Math for Programmers

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1. Introduction

2. Brief Introduction to Haskell

Part I.

Arithmetic and Combinatorics

3. Natural Numbers

3.1. What are numbers and what should they be?

In his famous article "What are numbers and what should they be?" German mathematician Richard Dedekind (1831 – 1916) described numbers as "free creations of the human mind". This idea is anything but self-evident. Dedekind's contemporary and fellow combatant for a new approach in math, namely set theory, Georg Cantor fiercely defended *mathematical platonism*, the idea that mathematical objects are real existing and that mathematical methods are tools that allow us to see these objects like, perhaps, telescopes, enable us to examine stars far away in the universe.

This view clashed with the view of the *constructivist* Leopold Kronecker, a resolute opponent to the novelties introduced by Cantor, Dedekind and others. In the view of the constructivists like Kronecker only those objects that can be constructed in a finite number of steps from a finite number of integers are valid objects of mathematics. Anything else is a chimaera. "The natural numbers are made by god", as he put it, "all else is the work of man."

Both views, constructivism and platonism, even though not mainlines of discussion anymore, are still influential today. The influence of platonism is visible, when it is said that some mathematician has "discovered" a concept, *e.g.* "Gauss discovered a construction of the heptadecagon". Mathematicians in the tradition of constructivism or its modern variant *intuitivism*, insist in a terminology that stresses that mathematical concepts are man-made. They would say Gauss has "invented", "developed" or, simply, "constructed".

It is consensus, however, that, independent of the question whether numbers are real existing things or invented concepts, there must be some kind of representation of numbers in our mind. It is also likely that these representations have some grounding in the nervous system and are thus part of our genetic legacy. A strong evidence therefor is the fact that some animals can distinguish sets of different numbers of objects. Birds, for instance, realise when an egg is missing in their nest, but do not necessarily realise that an egg was replaced by another one. That would suggest that the "creation" of numbers is not as "free" as it appears to be according to Dedekind. The ability to work with numbers would then be part of our biology, just like having language capability is part of our biology.

When we try to reduce numbers to the very heart of what they are, we quickly come to the

notion of counting. The assumption that numbers basically represent steps in the process of counting is supported by the fact that many number systems in use during history are modelled on something countable like fingers (leading to the decimal number system), fingers and toes (leading to the vigesimal system), finger bones (leading to systems based on multiples of 12) and so on. According to this view, relations between quantities (length, volume, weight, *etc.*) would represent more abstract concepts engineered in some way on top of the fundamental notion of counting. It is not so clear, however, if this hierarchy corresponds to the real historic development. Among the oldest known math problems are such as determining the quanitity of corn for baking bread in acient Egypt or investigating rhythmic patterns formed by single and multiple beats. Problems of this kind cannot be solved by mere counting, but demand comparison of *continuous* quantities like volume, weight or time intervals leading immediately to fractions. Numbers may hence be related to a human sense of quantity that is more general than counting.

Numbers appear to be very simple and clear concepts. But, in fact, together with the operations defined on them, addition, multiplication, subtraction and division, they quickly give rise to intricate problems. Those problems are not like add-ons to the numbers that were invented later to complicate things, they belong to the very core of the number concept. It appears, in fact, that those problems have already slumbered in the apparently simple concept, but needed some time to unfold themselves – centuries in some cases, millennia in others. Even if the number concept may have been created by the human mind, once fixed, all its consequences and complications are there. In this sense, every mathematician would agree to speak of "discovering" a problem in a certain field of math. Mathematicians may still disagree, however, on whether the solution to such a problem was "discovered" or "developed".

The most notorious group of long-standing problems is number theory, which deals mainly with questions of prime numbers, *i.e.* natural numbers that cannot be constructed from other natural numbers using multiplication. Prime numbers are investigated for some three thousand years now, but, still, many fundamental questions remain unanswered. Infamous are the number-theoretic theorems drafted, but not proven, by Pierre de Fermat in the 17^{th} century. Most of these theorems look quite innocent, but by proving them in the course of the following three centuries, math made huge progress developing methods that had been unknown to Fermat and his contemporaries.

The most insistent problem, known as *Fermat's Last Theorem*, was solved only in the late 20^{th} century by British mathematician Andrew Wiles. What makes this specific problem so mind-boggling is its extremely simple formulation – compared to the proof of some 200 pages. Indeed, Fermat doodled the theorem on the margin of a page in his copy of an acient math book, Diophantos' "Arithmetica", and, to the irritation of generations of mathematicians, he added: "I have found a really wonderful proof, but, unfortunately, this margin is too narrow to contain it." The theorem in question states that there is no solution for equations of the form: $a^z + b^z = c^z$ for a, b, c and z all natural numbers and $z > 2$. For $z = 2$, many solutions exist, the so called *Pythagorean triples*, *e.g.*: $3^2 + 4^2 = 5^2$.

Another set of difficult problems is algebra and the search for solutions of higher-degree equations, which caused a lot of headache throughout the centuries. The general solution for quadratic equations of the form $ax^2+bx+c=0$ was known since antiquity. Solutions for cubic and quartic equations were found by joint efforts and fierce competition in the 16^{th} century by Italian mathematicians, namely Niccolò Fontana Tartaglia, Gerolamo Cardano and Lodovico Ferrari. But it took until the early 19^{th} century, before mathematicians were able to make statements about quintic and even higher-degree equations. In fact, the investigations took an unpredicted direction and led to the development of a new branch of mathematics, *group theory*, which studies sets of objects – not only numbers – and operations defined on these objects, and *abstract algebra*, which generalises group theory making objects of completely different areas of mathematics comparable.

Group theory, which was developed during the 18^{th} and 19^{th} century by Joseph-Louis Lagrange, Paolo Ruffini, young Niels Henrik Abel and the incredible Évariste Galois, is a great helper in organising the number zoo that developed out of simpler notions of numbers in the course of the time. At the beginning – if we believe in numbers being fundamentally related to counting – there may have been only the concept of natural numbers, which was then extended by adding fractions to cope with relations between quantities and negative numbers to deal with negative quantities like debts. The investigations into geometry by Greek mathematicians led to the rise of irrational numbers – such as the number π – which, as a surprise to acient mathematicians, cannot be expressed in terms of ratios of integers. The studies in algebra led to complex numbers, which, in their turn, inspired the construction of hypercomplex numbers.

In this chapter, we will start with the simplest possible number concept, natural numbers, and we will stick to it as long as possible – perhaps longer. We then will make a big leap introducing negative numbers and fractions. Afterwards we enter the confusing world of irrational numbers and will probably understand the annoyance of mathematicians like Kronecker when faced with mathematical concepts dealing with these strange creatures. Later, complex numbers and even more exotic beasts will come into focus. On the way, we will introduce some basic group theory helping us to find our way in the number jungle.

3.2. Peano Numbers

The Italian mathematician Giuseppe Peano (1858 – 1932) defined an axiomatic system to describe the natural numbers and their properties. An axiomatic system consists of axioms, statements that are known or simply assumed to be true, and rules that describe how other statements can be derived from axioms, such that these new statements are true if the axioms are true. In practice the procedure is usually followed the other way round: one would try to find a sequence of applications of the rules that links a given statement with one or more axioms. This process is called a *proof* and is one of the main things mathematicians do to kill their time.

A major part of the discussions about the foundations of math in the first half of the 20^{th} century was about the idea formulated by David Hilbert to construct the whole of mathematics as an axiomatic system and, then, to prove that for every statement that is believed to be true a sequence of rule applications can be found that derives this statement from the axioms. The plan failed in the 1930ies, after releasing an incredible amount of mathematical creativity that resulted, among other things, in a theoretical model of a universal computing device, known today as the *Turing machine* and in the *lambda calculus*, which, as already discussed, is one of the foundations of functional programming. The first task of the Turing machine and of the lambda calculus was indeed to prove that Hilbert's plan is impossible.

Peano's objective was in Hilbert's line: to provide a foundation of natural numbers as a first step towards an axiomatisation of the whole field of arithmetic. In spite of this ambitious goal, Peano's axioms are quite simple. The basic idea is to define natural numbers by two elements: The explicitly defined number *Zero* and a recursive function *Successor* that defines any other number. Peano's axioms boil down to a formulation in Haskell like:

```
data Peano = Zero | S Peano
  deriving Show
```

This captures very well the process of counting. Instead of adding 1 to a given number, we just derive its successor:

```
succ :: Peano → Peano
succ p = S p
```

For instance, the successor of *Zero*, one, is: $S(Zero)$; two is $S(S(Zero))$, three is $S(S(S(Zero)))$ and so on.

We can also define a function to count backwards, *i.e.*:

```
pre :: Peano → Peano
pre Zero  = ⊥
pre (S p) = p
```

Zero, this is one of Peano's axioms, has no predecessor. The predecessor of any other number, is that number with one S removed. The predecessor of $S(S(S(S(S(Zero)))))$, five, for instance, is $S(S(S(S(Zero))))$, four.

We can also devise a simple addition function:

```
add :: Peano → Peano → Peano
add Zero a  = a
add a Zero  = a
add (S a) b = add a (S b)
```

Subtraction is implemented easily as well:

$$
\begin{aligned}
&sub :: Peano \to Peano \to Peano \\
&sub\ a\ Zero &&= a \\
&sub\ Zero\ a &&= \bot \\
&sub\ (S\ a)\ (S\ b) &&= sub\ a\ b
\end{aligned}
$$

Note that any number reduced by $Zero$ is just that number. $Zero$ reduced by any number but $Zero$, however, is undefined. For all other cases, we just reduce a and b by one S until we hit one of the base cases.

We could go on and define multiplication, division and all other arithmetic operations based on this notion of numbers. It is indeed convincing in its simplicity and used as a standard system for research into arithmetic until today. But it is not very convenient, especially when working with huge numbers. Peano numbers are unary numbers, that is, to represent the number 100, one has to write 100 symbols (in fact, 101 symbols: 100 S and 1 $Zero$). As a number system, Peano numbers are even less handy than the roman numerals, which introduce symbols at least for some greater values, such as V, X, L and C. A trick that is often used in literature to mitigate this shortcoming is to add a subscript number to the S symbol to make clear, how many Ses we would have to write to represent this value, for instance, $S_5(Zero)$ would be 5. But, of course, that makes the use of Peano numbers – as a number system – pointless. Already Peano was faced with the clumsiness of his axioms when used as a number system: he tried to use it as a didactic device in his teaching both at the university and at the military academy where he was working. In the case of the military academy, this led to desaster and, eventually, to his dismissal in 1901. His achievements as mathetmatician and logician, however, were respected in the scientific community worldwide.

Let us learn from Peano's didactic failure and look out for a more practical number system, one that allows us to use significantly fewer symbols than the value of the number we want to represent. A system that *scales* in this sense is our well-known decimal number system.

3.3. Decimal Numbers

A numeral system consists of a vocabulary of symbols, which we will call *digits*, rules that define how to compose digits to strings and a model that leads to an arithmetic interpretation of such strings. To make practical use of the numeral system, we must also define a set of basic operations, such as counting forward and backward, addition, subtraction, multiplication, division and whatever we want to do with our numbers.

We define the following vocabulary:

```
data Digit = Zero | One | Two   | Three | Four |
             Five | Six | Seven | Eight | Nine
  deriving (Show, Eq, Ord)
```

Numbers are lists of *Digit*s:

```
type Number = [Digit]
```

Some numbers that are used more often than all others are:

```
zero, unity, two, ten :: Number
zero  = [Zero]
unity = [One]
two   = [Two]
ten   = [One, Zero]
```

Here is the first basic operation, the *successor*, which we already know from Peano's axioms. To avoid confusion with the *succ* function in Haskell's *Prelude*, we will call it *next*:

```
next :: Number → Number
next [] = []
```

This is already the first important decision. We define the *next* of the empty list is the empty list. That implies that *nothing* is something different from *Zero*. We could enter difficult philosophical discussions about this statement. The decision, however, is mainly pragmatic: we need a base case for processing lists and this base case is just the empty list.

The next successors are straight forward:

```
next [Zero]  = [One]
next [One]   = [Two]
next [Two]   = [Three]
next [Three] = [Four]
next [Four]  = [Five]
next [Five]  = [Six]
next [Six]   = [Seven]
next [Seven] = [Eight]
next [Eight] = [Nine]
```

Now it gets interesting:

```
next [Nine] = [One, Zero]
```

Note that we need one more digit to represent the successor of the 10^{th} digit! The first place, read from right to left, returns to *Zero* and the second place goes up from *nothing* to *One*. This latter wording shows that our decision, concerning the empty list, is not

so innocent as it may appear at the first sight!

Now we have to define how to proceed with the successor of numbers consisting of more than one digit:

$$next\ (Zero : ds) = next\ ds$$

The first thing we do is to check if the head is $Zero$. In this case, we just reduce to the rest of the list, that is: a leading $Zero$ does not change the value of a number. In all other cases:

$$\begin{aligned} next\ ds = \textbf{case}\ last\ ds\ \textbf{of}& \\ Nine &\rightarrow next\ (init\ ds) +\!\!+ [Zero] \\ d &\rightarrow init\ ds\ +\!\!+ next\ [d] \end{aligned}$$

If the last digit of the number is $Nine$, we concatenate the successor of the number without the last digit ($init$) and $[Zero]$. The point is that the successor of $Nine$, as we have defined it above, is $[One, Zero]$. The last digit of the new number, hence, will be $Zero$ appended to the successor of the initial part. If the last number of the initial part is again $Nine$, we repeat the whole process on the number except the last digit. Example: the successor of the number $[Nine, Nine]$ is

$next\ [Nine] +\!\!+ [Zero]$
$[One, Zero] +\!\!+ [Zero]$
$[One, Zero, Zero]$.

For the case that the last digit is not $Nine$, the process is much simpler: we just replace the last digit by its successor. The successor of $[Nine, Eight]$, hence, is:

$[Nine] +\!\!+ next\ [Eight]$
$[Nine] +\!\!+ [Nine]$
$[Nine, Nine]$.

Note that this representation of numbers is not optimised for efficient processing. Haskell is not very good at accessing the last element of a list. There are many ideas to speed this up. An idea that suggests itself is to turn numbers around – relative to our usual reading direction – starting with the least siginificant digit, *e.g.* writing $[Zero, One]$ instead of $[One, Zero]$ to represent the number 10. We could also use a data type – such as the vector type – that allows for fast random access to all its elements. But this kind of optimisations would be better discussed in a Haskell tutorial.

The next basic operation is counting backwards. We start just as we started with $next$:

$$\begin{aligned} &prev :: Number \rightarrow Number \\ &prev\ [\,] = [\,] \end{aligned}$$

But we now have an important difference:

$$prev\ [Zero] = \bot$$

We cannot count below $Zero$! Any attempt to do so will result in an error. We have to take care of this in all operations we will design in the future.

Counting backwards for the digits from One to $Nine$, however, is straight backward:

$$\begin{aligned}
prev\ [One] &= [Zero] \\
prev\ [Two] &= [One] \\
prev\ [Three] &= [Two] \\
prev\ [Four] &= [Three] \\
prev\ [Five] &= [Four] \\
prev\ [Six] &= [Five] \\
prev\ [Seven] &= [Six] \\
prev\ [Eight] &= [Seven] \\
prev\ [Nine] &= [Eight]
\end{aligned}$$

But what happens with numbers with more than one digit? First we ignore leading $Zero$s:

$$prev\ (Zero : ds) = prev\ ds$$

For all other cases, we use a strategy very similar to the one we used for $next$:

$$\begin{aligned}
prev\ ds = \textbf{case}\ &last\ ds\ \textbf{of} \\
&Zero \rightarrow \textbf{case}\ init\ ds\ \textbf{of} \\
&\qquad\qquad [One] \rightarrow [Nine] \\
&\qquad\qquad ds' \quad\ \rightarrow prev\ ds' +\!\!+ [Nine] \\
&d \quad\ \rightarrow init\ ds +\!\!+ prev\ [d]
\end{aligned}$$

If the last digit is $Zero$, the last digit of the new number will be $Nine$ and the initial part of this number will be its predecessor. If the initial part is just $[One]$, its predecessor would be $zero$, which we can ignore for this case. The predecessor of $[One, Zero]$, hence, is $[Nine]$ (not $[Zero, Nine]$). If the number is $[One, Zero, Zero]$, the last digit will be $Nine$, which is then appended to the predecessor of $[One, Zero]$, whose predecessor, as we know already, is $[Nine]$. The result hence is $[Nine, Nine]$.

For the case that the last digit of the number is not $Zero$, we just append its predecessor to the initial part of the number and we are done. The predecessor of $[Nine, Nine]$, hence, is just

$[Nine] +\!\!+ prev\ [Nine]$
$[Nine] +\!\!+ [Eight]$
$[Nine, Eight]$.

Let us now look at how to add numbers. We start with the same logic we already encountered with Peano Numbers, *i.e.* we add by counting one number up and the other, simultaneously, down until we reach a base case:

```
add :: Number → Number → Number
add a []       = a
add [] b       = b
add a [Zero]   = a
add [Zero] b   = b
add a b        = next a `add` (prev b)
```

That is, any number added to $[Zero]$ (or to the empty list []) is just that number. In all other cases, addition of two numbers a and b is defined recursively as counting a up and b down. When we hit the base case, *i.e.* b reaches $[Zero]$, we have a result.

How many steps would we need to add two numbers this way? Well, that depends directly on the size of b. We will count a up, until b is $[Zero]$. For $b = 100$, we need 100 steps, for $b = 1\,000\,000$, we need $1\,000\,000$ steps. Is there a way to improve on that? Yes, of course! We can just apply the same logic we have used for *next* and *prev*, that is adding single-digit numbers and handling the carry properly. Here is a solution:

```
add2 :: Number → Number → Number
add2 as bs = reverse $ go (reverse as) (reverse bs)
  where go [] ys             = ys
        go xs []             = xs
        go (x : xs) (y : ys) = case add [x] [y] of
                                 [_, r] → r : go xs (go ys [One])
                                 [r]    → r : go xs ys
                                 _      → ⊥
```

We see at once that the logic has changed significantly. First, we suddenly appear to care for efficiency: we reverse the lists before processing them! This, however, is not only for efficiency. We now process the number digit by digit starting with the least significant one, that is we look at the number not as a number, but as a list of digits – we exploit the structure of the data type.

Accordingly, we do not handle the base case $[Zero]$ anymore, we are now concerned with the base case [], since this is the point, when we have consumed all elements of one of the lists. Until we reach the base case, we just add digit by digit. If the result is a number with two digits – note that we can never get to a number with more than two digits by adding two digits – we insert the less significant digit at the head of the list that will be created by continuing the process. We continue with the next step of *go*, but increase one of the numbers, the second one, by [One]. This is the *add carry* that takes care of the the most significant digit in the two-digit result. Again, by adding two digits, we will never get to a number with a digit greater than *One* in the first position. The greatest possible number, in fact, is $[Nine] + [Nine] = [One, Eight]$.

In the other case where the addition of the two digits results in a number with just one digit, we insert the result at the head of the list that is constructed by the regular continuation of *go* – here, we do not have to take care of any carry.

The final line of the function is just to avoid a warning from the Haksell compiler. It does not add any meaning to the definition of *add2*.

Let us look at an example, say, the addition of $765 + 998 = 1763$. We first reverse the lists, that is, we start with $[Seven, Six, Five]$ and $[Nine, Nine, Eight]$, but call *go* with $[Five, Six, Seven]$ and $[Eight, Nine, Nine]$:

$go\ [Five, Six, Seven]\ [Eight, Nine, Nine]$
$Three : go\ [Six, Seven]\ (go\ [Nine, Nine]\ [One])$
$Three : go\ [Six, Seven]\ (Zero : go\ [Nine]\ (go\ []\ [One])$
$Three : go\ [Six, Seven]\ (Zero : go\ [Nine]\ [One])$
$Three : go\ [Six, Seven]\ (Zero : Zero : go\ []\ (go\ []\ [One]))$
$Three : go\ [Six, Seven]\ (Zero : Zero : go\ []\ [One])$
$Three : go\ [Six, Seven]\ [Zero, Zero, One]$
$Three : Six : go\ [Seven]\ [Zero, One]$
$Three : Six : Seven : go\ []\ [One]$
$Three : Six : Seven : [One]$

The computation results in $[Three, Six, Seven, One]$, which reversed is $[One, Seven, Six, Three]$ and, hence, the correct resut.

So, how many steps do we need with this approach? We have one addition per digit in the smaller number plus one addition for the cases where the sum of two digits results in a two-digit number. For the worst case, we, hence, have $d + d = 2d$ steps, where d is the number of digits of the smaller number. When we add two numbers in the order of hundreds (or, more precisely, with the smaller number in the order of hundreds), we would have three additions (one for each digit of a number in the order of hundreds) plus, in the worst case, three add carries. Translated into steps of *next*, this would be for each single-digit addition in the worst case nine steps (any addition with $[Nine]$) and one *next* step per carry (since carry is always an addition of $[One]$). The worst case in terms of *next*, hence, is $9d + d = 10d$, for d the number of digits in the smaller number. For numbers in the order of millions, this amounts to $10 \times 7 = 70$ steps, compared to 1 000 000 steps for the naïve approach.

We could even go on and reduce the worst case of 9 *next* steps per addition to one single step, just by doing to the algorithm what they do to us in school: instead of using *next* for addition we can define addition tables for our 10 digits, *i.e.*

$add\ [Zero]\ a = a$
$add\ [One]\ [One] = [Two]$
$add\ [One]\ [Two] = [Three]$
$add\ [One]\ [Three] = [Four]$
...
But that approach is quite boring and, therefore, we will not go for it. Instead, we will look at subtraction. First, we implement the naïve approach that we need for subtraction

of digits:

```
sub :: Number → Number → Number
sub a [Zero]                = a
sub a b | a `cmp` b ≡ LT = ⊥
        | otherwise         = prev a `sub` (prev b)
```

Subtracting *zero* from any number is just that number. For other cases, we first have to compare the numbers. If the first number is less than the second, the result is undefined for natural numbers. Otherwise, we just count the two number down by one and continue until we hit the base case.

We will look at the comparison function *cmp* below. We will first define the more sophisticated version of subtraction for numbers with more than one digit:

```
sub2 :: Number → Number → Number
sub2 as bs | as `cmp` bs ≡ LT = ⊥
           | otherwise        = clean (reverse (go (reverse as)
                                                   (reverse bs)))
  where go xs []          = xs
        go [] _           = ⊥
        go (x : xs) (y : ys)
           | y > x        = let [r] = sub [One, x] [y]
                            in r : go xs (inc ys)
           | otherwise = let [r] = sub [x] [y] in r : go xs ys
```

As with *add*2, we reverse the lists to compute the result digit by digit starting with the least significant one using the function *go*. Note that we finally *clean* the list. *clean* removes leading *zero*s, a functionality that would certainly be very useful for real-world organisations too. Before we start the hard work entering *go*, we compare the values of the arguments and If the first one is smaller than the second one, the result is undefined.

In the *go* function, we distinguish two cases: if the first digit of the second argument is greater than that of the first one, we subtract y from $10 + x$ and increase ys by one. Otherwise, we just compute $x - y$.

The function *inc* is a variant of *next* for reversed numbers:

```
inc []          = [One]
inc (Nine : xs) = Zero : inc xs
inc (x : xs)    = next [x] ++ xs
```

Applied on the empty list, *inc* yields *unity*, which is a quite different behaviour than that of *next*. Applied on a list that starts with *Nine*, we insert *Zero* as the head of the *inc*'d tail of the list. Otherwise, we just substitute the head by its *next*.

Somewhat strange might be that we need that *cmp* function – we, apparently, do not need it in other cases. The point is that we have declared that we want to derive the

Digit data type from *Ord*. With this declaration, Haskell automatically imposes the order $Zero < One < Two < \cdots < Nine$. But that would not work for lists of digits. Haskell would assume that a list like $[Nine]$ is greater than $[One, Zero]$, which, as we know, is not the case. We have to tell the compiler explicity, how we want lists of digits to be handled. This is what the *cmp* function does:

```
cmp :: Number → Number → Ordering
cmp x y = case lencmp x y of
                GT → GT
                LT → LT
                EQ → go x y
  where go [] [] = EQ
        go (a : as) (b : bs) | a > b      = GT
                             | a < b      = LT
                             | otherwise = go as bs
        go _ _ = ⊥
```

The function goes through all possible cases, explaining that a longer number is always the greater one and that, in the case they are equally long, one must compare all digits until one is found that is greater or smaller than the digit at the same position in the other list.

Note that we use a special function, *lencmp*, to compare the length of two lists. We do this out of purity on one hand and for efficiency on the other. It would not appear *fair* to use the Prelude function *length*, since it is expressed in terms of a number type that is already much more complete than our humble *Number*s. We could, of course, define our own *length* function, for instance:

```
len :: [a] → Number
len = foldl' (λn _ → next n) zero
```

But, in fact, we are not too much interested in the concrete length of the two lists, we just want to know, which one, if any, is the longer one. It is not necessary to go through both lists separately in order to learn this, we can just run through both lists at the same time:

```
lencmp :: [a] → [a] → Ordering
lencmp [] []         = EQ
lencmp [] _          = LT
lencmp _ []          = GT
lencmp (_ : xs) (_ : ys) = lencmp xs ys
```

The *lencmp* function, bears a fundamental idea of comparing two sets: by assigning each member of one set to a member of the other until one of the sets is exhausted. The one that is not yet exhausted must be the greater one. Counting could be described in terms of this logic as a comparison of a set with the set of natural numbers. We assign the numbers $1, 2, \ldots$ until the first set is exhausted. The last number assigned is the size

of the first set. We will learn much more about this apparently simple principle in the future.

As we are already talking about little helpers, it is the right time to introduce some fundamental list functions that we will need to elaborate on the number type later. We will need variants of *take* and *drop*:

$$
\begin{aligned}
&nTake :: Number \to [a] \to [a] \\
&nTake\ [Zero]\ _\ \ = [\,] \\
&nTake\ _\ [\,]\ \ \ \ \ \ = [\,] \\
&nTake\ n\ (x : xs) = x : nTake\ (prev\ n)\ xs \\
&nDrop :: Number \to [a] \to [a] \\
&nDrop\ [Zero]\ xs = xs \\
&nDrop\ _\ [\,]\ \ \ \ \ \ = [\,] \\
&nDrop\ n\ (_ : xs) = nDrop\ (prev\ n)\ xs
\end{aligned}
$$

Very useful will be a function that turns all elements of a list into *Zero*s:

$$
\begin{aligned}
&toZero :: [a] \to [Digit] \\
&toZero = map\ (const\ Zero)
\end{aligned}
$$

We should also introduce the enumeration function that facilitates list definition, *i.e.* that gives us a list for a range of numbers of the form $[1\,..\,9]$ or, in terms of the *Number* type, $[[One]\,..\,[Nine]]$:

$$
\begin{aligned}
&enum :: Number \to Number \to [Number] \\
&enum\ l\ u \mid l\ `cmp`\ u \equiv GT = [\,] \\
&\qquad\quad \mid otherwise\ \ \ \ \ \ = go\ l\ u \\
&\quad \textbf{where}\ go\ a\ b \mid a\ `cmp`\ b \equiv EQ = [a] \\
&\qquad\qquad\qquad\ \ \mid otherwise\ \ \ \ \ \ = a : go\ (next\ a)\ b
\end{aligned}
$$

Finally, we also need the function *clean*, which is defined as:

$$
\begin{aligned}
&clean :: Number \to Number \\
&clean\ [Zero]\ \ \ \ \ = [Zero] \\
&clean\ (Zero : ds) = clean\ ds \\
&clean\ ds\ \ \ \ \ \ \ \ = ds
\end{aligned}
$$

We, hence, leave the number $[Zero]$ untouched. If the number starts with the digit $Zero$, but has more than just that one number, we ignore this leading $Zero$ and continue with the remainder of the list. (Note that, since the case of a list that consists of only the digit $Zero$ is already handled in the first case, ds in the second case will never be the empty list!) Finally, a list that does not start with $Zero$ is just given back as it is.

Now, let us turn to the model for our number type, that is how we interpret a list of digits. There are many ways to interpret numbers. A somewhat natural way is to indicate a function that, for any list of *Digit*s, gives us the numerical value of the number we intend to represent with this list. The, perhaps, most obvious way to do so is to convert the

list of *Digit*s into a string and then to read this string in again as integer. We would define a conversion function of the form

```
toString :: Number → String
toString = map toChar
  where toChar Zero = '0'
        toChar One  = '1'
```

and so on. But this approach is not very interesting. It does not give us any insight. What we would like to have instead is a model that explains how numeral systems work in general. The key to understand how such a model can be devised is to see that our system consists of 10 symbols. With one of these symbols, we, hence, can represent 10 different numbers. With two of these symbols, we represent 10×10 numbers, that is the numbers $0 \ldots 9$ plus the numbers $10 \ldots 19$ plus the numbers $20 \ldots 29$ plus $\ldots$ the numbers $90 \ldots 99$. With three of these symbols, we then can represent $10 \times 10 \times 10$ numbers, namely the numbers $0 \ldots 999$ and so on. In other words, the *weight* of a digit in a number represented in a numeral system with b symbols corresponds to a power of b, which we therefore call the *base* of that numeral system. A numeral system with 2 symbols would have the base 2, the weight of each digit would therefore be a power of 2. A numeral system with 16 symbols has the base 16 and the weight of each digit would be a power of 16.

The exponent, that is to which number we raise the base, is exactly the position of the digit in a number if we start to count positions with 0. The number 1763 has the value: $1 \times 10^3 + 7 \times 10^2 + 6 \times 10^1 + 3 \times 10^0 = 1000 + 700 + 60 + 3$:

3	2	1	0
1	7	6	3

We could devise a data type that represents those *weighted* digits nicely as:

```
type WDigit  = (Number, Digit)
type WNumber = [WDigit]
```

The *WDigit* is a tuple of *Number* and *Digit*, where the number is the exponent to which we have to raise the base. We can convert a *Number* easily to a *WNumber* by:

```
weigh :: Number → WNumber
weigh = go [Zero] ∘ reverse
  where go _ []       = []
        go n (d : ds) = (n, d) : go (next n) ds
```

The *weigh* function reverses the input in order to start with the least significant digit and then just passes through this list adding the exponent incrementing it by one in each step. Note that the order of a *WNumber* does not matter anymore, because the decisive information that is encoded in the position of each digit is now made explicit in the exponent.

The inverse of this function is:

```
unweigh :: WNumber → Number
unweigh = reverse ∘ map snd ∘ complete [Zero] ∘ sortW
  where complete _ [] = []
        complete n ((e, d) : xs)
          | n `cmp` e ≡ LT = (n, Zero) : complete (next n) ((e, d) : xs)
          | otherwise     = (e, d)     : complete (next n) xs
sortW :: WNumber → WNumber
sortW = sortBy (λx y → fst x `cmp` fst y)
```

We, first, sort the components of the *WNumber* in ascending order according to their exponents. We, then, *complete* the $WNumber$, *i.e.* we fill in $Zero$s for missing exponents such that the resuling $WNumber$ has a component for every exponent from 0 to the greatest one present. From this list, we extract the digits and reverse the result.

To build the model, we still need a function that converts digits into one-digit integers. This is straight forward:

```
digit2Int :: Digit → Int
digit2Int Zero  = 0
digit2Int One   = 1
digit2Int Two   = 2
digit2Int Three = 3
digit2Int Four  = 4
digit2Int Five  = 5
digit2Int Six   = 6
digit2Int Seven = 7
digit2Int Eight = 8
digit2Int Nine  = 9
```

To convert a $Number$ to an $Integer$, we first convert the $Number$ to a $WNumber$ and then convert the $WNumber$ to an $Integer$:

```
n2Integer :: Number → Integer
n2Integer []   = 0
n2Integer [n] = fromIntegral (digit2Int n)
n2Integer ns  = w2Integer    (weigh ns)
```

As a convention, we convert the empty list into 0. (We could raise an error for this case, but that does not appear to be necessary or even useful.) A one-digit number is simply converted by converting its single digit. Since *digit2Int* converts a digit to an *Int*, but we now want an *Integer*, we still have to call *fromIntegral* on the result, to convert from *Int* to *Integer*.

Numbers with many digits are converted to *WNumber* using *weigh* and then converted to *Integer* using *w2Integer*:

```
w2Integer :: WNumber → Integer
w2Integer          = sum ∘ map conv
  where conv w = let x = n2Integer (fst w)
                     d = fromIntegral (digit2Int (snd w))
                 in  d * 10 ↑ x
```

Weighted numbers are converted to *Integer* by summing up the single values of the digits, which are calculated in terms of powers of 10: $d \times 10^x$, where d is the digit converted to Int by *digit2Int* and then converted to $Integer$ by *fromIntegral*. x is the exponent converted to $Integer$ using *n2Integer*.

This looks like an infinite regress where we convert the exponent of the weighted number, which is a $Number$, to an Integer using $n2Integer$, which then calls *w2Integer*, which again calls *n2Integer* to convert the exponent, which, again, calls *w2Integer* and so on.

It is indeed very well possible that we have extremely large numbers with exponents that are many digits long, but even the greatest number will finally converge to an exponent that is smaller then 10. The incredibly large number $10^{100000000000}$, for example, has an exponent with 12 digits, which, represented as a *Number*, is

$[\,One, Zero, Zero, Zero, Zero, Zero, Zero, Zero, Zero, Zero, Zero, Zero\,]$.

The greatest exponent in this number, the exponent of the leading *One*, however, has just two digits: $[One, One]$, which, in the next conversion step, reduces to $[One]$ for the most significant digit and, hence, will be converted immediately to 1.

Of course, we do not need the data type *WNumber* for this conversion. We could very well have converted a number by reverting it and then pass through it with an inreasing *Integer* exponent starting from 0. The detour through weighted numbers, however, is a nice illustration of the model for our number system, and, perhaps, there will be use for this or a similar data type later during our journey.

3.4. Multiplication

As addition can be seen as a repeated application of counting up, multiplication can be seen as repeated addtion. A naïve implementation of multiplication, hence, is:

```
mul :: Number → Number → Number
mul _ [Zero] = [Zero]
mul [Zero] _ = [Zero]
mul a [One]  = a
mul [One] b  = b
mul a b      = a `add2` (a `mul` (prev b))
```

Notable, here, is that we have more base cases than with addition: Any number mul-

tiplied by *zero* is just *zero* and any number multiplied by *unity* is just that number. From here on, we add a to a and count b down simultaneously, until b reaches the base case $[One]$.

This simple implementation is not optimal in terms of computation complexity: we need b steps to multiply a and b. For a multiplication of two numbers in the range of millions, we need millions of single additions. As with addition, we can improve on this by multiplying digit by digit or, more precisely, by multiplying all digits of the first number by all digits of the second number. This, however, is somewhat more complicated than in the case of addition, because multiplication has effect on the weight of the digits. On two one-digit numbers, weight has no impact, since the weight of each digit is just 0: $(2 \times 10^0) \times (3 \times 10^0) = 6 \times 10^{0+0} = 6 \times 1 = 6$. But, if 2 and 3 above were digits of a number with more than one digit and, themselves not the least significant digits, *e.g.* 20×30, then we would have something of the form: $(2 \times 10^1) \times (3 \times 10^1) = 6 \times 10^2 = 600$.

We, therefore, have to take the weight of the digits into account. But what is the best way to do so? We could of course use the weighted number type we already defined in the previous section. The disadvantage of this approach, however, is that we have to perform additional arithmetic on the weight, potentially searching for equal weights in the resulting number or reordering it to bring equal weights together. We can avoid this overhead by reflecting the weight in numbers, simply by appending $n - 1 + m - 1$ *Zero*s to the result of multiplying the n^{th} digit of one number with the m^{th} digit of the other one and, eventually, adding up all these components.

We first implement a function that multiplies a digit with all digits of a number appending *Zero*s to each result and adding them up:

$$
\begin{array}{l}
mul1 :: Digit \to Number \to [Digit] \to Number \\
mul1\ _\ [\,]\ \ \ \

```
mulN :: Number → Number → [Digit] → Number
mulN []          _ _  = zero
mulN (x: _)      b []  = mul1 x b []
mulN (Zero : xs) b zs  = mulN xs b (tail zs)
mulN (x : xs)    b zs  = add2 (mul1 x b zs)
                              (mulN xs b (tail zs))
```

If the first number is exhausted, we just return *zero*. If the *Zero*s are exhausted, we apply *mul1*, *i.e.* we multiply one digit with b, and terminate. Note that the *Zero*s should be exhausted only if there is just one digit left in the numbers. Again, we ignore *Zero*, but respect its weight. In all other cases, we apply *mul1* on the first digit of the first number and add the result with the recursion on the tail of the first number and the tail of *Zero*s.

Finally, we apply *mulN* on two numbers creating the trail of *Zero*s:

```
mul2 :: Number → Number → Number
mul2 [] _ = zero
mul2 _ [] = zero
mul2 a b  = mulN a b ((toZero $ tail a) ++
                      (toZero $ tail b))
```

We handle the cases where one of the numbers is the empty list explicitly to avoid problems with the call of *tail* later on. We then call *mulN* for a and b and the trail of *Zero*s that results from converting all digits but one in a and b to *Zero*.

Note that this is exactly what we do, when we elaborate a multiplication with pen and paper. If we multiplied, say, 13×14, we would write the partial results aligned according to the number of zeros they would have:

13×14
$1 \times 1 = 100$
$1 \times 4 = 040$
$3 \times 1 = 030$
$3 \times 4 = 012$

Now we add up the partial results:

$100 + 040 = 140$
$030 + 012 = 042$
$140 + 042 = 182$

The grouping of additions chosen here corresponds to the additions performed in *mul1* and *mulN*: The first two additions are performed in *mul1*, the last line is done in *mulN*.

Let us look at how *mulN* works for $[One, Three] \times [One, Four]$. We start with

mulN $(One : [Three])\ [One, Four]\ [Zero, Zero] =$
add2 $(mul1\ One\ [One, Four]\ [Zero, Zero])\ (mulN\ [Three]\ [One, Four]\ [Zero])$.

The first term of *add2* is

$mul1\ \ One\ (One : [Four])\ [Zero, Zero] =$
$add2\ (([One]\ `mul`\ [One]) +\!\!+ [Zero, Zero])\ (mul1\ \ One\ [Four]\ [Zero])$.

The first term, here, reduces to

$[One] +\!\!+ [Zero, Zero] = [One, Zero, Zero]$,

which corresponds to **100** in the paper example above. The second term reduces to

$mul1\ \ One\ (Four : [])\ [Zero] =$
$add2\ (([One]\ mul\ [Four]) +\!\!+ [Zero])\ (mul1\ [One]\ []\ [])$,

which, in its turn, is just

$add2\ [Four, Zero]\ [Zero] = [Four, Zero]$

and corresponds to **40** in the manual calculation. We, hence, have

$add2\ [One, Zero, Zero]\ [Four, Zero] = [One, Four, Zero]$

at the end of the first round of *mul1*. This is the same result as we obtained in the first addition step in the manual process, *i.e.* **140**. Returning to the first equation, we now have:

$mulN\ (One : [Three])\ [One, Four]\ [Zero, Zero] =$
$add2\ ([One, Four, Zero])\ (mulN\ [Three]\ [One, Four]\ [Zero])$.

The second term of *add2*, here, produces:

$mulN\ (Three : [])\ [One, Four]\ [Zero] =$
$add2\ (mul1\ \ Three\ [One, Four]\ [Zero])\ (mulN\ []\ [One, Four]\ [])$.

The first term, the call to *mul1*, is:

$mul1\ \ Three\ (One : [Four])\ [Zero] =$
$add2\ (([Three]\ `mul`\ [One]) +\!\!+ [Zero])\ (mul1\ \ Three\ (Four : [])\ [])$.

The first term of this addition is $[Three, Zero]$, which corresponds to the same result **30** we had above in the third step of the manual multiplication, and the second term is:

$mul1\ [Three]\ (Four : [])\ [] = [Three]\ `mul`\ [Four] = [One, Two]$.

The result **12** we obtained before. Going back, we now have:

$mul1\ [Three]\ (One : [Four])\ [Zero] =$
$add2\ [Three, Zero]\ [One, Two] = [Four, Two]$

We now have the result of the second addition step in the paper multiplication, *i.e.* **42**. and, returning to the first equation, we get the final result:

$mulN\ (One : [Three])\ [One, Four]\ [Zero, Zero] =$
$add2\ [One, Four, Zero]\ [Four, Two] = [One, Eight, Two]$.

This is the last line of the addition: $140 + 42 = \mathbf{182}$.

How many steps do we need for multiplication with this approach? We, first, multiply all digits of one number with all digits of the other number and, thus, perform $n \times m$ one-digit multiplications, where n and m are the numbers of digits of the first and the second argument respectively. We then add all the $n \times m$ numbers together, resulting in $n \times m - 1$ multi-digit additions. Most of the multi-digit additions, though, add *Zero*s, which is just one comparison and, hence, quite unexpensive. We have, however, many of those simple steps, because we add numbers of the size $n + m - 1, n + m - 2, \ldots, 1$. This is the addition of all numbers from 1 to $n + m$, a type of problems, we will study in the next chapter.

Anyhow, the cost for *mul2* grows only in the size of the arguments, whereas the naïve *mul* grows directly in the value of the second number. The number of steps is $nmp+(nm-1)a$, where p is the cost for a single-digit multiplication and a that of an addition. For very, very large numbers, say, numbers with thousands or millions of digits, the approach, still, is too slow. There are many ways to multiply more efficiently, but that is not our focus here.

Multiplication appears to be such a tiny simple device, but it introduces huge complexity. If we just look at the patterns that *mul2* produces when processing two numbers $[a, b]$ and $[c, d]$: $[ac + ad + bc + bd]$, we see that multiplication is intimately involved with problems of combinatorics, which too will be a major topic of the next chapter. Imagine the multiplication of a number with itself, *i.e.* where c and d equal a and b, respectively:

$$[a, b] \times [a, b] = [aa + ab + ba + bb] = [a^2 + 2ab + b^2]. \tag{3.1}$$

Indeed, multiplying $[One, Two]$ with itself results in $[One, Four, Four]$ and $[One, Three]$ in $[One, Six, Nine]$. This pattern plays a role in many branches of mathematics, like algebra, combinatorics and probability theory, and is truly one of the most important facts you can learn about mathematics. Should equation 3.1 not be familiar to you already, you definitely should memorise it. The tiny device of multiplication, one could contemplate, is a focal point of many complications we will encounter on our journey – and this appears to me as one of the characteristics of mathematics: that small problems, such as multiplication, thought through, develop unforeseen impact on apparently completely different subjects.

A nice illustration of the patterns created by multiplication is the results of squaring numbers that consist only of 1s. Have a look at the following pyramid:

$$1 \times 1 = 1$$

$$11 \times 11 = 121$$

$$111 \times 111 = 12321$$

$$1111 \times 1111 = 1234321$$

$$11111 \times 11111 = 123454321$$

It is as if the digit in the centre of the number on the right-hand side of the equations wanted to tell us the size of the factors used to create it. When we try to fool the numbers, leaving some 1s out in one of the factors, they realise it immediately, and come up with "damaged" results like

$$1 \times 11 = 11$$

$$11 \times 111 = 1221$$

$$11 \times 1111 = 12221$$

$$11111 \times 1111111111 = 12345555554321.$$

Now, the central digits in the result tell us the size of the smaller number and their repetition tells us the difference to the greater factor, which is exactly one less than the number of repetitions.

3.5. Division and the Greatest Common Divisor

It is now time to introduce Euclid. Unusually little is known about this author. Important scholars of the time (about 300 BC) are usually mentioned by name in philosophical texts of other authors and often with some biographical detail. In the case of Euclid, this is different. Euclid is rarely mentioned by name – and when it happens, he is confused with an earlier philosopher of the same name – and nothing is told about his life but the fact that he was active in Alexandria for some time. This is particularly strange, since Euclid's work had a tremendous influence on the antiquity and on through the middle ages up to our days. This has led to the conjecture that Euclid was not a person, but a group of scholars at the university or library of Alexandria. This idea may be inspired by similar conjectures concerning the "person" of Homer or by the existence of groups named after fictional characters in later times like, in the 20^{th} century, the "Association of collaborators of Nicolas Bourbaki", a highly influential group of mathematicians dedicated to the formalisation of mathematics. Nicolas Bourbaki, even though he had an office at the École Normale Supérieure for some time, did not exist. He is a fictional character whose name was used for the publications of the Bourbaki collective.

Euclid – who or whatever he was – is the author of the *Elements*, the mother of all axiomatic systems and, certainly, one of the greatest intellectual achievements of the antiquity. The *Elements* lay out the acient knowledge on geometry, arithmetic and number

theory in fifteen books following a rigid plan starting with axioms, called "postulates", followed by theorems and their proofs based only on the axioms. There are some inaccuracies in the choice of the axioms and not all proofs are rock-solid according to modern standards. But, anyway, the rigidity of the Elements was not achieved again before the 19^{th} century, perhaps with the *Disquisitiones Arithmeticae* by 21-year-old Carl Friedrich Gauss.

Here, we are interested mainly in some of the content of book 7, which deals with issues of arithmetic and elementary number theory, in particular division and the greatest common divisor. According to Euclid, division solves equations of the form

$$a \; div \; b = q + r, \tag{3.2}$$

and fulfils the constraint

$$a = qb + r, 0 \leq r < b. \tag{3.3}$$

There is a kind of mismatch between this notion of division, usually called *division with remainder*, and multiplication in that multiplication of any two natural numbers results in a natural number, whereas division with remainder results in two numbers, the *quotient* q and the *remainder* r. The division of two numbers that are *divisible*, *i.e.* the division leaves no remainder, is just a special case of this operation like in $9 \; div \; 3 = 3+0$. In other cases, this does not work: $8 \; div \; 3 = 2+2$, since $2 \times 3 + 2 = 8$. We already have seen such a mismatch with addition and subtraction: the addition of any two natural numbers always produces a natural number; subtraction, however, does only produce a natural number when its second term is less than or, at most, equal to the first term. This will be an important topic in the progress of our investigations.

Euclid's algorithm to solve the equation goes as follows: Division by zero is not defined. Division of zero by another number (not zero) is zero. Otherwise, starting with the quotient $q = 0$ and the remainder $r = a$, if the remainder r is less than the divisor b, then the result is (q, r). Otherwise, we decrement the remainder by b and increment q by one:

```
quotRem :: Number → Number → (Number, Number)
quotRem _ [Zero] = error "division by zero"
quotRem [Zero] _ = (zero, zero)
quotRem a [One]  = (a, zero)
quotRem a b      = go a zero
  where go r q | r `cmp` b ≡ LT = (q, r)
               | otherwise      = go (r `sub2` b) (next q)
```

As you should realise at once, this algorithm is not efficient for large numbers a. If a is much larger than b, we will have to subtract lots of bs from it. In fact, the complexity of

this algorithm is $\lfloor a/b \rfloor$, since we need $\lfloor a/b \rfloor$ steps to bring a down to an r that is smaller than b. The complexity of the algorithm, hence, equals (a part of) its result!

As usual, we can improve by taking the structure of the numbers into account, namely by operating on digits instead of whole numbers. Have a look at the following, admittedly, scary-looking listing:

```
quotRem2 :: Number → Number → (Number, Number)
quotRem2 _ [Zero] = error "division by zero"
quotRem2 [Zero] _ = (zero, zero)
quotRem2 a [One]  = (a, zero)
quotRem2 a b      = go zero [] a
  where go q [] [] = (clean q, zero)
        go q c []  = (clean q, clean c)
        go q c r   = let x = clean (c ++ [head r])
                         y = tail r
                     in if x `cmp` b ≡ LT
                        then go (q ++ zero) x y
                        else let (q', r') = quotRem x b
                                 r2 | r' ≡ zero = []
                                    | otherwise = r'
                             in go (q ++ q') r2 y
```

We start, as usual, with the base cases: division by zero and not defined; zero divided by something else is zero. A number divided by one is just that number.

For all other cases, we call *go* with *zero* as quotient and a as remainder. There is an additional parameter, c, which takes care of carries. If we have exhausted, both the carries and the remainder, then the result is just $(q, zero)$, *i.e.* we have no remainder. If the remainder is exhausted, but not the carries, the carries together are the remainder. Otherwise, we proceed as follows: We take the head of of the remainder and concatenate it to previous carries starting with the empty list. If this number is less than b, we append a *Zero* to q and continue with x as carry and the *tail* of r. Note that, if this happens on the first digit, the *Zero*s appended to q will be cleaned off later. Only *Zero*s between digits are taken into account. This is exactly what we do, when we divide with pencil and paper: when, during the process, the next number in a cannot be divided by b, we append a zero to the partial result obtained so far and append the next number of a to the remainder of the previous calculation.

Otherwise, if x is not less than b, we divide these two numbers using the naïve *quotRem*. The quotient resulting from the application of *quotRem* is appended to the previous result q. The remainder, if not zero, is carried over. Since *quotRem* is applied, as soon as we arrive at a number that is equal to or greater than b appending one digit of a after the other, this number is at most 9 times as big as b. In other words, *quotRem* in this context, will never need more than 9 steps. Nevertheless, *quotRem* is the bottleneck of this implementation. With lookup tables for one-digit divisions, we could reach a

significant speed-up. But optimising, again, is not our prime concern here. Therefore, we will stick with this suboptimal solution.

An important aspect of the algorithm is that we chop off leading *Zero*s, whenever we go to use a sequence of digits as a number, in particular before we return the result and before calling *quotRem*. The algorithm handles numbers as sequence of digits that are as such meaningless. But whenever it operates on those sequences it takes care of handling them as proper numbers.

Let us look at a simple example, say, $[One, Two, Three]$ divided by $[Six]$. We start with

$go\ [Zero]\ []\ [One, Two, Three]$

and compute x as $clean\ ([] +\!\!+ [One])$ and y as $[Two, Three]$. Since x, which is $[One]$, is less than b, $[Six]$, we continue with

$go\ ([Zero] +\!\!+ [Zero])\ [One]\ [Two, Three]$.

This time x is $clean\ ([One] +\!\!+ [Two])$ and y is $[Three]$. x now is greater than b and therefore we compute

$(q', r') = quotRem\ [One, Two]\ [Six]$

where q' is $[Two]$ and r' is $[Zero]$. We then continue with

$go\ ([Zero, Zero] +\!\!+ [Two])\ []\ [Three]$

and compute x as $[Three]$ and y as $[]$. Since x, again, is less than b, we continue with

$go\ ([Zero, Zero, Two] +\!\!+ [Zero])\ [Three]\ []$,

which is the second base case of *go* leading to

$(clean\ [Zero, Zero, Two, Zero], clean\ [Three])$,

which in its turn is just $([Two, Zero], [Three])$ expressing the equation $6 \times 20 + 3 = 123$.

There are many interesting things to say about division and especially about the concept of the remainder. First, the remainder is an indicator for *divisibility*. A number b is said to divide a number a or a is divisible by b, $b \mid a$, if $a\ div\ b = (q, 0)$, *i.e.* if the remainder of the Euclidian division is 0. In Haskell, we can define the remainder as:

```
rem :: Number → Number → Number
rem a b = snd (quotRem2 a b)
```

The quotient, correspondingly, is

```
div :: Number → Number → Number
div a b = fst (quotRem2 a b)
```

Divisibility, then, is:

$$
\begin{aligned}
&\mathit{divides} :: \mathit{Number} \to \mathit{Number} \to \mathit{Bool}\\
&\mathit{divides}\ a\ b \mid \mathit{rem}\ b\ a \equiv \mathit{zero} = \mathit{True}\\
&\qquad\qquad\ \mid \mathit{otherwise} \qquad\ = \mathit{False}
\end{aligned}
$$

There are some rules (valid for natural numbers) that can be defined on divisibility, namely: For all numbers a: $1 \mid a$, that is: 1 divides all numbers, since $a\ \mathit{div}\ 1 = (a, 0)$.

It holds also that $a \mid b \wedge b \mid c \to a \mid c$. In other words: if a divides b and b divides c, then a also divides c. (The symbol "$\wedge$" means "AND" here.) This is because, if b divides c, then c is a multiple of b and, if a divides b, then b is a multiple of a and, in consequence, c is also a multiple of a. Any number divisible by 4, for instance, is also divisible by 2, since $2 \mid 4$.

Furthermore, if $a \mid b$ and $b \mid a$, then we can say that $a = b$, since, if a were greater than b, then a would not divide b and vice versa.

An interesting – and important – equality is also $a \mid b \wedge a \mid c \to a \mid (b + c)$. This rule says that the sum of any two numbers b and c, both divisible by another number a is also divisible by a. For the special case $a = 2$, this rule says that the sum of two even numbers is also even: $4 + 6 = 10$, $50 + 28 = 78$, $1024 + 512 = 1536$, ... This is true in general for all numbers a, *e.g.* 5: $10 + 15 = 25$, which is $2 \times 5 + 3 \times 5 = 5 \times 5$, or $35 + 625 = 660$, which is $7 \times 5 + 125 \times 5 = 132 \times 5$. We can go even further and say $a \times b + a \times c = a \times (b + c)$. This is called the distributive law and we have already used it implicitly when defining multiplication. We will come back to it very soon.

The remainder gives rise to an especially interesting concept, the concept of arithmetic *modulo* n. The term modulo refers just to the remainder of the Euclidian division. Most implementations in programming languages, including Haskell, distinguish the operator *mod* and *rem* according to the *signedness* of dividend and divisor. For the moment, that is not relevant for us, since we are working with natural numbers only, so, for the moment, we will treat *mod* and *rem* as being the same concept.

The most common example of modulo arithmetic is time measured with a 12 or 24 hours clock. At midnight, one can say it is 12 o'clock; since $12 \bmod 12 = 0$, we can also say, it is 0 o'clock. With the 24 hours clock, one hour after noon is 13:00 o'clock. $13 \bmod 12 = 1$, 13, thus, is just 1 in the 12 hours clock. This principle works for arbitrary large numbers, *e.g.* 36 is 12, since $36 \bmod 12 = 0$ and, since $36 \bmod 24 = 12$, we can say it is noon. 500 is 8 in the evening, since $500 \bmod 24 = 20$ and $20 \bmod 12 = 8$. With modular arithmetic, arbitrary large numbers modulo n are always numbers from 0 to $n - 1$ and any operation performed on numbers modulo n results in a number between 0 and $n - 1$. This apparently trivial fact is of huge importance. We will come back to it over and over again.

Especially interesting for programmers is arithmetic modulo 2, because any operation has either 0 or 1 as result, *i.e.* the vocabulary of binary number representation. Indeed, addition of the numbers 0 and 1 modulo 2 is just the *exclusive or* (XOR) operation: $0 + 0 = 0 \mod 2$, $1 + 0 = 1 \mod 2$, $1 + 1 = 0 \mod 2$, since $1 + 1 = 2$ and $2 \bmod 2 = 0$.

The XOR operation gives the same results: $0 \oplus 0 = 0$, $1 \oplus 0 = 1$, $1 \oplus 1 = 0$. Multiplication modulo 2 is equivalent to AND: $0 \times 0 = 0 \mod 2$, $0 \times 1 = 0 \mod 2$, $1 \times 1 = 1 \mod 2$. The truth values of the formula $p \wedge q$ are shown in the table below:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

One of the fundamental tools developed in the Elements is *gcd*, the *greatest common divisor*. As the name suggests, the *gcd* of two numbers a and b is the greatest number that divides both, a and b.

The algorithm given in the Elements is called *Euclidian algorithm* and is used with a small, but important variation until today. The original algorithm goes as follows: the gcd of any number a and 0 is a; the gcd of any number a with any number b is $\gcd(b, a - b)$, where $0 < b \leq a$. If $b > a$, we just turn the arguments around: $\gcd(b, a)$.

For large numbers, this is not efficient, especially, if a is much greater than b. The remarks on modulo above, however, hint strongly at a possible optimisation: the use of the remainder operation instead of difference:

$$
\begin{aligned}
&gcd :: Number \to Number \to Number \\
&gcd\ a\ [Zero] = a \\
&gcd\ a\ b \quad\ \ = gcd\ b\ (a\ `rem`\ b)
\end{aligned}
$$

Let us look at some examples:

$gcd\ [Nine]\ [Six] = gcd\ [Six]\ ([Nine]\ `rem`\ [Six])$,

which is $gcd\ [Six]\ [Three]$, which, in its turn, is

$gcd\ [Three]\ ([Six]\ `rem`\ [Three] = [Zero])$

and, hence

$gcd\ [Three]\ [Zero] = [Three]$.

More complicated is the *gcd* of [One,One] and [Six]:

$gcd\ [One, One]\ [Six] = gcd\ [Six]\ ([One, One]\ `rem`\ [Six])$,

which is

$gcd\ [Six]\ [Five] = gcd\ [Five]\ ([Six]\ `rem`\ [Five])$,

which is

$gcd\ [Five]\ [One] = gcd\ [One]\ ([Five]\ `rem`\ [One])$,

which leads to

$gcd\ [One]\ [Zero] = [One]$.

It is noteworthy that the algorithm always terminates. This is true because, since *rem* always reduces b to a value between *zero* and $a - 1$ and, with a getting smaller and smaller, we must at some point reach either *unity* (when b does not divide a) or *zero* (when b does divide a). If we reach *zero*, we have a result; otherwise, we will reach *zero* in the next step, because *unity*, as we have already discussed, divides any number.

Furthermore, if a is the smaller number, *gcd* will just flip the arguments, *e.g.* $gcd\ 10\ 100 = gcd\ 100\ (10\ `rem`\ 100)$ and, since $10\ div\ 100 = (0, 10)$, this corresponds to $gcd\ 100\ 10$.

We will analyse the running time of *gcd* later in chapter 3. For now, it may suffice that each step reduces the problem to $a \bmod b$, which is in the range of $0 \ldots b - 1$, while, with the original algorithm, the problem is reduced only to $a - b$ per step. With large numbers and, in particular, with a huge difference between a and b, this reduction is quite small. With the reduction by $a \bmod b$, the difference between the numbers and even the size of a do not matter. That is an effect of modular arithmetic.

An important insight related to the *gcd*, is *Euclid's lemma*, which states that if a divides cb, then a must share common factors with c or b. This is easy to see, since, that a divides cb means that there is a number n, such that $na = cb$. This number is $n = cb/a$. If a and cb did not share common factors, then cb/a would not be a natural number. For example 10 and 7 do not share factors with 3; there is thus no natural number n, such that $3n = 7 \times 10$. With 6 instead of 7, however, there is a common factor, namely 3 itself. Therefore, we can solve $3n = 6 \times 10 = 60$, simply by dividing 3 on both sides of the equation: $n = 60/3 = 20$.

Finally, we should mention a cousin of *gcd*, the *least common multiple*, *lcm*, the smallest number that is a multiple of two numbers, a and b. The obvious multiple of two numbers is the product of these numbers $a \times b$. But there may be a smaller number c, such that $a \mid c \wedge b \mid c$. How can we find that number? Well, if a and b have a gcd that is not 1, then any number divisible by a and divisible by b is also divisible by $\gcd(a, b)$. The product of a and b, hence, is divisible by $\gcd(a, b)$ and, since the *gcd* is the common divisor that reduces the product $a \times b$ most, that quotient must be the least common multiple, *i.e.*

$$lcm(a, b) = \frac{a \times b}{\gcd(a, b)}. \tag{3.4}$$

3.6. Powers, Roots and Logarithms

Multiplication can be seen as a kind of higher-order addition: one of the factors tells us how often we want to add the second factor to itself: $a \times b = b + b \ldots$ This relation can be expressed nicely with the summation notation $\sum$:

$$a \times b = \sum_{i=1}^{a} b$$

For instance, 2×3 is $\sum_{i=1}^{2} 3 = 3 + 3 = 6$ and 1×3 would just be $\sum_{i=1}^{1} 3 = 3$. For $a = 0$, summation is defined as 0.

In Haskell, for any a of type class *Num*, this is implemented as $sum :: [a] \to a$, which takes an argument of type $[a]$ and returns the sum of all elements in the input list. For our number type (which we have not yet defined as *Num*), this could be:

```
summation :: [Number] → Number
summation = foldr add2 zero
```

From this definition of multiplication as repeated addition, we can go further. We can introduce an operation that repeats multiplication of a number with itself. This operation is called *power*: $a^b = a \times a \times \ldots$ and can be captured with the product notation:

$$a^b = \prod_{i=1}^{b} a$$

a^2, for instance, is $\prod_{i=1}^{2} a = a \times a$. For $b = 0$, the product is defined as 1.

In Haskell, the product for any a of type class *Num* is implemented as $product :: [a] \to a$. For our number type, we could define:

```
nProduct :: [Number] → Number
nProduct = foldr mul2 unity
```

We can define *power* as:

```
power :: Number → Number → Number
power _ [Zero] = unity
power a [One]  = a
power a b      = a `mul2` power a (prev b)
```

This algorithm, of course, is not efficient, since it needs b steps to calculate the b^{th} power of any number. A common trick to accelerate the algorithm is *exponentiation by squaring* where we reduce b faster than by just decrementing it by one. Indeed, when we exponentiate a number with an even number b, the result is $a^{2\frac{b}{2}}$. What about odd bs? In this case, we reduce b by one, then we have an even number in the exponent, and multiply a once more: $a \times a^{2\frac{b}{2}}$. With this algorithm, we need, instead of b steps, a

logarithmic amount of steps (log base 2), which we will discuss in a second, plus one extra multiplication, when b is odd. In Haskell, this variant of power could be implemented as follows:

```
power2 :: Number → Number → Number
power2 _ [Zero] = unity
power2 a [One] = a
power2 [One] _ = unity
power2 a b      = case b `quotRem2` two of
                    (q, [Zero]) → power2 (a `mul2` a) q
                    (q, _)      → a `mul2`
                                  power2 (a `mul2` a) q
```

From *power*, we can go on further, introducing an operator that operates on powers, and, indeed, there is Knuth's *up-arrow* notation: $a \uparrow\uparrow b = a^{b^{b^{\cdots}}}$, which indicates that we raise a b times to the b^{th} power. When we have defined this, we can go on by introducing even more arrows: $a \uparrow\uparrow\uparrow b = a(\uparrow\uparrow (b \uparrow\uparrow (b \ldots)))$ and we can go on and on *ad infinitum*.

This approach gives us a lot of power to define huge numbers. But what about going backward? How can we invert the effect of power (not to mention Knuth's megapower)? There are in fact two ways to invert the power function. We may ask for the *root* a in $a^b = c$, if we know b and c, and we may ask for the *exponent* b, if a and c are known. The first operation is just called the *root*, whereas the latter is called the *logarithm* of c to base a.

Both these functions are again asymmetric in that any power of two natural numbers a^b results in a natural number, but not all natural numbers c have a natural numbered root a or a natural numbered logarithm b to base a. It is possible to define natural numbered approximations to the precise results. But, since we will not make any use of such functions, we will not implement them here. We come back to root and log algorithms later.

There are three bases whose logarithms are particularly interesting: the logarithm base 10 ($\log_{10}$) is intersting when we are working in the decimal system. The logarithm base 2 ($\log_2$) is interesting, when working with the binary system, but also in many areas where binarity plays a role, some of which we will explore later. Then there is the logarithm to the base e ($\log_e$), the so called *natural logarithm*. This number e, which is approximately 2.71828, is one of the most curious mathematical objects. It appears again and again in apparently unrelated problem areas such as number theory, calculus and statistics. It especially loves to appear, when you least expect it. We have no means to express this number with natural numbers, so we have to come back to it later to define it properly.

Unfortunately, there are different shorthands for these logarithms in different contexts. Computer scientists would write the binary logarithm log, because it is the most common in their field. This shorthand, however, usually means the natural logarithm in most

math publications and even many programming languages, including Haskell, use the symbol log for $\log_e$. To make it worse, in many engineering disciplines, $\log_{10}$ is considered the most common logarithm and, accordingly, log is considered to mean $\log_{10}$. There is an ISO standard, which apparently isn't followed by anybody, that gives the following reasonableconvention: $\log_2 = lb$, $\log_e = ln$ and $\log_{10} = lg$. But even these shorthands are often confused. The best way, therefore, appears to be to use explicit symbols with subscripts.

Logarithms adhere to very interesting arithmetic rules that often reduce computational complexity in dealing with huge numbers. The logarithm (base b) of the product of two numbers equals the sum of the logarithm (base b) of these numbers: $\log_b(n \times m) = \log_b(n) + \log_b(m)$. Example: $\log_2(4 \times 8) = \log_2(32) = 5$ and $\log_2(4) + \log_2(8) = 2 + 3 = 5$.

Accordingly, the logarithm of the quotient of two numbers equals the difference of the numerator and denominator: $\log_b(\frac{n}{m}) = \log_b(n) - \log_b(m)$, for instance $\log_2(\frac{32}{8}) = \log_2(4) = 2$ and $\log_2(32) - \log_2(8) = 5 - 3 = 2$.

The logarithm of a power of a number n equals the exponent multiplied with the logarithm of n: $\log_b(n^x) = x \times \log_b(n)$, *e.g.*: $\log_2(4^3) = \log_2(64) = 6$ and $3 \times \log_2(4) = 3 \times 2 = 6$.

Finally, the logarithm of a root of n equals the logarithm of n divided by the exponent: $\log_b(\sqrt[x]{n}) = \frac{\log_b(n)}{x}$, for example: $\log_2(\sqrt[3]{64}) = \log_2(4) = 2$ and $\frac{\log_2(64)}{3} = \frac{6}{3} = 2$.

We can also convert logarithms with different bases to each other. Let us assume we want to convert the logarithm base b of a number n to the logarithm base a of n; then $\log_a n = \frac{\log_b n}{\log_b a}$, *i.e.* we divide the logarithm $\log_b\ n$ by the logarithm $\log_b$ of a. We will later show why this rule holds.

3.7. Numbers as Strings

Until now we have looked at numbers as sequences of symbols, *i.e. strings*. In the next section that will end. We will then define our numbers as a fully-fledged Haskell number type. But before we do that, we will pause shortly to make the difference between the two viewpoints on numbers quite clear. Indeed, in many math problems, the representation of numbers as strings is relevant – especially in informatics. So, this viewpoint is not only related to the way how we happened to define our numbers, but is a genuine mathematical approach.

We have already seen some strange effects of multiplication on the characteristics of the resulting sequences of digits. A much simpler example that shows the properties of numbers as being strings is typing errors. There is no obvious numerical analogy between number pairs like 12 and 21, 26 and 62 or 39 and 93. But, obviously, there is a very simple function that produces these numbers, namely *reverse*:

$reverse\ [One, Two] = [Two, One]$
$reverse\ [Two, Six] = [Six, Two]$
$reverse\ [Three, Nine] = [Nine, Three]$

That is not a numeric property, but a property of any kind of sequence of symbols. Numbers as such, however, are not sequences of symbols. We rather make use of sequences of symbols to represent numbers. In some way, however, any formal system used to represent numbers will have the form of sequences of symbols and, as such, numbers exist in both worlds, a *purely* numerical and a symbolic world.

There is a well known sequence of natural numbers living on the very border between the numerical and the string side of numbers, the *look-and-say* sequence, which is often used in recreational math, but is also investigated by serious (even if playful) mathematicians, such as John H. Conway, co-author of the Book of Numbers. Can you guess how to continue the following sequence?

$1, 11, 21, 1211, 111221, \ldots$

The sequence starts just with one. The next number explains to us what its predecessor looks like: it is composed of one "one". This number, now, is composed of two "ones", which, in its turn, is composed of one "two" and one "one". This again is composed of one "one", one "two" and two "ones". Now, you are surely able to guess the next number.

There are some interesting questions about this sequence. What is the greatest digit that will ever occur in any number of this sequence? Well, we can easily prove that this digit is 3. The numbers of the sequence are composed of pairs of digits that describe groups of equal digits. The first digit of each pair says how often the second digit appears in this group. The number 111221, for instance, describes a number composed of three group: 11 12 21. The first group consists of one "one", the second group of one "two" and the last group of two "ones". Now, it may happen that the digit of the current group coincides with the number of digits in the next group. But the digit in that group must differ from the digits in the current group. Otherwise, it would belong to the current group. A good example is 11 12: if the forth number were 1, like 11 11, then we would have said 21 in the first place. Therefore, there will never be more than three equal numbers in a row and the greatest number to appear in any number is thus 3. □

How can we implement this sequence in Haskell? There seem to be two different principles: First, to describe a given number in terms of groups of digits and, second, to bootstrap a sequence where each number describes its predecessor. Let us implement these two principles separately. The first one is very simple:

```
say :: Number → Number
say xs = concat [len x ++ [head x] | x ← group xs]
```

The *group* function is defined in *Data.List* and groups a list according to repeated el-

ements, exactly what we need. On each element of its result set, we apply *len*, the *length* function for natural numbers we defined earlier, and concatenate this result with the head of that element. For instance, *group* $[One, Two, One, One]$ would give $[[One], [Two], [One, One]]$. The length of the first list is $[One]$ and concatenated with the head of $[One]$ gives $[One, One]$. The length of the second list, again, is $[One]$ and concatenate with the head of $[Two]$ gives $[One, Two]$. The length of the third list is $[Two]$ and concatenated with the head of $[One, One]$ is $[Two, One]$. Calling *concat* on these results gives $[One, One, One, Two, Two, One]$, which converted to an *Integer*, is 111221.

This function is more general than the sequence, however. We can apply it on any number, also on numbers we would never see in the look-and-say sequence. Applied on *unity*, *say* would just give $[One, One]$. Then, from *two* to $[Nine]$, the results are quite boring: $[One, Two], [One, Three], \ldots, [One, Nine]$. But applied on *ten*, it would result in $[One, One, One, Zero]$.

We will now use *say* to implement the look-and-say sequence starting from 1:

$$
\begin{aligned}
&says :: Number \to Number \\
&says\ [\,] &&= [\,] \\
&says\ [Zero] &&= [\,] \\
&says\ [One] &&= [One] \\
&says\ n &&= say\ (says\ (prev\ n))
\end{aligned}
$$

First, we handle the cases that are not part of the sequence: the empty list and *zero*. Then, we handle $[One]$, which is just $[One]$. Finally, we define the sequence for any number as *say* of *says* of the predecessor of that number. For instance:

$$
\begin{aligned}
say\ [Three] &= say\ (says\ (prev\ [Three])) \\
say\ [Three] &= say\ (says\ [Two]) \\
say\ [Three] &= say\ (say\ (says\ (prev\ [Two]))) \\
say\ [Three] &= say\ (say\ (says\ [One])) \\
say\ [Three] &= say\ (say\ [One]) \\
say\ [Three] &= say\ ([One, One]) \\
say\ [Three] &= [Two, One].
\end{aligned}
$$

Sometimes, the two sides of numbers, their numeric properties and their nature as sequences of digits, become entangled. This is the case with *narcissitic numbers*, a popular concept in recreational math – without further known applications in math or science. Narcissistic numbers are defined by the fact that they equal the sum of their digits raised to the power of the number of digits in the whole number. More formally, a narcissistic number n is a number for which holds:

$$n = \sum_{i=0}^{s} n_i^s,$$

where n_i is the digit of n at position i and s is the number of digits in n. In fact, we can define the property of being narcissistic much clearer as a test in Haskell using our number type:

$$
\begin{aligned}
&\mathit{narcissistic} :: \mathit{Number} \to \mathit{Bool} \\
&\mathit{narcissistic}\ n = \mathit{foldr}\ (\mathit{step}\ (\mathit{len}\ n))\ \mathit{zero}\ n \equiv n \\
&\quad \textbf{where}\ \mathit{step}\ s\ a\ b = b\ \text{`}\mathit{add2}\text{`}\ (\mathit{power2}\ [a]\ s)
\end{aligned}
$$

This property holds trivially for all numbers $< \mathit{ten}$. Then, they get rare. The narcissistic numbers between 10 and 1000 are: 153, 370, 371 and 407. 153, for instance, is narcissistic because $1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153$. Interesting is the pair 370 and 371: $370 = 3^3 + 7^3 + 0^3 = 27 + 343 + 0 = 370$. Now, if we add 1, *i.e.* $1^3 = 1$, 371 arises.

The number of narcissistic numbers in a given number system is limited. This is because for sufficient large ks, the smallest possible number of the form 10^{k-1}, *i.e.* the smallest number with k digits, is greater than the greatest number of the form $k \times 9^k$, *i.e.* the greatest number we can build by adding up the k^{th} powers of the digits of a k-digit-number. That means that, for large numbers, the numerical value will always be greater than the sum of the digits raised to the number of digits in that number. In the decimal system, this limit is reached with $k = 61$. 10^{60} is obviously the smallest number with 61 digits. The 61-digit number with which we can build the greatest sum of 61^{st} powers is the number $99\ldots9$ that consists of 61 9s. If we raise all these 9s to the 61^{st} power and sum the results, we will obtain a number with 60 digits. That number is clearly less than the least number we can represent with 61 digits. Therefore, no narcissistic numbers with more than 60 digits are possible. In practice, there are only 88 narcissistic numbers in the decimal number system and the greatest of those has 39 digits.

Another popular problem from recreational math is that of a 10-digit number, where each position tells how often the digit related to that position counted from left to right and from 0 to 9 is present in the number. If we represent such a number as in the following table

0	1	2	3	4	5	6	7	8	9
a	b	c	d	e	f	g	h	i	j

then a would tell how often 0 appears in that number, b, how often 1 appears in that number, c, how often 2 appears in that number and so on.

How can we tackle that problem? First, obviously, the numbers we write in the second line of the table must add up to 10, since these numbers tell how often the related digit appears in the whole number. Since the number has 10 digits, there must be in total 10 occurrences.

We can further assume that the most frequent digit that appears in the number is 0. Otherwise, if a greater digit appeared with high frequency, it would imply that also other numbers must appear more often, since every digit that appears in the number implies

another digit to appear. For instance, if 5 was the number with most occurrences, then some numbers must appear 5 times, namely those where we actually put the number 5.

So, let us just try. We could say that 0 occurs 9 times. We would have something like

0	1	2	3	4	5	6	7	8	9
9	0	0	0	0	0	0	0	0	1

This means that 0 appears 9 times and 9 appears once. But there are two problems with this solution: First, if 9 appears once, then 1 appears once as well, but, then, there are only 8 places left to put 0s in. Second, if 1 appearas once (to count 9), then we must put 1 below 1 in the table. But then 1 appears twice, so we must put 2 below 1 and, as a consequence, we must put 1 below 2. In fact, whatever the number of 0s is, for all solutions, we need at least two 1s and one 2, hence:

0	1	2	3	4	5	6	7	8	9
x	2	1	x	x	x	x	x	x	x

Let us think: We have to convert two of the xs into numbers, one into a number that we do not know yet and the other to 1 to count that unknown number. In other words, we will have 2, 1, 1 and some other number. Since we know that the numbers must add up to 10, we can just compute that unknown number as $x = 10 - (2 + 1 + 1) = 10 - 4 = 6$. The result then is 6 210 001 000:

0	1	2	3	4	5	6	7	8	9
6	2	1	0	0	0	1	0	0	0

Is this the only possible configuration, or are there others that fulfil the constraints? Let us assume there is another configuration. We already know that 7,8 and 9 do not work. So, instead of 6, we will have a number smaller than 6. This number could be 5. Then, we have to make up the difference between 6 and 5, since the numbers, at the end, must add up to 10. That means, we need one more 1. But, then, we need an additional number that occurs once to justify that additional 1. But, since there is only room for one additional number, that cannot be.

