1 Real Numbers

1.1 $\sqrt{2}$

Until now, we have looked at *discrete* numbers, that is numbers that are nicely separated from each other so that we can write them down unmistakebly and always know of which number we are currently talking. Now we enter a completely different universe. The universe of continuous numbers that cannot be written down in a finite number of steps. The representation of these numbers consists of infinitely many elements and, therefore, we will never be able to write the number down completely with all its elements. We may give a finite formula that describes how to compute the specific number, but we will never see the whole number written down. Those numbers are known since antiquity and, apparently, their existence came as a great surprise to Greek mathematicians.

The first step of our investigations into this kind of numbers, is to show that they *exist*, *i.e.*, there are contexts where they arise naturally. To start, we will assume that they are not necessary. We assume that all numbers are either natural, integral or fractional. Indeed, any of the fundamental arithmetic operations, $+, -, \times$ and $/$, applied on two natural numbers, results either in another natural number, a negative number or a fraction. We could therefore suspect that the result of any operation is a rational number, *i.e.* an integer or a fraction.

What about $\sqrt{2}$, the square root of 2? Let us assume that $\sqrt{2}$ is as well a fraction. Then we have two integers n and d, coprime to each other, such that

$$\sqrt{2} = \frac{n}{d} \tag{1.1}$$

and

$$2 = \left(\frac{n}{d}\right)^2 = \frac{n^2}{d^2}. \tag{1.2}$$

Any number can be represented as a product of primes. If $p_1 p_2 \ldots p_n$ is the prime factorsiation of n and $q_1 q_2 \ldots q_d$ is that of d, we can write:

$$\sqrt{2} = \frac{p_1 p_2 \ldots p_n}{q_1 q_2 \ldots q_d}. \tag{1.3}$$

It follows that

$$2 = \frac{p_1^2 p_2^2 \cdots p_n^2}{q_1^2 q_2^2 \cdots q_d^2}. \tag{1.4}$$

As we know from the previous chapter, two different prime numbers p and q squared (or raised to any integer) do not result in two numbers that share factors. The factorisation of p^n is just p^n and that of q^n is just q^n. They are coprime to each other. The fraction in equation 1.4, thus, cannot be an integer. There is only one way for such a fraction to result in an integer, *viz.*, when it is an integer itself. From this follows that, if the root of an integer is not an integer itself, it is not a rational number either.

But, if $\sqrt{2}$ is not a rational number, a number that can be represented as the fraction of two integers, what the heck is it then?

There are several methods approximate that number. The simplest and oldest is the *Babylonian* method, also called *Heron's* method for Heron of Alexandria, a Greek mathematician of the first century who lived in Alexandria.

The idea of Heron's method, basically, is to iteratively approximate the real value starting with a guess. We can use the root algorithm defined in the first chapter to generate a guess or start with some arbitrary value. This value, x, if it is not an integer, is either slightly too big or too small, *i.e.* we either have $xx > a$ or $xx < a$. So, on each step, we improve a bit on the value by taking the average of x and its counterpart a/x. If $xx > a$, then $a/x < x$ and, if $xx < a$, then $a/x > x$. The average of x and a/x is calculated as $(x + a/x)/2$. The result is used as guess for the next step. The more iterations of this kind we do, the better is the approximation. In Haskell:

```
heron :: Natural → Natural → Double
heron n s = let a = fromIntegral n in go s a (a / 2)
   where go 0 _ x = x
         go i a x = go (i − 1) a ((x + a / x) / 2)
```

The function takes two arguments. The first is the number whose square root we want to calculate and the second is the number of iterations. We then call *go* with s, the number of iterations, a, the *Double* representation of n and our first guess $a/2$.

In *go*, if we have reached $i = 0$, we yield the result x. Otherwise, we call *go* again with $i − 1$, a and the average of x and a/x.

