that is not 1. This sequence can be generated by means of

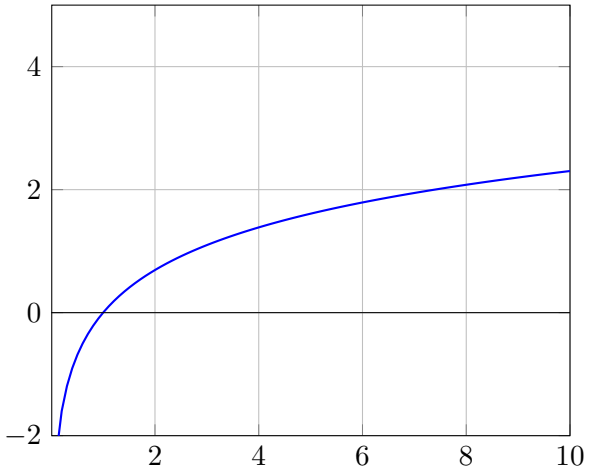
```
fastexp :: [Quoz]
fastexp = 1 : (Pos (1 % 2)) : go 1
  where go n = (16 * n − 4) : (4 * n + 1) : go (n + 1)
```

This fraction converges already after 7 steps to the value 2.718 281 828:

contfrac (*take* 7 *fastexp*).

The area of mathematics where e is really at home is analysis and its vast areas of application, which we will study in the third part of this series. The reason for the prominence of e in analysis stems from the *natural logarithm*, which we already introduced in the first chapter. The natural logarithm of a number n, usually denoted $\ln(n)$, is the exponent x, such that $e^x = n$.

The natural logarithm can be graphed as follows:



The curious fact that earned the natural logarithm its name is that, at $x = 1$, the curve has the slope 1. This might sound strange for the moment. We will investigate that later.

1.5 γ

The *harmonic series* is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \tag{1.30}$$

A harmonoic series with respect to a given number k, called the *harmonic number* H_k, then is

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \dots + \frac{1}{k} \tag{1.31}$$

This is easily implemented in Haskell as

```
harmonic :: Natural → Double
harmonic n = sum [1 / d | d ← map fromIntegral [1 .. n]]
```

Some harmonic numbers are (*map harmonic* $[1 \,..\, 10]$):

$1, 1.5, 1.8\overline{3}, 2.08\overline{3}, 2.28\overline{3}, 2.44\overline{9}, 2.5928, 2.7178, 2.8289, 2.9289.$

The harmonic series is an interesting object of study in its own right. Here, however, we are interested in something else. Namely, the difference of the harmonic series and the

natural logarithm:

$$\gamma = \lim_{n\to\infty} H_n - \ln(n). \tag{1.32}$$

We can implement this equation as

```
harmonatural :: Natural → Double
harmonatural n = harmonic n − ln n
  where ln = log ∘ fromIntegral
```

Applied on the first numbers with *map harmonatural* $[1\,..\,10]$, the function does not show interesting results:

$1.0, 0.8068, 0.7347, 0.697, 0.6738, 0.6582, 0.6469, 0.6384, 0.6317, 0.6263, \ldots$

Applied to greater numbers, however, the results approach a constant value:

harmonatural $100 = 0.58220$
harmonatural $1000 = 0.57771$
harmonatural $10000 = 0.57726$
harmonatural $100000 = 0.57722$

With even greater numbers, the difference converges to 0.577 21. This number, γ, was first mentioned by – surprise – Leonhard Euler and some years later by Italian mathematician Lorenzo Mascheroni (1750 – 1800) and is therefore called the Euler-Mascheroni constant.

This mysterious number appears in different contexts and, apparently, quite often as a difference or average. An ingenious investigation was carried out by Belgian mathematician Charles Jean de la Vallée-Poussin (1866 – 1962) who is famous for his proof of the Prime number theorem. Vallée-Poussin studied the quotients of a number n and the primes up to that number. If n is not prime itself, then there are some prime numbers p, namely those of the prime factorisation of n, such that $\frac{n}{p}$ is an integer. For others, this quotient is a rational number, which falls short of the next natural number. For instance, there are four prime numbers less than 10: 2, 3, 5 and 7. The quotients are

$5, 3.\overline{3}, 2, 1.\overline{428571}$.

5 and 2, the quotients of 2 and 5, respectively, are integers and there, hence, is no difference. The quotient $\frac{10}{3} = 3.\overline{3}$, however, falls short of 4 by $0.\overline{6}$ and $\frac{10}{7} = 1.\overline{428571}$ falls short of 2 by $0.\overline{57142828}$.

Vallée-Poussin asked what the average of this difference is. For the example 10, the average is about $0.3\overline{095238}$. One might think that this average, computed for many numbers or for very big numbers, is about 0.5, so that the probability for the quotient of n and a random prime number to fall into the first or the second half of the rational numbers between two integers is equal, *i.e.* 50% for both cases. It turns out it is not. For

huge numbers, de la Vallée-Poussin's average converges to 0.577 21, the Euler-Mascheroni constant.

It converges quite slow, however. If we implement the prime quotient as

$$
\begin{aligned}
&pquoz :: Natural \to [Double] \\
&pquoz\ n = [\, d\ /\ p \mid p \leftarrow ps\,] \\
&\quad \textbf{where}\ ps = map\ fromIntegral\ (takeWhile\ (<n)\ allprimes) \\
&\qquad\quad\ \ d\ \ = fromIntegral\ n
\end{aligned}
$$

and its average as

$$
\begin{aligned}
&pquozavg :: Integer \to Double \\
&pquozavg\ n = (sum\ ds)\ /\ (fromIntegral\ \$\ length\ ds) \\
&\quad \textbf{where}\ qs\ = pquoz\ n \\
&\qquad\quad\ \ ns = map\ (fromIntegral \circ ceiling)\ qs \\
&\qquad\quad\ \ ds = [\, n - q \mid (n, q) \leftarrow zip\ ns\ qs\,]
\end{aligned}
$$

we can experiment with some numbers like

$pquozavg\ 10 = 0.3\overline{095238}$
$pquozavg\ 100 = 0.548731$
$pquozavg\ 1000 = 0.5590468$
$pquozavg\ 10000 = 0.5666399$
$pquozavg\ 100000 = 0.5695143$
...

With greater and greater numbers, this value approaches γ. Restricting n to prime numbers produces good approximations of γ much earlier. From 7 on, *pquozavg* with primes results in numbers of the form $0.5\ldots$ $pquozavg\ 43 = 0.57416$ is already very close to γ. It may be mentioned that 43 is suspiciously close to 42.

With de la Vallée-Poussin's result in mind, it is not too surprising that γ is related to divisors and Euler's totient number. A result of Gauss' immediate successor in Göttingen, Peter Gustav Lejeune-Dirichlet (1805 – 1859), is related to the average number of divisors of the numbers $1\ldots n$. We have already defined a function to generate the divisors of a number n, namely *divs*. Now we map this function on all numbers up to n:

$$
\begin{aligned}
&divsupn :: Natural \to [[Natural]] \\
&divsupn\ n = map\ divs\ [1\,..\,n]
\end{aligned}
$$

Applied to 10, this function yields:

$$[[1], [1,2], [1,3], [1,2,4], [1,5], [1,2,3,6], [1,7], [1,2,4,8], [1,3,9], [1,2,5,10]]$$

For modelling Lejeune-Dirichlet's result, we further need to count the numbers of divisors of each number:

$$
\begin{aligned}
&ndivs :: Integer \to [Int] \\
&ndivs = map\ length \circ divsupn
\end{aligned}
$$

Applied again to 10, *ndivs* produces:

$[1, 2, 2, 3, 2, 4, 2, 4, 3, 4]$

Now we compute the average of this list using

```
dirichlet :: Integer → Double
dirichlet n   = s / l
  where ds = ndivs n
        l  = fromIntegral $ length ds
        s  = fromIntegral $ sum ds
```

For *dirichlet* 10 we see 2.7. This does not appear too spectacular. Greater numbers show:

dirichlet $100 = 4.759$
dirichlet $250 = 5.684$
dirichlet $500 = 6.38$
dirichelt $1000 = 7.069$

As we can see, the number is slowly increasing resembling a log function or, more specifically, the natural log. When we compare the natural log, we indeed see that the results are close:

$\ln 100 = 4.605$
$\ln 250 = 5.521$
$\ln 500 = 6.214$
$\ln 1000 = 6.907$

For greater and greater numbers, the difference of the *dirichlet* function and the natural logarithm approaches

$$0.154435 \approx 2\gamma - 1. \tag{1.33}$$

For the five examples above, the difference is still significantly away from that number:

$\Delta 100 = 0.2148$
$\Delta 250 = 0.1625$
$\Delta 500 = 0.1635$
$\Delta 1000 = 0.1612,$

but already $\Delta 2000 = 0.158$ comes close and $\Delta 4000 = 0.1572$ approaches the value even further.

An important constant derived from γ is e^{γ}, which is a limit often seen in number theory. One instance is the lower bound of the totient function. There is a clear upper bound, namely $n - 1$. Indeed, $\varphi(n)$ can never yield a value greater $n - 1$ and this upper bound is reached exclusively by prime numbers. There is no such linear lower bound. That is,

$\varphi(n)$ can assume values that are much smaller than the value seen for $n-1$ or other numbers less than n. But there is a lower bound that slowly grows with n. This lower bound is often given as $\frac{n}{\ln\ln n}$. This lower bound, however, is too big. There are some values that are still below that border. $\varphi(40)$, for instance, is 16. $\frac{40}{\ln\ln 40}$, however, is around 30. A better, even still not perfect approximation, is

$$\frac{n}{e^\gamma \ln\ln n}.$$

For $n = 40$ again, $\frac{40}{e^\gamma \ln\ln 40}$ is around 17 and, hence, very close to the real value.

We see that γ is really a quite mysterious number that appears in different contexts, sometimes in quite a subtle manner. The greatest mystery, however, is that it is not so clear that this number belongs here in the first place. Indeed, it has not yet been shown that γ is irrational. In the approximations, we have studied in this section, we actually have not seen the typical techniques to create irrational numbers like roots, continuous fractions and infinite series. If γ is indeed rational, then it must be the fraction of two really large numbers. In 2003, it has been shown that the denominator of such a fraction must be greater than 10^{242080}. A number big enough, for my taste, to speak of *irrational*.

1.6 Representation of Real Numbers

The set of real numbers $\mathbb{R}$ is the union of the rational and the irrational numbers. When we write real numbers on paper, we use the decimal notation. A number in decimal notation corresponds to an ordinary integer terminated by a dot called the decimal point; this integer corresponds to the part of the real number greater 1 or 0 of course. After the dot a stream of digits follows, which is not necessarily a number in the common sense, since it may start with zeros, *e.g.* 0.0001. In fact, one could say that the part after the dot corresponds to a a reversed integer, since the zero following this number have no impact on the value of the whole expression, *i.e.* $0.10 = 0.1$.

Any rational number can be expressed in this system. An integer corresponds just to the part before the dot: $1.0 = 1$. A fraction like $\frac{1}{2}$ is written as 0.5. We will later look at how this is computed concretely. Rationals in decimal notation can be easily identified: all numbers in decimal notation with a finite part after the dot are rational: 0.25 is $\frac{1}{4}$, 0.75 is $\frac{3}{4}$, 0.2 is $\frac{1}{5}$ and so on.

There are some rational numbers that are infinite. For example, $\frac{1}{3}$ is $0.333333\ldots$, which we encode as $0.\overline{3}$. Such periodic decimals are easy to convert to fractions. We just have to multiply them by a power of 10, such that there is a part greater 0 before the decimal point and that the first number of the repeating period is aligned to it. For $0.\overline{3}$, this is just $10 \times 0.\overline{3} = 3.\overline{3}$. For $0.1\overline{6}$, it would be $10 \times 0.1\overline{6} = 1.\overline{6}$. For $0.\overline{09}$, it would be $10^2 \times 0.\overline{09} = 9.\overline{09}$. We then subtract the original number from the result. If the original

number is x, we now have $10^n x - x = (10^n - 1)x$. For $x = 0.\overline{3}$ this is $9x$; for $x = 0.1\overline{6}$ this, too, is $9x$ and for $x = 0.\overline{09}$ this is $99x$. The results are 3, 1.5 and 9 respectively. We now build a fraction of this result as numerator and the factor (9 or 99) in the denominator. Hence, $0.\overline{3} = \frac{3}{9} = \frac{1}{3}$, $0.1\overline{6} = \frac{1.5}{9} = \frac{3}{18} = \frac{1}{6}$ and $0.\overline{09} = \frac{9}{99} = \frac{1}{11}$.

A curiosity resulting from this calculations is that $9 \times 1/3 = 3$ and $9 \times 0.\overline{3} = 2.\overline{9}$ are actually the same! A correct implementation must take this into account. Try it with Haskell, you will see that $9 * 0.3333333333333333$ is indeed 3. It will not work if you forget a three. The precision of the *Double* number type is 16 digits after the decimal point. With only 15 threes, the result will be 2.9999999999999997.

Irrational numbers in the decimal notation have infinite many digits after the decimal point. With this said, it is obvious that we cannot represent irrational numbers in this system. We can of course represent any number by some kind of formula like $\sqrt{5}$ and do some math in this way such as $\frac{1+\sqrt{5}}{2}$, *etc.* But often, when we are dealing with applied mathematics, such formulas are not very useful. We need an explicit number. But, unfortunately or not, we have only limited resources in paper, brainpower and time. That is, at some point we have to abandon the calculations and work with what can be achieved with the limited resources we have at our disposal.

The point in time at which we take the decision that we now have calculated enough is the measure for the precision of the real number type in question. On paper, we would hence say that we write only a limited number of digits after the decimal point. In most day-to-day situations where real numbers play a role, like in dealing with money, cooking, medication or travelling distances, we calculate up to one or two decimal places. Prices, for instances are often given as 4.99 or something, but hardly as 4.998. Recipes would tell that we need 2.5 pounds or whatever of something, using one decimal place. One would say that it is about 1.5km to somewhere, but hardly that it is 1.499961km. In other areas, especially in science much more precision is needed. We therefore need a flexible datatype.

A nice and clean format to represent real numbers uses two integers or, as the following definition, two natural numbers:

data *RealN* = *R Natural Natural*

The first number represents the integral part. You will remember that a number is a list of digits where every digit is multiplied by a power of ten according to the place of the digit. The first digit, counted from the right, is multiplied by $10^0 = 1$. The second is multiplied by 10^1, the third by 10^2 and so on.