Then, we could try 4. But 4 does not work either, since the difference now is 2 and we cannot just increase the ocurrences of 1 or 2 without justification. If we increased the occurrences of 1, we would have to add another number, to justify that additional 1. We need, in fact, three numbers, but we have only room for two.

Then 3 could be a solution: instead of one 6, we would have two 3s. But now, we must justify the second 3 and there is no room for another number appearing three times. So, since 1 and 2, obviously, will not work, we conclude, that 6 210 001 000 is the only possible configuration.

We are now very close to leave the world where we look at numbers mainly as strings. We will soon look at numbers in a completely different way. But before we do that, we still have to finalise the model of our number type, that is, we should define how

to convert an integer into our *Number*. We can of course just convert the integer into a string using *show* and then convert the string digit by digit to *Number*. But, again, that would be boring. We would not learn anything special about numbers, which is the main concern of all our exercises here.

Instead, we will think along the lines of decimal numbers being representations of powers of 10. We will ask: how many powers of 10 does a given number contain? How many powers of 10 are, for example, in the number 9827? To answer this question, we first have to find the floor of $\log_{10} 9827$, *i.e.* a number l such that $10^l \leq 9827$ and $10^{l+1} > 9827$: $\lfloor \log_{10} 9827 \rfloor$. For 9827, that is 3, since $10^3 = 1000$ and $10^4 = 10000$. To learn how many third powers of 10 are in the number, we divide the number by the third power of 10: $\left\lfloor \frac{9827}{10^3} \right\rfloor = 9$. We, hence, have 9 times the third power of 10 in 9827. The first digit is therefore *Nine*. To convert the whole number, we now apply the algorithm on the remainder of the division $\frac{9827}{10^3}$, which is 827.

But hold on: is this not quite expensive with a log operation and a division on each digit of the original integer? Yes, in fact, we can think much simpler in terms of modulo. A number in the decimal system is composed of the digits $0 \ldots 9$. Any number modulo 10 is one of these digits. The remainder of 9827 and 10, for instance, is 7, because the Euclidian division of 9827 and 10 is $(982, 7)$; the Euclidian division of 982 and 10 is $(98, 2)$; the result for 98 and 10 is $(9, 8)$ and that for 9 and 10 is just $(0, 9)$. In other words, we can just collect the remainders of the Euclidian division of the integer and 10 and convert each digit into our *Digit* type. Here is the code in Haskell:

```
integer2Num :: Integer → Number
integer2Num 0    = zero
integer2Num 1    = unity
integer2Num 2    = two
integer2Num 3    = [Three]
integer2Num 4    = [Four]
integer2Num 5    = [Five]
integer2Num 6    = [Six]
integer2Num 7    = [Seven]
integer2Num 8    = [Eight]
integer2Num 9    = [Nine]
integer2Num 10   = ten
integer2Num i    = go i
  where go n = case n `quotRem` 10 of
                 (0, r) →          integer2Num r
                 (q, r) → go q ++ integer2Num r
```

We start by handling all one-digit numbers and 10 explicitly This has two advantages: we speed up the processing for one-digit numbers and 10 and we do not need an extra conversion function for digits.

For all values of i not handled in the base cases, we compute quotient and remainder. If the quotient is 0, we are done with *go* and just yield the conversion of r, which must be a digit, since it is a remainder of division by 10. Otherwise, we continue with the quotient to which we append the conversion of the remainder.

For the example 9827, we would create the following sequence:

$go\ 9827 = go\ 827 +\!\!+ integer2Num\ 7$
$go\ 982 = go\ 82 +\!\!+ integer2Num\ 2 +\!\!+ integer2Num\ 7$
$go\ 98 = go\ 8 +\!\!+ integer2Num\ 8 +\!\!+ integer2Num\ 2 +\!\!+ integer2Num\ 7$
$go\ 9 = integer2Num\ 9 +\!\!+ integer2Num\ 8 +\!\!+ integer2Num\ 2 +\!\!+ integer2Num\ 7,$

which is

$[Nine] +\!\!+ [Eight] +\!\!+ [Two] +\!\!+ [Seven]$
$[Nine, Eight, Two, Seven].$

3.8. $\mathbb{N}$

We will now convert our *Number* type in a full-fledged Haskell *Num* type. This will allow us to use numeric symbols, *i.e.* the number $0 \ldots 9$, instead of the constructors *Zero*... *Nine*, for our type and we will be able to use the standard operators $+, -, *$. The first step is to define a **data** type – until now, we used only a type synonym for $[Digit]$:

```
data Natural = N Number
```

The new data type is called *Natural* and its only constructor is N receiving a *Number*, *i.e.* $[Digit]$, as parameter. The constructor N is named after the symbol for the set of natural numbers in math, which is $\mathbb{N}$.

We, then, make this data type instance of *Eq* and *Show*, where we use the previous defined functions *cmp* for comparisons and *n2Integer* for conversion to *Int* and subsequent *show*:

```
instance Eq Natural where
  (N a) == (N b) = cmp a b == EQ
instance Show Natural where
  show (N ns) = show (n2Integer ns)
```

Now we are ready to make *Natural* instance of *Num*. *Num* has the following methods we have to implement: $+$, $-$, $*$, *negate*, this would be a negative number, which we have not yet defined, so we leave this method undefined, *abs*, the absolute value of a number, *signum*, which is either 0 (for *zero*), 1 (for numbers > 0) or -1 (for numbers < 0), and *fromInteger*, a conversion function that turns instances of type class *Integral*, like *Int*

and *Integer*, into our data type. Here is the code:

```
instance Num Natural where
  (N as) + (N bs)                  = N (as `add2` bs)
  (N as) - (N bs) | cmp as bs ≡ LT = error "subtraction below zero"
     | otherwise                   = N (as `sub2` bs)
  (N as) * (N bs)                  = N (as `mul2` bs)
  negate  n                        = ⊥
  abs     n                        = n
  signum (N [Zero])                = 0
  signum n                         = 1
  fromInteger i                    = N (integer2Num i)
```

Two *Natural*s are added by adding the *Number*s of which they consists using *add2* and calling the constructor N on the result. Subtraction and multiplication are implemented accordingly using *sub2* and *mul2* respectively. *abs* n is just n, since Natural is always a positive number, we do not need to worry about negative numbers passed in to *abs*. *signum* for *zero* is just 0, for any other number, it is 1. Again, because of their absence, we do not need to handle negative numbers. For *fromInteger*, we finally use the conversion function *integer2Num*.

There are some other properties we would like our number type to have. First, numbers, in Haskell, are also *Enum*s, *i.e.* objects that can be enumerated. The class *Enum* defines the methods *succ* and *pred* – which we already know from Peano numbers – *toEnum*, the conversion of integrals, especially *Int*s, to our data type, and *fromEnum*, the opposite conversion:

```
instance Enum Natural where
  succ (N n)        = N (next n)
  pred (N [Zero])   = error "zero has no predecessor"
  pred (N n)        = N (prev n)
  toEnum            = N ∘ integer2Num ∘ fromIntegral
  fromEnum (N n) = fromIntegral (n2Integer n)
```

Numbers, additionally, have order. For every two numbers, we can say which of the two is greater or less than the other. This is captured by the type class *Ord*. The only method we have to implement for making *Natural* instance of *Ord* is compare:

```
instance Ord Natural where
  compare (N as) (N bs) = cmp as bs
```

We also want to be able to convert our numbers into real numbers. To do so, we make *Natural* an instance of *Real*:

```
instance Real Natural where
```

```
toRational (N ns) = fromIntegral $ n2Integer ns
```

Finally, our number type is a kind of integral, *i.e.* not a fraction. To express this in Haskell, we make *Natural* instance of the *Integral* class and implement the methods *quotRem* and *toInteger*. The code is quite obvious, no further explanations are necessary:

```
instance Integral Natural where
  quotRem (N as) (N bs) = let (q, r) = D.quotRem2 as bs in (N q, N r)
  div       a b         = fst $ quotRem a b
  toInteger (N ns)      = n2Integer ns
```

3.9. Abstract Algebra

When we look at the four fundamental arithmetic operations, addition, multiplication, subtraction and division, we see some striking differences between them. We can, more specifically, distinguish two groups of operations, namely addition and multiplication on one side and subtraction and division on the other.

for multiplication and addition, we can state that for any two natural numbers a and b, the result of the operations $a + b$ and $a \times b$ is again a natural number. For subtraction and division that is not true. As you may remember, for subtraction, we had to define an important exception, *viz.* that the second term must not be greater than the first one. Otherwise, the result is not a natural number.

Division according to Euclid, besides having the exception of *zero* in the denominator, differs completely, in that its result is not at all a number, but a pair of numbers (q, r). If we refer to division in terms of the *quot* operation (which returns only the quotient, not the remainder), then division would indeed behave similar to addition and multiplication (besides the division-by-zero exception). But that would leave us with a torso operation that falls behind the other operations in precision and universality.

Another property shared by addition and multiplication is the *associative law*:

$$a + (b + c) = (a + b) + c = a + b + c \tag{3.5}$$

$$a \times (b \times c) = (a \times b) \times c = a \times b \times c \tag{3.6}$$

This, again, is not true for subtraction and division, as can be easily shown by counter examples:

$$4 - (3 - 1) \neq (4 - 3) - 1,$$

since

$$4 - (3 - 1) = 4 - 2 = 2$$

and

$$(4 - 3) - 1 = 1 - 1 = 0.$$

For division, we cannot even state such an equality (or inequality), since the result of the Euclidian division, a pair of numbers, cannot serve as one of its arguments, *i.e.* a pair of numbers cannot be divided. If we, again, accept the *quot* operation as a compromise, we quickly find counter examples which show that the associative law does not hold for division either:

$$3/(2/2) \neq (3/2)/2,$$

since

$$3/(2/2) = 3/1 = 3$$

and

$$(3/2)/2 = 1/2 = 0.$$

Abstract Algebra uses such properties to define different classes of numbers and other "things" – of which we will soon see some examples. The first class of such things we can define is the *magma* or *groupoid*. A magma is a set together with a binary operation such that the set is closed under this operation. We will look at sets in more detail in the next section; for the moment, we can live with an informal intuition of sets being collections of things, here of certain types of numbers. That a set is closed under an operation is just the property we defined first, *i.e.* that c is a natural number if a and b are natural numbers in $a + b = c$. More formally, we can describe a magma as

$$M = (S, \cdot), \tag{3.7}$$

where S is a set and $\cdot$ is a binary operation, such that for all $a, b \in S$ (a and b are *element of* S, *i.e.* they are members of the set S), $a \cdot b \in S$, *i.e.* the result of the operation $a \cdot b$, too, is in S.

When we add the other property, the associative law, to the magma definition, we get a *semigroup*. A semigroup, hence, is a magma, where for the operation $\cdot$ the relation $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ holds.

Natural numbers with either addition or multiplication are clearly semigroups. We can be even more specific: Natural numbers are *abelian semigroups*, since the *commutative law* holds for them as well. The commutative law states that, for an operation $\cdot$ the relation: $a \cdot b = b \cdot a$ holds, which, again, is not true for subtraction and division.

The next property is the identity. This property states that there is an element e in S, for which holds that $a \cdot e = e \cdot a = a$. For addition and subtraction, this element e is *zero*. For multiplication, it is *unity*. For a division operation defined as *quot* this element would be *unity* as well.

A semigroup with identity is called a *monoid*. Natural numbers are hence abelian monoids, since the commutative law holds for them as well. An example for a non-abelian monoid is the set of all strings, STR, with the concatenation operation $+\!\!+$. First note that STR is closed under concatenation, since, for any strings a and b, it holds that (using Haskell syntax) $a +\!\!+ b$ is again a string, *e.g.* `"hello " ++ "world" ≡ "hello world"`.

Then, the associative law holds, since for any three strings, a, b and c: $a +\!\!+ (b +\!\!+ c) \equiv (a +\!\!+ b) +\!\!+ c \equiv a +\!\!+ b +\!\!+ c$, for instance: `"hello" ++ (" " ++ "world") ≡ ("hello" ++ " ") ++ "world" ≡ "hello world"`.

Next, there is an identity, *viz.* the empty string "", such that: $a +\!\!+$ `"" ≡ "" ++` $a \equiv a$, for instance: `"hello world" ++ "" ≡ "hello world"`.

Note, however, that the STR monoid is not commutative: $a +\!\!+ b \not\equiv b +\!\!+ a$, for instance: `"hello " ++ "world" ≠ "world" ++ "hello "`.

The term *semigroup* suggests that there is also something called a *group*, which, in some way, is more complete than a semigroup – and, indeed, there is. A group is a monoid with the addtional property of *invertibility*. Invertibility means that there is an element to invert the effect of an operation, such that for any a and b for which holds: $a \cdot b = c$, there is an element x, such that: $c \cdot x = a$. Note that this implies for b and its inverse element x: $b \cdot x = e$, where e is the identity.

Unfortunately for our poor natural numbers, there are no such elements with addition and multiplication. Note, however, that, if we had already introduced negative numbers and fractions, there would ineed exist such elements, namely for addition: $a + x = 0$, where obviously $x = -a$ and for multiplication: $a \times x = 1$ with $x = \frac{1}{a}$.

Let us summarise the fundamental properties of binary operations to keep track of all the properties that may hold for different types of objects:

	closure	associativty	identity	invertibility	commutativty
	$a \cdot b \in S$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$	$a \cdot e = e \cdot a = a$	$a \cdot \frac{1}{a} = e,$	$a \cdot b = b \cdot a$
magma	×				
semigroup	×	×			
monoid	×	×	×		
group	×	×	×	×	
abelian x	×	-	-	-	×

It is to be noted that any of the concepts magma, semigroup, monoid and group may have the property of being abelian. There are abelian magmas, semigroups, monoids and groups. Therefore, the *abelian x* is indifferent towards associativity, identitiy and invertibility. This depends entirely on the x, not on the x being abelian or not.

We will now introduce a major step. We have, so far, added additional propterties to magmas, semigroups and so on to create new kinds of objects. Now, we change the underlying definition to create something completely different, namely a *semiring*. A semiring is a set S together with **two** binary operations, denoted $\bullet$ and $\circ$:

$$R = (S, \bullet, \circ). \tag{3.8}$$

The operation $\bullet$ must form an abelian monoid with S and the operation $\circ$ must form a monoid (which may or may not be abelian) with S. These conditions are fulfilled for addition and multiplication on the natural numbers. Since both, addition and multiplication, form abelian monoids, for the moment, both may take either place in the definition. But there is one more property: the operations together must adhere to the *distributive law*, which states that

$$a \circ (b \bullet c) = (a \circ b) \bullet (a \circ c). \tag{3.9}$$

This, again, is true for natural numbers, if $\bullet$ corresponds to addition and $\circ$ to multiplication: $a \times (b + c) = (a \times b) + (a \times c)$. We can simplify this formula by leaving the parentheses out of course: $a \times b + a \times c$ and can even further simplify by adopting the usual convention that $a \times b = ab$: $ab + ac$.

A *ring* is a semiring, for which the additional property of invertibility holds on addition. A ring, hence, consists of an abelian group (addition in case of natural numbers) and a monoid (multiplication). Again, natural numbers do not form a ring, but only a semiring, since there are no negative numbers in natural numbers and there is thus no inverse for addition.

A ring, where multiplication is commutative, hence, a ring with an abelian group (addition) and an abelian monoid (multiplication) is called a *commutative ring*.

The most complete structure, however, is the *field*, where both operations, addition and

multiplication are abelian groups.

Here is the complete taxonomy:

	addition	multiplication
semiring	abelian monoid	monoid
ring	abelian group	monoid
commutative ring	abelian group	abelian monid
field	abelian group	abelian group

4. Induction, Series, Sets and Combinatorics

4.1. Logistics

Before we continue to investigate the properties of natural numbers, let us deviate from pure theory for a moment and have a look at a motivating example from my professional practice. It is quite a simple case, but, for me, it was one of the starting points to get involved with sequences, series, combinatorics and other things natural numbers can do.

I was working for a logistics service provider for media, mainly CDs, DVDs, video games, books and so on. The company did all the merchandise management for its customers, mainly retailers, and, for this purpose, ran a set of logistics centres. We got involved, when one of those logistics centres was completely renewed, in particular a new sorter system was installed that enabled the company to comfortably serve all their current customers and those expected according to steep growth rates in the near future.

A sorter is a machine that reorders things. Goods enter the warehouse ordered by the suppliers that actually sent the goods in lots of, for instance, 1.000 boxes of album A + 450 boxes of album B + 150 boxes of album C. These lots would go onto the sorter and the sorter would reorder them into lots according to customer orders, *e.g.*: customer I ordered: 150 boxes of album A + 30 boxes of album B + 10 boxes of album C, customer II ordered: 45 boxes of album A + 99 boxes of album B and so on.

Mechanically, the sorter consisted of a huge belt with carrier bins attached to it that went around in circles. Feeders would push goods onto the carrier bins and, at certain positions, the bins would drop goods into buckets on the floor beneath the belt, so called endpoints. At any time, endpoints were assigned to customers, so that each endpoint ended up with goods ordered by one specific customer.

Our software was responsible for the configuration of the machine. It decided, which customers were assigned to endpoints and how goods were related to customer orders. The really tricky task was optimising the process, but here I would like to focus on one single issue that, in fact, was much simpler than all that optimisation stuff, namely the allocation of customers to endpoints.

At any given time, the sorter had a certain allocation, that is an assignment of endpoints to customers. There were very big customers that received several lots per day and others

that would only receive lots on certain weekdays. Only those customers that would still receive a lot on the same day and within the current batch would actually have an allocation. The goods for those, currently not on the sorter, would fall in reserved endpoints, called "ragmen", for later batches or other weekdays. With this logic, the sorter was able to serve much more customers than it had endpoints, and what we wanted to know was how many ragmen we would need with respect to a given amount of customers.

Our idea for attacking the problem was the following: we started to assume naïvely that we could simply split the customers by some ratio in those currently *on* (assigned to an endpoint) and those currently *off* (not assigned to an endpoint). We would split, let us say, 1.000 customers into 500 allocated to some endpoint and 500 currently not allocated. But, unfortunately, we needed some endpoints to catch all the merchandise intended for those currently not *on*. So, we had to reserve a certain amount of endpoints as ragmen and subtract this number from the amount of endpoints available for allocated customers. A key variable was the number of customers *off* per ragman endpoint. We wanted this number, of course, to be as high as possible, because from this relation came the saving in endpoints that must be reserved as ragmen and it finally determined, how many customers the server could serve. On the other hand, we could not throw the goods for all customers currently not at the sorter into one single endpoint. This would have caused this endpoint to overflow every few minutes causing permant work in getting the merchandise to a waiting zone. This special point turned out to be quite complicated: small customers with small lots would need less ragman capacity than big ones; the problem was solved with a classification approach, but that does not matter here at all. For our purpose, it is just important that there actually was some value to determine this relation, let us say it was $c = 10$, meaning that we needed a ragman endpoint for every 10 customers not on the sorter.

We will now use the naïve assumption to compute the number of ragmen as $r = \lceil \frac{n-m}{c} \rceil$, where n is the number of customers and m the number of available endpoints. For our example of 1.000 customers and 500 endpoints, r is $\frac{1000-500}{10}$, hence, 50 ragman endpoints.

But this result cannot be true! We naïvley assumed that we have 500 endpoints. But in the very moment we reserve 50 endpoints as ragmen for customers not currently on the sorter, this number reduces instantly to $m - r$, that is 450 endpoints. We, therefore, have to reserve more ragmen, that is to say for those 50 customers that, now, have no endpoint on the sorter anymore. Since we need one ragman per 10 customers, this would give $50 + 5$ ragmen. But would this not reproduce the problem we wanted to solve in the first place? In the very moment, we add 5 more endpoints to the ragmen, we have to take away 5 from the available endpoints, reducing the number of available endpoints once again to $450 - 5 = 445$.

We end up with something called a series: the number of ragmen equals the number of endpoints divided by c plus this number divided by c plus this number divided by c and

so on. We can represent this with a nice formula as:

$$r = \left\lceil \frac{n-m}{c} \right\rceil + \left\lceil \frac{n-m}{c^2} \right\rceil + \dots \tag{4.1}$$

Or even nicer:

$$r = \sum_{k=1}^{\infty} \left\lceil \frac{n-m}{c^k} \right\rceil \tag{4.2}$$

You can easily convince yourself that dividing $n-m$ by c^2 is the same as dividing $\frac{n-m}{c}$ by c, because dividing a fraction by a natural number is equivalent to multiplying it with the denominator (we will look at this more carefully later). In the sum in equation 4.2, the k is therefore growing with each step.

But the equation, still, has a flaw. The inner division in the summation formula will leave smaller and smaller values that, at some point, become infinitesimally small. but, since we ceil the division result, these tiny values will always be rounded up to one, such that the formula produces an endless tail of ones, which is of course not what we want. Therefore, we should use the opposite of ceiling, floor, but should not forget to add one additional ragman to cope with the remainders:

$$r = 1 + \sum_{k=1}^{\infty} \left\lfloor \frac{n-m}{c^k} \right\rfloor \tag{4.3}$$

Now, when $\frac{n-m}{c^k}$ becomes less than one, the division result is rounded down to zero and the overall result of the summation converges to some integer value. For 1000 customers, the series converges already for $k = 3$; we, thus, need $50+5+1 = 56$ ragmen to cope with 1000 customers and will be able to serve 444 customers on the sorter. For, say, 2.000 customers, the series converges for $k = 4$, so we need $\lfloor \frac{1500}{10} \rfloor + \lfloor \frac{1500}{100} \rfloor + \lfloor \frac{1500}{1000} \rfloor + 1 = 167$ ragmen and will have 333 endpoints *on*. For 5.000 customers, the series, again, converges for $k = 4$ and we will need $\lfloor \frac{4500}{10} \rfloor + \lfloor \frac{4500}{100} \rfloor + \lfloor \frac{4500}{1000} \rfloor + 1 = 500$, which is just the amount of endpoints we have available in total. We, thus, cannot serve 5.000 customers with this configuration. We would need to increase c and accept more workload in moving goods into wating zones.

Let us look at a possible implementation of the above with our natural numbers. First, the notion of *convergence*, as we have used it above, appears to be interesting enough to define a function for it. The idea is that we sum up the results of a function applied to an increasing value until the result reaches zero and, in consequence, will not affect the cumulated result anymore:

$$\begin{aligned} &converge1 :: (Natural \to Natural) \to Natural \to Natural \\ &converge1\ f\ n = \textbf{let}\ r = f\ n \end{aligned}$$

$$\begin{array}{l} \quad\quad\quad\

4.2. Induction

The series we looked at in the previous section converge very soon, for realistic values, after 3 or 4 steps. But this may be different and then huge sums would arise that are costly to compute, since many, perhaps unfeasibly many additions had to be made. We already stumbled on such problems, when we looked at multiplication. It is therefore often desirable to find a *closed form* that leads to the same result without the necessity to go through all the single steps. Let us look at a very simple example. We could be interested in the value of the n first odd numbers summed up, *i.e.* for $n = 2$: $1 + 3 = 4$, $n = 3$: $1 + 3 + 5 = 9$, $n = 4$: $1 + 3 + 5 + 7 = 16$ and so on. With large values of n, we would have to go through many steps, *viz.* $n - 1$ additions.

First, let us think about how to express this as a formula. An odd number is a number that is not divisible by 2. Even numbers could be expressed as $2k$ for all ks from $1 \ldots n$, for instance the first even number, $n = 1$, is 2, the first two even numbers, $n = 2$, are 2 and 4, since $2 \times 2 = 4$, the first three even numbers, $n = 3$, are 2, 4 and 6, since $2 \times 3 = 6$ and so on. Odd numbers, correspondingly, can be described as: $2k - 1$. The first odd number, hence, is $2 \times 1 - 1 = 1$, the first two odd numbers, $n = 2$, are 1 and 3, since $2 \times 2 - 1 = 3$, the first three odd numbers, $n = 3$, are 1, 3 and 5, since $2 \times 3 - 1 = 5$ and so on. Correspondingly, the sum of the first n odd numbers can be properly described as:

$$\sum_{k=1}^{n} (2k - 1)$$

To convince ourselves that this formula is correct, let us go through some examples: If $n = 1$, then $2k - 1$ equals 1, for $n = 2$, this is the result of $n = 1$ plus $4 - 1$, hence $1 + 3 = 4$, for $n = 3$, the formula leads to $4 + 6 - 1 = 9$ and for $n = 4$, the result is $9 + 8 - 1 = 16$. The formula appears to be correct.

We can implement this formula literally by a simple Haskell program:

```
oddSum1 :: Natural → Natural
oddSum1 n = go 1
  where go k | k > n      = 0
             | otherwise = (2 * k - 1) + go (k + 1)
```

Now, is there a closed form that spares us from going through all the additions in the *go* function? When we look at the results of the first 9 numbers calling oddSum1 as

```
map oddSum1 [1..9]
```

we see that all the numbers are perfect squares: $1, 4, 9, 16, 25, 36, 49, 65, 81$. Indeed, the results suggest that the sum of the first n odd numbers equals n^2. But is this always

true or does it hold only for the first nine numbers we just happened to look at? Let us try a proof by *induction*.

Induction is an tremendously important technique, since it enables us to prove that a property holds for infinitely many numbers! A proof by induction proves that a property P holds for a base case and, by advancing from the base case to the following number, that it holds for all numbers we are interested in. Formally, we prove, for example, that $P(n) \rightarrow P(n+1)$, where $+1$ is a very common way to advance. With $+$, we actually prove that P holds for all $x > n$. But we can use induction also for with functions that advance at a different pace, for instance, we might want to prove that some property holds for even numbers, we would then advance with $+2$.

Proofs by induction consist of two parts: First, the proof that the property is true for the base case and, second, that it is still true when advancing from a number, for which we know that it is true, like the base case, to the next number.

For the example of the sum of the odd numbers, the base case, $n = 1$, is trivially true, since 1^2 and $\sum_{k=1}^{n}(2k-1)$ are both 1. Now, if we assume that, for a number n, it is true that the sum of the first n odd numbers is n^2, we have to show that this is also true for the next number $n+1$ or, more formally, that

$$\sum_{k=1}^{n+1}(2k-1) = (n+1)^2. \tag{4.4}$$

We can decompose the sum on the left side of the equal sign by taking the induction step $(n+1)$ out and get the following equation:

$$\sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 = (n+1)^2. \tag{4.5}$$

Note that the part broken out of the sum corresponds exactly to the formula within the sum for the case that $k = n+1$. Since we already now that the first part is n^2, we can simplify the expression on the left side of the equal sign to $n^2 + 2(n+1) - 1$, which, again simplified, gives:

$$n^2 + 2n + 1 = (n+1)^2 \quad \square \tag{4.6}$$

and, thus, concludes the proof. If you do not see that both sides are equal, multiply the right side out as $(n+1)(n+1)$, where $n \times n = \mathbf{n^2}$, $n \times 1 = \mathbf{n}$, $1 \times n = \mathbf{n}$ and $1 \times 1 = \mathbf{1}$. Summing this up gives $n^2 + 2n + 1$.

The *oddSum* function can thus be implemented in much more efficient way:

$$
\begin{aligned}
&\mathit{oddSum} :: \mathit{Natural} \to \mathit{Natural}\\
&\mathit{oddSum} = (\uparrow 2)
\end{aligned}
$$

For another example, let us look at even numbers. Formally, the sum of the first n even numbers corresponds to: $\sum_{k=1}^{n} 2k$. This is easily implemented in Haskell as

$$
\begin{aligned}
&\mathit{evenSum1} :: \mathit{Natural} \to \mathit{Natural}\\
&\mathit{evenSum1}\ n = \mathit{go}\ 1\\
&\quad \textbf{where}\ \mathit{go}\ k \mid k > n \quad\ = 0\\
&\qquad\qquad\quad \mid \mathit{otherwise} = 2 * k + \mathit{go}\ (k + 1)
\end{aligned}
$$

Applying $\mathit{evenSum1}$ to the test set [1..9] gives the sequence: $2, 6, 12, 20, 30, 42, 56, 72, 90$. These are obviously no perfect squares and, compared to the odd numbers $(1, 4, 9, 16, \dots)$, the results are slightly greater. How much greater are they? For $n = 1$, oddSum is 1, $\mathit{evenSum}$ is 2, $\mathit{evenSum}$ is hence $\mathit{oddSum} + 1$ for this case; for $n = 2$, the difference between the results 4 and 6 is 2; for $n = 3$, the difference between 9 and 12 is 3. This suggests a pattern: the difference between oddSum and $\mathit{evenSum}$ is exactly n. This would suggest the closed form $n^2 + n$ or, which is the same, $n(n+1)$. Can we prove this by induction?

For the base case $n = 1$, $\sum_{k=1}^{1} 2k$ and $n(n + 1)$ are both 2. Now assume that for some n, $\sum_{k=1}^{n} 2k = n(n+1)$ holds, as we have just seen for the base case $n = 1$, then we have to show that

$$\sum_{k=1}^{n+1} 2k = (n + 1)(n + 2). \tag{4.7}$$

Again, we decompose the sum on the left side of the equal sign:

$$\sum_{k=1}^{n} (2k) + 2(n + 1) = (n + 1)(n + 2). \tag{4.8}$$

According to our assumption, the summation now equals $n(n + 1)$:

$$n(n + 1) + 2(n + 1) = (n + 1)(n + 2). \tag{4.9}$$

The left side of the equation can be further simplified in two steps, first, to $n^2+n+2n+2$ and, second, to $n^2 + 3n + 2$, which concludes the proof:

$$n^2 + 3n + 2 = (n + 1)(n + 2) \quad \square \tag{4.10}$$

If you do not see the equality, just multiply $(n+1)(n+2)$ out: $n \times n = \mathbf{n^2}$, $n \times 2 = \mathbf{2n}$; $1 \times n = \mathbf{n}$, $1 \times 2 = \mathbf{2}$; adding all this up gives $n^2 + 2n + n + 2 = n^2 + 3n + 2$.

We can now define an efficient version of *evenSum*:

$$\begin{aligned} &\mathit{evenSum} :: \mathit{Natural} \to \mathit{Natural} \\ &\mathit{evenSum}\ n = n \uparrow 2 + n \end{aligned}$$

Now, of course, the question arises to what number the first n of both kinds of numbers, even and odd, sum up. One might think that this must be something like the sum of odd and even for n, but that is not true. Note that the sum of the first n either odd or even numbers is in fact number greater than the first n numbers, since, when we leave out every second number, then the result of counting n numbers is much higher than counting all numbers, *e.g.* for $n = 3$, the odd numbers are $1, 3, 5$ and the even are $2, 4, 6$. The first 3 numbers, however, are $1, 2, 3$.

The answer jumps into the eye when we look at the formula for the sum of even numbers: $\sum_{k=1}^{n} 2k$. This formula implies that, for each n, we take twice n. The sum of all numbers, in consequence, should be the half of the sum of the even, *i.e.* $\sum_{k=1}^{n} (k) = \frac{n(n+1)}{2}$, a formula that is sometimes humorously called *The Little Gauss*.

Once again, we prove by induction. The base case, $n = 1$, is trivially true: $\sum_{k=1}^{1} k = 1$ and $\frac{1*(1+1)}{2} = \frac{2}{2} = 1$. Now assume that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ holds for n; then, we have to prove that

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}. \tag{4.11}$$

As in our previous exercises, we take the induction step out of the summation formula and get $\sum_{k=1}^{n} (k) + (n+1)$. According to our assumption, we can reformulate this as $\frac{n(n+1)}{2} + (n+1)$. We have not yet discussed how to add fractions; to do this, we have to present both values as fractions with the same denominator, which is 2. To maintain the value of $n + 1$, when we divide it by 2, we have to multiply it with 2 at the same time, yielding the fraction $\frac{2(n+1)}{2} = \frac{2n+2}{2}$:

$$\frac{n(n+1)}{2} + \frac{2n+2}{2} = \frac{(n+1)(n+2)}{2} \tag{4.12}$$

After multiplying the numerator of the first fraction on the left side of the equation out $(n^2 + n)$ and then adding the two numerators we obtain, in the numerators, the formula we already know from the even numbers:

$$\frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2} \quad \square \tag{4.13}$$

The sum of the first n natural numbers in Haskell, hence is:

$$\begin{aligned}&natSum :: Natural \to Natural\\&natSum = (`div`2) \circ evenSum\end{aligned}$$

4.3. The Fibonacci Sequence

We have already discussed and analysed the run time behaviour of gcd. Let us look at an intriguing example, the gcd of, say, 89 and 55. As a reminder here the definition of gcd once again:

$$\begin{aligned}&gcd :: Natural \to Natural \to Natural\\&gcd\ a\ 0 = a\\&gcd\ a\ b = gcd\ b\ (a\ `rem`\ b)\end{aligned}$$

We start with $gcd\ 89\ 55$, which is $gcd\ 55\ (89\ `rem`\ 55)$ after one step. What is the remainder of 89 and 55? 89 divided by 55 is 1 leaving the remainder $89 - 55 = 34$. The next round, hence, is $gcd\ 55\ 34$. The remainder of 55 and 34 is $55 - 34 = 21$. We recurse once again, this time with $gcd\ 34\ 21$. The remainder of 34 and 21 is $34 - 21 = 13$. The next step, hence, is $gcd\ 21\ 13$, which leads to the remainder $21 - 13 = 8$. As you see, this gets quite boring, but we are not done yet, since the next round $gcd\ 13\ 8$ forces us to call the function again with 8 and $13 - 8 = 5$, which then leads to $gcd\ 5\ 3$, subsequently to $gcd\ 3\ 2$ and then to $gcd\ 2\ 1$. The division of 2 by 1 is 2 leaving no remainder and, finally, we call $gcd\ 1\ 0$, which reduces immediately to 1.

Apparently, we got in some kind of trap. The first pair of numbers, 89 and 55, leads to a sequence of numbers, where every number is the sum of its two predecessors: $1 + 2 = 3, 2 + 3 = 5, 3 + 5 = 8, 5 + 8 = 13, 8 + 13 = 21, 13 + 21 = 34, 21 + 34 = 55, 34 + 55 = 89$. We entered with 89 and 55 and computed the remainder. Since the difference of 89 and 55 is less than 55, the remainder between these two number is just the difference $89 - 55$. That way, we got to the next pair, 55 and 34, for which the same is true, *viz.* that the remainder is just the difference between the two and so we continued step for step until we finally reached $(2, 1)$.

This sequence is well known. It was used by the Italian mathematician Leonardo Pisano, better known as Fibonacci (*Filius*, that is, son of Bonaccio), as an arithmetic exercise in his *Abacus* ("calculating") book, which was published in 1202. The sequence is the solution to an exercise with the following wording: "How many pairs of rabbits can be produced from a single pair in a year's time if every fertile pair produces a new pair of offsping per month and every pair becomes fertile in the age of one month?" We start with 1 pair, which produces one offspring after one month, yielding 2 pairs; in the second month, the first pair produces one more offsping, hence we have 3 pairs. In the third month, we have 2 fertile pairs producing each 1 more pair and we, hence, have 5 pairs. This, quickly, becomes confusing. Here a table that gives an overview of what happens

during the first year:

month	1	2	3	4	5	6	7	8	9	10	11	12
new pairs	1	1	2	3	5	8	13	21	34	55	89	144
total	1	2	4	7	12	20	33	54	88	143	232	376

This means that, in month 1, there is 1 new pair; in month 2, there is another new pair; in month 3, there are 2 new pairs; in month 4, there are 3 new pairs; in month 5, there are 5 new pairs; ...; in month 12, there are 144 new pairs. This is the Fibonacci sequence, whose first 12 values are given in the second row. The answer to Fibonacci's question consists in summing the sequence up: $\sum_{k=2}^{12} F_k = 375$. This can be seen in the third row of the table, which shows the total number of rabbit pairs for each month. Since this sum includes the first pair, which was already there, we must subtract one from the values in this row to come to the correct result.

The Fibonacci function can be defined as:

$$F_n = \begin{cases} 0 & \text{if n} = 0 \\ 1 & \text{if n} = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

The Fibonacci sequence is explicitly defined for 0 and 1 (since, of course, 0 and 1 do not have two predecessor from which they could be derived) and for all other numbers recursively as the sum of the Fibonacci numbers of its two predecessors. In Haskell this looks like:

$$\begin{aligned} &\mathit{fib} :: \mathit{Natural} \to \mathit{Natural} \\ &\mathit{fib}\ 0 = 0 \\ &\mathit{fib}\ 1 = 1 \\ &\mathit{fib}\ n = \mathit{fib}\ (n-1) + \mathit{fib}\ (n-2) \end{aligned}$$

Many people have studied the Fibonacci sequence, following Fibonacci, but also independently and even before it was mentioned in the *Abacus Book*. The sequence has the astonishing habit of popping up in very different contexts in the study of mathematics, nature, arts and music. Many mathematical objects, however, have this surprising – or, for some at least, annoying – property.

The first known practical application of the Fibonacci sequence in Europe appeared in an article of the French mathematician Gabriel Lamé in 1844 that identified the worst case of the Euclidian gcd algorithm as any two subsequent numbers of the Fibonacci sequence. This remarkable paper is also considered as the earliest study in computational complexity theory, a discipline that would be established only 120 years later.[1]

[1]There are of course always predecessors. The relation between gcd and the Fibonacci sequence was,

Lamé gave the worst case of the algorithm as n for $\gcd(F_{n+2}, F_{n+1})$. Let us check if this is true for our example above. We started with 89 and 55, which correspond to F_{11} and F_{10}. According to Lamé, we would then need 9 steps to terminate the algorithm. The pairs of numbers we applied are: $(89, 55), (55, 34), (34, 21), (21, 13), (13, 8)$, $(8, 5), (5, 3), (3, 2), (2, 1), (1, 0)$, which are 10 pairs of numbers and indeed 9 steps. But why is this so and is it always true or just for the sequence, we happen to look at?

We can answer the first question by observing the recursive nature of the Euclidian algorithm. When there is a pair of subsequent Fibonacci numbers that needs n steps, then the pair of the next subsequent Fibonacci numbers will reduce to the first pair after one round of gcd and, thus, needs $n + 1$ steps: all the steps of the first pair plus the one for itself. This is just the structure of mathematical induction, which leads us to the proof. We choose $(2, 1)$ as the base case, which is (F_3, F_2) and, as we have seen, needs one step to result in the trivial case $(1, 0)$. If the proposition that $\gcd(F_{n+2}, F_{n+1})$ needs n steps is true, then, for the case $n = 1$, F_{n+2} is F_3, which is 2, and F_{n+1} is F_2, which is 1. Therefore, the base case $(2, 1)$ fulfils the rule.

Now assume that we have a case for which it is true that the number of steps of $\gcd(F_{n+2}, F_{n+1})$ is n. Then we have to show that the number of steps of $\gcd(F_{n+3}$, $F_{n+2})$ is $n+1$. According to its definition, gcd for a pair of numbers (a, b) is $(b, a \bmod b)$. For subsequent Fibonacci numbers (as we have already shown informally above), $a \bmod b$ is identical to $a - b$ (except for the case where $b = 1$). After one step with $a = F_{n+3}$ and $b = F_{n+2}$, we therefore have:

$$\gcd(F_{n+2}, F_{n+3} - F_{n+2}).$$

We can substitute F_{n+3} in this formula according to the definition of the Fibonacci sequence, $F_n = F_{n-2} + F_{n-1}$, by $F_{n+1} + F_{n+2}$:

$$\gcd(F_{n+2}, F_{n+1} + F_{n+2} - F_{n+2}),$$

which, of course, simplifies to

$$\gcd(F_{n+2}, F_{n+1}).$$

This shows that we can reduce $\gcd(F_{n+3}, F_{n+2})$ to $\gcd(F_{n+2}, F_{n+1})$ in one step and that concludes the proof.

There is much more to say about this delightful sequence, and we are even far away from the conclusions of Lamé's paper. Unfortunately, we have not yet acquired the tools to

according to Knuth, already discussed in a paper by a French mathematician called Léger in 1837, and an analysis of the run time behaviour of the Euclidan algorithm was already presented in a paper by another French mathematician called Reynaud in 1811.

talk about these upcoming issues in a meaningful way. But, very soon, we will have. In the meanwhile, you might try to discover the Fibonacci sequence in other objects we will meet on our way, for example, in a certain very strange triangle.

4.4. Factorial

A fundamental concept in mathematics, computer science and also real life is the idea of *permutation*, variations in the order of a sequence of objects. Shuffling a card deck would for instance create permutations of the original arrangement of the cards. The possible outcomes of a sports event, the order in which the sprinters in a race arrive or the final classification of a league where all teams play all others, is another example.

For the list $[1, 2, 3]$ (in Haskell notation), the following permutations are possible: $[1, 2, 3]$ (this is the identity), $[2, 1, 3]$, $[2, 3, 1]$, $[1, 3, 2]$, $[3, 1, 2]$ and $[3, 2, 1]$.

Let us look at how to construct all permutations of a given sequence. The simplest case is the empty list that allows only one arrangement: $permutations\ [\,] = [[\,]]$. From this base case on, we can easily create permutations of longer lists, simply inserting new elements at every possible position within the permutations. The permutations of a list with one elmenent, for instance, would be constructed by inserting this element, say x, in all possible positions of all possible permutations of the empty list, trivially yielding: $[[x]]$. Now, when we add one more element, we get: $[[y, x], [x, y]]$, first adding the new element y in front of the existing element x and, second, adding it behind x. We now easily create the permutations of a list with three elements by simply inserting the new element z in all possible positions of these two sequences, which, for the first, gives: $[z, y, x], [y, z, x], [y, x, z]$ and for the second: $[z, x, y], [x, z, y], [x, y, z]$. Compare this pattern to the permutations of the list $[1, 2, 3]$ above with $z = 1, y = 2$ and $x = 3$.

Let us implement the process of inserting a new element at any possible position of a list in Haskell using the *cons* operator (:):

$$
\begin{aligned}
&insall :: a \to [a] \to [[a]] \\
&insall\ p\ [\,] \qquad = [[p]] \\
&insall\ p\ (x : xs) = (p : x : xs) : (map\ (x:)\ \$\ insall\ p\ xs)
\end{aligned}
$$

As base case, we have p, the new element, added to the empty list, which trivially results in $[[p]]$. From here on, for any list of the form $x{:}xs$, we add p in front of the list $(p{:}x{:}xs)$ and then repeat the process for all possible reductions of the list until we reach the base case. In each recursion step, we add x, the head of the original list, in front of the resulting lists. Imagine this for the case $p = 1$, $x = 2$ and $xs = \{3\}$: We first create $[1, 2, 3]$ by means of $p : x : xs$; we then enter *insall* again with $p = 1$, $x = 3$ and $xs = \{\}$, which creates $1 : 3 : [\,]$, to which later, when we return, 2, the x of the previous step, is inserted, yielding $[2, 1, 3]$. With the next step, we hit our base case $insall\ 1\ [\,] = [[1]]$. Returning to the step with $x = 3$, mapping $(x:)$ gives $[3, 1]$ and, one step further back,

$[2,3,1]$. We, hence, have created three cases: $[[1,2,3],[2,1,3],[2,3,1]]$ inserting 1 in front of the list, in the middle of the list and at the end.

To generate all possible permutations we would need to apply *insall* to *all* permutations of the input list, that is not only to $[2,3]$ as above, but also to $[3,2]$. This is done by the following function:

$$
\begin{aligned}
&perms :: [a] \to [[a]] \\
&perms\ [\,] = [[\,]] \\
&perms\ (x : xs) = concatMap\ (insall\ x)\ \$\ perms\ xs
\end{aligned}
$$

Called with $[1,2,3]$, the function would map *insall* 1 on the result of *perms* $2:[3]$. This, in its turn, would map *insall* 2 onto *perms* $3:[\,]$. Finally, we get to the base case resulting in $[[\,]]$. Going back the call tree, *insall* 3 would now be called on the empty set yielding $[3]$; one step further back, *insall* 2 would now result in $[[2,3],[3,2]]$. Mapping *insall* 1 finally on these two lists leads to the desired result.

You will have noticed that we are using the function *concatMap*. The reason is that each call of *insall* creates a list of lists (a subset of the possible permutations). Mapping *insall* 1 on the permutations of $[2,3]$, for instance, creates two lists, one for each permutation ($[2,3]$ and $[3,2]$): $[[1,2,3],[2,1,3],[2,3,1]]$ and $[[1,3,2],[3,1,2],[3,2,1]]$. We could use the function *concat* to merge the lists together, like: $concat\backslash\$\, map\ (insall\ x)\backslash\$\ perms\ xs$; *concatMap* is much more convenient: it performs mapping and merging in one step.

We have not yet noted explicitly that, when talking about permutations, we treat sequences as Haskell lists. Important is that the elements in permutation lists are distinct. In a list like $[1,2,2]$, we cannot distinguish the last two elements leading to errors in counting possible permutations. In fact, when we say *sequence*, we mean an ordering of the elements of a *set*. Sets, by definition, do not contain duplicates. We will look at sets more closely in the next section.

So, how many possible permutations are there for a list with n elements? We have seen that for the empty list and for any list with only one element, there is just one possible arrangement. For a list with two elements, there are two permuntations ($[a,b],[b,a]$). For a list with three elments, there are six permutations. Indeed, for a list with three elements, we can select three different elements as the head of the list and we then have two possible permutations for the tail of each of these three list. This suggests that the number of permutations is again a recursive sequence of numbers: for a list with 2 elements, there are 2×1 possible permutations; for a list with 3 elements, there are 3×2 possible permutations or, more generally, for a list with n elements, there are n times the number of possiblilities for a list with $n-1$ elements. This function is called *factorial* and is defined as:

$$n! = \prod_{k=1}^{n} k. \tag{4.14}$$

We can define factorial in Haskell as follows:

$$\begin{aligned} &fac :: (Num\ a, Eq\ a) \Rightarrow a \rightarrow a \\ &fac\ 0 = 1 \\ &fac\ n = n * fac\ (n-1) \end{aligned}$$

There is sometimes confusion about the fact that 0! is 1 and not, as one might expect, 0. There are however good arguments for this choice. The first is that the empty list is something that we can present as an input to a function creating permutations. If the output were nothing, then the empty list would have vanished by some mysterious trick. The output should therefore be the empty list again and, thus, there is exactly one possible permutation for the empty list.

Another argument is that, if 0! were 0, we could not include 0 into the recursive definition of factorial. Otherwise, the result of any factorial would be zero! The inversion of factorial, *i.e.*

$$n! = \frac{(n+1)!}{n+1}, \tag{4.15}$$

would not work either. 4! is for instance $\frac{5!=120}{5} = 24$, 3! is $\frac{4!=24}{4} = 6$, $2! = \frac{3!=6}{3} = 2$, $1! = \frac{2!=2}{2} = 1$ and, finally, $0! = \frac{1!=1}{1} = 1$.

The first factorials, which you can create by *map fac* $[0\,..\,7]$, are: $1, 1, 2, 6, 24, 120,$ $720, 5040$. They, then, increase very quickly, 10!, for instance, is 3 628 800. Knuth mentions that this value is a rule of thumb for the limit of what is reasonably computable. Algorithms that need more than 10! steps, quickly get out of hand, consuming too much space or time. Techniques to increase the available computational power may push this limit a bit ahead, but factorial grows even faster than Moore's law, drawing a definite line for computability.

Unfortunately, no closed form of the factorial function is known. There are approximations, at which we will look later in this book, but to obtain the precise value, a recursive computation is necessary, making factorial an expensive operation.

But let us have another look at permutations, which are very interesting beast. In the literature, different notations are used to describe permutations. A very simple, but quite verbose one is the *two-line* notation used by the great French mathematician Augustin-Louis Cauchy (1789 – 1857). In this notation, the original sequence is given in one line and the resulting sequence in a second line, hence, for a permutation σ:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}.$$

According to this definition, the permutation σ would substitute 2 for 1, 5 for 2, 4 for 3 and 3 for 4 and, finally, 1 for 5. The alternative *tuple notation* would just give the second line as (2,5,4,3,1) and assume a *natural* ordering for the original sequence. This notation is useful, when several permutations on the same sequence are discussed. The original sequence would be introduced once, and afterwards only the variations are given.

More elegant, however, is the *cycle notation*, which describes the effect of subsequent applications of σ. In the example above, you see, for instance, that one application of σ on 1 would yield 2, *i.e.* 2 takes the place of 1. Another application of σ, *i.e.* the application on 2 in the second line, would result in 5 (since $\sigma(2) = 5$). The next application, this time on 5, would put 1 back into place (since $\sigma(5) = 1$). These subsequent applications describe an *orbit* of the permutation σ. Each orbit is presented as a sequence of numbers in parentheses of the form $(x\ \sigma(x)\ \sigma(\sigma(x))\ \sigma(\sigma(\sigma(x)))\ \ldots)$, leaving out the final step where the cycle returns to the original configuration. An element that is fixed under this permutation, *i.e.* that remains at its place, may be presented as an orbit with a single element or left out completely. The permutation σ above in cycle notation is (1 2 5)(3 4). The first orbit describes the following relations: $\sigma(1) = 2$, $\sigma(2) = 5$ and $\sigma(5) = 1$, restoring 1 in its original place. The second orbit describes the simpler relation $\sigma(3) = 4$ and $\sigma(4) = 3$. This describes the permutation σ completely.

Can we devise a Haskell function that performs a permutation given in cycle notation? We first need a function that creates a result list by replacing elements in the original list. Since orbits define substitutions according to the original list, we need to refer to this list, whenever we make a substitution in the result list. Using the result list as a reference, we would, as in the case of 2, substitute a substitution, *e.g.* 2 for 5 at the first place instead of the second place. Here is the *replace* function:

```
replace :: (Eq a) ⇒ a → a → [a] → [a] → [a]
replace _ _ [] _ = []
replace p s (y : ys) (z : zs) | y ≡ p       = s : zs
                              | otherwise = z : replace p s ys zs
```

In this function, p is the element from the original list that will be substituted, the substitute is s. We pass through the original list and the result list in parallel assuming that the result list is initially equal to the original list. When p is found in the original, s is placed at its postition and the function terminates. (Since Haskell lists, in this case, represent sequences that do not contain duplicates, we just terminate after the first substitution.) Otherwise, the value already there at this position in the resulting list is preserved and the search continues.

We will use *replace* in the definition of a function creating permutations according to a

definition in cycle notation. Cycle notation is translated to Haskell as a list of lists, each inner list representing one orbit:

```
type Perm a = [[a]]
```

The *permute* function takes such a *Perm* and a list on which to perform the permutation. An orbit consisting of the empty list or of only one element is the identity and, hence, ignored. Otherwise, one orbit after the other is processed:

```
permute :: (Eq a) => Perm a -> [a] -> [a]
permute []       xs = xs
permute ([]:ps)  xs = permute ps xs
permute ([p]:ps) xs = permute ps xs
permute (p:ps)   xs = permute ps $ orbit (head p) xs p
  where orbit _ rs []          = rs
        orbit x rs [u]         = replace u x xs rs
        orbit x rs (p1:p2:pp) = orbit x (replace p1 p2 xs rs) (p2:pp)
```

For every orbit (that contains more than one element), *permute* is applied to the result of the function *orbit*, which takes the first element of the current orbit, the input list and the current orbit as a whole. The function processes the orbit by replacing the first element by the second, the second by the third and so on. The last element is replaced by the head of the orbit, which, for this purpose, is explicitly passed to the function.

Note that each call to *orbit* and, hence, each recursion step of permute creates a result list, which is then used for the next recursion step. Since orbits do not share elements, no change in the result list made according to one orbit will be touched when processing another orbit; only elements not yet handled by the previous orbits will be changed. It is therefore safe to substitute the input list by the list resulting from processing the previous orbits.

The cyclic notation introduces the idea of composing permutations, *i.e.* applying a permutation on the result of another. The permutation above applied to itself, for instance, would yield $[5,1,3,4,2]$; applying it once again results in $[1,2,4,3,5]$. Six subsequent applications would return to the original list:

let *sigma* = *permute* $[[1,2,5]\ [3,4]]$
sigma $[1,2,3,4,5]$ is $[2,5,4,3,1]$.
sigma $[2,5,4,3,1]$ is $[5,1,3,4,2]$.
sigma $[5,1,3,4,2]$ is $[1,2,4,3,5]$.
sigma $[1,2,4,3,5]$ is $[2,5,3,4,1]$.
sigma $[2,5,3,4,1]$ is $[5,1,4,3,2]$.
sigma $[5,1,4,3,2]$ is $[1,2,3,4,5]$.

The result in the third line is funny: It is almost identical to the original list, but with 3 and 4 swapped. The two orbits of the permutation σ appear to move at different

speed: the first orbit with three elements needs three applications to return to the original configuration; the second orbit with two elements needs only two applications. Apparently, 2 does not divide 3; the orbits are therefore out of sink until the permutation was performed $2 \times 3 = 6$ times.

One could think of systems of permutations (and people have actually done so), such that the application of the permutations within this system to each other, *i.e.* the composition of permutations (denoted: $a \cdot b$), would always yield the same set of sequences. Trivially, all possible permutations of a list form such a system. More interesting are subsets of all possible permutations. Let us simplifiy the original list above to $[1, 2, 3, 4]$, which has 4 elements and, hence, $4! = 24$ possible permutations. On this list, we define a set of permutations, namely

$$e = (1)(2)(3)(4) \tag{4.16}$$
$$a = (1\ 2)(3)(4) \tag{4.17}$$
$$b = (1)(2)(3\ 4) \tag{4.18}$$
$$c = (1\ 2)(3\ 4) \tag{4.19}$$

The first permutation e is just the identity that fixes all elements. The second permutation, a, swaps 1 and 2 and fixes 3 and 4. One application of a would yield $[2, 1, 3, 4]$ and two applications $(a \cdot a)$ would yield the original list again. The third permutation, b, fixes 1 and 2 and swaps 3 and 4. One application of b would yield $[1, 2, 4, 3]$ and two applications $(b \cdot b)$ would yield the original list again. The fourth permutation, c, swaps 1 and 2 and 3 and 4. It is the same as $a \cdot b$, thus creating $[2, 1, 4, 3]$ and $c \cdot c$ would return to the original list. We can now observe that all possible compositions of these permutations, create permuations that are already part of the system:

$$\begin{aligned} a \cdot a = b \cdot b &= e \\ b \cdot a \cdot b \cdot a &= e \\ a \cdot b = b \cdot a &= c \\ c \cdot c &= e \\ c \cdot a \cdot b &= e \end{aligned}$$

You can try every possible combination, the result is always a permutation that is already there. This property of composition of the set of permutations above bears some similarity with natural numbers together with the operations addition and multiplication: The result of an addition or multiplication with any two natural numbers is again a natural number, and the result of a composition of any two permutations in the system is again in the system.

Such systems of permutations, hence, are magmas (as defined in the previous chapter) where the carrier set is the set of permutations and the binary operation is composition. Furthermore, the permutation system fulfils associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c = a \cdot b \cdot c$. So, it is also a semigroup. Since the identity permutation is part of the system, the system is also a monoid and, to be more specific, an abelian monoid, since commutativity, as well, is a property of the composition permutations.

Since we designed the system in a way that every permutation, applied to itself, restores the original sequence, such that $x \cdot y = y \cdot x = e$, there is also an inverse element to every element in the system: the inverse of a permutation is the composition with itself! $a \cdot a = e$. This means that we have found a group!

The set of all possible permutations of a sequence, trivially, is always a group and called the *symmetric group*: since all possible permutations are in the group, every possible composition of two permutations leads to a permutation that is in the group as well, so it is closed under composition; composition, as we have seen, is associative; since, again, all permutations are in the group, there is an identiy element (the permutation that fixes all elements) and, since all possible permutations are in the group, there is for each permutation a permutation that returns to the original configuration, the inverse element. These properties make the symmetric group a group.

But, of course, not all possible subsets of the symmetric group are groups. Subsets of the symmetric group that do not contain the identity are not groups; sets containing permutations that, composed with each other, yield a permutation that is not part of the set are not groups either.

4.5. Random Permutations

We have discussed how we can generate all permutations of a given sequence. But we have not discussed the much more frequent task of creating a *random* permutation of a given sequence.

Algorithms creating random permutations are relatively simple compared to those creating all permutations – if there were not the adjective *random*. Randomness, in fact, is quite a difficult issue in particular when we are thinking of ways to achieve real randomness. Randomness in mathematics is usually defined in terms of a sequence. According to the definition of the great Russian mathematician Andrey Kolmogorov (1903 – 1987) who actually axiomatised and thereby modernised probability theory, a sequence is random, when it cannot be created by a program that, interpreted as a string, is shorter than that sequence. For instance, we could look at any sequence of numbers such as $1, 2, 3, \ldots$ A program to create such a sequence is just $genSeq\ n = n : genSeq\ (n + 1)$ and, obviously, much shorter than the resulting sequence.

When you think of it, it is indeed difficult to create a sequence without any *patterns* in

it, such as regular distances between elements, periodic repetitions and so on. You may think of any of the sequences we have looked at so far: there was always a pattern that led to a way to define a program to generate that sequence and the program was always represented as a finite string of Haskell code that was much shorter than the sequence, which, usually, was infinite. For instance, the definitions of the Fibonacci sequence or of Factorials are much shorter than that sequences, which are infinite. But, even with finite sequences, we have the same principle. Look, for instance, at the sequence $5, 16, 8, 4, 2, 1$, which, on the first sight, appears completely random. However, there is a program that generates this as well as many other similar sequences, namely the *hailstone* algorithm:

$$
\begin{aligned}
&\mathit{hailstone} :: \mathit{Integer} \to [\mathit{Integer}] \\
&\mathit{hailstone}\ 1 = [1] \\
&\mathit{hailstone}\ n \mid \mathit{even}\ n \quad\ = n : \mathit{hailstone}\ (n\ \text{`}\mathit{div}\text{`}\ 2) \\
&\qquad\qquad \mid \mathit{otherwise} = n : \mathit{hailstone}\ (3 * n + 1)
\end{aligned}
$$

One may argue that this code is in fact longer than the resulting sequence. But it would be very easy to encode it in a more concise way, where, for instance, numerical codes represent the tokens of the Haskell language. Furthermore, the code implements the general case that creates the hailstone sequence for any number > 1. For $n = 11$, it is already a bit longer: $11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1$.

The hailstone algorithm, by the way, always terminates with 1, independent of the number n we start with. This is the *Collatz conjecture*, named after the German mathematician Lothar Collatz (1910 – 1990), who posed the problem in 1937. It is unproven and it might be undecidable according to results from John Conway. But that is another story.

Kolmogorov randomness does not only apply to numerical sequences. When we have a sequence of symbols like $a, b, c, d, \ldots$, there is either some regularity or it is not possible to define a program that does not contain the sequence itself and, hence, has no potential to be shorter than the sequence in the first place. The question arises: how do we generate a random sequence, if there is no program that generates it and is significantly shorter than that sequence? Would that not mean that, to generate n bits of randomness, we would need a program that is at least n bits long? Yes, that is basically the case. Any short deterministic program, however this program is implemented, will follow some rules and will eventually create a sequence that still bears traces of that regularity.

The only way to generate true randomness is to pick up numbers from outside of the current problem domain, that is we have to look around to find numbers from other contexts. But, careful: many numbers you see around you still contain regularities. For instance, all numbers generated with the current date as input bear regularity related to the date. It would not be a good idea to use such a date-related number to create, say, a session key for securely encrypted communication through an open channel.

Random number generators implemented in modern operating systems collect numbers that are created by the system while operating. A typical source of randomness is

keystrokes. Every single keystroke creates some data that is stored in a pool for randomness from which other programs can later request some bits of randomness. To get access to true random data, thus, implies that the program requesting those data needs to interact with the operating system. Therefore, whenever we need randomness in Haskell, we need the *IO Monad*. This adds some complexity to our code; but, in fact, this complexity just reflects reality: randomness *is* complex.

In Haskell, there is a module called *System.Random* that provides functions to create random numbers, both *pseudo-random* numbers, which create sequences that appear random on the first sight, but are generated by deterministic algorithms, and true random numbers. Interesting for us in this module is the function *randomRIO*, which creates random objects within a range defined as a tuple. The call *randomRIO* $(0, 9)$, for instance, would create a random number between 0 to 9 (both included). Since *randomRIO* does not know our number type *Natural*, we would have to define a way for *randomRIO* to create random *Natural*s. It is much simpler, however, to use a type known to *randomRIO* and to convert the result afterwards. Here is a simple implementation of a function *randomNatural* that generates a random natural number:

$$
\begin{aligned}
&\mathit{randomNatural} :: (\mathit{Natural}, \mathit{Natural}) \to \mathit{IO\ Natural} \\
&\mathit{randomNatural}\ (l, u) = \mathbf{let}\ \mathit{il} = \mathit{fromIntegral}\ l \\
&\qquad\qquad\qquad\qquad\quad \mathit{iu} = \mathit{fromIntegral}\ u \\
&\qquad\qquad\qquad\quad\ \mathbf{in}\ \ \mathit{fromIntegral} <\$> \\
&\qquad\qquad\qquad\qquad\quad (\mathit{randomRIO}\ (\mathit{il}, \mathit{iu}) :: (\mathit{IO\ Integer}))
\end{aligned}
$$

The range we want the result to lie in is defined by the tuple (l, u), for *lower* and *upper*. We convert the elements of the tuples to *li* and *iu*, which, as we see in an instance, are of type *Integer*. We then call the random number generator with the type signature *IO Integer* defining the output type. This output is finally converted back to natural using *fromIntegral*.

The canonical algorithm for generating random numbers is called *Fisher-Yates shuffle*, after its inventors Ronald Fisher (1890 – 1962) and Frank Yates (1902 –1994), but is also called *Knuth shuffle*, because it became popular through Knuth's masterpiece. The algorithm goes through the sequence we want to permute and, for each index i, that is the place of the element in the sequence starting from 0, it generates a random number j between 0 and $n - 1$, where n is the number of elements in the sequence. If this number is different from the current index, it swaps the elements at positions i and j.

Until now, we have worked only with lists. Lists are extremely efficient, when passing through from the head to the last. Now, however, we need to refer to other places in the list that may be ahead to the end of the sequence or behind closer to its head, depending on the value of j. Also, we have to change the list by going through it. This is essential, because, we might change the same place more than once. For the *fold*-kind of processing that was so typical for the functions we have studied so far, this would be extremely ineffecient. We therefore use another data type, a mutable vector, defined

in *Data.Vector.Mutable*. First, we will look at a function that creates a mutable vector from a list:

```
createVector :: [a] → IO (V.IOVector a)
createVector xs = do v ← V.new (length xs)
                     initV v 0 xs
                     return v
  where initV _ _ []     = return ()
        initV v i (z : zs) = V.unsafeWrite v i z
                           ≫ initV v (i + 1) zs
```

We first create a new vector of the size of the list. Then we initialise this vector just passing through the list in a *map* fashion, but incrementing the index i at each step. We use the vector function *unsafeWrite*, which takes a vector, v, an index, i, and the value to write, z. The function is called *unsafe* because it does not perform a boundary check (and is, as such, much faster than its safe cousin). Since we are careful to move within the boundaries, there is no huge risk involved in using the *unsafe* version of this operation. Finally, we just return the initialised vector.

The next function does the opposite: it converts a vector back to a list:

```
vector2list :: V.IOVector a → Int → IO [a]
vector2list v n = go 0
  where go i | i ≡ n      = return []
             | otherwise = do x ← V.unsafeRead v i
                              (x:) <$> go (i + 1)
```

The function is quite simple. It goes through the vector reading one position after the other and, when it reaches n, just returns the empty list. On each step, the value at position i is read and inserted as the head of the list that results from recursing on *go*. Now we are ready to actually implement the *kshuffle*:

```
kshuffle :: [a] → IO [a]
kshuffle xs = do let n = length xs
                 vs ← createVector xs
                 is ← randomidx n 0
                 go 0 is vs
                 vector2list vs n
  where randomidx n k | k ≡ n      = return []
                      | otherwise = do i ← randomRIO (0, n − 1)
                                       (i:) <$> randomidx n (k + 1)
        go _ [] _       = return ()
        go k (i : is) vs = when (k ≢ i) (V.unsafeSwap vs k i)
                         ≫ go (k + 1) is vs
```

We start by creating the vector using the function *createVector* defined above. Note

that, since we need it more than once, we initially store the size of the list in the variable n. Since we compute it again in *createVector*, there is potential for improvement.

In the next step, we create a list of n numbers using *randomidx*. *randomidx* calls *randomRIO* n times making each result head of the list that is constructed by recursion. Note that we do not use *randomNatural*. We will see in *go* that the results of *randomidx* are used as vector indices and, since vector indices are of type *Int*, we spare some forth and back conversions. *go* expects three arguments: an *Int* called k, a list of *Int*, these are the random indices just created, and the vector on which we are operating. For each index in the list, we swap the value at position k in the vector, which is the index in the natural ordering starting from 0, with the value at position i and continue with the recursion on *go* with $k + 1$ and the tail of the list of random indices. Finally, we call *vector2list* on the manipulated vector yielding a permutation of the input list.

One may be tempted to say that the permutation is generated by a permutation of the indices of the initial list. But do not be fooled! The random indices we are generating do not consitute, at least not necessarily, a valid permutation of the natural ordering of the input list. Each index is generated randomly – completely *independent* of the other indices. In consequence, some of the values we get back from *randomRIO*, in fact, at least theoretically, all of them, may be equal – and this is the whole point of this shuffle.

Consider the input list a, b, c, d, e with the natural ordering of positions $0, 1, 2, 3, 4$, *i.e.* at postion 0, we have a, at position 1, we have b, at position 2, we have c and so on. *randomidx* could result in a list of random indices like, for example, $2, 0, 1, 3, 4$, which would be a permutation of the natural order. However, it may also result in a list like $2, 0, 1, 1, 4$, which is not a permutation. The *kshuffle* algorithm does not require the constraint that the indices we create form a permutation of the initial order. It guarantees that the overall result is actually a permutation of the input list without such a constraint. This saves us from the trouble of checking the result of the random number generator and calling it again each time, there is a collision.

Imagine *randomidx* would create the list $1, 1, 1, 1, 1$, which we could obtain with a probability of $\frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} = \frac{1}{5^5} = \frac{1}{3125}$. We now *go* through the natural positions k, $0 \ldots 4$ and the vector initially representing the list a, b, c, d, e. It is essential to realise that operations on a mutable vector are *destructive*, that is all operations are performed on the current state of the vector, which changes from step to step, such that the output of each step is the input to the next step. What happens is the following:

1. We swap position 0 and 1 resulting in b, a, c, d, e;
2. We do not do anything, because the indices k and i are both 1 in the second step, maintaining b, a, c, d, e;
3. We swap positions 2 and 1 resulting in b, c, a, d, e;
4. We swap positions 3 and 1 resulting in b, d, a, c, e;

5. We swap positions 4 and 1 resulting in b, e, a, c, d,

resulting overall in a valid permutation of the input list.

4.6. Binomial Coefficients

Closely related to permutations are problems of selecting a number of items from a given set. Whereas permutation problems have the structure of shuffling cards, selection problems have that of dealing cards. This analogy leads to an intuitive and simple algorithm to find all possible selections of k out of a set with n elements by taking the first k objects from all possible permutations of this set and, afterwards, removing the duplicates. Consider the set $\{1, 2, 3\}$ and let us find all possible selections of two elements of this set. We start by choosing the first two elements of the given sequence and get $\{\{1, 2\}\}$. Now we create a permutation: $\{2, 1, 3\}$ and, again, take the first two elements. The result set is now: $\{\{1, 2\}, \{2, 1\}\}$. We continue with the next permuation $\{2, 3, 1\}$, which leads us to the result set $\{\{1, 2\}, \{2, 1\}, \{2, 3\}\}$. Going on this way – and we already have defined an algorithm to create all possible permutations of a set in the previous section – we finally get to the result set $\{\{1, 2\}, \{2, 1\}, \{2, 3\}, \{3, 2\}, \{3, 1\}, \{1, 3\}\}$. Since, as we know from the previous section, there are $3! = 6$ permutations, there are also six sequences with the first k elements of these six permutations. But, since we want unique selections, not permutations of the same elements, we now remove the duplicates from this result set and arrive at $\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$, that is three different selections of two elements out of three.

This algorithm suggests that the number of k selections out of n elements is somehow related to the factorial function. But, obviously, the factorial is too big a result, we have to reduce the factorial by the number of the permutations of the results. Let us think along the lines of permutation: we have 3 ways to select 1 object out of 3: $\{\{1\}, \{2\}, \{3\}\}$. For the factorial, we said that we now combine all permutations of the remaining 2 objects with this 3 possible solutions and compute the number of these permutations as 3×2. However, since order does not matter, the first selection conditions the following selections. After the first step, we seemingly have two options for each of the first selections in step 2:

step 1	step 2
1	$\{2, 3\}$
2	$\{1, 3\}$
3	$\{1, 2\}$

But note that, when we select 2 in the first row, the option 1 in the second row will vanish, since we already selected $\{1, 2\}$, which is the same as $\{2, 1\}$. Likewise, when we select 3 in the first row, we cannot select 1 in the third row because, again, $\{1, 3\}$ is the

same as $\{3, 1\}$. It, therefore, would be much more appropriate to represent our options as in the following table:

step 1	step 2
1	$\{2, 3\}$
2	$\{3\}$
3	$\{\}$

At the beginning, we are completely free to choose any element, but when we come to the second, the options are suddenly reduced and at the third step there are no options left at all. For the case 2 out of 3, we see that the first selection halves our options in the second step. This suggests that we have to divide the number of options per step. With permutation, we had $n \times (n-1)$, but with selection, we apparently have something like $n \times \frac{n-1}{2}$, which, for the case 2 out of 3, is $3 \times \frac{3-1}{2} = 3 \times \frac{2}{2} = 3 \times 1 = 3$. When we continue this scheme, considering that each choice that was already made conditions the next choice, we get a product of the form: $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \ldots$ Selecting 3 out of 5, for instance, is: $\frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} = 10$. This leads to the generalised product for k out n: $\frac{n}{1} \times \frac{n-1}{2} \times \cdots \times \frac{n-(k-1)}{k}$. This product is known as the binomial coefficient $\binom{n}{k}$ pronounced *n choose k*.

We easily see that the part below the fraction line is $k!$ The part above the line is a partial factorial of n, called falling factorial or *to-the-k^{th}-falling*:

$$n^{\underline{k}} = n \times (n-1) \times \cdots \times (n-k+1) = \prod_{j=1}^{k} n + 1 - j. \tag{4.20}$$

We, therefore, can represent the binomial coefficient as either:

$$\binom{n}{k} = \prod_{j=1}^{k} \frac{n+1-j}{j} \tag{4.21}$$

or:

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}. \tag{4.22}$$

But there is still another formula, which, even though less efficient in terms of computational complexity, is often used to ease proofs involving binomial coefficients and which is closer to our first intuition that the selection is somehow related to factorials reduced by some value:

$$\binom{n}{k} = \frac{n!}{k! \times (n-k)!}. \tag{4.23}$$

It can be seen immediately that this formula is equivalent to equation 4.22, whenever $k \leq n$, since the values of $n!$ in the numerator cancel out with the values of $(n-k)!$ in the denominator. Indeed, $n!$ could be split into two halves (which are not necessarily equal of course), the upper product $n^{\underline{k}}$ $(n \times (n-1) \times \cdots \times (n-k+1))$ and the lower product $(1 \times 2 \times \cdots \times (n-k))$. By cancelling out the lower half, we remove the lower product from numerator and denominator and are left with the falling factorial in the numerator.

We could have derived equation 4.22 much more easily with a different kind of reasoning: Given a set with n elements, there are $n^{\underline{k}}$ permutations of k elements of this set. There are n ways to choose the first element, $n-1$ ways to choose the second element and so on and $n-k+1$ ways to choose the k^{th} element. Obviously, we could reach the same result, all permutations of k elements out of n, by first selecting these k elements and then create all possible permutations of these k elements. The number of possibilities of choosing k out of n is the binomial coefficient, $\binom{n}{k}$, which we would like to derive. The possible permutations of these k elements is of course $k!$ We now have to combine these two steps: We have for any selection of k elements out of n $k!$ permutations, that is $\binom{n}{k} \times k!$ Since this processing has the same result as choosing all permutations of k out of n in the first place, we come up with the equation:

$$n^{\underline{k}} = \binom{n}{k} \times k! \tag{4.24}$$

To know what the expression $\binom{n}{k}$ is we just divide $k!$ on both sides of the equation and get equation 4.22:

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}. \tag{4.25}$$

Let us look at some concrete values of the binomial coefficients: $\binom{n}{0} = \binom{n}{n} = 1$ and for $k < 0$ or $k > n$: $\binom{n}{k} = 0$. For $0 \leq k \leq n$, for instance: $\binom{3}{2} = 3$, $\binom{4}{2} = 6$, $\binom{4}{3} = 4$, $\binom{5}{2} = 10$, $\binom{5}{3} = 10$. We can arrange the results in a structure, called Pascal's Triangle, after the great French mathematician and philosopher Blaise Pascal (1623 – 1662) who used binomial coefficients to investigate probabilities and, in the process, created a new branch of mathematics, namely probability theory:

| | | |

deed, this relation is one of the most important theorems in mathematics, the *binomial theorem*, which we will formulate in a second. First, let us look at the multiplication pattern. The distributive law tells us that

$$(a+b)(c+d) = ac + ad + bc + bd. \tag{4.28}$$

Now, what happens if $a = c$ and $b = d$? We would then get:

$$(a+b)(a+b) = aa + ab + ba + bb, \tag{4.29}$$

which is the same as

$$(\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b}) = \mathbf{a^2} + \mathbf{2ab} + \mathbf{b^2}. \tag{4.30}$$

When we now multiply $(a+b)$ with this result, we get:

$$\begin{aligned}(a+b)(a^2+2ab+b^2) = a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3 = \\ a^3 + 2a^2b + ba^2 + ab^2 + 2ab^2 + b^3 = \\ \mathbf{a^3 + 3a^2b + 3ab^2 + b^3}\end{aligned} \tag{4.31}$$

Multiplied with $(a+b)$ once again:

$$\begin{aligned}(a+b)(a^3+3a^2b+3ab^2+b^3) = \\ a^4 + 3a^3b + 3a^2b^2 + ab^3 + ba^3 + 3a^2b^2 + 3ab^3 + b^4 = \\ a^4 + 3a^3b + ba^3 + 3a^2b^2 + 3a^2b^2 + ab^3 + 3ab^3 + b^4 = \\ \mathbf{a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4}\end{aligned} \tag{4.32}$$

The coefficients in these formulas, as you can see, equal the binomial coefficients in Pascal's Triangle. The Triangle can thus be interpreted as results of power functions:

$$\begin{aligned}(a+b)^0 &= 1 \\ (a+b)^1 &= 1a + 1b \\ (a+b)^2 &= 1a^2 + 2ab + 1b^2 \\ (a+b)^3 &= 1a^3 + 3a^2b + 3ab^2 + 1b^3 \\ (a+b)^4 &= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4\end{aligned}$$

$$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

$$\dots$$

This, in general, is the binomial theorem:

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \dots + \binom{n}{n}x^0 y^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\end{aligned} \tag{4.33}$$

But why is this so? According to multiplication rules, the multiplication of two factors $(a+b)(c+d)$ yields a combination of each of the terms of one of the factors with the terms of the other factor: $ac + ad + bc + bd$. If $a = c$ and $b = d$, we will create combinations of terms with themselves: $aa + ab + ba + bb$. How many ways are there to combine a with a in $(a + b)(a + b)$? There is exactly one way, because the a of the first factor will find exactly one a in the second factor. But how many ways are there to combine a and b? Well, the a in the first factor will find one b in the second, and the b in the first factor will find one a in the second. There are hence two ways to combine a and b and we could interpret these two combinations as two different *strings*, the string ab and the string ba. We know that there are $\binom{2}{1} = 2$ different ways to select one of these strings: either ab or ba. Since these strings represent products of a and b and, according to the commutative law, the order of the factors does not matter, we can just add them up, which leaves us with a coefficient that states exactly how many strings of homogeneous as and bs there are in the sum.