Let us compute $\sqrt{2}$ following this approach for, say, five iterations. We first have

go 5 2 1 = *go* 4 2 ((1 + 2) / 2)
go 4 2 1.5 = *go* 3 2 ((1.5 + 2 / 1.5) / 2)
go 3 2 1.416666 = *go* 2 2 ((1.416666 + 2 / 1.416666) / 2)
go 2 2 1.414215 = *go* 1 2 ((1.414215 + 2 / 1.414215) / 2)
go 1 2 1.414213 = *go* 0 2 ((1.414213 + 2 / 1.414213) / 2)
go 1 2 1.414213 = 1.414213.

Note that we do not show the complete *Double* value, but only the first six digits. The results of the last two steps, therefore, are identical. They differ, in fact, at the twelfth digit: 1.4142135623746899 (*heron* 2 4) versus 1.414213562373095 (*heron* 2 5). Note that the result of five iterations has one digit less than that of four iterations. This is because that after the last digit, 5, the digit 0 follows and then the *precision* of the Double number type is exhausted. *Irrational* numbers, this is the designation of the type of numbers we are talking about, consist of infinitely many digits. Therefore, the last digit in the number presented above is in fact not the last digit of the number. With slightly higher precision, we would see that the number continues like 03 . . .

The result of (*heron* 2 5) ↑ 2 is fairly close to 2: 1.9999999999999996. It will not get any closer using *Double* representation.

1.2 Φ

Another interesting irrational number is τ or Φ, known as the *divine proportion* or *golden ratio*. The golden ratio is the relation of two quantities, such that the greater relates to the lesser as the sum of both to the greater. This may sound confusing, so here are two symbolic representations:

$$\frac{a}{b} = \frac{a + b}{a}. \tag{1.5}$$

In this equation, a is the greater number and b the lesser. The equation states that the relation of a to b equals the relation of $a + b$ to a. Alternatively we can say

$$\frac{b}{a} = \frac{a}{b - a}. \tag{1.6}$$

Here b is the sum of the two quantities and a is the greater one. The equation states that the relation of b to a is the same as the relation of a to $b - a$, which, of course, is then the smaller quantity.

The golden ratio is known since antiquity. The symbol Φ is an homage to ancient Greek sculptor and painter Phidias (480 – 430 BC). Its first heyday after antiquity had the divine proportion during Renaissance and then again in the 18th and 19th centuries. Up to our times, artists, writers and mathematicians have repeatedly called the golden ratio

especially pleasing. But, here, of course, we will refrain from aesthetical judgements outside the realm of mathematics.

From equation 1.6 we can derive the numerical value of Φ in the form $\frac{\Phi}{1}$. We can then calculate b, for any value a, as $a\Phi$. The derivation uses some algebraic methods, which we will study more closely in the next part, especially *completing the square*. As such this section is also an algebra teaser.

We start by setting $\Phi = \frac{b}{a}$ with $a = 1$. We then have on the left side of the equation $\Phi = \frac{b}{1} = b$. On the right-hand side, we have $\frac{a=1}{\Phi-1}$:

$$\Phi = \frac{1}{\Phi - 1}. \tag{1.7}$$

Multiplying both sides by $\Phi - 1$, we get

$$\Phi^2 - \Phi = 1. \tag{1.8}$$

This is a quadratic equation and, with some phantasy, we can even recognise the fragment of a binomial formula on the left side. A complete binomial formula would be

$$(a - b)^2 = a^2 - 2ab + b^2. \tag{1.9}$$

We try to pattern match the fragment above such that Φ^2 is a^2 and $-\Phi$ is $-2ab$. That would mean that the last term, b^2 would correspond to the square half of the number that is multiplied by Φ to yield $-\Phi$. $-\Phi$ can be seen as $-1 \times \Phi$. Half of that number is $-\frac{1}{2}$. That squared is $\frac{1}{4}$. So, if we add $\frac{1}{4}$ to both sides of the equation, we would end up with a complete binomial formula on the left side:

$$\Phi^2 - \Phi + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}. \tag{1.10}$$