The digit multiplied by 10^0, is the last digit before the decimal point. If we wanted to push it to the right of the decimal point, we would need to reduce the exponent. So, we would multiply it not by 10^0, but by 10^{-1} to push it to the first decimal place. This is the function of the second number in the datatype above. It represents the value of the least significant bit in terms of the exponent to which we have to raise 10 to obtain

the number represented by this datatype. Since our datatype uses a natural number, we have to negate it to find the exponent we need.

For instance, the number $R\ 25\ 2$ corresponds to 25×10^{-2}, which we can reduce stepwise to 2.5×10^{-1} and 0.25×10^{0}. A meaningful way to show this datatype would therefore be:

```
instance Show RealN where
  show (R a e) = show a ++ "*10^(-" ++ show e ++ ")"
```

There are obviously many ways to represent the same number with this number type. 1, for instance, can be represented as

```
one = R 1 0
one = R 10 1
one = R 100 2
one = R 1000 3
...
```

To keep numbers as concise as possible, we define a function to simplify numbers with redundant zeros:

```
simplify :: RealN → RealN
simplify (R 0 _) = R 0 0
simplify (R a e) | e > 0 ∧
                   a `rem` 10 ≡ 0 = simplify (R (a `div` 10) (e − 1))
                 | otherwise      = R a e
```

As long as the exponent is greater 0 and the base a is divisible by 10, we reduce the exponent by one and divide a by 10. In other words, we remove unnecessary zeros. The following constructor uses *simplify* to create clean real numbers:

```
real :: Natural → Natural → RealN
real i e = simplify (R i e)
```

1.7 $\mathbb{R}$

We now define how to check two real numbers for equality:

```
instance Eq RealN where
  r1@(R a e1) ≡ r2@(R b e2) | e1 ≡ e2 = a ≡ b
                            | e1 > e2 = r1 ≡ blowup e1 r2
                            | e1 < e2 = blowup e2 r1 ≡ r2
```

If the exponents are equal, then we trivially compare the coefficients. Otherwise, we first expand the number with the smaller exponent using *blowup*:

```
blowup :: Natural → RealN → RealN
blowup i (R r e) | i ⩽ e       = R r e
                 | otherwise = R (r * 10↑(i − e)) i
```

That is simple! If the target i is greater than the current exponent of the number, we just multiply the coefficient by 10 raised to the difference of the target exponent and the current exponent and make the target the new exponent. Otherwise, nothing changes.

We continue with comparison, which follows exactly the same logic:

```
instance Ord RealN where
  compare r1@(R a e1) r2@(R b e2) | e1 ≡ e2 = compare a b
                                  | e1 > e2 = compare r1 (blowup e1 r2)
                                  | e1 < e2 = compare (blowup e2 r1) r2
```

Now we make *RealN* instance of *Num*:

```
instance Num RealN where
  (R a e1) + (R b e2) | e1 ≡ e2   = simplify $ R (a + b) e1
                      | e1 > e2   = simplify $ R (a + b * 10↑(e1 − e2)) e1
                      | otherwise = simplify $ R (a * 10↑(e2 − e1) + b) e2
  (R a e1) − (R b e2) | e1 ≡ e2 ∧
                        a ⩾ b     = simplify $ R (a − b) e1
                      | e1 ≡ e2   = error "subtraction beyond zero!"
                      | e1 > e2   = simplify $ (R a e1) − (R (b * 10↑(e1 − e2)) e1)
                      | otherwise = simplify $ (R (a * 10↑(e2 − e1)) e2) − (R b e2)
  (R a e1) * (R b e2) = real (a * b) (e1 + e2)
  negate  r       = r   -- we cannot negate natural numbers
  abs     r       = r
  signum  r       = r
  fromInteger i = R (fromIntegral i) 0
```

Addition is again the same logic. For two numbers with equal exponents, we just add the coefficients. If the exponents differ, we first convert the smaller number to the greater exponent.

For subtraction, note that we define *RealN* like numbers before without negatives. To consider signedness, we still have to use the datatype *Signed RealN*. Consequently, we have to rule out the case where the first number is smaller than the second one.

Multiplication is interesting. We multiply two real numbers by multiplying the coefficients and adding the exponents. We have already seen this logic, when defining the natural number type. Some simple examples may convince you that this is the right way to go. 1×0.1, for instance, is 0.1. In terms of our *RealN* type, this corresponds to $(R\ 1\ 0) * (R\ 1\ 1) \equiv (R\ (1 * 1)\ (0 + 1)$.

The next task is to make *RealN* instance of *Fractional*:

```
instance Fractional RealN where
  (/) = rdiv 17
  fromRational r = (R (fromIntegral $ R.numerator r) 0) /
                   (R (fromIntegral $ R.denominator r) 0)
```

The method *fromRational* is quite simple. We just create two real numbers, the numerator of the original fraction and its denominator, and the we divide them. What we need to do this, of course, is division. Division, as usual, is a bit more complicated than the other arithmetic operations. We define it as follows:

```
rdiv :: Natural → RealN → RealN → RealN
rdiv n r1@(R a e1) r2@(R b e2) | e1 < e2 =
                                 rdiv n (blowup e2 r1) r2
                               | a < b ∧ e1 ≡ e2 =
                                 rdiv n (blowup (e2 + 1) r1) r2
                               | otherwise =
                                 simplify (R (go n a b) (e1 − e2 + n))
  where go i x y | i ⩽ 0 = 0
                 | otherwise =
                   case x `quotRem` y of
                     (q, 0) → 10 ↑ i ∗ q
                     (q, r) → let (r′, e)      = borrow r y
                                  q′           = 10 ↑ i ∗ q
                              in if e > i then q′
                                          else q′ + go (i − e) r′ y
```

rdiv has one more argument than the arithmetic operations seen before. This additional argument, n, defines the precision of the result. This is necessary, because, as we will see, the number of iterations the function has to perform depends on the precision the result is expected to have.

If the first number is smaller than the second, either because its exponent or its coefficient is smaller, we blow it up so that it is at least the same size. Then we calculate the new coefficient by means of *go* and the new exponent as the difference of the first and the second exponent plus the expected precision. Note that division has the inverse effect on the size of the exponents as multiplication. When we look again at the example 1 and 0.1, we have $1/0.1 = 10$, which translates to $(R\ 1\ 0)\ /\ (R\ 1\ 1) = R\ (1\ /\ 1)\ (0 - 1)$, which of course is the same as $R\ 10\ 0$.

The inner function *go* proceeds until i, which initially is n, becomes 0 or smaller. In each step, we divide x, initially the coefficient of the first number, and y, the coefficient of the second number. If the result leaves no remainder, we are done. We just raise q to the power of the step in question. Otherwise, we continue dividing the remainder r by y. But before we continue, we borrow from y, that is, we increase r until it is at least y. i is then decremented by the number of zeros we borrowed this way. If we run out of i, so to speak, that is if $e > i$, then we terminate with q raised to the current step.

borrow is just the same as *blowup* applied to two natural numbers:

$$
\begin{aligned}
&\mathit{borrow} :: \mathit{Natural} \to \mathit{Natural} \to (\mathit{Natural}, \mathit{Natural}) \\
&\mathit{borrow}\ a\ b \mid a \geqslant b \quad\ = (a, 0) \\
&\qquad\qquad \mid \mathit{otherwise} = \mathbf{let}\ (x, e) = \mathit{borrow}\ (10 * a)\ b\ \mathbf{in}\ (x, e + 1)
\end{aligned}
$$

We now make *RealN* instance of *Real*. We need to define just one method, namely how to convert *RealN* to *Rational*, which we do just by creating a fraction with the coefficient of the real number in the numerator and 10 raised to the exponent in the denominator. The rest is done by the *Rational* number type:

$$
\begin{aligned}
&\mathbf{instance}\ \mathit{Real}\ \mathit{RealN}\ \mathbf{where} \\
&\quad \mathit{toRational}\ (R\ r\ e) = i \,\%\, (10 \uparrow x) \\
&\qquad \mathbf{where}\ i\ = \mathit{fromIntegral}\ r :: \mathit{Integer} \\
&\qquad\qquad\ \ x = \mathit{fromIntegral}\ e :: \mathit{Integer}
\end{aligned}
$$

We further add a function to convert real numbers to our *Ratio* type:

$$
\begin{aligned}
&\mathit{r2R} :: \mathit{RealN} \to \mathit{Ratio} \\
&\mathit{r2R}\ (R\ a\ e) = \mathit{ratio}\ a\ (10 \uparrow e)
\end{aligned}
$$

We also add a function to convert our real number type to the standard *Double* type:

$$
\begin{aligned}
&\mathit{r2d} :: \mathit{RealN} \to \mathit{Double} \\
&\mathit{r2d}\ r@(R\ a\ e) \mid e > 16 \quad\ = \mathit{r2d}\ (\mathit{roundr}\ 16\ r) \\
&\qquad\qquad\qquad\ \mid \mathit{otherwise} = (\mathit{fromIntegral}\ a)\ /\ 10 \uparrow e
\end{aligned}
$$

An inconvenience is that we have to round a number given in our number type so it fits into a *Double*. For this, we assume that the *Double* has a precision of 16 decimal digits. This is not quite true. The *Double* type has room for 16 digits. But if the first digits after the decimal point are zeros, the *Double* type will present this as a number raised to a negative exponent, just as we do with our real type. In this case, the *Double* type may have a much higher precision than 16. For our purpose, however, this is not too relevant.

So, here is how we round:

$$
\begin{aligned}
&\mathit{roundr} :: \mathit{Natural} \to \mathit{RealN} \to \mathit{RealN} \\
&\mathit{roundr}\ n\ (R\ a\ e) \mid n \geqslant e \quad\ = R\ a\ e \\
&\qquad\qquad\qquad\ \mid \mathit{otherwise} = \mathbf{let}\ b\ = a\ \text{`}\mathit{div}\text{`}\ 10 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\ \ l\ = a - 10 * b \\
&\qquad\qquad\qquad\qquad\qquad\qquad\ \ d \mid l < 5 = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \mid \mathit{otherwise} = 1 \\
&\qquad\qquad\qquad\qquad\qquad\quad\ \mathbf{in}\ \mathit{roundr}\ n\ (R\ (b + d)\ (e - 1))
\end{aligned}
$$

That is, we first get the least significant digit l as $l = a - 10 * (\mathit{div}\ a\ 10)$, where, if *div* is the Euclidian division, the last digit of a remains. If the last digit is less than 5, we just drop it off. Otherwise, we add 1 to the whole number. If we now define

$one = R\ 1\ 0$
$three = R\ 3\ 0$
$third = one\ /\ three$,

then *roundr* 16 (*three* $*$ *third*) yields 1, as desired.

Finally, we define

type *SReal* = *Signed RealN*

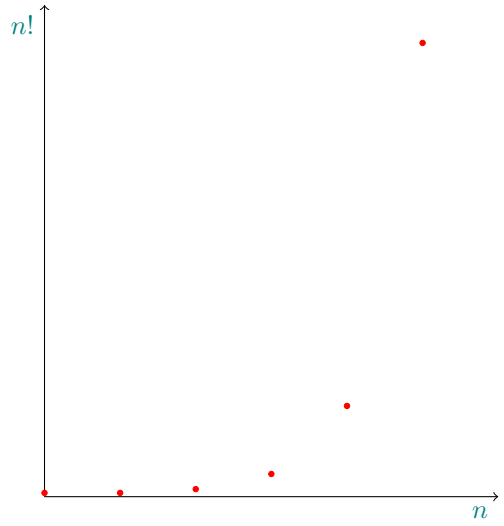
and have a fulfledged real number datatype.

1.8 Real Factorials

As we have already done in the domain of integers, we will now try to generalise some of the combinatorial sequences to the domain of real numbers. We did this with integers for the binomial coefficients. Before we can introduce real binomials, however, we need to look at factorials. The factorial of n, $n!$, is the number of possible permutations of a sequence of n elements. This is a very concrete and easy to grasp concept. A function, however, that results in a fraction or even an irrational number does not count anything similar to that. It may measure a *continuous* quantity (a weight or distance, for example), but certainly not a *discrete* value like a counting result.

This leads to a didactical dilemma that often arises in modern mathematics. Mathematics usually aims to generalise concepts and often independent of concrete applications of this generalised concept. The applications often follow, sometimes years or even centuries later. We all know geometry in the two-dimensional plane. Mathematicians have generalised the concepts of two-dimensional geometry to n dimensions. From a naïve perspective looking only at immediate applicability, there is not much sense in such geometries beyond three or, with Einstein in mind, four dimensions. However, the areas of mathematics that study *space* with more than four or even infinitely many dimensions (such as linear algebra and complex analysis), actually have many applications in statistics, engineering, physics and other rather practical domains. Furthermore, applications is an important, but not the only motivation for mathematical investigation. Mathematics studies its fundamental concepts (like numbers, sets or the space in which we exercise geometry) to arrive at general theorems. So, even when there is no immediate application yet, for the sake of better understanding of the concepts involved, mathematicians do not hesitate to ask questions that appear to be absurd or meaningless to "ordinary" people (whoever those are).

An obvious way to look at real factorials is to represent them in the Cartesian plane, *i.e.* in the coordinate system. We can sketch the factorials of natural numbers as:



where the factorials of n are shown on the vertical axis (the y-axis) with a scale of $1:10$ in relation to the values for n on the horizontal axis (the x-axis). The diagram shows the factorials of the numbers $n \in \{0, 1, 2, 3, 4, 5\}$, which are $1, 1, 2, 6, 24, 120$. The factorials are drawn as dots with the coordinates $(n, n!)$. The space between these dots is empty. A continuous interpretation of the factorials would ask for the values within this white space and aim to find a curve that connects the discrete dots in the picture.

One obvious requirement for such a function f is that for any $n \in \mathbb{N}, f(n) = n!$. A function that (almost) fulfils this requirement is the *Gamma function*, Γ. For any integer $n > 1$, Γ is

$$\Gamma(n) = (n-1)\Gamma(n-1) \tag{1.34}$$

with $\Gamma(1) = 1$. We can model this in Haskell as

$$\begin{aligned}
&\mathit{gamman} :: \mathit{Natural} \to \mathit{Natural} \\
&\mathit{gamman}\ 0 = \bot \\
&\mathit{gamman}\ 1 = 1 \\
&\mathit{gamman}\ n = (n-1) * \mathit{gamman}\ (n-1)
\end{aligned}$$

When we apply this to the numbers $1 \ldots 10$, $\mathit{map}\ \mathit{gamman}\ [1\,..\,10]$, we see:

$[1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880]$

There is a snag. Here are the factorials, created with $\mathit{map}\ \mathit{fac}\ [1\,..\,10]$:

$[1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800]$

The Γ function, hence, creates the factorials, but shifted down by one. Indeed, we have

$$\Gamma(n+1) = n! \tag{1.35}$$

With $[gamman\ (n+1) \mid n \leftarrow [1\,..\,10]]$, we finally see the factorials in their correct places in the sequence:

$[1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800]$

That the Γ function is defined like this, is for historical reasons and does not need to bother us here. We only have to keep in mind that, whenever we want to make the connection from Γ to factorial, we need to increment n by 1.