There is a nice illustration of this argument: Let us look at the set of the two numbers $\{1, 2\}$. There are two possibilities to select one of these numbers: 1 or 2. Now, we could interpret these numbers as answer to the question "What are the positions where one of the characters 'a' and 'b' can be placed in a two-character string?" The answer is: either at the beginning or at the end, *i.e.* either $\mathbf{a}b$ or $b\mathbf{a}$. For $(a + b)^3$, this is even more obvious. Compare the positions of the a's in terms with two a's with the possible selections $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ of two out of the set $\{1, 2, 3\}$: $(a+b)(aa+ab+ba+bb) = aaa + \mathbf{aa}b + a\mathbf{a}b + abb + b\mathbf{aa} + bab + bba + bb$.

This is a subtle argument. To assure ourselve that the theorem really holds for all n, we should try a proof by induction. We have already demonstrated that it indeed holds for several cases, like $(a + b)^0$, $(a + b)^1$, $(a + b)^2$ and so on. Any of these cases serves as base case. Assuming the base case holds, we will show that

$$(a+b)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n+1-k}. \tag{4.34}$$

We start with the simple equation

$$(a+b)^{n+1} = (a+b)^n(a+b) \tag{4.35}$$

and then reformulate it replacing $(a+b)^n$ by the base case:

$$(a+b)^{n+1} = \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\right)(a+b). \tag{4.36}$$

We know that, to multiply a sum with another sum, we have to distribute all the terms of one sum over all terms of the second sum. This is, we multiply a with the summation and then we multiply b with the summation. In the first case, the exponents of a within the summation are incremented by one, in the second case, the exponents of b are incremented by one:

$$(a+b)^{n+1} = \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k}. \tag{4.37}$$

The second term looks already quite similar to the case in equation 4.34, both have $a^k b^{n+1-k}$. Now, to make the first term match as well, we will use one of those *tricks* that make many feel that math is just about pushing meaningless symbols back and forth. Indeed, since we are working with sums here, the proof involves much more technique than the proofs we have seen so far. The purpose, however, is still the same: we want to show that we can transform one formula into another by manipulating these formulas according to simple grammar rules. That this has a very technical, even *tricky* flavour is much more related to the limitations of our mind that does not see through things as simple as numbers, but has to create formal apparatus not to get lost in the dark woods of reasoning.

Well, what is that trick then? The trick consists in raising the k in the summation index and to change the terms in the summation formula accordingly, that is, instead of a^{k+1}, we want to have a^k and we achieve this, by not letting k run from 0 to n, but from 1 to $n+1$:

$$(a+b)^{n+1} = \sum_{k=1}^{n+1} \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k}. \tag{4.38}$$

Please confirm for yourself with pencil and paper that the first summation in equations 4.37 and 4.38 is the same:

$$\sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k}=\sum_{k=1}^{n+1}\binom{n}{k-1}a^{k}b^{n+1-k}$$

All we have done is pushing the index of the summation one up and, to maintain the value of the whole, reducing k by one in the summation formula.

Now we want to combine the two sums, but, unfortunately, after having pushed up the summation index, the two sums do not match anymore. Apparently, while trying to solve one problem, we have created another one. But hold on! Let us try a bit and just take the case $k = n + 1$ in the first term and the case $k = 0$ in the second term out. The case $k = n + 1$ corresponds to the expression $\binom{n}{n+1-1}a^{n+1}b^{n+1-(n+1)}$, which, of course, is simply a^{n+1}, since $\binom{n}{n+1-1} = \binom{n}{n} = 1$ and $b^{n+1-(n+1)} = b^{n+1-n-1} = b^0 = 1$. Accordingly, the case $k = 0$ in the second term corresponds to $\binom{n}{0}a^0b^{n+1-0} = b^{n+1}$. When we combine all those again, we get to:

$$(a+b)^{n+1}=a^{n+1}+\sum_{k=1}^{n}\binom{n}{k-1}a^{k}b^{n+1-k}+\sum_{k=1}^{n}\binom{n}{k}a^{k}b^{n+1-k}+b^{n+1}. \tag{4.39}$$

The next step provides you with a test of how well you have internalised the distributive law. The sum of the two summations has the form: $(\alpha c + \alpha d) + (\beta c + \beta d)$, where $\alpha = \binom{n}{k-1}$ and $\beta = \binom{n}{k}$ and c and d represent different steps of the summations, *i.e.* c is $a^k b^{n+1-k}$ for $k = 1$ and d the same for $k = 2$ and so on. Please make sure that you see this analogy!

By applying the distributive law once, we get to $\alpha(c + d) + \beta(c + d)$. This is really fundamental – please make sure you get to the same result by distributing α and β over their respective $(c + d)$!

Now we apply the distributive law once again taking $(c + d)$ out: $(\alpha + \beta)(c + d)$. Please make sure again that this holds for you by distributing $(c + d)$ over $(\alpha + \beta)$!

When we substitute α and β by the binomial coefficients, we get $(\binom{n}{k-1} + \binom{n}{k})(c + d)$, right? In the next equation, we have just applied these little steps:

$$(a+b)^{n+1}=a^{n+1}+\sum_{k=1}^{n}\left(\binom{n}{k-1}+\binom{n}{k}\right)a^{k}b^{n+1-k}+b^{n+1}. \tag{4.40}$$

Now you might recognise Pascal's rule given in equation 4.27 above. Indeed, the Almighty Triangle tells us that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. In other words, we can simplify the equation to

$$(a+b)^{n+1} = a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} + b^{n+1}. \tag{4.41}$$

Finally, we integrate the special cases $k = 0$ and $k = n + 1$ again, just by manipulating the summation index:

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k} \tag{4.42}$$

and we are done, since – using some mathematical trickery – we have just derived equation 4.34.

Coming back to the question of how to implement binomial coefficients efficiently, we should compare the two alternatives we have already identified as possible candidates, *viz.* equations 4.21 and 4.22, which are repeated here for convenience:

$$\binom{n}{k} = \prod_{j=1}^{k} \frac{n+1-j}{j}, \tag{4.43}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}. \tag{4.44}$$

The first option performs k divisions and $k-1$ multiplications: one division per step and the multiplications of the partial results, that is $k + k - 1 = 2k - 1$ operations in total, not counting the sum $n + 1 - j$, which is a minor cost factor.

The second option performs $k-1$ multiplications for $n^{\underline{k}}$, $k-1$ multiplications for $k!$ and one division, hence, $k - 1 + k - 1 + 1 = 2k - 1$ operations. That looks like a draw.

In general terms, there is an argument concerning implementation strategy in favour of the first option. With the second option, we first create two potentially huge values that must be kept in memory, namely $n^{\underline{k}}$ and $k!$. When we have created these values, we reduce them again dividing one by the other. The first option, in contrast, builds the final result by stepwise incrementation without the need to create values greater than the final result. So, let us implement the first option:

```
choose :: Natural → Natural → Natural
choose n 0 = 1
choose n 1 = n
choose n k | k > n      = 0
           | 2 * k > n  = choose n (n − k)
           | otherwise = go 1 1
```

$$\begin{aligned}\textbf{where } go\ m\ i \mid i > k &= m \\ \mid otherwise &= go\ (m * (n - k + i))\ `div`\ i)\ (i + 1)\end{aligned}$$

The implementation is straight forward. The function *choose* is defined over natural numbers, so we do not have to deal with negative numbers. The first parameter corresponds to n, the second one to k in $\binom{n}{k}$. Whenever $k = 0$, the result is 1 and if $k = 1$, the result is n. For all other cases, we first test if $k > n$; if so, the result is just 0. Otherwise, we make the distinction $2k > n$ and if so, we calculate *choose* for n and $n - k$, *e.g.* $\binom{5}{4} = \binom{5}{1}$. Otherwise we build the product using the function *go* that is defined as follows: If $i > k$, we use m, otherwise we recurse with the result of $\frac{m \times (n-k+i)}{i}$ and $i + 1$.

Let us look at the example $\binom{5}{3}$. We start with *go* 1 1, which expands to

$go\ (1 * (5 - 3 + 1)\ `div`\ 1)\ (1 + 1)$,

which is *go* 3 2. This, in its turn, expands to

$go\ (3 * (5 - 3 + 2)\ `div`\ 2)\ (2 + 1)$,

which equals *go* 6 3, expands to

$go\ (6 * (5 - 3 + 3)\ `div`\ 3)\ (3 + 1)$

and results in *go* 10 4. Since i is now greater than k, $4 > 3$, we just get back m, which is 10 and, thus, the correct result.

A word of caution might be in place here. The *choose* function above does not implement the product in the formula one-to-one. There is a slight deviation, in that we multiply the result of the previous step with the sum of the current step, before we apply the division. The reason becomes obvious, when we look at $\binom{6}{3}$, for instance. According to the formula, we would compute $\frac{6+1-1=6}{1} \times \frac{6+1-2=5}{2} \times \ldots$. The second factor, $\frac{5}{2}$, is not a natural number – we cannot express this value with the only tool we own so far. The result however is the same, which you can prove to yourself simply by completing the product above and comparing your result with the All-knowing Triangle. We will investigate binomial coefficients more deeply, especially the question why they always result in an integer in spite of division being involved.

4.7. Combinatorial Problems with Sets

Many real-world problems can be modelled in terms of *sets*. We have already used sets informally and, indeed, they are of tremendous importance in the whole field of mathematics – set theory is even believed to provide a sound fundamentation for most areas of mathematics. This, however, is strongly contested since more than hundred years now and today there are other candidates for this role besides set theory. But today many, if not most mathematicians, after long battles over the foundations of math

mainly during the first half of the 20^{th}, are tired of discussing these issues.

Anyway, what is a set in the first place? Georg Cantor (1845 – 1918), one of the main inventors of set theory, provided several definitions, for instance: "a set is a Many that allows being thought of as a One" or: "a collection of distinct objects". Both definitions are quite abstract, but, in this respect, they express a major aspect of set theory quite well.

The second definition, "collection of distinct objects", serves our purposes well enough. A set can consist of any kind of objects, as long as these objects can be clearly distinguished. Examples are: The set of all green things, the set of all people, The set of Peter, Paul and Mary, the set of all animals that belong to the emperor, the set of the natural numbers from 1 to 9 the set of all natural numbers and so on.

There are different ways to define sets. We can first give a definition: the set of the natural numbers from 1 to 9, the set of all people, the members of the Simpsons family, *etc.* But we can also name the members explicitly: Homer, Marge, Bart, Lisa and Maggie or $1, 2, 3, 4, 5, 6, 7, 8, 9$.

The first way to define a set is called by *intension*. The intension of a set is what it implies, without referring explicitly to its members. The second way is called by *extension*. The extension of a set consists of all its members.

This distinction is used in different ways of defining lists in Haskell. One can define a list by extension: `[1,2,3,4,5]` or by intension: `[1..5]`, `[1..]`. A powerful tool to define lists by intension is list comprehension, for instance:

$[x \mid x \leftarrow [1\,..\,100], even\ x, x\ `mod`\ 3 \not\equiv 0]$,

which would contain all even numbers between 1 and 100 that are not multiples of 3.

Defining sets by intension is very powerful. The overhead of constructing a set by extension, *i.e.* by naming all its members, is quite heavy. If we had to mention all numbers we wanted to use in a program beforehand, the code would become incredibly large and we would need to work on it literally an eternity. Instead, we just define the kind of objects we want to work with. However, intension bears the risk of introducing some mind-boggling complications, one of which is infinite sets. For the time being, we will steer clear of any of these complications. We have sufficient work with the kind of math that comes without fierce creatures like infinity.

Sets, as you have already seen, are written in braces like, for instance: $\{1, 2, 3\}$. The members of a set, here the numbers 1, 2 and 3, are called elements of this set, it holds true, for example, that $1 \in \{1, 2, 3\}$ and $0 \notin \{1, 2, 3\}$.

A similar relation is *subset*. A set A is subset of another set B, iff all elements of A are also in B: $\{1\} \subseteq \{1, 2, 3\}$, $\{1, 3\} \subseteq \{1, 2, 3\}$ and also $\{1, 2, 3\} \subseteq \{1, 2, 3\}$. The last case is interesting, because it asserts that every set is subset of itself. To exclude this case and only talk about subsets that are smaller than the set in question, we refer to the

proper or *strict subset*, denoted as $A \subset B$. An important detail of the subset relation is that there is one special set that is subset of any set, *viz.* the *empty set* $\{\}$ that does not contain any element and which is often denoted as $\varnothing$,

As you can see in the example $\{1, 2, 3\}$, a set may have many subsets. The set of all possible subsets of a set is called the *powerset* of this set, often written $P(S)$ for a set S. The powerset of $\{1, 2, 3\}$, for example, is: $\{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Does this remind you of something? Perhaps not yet. What if I was to ask: how many elements are there in the powerset of a set with n elements? Well, there is the empty set, the set itself, then sets with one element, sets with two elements and so on. How many sets with k elements are there in the powerset of a set with n elements? The answer is: there are as many sets of k elements as there are ways to select k items out of n. In other words, the size of the powerset equals the sum of all binomial coefficients $\binom{n}{k}$ for one specific value of n, *i.e.* the sum of all values in one row of Pascal's Triangle. For $n = 0$, n is the number of elements of the set, we have: $\binom{0}{0} = 1$, since the only subset of $\varnothing$ is $\varnothing$. For $n = 1$, we have: $\binom{1}{0} + \binom{1}{1} = 2$. For $n = 2$, we have: $\binom{2}{0} + \binom{2}{1} + \binom{2}{2}$, which is $1 + 2 + 1 = 4$. For $n = 3$, we have: $\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$, which is $1 + 3 + 3 + 1 = 8$, for $n = 4$, we have: $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$, which is $1 + 4 + 6 + 4 + 1 = 16$.

Probably, you already see the pattern. For 5 elements, there are 32 possible subsets; for 6 elements, there are 64 subsets, for 7, there are 128 and for 8 there are 256 subsets. In general, for a set with n elements, there are 2^n subsets and, as you may confirm in the Triangle in the previous section, the sum of all binomial coefficients in one row of the Triangle is also 2^n. This, in its turn, implies that the sum of the coefficients in an expression of the form $a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$, as well, is 2^n.

Is there a good algorithm to construct the powerset of a given set? There are in fact many ways to build the powerset, some more efficient or more elegant than others, but really *good* in the sense that it efficiently creates powersets of arbitrarily large sets is none of them. The size of the powerset increases exponentially in the size of the input set, which basically means that it is not feasible at all to create the powerset in most cases. The powerset of a set of 10 elements, for instance, has 1024 elements. That of a set of 15 elements has already 32 768 and a set of 20 elements has more than a million.

Here is a Haskell implementation of a quite elegant and simple algorithm:

```
ps :: (Eq a) ⇒ [a] → [[a]]
ps [] = [[]]
ps (x : xs) = ps xs ++ map (x:) (ps xs)
```

Note that we use lists instead of sets. There is a set module in Haskell, but since we will not work too much with sets, we stick to lists with the convention that there should be no duplicates in lists that represent sets.

Let us see how the *ps* function works for the input $\{1, 2, 3\}$. We start with *ps* $(1 : [2, 3])$

and immediately continue with $ps\ (2 : [3])$ and, in the next round, with $ps\ (3 : [])$, which then leads to the base case $ps\ [] = [[]]$. On the way back, we then have $[[]] +\!\!+ map\ (3:)\ [[]]$, which leads to the result $[[], [3]]$. This, one step further back, leads to $[[], [3]] +\!\!+ map\ (2:)\ [[], [3]]$, which results in $[[], [3], [2], [2, 3]]$. In the previous step, we then have: $[[], [3], [2], [2, 3]] +\!\!+ map\ (1:)\ [[], [3], [2], [2, 3]]$, resulting in $[[], [3], [2], [2, 3], [1], [1, 3], [1, 2], [1, 2, 3]]$ and we are done.

A completely different approach is based on the observation that there are 2^n possible subsets of a set with n elements and this happens to be the number of values one can represent with a binary number of length n, namely the values $0 \ldots n - 1$. Binary numbers, which we will discuss in more detail later, use only the digits 0 and 1 instead of the digits $0 \ldots 9$ as we do with decimal numbers. In binary numbers we would count like

binary	decimal
0	0
1	1
10	2
11	3
100	4
101	5
110	6
111	7
1000	8
1001	9
1010	10
1011	11
1100	12
1101	13
1110	14
1111	15

Indeed, there are 10 kinds of people in the world: those who understand binary numbers and those who do not.

To construct the powerset of a set of n elements, we can use binary numbers with n digits. We would loop over this numbers starting from 0 and move up to the greatest number representable with n digits. For $n = 3$, we would loop through: 000, 001, 010, 011, 100, 101, 110, 111. Each of these numbers describes one subset of the input set, such that the k^{th} digit of each number would tell us, whether the k^{th} element of the input set is part of the current subset. The number 000 would indicate the empty set. The number 001 would indicate that the first element is in the set: $\{1\}$. The number 010 would indicate that the second element is in the set: $\{2\}$. The number 011 would indicate that the first and the second element are in the set: $\{1, 2\}$ and so on.

An important issue to gain any speed advantage by this scheme is how to map the binary numbers to elements in the input set. We could naïvely use an underlying representation of binary numbers like lists – as we have done for our natural numbers – iterate through these lists and, every time we find a 1, add the corresponding element of the input set to the current subset. But this would mean that we had to loop through 2^n lists of length n. That does not sound very efficient.

The key is to realise that we are talking about numbers. We do not need to represent binary numbers as lists at all. Instead, we can just use decimal numbers and extract the positions where, in the binary representation of each number, there is a 1.

To illustrate this, remember that the value of a decimal number is computed as a sum of powers of 10: $1024 = 1 \times 10^3 + 0 \times 10^2 + 2 \times 10^1 + 4 \times 10^0$. The representation of 1024 as powers of two is of course much simpler: $1024 = 1 \times 2^{10} + 0 \times 2^9 + 0 \times 2^8 + \cdots + 0 \times 2^0$ or, for short: $1024 = 2^{10}$. Let us look at a number with a simple decimal representation like 1000, which, in powers of 10, is simply: 10^3. Represented as powers of two, however: $2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3$, which is $512 + 256 + 128 + 64 + 32 + 8 = 1000$.

The point is that the exponents of 1000 represented as powers of two indicate where the binary representation of 1000 has a 1. 1000 in the binary system, indeed, is: 1111101000, whereas 1024 is 10000000000. Let us index these numbers, first 1000:

10	9	8	7	6	5	4	3	2	1	0
0	1	1	1	1	1	0	1	0	0	0

and 1024:

10	9	8	7	6	5	4	3	2	1	0
1	0	0	0	0	0	0	0	0	0	0

You see that the indexes in the first row that have a 1 in the second row correspond to the exponents of the powers of two that sum up to the respective number, *i.e.* 9, 8, 7, 6, 5 and 3 for 1000 and 10 for 1024. We can, hence, interpret the exponents of the powers of two as indexes into the set for which we want to construct the powerset. Think of the input set as an array in a language like C, where we can refer to an element of the set directly by addressing the memory cell where is resides: `x = set[0];` for instance, would give us the first element of the set.

When we look at how a number, say d, is computed as sum of powers of two, we can derive the following algorithm: compute the greatest power of two that is smaller than or equal to d and than do the same with the difference between d and this power of two until the difference, which in analogy to division, we may call the remainder, is zero. The greatest power of two $\leq d$ is just the log base 2 of this number rounded down to the next natural number. A function implementing this in Haskell would be (cheating on our number type by using floating point numbers):

$$\begin{aligned}&\mathit{natLog} :: \mathit{Natural} \to \mathit{Natural} \to (\mathit{Natural}, \mathit{Natural})\\&\mathit{natLog}\ b\ n = \textbf{let}\ e = \mathit{floor}\ \$\ \mathit{logBase}\ (\mathit{fromIntegral}\ b)\end{aligned}$$

```
                                        (fromIntegral n)
        in (e, n - b ↑ e)
```

This function takes a natural number b, the base, and a natural number n. The result consists of two numbers (e, r) that shall fulfil the condition $n = b^e + r$.

We can use this function to obtain all exponents of the powers of two that sum up to a given number:

```
binExp :: Natural → [Natural]
binExp 0 = []
binExp 1 = [0]
binExp n = let (e, r) = natLog 2 n in e : binExp r
```

That is, for input 0 and 1, we explicitly define the results [] and [0]. Here, [] means that there is no 1 in the binary representation and [0] means that there is 1 at the first position (indexed by 0). For any other number n, we calculate the exponent e of the greatest power of two $\leq n$ and the remainder r and add e to the list that will result from applying *binExp* to r. Let us look at the example 1000. We start with *natLog* 2 1000:

binExp $1000 = (9, 1000 - 2^9 = 1000 - 512 = 488)$
$9 :$ *binExp* $488 = (8, 488 - 2^8 = 488 - 256 = 232)$
$9 : 8 :$ *binExp* $232 = (7, 232 - 2^7 = 232 - 128 = 104)$
$9 : 8 : 7 :$ *binExp* $104 = (6, 104 - 2^6 = 104 - 64 = 40)$
$9 : 8 : 7 : 6 :$ *binExp* $40 = (5, 40 - 2^5 = 40 - 32 = 8)$
$9 : 8 : 7 : 6 : 5 :$ *binExp* $8 = (3, 8 - 2^3 = 8 - 8 = 0)$
$9 : 8 : 7 : 6 : 5 : 3 :$ *binExp* $0 = []$
$9 : 8 : 7 : 6 : 5 : 3 : []$,

which, indeed, is the list of the exponents of the powers of two that add up to 1000.

Now we need a function that loops through all numbers $0 \ldots 2^n - 1$, calculates the exponents of the powers of two for each number and then retrieves the elements in the input set that corresponds to the exponents:

```
ps2 :: (Eq a) ⇒ [a] → [[a]]
ps2 [] = [[]]
ps2 xs = go (2 ↑ (length xs) - 1) 0
  where go n i | i ≡ n       = [xs]
               | otherwise = let s = map exp2idx $ binExp i
                             in s : go n (i + 1)
        exp2idx x = xs !! (fromIntegral x)
```

The function *ps*2 returns just a set that contains the empty set when called with the empty set. Otherwise, it enters a loop with two parameters: $2^{(length\ xs)} - 1$, which is

the greatest number that can be represented with a binary number with n digits, when we start to count at 0, and the number we start with, namely 0. For each number i: if we have reached the last number, we just know the corresponding subset is the input set itself. Otherwise, we map the function $exponent \rightarrow index$ to the result of $binExp$ applied to the current number i. The mapping function, $exp2idx$, uses the list index operator `!!` to get the element of the input list xs at the position x, which is just an exponent. (Note that we have to convert x from $Natural$ to Int, since `!!` expects an Int value.)

This algorithm exploits a fascinating *isomorphism* – an analogous structure – between binary numbers and powersets. With an appropriate data structure to represent sets, like $Vector$, and, of course, a more efficient number representation than our humble natural numbers, the algorithm definitely beats the one we implemented as ps. Furthermore, this algorithm can be paralellised according to number ranges, which is not possible with the previous algorithm, since, there, results depend on inermediate results, such that each step builds on a predecessor.

Unfortunately, lists show very bad performance with random access such as indexing. Therefore, $ps2$ is slower than ps. But using Haskell vectors (implemented in module $Data.Vector$) and Integers instead of our $Natural$, $ps2$ is indeed faster. The changes, by the way, are minimal. Just compare the implementation of $ps2$ and psv:

```
psv :: (Eq a) ⇒ [a] → [[a]]
psv [] = [[]]
psv xs = let v = V.fromList xs
         in go v (2 ↑ (length xs) − 1) 0
  where go v n i | i ≡ n      = [xs]
                 | otherwise = let s = map (exp2idx v) (binExp i)
                               in s : go v n (i + 1)
        exp2idx v x = v ! (fromIntegral x)
```

The changes to the code of $ps2$ relate to the introduction of v, a vector created from xs by using the $fromList$ function from the vector module, which is qualified as V.

In practical terms, however, the performance of the powerset function does not matter too much, since, as already said, it is not feasible to compute the powerset of large sets anyway. Nevertheless, problems related to subsets are quite common. An infamous example is the *set cover* problem.

The challenge in the set cover problem is to combine given subsets of a set A so that the combined subsets together equal A. This involves an operation on sets we have not yet discussed. Combining sets is formally called *union*: $A \cup B$. The union of two sets, A and B, contains all elements that are in A or B (or both), for example: $\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$.

Two other important set operations are intersection and difference. The intersection

of two sets A and B, $A \cap B$, contains all elements x, such that $x \in A$ and $x \in B$. To continue with the example used above: $\{1,2,3\} \cap \{3,4,5\} = \{3\}$. The intersection of the union of two sets with one of these sets is just that set, $(A \cup B) \cap A = A$: $(\{1,2,3\} \cup \{3,4,5\}) \cap \{1,2,3\} = \{1,2,3,4,5\} \cap \{1,2,3\} = \{1,2,3\}$.

The difference of two sets A and B, $A \setminus B$, contains all elements in A that are not in B, for example: $\{1,2,3\} \setminus \{3,4,5\} = \{1,2\}$. If B is a subset of A, then the difference $A \setminus B$ is called the *complement* of B in A.

Now let us model the three set operations union, intersection and difference with Haskell lists. The simplest case is difference, since, assuming that we always use lists without duplicates, we can just use the predefined list operator $\backslash\backslash$. Union is not too difficult either using the function *nub*, which removes duplicates from a list:

$$
\begin{aligned}
&union :: (Eq\ a) \Rightarrow [a] \to [a] \to [a] \\
&union\ a\ b = nub\ (a +\!\!+ b)
\end{aligned}
$$

Using *nub* is necessary, since merging the two lists will introduce duplicates for any $x \in a$ and $x \in b$.

Intersect is slightly more difficult. We could implement intersect by means of *nub*; we used *nub* in *union* to remove the duplicates of exactly those elements that we want to have in intersect. The intersect, hence, could be implemented as $a +\!\!+ b \setminus\setminus nub\ (a +\!\!+ b)$. This would define the intersect as the difference of the concatenation of two lists and the union of these two lists. Have a look at the example $A = \{1,2,3\}$ and $B = \{3,4,5\}$:

$A +\!\!+ B \setminus\setminus nub\ (A +\!\!+ B) =$
$[1,2,3] +\!\!+ [3,4,5] \setminus\setminus nub\ ([1,2,3] +\!\!+ [3,4,5]) =$
$[1,2,3,3,4,5] \setminus\setminus nub\ ([1,2,3,3,4,5]) =$
$[1,2,3,3,4,5] \setminus\setminus [1,2,3,4,5] =$
$[3]$.

This implementation, however, is not very efficient. Preferable is the following one:

$$
\begin{aligned}
&intersect :: (Eq\ a) \Rightarrow [a] \to [a] \to [a] \\
&intersect\ [\,]\ _ = [\,] \\
&intersect\ _\ [\,] = [\,] \\
&intersect\ (a : as)\ bs \mid a \in bs \quad\ = a : intersect\ as\ bs \\
&\qquad\qquad\qquad\quad\ \mid otherwise = \quad intersect\ as\ bs
\end{aligned}
$$

We first define the intersection of the empty set with any other set as the empty set. (Note the similarity of the role of the $\varnothing$ in union and intersect with that of 0 in addition and multiplication!) For other cases, we start with the first element of the first list, a, and check if it is also in bs; if so, we add a to the result set and continue with *intersect* on the tail of the first list; otherwise, we continue without adding anything in this round.

We now can state the set cover problem more formally: We have a set U, called the *universe*, and a set $S = \{s_1, s_2, \ldots, s_n\}$ of subsets of U, $s_1 \subseteq U, s_2 \subseteq U, \ldots, s_n \subseteq U$, such that the union of all the sets in S equals U, $s_1 \cup s_2 \cup \cdots \cup s_n = U$. What is the least expensive union of a subset of S that yields U?

Least expensive may be interpreted in different ways. In the pure mathematical sense, it usually means the smallest number of sets, but in real world problems, least expensive may refer to lowest cost, shortest time, fewest people involved, *etc.* The problem is in fact very common. It comes up in scheduling problems where the members of S represent sets of threads assigned to groups of processors; very typical are problems of team building where the sets in S represent teams of people with complementing skills; but there are also problems similar to the *travelling salesman* problem where the sets in S represent locations that must be visited during a round trip.

So, how many steps do we need to solve this problem? To find the optimal solution, we basically have to try out all combinations of subsets in S. For $S = \{a, b, c\}$, $\{a\}$ may be the best solution, $\{b\}$ may be, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ and, of course, $\{a, b, c\}$. As you should see, that are 2^n possibilities, *i.e.* the sum of all binomial coefficients $\binom{n}{k}$ where n is the size of S. That, as we know, is not feasible to compute with large S's. There are, however, solutions for specific problems using heuristics.

Heuristics are helpers in otherwise exponential search problems. In practice, heuristics may be derived from the concrete problem domain. With respect to the examples mentioned above, it is often obvious that we do not want to combine threads on one processor that better work in parallel; concerning problems with teams, we could exclude combinations of people who do not like each other or we may want to construct gender balanced teams. Such restrictions and insights can be used to drastically reduce the number of possible solutions and, thus, making computation feasible. But think, for instance, of a general purpose operating system that does not have any previous knowledge about the user tasks it should run. No real-world heuristics are available for the kernel to find an optimal balance.

There are purely mathematical heuristics that may come to aid in cases where the problem domain itself does not offer reasonable simplifications. For the set cover problem, a known heuristic that reduces computational complexity significantly, is to search for local optimums instead of the global optimum. That is, we do not try to find the solution that is the best compared with all other solutions, but, instead, we make optimal decisions in each round. For example, if we had the universe $U = \{1, 2, 3, 4, 5, 6\}$ and $S = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 4\}, \{3, 5\}, \{1, 6\}\}$, the optimal solution would be $\{\{1, 2, 4\}, \{3, 5\}, \{1, 6\}\}$. The key to find this solution is to realise that the second set in S, $\{1, 2, 4\}$, is the better choice compared to the first set $\{1, 2, 3\}$. But to actually realise that, we have to try all possible combinations of sets, which are 2^n and, hence, too many. An algorithm that does not go for the global optimum, but for local optimums, would just take the first set, because, in the moment of the decision, it is one of two equally good options and there is nothing that would hint to the fact that, with the second set, the

overall outcome would be better. This *greedy* algorithm will consequently find only a suboptimal solution, *i.e.* $\{1,2,3\}$, $\{1,4\}$ or even $\{1,2,4\}$, $\{3,5\}$ and $\{1,6\}$. It, hence, needs one set more than the global optimum.

In many cases, local optimums are sufficient and feasible to compute. This should be motivation enough to try to implement a greedy solution for the set cover problem. The algorithm will in each step take the set that brings the greatest reduction in the distance between the current state and the universe. We, first, need some way to express this distance and an obvious notion for distance is just the size of the difference between the universe and another set:

```
dist :: (Eq a) ⇒ [a] → [a] → Int
dist a b = length (a \\ b)
```

Now, we need a function, say, *best* that uses *dist* to find the set in S with the least distance to the universe and another function that repeatedly finds the local minimum using *best*, until either all sets in S have been used or no set in S is able to reduce the distance to the universe anymore. Here are these functions:

```
greedySetCover :: (Eq a) ⇒ [a] → [[a]] → [[a]]
greedySetCover u s = loop (length u) [] s
  where loop _ _ []     = []
        loop m rs xs = let (m', p) = best m rs [] xs
                       in if m' < m
                            then p : loop m' (p `union` rs) (delete p xs)
                            else []
        best m _ p []       = (m, p)
        best m r p (x : xs) = let m' = dist u (x `union` r)
                              in if m' < m then best m' r x xs
                                           else best m  r p xs
```

The measure for the current optimum is the variable m used in *loop* and *best*. The whole algorithm starts with $m = length(u)$, which is the worst possible distance, *viz.* the distance between $\varnothing$ and the universe.

The second parameter passed to *loop*, rs, is the union of partial results. It is initially empty. The third parameter is the set of subsets we are working on starting with S. With an empty S, *loop* is just $\varnothing$. Otherwise, it uses *best* to get the local optimum, which is the tuple (m', p), where p is the best choice for the local optimum and m' the distance of this set to the universe. If $m' < m$, we actually have found a solution that improves on the current state and we continue adding p to the result set, which results from the recursion of *loop* with m' as current optimum, the union of p and the partial result rs and the current instance of S without p. Otherwise, the result is just the empty set.

The function *best* simply goes through all elements of the current instance of S. If *best*

arrives at the end of the list, it just returns the previously identified optimum (m, p). Otherwise, for each element of the current set of subsets, it computes the distance and, should the current distance improve on the result, continues with this current optimum, if it does not, it continues with the old parameters.

The fact that we do not go back in the *loop* function to test other options, but always stick with a solution once it was found makes this algorithm *greedy*: It takes the money and runs. What is the speed-up we obtain with this apparently ugly strategy? One call of *best* passes through the whole list, which, initially, is S. *loop*, if *best* has found an optimum that improves on the old result, removes the corresponding element from the list and repeates the process. This time, *best* will go through a list of $n - 1$ elements, where n is the size of S. If it finds a new mimimum again, the corresponding element is removed, and we get a list of $n - 2$ elements. The process repeats, until *best* does not find a new optimum anymore. In the worst case, this is only after all elements in the list have been consumed. The maximum number of steps that must be processed, hence, is $n+n-1+n-2+\cdots+1$ or simply the series $\sum_{k=1}^{n} k$, which, as we already know as the Little Gauss, is $\frac{n^2+n}{2}$. For a set S with 100 elements, we would need to consider 2^{100} possible cases to compute the global optimum, which is 1 267 650 600 228 229 401 496 703 205 376. With the local optimum, we can reduce this number to $\frac{100\times 101}{2} = 5050$ steps. For some cases, the local minimum is therefore the preferred solution.

4.8. Stirling Numbers

We saw that the number of all permutations of a set equals the factorial of the number of elements in that set. We also looked at the cycle notation where permutations are encoded as results of subsequent applications of this permutation; (1 2 5)(3 4), for instance, applied once to the sequence (1 2 3 4 5), would yield (2 5 4 3 1). It is now quite natural to ask – at least for a mathematician – how many permutations there are for a given number of orbits in cycle notation.

To answer this question, we first have to know how many different combinations of a given number of orbits we actually may have. The orbits in cycle notation, in fact, are *partitions* of a set. Partitions are non-empty, distinct subsets. The union of the partitions of a complete partitioning of a set is just the original set. The orbits of the permutation given above, for instance, (1 2 5)(3 4) can be seen as subsets that, obviously, are not empty and, since they have no element in common, are distinct. Their union $\{1, 2, 5\} \cup \{3, 4\}$, as you can easily verify, equals the original set $\{1, 2, 3, 4, 5\}$.

We could, hence, think in the lines of the powerset to generate partitions. We just leave out the empty set and, eventually, pick only groups of sets that, together, add up to the whole set. It is in fact somewhat more complicated. To illustrate that let us look at an algorithm that generates all possibilities to partition a set into two partitions:

$$twoPartitions :: (Eq\ a) \Rightarrow [a] \rightarrow [([a], [a])]$$

$$
\begin{aligned}
&\mathit{twoPartitions} = \mathit{fltr} \circ \mathit{p2} \\
&\quad \textbf{where } \mathit{p2}\ [\,] \quad\ = [([\,],[\,])] \\
&\qquad\quad \mathit{p2}\ (x : xs) = [(x : a, b) \mid (a, b) \leftarrow \mathit{p2}\ xs] +\!\!+ \\
&\qquad\qquad\qquad\quad \mathit{fltr}\ [(a, x : b) \mid (a, b) \leftarrow \mathit{p2}\ xs] \\
&\qquad\quad \mathit{fltr} = \mathit{filter}\ (\lambda p \rightarrow \neg\ (\mathit{null}\ (\mathit{fst}\ p) \vee \\
&\qquad\qquad\qquad\qquad\qquad\quad \mathit{null}\ (\mathit{snd}\ p)))
\end{aligned}
$$

This innocent looking lines of code are quite tricky. The basic idea is implemented in *p2*: For an empty set, *p2* returns a pair of empty sets. For any other set, it applies *p2* twice on the *tail* of the list and adds the *head* once to the first of the pair and once to the second of the pair. This sounds easy, but there is an issue: The intermediate result sets will contain empty sets. In the first result, the second set is empty and, in the second result, the first one is empty. To solve this problem, we explicitly filter empty sets out (using *fltr*). If we applied the filter once on the overall result, we would get the following pairs for the set $\{1, 2, 3\}$:

$([1, 2], [3])$
$([1, 3], [2])$
$([1], [2, 3])$
$([2, 3], [1])$
$([2], [1, 3])$
$([3], [1, 2])$.

All results are correct, but there are too many of them. More specifically, some of the results are repeated. $([1, 2], [3])$ and $([3], [1, 2])$ are different tuples of course, but they describe the same partitioning consisting of the subsets $\{1, 2\}$ and $\{3\}$.

In the code above, this issue is solved by applying the filter once again, *viz.* on the second intermediate result. Let us look at how the second list comprehension develops. After the first application of *p2*, we have a tuple of two empty sets:

$\mathit{p2}\ (3 : [\,]) = [(a, 3 : b) \mid (a, b) \leftarrow [([\,],[\,])]]$
$\mathit{p2}\ (3 : [\,]) = [([\,], [3])]$.

This result is filtered out, because the first set is empty. In the next round we consequently have, as input, only the result of the first, unfiltered, list comprehension:

$\mathit{p2}\ (2 : [3]) = [(a, 2 : b) \mid (a, b) \leftarrow [([3],[\,])]]$
$\mathit{p2}\ (2 : [3]) = [([3], [2])]$,

which is preserved. We then get to the final round where the input is now the result of the first comprehension plus the one created above:

$\mathit{p2}\ (1 : [2, 3]) = [(a, 1 : b) \mid (a, b) \leftarrow [([3], [2]), ([2, 3], [\,])]]$
$\mathit{p2}\ (1 : [2, 3]) = [([3], [1, 2]), ([2, 3], [1])]$.

The result of the first comprehension in the last round consists of the pairs: $([1,3],[2])$, which results from the input $([3],[2])$, and $([1,2,3],[])$, which is removed by the filter on the final result of *p2*. This gives the correct result:

$([1,3],[2])$
$([2,3],[1])$
$([3],[1,2])$.

As long as we create only two partitions, we can use pairs and the nice list comprehension to make the code clear. For the generation of k partitions, the code becomes somewhat more obscure. It is in particulare not sufficient anymore to call the filter twice. Instead, we need an additional function that removes the permutations of sets of partitions. To partition the set $\{1,2,3,4\}$ into three partitions, for example, we need to remove all but one of

$\{1,2\},\{3\},\{4\}$,
$\{1,2\},\{4\},\{3\}$,
$\{4\},\{1,2\},\{3\}$,
$\{3\},\{1,2\},\{4\}$,
$\{4\},\{3\},\{1,2\}$ and
$\{3\},\{4\},\{1,2\}$.

We will not develop this algorithm here, because it is highly inefficient. We would create much more sets than necessary and, then, we still have to generate permutations to remove those sets that are superfluous. Fortunately, there is an alternative very similar to that we have used to create powersets. For powersets, we used all binary numbers from 0 to $2^n - 1$, where n is the number of elements in the set. To generate partitions, we have to modify this idea in two respects.

First, we do not have the simple decision whether an element is in the current subset or not, but in which of k partitions it is. To get this information, we need a number system with the base k. For two partitions, this is just the binary number system. For three partitions, it would be a number system with base 3. For four partitions, we would use a number system with base 4 and so on.

Second, we have to restrict the numbers in a way that the result set points only to distinct partitionings. Imagine we want to know how to partition the set $\{1,2,3,4,5\}$ into three subsets. We would then need numbers of the form: 01200 or 01002. The first number, 01200, would point to the partitions $\{1,4,5\},\{2\},\{3\}$ and the second number, 01002, would point to the partitions $\{1,3,4\},\{2\},\{5\}$. In other words, the digits of the number indicate the partition in which the element at this position (in the original sequence) would be starting to count at index 0 for the first partition. Obviously, the number 01000 would be no good, because it does not describe a set of three partitions, but only one of two partitions. Also, with number 00012 already in the result, we do not want to generate the number 22210, because the corresponding sequences of partitions $\{1,2,3\},\{4\},\{5\}$ and $\{5\},\{4\},\{1,2,3\}$ are permutations of each other and, thus, describe

the same set of partitions.

There is a simple trick to avoid such duplications and this trick has a name: *Restricted Growth String*s, RGS for short. RGS are similar to numbers, but have leading zeros – they are therefore strings rather than proper numbers. The length of RGS depends on the purpose for which they are used. In our case, we want their length to equal the number of elements in the original set.

When counted up, RGS grow in an ordered fashion, such that lesser digits appear before greater ones. For instance, we allow numbers like 0012 and 0102, but do not allow such like 0201 or 1002. This implies that each new digit is at most one greater than the greatest digit already in the number, *i.e.* $a_i \leq 1 + \max a_1, a_2, ...a_{i-1}$. This restriction rules out numbers with a combination of digits that has already appeared with smaller RGS before. Of the strings 0123, 0132, 0213, 0231, 0312 and 0321 only the first is a valid RGS. With the others, either 3 or 2 appear before 1, violating the restriction that no digit must be greater than the greatest number appeared so far plus 1. You can easily verify that all those strings point to the same partitioning of set $\{1, 2, 3, 4\}$, namely permutations of the set $\{\{1\}, \{2\}, \{3\}, \{4\}\}$.

The ordered growth also implies that the first digit in an RGS is always 0. Otherwise, if 0 did not appear in the string at all, the first partition would be empty and the partitioning would, hence, be invalid; if 0 did appear later in the string, the string would not be ordered, *i.e.* a greater number would appear before the smallest possible number *zero*.

To be sure that the RGS-technique effectively avoids duplication of subset by suppressing permuations, we should at least sketch a proof of the concept. We should prove that restricted growth makes complementing groups of digits impossible, such that all digits k_1 and k_2 swap their positions from one string to the other. The following diagram shows four positions in a string where, at positions i and $i+1$, there is the digit k_1 and, at positions j and $j+1$, there is the digit k_2:

...	i	$i+1$	...	j	$j+1$	...
...	k_1	k_1	...	k_2	k_2	...

We assume that $j > i + 1$ and we assume that all occurences of k_1 and k_2 in the string are shown. In other words, this partial string shows the partitions $k_1 = \{i, i + 1\}$ and $k_2 = \{j, j + 1\}$.

We prove by contradiction on restricted growth and assume that this string is possible with both cases, $k_1 < k_2$ or, alternatively, $k_2 < k_1$. Consider the case $k_2 < k_1$. In this case k_2 must appear in a position $p < i$, otherwise, ordering would be violated. But this contradicts the assumption that $k_2 = \{j, j + 1\}$. So, either we violate ordering or k_2 is not shown completely in the diagram above and, then, the subsets are not complementing. Ordering is violated because it implies that any digit a_i in the string is at most $1 + \max a_1, a_2, \ldots, a_{i-1}$. i is either 0, then k_1, per definition, is 0 as well and

no (natural) number is less than 0, hence k_2 cannot be less than k_1; or i is not 0, then there must be a digit $k_0 = k_1 - 1$. If we assume that $k_2 < k_1$, we must assume that $k_1 - 1 < k_2 < k_1$ and, hence, that $0 < k2 < 1$. But that cannot be, since k_2 is still a natural number. □

To implement the RGS analogy, we first need a function that converts decimal numbers into numbers with base b. We have already looked at such functions, for $b = 2$ in the previous section, which we called *binExp*, and, for $b = 10$, in the previous chapter in the context of the conversion function *integer2Num*. *binExp* was tailored for binary numbers, since it yielded only the positions where the binary result would have a 1. That information is obviously not sufficient for number systems with $b > 2$, where the decision which number to put at a given position is not binary.

Let us recall how we converted integer to our natural number type. We divided the number by 10, collecting the remainders and continuing on the quotient, like in the following example:

$1000\ `quotRem`\ 10 = (100, 0)$
$100\ `quotRem`\ 10 = (10, 0)$
$10\ `quotRem`\ 10 = (1, 0)$
$1\ `quotRem`\ 10 = (0, 1)$.

Now, the remainders of the subsequent divisions bottom-up would read $1, 0, 0, 0$, which are just the components of the decimal representation of the number 1000. If we do this with $b = 2$, we would see:

$1000\ `quotRem`\ 2 = (500, 0)$
$500\ `quotRem`\ 2 = (250, 0)$
$250\ `quotRem`\ 2 = (125, 0)$
$125\ `quotRem`\ 2 = (62, 1)$
$62\ `quotRem`\ 2 = (31, 0)$
$31\ `quotRem`\ 2 = (15, 1)$
$15\ `quotRem`\ 2 = (7, 1)$
$7\ `quotRem`\ 2 = (3, 1)$
$3\ `quotRem`\ 2 = (1, 1)$
$1\ `quotRem`\ 2 = (0, 1)$.

1000, in the binary system, hence, is 1111101000, the same result we have already obtained in the previous section. We can implement this procedure in Haskell as:[2]

```
toBaseN :: Int → Int → [Int]
toBaseN b = reverse ∘ go
  where go x = case x `quotRem` b of
```

[2]We use *Int* instead of *Natural* here, because, in the following, we will need list functions like *length* or *take* quite often; with *Natural*, we would have to add a lot of conversions, which is much harder to read.

$$(0, r) \to [r]$$
$$(q, r) \to r : go\ q$$

The *go*-part of this function applied to base $b = 3$ and, say, 1024 would develop as follows:

$go\ 1024 = 1024\ `quotRem`\ 3$
$1 : go\ 341 = 341\ `quotRem`\ 3$
$1 : 2 : go\ 113 = 113\ `quotRem`\ 3$
$1 : 2 : 2 : go\ 37 = 37\ `quotRem`\ 3$
$1 : 2 : 2 : 1 : go\ 12 = 12\ `quotRem`\ 3$
$1 : 2 : 2 : 1 : 0 : go\ 4 = 4\ `quotRem`\ 3$
$1 : 2 : 2 : 1 : 0 : 1 : go\ 1 = [1]$
$1 : 2 : 2 : 1 : 0 : 1 : 1,$

which, reversed, is $[1, 1, 0, 1, 2, 2, 1]$ and the correct representation of 1000 in the ternary system.

Now we need some functions to convert the number given in the b-ary system into an RGS. First we fill the number with leading zeros until it has the desired size n:

```
rgs :: Int → Int → Int → [Int]
rgs b n i = let r = toBaseN b i
                d = n − length r
                p = if d > 0 then take d (repeat 0) else [ ]
            in if d < 0 then [ ] else p ++ r
```

Note that, if the length of the result of *toBaseN* exceeds n, the function yields the empty list. This, in fact, is an error case that could be handled explicitly. On the other hand, returning the empty list is a good enough indication for an error and we could check for this error in code using *rgs* later.

We now define a wrapper around this conversion function to apply the restrictions:

```
toRgs :: ([Int] → Bool) →
  Int → Int → Int → (Int, [Int])
toRgs rst b n i = go (rgs b n i)
  where go r | ¬ (rst r)  = toRgs rst b n (i + 1)
             | otherwise = (i, r)
```

This function converts a decimal number to an RGS and checks if the result obeys the restrictions, which are passed in as a boolean function. If it does not, then the input is incremented by one and the function is called again. Otherwise, the function yields a tuple consisting of the decimal number that we have eventually reached and the RGS.

We define the growth restriction as follows:

```
rGrowth :: [Int] → Bool
rGrowth  []       = True
rGrowth  (x : xs) = go x xs
  where go _ []       = True
        go d (z : zs) = if z − d > 1 then False
                        else let d′ = max d z in go d′ zs
```

Since we want to see only partitionings with b subsets, we, still, need another restriction. We do not want to see RGS of the form 01000, when we ask for three partitions, or 01230, when we ask for five. For this end, we need the restriction that there must be b different digits in the resulting RGS. The restriction is easily implemented as:

```
hasN :: Int → [Int] → Bool
hasN b r = length (nub r) ≡ b
```

We apply this restrictions in yet another wrapper to call *toRgs*:

```
toRgsN :: Int → Int → Int → (Int, [Int])
toRgsN b = toRgs rst b
  where rst r = rGrowth r ∧ hasN b r
```

Finally, we can implement a loop that counts RGS up from 1 to the last number with leading 0:

```
countRgs :: Int → Int → [[Int]]
countRgs 1 n  = [rgs 1 n 1]
countRgs b n  | b ≡ n      = [[1 .. n − 1]]
              | b > n      = []
              | otherwise = go 1
  where go i = let (j, r) = toRgsN b n i
               in if head r ≢ 0 then [] else r : go (j + 1)
```

Note that we jump over numbers that do not obey the restrictions: we continue always with *go* applied to j, the first return value of *toRgsN*. j does not necessarily equal i; it depends on how many numbers have been ignored by *toRgs* because they did not obey the restrictions. When we call countRgs on 3, the numbers of partitions we want to have, and 4, the number of elements in the original set, we get the following RGS:

[0, 0, 1, 2]
[0, 1, 0, 2]
[0, 1, 1, 2]
[0, 1, 2, 0]
[0, 1, 2, 1]
[0, 1, 2, 2],

which correspond to the partitionings of $\{1, 2, 3, 4\}$:

$\{1, 2\}, \{3\}, \{4\}$
$\{1, 3\}, \{2\}, \{4\}$
$\{1\}, \{2, 3\}, \{4\}$
$\{1, 4\}, \{2\}, \{4\}$
$\{1\}, \{2, 4\}, \{3\}$
$\{1\}, \{2\}, \{3, 4\}$.

This analogy between RGS and partitions is implemented as:

```
rgs2set :: (Eq a) ⇒ [Int] → [a] → [[a]] → [[a]]
rgs2set [] _ ps              = ps
rgs2set _ [] ps              = ps
rgs2set (r : rs) (x : xs) ps = rgs2set rs xs (ins r x ps)
  where ins _ _ []           = ⊥
        ins 0 p (z : zs)     = (p : z) : zs
        ins i p (z : zs)     = z : ins (i − 1) p zs
```

The function receives three arguments: The RGS, the original set we want to partition and the result set, which initially should contain k empty lists with k the number of partitions we want to obtain. (This pre-condition, actually, is not enforced in this code.) Note that the logic of using k empty lists is very similar to the trick we used in *twoParatitions* above.

When we have exhausted either the RGS or the original set, the result is just *ps*, the result set we passed in. Otherwise, we recurse with the tails of the RGS and the set (this way establishing the analogy) inserting the element of the set that corresponds to the current position of the RGS to the partition that, in its turn, corresponds to the digit of the RGS at this position. If the digit is 0, we just insert the element into the first list, otherwise we recurse (on *ins*) decrementing the digit by 1. Note that *ins* is $\perp$ for the case that the result set is empty before we reach 0. This, obviously, would hint to an erroneous RGS and, hence, to a coding error.

We now can put everything together:

```
nPartitions :: (Eq a) ⇒ Int → [a] → [[[a]]]
nPartitions _ [] = []
nPartitions 1 xs = [[xs]]
nPartitions k xs | k ⩾ length xs = [[[x] | x ← xs]]
                 | otherwise     = go (countRgs k (length xs))
  where go []       = []
        go (r : rs) = rgs2set r xs (take k (repeat [])) : go rs
```

The function *nPartitions* receives the number of partitions we would like to have, k, and the original set, *xs*. If the set is empty, the result is empty, too. If we just want one

partition, we return the set as its only partition. If k equals or exceeds the size of the set, we just return each element in its own set. (We could return an error for the case that k exceeds the size of the set, but, for sake of simplicity, we allow this case, returning an incomplete result.) Otherwise, we call *countRgs* and apply *rgs2set* to all elements of the result. The set of empty lists we need to start with *rgs2set* is created by *repeat* [] which creates an infinite list containing the empty list – $[[], [], [], ...]$ – from which we just take k, *i.e.* the number of partitions. The call *nPartitions* 2 $[1, 2, 3]$ would yield the expected result of all possibilities to partition $[1, 2, 3]$ in 2 subsets:

$[2, 1], [3]$
$[3, 1], [2]$
$[1], [3, 2]$.

This result corresponds to the RGS:

001,
010 and
011.

The order of elements within partitions is explained by the fact that *ins* (in *rgs2set*) adds new elements using ":" – the element that was first inserted into the list is therefore the last one in the resulting partition.

It should be noted that, as for the powerset, we can optimise this code by using other data structures than lists. Since, as for the powerset as well, the number of possible partitionings of huge sets is incredibly large, it is not feasible to computate all partitionings of great sets anyway. We, therefore, leave it with a non-optimal implementation.

Now, that we have arrived here, the mathematically natural question occurs of how many partitions there are for a set of size n. Well, we have a tool to try that out. The following – *er* – triangle shows results of calls of *nPartitions*. Each line corresponds to the call $[length\ (nPartitions\ k\ [1\,..\,n])\ |\ k \leftarrow [1\,..\,n]]$, where n starts with 1 at the top of the triangle and is counted up until 7 at its bottom.

1						1						
2					1		1					
3				1		3		1				
4			1		7		6		1			
5		1		15		25		10		1		
6	1		31		90		65		15		1	
7		63		301		350		140		21		1

On the first sight, the values in this triangle besides the ones in the outer diagonals appear less regular than those in Pascal's nice and tidy triangle. Nevertheless, already the second sight reveals some curious relations. The second diagonal from top-right to left-bottom, which reads 1,3,7,15,31,63, corresponds to the values $2^n - 1$. The second diagonal from top-left to right-bottom shows other numbers we already know, namely

1,3,6,10,15,21. If you take the differences, you will observe that each number is the sum of n and its predecessor. In other words, this diagonal contains the sum of all numbers from 1 to n, where n is the line number.

The triangle overall shows the *Stirling set numbers*, also known as the *Stirling numbers of the second kind*, which are denoted by

$$S(n,k) = \left\{ {n \atop k} \right\}. \tag{4.45}$$

The formula to compute Stirling numbers remarkably resembles Pascal's rule:

$$\left\{ {n \atop k} \right\} = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ k \times \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} & \text{otherwise.} \end{cases} \tag{4.46}$$

This translates into Haskell as:

```
stirling2 :: Natural → Natural → Natural
stirling2 0 _ = 0
stirling2 _ 0 = 0
stirling2 1 1 = 1
stirling2 n k | k > n      = 0
              | otherwise = k * (stirling2 (n - 1) k) +
                                (stirling2 (n - 1) (k - 1))
```

The code is almost a one-to-one translation of the mathematical formulation. However, there is an extra line, namely the base case *stirling2* _ 0 = 0. This base case must be introduced to avoid that we go below zero for k with the rule part *stirling2* $(n-1)$ $(k-1)$. The natural numbers are not defined for the range less than zero and so we have to stop here explicitly.

The sum of all Stirling numbers of the same row, *i.e.* $\sum_{k=1}^{n} \left\{ {n \atop k} \right\}$, is called the *Bell number* of n. The first 7 Bell numbers are: 1, 2, 5, 15, 52, 203, 877.

Since Stirling numbers of the second kind count the ways to partition a set into a given number of subsets, Bell numbers indicate all ways to partition a set, not restricting the number of subsets we want to obtain. The following table shows this relation for the set $\{1,2,3,4\}$:

$\left\{ {4 \atop 1} \right\} = 1$	$\left\{ {4 \atop 2} \right\} = 7$	$\left\{ {4 \atop 3} \right\} = 6$	$\left\{ {4 \atop 4} \right\} = 1$
$\{1,2,3,4\}$	$\{1,2,3\},\{4\}$	$\{1,2\},\{3\},\{4\}$	$\{1\},\{2\},\{3\},\{4\}$
	$\{1,2,4\},\{3\}$	$\{1,3\},\{2\},\{4\}$	
	$\{1,3,4\},\{2\}$	$\{1,4\},\{2\},\{3\}$	
	$\{2,3,4\},\{1\}$	$\{1\},\{2,3\},\{4\}$	
	$\{1,2\},\{3,4\}$	$\{1\},\{2,4\},\{3\}$	
	$\{1,3\},\{2,4\}$	$\{1\},\{2\},\{3,4\}$	
	$\{1,4\},\{2,3\}$		

If you count the partitionings in one column, you get the Stirling number for that column; the number of all partitions in all columns equals the Bell number of 4, which is 15.

The inventors – or discoverers, depending on your philosophical view – of Bell numbers and Stirling numbers are two very interesting characters. Eric Temple Bell (1883 – 1960) was professor of mathematics in the United States for most of his life. But he was also an early science fiction writer and a math historian. His science fiction reached a higher level of science than most other publications in this genre at his time, but was often critised as poorly written and, in particular, for its weak characterisation of protagonists. His contributions to math history were even more fiercely critised as fictitious and romantic (as in the case of his biographical sketch of Évarist Galois) or as stereotypical (as in the case of his description of the life of Georg Cantor).

James Stirling (1692 – 1770) was from Scotland. He studied and taught in Oxford for some years, but had to flee from England, when he was accused of conspiracy based on his correspondence with Jacobites, that is supporters of the catholic kings, in particular James II who was deposed in 1688. After ten years of exile in Venice, he started to fear for his life again, because he discovered a trade secret of the glassmakers of Venice and returned to England with the help of his friend Isaac Newton. Much of Stirling's work is in fact tightly coupled with that of Newton. Stirling very much promoted Newton's discoveries and methods, for instance in his book *Methodus differentialis*. During the last years of his life, he was manager of the Scots Mining Company. During this period, he published mainly on topics of applied mathematics.

But let us return to the intial question. We were investigating the possible permutations with a given number of orbits in the cycle notation and have just learnt how to generate all possible k orbits of a set with n elements. We have found out how to partition a set of n elements into k subsets. However, the distinct subsets are not yet sufficient to generate all possible permutations, since the permutation of (1 2 3 4 5) $\sigma 1 = (1\ 2\ 5)(3\ 4)$ is not the same as $\sigma 2 = (1\ 5\ 2)(3\ 4)$:

$$\sigma 1 = 2\ 5\ 4\ 3\ 1$$

$$\sigma 2 = 5\ 1\ 4\ 3\ 2$$

So, do we need all permutations of the subsets? (Was our survey of RGS in vain?) Apparently not, since $\sigma 1$ is just the same permutation as $\sigma 3 = (1\ 2\ 5)(4\ 3)$. It is also the same as $(5\ 1\ 2)(3\ 4)$. In fact, the cycle notation is indifferent concerning the starting point – for this reason, it is called cyclic. Therefore, not all permutations are relevant, but only those that change the order after the first element. An orbit permutating function aware of this peculiarity is:

```
permOrbits :: (Eq a) ⇒ Perm a → [Perm a]
permOrbits []       = [[]]
permOrbits (o : oo) = concat [map (:x) (oPerms o) | x ← permOrbits oo]
  where oPerms []       = []
        oPerms (x : xs) = [x : ps | ps ← perms xs]
```

This function just passes through all orbits of the input permutation creating permutations of each one using *oPerms*. It looks a bit weird that we do not just use *map* to create that result, but a list comprehension to which we even further apply *concat*. However, *map* does not yield the desired result. *map* would just create a list of permutated orbits – but we want to obtain complete cycles each of which may consist of more than one orbit. For this reason, we create permutations of the head and insert all permutations of the head to all results of the recursion of *permOrbits*.

The function we use for creating permutations of orbits is *oPerms*, which applies *perms*, all permutations of a list, to the tail of the input list. The head of the list, hence, remains always the same. For instance, *oPerms* $[3, 4]$ is just $[3, 4]$. *oPerms* $[1, 2, 5]$, however, yields $[1, 2, 5]$ and $[1, 5, 2]$.

Now, for one possible cycle, we can just apply all permutations resulting from *permOrbits*:

```
permsOfCycle :: (Eq a) ⇒ Perm a → [a] → [[a]]
permsOfCycle os xs = [permute o xs | o ← permOrbits os]
```

This function creates all permutations that are possible given one partitioning of the input set. We now map this function on all possible partitionings with k subsets:

```
permsWithCycles :: (Eq a) ⇒ Int → [a] → [[a]]
permsWithCycles k xs = concat [
  permsOfCycle x xs | x ← nPartitions k xs]
```

Applied on sets with n elements, for $n = 1 \ldots 7$, and $k = 1 \ldots n$, *permsWithCycles* yields results of length:

1							1						
2						1		1					
3					2		3		1				
4				6		11		6		1			
5			24		50		35		10		1		
6		120		274		225		85		15		1	
7	720		1764		1624		735		175		21		1

These, as you may have guessed already, are the Stirling numbers *of the first kind*, also known as *Stirling cycle numbers*, since they count the number of possible permutations with a given number of orbits in the cycle notation. They are denoted by

$$s(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} \tag{4.47}$$

and can be calculated as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ (n-1) \times \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right] + \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right] & \text{otherwise.} \end{cases} \tag{4.48}$$

In Haskell, this would be:

```
stirling1 :: Natural → Natural → Natural
stirling1 0 _ = 0
stirling1 _ 0 = 0
stirling1 1 1 = 1
stirling1 n k | k > n      = 0
              | otherwise = (n − 1) ∗ (stirling1 (n − 1) k) +
                            (stirling1 (n − 1) (k − 1))
```

We have seen that the sum of all Stirling numbers of the second kind in one row, *i.e.* $\sum_{k=1}^{n} \left\{ {n \atop k} \right\}$, is the Bell number of n. Can you guess what this sum is for Stirling numbers of the first kind?

Remember that each Stirling number of the form $\left[{n \atop k} \right]$ shows the number of permutations with a given number of orbits in cycle notation. If you add up all possible permutations of all numbers of orbits $1 \ldots n$, what do you get? Let us see:

For $n = 1$, we trivially get 1.
For $n = 2$, we get $1 + 1 = 2$.
For $n = 3$, we get $2 + 3 + 1 = 6$.
For $n = 4$, we get $6 + 11 + 6 + 1 = 24$.
For $n = 5$, we get $24 + 50 + 35 + 10 + 1 = 120$.

We know these numbers: these are the factorials of $n = n!$. Since the factorial counts the number of all possible permutations of a set, it is just natural that the Stirling numbers of the first kind, which count the possible permutations of a set with a given number of orbits, add up to the number of all possible permutations, *i.e.* the factorial of the size of the set. The triangle itself hints to that. The outer left diagonal, actually, shows the factorials! It shows the factorials of $n - 1$ though (with n indicating the row). If you think of how we create permutations of orbits – *viz.* as permutations of the tail of the orbit, without touching the head – it becomes immediately clear why the Stirling number $\left[{n \atop 1}\right]$, the one with only one partition, equals $(n - 1)!$.

There is still a lot to say about Stirling numbers. But that may involve concepts we have not yet discussed. So, we will have to come back to this topic later.

4.9. Eulerian Numbers

When we discussed Haskell, we studied a beautiful sorting algorithm, Hoare's *quicksort*. We already mentioned that *quicksort* is not the best algorithm in practice and it is this question to which we are coming back now.

The reason why *quicksort* is not optimal is that it looks at the world in black and white: a sequence is either sorted or it is not. But reality is not like this. Complete order, that is to say, a sequence where every element is at its place according to the order of the data type, starting with the least element in the sequence and each element being greater than its predecessor, is very rare, of course, and usually an effort is necessary to create complete order in this sense. Complete disorder, however, that is a sequence where no element is at its place, is quite rare too. In fact, it is as rare as complete order: for any sequence of unique elements, there is exactly one permutation showing complete order and one permutation showing complete disorder. Order and disorder balance each other.

For instance, the sequence $1, 2, 3, 4, 5$, is completely ordered; the permutation showing complete disorder would be $5, 4, 3, 2, 1$ and that is the original sequence reversed obeying as such another kind of order, *viz.* $\leq$ instead of $\geq$. All other permutations are in between. That is they show some order with a leaning either to the original sequence or to its reverse. For example: $5, 2, 4, 3, 1$ is close to the opposite order; reversed, it would be $1, 3, 4, 2, 5$ and close to order.

A sorting algorithm that exploits this pre-existing order in any input is *mergesort*. The underlying idea of *mergesort* is to merge two ordered lists. This can be implemented in Haskell simply as

$$
\begin{aligned}
&\mathit{merge} :: (\mathit{Ord}\ a) \Rightarrow [a] \to [a] \to [a] \\
&\mathit{merge}\ [\,]\ xs = xs \\
&\mathit{merge}\ xs\ [\,] = xs \\
&\mathit{merge}\ (x : xs)\ (y : ys) \mid x \leqslant y \quad\ = x : \mathit{merge}\ xs\ (y : ys) \\
&\qquad\qquad\qquad\qquad\ \mid \mathit{otherwise} = y : \mathit{merge}\ (x : xs)\ ys
\end{aligned}
$$

We first treat the base cases where one of the lists is empty, just yielding the respective other list. Then we compare the heads of the lists. The smaller one goes to the head of the merged list and we recurse with what remains.

We now want to use *merge* on an unordered, that is to say, incompletely ordered list. Therefore, we split the input into a list of ordered sublists. Ordered sublists are often referred to as *run*s. The positions where one ordered list ends and another one begins are called *stepdown*s, reflecting the idea that the sequence of increasing elements is interrupted by stepping down to a smaller element. Here is a Haskell function that splits a list into runs:

$$
\begin{aligned}
&\mathit{runs} :: (\mathit{Ord}\ a) \Rightarrow [a] \to [[a]] \\
&\mathit{runs}\ [\,] \quad\ = [\,] \\
&\mathit{runs}\ xs \quad = \mathbf{let}\ (r, zs) = \mathit{run}\ xs\ \mathbf{in}\ r : \mathit{runs}\ zs \\
&\quad \mathbf{where}\ \mathit{run}\ [\,] \quad = ([\,], [\,]) \\
&\qquad\qquad \mathit{run}\ [x] = ([x], [\,]) \\
&\qquad\qquad \mathit{run}\ (x : y : zs) \mid x > y \quad\ = ([x], y : zs) \\
&\qquad\qquad\qquad\qquad\qquad\ \mid \mathit{otherwise} = \mathbf{let}\ (r, ys) = \mathit{run}\ (y : zs)\ \mathbf{in}\ (x : r, ys)
\end{aligned}
$$

The function applied to the empty list just yields the empty list. Applied to a non-empty list is calls the helper function *run* that returns a tuple consisting of two lists of as. The first element is then the head of the list resulting from *runs* applied to the second list.

The helper function *run* applied on the empty list yields a tuple of twice the empty list. Applied to a list containing only one element, it yield a tuple containing this list and the empty list. Otherwise, the first two elements of the list, x and y, are compared. If $x > y$, x goes to the first list of the resulting tuple, y goes to the rest of the list. This is a stepdown: the first element x is greater than its successor y and, hence, the natural order of the sequence is violated. Otherwise, we continue with $y : zs$ and insert x as the head of the resulting r, the first of the tuple. This is the case, where the run continues.

Let us look at an example: the permutation we already used above $1, 3, 4, 2, 5$. When we call *run* for the first time, we have:

$\mathit{run}\ (1 : 3 : [4, 2, 5])$

and we enter the *otherwise* alternative with $x = 1$:

$\mathit{run}\ (3 : 4 : [2, 5])$.

We again enter *otherwise*, this time $x = 3$:

run $(4 : 2 : [5])$.

But this time we have $4 > 2$ and enter the first branch, that is we yield

$([4], 2 : [5])$.

Going backwards, we see:

$(3 : [4], [2, 5])$
$(1 : [3, 4], [2, 5])$,

which finally appears as result of the first call to *run* in *runs*, which leads to $[1, 3, 4] :$ *runs* $[2, 5]$. $[2, 5]$ is now handled in the same way and we obtain the overall result $[[1, 3, 4], [2, 5]]$. This way, thee entire list is split into two runs.

When we look at the other example, the permutation that was closer to disorder, there are more runs: $5|24|3|1$, which is the proper mathematical notation for the list $[[5], [2, 4], [3], [1]]$ consisting of four runs. If we reverse the list, however, the number of runs reduces and we get the list with only two runs above.

Can we exploit the fact that we can reduce the number of runs by reverting the list? Yes, of course, otherwise the question would not have been asked. It turns out that, for lists with unique elements, if the number of runs r is greater than $\frac{n}{2} + 1$, then r', the number of runs of the reversed list is less than r. In other words, we could check the number of runs, before we start to sort, and revert the list if the number of runs is greater than the half of the length of the list plus 1.

Note that in the real world we often see lists with repeated elements. The repetitions, however, would not spoil the result, since repetitions would just reduce the possible number of runs. In consequence, it may happen that reverting the list, even though the number of runs is less than the half of the list size plus 1, would improve performance. But without reverting, the algorithm is still good, *viz.* comparable to the performance of a cousin of the same size without repetitions.

Let us look at an implementation of *mergesort* that exploits order in the input in this sense. First, we need a function that merges the runs, that is a merge for a list of lists. Let us call this function *multimerge*:

$$
\begin{aligned}
&\mathit{multimerge} :: (\mathit{Ord}\ a) \Rightarrow [[a]] \to [a] \\
&\mathit{multimerge}\ [\,] &&= [\,] \\
&\mathit{multimerge}\ [xs] &&= xs \\
&\mathit{multimerge}\ (x : y : zs) &&= \mathit{merge}\ (\mathit{merge}\ x\ y)\ (\mathit{multimerge}\ zs)
\end{aligned}
$$

The function reduces a list of lists of a to a plain list of a. For the empty list, it yields the empty list. For a list that contains only one single list, it yields just that list. For a list with more elements, it merges the first two lists and merges the resulting list with the list that results from *multimerging* the rest. With an example that will become clearer. But let us not use the list with two runs, because that would not let us see the

recursion on *multimerge*. Instead, we use the non-optimal list with four runs. We would start with:

multimerge $[[5] : [2, 4] : [[3], [1]]$

and perform

merge (*merge* $[5]$ $[2, 4]$) (*multimerge* $[[3], [1]]$),

which is

merge $[2, 4, 5]$ (*multimerge* $[[3], [1]]$)

and results in the call to *multimerge*:

multimerge $[[3], [1]]$.

This reduces to

merge (*merge* $[3]$ $[1]$) (*multimerge* $[]$),

which is

merge $[1, 3]$ $[]$,

which, in its turn, is $[1, 3]$. Going backwards, we obtain

merge $[2, 4, 5]$ $[1, 3]$,

which is $[1, 2, 3, 4, 5]$.

We can now put everything together:

```
mergesort :: (Ord a) ⇒ [a] → [a]
mergesort l = let  rs = runs l
                   n' = length l
                   n  = if even n' then n' else n' + 1
              in if length rs > (n `div` 2) + 1
                 then multimerge $ runs (reverse l)
                 else multimerge rs
```

We first create the runs for the input list l. We then compute the length of l n'. If n' is not even, we add one, just to be sure, we later refer to at least the half of n'. Then, if the length of the runs is more than half of n plus 1, then we run *multimerge* on the runs of the reversed input list, otherwise, we run *multimerge* on the runs of l.

There is a well-known mathematical concept that is closely related to the conept of runs: the *Eulerian numbers*, which, as the Stirling numbers, come in two flavours ingeniously called Eulerian numbers of the first kind and Eulerian numbers of the second kind.

The Eulerian numbers of the first kind, denoted by $\left\langle {n \atop m} \right\rangle$, count the number of permu-

tations of a set of n distinct elements where m numbers are greater than the previous element. They, hence, do not count the number of runs directly. They, first, use the number of elements that are actually member of a run without the first element of that run (which is a stepdown). This, actually, is the value of m, which counts, in other words, how many of the elements are not stepdowns. They, second, count the number of permutations that have m elements in such a configuration. For instance, the set $\{1, 2, 3, 4, 5\}$ has $5! = 120$ permutations. There is exactly one permutation where 4 elements are greater than their predecessor, namely the permutation $1, 2, 3, 4, 5$. Hence: $\left\langle {5 \atop 4} \right\rangle = 1$. There is also only one permutation with no element greater than its predecessor, namely $5, 4, 3, 2, 1$. Hence: $\left\langle {5 \atop 0} \right\rangle = 1$. There are 26 permutations with only 3 elements greater than their predecessor and 66 with only 2 elements greater than their predecessor.

You perhaps see immediately that Eulerian numbers can be used to compute the average running time of *mergesort* for input of a given size n. The Eulerian numbers allow us to compute the probability of that input having 1 run, 2 runs, ..., n runs. For the example above, we have the probabiliy $\frac{1}{5!=120}$ that there is no stepdown at all and that we, hence, do not have to do anything; we have the probability $\frac{26}{120} = \frac{13}{60}$ that we have to do only one merge step and the probability of $\frac{66}{120} = \frac{11}{20}$ that we have to do two merge steps and so on. (Knuth provides an extensive analysis in the third volume of his masterpiece.)