We can apply the binomial theorem on the left side and get

$$\left(\Phi - \frac{1}{2}\right)^2 = \frac{5}{4}. \tag{1.11}$$

Now we take the square root:

$$\Phi - \frac{1}{2} = \frac{\pm\sqrt{5}}{2}. \tag{1.12}$$

4

Note that the $\pm$ in front of $\sqrt{5}$ reflects the fact that the square root of 5 (or any other number) may be positive or negative. Both solutions are possible. However, since we are looking for a positive relation, we only consider the positive solution and ignore the other one. We can therefore simplify to

$$\Phi - \frac{1}{2} = \frac{\sqrt{5}}{2}. \tag{1.13}$$

Finally, we add $\frac{1}{2}$ to both sides:

$$\Phi = \frac{1 + \sqrt{5}}{2} \tag{1.14}$$

and voilá that is Φ. The numerical value is approximately 1.618033988749895. We can define it as a constant in Haskell as

$$phi :: Double$$
$$phi = 0.5 * (1 + sqrt\ 5)$$

We can now define a function that, for any given a, yields b:

$$golden :: Double \rightarrow Double$$
$$golden\ a = a * phi$$

$golden$ 1, of course is just Φ, *i.e.* 1.618033988749895. $golden$ 2 is twice that value, namely 3.23606797749979. Furthermore, we can state that 2 / ($golden$ 2 $-$ 2), which is $\frac{a}{b-a}$, is approximately the same, namely, again, an approximation to Φ.

That is very nice. But there is more to come. Φ, in fact, is intimately connected to the Fibonacci Sequence, as we will se in the next chapter.

1.3 π

π is probably the most famous irrational number. We, therefore, do not need much of an explanation to introduce that number. π emerged in antique mathematics in studying the circle. π expresses the relation between the diameter (depicted in red in the image below) and the circumference (depicted in black):
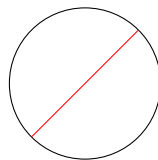
Since, often, the radius, which is half the diameter, is much more important in mathematics, it has been proposed to use $\tau = 2\pi$ where π is used today. But π has survived through history and, even though slightly suboptimal in some situations, it is still in use today.

The reason why the perimeter instead of the radius was used to define the circle constant is probably because classic approaches to approximate π take the perimeter as basis. They start by drawing a square with side length 1 and inscribe a circle into the square with perimeter 1:
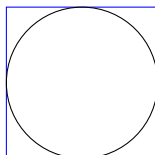
Since the square has side length 1, its perimeter, the sum of all its sides is $1+1+1+1 = 4$ and, as we can see clearly in the picture above, the perimeter is greater than that of the circle. 4, hence, is an upper bound for the perimeter of the circle with perimeter 1. A lower bound would then be given by a square inscribed in the circle, such that the distance between its opposing corners (red) is 1, the perimeter of the circle:
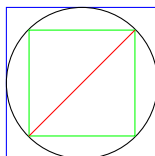
We see two right triangles with two green sides on a red basis. The basis is the perimeter of the circle, of which we know that its length is 1. You certainly know the Pythagorean theorem, probably the most famous or notorious theorem of all mathematics, which states that, in a right triangle, one with a right angle, an angle of $90°$, the sum of the squares of the sides to the left and right of that angle (the green sides) equals the square of the hypothenuse, the red side, which is opposite to the right angle. This can be stated as:

$$a^2 = b^2 + c^2, \tag{1.15}$$

where a is the red side, whose length we know, namely 1. We further know that the green sides are equal. We hence have:

$$1^2 = 2b^2. \tag{1.16}$$

and further derive

$$1 = \sqrt{2b^2}, \tag{1.17}$$

which is

$$1 = \sqrt{2}b. \tag{1.18}$$

b, the side length of the green square, hence, is $\frac{1}{\sqrt{2}}$, which is approximately 0.707. The perimeter of the inner square is thus 4×0.707, which is approximately 2.828. Thus π is some value between 2.828 and 4.

That result is not very satifactory, of course. There is room for a lot of numbers between 2.828 and 4. The method was therefore extended by choosing polygons with more than four sides to come closer to the real value of π. The antique record holder for approximating π is Archimedes who started off with a hexagon, which is easy to construct with compass and ruler:
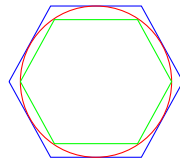
Then he subsequently doubled the number of sides of the polygon, so that he obtained polygons with 12, 24, 48 and, finally, 96 sides. With this approach he concluded that $\frac{223}{71} < \pi < \frac{22}{7}$, which translates to a number between 3.1408 and 3.1428 and is pretty close to the approximated value 3.14159.