Well, the *gamman* function above shows us just another way to express factorials for natural numbers. But we wanted to find a function for real numbers. There are indeed many ways to define the Γ function in that domain. The canonical way is the *Euler integral of the second kind*, but, since we have not yet introduced integrals, we choose another way that we already know, namely infinite products.

We first look at the following product, which was found already by Euler:

$$\Gamma(x) = \frac{1}{x} \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^x}{1+\frac{x}{n}} \tag{1.36}$$

We can reformulate this equation in Haskell as:

```
gammal :: Natural → RealN → RealN
gammal i x = (1 / x) * product [(1 + 1 / n) ** x / (1 + x / n) | n ← [1 .. m]]
  where m = fromIntegral i
```

The function receives a *Natural* and a *RealN*. The *Natural*, i, is just the number of iterations we want to perform, since we do not have the time to go through all iterations of the infinite product. The *go* function implements the product itself. Finally we multiply $\frac{1}{x}$ and we are done.

Let us look at how precise this function can mimic $n!$ for a given number of iterations. When we apply *gammal* on $x = 1$, then we obviously get 1, since we compute

$$\frac{1}{1} \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^1}{1+\frac{1}{n}} = \prod_{n=1}^{\infty} \frac{1+\frac{1}{n}}{1+\frac{1}{n}} = 1 \times 1 \times 1 \times \cdots = 1.$$

Indeed, *gammal* 1 1 immediately yields 1. Next, we try *gammal* 1 2 and get

$$0.\overline{6}$$

That is far off the expected value 1. We increase the number of iterations and try *gammal* 10 2:

$$0.91666666\ldots$$

Already better, but still not 1. We try *gammal* 100 2 and see:

$$0.99019607\ldots$$

and *gammal* 1000 2:

$$0.99900199\ldots$$

We see, the function converges slowly. When we try the factorials for $3\ldots6$ with 1000 iterations, we see:

$$1.99401993\ldots, 5.96418512\ldots, 23.76178831\ldots, 118.21843985\ldots$$

The first two values are fairly close to the expected results. $\Gamma(5)$ and $\Gamma(6)$, which should be 24 and 120, however, are clearly off. We try *gammal* 10000 5 and see:

$$23.97601798\ldots$$

Not good, but much better. What about *gammal* 10000 6? Here it is:

$$119.82018583\ldots$$

Still more than 0.1 off the expected result 120. We try again with 100 000 iterations:

$$119.98200185\ldots$$

That was still not enough! Let us try with one million iterations:

$$119.99820001\ldots$$

We see that the infinite product slowly approaches the expected results of the Γ function, but the stress here is on "slowly". Already for $x = 6$, we need a lot of iterations to achieve a deviation of less than 0.01.

Let us look at another infinite product. It is not faster than the one we looked at – on the contrary, it is even slower – but it is a nice formula:

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)^{-1} e^{\frac{x}{n}}, \tag{1.37}$$

where γ is the Euler-Mascheroni constant and e, the Euler-Napier constant. This formula was found by German mathematician Karl Weierstrass (1815 – 1897) who was instrumental in the foundations of modern analysis. In Haskell, his definition of the Γ function may look like:

```
gammae :: Natural → RealN → RealN
gammae i x = f * product [e ** (x / n) * (1 + (x / n)) ** (-1) | n ← [1 .. m]]
  where f = e ** ((-1) * gamma * x) / x
        m = fromIntegral (i - 1)
```

An important difference to the first formula is that, from this one, it is not obvious that it should result in 1 for $x = 1$. Let us give it a try. *gammae* 1 1:

$$0.56145836\ldots$$

That is an ugly result! Already for 1, it diverges from the expected result by almost $\frac{1}{2}$! Let us see how many iterations we need to approach 1:

gammae 10 1 = 0.95043595...
gammae 100 1 = 0.99500219...
gammae 1000 1 = 0.99949804...

and so on. What about 2?

gammae 1 2 = 0.15761774...
gammae 10 2 = 0.82120772...
gammae 100 2 = 0.98022710...
gammae 1000 2 = 0.99799833...

It definitely converges slower than Euler's formula. For the remainder of this section, we will therefore stick to Euler's solution.

Let us look at some other numbers, not positive integers, for instance:

$$\Gamma(0) = \infty,$$
$$\Gamma(-1) = -\infty,$$

So, $\Gamma(0)$ and $\Gamma(-1)$ yield $\pm\infty$. What about $\frac{1}{2}$? Using *gammal*, we see the following results:

gammal 1 0.5 = 1.8856...
gammal 10 0.5 = 1.7927...
gammal 100 0.5 = 1.7746...
gammal 1000 0.5 = 1.7726...

It will finally approach:

$$\Gamma(0.5) = 1.77267520\ldots$$

Is it possible that $\Gamma(0.5)$ yields such a boring number? Well, is it such boring? Look what happens (with one million iterations):

gammal 1000000 0.5 $*$ *gammal* 1000000 0.5 = 3.14159...

That is π! So $\Gamma(0.5) = \sqrt{\pi}$. Not bad! The occurrence of both, e and γ, in Weierstrass' formula already looked somewhat suspicious. It was only a matter of time, when we would meet π in applying the function to some values. In fact, there are many values for which Γ produces a product of $\sqrt{\pi}$ with some fraction. For instance:

$$\begin{aligned}
\Gamma\left(\frac{3}{2}\right) &= \frac{1}{2}\sqrt{\pi}, \\
\Gamma\left(\frac{5}{2}\right) &= \frac{3}{4}\sqrt{\pi}, \\
\Gamma\left(\frac{7}{2}\right) &= \frac{15}{8}\sqrt{\pi}, \\
\Gamma\left(\frac{9}{2}\right) &= \frac{105}{16}\sqrt{\pi}, \\
&\ldots
\end{aligned}$$

These results suggest a pattern for odd numbers n. Apparently, $\Gamma(\frac{n}{2})$ yields a product of the form $\frac{k}{2^{(n-1)/2}}\sqrt{\pi}$, where $k = (n-2)(k_{n-2})$.

Can we say more about the factor k? For the odd numbers $1, 3, \ldots 11$, k is 1, 3, 15, 105, 945, 10395. These are the *double factorials*, $n!!$, for the odd numbers, *i.e.* the products of all odd numbers $1, 3, .., n$. We, hence, have for odd numbers n:

$$\Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!!}{2^{\frac{n-1}{2}}}\sqrt{\pi} \tag{1.38}$$

which can be implemented in Haskell as

$$
\begin{aligned}
&\mathit{gammaho} :: \mathit{Natural} \to \mathit{RealN}\\
&\mathit{gammaho}\ 1 = \mathit{sqrt}\ \mathit{pi}\\
&\mathit{gammaho}\ n \mid \mathit{even}\ n \quad\ = \mathit{error}\ \texttt{"not an odd number!"}\\
&\qquad\qquad \mid \mathit{otherwise} = \mathit{rff}\ (n-2)\ /\ 2 \ast\ast\ i\\
&\qquad\qquad\qquad\qquad \ast\ \mathit{sqrt}\ \mathit{pi}\\
&\quad \textbf{where}\ \mathit{rff} = \mathit{fromIntegral} \circ \mathit{facfac}\\
&\qquad\quad\ \ i \quad = \mathit{fromIntegral}\ ((n-1)\ `\mathit{div}`\ 2)
\end{aligned}
$$

The Γ function shows many of such suprising properties. It has been and is still being extensively studied and a lot of relations to other functions, such as the Riemann zeta function, have been found.

But let us now go on to the definition of real binomial coefficients using the Γ function. To this end, we define a new *choose* function, namely:

$$
\begin{aligned}
&\mathit{chooser} :: \mathit{Natural} \to \mathit{RealN} \to \mathit{RealN} \to \mathit{RealN}\\
&\mathit{chooser}\ i\ n\ k = \mathit{gammal}\ i\ (n+1)\ /\\
&\qquad\qquad\qquad (\mathit{gammal}\ i\ (k+1) \ast \mathit{gammal}\ i\ (n-k+1))
\end{aligned}
$$

Again, we try this function on integers. For instance:

$\mathit{choose}\ 2\ 1 = 2$
$\mathit{choose}\ 3\ 1 = 3$
$\mathit{choose}\ 3\ 2 = 3$
$\mathit{choose}\ 5\ 2 = 10$
$\mathit{choose}\ 5\ 3 = 10$
$\mathit{choose}\ 7\ 2 = 21$
$\mathit{choose}\ 7\ 3 = 35$

We start with $\binom{2}{1} = 2$, using *chooser*

$\mathit{chooser}\ 1\ 2\ 1 = 1.5$

Far off. So, again, we increase the number of iterations:

$\mathit{chooser}\ 1\ 2\ 1 = 1.5$
$\mathit{chooser}\ 10\ 2\ 1 = 1.8461...$
$\mathit{chooser}\ 100\ 2\ 1 = 1.9805...$
$\mathit{chooser}\ 1000\ 2\ 1 = 1.9980...$
$\mathit{chooser}\ 10000\ 2\ 1 = 1.9998...$

After 10 000 iterations, we come pretty close. Let us try the other examples with 10 000 iterations:

$\mathit{chooser}\ 10000\ 3\ 1 = 2.9994...$
$\mathit{chooser}\ 10000\ 3\ 2 = 2.9994...$
$\mathit{chooser}\ 10000\ 5\ 2 = 9.9940...$
$\mathit{chooser}\ 10000\ 5\ 3 = 9.9940...$
$\mathit{chooser}\ 10000\ 7\ 2 = 10.9790...$
$\mathit{chooser}\ 10000\ 7\ 3 = 34.9580...$

which is fairly close for all these numbers.

The resulting function has been little studied. It is known that many of the binomial identities fail for real numbers. The behaviour for different values of n and k not integers is very complex. We will not go into details here. But we will certainly come back to the Γ-function and its applications later on.

1.9 The Stern-Brocot Tree

Achille Brocot (1817 – 1878) was part of a clockmaker dynasty in Paris started by his father and continuing after his death. The Brocots had a strong emphasis on engineering. Under the pressure of cheap low-quality imports mainly from the USA, they innovated clockmaking with the aim to reduce production cost without equivalent degradation in quality. The constant engineering work manifested in a considerable number of patents hold by family members. The most productive, in terms of engineering, however, was Achille who improved many of his father's inventions and developed new ones. He also introduced a novelty to mathematics, which, surprisingly, has not only practical, but also theoretical value.

In clockmaking, as in machine construction in general, determining the ratio of components to each other, for instance, gear ratios, is a very frequent task. As often in practice, those ratios are not nice and clean, but very odd numbers with many decimal digits. Brocot developed a way to easily approximate such numbers with arbitrary precision and, in consequence, to approximate any real number with arbitrary precision. In the process, he developed yet another way to list all rational numbers.

Brocot's method can be described in terms of finite continued fractions. Recall that we can use lists of the form

$$[n; a, b, c, \dots]$$

to encode continued fractions like

$$n + \frac{1}{a + \frac{1}{b + \frac{1}{c + \dots}}}.$$

In contrast to continued fractions we have seen so far, we now look at finite continued fractions that actually result in rational numbers. The process to compute such a continued fraction can be captured in Haskell as:

$$
\begin{aligned}
&\mathit{contfracr} :: [\mathit{Ratio}] \to \mathit{Ratio}\\
&\mathit{contfracr}\ [\,] \quad\ = 0\\
&\mathit{contfracr}\ [i] \quad\ = i\\
&\mathit{contfracr}\ (i : \mathit{is}) = i + (\mathit{invert}\ \$\ \mathit{contfracr}\ \mathit{is})
\end{aligned}
$$

Here, *invert* is a function to create the multiplicative inverse of a fraction, *i.e.* $\mathit{invert}(\frac{n}{d}) = \frac{d}{n}$ or in Haskell:

$$
\begin{aligned}
&\mathit{invert} :: \mathit{Ratio} \to \mathit{Ratio}\\
&\mathit{invert}\ (Q\ n\ d) = Q\ d\ n
\end{aligned}
$$

As you will see immediately, the expression $(\mathit{invert}\ \$\ \mathit{contfracr}\ \mathit{is})$, corresponds to

$$\frac{1}{\mathit{contfracr}\ \mathit{is}}.$$

The definition above, hence, creates a continued fraction that terminates with the last element in the list.

Now we introduce a simple rule to create from any continued fraction given in list notation two new continued fractions:

$$
\begin{aligned}
&\mathit{brocotkids} :: [\mathit{Ratio}] \to ([\mathit{Ratio}], [\mathit{Ratio}])\\
&\mathit{brocotkids}\ r = \mathbf{let}\ h \ \ = \mathit{init}\ r\\
&\qquad\qquad\qquad\qquad\ \ l \ \ = \mathit{last}\ r\\
&\qquad\qquad\qquad\qquad\ \ s \ \ = \mathit{length}\ r\\
&\qquad\qquad\qquad\qquad\ \ \mathit{k1} = h +\!\!+ [l + 1]\\
&\qquad\qquad\qquad\qquad\ \ \mathit{k2} = h +\!\!+ [l - 1, 2]\\
&\qquad\qquad\quad \mathbf{in\ if}\ \mathit{even}\ s\ \mathbf{then}\ (\mathit{k1}, \mathit{k2})\ \mathbf{else}\ (\mathit{k2}, \mathit{k1})
\end{aligned}
$$

This function yields two lists, k_1 and k_2. They are computed as the initial part of the input list, to which, in the case of k_1, one number is appended, namely the last element of the input list plus 1, or, in the case of k_2, two numbers are appended, namely the last element minus 1 and 2. For the input list $[0, 1]$, which is just 1, for instance, k_1 is $[0, 2]$, which is $\frac{1}{2}$, and k_2 is $[0, 0, 2]$, which is $\frac{1}{\frac{1}{2}} = 2$.

When we compare the paritiy of the length of the lists, we see that k_1 has the same parity as the input list and k_2 has the opposite parity. In particular, if the input list is even, then k_1 is even and k_2 is odd; if it is odd, then k_1 is odd and k_2 is even.