Let us look at a smaller set, where we can actually look at all permutations, say $\{1, 2, 3\}$. There is of course 1 permutation for 2 non-stepdowns and also 1 for no non-stepdown at all: $\left\langle {3 \atop 0} \right\rangle = \left\langle {3 \atop 2} \right\rangle = 1$. For 1 element greater than its predecessor, there are $\left\langle {3 \atop 1} \right\rangle = 4$ possibilities, namley: $213, 231, 132, 312$. The sum of all values is of course the number of all permutations, *i.e.* $3! = 6$: $\left\langle {3 \atop 0} \right\rangle + \left\langle {3 \atop 1} \right\rangle + \left\langle {3 \atop 2} \right\rangle = 1 + 4 + 1 = 6$.

Here are some values arranged in – oh, you guessed it already:

1							1							
2						1			1					
3					1			4			1			
4				1			11			11			1	
5			1			26			66			26			1
6		1			57			302			302			57			1
7	1			120			1191			2416			1191			120			1

Make sure yourself that, for each line n, the following identity holds:

$$n! = \sum_{m=0}^{n-1} \left\langle {n \atop m} \right\rangle \tag{4.49}$$

There is a recursive formula to compute the Eulerian numbers, which is

$$\left\langle {n \atop m} \right\rangle = (n-m)\left\langle {n-1 \atop m-1} \right\rangle + (m+1)\left\langle {n-1 \atop m} \right\rangle, \tag{4.50}$$

with $\left\langle {0 \atop m} \right\rangle = \left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1$. In Haskell, this can be implemented as:

```
eulerian1 :: Natural → Natural → Natural
eulerian1 0 _  = 1
eulerian1 _ 0  = 1
eulerian1 n m | m ≡ n − 1 = 1
              | otherwise = (n − m) ∗ eulerian1 (n − 1) (m − 1)
                          + (m + 1) ∗ eulerian1 (n − 1) m
```

There is also a closed form to compute the nth Eulerian number and this closed form reveals a relation of the Eulerian numbers with the binomial coefficients:

$$\left\langle {n \atop m} \right\rangle = \sum_{k=0}^{m} (-1)^k \binom{n+1}{k} (m+1-k)^n. \tag{4.51}$$

This expands into a series where the results of the products $\binom{n+1}{k}(m+1-k)^n$ are alternately added or subtracted. For instance for $\left\langle {5 \atop 2} \right\rangle$:

$$\left\langle {5 \atop 2} \right\rangle = (-1)^0 \times \binom{6}{0} \times (2+1-0)^5 + (-1)^1 \times \binom{6}{1} \times (2+1-1)^5 + (-1)^2 \times \binom{6}{2} \times (2+1-2)^5, \tag{4.52}$$

which is

$$\begin{array}{rrcrcrcr} & 1 & \times & 3^5 & & & = & 243 \\ - & 6 & \times & 2^5 & = & 6 \times 32 & = & 192 \\ + & 15 & \times & 1^5 & & & = & 15, \end{array}$$

hence: $243 - 192 + 15 = 66$.

It perhaps helps to get the closed form right to look at it in Haskell:

```
eu1closed :: Natural → Natural → Natural
eu1closed n m = go 0
  where go k = let a = (−1) ↑ k
                   b = choose (n + 1) k
                   c = (m + 1 − k) ↑ n
               in (a ∗ b ∗ c) + if k ≡ m then 0
                                         else go (k + 1)
```

The Eulerian numbers of the second kind are as well quite interesting. They deal with *multisets*, that is sets with repeated elements and, as such, they may pave the way to a theory for real world input. In concrete, they count the number of permutations of a multiset with n different elements with m ascents, where an ascent occurs whenever a number is greater than its predecessor. The concept of an ascent, hence, is stronger than that of a non-stepdown. A non-stepdown would include a number that equals its predecessor, but that is not an ascent.

To understand Eulerian numbers of the second kind, we first have to understand how many permutations there are for multisets. It is not the factorial, but the *doublefactorial*, also called the *semifactorial*, of $2n-1$, where n is the number of unique elements in the multiset. For instance, the multiset $\{1,1,2,2,3,3\}$ has $n=3$ unique elements, namely 1, 2 and 3. The doublefactorial of $2n-1=5$, denoted by $n!!$, is 15.

The doublefactorial of n is the product of all numbers $1\ldots n$ that have the same parity as n. If n is even, we multiply all even numbers $2\ldots n$, otherwise, if n is odd, we multiply the odd numbers $1\ldots n$. In Haskell, this may look like this:

$$
\begin{aligned}
&\mathit{doublefac} :: \mathit{Natural} \to \mathit{Natural}\\
&\mathit{doublefac}\ 0 = 1\\
&\mathit{doublefac}\ 1 = 1\\
&\mathit{doublefac}\ n = n * \mathit{doublefac}\ (n-2)
\end{aligned}
$$

For instance, $5!! = 1 \times 3 \times 5 = 15$ and $6!! = 2 \times 4 \times 6 = 48$.

The permutations of the multiset $\{1,1,2,2,3,3\}$ are

332211,
221133, 221331, 223311, 233211, 113322, 133221, 331122, 331221,
112233, 122133, 112332, 123321, 133122, 122331.

In the first line, there is one permutation with no ascent, in the second line, there are 8 permutations with one ascent and in the third line, there are 6 permutations with two ascents. The Eulerian numbers of the second kind, denoted by $\left\langle\!\left\langle {n \atop m} \right\rangle\!\right\rangle$, for a multiset with $n=3$ different elements, thus, are: $\left\langle\!\left\langle {3 \atop 0} \right\rangle\!\right\rangle = 1$, $\left\langle\!\left\langle {3 \atop 1} \right\rangle\!\right\rangle = 8$ and $\left\langle\!\left\langle {3 \atop 2} \right\rangle\!\right\rangle = 6$. Here are some values arranged in the last triangle you will see for some time:

1						1							
2					1		2						
3				1		8		6					
4			1		22		58		24				
5		1		52		328		444		120			
6	1		114		1452		4400		3708		720		
7	1	240		5610		32120		58140		33984		5040	

Again, this triangle shows some interesting properties. The sum of each row n is of course the doublefactorial of $2n-1$, *e.g.* $\left\langle\!\left\langle {3 \atop 0} \right\rangle\!\right\rangle + \left\langle\!\left\langle {3 \atop 1} \right\rangle\!\right\rangle + \left\langle\!\left\langle {3 \atop 2} \right\rangle\!\right\rangle = 5!! = 15$ and $\left\langle\!\left\langle {4 \atop 0} \right\rangle\!\right\rangle + \left\langle\!\left\langle {4 \atop 1} \right\rangle\!\right\rangle +$

$\left\langle\!\left\langle {4 \atop 2} \right\rangle\!\right\rangle + \left\langle\!\left\langle {4 \atop 3} \right\rangle\!\right\rangle = 7!! = 105$. Neatly, the oughter right-hand diagonal shows the factorials. The last value of each row, $\left\langle\!\left\langle {n \atop n-1} \right\rangle\!\right\rangle$, hence, is $n!$.

Here comes the formula to compute the Eulerian numbers of the second kind recursively:

$$\left\langle\!\left\langle {n \atop m} \right\rangle\!\right\rangle = (2n - m - 1) \left\langle\!\left\langle {n-1 \atop m-1} \right\rangle\!\right\rangle + (m+1) \left\langle\!\left\langle {n-1 \atop m} \right\rangle\!\right\rangle, \tag{4.53}$$

with $\left\langle\!\left\langle {0 \atop 0} \right\rangle\!\right\rangle = 1$ and $\left\langle\!\left\langle {0 \atop m} \right\rangle\!\right\rangle = 0$. In Haskell this is:

```
eulerian2 :: Natural → Natural → Natural
eulerian2 0 0  = 1
eulerian2 0 _  = 0
eulerian2 n m = (2 * n - m - 1) * eulerian2 (n - 1) (m - 1)
              + (m + 1)         * eulerian2 (n - 1) m
```

The Eulerian numbers, as many other things in mathematics, are named after Leonhard Euler (1707 – 1783), a Swiss mathematician and one of the most important mathematicians of all time. He is certainly the most important mathematician of his own time, and, for sure, the most productive one ever. He turned his family into a kind of math factory that produced thousands of papers in number theory, analysis, mechanics, optics, astronomy, ballistics and even music theory. He is also regarded as the founder of graph theory and topology. Modern terminology and notation is strongly based on Euler. He proved many theorems and made even more conjectures. But he was also a great math teacher, as his *Letters to a German Princess* show in which he lectured on mathematical subjects to non-mathematicians.

5. Primes

5.1. Gaps

The natural numbers are very simple in the sense that there are very simple ways to enumerate them all (given that we have infinite time and patience to do so, of course). Given a starting point, such as 0, 1 or any other number, we can generate all numbers from this point on just by counting up. This fact is so obvious that it appears ridiculous to visualise it like this:

1	2	3	4	5	6	7	8	9	10	...
1	2	3	4	5	6	7	8	9	10	...

A formula to generate this sequence might be: $f(n) = f(n-1) + 1$, *i.e.* the value for n equals the value of $n-1$ plus 1, which can be simplified to the closed form n. In fact, the function $f(n-1)+1$, starting with $n = 0$ is just the definition of natural numbers. The number used for counting, 1, is therefore called *unity*.

A slightly more interesting sequence is $f(n) = f(n-1) + 2$, that is is the multiplication table of 2, *i.e.* $f(n) = 2 \times n$:

1	2	3	4	5	6	7	8	9	10	...
2	4	6	8	10	12	14	16	18	20	...

This table shows exactly half of all numbers, *viz.* the even numbers. So, let us look at the second half of the numbers, the odd ones. The following table, correspondingly, shows a third of all numbers, namely those, divisible by 3:

1	2	3	4	5	6	7	8	9	10	...
3	6	9	12	15	18	21	24	27	30	...

Every second number in this table is even and, hence, already appears in the previous table. The logically next table, the one containing multiples of 4, would contain a quarter of all numbers. But it is not very interesting, since all numbers in that table already appear in the multiplication table for 2. With 5, however, we could get some new numbers to fill up the second half:

1	2	3	4	5	6	7	8	9	10	...
5	10	15	20	25	30	35	40	45	50	...

We see that every second number was already in the multiplication table for 2 and every

third number in that for 3. So, it seems it is not too easy to get all numbers together – there is a lot of repetition in multiplication tables!

We can safely jump over the logically next table, 6, because all numbers in that table are already in the second table and, since 6 is a multiple of 3, in the third table as well. In the hope to find some more numbers of the second half, we continue with 7:

1	2	3	4	5	6	7	8	9	10	...
7	14	21	28	35	42	49	56	63	70	...

Every second number is in the table for 2, every third number appears also in the third table and every fifth number appears in the table for 5. The first new number we see is $7 \times 7 = 49$. It appears that we are running short of novelties! Indeed, we now have to skip 8 (since it is even), 9 (since it is a multiple of 3) and 10 (since it is not only even, but also a multiple of 5). The first number with some potential to bring something new is 11:

1	2	3	4	5	6	7	8	9	10	...
11	22	33	44	55	66	77	88	99	110	...

With some disappointment, we have to admit that there is no number up to $n = 10$ that we have not seen so far (besides 11 itself). With 12 we will not have more luck, since 12 is even. So let us have a look at 13 before we give up:

1	2	3	4	5	6	7	8	9	10	...
13	26	39	52	65	78	91	104	117	130	...

So, filling up the second half of the numbers does not appear to be an easy task. Very few numbers are really "new" in the sense that they are not multiples of numbers we have already seen. On second thought, this fact is not so curious anymore. The numbers that appear in a table for k have the form $n \times k$ and, for all $n < k$, we, of course, have seen n already in the tables for n, since $n \times k$ is the same as $k \times n$. In the table for 7, 7×7 was therefore the first number we had not yet seen. For numbers greater than k, the same is true for all multiples of earlier numbers; so 8×7, for instance, appears in the multiplication table of 4, since 8 is a multiple of 4. 9×7 appears in the table for 3, since 9 is a multiple of 3 and so on.

The really curious fact in this light is another one: that there, at all, are numbers that do not appear in tables seen so far. In fact, the numbers 2, 3, 5, 7, 11 and 13 never appear in any table, but their own. For 2 and 3, this is obvious, because 2 is the number we are starting with, so there simply is no table of a smaller number in which 2 could appear. Since 3, 5, 7 and so on are all odd, they cannot appear in the table for 2. But could they not appear in a later table? 5, obviously, cannot appear in the table for 3, since 5 is not a multiple of 3. The same holds for 7 and 7 is not a multiple of 5 either, so it will not appear in that table. We can go on this way with 11, 13 and any other number not seen so far and will always conclude that it cannot have appeared in an earlier table, since it is not a multiple of any number we have looked at until now.

We may think that this is a curiosity of small numbers until, say, 10 or so. But, when we go further, we always find another one: 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. These unreachable numbers are called *primes* as opposed to *composites*, which are composed of other numbers in terms of multiplication. A prime, by contrast, is a number that is divisible only by 1 and itself.

Until now, we did not make a great effort to list prime numbers. In fact, almost every second number so far was prime. But it is hard to predict the next prime for any given n. For instance, what is the next prime after 47? Since, $48 = 2 \times 2 \times 2 \times 2 \times 3$, $49 = 7 \times 7$, $50 = 2 \times 5 \times 5$, $51 = 3 \times 17$ and $52 = 2 \times 2 \times 13$ none of these numbers is prime. The next one is only 53. So, what is the next prime after 6053 (which itself is prime)? Is there any more prime at all after this one or any other prime greater than the last prime we found so far in general? Let us examine how to find primes and which number is the last prime.

5.2. Finding Primes

In the previous section, we have already adopted a classical method to find primes, namely a *sieve*. We started by picking 2 and generated all multiples of 2. We then took the first number greater than 2 that was not in the list of multiples of 2, 3, and generated all its multiples. We continued by picking the first number greater than 3 that was neither in the list of multiples of 2 nor in that of multiples of 3, *viz.* 5. As a result, we identify primes as the heads of these list, $2, 3, 5, 7, 11, 13, \ldots$.

This method was invented by Eratosthenes, a polymath of the 3^{rd} century BC, who was librarian of the library of Alexandria, which we already visited discussing Euclid. Eratosthenes wrote books on history, poetry, music, sports, math and other topics; he is considered founder of geography and he estimated the circumference of the earth remarkably close to the accurate value of about 40 000km known today. The exact precision of his estimate is subject to dispute – scholars refer to values he may have given between 39 500km, which would correspond to an error of only 1.5%, and 46 500km, which would be a more likely deviation of about 15%.

The *Sieve of Eratosthenes* goes as follows: write all numbers from 2 up to n, the upper limit of the range, for which you want to calculate the prime numbers. Eliminate all multiples of 2. Take the first number greater than 2 that was not yet eliminated and eliminate all its multiples. Take the first number greater than that number and proceed as before until the next number is n.

The following code shows an implementation in Haskell without upper limit, *i.e.* it finds all prime numbers exploiting lazy evaluation:

$$\begin{aligned} &\mathit{erato} :: [\mathit{Natural}] \\ &\mathit{erato} = \mathit{sieve}\ 2\ [2..] \end{aligned}$$

$$\begin{array}{l}\textbf{where } sieve\ x\ xs = \textbf{case } filter\ (\lambda p \to p\ `rem`\ x \not\equiv 0)\ xs\ \textbf{of}\\ \qquad\qquad\qquad\qquad [\,] \to [x]\\ \qquad\qquad\qquad\qquad ps \to x : sieve\ (head\ ps)\ ps\end{array}$$

The function *sieve* has two arguments: the current number (starting with 2) and the list of all numbers starting with 2. The function filters all numbers out that leave remainder 0 divided by 2. If we have reached the last number, *i.e.* there are no more numbers in the list of all numbers, we just return the current number (which to know, if this case ever manifests, would be of the utmost interest). Otherwise, we insert the current number as head of the recursion on *sieve* that takes the head of the filtered list as the current number and the filtered list itself. In the first round we would get:

$sieve\ 2\ [2,3,4,5,6,7,8,9,...] = 2 : sieve\ 3\ [3,5,7,9,...]$

and continue with:

$sieve\ 3\ [3,5,7,9,...] = 3 : sieve\ 5\ [5,7,...]$
$sieve\ 5\ [7,...] = 5 : sieve\ 7\ [...]$

and so on.

If you call *erato* just like that, the function goes on forever, that is until the resources of your computer are exhausted or you interrupt the program. A call like *take n erato* would show the first n primes; a call like *takeWhile* $(<n)$ *erato* would show all primes less than n; be careful: a call like *takeWhile* $(\not\equiv n)$ *erato*, where n is not itself a prime, will run forever!

The principal merit of *erato* is its simplicity and conciseness; it is not very efficient however. Much more efficient is a very nice variation of the sieve of Eratosthenes given in the *Haskell Road*. This implementation is based on a primality test to decide whether a number enters the list of primes or not. The test itself uses a list of primes:

$$\begin{array}{lll}nextPrime :: [Natural] \to Natural \to Natural & & \\ nextPrime\ [\,]\ n & = n & \\ nextPrime\ (p : ps)\ n & \mid rem\ n\ p \equiv 0 & = p\\ & \mid p \uparrow 2 > n & = n\\ & \mid otherwise & = nextPrime\ ps\ n\end{array}$$

nextPrime receives a list of primes and a number, n, to be tested for primality; if n is a prime, this number is returned, otherwise, the first prime dividing that number is returned.

If the first prime in the list, p, divides n, then p is returned; otherwise, if n is less than p^2, n is returned. Since p does not divide n and n is smaller than p^2 and p is the smallest prime remaining in the list, there will be no two primes in the list that multiplied with each other yield n. Because the primes remaining in the list are all greater than p, any

product of two of them will obviously be greater than p^2 and, hence, greater than n. Therefore, n must be a prime. Otherwise, if n is greater than p^2, we continue the search with the next prime in the list.

The following code sequence turns *nextPrime* into a Boolean primality test:

$$
\begin{aligned}
&prime :: Natural \to Bool \\
&prime\ n \mid n \equiv 1 \quad\ = False \\
&\qquad\ \mid otherwise = ldp\ n \equiv n \\
&ldp :: Natural \to Natural \\
&ldp = nextPrime\ allprimes
\end{aligned}
$$

ldp stands for *least dividing prime* and yields the first prime number that divides n. It calls *nextPrime* with a list of all primes. The function is used in the test function *prime*, which compares n with *ldp* n, *i.e.* if the first prime that divides n is n, then n is a prime itself.

Now, where does the list of all primes come from? This is the beautiful part of the code. It is created in terms of *prime* used as a filter on the natural numbers:

$$
\begin{aligned}
&allprimes :: [Natural] \\
&allprimes = 2 : filter\ prime\ [3\,..]
\end{aligned}
$$

Note that it is essential here to add 2 explicitly to the result, since it is 2 that bootstraps the algorithm. If we created *allprimes* as *filter prime* $[2\,..]$, we would introduce an infinite regress: *allprimes* would try to filter prime numbers using *prime*, which, in its turn, uses *nextPrime* with *allprimes*, which, again calls *prime* to get a prime out of the list of numbers. With 2 already in that list, *prime* will first test primality of n, which is 3 in the first round, with the head of *allprimes*, *i.e.* 2. Since 2 does not divide 3 and $2^2 > 3$, 3 is returned and the algorithm is up and running.

A lot of sieves have been developed since Eratosthenes and many of them are much more efficient than Eratosthenes' sieve. A particular interesting one is the *Sieve of Sundaram*, which was developed by an Indian math student in the 1930ies. Sundaram's sieve finds the odd primes up to a limit of $2n+2$ with the minor drawback that 2 is not in the list. Since 2 is the only even prime, this issue is easily solved by just adding 2 explicitly.

The algorithm is based on the fact that odd composites have odd factors. As we have already seen in the previous chapter, odd numbers can be represented as $2n+1$. Odd composites, therefore, have factors of the form $(2i+1)(2j+1)$. If we multiply this out, we obtain $4ij+2i+2j+1$. We can split this sum into two terms of the form $(4ij+2i+2j)+1$ using the associative law. We move 2 out of the first term yielding $2(2ij+i+j)+1$. Sundaram's algorithm cleverly removes all numbers of the form $2ij+i+j$ from the list of all numbers up to a given limit, doubles the remaining numbers and adds 1 to the result. Since all resulting numbers are again of the form $2n+1$, they are all odd and,

since all numbers of the form $2ij + i + j$ have been removed, we know that none of the resulting odd numbers $2n + 1$ is composite.

Here is a possible implementation:

```
sund :: Natural → [Natural]
sund n = 2 : [2 * x + 1 | x ← [1 .. n] \\
                          [i + j + 2 * i * j | i ← [1 .. lim 0],
                                               j ← [i .. lim i]]]
  where lim 0 = floor $ sqrt (fromIntegral n / 2)
        lim i = floor $ fromIntegral (n − i) / fromIntegral (2 * i + 1)
```

We create the list of all numbers $[1 .. n]$ and subtract from it the list of all numbers of the form $i + j + 2ij$. the numbers i and j are generated as $[1 .. lim\ 0]$ and $[i .. lim\ i]$ respectively.

With the limits *lim*, we avoid multiplying pairs of numbers twice, such as 2×3 and 3×2. To achieve this, we generate i starting from 1 up to the greatest whole number less than the square root of half of n. The reasoning for this limit is that we do not want to generate too many is beyond n, since, at the end, we want to have primes only up to $2n + 2$, *i.e.* $i + j + 2ij <= n$ (since, at the end, we still multiply by 2). The smallest j we will use is i, which, injected into the above formula, yields $2i^2 + 2i$. For huge is, the second part is negligible, so instead of using this formula, we simplify it to $2i^2$ and generate is from 1 to a number x that squared and duplicated is at most eqal to n. This number, obviously, is the square root of half of n.

For j, the lower limit is i; the upper limit, $i + j + 2ij <= n$, can be expressed in terms of n and i: First we bring i on the other side: $j + 2ij <= n - i$; now we factor j out: $j(1 + 2i) <= n - i$, divide by $1 + 2i$: $j <= \frac{n-i}{2i+1}$ and get the limit defined in the code as *lim i*.

A bit confusing might be that we use *lim* 0 to calculate the limit for i. This is just a trick to use the same function for i and j. In fact, the limit for i is a constant relative to n. We could have it defined without an argument at all. But this way, using the same function for i and j, it looks nicer.

Sieves do a great job in creating lists of prime numbers. They are weak, when it comes to finding new prime numbers. Since sieves depend on primes discovered so far, any search for new prime numbers must start at the beginning, *i.e.* with 2. How far we will get, depends on available time and computing power. Those are serious limits in finding new primes, which may lie far ahead on the number ray. Instead of going forward step by step, as sieves do, we might want to make huge leaps forward ignoring thousands and millions of numbers.

A very simple method to guess a prime is based on the observation that primes often come in pairs. This, obviously, does not introduce huge leaps leaving thousands and

millions of numbers out, it just leaves one number out. For instance 3 is prime and $3 + 2 = 5$ is prime too. Now, 5 is prime and $5 + 2 = 7$ is prime as well. 11 is prime and $11 + 2 = 13$ is prime too. So are 17 and 19, 29 and 31, 41 and 43, 59 and 61 and 71 and 73. But there are also many primes without a twin, *e.g.* 37, 53, 67, 83 and 89. In fact, if all primes came as twins, every second number would be prime and that, definitely, is not true. How many prime pairs there are and whether there are infinitely many of them is not known today. It is an unresolved problem in mathematics.

Bigger leaps are introduced by so called *Mersenne primes*, which have the form $2^n - 1$. This method of finding primes is based on the fact that many primes are powers of 2 minus 1, for instance: $3 = 2^2 - 1, 7 = 2^3 - 1, 31 = 2^5 - 1, 127 = 2^7 - 1$. Not all powers of 2, however, lead to Mersenne primes. $2^4 - 1 = 15$ is the composite number 3×5. $2^6 - 1 = 63$ is composite as well. So are $2^8 - 1 = 255 = 3 \times 5 \times 17, 2^9 - 1 = 511 = 7 \times 73, 2^{10} - 1 = 1023 = 3 \times 11 \times 31$. It turns out that all numbers of the form $2^n - 1$, where n is composite, are composite numbers as well. Mersenne primes are hence restricted to $2^p - 1$ with p a prime number. But even under this condition, not all Mersenne numbers are indeed primes, for instance $2^{11} - 1 = 2047 = 23 \times 89$. It is not known how many Mersenne primes there are and if there are infinitely many of them. This, again, is still an open problem in mathematics.

Most huge primes found today are Mersenne Primes. The search is assisted by the *Great Internet Mersenne Prime* Search (GIMP), which already found more than a dozen primes, most of them are also the greatest prime numbers known so far. The greatest of them is more than 10 million digits long.

Marin Mersenne (1588 – 1648), after whom Mersenne primes are named, was a priest who taught theology and philosophy in France. He wrote on philosopy and theology, but also on math and acoustics and he edited works of acient mathematicians such as Euclid and Archimedes. Mersenne was also a great organiser of science who corresponded with many mathematicians and scientists of his time including René Descartes, Galileo, Pierre Fermat and Etienne Pascal, the father of Blaise.

The main contribution to math is a list of Mersenne primes he compiled up to the exponent 257. There are some flaws in the list, he missed some primes and added some composite numbers. However, the effort is still impressive considering that all the math was done by hand.

Even greater leaps are introduced by searching for *Fermat primes*, named after our friend Pierre Fermat. These primes have the form $2^{2^n} + 1$. 3, for example, is a Fermat prime, since $2^{2^0} + 1 = 2^1 + 1 = 2 + 1 = 1$. 5, as well is a Fermat prime, since $2^{2^1} + 1 = 2^2 + 1 = 4 + 1 = 5$. The next Fermat prime is $2^{2^2} + 1 = 2^4 + 1 = 16 + 1 = 17$. The next is $2^{2^3} + 1 = 2^8 + 1 = 256 + 1 = 257$. The next, as you may have already guessed, is $2^{2^4} + 1 = 2^{16} + 1 = 65536 + 1 = 65537$. $2^{2^5} + 1 = 4294967297$, however, is not a prime, since $4294967297 = 641 \times 6700417$. The next one $2^{2^6} + 1 = 18446744073709551617$, as well, is composite, since $18446744073709551617 = 274177 \times 67280421310721$. As with Mersenne primes, we see that not all numbers constructed following the Fermat prime

formula are actually primes. There is just a certain probability (which, hopefully, is greater than that of randomly picking a number) that a Fermat prime is indeed a prime. The largest Fermat prime known today is actually $2^{16}+1$ and there is evidence that the number of Fermat primes is finite.

5.3. The fundamental Theorem of Arithmetic

The theorem with the somewhat bombastic name *fundamental theorem of arithmetic* states that every natural number other than unity either is a prime or can be expressed as a unique product of primes, its *prime factorisation*. "Unique", here, means that, for every natural number, there is exactly one prime factorisation.

On the first sight, this might appear trivial, but consider an example: 24 can be expressed as product in different ways, for instance: $24 = 3 \times 8$ and $24 = 4 \times 6$. 4, 6 and 8, however, are not prime numbers: $4 = 2 \times 2$, $6 = 2 \times 3$ and $8 = 2 \times 2 \times 2$ and, in consequence, the different products reduce to $24 = 2 \times 2 \times 2 \times 3$. In other words, there are many ways to obtain a number by multiplying other numbers, but there is only one way to obtain this number by multiplying prime numbers.

Let us have a look at some prime factorisations: 2 is prime and its prime factorisation is just 2. 3 is prime and its prime factorisation is just 3. 4 is composite and its prime factorisation is $2 \times 2 = 2^2$. Here are the numbers 5 to 20 and their prime factorisation:

$5 = \{5\}$
$6 = \{2, 3\}$
$7 = \{7\}$
$8 = \{2^3\}$
$9 = \{3^2\}$
$10 = \{2, 5\}$
$11 = \{11\}$
$12 = \{2^2, 3\}$
$13 = \{13\}$
$14 = \{2, 7\}$
$15 = \{3, 5\}$
$16 = \{2^4\}$
$17 = \{17\}$
$18 = \{2, 3^2\}$
$19 = \{19\}$
$20 = \{2^2, 5\}$

The facts constituting the fundamental theorem are known since antiquity and are outlined and proven in Book VII of the Elements. Many other proofs have been suggested since approaching the theorem from different angles. Indeed, its name already suggests that it is not an ordinary theorem, but is right in the centre of a whole bunch of problems

in arithmetic.

In this section, we will look at one proof. We will basically establish that 1. every number can be factored into primes and 2. for any number, this factorisation is unique. We will also see, as a corollary to 1, that there are infinintely many primes.

The first theorem – that every number can be factored into primes – is quite simple. We distinguish two cases: the number is either prime or composite. If it is prime, the factorisation is simply that number. Otherwise, if it is not prime, there are two numbers, a and b such that $a \times b = n$, where n is the number in question. Now, a and b either are prime numbers or there are other numbers a_1 and a_2, such that $a_1 \times a_2 = a$, and b_1 and b_2, such that $b_1 \times b_2 = b$. Indeed, that a number is composite means essentially that there exist two other numbers that divide this number. Therefore, every number can be expressed as a product of prime numbers. □

Now we will prove the fundamental theorem of arithmetic. We will use a proof technique called *indirect proof* or *proof by contradiction*. We have already used this type of proof silently and it is in fact a quite common tool in reasoning. The strengths of indirect proofs are that they are often very simple, much simpler than direct proofs, and that they can prove things that we cannot demonstrate. The latter is of major importance, in particular when talking about infinitely big or small things. We can prove, for example, that there are infinitely many primes without the need to construct infinitely many primes.

Indirect proofs also have some drawbacks, though. The most important one is that they do not provide a method to compute the result. An indirect proof may be used to prove the existence of something, but does not provide a method to construct that "something" (such as the result of a given function or the greatest prime number). They are therefore sterile in the sense that we obtain only the abstract knowledge that some theorem is true, but no further insight into the concepts under investigation and no new methods to work with these concepts. That is quite poor and, indeed, many mathematicians have expressed their inconvenience or even disgust when confronted with indirect proofs. There is even a philosophical tradition within mathematics, *mathematical constructivism*, that aims to find direct proofs for all mathematical theorems for which only indirect proofs are known today. Without taking side in the philosophical debate, most mathematicians would agree today that an indirect proof should be the last resort, *i.e.*, when there is a direct proof, it should be preferred.

So, what is an indirect proof in the first place? The proof of a theorem A works by demonstrating that the assumption $\neg A$ leads to a contradiction. We therefore start the proof by stating what we assume to be true. Let us look, as a simple example, a variant of Euclid's proof that there are infinitely many primes:

Assume that there is a finite number of primes. Then we can enumerate the set P of all primes as $P = \{2, 3, 5, \ldots, p\}$, where p is the last prime. The product of the primes in this set is a composite number: $n = 2 \times 3 \times 5 \times \cdots \times p$. So, what about $n + 1$?

This number is either prime, then P was incomplete, which immediately contradicts our assumption; or it is composite and then it has a prime factorisation. But none of the primes in P can be part of that factorisation, because no number greater than 1 divides both n and $n+1$: 2 divides n and $n+2$, but not $n+1$; 3 divides n and $n+3$, but not $n+1$, p divides n and $n+p$, but not $n+1$. Therefore, there must be at least one prime that is not in P, which, again, contradicts our assumption. □

There are, hence, infinitely many primes.

We now prove the fundamental theorem of arithmetic, *i.e.* that there is only one way to factor any given number into primes. We prove this by contradiction and assume that there is at least one number for which it is actually possible to find more than one prime factorisation. We must be very cautious about such assumptions, since we want the contradiction to hit the right place. We, therefore, assume that there is *at least* one number without assuming anything further – there may be just that one number, there may be many or it may be even true for all composites that there is more than one prime factorisation.

In the following, however, we will talk about just one such number. If there is only one number with that property, we talk about that one. Otherwise, if there are many, we talk about the smallest number with that property. We can simply verify for small numbers, in particular 4, the smallest composite number, that there is only one prime factorisation, *i.e.* 2×2. If there is a number with more than one factorisation, it is definitely greater than 4.

We call this smallest number for which more than one prime factorisation exist m:

$$m = p_1 \times p_2 \times \cdots \times p_r \tag{5.1}$$

$$m = q_1 \times q_2 \times \cdots \times q_s \tag{5.2}$$

The ps and qs in these equations are all primes. Also, the ps and qs differ, such that at least one p is not in the list of qs and vice versa. To illustrate this, the ps could be 3 and 8 (if 8 was a prime number) and the qs could be 4 and 6 (if 4 and 6 were prime numbers) in the factorisations of 24. 4, 6 and 8, of course, are not prime numbers. But what we claim (to, hopefully, create a contradiction) is that there are numbers for which different decompositions are possible even with prime numbers.

We further assume that the two factorisations, the ps and qs above are ordered, such that

$$p_1 \leq p_2 \leq \cdots \leq p_r$$

and

$$q_1 \leq q_2 \leq \cdots \leq q_s.$$

Now, p_1 and q_1 must be different, *i.e.* either $p_1 < q_1$ or $q_1 < p_1$, for, if $p_1 = q_1$, we could divide both sides, the p-factorisation and the q-factorisation, by p_1 and obtain a number with two factorisations that is actually smaller than m – but we assume that m is the smallest number with that property. This assumption forces us to also assume that either $p_1 < q_1$ or $q_1 < p_1$. Let us say that p_1 is the smaller one. (It is irrelevant which one it actually is. If we chose q_1 to be the smaller one, we would just swap ps and qs in the following equations.) We can now compute a number m':

$$m' = m - (p_1 \times q_2 \times \cdots \times q_s), \tag{5.3}$$

for which it, obviously, holds that $0 < m' < m$, *i.e.* m' is smaller than m and, hence, has a unique prime factorisation (since m is the smallest number with the property that it has more than one prime factorisation).

Now, by substituting for m, we derive:

$$m' = (p_1 \times p_2 \times \cdots \times p_r) - (p_1 \times q_2 \times \cdots \times q_s) \tag{5.4}$$

and, by factoring p_1 out, we get:

$$m' = p_1 \times (p_2 \times p_3 \times \cdots \times p_r - q_2 \times q_3 \times \cdots \times q_s) \tag{5.5}$$

and clearly see that p_1 is a factor of m'. But we can also derive

$$m' = (q_1 \times q_2 \times \cdots \times q_s) - (p_1 \times q_2 \times \cdots \times q_s), \tag{5.6}$$

from which, by dividing by $q_2 \times q_3 \times \cdots \times q_s$, we can further derive

$$m' = (q_1 - p_1) \times (q_2 \times q_3 \times \cdots \times q_s). \tag{5.7}$$

Since p_1 is a factor of m', it must be a factor of either $q_1 - p_1$ or $q_2 \times q_3 \times \cdots \times q_s$. (Remember that there is only one way to factor m' into primes, since it is smaller than m, the smallest number with more than one prime factorisation.) It cannot be a factor of $q_2 \times q_3 \times \cdots \times q_s$, since all the qs are primes and greater than p_1. So, it must be a factor of $q_1 - p_1$. In other words, there must be a number, say, h for which it holds that

$$q_1 - p_1 = p_1 \times h \tag{5.8}$$

By adding p_1 to both sides we get $q_1 = p_1 \times h + p_1$. By factoring p_1 out on the right-hand side of the equation we obtain:

$$q_1 = p_1 \times (h + 1) \tag{5.9}$$

In other words, p_1 is a factor of q_1. But this is a contradiction, since q_1 is prime. □

5.4. Factoring

In the previous section, we have made heavy use of factoring, *i.e.* of decomposing a number into its prime factors. But we did so without indicating an algorithm. We have just stated that $9 = 3 \times 3$ or $21 = 3 \times 7$. For such small numbers, that is certainly acceptable – we have learnt our multiplication tables in school and can immediately say that $32 = 2^5$. With bigger numbers, this becomes increasingly difficult and, therefore, we clearly need an algorithm.

Unfortunately, no efficient factorisation algorithm is known. In fact, there is one that is sufficiently fast, but that one does not run on traditional computers. It is a quantum algorithm. It was developed by the American mathematician Peter Shor in 1994. It has already been implemented on real quantum computers several times and in 2012 it was used to factor 21. That is not the greatest number factored by a quantum computer so far. With another approach, not involving Shor's algorithm, but a simulation method called *abbiatic quantum computation*, a Chinese team achieved to factor 143 as well in 2012.

We will not enter the quantum world here, but rather stick to classical algorithms, even if this leaves us with highly inefficient programs that need exponential time to factor numbers. There is something good about the fact that factorisation is hard: many algorithms in cryptography are based on it. So, would factorisation be made much simpler overnight, our online banking passwords and other data would not be secure anymore. On the other hand, there is no proof that factorisation will remain a hard problem forever. It is not even known to which *complexity class* factorisation belongs.

Complexity classes, in theoretical computer science, descibe the level of difficulty of solving a problem. There are a lot of complexity classes; for the moment, two are sufficient: P and NP. P stands for *polynomial time*. Polynomial refers to formulas of the form $n^a + c$ or similar, where n is the size of the input (for example the number we want to factor) and a and c are constants or even other – but similar – formulas. Essential is that, in formulas that describe the cost of algorithms that are solutions to P-problems, n does never appear as exponent. Algorithms, whose cost is described by formulas where n appears in exponents, *e.g.* a^n, are exponential. Such algorithm are considered unfeasible for input of relevant size.

The interesting point about NP, now, is that solutions for problems in this class need an incredible amount of steps, such as exponential time where n appears in the exponent, but, if a potential answer is known, it is extremely easy to verify if this answer is correct. The acronym NP means *non-deterministic polynomial time*. The name refers to the fact that if an answer is known it can be verified in polynomial time (the verification, hence, is a P-problem). Where the answer comes from, however, is unclear – it appears out of the blue, in a non-determinstc way. It could be chosen randomly, for instance, or a magus, like Merlin, could have suggested it. Notice that this is definitely a characteristic of factorisation. Given a number such as 1771, it may be hard to say what its prime factors are. If we were told, however, that 7 is one of the factors, we can use division to verify that $1771 \bmod 7 = 0$ and even to reduce the problem to finding the factors of $1771/7 = 253$.

Today, factorisation is not considered to be in NP. Shor's algorithm is a quantum probabilistic algorithm (belonging to class BQP – *bounded-error quantum probabilistic*) and, therefore, it is assumed that it may belong also to a classic probabilistic class (such as BPP – *bounded-error probabilistic polynomial*). This assumption is supported by the fact that there appears to be a consistent distribution of primes – but more on that later.

Factorisation is an area with extensive research. To that effect, there are many algorithms available, most of them exploiting probabilistic in some way or another. We will stick to an extremely simple approach that, basically, uses a trial-and-error method. We simply go through all prime numbers, for this purpose we can use one of the prime number sieves, and try to divide the input number by each one until we find a prime that divides this number. If we do not find such a prime, the number must be prime and its factorisation is just that number.

Here is the searching algorithm:

$$
\begin{aligned}
&\mathit{findf} :: \mathit{Natural} \to \mathit{Natural} \to [\mathit{Natural}] \to (\mathit{Natural}, \mathit{Natural}) \\
&\mathit{findf}\ n\ _\ [\,] = (1, n) \\
&\mathit{findf}\ n\ l\ (p : ps) \mid p > l \qquad = (1, n) \\
&\qquad\qquad\qquad\quad \mid \mathit{otherwise} = \mathbf{case}\ n\ `\mathit{quotRem}`\ p\ \mathbf{of} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (q, 0) \to (p, q) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (_, r) \to \mathit{findf}\ n\ l\ ps
\end{aligned}
$$

The function *findf* receives three arguments: the input number, n, an upper limit, l, and the list of primes. It yields a pair of numbers: the first is the prime number that divides n, the second is the quotient of n divided by the prime. If the list of primes is exhausted, a case that, as we know from the previous section, is extremely rare, we just return $(1, n)$ to indicate that we have not found a proper solution. Otherwise, we check if we have reached the upper limit. In this case, we again yield $(1, n)$ to signal that no proper solution was found. Otherwise, we divide n by the first prime in the list and, if the remainder is zero, we yield this prime and the quotient. Otherwise, we just continue with the remainder of the prime list.

The result of this function, hence, is one prime factor and another number that, multiplied by the factor is n. If this other number is prime as well, we are done. Otherwise, we must continue factoring this other number:

```
trialfact :: Natural → [Natural]
trialfact 0 = []
trialfact 1 = []
trialfact n = let l = fromIntegral $ floor $ sqrt (fromIntegral n)
              in case findf n l allprimes of
                   (1, _) → [n]
                   (p, q) → if prime q then [p, q]
                                       else p : trialfact q
```

This function receives the number to be factored and yields the list of factors of this number. 0 and 1 cannot be factored. The result in this case, hence, is simply the empty list. For all other cases, we first determine the upper limit as the greatest natural number less than the square root of n. We already discussed the reasoning for this upper limit in the context of the Sundaram sieve: we assume an ordered list of primes and, when the current prime is greater than this limit, the square root of n, the product of any two primes greater than this prime will necessarily be greater than n. There is hence no need to continue the search.

We then call *findf* with n, the limit l and *allprimes*. If the result is $(1, _)$, *i.e.* if *findf* has not found a proper solution, then n must be prime and we return $[n]$ as the only factor. For other results, we know that p, the first of the pair, is a prime. We do not know this for the second of the pair, which may or may not be prime. If it is prime, we yield $[p, q]$. Otherwise, we call *trialfact* with q and add p to the result.

We could skip the primality test and just continue with *trialfact* q, since, if q is prime, *findf* will yield $(1, q)$ and then *trialfact* would yield $[q]$ anyway. In the hope of finding a primality test that is more efficient then the test we have defined so far (which, itself, uses a sieve to construct *allprimes*), we use this explicit test to obtain some speed-up in the future.

We can use *trialfact* to investigate the distribution of primes further. We start with the numbers $2 \ldots 32$ and create the list of factorisations of these numbers calling
map trialfact $[2 \ldots 32]$:

[2]
[3]
[2, 2]
[5]
[2, 3]
[7]
[2, 2, 2]
[3, 3]
[2, 5]
[11]
[2, 2, 3]
[13]
[2, 7]
[3, 5]
[2, 2, 2, 2]
[17]
[2, 3, 3]
[19]
[2, 2, 5]
[3, 7]
[2, 11]
[23]
[2, 2, 2, 3]
[5, 5]
[2, 13]
[3, 3, 3]
[2, 2, 7]
[29]
[2, 3, 5]
[31]
[2, 2, 2, 2, 2]

What jumps immediately into the eyes is the fact that the lists appear to grow in length – with sporadic prime numbers to appear in between slowing down the growth. Here is the list of the sizes of the factorisations $2 \ldots 32$:

1, 1, 2, 1, 2, 1, 3, 2, 2, 1, 3, 1, 2, 2, 4, 1, 3, 1, 3, 2, 2, 1, 4, 2, 2, 3, 3, 1, 3, 1, 5.

Most factorisations ($2 \ldots 32$) are of size 1 or 2, 1 being the size of prime number factorisations, which consists only of that prime number. Greater numbers appear sporadically, 3, 4 and 5, and seem to grow – in-line with our previous observation. Those greater numbers are certainly caused by repetition of prime numbers, such as $2 \times 2 \times 2 \times 2 = 16$. How would it look if we counted only unique primes? Let us have a try:

1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 2, 1, 2, 2, 1, 1, 2, 1, 2, 2, 2, 1, 2, 1, 2, 1, 2, 1, 3, 1, 1.

The factorisations, now, grow much slower. The first factorisation with more than 3 distinct primes appears only with 30 ($2 \times 3 \times 5$). All other factorisations have either size 1 or 2. Factorisations of size 1 are those of the primes and those containing only repeated primes. But there are no clear patterns that would reveal some regularity among factorisations. Perhaps, it could be helpful to look at the distinction odd versus even sized factorisations? To do that, we should distinguish between factorisations with and without repeated primes. We could, for instance, say that factorisations with repeated primes have the value 0; odd-sized factorisations have value -1 and even-sized factorisations have value 1.

This rule describes the Möbius function, which we could define as:

$$
\begin{aligned}
&\mathit{moebius} :: \mathit{Natural} \to \mathit{Integer} \\
&\mathit{moebius} = \mathit{chk} \circ \mathit{trialfact} \\
&\quad \textbf{where } \mathit{chk}\ f \mid f \not\equiv \mathit{nub}\ f \quad\ = 0 \\
&\qquad\qquad\quad \mid \mathit{even}\ (\mathit{length}\ f) = 1 \\
&\qquad\qquad\quad \mid \mathit{otherwise} \quad\ = -1
\end{aligned}
$$

The *moebius* function for a number n checks whether the factorisation of that number contains repeated primes ($f \not\equiv \mathit{nub}\ f$); if so, the result is 0. Otherwise, if the number of primes in the factorisation is even, the result is 1. Otherwise, the result is -1. Notice that we use *Integer* as output data type, since -1 is not a natural number.

Here are the values of the Möbius function for the numbers $2 \ldots 32$:

-1, -1, 0, -1, 1, -1, 0, 0, 1, -1, 0, -1, 1, 1, 0, -1, 0, -1, 0, 1, 1, -1, 0, 0, 1, 0, 0, -1, -1, -1, 0.

It is still difficult to see regularities. What, if we defined an accumulated Möbius function where each value corresponds to the sum of the values of the Möbius function up to the the current number:

$$
\begin{aligned}
&\mathit{mertens} :: \mathit{Natural} \to \mathit{Integer} \\
&\mathit{mertens}\ n = \mathit{sum}\ (\mathit{map}\ \mathit{moebius}\ [1 \mathinner{..} n])
\end{aligned}
$$

Mapped on the numbers $2 \ldots 32$, this function gives:

0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, -3, -2, -1, -1, -2, -2, -3, -3, -2, -1, -2, -2, -2, -1, -1, -1, -2, -3, -4, -4.

This still does not reveal convincing patterns. Apparently, most numbers have a negative value, but this is true only for the small section we are looking at. The result for *mertens* 39 and *mertens* 40, for instance, is 0. The values for 94 to 100 are all positive peaking with 2 at 95 and 96.

This investigation appears to remain fruitless and we should give it up at least for the moment. We will come back to Möbius and Mertens, however. Even if not visible on the surface, there is something about the concept.

The Möbius function was invented by August Ferdinand Möbius (1790 – 1868), a German mathematician and astronomer, student of Gauss in Göttingen. There are many unusual concepts discovered or developed by this man, for instance, the famous *Möbius strip*, a two-dimensional surface with the uncommon property of having only one side in three-dimensional space.

Franz Mertens (1840 – 1927), after whom the Mertens function is named, is less known. He proposed the Mertens function together with a conjecture concerning its growth that, if proven correct, could have been used to prove the *Riemann Hypothesis* on the distribution of primes. But, unfortunately, Meterns' conjecture was proven wrong and the Riemann Hypothesis remains an enigma until today.

5.5. Factoring Factorials

In the next sections, we will dive into group theory related to primes. There is a problem that makes a very nice link between factoring and the upcoming investigations: factoring factorials.

The factorial of a number n is defined as the product of all numbers $1 \dots n$: $1 \times 2 \times 3 \times \dots \times n$. The immediate consequence of this definition is that all primes in the range $1 \dots n$ are factors of $n!$ and that all factors of $n!$ are primes in the range $1 \dots n$. The first fact is easy to see: since we multiply all numbers $1 \dots n$ with each other, all primes in this range must be part of the product. Furthermore, all composites in this range are products of prime factors in this range and, hence, $n!$ is the product of products of the primes between 1 and n.

For the second fact to become clear, assume for a moment that there were a prime $p > n$ that is a prime factor of $n!$ That would mean that some product of primes $< n$ would result in that p. But that is impossible, since p is a prime. It is not a product of other primes and can therefore not result from multiplying other primes and can thus not be a prime factor of $n!$

To find the prime factors of $n!$ (and, hence, $n!$ itself), we have to ask how often each prime appears in the factorisation of $n!$? This leads to the question how many numbers in the range $1 \dots n$ are actually divisible by a given prime. This is easily answered, when we realise that the range $1 \dots n$ consists of n consecutive numbers. For any number a, every a^{th} number is divided by a. There are, hence, $\lfloor n/p \rfloor$ numbers that are divided by p. Let us look at the example $n = 6$ and $p = 2$. The product $n!$ consists of six numbers: $1, 2, 3, 4, 5, 6$. Every second number is even, namely 2, 4 and 6 itself. This is $6/2 = 3$ numbers. Therefore, 2 must appear at least 3 times as factor in $n!$

But wait: 2 appears 2 times in 4, since the factorisation of 4 is 2^2. How can we tell how many of the numbers in the range are divided by 2 more than once? Well, we just do the same, we divide 6 by 4, since every fourth number is divided by 4. Since $\lfloor 6/4 \rfloor = 1$,

there is only one number in the range $1 \ldots 6$ that is divided by 4 and, hence, divided twice by 2, *viz.* 4 itself. When we add the two results $\lfloor 6/2 \rfloor = 3$ and $\lfloor 6/4 \rfloor = 1$, we get 4. In other words, there are 3 numbers divided by 2 and 1 number divided by 2 twice. Therefore, 2 appears 4 times in the prime factorisation of 6! Let us check if this result is correct: $6! = 720$. The prime factorisation of 720 is *trialfact* 720: $[2, 2, 2, 2, 3, 3, 5]$. So the result is correct.

Let us try to confirm the result for the other primes ≤ 6, namely 3 and 5. $\lfloor 6/3 \rfloor$ is 2; there are hence two numbers divided by 3 and these numbers are 3 and 6. Since $3^2 = 9$ is already greater than 6, there is no number in the range of interest that is divided by 3 twice. Therefore, there are two occurrences of 3 in the prime factorisation of 6! $\lfloor 6/5 \rfloor$ is 1, which means there is only one number divided by 5, namely 5 itself. 5, therefore appears once in the factorisation of 6! With this approach, we arrive at the correct result: $6! = 2^4 \times 3^2 \times 5 = 720$.

We can implement this approach in Haskell to get a speed-up on the factorial computation compared to the laborious multiplication of all numbers $1 \ldots n$. We first implement the logic to find the number of occurrences for one prime:

```
pInFac :: Natural → Natural → Natural
pInFac n p = p ↑ (go p 1)
  where go q e = let t = n `div` q
                 in if t ⩽ 1 then t
                           else let e' = e + 1 in t + go (p ↑ e') e'
```

In *go*, which is called with a prime number p and and expononent $e = 1$, we first compute the quotient $\lfloor n/q \rfloor$. If this quotient is 1 or 0, we immediately yield this number. Otherwise, we continue with p raised to $e + 1$. That is, if $n = 6$ and $p = 2$, then we first compute $t = 6$ `` `div` `` 3, which is 2 and hence greater than 1. We now increment e, which, initially is 1, and call *go* with $2^2 = 4$ and $e = 1 + 1 = 2$.

In the next round t is 6 `` `div` `` 4, which is 1. We return immediately and add 1 to the previous value of t, which was 3, obtaining 4. The function overall yields p raised to the result of *go*, hence $2^4 = 16$.

We call this function in the following code:

```
facfac :: Natural → [Natural]
facfac n = go allprimes
  where go (p : ps) = let x = pInFac n p
    in if x ≡ 1 then [] else x : go ps
```

The function *facfac* results in a list of *Natural*, *i.e.* it yields the factors of $n!$, sucht that *product* (*facfac* n) is $n!$ It calls the internal function *go* on *allprimes*. On the first prime, it calls *pInFac* n p. If the result is 1, *i.e.* if we raised p to zero, we terminate with the empty list. Otherwise, we continue with the tail of *allprimes*.

Here are the results for the numbers $2 \ldots 12$:

$[2]$
$[2, 3]$
$[8, 3]$
$[8, 3, 5]$
$[16, 9, 5]$
$[16, 9, 5, 7]$
$[128, 9, 5, 7]$
$[128, 81, 5, 7]$
$[256, 81, 25, 7]$
$[256, 81, 25, 7, 11]$
$[1024, 243, 25, 7, 11]$

It would be very interesting, of course, to know how much faster *facfac* is compared to the ordinary recursive *fact*. Well, *fact* multiplies n numbers with each other. There are hence $n - 1$ multiplications. By contrast, *facfac* calls *pInFac* for every prime less than or equal to n. *pInFac* is somewhat more complex in computation than multiplication, but when the difference between n and the number of primes up to n is significant, then the difference between the cost of multiplication and that of *pInFac* does not matter. The question remains: how many primes are there among the first n numbers?

5.6. Arithmetic modulo a Prime

On the first sight, there is nothing special about arithmetic modulo a prime. It is plain modular arithmetic where the number to which all the number operations are taken modulo happens to be a prime. However, as we will see in this section, something very significant changes, when it actually is a prime. Let us first recall the properties of modular arithmetic and define a Haskell data type to model it.

As we have seen before, numbers modulo a number n repeat cyclicly, that is any number modulo n is a number in the range $0 \ldots n - 1$. When we think of the clock again, any number, independent of its size, reduces to a number $0 \ldots 11$ taken modulo 12: 1 mod 12 is 1, 5 mod 12 is 5, 10 mod 12 is 10 and 13 mod 12 is again 1, 17 mod 12 is 5 and 22 mod 12 is just 10 again. In other words, the numbers modulo a number n form a finite groupoid (or magma) whith addition and multiplication, that is, addition and multiplication are closed under natural numbers modulo n:

$5 + 3 = 8 \bmod 12 = 8$
$5 + 10 = 15 \bmod 12 = 3$
$13 + 15 = 28 \bmod 12 = 4$
$15 + 17 = 32 \bmod 12 = 8$

and

$5 \times 3 = 15 \bmod 12 = 3$
$5 \times 10 = 50 \bmod 12 = 2$
$13 \times 15 = 195 \bmod 12 = 3$
$15 \times 17 = 255 \bmod 12 = 3.$

If we consider more complex sums and products of the form $a+b+\cdots+c$ and $a\times b\times\cdots\times c$, it becomes apparent that it is more efficient to take the terms and factors modulo n before applying the operation:

$13 \bmod 12 + 15 \bmod 12 = 1 + 3 = 4$
$15 \bmod 12 + 17 \bmod 12 = 3 + 5 = 8$

or:

$13 \bmod 12 \times 15 \bmod 12 = 1 \times 3 = 3$
$15 \bmod 12 \times 17 \bmod 12 = 3 \times 5 = 15 \bmod 12 = 3.$

A short note on terminology may be in order here. You may have realised that two different operators for the modulo are used: one is a binary operator, which we use to indicate that a number is taken modulo another number: $a \bmod n$. The other is an indicator that an operation is taken modulo: $a + b \pmod n$ (with a bigger gap between b and "mod"). The point is to avoid confusion between $a + b \bmod n$, that would be a number a not modulo another number $+b$ modulo n. For example: $15 + 28 \bmod n = 19$, but $15 + 28 \pmod n = 7$ where the whole operation is taken modulo n. Since the distinction between "mod" and " mod " is quite subtle, we will use the convention that, if not obvious from the context or indicated otherwise, we usually take operations modulo n.

Modular arithmetic looks a bit weired at the beginning, for instance $13+15 = 4$ definitely looks wrong. But, in fact, nothing special has changed. We see that the associativity law holds:

$13 + (15 + 17) \pmod{12} = (13 + 15) + 17 \pmod{12} = 9,$
$13 \times (15 \times 17) \pmod{12} = (13 \times 15) \times 17 \pmod{12} = 3.$

There is also an identity for each, addition and multiplication. For addition this is 0 (and, thus, all integer multiples of 12):

$0 \bmod 12 = 0$
$24 \bmod 12 = 0$
$36 \bmod 12 = 0$
$\ldots$

For multiplication, the identity is 1 (and, hence, all multiples of 12 plus 1):

$1 \bmod 12 = 1$
$25 \bmod 12 = 1$
$37 \bmod 12 = 1$

...

It holds of course that for any a divisible by 12:

$$a + b \mod n = b$$

and

$$(a + 1) \times b \mod n = b.$$

Of course, also the distributive law holds:

$$a \times (b + c) \mod n = ab + ac \mod n.$$

This altogether means that numbers modulo n form a semiring with addition and multiplication just as the natural numbers. The difference between natural numbers and numbers modulo n is that the set of natural numbers is infinite whereas the set $1 \ldots n-1$ is of course finite.

Let us have a look at how we can model such a modular semigroup with Haskell. We first define a data type *Module*:

```
data Module = Module Natural Natural
```

A *Module* according to this definition is created by two natural numbers. The first is the modulus n to which we take the second modulo. For instance, *Module* 12 13 is 13 mod 12. To enforce modular arithmetic from the beginning, we provide a constructor:

```
tomod :: Natural → Natural → Module
tomod n a = Module n (a `rem` n)
```

For all (binary) operations on *Module*s, we want to ensure that both parameters have the same modulus. Operating on two *Module*s with different moduli leads to wrong results. For this reason, we will use a guard on all operations:

```
withGuard :: (Module → Module → r) → Module → Module → r
withGuard o x@(Module n  a)
          y@(Module n' b) | n ≢ n'     = error "different moduli"
                          | otherwise = x `o` y
```

Now, we make *Module* instance of some type classes:

```
instance Show Module where
  show (Module n a) = show a
instance Eq Module where
  (≡) = withGuard (λ(Module _ a) (Module _ b) → a ≡ b)
instance Ord Module where
  compare = withGuard (λ(Module _ a) (Module _ b) → compare a b)
```

We *show* the number actually taken modulo n and consider n known in the context. This is much more convenient to read, even if some information is lost.

To check for equality and to compare two *Module*s we apply the guard to the operations, $(\equiv)$ and *compare* respectively.

The next listing shows addition and multiplication:

```
add :: Module → Module → Module
add (Module n a) (Module _ b) = Module n ((a + b) `rem` n)
mul :: Module → Module → Module
mul (Module n a) (Module _ b) = Module n ((a * b) `rem` n)
```

Since the numbers a and b are already modulo n, taking the results of the computations modulo n is an inexpensive operation. Since the maximum for both, a and b, is $n-1$, $a+b$ is at most $2n-2$ and, taking this modulo n, is just $2n-2-n = n-2$.

The greatest value the product $a \times b$ can achieve is $(n-1)(n-1) = n^2 - 2n + 1$. To reduce this value modulo n, one division step is needed and that is indeed the worst case for modular multiplication.

Now, what about subtraction? Subtracting two numbers modulo n should also be in the range $0 \ldots n-1$, but what happens, when the second number is greater than the first one? The normal subtraction beyond zero gives $a - b = -(b - a)$. In modular arithmetic, a negative number $-k$ is interpreted as $n-k$, *i.e.* the minus sign is interpreted as counting down from n, which in fact is the same as counting down from 0, since $n \bmod n$ is just 0.

We can therefore, when we have a negative number in the range $-(n-1) \ldots 0$, just add n to the result: $a - b = -(b - a) + n = n - (b - a)$. For instance for $n = 13$: $3 - 9 + 12 = -6 + 12 = 6$. Note that subtraction handled like this is the inverse of addition: $3 - 9 = 6 \mod 12$ and $6 + 9 = 3 \mod 12$. The point is that addition modulo n with two numbers already modulo n is at most $2n-2$. The remainder of any number up to $2n-2$ is just this number minus n: $6 + 9 = 15 - 12 = 3$. For subtraction, a similar is true: the smallest value, subtraction can produce is $n-1$ (in the case of $0-11$, for instance). The inverse of this operation for negative numbers is then just adding n instead of subtracting it.

We, hence, can implement subtraction as:

$$
\begin{aligned}
&\mathit{sub} :: \mathit{Module} \to \mathit{Module} \to \mathit{Module}\\
&\mathit{sub}\ (\mathit{Module}\ n\ a)\ (\mathit{Module}\ _\ b) \mid a < b \quad\ = \mathit{Module}\ n\ (a + n - b)\\
&\qquad\qquad\qquad\qquad\qquad\quad \mid \mathit{otherwise} = \mathit{Module}\ n\ (a - b)
\end{aligned}
$$

Note that we change the order of the operations for the case that $a < b$. If we performed $a - b$ first, we would subtract beyond zero. Even if the overall result is again a natural number, the intermediate result is not. Therefore, we first add a and n and then we subtract b. In spite of handling problems of natural numbers only, we are on the verge of entering new territory. But we can state a very exciting result: In modular arithmetic, natural numbers and addition form a group: addition is closed, addition adheres to the associativity law, there is an identity, namely 0 (and all multiples of the modulus n), and, for any number modulo n, there is an inverse element.

With addition, subtraction and multiplication defined, we can now make *Module* an instance of *Num*:

$$
\begin{aligned}
&\textbf{instance}\ \mathit{Num}\ \mathit{Module}\ \textbf{where}\\
&\quad (+) \quad = \mathit{withGuard}\ \mathit{add}\\
&\quad (-) \quad = \mathit{withGuard}\ \mathit{sub}\\
&\quad (*) \quad = \mathit{withGuard}\ \mathit{mul}\\
&\quad \mathit{abs}\ a = a\\
&\quad \mathit{signum}\ (\mathit{Module}\ n\ a) = (\mathit{Module}\ n\ (\mathit{signum}\ a))\\
&\quad \mathit{fromInteger}\ i = \mathit{Module}\ (\mathit{fromInteger}\ (i + 1))\ (\mathit{fromInteger}\ i)
\end{aligned}
$$

The basic arithmetic operations are defined as *add*, *sub* and *mul* with the guard to avoid arithmetic on different moduli. As with natural numbers, we ignore *abs*, since all natural numbers are positive. *signum* is just the *signum* of a, *i.e.* a *Module* with 0 for 0 and 1 for any number greater than 0.

fromInteger is a bit tricky. We cannot convert an integer to a *Module* "as such". To convert an integer, we must know to which n the number should be taken modulo. When we say that, by default, an integer i is $i \bmod (i + 1)$, the value of that module is always i, for instance: $2 \bmod 3 = 2$. This appears to be a reasonable default value.

Division, as usual, is not so easy. We would like to define division in a way that it serves as inverse of multiplication, *e.g.* $a \times b/a = b$. That means that for any number a, we want a number a', such that $a \times a' = 1$. Consider the example $n = 6$ and $a = 3$:

$3 \times 0 = 0 \mod 6$
$3 \times 1 = 3 \mod 6$
$3 \times 2 = 0 \mod 6$
$3 \times 3 = 3 \mod 6$
$3 \times 4 = 0 \mod 6$
$3 \times 5 = 3 \mod 6.$

This looks strange: any multiplication of 3 modulo 6 creates either 0 or 3, but not 1,2,4 or 5. The point is that 3 divides 6, in particular $2 \times 3 = 6$. For this reason, 6 divides

every second product of 3 greater than 3, *i.e.* $6, 12, 18, 24, \ldots$ The other half of multiples are just those that leave a remainder of 3 divided by 6.

What, if a does not divide n, like for example with $a = 6$ and $n = 9$?

$6 \times 0 = 0 \mod 9$
$6 \times 1 = 6 \mod 9$
$6 \times 2 = 3 \mod 9$
$6 \times 3 = 0 \mod 9$
$6 \times 4 = 6 \mod 9$
$6 \times 5 = 3 \mod 9$
$6 \times 6 = 0 \mod 9$
$6 \times 7 = 6 \mod 9$
$6 \times 8 = 3 \mod 9.$

We see more variety, but, still, we have only 3 numbers out of 9 possible. Now, every third multiple of six is divisible by 9. This is, of course, because 3 divides 9 and also divides 6. In consequence every third multiple of 6 is divisible by 9. When we think this through, we see that there are indeed many numbers n divisible by numbers $1 \ldots n-1$. Only if n is *coprime* to those numbers, all of them would appear as result of multiplication of any two numbers $a, b \in 1 \ldots n-1$. That two numbers, a and b, are coprime means that they have no common factors, *i.e.*: $\gcd(a, b) = 1$. If we look at an example, where n is coprime of all numbers $1 \ldots n-1$, we see that all numbers $0 \ldots n-1$ actually appear as results of multiplications of two numbers in the range:

$3 \times 0 = 0 \mod 5$
$3 \times 1 = 3 \mod 5$
$3 \times 2 = 1 \mod 5$
$3 \times 3 = 4 \mod 5$
$3 \times 4 = 2 \mod 5.$

The point is that 5 has no common divisor with any of the numbers $0 \ldots 4$. We know that for sure, because 5 is a prime number. In consquence, no multiple of any number a from the range $1 \ldots 4$ will be divisible by 5 but those that are also multiples of 5. For this reason, any multiple of a number in the range will again leave a remainder in the range when divided by 5. The only exceptions are multiples of the form ka where k is a multiple of 5. At the same time, the results of the multiplications of any two remainders must be different, that is, for any distinct $a, b, c, \in 1 \ldots 4$, if $ab = d$ and $ac = e$, then $d \neq e$. Otherwise, ab and ac would leave the same remainder with 5, which cannot be, since that would mean that there was a number k, such that $5k + ab = 5k + ac$, boiling down to $ab = ac$ and, by dividing a, $b = c$.

This is a significant result. It implies that, if we could devise an algorithm that finds the inverse of any $a \mod n$ (for n being prime), we would not only have a division algorithm, but we would have defined a multiplication group over natural numbers – and this is where arithmetic modulo a prime is different from arithmetic modulo a composite.

We find such an algorithm if we go to the heart of the matter. It is related to a special property of gcd. We know that gcd proceeds by applying the modulo operation: $\gcd(a, b) = \gcd(b, a \bmod b)$. If c is the remainder, *i.e.* $c = a \bmod b$, then we have

$$c = a - qb, \tag{5.10}$$

where q is the quotient, *i.e.* the greatest number that multiplied with b is equal or less than a. This equation is equivalent to the following:

$$c = ka + lb, \tag{5.11}$$

where $k = 1$ and $l = -q$. We now prove by induction on gcd that, for any d such that $d = \gcd(a, b)$, there are two integers k and l, positive or negative, for which holds $d = ka + lb$. The base case is equation 5.11. We have to prove that, if 5.11 shows the n^{th} recursion step of gcd, then, in the $(n+1)^{th}$ rescursion step, it still holds for the remainder of the arguments in that iteration, d, that there are two integers m and n, such that $d = ma + nb$. Since the final result of gcd is the remainder of the previous recursion step, this proves that there are always two integers k and l such that $\gcd(a, b) = ka + la$.

If c in the equation $c = ka + lb$ represents the remainder in the n^{th} recursion step of *gcd*, then c is the second argument in the next recursion step and d in $d = b - qc$ represents the remainder in this step. We substitute the base case 5.11 for c:

$$d = b - qc = b - q(ka + lb) \tag{5.12}$$

Let us distinguish the quotient in equation 5.10 and the one in 5.12 by adding the subscripts: $c = a - q_1 b$ and $d = b - q_2 c$. Since $c = a - q_1 b$, we can state that $d = b - q_2(a - q_1 b)$ or, according to the base case 5.11:

$$d = b - q_2(ka + lb). \tag{5.13}$$

We multiply this out to get:

$$d = b - (q_2 ka + q_2 lb), \tag{5.14}$$

which is just

$$d = b - q_2 ka - q_2 lb. \tag{5.15}$$

By regrouping and adding $-q_2 lb + b$, we obviously get

$$d = -q_2ka - (q_2l - 1)b. \tag{5.16}$$

We set $m = -q_2k$ and $n = -(q_2l - 1)$ and obtain the desired result:

$$d = ma + nb. \quad \square \tag{5.17}$$

To illustrate this with an example, we claim that the remainder in the second iteration of $\gcd(21, 15)$ is $-21q_2k - 15(q_2l - 1)$, where k and l fulfil the equation $21k + 15l = (21 \bmod 15)$ and $l = -q_1$. the quotient of 21 and 15, which is 1. So we have $21k-15 = 6$, which becomes true if $k = 1$.

In the next round, we have $\gcd(15, 6)$. The quotient of 15 and 6, q_2, is 2. We, hence, claim that $21 \times -2 \times 1 - 15 \times (2 \times -1 - 1) = 15 \bmod 6 = 3$. Let us see if this is true. We simplify to $-42 - 15(-2 - 1)$, which in its turn is $-42 - 15(-3)$ or $-42 + 45 = 3$, which is indeed the expected result.

Now we will look at the special case that the gcd of two numbers a and b is 1. There still must be two numbers k and l, such that

$$1 = ka + lb. \tag{5.18}$$

From this, we can prove as a corollary, that if a prime p divides ab, it must divide either a or b, a fact that we took for granted, when we proved the fundamental theorem of arithmetic. If p does not divide a, then we have $\gcd(a, p) = 1$ and, hence, $1 = ka + lp$. Multiplying by b, we get $b = b(ka + lp)$ or $b = kab + lbp$. The fact that p divides ab means that there is a number r, such that $ab = rp$. So, we can also say $b = krp + lbp$ or $b = p(kr + lb)$, where p clearly appears as a factor of b. $\square$

Let us come back to our division problem. The relevant information is equation 5.18, *i.e.* that, if $\gcd(a, b) = 1$, then there are two integers k and l, such that $1 = ka+lb$. We want to get something for a that looks like $\frac{1}{a}$, since $a \times \frac{1}{a} = 1$. In other words: $\frac{1}{a}$ is the inverse of a. A function that would serve as such an inverse for a would be $f(x) = kx + lb$, for the two integers k and l.

If we compute $\gcd(a, n)$, where n is a prime, we know we get 1 back and we know there must be two integers k and l such that $1 = ka + ln$. We can transform this equation by subtracting ln to $ka = 1 - ln$. Since ln is a multiple of n, $1 - ln$, which is the same as $-ln + 1$, would leave the remainder 1 on division by n, *i.e.* $-ln + 1 = 1 \mod n$. In other words: $ka = 1 \mod n$. That is actually what we are looking for: a number that, multiplied by a, is 1. k, hence, is the wanted inverse of a modulo n. The question now is: how to get to k?

There is a well known algorithm that produces not only the greatest common divisor, but also k and l. This algorithm is called the *extended greatest common divisor* or xGCD:

$$
\begin{aligned}
&xgcd :: Integer \to Integer \to (Integer, (Integer, Integer))\\
&xgcd\ a\ b = go\ a\ b\ 1\ 0\ 0\ 1\\
&\quad \textbf{where}\ go\ c\ 0\ uc\ vc\ _\ _ \quad = (c, (uc, vc))\\
&\qquad\quad go\ c\ d\ uc\ vc\ ud\ vd = \textbf{let}\ (q, r) = c\ `quotRem`\ d\\
&\qquad\qquad\qquad\qquad\qquad\quad \textbf{in}\ go\ d\ r\ ud\ vd\ (uc - q * ud)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (vc - q * vd)
\end{aligned}
$$

The listing, admittedly, looks somewhat confusing at the first sight. However, it bears the classic *gcd*. If you ignore the four additional parameters of *go*, you see that *go* calls *quotRem*, instead of just *rem* as *gcd* does, and it then recurses with $go\ d\ r$, d being initially b and r being the remainder – that is just *gcd*. But *go* additionally computes $uc - q \times ud$ and $vc - q \times vd$. We start with $uc = 1$, $vc = 0$, $ud = 0$ and $vd = 1$, hence: $1 - q \times 0 = 1$ and $0 - q \times 1 = -q$. These are just k and l after the first iteration. In the next iteration, we will have $uc = 0$, $vc = 1$, $ud = 1$ and $vd = -q_1$ and compute $0 - q_2 \times 1$ and $1 - q_1 \times -q_2$, which you will recognise as m and n from equation 5.16. The algorithm is just another formulation of the proof we have discussed above.

Let us look at *xgcd* with the example above, $a = 21$ and $b = 15$. We start to call *go* as *go* 21 15 1 0 0 1, which is

$(1, 6) = 21\ `quotRem`\ 15$

in

$go\ 15\ 6\ 0\ 1\ (1 - 1 * 0)\ (0 - q * 1)$
$go\ 15\ 6\ 0\ 1\ 1\ (-1)$.

This leads to

$(2, 3) = 15\ `quotRem`\ 6$

in

$go\ 6\ 3\ 1\ (-1)\ (0 - 2 * 1)\ (1 - 2 * (-1))$
$go\ 6\ 3\ 1\ (-1)\ (-2)\ 3$.

In the next round we have

$(2, 0) = 6\ `quotRem`\ 3$

and we now call, ignoring *ud* and *vd*:

$go\ 3\ 0\ (-2)\ 3\ _\ _$

and, since $d = 0$, just yield $(3, (-2, 3))$, where the k we are looking for is -2, *i.e.* the first of the inner tuple.

Since 15 and 21 in the example above are not coprime the remainder is not 1 but 3 (since $\gcd(21, 15) = 3$), When we use the function with a prime number p and any number $a < p$, the remainder is 1. The resulting k is the inverse of a mod p and we can therefore

use this k to implement division. But, actually, we are not in Kansas anymore: k may be negative. That is, even if we are still discussing problems of natural numbers, we have to refer to negative numbers and, thus, use a number type we have not yet implemented.

Technically, this is quite simple – it hurts of course that we have to cheat in this way. Anyway, here is a simple solution:

$$
\begin{aligned}
&\mathit{nxgcd} :: \mathit{Natural} \to \mathit{Natural} \to (\mathit{Natural}, \mathit{Natural}) \\
&\mathit{nxgcd}\ a\ n = \mathbf{let}\ a' \quad\quad\ = \mathit{fromIntegral}\ a \\
&\qquad\qquad\qquad\quad\ n' \quad\quad\ = \mathit{fromIntegral}\ n \\
&\qquad\qquad\qquad\quad\ (r, (k, _)) = \mathit{xgcd}\ a'\ n' \\
&\qquad\qquad\quad \mathbf{in\ if}\ k < 0\ \mathbf{then}\ (\mathit{fromIntegral}\ r, \mathit{fromIntegral}\ (k + n')) \\
&\qquad\qquad\qquad\qquad\quad \mathbf{else}\ (\mathit{fromIntegral}\ r, \mathit{fromIntegral}\ k)
\end{aligned}
$$

This function is somewhat difficult to look through because of all the conversions. First we have to convert the natural numbers to integers using *fromIntegral*, then we have to convert the result back to a natural number, again, using *fromIntegral*. This works because both types, *Integer* and *Natural*, belong to class *Integral*.

After conversion, we apply *xgcd* on the integers ignoring l, just using the remainder and k. (The reason that we do not throw away the remainder as well is that we will need the remainder sometimes to check that $\mathit{xgcd}\ a\ b \equiv 1$.) Now, if k is a negative number, we add the modulus n to it, as we have learnt, when we studied subtraction.