In modern times, mathematicians started to search for approximations by other means than geometry, in particular by infinite series. One of the first series was discovered by Indian mathematician Nilakantha Somayaji (1444 – 1544) in the 15th century. It goes like

$$\pi = 3 + \frac{4}{2 \times 3 \times 4} - \frac{4}{4 \times 5 \times 6} + \frac{4}{6 \times 7 \times 8} - \ldots \tag{1.19}$$

We can implement this in Haskell as

```
nilak :: Int → Double
nilak i    | even i    = nilak (i + 1)
           | otherwise = go i 2 3 4
    where go 0 _ _ _ = 3
          go n a b c = let k | even n = -4
                             | otherwise = 4
                       in (k / (a * b * c)) + go (n − 1) c (c + 1) (c + 2)
```

Here we use a negative term, whenever n, the counter for the step we are performing, is even. Since, with this approach, an even number of steps would produce a bad approximation, for i even, we perform $i + 1$, and odd number of steps. This way, the series converges to 3.14159 after about 35 steps, *i.e. nilak* 35 is some number that starts with 3.14159.

An even faster convergence is obtained by the beautiful series discovered by French mathematician François Viète (1540 – 1603) in 1593:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2 + \sqrt{2}}}{2} \times \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \times \ldots \tag{1.20}$$

In Haskell this gives rise to a very nice recursive function:

```
vietep :: Int → Double
vietep i = 2 / (go 0 (sqrt 2))
    where go n t | n ≡ i    = 1
                 | otherwise = (t / 2) * go (n + 1) (sqrt (2 + t))
```

There are many other series, some focusing on early convergence, others on beauty. An exceptionally beautiful series is that of German polymath Gottfried Wilhelm Leibniz (1646 – 1716), who we will get to know more closely later on:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \tag{1.21}$$

In Haskell this is, for instance:

```
leipi :: Int → Double
leipi i = 4 * go 0 1
    where go n d | n ≡ i    = 0
                 | otherwise = let x | even n = 1
                                     | otherwise = -1
                               in x / d + go (n + 1) (d + 2)
```

This series converges really slowly. We reach 3.14159 only after about 400 000 steps.

π appears quite often in mathematics, particularly in geometry. But there are also some unexpected entries of this number. The inevitable Leonhard Euler solved a function,

which today is called *Riemann zeta function*, for the special case $s = 2$:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.22}$$

Euler showed that, for the special case $s = 2$, ζ is $\frac{\pi^2}{6}$. This is surprising, because the zeta function is not related to circles, but to number theory. It appears for example, when calculating the probability of two numbers being coprime to each other. Two numbers are coprime if they do not share prime factors. The probability of a number being divisible by a given prime p is $\frac{1}{p}$, since every p^{th} number is divisible by p. For two independently chosen number, the probability that both are not divisible by prime p is therefore $\frac{1}{p} \times \frac{1}{p} = \frac{1}{p^2}$. The reversed probability that both are not divisible by that prime, hence, is $1 - \frac{1}{p^2}$. The probablitiy that there is no prime at all that divides both is then

$$\prod_{p}^{\infty} 1 - \frac{1}{p^2}. \tag{1.23}$$

To cut a long story short, this equation can be transformed into the equation

$$\frac{1}{1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots} = \frac{1}{\zeta(2)} = \frac{1}{\frac{\pi^2}{6}} = \frac{6}{\pi^2} = 0.607 \approx 61\% \tag{1.24}$$

and with this π appears as a constant in number theory expressing the probability of two randomly chosen number being coprimes.

1.4 e

1.5 γ

1.6 Arithmetic with Real Numbers

1.7 Representation of Real Numbers

1.8 $\mathbb{R}$

1.9 Real Factorials

1.10 Real Binomial Coefficients

1.11 The Continuum

1.12 Review of the Number Zoo