Now, we see for an even list like $[a, b]$ that there is an integer, a, to which the inverse of the second number, b is added. If b grows, then the overall result shrinks. The structure of an odd list is like $[a, b, c]$. Again, the integer a is added to the inverse of what follows in the list. But this time, if c grows, the inverse of c, $\frac{1}{c}$, shrinks and, as such, the value of $\frac{1}{b+1/c}$ grows. Therefore, if the number of elements is even, the value of k_1 is less than the value of the input list and, if it is odd, then the value is greater. You can easily convince yourself that for k_2 this is exactly the other way round. In consequence, the

numerical value of the left list returned by *brocotkids* is always smaller than that of the input list and that of the right one is greater.

Using this function, we can now approximate any real number. The idea is that, if the current continued fraction results in a number greater than the number in question, we continue with the left kid; if it is smaller, we continue with the right kid. Here is an implementation:

```
approx :: Natural → RealN → [Ratio]
approx i d = go i [0, 1]
  where go :: Natural → [Ratio] → [Ratio]
        go 0 _ = [ ]
        go j r = let k@(Q a b) = contfracr r
                     d' = (fromIntegral a) / (fromIntegral b)
                     (k1, k2) = brocotkids r
                 in if d' ≡ d then [Q a b]
                              else if d' < d then k : go (j − 1) k2
                                             else  k : go (j − 1) k1
```

This function takes two arguments. The first, a natural number, defines the number of iterations we want to do. The second is the real number we want to approximate. We start the internal *go* with i and the list $[0, 1]$. In *go*, as long as $j > 0$, we compute the rational number that corresponds to the input list; then we compute the corresponding real number. If the number we computed this way equals the input (*i.e.* the input is rational), we are done. Otherwise, if it is less than the input, we continue with k_2; if it is greater, we continue with k_1.

The function yields the whole trajectory whose last number is the best approximation with n iterations. The result of *approx* 10 *pi*, for instance is:

$$1, 2, 3, 4, \frac{7}{2}, \frac{10}{3}, \frac{13}{4}, \frac{16}{5}, \frac{19}{6}, \frac{22}{7}.$$

The last fraction $\frac{22}{7}$ is approximately 3.142857, which still is a bit away from 3.141592. We reach 3.1415 with *approx* 25 *pi*, for which the last fraction is $\frac{333}{106} = 3.141509$. This way, we can come as close to π as we wish.

Since, with the *brocotkids* function, we always create two follow-ups for the input list, we can easily define a binary tree where, for each node, k_1 is the left subtree and k_2 is the right subtree. This tree, in fact, is well-known and is called *Stern-Brocot tree* in honour of Achille Brocot and Moritz Stern, the German number theorist we already know from the discussion of the Calkin-Wilf tree.

The Stern-Brocot tree can be defined as

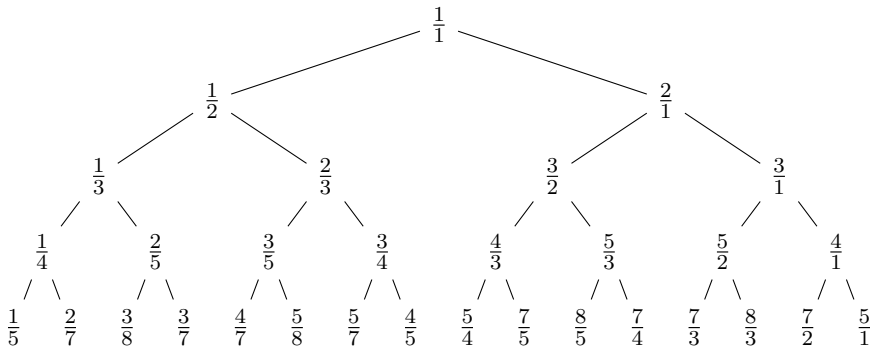
```
type SterBroc = Tree [Ratio]
sterbroc :: Zahl → [Ratio] → SterBroc
sterbroc i r | i ≡ 0       = Node r []
             | otherwise = let (k1, k2) = brocotkids r
                           in Node r [sterbroc (i − 1) k1,
                                      sterbroc (i − 1) k2]
```

The function *sterbroc* takes an integer argument to define the number of generations we want to create and an initial list of *Ratio*. If we have exhausted the number of generations, we create the final *Node* without kids. Otherwise, we create the *brocotkids* and continue with *sterbroc* on k_1 and k_2. If we start with a negative number, we will generate infinitely many generations.

We can now convert the continued fractions in the nodes to fractions by *fmap*ing *contfracr* on them. Here is a function that creates the Stern-Brocot Tree from root node $[0, 1]$ labled with fractions:

```
sterbrocTree :: Zahl → Tree Ratio
sterbrocTree i = fmap contfracr (sterbroc i [0, 1])
```

For the first five generations, this tree is



As you can see at once, this tree has many properties in common with the Calkin-Wilf tree. First and trivially, the left kid of a node k is less than k and the right kid of the same node is greater than k. The left-most branch of the tree contains all fractions with 1 in the numerator like $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and so on. The right-most branch contains the integers $\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}$ and so on.

Furthermore, the product of each generation is 1. For instance, $\frac{1}{1} = 1$, $\frac{1}{2} \times \frac{2}{1} = 1$, $\frac{1}{3} \times \frac{2}{3} \times \frac{3}{2} \times \frac{3}{1} = 1$ and so on. In fact, we see in each generation the same fractions we would also see in the Calkin-Wilf tree. The order of the fraction, however, is different. More precisely, the order of the inner fractions differs, since, as we have seen, the left-most and right-most numbers are the same.

We could hence ask the obvious question: how can we permute the generations of the

Stern-Brocot tree to obtain the generations of the Calkin-Wilf tree and vice versa? Let us look at an example. The 4^{th} generation of the Calkin-Wilf tree is
getKids 4 (*calWiTree* 4 (Q 1 1)):

$$\frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}.$$

The 4^{th} generation of the Stern-Brocot tree is
getKids 4 (*sterbroctree* 4):

$$\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, \frac{4}{1}.$$

We see that only some fractions changed their places and the changes are all direct swaps, such that the second position in the Calkin-Wilf tree changed with the fifth position and the fourth position changed with the seventh position. The other positions, the first, third, sixth and eighth, remain in their place. We could describe this in cyclic notation, using indexes from 0 – 7 for the eight positions:

$$(1, 4)(3, 6).$$

In other words, we represent the generations as arrays with indexes 0 – 7:

	0	1	2	3	4	5	6	7
Calkin-Wilf	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{5}$	$\frac{5}{2}$	$\frac{2}{5}$	$\frac{5}{3}$	$\frac{3}{4}$	$\frac{4}{1}$
Stern-Brocot	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{5}{2}$	$\frac{4}{1}$

So, what is so special about the indexes 1, 3, 4 and 6 that distinguishes them from the indexes 0, 2, 5 and 7? When we represent these numbers in binary format with leading zeros, so that all binary numbers have the same length, we have

0	1	2	3	4	5	6	7
000	001	010	011	100	101	110	111

When we look at the indexes whose fractions do not change, *i.e.* 0, 2, 5 and 7, we see one property that they all have in common: they are all symmetric. That is, when we reverse the bit strings, we still have the same number. $0 = 000$ reversed is still $000 = 0$; $2 = 010$ reversed is still $010 = 2$; $5 = 101$ reversed is still $101 = 5$ and $7 = 111$ reversed is still $111 = 7$. $1 = 001$ reversed, however, is $100 = 4$ and vice versa and $3 = 011$ reversed

is $110 = 6$. This corresponds exactly to the permutation $(1, 4)(3, 6)$ and is an instance of a bit-reversal permutation.

Let us try to implement the bit-reversal permutation. First we implement the bit-reverse of the indexes. To do so, we first need to convert the decimal index into a binary number; then we add zeros in front of all binary numbers that are shorter than the greatest number; then we simply reverse the lists of binary digits, remove the leading zeros and convert back to decimal numbers. This can be nicely expressed by the function

$$
\begin{aligned}
&\mathit{bitrev} :: \mathit{Int} \to \mathit{Int} \to \mathit{Int} \\
&\mathit{bitrev}\ x = \mathit{fromIntegral} \circ \mathit{fromBinary} \circ \\
&\qquad\qquad \mathit{cleanz} \circ \mathit{reverse} \circ \mathit{fillup}\ x\ 0 \circ \\
&\qquad\qquad \mathit{toBinary} \circ \mathit{fromIntegral}
\end{aligned}
$$

where *fillup* is defined as

$$
\begin{aligned}
&\mathit{fillup} :: \mathit{Int} \to \mathit{Int} \to [\mathit{Int}] \to [\mathit{Int}] \\
&\mathit{fillup}\ i\ z\ \mathit{is} \mid \mathit{length}\ \mathit{is} \equiv i = \mathit{is} \\
&\qquad\qquad\qquad \mid \mathit{otherwise} \quad\ = \mathit{fillup}\ i\ z\ (z : \mathit{is})
\end{aligned}
$$

and *cleanz* as

$$
\begin{aligned}
&\mathit{cleanz} :: [\mathit{Int}] \to [\mathit{Int}] \\
&\mathit{cleanz}\ [\,] \quad\ = [\,] \\
&\mathit{cleanz}\ [0] \quad\ = [0] \\
&\mathit{cleanz}\ (0 : \mathit{is}) = \mathit{cleanz}\ \mathit{is} \\
&\mathit{cleanz}\ \mathit{is} \quad\ = \mathit{is}
\end{aligned}
$$

To apply this, we first have to calculate the size of the greatest number in our set in binary format. If we assume that we have a list of consecutive numbers from $0 \ldots n-1$, then the size of the greatest number is just $\log_2 n$, the binary logarithm of n. For $n = 8$, for instance, this is 3. With this out of the way, we can define a bit reversal of the indexes of any set $[a]$ as:

$$
\begin{aligned}
&\mathit{idxbitrev} :: [a] \to [\mathit{Int}] \\
&\mathit{idxbitrev}\ \mathit{xs} = \mathbf{let}\ l\ = \mathit{fromIntegral}\ \$\ \mathit{length}\ \mathit{xs} \\
&\qquad\qquad\qquad\qquad\ \ x = \mathit{round}\ \$\ \mathit{logBase}\ 2\ (\mathit{fromIntegral}\ l) \\
&\qquad\qquad\qquad\ \ \mathbf{in}\ [\mathit{bitrev}\ x\ i \mid i \leftarrow [0\,..\,l-1]]
\end{aligned}
$$

and use this function to permute the original input list:

$$
\begin{aligned}
&\mathit{bitreverse} :: [a] \to [a] \\
&\mathit{bitreverse}\ \mathit{xs} = \mathit{go}\ \mathit{xs}\ (\mathit{idxbitrev}\ \mathit{xs}) \\
&\quad \mathbf{where}\ \mathit{go}\ \_\ [\,] \quad\ = [\,] \\
&\qquad\qquad \mathit{go}\ \mathit{zs}\ (p : \mathit{ps}) = \mathit{zs}\ !!\ p : \mathit{go}\ \mathit{zs}\ \mathit{ps}
\end{aligned}
$$

Applied on the list $[0, 1, 2, 3, 4, 5, 6, 7]$, we see exactly the $(1, 4)(3, 6)$ permutation we saw above, namely $[0, 4, 2, 6, 1, 5, 3, 7]$. Applied on a generation from the Calkin-Wilf tree, we see the corresponding generation from the Stern-Brocot tree.

Let $t = calWiTree\ (-1)\ (1 \,\%\, 1)$, we see for

$getKids\ 3\ t$	$\frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}$
$bitreverse\ (getKids\ 3\ t)$	$\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{1}$
$getKids\ 4\ t$	$\frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}$
$bitreverse\ (getKids\ 4\ t)$	$\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, \frac{4}{1}$

Since the generations of the Stern-Brocot tree are nothing but permutations of the Calkin-Wilf tree, we can derive a sequence from the Stern-Brocot tree that lists all rational numbers. We create this sequence in exaclty the same way we did for the CalkinWilf tree, namely

```
enumQsb :: [Ratio]
enumQsb = go 1 $ sterbrocTree (-1)
  where go i t = getKids i t ++ go (i + 1) t
```

The numerators of the sequence derived in this way from the Calkin-Wilf tree equal the well-known Stern sequence. Is there another well-known sequence that is equivalent to the numerators of the Stern-Brocot tree sequence? Let us ask the On-line Encyclopedia with the first segment of that sequence generated by *map numerator* (*take* 20 *enumQsb*):

$$1, 1, 2, 1, 2, 3, 3, 1, 2, 3, 3, 4, 5, 5, 4, 1, 2, 3, 3, 4, 5, 5, 4, 5, 7.$$

The Encyclopedia tells us that this is the numerators of the *Farey sequence*. This sequence, named for British geologist John Farey (1766 – 1826), has a lot of remarkable properties. The Farey sequence of n lists all fractions in canoncial form between 0 and 1, usually included, with a denominator less than or equal to n. For instance, the Farey sequence of 1, designated F_1 just contains $0, 1$; F_2 contains $0, \frac{1}{2}, 1$; F_3 contains $0, \frac{1}{3}, \frac{2}{3}, 1$ and so on.

A direct way to implement this could be to combine all numbers from $0 \ldots n$ in the numerator with all numbers $1 \ldots n$ in the denominator that are smaller than 1 and to sort and *nub* the resulting list, like this:

```
farey2 :: Natural → [Ratio]
farey2 n = sort (nub $ filter (⩽ 1) $
                    concatMap (λx → map (x%) [1 .. n]) [0 .. n])
```

With this approach, we create a lot of fractions that we do not need and that we filter out again afterwards. A more interesting approach, also in the light of the topic of this section, is the following:

$$
\begin{aligned}
&\mathit{farey} :: \mathit{Natural} \to [\mathit{Ratio}] \\
&\mathit{farey}\ n = 0 : \mathit{sort}\ (\mathit{go}\ 1\ \$\ \mathit{sterbrocTree}\ (-1)) \\
&\quad \textbf{where}\ \mathit{go}\ k\ t = \textbf{let}\ g = \mathit{getKids}\ k\ t \\
&\qquad\qquad\qquad\qquad\quad l = \mathit{filter}\ \mathit{fltr}\ g \\
&\qquad\qquad\qquad\quad \textbf{in if}\ \mathit{null}\ l\ \textbf{then}\ l\ \textbf{else}\ l +\!\!+ \mathit{go}\ (k+1)\ t \\
&\qquad\quad \mathit{fltr}\ k\ = k \leqslant 1 \wedge n \geqslant \mathit{denominator}\ k
\end{aligned}
$$

Here, we iterate over the generations of the Stern-Brocot tree removing the fractions that are greater than 1 or have a denominator greater n. When we do not get results anymore, *i.e.* all denominators are greater than n, we are done.