Let us specialise *nxgcd* for the case that we only want to have the inverse:

$$
\begin{aligned}
&\mathit{inverse} :: \mathit{Natural} \to \mathit{Natural} \to \mathit{Natural} \\
&\mathit{inverse}\ a = \mathit{snd} \circ \mathit{nxgcd}\ a
\end{aligned}
$$

The inverses of the numbers modulo 5 are for instance:

$1 : \mathit{inverse}\ 1\ 5 = 1$
$2 : \mathit{inverse}\ 2\ 5 = 3$
$3 : \mathit{inverse}\ 3\ 5 = 2$
$4 : \mathit{inverse}\ 4\ 5 = 4.$

Note that there is no inverse for 0, since any number multiplied by 0 is just 0. Another way to state this is that $1/0$ is undefined.

Any other number a and its inverse a' behave as follows: $a \times a' = 1$ and, of course, $a \times b \times a' = b$. For instance:

$1 \times 1 = 1 \mod 5$
$2 \times 3 = 1 \mod 5$
$3 \times 2 = 1 \mod 5$
$4 \times 4 = 1 \mod 5.$

We can also play around like:

$2 \times 4 \times 3 = 4 \mod 5$
$3 \times 4 \times 4 = 3 \mod 5$
$3 \times 1001 \times 2 = 1001 = 1 \mod 5.$

Division, the inverse operation to multiplication, is now easily implemented as:

```
mDiv :: Module → Module → Module
mDiv (Module n a1) (Module _ a2) = Module n (((inverse a2 n) * a1) `rem` n)
```

With this function, we have a way to make *Module* member of the *Integral* class, but, before we can do that, we have to make it instance of the *Enum* and the *Real* classes, which is straight forward:

```
instance Enum Module where
  fromEnum (Module _ a) = fromIntegral a
  toEnum i              = tomod (fromIntegral (i + 1)) (fromIntegral i)
instance Real Module where
  toRational (Module _ a) = fromIntegral a
```

The *Integral* instance is now simply defined as:

```
instance Integral Module where
  quotRem x@(Module n _) y = (withGuard mDiv x y, Module n 0)
  toInteger (Module _ a)   = fromIntegral a
```

For *quotRem*, *mDiv* is used to compute the quotient and, since we have defined division in terms of multiplication, we know that there is never to be a remainder different from 0. We, hence, just return a *Module* with the value 0 as remainder of *quotRem*.

We could now go even further and define an instance for *Fractional*:

```
instance Fractional Module where
  (/) = mDiv
  fromRational = ⊥
```

It is nice to have the division operator available for modules, so we can do things like $a \,/\, b$, where a and b are of type *Module*. For *fromRational*, however, which is mandatory for defining the *Fractional* class, we have, for the time being, no good implementation.

There is an important corollary that follows from the invertibility of numbers modulo a prime, namely that any number in the range $1 \ldots p-1$ can be created by multiplication of other numbers in this range and for any two number a and n, there is unique number b that fulfils the equation

$$ax = n. \tag{5.19}$$

In other words: There are no primes modular a prime. For natural numbers with ordinary arithmetic, this is clearly not true. There is for instance no solution for equations

like $3x = 2$ or $3x = 5$. In arithmetic modulo a prime, however, you always find a solution, for instance: $3x = 2 \mod 5$ has the solution 4, since $3 \times 4 = 12 = 2 \mod 5$.

This follows immediately from invertibility, since we only have to multiply n to the inverse of a to find x. If we have the inverse a' of a, such that

$$aa' = 1, \tag{5.20}$$

we just multiply n on both sides and get:

$$naa' = 1n. \tag{5.21}$$

For the example above, $3x = 2 \mod 5$, we can infer x from

$$3 \times 2 = 1 \mod 5 \tag{5.22}$$

by multiplying 2 on both sides:

$$2 \times 3 \times 2 = 2 \mod 5. \tag{5.23}$$

$ax = 2 \mod 5$, hence, has the solution $x = 4$. Here is a kind of magic square for numbers modulo 5:

	1	2	3	4
1	1	3	2	4
2	2	1	4	3
3	3	4	1	2
4	4	2	3	1

The leftmost column shows a multiplication result. The multiplication is defined as: $row_1 \times row_n$. The second row, with 1 in the first column, shows the inverse for each number: The inverse of 1 is 1; the inverse of 2 is 3; the inverse of 3 is 2 and the inverse of 4 is 4. The next row shows the multiplications resulting in 2: 1×2, 2×1, 3×4 and 4×3.

Let us summarise what we have learnt. Arithmetic modulo a prime p constitutes a finite field of the numbers $0 \ldots p-1$. The arithmetic operations on numbers modulo p always yield a number in that range, *i.e.* the operations are closed modulo p. Additionally to the properties we had already seen for natural numbers, associativity, identity and commutativity, we saw that operations modulo p are invertible for both addition and multiplication. In spite of the observation that all numbers we are dealing with are

positive integers, *i.e.* natural numbers, subtraction and division are closed and every number modulo a prime has an inverse number for addition and multiplication. For the multiplicative group of the field, 0 must be excluded, since there is no number k such that $0 \times k = 1$ or, stated differently, $1/0$ is undefined. This, however, is true for all multiplicative groups.

5.7. Congruence

There is an important fact that, in the light of modular arithmetic, appears to be completely trivial, namely that all numbers $0 \ldots n-1$ leave the same remainder divided by n as infinitely many other numbers $\geq n$. For instance, 0 leaves the same remainder as n divided by n; 1 leaves the same remainder as $n+1$; 2 leaves the same remainder as $n+2$ and so on. Furthermore, 1 leaves the same remainder as $2n+1$, $3n+1$, $4n+1$, ... This relation, that two numbers leave the same remainder divided by another number n, is called congruence and is written:

$$a \equiv b \pmod{n}.$$

We have for example:

$$1 \equiv mn + 1 \pmod{n} \tag{5.24}$$

$$2 \equiv mn + 2 \pmod{n} \tag{5.25}$$

$$k \equiv mn + k \pmod{n} \tag{5.26}$$

An important congruence system is Fermat's *Little Theorem*, which is called like this to distinguish it from the other famous theorem by Pierre Fermat, his *Last Theorem*, which, in its turn, is named this way, because, for many centuries, it was the last of Fermat's propositions that was not yet proven.

Fermat's little theorem states that, for any integer a and any prime number p:

$$a^p \equiv a \pmod{p}, \tag{5.27}$$

which is the same as:

$$a^{p-1} \equiv 1 \pmod{p}. \tag{5.28}$$

That the two equations are equivalent is seen immediately, when we multiply both sides of the second equation with a: $a \times a^{p-1} = a^p \equiv a \times 1 = a \pmod{p}$.

One of the proofs of the little theorem brings two major themes together that we have already discussed, namely binomial coefficients and modular arithmetic. You may have observed already that all binomial coefficients $\binom{p}{k}$, where p is prime and $0 < k < p$, are multiples of p. For instance:

$\binom{3}{2} = 3$
$\binom{5}{2} = 10$
$\binom{5}{3} = 10$
$\binom{7}{2} = 21$
$\binom{7}{3} = 35$
$\binom{7}{4} = 35$
$\binom{7}{5} = 21.$

This is not true for coefficients where p is not prime. For instance:

$\binom{4}{2} = 6$
$\binom{6}{2} = 15$
$\binom{8}{2} = 28$
$\binom{8}{4} = 70.$

To prove this, we first observe that all binomial coefficients are integers. Since we hardly know anything but natural numbers, we will not prove this fact here, but postpone the discussion to the next chapter. One way to define binomial coefficients is by means of factorials:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{5.29}$$

We transform this equation multiplying $k!(n-k)!$ on both sides:

$$n! = \binom{n}{k} k!(n-k)!, \tag{5.30}$$

which shows that n must divide either $\binom{n}{k}$ or $k!(n-k)!$, because the product of these two factors equals $n!$, which is a multiple of n by definition. Note that this is just the application of Euclid's lemma to a prime number. Since a prime has no factors to share with other numbers but itself, it must divide at least one of the factors of a product that it divides.

Let us check if n divides $k!(n-k)!$ Again, to divide the whole, n must divide one of the factors, either $k!$ or $(n-k)!$ But, if n is prime and $0 < k < n$ and $0 < n-k < n$, it cannot divide either of them, since none of the factors of $k!$ $(1 \times 2 \times \cdots \times k)$ and none of the factors of $(n-k)!$ $(1 \times 2 \times \cdots \times (n-k))$ is divided by n or divides n. One cannot compose a number that is divisible by a prime by multiplying only numbers that are

smaller than that prime. We could do so easily for composites. $4! = 24$, for instance, is divisible by 8. But no number smaller than a given prime multiplied by another number smaller than that prime, will ever be divided by that prime. So, obviously, n must divide $\binom{n}{k}$ or, in other words, $\binom{n}{k}$ is a multiple of n. □

To the delight of every newcomer, it follows immediately from this fact that, if p is prime:

$$(a+b)^p \equiv a^p + b^p \pmod{p}. \tag{5.31}$$

This identity, for understandable reasons, is sometimes called *Freshman's Dream*. The binomial theorem, which we have already proven, states:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}. \tag{5.32}$$

Since all $\binom{n}{k}$ for $0 < k < n$ are multiples of n if n is prime, they are all $0 \bmod n$. (Remember that, in modular arithmetic, we can take the modulo at any point when calculating a complex formula!) In the summation, hence, all terms for the steps $0 < k < n$ are 0 and only the first case, $k = 0$, and the last case, $k = n$, remain, whose coefficients are $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$. The resulting formula, hence, is

$$(a+b)^n = \binom{n}{n} a^n + \binom{n}{0} b^{n-0} = a^n + b^n \quad \square$$

We will now prove Fermat's little theorem by induction. We choose the base case $a = 1$. Since $1^p = 1$, it trivially holds that $1^p \equiv 1 \pmod{p}$. We now have to prove that, if $a^p \equiv a \pmod{p}$ holds, it also holds that

$$(a+1)^p \equiv a + 1 \pmod{p}. \tag{5.33}$$

$(a+1)^p$ is a binomial formula with a prime exponent. We have already shown that $(a+b)^p = a^p + b^p \mod p$ and we can therefore conlcude $(a+1)^p = a^p + 1^p \mod p$, which, of course, is just $a^p + 1$. From the base case we know that $a^p \equiv a \pmod{p}$ and can therefore further conclude that $a^p + 1 \equiv a + 1 \pmod{p}$. □

Another interesting congruence system with tremendous importance in cryptography is the *Chinese Remainder Theorem*. The funny name results from the fact that systems related to this theorem were first investigated by Chinese mathematicians, namely Sun Tzu, who lived between the 3^{rd} and the 5^{th} century, and Qin Jiushao, who provided a complete solution in his "Mathematical Treatise in Nine Sections" published in the mid-13^{th} century.

The theorem deals with problems of congruence systems where the task is to find a number x that leaves given remainders with given numbers. We can state such systems in general as follows:

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\dots \\ x &\equiv a_r \pmod{n_r} \end{aligned}$$

The theorem now states for x, $a_1 \dots a_r$ and $n_1 \dots n_r \in \mathbb{N}$ that, if the numbers $n_1 \dots n_r$ are coprime, then there is always a solution for x, which even further is unique modulo $\prod n_i$, the product of all the ns and, since the ns are coprime to each other, their least common multiplier.

Before we prove this theorem, let us look at potential algorithms to solve such systems. The first account would be "common sense": we would just search "brute-force" for the proper solutions among candidates. Candidates are all numbers congruent to $a_1 \dots a_r$ modulo $n_1 \dots n_r$. For each pair of (a_i, n_i), we would create a list of congruences. Solutions would be the numbers that are in all such lists, *i.e.* the intersection of those lists. We would start by creating lists of congruences. The first element in the list of congruences for a pair (a_i, n_i) would be a_i, the next would be $a_i + n_i$ (since that number leaves the same remainder as a_i divided by n_i):

```
congruences :: Natural → Natural → [Natural]
congruences a n = a : congruences (a + n) n
```

This function will create an infinite list of all numbers leaving the same remainder with n (which is n_i) as a (which is a_i). For $a = 2$ and $n = 3$, *i.e.* the congruence $x \equiv 2 \pmod 3$, we would generate the list:

$[2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, ..]$

We would then devise a function to apply this generator to all pairs of (a_i, n_i) in the congruence system at hand:

```
mapCongruences :: Int → [Natural] → [Natural] → [[Natural]]
mapCongruences l as ns = [take l (congruences a n) | (a, n) ← zip as ns]
```

This function receives three arguments: l of type *Int* (which should rather be *Natural*, but *Int* is chosen for convenience, since it is used with *take*) and *as* and *ns* both of type [*Natural*]. The function simply applies all pairs of (a, n) to the *congruences* generator. It limits the length of resulting lists from the generator to l. Otherwise, the list comprehension would never come to a result that we could then use to find the solution. Calling *mapCongruences* for the simple congruence system

$$\begin{aligned} x &\equiv 2 \pmod 3 \\ x &\equiv 3 \pmod 4 \\ x &\equiv 1 \pmod 5 \end{aligned}$$

with $l = 10$ yields three lists:

$[2, 5, 8, 11, 14, 17, 20, 23, 26, 29]$
$[3, 7, 11, 15, 19, 23, 27, 31, 35, 39]$
$[1, 6, 11, 16, 21, 26, 31, 36, 41, 46]$.

Now we just have to intersect these lists:

```
chinese1 :: [Natural] → [Natural] → [Natural]
chinese1 as ns = case mapCongruences (fromIntegral $ product ns) as ns of
                   [] → []
                   cs → foldr intersect (head cs) (tail cs)
```

This function receives two arguments, the list of $a_1 \dots a_r$ and the list of $n_1 \dots n_r$. On these lists, it calls *mapCongruences* with $l = \prod n_i$. The idea behind this choice will become clear in a minute. If the result of this application is an empty list, we return an empty list. This is just a trick to avoid an exception in the case where the input consists of empty lists. Otherwise, we fold the result list with *intersect* and *head cs* as the base case for *foldr*. The result for *chinese1* $[2, 3, 1]$ $[3, 4, 5]$ is

$[11, 71, 131]$.

If the theorem is correct, the three numbers in the resulting list should be congruent to each other modulo $3 \times 4 \times 5 = 60$. The call *map* (‘*rem*‘60) $[11, 71, 131]$, indeed, yields $[11, 11, 11]$.

The fact that the solution is unique modulo the product of all ns implies that there must be at least one solution in the range $a_s \cdots \prod n_i$, where a_s is the smallest of the as in the system. There is not necessarily a solution in the range of any particular n. But if there was no solution less than the product $\prod n_i$, there trivially would be no solution at all.

To guarantee that we look at all candidates up to $\prod n_i$, we take lists of the length $\prod n_i$. This is certainly exaggerated, since in lists with that number of elements, there are already much greater numbers. But it guarantees that we will find a solution.

This brute-force algorithm is quite instructive, since it shows the structure of the problem quite well. On the other hand, it is inefficient in terms of computational complexity. For big systems and, in particular, for large ns the congruence lists become unmanageably large. We should look for an algorithm that exploits our knowledge on modular arithmetic.

To start with, we observe that, obviously, all ns divide $\prod n_i$ (which we will call pN in

the remainder of this section). But, since the ns are coprime to each other, n_i would not divide the product of all numbers but itself, that is $\frac{pN}{n_i}$. In other words $gcd(\frac{pN}{n_i}, n_i) = 1$. There, hence, exist two integers, k and l, such that $k \times \frac{pN}{n_i} + ln_i = 1$. Since ln_i is a multiplie of n_i, this means that $k \times \frac{pN}{n_i} = 1 \bmod n_i$. The number k is thus the inverse of $\frac{pN}{n_i} \bmod n_i$. Let us call the product of k and $\frac{pN}{n_i}$ (of which we know that it is 1 mod n_i) e_i: $e_i = k \times \frac{pN}{n_i}$.

We could now write the ridiculous formula $x \equiv e_i \times a_i \pmod{n_i}$ where we multiply e_i with a_i in the corresponding line of the congruence system. The formula is ridiculous, because we already know that $e_i \bmod n_i = 1$, the formula, hence, says $x \equiv 1 \times a_i \pmod{n_i}$, which adds very little to the original formulation $x \equiv a_i \pmod{n_i}$. But, actually, this stupid formula leads directly to the solution.

Note that for any $n_j, j \neq i$, n_j divides $\frac{pN}{n_i}$ and all its multiples including $e_i = k \times \frac{pN}{n_i}$. That is, for any e_i: $e_i \equiv 0 \pmod{n_j}, j \neq i$. So we could create the following, equally ridiculous equation:

$$x \equiv \sum_{j=1}^{r} e_j \times a_i \pmod{n_i}. \tag{5.34}$$

Since, for any specific i, all $e_j, j \neq i$, are actually 0 and for the one case, where $j = i$, e_i is 1, this equation is trivially true for any line in the system. It just states $x \equiv a_i \pmod{n_i}$. In other words: this sum fits all the single lines of the congruence system.

This trivially magic sum not taken modulo to any of the individual ns is of course a number that is much larger – or, much smaller, *i.e.* a large negative number – than the smallest number that would fulfil all congruences in the system. The unique solution, however, is this number taken modulo pN. Since pN is just a multiple of any of the n_i in the system, all numbers leaving the same remainder modulo pN will leave the same remainder modulo a specific n_i.

This approach to Chinese remainder systems is much more efficient than the brute-force logic we implemented before. To implement it in Haskell, we first implement the function that finds e_i, using the extended *gcd*:

```
inv :: Integer → Integer → Integer
inv n pN = let b           = pN `div` n
               (_, (k, _)) = xgcd b n
           in  k * b
```

Then we call this function for each pair (a_i, n_i) in the system, sum the products $a_i \times e_i$ and yield the result modulo pN:

$$
\begin{aligned}
&chinese :: [Integer] \to [Integer] \to Natural \\
&chinese\ as\ ns = \mathbf{let}\ pN = product\ ns \\
&\qquad\qquad es\ \ = \quad [inv\ n\ pN \mid n \quad \leftarrow ns] \\
&\qquad\qquad e\ \ = sum\ [a * e \quad \mid (a, e) \leftarrow zip\ as\ es] \\
&\qquad\ \mathbf{in}\ \ fromIntegral\ (e\ `nmod`\ pN)
\end{aligned}
$$

Since we are working with integers here instead of natural numbers – giving up to pretend that we can solve all problems related to natural numbers with natural numbers alone – we use a *mod* operator that is modelled on the *Module* data type defined in the previous section:

$$
\begin{aligned}
&nmod :: Integer \to Integer \to Integer \\
&nmod\ x\ n \mid x < 0 \quad\ = n - ((-x)\ `rem`\ n) \\
&\qquad\ \ \mid otherwise = x\ `rem`\ n
\end{aligned}
$$

Consider the example we already used above:

$$
\begin{aligned}
x &\equiv 2 \pmod 3 \\
x &\equiv 3 \pmod 4 \\
x &\equiv 1 \pmod 5.
\end{aligned}
$$

We would solve this system by calling $chinese\ [2,3,1]\ [3,4,5]$. The results for *inv* are:

$inv\ 3\ 60 = -20$
$inv\ 4\ 60 = -15$
$inv\ 5\ 60 = -24.$

We would now call:

$sum\ [2 * (-20), 3 * (-15), 1 * (-24)] = sum\ [-40, -45, -24] = -109$

and take the result modulo 60: $(-109)\ `nmod`\ 60 = 11$. Confirm that this result fulfils the system:

$$
\begin{aligned}
11 &\equiv 2 \pmod 3 \\
11 &\equiv 3 \pmod 4 \\
11 &\equiv 1 \pmod 5.
\end{aligned}
$$

5.8. Quadratic Residues

Quadratic residues of a number n are natural numbers congruent to a perfect square modulo n. In general, q is a quadratic residue modulo n if:

$$x^2 \equiv q \pmod{n}. \tag{5.35}$$

A simple function to test whether a number q is indeed a quadratic residue with respect to another number x could look like this:

$$\begin{aligned}&\mathit{isResidue} :: \mathit{Natural} \to \mathit{Natural} \to \mathit{Natural} \to \mathit{Bool}\\ &\mathit{isResidue}\ q\ n\ x = (x \uparrow 2)\ \text{`}\mathit{rem}\text{`}\ n \equiv q\end{aligned}$$

This function is a nice test, but it does not help us to find residues. The following function does that:

$$\begin{aligned}&\mathit{residues} :: \mathit{Natural} \to [\mathit{Natural}]\\ &\mathit{residues}\ n = \mathit{sort}\ \$\ \mathit{nub}\ \$\ \mathit{map}\ (\lambda x \to (x \uparrow 2)\ \text{`}\mathit{rem}\text{`}\ n)\ [0\,..\,n-1]\end{aligned}$$

residues finds all residues modulo n by simply taking the remainder of the squares of all numbers $0 \ldots n-1$. Note that these are all remainders of squares modulo this number. Any other square, for instance the square $(n+1)^2$, will reduce to one of the remainders in the range $0 \ldots n-1$. $(n+1)^2$ would just reduce to the remainder 1; $(n+2)^2$ would reduce to the remainder 2; likewise $(mn+1)^2$ would reduce to 1 or, in general, any number of the form $(mn+r)^2$, where r is a number from the range $0 \ldots n-1$, will always reduce to r.

Since *residues* tests all numbers in the range $0 \ldots n-1$, some numbers may appear more than once. The residues of 6, for instance, are: $0, 1, 4, 3, 4, 1$, since

$$\begin{aligned}0^2 = 0 &\equiv 0 \pmod{6}\\ 1^2 = 1 &\equiv 1 \pmod{6}\\ 2^2 = 4 &\equiv 4 \pmod{6}\\ 3^2 = 9 &\equiv 3 \pmod{6}\\ 4^2 = 16 &\equiv 4 \pmod{6}\\ 5^2 = 25 &\equiv 1 \pmod{6}\end{aligned}$$

The function, therefore, *nub*s the result and sorts it for convenience.

Let us look at the residues of some small numbers:

$$\begin{aligned}&\mathit{residues}\ 9 = [0, 1, 4, 7]\\ &\mathit{residues}\ 15 = [0, 1, 4, 6, 9, 10]\\ &\mathit{residues}\ 21 = [0, 1, 4, 7, 9, 15, 16, 18]\end{aligned}$$

What happens, when the modulus is prime? Some examples:

$$\begin{aligned}&\mathit{residues}\ 3 = [0, 1]\\ &\mathit{residues}\ 5 = [0, 1, 4]\end{aligned}$$

$residues\ 7 = [0, 1, 2, 4]$
$residues\ 11 = [0, 1, 3, 4, 5, 9]$
$residues\ 13 = [0, 1, 3, 4, 9, 10, 12]$

Apparently, the number of residues per modulus is constantly growing. In the case of prime moduli, however, there appears to be a strict relation between the modulus and the number of residues. There seem to be roughly $p/2$ residues for a prime modulus p or, more precisely, there are $(p + 1)/2$ residues (if we include 0). This is not the fact with composite moduli. Let us devise a function that may help us to further investigate this fact:

$$\begin{aligned}&countResidues :: Natural \to Int\\&countResidues = length \circ residues\end{aligned}$$

Applied to a random sequence of composite numbers the result appears to be random too (besides the fact that the number of residues is slowly growing together with the moduli):

9	15	21	25	26	30	32	35	90	100	150	500
4	6	8	11	14	12	7	12	24	22	44	106

Applied on prime numbers the result is always $(p + 1)/2$:

3	5	7	11	13	17	19	23	29	31	37	41
2	3	4	6	7	9	10	12	15	16	19	21

and so on. There is, however, one remarkable exception, namely 2: $residues\ 2 = [0, 1]$ and, hence, $countResidues\ 2 = 2$. The general rule is therefore that, for an **odd** prime p, there are $(p + 1)/2$ residues and $(p - 1)/2$ nonresidues.

When we look at the residues of primes above, we see that some numbers appear more than once. For instance, 4 is residue of 5, 7, 11 and 13; 2, by contrast, appears only once; 3 appears twice. An interesting line of investigation could be, which prime moduli have a certain residue and which have not. The following function is a nice tool for this investigation:

$$\begin{aligned}&hasResidue :: Integer \to Integer \to Bool\\&hasResidue\ n\ q \quad \mid q < 0 \wedge abs\ q > n \quad = hasResidue\ n\ (q\ `rem`\ n)\\&\qquad\qquad\ \mid q < 0 \qquad\qquad\quad = hasResidue\ n\ (n + q)\\&\qquad\qquad\ \mid q \equiv 0 \qquad\qquad\quad = True\\&\qquad\qquad\ \mid otherwise \qquad\quad = check\ 0\\&\quad \textbf{where}\ check\ x \mid x \equiv n \qquad\qquad\quad = False\\&\qquad\qquad\ \mid (x \uparrow 2)\ `rem`\ n \equiv q = True\\&\qquad\qquad\ \mid otherwise \qquad\quad = check\ (x + 1)\end{aligned}$$

There is something special about this function that should be explained. First thing to notice is that it does not operate on *Natural*, but on *Integer* and, indeed, the first two guards immediately take care of negative residues (q). In the first line, a negative q with

an absolute value greater than n is reduced to the negative remainder; for a negative remainder already reduced to a remainder modulo n, the function is simply called again on $n + q$, that is n minus the absolute value of q. This is *negative congruence* that we already encountered, when we started to discuss arithmetic modulo a prime. It is a way to generalise the case of $n - a$, where we are not looking for a fixed number, but for a residue defined relative to n, *e.g.* $n - 1$, which would just be -1.

For $q = 0$, the function simply yields *True*, since 0 is residue of any number. For positive integers, *hasResidue* calls *check* 0. *check*, as can be seen in the third line, counts the $x$es up to n. When it reaches n, it yields *False* (first line). Should it encounter a case where x^2 equals q mod n, it terminates yielding *True*.

We can test the function asking for 4 in 9, 15 and 21 and will see that in all three cases, the result is *True*. Now we would like to extend this to learn if all numbers starting from 5 (where it appears for the first time) have the residue 4:

```
haveResidue :: Integer → [Integer] → [Integer]
haveResidue _ [] = []
haveResidue q (n : ns) | n `hasResidue` q = n : haveResidue q ns
                       | otherwise        =     haveResidue q ns
```

This function searches for numbers in a given list (second argument) that have q as a residue. We could, for instance, call this function on all numbers from 5 onwards (restricting the result to 10): *take* 10 (*haveResidue* 4 [5..]). The result is indeed $[5, 6, 7, 8, 9, 10, 11, 12, 13, 14]$.

Do not let yourself be confused by the fact that 4 is a quadratic residue of all numbers greater than 4. It is just trivial; in fact, all perfect squares are residues of numbers greater than these squares. The same is true for 9, which is residue of $10, 11, 12, \ldots$; 16 is residue of $17, 18, 19, \ldots$ and in general any number of the form x^2 is residue of any number $n > x^2$. This is just the definition of quadratic residue: $x^2 \equiv q \pmod{n}$, for the special case where $q = x^2$, which is trivially true, whenever $n > x^2$.

To continue the investigation into residues of primes, we specialise *haveResidue* to odd primes:

```
primesWithResidue :: Integer → [Integer]
primesWithResidue = (`haveResidue`(drop 1 intAllprimes))
  where intAllprimes = map fromIntegral allprimes
```

Since we have already seen that all numbers from 5 on have 4 as a residue, *primesWithResidue* 4, will just give us the primes starting from 5. A more interesting investigation is in fact -1, *i.e.* all primes p that have $p - 1$ as a residue:

primesWithResidue $(-1) = [5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97..]$.

Is there something special about this list of primes? Not, perhaps, on the first sight. However, try this: *map* (`*rem*`4) (*primesWithResidue* (-1)) and you will see an endless

list: $1, 1, 1, 1, 1, 1, \ldots$. In other words, all these primes are $\equiv 1 \pmod 4$. If this is true, the following function should create exactly the same list of primes:

```
minus1Residues :: [Integer]
minus1Residues = go (drop 1 intAllprimes)
  where go []   = []
        go (p : ps) | p `rem` 4 ≡ 1 = p : go ps
                    | otherwise     =     go ps
        intAllprimes = map fromIntegral allprimes
```

And, indeed, it does:

$minus1Residues = [5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97 \ldots]$.

This fact is quite important. It is known as the *first supplement* to the *law of quadratic reciprocity*. We will start the proof of the first supplement with an apparently unrelated theorem, namely *Wilson's Theorem*, which states that if and only if n is prime, then it holds that

$$(n-1)! \equiv -1 \pmod n. \tag{5.36}$$

To check this quickly with some small primes:

$$\begin{aligned} 2! = 2 &\equiv -1 \pmod 3 \\ 4! = 24 &\equiv -1 \pmod 5 \\ 6! = 720 &\equiv -1 \pmod 7 \\ 10! = 3628800 &\equiv -1 \pmod{11} \end{aligned}$$

and some small composites:

$$\begin{aligned} 3! = 6 &\equiv 2 \pmod 4 \\ 5! = 120 &\equiv 0 \pmod 6 \\ 7! = 5040 &\equiv 0 \pmod 8 \end{aligned}$$

The proof is rather simple. We first prove that Wilson's theorem holds for all primes, then we prove that it does not hold for composites.

To prove that it holds for primes, remember from modular arithmetic that, with a prime modulus n, every number $a \in 1 \ldots n-1$ has an inverse $a' \in 1 \ldots n-1$, such that $a \times a' = 1 \mod n$. For instance, the numbers $1 \ldots 6$ and their inverses modulo 7 are:

$$(1,1),(2,4),(3,5),(6,6)$$

For all pairs of numbers (a, a') where $a \neq a'$, the product of the pair is just 1. The example above seems to suggest that for all numbers, but 1 and $n-1$, $a \neq a'$. If this is true, then the factorial of $n-1$ modulo n, where n is prime, would translate into $1 \times 1 \times \cdots \times n-1$, which obviously is $\equiv (n-1) \pmod{n}$. But does it actually hold for all primes? Let us look: if a is its own inverse, *i.e.* $a = a'$, we would have $a^2 \equiv 1 \pmod{n}$. In other words: a^2 would leave a remainder of 1 divided by n. a^2-1, hence, should leave no remainder, thus: $a^2 - 1 \equiv 0 \pmod{n}$. We can factor $a^2 - 1$ into $(a+1)(a-1)$. The factors help us to find numbers that substituted for a would make the whole expression 0. One possibility is obviously 1: $(1+1) \times (1-1) = 2 \times 0 = 0$. Another possibility, however, is $n-1$:

$$(n-1+1)(n-1-1) =$$
$$n(n-2) = n^2 - 2n$$

and, since $n^2 - 2n$ contains only multiples of n:

$$n^2 - 2n \equiv 0 \pmod{n}.$$

This does not hold for any other number from the range $2 \ldots n-1$, since there will be always a remainder that does not reduce to a multiple of n, for instance: $(n-2+1)(n-2-1) = (n-1)(n-3) = n^2 - 4n + 3$, which is congruent to $3 \pmod{n}$; $(2+1)(2-1) = 3 \times 1 = 3$, which, again, is congruent to $3 \pmod{n}$ and in general $(n-a+1)(n-a-1) = n^2 - 3an + a^2 - 1$, which, modulo n, is $a^2 - 1$. If $a \neq 1$, this is not congruent $0 \pmod{n}$. It therefore holds for all primes n that $(n-1)! \equiv -1 \pmod{n}$. □

Now the second part of the proof: That Wilson's theorem is never true when n is composite. We prove by contradiction and assume that there is a composite n, such that $(n-1)! \equiv -1 \pmod{n}$. That n is composite means that there is a prime number p that divides n. This p is one of the prime factors of n, *i.e.*: $n = mp$, for some integer m. This also means that p is smaller than n and in the range $2 \ldots n-1$. Therefore, p must also divide $(n-1)!$, because it appears as one of the factors in $1 \times 2 \times \cdots \times n-1$. In other words: $(n-1)! \equiv 0 \pmod{p}$.

But we also have $n = mp$. From modular arithmetic we know that if $a \equiv b \pmod{n}$, then also: $a \equiv b \pmod{mn}$. So, from $(n-1)! \equiv 0 \pmod{p}$, it follows that also $(n-1)! \equiv 0 \pmod{mp}$ and, since $n = mp$, $(n-1)! \equiv 0 \pmod{n}$. This contradicts our assumption that there is a composite n, such that $(n-1) \equiv -1 \pmod{n}$. □

Let us come back to the first supplement, which claims that -1 is a residue of an odd prime if and only if that prime is congruent 1 modulo 4. Any number not divided by 4

is either 1, 2 or 3 modulo 4. Since we are dealing only with odd primes, we can ignore the case 2, because no odd number will ever leave the remainder 2 divided by 4. That does only happen with even numbers not divided by 4, like 6, 10, 14, *etc.* We, hence, distinguish two cases: $p \equiv 1$ and $p \equiv 3$ both modulo 4. We first prove that an odd prime $p \equiv 1 \pmod 4$ has residue -1 and then that an odd prime $p \equiv 3 \pmod 4$ does not have residue -1.

We start with the observation that $p \equiv 1 \pmod 4$ implies $(p-1) \equiv 0 \pmod 4$, *i.e.* that $p-1$ is divisible by 4. This, in its turn, implies that we can group the remainders $1 \ldots p-1$ into two sets with the same even number of elements. The remainders of the prime 5, for example, are $1, 2, 3, 4$ and we can group them into $\{\{1,2\},\{3,4\}\}$. For 13, these groups would be $\{\{1,2,3,4,5,6\},\{7,8,9,10,11,12\}\}$.

Now we rewrite the second group in terms of negative congruences:

$$\{\{1,2,3,4,5,6\},\{-6,-5,-4,-3,-2,-1\}\}$$

and then organise the groups as pairs of equal absolute values:

$$\{(1,-1),(2,-2),(3,-3),(4,-4),(5,-5),(6,-6)\}.$$

To compute the factorial of $p-1$, we could first multiply the members of each pair: $\{-1,-4,-9,-16,-25,-36\}$. It is essential to realise that the number of negative signs is even because the number of elements of each group is even. They, hence, cancel out on multiplication. This would be the same as squaring the members of the first group (1 to 6) before multiplying them: $\{1,4,9,16,25,36\}$. We can do this the other way round as well: first, we multiply the two halfs out, creating the factorial for each group, and then multiply the two equal results, which is the same as squaring one of the results. In other words, if $p \equiv 1 \pmod 4$, then

$$(p-1)! = \left(\frac{p-1}{2}\right)!^2. \tag{5.37}$$

The right-hand side of this equation is a formal description of what we did above. We split the numbers $1 \ldots p-1$ into halves: $1 \ldots \frac{p-1}{2}$ and $-\frac{p-1}{2} \cdots -1$, computed the factorial of each half and then multiplied the results, which of course are equal and multiplying them is thus equivalent to squaring.

For a prime p with the residue -1, there must be one number a, such that $a^2 \equiv -1 \pmod p$. The equation above shows that $(\frac{p-1}{2})!^2$ is actually $(p-1)!$, which, according to Wilson's theorem, is $-1 \pmod p$. $\frac{p-1}{2}!$, which squared is $(p-1)!$ and, according to Wilsons' theorem, -1. $\frac{p-1}{2}!$ is therefore such a number a. This, as shown above, is the case, if we can split the sequence of numbers $1 \ldots p-1$ into two halves with an even

number of members each. This, however, is only possible for an odd prime p, if $p-1 \equiv 0 \pmod 4$, which implies that $p \equiv 1 \pmod 4$. □

We will now show that primes of the form $p \equiv 3 \pmod 4$ do not have the residue -1 to complete the proof. Let us assume there is a number a, such that $a^2 \equiv -1 \pmod p$ and $p \equiv 3 \pmod 4$.

We start with the equation

$$a^2 \equiv -1 \pmod p \tag{5.38}$$

and raise both sides to the power of $\frac{p-1}{2}$:

$$a^{2\frac{p-1}{2}} \equiv -1^{\frac{p-1}{2}} \pmod p. \tag{5.39}$$

This can be simplified to

$$a^{p-1} \equiv -1^{\frac{p-1}{2}} \pmod p. \tag{5.40}$$

Note that $p-1$ is even (since p is an odd prime). But, since it is not divisisble by 4, since otherwise $p \equiv 1 \pmod 4$, $(p-1)/2$ must be odd. An example is $p = 7$, for which $(p-1)/2 = 3$. -1 raised to an odd power, however, is -1 and therefore we have:

$$a^{p-1} \equiv -1 \pmod p. \tag{5.41}$$

But that cannot be true, because Fermat's little theorem states that

$$a^{p-1} \equiv 1 \pmod p \tag{5.42}$$

There is only one p for which both equations are true at the same time, namely 2: $a^1 \equiv 1 \pmod 2$. This is actually true for any odd a.

But we are looking at odd primes and 2 is not an odd prime. Therefore, one of the equations must be wrong. Since we know for sure that Fermat's theorem is true, the wrong one must be 5.41. Therefore, -1 cannot be a residue of primes of the form $p \equiv 3 \pmod 4$ □.

There is also a *second supplement* to the law of reciprocity, which happens to deal with 2 as residue. When we look at these numbers, using *primesWithResidue* 2, we get: 7, 17, 23, 31, 41, 47, 71, 73, 79, 89, 97, 103,... What do these primes have in common? They are all one off numbers divisible by 8:

$$
\begin{aligned}
7 &\equiv -1 \pmod 8 \\
17 &\equiv 1 \pmod 8 \\
23 &\equiv -1 \pmod 8 \\
31 &\equiv -1 \pmod 8 \\
41 &\equiv 1 \pmod 8 \\
47 &\equiv -1 \pmod 8 \\
&\dots
\end{aligned}
$$

The second supplement indeed states that ± 2 is residue of an odd prime p if and only if $p \equiv \pm 1 \pmod 8$.

We will not prove this theorem here. Instead, we will look at a general criterion to decide quickly whether a number is residue of an odd prime, namely *Euler's Criterion*, which states that, if p is an odd prime, then:

$$a^{\frac{p-1}{2}} \equiv 1 \pmod p \text{ iff } a \text{ is a residue of } p \tag{5.43}$$

$$a^{\frac{p-1}{2}} \equiv -1 \pmod p \text{ iff } a \text{ is a nonresidue of } p \tag{5.44}$$

We can translate this criterion into Haskell as:

```
euCriterion :: Integer → Integer → Bool
euCriterion a p | a < 0 ∧ abs a > p = euCriterion (a `rem` p) p
                | a < 0             = euCriterion (p + a) p
                | otherwise         = let n = (p − 1) `div` 2
                                      in (a ↑ n) `rem` p ≡ 1
```

As we did before, we first handle the cases of negative congruence: a negative number is reduced to a negative number with an absolute value in the range of $1 \dots p-1$ and then p is added. A positive number is just raised to the power of $(p-1)/2$. If the remainder of this number is 1, this number is indeed a residue of p.

You see that in Euler's Criterion there appears a formula that we already know from the first supplement and, indeed, the proof of the Criterion with the background of the first supplement is quite simple.

We start by considering the case where a is a nonresidue: $a^{\frac{p-1}{2}} \equiv -1 \pmod p$. For equations of the form $bx \equiv a \pmod p$, as we have discussed, when we introduced arithmetic modulo a prime number, there is a unique solution b' such that $b \times b' \equiv a \pmod p$. Notice that $b \neq b'$, since, otherwise we would have the case $b^2 \equiv a \pmod p$ and, in consequence, a would be a residue of p, but we are discussing the case that a is a nonresidue. We now apply the same technique as above: we build pairs of bs and b's to simplify the

computation of the factorial. For each pair (b, b'), where $b \neq b'$, we have $b \times b' \equiv a \pmod{p}$. Since there is exactly one b' for any b in $1 \ldots p-1$, there are $\frac{p-1}{2}$ pairs of (b, b'). For instance:

$1 \times 6 \equiv 6 \pmod{7}$
$2 \times 3 \equiv 6 \pmod{7}$
$4 \times 5 \equiv 6 \pmod{7}$.

When we compute the factorial, we multiply these pairs out, each of which gives a. So, the factorial is $a \times a \times \cdots \times a$ and, since there are $(p-1)/2$ such pairs, $a^{\frac{p-1}{2}}$:

$$(p-1)! \equiv a^{\frac{p-a}{2}} \pmod{p}. \tag{5.45}$$

From Wilson's theorem, we know that $(p-1)! \equiv -1 \pmod{p}$. We, therefore, conclude that $a^{\frac{p-a}{2}} \equiv -1 \pmod{p}$. □

Now we consider the other case, *i.e.* that a is a quadratic residue. In this case, we actually have a solution for $x^2 \equiv a \pmod{p}$.

Consider Fermat's little theorem again:

$$a^{p-1} \equiv 1 \pmod{p}. \tag{5.46}$$

We subtract 1 from both sides:

$$a^{p-1} - 1 \equiv 0 \pmod{p} \tag{5.47}$$

and force in our magic formula $\frac{p-1}{2}$ by factoring the left-hand side. What we are doing step-by-step is

$$a^{\frac{p-1}{2}} \times a^{\frac{p-1}{2}} = a^{\frac{p-1}{2}+\frac{p-1}{2}} =$$
$$a^{\frac{(p-1)+(p-1)}{2}} = a^{\frac{2p-2}{2}} = a^{p-1}.$$

We can reformulate equation 5.47 accordingly:

$$(a^{\frac{p-1}{2}} - 1)(a^{\frac{p-1}{2}} + 1) \equiv 0 \pmod{p}. \tag{5.48}$$

Since, to make a product 0, one of the factors must equal 0, $a^{\frac{p-1}{2}}$ must take either the value 1 or -1.

If a is a quadratic residue, then we have some integer x, such that $a \equiv x^2$. We, hence, could write:

$$(x^{2^{\frac{p-1}{2}}} - 1)(x^{2^{\frac{p-1}{2}}} + 1) \equiv 0 \pmod{p}. \tag{5.49}$$

That would mean that, to make the first factor 0, x^2 must be 1 modulo p and, to make the second factor 0, x^2 must be -1 modulo p. We, hence, want either:

$$x^{2^{\frac{p-1}{2}}} \equiv 1 \pmod{p} \tag{5.50}$$

or

$$x^{2^{\frac{p-1}{2}}} \equiv -1 \pmod{p} \tag{5.51}$$

Since x^{a^b} is just x^{ab}, we can simplify to:

$$x^{p-1} \equiv 1 \pmod{p}, \tag{5.52}$$

for equation 5.50 and

$$x^{p-1} \equiv -1 \pmod{p}, \tag{5.53}$$

for equation 5.51. The second result contradicts Fermat as already seen above and, thus, cannot be the case. One may be tempted to say immediately that the second factor cannot be 0, since then we would have $x^2 = -1$, which cannot be true for any integer x. With modular arithmetic, however, this is not true. A counterexample is $2^2 = 4$, which is -1 modulo 5. We therefore need the reference to Fermat's little theorem to actually show that equation 5.51 leads to a contradiction with a proven theorem.

The simplification of equation 5.50, however, just yields the little theorem. We, thus, can derive Euler's criterion directly from Fermat and that proves that, if a is a quadratic residue of p, we always have $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$. □

The French mathematician Adrien-Marie Legendre (1752 – 1833) defined a function using Euler's Criterion and a nice notation to express this function known as the *Legendre Symbol*. It can be defined as:

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}. \tag{5.54}$$

For $a \in \mathbb{Z}$ and p an odd prime, $\left(\frac{a}{p}\right) \in \{1, -1, 0\}$. More specifically, if a is a residue of p, then $\left(\frac{a}{p}\right)$ is 1, if it is a nonresidue, it is -1, and if $a \equiv 0 \pmod{p}$, it is 0.

The Legrende Symbol has some interesting properties. We will highlight only two of them here. The first is that if $a \equiv b \pmod{p}$, then (trivially)

$$\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right). \tag{5.55}$$

More interesting is multiplicativity:

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right). \tag{5.56}$$

For example, $\left(\frac{3}{11}\right) = 1$ and $\left(\frac{5}{11}\right) = 1$. So $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = 1$ and $\left(\frac{3\times5=15}{11}\right)$ is 1 as well. $\left(\frac{6}{11}\right) = -1$ and $\left(\frac{3\times6=18}{11}\right)$ is -1. $\left(\frac{2}{11}\right) = -1$ and $\left(\frac{2\times6=12}{11}\right) = 1$. A final example with 0: $\left(\frac{22}{11}\right) = 0$ and, with any other number, *e.g.*: $\left(\frac{2\times22=44}{11}\right) = 0$.

We can implement the Legendre Symbol easily in Haskell as:

```
legendre :: Integer → Integer → Integer
legendre a p = let n = (p − 1) `div` 2
               in case (a ↑ n) `rem` p of
                    0 → 0
                    1 → 1
                    x → x − p
```

For some time now, we are beating around the *law of quadratic reciprocity* and it appears to be high time to finally explain what this law is all about. The law is about two primes, p and q, and claims that if the product

$$\frac{p-1}{2} \times \frac{q-1}{2} = \frac{(p-1)(q-1)}{4}$$

is even, then, if p is a residue of q, q is also a residue of p. Otherwise, if the above product is odd, then, if p is a residue of q, q is nonresidue of p.

This can be formulated much more clearly using the Legendre symbol:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}} \tag{5.57}$$

If $\frac{(p-1)(q-1)}{4}$ is even, then the right-hand side of the equation becomes 1, otherwise it is -1. To become 1, the Legendre Symbols on the left-hand side of the equation must be either both negative or both positive. They are both positive, namely 1, if p is residue of q and q is residue of p. They are both negative, namely -1, if neither p is residue of q nor q of p.

For the right-hand side to become -1, one of the Legendre Symbols must be negative and the other positive. This is the case if either p is residue of q, but q is not residue of p or if q is a residue of p, but p is not a residue of q.

For example, look at the primes 7 and 11. The residues of 7 are $\{0, 1, 2, 4\}$ and, since $11 \equiv 4 \pmod 7$, 11 is a residue of 7. The residues of 11 are $\{0, 1, 3, 4, 5, 9\}$. 7, hence, is not a residue of 11. Now look at the fraction

$$\frac{(7-1)(11-1)}{4} = \frac{60}{4} = 15,$$

which is odd. Therefore 7 can only be a residue of 11, if 11 is a nonresidue of 7 and vice versa. 11 is a residue of 7, therefore 7 is not a residue of 11.

What about 7 and 29? Look at the magic fraction:

$$\frac{(7-1)(29-1)}{4} = \frac{168}{4} = 42.$$

42 is even, therefore 7 can only be residue of 29 if 29 is a residue of 7. The residues of 29 are: $\{0, 1, 4, 5, 6, 7, 9, 13, 16, 20, 22, 23, 24, 25, 28\}$. Since 7 is included (and 7 hence is a residue of 29), 29 must also be a residue of 7. Since $29 \equiv 1 \pmod 7$ and 1 is indeed residue of 7, 29 is a residue of 7.

The residues of 5 are $\{0, 1, 4\}$. For 5 and 7, the magic formula is even:

$$\frac{(5-1)(7-1)}{4} = \frac{24}{4} = 6.$$

So, since 5 is a nonresidue of 7, 7 must also be a nonresidue of 5.

The law had already been conjectured by Euler and Legendre, when Gauss finally proved it in the *Disquisitiones*. Gauss called the theorem the *Golden Rule* and, interestingly, the *fundamental theorem of arithmetic* highlighting the value he attached to it. During his life he provided eight different proofs. Many more proofs have been devised since Gauss. According to the *Book*, there were 196 different proofs in the year 2000. We will not go through them here.

5.9. Generators and Subgroups

Let us look at powers of numbers modulo a prime from another angle. In the previous section, we looked at remainders that are squares. Now we look at what happens to remainders, when we raise them to exponents:

$$x^1, x^2, x^3, \ldots, x^{p-1} \mod p.$$

What do we expect to happen? We first can predict that, for any number x, there is an exponent k, such that $x^k = 1$. In other words, the set that we create in this way always contains the identity. One case is x^{p-1} for which we know from Fermat's little theorem that it is congruent to 1 for any number x. For instance $3^6 = 1 \pmod 7$. We also know that, if x is a residue, then $x^{\frac{p-1}{2}} = 1$. There, hence, are numbers that result in a smaller set of numbers, since, once $x^k = 1$, the sequence will just repeat with $x^{k+1} = x$, $x^{k+2} = x^2$ and so on.

If x is a nonresidue, we know that $x^{\frac{p-1}{2}} = -1$. Then, $x^{\frac{p-1}{2}} x^{\frac{p-1}{2}} = x^{p-1} = 1$. Since, in the group of any odd prime there are residues and nonresidues, we know for sure that some numbers create the whole group and others do not.

For 7, the powers of 3, for instance, yield the whole group:

$$\begin{aligned}
3^1 &\equiv 3 \pmod 7 \\
3^2 &\equiv 2 \pmod 7 \\
3^3 &\equiv 6 \pmod 7 \\
3^4 &\equiv 4 \pmod 7 \\
3^5 &\equiv 5 \pmod 7 \\
3^6 &\equiv 1 \pmod 7
\end{aligned}$$

The powers of 4, which is a residue of 7, do not:

$$\begin{aligned}
4^1 &\equiv 4 \pmod 7 \\
4^2 &\equiv 2 \pmod 7 \\
4^3 &\equiv 1 \pmod 7.
\end{aligned}$$

Second, we observe that we create a set of numbers with certain relations among them:

$$\begin{aligned}
a &= x^1 \\
b &= x^2 \\
c &= x^3 \\
&\ldots \\
1 &= x^{p-1}
\end{aligned}$$

Any multiplication of two numbers in the set results in another number in the set. Therefore, any power of a number in the set will result in another number in the set. Since $b = aa$ and $c = aaa$, it also holds that $c = ab$. We can go on this way by observing that every number n_i in the set is the result of multiplying the first number in the set a, which is just $x^1 = x$, with its predecessor n_{i-1} or the second number $x^2 = aa$ with n_{i-2} and so on. The set, hence, is closed under multiplication. Furthermore, at some step, n_i becomes 1 and, from any number in the set, we can get to 1 by multiplying another number in the set. This is trivially true for $1 \times x^k = 1$, if k is the number such that $x^k \equiv 1 \pmod{n}$; it is also true for $x^1 x^{k-1} = 1$ and it is in general true for any number $x^i x^{k-i}, 0 \leq i \leq k$. When we have, for instance $k = 3$, then $1 \times aaa = aaa = 1$. $a \times aa = aaa = 1$ and $aaa \times aaa = 1 \times 1 = 1$. In other words, for every element a in the set, there is also its inverse a' in the set, such that $aa' = 1$. That means that the resulting set is again a multiplicative group.

We call x, the number that generates a group G, the *generator* for G. It is often also called a *primitive element* of G. If the group H generated by a number g modulo p is not the whole group G of p, *i.e.* the numbers $\{1, \ldots, p-1\}$, then we call H a *proper subgroup* of G. A subgroup H of a group G is a group that contains only numbers that are also in G. G, hence, is a subgroup of G itself. A proper subgroup, H, of a group G is a subgroup, where not all members of G are also in H. A proper subgroup G is therefore smaller than G. The group generated by the generator 3 modulo 7, for instance, is a subgroup of G (it is in fact identical to G). The group generated by the generator 4 modulo 7, too, is a subgroup of G, but it is a proper subgroup, since all elements in this group are also in G, but not all elements in G are in this subgroup. This is the same concept as the subset in set theory.

We can devise a simple function to generate a group, given p, the prime, and g, the generator:

```
generate :: Natural → Natural → [Natural]
generate p g = sort $ nub (map (λa → (g ↑ a) `rem` p) [1 .. p − 1])
```

Note that we *nub* the result to restrict the resulting set to the group itself. For the case where g generates a proper subgroup of the entire group, we otherwise would get repetitions. We also *sort* the groups to get a canonical order, *i.e.* $[1, 2, 4]$ instead of $[4, 2, 1]$.

Let us look at the groups generated by all the numbers $\{1 \ldots 6\}$ modulo 7:

generate 7 1 = $[1]$
generate 7 2 = $[1, 2, 4]$
generate 7 3 = $[1, 2, 3, 4, 5, 6]$
generate 7 4 = $[1, 2, 4]$
generate 7 5 = $[1, 2, 3, 4, 5, 6]$
generate 7 6 = $[1, 6]$.

We see 4 different groups. Two of these groups are quite trivial: $g = 1$ generates a group with just 1 element, since $1 \times 1 = 1$; $g = 6$ generates a group with two elements, since $1 \times 6 = 6$ and $6 \times 6 = 1$. These two trivial groups exist for any prime greater 2, since $1 \times 1 = 1$, trivially, holds for any modulus and $(p-1)(p-1) = 1$ holds for any prime modulus. 2 is an exception, because, with 2, we have $1 = p - 1$ and, therefore, 2 has only one trivial group.

The other subgroups modulo 7 are: $\{1, 2, 4\}$ generated by 2 and 4 and the complete group $\{1 \ldots p-1\}$ generated by 3 and 5. The size of these groups are 1 and 2 (for the trivial ones) and 3 and 6 for the non-trivial ones. We call the size of a group its *order* and write $|G|$ for the order of group G. The order of the complete prime group is, as we know, $p - 1$. What about the order of the other groups? Is there a pattern too?

To further investigate, we define a function that shows all subgroups of a prime:

$$\begin{aligned}&\mathit{allGroups} :: \mathit{Natural} \to [[\mathit{Natural}]]\\&\mathit{allGroups}\ p = \mathit{map}\ (\mathit{generate}\ p)\ [2\,..\,p-2]\end{aligned}$$

Note that we leave out the trivial groups 1 and $\{1, p-1\}$; we know that they exist for any p, so there is not much information added by showing them.

These are the results for *allGroups* 13 (with duplicates already removed):

$[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$
$[1, 3, 9]$
$[1, 3, 4, 9, 10, 12]$
$[1, 5, 8, 12]$

We see again 4 groups. They have the orders 3, 4, 6 and 12. There are also the two trivial groups with order 1 and 2 of course. A striking peculiarity is that for both 7 and 13, all subgroups have orders that divide the order of the main group. For 7, the order of the main group is 6 and the proper subgroups have the orders 1, 2 and 3. For 13, the order of the main group is 12 and the proper subgroups have the orders 1, 2, 3, 4 and 6. We will use the following function to investigate this further:

$$\begin{aligned}&\mathit{orders} :: \mathit{Natural} \to [\mathit{Int}]\\&\mathit{orders} = \mathit{nub} \circ \mathit{map}\ \mathit{length} \circ \mathit{allGroups}\end{aligned}$$

The result for *orders* 13 is $[12, 3, 6, 4]$.

We now map this function on the primes; we drop the first two primes, 2 and 3, because they have only the trivial subgroups and start by looking at the first 9 primes starting with 5 calling *map orders* (*take* 9 (*drop* 2 *allprimes*)):

5	7	11	13	17	19	23	29	31
4	3,6	5,10	3,4,6,12	4,8,16	3,6,9,18	11,22	4,7,14,28	3,5,6,10,15,30

The suspicion is confirmed: The orders of the subgroups always divide the order of the prime group. Indeed, this fact is known as *Lagrange's theorem*. It was proven by Joseph

Louis Lagrange, 1736 – 1813, an Italian mathematician who lived and taught in Turin, Berlin and Paris. Lagrange proved another important theorem that we already met: Wilson's theorem.

Lagrange's theorem is a kind of crossroads between different branches of mathematics including group theory, number theory, set theory and algebra. It is as such a quite deep theorem and to appreciate its full meaning, we need much more mathematical machinery than we have available right now. We will come back to Lagrange's theorem and provide a complete proof. Here, we will provide a quite simple proof sufficient for the current context concerning finite multiplicative groups. But even this proof provides surprisingly deep insight into the structure of groups.

Lagrange's theorem states that for any group G and any of its subgroups H: $|H|$ divides $|G|$. We start the proof by considering an arbitrary group modulo a prime. Such a group is generated by a sequence of powers of a: $a^1, a^2, \ldots, a^k$, where $a^k = 1$. For sake of explicitness, let us consider a concrete example, say, the group modulo 7, which has order 6. Let the sequence of powers of a, any primitive element of that group, be the sequence:

$$a, b, p-1, b', a', 1.$$

In this group, the placement of b and its inverse b' is arbitrary. The placement of $a = a^1$, $p-1$, a' and 1, however, is on purpose and respects the order in which these numbers necessarily appear, when the numbers reflect the numbers generated by a^1, a^2 and so on. The first number, a^1, trivially is a. The last number a^k is 1. The last but one number is that number in the group that multiplied by a results in 1, *i.e.* the inverse of a. Since a^6, in this example, is 1, a^3 must be its own inverse, *i.e.* a number that multiplied by itself, $a^{3+3} = a^6$, is 1. We know that $p-1$ is the only number, besides 1 itself of course, that is its own inverse.

Since the sequence terminates with $a^6 = 1$, it would repeat with $a^7 = a$, when we continue. We show this in the following table up to exponent $k = 12$:

1	2	3	4	5	6	7	8	9	10	11	12
a	b	p-1	b'	a'	1	a	b	p-1	b'	a'	1

Based on this information, try to imagine the group generated by b. b is aa, so we get (with the headline indicating the exponents of a, not of b!):

1	2	3	4	5	6	7	8	9	10	11	12
	b		b'		1		b		b'		1

Indeed, we know that $b = a^2$ and that $b' = a^4$. Consequently, $bb = aaaa = a^4 = b'$ and, even further, $bbb = a^6 = 1$. We therefore see a group with three members.

The inverse of b would generate another group with three members, but since the exponent of b', which is 4, does not divide the exponent of 1, 6, we need more than one cycle to terminate the group:

1	2	3	4	5	6	7	8	9	10	11	12
			b'				b				1

Continuing this scheme, we can easily imagine the group generated by $p-1$:

1	2	3	4	5	6	7	8	9	10	11	12
		p-1			1			p-1			1

and that generated by 1:

1	2	3	4	5	6	7	8	9	10	11	12
					1						1

In other words, the periodicity of element 1, which appears every 6 as, and the relation among numbers, that is $b = aa$ and $p-1 = aaa$ in this example, determine all possible group orders. If $a^i = p-1$, then $a^{2i} = 1$ determines the order of the group generated by a. If b is member of that group and $b = a^j$, then j must divide either $2i$ or a multiple of that number – a', which is a^5, for instance, would only return to 1 at index 30. Since only every j^{th} element is in the group of b, the group has $\frac{lcm(2i,j)}{j}$ members. For b, we have $j = 2$ and $\frac{lcm(6,2)}{2} = 3$ group members. For b', we have $j = 4$ and $\frac{lcm(6,4)}{4} = 3$ group members. For $p-1$, we have $j = 3$ and $\frac{lcm(6,3)}{3} = 2$ group members. For 1, we have $j = 6$ and $\frac{lcm(6,6)}{6} = 1$ group member. For a' we have $j = 5$ and $\frac{lcm(6,5)}{5} = 6$ group members. For a, finally, we have $j = 1$ and $\frac{lcm(6,1)}{1} = 6$ group members. The number $\frac{lcm(2i,j)}{j}$, for sure, divides $2i$, since $lcm(2i, j)$ is a multiple of both, $2i$ and j. That is all what we wanted to prove. □

For our case at hand, this proof (even though a bit sloppy using an example) is sufficient. The theorem, however, is not limited to remainders of primes. We will see other examples soon and, indeed, we already saw an example with very similar effects. In the previous section, we discussed composition of permutations and there we saw that a permutation with orbits of different size n and m, need $lcm(n, m)$ applications to come back to the original sequence.

But let us come back to problems of primes. When we look at the table of subgroup orders above, we see some primes, such as 11 and 23, with strikingly fewer subgroups than the primes in their surrounding. 11 and 23, both, have two subgroups, while 13, 19 and 29 all have four subgroups. Is this a pattern or is it just one of the curiosities that arise with small numbers? Here is a list of six more primes generated by *orders* (*take* 6 (*drop* 11 *allprimes*)):

37	41	43	47	53	59
3,4,6,9,12,18,36	4,5,8,10,20,40	3,6,7,14,21,42	23,46	4,13,26,52	29,58

Most primes we see in this sequence have 6 or 7 subgroups and one, 53, has 4. The outliers are 47 and 59 with 2 subgroups each. So, what is special about the primes 11, 23, 47 and 59?

Let us examine their subgroup orders:

11	23	47	59
5,10	11,22	23,46	29,58

All of these primes have the subgroup with order $p-1$ and a subgroup with order $\frac{p-1}{2}$, *i.e.* the half of the order of the main group. In all cases, the half of the order of the main group is again a prime number: 5, 11, 23 and 29. When the order of the main group has only two prime factors, namely 2 and q, where q is again prime, then, since the order of any subgroup must divide the order of the main group, there cannot be another subgroup besides the trivial ones with order 1 and 2. This fact has huge importance for cryptography, especially the Diffie-Hellman key exchange protocol and the Schnorr signature. These cryptosystems make use of primes of the form $2p+1$ where p is also prime to guarantee that any element chosen but 1 and $p-1$ is member of a huge subgroup. We will discuss this later in this chapter.

Primes of the form $2p+1$ (with p prime) are called *safe primes*. The other prime, the p in the safe prime formula, is called *Sophie Germain* prime after the great French mathematician Sophie Germain (1776 – 1831). Germain began to study math with about 13 years of age. Later, when the École Polytechnqiue opened in Paris during the French Revolution, she started to send essays to the teachers there and one of them, again Lagrange, recognised her talent.

Since it was not allowed for women to study at the Polytechnique at that time, Germain used the name of a former student, Antoine-August Leblanc, when she presented her papers. Lagrange, however, was quite excited about the quality of these essays and was eager to get to know this talented student. So, Germain was forced to reveal her identity, when Lagrange invited the person he believed to be Leblanc. Fortunately, Lagrange continued to support Germain and she was able to present important results in mathematics including number theory and won prestigious prizes from the Paris Academy of Sciences.

Germain would also correspond with Gauss, again under the name Leblanc; when the Napoleon army occupied Braunschweig, where Gauss lived at that time, she asked a friend of the family who was actually a general of the French army to see after Gauss' safety during the occupation. On this occasion, Gauss learnt who his French correspondent was and wrote later:

> How can I describe my astonishment and admiration on seeing my esteemed correspondent M leBlanc metamorphosed into this celebrated person... when a woman, because of her sex, our customs and prejudices, encounters infinitely more obstacles than men in familiarising herself with knotty problems, yet overcomes these fetters and penetrates that which is most hidden, she doubtless has the most noble courage, extraordinary talent, and superior genius.

Let us devise a fuction in the honour of Sophie Germain to list the primes that bear her name:

$$sophieprimes :: [Natural]$$
$$sophieprimes = filter\ (\lambda p \to prime\ (2 * p + 1))\ allprimes$$

The first 16 Sophie Germain primes, listed with *take* 16 *sophieprimes* are:

2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233.

Let us confirm that the primes of the form $2p + 1$ that correspond to these Sophie Germain primes all have only two non-trivial groups. We use the function:

$$safeprimes :: [Natural]$$
$$safeprimes = map\ (\lambda q \to 2 * q + 1)\ sophieprimes$$

These are 16 safe primes greater than 59 listed with *take* 16 (*dropWhile* ($\leqslant 59$) *safeprimes*):

83, 107, 167, 179, 227, 263, 347, 359, 383, 467, 479, 503, 563, 587, 719, 839.

The subgroup orders of the first 9 of them are:

83	107	167	179	227	263	347	359	383
41,82	53,106	83,166	89,178	113,226	131,262	173,346	179,358	191,358

and we see that, indeed, all safe primes have exactly two non-trivial groups.

The concept can be extended to primes of the form $p = nq+1$ where n is even. This way we obtain a p with potentially much more subgroups than just the group of order $\frac{p-1}{2}$, *viz.* one subgroup per prime factor of n. The group we want to use, that of size $\frac{p-1}{n}$, is called the *Schnorr group* of p after the German mathematician and cryptographer Claus Schnorr, who studied the mathematical properties of this group extensively and invented a cryptographic authentication system based on it, the *Schnorr signature*. We will come back to that soon.

5.10. Primality Tests

Primality tests are algorithms that test whether a given number is prime. Until now, we have only one primality test. This test, basically, creates one prime after the other until it finds one that divides the number in question. If the first number found is

that number itself, then this number is prime. The test finds an answer after having, in the worst case, examined $\sqrt{n}$ numbers. For small numbers, this is great. For large numbers, however, this may turn out to be very expensive. To be honest, in spite of all its ingenuity, this algorithm is quite shabby. It is just a brute force attack that tries to solve the problem by looking at all primes that stand between us and $\sqrt{n}$.

In the previous sections, we have learnt a lot of facts concerning prime numbers. Perhaps some of those facts may help us to distinguish between composites and primes. The first candidate is Fermat's little theorem, which states that, for any integer a and any prime p:

$$a^{p-1} \equiv 1 \pmod{p}. \tag{5.58}$$

This way, Fermat's theorem provides a criterion for a number being prime (or, more precisely, being not prime) that can be easily tested. Power is still a heavy operation with large numbers, but we need to apply the opertation only once and that is indeed much cheaper than going through millions of primes to test just one number for primality. A test based on Fermat could be implemented like this:

$$\begin{aligned}
&\mathit{fprime} :: \mathit{Natural} \to \mathit{Bool}\\
&\mathit{fprime}\ 0 = \mathit{False}\\
&\mathit{fprime}\ 1 = \mathit{False}\\
&\mathit{fprime}\ 2 = \mathit{True}\\
&\mathit{fprime}\ p = (2 \uparrow (p-1))\ \text{`}\mathit{rem}\text{`}\ p \equiv 1
\end{aligned}$$

Since the condition should hold for any integer a, we just choose 2 and check if it, raised to $p-1$, leaves the remainder 1 divided by p. (We do not choose 1, of course, since 1 trivially leaves 1 with any number p.) We could test this new primality test with the old one comparing their results, like this: $[(n, \mathit{prime}\ n, \mathit{fprime}\ n) \mid n \leftarrow [1\,..]]$. The result look like this:

(1, *False*, *False*)
(2, *True*, *True*)
(3, *True*, *True*)
(4, *False*, *False*)
(5, *True*, *True*)
...

and without much surprise, we see that *fprime* produces the same results as *prime*. But be careful: Fermat's little theorem claims that all primes adhere to the rule, but it does not make any statement on composites. According to the theorem, composites may or may not leave a remainder of 1 with a^{p-1}. What the criterion establishes is therefore not that p is prime, but that, if the condition is not fulfilled, p is not prime. Is this relevant? Well, let us see what happens, when we continue the listing above:

...

(339, *False*, *False*)
(340, *False*, *False*)
(341, *False*, *True*)
...

The primality tests disagree on 341! Which one is right? First let us look at the Fermat test:

$$2^{340} \equiv 1 \pmod{341}.$$

According to this test, 341 appears to be prime. So it must not have any factors. However, *trialfact* 341:

$[11, 31]$

Apparently, 341 has two factors, 11 and 31, since $11 \times 31 = 341$. So, 341 is definitely not a prime. In fact, there are composites that pass certain primality tests, so called *pseudoprimes*. We could have actually avoided falling into this trap by chosing a different a, for instance:

$$3^{340} \equiv 56 \pmod{341}.$$

However, there are still 98 of 338 numbers in the range $3 \ldots 340$ for which the Fermat test would have succeeded. We could repair the Fermat test by making it stronger: we could demand that all numbers $2 \ldots 340$ must pass the test, before we accept p being prime. This, however, would make the Fermat test quite expensive – and that it is inexpensive was the main reason we have chosen it in the first place. As an alternative, we could demand that there should be a certain amount of numbers for which the Fermat test should not fail. But even this would not help a lot, since there are numbers where indeed most $a \in \{2 \ldots p-1\}$ would pass the test, namely, the *Carmichael numbers*:

561, 1105, 1729, 2465, 2821, 6601, 8911, ...

A Carmichael number n is a composite number such that $a^{n-1} \equiv 1 \pmod{n}$ for every a coprime to n. When Robert Carmichael discovered the first of these numbers in 1910, the notion already existed, but under another name and with another definition. Already in 1899, the German mathematician Alwin Korselt defined numbers n, such that n is squarefree (no prime factor appears more than once in the prime factorisation of that number) and that, for every prime factor p, it holds that $(p-1)|(n-1)$. It turned out that both definitions are equivalent. 561, for example, has the factorisation $\{3, 11, 17\}$. Trivially, $3 - 1 = 2$ divides $561 - 1 = 560$; $11 - 1 = 10$ also divides 560 and, finally, $17 - 1 = 16$ divides 560, since $16 \times 35 = 560$.

Numbers coprime to 561 in the range $1 \ldots 560$ are all numbers not multiples of the prime factors 3, 11 or 16. 242 of the 560 numbers $1 \ldots 560$ are actually multiples of (at least)

one of the prime factors. All other numbers are coprime to 561. In other words, more than half of the numbers will pass the Fermat test. Carmichael numbers are therefore hard to distinguish from primes by means of tests that avoid testing all remainders of n.

There are candidates for much stronger primality tests, however. Wilson's theorem, for instance, provides a criterion that holds for primes only, namely:

$$(p-1)! \equiv -1 \pmod{p}. \tag{5.59}$$

A primality test that would not fall for Carmichael numbers and other pseudoprimes could be based on Wilson's theorem:

$$\begin{aligned}
&\mathit{wprime} :: \mathit{Natural} \to \mathit{Bool}\\
&\mathit{wprime}\ 0 = \mathit{False}\\
&\mathit{wprime}\ 1 = \mathit{False}\\
&\mathit{wprime}\ 2 = \mathit{True}\\
&\mathit{wprime}\ n = (\mathit{fac}\ (n-1))\ `\mathit{rem}`\ n \equiv (n-1)
\end{aligned}$$

And, indeed, *wprime* gives *False* for 341 as it does for any of the Carmichael numbers. Unfortunately, factorial is a very expensive operation rendering *wprime* as ineffective as *prime*.

Another idea is to base primality tests on the observation that only for a prime p it holds that for any $0 < k < p$:

$$\binom{p}{k} \equiv 0 \pmod{p}. \tag{5.60}$$

But, again, we have to test this for all ks. In the case of 561, for instance, 446 of the 560 possible ks fulfil the equation. But computing all binomial coefficients $\binom{p}{1}, \binom{p}{2}, \ldots \binom{p}{p-1}$, is not feasible for large numbers.

The next bunch of ideas would use an implication of that fact, such as the *freshman's dream*:

$$(a+b)^p \equiv a^p + b^p \pmod{p} \tag{5.61}$$

Unfortunately, there is a proof (by Ghatage and Scott) that this condition is true exactly if p is prime or ... a Carmichael number. To distinguish primes from Carmichael numbers, it is again necessary to test for all numbers a.

There is finally an efficient test that builds on a variant of freshman's dream, namely the Agrawal-Kayal-Saxena (AKS) test, which exploits the fact that

$$(x-a)^p \equiv (x^p - a) \pmod{p}. \tag{5.62}$$

Agrawal, Kayal and Saxena, a group of Indian mathematicians and computer scientists won the Gödel Prize for their paper "PRIME is in P" where they actually presented the AKS test. The test adopts algebraic methods to avoid computing all possible xs and as to establish that a number is prime. Since we have not looked into algebra yet, we have to postpone the discussion of this algorithm. In practical terms, this is not an issue, since there are algorithms that establish the primality of a number with sufficiently high probability.

It is a very common approach in math and computer science to accept algorithms with bounded error probability if those algorithms are significantly faster or simpler than their deterministic cousins and if an error bound can be given. This bound can then be used to compute the number of repetitions necessary to make the probability of failure small enough to be ignored.

A test able to establish the primality of a number greater than 2 with sufficient probability is the Rabin-Miller test. The mathematical idea is based on Fermat's little theorem, but with some refinement. The reasoning starts with the observation that $p-1$ in a^{p-1} in Fermat's equation is even, since p is prime, as required by the theorem, and $p > 2$, as required by Rabin-Miller. We could therefore represent that number as a series of squares of the form

$$a^{s^{2^{2^{\cdots}}}},$$

for some odd integer s. This is of course equivalent to

$$a^{s\times 2\times 2\times \cdots}.$$

If p is, say, 5, then the Fermat equation would look like $a^4 \equiv \pmod 5$, which we could write as $a^{1\times 2\times 2}$, where $s = 1$. For $p = 7$, this would look like $a^{3\times 2}$ and $s = 3$.

Let us look at the "last" square $(a^d)^2$ independent of whether d is even or odd. We know this square must make the equation congruent to 1 modulo p. Let us examine this last exit before $p-1$, a^d, and call this number x. We then have:

$$x^2 \equiv 1 \pmod p,$$

From here, we can apply the same technique we have already used to prove Wilson's theorem; we first subtract 1 on both sides and we get

$$x^2 - 1 \equiv 0 \pmod p.$$

$x^2 - 1$ can be factored into $(x+1)(x-1)$ and we get the congruence

$$(x+1)(x-1) \equiv 0 \pmod{p}.$$

For the product on the left-hand side to become 0, one of its factors must be 0. For the first to be 0, x must equal -1; for the second, x must equal 1. This is the same result we have already obtained, when proving Wilson's theorem.

We substitute a^d back for x and see that the last exit before the last square must be either $a^d \equiv -1 \pmod{p}$ or $a^d \equiv 1 \pmod{p}$. The second case, however, where a^d is 1, can only occur if a^s was 1 or $p-1$ right from the beginning or if somewhere on the way from a^s to a^d, the whole expression becomes $p-1$. We know this for sure from Wilson's theorem: For every number a in the range $1 \ldots p-1$, there is an inverse a', such that $aa' \equiv 1 \pmod{p}$ and there are only two numbers for which $a = a'$, namely 1 and $p-1$. Squaring a number that is neither 1 nor $p-1$, therefore, cannot result in 1. There are thus only two ways for a^{2d} to become 1: either a^s was 1 right from the beginning, then squaring will not change anything; or, at some point, a^d is $p-1$ (which may be obtained by squaring two numbers) and then, in the next step, 1. It is impossible, however, that, if a^s was not 1 nor $p-1$, that 1 pops up on the way, without $p-1$ having occurred before.

This is the idea of Rabin-Miller: It finds an odd number s and a number t that tells us how often we have to square a^s to get to a^{p-1}. Then it checks if a^s is either 1 or $p-1$. If not, it checks if any of a^s, a^{2s}, a^{4s}, ..., a^{ts} is $p-1$. If this is not the case, p is composite.

The advantage of this method over the simple Fermat test is that it reduces the set of remainders per number that actually pass. This also reduces the probability of the test to actually go wrong for a specific number. If this probability is significantly less than 50%, we can reach a correct result with high probability by applying the test more than once. But let us postpone the probability reasoning for a short while. First, we will have a look at the implementation of the algorithm.

