

# 1 Real Numbers

## 1.1 $\sqrt{2}$

Until now, we have looked at *discrete* numbers, that is numbers that are nicely separated from each other so that we can write them down unmistakably and always know of which number we are currently talking. Now we enter a completely different universe. The universe of continuous numbers that cannot be written down in a finite number of steps. The representation of these numbers consists of infinitely many elements and, therefore, we will never be able to write the number down completely with all its elements. We may give a finite formula that describes how to compute the specific number, but we will never see the whole number written down. Those numbers are known since antiquity and, apparently, their existence came as a great surprise to Greek mathematicians.

The first step of our investigations into this kind of numbers, is to show that they *exists*, *i.e.* there are contexts where they arise naturally. To start, we will assume that they are not necessary. We assume that all numbers are either natural, integral or fractional. Indeed, any of the fundamental arithmetic operations,  $+$ ,  $-$ ,  $\times$  and  $/$ , applied on two natural numbers, results either in another natural number, a negative number or a fraction. We could therefore suspect that the result of any operation is a rational number, *i.e.* an integer or a fraction.

What about  $\sqrt{2}$ , the square root of 2? Let us assume that  $\sqrt{2}$  is as well a fraction. Then we have two integers  $n$  and  $d$ , coprime to each other, such that

$$\sqrt{2} = \frac{n}{d} \tag{1.1}$$

and

$$2 = \left(\frac{n}{d}\right)^2 = \frac{n^2}{d^2}. \tag{1.2}$$

Any number can be represented as a product of primes. If  $p_1 p_2 \dots p_n$  is the prime factorsiation of  $n$  and  $q_1 q_2 \dots q_d$  is that of  $d$ , we can write:

$$\sqrt{2} = \frac{p_1 p_2 \dots p_n}{q_1 q_2 \dots q_d}. \tag{1.3}$$

It follows that

$$2 = \frac{p_1^2 p_2^2 \cdots p_n^2}{q_1^2 q_2^2 \cdots q_d^2}. \quad (1.4)$$

As we know from the previous chapter, two different prime numbers  $p$  and  $q$  squared (or raised to any integer) do not result in two numbers that share factors. The factorisation of  $p^n$  is just  $p^n$  and that of  $q^n$  is just  $q^n$ . They are coprime to each other. The fraction in equation 1.4, thus, cannot be an integer. There is only one way for such a fraction to result in an integer, *viz.* when it is an integer itself. From this follows that, if the root of an integer is not an integer itself, it is not a rational number either.

But, if  $\sqrt{2}$  is not a rational number, a number that can be represented as the fraction of two integers, what the heck is it then?

**1.2  $\Phi$**

**1.3  $\pi$**

**1.4  $e$**

**1.5  $\gamma$**

**1.6 Arithmetic with Real Numbers**

**1.7 Representation of Real Numbers**

**1.8  $\mathbb{R}$**

**1.9 Real Factorials**

**1.10 Real Binomial Coefficients**

**1.11 The Continuum**

**1.12 Review of the Number Zoo**