Let us try this algorithm on some numbers:

$$F_4 = \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\} \tag{1.39}$$

$$F_5 = \left\{0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1\right\} \tag{1.40}$$

$$F_6 = \left\{0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, 1\right\} \tag{1.41}$$

We see some interesting properties. First and this should be obvious, we see n as a denominator in sequence F_n exaclty $\varphi(n)$ times. For F_6, for instance, we could create the fractions $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$ and $\frac{5}{6}$. The fractions $\frac{2}{6} \ldots \frac{4}{6}$, however, are not in canonical form, since the numerators $2 \ldots 4$ all share divisors with 6. Since there are $\varphi(n)$ numerators that do not share divisors with n, there are only $\varphi(n)$ fractions less than 1 whith n in the denominator.

Another property is that, for two consecutive fractions in the Farey sequence, $\frac{a}{b}$ and $\frac{c}{d}$, the cross products ad and cb are consecutive integers. In again F_6, for the fractions $\frac{1}{6}$ and $\frac{1}{5}$, the cross products, trivially, are 5 and 6. More interesting are the fractions $\frac{3}{5}$ and $\frac{2}{3}$ whose cross products are $3 \times 3 = 9$ and $5 \times 2 = 10$.

Even further, for any three consecutive fractions in a Farey sequence, the middle one, called the mediant fraction, can be calculated from the outer ones as $\frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d}$. For instance in F_6:

$$\frac{1+1}{6+4} = \frac{2}{10} = \frac{1}{5}, \tag{1.42}$$

$$\frac{2+3}{5+5} = \frac{5}{10} = \frac{1}{2} \tag{1.43}$$

and

$$\frac{3+5}{4+6} = \frac{8}{10} = \frac{4}{5}. \tag{1.44}$$

This property can be used to compute F_{n+1} from F_n. We just have to insert those mediant fractions of two consecutive fractions in F_n, for which the denominator is $n+1$. In F_6 we would insert

$$\frac{0+1}{1+6}, \frac{1+1}{4+3}, \frac{2+1}{5+2}, \frac{1+3}{2+5}, \frac{2+3}{3+4}, \frac{5+1}{6+1}$$

resulting in

$$F_7 = \left\{0, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, 1\right\} \tag{1.45}$$

We can implement this as

```
nxtFarey :: Natural → [Ratio] → [Ratio]
nxtFarey n []    = []
nxtFarey n [r]   = [r]
nxtFarey n (a : b : rs) | denominator a +
                          denominator b ≡ n = nxtFarey n (a : x : b : rs)
                        | otherwise         = a : nxtFarey n (b : rs)
  where x = let n1 = numerator a
                n2 = numerator b
                d1 = denominator a
                d2 = denominator b
            in (n1 + n2) % (d1 + d2)
```

In fact, we can construct the Stern-Brocot tree by means of mediant fractions. The outer fractions, in this algorithm are the predecessors of the current node, namely the direct predecessor and either the predecessor of the predecessor or the sibling of that node. For instance, the second node in the third generation is $\frac{2}{3}$. Its kids are $\frac{3}{5}$ and $\frac{3}{4}$. $\frac{3}{5}$ is $\frac{2+1}{3+2}$ and, thus, the sum of $\frac{2}{3}$ and its predecessor; $\frac{4}{3}$, however, is $\frac{2+1}{3+1}$ and, hence, the sum of the $\frac{2}{3}$ and the predecessor of its predecessor.

The question now is how to bootstrap this algorithm, since the root node does not have predecessors. For this case, we imagine two predecessors, namely the fractions $\frac{0}{1}$ and $\frac{1}{0}$, the latter of which, of course, is not a proper fraction. The assumption of such nodes, however, helps us derive the outer branches, where, on the left side, the numerator does not change, hence is constructed by addition with 0, and, on the right side, the denominator does not change and is likewise constructed by addition with 0.

We implement this as

```
mSterbroctree :: Zahl → Natural → Natural →
                          Natural → Natural → Ratio → Tree Ratio
mSterbroctree 0 _ _ _ _ r = Node r []
mSterbroctree n a b c d r =
  let rn = numerator r
      rd = denominator r
      k1 = (a + rn) % (b + rd)
      k2 = (c + rn) % (d + rd)
  in  if k1 < k2
      then Node r [mSterbroctree (n − 1) a b rn rd k1,
                   mSterbroctree (n − 1) c d rn rd k2]
      else  Node r [mSterbroctree (n − 1) c d rn rd k2,
                   mSterbroctree (n − 1) a b rn rd k1]
```

Note that we have to use two pairs of natural numbers instead of two fractions to encode the predecessors. This is because we have to represent the imagined predecessor $\frac{1}{0}$, which is not a proper fraction. Finally, we check for the smaller of the resulting numbers k_1 and k_2 to make sure that the smaller one always goes to the left and the greater to the right. This implementation now gives exactly the same tree as the implementation using continued fractions introduced at the beginning of the section.

1.10 The Closed Form of the Fibonacci Sequence

There is a pending problem from the second chapter: is there a closed form of the Fibonacci sequence? Meanwhile, we have learnt almost everything to answer this question – and what we have not yet learnt, well, we just ignore it.

The method we choose is *ordinary generating funcions* (OGF). As you may remember, an OGF turns an infinite sequence of the form $a_1, a_2, \ldots$ into an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n.$$

The sequence of the Fibonacci numbers begins with

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...,

where each number is the sum of its two predecessors bootstrapping with 0, 1.

We call the n^{th} Fibonacci number F_n; we have for example

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13$$

and so on.

The OGF for this sequence is

$$\sum_{n=0}^{\infty} F_n x^n,$$

i.e.:

$$F_0 + F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \dots$$

In the following we will try to find other series that can express the same result, such that

$$\sum_{n=0}^{\infty} F_n x^n = \sum_{n=0}^{\infty} a_n x^n, \tag{1.46}$$

so that we can conjecture (and later prove) that $F_n = a_n$.

We start our journey by defining our generating function G:

$$G(x) = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \dots \tag{1.47}$$

This G that we can carry around like a bag. As before with geometric series, we multipl G by x and get:

$$xG(x) = F_0 x + F_1 x^2 + F_2 x^3 + F_3 x^4 + \dots \tag{1.48}$$

As a result, the exponents of the $x$es increase. Before we had F_0, now we have $F_0 x$; we had $F_1 x$, now we have $F_1 x^2$ and so on. Because that is so much fun, we just repeat the process and multiply xG once again by x yielding $x^2 G$:

$$x^2 G(x) = F_0 x^2 + F_1 x^3 + F_2 x^4 + F_3 x^5 + \dots \tag{1.49}$$

After that we subtract:

$$G(x) - xG(x) - x^2 G(x) = (1 - x - x^2)G(x). \tag{1.50}$$

What happens to the terms now? In the following equation, terms with equal $x$es arranged together:

$$
\begin{array}{lllllll}
(1-x-x^2)G(x) = & (F_0 & +F_1x & +F_2x^2 & +F_3x^3 & +\dots) & - \\
& (& F_0x & +F_1x^2 & +F_2x^3 & +\dots) & - \\
& (& & +F_0x^2 & +F_1x^3 & +\dots) &
\end{array}
$$

We see as a result that terms with equal $x$es form groups with three Fibonacci numbers: in the first line (starting with the column for x^2), we have F_n, in the second line, we have at the same position F_{n-1} and, in the third line at this position, we have F_{n-2}. Now, $F_n = F_{n-1} + F_{n-2}$. But, here, we compute $F_n - F_{n-1} - F_{n-2}$. So, what we do is $F_{n-1} + F_{n-2} - F_{n-1} - F_{n-2}$. In other words, we eliminate F_n. We are therefore left with

$$(1-x-x^2)G(x) = F_0 + F_1x - F_0x. \tag{1.51}$$

But since $F_0 = 0$, this is just

$$(1-x-x^2)G(x) = x. \tag{1.52}$$

We solve for G, *i.e.* we divide by $1-x-x^2$ and get

$$G(x) = \frac{x}{1-x-x^2}. \tag{1.53}$$

That is a nice looking formula! But not the end of the story. The next thing we do is to factor the denominator. Since this is a polynomial of 2^{nd} degree, we can do this by *completing the square* as we did before in a certain section of this chapter whose title is a spoiler to the punch line of what we are doing here.

Completing the square is a method that gives us the *roots* of the polynomial, *i.e.* the values of x for which the whole expression becomes zero. For reasons that will be discussed at length in the next chapter, for any root r of a polynomial, $(x-r)$ is a factor.

Anyway, let us complete the square. We have the equation:

$$-x^2 - x + 1 = 0. \tag{1.54}$$

We bring 1 to the right-hand side and get:

$$-x^2 - x = -1. \tag{1.55}$$

It is tempting to multiply by -1 to make everything positive. But remember that this equation is the denominator of a fraction. We cannot simply multiply the denominator by -1 without doing the same with the numerator. That would change the sign of the fraction. We therefore factor -1 out and make a mental note that we have to multiply it in again at the end of the calculations. With -1 factored out we get:

$$x^2 + x = 1. \tag{1.56}$$

Now, we add the missing term, which is half of the second coefficient squared. The second coefficient, the one before the x, is just 1. Half of it is $\frac{1}{2}$ and squared that is $\frac{1}{4}$:

$$x^2 + x + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}. \tag{1.57}$$

We take the square root of both sides and get:

$$x + \frac{1}{2} = \frac{\pm\sqrt{5}}{2}. \tag{1.58}$$

The last step is to bring $\frac{1}{2}$ to the right-hand side:

$$x = \frac{\pm\sqrt{5} - 1}{2}. \tag{1.59}$$

Bang!

No bang? OK, let us examine the beast on the right-hand side. When we take the negative root, we have:

$$\frac{-1 - \sqrt{5}}{2}.$$

You might remember the number $\frac{1+\sqrt{5}}{2}$, which is called Φ or the *golden ratio*. Well, the number we see for the negative root is $-\Phi$, the additive inverse of the golden ratio.

What about the positive root? That is

$$\frac{-1 + \sqrt{5}}{2}$$

and, thus, the negative of the conjugate of Φ, $\overline{\Phi}$. The conjugate of Φ is

$$\overline{\Phi} = \frac{1-\sqrt{5}}{2}. \tag{1.60}$$

Two nice properties that we will use later follow immediately:

$$\Phi + \overline{\Phi} = 1 \tag{1.61}$$

and

$$\Phi - \overline{\Phi} = \sqrt{5}. \tag{1.62}$$

By completing the square, we found two roots, namely $-\Phi$ and $-\overline{\Phi}$. That implies that $(x+\Phi)$ and $(x+\overline{\Phi})$ are factors of $(1-x-x^2)$. Let us check if this is true. We multiply $(x+\Phi)(x+\overline{\Phi})$ and get

$$x^2 + \overline{\Phi}x + \Phi x + \overline{\Phi}\Phi.$$

According to the first property above, $\overline{\Phi} + \Phi = 1$. But what about $\overline{\Phi} \times \Phi$? Let us look:

$$\Phi\overline{\Phi} = \frac{1+\sqrt{5}}{2} \times \frac{1-\sqrt{5}}{2} = \frac{(1+\sqrt{5})(1-\sqrt{5})}{4} = \frac{1-\sqrt{5}+\sqrt{5}-5}{4} = \frac{-4}{4} = -1. \tag{1.63}$$

The overall product of the factors, hence, is

$$x^2 + x - 1.$$

Oops! We forgot the -1 we factored out above! The correct result rather is

$$-(x^2 + x - 1).$$

We can now rewrite equation 1.53 as

$$G(x) = \frac{x}{-(x+\Phi)(x+\overline{\Phi})}, \tag{1.64}$$

where the denominator has been replaced by the product of its factors. If this is not entirely clear to you, do not worry. The relation of roots and factors of polynomials is one of the main topics of the next chapter.

Now comes a very cute step. We will split the fraction into two fractions. The reason why we do it is that we want to make each of them look like a geometric series.

The way how we do it is the inverse of adding fractions. When we add fractions, we multiply the denominator of one fraction by the denominator of the other. (In fact, we use the greatest common divisor of the two denominators.) For instance:

$$\frac{DA+CB}{CD} = \frac{A}{C} + \frac{B}{D}. \tag{1.65}$$

When we apply this to equation 1.64, we get

$$G(x) = \frac{x}{-(x+\Phi)(x+\overline{\Phi})} = \frac{A}{x+\Phi} + \frac{B}{x+\overline{\Phi}} \tag{1.66}$$

To get to know A and B, we multiply both sides by the denominator $(x+\Phi)(x+\overline{\Phi})$ and get

$$-x = A(x+\overline{\Phi}) + B(x+\Phi). \tag{1.67}$$

Note that we moved the minus sign up to the numerator, so the right-hand side of the equation keeps clear.

In order to solve for A we set $x = -\Phi$ to let B disappear:

$$\Phi = A(-\Phi+\overline{\Phi}). \tag{1.68}$$

Note the effect of the minus sign on the left side of the equation.

The second property introduced above leads to $-\Phi+\overline{\Phi} = -\sqrt{5}$ and, after dividing this, we get

$$A = \frac{\Phi}{-\sqrt{5}} = -\frac{\Phi}{\sqrt{5}}. \tag{1.69}$$

In order to solve for B we set $x = -\overline{\Phi}$ to let A disappear:

$$\overline{\Phi} = B(-\overline{\Phi}+\Phi). \tag{1.70}$$

Since $\Phi - \overline{\Phi} = \sqrt{5}$, we get this time:

$$B = \frac{\overline{\Phi}}{\sqrt{5}}. \tag{1.71}$$

We, hence, can rewrite equation 1.66 (with $\frac{1}{\sqrt{5}}$ factored out) as

$$G(x) = \frac{1}{\sqrt{5}} \left(-\frac{\Phi}{x + \Phi} + \frac{\overline{\Phi}}{x + \overline{\Phi}} \right) \tag{1.72}$$

To see the progress we made, remember that our intention is to make the resulting formula look like geometric series. A geometric series, in its most basic form, looks like

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1 - r}, \tag{1.73}$$

for $a_0 = 1$.