To start, we need a function that gives us s, the odd number after taking all squares out of $p-1$, and t, the number that tells us how many squares we have actually taken out to reach s. It then holds that $2^t s = p-1$. In the lack of a useful name for that function, we call it *odd2t*:

```
odd2t :: Natural → Natural → (Natural, Natural)
odd2t s t | even s     = odd2t (s `div` 2) (t + 1)
          | otherwise = (s, t)
```

For 16, *odd2t* would give $(1, 4)$, since 16 is a power of 2, *i.e.* divided subsequently by 2, it will reach 1. One has to multiply 1 4 times by 2 to get 16 back: $1 \times 2^4 = 16$. For 18, *odd2t*, accordingly, would yield $(9, 1)$, since $9 \times 2^1 = 18$.

The next function is the primality test itself:

```
rmPrime :: Natural → Natural → Natural → Natural → Bool
rmPrime p a s t = case (a ↑ s) `rem` p of
                    1 → True
                    v → if v ≡ p − 1 then True
                                     else go t v
  where go 1 _ = False
        go t v = case (v ↑ 2) `rem` p of
                   1 → False
                   v′ → if v′ ≡ p − 1 then True
                                      else go (t − 1) v′
```

The function receives four arguments: The number to test for primality, the test candidate a, also called a witness for the primality of p, and s and t obtained from *odd2t*. The function raises a to the power of s. If the result is 1 or $p-1$ modulo p, p has already passed the test. Otherwise, we loop through *go*. *go* receives two argument t and v (which initially is a^s). If t is 1, we have exhausted all the squares in $p-1$ without having seen $p-1$. The test has failed. Otherwise, we create the next square modulo p. If this square is 1, something is wrong: we know that v was neither 1 nor $p-1$, so squaring it cannot result in 1, if p is prime, because only 1 and $p-1$ are their own inverses. Otherwise, if the result is $p-1$, we are done. Further squaring will yield 1 and all conditions for p being a prime are fulfilled. Otherwise, we continue with the next square, reducing the square counter t by 1.

The function that brings these bits together needs randomness to choose as. To this end, we use the function *randomNatural* defined in the previous chapter:

```
rabinMiller :: Natural → Natural → IO Bool
rabinMiller _ 0 = return False
rabinMiller _ 1 = return False
rabinMiller _ 2 = return True
rabinMiller k p | even p     = return False
                | otherwise = let (s, t) = odd2t (p − 1) 0 in go k s t
  where go 0 _ _ = return True
        go i s t  = do a ← randomNatural (2, p − 1)
                       if rmPrime p a s t then go (i − 1) s t
                                          else return False
```

The function receives two arguments: k and p. p is the number under test. k tells the function how often it has to repeat the test until the expected probability is reached. If k is exhausted, *i.e.* $k = 0$, we return *True* (in *go*) and p has passed the complete test.

At the beginning, we take care of some trivial cases, such as 0 and 1, which are never prime, and 2, which actually is prime. With the exception 2, no even number is prime. Then we start the hard work: we first find s and t using *odd2t*; then we enter *go*. We generate an a from the range $2 \ldots p-1$, using *randomNatural*. Then we apply the test.

If the test fails, we immediately return False. If the test passes, we repeat until $i = 0$.

To reason about the probability for the test to fail, let us look at some examples. The following simple function can be applied to a number to show the results of the Fermat test. It returns a list of tuples where the first element is one of the numbers $2 \ldots n-1$ and the second is this number raised to $n-1$ mod n:

$$
\begin{aligned}
&\mathit{rest} :: \mathit{Natural} \to [(\mathit{Natural}, \mathit{Natural})] \\
&\mathit{rest}\ n = \mathit{zip}\ \mathit{rs}\ \$\ \mathit{map}\ (\lambda a \to (a \uparrow (n-1))\ `\mathit{rem}`\ n)\ \mathit{rs} \\
&\quad \textbf{where}\ \mathit{rs} = [2\,..\,n]
\end{aligned}
$$

rest 9, for instance, yields:

$[(2,4),(3,0),(4,7),(5,7),(6,0),(7,4),(8,1)]$

We see that, for most of the numbers, the Fermat test would fail. For 8, however, it would pass, since $8^8 \equiv 1 \pmod 9$. 8 is therefore a *liar* concerning the primality (or, more precisely, for the compositeness) of 9. Unfortunately, Rabin-Miller would not help us in this case, since *odd2t* 8 0 = (1, 3); 8^1, however, is $n-1$ and the test would immediately pass. 8, hence, is a *strong liar* for 9.

Let us look at another example: 15. *odd2t* for 15 gives $(7,1)$, since $14\ `\mathit{div}`\ 2$ is 7, which is odd. *rest* 15 yields:

$[(2,4),(3,9),(4,1),(5,10),(6,6),(7,4),(8,4),(9,6),(10,10),(11,1),(12,9),(13,4),(14,1)]$.

There are several liars: 4, 11 and 14. 14, again is a strong liar, since $14^7 \bmod 15 = 14$, which is $n-1$. 11 and 4, however, are ruled out by Rabin-Miller: $11^7 \bmod 15 = 11$, which squared would never be 1, if 15 were prime; $4^7 \bmod 15 = 4$, which squared, again, would not result in 1, if 15 were prime.

Let us devise a function that counts the occurrences of (Fermat) liars and strong liars for any given composite n. The first function is called *liars* and quite simple:

$$
\begin{aligned}
&\mathit{liars} :: \mathit{Natural} \to \mathit{Int} \\
&\mathit{liars} = \mathit{length} \circ \mathit{filter}\ (\equiv 1) \circ \mathit{map}\ \mathit{snd} \circ \mathit{rest}
\end{aligned}
$$

That is we start with *rest*, ignore the first element of each tuple, filter the 1s and count the elements of the resulting list. (The return type of the function is *Int*, rather than *Natural*, because we use *length*, which returns an *Int* anyway.)

The strong liar function is a bit more tricky:

```
strongLiars :: Natural → Int
strongLiars n | even n     = 0
              | otherwise =
  let (s, t) = odd2t (n − 1) 0
      sl     = foldr (λa l → detector l a s t) [] [2 .. n − 1]
  in  length sl
  where detector l a s t | rmPrime n a s t = a : l
                         | otherwise       = l
```

For finding strong liars, we implement a part of the Rabin-Miller test and, therefore, we ignore even numbers (just yielding 0). For even numbers, the (s, t) values would not make any sense, since $n - 1$ is odd! Then, we apply the *rmPrime* test we implemented for Rabin-Miller to all remainders, adding those that pass to the result list. Finally, we just yield the length of that list.

If we apply *liars* to a prime number p, all witnesses $2 \ldots p - 1$ are counted as liars, *e.g.* *liars* 11: 9. The same is true for *strongLiars*, since the fact that all witnesses are strong liars could be a definition of primalitiy. Thus, *strongLiars* 11: 9.

Applied to 9, *liars* and *strongLiars* yield 1. Applied to 15, *liars* yields 3; *strongLiars*, however, yields only 1. This is in-line with our investigation above. Here are some more examples for numbers between 21 and 75:

21	25	27	33	35	39	45	49	51	55	57	63	65	69	75
3,1	3,3	1,1	3,1	3,1	3,1	7,1	5,5	3,1	3,1	3,1	3,1	15,5	3,1	3,1

We see a quite colourful picture. Many numbers have 3 liars and 1 strong liar; for some numbers, there is no difference in liars and strong liars, for instance 25 and 27, both have the same numbers of liars and strong liars, namely 3 and 1. Other numbers, *e.g.* 45, show a strong reduction in going from liars to strong liars. There are some peaks, *e.g.* 65 has 15 liars and 5 strong liars, much more liars than most other numbers. If we continue up to 99, the greatest number we will see is $(35, 17)$ for 91. The nasty number 341 has 99 liars and 49 strong liars. Finally, here are the dreadful Carmichael numbers:

561	1105	1729	2465	2821	6601	8911
319,9	767,29	1295,161	1791,69	2159,269	5279,329	7127,1781

For many Carmichael numbers, the reduction of liars is significant – for some, the reduction is about factor 10 – 30. There are some exceptions with reduction of a factor of "only" 6 like 8911. In general, the number of strong liars is very low compared to n, the prime candidate. It can be shown in fact that the number of strong liars for an odd composite n is at most $\frac{n}{4}$. (This has actually been shown with contributions, among others, by the legendary Paul Erdős, 1913 – 1996.) The arguments, however, are much beyond our scope.

The ratio $\frac{n}{4}$ implies that the probability of hitting a strong liar, when performing *rmPrime* on a randomly chosen witness for n, is $\frac{1}{4}$. In other words, one has to try

four times in average to get a strong liar by chance. When we repeat the test several times, we reduce the probability that **all** witnesses we have used are strong liars. It is important to notice that the test yields *False* immediately, when we find a witness for compositeness. We continue only if all tests so far have been witnesses for primality.

The probability is therefore computed as $\frac{1}{4^k}$, where k is the number of repetitions. The probability to obtain two liars in two applications is $\frac{1}{4^2} = \frac{1}{16}$. We, hence, would have to call a Rabin-Miller Test with two repetitions 16 times in average to test only on strong liars once. With four repetitions, the denominator is $4^4 = 64$, with eight, it is 65 536, with sixteen, it is 4 294 967 296 and so on. A reasonable value for k to defend against malicious attacks given, for example, in *Cryptographic Engineering* is $k = 64$. In average, one has a chance of 1 out of 4^{64} or 340 282 366 920 938 463 463 374 607 431 768 211 456 to hit only strong liars in testing a number for primality. That, indeed, appears to be reasonable. A final version of the Rabin-Miller Test could then look like this:

```
rmptest :: Natural → IO Bool
rmptest 0 = return False
rmptest 1 = return False
rmptest 2 = return True
rmptest n | even n     = return False
          | n < 64     = return (wprime (fromIntegral n))
          | otherwise = rabinMiller 64 n
```

5.11. Primes in Cryptography

To say this right at the beginning: this is not an introduction to cryptography! We will go through some examples of how primes are used in cryptography, in particular the Diffie-Hellman key exchange protocol and RSA. But we will not discuss pitfalls, common errors or other issues you have to take care of when implementing cryptosystems. If you want to learn about cryptography, you definitely have to study the literature on the topic. Cryptography poses very hard engineering challenges and we are far from adressing them here. Again: this is not an introduction to cryptography! We do not even use a big number library that would be able to cope with numbers with hundreds or thousands of digits. When you use the algorithms presented here (like the Rabin-Miller test) on numbers of that size, you will probably have to wait minutes or even hours for results. Raising a number of thousands of digits to another number of thousands of digits requests special handling. You cannot do that with our *Natural* data type or even the Haskell *Integer* data type. So, once again: this is not an introduction to cryptography.

Primes come into play in cryptography, typically, in very special applications. Usually, to encrypt and decrypt a message, secret keys are used that are known to both sides of the communication, traditionally called Alice and Bob in the literature. The algorithms to convert the plain message into the *cyphertext* have in most cases nothing to do with primes, but are rather combinations of basic operations like XOR and *bit shuffle* involving

the original message and the key material.

The weakest link in this kind of cryptography is the key itself. All parties that take part in the secure communication must know the key and, in consequence, the key must be shared among them in a secure way. How to share the key safely is indeed a major challenge. For the German submarines in World War II, just to name a popular example, the loss of a codebook was at least as challenging as the cryptoanalysis by British specialists at Bletchley Park. A new codebook could hardly be distributed among all submarines on the ocean in due time.

The challenge is an instance of the bootstrapping problem: we have to start a process, namely secure communication, without having the means to run this process, namely secret keys distributed among all parties. One solution is to use publicly available information to convert a plain message into a cyphertext by means of a *one-way function*, *i.e.* a function that is easy to calculate, but difficult to revert. Here is where primes come in.

A one-way function f, is a function that computes a result from a given input, such as $f(x) = y$, for which no inverse f' is known, such that $f'(y) = x$ or $f'(f(x)) = x$. Plain multiplication, for instance, is not a good one-way function, since with the result y known, we simply can revert the effect by division. If we have a function $f(x) = ax$, and a cyphertext $f(x) = y$. We can reconstruct x, simply, by $f'(y) = y/a$. Multiplication would be a good one-way function, however, if both a and x were unknown. The inverse, would then be factoring – and factoring of large numbers is indeed a hard problem.

Before we have a closer look at concrete examples, we have to come back to a question we have already discussed (but without satisfying results), namely how to generate huge primes. We have seen Mersenne primes and Fermat primes, but little is known about this kind of numbers and in particular, no recipe is known how to find new ones. Practial algorithms are in fact much simpler in terms of mathematical ideas. Prime generators usually generate a random number in a given range, test whether it is prime and, if it is, return this number or, if it is not, try again.

A reasonable prime generator for natural numbers, using once again the random number generator *randomNatural*, could look like this:

```
generatePrime :: Natural → IO Natural
generatePrime k = do n ← randomNatural (2 ↑ (k − 1), 2 ↑ k − 1)
                     t ← rmptest n
                     if t  then return n
                           else generatePrime k
```

As you can see, this is just a trial-and-error approach using the Rabin-Miller test to check whether a given number is prime. Is there not a risk of running a lot of time depending on the range we have chosen? Indeed, it can take a lot of time, before we actually find a prime. However, we know some things about the distribution of primes, so that we are guaranteed to find a prime within a reasonably chosen range. We will

discuss some of the relevant aspects concerning the distribution of primes in the next section. For the moment, it may suffice to mention that a range like $k \ldots k + c$, where c is some constant smaller than k, is obviously not sufficient. The range above, however, is chosen in terms of powers of 2 and, assuming that k is at least 10, we are guaranteed to find some prime numbers in such a range like, for $k = 10$, $512 \ldots 1023$. It is quite common, by the way, to give the size of the prime wanted in bits, *i.e.* in terms of the number of digits in binary representation, rather than in decimal representation.

The first example of prime-based key exchange is the Diffie-Hellman protocol developed by Whitfield Diffie, Martin Hellman and Ralph Merkle in the 70ies. There was another group of scientists that developed a similar algorithm shortly before Diffie and Hellman published their results. Those scientists, unfortunately, were working for the British secret service, GCHQ, at the time and were not allowed to publish their results. They, thus, escaped eternity.

The Diffie-Hellman protocol is very simple and elegant, but has some pitfalls that must be addressed. Its main purpose is to exchange a secret key between Alice and Bob without Eve, another fictional character famous in cryptography literature, getting to know that key by eavesdropping. Before the Diffie-Hellman protocol actually starts, Alice and Bob have agreed somehow on a public key that may be known to anyone including Eve. This public key may be assigned to either Alice or Bob and may be registered in a kind of phone book or it may be agreed upon in the *handshaking* phase of the protocol.

The public key consists of two parts: a prime number p and a generator g. When Alice initiates the protocol, she chooses a number x from the range $2 \ldots p-2$, computes $g^x \bmod p$ and sends this number to Bob. Bob, in his turn, chooses a number y from the same range, computes $g^y \bmod p$ and sends the result back to Alice. Evil Eve knows p and g and may see g^x and g^y. But none of these values is actually the key. The key, instead, is g^{xy}. At the end of the protocol, this number is known to Alice and Bob. Each of them just has to raise the number he or she receives from the other by his or her own number. So Alice chooses x and sends g^x to Bob. Bob chooses y and sends g^y. Alice computes the key as $k = g^{y^x}$, *i.e.* she raises the number she receives from Bob to her own number; Bob computes the key, accordingly, as g^{x^y}, *i.e.* he raises the number he receives from Alice to his own number. That is all.

For a simple example, let us assume the public key is $p = 11$ and $g = 6$. Now, Alice chooses a random number from the range $2 \ldots 9$, say, 4, and Bob likewise, say, 3. Alice computes $6^4 \bmod 11 = 9$ and sends 9 to Bob. Bob computes $6^3 \bmod 11 = 7$ and sends 7 to Alice. Alice computes the key as $k = 7^4 \bmod 11 = 3$ and Bob computes it as $k = 9^3 \bmod 11 = 3$. The result is the same for both, of course, because, eventually, both have performed the same operation: g^{xy}.

The security is based on the difficulty to solve the equation

$$k = g^{xy}, \tag{5.63}$$

where g, g^x and g^y are known. This is an instance of the *discrete logarithm* problem, the logarithm in a finite field. If k, g and xy were ordinary real numbers, we could solve the equation in three steps: $x = log_g(g^x)$, $y = log_g(g^y)$ and, finally, $k = g^{xy}$. For the discrete logarithm, however, no efficient solution is known today.

In a trivial examples like the one we used above, Eve can simply try out all possible combinations. In real cryptography applications, we therefore have to use very large primes. But the security of the algorithm also depends on the order of g. For instance, if we want a security that corresponds to 1000 bits (a number with more than 300 digits in decimal representation), then the order of g should be much more than some hundreds or thousands or even millions of numbers generated by that g. To choose a proper g is therefore essential for the strength of the protocol. The question now is: how to choose a proper g?

If you have carefully read the previous sections, you already know the answer: we must use a g from the Schnorr group of a safe prime. Indeed, knowing the order of g is again a hard problem. We have to solve the equation $g^k = 1 \mod p$, which is again an instance of the discrete logarithm. We can circumvent this problem of finding an appropriate group by finding an appropriate prime. With a safe prime, the selection of an appropriate g is indeed simple. We just have to avoid one of the trivial groups, that is we have to avoid 1 and $p - 1$, then we are guaranteed that g is in the group of the Sophie Germain prime q or in the larger group of the safe prime $2q + 1$.

Sophie Germain primes, hence, give us a means to reduce the difficult problem of finding a proper g to the much simpler problem of selecting a proper prime:

```
safePrime :: Natural → IO Natural
safePrime k = generatePrime k ≫= λq → let p = 2 ∗ q + 1 in do
  t ← rmptest p
  if t then return p
       else safePrime k
```

This function generates a *safe prime*, *i.e.* a prime of the form $2q+1$, where q is a Sophie Germain prime. It just generates a random prime q, doubles it and adds 1 and checks whether the resulting number p is again prime and, if not, repeats the process. From the group of this prime, we can then take a random g, $g \neq 1, g \neq p-1$, and this g belongs in the worst case to the group q of order $\frac{p-1}{2}$. Since q was generated according to security level k, the group of q is exactly what we need and our main issue is solved.

There is still a problem, though. From the communication between Alice and Bob, Eve sees g^x and g^y. If she has read the section on quadratic residues, she can determine whether g is a square (modulo p) using the Legendre symbol. If g is a nonresidue, then she has an attack: she can repeat the test on the numbers she sees, namely, g^x and g^y.

If a number of the form g^z is a nonresidue, where g as well is a nonresidue, then z is odd, otherwise, z is even. This is because even exponents are just repeated squares. So, if z is even, g^z is a residue and, otherwise, it is not.

We should avoid this leakage. Even though Eve does not learn the whole number, she gets an information she is not entitled to have. The leakage effectively reduces the security level by the factor 2, since Eve learns the least significant digit of the number in binary representation: odd numbers in binary representation end on 1, even numbers on 0. We can avoid this problem simply by chosing a residue right from the beginning. The function to generate an appropriate g could then look like this:

```
generator :: Natural → IO Natural
generator p = do a ← randomNatural (2, p − 2)
                 let g = (a ↑ 2) `rem` p
                 if g ≡ p − 1 then generator p
                              else return g
```

We select an a from the range $2 \ldots p-2$ and square it. The number, hence, is guaranteed to be a residue eliminating the problem discussed above. We test if $g = p - 1$, which is forbidden, since it is in a trivial group. a^2, as you know, can become $p - 1$, if $p \equiv 1 \pmod 4$. You may argue that this is not the case, when p is a safe prime, because, if $\frac{p-1}{2}$ is prime, then $p - 1$ cannot be a multiple of 4. It is a good practice, however, to keep different components of the security infrastructure independent of each other. In spite of the fact that we usually use *generator* with safe primes, it could happen that someone uses it with another kind of prime (for example a prime of the form $nq + 1$). This test simply avoids that anything bad happens under such circumstances.

Note that we really do not need to check for $g = 1$, since a^2, with a chosen from the range $2 \ldots p - 2$, excluding both 1 and $p - 1$, can never be 1.

The next function initialises the protocol. It is assumed that p and g are known already to all involved parties when it is called:

```
initProtocol :: Chan Natural → Chan Natural →
                Natural → Natural → IO Natural
initProtocol inch outch p g = do
  x  ← randomNatural (2, p − 2)
  let gx = (g ↑ x) `rem` p
  writeChan outch gx
  gy ← readChan inch
  unless (checkG p gy) $ error ("suspicious value: " ++ show gy)
  return ((gy ↑ x) `rem` p)
```

The function receives four arguments: An input channel and an output channel and p and g. It starts by generating a random x from the range $2 \ldots p - 2$. It then computes $g^x \bmod p$ and sends the result through the outgoing channel. Then it waits for an

answer through the incoming channel. When a response is received, the value is checked by *checkG*, at which we will look in an instant. If the value is accepted, the function returns this value raised to x modulo p. This is the secret key.

It is actually necessary to check incoming values, since Eve may have intercepted the communication and may have sent a value that is not in the Schnorr group. To protect against this *man-in-the-middle* attack, both sides apply the following tests:

```
checkG :: Natural → Natural → Bool
checkG p x = x ≢ 1 ∧
             x < p ∧
             legendre (fromIntegral x)
                      (fromIntegral p) ≡ 1
```

For any value x received through the channel, it must hold that $x \neq 1$ (because 1 is in the wrong group), it must also hold that $x < p$, *i.e.* it must be a value modulo p, and x must be a residue of p. This is because we have chosen g to be a square and any square raised to some power is still a residue of p. So, if one of these conditions does not hold, something is wrong and we immediately abort the protocol.

Now, we look at the other side of the communication:

```
acceptProtocol :: Chan Natural → Chan Natural →
                  Natural → Natural → IO Natural
acceptProtocol inch outch p g = do
  gx ← readChan inch
  unless (checkG p gx) $ error ("suspicious value: " ++ show gx)
  y  ← randomNatural (2, p − 2)
  let gy = (g ↑ y) `rem` p
  writeChan outch gy
  return ((gx ↑ y) `rem` p)
```

The function starts by waiting on input. When it receives some input, it checks it using *checkG*. If the value is accepted, the function generates a random y from the range $2 \ldots p-2$, computes $g^y \bmod p$ and sends it back; finally the key is returned.

Now, the protocol has terminated, both sides, Alice and Bob, know the key and they can start to use this key to encrypt the messages exchanged on the channel. Since, as mentioned several times, this is not an introduction to cryptography, we have omitted a lot of details. One example is a robust defence against *denial-of-service* attacks. In the code above, we wait for input forever. This is never a good idea. An attacker could initiate one protocol after the other without terminating any of them. The server waiting for requests would quickly run out of resources.

A cryptosystem with somewhat different purposes than Diffie-Hellman is RSA, named after its inventors Ronald Rivest, Adi Shamir and Leonard Adleman and published in 1978. Similar to Diffie-Hellman, RSA aims to provide an encryption system for key

exchange, but, beyond encryption, RSA is also designed as an authentication system based on electronic signatures. The latter implies that the public key is assigned to a person. Furthermore, an infrastructure is needed that inspires the confidence that a given public key is really the key of the person one believes one is communicating with. There are many ways to create such an infrastructure. It may be community-based, for example, so that people one trusts guarantee for people they trust; another way is that trusted organisations provide phone books where signatures can be looked up. All possible implementations depend at some point on trust. In fact, trust is just the other side of the security coin: without trust, there is no security.

When Alice uses RSA to send a secret message to Bob, she uses Bob's public key to encrypt the message, and Bob, later, uses his private key to decrypt the message. The design of RSA guarantees that it is extremely difficult to decrypt the cyphertext without knowledge of Bob's private key. If Alice, additionally, wants to sign the message to make sure that Eve cannot exchange her messages by other ones, she uses her own private key to create a signature on the message (usually by first creating a *hash* of that message). Bob can then assure himself that the message was really sent by Alice by verifying the signature using Alice's public key. The role of public and private key, hence, is swapped in encryption and signing. Encryption is done with the addressee's pupblic key and undone by the addresse's private key; signing is done with the sender's private key and approved with the sender's public key.

Just as Diffie-Hellman, RSA is based on modular arithmetic – but with a composite modulus. With a composite modulus, some care must be taken, of course, since it does behave as a prime only with respect to those numbers that are coprime to it. The modulus n is created by multiplying two large primes, p and q. This guarantees that n behaves like an "ordinary" prime with respect to most numbers in the range $1 \dots n-1$, *viz.* all numbers that are not multiples of p and q.

We also need a number t, such that for numbers a coprime to n, it holds that $a^t \equiv 1 \pmod{n}$. For a prime number p, as we know from Fermat's little theorem, this t would be $p-1$. For the prime factors of n, hence, $p-1$ and $q-1$ would do the trick. But any multiple of $p-1$ and $q-1$ would do the trick as well, including of course $(p-1)(q-1)$ or $\operatorname{lcm}(p-1, q-1)$, the least common multiple of $p-1$ and $q-1$.

Now, consider Fermat's theorem written like this:

$$a^p \equiv a \pmod{p} \tag{5.64}$$

and try to solve the following congruence system:

$$\begin{aligned} x &\equiv a \pmod{p} \\ x &\equiv a \pmod{q}. \end{aligned}$$

An obvious solution to this system, according to the Chinese Remainder theorem, is the number x that is congruent to a modulo pq. Let us write x as a^t, where $t = \text{lcm}(p-1, q-1)$:

$$a^t \equiv a \pmod{p}$$
$$a^t \equiv a \pmod{q}.$$

An obvious solution to this system, still according to the Chinese Remainder theorem, is $a^t \equiv a \pmod{pq}$. Let us look at an example: $p = 7$, $q = 11$ and $n = pq = 77$. The lcm of $p - 1 = 6$ and $q - 1 = 10$ is 30. Therefore, for any number a: $a^{30} \equiv 1 \pmod{7}$ and $a^{30} \equiv 1 \pmod{11}$, but also: $a^{31} \equiv a \pmod{7}$ and $a^{31} \equiv a \pmod{11}$ and, this is important, $a^{30} \equiv 1 \pmod{77}$ and $a^{31} \equiv a \pmod{77}$. For instance $a = 3$: $3^{30} \equiv 1 \pmod{7}$, $3^{31} \equiv 3 \pmod{7}$ and $3^{30} \equiv 1 \pmod{11}$ and $3^{31} \equiv 3 \pmod{11}$. But also: $3^{30} \equiv 1 \pmod{77}$ and $3^{31} \equiv 3 \pmod{77}$.

Note that this is nothing new. We just applied the Chinese Remainder theorem to a quite trivial case. However, from this trivial case (and with some help from Euler as we will see later), an important theorem follows, namely *Carmichael's theorem* that can be stated in the scope of our problem here as: the least number t fulfilling the congruence $a^t \equiv 1 \pmod{n}$ for any number a coprime to n is the lcm of $p_1 - 1, p_2 - 1, \ldots p_s - 1$, where $p_1 \ldots p_s$ are the prime factors of n. Since our number n has only two prime factors, namely p and q, our t is $\text{lcm}(p-1, q-1)$. So, finally, this fellow Carmichael is not only bugging us with crazy numbers, but he actually lends a hand to solve a problem once in a while! Thanks Robert!

The importance for the RSA system is related to the fact that we need a public number to compute the cyphertext and a private number to reconstruct the original message. The method to compute the cyphertext is exponentiation. For this purpose, we need an exponent called e. The number e is chosen such that it is a small odd number, $1 < e < t$ and e does not divide t nor n. Now we find the inverse e' of e mod t, such that $ee' \equiv 1 \pmod{t}$.

When Alice encrypts a message m, she computes $c = m^e \mod n$. To decrypt the cyphertext c, Bob computes $c^{e'}$, which is $m = m^{ee'} = m^{ee'} \mod n$. Modulo t $m^{ee'}$ would just be $m^1 = m$, since $ee' \equiv 1 \pmod{t}$. Modulo n, this number is some multiple of t plus 1: $ee' \equiv kt + 1 \pmod{n}$. We, hence, have $m^{kt+1} = m^{kt} \times m = m^{t^k} \times m$. But since $m^t \equiv 1 \pmod{n}$, due to the Carmichael theorem, we have $1^k \times m = 1 \times m = m$. Again, Mr Carmichael, thank you!

To resume what we need for RSA: We have a public key (n, e) and a private key (p, q, t, e'), where $n = pq$, $t = \text{lcm}((p-1), (q-1))$ and $ee' \equiv 1 \pmod{t}$. It is essential that all components of the private key remain secret. Any component that leaks out helps Eve reveal the entire private key. The core of the secret is e', since it can be used directly to decrypt encrypted messages and to sign messages in the name of its

owner. With t revealed, e' can be simply computed with the extended gcd algorithm; with one of p or q revealed, the respective other factor of n can be simply computed by $q = n/p$ or $p = n/q$. With p and q both known, however, t can be computed by means of the lcm. This boils down to the fact that the security of RSA depends on the difficulty of the discrete logarithm and the factoring of large numbers.

Since, in any concrete implementation of RSA, we have to refer to the components of the keys quite often, let us define data types to encapsulate the public and the private information:

```
type PublicK = (Natural, Natural)
type PrivateK = (Natural, Natural, Natural, Natural)

pubN, pubE :: PublicK → Natural
pubN = fst
pubE = snd

privP, privQ, privT, privD :: PrivateK → Natural
privP (p, _, _, _) = p
privQ (_, q, _, _) = q
privT (_, _, t, _) = t
privD (_, _, _, d) = d
```

This is just two type synonyms for private and public key and a set of accessor functions. Note that we call the inverse of e in the private key d as it is often referred to in the literature.

As for Diffie-Hellman, chosing good values for public and private key is an essential part of the system. The following function is a reasonable key generator:

```
generateKeys :: Natural → IO (PublicK, PrivateK)
generateKeys k = do p    ← genPrime 1
                    q    ← genPrime p
                    let t = lcm (p − 1) (q − 1)
                    pair ← findED 1000 t
                    case pair of
                         Nothing    → generateKeys k
                         Just (e, d) → if e ≡ d ∨
                                          e ≡ p ∨ e ≡ q ∨
                                          d ≡ p ∨ d ≡ q
                                       then generateKeys k
                                       else let pub = (p ∗ q, e)
                                                priv = (p, q, t, d)
                                            in return (pub, priv)
  where genPrime p = do q ← generatePrime (k `div` 2)
                        if p ≡ q then genPrime p
                                 else return q
```

We start by generating two primes using *genPrime*. *genPrime* incorporates a test to ensure that we do not accidently choose the same prime twice. When we call *genPrime* for p, we pass a number that is certainly not equal to the generated prime, namely one. In the second call to generate q, we pass p. Then we create t as $\text{lcm}(p-1, q-1)$ and then we find the pair (e, d) using *findED*, at which we will look next. *findED* returns a *Maybe* value. If the result is *Nothing*, we start all over again. Otherwise, we ensure that e and d differ and that e and d differ from p and q. This is to ensure that we do not accidently publish one of the secrets, namely e or one of the prime factors of n. Finally, we create the public key as (pq, e) and the private key as (p, q, t, d).

The most intriguing part of the key generator is finding the pair (e, d). Since t is not a prime number, we are not guaranteed to find an inverse for any $e < t$. So, we may need several tries to find an e. But it might even be that there is no such pair at all for t. Since t depends on the primes p and q, in such a case, we have to start from the beginning. As an example, consider the primes $p = 5$ and $q = 7$; t in this case is 12. If we try all combinations of the numbers $2 \ldots 10$, we see that the only pairs of numbers whose product is 1 are 5×5 and 7×7. In this case: $e = d$, an option that any attacker will probably try first and should therefore be avoided. Here is an implementation of *findED*:

```
findED :: Natural → Natural → IO (Maybe (Natural, Natural))
findED 0 _ = return Nothing
findED i t = randomNatural (7, t − 2) >>= λx →
  let z | even x      = x − 1
        | otherwise   = x
    (e, d)            = tryXgcd z t
  in if (e ∗ d) `rem` t ≡ 1 then if e > t `div` 2 then return (Just (d, e))
                                                  else return (Just (e, d))
                            else findED (i − 1) t
  where tryXgcd a t = case nxgcd a t of
                        (1, k) → (a, k)
                        _      → (0, 0)
```

The function takes two arguments. The first is a counter that, when expired, indicates that we give up the search with the given t. As you can see above in *generateKeys*, the counter is defined as 1000. That is just some randomly chosen value – there may be better ones. We could try to invent some ratio for t such as half of the odd numbers smaller than t. But t is a very large number. Any ratio would force us to test single numbers for hours if we are unlucky.

We then choose a random number from the range $7 \ldots t-2$ as a candidate for e. We do not want $e = t - 1$, since the inverse in this case is likely to be e itself. We start with 7, since we want to avoid very small values for e. We also want e to be odd. Otherwise, it will not be coprime to t, which is even.

We then compute (e, d) by means of *tryXgcd*. This function computes the *gcd* and the

inverse by means of *nxgcd*, which we defined in the section on modular arithmetic. If the *gcd* is not 1, then we are unlucky, since e and t must be coprime. In this case, we return $(0, 0)$, a pair that certainly will not pass the test $ed \equiv 1 \pmod{t}$ and, this way, we cause the next try of *findED*. Otherwise, we return the pair (a, k), e and its inverse modulo t.

If we have found a suitable pair in *findED*, we return this pair either as (e, d) or as (d, e), if $e > t$`div`2. The reasoning is that encryption will be more efficient with a small e. On the other hand, we do not want to impose any explicit property on d (such as $d > e$), since that would be a hint that reduces the security level.

The key generator is usually not used, when we establish a secure communication. Instead, the key pair is considered to be stable as long as it has not been compromised by loss or by an attack on the server where key pairs are stored. Some stability is necessary for the authentication part of RSA. If public keys changed frequently, it would be more difficult to be sure that a given key really belongs to the person one thinks it belongs to.

Once the keys are available and the public key has been published, Alice can encrypt a message to Bob using the encryption function:

```
encrypt :: PublicK → Natural → Natural
encrypt pub m = (m ↑ (pubE pub)) `rem` (pubN pub)
```

Alice uses this function with Bob's public key and a message m represented as a (large) integer value. The function raises the message m to e and takes the result modular n. Bob can decrypt the cyphertext using his private key:

```
decrypt :: PublicK → PrivateK → Natural → Natural
decrypt pub priv c = (c ↑ (privD priv)) `rem` (pubN pub)
```

To sign a message, Alice would use her private key:

```
sign :: PublicK → PrivateK → Natural → Natural
sign pub priv m = (m ↑ (privD priv)) `rem` (pubN pub)
```

Signing, hence, is just the same as decryption, but on a message that was not yet encrypted. To verify the signature, Bob uses Alice's public key with a function similar to encryption:

```
verify :: PublicK → Natural → Natural
verify pub s = (s ↑ (pubE pub)) `rem` (pubN pub)
```

In this form, RSA is not safe, however. There are a lot of issues that must be addressed. In particular, we omitted everything related to *padding* messages before they are encrypted and to *hashing* messages before they are signed. These steps are essential for RSA to be secure. But since these steps have little relation to the mathematics of primes, they are not relevant to this chapter. After all, this is not an introduction to cryptography.

5.12. Open Problems

In the summer of 1742, Christian Goldbach (1690 – 1764), a mathematician and diplomat in the service of the Russian Czar, wrote a letter to Leonhard Euler. In this letter he told Euler of an observation he made: every even natural number greater than 2 can be represented as a sum of two primes. He had tested this with a lot of numbers – basically, with any number he could find – but was unable to prove the conjecture. Euler was excited about the observation and answered that he was sure that the conjecture must be true but that he was unable to demonstrate it either. This, basically, is still the state of affairs today: There is a lot of evidence that Goldbach's conjecture is true, much more than in the times of Euler and Goldbach, but there is no rigorous proof.

Let us see some examples:

$4 = 2 + 2,$
$6 = 3 + 3,$
$8 = 3 + 5,$
$10 = 3 + 7 = 5 + 5,$
$12 = 5 + 7,$
$14 = 7 + 7 = 3 + 11,$
$16 = 3 + 13$

and so on.

Here is a function to compute for any number n one of the sums of two primes that equal n:

```
goldbach :: Natural → (Natural, Natural)
goldbach n | odd n      = error "not even"
           | n ≡ 2      = error "not greater 2"
           | otherwise = go allprimes
  where go [] = error "The end is ny!"
        go (p : ps) | p > n − 2 = error ("disproved: " ++ show n)
                    | otherwise = let q = n − p
                                  in if prime q then (p, q)
                                                else go ps
```

The really interesting point is the second error in the *go* function. When you can find a number, for which this error occurs, there is certainly some mathematical honour you can earn.

One can think of many ways to find numbers to which to apply the *goldbach* function, the result is always a prime tuple, *e.g.* powers of 2:

$(2, 2), (3, 5), (3, 13), (3, 29), (3, 61), (19, 109), (5, 251), (3, 509), (3, 1021),$

the year of the first known formulation of the conjecture: $goldbach\ 1742 = (19, 1723)$, safe primes: $map\ goldbach\ (map\ (+1)\ safeprimes)$:

$(7, 257), (11, 337), (7, 353), (5, 379), (5, 463), (13, 467), (5, 499), (7, 557), (11, 577), \ldots$

or the ASCII code of the word "goldbach":
goldbach \$ *read* \$ *concatMap* (*show* ∘ *ord*) `"goldbach"`: ... (This will take a while to compute, so let us continue ...)

There is so much evidence in favour of the correctness of the conjecture that it is considered to be true by most mathematicians today. But there is still no rigorous proof. We cannot claim the truth of the conjecture just by applying it to a finite set of numbers, even if that set is incredible large. The point is of course that there are infinitely many numbers. For any number n up to which we may have tested the property, there may be a larger number $n+k$ for which the property does not hold. By just computing concrete results for given numbers, we can therefore not prove the theorem; we can only disprove it this way by finding a number n for which the property does not hold.

The Goldbach property, *i.e.* the property of a number to be representable by the sum of two primes, is so prototypical for many problems in mathematics that properties of this kind are often called *Goldbach-like* in mathematical logic. Goldbach-like properties can be easily calculated for any given number n, but there is no way of proving that such a property holds for all numbers other than going through all of them and actually calculating it for each and every one. Since we cannot go through all numbers in a finite number of steps, this kind of statements, even though calculable for any given instance, cannot be calculated from the axioms of a given formal system.

There is nothing special about the Goldbach conjecture itself that would yield this characteristic. In fact, many unproven conjectures share it. That every number can be factored into primes, for instance, is a Goldbach-like statement too. For this statement, we actually have a simple proof. We even have a proof for the much stronger fundamental theorem of arithmetic that states that every number has a *unique* prime factorisation (which is not a Goldbach-like statement). If we did not have a proof of the fact that every number can be factored into primes, the only technique we would have at hand to prove or disprove the statement would be to find a counterexample. Testing numbers for this property as such would be quite simple, since we just have to factor given numbers and say whether they can be factored into primes or not. Until now, however, the property has turned out to be true for any number we have looked at. If we not had the proof of the theorem, we could imagine that there might be a counterexample lurking among the infintely many numbers we have not yet examined. But there are not enough computing resources to look at all of them in finite time.

A simple property that is not a Goldbach-like statement is the twin prime conjecture, which we already encountered, stating that there is an infinite number of pairs of primes p and q, such that $q = p + 2$. Examples are 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31 and many other number pairs. Unlike the Goldbach conjecture, we cannot disprove the twin prime conjecture by finding a counterexample. The conjecture states that there are infinitely many twin primes. There is thus no criterion to conclude from the fact that a pair of numbers (n, m) does not have the property of being twin primes

that the conjecture is false. We could, for instance, get excited about the fact that 23 has no twin and declare 17 and 19 the last pair of primes. However, when we go on, we will find 29 and 31. The only way to disprove the twin prime conjecture is therefore to go through all numbers, which, again, is not possible in a finite number of steps.

A similar is true for the fundamental theorem of arithmetic. We cannot decide this theorem even for a single number. For any number we factor, we additionally have to compare the obtained factorisation with those of all other numbers (including those we have not yet examined) to decide if the factorisation is unique. Establishing the property for a single number already includes infinitely many steps and is therefore not possible in practical terms. Apparently, there is no way to prove this theorem but in an indirect fashion.

Another complex of open problems is factoring itself for which, as we have seen, no efficient algorithm is known. There are ways to find the prime factors of a given number, of course. But with large numbers, factoring becomes resource-intensive. We need a lot of steps to find the factors.

The factoring problem is tightly coupled with the problem of solving the discrete logarithm. We got an idea of this coupling already, when we looked at the Carmichael theorem: when we know the factors of a number, we can easily find a number t, such that $a^t \equiv 1 \pmod{n}$, where a and n are coprime. Shor's quantum factoring algorithm, actually, exploits this coupling the other way round. It finds the number t and uses t to find two prime factors of n.

In the world of classic, non-quantum computing, we know of only one way to find t, the order of the group generated by a, namely to raise a to the sequence of numbers $1, 2, \ldots, n-1$ until a raised to one of these numbers is 1 (mod n). Here is a Haskell function that uses this logic to find t for a given a:

```
order :: Natural → Natural → Natural
order n a = go 2 (a * a)
  where go t a' | a' `rem` n ≡ 1 = t
                | t ⩾ n          = error "group exhausted"
                | otherwise      = go (t + 1) (a' * a)
```

Note that we introduce a guard that saves us from looping eternally: when we reach $t = n$, we abandon the function. We will see why we do this in a second.

Once we have found t, it is easy to find the factors. We start – once again – with the equation

$$a^t \equiv 1 \pmod{n} \tag{5.65}$$

and subtract 1 from both sides yielding

$$a^t - 1 \equiv 0 \pmod{n}. \tag{5.66}$$

Then we factor the left-hand side of the equation:

$$(a^{\frac{t}{2}} - 1)(a^{\frac{t}{2}} + 1) \equiv 0 \pmod{n}. \tag{5.67}$$

In other words, the product of $a^{\frac{t}{2}} - 1$ and $a^{\frac{t}{2}} + 1$ is congruent to 0 modulo n, *i.e.* this product is a multiple of n. That, in its turn, means that n and this product must have common factors. Consequently, $\gcd(a^{\frac{t}{2}} - 1, n)$ or $\gcd(a^{\frac{t}{2}} + 1, n)$ will produce at least one factor of n.

This does not work in all cases, however. First, t must be even; otherwise $\frac{t}{2}$ would not be natural number. If t is odd, we therefore have to look for another a. Second, a must be coprime to n. On the other hand, If a is not coprime to n, then we have already found a factor of n, namely the gcd of n and a. Third, $a^{\frac{t}{2}}$ should not be $n-1$, since in that case a^t is trivially 1 (mod n). Finally, n should be squarefree. If n is not squarefree, some numbers a will fulfil the equation $a^t \equiv 0 \pmod{n}$. That is, there are remainders of n that, multiplied by themselves, will yield a multiple of n. In that case, we will hit the error "group exhausted" in the *order* function above. The property of a number being squarefree or not, however, is hard to establish. If we introduce a test at the beginning of the algorithm, we are back where we started: an algorithm that takes an exponential number of steps. We therefore have to accept that an error may occur for any number on which we apply Shor's algorithm.

Here is a simple Haskell implementation of the classical part of Shor's algorithm using the order function defined above:

```
shorfact :: Natural → IO [Natural]
shorfact 0 = return []
shorfact 1 = return []
shorfact 2 = return [2]
shorfact n | even n     = do fs ← shorfact (n `div` 2)
                             return (2 : fs)
           | otherwise = do p ← rabinMiller 16 n
                             if p then return [n] else loop
  where loop = do a ← randomNatural (3, n − 2)
                  case gcd n a of
                       1 → check a (order n a)
                       f → do fs1 ← shorfact (n `div` f)
                              fs2 ← shorfact f
                              return (fs1 ++ fs2)
        check a t  | odd t = loop
                   | (a ↑ (t `div` 2)) `rem` n ≡ n − 1 = loop
                   | otherwise = let f1 = gcd n (a ↑ (t `div` 2) − 1)
                                     f2 = gcd n (a ↑ (t `div` 2) + 1)
                                     f | f1 ≡ 1     = f2
                                       | otherwise = f1
                                 in do fs2 ← shorfact (n `div` f)
                                       fs1 ← shorfact f
                                       return (fs1 ++ fs2)
```

As usual, we start with some trivial base cases: 0 and 1 have no factors; 2 is the first prime and has only one factor, namely 2 itself. Then, if n is even, we apply the algorithm to half of n and add 2 to the resulting list of factors. Otherwise, we first check if n is prime using the Rabin-Miller test (with a relaxed repetition value). If it is prime, there is only one factor, namely n. Otherwise, we enter *loop*.

Here, we start by choosing a random number that is not in one of the trivial groups of n and test if we have been lucky by checking if $\gcd(n, a) = 1$. If it is not 1, then we have found a factor by chance. We continue with *shorfact* on n dividied by f and on f, which still could be a composite factor of n. Finally, we merge the two resulting lists of factors.

Otherwise, we call *check*. This function decides how to continue depending on the result of *order*: we may need to start again, if t does not fulfil the preconditions, or we continue with the gcds. If t is odd, a turns out to be useless and we start again with another a. If $a^{\frac{t}{2}}$ is $n - 1$, a is again useless and we start with another a. Otherwise, we compute the gcds of $a^{\frac{t}{2}} \pm 1$. We take one of the results to continue just making sure that, should one of the values be 1, we take the other one. That is a bit sloppy of course, since the case that both results are 1 is not handled. If that happens, however, something must be wrong in our math and, then, we cannot guarantee that the result is correct at all.

The quantum magic enters in the *order* subroutine. The algorithm that finds the order of a uses techniques that are far from what we have learnt so far, in particular *Fourier analysis*. Fourier analysis is a technique that represents complicated functions in terms of simpler and well-understood functions, namely trigonometric functions. The idea is that the quantum processor is initialised with a *superposition* of the period of the function $f(x) = x^t \bmod n$, which is then simplified reducing f by Fourier analysis.

This sounds like alchemy – and, yes, basically it is alchemy, quantum alchemy to be specific. We will look at Fourier analysis later, when we have studied the calculus. But I cannot promise that you will understand quantum alchemy when you will have understood Fourier. I am not sure, in fact, if even quantum alchemists understand quantum alchemy.

The result of the quantum Fourier analysis, as always in quantum computing, is correct with a certain probability. The whole algorithm, therefore, must be repeated and we must ensure that there is one result that appears significantly more often than others. The reasoning here is similar to the reasoning we have already applied to the Rabin-Miller test. So, this part of the algorithm (which we have left out above) should not be shocking. Shocking is rather the fact that there actually is a quantum algorithm that solves mathematical problems that cannot be solved (yet) on classic computers. This may be a hard boundary that cannot be passed, so that we have to accept that there are problems that can be solved in the classic world and others than can be solved only in the quantum world. This is an open and, indeed, very deep question in modern mathematics and computer science. But let us come back to primes, which, for my taste, are spooky enough.

The most fundamental unsolved problems deal with the distribution of primes and the *prime number theorem*. To investigate the distribution of primes, we can start by counting primes up to a given number n, a function usually called $\Pi(n)$. We can implement Π in Haskell as:

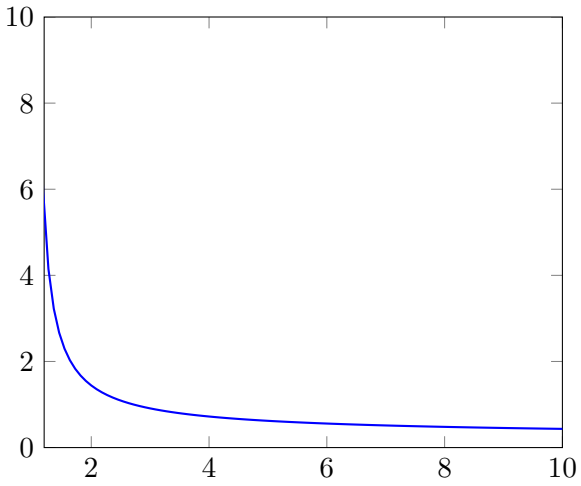
```
countp :: Natural → Int
countp n = length (takeWhile (<n) allprimes)
```

$\Pi(10)$ or, in Haskell, *countp* 10 gives 4, since there are 4 primes less than 10: 2, 3, 5 and 7. Here are the values for some small powers of 10:

10^1	=	4
10^2	=	25
10^3	=	168
10^4	=	1229
10^5	=	9592

Such tables of values of Π were already compiled in the early 18^{th} century. From some

of such tables Legendre conjectured that Π is somehow related to the natural logarithm (a beast we already met without much explanations in the first chapter). Specifically, he proposed that $\Pi(n) = \frac{n}{\ln(n)+c}$, where c is some constant. Gauss and later his student Peter Gustav Lejeune Dirichlet (1805 – 1859) improved this conjecture with the *logarithmic integral*, a concept from calculus and, as such, far ahead on our way. Intuitively, the integral yields the area under a curve; the logarithmic integral is the area under the curve described by the function $\frac{1}{\ln(n)}$, which looks like this:



Let us compare the precision of these two approximations. The following table lists the values for $\Pi(n)$ and the differences $\frac{n}{\ln(n)} - \Pi(n)$ and $li(n) - \Pi(n)$, where $li(n)$ is the logarithmic integral of n:

n	$\Pi(n)$	$n/\ln(n)$	$li(n)$
10	4	0	2
10^2	25	-3	5
10^3	168	-23	10
10^4	1229	-143	17
10^5	9592	-906	38
10^6	78498	-6116	130
10^7	664579	-44158	339
10^8	5761455	-332774	754
10^9	50847534	-2592592	1701
...	...	...	...

The error of the logarithmic integral grows much slower than that of Legendre's conjecture. For small numbers, the error of the li variant is still greater. But already for $n = 10^3$, Legendre overtakes li and is then increasing orders of magnitude faster than li.

This is not the last word, however. There is a function that is still more precise in most cases. In his famous paper, "On the Number of Primes less than a given Magnitude", the ingenious mathematician Bernhard Riemann (1826 – 1866) not only introduced the most tantalizing of all unsolved mathematical problems, the *Riemann Hypothesis*, but he also proposed a refinement to Gauss/Dirichlet's Π-approximation. He conjectured that Π corresponds roughly to the infinite series

$$R(n) = li(n) - \frac{1}{2}li(n^{\frac{1}{2}}) - \frac{1}{3}li(n^{\frac{1}{3}}) - \frac{1}{5}li(n^{\frac{1}{5}}) + \frac{1}{6}li(n^{\frac{1}{6}}) - \ldots \tag{5.68}$$

That the number of primes is related to the natural logarithm is already an astonishing fact. But, now, Riemann goes even further. To see what it is that is so surprising about Riemann's improvement, we present the equation above in a more compact form:

$$R(n) = \sum_{k=1}^{\infty} \frac{\mu(n)}{k} li(n^{\frac{1}{k}}) \tag{5.69}$$

The function $\mu(n)$ appearing in the formula is the Möbius function, which yields 0, 1 or -1 depending on n being squarefree and the number of prime factors of n being even or odd. To remind you of the first values of the Moebius function: $\mu(1) = 1$, $\mu(2) = -1$, $\mu(3) = -1$, $\mu(4) = 0$, $\mu(5) = -1$ and $\mu(6) = 1$ leading to the first terms of the summation above: $\frac{1}{1}li(n^{\frac{1}{1}}) = li(n)$, $\frac{-1}{2}li(n^{\frac{1}{2}})$, $\frac{-1}{3}li(n^{\frac{1}{3}})$, $\frac{0}{4}li(n^{\frac{1}{4}}) = 0$, $\frac{-1}{5}li(n^{\frac{1}{5}})$ and $\frac{1}{6}li(n^{\frac{1}{6}})$. In other words: the Möbius function is intimately related to the concept of the distribution of primes through the prime number theory, even if it does not reveal any regularity at the first and even the second sight.

For many values of n, the Riemann refinement is much closer to the real value of Π than either Legendre's try and Gauss/Dirichlet's improvement. It is esitmated that it is better 99% of the time. Occasionally, however, Riemann's value is worse than that of Gauss/Dirichlet. The latter fact is known only theoretically, no specific n is known that makes Riemann worse. In most cases, Riemann's value is even much better, for instance: for 10^9, Gauss/Dirichlet's deviation is 1701 while Riemann's difference is only -79; The difference of Gauss/Dirichlet for 10^{16} is more than 3 million; Riemann's difference is 327 052 and thus 10 times better.

The prime number theorem is usually stated as an asymptotic law of the form:

$$\lim_{n\to\infty} \frac{\Pi(n)}{n/\ln(n)} = 1, \tag{5.70}$$

which means that for large n, the relation of $\Pi(n)$ and $\frac{n}{\ln(n)}$ approximates 1. A first attempt to prove the theorem was made by the Russian mathematician Pafnuty Lvovich Chebyshev (1821 – 1894). Even though his paper failed to strictly prove the theorem, he

could prove Bertrand's postulate, which states that there is at least one prime number between n and $2n$ for $n \geq 2$ and on which we have already relied when looking at cryptography. A proof was finally given at the the end of the 19^{th} century independently by the French mathematicians Jacques Hadamard (1865 – 1963) and Charles Jean de la Vallée-Poussin (1866 – 1962).

It is unknown until today, though, if strict error margins for the approximation of Π can be given and if the available approximations can be further improved. Many of these questions are related to Riemann's Hypothesis. But mathematicians today appear to be far from producing a proof (or refutation) of this mega-hypothesis.

6. The Inverse Element

6.1. Inverse Elements

In a group, any element a has an inverse a', such that $a \cdot a' = e$, where e is the neutral element of that group. We have already met some groups, permutation groups and finite groups of modular arithmetic. We will now look at infinite groups, namely the additive group over $\mathbb{Z}$, the integers, and the multiplicative group over $\mathbb{Q}$, the rationals.

As you may have guessed already, the inverse element of a natural number n in the additive group over integers is the *negation* of n, $-n$. It is easy to see that n plus its inverse, $-n$, is just 0, the neutral element of addition: $n + (-n) = n - n = 0$. Since it is, of course, also true that $-n + n = n - n = 0$, the inverse element of a negative number $-n$ is its positive counterpart n. With the law of double negation, $-(-n) = n$, that is the negation of the negation of n is n, the formula $n + (-n) = 0$ is universally true for both positive and negative numbers. When we are talking about n, the positive, we just have : $n + (-n) = n - n = 0$. When we are talking about $-n$, the negative, we have: $-n + (-(-n)) = -n + n = n - n = 0$.

Note that with this identity, we can solve any equation of the form $a + x = b$ in the additive group of $\mathbb{Z}$. We just add the inverse of a to both sides of the equation: $a + (-a) + x = b + (-a)$, which, of course, is $x = b - 1$. Without negative numbers, there were equations we could not solve in this way, for instance $4 + x = 3$. We had gaps, so to speak, in our additive equations. But now, in the group over integers, there are no such gaps anymore: we just add the inverse of 4 on both sides and get $x = 3 - 4 = -1$.

What is the inverse element of integers in the multiplicative group? Well, in this group, it must still hold that $a \cdot a' = e$, where $\cdot$ is multiplication and e, the neutral element, is unity. We, hence, have $a \times a' = 1$. We easily find a solution with division. We divide a on both sides and get $a' = \frac{1}{a}$ and that is the answer: the inverse element of a natural number n in the multiplicative group is $\frac{1}{n}$.

We see that we now can solve any multiplicative equation of the form $ax = b$, just by multiplying the inverse of a on both sides: $ax\frac{1}{a} = b\frac{1}{a}$, which is $x\frac{a}{a} = \frac{b}{a}$. The left-hand side reduces to $1x = x$ and we have $x = \frac{b}{a}$. As we have already seen in finite groups, prime numbers simply "disappear" with fractions, since we can now reach them with multiplication, for instance: $3x = 5$ is equivalent to $x = \frac{5}{3}$.

Let us go on and build something bigger: a field consisting of addition and multiplication

where the distributive law holds. We already know that the distributive law holds in the world of natural numbers: $a(b+c) = ab + ac$. But what, when we have creatures like $-n$ and $\frac{1}{n}$? We obviously need rules to add fractions and to negate products.

Let us start with fractions. We may add fractions with the same denominator by simply adding the numerators, *i.e.* $\frac{a}{n} + \frac{b}{n} = \frac{a+b}{n}$. With different denominators, we first have to manipulate the fractions in a way that they have the same denominator, but without changing their value. The simplest way to do that is to multiply one denominator by the other and to multiply the numerators correspondingly: $\frac{a}{n} + \frac{b}{m} = \frac{am}{nm} + \frac{bn}{mn} = \frac{am+bn}{mn}$.

We can reduce the computational complexity of this operation in most cases by using the lcm instead of the product nm. As you may remember, the lcm of two numbers, a and b is $\frac{ab}{\gcd(a,b)}$. We would still multiply the numerator by the value by which the denominator changes, *i.e.* the lcm divided by the denominator and would get

$$\frac{a}{n} + \frac{b}{m} = \frac{a \times \frac{lcm(n,m)}{n} + b \times \frac{lcm(n,m)}{m}}{lcm(n,m)}. \tag{6.1}$$

That looks complicated, but is simple when we take two concrete numbers: $\frac{1}{6} + \frac{2}{9}$. We first compute the $lcm(6,9)$ as

$$\frac{6 \times 9}{\gcd(6,9)}.$$

The gcd of 6 and 9 is 3, the product $6 \times 9 = 54$ and $54/3 = 18$. So, we have:

$$\frac{1 \times \frac{18}{6} + 2 \times \frac{18}{9}}{lcm(6,9) = 18} = \frac{3+4}{18} = \frac{7}{18}.$$

It looks simpler now, but still it seems that we do need more steps than by just multiplying the respective other denominator to numerator and denominator of both fractions. However, when we do this, we have to operate with greater numbers and, at the end, reduce the fractions to their *canonical form*, which is

$$\frac{a}{b} = \frac{a/\gcd(a,b)}{b/\gcd(a,b)}. \tag{6.2}$$

For the example, this would mean

$$\frac{1 \times 9}{6 \times 9} + \frac{2 \times 6}{9 \times 6} = \frac{9}{54} + \frac{12}{54} = \frac{21}{54}.$$

Now, we reduce to canonical form:

$$\frac{21/\gcd(21,54)}{54/\gcd(21,54)},$$

which is

$$\frac{\frac{21}{3}=7}{\frac{54}{3}=18}.$$

How to multiply negative numbers of the form $-a(b+c)$? Let us look at an example: the additive inverse of 6 is -6. We would therefore expect $-6+6=0$. Furthermore, we have $2\times 3=6$. Now, if we add the inverse of 3 to 3 once, we get 0: $-3+3=0$. What should we get, if we add the inverse of 3 twice to twice 3, *i.e.* $2\times 3+2\times -3$? We expect it to be 0, correct? Therefore and since 2×3 is 6, 2×-3 must be the inverse of 6, hence, -6.

The same is true, the other way round, thus $2\times 3+(-2)\times 3=0$. That means that we can move the minus sign in a product, such that $a\times -b=-a\times b$. To any such product we can simply add the factor 1 without changing the result: $a\times b=1\times a\times b$. We, therefore, have $1\times a\times -b=1\times -a\times b=-1\times a\times b$. This facilitates life a bit: we can handle one minus sign as the additional factor -1. In other words, multiplying by -1 has the same effect has negating: $-1n=-n$.

Going back to the question of how to handle products of the form $-a(b+c)$, we now can say that $-a(b+c)=-1a(b+c)$. Multiplying a out in terms of the distributive law, we get: $-1(ab+ac)$. Now, we multiply -1, just as we did above: $(-1)ab+(-1)ac$ and, since we know that $-1n=-n$, we derive $-ab-ac$.

What if we have more than one minus sign in a product like, for instance: -2×-3? We saw above that $2\times -3+2\times 3=0$ and we can reformulate this as $-1\times 2\times 3+2\times 3=0$. Now, if we have two negative factors, we add one more minus sign, *i.e.* one more factor -1: $-1\times -1\times 2\times 3+2\times 3=$? We just substitute one -1 after the other by negation. We first get $-1\times -(2\times 3)\ldots$ and then $-(-(2\times 3))$. We now see clearly that this should be the negation, *i.e.* the inverse of $-(2\times 3)$. The inverse of $-(2\times 3)$, however, is just 2×3 and that is 6. Therefore: $-2\times -3+2\times 3=12$.

So, what happens if we multiply negative numbers and fractions? When we just follow multiplication rules we get $-1\times\frac{1}{n}=\frac{-1}{n}$. We, hence, would say, according to the rules derived above, that $\frac{-1}{n}$ is the additive inverse of $\frac{1}{n}$. What about $\frac{1}{-n}$? This should be the multiplicative inverse of $-n$, such that $-n\times\frac{1}{-n}=1$. We again can move the minus sign around to get $-1\times n\times\frac{1}{-n}$ leading to $-1\times\frac{n}{-n}=1$. Dividing n on both sides gives $-1\times\frac{1}{-n}=\frac{1}{n}$ and multiplying -1 gives $\frac{1}{-n}=\frac{-1}{n}$. In other words, a fraction with a minus sign in it, independently of where it appears, in the numerator or the denominator, is the additive inverse of the same fraction without the minus sign: $\frac{a}{b}+\frac{-a}{b}=\frac{a}{b}+\frac{a}{-b}=\frac{a}{b}-\frac{a}{b}=0$.

In summary, we can define a set of rules on multiplication that are independent of whatever a and b are:

$$\begin{array}{rcrcr} a & \times & b & = & ab \\ -a & \times & -b & = & ab \\ -a & \times & b & = & -ab \\ a & \times & -b & = & -ab \end{array}$$

With this, we have established an important theoretical result, namely that $\mathbb{Q}$, including negative numbers, is an infinite field with addition and multiplication. But let us go on. There are still operations we have seen for natural numbers combining multiplication and addition that may and should have an interpretation with integers and fractions, namely exponentiation.

From the table above, we see immediately that products with an even number of negative numbers are positive – a fact that we already used when discussing prime numbers. In general, any number raised to an even exponent is positive independent of that number being positive or negative.

This leads to a difficulty with the root operation, since even roots may have two different results: a positive number or its additive inverse. For instance, $\sqrt{4}$ could be 2 and -2. Even further, the operation cannot be applied to a negative number: $\sqrt{-1}$ has no meaning – at least not with the creatures we have met so far. There may be an object i that fulfil equations of the form $i = \sqrt{-1}$, but such objects are beyond our imagination at this point in time.

Now, what is the effect of having negative numbers or fractions in the exponent?

We will first look at fractions as exponents and investigate powers of the form $a^{\frac{1}{n}}$. To clarify the meaning of such expression, we will use the rules we know so far to observe what happens, when we multiply two powers with the same base and with fractional exponents:

$$x = a^{\frac{1}{n}} \times a^{\frac{1}{m}} = a^{\frac{1}{n}+\frac{1}{m}}. \tag{6.3}$$

Let us look at the special case of the exponent $\frac{1}{2}$:

$$x = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}}. \tag{6.4}$$

Obviously, $\frac{1}{2} + \frac{1}{2} = 2 \times \frac{1}{2} = 1$. In other words, x, in this case, is just a. Furthermore, we see that there is a number $n = a^{\frac{1}{2}}$, such that $n \times n = n^2 = x$. For which number does

this hold? Well, it is just the definition of the square root: $\sqrt{x} \times \sqrt{x} = (\sqrt{x})^2 = x$. We would conclude that $a^{\frac{1}{n}}$ is equivalent to $\sqrt[n]{a}$:

$$a^{\frac{1}{n}} = \sqrt[n]{a}. \tag{6.5}$$

This, in fact, makes a lot of sense. We would expect, for instance, that $(a^{\frac{1}{n}})^n$ is just a, since $(\sqrt[n]{a})^n = a$. When we multiply it out, we get indeed $a^{\frac{1}{n} \times n} = a^1 = a$. We would also expect that $(a^{\frac{1}{n}})^{\frac{1}{m}}$ is $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$, *e.g.* $(a^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{\sqrt{a}} = \sqrt[4]{a}$. When we multiply it out again, we indeed obtain $a^{\frac{1}{2} \times \frac{1}{2}} = a^{\frac{1}{4}}$.

Now, what about negative exponents? We adopt the same technique, *i.e.* we multiply two powers with the same base:

$$a \times a^{-1}.$$

We can write this as $a^1 \times a^{-1}$ and this is

$$a^1 \times a^{-1} = a^{1-1} = a^0 = 1.$$

We see a^{-1} is the multiplicative inverse of a. But we already know that the inverse of a is $\frac{1}{a}$. We conclude that

$$a^{-n} = \frac{1}{a^n}. \tag{6.6}$$

This conjecture would imply that $a^{-n} \times a^n = 1$, since $\frac{1}{a^n} \times a^n = 1$ and, indeed: $a^{-n} \times a^n = a^{-n+n} = a^0 = 1$. It would also imply that $(a^{-n})^{-1} = a^n$, since $a^{-n} = \frac{1}{a^n}$, whose inverse is a^n. Indeed, we have $(a^{-n})^{-1} = a^{-n \times -1} = a^n$.

A side effect of this rule is that we now have a very nice notation for the multiplicative inverse. Until now, we have used the symbol a' to denote the inverse of a. Since $'$ is also used in other contexts, the notation a^{-1} is much clearer and we will stick to it from now on.

6.2. $\mathbb{Z}$

We will now implement a number type that takes signedness into account. We will do so in a way that allows us to negate objects of different kind, basically any type of number. We therefore start by defining a parametrised data type:

```
data Signed a = Pos a | Neg a
  deriving (Eq, Show)
```

The data type has two constructors, *Pos* and *Neg* for a positive and a negative a respectivley. The expression **let** $x = Neg\ 1$ would assign the value -1 to x. We would instantiate a concrete data type by giving a concrete type for the type parameter, *e.g.*

```
type Zahl = Signed Natural
```

This type is called *Zahl*, the German word for *number*, which was used for the designation of the set $\mathbb{Z}$ of the integers. When Abstract Algebra started to be a major field of mathematics, the University of Göttingen was the gravitational centre of the math world and, since it was not yet common to use English in scientific contributions, many German words slipped into math terminology.

For convenience, we add a direct conversion from *Zahl* to *Natural*:

```
z2n :: Zahl → Natural
z2n (Pos n) = n
z2n (Neg _) = ⊥
```

Another convenience function should be defined for *Neg*, namely to guarantee that 0 is always positive. Otherwise, we could run into situations where we compare *Pos* 0 and *Neg* 0 and obtain a difference that does not exist. We therefore define

```
neg0 :: (Eq a, Num a) ⇒ a → Signed a
neg0 0 = Pos 0
neg0 x = Neg x
```

We now make *Signed* an instance of *Ord*:

```
instance (Ord a) ⇒ Ord (Signed a) where
  compare (Pos a) (Pos b) = compare a b
  compare (Neg a) (Neg b) = compare b a
  compare (Pos _) (Neg _) = GT
  compare (Neg _) (Pos _) = LT
```

The difficult cases are implemented in the first two lines. When a and b have the same sign, we need to compare a and b themselves to decide which number is greater than the other. If the sign differs, we can immediately decide that the one with the negative sign is smaller.

Signed is also an instance of *Enum*:

```
instance (Enum a) ⇒ Enum (Signed a) where
  toEnum i | i ⩾ 0    = Pos $ toEnum i
           | i < 0    = Neg $ toEnum i
  fromEnum (Pos a) = fromEnum a
  fromEnum (Neg a) = negate (fromEnum a)
```

With this definition some more semantics comes in. We explicitly define that, converting an integer greater or equal to zero, we use the *Pos* constructor; converting an integer less than zero, we use the *Neg* constructor. Furthermore, when we convert in the opposite direction, *Pos a* is just an a, whereas *Neg a* is the negation of a.

Now we come ot arithmetic, first addition:

$$
\begin{aligned}
&\textbf{instance}\ (\mathit{Ord}\ a, \mathit{Num}\ a) \Rightarrow \mathit{Num}\ (\mathit{Signed}\ a)\ \textbf{where}\\
&\quad (\mathit{Pos}\ a) + (\mathit{Pos}\ b) = \mathit{Pos}\ (a + b)\\
&\quad (\mathit{Neg}\ a) + (\mathit{Neg}\ b) = \mathit{neg0}\ (a + b)\\
&\quad (\mathit{Pos}\ a) + (\mathit{Neg}\ b) \mid a > b = \mathit{Pos}\ (a - b)\\
&\qquad\qquad\qquad\qquad\ \mid a < b = \mathit{Neg}\ (b - a)\\
&\qquad\qquad\qquad\qquad\ \mid a \equiv b = \mathit{Pos}\ 0\\
&\quad (\mathit{Neg}\ a) + (\mathit{Pos}\ b) \mid a > b = \mathit{Neg}\ (a - b)\\
&\qquad\qquad\qquad\qquad\ \mid a < b = \mathit{Pos}\ (b - a)\\
&\qquad\qquad\qquad\qquad\ \mid a \equiv b = \mathit{Pos}\ 0
\end{aligned}
$$

The addition of two positive numbers is a positive sum. The addition of two negative numbers ($-a + (-b) = -a - b$) is a negative sum. The addition of a negative and a positive number results in a difference, which may be positive or negative depending on which number is greater: the negative or the positive one.

We define subtraction in terms of addition: subtracting a positive number b from any number a is the same as adding the negation of b to a. Vice versa, subtracting a negative number b is the same as adding a postive number.

$$
\begin{aligned}
&a - (\mathit{Pos}\ b) = a + (\mathit{neg0}\ b)\\
&a - (\mathit{Neg}\ b) = a + (\mathit{Pos}\ b)
\end{aligned}
$$

Multiplication:

$$
\begin{aligned}
&(\mathit{Pos}\ a) * (\mathit{Pos}\ b) = \mathit{Pos}\ (a * b)\\
&(\mathit{Neg}\ a) * (\mathit{Neg}\ b) = \mathit{Pos}\ (a * b)\\
&(\mathit{Pos}\ 0) * (\mathit{Neg}\ _) = \mathit{Pos}\ 0\\
&(\mathit{Pos}\ a) * (\mathit{Neg}\ b) = \mathit{Neg}\ (a * b)\\
&(\mathit{Neg}\ _) * (\mathit{Pos}\ 0) = \mathit{Pos}\ 0\\
&(\mathit{Neg}\ a) * (\mathit{Pos}\ b) = \mathit{Neg}\ (a * b)
\end{aligned}
$$

This is a straight forward implementation of the rules we have already seen above: the product of two positive numbers is positive; the product of two negative numbers is positive; the product of a positive and a negative number is negative.

The next method is *negate*. There is one minor issue we have to handle: what do we do if the number is 0? In this case, we assume the number is positive. But that is a mere convention. Without this convention, we would have to introduce a constructor for 0 that is neither postive nor negative.

```
negate (Pos a) | signum a ≡ 0 = Pos a
               | otherwise    = Neg a
negate (Neg a) = Pos a
```

Finally, we have *abs*, *signum* and *fromInteger*. There is nothing new in the implementation of these methods:

```
abs (Pos a)       = Pos a
abs (Neg a)       = Pos a
signum (Pos a)    = Pos (signum a)
signum (Neg a)    = Neg (signum $ negate a)
fromInteger i     | i ⩾ 0 = Pos (fromInteger i)
                  | i < 0 = Neg (fromInteger $ abs i)
```

We make *Signed* an instance of *Real*:

```
instance (Real a) ⇒ Real (Signed a) where
  toRational (Pos i) = toRational i
  toRational (Neg i) = negate (toRational i)
```

We also make *Signed* an instance of *Integral*:

```
instance (Enum a, Integral a) ⇒ Integral (Signed a) where
  quotRem (Pos a) (Pos b) = let (q, r) = quotRem a b in (Pos q, Pos  r)
  quotRem (Neg a) (Neg b) = let (q, r) = quotRem a b in (Pos q, neg0 r)
  quotRem (Pos a) (Neg b) = let (q, r) = quotRem a b in (neg0 q, Pos r)
  quotRem (Neg a) (Pos b) = let (q, r) = quotRem a b in (neg0 q, neg0 r)
  toInteger (Pos a) = toInteger a
  toInteger (Neg a) = negate (toInteger a)
```

The implementation of *toInteger* contains nothing new. But have a look at the definition of *quotRem*. The first case, where both numbers are positive, is easy: we return a positive quotient and a positive remainder. When both numbers are negative, the quotient is postive and the remainder is negative. Indeed, when we have the equation $a = qb + r$ and both, a and b, are negative, then a postive q will bring b close to a. With $a = -5$ and $b = -2$, for instance, the quotient would be 2: $2 \times -2 = -4$. Now, what do we have to add to -4 to reach -5? Obviously, -1. Therefore the remainder must be negative.

Now, when we have a postive a and a negative b, for instance: $a = 5$ and $b = -2$; then a negative quotient will bring us close to a: $-2 \times -2 = 4$. Missing now is the positive remainder 1. Finally, when a is negative and b is positive, we need a negative quotient, *e.g.*: $a = -5$ and $b = 2$; then $-2 \times 2 = -4$. Missing, in this case, is again a negative remainder, namely -1.

In the future, there may arise the need to make number types signed that do no fit into the classes from which we derived *Signed* so far. In particular, a signed fraction should inherit from Fractional. We therefore make *Signed* an instance of Fractional:

```
instance (Eq a, Ord a, Fractional a) ⇒ Fractional (Signed a) where
  (Pos a) / (Pos b) = Pos (a / b)
  (Neg a) / (Neg b) = Pos (a / b)
  (Pos 0) / (Neg b) = Pos (0 / b)
  (Pos a) / (Neg b) = Neg (a / b)
  (Neg a) / (Pos b) = Neg (a / b)
```

6.3. Negative Binomial Coefficients

The aim of this section is to generalise binomial coefficients to integers using the new data type *Zahl*. There are many ways how to approach this goal, which perhaps can be grouped into two main approaches: first, we can look for an application of such a generalisation in the "real world" and search an appropriate mathematical tool for this problem. We started with binomial coefficients by discussing combinatorial issues, but, of course, combinatorial problems like the ways to choose k objects out of n provide no meaningful interpretation for negative numbers – what should a negative number of possibilities mean? But there are other applications, such as the multiplication of sums.

The second basic approach is not to look for applications, but to investigate the formalism asking "what happens, when we change it?" This may appear much less interesting at the first sight, since there is no obvious use case for such an altered formalism. However, such formal approaches do not only help deepening the insight into specific mathematical problems, but they also lead to new tools, which may find their applications in the future. This has happened more than once in the history of mathematics. When David Hilbert, the champion of the formalistic approach, redefined geometry extending it from the two-dimensional plane and the three-dimensional space to n-dimensional manifolds, there was no concrete application in sight. It took only a short while, however, before John von Neumann and others started using the concepts of Hilbert's geometry to model quantum physics.