So, let us try to get rid of the numerators. We can achieve that, by dividing both, numerator and denominator by the numerator:

$$G(x) = \frac{1}{\sqrt{5}} \left(-\frac{1}{\frac{1}{\Phi}x + 1} + \frac{1}{\frac{1}{\overline{\Phi}}x + 1} \right) \tag{1.74}$$

There is another nice property of Φ and $\overline{\Phi}$ that we have not yet mentioned. The term $\frac{1}{\Phi}$ is the multiplicative inverse of Φ. We have seen above that $\Phi\overline{\Phi}$ is -1. The multiplicative inverse of Φ must therefore be the additive inverse of $\overline{\Phi}$:

$$\frac{1}{\Phi} = \frac{1}{\frac{1+\sqrt{5}}{2}} = \frac{2}{1 + \sqrt{5}} = -\overline{\Phi}. \tag{1.75}$$

Correspondingly, the multiplicative inverse of $\overline{\Phi}$ is the additive inverse of Φ, which, of course, is $-\Phi$.

We, hence, can reduce the equation above to

$$G(x) = \frac{1}{\sqrt{5}} \left(-\frac{1}{1 - \overline{\Phi}x} + \frac{1}{1 - \Phi x} \right) \tag{1.76}$$

This, now, really looks like two geometric series, one with $r = \overline{\Phi}x$, the other with $r = \Phi x$. So, now, finally, here comes the punch line:

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{\sqrt{5}} \left(\sum_{n=0}^{\infty} (\Phi x)^n - \sum_{n=0}^{\infty} (\overline{\Phi} x)^n \right) = \sum_{n=0}^{\infty} \frac{(\Phi^n - \overline{\Phi}^n)}{\sqrt{5}} x^n. \tag{1.77}$$

We, here, have two series that are supposed to be equal. We, therefore, conjecture that coefficients must be equal:

$$F_n = \frac{\Phi^n - \overline{\Phi}^n}{\sqrt{5}}. \tag{1.78}$$

We can use this implement a much more efficient implementation of *fib*. The naïve version we implemented in chapter 2 went like this:

$$\begin{aligned}
&\mathit{fib} :: \mathit{Natural} \to \mathit{Natural} \\
&\mathit{fib}\ 0 = 0 \\
&\mathit{fib}\ 1 = 1 \\
&\mathit{fib}\ n = \mathit{fib}\ (n-1) + \mathit{fib}\ (n-2)
\end{aligned}$$

The formula above clearly indicates that the result is a real number. The implementation in Haskell needs to take that into account:

$$\begin{aligned}
&\mathit{fir} :: \mathit{Natural} \to \mathit{RealN} \\
&\mathit{fir}\ n = (\mathit{Phi} \uparrow n - \mathit{Phi}' \uparrow n)\ /\ \mathit{sqrt}\ 5
\end{aligned}$$

We apply the function to some numbers like this: $\mathit{map}\ \mathit{fir}\ [0\,..\,9]$ and see

$[0.0, 1.0, 1.0, 2.0, 3.0, 5.0, 8.0, 13.0, 21.0, 34.0]$.

Until here everything is as expected. But when we go on ($\mathit{map}\ \mathit{fir}\ [10\,..\,14]$):

$[54.999, 89.0, 143.999, 232.999, 377.00000000000006]$

We see that some numbers are slightly off the expected result; sometimes above sometimes below. Indeed, why should we expect clear-cut integers in the first place?

Let us look at the small numbers to better understand what happens. For $n = 0$, we get $(1-1)/\sqrt{5}$. That is just zero. For $n = 1$, we get $\sqrt{5}/\sqrt{5}$, which is 1. For $n = 2$ we get, a bit surprisingly, $\Phi^2 = 2.6180\ldots$ and $\overline{\Phi}^2 = 0.3819\ldots$ Now, $\Phi^2 - \overline{\Phi}^2 = 2.2360\ldots$, which happens to be $\sqrt{5}$ again and, thus, we get 1.

For $n = 3$, we get $\Phi^3 - \overline{\Phi}^3 = 4.4721\ldots$, which happens to be $2\sqrt{5}$. We, hence, get exactly 2. Here are the next values:

$$\begin{aligned}
&\Phi^4 - \overline{\Phi}^4 = 3\sqrt{5} \\
&\Phi^5 - \overline{\Phi}^5 = 5\sqrt{5} \\
&\Phi^6 - \overline{\Phi}^6 = 8\sqrt{5} \\
&\Phi^7 - \overline{\Phi}^7 = 13\sqrt{5}.
\end{aligned}$$

In summary, we have

$$\Phi^n - \overline{\Phi}^n = F_n\sqrt{5}, \tag{1.79}$$

which is exactly according to the equation we have found. But, of course, we are working with limited precision and thus get slightly off with growing numbers. The solution is just to round to the nearest integer. Once we do that, we can consider a simplification. Since $|\overline{\Phi}|$, the absolute value of the conjugate of Φ, is a number less than 1, its powers with growing exponents become smaller and smaller and, thus do not affect the result, which is rounded to the nearest integer anyway. Therefore, we can leave it out. The simplified formula would be

$$\frac{\Phi^n}{\sqrt{5}}.$$

We need to be careful with small numbers, though. The first results with this formula are

$$[0.447, 0.723, 1.170, 1.894, 3.065, 4.959, 8.024, 12.984, 21.009, 33.994, 55.003].$$

They are close enough to the expected value 0, 1, 1, 2, 3, 5, 8, 13, 21, 34 and 55 to yield the correct result rounding to the nearest integer. The implementation of the closed form of the Fibonacci sequence in Haskell finally is:

```
fi :: Natural → Natural
fi n = round (phi ↑ n / sqrt 5)
```

Compare the speed of *fib* and *fi* applied to big numbers.

But, again, how can it be that a formula involving things like the $\sqrt{5}$ always results in an integer? To answer this question, we may observe what is going on in the formula algebraicly. When we create powers of Φ, we compute

$$\Phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) \tag{1.80}$$

When we treat the integers and the roots as distinct quantities that cannot be mixed, we will see the coefficients of those quantities as discrete objects next to each other. We could express the formula above as

$$\left(\frac{a+b}{c}\right)\left(\frac{d+e}{f}\right) = \frac{ad+5be+ad+be}{cf}.$$

The point is that a and d are just normal integers, whereas b and e are multiples of $\sqrt{5}$. When we multiply a and d, we get back an integer. When we multiply a and e or b and d, we get back a multiple of $\sqrt{5}$. When we multiply b and e, which both are multiples of $\sqrt{5}$, we get back an ordinary integer, since $\sqrt{5} \times \sqrt{5} = 5$.

We can model this in Haskell quite easily:

```
data Phi a = Phi a a a
  deriving (Show, Eq)
```

This is just a data type consisting of three components. Here is how we would model Φ and $\overline{\Phi}$:

```
one :: Phi Integer
one = Phi 1 1 2

one' :: Phi Integer
one' = Phi 1 (-1) 2
```

The first component represents the multiples of the integer; the second component represents the multiples of $\sqrt{5}$; the last component represents the denominator. Here is a clean constructor for this data type:

```
mkPhi :: (Num a, Integral a) => a -> a -> a -> Phi a
mkPhi a b c = Phi (a `div` g) (b `div` g) (c `div` g)
  where k = gcd a c
        m = gcd b c
        g = gcd k m
```

This function just reduces the fraction to the canonical form where numerator and denominator do not share divisors. Here is how we add two of these beasts:

```
add :: (Num a, Integral a) => Phi a -> Phi a -> Phi a
add (Phi a b c) (Phi d e f) = mkPhi (f * a + c * d) (f * b + c * e) (c * f)
```

and how we negate one of those:

```
neg :: (Num a, Integral a) => Phi a -> Phi a
neg (Phi a b c) = mkPhi (-a) (-b) c
```

The multiplication formula already introduced above is implemented like this:

```
mul :: (Num a, Integral a) => Phi a -> Phi a -> Phi a
mul (Phi a b c) (Phi d e f) = mkPhi (a * d + 5 * b * e) (a * e + b * d) (c * f)
```

Power is now simply built on top of *mul*:

```
pow :: Phi Integer -> Int -> Phi Integer
pow p n = foldl' mul p $ take (n - 1) (repeat p)
```

Let us test:

pow one 1: *Phi* 1 1 2 (this is just one)
pow one 2: *Phi* 3 1 2
pow one 3: *Phi* 2 1 1
pow one 4: *Phi* 7 3 2

and so on.

pow one′ 1: *Phi* 1 (−1) 2
pow one′ 2: *Phi* 3 (−1) 2
pow one′ 3: *Phi* 2 (−1) 1
pow one′ 4: *Phi* 7 (−3) 2.

The results are identical except for the negative sign before the second component, the multiples of $\sqrt{5}$. Of course, when we negate the powers of *one′*, we will move the minus sign from the second to the first component:

neg (*pow one′* 1): *Phi* (−1) 1 2
neg (*pow one′* 2): *Phi* (−3) 1 2
neg (*pow one′* 3): *Phi* (−2) 1 1
neg (*pow one′* 4): *Phi* (−7) 3 2.

Now, we devise a function that builds triples of the form $(\Phi^n, \overline{\Phi}^n, \Phi^n - \overline{\Phi}^n)$:

```
triple :: Int → (Phi Integer, Phi Integer, Phi Integer)
triple n = (p, q, d)
  where p = pow one n
        q = pow one′ n
        d = add p (neg q)
```

The following output was generated with a pretty printer (you are certainly able to implement yourself): mapping *triple* on [1 .. 20]

```
(+00001 +00001 +00002) (+00001 -00001 +00002) (+00000 +00001 +00001)
(+00003 +00001 +00002) (+00003 -00001 +00002) (+00000 +00001 +00001)
(+00002 +00001 +00001) (+00002 -00001 +00001) (+00000 +00002 +00001)
(+00007 +00003 +00002) (+00007 -00003 +00002) (+00000 +00003 +00001)
(+00011 +00005 +00002) (+00011 -00005 +00002) (+00000 +00005 +00001)
(+00009 +00004 +00001) (+00009 -00004 +00001) (+00000 +00008 +00001)
(+00029 +00013 +00002) (+00029 -00013 +00002) (+00000 +00013 +00001)
(+00047 +00021 +00002) (+00047 -00021 +00002) (+00000 +00021 +00001)
(+00038 +00017 +00001) (+00038 -00017 +00001) (+00000 +00034 +00001)
(+00123 +00055 +00002) (+00123 -00055 +00002) (+00000 +00055 +00001)
(+00199 +00089 +00002) (+00199 -00089 +00002) (+00000 +00089 +00001)
(+00161 +00072 +00001) (+00161 -00072 +00001) (+00000 +00144 +00001)
(+00521 +00233 +00002) (+00521 -00233 +00002) (+00000 +00233 +00001)
(+00843 +00377 +00002) (+00843 -00377 +00002) (+00000 +00377 +00001)
(+00682 +00305 +00001) (+00682 -00305 +00001) (+00000 +00610 +00001)
(+02207 +00987 +00002) (+02207 -00987 +00002) (+00000 +00987 +00001)
(+03571 +01597 +00002) (+03571 -01597 +00002) (+00000 +01597 +00001)
(+02889 +01292 +00001) (+02889 -01292 +00001) (+00000 +02584 +00001)
(+09349 +04181 +00002) (+09349 -04181 +00002) (+00000 +04181 +00001)
(+15127 +06765 +00002) (+15127 -06765 +00002) (+00000 +06765 +00001)
```

The powers of Φ and $\overline{\Phi}$, as already mentioned, are equal with the exception of the sign

of the multiples of $\sqrt{5}$. When we subtract $\overline{\Phi}$ from Φ, the integers will disappear and we will *add* the absolute values of the multiples of $\sqrt{5}$. When we add two equal numbers, we obtain an even number. Observe that, for most cases, already Φ and its conjugate have a fibonacci number as multiple of $\sqrt{5}$. In those cases, the denominator is 2. We, hence, add two Fibonacci numbers to obtain $2F_n$, which, divided by 2, results in F_n. In some cases, we do not see a Fibonacci number. That Occurs in exactly those instances where the Fibonacci number itself is even. In all those cases, the denominator is 1 – and, thus, we get an even Fibonacci number. In fact, every third Fibonacci number is even. (Why?) When you look at the denominators of the powers of Φ, you see the pattern $2, 2, 1$ repeating over and over again. Where you see 1, you see an even Fibonacci number.

But why is that so? Have we not just swapped one enigma for the other?

When we push this analysis forward, we will see that everything boils down to combinations of terms in the distribution law and, hence, to the binomial theorem. Indeed, we can express Fibonacci numbers in terms of binomial coefficients of the form:

$$F_n = \sum_{k=0}^{\frac{n-1}{2}} \binom{n-k-1}{k} \tag{1.81}$$

Well, that is possible. Here is a last try to explain what the Golden Ratio and the Fibonacci sequence have in common.

```
fratio :: Integer → RealN
fratio n = np / nn
  where np = fromInteger (fi (n + 1))
        nn = fromInteger (fi n)
```

1.11 Field Extension

A very common activity of mathematicians is solving equations. They usually solve equations with coefficients of a certain type of numbers (like integers or rationals) assuming that the solution is of that same number type. A typical example is Diophantine equations, named after the late-antique mathematician Diophantus of Alexandria who lived in the third century. He studied equations and is the first mathematician to be known to have introduced abstract symbols for numbers. Diophantine equations operate over the integers. The known values, *i.e.* the coefficients, as well as the unknown values, *i.e.* the solutions, must be integers. The most famous result from the study of Diophantine equations is perhaps the proof of Fermat's Last Theorem, which states that there are no solutions for $z > 2$ in equations of the form

$$a^z + b^z = c^z \tag{1.82}$$

where a, b, c and z are all integers. Fermat scribbled his conjecture in the margin of his copy of Diophantus' "Arithmetica". It turned into his last *theorem* only when Andrew Wiles proved it in the 90ies of the 20^{th} century using concepts that went far beyond the knowledge of Fermat and his contemporaries.

In modern times, equations are typically studied in a *field* and you might remember that a field is a structure defined over a set of numbers with two operations. Both operations establish an *Abelian group* with that set of numbers where one operation (called *multiplication*) distributes over the other (called *addition*). More formally, a field is defined as a structure

$$(S, +, \times),$$

where S is the set of numbers, "+" the addition operation and "×" multiplication. For both operations, the following properties must hold:

1. **Closure**: for all $a, b \in S : a \circ b \in S$.
2. **Associativity**: $a \circ (b \circ c) = (a \circ b) \circ c = a \circ b \circ c$.
3. **Identity**: there is exactly one element $e \in S$, called the identity, such that for all $a \in S : a \circ e = e \circ a = a$.
4. **Invertibility** for each element $a \in S$, there is an element a', such that $a \circ a' = e$.
5. **Commutativity** $a \circ b = b \circ a$.

Properties 1 – 4, as you will have realised, are just the group laws. Property 5, commutativity, makes the group *Abelian*.

Furthermore, multiplication **distributes** over addition, i.e.

$$a \times (b + c) = ab + ac.$$

When all these properties hold, then we have a field. We have already seen that $\mathbb{Q}$, the rational numbers, is a field. Historically, this field was important for the theory of solving equations. For the special case of linear equations, that is equations without exponents, rational coefficients lead to rational solutions. The reason is that the operations we need to solve linear equations are just the four arithmetic operations addition, subtraction, multiplication and division. A simple equation of the form

$$ax + b = 0 \tag{1.83}$$

is solved by first subtracting b from both sides of the equation and then dividing both sides by a leading to

$$x = -\frac{b}{a}. \tag{1.84}$$

Note that the solutions are not necessarily integers, like in Diophantine equations, since not every integer has a multiplicative inverse. In other words, integers do not constitute a group over multiplication and integers, therefore, do not form a field. In the field $\mathbb{Q}$ of rational numbers, however, there is a solution (and exactly one solution) that lies within that field.

But, of course, there are equations that cannot be solved by applying the four fundamental arithmetic operations alone, quadratic equations, for instance:

$$x^2 - 2 = 0. \tag{1.85}$$

We proceed like for the linear equation: we subtract -2 on both sides and then, instead of dividing by something, we take the square root leading to

$$x = \sqrt{2}. \tag{1.86}$$

Unfortunately, $\sqrt{2}$ is, as we already know, not in the field $\mathbb{Q}$. It is irrational. We can of course redefine the field in which we started to solve this equation in the first place assuming $\mathbb{R}$ instead of $\mathbb{Q}$. But that is a sloppy solution. A typical question for a mathematician is: what is the *smallest* field comprising both, the coefficients and the solutions of this kind of equations? A possible answer is: the field $\mathbb{Q}$ with the solution $\sqrt{2}$ added to it. That is, we *extend* the field $\mathbb{Q}$ by *adjoining* $\sqrt{2}$. We should then get a new field, called $\mathbb{Q}(\sqrt{2})$, of which the original field $\mathbb{Q}$ is a subfield, *i.e.*

$$\mathbb{Q} \subset \mathbb{Q}\left(\sqrt{2}\right).$$

What does this new field look like? Of course, we cannot just extend the underlying set, such like:

$$S = \left\{0, 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \ldots, \infty, \sqrt{2}\right\}.$$

This, obviously, would not lead to a field, since not for every $a \in S$ $a\sqrt{2} \in S$. $2 \times \sqrt{2}$, for instance, is not in the field; $\frac{1}{2} \times \sqrt{2}$, too, is not in the field and so on. In fact, for almost no $a \in S$, closure is fulfilled. It is fulfilled only for 0, the identity and $\sqrt{2}$ itself, since $0 \times \sqrt{2} = 0 \in S$, $1 \times \sqrt{2} = \sqrt{2} \in S$ and $\sqrt{2} \times \sqrt{2} = 2 \in$S.

So, we need a better approach. The following formalism defines the *smallest* field that contains $\mathbb{Q}$ and the square root of any number $r \in \mathbb{Q}$. We first define a new number type consisting of a tuple (a, b), for $a, b \in \mathbb{Q}$. The natural interpretation of this tuple is

$$(a, b) = a + b\sqrt{r}. \tag{1.87}$$

The set underlying the field $\mathbb{Q}(\sqrt{r})$, thus, consists of the numbers

$$\left\{a + b\sqrt{r} | a, b \in \mathbb{Q}\right\},$$

with r being a constant rational number, *i.e.* $r \in \mathbb{Q}$. If you want to look at the concrete numbers, you may reformulate this in terms of Haskell list comprehension:

```
[(fromRational a) +
 (fromRational b) * (sqrt r) | a ← enumQ,
                               b ← enumQ]
```

Do not be confused by the fact that, in Haskell, we have to convert a and b to real numbers. Haskell has no built-in notion of field extension. Numbers are either rational or real. Since *sqrt r* is *RealN*, we have to convert everything to *RealN*. The result, however, shows what the numbers in the new field look like "in reality", whatever that is supposed to mean.

Now we define the arithmetic operations in a way that fulfils the group properties. First, addition is

$$(a, b) + (c, d) = (a + c, b + d). \tag{1.88}$$

That is easy and, in fact, follows from basic properties of addition in the field $\mathbb{Q}$. If we have something like $a + bx + c + dx$, we usually simplify to $a + c + (b + d)x$. That is just the same as we did above.

Multiplication is a bit more complicated. Let us first ask, how the product of two expressions of the form $a + bx$ and $c + dx$ looks like:

$$(a + bx)(c + dx) = ac + adx + bcx + bdx^2.$$

Since, x in our case is the square root of r, $x^2 = \sqrt{r} \times \sqrt{r} = r$ and we, hence, get

$$ac + rbd + (ad + bc)\sqrt{r}.$$

From this we can derive the general rule

$$(a,b)(c,d) = (ac + rbd, ad + bc), \tag{1.89}$$

where r is the number, whose square root is adjoint to our base field. For $\mathbb{Q}(\sqrt{2})$, this is

$$(a,b)(c,d) = (ac + 2bd, ad + bc). \tag{1.90}$$

This construction of the field extension guarantees that for any addition and any multiplication of two elements in this new field, the result, again, is an element of this field. The rules also guarantee associativity, as you may easily convince yourself. But what about the identities of addition and multiplication?

In general, a rationl number a is, in the new field, represented as $(a, 0)$. This is easy to see, because $(a, 0) = a + 0 \times \sqrt{r} = a$. Since the additive identity is 0 in $\mathbb{Q}$, the identity should be $(0, 0)$. We just follow rule 1.88 to prove that:

$$(a + b) + (0, 0) = (a + 0, b + 0) = (a + b) \quad \square. \tag{1.91}$$

What about the multiplicative identity? We expect it to be the representation of 1 in the new field, which is $(1, 0)$, since $1 + 0\sqrt{r} = 1 + 0 = 1$. Let us check:

$$(a + b)(1, 0) = (1a + 2b \times 0, a \times 0 + 1b) = (a, b), \tag{1.92}$$

which, indeed, fulfils the identity property. $\square$

The next question is how the inverse will look like in the new field. For the additive inverse that is not difficult to answer. Since, for any number (a, b), the additive inverse $-(a, b)$ should fulfil the property $(a, b) + -(a, b) = (0, 0)$, the inverse must therefore be

$$-(a, b) = (-a, -b). \tag{1.93}$$

We can check this quickly using again 1.88:

$$(a, b) + (-a, -b) = (a - a, b - b) = (0, 0). \quad \square \tag{1.94}$$

Concerning multiplication, which, as usual, is a bit more complicated than addition, the inverse is

$$(a, b)^{-1} = \left(\frac{a}{a^2 - rb^2}, -\frac{b}{a^2 - rb^2} \right). \tag{1.95}$$

Here is the proof using 1.89:

$$(a,b)\left(\frac{a}{a^2-rb^2},-\frac{b}{a^2-rb^2}\right)=\left(\frac{a^2}{a^2-rb^2}-\frac{rb^2}{a^2-rb^2},\frac{ab}{a^2-rb^2}-\frac{ab}{a^2-rb^2}\right). \quad (1.96)$$

The scary looking formula on the right-hand side of the equation can be simplified. The first component is

$$\frac{a^2}{a^2-rb^2}-\frac{rb^2}{a^2-rb^2},$$

which can be reduced to one fraction, since the denominators are equal:

$$\frac{a^2-rb^2}{a^2-rb^2}=1.$$

We see a fraction with identical numerator and denominator. The fraction, hence, can be further reduced to 1.

The second component is

$$\frac{ab}{a^2-rb^2}-\frac{ab}{a^2-rb^2}=0.$$

We finally get

$$(a,b)\left(\frac{a}{a^2-rb^2},-\frac{b}{a^2-rb^2}\right)=(1,0), \quad (1.97)$$

which proves that the beast in 1.95 fulfils the invertibility property. □

The final piece in showing that $\mathbb{Q}(\sqrt{r})$ is indeed a field considering the rules 1.88 and 1.89 is distributivity. Distributivity requires that

$$(a,b)((c,d)+(e,f))=(a,b)(c,d)+(a,b)(e,f). \quad (1.98)$$

Multiplying the left side out, we get

$$(ac+rbd,ad+bc)+(ae+rbf,af+be),$$

which, when added, is

$$(ac + rbd + ae + rbf, ad + bc + af + be).$$

We show that this is true by multiplying

$$(a + b\sqrt{r})(c + d\sqrt{r} + e + f\sqrt{r}).$$

We regroup the second part:

$$c + e + d\sqrt{r} + f\sqrt{r}$$

and distribute, first a:

$$ac + ae + ad\sqrt{r} + af\sqrt{r}$$

and then $b\sqrt{r}$:

$$cb\sqrt{r} + eb\sqrt{r} + d\sqrt{r}b\sqrt{r} + f\sqrt{r}b\sqrt{r}.$$

This second term simplifies to

$$cb\sqrt{r} + eb\sqrt{r} + bdr + fbr.$$

Now we bring the two terms together and get

$$ac + ae + rbd + rbf + (ad + bc + af + be)\sqrt{r} = (ac + ae + rbd + rbf, ad + bc + af + be)$$

as desired. □

We have defined the smallest field that extends $\mathbb{Q}$ by adjoining $\sqrt{r}$ for any rational number r. This is nice, because it allows us to add the square roots of any rational number using the same recipe. We can even go further and extend the extended field by adjoining the square roots of square roots on top of the extension already containing the square roots, *i.e.*:

$$\mathbb{Q}(\sqrt{r}, \sqrt[4]{r}).$$

We can go still further and add the square roots of the square roots of the square roots:

$$\mathbb{Q}(\sqrt{r}, \sqrt[4]{r}, \sqrt[8]{r})$$

and then the square roots of the square roots of the square roots of the square roots and so on *ad infinitum*. There are indeed classic problems, as we will see later, that can be solved in exactly this field: $\mathbb{Q}$ extended by the n^{th}-roots, where n is any power of 2.

Extensions resulting from building extensions on top of extensions are sometimes called *towers of fields* where one field is put on the top of another field yielding a batch of pancakes that slowly grows higher and higher. This technique is often used in algebra, more specifically in *Galois Theory*, to study equations of higher degrees. For instance, the equation

$$x^4 - 4x^3 - 4x^2 + 8x - 2 = 0 \tag{1.99}$$

has four solutions, namely

$$\begin{aligned} x_1 &= 1 + \sqrt{2} + \sqrt{3 + \sqrt{2}} \\ x_2 &= 1 + \sqrt{2} - \sqrt{3 + \sqrt{2}} \\ x_3 &= 1 - \sqrt{2} + \sqrt{3 - \sqrt{2}} \\ x_4 &= 1 - \sqrt{2} - \sqrt{3 - \sqrt{2}} \end{aligned}$$

We can build a tower of extensions of $\mathbb{Q}$ by stepwise adjoining the irrational components of the solutions:

$$\begin{aligned} &\mathbb{Q} \\ &\mathbb{Q}\left(\sqrt{2}\right) \\ &\mathbb{Q}\left(\sqrt{2}, \sqrt{3 + \sqrt{2}}\right) \\ &\mathbb{Q}\left(\sqrt{2}, \sqrt{3 + \sqrt{2}}, \sqrt{3 - \sqrt{2}}\right) \end{aligned}$$

The interesting question presents itself whether it is possible to build a tower from $\mathbb{Q}$ to $\mathbb{R}$. It is indeed possible. But it is not trivial. $\mathbb{R}$ is in fact much bigger than $\mathbb{Q}$. How much bigger, we will soon see.

The difficulty arising in building that tower is that we need different definitions for different things we adjoin to $\mathbb{Q}$ and its extensions. Until now, we have only looked at

square roots. But how to add n-roots where n is not a power of two? $\mathbb{Q}(\sqrt[3]{r})$, for instance, can not be represented by the formulas above. This is because $\sqrt[3]{r} \times \sqrt[3]{r}$ is not a rational number and is therefore not in the field. The *degree* of this extension is not the same as that of $\mathbb{Q}(\sqrt{r})$. The degree of $\mathbb{Q}(\sqrt{r})$ is 2, since it can be represented by a pair of numbers (a, b). $\mathbb{Q}(\sqrt[3]{r})$, however, cannot be represented by a pair; a triple is needed, namely the triple

$$(a, b, c) = a + b\sqrt[3]{r} + c\sqrt[3]{r^2}.$$

The degree of $\mathbb{Q}(\sqrt[3]{r})$ is therefore 3.

What does the field $\mathbb{Q}(\sqrt[3]{r})$ look like? For addition, it looks very similar to the field $\mathbb{Q}(\sqrt{r})$. The addition rule is

$$(a, b, c) + (d, e, f) = (a + d, b + e, c + f). \tag{1.100}$$

The additive identity, trivially, is $(0, 0, 0)$. The inverse $-(a, b, c)$ is $(-a, -b, -c)$.