Anyhow, we start with the second approach. Still, there are many ways to go. We can start with the formula $\binom{n}{k}$ and ask ourselves what happens if $n < 0$ or $k < 0$. To begin with, let us assume $n < 0$ and $k \geq 0$. If we just apply the formula $\frac{n(n-1)\ldots(n-k+1)}{k!}$, we get for $\binom{-n}{k}$ a kind of falling factorial in the numerator that looks like a rising factorial with minus signs in front of the numbers, *e.g.* $\binom{-6}{3}$ is $\frac{-6\times-7\times-8}{6}$, which is $-1\times-7\times-8 = -56$. Obviously, the signedness of the result depends on whether k is even or odd. Since there are k negative factors in the numerator, the product will be positive if k is even and negative otherwise.

Here is a Haskell implementation for this version of negative binomial coefficients:

$$
\begin{aligned}
&\mathit{chooseNeg} :: \mathit{Zahl} \to \mathit{Natural} \to \mathit{Zahl} \\
&\mathit{chooseNeg}\ n\ k \mid n \geqslant 0 \wedge k \geqslant 0 = \mathit{Pos}\ (\mathit{choose}\ (\mathit{z2n}\ n)\ k) \\
&\qquad\qquad\quad \mid n \equiv k \qquad\quad = 1 \\
&\qquad\qquad\quad \mid k \equiv 0 \qquad\quad = 1 \\
&\qquad\qquad\quad \mid k \equiv 1 \qquad\quad = n \\
&\qquad\qquad\quad \mid \mathit{otherwise} \quad = (n > \mid (\mathit{Pos}\ k))\ `\mathit{div}`\ (\mathit{fac}\ (\mathit{Pos}\ k))
\end{aligned}
$$

This function accepts two arguments, one of type *Zahl* and the other of type *Natural*. For the moment, we want to avoid negative ks and to rule negative values out right from the beginning, we choose *Natural* as data type for k.

When both, n and k, are positive, we just use the old *choose* converting n to *Natural*. Then we handle the trivial cases. Finally, we just implement one of the formulas for binomial coefficients. Here are some values:

$\mathit{map}\ (\mathit{chooseNeg}\ (-1))\ [0\,..\,9]$
$[1, -1, 1, -1, 1, -1, 1, -1, 1, -1]$

$\mathit{map}\ (\mathit{chooseNeg}\ (-2))\ [0\,..\,9]$
$[1, -2, 3, -4, 5, -6, 7, -8, 9, -10]$

$\mathit{map}\ (\mathit{chooseNeg}\ (-3))\ [0\,..\,9]$
$[1, -3, 6, -10, 15, -21, 28, -36, 45, -55]$

$\mathit{map}\ (\mathit{chooseNeg}\ (-4))\ [0\,..\,9]$
$[1, -4, 10, -20, 35, -56, 84, -120, 165, -220]$

$\mathit{map}\ (\mathit{chooseNeg}\ (-5))\ [0\,..\,9]$
$[1, -5, 15, -35, 70, -126, 210, -330, 495, -715]$

$\mathit{map}\ (\mathit{chooseNeg}\ (-6))\ [0\,..\,9]$
$[1, -6, 21, -56, 126, -252, 462, -792, 1287, -2002]$

We see that the coefficients for each n alternate between positive and negative values depending on k being even or odd. We also see that there is no limit anymore from which on the coefficients are all zero. In the original definition, we had $\binom{n}{k} = 0$ for $k > n$. But now we have to give up that rule, because for $n < 0$ and $k \geq 0$, $k > n$ trivially holds for all cases. In consequence, we lose the nice symmetry we had in the original triangle. Of course, we can restore many of the old characteristics by changing the definition of *chooseNeg* for negative ns to $(n \mid > (\mathit{Pos}\ k))\ `\mathit{div}`\ (\mathit{fac}\ (\mathit{Pos}\ k))$. In this variant, let us call it *chooseNeg2*, we use the rising factorial, such that the absolute values of the numbers in the numerator equal the numbers in the numerator for $n \geq 0$. For instance:

$\mathit{map}\ (\mathit{chooseNeg2}\ (-2))\ [0\,..\,9]$
$[1, -2, 1, 0, 0, 0, 0, 0, 0, 0]$

map (*chooseNeg2* (-3)) $[0\,..\,9]$
$[1,-3,3,-1,0,0,0,0,0,0]$

map (*chooseNeg2* (-4)) $[0\,..\,9]$
$[1,-4,6,-4,1,0,0,0,0,0]$

map (*chooseNeg2* (-5)) $[0\,..\,9]$
$[1,-5,10,-10,5,-1,0,0,0,0]$

map (*chooseNeg2* (-6)) $[0\,..\,9]$
$[1,-6,15,-20,15,-6,1,0,0,0]$

The first solution, *chooseNeg*, however, is more faithful to the original definition of the binomial coefficients, even if its results do not resemble the original results. One of the characteristics that is preserved is Pascal's rule:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{6.7}$$

Indeed, we could use Pascal's rule to find negative coefficients in the first place. If we are interested in the coefficient $\binom{n-1}{k}$, we just subtract $\binom{n-1}{k-1}$ from both sides and get

$$\binom{n-1}{k} = \binom{n}{k} - \binom{n-1}{k-1}. \tag{6.8}$$

We can use this equation to search the unknown territory of negative coefficients starting from the well-known territory of positive coefficients using each result as a base camp for further expeditions. We start with $\binom{0}{0}$ and want to know the value for $\binom{-1}{0}$. Since coefficients for $k = 0$ are defined as 1, this case turns out to be trivial: $\binom{-1}{0} = 1$. The next is $\binom{-1}{1}$, which is $\binom{0}{1} - \binom{-1}{0}$, hence $0-1 = -1$. The next is $\binom{-1}{2} = \binom{0}{2} - \binom{-1}{1}$, which is $0-(-1) = 0+1 = 1$. Next: $\binom{-1}{3} = \binom{0}{3} - \binom{-1}{2}$, which is $0-1 = -1$ and so on. Indeed, the coefficients of -1, as we have seen before, are just alternating positive and negative 1s:

map (*chooseNeg* (-1)) $[0\,..\,9]$
$[1,-1,1,-1,1,-1,1,-1,1,-1]$

On the basis of this result, we can investigate the coefficients of -2. We know that $\binom{-2}{0}$ is 1 and continue with $\binom{-2}{1}$, which is $\binom{-1}{1} - \binom{-2}{0}$, which is $-1-1 = -2$. The next is $\binom{-2}{2} = \binom{-1}{2} - \binom{-2}{1}$ or $1-(-2) = 1+2 = 3$. We continue with $\binom{-2}{3} = \binom{-1}{3} - \binom{-2}{2}$, which is $-1-3 = -4$. It turns out that the coefficients for $n = -2$ are

map (*chooseNeg* (-2)) $[0\,..\,9]$
$[1,-2,3,-4,5,-6,7,-8,9,-10]$,

which is a beautiful result. If we go on this way, we will reproduce the values observed above using *chooseNeg*.

Now, what about negative ks? There is no direct way to implement something like *chooseNeg* for negative ks, because we do not know what a negative factorial would mean. Of course, we can apply a trick very similar to that we used for *chooseNeg2*, *i.e.* we compute the factorial not as $fac\ n = n * fac\ (n-1)$, but as $fac\ n = n * fac\ (n+1)$ going up towards zero. In this case, we also have to take care of the rising or falling factorial of the numerator. For instance, we can use the falling factorial with the inverse of k, such that the value of the numerator remains the same independent of k being positive or negative. The result would resemble that of *chooseNeg2*, *i.e.* we would have alternating positive and negative coefficients for $n > 0$ and positive coefficients for $n < 0$.

More interesting is using Pascal's rule to create coefficients for negative ks. But we have to be careful. In equation 6.8, we used a coefficient for $k-1$ to find the coefficient for k. This will not work. When we look for $\binom{n}{-k}$, we do not yet know the value for $\binom{n}{-(k+1)}$, since we are entering the territory of negative numbers from above. Therefore, we need a variant of equation 6.8. Instead of subtracting $\binom{n-1}{k-1}$ from Pascal's rule, we subtract the other term $\binom{n-1}{k}$ and get

$$\binom{n-1}{k-1} = \binom{n}{k} - \binom{n-1}{k}. \tag{6.9}$$

It is obvious that any coefficient resulting from a positive n and a negative k in this way equals 0, since any coefficient with $k = 0$ is 1. So, $\binom{n}{-1} = 1 - 1 = 0$. However, we also have $\binom{0}{k} = 0$ for any k and $\binom{n}{n} = 1$. For $\binom{-1}{-2}$, we therefore have $\binom{0}{-1} - \binom{-1}{-1} = 0 - 1 = -1$. For $\binom{-1}{-3}$, we get $0 - \binom{-1}{-2} = 0 - (-1) = 0 + 1 = 1$. The next coefficient $\binom{-1}{-4}$, froreseeably, is $0 - \binom{-1}{-3}$, which is $0 - 1 = -1$. Again, for $n = -1$, we get a sequence of alternating negative and positive ones.

For $\binom{-2}{k}$, we get $\binom{-2}{-2} = 1$, $\binom{-2}{-3}$ is $\binom{-1}{-2} - \binom{-2}{-2} = -1 - 1 = -2$. $\binom{-2}{-4}$ then is $\binom{-1}{-3} - \binom{-2}{-3} = 1 - (-2) = 3$. The next coefficient is $\binom{-2}{-5} = \binom{-1}{-4} - \binom{-2}{-3}$, which is $-1 - 3 = -4$ and so on. The binomial coefficients for -2 with negative ks, thus, is just the sequence $1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \ldots$, which we have already seen for positive ks above. The symmetry of the triangle, hence, is reinstalled. For coefficients with $n = -2$ and $k = -9, -8, \ldots, -1, 0, 1, \ldots, 7$, we get the sequence

$$-8, 7, -6, 5, -4, 3, -2, 1, 0, 1, -2, 3, -4, 5, -6, 7, -8.$$

To confirm this result, we implement the backward rule as

$$
\begin{aligned}
&\mathit{pascalBack} :: \mathit{Zahl} \to \mathit{Zahl} \to \mathit{Zahl}\\
&\mathit{pascalBack}\ n\ 0 = 1\\
&\mathit{pascalBack}\ n\ k \mid k \equiv n \qquad = 1\\
&\qquad\qquad\quad \mid k \equiv 0 \qquad = 1\\
&\qquad\qquad\quad \mid n \equiv 0 \qquad = 0\\
&\qquad\qquad\quad \mid n > 0 \wedge k < 0 = 0\\
&\qquad\qquad\quad \mid n > 0 \wedge k \geqslant 0 = \mathit{Pos}\ (\mathit{choose}\ (\mathit{z2n}\ n)\ (\mathit{z2n}\ k)\\
&\qquad\qquad\quad \mid k < 0 \qquad = \mathit{pascalBack}\ (n+1)\ (k+1) - \mathit{pascalBack}\ n\ (k+1)\\
&\qquad\qquad\quad \mid \mathit{otherwise} = \mathit{pascalBack}\ (n+1)\ k \qquad - \mathit{pascalBack}\ n\ (k-1)
\end{aligned}
$$

We first handle the trivial cases $n = k$, $k = 0$ and $n = 0$. When $n > 0$ and $k < 0$, the coefficient is 0. For n and k both greater 0, we use *choose*. For $k < 0$, we use the rule in equation 6.9 and for $n < 0$, with $k > 0$, we use the rule in equation 6.8. Now we map *pascalBack* for specific ns to a range of ks:

$\mathit{map}\ (\mathit{pascalBack}\ (-1))\ [-9, -8\,..\,9]$
$[1, -1, 1, -1, 1, -1, 1, -1, 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1]$

$\mathit{map}\ (\mathit{pascalBack}\ (-2))\ [-9, -8\,..\,9]$
$[-8, 7, -6, 5, -4, 3, -2, 1, 0, 1, -2, 3, -4, 5, -6, 7, -8, 9, -10]$

$\mathit{map}\ (\mathit{pascalBack}\ (-3))\ [-9, -8\,..\,9]$
$[28, -21, 15, -10, 6, -3, 1, 0, 0, 1, -3, 6, -10, 15, -21, 28, -36, 45, -55]$

$\mathit{map}\ (\mathit{pascalBack}\ (-4))\ [-9, -8\,..\,9]$
$[-56, 35, -20, 10, -4, 1, 0, 0, 0, 1, -4, 10, -20, 35, -56, 84, -120, 165, -220]$

$\mathit{map}\ (\mathit{pascalBack}\ (-5))\ [-9, -8\,..\,9]$
$[70, -35, 15, -5, 1, 0, 0, 0, 0, 1, -5, 15, -35, 70, -126, 210, -330, 495, -715]$

The symmetry of positive and negative ks is not perfect. The coefficients for $n < 0$ and $n < k < 0$ are all 0. The sequence, in consequence, is mirrowed with a delay of $|n| - 1$ ks, such that the coefficient that corresponds to the coefficient $\binom{n}{k}$ is not $\binom{n}{-k}$, but rather $\binom{n}{n-k}$. For instance: $\binom{-2}{4} = 5 = \binom{-2}{-6}$, $\binom{-3}{5} = -21 = \binom{-3}{-8}$ and $\binom{-5}{2} = 15 = \binom{-5}{-7}$.

Looking at the numbers of negative ns, I have the strange feeling that I already saw those sequences somewhere. But where? These are definitely not the rows of Pascal's triangle, but perhaps something else? Let us look at the triangle once again:

0																1																
1														1				1														
2											1				2				1												
3									1				3				3				1										
4							1				4				6				4				1								
5						1			5				10				10				5				1						
6					1				6				15				20				15				6				1		
7				1			7				21				35				35				21				7				1
8			1			8			28				56				70				56				28				8				1
9		1			9			36			84				126				126				84				36				9				1

Now follow the diagonals from the upper vertex left and right downwards. On both sides you see first the 1s, then the counting numbers $1, 2, 3, \ldots$, then the sequence we saw with $\binom{-3}{k}$, then the sequence we saw with $\binom{-4}{k}$ and so on. In other words, if we turn the triangle by 90° counterclockwise, we obtain the sequences for negative n; the same also works with a clockwise turn, but then we read the sequence from right to left.

This boils down to the equations

$$\left|\binom{-n}{1}\right| = \binom{n}{1} = \binom{n}{n-1} \tag{6.10}$$

and

$$\left|\binom{-n}{k+1}\right| = \binom{n+k-1}{k+1} = \binom{n+k-1}{n-1}. \tag{6.11}$$

For instance:

$$\begin{aligned}
\left|\binom{-3}{1}\right| &= \binom{3}{1} = \binom{3}{2} = 3 \\
\left|\binom{-3}{2}\right| &= \binom{4}{2} = \binom{4}{2} = 6 \\
\left|\binom{-3}{3}\right| &= \binom{5}{3} = \binom{5}{2} = 10 \\
\left|\binom{-3}{4}\right| &= \binom{6}{4} = \binom{6}{2} = 15 \\
\left|\binom{-3}{5}\right| &= \binom{7}{5} = \binom{7}{2} = 21 \\
\left|\binom{-3}{6}\right| &= \binom{8}{6} = \binom{8}{2} = 28
\end{aligned}$$

This result may look a bit surprising at the first sight. But when we look at the formula that actually generates the value, it is obvious:

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}. \tag{6.12}$$

When we have a negative n, the falling factorial in the numerator is in fact a rising factorial with negative numbers:

$$-n^{\underline{k}} = -n \times -(n+1) \times \cdots \times -(n+k-1).$$

Each number in the product is one less than its predecessor, but, since the numbers are negative, the absolute value is greater than its predecessor. If we eliminate the minus signs, we obtain the rising factorial for n:

$$n^{\overline{k}} = n \times (n+1) \times \cdots \times (n+k-1),$$

which is just the falling factorial for $n+k-1$:

$$(n+k-1)^{\underline{k}} = (n+k-1) \times (n+k-2) \times \cdots \times n.$$

We can therefore conclude that

$$\left|\binom{-n}{k}\right| = \frac{(n+k-1)^{\underline{k}}}{k!} = \binom{n+k-1}{k}. \tag{6.13}$$

That $\binom{n+k-1}{k}$ also equals $\binom{n+k-1}{n-1}$ results from the fact that $n-1$ and k maintain the same distance from one of the sides of the triangle, *i.e.* from either $\binom{n+k-1}{0}$ or $\binom{n+k-1}{n+k-1}$. k is trivially k coefficients away from any $\binom{n}{0}$, whereas $n-1$ is $(n+k-1)-(n-1)$ away from $\binom{n+k-1}{n+k-1}$, which is $n+k-1-n+1=k$. This is just an implication of the triangle's symmetry.

Somewhat more difficult is to see the relation to Pascal's rule. In fact, we have never proven that Pascal's rule follows from the fraction $\frac{n^{\underline{k}}}{k!}$. If we can establish this relation, the backward rule will follow immediately. So, we definitely should try to prove Pascal's rule.

We want to establish that

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{6.14}$$

and do this directly using the definitions

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} \tag{6.15}$$

$$\binom{n}{k-1} = \frac{n^{\underline{k-1}}}{(k-1)!} \tag{6.16}$$

Our claim is that

$$\binom{n+1}{k} = \frac{n^{\underline{k}}}{k!} + \frac{n^{\underline{k-1}}}{(k-1)!} \tag{6.17}$$

In other words, when we can deduce the left-hand side of this equation from the right-hand side, we are done. So let us just add the two fractions on the right-hand side. We first convert them to a common denominator. We can do this simply by multiplying the second fraction by k:

$$\frac{kn^{\underline{k-1}}}{k(k-1)! = k!}$$

We can join the two fractions with the common denominator and obtain a sum in the numerator:

$$\frac{n^{\underline{k}} + kn^{\underline{k-1}}}{k!}$$

If we extend the falling factorials in the numerator we get

$$n(n-1)\dots(n-k+1) + n(n-1)\dots(n-(k-1)+1)k.$$

The terms have the same number of factors, where the last factor in the first term is $n-k+1$ and the one in the second term is k. The factor before the last in the second term is $n-(k-1)+1$, which is $n-k+1+1 = n-k+2$, and this term is also the last but second factor in the first term. In other words, the factors of the terms are equal with the exception of the last factor. We can factor the equal factors out. If we do this stepwise, this looks like (using brackets to indicate what remains within the sum):

$$n[(n-1)(n-2)\dots(n-k+1)+(n-1)(n-2)\dots(n-k+2)k]$$
$$n(n-1)[(n-2)\dots(n-k+1)+(n-2)\dots(n-k+2)k]$$
$$\dots$$
$$n(n-1)\dots(n-k+2)[n-k+1+k].$$

The remaining sum can now be simplified: $n-k+1+k=n+1$ and with this we recognise in the whole expression the falling factorial of $n+1$. The whole fraction is now

$$\frac{(n+1)^{\underline{k}}}{k!},$$

which is the definition of the binomial coefficient $\binom{n+1}{k}$ and that concludes the proof. □

The proof establishes the relation between the definition of binomial coefficients and Pascal's rule. This spares us from going through the laborious task of establishing the relation expressed in equation 6.11 using only Pascal's rule.

We now switch to the first approach mentioned at the beginning of this section, *i.e.* trying to find a mathematical formalism for a practical problem. The practical problem is multiplication. We want to know what happens with negative as or bs in products of the form

$$(a+b)^n.$$

Negative ns are not too interesting here, since $(a+b)^{-n}$ is just the inverse of $(a+b)^n$, which is $\frac{1}{(a+b)^n}$, without any effect on the coefficients themselves.

We could, of course go on by trying out this formula with concrete numbers a and b. It appears much more promising, however, to choose a symbolic approach manipulating strings of the form `"a"` and `"b"`. The idea is to use string operations to simulate multiplication and addition. We do so in two steps: first we combine strings, then we simplify them in a way mimicking the rules of addition and multiplication. Since we want to see the differences between positive and negative coefficients, we need a means to negate strings simulating negative numbers. To this end, we define the simple data type

```
data Sym = P String | N String
  deriving (Eq, Show)
```

where, as you may have guessed, the P-constructor represents positive strings and the N-constructor represents negative strings. We now combine two symbols to simulate multiplication:

$$
\begin{aligned}
&comb :: Sym \to Sym \to Sym \\
&comb\ (P\ a)\ (P\ b) = P\ (a +\!\!+ b) \\
&comb\ (P\ a)\ (N\ b) = N\ (a +\!\!+ b) \\
&comb\ (N\ a)\ (P\ b) = N\ (a +\!\!+ b) \\
&comb\ (N\ a)\ (N\ b) = P\ (a +\!\!+ b)
\end{aligned}
$$

You probably realise the pattern we already used to define the number type *Zahl*: two numbers of equal signedness result in a positive number and two numbers of different signedness result in a negative number. Multiplication itself is just the concatenation of the two strings. $comb\ (P$ `"a"`$)\ (P$ `"b"`$)$, hence, is P `"ab"`; $comb\ (N$ `"a"`$)\ (P$ `"b"`$)$ is N `"ab"`.

We represent addition as lists of strings. We then can formulate the distributive law as

$$
\begin{aligned}
&combN :: Sym \to [Sym] \to [Sym] \\
&combN\ x = map\ (comb\ x)
\end{aligned}
$$

where we multiply a number, x, with a sum by multiplying that number with each term of the sum, mapping the basic combinator operation *comb* on the list representing the sum. Based on *combN*, we can now define the multiplication of a sums by itself:

$$
\begin{aligned}
&combine :: [Sym] \to [Sym] \\
&combine\ xs = concat\ [combN\ x\ xs \mid x \leftarrow xs]
\end{aligned}
$$

Let us look at an example to get a grip on *combine*:

let $a = P$ `"a"`
let $b = P$ `"b"`
$combine\ [a, b] = concat\ [combN\ x\ [a, b] \mid x \leftarrow [a, b]]$.

The list comprehension will first apply $combN\ a\ [a, b]$, resulting in $[aa, ab]$, and then it will apply $combN\ b\ [a, b]$ resulting in $[ba, bb]$. These two lists are then merged using *concat* resulting in $[aa, ab, ba, bb]$. We will later have to simplify this list, since, as we know, ab and ba are equal and can be written $2ab$. We come back to this immediately, but first we want to implement one more function, namely a combinator that applies *combine* n times:

$$
\begin{aligned}
&combinator :: [Sym] \to Natural \to [Sym] \\
&combinator\ _\ 0 = [\,] \\
&combinator\ xs\ n = go\ xs\ (n-1) \\
&\quad \textbf{where}\ go\ ys\ 0 = ys \\
&\qquad\qquad go\ ys\ n = go\ (concat\ [combN\ x\ ys \mid x \leftarrow xs])\ (n-1)
\end{aligned}
$$

Note that this function does not reuse *combine*, but implements it anew. The reason is that we do not want to combine the original input with itself, but with the result of the previous recursion. For instance, if $n = 3$, then we start with $[combN\ x\ [a, b] \mid x \leftarrow [a, b]]$ resulting in $[aa, ab, ba, bb]$. In the next round, we multiply this result by $[a, b]$: $[combN\ x\ [aa, ab, ba, bb] \mid x \leftarrow [a, b]]$ resulting in $[aaa, aab, aba, abb, baa, bab, bba, bbb]$

and corresponding to the expression $(a + b)^3$.

Another interesting aspect of *combinator* is the base case for $n = 0$, which is just defined as being the empty list. In this context, the empty list would, hence, represent the number 1, since 1, as we know, is the coefficient $\binom{0}{0}$.

Now we want to simplify results, such that $[aa, ab, ba, ba] = [aa, 2 * ab, ba]$. First, we need to express that `"ab"` and `"ba"` are the same thing. Therefore, we sort the string:

```
sortStr :: Sym → Sym
sortStr (P a) = P (sort a)
sortStr (N a) = N (sort a)
```

leading to a canonical form for strings. The call *sortStr* (*P* `"ba"`) would result in *P* `"ab"`.

To simplify a complete list, we compare all symbols in the lists and consolidate symbols with equal strings resulting in lists of the type (*Natural*, *Sym*), where the natural number counts the ocurrences of that symbol in the list:

```
simplify :: [Sym] → [(Natural, Sym)]
simplify xs = go (sort $ map sortStr xs) 1
  where go [] _ = []
        go [x] n = [(n, x)]
        go ((P x) : (P y) : zs)  n | x ≡ y      = go ((P y) : zs) (n + 1)
                                   | otherwise = (n, P x) : go ((P y) : zs) 1
        go ((N x) : (N y) : zs) n | x ≡ y      = go ((N y) : zs) (n + 1)
                                   | otherwise = (n, N x) : go ((N y) : zs) 1
        go ((N x) : (P y) : zs)  n | x ≡ y      = go zs 1
                                   | otherwise = (n, N x) : go ((P y) : zs) 1
        go ((P x) : (N y) : zs)  n | x ≡ y      = go zs 1
                                   | otherwise = (n, P x) : go ((N y) : zs) 1
```

We start by first transforming all symbols into the canonical form sorting their strings. Then we sort the list of symbols itself. The idea is that all strings of the same kind are listed in a row, such that we can easily count them. But, of course, we do not want to have the negative and positive symbols separated, we want all symbols with the same string in a row, independently of these symbols being positive or negative. We have to implement this notion of comparison ignoring the sign. To this end, we make *Sym* an instance of *Ord*:

```
instance Ord Sym where
  compare (P a) (P b) = compare a b
  compare (P a) (N b) = compare a b
  compare (N a) (P b) = compare a b
  compare (N a) (N b) = compare a b
```

We now *go* through the list of symbols sorted in this sense of comparison and check on

the first and the second of the remaining list on each step. If the signedness of first and second are equal and their strings are equal, we increment n and store (n, x), x the head of the list, whenever they differ starting the rest of the list with $n = 1$. Otherwise, when the signs are not equal, but the strings are, then we drop both symbols and continue with the rest of the list after the second.

Applied step for step on the list

$[aaa, aab, aba, abb, baa, bab, bba, bbb]$,

this would advance as follows. We first sort all strings resulting

$[aaa, aab, aab, abb, aab, abb, abb, bbb]$.

We next sort the symbols:

$[aaa, aab, aab, aab, abb, abb, abb, bbb]$.

We then weight according to the number of their appearance:

$[(1, aaa), (3, aab), (3, abb), (1, bbb)]$.

In this example, we ignore signedness, since all terms are positive anyway. The interesting thing thus is still to commence: the application to sums with negative terms. We do this with the simple function

$$\begin{aligned} &\mathit{powsum} :: [\mathit{Sym}] \to \mathit{Natural} \to [(\mathit{Natural}, \mathit{Sym})] \\ &\mathit{powsum}\ \mathit{xs} = \mathit{simplify} \circ \mathit{combinator}\ \mathit{xs} \end{aligned}$$

We now define two variables, a positive string a and a negative one b: **let** $a = P$ `"a"` and **let** $b = N$ `"b"` and just apply *powsum* with increasing ns:

powsum $[a, b]$ 1
$[(1, P\ \texttt{"a"}), (1, N\ \texttt{"b"})]$

powsum $[a, b]$ 2
$[(1, P\ \texttt{"aa"}), (2, N\ \texttt{"ab"}), (1, P\ \texttt{"bb"})]$

powsum $[a, b]$ 3
$[(1, P\ \texttt{"aaa"}), (3, N\ \texttt{"aab"}), (3, P\ \texttt{"abb"}), (1, N\ \texttt{"bbb"})]$

It appears that the absolute values of the coefficients do not change. We have, for instance, $\binom{3}{0} = 1$, $\binom{3}{1} = -3$, $\binom{3}{2} = 3$, $\binom{3}{3} = -1$. What, if we swap the negative sign: **let** $a = N$ `"a"` and **let** $b = P$ `"b"`?

powsum $[a, b]$ 1
$[(1, N\ \texttt{"a"}), (1, P\ \texttt{"b"})]$

powsum $[a, b]$ 2
$[(1, P\ \texttt{"aa"}), (2, N\ \texttt{"ab"}), (1, P\ \texttt{"bb"})]$

powsum $[a, b]$ 3
$[(1, N$ `"aaa"`$), (3, P$ `"aab"`$), (3, N$ `"abb"`$), (1, P$ `"bbb"`$)]$

The result is just the same for even exponents. For odd exponents, the signs are just exchanged: $\binom{3}{0} = -1$, $\binom{3}{1} = 3$, $\binom{3}{2} = -3$, $\binom{3}{3} = 1$.

What if both terms are negative: **let** $a = N$ `"a"` and **let** $b = N$ `"b"`?

powsum $[a, b]$ 1
$[(1, N$ `"a"`$), (1, N$ `"b"`$)]$

powsum $[a, b]$ 2
$[(1, P$ `"aa"`$), (2, P$ `"ab"`$), (1, P$ `"bb"`$)]$

powsum $[a, b]$ 3
$[(1, N$ `"aaa"`$), (3, N$ `"aab"`$), (3, N$ `"abb"`$), (1, N$ `"bbb"`$)]$

Now, wonder of wonders, even exponents lead to positive coefficients, while odd exponents lead to negative coefficients: $\binom{3}{0} = -1$, $\binom{3}{1} = -3$, $\binom{3}{2} = -3$, $\binom{3}{3} = -1$.

This result appears quite logical, since, with even exponents, we multiply an even number of negative factors, while, with odd exponents, we multiply an odd number of negative factors.

To confirm these results with more data, we define one more function to extract the coefficients and, this way, making the output more readable:

$$\begin{aligned}&\mathit{coeffs} :: [\mathit{Sym}] \to \mathit{Natural} \to [\mathit{Natural}]\\&\mathit{coeffs}\ \mathit{xs} = \mathit{map}\ \mathit{getCoeff} \circ \mathit{powsum}\ \mathit{xs}\end{aligned}$$

where *getCoeff* is

$$\begin{aligned}&\mathit{getCoeff} :: (\mathit{Natural}, \mathit{Sym}) \to \mathit{Natural}\\&\mathit{getCoeff}\ (n, P\ _) = n\\&\mathit{getCoeff}\ (n, N\ _) = -n\end{aligned}$$

We now map *coeffs* on the numbers $[0\,..\,9]$ and, with a and b still both negative, see Pascal's triangle with signs alternating per row:

$[\,]$
$[-1, -1]$
$[1, 2, 1]$
$[-1, -3, -3, -1]$
$[1, 4, 6, 4, 1]$
$[-1, -5, -10, -10, -5, -1]$
$[1, 6, 15, 20, 15, 6, 1]$
$[-1, -7, -21, -35, -35, -21, -7, -1]$
$[1, 8, 28, 56, 70, 56, 28, 8, 1]$
$[-1, -9, -36, -84, -126, -126, -84, -36, -9, -1]$

For one of the value a and b positive and the other negative, we see the coefficients in one row alternating in signedness. For a positive and b negative, we see:

$[]$
$[1, -1]$
$[1, -2, 1]$
$[1, -3, 3, -1]$
$[1, -4, 6, -4, 1]$
$[1, -5, 10, -10, 5, -1]$
$[1, -6, 15, -20, 15, -6, 1]$
$[1, -7, 21, -35, 35, -21, 7, -1]$
$[1, -8, 28, -56, 70, -56, 28, -8, 1]$
$[1, -9, 36, -84, 126, -126, 84, -36, 9, -1]$

The other way round, a negative and b positive, we see just the same triangle where, for odd exponents, the minus signs are swapped. The triangle, hence, is the same as the one before with each row reversed:

$[]$
$[-1, 1]$
$[1, -2, 1]$
$[-1, 3, -3, 1]$
$[1, -4, 6, -4, 1]$
$[-1, 5, -10, 10, -5, 1]$
$[1, -6, 15, -20, 15, -6, 1]$
$[-1, 7, -21, 35, -35, 21, -7, 1]$
$[1, -8, 28, -56, 70, -56, 28, -8, 1]$
$[-1, 9, -36, 84, -126, 126, -84, 36, -9, 1]$

6.4. $\mathbb{Q}$

We now turn our attention to fractions and start by implementing a rational data type:

```
data Ratio = Q Natural Natural
  deriving (Show, Eq)
```

A *Ratio* has a constructor Q that takes two natural numbers. The name of the constructor is derived from the symbol for the set of rational numbers $\mathbb{Q}$ that was introduced by Giuseppe Peano and stems from the Italian word *Quoziente*.

It would be nice of course to have a function that creates a rational in its canonical form, *i.e.* reduced to two natural numbers that are coprime to each other. This is done by *ratio*:

```
ratio :: Natural → Natural → Ratio
ratio _ 0 = error "division by zero"
ratio a b = reduce (Q a b)
```

for which we define the infix %:

```
infix %
(%) :: Natural → Natural → Ratio
(%) = ratio
```

so that we can create ratios with expressions like 5 % 2, 8 % 4 and so on. The function *reduce* called in *ratio* is defined as follows:

```
reduce :: Ratio → Ratio
reduce (Q _ 0) = error "division by zero"
reduce (Q 0 _) = Q 0 1
reduce (Q n d) = Q (n `div` gcd n d)
                   (d `div` gcd n d)
```

which reduces numerator and denominator to the quotient of the greatest common divisor of numerator and denominator. If numerator and denominator are coprime, the *gcd* is just 1 and the numbers are not changed at all.

We now make *Ratio* an instance of *Ord*:

```
instance Ord Ratio where
  compare x@(Q nx dx) y@(Q ny dy) | dx ≡ dy   = compare nx ny
                                  | otherwise = let (x', y') = unify x y
                                                in compare x' y'
```

If the denominators are equal, we just compare the numerators, *i.e.* $\frac{1}{5} < \frac{2}{5} < \frac{3}{5}$ and so on. Otherwise, if the denominators differ, we must convert the fractions to a common denominator before we actually can compare them. This is done using *unify*:

```
unify :: Ratio → Ratio → (Ratio, Ratio)
unify (Q nx dx) (Q ny dy) = (Q (nx * (lcm dx dy) `div` dx) (lcm dx dy),
                             Q (ny * (lcm dx dy) `div` dy) (lcm dy dx))
```

This is the implementation of the logic already described before: we convert a fraction to the common denominator defined by the *lcm* of both denominators and multiply the numerators by the number we would have to multiply the denominator by to get the *lcm*, which trivially is the *lcm* divided by the denominator: $lcm(dx, dy) = dx \times \frac{lcm(dx,dy)}{dx}$. This may appear a bit complicated, but it is much faster, whenever the denominators are not coprime to each other.

The next step is to make *Ratio* an instance of *Enum*:

```
instance Enum Ratio where
  toEnum i = Q (toEnum i) (toEnum 1)
  fromEnum (Q n d) = fromEnum (n `div` d)
```

which implies a conversion from and to an integer type. A plain integral number i is converted to a *Ratio* using the denominator 1. For the backward conversion, *div* is used leading to the loss of precision if the denominator does not divide the numerator.

Now we come to the heart of the data type making it instance of *Num*:

```
instance Num Ratio where
  x@(Q nx dx) + y@(Q ny dy) | dx ≡ dy   = (nx + ny) % dx
                            | otherwise = let (x', y') = unify x y
                                          in (x' + y')
  x@(Q nx dx) - y@(Q ny dy) | x ≡ y = Q 0 1
                            | x > y ∧ dx ≡ dy = (nx - ny) % dx
                            | x > y = let (x', y') = unify x y
                                      in x' - y'
                            | otherwise = error "Subtraction beyond zero!"
  (Q nx dx) * (Q ny dy) = (nx * ny) % (dx * dy)
  negate a = a
  abs a = a
  signum (Q 0 _) = 0
  signum (Q _ _) = 1
  fromInteger i = Q (fromIntegral i) 1
```

We add two fraction with the same denominator by reducing the result of $\frac{nx+ny}{d}$. If we add two fractions in canonical form, such as $\frac{1}{9}$ and $\frac{5}{9}$, we may arrive at a result that is not in canonical form like, in this example, $\frac{6}{9}$, which should be reduced to $\frac{2}{3}$. Otherwise, if the denominators differ, we first convert the fractions to a common denominator before we add them.

Since we have defined *Ratio* as a fraction of two natural numbers (and not of two integers), we have to be careful with subtraction. If the two fractions are equal, the result is zero, which is represented as $\frac{0}{1}$. If $x > y$, we use the same strategy as with addition. Otherwise, if $y > x$, subtraction is undefined.

Multiplication is easy: we just reduce the result of multiplying the two numerators and denominators by each other. The other functions do not add anything new. We just define *negate*, *abs* and *signum* as we have done before for plain natural numbers and we define the conversion from integer as we have done for *Enum* already.

The next step, however, is unique: we define *Ratio* as an instance of *Fractional*. The core of this is to define a division function and do so defining division as the inverse of multiplication:

```
rdiv :: Ratio → Ratio → Ratio
rdiv (Q nx dx) (Q ny dy) = (Q nx dx) * (Q dy ny)
```

The division of a fraction $\frac{nx}{dx}$ by another $\frac{ny}{dy}$ is just the multiplication of that fraction with the inverse of the second one, which is $\frac{dy}{ny}$. The complete implementation of the

Fractional type then is

```
instance Fractional Ratio where
  (/)          = rdiv
  fromRational r = Q (fromIntegral $ numerator r)
                     (fromIntegral $ denominator r)
```

This is not a complete definition of $\mathbb{Q}$, however. $\mathbb{Q}$ is usually defined on top of the integers rather than on top of natural numbers. So, our data type should be signed. That, however, is quite easy to achieve:

```
type Quoz = Signed Ratio
```

6.5. Zeno's Paradox

The ancient Greek philosopher Zeno of Elea, who lived in the 5^{th} century BC, devised a number of paradoxes that came upon us indirectly through the work of Aristotle and its commentators. Zeno designed the paradoxes to defend the philosophy of the *One* developed by Zeno's teacher Parmenides. According to this philosophy everything is One, undivisible, motionless, eternal, everywhere and nowhere. That we actually see motion, distinguish and divide things around us, that everything "in this world" is volatile and that everything has its place, is either here or there, is, according to Parmenides, just an illusion.

Plato discusses the philosophy of Parmenides in one of his most intriguing dialogs, "Parmenides", where young Socrates, Zeno and Parmenides himself analyse contradictions that arise both in Plato's *theory of forms* as well as in Parmenides' theory of the One. The Parmenides dialog had a deep influence on European philosophy and religion. It was the main inspiration for the late-ancient *neoplatonism* and, for many centuries, it shaped the interpretation of ancient philosophy by medieval thinkers. Scholars today, however, are not so sure anymore what the meaning of this dialog may be. Some see in it a critical discussion of the theory of forms, others hold it is a collection of exercises for students of Plato's academy, and again others consider the dialog as highly ironic, actually criticising Parmenides and other philosophers for using terms that they know from everyday life in a context where the ideas associated with this terms do not hold anymore – very similar to the therapeutic approach of Ludwig Wittgenstein.

It is tempting to relate the philosophy of the One with the philosophical worldview of mathematical platonism. Constructivists would see the world as dynamic, as a chaotic process without meaning in itself or, pessimistically, as a thermodynamic process that tends to entropy. It is an effort of human beings to create order in this dynamic and perhaps chaotic world. Therefore, prime numbers – or any other mathematical object – do not exist, we define them and we have to invest energy to construct them. By contrast, mathematical platonists would hold that there is a static eternal structure

that does not change at all. The mathematical objects are out there, perhaps like in a gigantic lattice behind the universe. It would then be absurd to say that we construct prime numbers. We find them travelling along the eternal metaphysical structure that is behind of what we can perceive directly with our senses.

Be that as it may, we are here much more interested in the math in Zeno's paradoxes. The most famous one is the race of Achilles and the tortoise. Achilles gives the tortoise a lead of, say, hundred meters. The question now is when Achilles will actually catch up with the tortoise. Zeno says: never, for it is impossible. To catch up, he must reach a point where the tortoise has been shortly before. But when he gets there, the tortoise is already ahead. Perhaps just a few metres, but definitely ahead. So, again, to reach that point, Achilles will need some time. When he reaches the point where the tortoise has been a second before, the same is already a bit further. To reach that point, Achilles again needs some time and in this time the tortoise again makes some progress and so it goes on and on.

A more concrete version of this paradox is given in the so called *Dichotomy* paradox. It states that, in general, it is impossible to move from A to B. Since, to do so, one has first to make half of the way arriving at a point C. To move from C to B, one now has to first make half of the way arriving at a point D. To move from D to B, one now has to first make half of the way arriving at yet another point and so on and so on.

This paradox appears to be at odds with what we observe in the physical world where it indeed appears to be possible to move from A to B quite easily. The paradox, however, draws our attention to the fact that, between any two rational numbers, there are infinitely many other rational numbers. Between 0 and 1, for instance, there is $\frac{1}{2}$. Between 0 and $\frac{1}{2}$, there is $\frac{1}{4}$. Between 0 and $\frac{1}{4}$, there is $\frac{1}{8}$. Between 0 and $\frac{1}{8}$, there is $\frac{1}{16}$ and, in general, between 0 and any number of the form $\frac{1}{2^k}$, there is a number $\frac{1}{2^{k+1}}$.

So, following these points as in Zeno's paradox, how close to B would we get after k steps? The problem can be represented as a sum of the form

$$\sum_{i=1}^{k} \frac{1}{2^i}$$

After $k = 1$ step, we would be $\frac{1}{2}$ of the way away from B. After $k = 2$ steps, the distance would be

$$\frac{1}{2} + \frac{1}{2^2}.$$

We convert the fractions to a common denominator multiplying the first fraction by 2 and arrive at

$$\frac{2+1}{4} = \frac{3}{4}.$$

For $k = 3$ steps, we have

$$\frac{3}{4} + \frac{1}{2^3} = \frac{6+1}{8} = \frac{7}{8}.$$

These tests suggest the general formula

$$\sum_{i=1}^{k} \frac{1}{2^i} = \frac{2^k - 1}{2^k}. \tag{6.18}$$

This equation cries out for an induction proof. Any of the examples above serves as base case. We then have to prove that

$$\frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}. \tag{6.19}$$

We convert the fractions to a common denominator multiplying the first fraction by 2:

$$\frac{2(2^k - 1) + 1}{2^{k+1}}.$$

We simplify the numerator: $2 \times 2^k = 2^{k+1}$ and $2 \times (-1) = -2$; we, hence, have in the numerator $2^{k+1} - 2 + 1$, which can be simplified to $2^{k+1} - 1$ and leads to the desired result

$$\frac{2^{k+1} - 1}{2^{k+1}}. \quad \square$$

That was easy! Can we generalise the result for any denominator n, such that

$$\frac{1}{n^k} + \frac{1}{n^{k+1}} = \frac{n^{k+1} - 1}{n^{k+1}}? \tag{6.20}$$

If we went a third of the way on each step instead of half of it, we had $\frac{1}{3^k} + \frac{1}{3^{k+1}}$, for instance, $\frac{1}{3} + \frac{1}{9}$. We convert the fraction to a common denominator multiplying the first by 3: $\frac{3+1}{9} = \frac{4}{9}$. So, equation 6.20 seems to be wrong. The nice and clean result with the denominator 2 appears to be one of those deceptions that are so common for small numbers, which often behave very differently from greater numbers.

But let us stop moaning. What actually is the rule for $n = 3$? After the next step, we would have

$$\frac{4}{9} + \frac{1}{27}.$$

We multiply the first fraction by 3 and have

$$\frac{12+1}{27} = \frac{13}{27}.$$

For $k = 4$, we would have

$$\frac{13}{27} + \frac{1}{81} = \frac{39+1}{81} = \frac{40}{81}.$$

The experiments this time suggest the rule

$$\sum_{i=1}^{k} \frac{1}{3^i} = \frac{(3^k - 1)/2}{3^k}. \tag{6.21}$$

We prove again by induction with any of the examples serving as base case. We have to prove that

$$\frac{(3^k - 1)/2}{3^k} + \frac{1}{3^{k+1}} = \frac{(3^{k+1} - 1)/2}{3^{k+1}}. \tag{6.22}$$

We multiply the first fraction by 3 in numerator and denominator and get in the numerator $\frac{3(3^k-1)}{2} = \frac{3^{k+1}-3}{2}$. We can now add the two fractions:

$$\frac{(3^{k+1} - 3)/2 + 1}{3^{k+1}}.$$

To add 1 to the fraction in the numerator we have to convert 1 to a fraction with the denominator 2, which, of course, is $\frac{2}{2}$. We, hence, have in the numerator $\frac{3^{k+1}-3+2}{2}$ and this leads to the desired result:

$$\frac{(3^{k+1} - 1)/2}{3^{k+1}}. \quad \square \tag{6.23}$$

Before we dare to make a new conjecture based on equations 6.18 and 6.21, let us collect some more data. Since $n = 4$ is closely related to $n = 2$, we will immediately go to $n = 5$. For $k = 2$ we have

$$\frac{1}{5}+\frac{1}{25}=\frac{5+1}{25}=\frac{6}{25}.$$

For $k=3$ we have

$$\frac{6}{25}+\frac{1}{125}=\frac{30+1}{125}=\frac{31}{125}.$$

For $k=4$ we have

$$\frac{31}{125}+\frac{1}{625}=\frac{155+1}{625}=\frac{156}{625}.$$

In these examples, we see the relation

$$\sum_{i=1}^{k}\frac{1}{5^i}=\frac{(5^k-1)/4}{5^k}. \tag{6.24}$$

We prove easily by induction using any of the examples as base case. We have to show that

$$\frac{(5^k-1)/4}{5^k}+\frac{1}{5^{k+1}}=\frac{(5^{k+1}-1)/4}{5^{k+1}}. \tag{6.25}$$

We multiply the first fraction by 5, yielding the numerator $\frac{5^{k+1}-5}{4}$ and, when adding 1, we get $\frac{5^{k+1}-5}{4}+\frac{4}{4}$, which, of course, leads to the desired result. □

To summarise: with $n=2$, we see $\frac{n^k-1}{n^k}$; with $n=3$, we see $\frac{(n^k-1)/2}{n^k}$; with $n=5$, we see $\frac{(n^k-1)/4}{n^k}$. This suggests the general form

$$\sum_{i=1}^{k}\frac{1}{n^i}=\frac{(n^k-1)/(n-1)}{n^k}, \tag{6.26}$$

which would nicely explain why we overlooked the division in the numerator for the case $n=2$, since, here, $n-1=1$ and anything divided by 1 is just that anything.

It, again, does not appear to be too difficult to prove the result. We have a lot of base cases already and now want to prove that

$$\frac{(n^k-1)/(n-1)}{n^k}+\frac{1}{n^{k+1}}=\frac{(n^{k+1}-1)/(n-1)}{n^{k+1}}. \tag{6.27}$$

We multiply the first fraction by n in numerator and denominator and get in the numerator

$$\frac{n(n^k-1)}{n-1} = \frac{n^{k+1}-n}{n-1}.$$

We now add 1 represented as the fraction $\frac{n-1}{n-1}$:

$$\frac{n^{k+1}-n}{n-1} + \frac{n-1}{n-1},$$

leading to

$$\frac{n^{k+1}-n+n-1}{n-1} = \frac{n^{k+1}-1}{n-1},$$

which is the desired result

$$\frac{(n^{k+1}-1)/(n-1)}{n^{k+1}}. \quad \square \tag{6.28}$$

6.6. Linear Equations

6.7. Binomial Coefficients are Integers

To prove that binomial coefficients are integers is quite easy. We will make our argument for coefficients of the form $1 \leq k \leq n$ in $\binom{n}{k}$. For cases outside of this range, the coefficients are defined as 1 for $k = 0$, and as 0 for $k > n$. So, there is nothing to prove in these cases.

We have the equation

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}. \tag{6.29}$$

We can prove that $\binom{n}{k}$ is an integer for any n and any $k \leq \frac{n}{2}$ by induction. The induction argument holds, until we have $k > \frac{n}{2}$. At this moment, the factors in the numerator and denominator begin to intersect and the factors now common in numerator and denominator are cancelled out leading to a corresponding case in the lower half of ks, namely the case $k = n - k$. You can easily convince yourself by trying some examples that this is the reason for the symmetry in Pascal's triangle.

Since, in the proof, we have already handled the cases in the lower half of ks by induction up to $\frac{n}{2}$, there is nothing that still needs to be proven. The proof, therefore, consists in the induction argument that if $\binom{n}{k}$ is an integer, then $\binom{n}{k+1}$ is an integer too for $k \leq \frac{n}{2}$.

We first handle the trivial case $k = 1$. Here, we have $\frac{n}{1!=1}$, which, trivially, is an integer. Note that, for this case, the falling factorial, which is defined as the product of the consecutive numbers n to $n - k + 1$, is just n, since $n - 1 + 1 = n$.

For $k = 2$, we have

$$\frac{n(n-1)}{2}.$$

In the numerator we have a product of two consecutive numbers, one of which must be even. We, hence, divide the even one and the denominator by 2 and have an integer.

For $k = 3$, we have

$$\frac{n(n-1)(n-2)}{6}.$$

We now have three consecutive numbers as factors in the numerator and $3! = 6$ in the denominator. One of three consecutive numbers must be divided by 3, since there are only two numbers between any two multiples of 3. If, for instance, $n = 11$, then n and $n-1 = 10$ are not divided by 3, but $n-2 = 9$ is. So we reduce the problem to $k-1 = 2$. But note that there is a difference between the previous case $k = 2$ and the case $k - 1$ at which we are arriving now: In the previous case, we had two consecutive numbers in the numerator, but now we have three numbers that are not necessarily consecutive anymore. It may have been the middle number, $n - 1$, that we divided by 3; then, if n is odd, $n - 2$ is odd as well. However, if n and $n - 2$ are not even, then $n - 1$ must have been and then it must have been divisible by 3 and 2. In consequence, even though the numbers are not consecutive anymore, the divisibility argument still holds.

This is how the induction argument works: for any $k + 1$, we have $k + 1$ consecutive factors in the numerator and we have $(k + 1) \times k!$ in the denominator. Since there are $k + 1$ consecutive numbers in the numerator, all greater $k + 1$, one of them must be divided by $k + 1$. By dividing this number and the denominator by $k + 1$, we reduce the problem to k with k factors in the numerator and $k!$ in the denominator. Now, the factors are not consecutive anymore, but that does not affect the argument: either the number that was reduced by dividing by $k + 1$ is divided by k as well and then we have reduced the problem to $k - 2$ already, or, if it is not divided by k, then one of the other numbers must be, because we started with $k + 1$ numbers in the first place. □

Let us quickly look at the next example, $k = 4$, just to illustrate the argument once again. With $k = 4$ we have the fraction

$$\frac{n(n-1)(n-2)(n-3)}{24}.$$

There are four consecutive numbers in the numerator, one of which must be divided by 4. We divide this number by 4 and the problem is reduced to the case $k = 3$. Again, the numbers are not consecutive anymore. But if the number that we reduce by dividing by 4 is divisible by 3, then we would have reduced the problem to $k = 2$ already. Otherwise, that number was just a number between two multiples of 3 and the argument does not suffer.

A concrete example makes the argument entirely clear. Consider $\binom{11}{4} = \frac{11\times10\times9\times8}{24}$. One of the numbers in the numerator must divide 4. 11 does not, 10 and 9 either, but 8 does. We divide numerator and denominator by 4 and, with that, reduce the problem to $\frac{11\times10\times9\times2}{6}$. There are 3 consecutive numbers in the numerator, one of which must be divided by 3. 11 and 10 are not divided by 3, but 9 is and, hence, we reduce the problem to $\frac{11\times10\times3\times2}{2}$, which was our base case. We now have the choice to either cancel 2 in the numerator and 2 in the denominator or to divide 10 by 2 in the numerator and cancel 2 in the denominator. If we choose the latter, we divide 10 by 2 and obtain $\frac{11\times5\times3\times2}{1} = 11 \times 5 \times 3 \times 2$, which is $55 \times 6 = 330$.

When calculating binomial coefficients by hand, we see that the main activity in this process is to cancel numbers in the numerator and the denominator. The question arises if we could not spare a lot of computing by using numbers that are already reduced before we start working with them. In particular, we see that the reduction of the numerator tends towards the prime factors of the binomial coefficient. We know that finding the prime factors is a very hard problem. But could there not be a shortcut to the binomial coefficient by using prime factors?

It turns out there is. The solution, however, sounds a bit surreal, since it combines facts that, on the first sight, are completely unrelated. The point is that there is a way to quickly decide for any prime number p whether it is part of the prime factorisation of a binomial coefficient and to even determine how often it occurs in its prime factorisation by counting the number of *borrows* we have to make subtracting k from n in the numeral system of base p.

To determine how often (or if at all) 2 appears in the prime factorisation of $\binom{6}{2}$, we would perform the subtraction $6 - 2$ in binary format. 6 in binary format is 110 and 2 is just 10. So, we subtract:

$$\begin{array}{cccc} & 1 & 1 & 0 \\ - & & 1 & 0 \\ = & 1 & 0 & 0 \end{array}$$

The final result 100 is 4 in decimal notation and, hence, correct. Since we have not borrowed once, 2 is not a factor of $\binom{6}{2}$. We check the next prime number 3. 6 in base 3 is 20 and 2 is just 2:

$$\begin{array}{cc} & 2 \quad 0 \\ - & \quad 2 \end{array}$$

Now we have to borrow to compute 0 - 2, so we have:

$$\begin{array}{cc} & 2 \quad 0 \\ - & \underline{1} \quad 2 \\ = & 1 \quad 1 \end{array}$$

11 base 3 is 4 in the decimal system, so the result is correct. Furthermore, we had to borrow once to compute this result. We, therefore, claim that 3 appears once in the prime factorisation of $\binom{6}{2}$.

We look at the next prime, 5. 6 in base 5 is 11 and 2 in base 5 is just 2 again. So we have:

$$\begin{array}{cc} & 1 \quad 1 \\ - & \quad 2 \end{array}$$

Again, we have to borrow to compute 1 - 2:

$$\begin{array}{cc} & 1 \quad 1 \\ - & \underline{1} \quad 2 \\ = & 0 \quad 4 \end{array}$$

Since 4 in base 5 is just decimal 4, the result, again, is correct. To reach it, we had to borrow once and we, therefore, claim that 5 appears once in the prime factorisation of $\binom{6}{2}$.

The next prime would be 7, but we do not need to go on, since 7 is greater than $n = 6$. Because we multiply n only by values less than n (namely: $n-1, n-2, \ldots, n-k+1$), 7 cannot be a factor of such a number. Our final result, thus, is: $\binom{6}{2} = 3^1 \times 5^1 = 3 \times 5 = 15$. Let us check this result against our usual method to compupte the binomial coefficient: $\frac{6 \times 5}{2} = 3 \times 5 = 15$. The result is correct.

But what, on earth, has the factorisation of binomial coefficients to do with the borrows in $n - k$? The link is the following theorem:

$$\binom{n}{k} \equiv \prod_{i=0}^{r} \binom{a_i}{b_i} \pmod{p}, \tag{6.30}$$

where the as and bs are the coefficients in the representation of n and k in base p:

$$n = a_r p^r + a_{r-1} p^{r-1} + \cdots + a_1 p + a_0 \tag{6.31}$$

and

$$k = b_r p^r + b_{r-1} p^{r-1} + \cdots + b_1 p + b_0. \tag{6.32}$$

The theorem, hence, claims that $\binom{n}{k}$ is congruent to the product of the coefficients in the representation base p modulo that p. We are back to congruences and modular arithmetic!

The theorem is a corollary of *Lucas' theorem*, which we will now introduce as a lemma to prove the theorem above. Lucas' theorem states that

$$\binom{n}{k} \equiv \binom{\lfloor n/p \rfloor}{\lfloor k/p \rfloor} \binom{n \bmod p}{k \bmod p} \pmod{p}, \tag{6.33}$$

which is exceptionally beautiful, since it decomposes n and k into the two parts of the Euclidian division, the quotient $\lfloor n/p \rfloor$ and the remainder $n \bmod p$. Let us rename the quotient and remainder of n and k, because we will refer to them quite often in this section: let $u = \lfloor n/p \rfloor$ and $v = n \bmod p$, such that $n = up + v$, and let $s = \lfloor k/p \rfloor$ and $t = k \bmod p$, such that $k = sp + t$. We can now rewrite the usual computation of the coefficient

$$\binom{n}{k} = \frac{n}{k} \times \frac{n-1}{k-1} \times \cdots \times \frac{n-k+1}{1} \tag{6.34}$$

as

$$\binom{n}{k} = \frac{up+v}{sp+t} \times \frac{up+v-1}{sp+t-1} \times \cdots \times \frac{up+v-k+1}{1}. \tag{6.35}$$

This formula leads to a cyclic repetition of denominators of the form

$$sp + t - 1, sp + t - 2, \ldots, sp + t - t.$$

We have to be careful with the denominators of the form $sp + t - t = sp$, since, modulo p, they are just zero and the corresponding fraction is thus undefined. But before we get into it, let us look at the t very first numbers, that is the fractions, before the formula reaches a multiple of p for the first time. These fractions are:

$$\frac{up+v}{sp+t} \times \frac{up+v-1}{sp+t-1} \times \cdots \times \frac{up+v-t+1}{sp+t-t+1},$$

which modulo p is

$$\frac{v}{t} \times \frac{v-1}{t-1} \times \cdots \times \frac{v-t+1}{1}.$$

This, in its turn, is just the usual way to define the binomial coefficient for v and t:

$$\binom{v}{t} = \frac{v}{t} \times \frac{v-1}{t-1} \times \cdots \times \frac{v-t+1}{1}. \tag{6.36}$$

But v and t are $n \bmod p$ and $k \bmod p$ respectively and substituting back these values for v and t in the equation leads to

$$\binom{n \bmod p}{k \bmod p} = \frac{n \bmod p}{k \bmod p} \times \frac{(n \bmod p)-1}{(k \bmod p)-1} \times \cdots \times \frac{(n \bmod p)-(k \bmod p)+1}{1} \tag{6.37}$$

and we conclude

$$\binom{n}{k} \equiv \binom{n \bmod p}{k \bmod p} X \pmod{p}, \tag{6.38}$$

where X is the rest of the product after the first t factors we are looking at right now.

Consider the example $\binom{90}{31}$ and the prime 7. For n, we get $u = 90/7 = 12$ and, since $12 \times 7 = 84$, we have $v = 90 \bmod 7 = 6$. For k, we get $s = 31/7 = 4$ and, since $4 \times 7 = 28$, we have $t = 31 \bmod 7 = 3$. The first t factors, we have looked at so far are

$$\frac{90 \times 89 \times 88}{31 \times 30 \times 29}.$$

We take all factors in numerator and denominator modulo 7:

$$\frac{6 \times 5 \times 4}{3 \times 2 \times 1},$$

which, as you can see, is just $\binom{6}{3} = \binom{90 \bmod 7}{31 \bmod 7}$.

Now we will look at the ominous X. Since X is the product with the first $k \bmod p$ factors cut off and the number of factors in the entire product is k, the number of the remaining factors is a multiple of p. These factors fall into $\frac{k}{p}$ groups each of which contains in the numerator and the denominator one multiple of p and $p-1$ remainders of p. Let us look at the denominators of such a group:

$$\frac{\cdots}{sp} \times \frac{\cdots}{sp+1} \times \cdots \times \frac{\cdots}{sp+p-1}.$$

Since the whole is a product, these values are multiplied with each other and, as we know from Wilson's theorem, the factorial of $p-1$ is $(p-1)! \equiv p-1 \pmod{p}$. We certainly have the same remainders in the numerators modulo p, which, again according to Wilson, are $p-1$ modulo p and, therefore, cancel out. We are then left with the factors that are multiples of p.

Continuing the example, the first of such groups would be

$$\frac{87 \times 86 \times 85 \times 84 \times 83 \times 82 \times 81}{28 \times 27 \times 26 \times 25 \times 24 \times 23 \times 22}$$

Here, 84 in the numerator and 28 in the denominator are multiples of 7. All other numbers are remainders. In the denominator, we have a complete group of remainders $22 \ldots 27$, which are $1 \ldots 6$ modulo 7. These multiplied with each other are, according to Wilson's theorem, congruent to 6 modulo 7. In the numerator, we do not see the whole group at once. Instead, we see two different parts of two groups separated by 84, the multiple of 7, in the middle: $85, 86, 87$, which are congruent to 1, 2 and 3 modulo 7, and $81, 82, 83$, which are congruent to 4, 5 and 6 modulo 7. Multiplying these remainders is, again according to Wilson's theorem, congruent to 6 modulo 7. So, we cancel all these values in numerator and denominator and keep only

$$\frac{84}{28}.$$

As already observed, we have $\lfloor k/p \rfloor = \lfloor 31/7 \rfloor = 4$ of such groups. We are therefore left with 4 fractions with a multiple of 7 in the numerator and the denominator, namely the factors:

$$\frac{84}{28} \times \frac{77}{21} \times \frac{70}{14} \times \frac{63}{7}.$$

When we divide numerator and denominator by 7, we get

$$\frac{12}{4} \times \frac{11}{3} \times \frac{10}{2} \times \frac{9}{1}.$$

and see by this simple trick of black magic that the result is

$$\binom{12}{4} = \binom{\lfloor 90/7 \rfloor}{\lfloor 31/7 \rfloor} = \binom{\lfloor n/p \rfloor}{\lfloor k/p \rfloor}.$$

Unfortunately, there are very few scholars left who would accept magic as proof and so we must continue with the abstract reasoning. Note again that we have taken the first $t = k \bmod p$ terms out in the previous step. The denominators we are left with, when we

arrive at fractions with multiples of p in the numerator and denominator, are therefore $k-(k \bmod p), k-(k \bmod p)-p, k-(k \bmod p)-2p, \ldots, 1$. In the example above, 28 corresponds to $k-(k \bmod p)$: $31-3=28$, 21 corresponds to $k-(k \bmod p)-p$ and so on.

The numerators are not so clean, but very similar: $n-(k \bmod p)-x, n-(k \bmod p)-x-p, \ldots$ The x in this formula results from the fact that $n-(k \bmod p)$ does not necessarily result in a multiple of p. For instance, $90-3=87$ is not a multiple of p. x in this case is 3, since $90-3-3=84$, which is a multiple of p. In fact, we can determine the value of x more specifically as $(n \bmod p)-(k \bmod p)$, which is $6-3=3$, but we do not need to make use of this fact. It is sufficient to realise that each value must be divisible by p and, hence, $(k \bmod p)+x<p$. When we now divide by p, we get for each factor

$$\frac{\lfloor (n-(k \bmod p)-x_i-a_i p)/p \rfloor}{\lfloor (k-(k \bmod p)-b_i p)/p \rfloor},$$

where the as and bs run from 0 to the number of groups we have minus 1, *i.e.* $\lfloor k/p \rfloor - 1$.

Since the second term of the differences in numerator and denominator are remainders of p that, together with n and, respectively, k, are multiples of p, this is just the same as saying

$$\frac{\lfloor (n-a_i p)/p \rfloor}{\lfloor (k-b_i p)/p \rfloor},$$

which of course is

$$\frac{\lfloor n/p \rfloor - a_i}{\lfloor k/p \rfloor - b_i}.$$

Since the as and bs run from 0 to the number of the last group, we get this way the product

$$\frac{\lfloor n/p \rfloor}{\lfloor k/p \rfloor} \times \frac{\lfloor n/p \rfloor - 1}{\lfloor k/p \rfloor - 1} \times \frac{\lfloor n/p \rfloor - 2}{\lfloor k/p \rfloor - 2} \times \cdots \times \frac{\lfloor n/p \rfloor - \lfloor k/p \rfloor + 1}{\lfloor k/p \rfloor - \lfloor k/p \rfloor + 1},$$

which we immediately recognise as the computation for

$$\binom{\lfloor n/p \rfloor}{\lfloor k/p \rfloor}.$$

You, hopefully, remember that this is the X, we left over in equation 6.38. Substituting for X we derive the intended result:

$$\binom{n}{k} \equiv \binom{n \bmod p}{k \bmod p}\binom{\lfloor n/p \rfloor}{\lfloor k/p \rfloor} \pmod{p} \tag{6.39}$$

and this completes the proof. □

But we have to add an important remark. Binomial coefficients with $k > n$ are defined to be zero. The equation, thus, tells us that the prime p divides the coefficient if $k \bmod p > n \bmod p$. For instance $\binom{8}{3}$, which is $\frac{8\times7\times6}{6} = 8 \times 7 = 56$, is divided by 7, since 7 appears as a factor in the numerator and, indeed: $\binom{8 \bmod 7}{3 \bmod 7} = \binom{1}{3}$. This would also work with $\binom{9}{3}$, which is $\frac{9\times8\times7}{6} = 3 \times 4 \times 7 = 84$, where 7, again, appears as a factor in the numerator and $\binom{9 \bmod 7}{3 \bmod 7} = \binom{2}{3}$. It does not work with $\binom{9}{2}$, which is $\frac{9\times8}{2} = 9 \times 4 = 36$, since $\binom{9 \bmod 7}{2 \bmod 7} = \binom{2}{2} = 1$ and, indeed: $36 \bmod 7 = 1$. Let us memorise this result: a prime p divides a binomial coefficient $\binom{n}{k}$, if $k \bmod p > n \bmod p$.

We, finally, come to the corollary, which we wanted to prove in the first place. We need to prove that

$$\binom{n}{k} \equiv \prod_{i=0}^{r} \binom{a_i}{b_i} \pmod{p}, \tag{6.40}$$

where the as and bs are the coefficients in the representation of n and k base p. We now calculate u, v, s and t, as we have done before, as $u = \lfloor n/p \rfloor$, $v = n \bmod p$, which is just the last coefficient in the p-base representation of n a_0, $s = \lfloor k/p \rfloor$ and $t = k \bmod p$, which is just the last coefficient b_0.

The p-base representations of n and k are $n = a_r p^r + \cdots + a_1 p + a_0$ and $k = b_r p^r + \cdots + b_1 p + b_0$. If we divide those by p, we get $u = a_r p^{r-1} + \cdots + a_1$ and $s = b_r p^{r-1} + \cdots + b_1$ with a_0 and b_0 as remainders.

From Lucas' theorem we conclude that

$$\binom{n}{k} \equiv \binom{u}{v}\binom{a_0}{b_0} \pmod{p}. \tag{6.41}$$

Now we just repeat the process for u and v:

$$\binom{n}{k} \equiv \binom{\lfloor u/p \rfloor}{\lfloor v/p \rfloor}\binom{a_1}{b_1}\binom{a_0}{b_0} \pmod{p} \tag{6.42}$$

and continue until we have

$$\binom{n}{k} \equiv \binom{a_r}{b_r} \cdots \binom{a_1}{b_1}\binom{a_0}{b_0} \pmod{p}, \tag{6.43}$$

which then concludes the proof. □

We see immediately that p divides $\binom{n}{k}$, when at least one digit of k in the p-base representation is greater than the corresponding digit of n, because, in this case, the corresponding binomial coefficient is zero and, in consequence, the whole product modulo p becomes zero.

That a digit of k is greater than the corresponding digit of n implies that, on subtracting k from n, we have to borrow from the next place. Therefore, if we have to borrow during subtraction, then p divides $\binom{n}{k}$ and, thus, is a prime factor of that number.

So, we divide $\binom{n}{k}$ by p leading to the product in equation 6.43 with one pair of digits removed and search again for a pair of digits where $k > n$. If we find one, the number $\binom{n}{k}$ is divided twice by p, so p appears twice in the factorisation of that number. We again divide by p and repeat the process until we do not find a pair $k > n$ anymore. Then we know how often p appears in the prime factorisation of $\binom{n}{k}$. If we do this for all primes $\leq n$, we learn the complete prime factorisation of $\binom{n}{k}$.

We now will implement this logic in Haskell. We start with the notion of borrows:

```
borrows :: Natural → Natural → Natural → Natural → Natural
borrows _ 0 _ _ _ = 0
borrows p u v s t | v < t = 1 + borrows p (u `div` p) (u `rem` p)
                                (s `div` p) ((s `rem` p) + 1)
                  | otherwise = borrows p (u `div` p) (u `rem` p)
                                (s `div` p) (s `rem` p)
```

The function *borrows* takes five arguments all of our old type *Natural*. The first argument is the prime; the next four arguments are u, v, s and t, that is $u = \lfloor n/p \rfloor$, $v = n \bmod p$, $s = \lfloor k/p \rfloor$ and $t = k \bmod p$.

If $u = 0$, we are through and no more borrows are to be found. Otherwise, if $v < t$, we have to borrow. The borrow is actually seen in the recursive call of borrows, where we increment s mod p by 1. We also add 1 to the overall result. Otherwise, we call borrows with the quotients and remainders of u and s. The recursion implements the logic we see in equation 6.43: we reduce the product factor by factor by dividing by p; on each step, we check if a borrow occurs and continue with the next step.

This is the heart of our algorithm. However, we can improve on this. There are cases we can decide immediately without going through the whole process. Consider a prime $p \leq n - k$. Since the numerator in the computation of the binomial coefficient runs from $n \ldots (n - k + 1)$, this prime will not appear directly in the numerator. It could of course still be a factor of one of the numbers $n, n - 1, \ldots, n - k + 1$ and, as such, be a factor of the resulting number. But then it must be less than or at most equal to the half one of those numbers. Otherwise, there would be no prime number by which we could multiply p to obtain one of those number. In consequence, when $p \leq n - k$ and $p > n/2$, p cannot divide $\binom{n}{k}$.

On the other hand, if $p > n - k$, then it appears in the numerator; if also $p > k$, then it will not appear in the denominator. In consequence, if $p > k$ and $p > n - k$, then it will not be cancelled out and is therefore prime factor of the binomial coefficient. Furthermore, it can appear only once in the numerator. There are only k consecutive numbers in the numerator and $p > k$. If we assume there is a factor ap with $a \geq 1$, then we will reach n in one direction and $n - k + 1$ in the other direction, before we reach either $(a + 1)p$ or $(a - 1)p$. Therefore, $a = 1$ or, in other words, p appears only once in the prime factorisation.

Finally, if $n \bmod p < k \bmod p$, we know for sure that p divides the number at least once. If $p^2 > n$, then we know that p divides $\binom{n}{k}$ exactly once.

We will implement these one-step decisions as filters in a function that calls *borrows*:

```
powOfp :: Natural → Natural → Natural → Natural
powOfp n k p | p ⩽ n − k ∧ p > n `div` 2          = 0
             | p > k ∧ p > n − k                  = 1
             | p ∗ p > n ∧ n `rem` p < k `rem` p = 1
             | otherwise = borrows p (n `div` p) (n `rem` p)
                                     (k `div` p) (k `rem` p)
```

Now we implement a new variant of *choose* making use of borrows:

```
choose3 :: Natural → Natural → Natural
choose3 n k  = product (map f ps)
  where ps   = takeWhile (⩽ n) allprimes
        f p = p ↑ (powOfp n k p)
```

The implementation is quite simple and there is certainly room for optimisation. It just maps $p^{(\text{powOfp } n\ k\ p)}$ on all primes up to n and builds the product of the result. A possible improvement is to use *fold* instead of *map* and not to add primes to the result for which *powOfp* yields 0. That would reduce the size of the resulting list of primes drastically. With *map*, there are a lot of 1s, in fact, most of the elements are 1 in most cases.

But let us investigate the running time of *choose3* compared to that of $\frac{n^k}{k!}$ in more general terms. The running time of the fraction is a function of k, such as $2k - 1$, since we have $k-1$ multiplications in numerator and denominator and one division, as already discussed in a previous chapter. Since we are not too much interested in the details of the operations, *i.e.* the cost of single multiplications and divisions and the cost of *borrows* for one prime, we will use the *big-O* notation, for instance: $\mathcal{O}(2k - 1)$. This notation tells us that there is a function of the form $2k - 1$, which is a limit for our function, the running time of the fraction. In other words, for huge input values, the function within the $\mathcal{O}$ is equal to or greater than our function.

The running time of *choose3*, by contrast, is a function of $\Pi(n)$, that is the number of primes up to n, approxmimately $n/\ln n$. In big-O notation, we state that the running

time of *choose3* is $\mathcal{O}(n/\ln n)$.

For large ns and small ks, *e.g.* $\binom{1000000}{2}$, the fraction appears to be much better. Of course, the multiplications in the numerator are heavy, since the complexity of multiplication grows with the number of digits of the factors, but there are still only few such multiplications. The multiplications in the denominator, in compensation, are trivial with small ks.

The complexity of *choose3* for this specific number is $1000000/\ln 1000000 \approx 72382$ and, as such, of course much worse. With larger ks, however, the picture changes. Even though we can reduce the complexity of the fraction for cases where $k > n/2$ to cases with $k < n/2$, as discussed before, there is sufficient room for ks around $n/2$ that makes *choose3* much more efficient than the fraction. In general, we can say that *choose3* is more efficient, whenever $\frac{n}{2\ln n} + 1 < k < n - \frac{n}{2\ln n} - 1$. For the specific example of $n = 1000000$, *choose3* is more efficient for $36192 < k < 963807$.

The fact underlying the algorithm, the relation between the number of occurrences of a prime p in the factorisation of a binomial coefficient and the number of borrows in the subtraction of n and k in base-p, was already known in the 19^{th} century, but was apparently forgotten during the 20^{th} century. It was definitely mentioned by German mathematician Ernst Kummer (1810 – 1893), but described as an algorithm apparently only in 1987 in a short, but nice computer science paper by French mathematician Pascal Goetgheluck who rediscovered the relation by computer analysis.

Lucas' theorem is named for Édouard Lucas (1842 – 1891), a French mathematician who worked mainly in number theory, but is also known for problems and solutions in recreational math. The *Tower of Hanoi* is of his invention. Lucas died of septicemia in consequence of a household accident where he was cut by a piece of broken crockery.

6.8. Euler's Totient Function

The constructor function *ratio* of our *Ratio* data type reduces fractions to their canonical form by dividing numerator and denominator by their *gcd*. If the *gcd* is just 1, numerator and denominator do not change. If the *gcd* is not 1, however, the numbers actually do change. For instance, the fraction $\frac{1}{6}$ is already in canonical form. The fraction $\frac{3}{6}$, however, is not, since $gcd(6,3)$ is 3 and so we reduce: $\frac{3/3=1}{6/3=2} = \frac{1}{2}$.