Harder, however, is multiplication. Let us investigate the multiplication rule. We try to multiply two numbers in the new field

$$(a, b, c)(d, e, f).$$

This corresponds to the expression

$$(a + b\sqrt[3]{r} + c\sqrt[3]{r^2})(d + e\sqrt[3]{r} + f\sqrt[3]{r^2}).$$

We distribute the first sum term by term over the second sum. Distributing a gives

$$ad + ae\sqrt[3]{r} + af\sqrt[3]{r^2};$$

Distributing $b\sqrt[3]{r}$ gives

$$bd\sqrt[3]{r} + be\sqrt[3]{r}\sqrt[3]{r} + bf\sqrt[3]{r}\sqrt[3]{r^2}$$

and distributing $c\sqrt[3]{r^2}$ gives

$$cd\sqrt[3]{r^2} + ce\sqrt[3]{r^2}\sqrt[3]{r} + cf\sqrt[3]{r^2}\sqrt[3]{r^2}.$$

Now, we represent the roots as fractional exponents and get:

$$ad + aer^{\frac{1}{3}} + afr^{\frac{2}{3}}$$

for the first component,

$$bdr^{\frac{1}{3}} + ber^{\frac{1}{3}}r^{\frac{1}{3}} + bfr^{\frac{1}{3}}r^{\frac{2}{3}}$$

for the second and

$$cdr^{\frac{2}{3}} + cer^{\frac{2}{3}}r^{\frac{1}{3}} + cfr^{\frac{2}{3}}r^{\frac{2}{3}}$$

for the third. We can simplify the second component to

$$bdr^{\frac{1}{3}} + ber^{\frac{2}{3}} + bfr$$

and the third to

$$cdr^{\frac{2}{3}} + cer + cfr^{\frac{4}{3}}.$$

The last term in this expression is not very nice. It, apparently, introduces a new element that we do not yet know. However, we can transform it:

$$r^{\frac{4}{3}} = r^{\frac{3}{3}+\frac{1}{3}} = rr^{\frac{1}{3}}$$

resulting in a product of two elements we do know already, namely r, which is a rational number, and $r^{\frac{1}{3}}$, which is just $\sqrt[3]{r}$. The last component, hence, is

$$cdr^{\frac{2}{3}} + cer + cfrr^{\frac{1}{3}}.$$

We will now group the terms in the components according to their exponents. First the terms without exponent

$$ad + bfr + cer,$$

then those with exponent $\frac{1}{3}$:

$$aer^{\frac{1}{3}} + bdr^{\frac{1}{3}} + cfrr^{\frac{1}{3}}$$

and, finally, those with exponent $\frac{2}{3}$:

$$afr^{\frac{2}{3}} + ber^{\frac{2}{3}} + cdr^{\frac{2}{3}}.$$

When we convert the exponents back to roots, we see that we have three groups: one consisting of rational numbers only, one consisting of rational numbers multiplied by $\sqrt[3]{r}$ and, finally, one consisting of rational numbers multiplied by $\sqrt[3]{r^2}$. We conclude that the multiplication formula in this field is

$$(a, b, c)(d, e, f) = (ad + bfr + cer, ae + bd + cfr, af + be + cd), \tag{1.101}$$

where r is the rational number, whose third root was adjoint to $\mathbb{Q}$.

The multiplicative identity should be $(1, 0, 0)$ and, indeed, $(a, b, c)(1, 0, 0)$ gives according to 1.101:

$$(a + 0 + 0, 0 + b + 0, 0 + 0 + c) = (a, b, c). \quad \square$$

What, however, is the multiplicative inverse? Well, answering this questions corresponds to solving the equation

$$ad + bfr + cer + (ae + bd + cfr)\sqrt[3]{r} + (af + be + cd)\sqrt[3]{r^2} = 1. \tag{1.102}$$

Solving such equations is a major topic of the next part. With the techniques we have at our disposal now, this is not easy. We will come back to that question later. Anyway, what should be clear from the exercise is that it is possible to extend the field $\mathbb{Q}$ step by step including always more irrational numbers until we reach $\mathbb{R}$. But this process is not trivial. It involves a lot of algebra. It is a true Tower of Babel. And, until here, we have only looked at irrational numbers that are roots of rational numbers. We have not yet discussed how to extend fields by *transcendental numbers*, *i.e.* numbers that are not roots of rational numbers and, even further, do not appear as solutions of equations with rational coefficients at all. Our friends π and e are examples of such numbers.

1.12 The Continuum

So. How many real numbers are there? As before, we will try to *count* the real numbers by building a set of tuples (r, n) for all numbers $r \in \mathbb{R}$ and $n \in \mathbb{N}$. In order to do this, we need to first establish a sequence of real numbers. We start by investigating the real numbers in the range $0 \ldots 1$. We construct a sequence of real numbers in this interval. We start by creating some sequence like the following:

$$0.10001\ldots$$

$$0.01001\ldots$$

$$0.00101\ldots$$

$$0.00011\ldots$$

$$0.00001\ldots$$

$$\ldots$$

We now have an infinite sequence of infinite sequences of digits. Can we enumerate this sequence and assign natural numbers to each single real number to count the whole sequence? There is an issue. For any given sequence, we can easily construct a new number, not yet in the sequence, but in the same interval (the numbers $0\ldots1$). The method is called Cantor's (second) *diagonal argument*. It goes like this: For each number in the sequence, from the first number take the first digit, from the second number take the second digit, from the third number take the third digit and so on. Since the numbers in the sequence are distinct, the new number, constructed in this way, is different from all other numbers in the sequence. For instance:

$$0.\mathbf{1}0001\ldots$$

$$0.0\mathbf{1}001\ldots$$

$$0.00\mathbf{1}01\ldots$$

$$0.000\mathbf{1}1\ldots$$

$$0.0000\mathbf{1}\ldots$$

$$\ldots$$

$$0.\mathbf{11111}\ldots$$

Note that the sequence above was not particularly designed to hold for this method. As long as the numbers are irrational, *i.e.* each of them consists of an infinite non-repeating sequence of digits, we will, following the method, always construct a number that is not yet in the list.

The point is that we can do this with any sequence of irrational numbers one may come up with. In consequence, when we construct an enumeration of the real numbers, as we did for rational numbers using the Calkin-Wilf or the Stern-Brocot tree, there is always a number that we can introduce between any two numbers in the sequence. Any possible sequence of the real numbers, hence, is necessarily incomplete. We, therefore, arrive at the strange conclusion that $|\mathbb{R}| \neq |\mathbb{N}|$ or, in other words, $\mathbb{R}$ is not *countable*.

Cantor says that the sets are both infinite, but they are infinite in different ways: $\mathbb{N}$ is countably infinite, while $\mathbb{R}$ is *uncountably* infinite. There are thus different infinite cardinalities. That of $\mathbb{N}$ is $\aleph_0$; that of other sets may be $\aleph_1, \aleph_2, \aleph_3, \ldots$ leading to a whole new universe of numbers that express different ways of being infinite.

Cantor conjectured that the cardinality of $\mathbb{R}$ is the cardinality of the *powerset* of $\mathbb{N}$, which is, as you may remember, 2^n for a set with n elements. The cardinality of $\mathbb{R}$, would then be $2^{\aleph_0}$.

This makes a lot of sense, when you consider the diagonal method above. Indeed, when we created the powerset of a given set, we used binary numbers to encode the presence or absence of a given element in one of the sets in the powerset. For the set $S = \{a, b, c\}$, the number 100_2 would encode the set $\{a\} \in P(S)$. Now, when you consider that S is infinite, you have an infinite sequence of such numbers and, as we did above illustrating the diagonal argument, we can introduce for any given such sequence a new number, *i.e.* a new element of $P(S)$.

Cantor further conjectured that $\aleph_1 = 2^{\aleph_0}$. That would mean that there is no infinite cardinality "between" that of $\mathbb{N}$ and that of $\mathbb{R}$. Cantor's conjecture is very famous under the name *Continuum Hypothesis* (CH). In the early 20^{th} century it was important enough for Hilbert to include it into his equally famous 23 problems that he assumed to be the most important math problems to be solved. It was, in fact, the first of these 23 problems.

The hypothesis as such is still unsolved today. In 1940, however, Kurt Gödel showed that CH cannot be disproven based on the standard axiomatic system, the Zermelo-Fraenkel set theory (ZF). In 1963, again, Paul Cohen showed that it cannot be proven in ZF either. Mathematicians today say that CH is *independent* from ZF. This may perhaps be translated into sloppy common speech as CH is irrelevant, at least in the context of the standard axiomatic system.

Without out too much phantasy, we can go beyond CH and suspect that there is a general rule that for any $n \in \mathbb{N} : \aleph_n = 2^{\aleph_{n-1}}$. $\aleph_1 = 2^{\aleph_0}$ would then be no exception. It would just be the way to count in infinity. This hypothesis is called the *Generalised Continuum Hypothesis* (GCH). The idea is extremely beautiful and, again, it makes a lot of sense. We cannot, of course, extend infinity by just adding an element; that would still be infinity. Between a set and its powerset, however, there is a *structural* difference that still holds, as we have seen, with infinite sets. There is no one-to-one mapping from the set to its powerset or, more formally worded, "there is no *surjection* from a set to its powerset". This is known as *Cantor's theorem* and Cantor's diagonal argument demonstrates that the relation between the sets $\mathbb{N}$ and $\mathbb{R}$ is an instance of this theorem.

The GCH, just as the weaker CH, is also independent from ZF. There are, however, some stronger implications when assuming the truth of GCH, as shown, again, by Kurt Gödel and by Polish mathematician Wacław Sierpiński (1882 – 1969). That, however, would lead us deeply into mathematical logic, which is not our topic here. More interesting appears to be the question what actually causes the difference between the cardinalities of $\mathbb{N}$ and $\mathbb{R}$.

This question is discussed in Cantor's first article on set theory, a tremendously important document in the history of math. In this short article – it has less than five pages

– Cantor first proves that the set of real *algebraic* numbers is countable and then that the set of real numbers is not countable providing this way a new proof for the existence of the *transcendental* numbers and demonstrating that it is the transcendental numbers that make the set of real number uncountable.

To enumerate the algebraic numbers, Cantor uses a sophisticated trick, that involves mathematical machinery that is not yet at our disposal. He uses *polynomials* – those beasts will be a major topic of the next part – that are *irreducible* over $\mathbb{Q}$. He orders these polynomials and interprets algebraic numbers as the *roots*, *i.e.* the solutions, of the polynomials. The first ten polynomials and their roots are:

Polynomial	Root
x	0
$x+1$	-1
$x-1$	1
$x+2$	-2
$2x+1$	$-\frac{1}{2}$
$2x-1$	$\frac{1}{2}$
$x-2$	2
$x+3$	-3
x^2+x-1	$\frac{-1-\sqrt{5}}{2}$
x^2-2	$-\sqrt{2}$
...	...

The column "Root" contains the components of the enumeration of the algebraic numbers, which, hence, goes like

$$0, -1, 1, -2, -\frac{1}{2}, \frac{1}{2}, 2, -3, \frac{-1-\sqrt{5}}{2}, -\sqrt{2}, \ldots$$

The enumeration technique is better than the one he used to enumerate the rationals, since it contains each algebraic number only once. Perhaps you remember that, using Cantor's original technique for enumerating the rationals, we had to filter out duplicates.

We see further that the sequence enumerates an extension of $\mathbb{Q}$ containing roots of rational numbers and more complex formulas including such roots. Since Cantor used the property of a real number being algebraic, *i.e.* appearing as a solution of a polynomial

with rational coefficients, he guarantees that the resulting enumeration contains exactly all algebraic numbers. He, thus, proves that the cardinality of the set of algebraic numbers equals $|\mathbb{N}| = \aleph_0$. Since, as we have already seen, the set of real numbers has not the cardinality $\aleph_0$, this means that it is indeed the transcendental numbers that make $\mathbb{R}$ uncountable.

1.13 Review of the Number Zoo

We have, in the previous chapters, studied properties of numbers and problems that are related to things that are countable. On the way, we repeatedly met concepts related to sets with operations that show certain properties, in particular closure, associativity, identity and invertibility. We found such structures not only among numbers, but also in relation with other objects like strings and permutations.

We called a structure of the form

$$(S, \circ)$$

consisting of a set S and an operation $\circ$ a magma, if the operation is closed over S, *i.e.*

$$a, b \in S \rightarrow a \circ b \in S.$$

A magma is a universal structure found in many contexts, not only numbers. But we saw that the natural numbers, $\mathbb{N}$, form a magma together with both, addition and multiplication. We can even go further and predict that all number types that we have constructed by extending the notion of *natural number*, namely $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, are magmas: they are all closed under the operations $+$ and $\times$.

But there are other properties, seen with $+$ and $\times$ together with any of our number types, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. First associativity:

$$(a \circ b) \circ c = a \circ (b \circ c) = a \circ b \circ c.$$

This property makes all our number types semigroups. Furthermore, they all have an identity, e, together with either of the operations, namely 0 for addition and 1 for multiplication, such that for any $a \in S$:

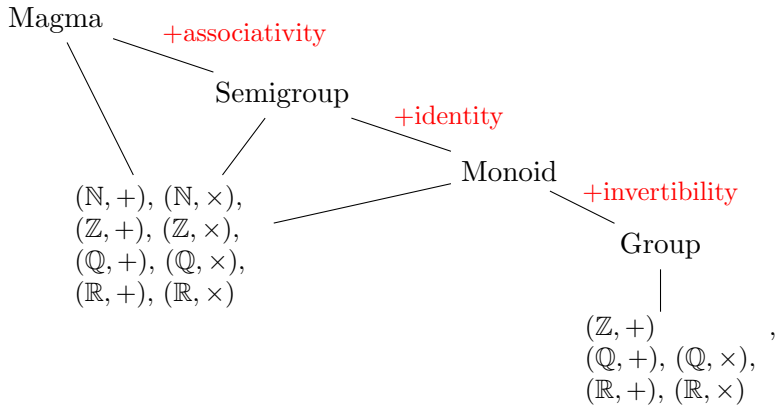
$$a \circ e = e \circ a = a.$$

This property makes all the number types together with either of the operations monoids.

Now, we have seen that some of the number types and operations, but not all of them, have yet another property, namely invertibility, *i.e.* the property that, for any $a \in S$, there is an element $a' \in S$, such that

$$a \circ a' = e.$$

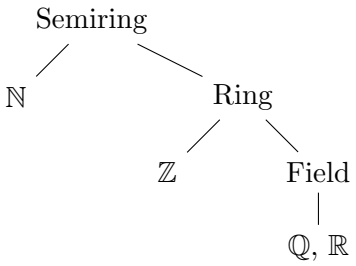
This property holds for $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{Q}, \times)$ and $(\mathbb{R}, +)$ as well as $(\mathbb{R}, \times)$. Invertibility makes these structures groups. The following sketch summarises this result:



On top of these definitions, a different kind of structures is defined, that serves to distinguish different types of numbers. These new structures consist of a set and two operations, called addition and multiplication, respectively:

$$(S, +, \times).$$

If both operations form monoids over S, then we call this structure a **semiring**. An example of a semiring is $(\mathbb{N}, +, \times)$, since this structure consists of two monoids. If addition forms a group over S and multiplication forms a monoid, then we call this structure a **ring**. An example of a ring is $(\mathbb{Z}, +, \times)$, because addition in this structure is a group and multiplication is a monoid. If both, addition and multiplication, form a group over S, we call the resulting strucure a **field**. Examples for fields are $(\mathbb{Q}, +, \times)$ and $(\mathbb{R}, +, \times)$, since these structures have groups for both addition and multiplication. Here is an overview over our number types:


Semiring
ℕ
Ring
ℤ
Field
ℚ, ℝ