This leads to the observation that not all fractions possible with one denominator manifest with the *Ratio* number type. For the denominator 6, for example, we have only $\frac{1}{6}$ and $\frac{5}{6}$. For other numerators, we have $\frac{2}{6} = \frac{1}{3}$, $\frac{3}{6} = \frac{1}{2}$ and $\frac{4}{6} = \frac{2}{3}$. We could write a function that shows all proper fractions for one denominator, *e.g.*

$$\begin{aligned}&\mathit{fracs1} :: \mathit{Natural} \to [\mathit{Ratio}]\\ &\mathit{fracs1}\ n = [x \,\%\, n \mid x \leftarrow [1\,..\,n-1]]\end{aligned}$$

fracs1 6, for instance, gives:

$[Q\ 1\ 6, Q\ 1\ 3, Q\ 1\ 2, Q\ 2\ 3, Q\ 5\ 6]$,

which is easier to read in mathematical notation:

$$\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}.$$

How many proper fractions are there for a specific denominator? In the example above, there are only two proper fractions with the denominator 6. We devise a function to filter out the fractions that actually preserve the denominator 6:

$$\begin{aligned}&\mathit{fracs2} :: \mathit{Natural} \to [\mathit{Ratio}]\\ &\mathit{fracs2}\ n = \mathit{filter}\ (\lambda(Q\ _\ d) \to d \equiv n)\ (\mathit{fracs1}\ n)\end{aligned}$$

This function, again applied to 6 would give (in mathematical notation):

$$\frac{1}{6}, \frac{5}{6}.$$

Applied to 12, it would give:

$$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}.$$

It is easy to see that the numerators of the fractions with a given denominator n, correspond to the group of numbers $g < n$ coprime to n. We, hence, could also create a function that just finds the coprimes from the range $0 \ldots n-1$:

$$\begin{aligned}&\mathit{coprimes} :: \mathit{Natural} \to [\mathit{Natural}]\\ &\mathit{coprimes}\ n = [x \mid x \leftarrow [0\,..\,n-1], \mathit{gcd}\ n\ x \equiv 1]\end{aligned}$$

With little surprise, we see that *coprimes* 12 gives

$$1, 5, 7, 11.$$

Mathematicians like to quantify things and so it is no wonder that there is a well known function, Euler's *totient* function, often denoted $\varphi(n)$, that actually counts the coprimes. This is easily implemented:

$$\begin{aligned}&\mathit{tot} :: \mathit{Natural} \to [\mathit{Natural}]\\ &\mathit{tot} = \mathit{fromIntegral} \circ \mathit{length} \circ \mathit{coprimes}\end{aligned}$$

Let us have a look at the results of the totient function for the first 20 or so numbers:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	...
1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16	6	18	...

We first see that the totient of a prime p is $p-1$: $\varphi(2) = 1$, $\varphi(3) = 2$, $\varphi(5) = 4$, $\varphi(7) = 6$, $\varphi(11) = 10, \ldots$ Of course, that is the group of remainders of that prime – the previous chapter was dedicated almost entirely to that group!

The totients of composites are different. They slowly increase, but many numbers have the same totient. Since the difference between primes and composites lies in the fact that composites have divisors different from 1 and themselves, it is only natural to suspect that there is a relation between the totients of composites and their divisors. For instance, 2 has one divisor: 1. The totien of 1 is $\varphi(1) = 1$ and that of 2, $\varphi(2)$ is 1 as well. It may just be one of those peculiarities we see with small numbers, but what we see is the curious relation $\varphi(1) + \varphi(2) = 2$. We could formulate the hypothesis that the sum of the totients of the divisors of a number n is that number n. $n = \varphi(1)+\varphi(2)+\cdots+\varphi(n)$ or, more elegantly:

$$n = \sum_{d|n} \varphi(d). \tag{6.44}$$

Let us check this hypothesis. From 2, we go on to 3, but 3 is prime and has only the divisors 1 and 3 and, trivially, $\varphi(1) + \varphi(3) = 3$, since, for any prime p: $\varphi(p) = p - 1$. We go on to 4. The divisors of 4 are 1,2 and 4: $\varphi(1) + \varphi(2) + \varphi(4) = 1 + 1 + 2 = 4$. The next number, 5, again is prime and we go on to 6, which has the divisors 1, 2, 3, 6: $\varphi(1) + \varphi(2) + \varphi(3) + \varphi(6) = 1 + 1 + 2 + 2 = 6$. Until here, the equation is confirmed. Let us jump a bit forward: 12 has the divisors 1, 2, 3, 4, 6, 12: $\varphi(1) + \varphi(2) + \varphi(3) + \varphi(4) + \varphi(6) + \varphi(12) = 1 + 1 + 2 + 2 + 2 + 4 = 12$. The equation appears to be true. But can we prove it?

In fact, it follows from a number of fundamental, but quite simple theorems that one would probably tend to take for granted on first encounter. One of these theorems is related to cardinality of set union. The theorem states that

$$|S1 \cup S2| = |S1| + |S2| - |S1 \cap S2| \tag{6.45}$$

that is: the cardinality of the union of two sets equals the sum of the cardinalities of the two sets minus the cardinality of the intersection of the two sets.

Proof: The intersection of two sets $S1$ and $S2$ contains all elements that are both in $S1$ and $S2$. The union of two sets contains all elements of $S1$ and $S2$. But those elements that are in both sets will appear only once in the union, since this is the definition of the very notion of *set*. We can therefore first build a collection of all elements in the sets including the duplicates and then, in a second step, remove the duplicates. The elements that we remove, however, are exactly those that are also elements of the intersection. The number of elements in the union, hence, is exactly the sum of the numbers of elements of the individual sets minus the number of duplicates. □

There are two corollaries that immediately follow from this theorem. First, for two disjoint sets $S1$ and $S2$, *i.e.* sets for which $S1 \cap S2 = \varnothing$, the equation above simplifies to:

$$|S1 \cup S2| = |S1| + |S2|. \tag{6.46}$$

This is trivially true, since $|\varnothing| = 0$.

Second, for sets that are pairwise disjoint (but only for those!), we can derive the general case:

$$\left| \bigcup_{i=1}^{n} S_i \right| = \sum_{i=1}^{n} |S_i|, \tag{6.47}$$

where $\bigcup$ is the union operator for n sets, where n is not necessarily 2. It, hence, does for $\cup$ what $\sum$ does for $+$. Systems of such operators that are applied to an arbitrary number of operands are called σ-algebras. But, for the time being, that is just a fancy word.

The next fundamental theorem states that the sum of the divisors of a number n equals the sum of the fractions $\frac{n}{d_i}$, where $d_i = d_1, d_2, \ldots, d_r$ are the divisors of n. More formally, the theorem states:

$$\sum_{d|n} d = \sum_{d|n} \frac{n}{d}. \tag{6.48}$$

The point, here, is to see that if d is a divisor of n, then $\frac{n}{d}$ is a divisor too. That d is a divisor means exactly that: n divided by d results in another integer m, such that $d = \frac{n}{m}$ and $dm = n$. Since the set of divisors of n contains all divisors of n and the set of quotients $\frac{n}{d}$ contains quotients with all divisors, the two sets are equal. The only aspect that changes, when we see these sets as sequences of numbers, is the order. Since order has no influence on the result of the sum, the two sums are equal. □

For the example $n = 12$, the divisors are $1, 2, 3, 4, 6, 12$. The quotients generated by dividing n by the divisors are $12, 6, 4, 3, 2, 1$. The sum of the first sequence is $1 + 2 + 3 + 4 + 6 + 12 = 28$. The sum of the second sequence is $12 + 6 + 4 + 3 + 2 + 1 = 28$.

It remains to note that, when we have a sum of functions, then still $\sum_{d|n} f(\frac{n}{d}) = \sum_{d|n} f(d)$, since the values to which the function is applied are still the same in both sets.

Equipped with these simple tools, we return to the sum of the totients of the divisors. We start by defining a set S_d that contains all numbers, whose *gcd* with n is d:

$$S_d = \{m \in \mathbb{N} : 1 \leq m \leq n, \gcd(n, m) = d\}. \tag{6.49}$$

In Haskell this would be:

$s :: Natural \to Natural \to [Natural]$
$s\ n\ d = [m \mid m \leftarrow [1\,..\,n], gcd\ n\ m \equiv d]$

When we map this function with $n = 12$ on the divisors of 12, $map\ s\ 12\ [1, 2, 3, 4, 6, 12]$, we get:

$[1, 5, 7, 11]$
$[2, 10]$
$[3, 9]$
$[4, 8]$
$[6]$
$[12]$.

We see six pairwise disjoint sets whose union equals the numbers $1 \ldots 12$. The first set contains the coprimes of 12, since we ask for m, such that $\gcd(12, m) = 1$. The next set contains the numbers, such that $\gcd(12, m) = 2$, the next, the numbers, such that $\gcd(12, m) = 3$ and so on. In other words, these lists together contain all numbers $1 \ldots 12$ partitioned according to their greatest common divisor with $n = 12$. Note that the lists together necessarily contain all the numbers in the range $1 \ldots n$, since, either a number does not have common divisors with n, then it is in the first set for $\gcd(n, m) = 1$, or it has a common divisor with n. Then it is in one of the other sets. This is just what the set S_d mapped on the divisors of n is about.

The sets are also necessarily disjoint from each other, since no number m would, on one occasion, have a $gcd\ d_1$ with n and, on another, a distinct $gcd\ d_2$ with the same n. It either shares d_1 as greatest common divisor with n or divisor d_2. It, hence, is either in set S_{d_1} or in set S_{d_2}.

But there is more. The set for divisor 2 contains 2 and 10. These numbers divided by 2 give 1 and 5. $\frac{12}{2}$ is 6 and 1 and 5 are the coprimes of 6. The set for divisor 3 contains 3 and 9; these numbers divided by 3 are 1 and 3. $\frac{12}{3}$ is 4 and 1 and 3 are the coprimes of 4 and so on. In other words, the sets that we see above contain numbers m that, divided by the corresponding divisor d, $\frac{m}{d}$, are the coprimes of $\frac{n}{d}$. This results from the fact that these numbers and the divisor are related to each other by the gcd. When we have two numbers m and n and we compute their gcd: $d = \gcd(n, m)$, then $\frac{n}{d}$ is coprime to $\frac{m}{d}$, since we divide them by the biggest number that divides both. Therefore, all numbers in the set S_d are necessarily coprimes of $\frac{n}{d}$.

Can there be a coprime of $\frac{n}{d}$ (less than $\frac{n}{d}$) that is not in the set S_d? We created the list of coprimes by first computing m, such that $\gcd(n, m) = d$, and then $c = \frac{m}{d}$. Now, let us assume that there is a coprime c that escapes this filter. In other words, there is another number $k \neq d$, such that $\gcd(n, m) = k$ and $c = \frac{m}{k}$. To be a coprime of interest,

we must have $\frac{m}{k} < \frac{n}{d}$. Since $\frac{dn}{d} = n$, we must have $\frac{dm}{k} < n$. This number must therefore appear in one of the S_{d_i}. We can ask: in which one? The answer is $\gcd(n, \frac{dm}{k})$. There are two candidates: d and $\frac{m}{k}$. But $\frac{m}{k}$ cannot be a divisor of n, since k is the greatest divisor m and n have in common. They do not share any other divisor, not even $\frac{m}{k}$. Therefore, d must be the greatest common divisor of n and $\frac{dm}{k}$. But then this number appears in S_d and $\frac{m}{k}$ does not escape our filter.

It follows that each of the sets S_d contains exactly those numbers that divided by d are the coprimes of $\frac{n}{d}$. The size of each of these sets is thus the totient number of $\frac{n}{d}$:

$$|S_d| = \varphi\left(\frac{n}{d}\right). \tag{6.50}$$

To complete the proof, we now have to extend the relation between one of those sets and the totient of one $\frac{n}{d}$ to that between the union of all the S_{d_i} and the sum of the totient numbers of the divisors. From cardinality of disjoint sets (equation 6.47) we know that the cardinality of the union of disjoint sets is the sum of the cardinality of each of the sets, so we have:

$$\left|\bigcup_{d|n} S_d\right| = \sum_{d|n} \varphi\left(\frac{n}{d}\right). \tag{6.51}$$

From sum of divisors (equation 6.48) we know even further that the sum of $\frac{n}{d}$ equals the sum of d, therefore:

$$\left|\bigcup_{d|n} S_d\right| = \sum_{d|n} \varphi(d). \tag{6.52}$$

We have seen that the union of the S_{d_i} for a given n contains all numbers in the range $1 \ldots n$:

$$\bigcup_{d|n} S_d = \{1 \ldots n\}. \tag{6.53}$$

Since the set $\{1 \ldots n\}$ contains n numbers, we can conclude that

$$\left|\bigcup_{d|n} S_d\right| = n, \tag{6.54}$$

from which, together with equation 6.52, we then can conclude that

$$\sum_{d|n} \varphi(d) = n. \quad \square \tag{6.55}$$

We could define a recursive function very similar to Pascal's rule that exploits this relation. We first define a function to get the divisors

$$\begin{aligned}&\mathit{divs} :: \mathit{Natural} \to [\mathit{Natural}]\\ &\mathit{divs}\ n = [\, d \mid d \leftarrow [1\,..\,n], \mathit{rem}\ n\ d \equiv 0\,]\end{aligned}$$

Then we add up the totients of these numbers (leaving n out, because that is the one we want to compute) and subtract the result from n and, this way, obtain the totient number of n:

$$\begin{aligned}&\mathit{divsum} :: \mathit{Natural} \to [\mathit{Natural}]\\ &\mathit{divsum}\ 1 = 1\\ &\mathit{divsum}\ 2 = 1\\ &\mathit{divsum}\ n = n - \mathit{sum}\,[\,\mathit{divsum}\ d \mid d \leftarrow \mathit{divs}\ n, d < n\,]\end{aligned}$$

Another property of the totient function is multiplicity of totients of coprimes, that is

$$\varphi(a) \times \varphi(b) = \varphi(ab), \text{if}\, \gcd(a,b) = 1. \tag{6.56}$$

For instance, the coprimes of 3 are 1 and 2; those of 5 are 1, 2, 3 and 4. $\varphi(3)$, hence, is 2 and $\varphi(5)$ is 4. $\varphi(3 \times 5 = 15)$ is 8, which also is 2×4. Indeed, the coprimes of 15 are 1, 2, 4, 7, 8, 11, 13 and 14. An example of two coprimes that are not both primes is 5 and 8. $\varphi(5) = 4$ and $\varphi(8) = 4$. $\varphi(5 \times 8 = 40) = 16$, which also is 4×4.

This property might look surprising at the first sight, but is becomes almost trivial in the light of the Chinese Remainder theorem. For two coprimes a and b and their sets of coprimes A and B, we can, for any $a_i \in A$ and $b_j \in B$ create congruence systems of the form

$$\begin{aligned} x &\equiv a_i \pmod{a}\\ x &\equiv b_j \pmod{b}\end{aligned}$$

The Chinese Remainder theorem guarantees that, for every case, their is a solution that is unique modulo ab, *i.e.* there are no two different systems with the same solution and there is no system without a solution. Since the solutions are unique modulo ab, there must be exactly one number in the group of coprimes of ab for any combination of a_i and b_j. Since there are $|A| \times |B|$ combinations of all elements of A and B, $\varphi(ab)$ must be $\varphi(a) \times \varphi(b)$.

To illustrate that, we can create all the combinations of as and bs and then apply the Chinese remainder on all of them. First we create all combinations of as and bs:

```
consys :: Natural → Natural → [[Natural]]
consys a b = concatMap mm (coprimes b)
  where mm y = [[x, y] | x ← coprimes a]
```

The result for *consys* 5 8, for instance, is:

$[1,1],[2,1],[3,1],[4,1],$
$[1,3],[2,3],[3,3],[4,3],$
$[1,5],[2,5],[3,5],[4,5],$
$[1,7],[2,7],[3,7],[4,7]$

Now we map *chinese* on this:

```
china :: Natural → Natural → [Natural]
china a b = map (esenihc [a, b]) (consys a b)
  where esenihc = flip chinese
```

Note that, to map *chinese* on *consys*, we have to flip it. *chinese* expects first the congruences and then the moduli, but we need it the other way round.

This is the result for *china* 5 8:

$1, 11, 21, 31, 17, 27, 37, 7, 33, 3, 13, 23, 9, 19, 29, 39$

and this is the result for *coprimes* 40:

$1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39.$

When you sort the result of *china* 5 8, you will see that the results are the same.

We are now approaching the climax of this section. There is a little fact we need, before we can go right to it, which may appear a tiny curiosity. This curiosity is yet another property of the totient function concerning the totient of powers of prime numbers:

$$\varphi(p^k) = p^k - p^{k-1}. \tag{6.57}$$

That is, if p is prime, then the totient of p^k equals the difference of this number p^k and the previous power of that prime p^{k-1}. An example is 27, which is 3^3. We compute $27 - 3^2 = 27 - 9 = 18$, which is indeed $\varphi(27)$.

The proof is quite simple. Since p is prime, its powers have only one prime factor, namely p. When we say *prime power*, we mean exactly this: a number whose factorisation consists of one prime raised to some $k \geq 1$: p^k. Therefore, the only numbers that share divisors with p^k are multiples of p less than or equal to p^k: p, $2p$, $3p$, $\ldots$, p^k. The 9 numbers that share divisors with 27 are:

$$3, 6, 9, 12, 15, 18, 21, 24, 27.$$

How many multiples of p less than or equal to p^k are there? There are p^k numbers in the range $1 \dots p^k$, every p^{th} number of which is a multiple of p. There, hence, are $\frac{p^k}{p} = p^{k-1}$ numbers that divide p^k. The number of coprimes in this range is therefore p^k minus that number and, thus, $p^k - p^{k-1}$. □

We can play around a bit with this formula. The most obvious we can do is to factor p^{k-1} out to get $p^{k-1}(p-1)$. So, we could compute $\varphi(27) = 9 \times 2 = 18$. Even more important for the following, however, is the formula at which we arrive by factoring p^k out. We then get

$$\varphi\left(p^k\right) = p^k\left(1 - \frac{1}{p}\right) \tag{6.58}$$

This formula leads directly to a closed form for the totient function, namely *Euler's product formula*. Any number can be represented as a product of prime powers: $n = p_1^{k_1} p_2^{k_2} \dots$ Since the ps in this formula are all prime, the resulting prime powers are for sure coprime to each other. That means that the multiplicity property of the totient function applies, *i.e.*

$$\varphi(n) = \varphi\left(p_1^{k_1}\right)\varphi\left(p_2^{k_2}\right)\dots.$$

We now can substitute the totient computations of the prime powers on the right-hand side by the formula in equation 6.58 resulting in

$$\varphi(n) = p_1^{k_1}\left(1 - \frac{1}{p_1}\right) p_2^{k_2}\left(1 - \frac{1}{p_2}\right)\dots$$

We regroup the formula a bit to get:

$$\varphi(n) = p_1^{k_1} p_2^{k_2} \dots \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots$$

and see that we have all the prime factors of n and then the differences. The prime factors multiplied out result in n, so we can simplify and obtain Euler's product formula:

$$\varphi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right). \tag{6.59}$$

Consider again the example $n = 12$. The formula claims that

$$\varphi(12) = 12 \times \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right),$$

since the prime factors of 12 are 2 and 3. Let us see: $1 - \frac{1}{2}$ is just $\frac{1}{2}$, $1 - \frac{1}{3}$ is $\frac{2}{3}$. We therefore get $12 \times \frac{1}{2} \times \frac{2}{3}$. $12 \times \frac{1}{2} = 6$ and $6 \times \frac{2}{3} = \frac{12}{3} = 4$, which, indeed, is $\varphi(12)$.

We can use this formula to implement a variant of *tot* in Haskell:

```
ptot :: Natural → Natural
ptot n = let r = ratio n 1
         in  case r * product [ratio (p − 1) p | p ← nub (trialfact n)] of
             Q t 1 → t
             _     → error "Totient not an integer"
```

Note that we use the *Ratio* number type. To convert it back to *Natural*, we check if the denominator is 1, *i.e.* if the resulting fraction is indeed an integer. If so, we return the numerator. Otherwise, we create an error, which, of course, should not occur if Euler was right.

But what is so special about this function? Well, it leads us quickly to an old friend. Consider a squarefree number n, that is a number where the exponents in the prime factorisation are all 1: $n = p_1 p_2 \dots p_r$. We perform some simple transformations on 6.59. First, we transform the factors $1 - \frac{1}{p}$ to $\frac{p}{p} - \frac{1}{p} = \frac{p-1}{p}$. Then we have:

$$\varphi(n) = n\frac{p-1}{p}\frac{q-1}{q}\dots \tag{6.60}$$

As you may have noticed, this is the form in which we implemented Euler's formula in *ptot*. Now we multiply this out:

$$\varphi(n) = \frac{n(p-1)(q-1)\dots}{pq\dots} \tag{6.61}$$

Since n is squarefree, the denominator, the product of the prime factors, equals n. n and the entire denominator, hence, cancel out. Now we have:

$$\varphi(n) = (p-1)(q-1)\dots \tag{6.62}$$

That is, the totient number of a squarefree number is the product of the prime factors, each one reduced by one, a result that follows from multiplicity of totients of numbers that are coprime to each other.

The formula is quite interesting. It is very close to the formula we used for the RSA algorithm to find a number t for which

$$a^t \equiv 1 \pmod{n}.$$

It will probably not come as a surprise that there is indeed *Euler's theorem*, which states that, for a and n coprime to each other:

$$a^{\varphi(n)} \equiv 1 \pmod{n}, \tag{6.63}$$

which, as you will at once recognise, is a generalisation of Fermat's little theorem. Fermat's theorem expresses the same congruence for a prime number. Since, as we know, the totient number of a prime number p is $p-1$, Fermat's theorem can be reduced to Euler's theorem. In fact, Euler's theorem is nothing new at all. We already know from the previous chapter that the powers of a up to a^k, such that $a^k \equiv 1 \pmod{n}$, establish a group or subgroup of the numbers coprime to n. Since $\varphi(n)$ is the size of the group of coprimes of n, any subgroup of n has either size $\varphi(n)$ or a size that divides $\varphi(n)$. But for sure, for any coprime a the group is at most $\varphi(n)$, since there are only $\varphi(n)$ elements in the group.

For the example 12, the groups are all trivial. The coprimes of 12 are 1, 5, 7 and 11. 1 establishes the trivial group $\{1\}$. 5 is in the likewise trivial group $\{1, 5\}$, since $5^2 = 25$ and, hence, $5^2 \equiv 1 \pmod{12}$. Since $7^2 = 49 \equiv 1 \pmod{12}$, the group of 7 is also trivial. Finally, since $11^2 = 121 \equiv 1 \pmod{12}$, 11 is in a trivial group as well.

A more interesting case is 14. Let us look at the groups of 14 using *generate*, which we defined in the previous chapter, like *map* (*generate* 14) (*coprimes* 14):

$[1]$
$[3, 9, 13, 11, 5, 1]$
$[5, 11, 13, 9, 3, 1]$
$[9, 11, 1]$
$[11, 9, 1]$
$[13, 1]$

We see four different groups. The trivial groups 1 and $n-1$ and the non-trivial groups $\{1, 3, 5, 9, 11, 13\}$, identical to the coprimes of 14, and $\{1, 9, 11\}$, a subgroup with three elements.

From these examples it should be clear that not for all coprimes a $\varphi(n)$ is the first number for which the congruence in Euler's theorem is established. In fact, in many cases, there are smaller numbers k that make $a^k \equiv 1 \pmod{n}$. For $\varphi(n)$, however, it is guarenteed for any a coprime to n that the congruence holds.

But, still, $\varphi(n)$ is not necessarily the smallest number that guarantees the congruence. In some cases, there is a smaller number that does the job and this number can be calculated by the *Carmichael function*, of which we have already used a part, when we discussed RSA.

The Carmichael function is usually denoted $\lambda(n)$ (but has nothing to do with the λ-calculus!). It is a bit difficult to give its definition in words. It is much easier, actually,

to define it in Haskell:

```
lambda :: Natural → Natural
lambda 2 = tot 2
lambda 4 = tot 4
lambda n | twopower n     = (tot n) `div` 2
         | primepower n   = tot n
         | even n ∧ primepower (n `div` 2) = tot n
         | otherwise      = let ps = map lambda (simplify $ trialfact n)
                            in foldl' lcm 1 ps
  where simplify = map product ∘ group ∘ sort
```

There are two base cases stating that $\lambda(2)$ and $\lambda(4)$ equal $\varphi(2)$ and $\varphi(4)$ respectively. Otherwise, if n is a power of 2, then $\lambda(n)$ equals the half of $\varphi(n)$. An example is 8. We would expect that any group generated with coprimes of 8 has at most two members, since $\varphi(8) = 4$, and $\lambda(8) = 2$. We generate the groups with *map* (*generate* 8) (*coprimes* 8) and see:

[1]
[3, 1]
[5, 1]
[7, 1].

The prediction, hence, is correct. We saw a similar result for 12, but that has other reasons as we will see below.

The function *twopower*, by the way, is defined as

```
twopower :: Natural → Bool
twopower 1 = True
twopower 2 = True
twopower n | even n     = twopower (n `div` 2)
           | otherwise = False
```

The next line states that for any primepower n, $\lambda(n) = \varphi(n)$.

The function *primepower* is defined as

```
primepower :: Natural → Bool
primepower n = length (nub $ trialfact n) ≡ 1
```

In other words, a primepower is a number whose prime factorisation consists of only one prime (which itself, however, may appear more than once). Since powers of 2 are handled in the previous guard, we are dealing here only with powers of odd primes. An example is 9, which is 3^2. The coprimes of 9 are 1, 2, 4, 5, 7 and 8. The totient number, hence, is 6. The groups are *map* (*generate* 9) (*coprimes* 9):

[1]
[2, 4, 8, 7 5, 1]
[4, 7, 1]
[5, 7, 8, 4, 2, 1]
[7, 4, 1]
[8, 1].

We see four different groups. The two trivial groups and two groups with 3 and 6 members respectively. Again, the prediction is correct.

The next line states that, if n is even and half of n is a primepower, then again $\lambda(n) = \varphi(n)$. An example is 18, since 18 is even and the half of 18 is 9, which is a power of 3. $\varphi(18)$ is 6, so we would expect to see groups with at most 6 elements. Here is the result for *map* (*generate* 18) (*coprimes* 18):

[1]
[5, 7, 17, 13, 11, 1]
[7, 13, 1]
[11, 13, 17, 7, 5, 1]
[13, 7, 1]
[17, 1].

We see, again, four groups, the two trivial groups 1 and $n-1$ and two non-trivial groups with 3 and 6 members respectively.

We come to the *otherwise* guard. If n is not 2 or 4, not a power of 2 nor a power of another prime and not twice a power of a prime, then we do the following: we compute the factorisation, order the factors, group by equal factors, compute the primepower that corresponds to each group of factors and map λ on the resulting numbers. Then we compute the *lcm* of the results. In short: $\lambda(n)$ is the *lcm* of the λ mappend to the prime powers in the prime factorisation of n.

An example for a number that is not a primepower nor twice a primepower is 20. The factorisation of 20 is $\{2, 2, 5\}$. We compute the primepowers resulting in $\{4, 5\}$. When we map λ on them, we should get $\lambda(4) = \varphi(4) = 2$ and $\lambda(5) = \varphi(5) = 4$. The *lcm* of 2 and 4 is 4 and, hence, $\lambda(20) = 4$. We, thus, should not see a group with more than 4 elements. We call *map* (*generate* 20) (*coprimes* 20) and see:

[1]
[3, 9, 7, 1]
[7, 9, 3, 1]
[9, 1]
[11, 1]
[13, 9, 17, 1]
[17, 9, 13, 1]
[19, 1].

We see 6 different groups, the two trivial groups 1 and $n-1$ and four non-trivial groups

with 2 and, respectively, 4 members.

The factorisation of 12 is $\{2, 2, 3\}$, so we apply λ on the numbers 4 and 3, which for both cases is 2. The *lcm* of 2 is just 2 and, therefore, we do not see groups with more than 2 members with the coprimes of 12.

Now, as you may have guessed, *Carmichael's theorem* states that, if a and n are coprime to each other, then

$$a^{\lambda(n)} \equiv 1 \pmod{n}. \tag{6.64}$$

For primes, the theorem is identical to Fermat's little theorem. For powers of odd primes, it reduces to Euler's theorem. The *lcm* of primepowers under the *otherwise*-guard is a consequence of the Chinese Remainder theorem and the very notion of the *lcm*. We know that, if $x \equiv 1 \pmod{n}$, then also $x \equiv 1 \pmod{mn}$. However, mn is not necessarily the first multiple of n and m that establishes the congruence. Any number that is a multiple of both, n and m, would have the same effect and the first number that is a multiple of both is $lcm(m, n)$.

The totient number of twice the power of an odd prime, $2p^k$, is the same as the totient number of that odd prime power, p^k: $\varphi(p^k) = \varphi(2p^k)$. The coprimes of p^k are all numbers from 1 to p^k that are not multiples of p, including all even numbers. Since twice that primepower is an even number, the even numbers are not part of the coprimes of that number. So, the coprimes of $2p^k$ in the range $1 \ldots p^k$, are exactly half of the coprimes of p^k. But now, there are the coprimes in the second half $p^k \ldots 2p^k$. Since the interval is the same in size and we eliminate the same number of numbers in that range as in the first half, namely the even numbers and the multiples of p, we end up with two sequences, each containing half as many numbers as the original sequence of coprimes of p^k. The two halfs together, therefore, make for the same amount of coprimes of p^k and $2p^k$. So, we can handle these cases in the same way.

The general rule, however, would produce the same result. According to the general rule, we would first compute λ for the individual primepowers and then the *lcm* of these values. The factorisation of a number that is twice a primepower contains the factor 2 and the primepower. The value for $\lambda(2)$ is $\varphi(2)$, which is 1. The *lcm* of 1 and another number is that other number. There, hence, is no difference between this rule and the general rule.

Now, what about the powers of 2 greater 4? To show that the greatest group of a power of 2 is half the totient of that number is quite an interesting exercise in group theory. The coprimes of a power of 2 have a quite peculiar structure, namely

$$1, \ldots, m_1, m_2, \ldots, n-1.$$

Interesting are the middle numbers m_1 and m_2. They both are their own inverses,

such that $m_1 m_1 \equiv 1 \pmod{n}$ and $m_2 m_2 \equiv 1 \pmod{n}$. The set of coprimes, therefore, consists of two symmetric halves, each starting and ending with a number that is its own inverse: $1 \ldots m_1$ is the first half, $m_2 \ldots n-1$ is the second half.

The number of coprimes is of course even since they consist of all odd numbers $1 \ldots 2^k - 1$. Therefore, we do not have one central number, but the two middle numbers m_1 and m_2, which are one off the half of 2^k, that is $m_1 = 2^{k-1} - 1$ and $m_2 = 2^{k-1} + 1$. The following calculation shows that both m_1 and m_2 squared are immediately 1 modulo 2^k, for any 2^k with $k > 2$. For m_1 we have:

$$(2^{k-1} - 1)(2^{k-1} - 1).$$

When we multiply this out we get the terms $2^{k-1+k-1}$, which simplifies to 2^{2k-2}, $-2^{k-1} - 2^{k-1}$, which simplifies to -2^k, and 1:

$$2^{2k-2} - 2^k + 1.$$

We can factor 2^k out of the the first two terms:

$$2^k(2^{k-2} - 1) + 1,$$

and see clearly that the first remaining term is divided by 2^k and, thus, disappears modulo 2^k. We are left with 1 and this shows that m_1 is its own inverse.

For m_2, the proof is very similar, with the difference that we are left over with $2^k(2^{k-2}+1)$ for the first term. However, this term is a multiple of 2^k as well, and we are again left with 1.

Now, we can select a random generator of the group, say, a and look what it generates. For explicitness, we consider the case of $2^4 = 16$, whose coprimes are

$$1, 3, 5, 7, 9, 11, 13, 15$$

This group has the form (with arbitrary placement of the inverses of a and b):

$$1, a, b, m_1, m_2, b', a', n-1.$$

We see at once that no generator will create a sequence with 8 elements. Any sequence generated by exponentiation of a can contain only one of the elements m_1, m_2 and $n-1$. Since, if $a^k = m_2$, $a^{2k} = 1$ and, afterwards, the whole cycle repeats. If some a^l, for $k < l < 2k$, was m_1 or $n-1$, then a^{2l} would be 1 again. But that cannot be, since $2l > 2k$ and, therefore, a^{2l} is part of the second cycle, which has to be exactly the same

as the first. But, obviously, there was only one 1 in the first cycle, namely at a^{2k} and there must be only one 1 in the second cycle, namely at a^{4k}. Therefore, there can be only one of the elements m_1, m_2 and $n-1$ in the sequence and this reduces the longest possible sequence for this example to 6, for instance:

1	2	3	4	5	6
a	b	m_2	b'	a'	1

Until here it looks fine. But observe that we now have one set with six numbers, the group generated by a, which we will call G, and its complement relative to the set of coprimes, which contains 2 elements. For the group above that containts m_2, the complement consists of m_1 and $n-1$.

Now, we will construct what is called a *coset*. A coset of G, in our context here, is a set of numbers resulting from one element of the complement of G, multiplied by all numbers of G. Let us say, this element is m_1. Then the coset of G created by m_1 denoted m_1G is

$$m_1G = \{m_1a, m_1b, m_1m_2, m_1b', m_1a', m_1\} \tag{6.65}$$

Note that this set contains six numbers. These numbers are necessarily different from all numbers in G, since the numbers in G form a group, the product of two members of which result in another member of it and, for each pair of members of the group c and d, there is one number x in the group, such that $xc = d$. m_1, however, is not member of the group and, if m_1 multiplied by c resulted in another member d, then we would have the impossible case that d is the result of multiplications of c for two different numbers: m_1 and x. Therefore, no number in m_1G can possibly equal any number in G.

But we do not have six numbers! We only have two numbers, namely m_1 and $n-1$. Therefore, no group that we create on a set of coprimes with such a structure can be greater than half of the number of coprimes. In our example that is four. With four numbers in G and four elements in the complement of G, we would have no problem at all. But, definitely, a group with six members does not work. □

A corollary of this simple but important argument is that the order of any subgroup of a group of coprimes must divide the number of coprimes. This extends the proof of Lagrange's theorem for prime groups to composite groups. We will extend it even further in the future. For the moment, however, we can be satisfied with the result. We have proven Carmichael's theorem.

6.9. How many Fractions?

How many negative numbers are there? That is a strange question! How do you want me to answer? I can tell you how many numbers of a specific kind are there only in

relation to another kind of numbers. The words "how many" clearly indicate that the answer is again a number.

Let us try to state the question more pecisely: are there more or fewer negative numbers than natural numbers or are there exactly as many negative numbers as natural numbers?

To answer the question, I would like to suggest a way to compare two sets. To compare the size of two sets, A and B, we create a third set, which consists of pairs (a, b), such that $a \in A$ and $b \in B$. The sets are equal, if and only if every $a \in A$ appears exactly once in the new set and every $b \in B$ appears exactly once in the new set. If there is an $a \in A$ that does not appear in the new set, but all $b \in B$ appear exactly once, then B is greater than A. If there is a $b \in B$ that does not appear, but all $a \in A$ appear, then A is greater than B.

Furthermore, I suggest a way of counting a set A. We count a set by creating a new set that consists of pairs (a, n), such that $a \in A$ and $n \in \mathbb{N}$. For n, we start with 0 and, for each element in A, we increase n by 1 before we put it into the new set, like this:

$$
\begin{aligned}
&\mathit{count} :: [a] \to [(a, \mathit{Natural})] \\
&\mathit{count} = \mathit{go}\ 0 \\
&\quad \textbf{where}\ \mathit{go}\ _\ [\,] = [\,] \\
&\qquad\quad \mathit{go}\ n\ (x : xs) = (x, n + 1)\ \mathit{go}\ (n + 1)\ xs
\end{aligned}
$$

The greatest number n, we find in the pairs of this set is the number of elements in A. Let us see if we can count the negative numbers in this manner. We count them by creating the set $\{(-1, 1), (-2, 2), (-3, 3), \dots\}$. Do we ever run out of negative or natural numbers? I don't think so. Should we ever feel that we run out of negative numbers, then we just take the current natural number and put a minus sign before it. Should we ever feel that we run out of natural numbers, then we simply take the current negative number and remove the negative sign. This proves, I guess, that there is a way to assign each negative number to exactly one natural number and vice versa. There are hence as many negative numbers as natural numbers.

Well, how many numbers are that? That are $|\mathbb{N}|$ numbers. If you want a word for that, call it *aleph-zero* and write it like this: $\aleph_0$. A set with this cardinality is infinite, but countable. Calling *count* on it, we will never get a final answer. But we will have a partial result at any given step.

What about all the integers? There should be twice as much as natural numbers, right? Let us see. We first create a set to count the natural numbers:

$$\{(1, 1), (2, 2), (3, 3), \dots\}$$

Then we insert a negative number before or behind each positive number:

$$\{(1,1),(-1,2),(2,3),(-2,4),(3,5),(-3,6),\dots\}$$

Again, it appears that we do not run out of natural numbers to count all the integers. The set of integers, hence, has cardinality $\aleph_0$ too.

What about fractions? On the first sight, fractions look very different. There are infinitely many of them between any two natural numbers as we have seen with Zeno's paradox. But now comes Cantor and his first diagonal argument to show that fractions are countable and, therefore, that the set of fractions has cardinality $\aleph_0$.

Cantor's proof, his first diagonal argument, goes as follows. He arranged the fractions in a table, such that the first column contained the integers starting from 1 in the first row and counting up advancing row by row. The integers correspond to fractions with 1 as denominator. So, we could say, the first column of this table is dedicated to denominator 1. The second column, correspondingly, is dedicated to denominator 2; the third to denominator 3 and so on. Then, the rows are dedicated likewise to numerators. The first row contains numerator 1, the second contains numerator 2, the third numerator 3 and so on. Like this:

$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\dots$
$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\dots$
$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\dots$
$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\dots$
$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\dots$
$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

This table is a brute-force approach to list all fractions in two dimensions. Obviously, this table contains all possible fractions, since, for any possible pair of numbers, it contains a fraction, which, however, is not necessarily in canonical form, so the table contains duplicates.

Then, using this table, he created a sequence. He started with the first cell containing $\frac{1}{1}$. From there he went to the next row $\frac{2}{1}$. Then he applied the rule *up*, that is he went up following a diagonal line to the first row, so he would eventually reach $\frac{1}{2}$. He went one to the right and then applied rule *down*, that is he went down following a diagonal line to the first column, so he would eventually reach $\frac{3}{1}$. Now he continued to the next row and repeated the process of going *up* and *down* in diagonal lines infitely adding the number of each cell he crossed to the sequence.

The sequence evolves as follows: Cantor starts with $\frac{1}{1}$, adds $\frac{2}{1}$, applies rule *up* and adds $\frac{1}{2}$, goes to the right, adds $\frac{1}{3}$ and applies rule *down* adding $\frac{2}{2}$; then he goes to the next row adding $\frac{4}{1}$ and goes *up* again adding $\frac{3}{2}$, $\frac{2}{3}$ and $\frac{1}{4}$ and so he goes on forever.

We can reformulate this rule in Haskell, which will make the process clearer:

```
cantor1 :: [Ratio]
cantor1 = (1 % 1) : go 2 1
  where go   n 1 = up   n 1 ++ go 1 (n + 1)
        go   1 d = down 1 d ++ go (d + 1) 1
        down n 1 = [n % 1]
        down n d = (n % d) : down (n + 1) (d - 1)
        up   1 d = [1 % d]
        up   n d = (n % d) : up   (n - 1) (d + 1)
```

When we look at rule *up*, starting at the bottom of the code, we see the base case where the numerator is 1. In this case, we just yield $[1 \% d]$. Otherwise, we call *up* again with the numerator n decremented by 1 and the denominator incremented by 1. *up* 4 1, thus, is processed as follows:

```
up 4 1 = (4 % 1) : up 3 2
up 3 2 = (3 % 2) : up 2 3
up 2 3 = (2 % 3) : up 1 4
up 1 4 = [1 % 4]
```

yielding the sequence $\frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}$. *go*, after calling *up*, proceeds with *go* 1 $(n + 1)$. We, hence, would continue with *go* 1 5, which calls *down*. The base case of *down* is the case where the denominator is 1. Otherwise, we increment the numerator by 1 and decrement the denominator by 1. *down* 1 5, hence, is processed as follows:

```
down 1 5 = (1 % 5) : down 2 4
down 2 4 = (2 % 4) : down 3 3
down 3 3 = (3 % 3) : down 4 2
down 4 2 = (4 % 2) : down 5 1
down 5 1 = [5 % 1]
```

yielding the sequence $\frac{1}{5}, \frac{2}{4}, \frac{3}{3}, \frac{4}{2}, \frac{5}{1}$. When we put the sequence together, including the first steps, we see

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{4}, \frac{3}{3}, \frac{4}{2}, \frac{5}{1}, \ldots$$

When we reduce all fractions to canonical form, we see a lot of repetitions:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{1}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{5}{1}, \ldots$$

We see the numbers $\frac{1}{1} = 1$, $\frac{2}{1} = 2$ and $\frac{1}{2}$ repeated several times. They will continue to appear over and over again and, even worse, other numbers will start to reappear too. That is, before we can use this sequence to count fractions, we need to filter duplicates out leading to the sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \frac{5}{1}, \dots$$

But now we see that we can enumerate, that is count, the fractions creating the sequence

$$\left(\frac{1}{1}, 1\right), \left(\frac{2}{1}, 2\right), \left(\frac{1}{2}, 3\right), \left(\frac{1}{3}, 4\right), \left(\frac{3}{1}, 5\right), \left(\frac{4}{1}, 6\right), \left(\frac{3}{2}, 7\right), \left(\frac{2}{3}, 8\right), \left(\frac{1}{4}, 9\right), \dots$$

This clearly shows that the cardinality of the set of fractions is $\aleph_0$. □

This result may feel a bit odd on the first sight. We clearly have the feeling that there must be more fractions than integers, because, between any two integers, there are infinitely many fractions. When we think of a visualisation with a pair of balances, with the fractions being in one balance and the integers in the other, then, what we would see at any given instance, clearly indicates that there must be more fractions than integers. However, our feeling betrays us, when it comes to infinity. Indeed, our feeling was not made for infinity. Therefore, at least if we accept the notions of comparison and counting outlined above, then we have to accept the result of Cantor's argument. Even further, I would say that the fact that this argument shows things in a way that contradicts our spontaneous way to see these things, underlines the extraordinary quality of this argument. Cantor lets us see things that are usually hidden from our perception. This makes Cantor, who was seen by his contemporary opponents as a kind of sorcerer, a true magus.

It is, by the way, quite simple to extend the argument to negative fractions. We just have to insert behind each number its additive inverse, resulting in the sequence:

$$\left(\frac{1}{1}, 1\right), \left(-\frac{1}{1}, 2\right), \left(\frac{2}{1}, 3\right), \left(-\frac{2}{1}, 4\right), \left(\frac{1}{2}, 5\right), \left(-\frac{1}{2}, 6\right), \left(\frac{1}{3}, 7\right), \left(-\frac{1}{3}, 8\right), \dots$$

Indeed, there appears to be a lot of room for new numbers, once we are dealing with infinity. This led the great David Hilbert to the analogy of the hotel, which is today named after him *Hilbert's Hotel*. In this analogy, there is a hotel with the uncommon property of having infinitely many rooms. This hotel will never run out of rooms for new guests. If a new guest arrives and there are already infinitely many guests, the manager just asks the guests to move one room up, *i.e.* the guest currently in room 1 moves to room 2, the guest in room 2 moves to room 3 and so on. At the end of the process,

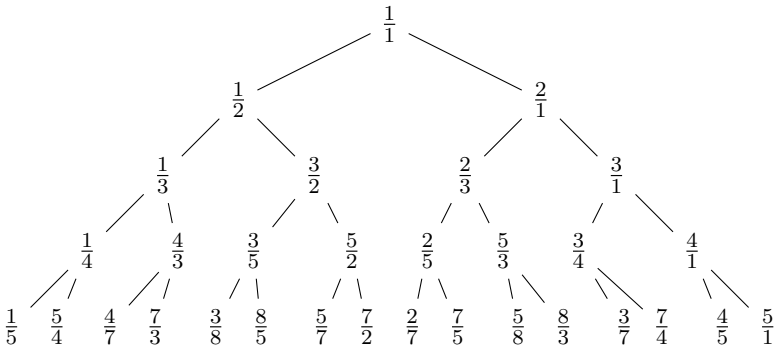
room 1 is free for the new arriving guest. Since there are infinitely many rooms, there is no guest, even though there are infinitely many of them already, who would not find a room with a room number one greater than his previous room number.

This approach works for any number of finitely many guests arriving. But even when infinitely many new guests arrive, the manager, still, has resources. In this case, he could ask the guests to move to a room with a number twice the number of his current room, *e.g.* the guest in room 1 would move to room 2, the guest in room 2 would move to room 4, the guest in room 3 would move to room 6 and so on leaving infinitely many rooms with odd room numbers unoccupied and making room for infinitely many new guests.

But let us go back to more technical stuff. In spite of its ingenuity, Cantor's argument is not perfect. In particular, the sequence and how it is created is quite uggly, but, apparently, nobody, for more than hundred years, cared too much about that. Then, in 2000, Neil Calkin and Herbert Wilf published a paper with a new sequence with a bunch of interesting properties that make this sequence for enumerating the fractions much more attractive than Cantor's original sequence. The beginning of the sequence is

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \ldots$$

The sequence, as we will show in a minute, corresponds to a *binary tree* of the form



When you have a closer look at the tree, you see that the kids of each node are created by a simple formula. If the current node has the form $\frac{n}{d}$, then the left kid corresponds to $\frac{n}{n+d}$ and the right kid corresponds to $\frac{n+d}{d}$. For instance, the kids of $\frac{1}{1}$ are $\frac{1}{1+1}$ and $\frac{1+1}{1}$. The kids of $\frac{3}{2}$ are $\frac{3}{3+2}$ and $\frac{3+2}{2}$.

We can easily create this tree in Haskell. First, we need a data type to represent the tree:

```
data Tree a = Node a [Tree a]
  deriving (Eq, Show)
type CalWiTree = Tree Ratio
```

The *Tree* data type is in fact not a binary tree, but a generic tree with an arbitrary number of nodes. But it is simple to implement and serves our purpose. The data type is parametrised, so we define a specialised data type *Tree Ratio* and the type synonym *CalWiTree* referring to this type. Now we create the tree with:

$$
\begin{aligned}
&\mathit{calWiTree} :: \mathit{Zahl} \to \mathit{Ratio} \to \mathit{CalWiTree} \\
&\mathit{calWiTree}\ 1\ r \qquad\qquad\quad = \mathit{Node}\ r\ [\,] \\
&\mathit{calWiTree}\ i\ r@(Q\ n\ d) = \mathit{Node}\ r\ [\mathit{calWiTree}\ (i-1)\ (n\ \%\ (n+d)), \\
&\qquad\qquad\qquad\qquad\qquad\quad \mathit{calWiTree}\ (i-1)\ ((n+d)\ \%\ d)]
\end{aligned}
$$

The function takes two arguments, a *Zahl* and a *Ratio*. The *Ratio* is the starting point and the *Zahl* is the number of generations we want to create. Often we do not want the function to create the entire sequence – for that a lot of patience and memory resources would be necessary – but only a tiny part of it. In this case, we set the *Zahl* to $i \geq 1$. If i reaches 1, we create the current node without kids. If we want the function to create the entire infinite tree, we just assign a value $i < 1$. If $i \neq 1$, we create a node with r as data and the kids resulting from calling *calWiTree* on $\frac{n}{n+d}$ and $\frac{n+d}{d}$ with i decremented by 1.

This shows that there is a very simple algorithm to generate the tree. We will now show that Calkin-Wilf tree and Calkin-Wilf sequence are equivalent. We do so by creating an algorithm that converts the tree to the sequence.

We may be tempted to do this with a typical recursive function that does something with the current node and then adds the result of the operation to the partial sequences that result from recursively calling the function on the left and the right kid. This approach, however, is *depth-first*. The resulting sequences would follow the branches of the tree. It would create partial sequences like, for instance, $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ But what we need is partial sequences that cover generation by generation, *i.e.* $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}$ and so on. In other words, we need a *breadth-first* approach.

Since this is a generic problem, we can define a function on the level of the generic *Tree* data type that creates a sequence composed of subsequences corresponding to tree generations:

$$
\begin{aligned}
&\mathit{getKids} :: \mathit{Natural} \to \mathit{Tree}\ a \to [a] \\
&\mathit{getKids}\ 1\ (\mathit{Node}\ r\ _) \qquad\quad = [r] \\
&\mathit{getKids}\ n\ (\mathit{Node}\ r\ [\,]) \qquad\quad = [\,] \\
&\mathit{getKids}\ n\ (\mathit{Node}\ r\ (x : xs)) = \mathit{getKids}\ (n-1)\ x +\!\!+ \\
&\qquad\qquad\qquad\qquad\qquad\quad \mathit{getKids}\ n\ (\mathit{Node}\ r\ xs)
\end{aligned}
$$

The function receives two arguments, a *Natural*, which determines the generation we want to obtain, and the tree on which we are operating. If the generation is 1, we just give back the data of the current node. Otherwise, we distinguish two cases: If the current node has no kids, then the result is the empty list. This indicates that we have exhausted the current (finite) tree. Otherwise, we advance recursively on the head and tail of the list of kids. Decisive is that we do not add anything to the resulting

sequence, before we have reached the intended depth n. This way, the function produces a sequence containing all kids on level n. We now just apply *getKids* to all generations in the tree:

$$
\begin{aligned}
&\mathit{calWiTree2Seq} :: \mathit{CalwiTree} \to [\mathit{Rational}] \\
&\mathit{calWiTree2Seq}\ t = \mathit{go}\ 1 \\
&\quad \textbf{where}\ \mathit{go}\ n = \textbf{case}\ \mathit{getKids}\ n\ t\ \textbf{of} \\
&\qquad\qquad\qquad [\,] \to [\,] \\
&\qquad\qquad\qquad \mathit{sq} \to \mathit{sq} +\!\!+ \mathit{go}\ (n+1)
\end{aligned}
$$

We have shown that there is a simple algorithm to generate the tree and that there is a simple algorithm to convert the tree into the sequence. The latter is quite useful, since it means that tree and sequence are equivalent. This allows us to prove some crucial properties of the sequence using the tree, which is much simpler than proving them on the sequence directly.

Already a quick glance at the tree reveales some interesting properties, for instance, that, in all cases, the left kid is smaller and the right kid is greater than the parent node. This, of course, is just a consquence of the generating algorithm. We also see that the integers are all in the right-most branch, which equals the first column in Cantor's table. The left-most branch equals the first row in Cantor's table: it contains all fractions with 1 in the numerator. We also see that all fractions are in canonical form, different from Cantor's table. Also, no fraction repeats and, as far as we can tell, the fractions appear to be complete.

The crucial properties, those that we need to show that the Calkin-Wilf sequence contains exactly all fractions and, thus, can be used for Cantor's argument in place of the old sequence, are:

1. All fractions in the tree are in canonical form;
2. Every possible fraction (in canoncial form) appears in the tree;
3. Every fraction appears exactly once.

The first property is simple to prove. We observe that the first fractions appearing in the tree that do not have 1 in numerator or denominator have there either 2 or 3. 2 and 3 a coprime to each other. Summing two coprimes, a and b, will lead to a number that again is coprime to both a and b. Therefore, if we have at a node n a fraction whose numerator and denominator are coprime to each other, the whole subtree below n will contain fractions whose numerator and denominator are coprime to each other. Therefore, the subtrees below $\frac{3}{2}$ and $\frac{2}{3}$ will only contain fractions in canonical form.

The fractions that actually have 1 in numerator or denominator, however, will necessarily lead to a fraction whose numerator and denominator are coprime to each other, since no number n, except $n = 1$, shares any divisor with $n + 1$. Since we start the tree with $\frac{1}{1}$, the whole tree can only contain fractions in canonical form. □

We prove the second property by contradiction. Let us assume that there are fractions that do not appear in the tree. Then, there is a number of fractions that have the smallest denominator and, among those, there is one with the smallest numerator. Let $\frac{n}{d}$ be this fraction. If $n > d$, then $\frac{n-d}{d}$ cannot appear either, since $\frac{n}{d}$ would be its right kid. But that is a contradiction to our assumption that $\frac{n}{d}$ was the nonappearing fraction with the smallest numerator among those fractions with denominator d, but, obviously, $n - d$ is an even smaller numerator. If $n < d$, then $\frac{n}{d-n}$ cannot appear either, since, $\frac{n}{d}$ would be its left kid. But that again is a contradiction, since the denominator is smaller than the denominator of one of the fractions we assumed to be those with the smallest denominator. The only way to solve this dilemma is to assume that n and d are equal. Then, indeed, $\frac{n-d}{d} = 0$ and $\frac{n}{n-d} = \perp$ would not be in the tree. But such a fraction with $n = d$ is irrelevant, since it reduces to $\frac{1}{1}$, which already is in the tree. □

To prove the third property, we first observe that 1 is the root of the tree. Since any fraction below 1 is either $\frac{n}{n+d}$ or $\frac{n+d}{d}$, there cannot be a fraction with $n = d$. With this out of the way, we can argue as we did for the second property: we assume there are fractions that appear more than once. From all these fractions, there is a group that shares the smallest denominator and, among this group, one with the smallest numerator. But this fraction is then either the left kid of two fractions of the form $\frac{n}{d-n}$, making the denominator of these fractions even smaller, or the right kid of two fractions of the form $\frac{n-d}{d}$, making the numerator of these fractions even smaller. In both cases we arrive at a contradiction. □

We can also prove directly – but the argument may be more subtle or, which is the same, almost self-evident. We know that all nodes are of either of the forms $\frac{n}{n+d}$ or $\frac{n+d}{d}$. Since, except for the root node, we never have $n = d$, no node derived from one of those fractions can ever equal one derived from the other. To say that two fractions derived from such nodes are equal, would mean that we could have two numbers n and d, such that $n = n + d$ and $d = n + d$. That would only work if $n = 0$ and $d = 0$. But that case cannot occur. □

That taken all together shows that the Calkin-Wilf sequence contains all fractions exactly once. Since we can enumerate this sequence, we can enumerate all fractions and, hence, $|\mathbb{Q}| = \aleph_0$. □

There is still another advantage of this sequence over Cantor's original one. There is a simple, yet quite exciting way to compute which is the n^{th} fraction. The key is to realise that we are dealing with a structure, namely a binary tree, that stores sequences of binary decisions. At any given node in the tree, we can go either right or left. We could, therefore, describe the n^{th} position as a trajectory through the tree, where at each node, we take a binary decision: going right or going left. An efficient way to encode a sequence of binary decisions is a binary number, and, indeed, the position in the sequence, represented as a binary number leads to the fraction at that position. Here is a function that, given a natural number n, returns the fraction in the Calkin-Wilf sequence at position n:

```
calwiR :: Natural → Rational
calwiR = go (0 % 1) ∘ toBinary
  where go r []                 = r
        go r@(Q n d) (0 : xs) = go (n % (n + d)) xs
        go r@(Q n d) (1 : xs) = go ((n + d) % d) xs
```

We start by converting n to binary format (*i.e.* a list of 0s and 1s). Then we call *go* starting with 0, since, for the first number 1, we want position 1. If we have exhausted the binary number, the result is just r, the rational number we pass to *go*. Otherwise, we distinguish two cases: the head of the binary number being 0 or 1. If it is 0, we follow the left branch, which has the fraction $\frac{n}{n+d}$ at its top; if it is 1, we follow the right branch with the fraction $\frac{n+d}{d}$.

Consider $n = 25$ as an example. 25 in binary format is 11001. We go through the steps:

```
go (Q 0 1) [1,1,0,0,1] = go (1 % 1) [1,0,0,1]
go (Q 1 1) [1,0,0,1] = go (2 % 1) [0,0,1]
go (Q 2 1) [0,0,1] = go (2 % 3) [0,1]
go (Q 2 3) [0,1] = go (2 % 5) [1]
go (Q 2 5) [1] = go (7 % 5) []
go (Q 7 5) [] = Q 7 5
```

Let us check if this is true; *take* 1 \$ *drop* 24 \$ *calWiTree2Seq* (*calWiTree* 5 (1 % 1)) gives

[*Q* 7 5],

which corresponds to the correct result $\frac{7}{5}$. With this we can create the sequence much simpler without deviating through the tree:

```
calwis :: [Ratio]
calwis = map calwiR [1..]
```

We can also do the opposite: compute the position of any given fraction. For this, we just have to turn the logic described above around:

```
calwiP :: Ratio → Natural
calwiP = fromBinary ∘ reverse ∘ go
  where go (Q 0 _) = []
        go (Q n d) | n ⩾ d      = 1 : go ((n − d) % d)
                   | otherwise = 0 : go (n % (d − n))
```

We look at the fraction and, if $n \geq d$, then we add 1 to the digits of the resulting binary number, otherwise, we add 0. Note that there is only one case where $n = d$ (as we have proven above), namely the root node of the tree. In all other cases, we either have $n > d$ or $n < d$. In the first case, we know that the fraction is a right kid and in the second case, we know it is a left kid. When we call this function on $\frac{7}{5}$, we see the steps:

$go\ (Q\ 7\ 5) = 1 : go\ (2 \,\%\, 5)$
$go\ (Q\ 2\ 5) = 0 : go\ (2 \,\%\, 3)$
$go\ (Q\ 2\ 3) = 0 : go\ (2 \,\%\, 1)$
$go\ (Q\ 2\ 1) = 1 : go\ (1 \,\%\, 1)$
$go\ (Q\ 1\ 1) = 1 : go\ (0 \,\%\, 1)$
$go\ (Q\ 0\ 1) = [\,]$,

which leads to the list $1 : 0 : 0 : 1 : 1 : [\,]$. This evaluated and reversed is $1, 1, 0, 0, 1$, which, converted to decimal representation, is 25.

Surprisingly or not, the Calkin-Wilf sequence is not completely new. A part of it was already studied in the 19^{th} century by German mathematician Moritz Stern (1807 – 1894), successor of the early deceased sucessor of Gauss at the University of Göttingen, Lejeune-Dirichlet, and professor of Bernhard Riemann. The numerators of the Calkin-Wilf sequence correspond to *Stern's diatomic sequence*. Using the Calkin-Wilf sequence, we can produce Stern's sequence with the function

```
stern :: [Natural]
stern = map numerator calwis
  where numerator (Q n _) = n
```

take 32 *stern* shows:

$$1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, 3, 7, 4, 5, 1.$$

Mapping *denominator* defined as $denominator\ (Q\ _\ d) = d$ on the Calkin-Wilf sequence would give a very similar result: the Stern sequence one ahead, *i.e.*:

$$1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, 3, 7, 4, 5, 1, 6.$$

Edsgar Dijkstra, the great pioneer of the art of computer programming, studied this sequence not knowing that it had already been studied before. He called it the *fusc* sequence and generated it with the function

```
fusc :: Natural → Natural
fusc 0 = 0
fusc 1 = 1
fusc n | even n     = fusc (n `div` 2)
       | otherwise = let k = (n − 1) `div` 2
                     in fusc k + fusc (k + 1)
```

From the definition of the *fusc* function, we can read some of the many properties of Stern's sequence (and the Calkin-Wilf tree). First, an even number has the same value as half of that number, for instance *fusc* 3 is 2 and so is *fusc* 6, *fusc* 12, *fusc* 24 and so on. Even numbers in binary format end on zeros. For instance, 3 in binary notation is 11. 2×3 is 110, 2×6 is 1100, 2×12 is 11000 and so on. The binary format clearly

indicates that, after having reached the number before the trail of zeros at the end, we go down in a straight line following the left branch of that node in the Calkin-Wilf tree. Since, in the left path, the numerator never changes, the result of $fusc(n)$ equals the result of $fusc(2n)$.

We also see that for any power of 2, *fusc* equals *fusc* 1, which is 1; *map fusc* $[2, 4, 8, 16, 32, 64, 128, 256]$, hence, gives $[1, 1, 1, 1, 1, 1, 1, 1]$. Note that, looking at the Calkin-Wilf tree, this is immediate obvious, since powers of 2 in binary representation are numbers of the form $1, 10, 100, 1000, \ldots$ Those numbers indicate that we navigate through the tree in a straight line following the left branch of the root node $\frac{1}{1}$.

The *fusc* results of powers of two minus one $(1, 11, 111, 1111, \ldots)$ equal the number of digits of this number in binary form. This is the right outer branch of the tree with the integers.

The *fusc* results of powers of two plus one $(1, 11, 101, 1001, \ldots)$ also equal the number of digits in the binary representation of that number. These numerators appear in the immediate neighbours of the powers of two in the left outer branch of the tree, for instance $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots$

What about numbers with an alternating sequence of 1s and 0s, like 101010101? Those numbers are not in the outer branches and not even close to them. Indeed, they tend to the horizontal centre of the tree. The first 1 leads to node $\frac{1}{1}$. We now go left, that is, we add the numerator to the denominator leading to $\frac{1}{2}$; we then add the denominator to the numerator leading to $\frac{3}{2}$; then we add the numerator to the denominator again leading to $\frac{3}{5}$ and so we go on and obtain the fractions $\frac{8}{5}, \frac{8}{13}, \frac{21}{13}, \frac{21}{34}, \frac{55}{34}, \ldots$ Do you see the point? All the numerators and denominators are Fibonacci numbers! Well, what we did above, adding the two numbers we obtained before starting with the pair $(1, 1)$, is just the recipe to create the Fibonacci numbers.

An amazing property of *fusc*, found by Dijkstra, is the fact that two numbers whose binary representations are the reverse of each other have the same *fusc* result. 25, for instance, is 11001. The reverse, 10011, is 19, and *fusc* $19 =$ *fusc* $25 = 7$.

For the Calwin-Wilf tree this means that, when we have two trajectories through the tree, where each step after the root node is the opposite of the simultaneous step of the other one, we arrive at two fractions with the same numerator. The trajectory defined by the binary sequence $1, 1, 0, 0, 1$ leads, first, to the root node $\frac{1}{1}$, then through $\frac{2}{1}$, $\frac{2}{3}$ and $\frac{2}{5}$ to $\frac{7}{5}$. The reverse of this sequence, $1, 0, 0, 1, 1$ leads, first, to the root node $\frac{1}{1}$ and then through $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{4}{3}$ to $\frac{7}{3}$.

A similar is true for two numbers, whose binary sequences can be transformed into one another by inverting the inner bits. For instance, 11001, 25, can be transformed into 10111, inverting all bits, but the first and the last one. 10111 is 23 and *fusc* $19 =$ *fusc* $23 =$ *fusc* 25. What about the bit inverse of 19? That is 11101, the reverse of 10111 and 29 in decimal notation. Therefore *fusc* $19 =$ *fusc* $23 =$ *fusc* $25 =$ *fusc* 29.

We can reformulate this result in terms of group theory. We have three basic transformations i, the identity, ρ, the reverse, and β, the bit inverse. We add one more transformation, the composition of ρ and β and call it $\sigma = \rho \cdot \beta$. The operation defined over this set is composition. We see that the identity is part of the set; for each transformation, its inverse is in the set, too, because $\rho \cdot \rho = i$, $\beta \cdot \beta = i$ and $\sigma \cdot \sigma = i$. To illustrate the logic of this group with the numbers above, we define it on the base string 19, which is 10011:

$i = 10011$
$\rho = 11001$
$\beta = 11101$
$\sigma = 10111$.

Now we can play around and see that we will never generate a string that is not already in the group:

$\rho \cdot i = 11001$
$\rho \cdot \rho = i = 10011$
$\beta \cdot \beta = i = 10011$
$\sigma \cdot \sigma = i = 10011$
$\rho \cdot \beta = \sigma = 10111$
$\beta \cdot \rho = \sigma = 10111$
$\sigma \cdot \rho = \beta = 11101$
$\sigma \cdot \beta = \rho = 11001$
...

All elements of one group are in the same generation of the Calkin-Wilf tree, since they all have the same number of digits. Numbers with a symmetric binary representation, such that $\rho = i$, lead to groups with only two distinguishable members, for instance $fusc(2^n+1) = fusc(2^{n+1}-1)$. The same is true for numbers with a binary representation such that $\rho = \beta$, for instance, 101011 (43) = 110101 (53) and $fusc(43) = fusc(53) = 13$.

There are infinitely many numbers with the same *fusc* result. Most of these numbers have trailing zeros and, as such, are in the long shadow thrown by one of the original odd numbers with the same result. One example of such a shadow is the outer left branch, which maintains the numerator of the root node $\frac{1}{1}$ and also maintains the leading 1 in the binary representation of its positions, merely adding more and more 0s to it.

How many "original" numbers in this sense are there for a given *fusc* result n? The answer is simple if we consider two facts: 1. The fractions in the Calkin-Wilf tree are in canonical form, *i.e.* numerator and denominator do not share divisors, and 2. Any position number whose binary representation ends with 1, points to a right kid and, for all fractions $\frac{n}{d}$ that are right kids: $n > d$. Binary numbers, however, that end with 0, point to a left kid and, therefore, $n < d$. In other words, the number of original numbers for a given numerator n is $\varphi(n)$, the totient number of n. The denominators of the original fractions are the coprimes of n.

The numerator 7, for instance, appears in six positions: 19, 23, 25, 29, 65 and 127. The denominators of the fractions at those positions are 3, 2, 5, 4, 6 and 1. For numerator 8, there are only four such numbers: 21, 27, 129 and 255. The denominators at those positions are 5, 3, 7 and 1. Note that 21 in binary format is 10101, which is its own reverse, and 27 is the bit inverse of 21, namely 11011, which also is its own reverse.

Furthermore, those numbers appear in groups with 2 or 4 members, depending on the properties of the binary representation. The number of such groups, hence, is $\frac{\varphi(n)}{k}$, where k is some integer that divides $\varphi(n)$. For 7: $k = 3$, since there are two groups, one containing 4 elements, the other containing 2. For 8: $k = 2$, since there are two groups, both containing 2 elements.

The last group is the one consisting of binary numbers with n bits, *i.e.* $2^{n-1}+1$ and 2^n-1. The other groups appear in generations of the Calkin-Wilf tree before the generation with that final group. For 7, the generation of the group with four members is the fifth generation and the generation with the final group is of course the seventh generation. In other cases, the groups can be many generations apart. The numerator 55, for instance, appears for the first time in generation 10, namely in the fraction $\frac{55}{34}$ (both Fibonacci numbers). This is far off from generation 55 with the group consisting of $\frac{55}{1}$ and $\frac{55}{54}$.

Interestingly, Dijkstra was not aware of the relation of the *fusc* algorithm to the Stern sequence, and the Calkin-Wilf tree was not even around at that time. Dijkstra describes *fusc* as a state automaton that parses strings consisting of 1s and 0s. The parsing result would be a number, namely the result of *fusc*. We could now say that the Calkin-Wilf tree is a model that gives meaning to the strings in terms of trajectories through the tree.

A final remark relates to the product of one generation in the tree. Each generation consists of fractions whose numerators and denominators were created by adding the numerators and denominators of the fractions of the previous generation. We start with the fraction $\frac{1}{1}$. In consequence, in any generation, there is for any fraction $\frac{n}{d}$ a fraction $\frac{d}{n}$. The fractions in the fifth generation for example, the one containing the fractions at positions 19, 23, 25 and 29 in the Calkin-Wilf sequence, can be paired up in the following way:

$$\left(\frac{1}{5},\frac{5}{1}\right),\left(\frac{5}{4},\frac{4}{5}\right),\left(\frac{4}{7},\frac{7}{4}\right),\left(\frac{7}{3},\frac{3}{7}\right),\left(\frac{3}{8},\frac{8}{3}\right),\left(\frac{8}{5},\frac{5}{8}\right),\left(\frac{5}{7},\frac{7}{5}\right),\left(\frac{7}{2},\frac{2}{7}\right).$$

The product of each pair is 1. The product of all fractions in one generation is therefore 1 as well. You can try this out with the simple function

```
genprod :: Natural → CalwiTree → Rational
genprod n t = product (getKids n t)
```

A sequence with so many nice properties, one might feel to say with some poetic fervour, cannot be meaningless. Isn't there anything in the (more or less) real world that these

numbers would actually count? It turns out there is. There are in fact many things the Stern sequence actually counts. Just to mention two things: It counts odd binomial coefficients of the form $\binom{n-r}{r}, 0 \leq 2r \leq n$. That is the odd numbers in the first half of the lines $n - r$ in Pascal's triangle. This would translate to a function creating a sequence of numbers of the form

```
oddCos :: Natural → [Natural]
oddCos n = filter odd [choose (n − r) r | r ← [0 .. (n `div` 2)]]
```

fusc $(n + 1)$ is exactly the size of that sequence. For instance: *oddCos* 5 is $[1, 3]$ and *fusc* 6 is 2; *oddCos* 6 is $[1, 5, 1]$ and *fusc* 7 is 3; *oddCos* 7 is $[1]$ and *fusc* 8, which is a power of 2, is 1; *oddCos* 8 is $[1, 7, 15, 1]$ and *fusc* 9 is 4; *oddCos* 9 is $[1, 21, 5]$ and *fusc* 10 is 3.

Moritz Stern arrived at his sequence, when studying ways to represent numbers as powers of 2. Any number can be written as such a sum and most numbers even in various ways. For instance, $2 = 2^0 + 2^0$, $3 = 2^0 + 2^1 = 2^0 + 2^0 + 2^0$, $4 = 2^1 + 2^1 = 2^0 + 2^0 + 2^1$, $5 = 2^0 + 2^2 = 2^0 + 2^1 + 2^1$ and so on. Stern focussed on so called *hyperbinary* systems, that is sums of powers of 2, where any power appears at most twice. For instance, $3 = 2^0 + 2^1$ is such a system, but $3 = 2^0 + 2^0 + 2^0$ is not. Stern's sequence counts the number of ways this is possible for any number $n - 1$. In other words, *fusc* $(n + 1)$ is exactly the number of hyperbinary systems for n. For 3, as an example, there is only one way and *fusc* 4 is 1; for 4, there are 3 such systems: $2^0 + 2^0 + 2^1$, $2^1 + 2^1$ and 2^2 and *fusc* 5 is 3; for 5, there are only 2 such systems: $2^0 + 2^1 + 2^1$ and $2^0 + 2^2$ and *fusc* 6 is 2.

Finding all hyperbinary systems for a number n is quite an interesting problem in its own right. A brute-force and, hence, inefficient algorithm could apply the following logic. We first find all powers of 2 less than or equal to n:

```
pows2 :: Natural → [Natural]
pows2 n = takeWhile (⩽ n) [2 ↑ x | x ← [0 ..]]
```

We then create all permutations of this set and try to build sums that equal n:

```
hyperbin :: Natural → [[Natural]]
hyperbin n = nub (go $ perms $ pows2 n)
  where go pss = filter (λk → sum k ≡ n)
                        [sort (sums 0 ps ps) | ps ← pss]
        sums _ [] []       = []
        sums s [] ds       = sums s ds []
        sums s (p : ps) ds | s + p > n = sums s ds ps
                           | s + p ≡ n = [p]
                           | otherwise = p : sums (s + p) ps ds
```

Note that we pass the available pool of powers of $2 \leq n$ twice to *sums*. When the first instance is exhausted or $s + p > n$, we start to use the second instance of the pool. This reflects the fact that we are allowed to use every number twice. If the sum $s + p$ equals n,

we have found a valid hyperbinary system. Otherwise, if $s + p < n$, we continue adding the current power to the result set. In *go*, we try *sums* on all permutations of the powers filtering the resulting sets for which *sum* equals n. We sort each list to make lists with equal elements equal and to, thus, be able to recognise duplicates and to remove them with *nub*.

7. Real Numbers

7.1. $\sqrt{2}$

Until now, we have looked at *discrete* numbers, that is numbers that are nicely separated from each other so that we can write them down unmistakebly and always know of which number we are currently talking. Now we enter a completely different universe. The universe of continuous numbers that cannot be written down in a finite number of steps. The representation of these numbers consists of infinitely many elements and, therefore, we will never be able to write the number down completely with all its elements. We may give a finite formula that describes how to compute the specific number, but we will never see the whole number written down. Those numbers are known since antiquity and, apparently, their existence came as a great surprise to Greek mathematicians.

The first step of our investigations into this kind of numbers, is to show that they *exist*, *i.e.*, there are contexts where they arise naturally. To start, we will assume that they are not necessary. We assume that all numbers are either natural, integral or fractional. Indeed, any of the fundamental arithmetic operations, $+, -, \times$ and $/$, applied on two rational numbers results always in a rational number, *i.e.* an integer or a fraction. We could therefore suspect that the result of any operation is a rational number, *i.e.* an integer or a fraction.

What about $\sqrt{2}$, the square root of 2? Let us assume that $\sqrt{2}$ is as well a fraction. Then we have two integers n and d, coprime to each other, such that

$$\sqrt{2} = \frac{n}{d} \tag{7.1}$$

and

$$2 = \left(\frac{n}{d}\right)^2 = \frac{n^2}{d^2}. \tag{7.2}$$

Any number can be represented as a product of primes. If $p_1 p_2 \dots p_n$ is the prime factorsiation of n and $q_1 q_2 \dots q_d$ is that of d, we can write:

$$\sqrt{2} = \frac{p_1 p_2 \dots p_n}{q_1 q_2 \dots q_d}. \tag{7.3}$$

It follows that

$$2 = \frac{p_1^2 p_2^2 \cdots p_n^2}{q_1^2 q_2^2 \cdots q_d^2}. \tag{7.4}$$

As we know from the previous chapter, two different prime numbers p and q squared (or raised to any integer) do not result in two numbers that share factors. The factorisation of p^n is just p^n and that of q^n is just q^n. They are coprime to each other. The fraction in equation 7.4, thus, cannot represent an integer, such as 2. There is only one way for such a fraction to result in an integer, *viz.*, when the numerator is an integer and the denominator is 1, which is obviously not the case for $\sqrt{2}$. It follows that, if the root of an integer is not an integer itself, it is not a rational number either.

But, if $\sqrt{2}$ is not a rational number, a number that can be represented as the fraction of two integers, what the heck is it then?

There are several methods to approximate the number $\sqrt{n}$. The simplest and oldest is the *Babylonian* method, also called *Heron's* method for Heron of Alexandria, a Greek mathematician of the first century who lived in Alexandria.

The idea of Heron's method, basically, is to iteratively approximate the real value starting with a guess. We can start with some arbitrary value. If the first guess, say g, does not equal $\sqrt{n}$, *i.e.* $gg \neq n$, then g is either slightly too big or too small. We either have $gg > n$ or $gg < n$. So, on each step, we improve a bit on the value by taking the average of g and its counterpart n/g. If $gg > n$, then clearly $n/g < g$ and, if $gg < n$, then $n/g > g$. The average of g and a/g is calculated as $(g + a/g)/2$. The result is used as input for the next round. The more iterations of this kind we do, the better is the approximation. In Haskell:

```
heron :: Natural → Natural → Double
heron n s = let a = fromIntegral n in go s a (a / 2)
  where go 0 _ x = x
        go i a x | xx = a     = x
                 | otherwise = go (i − 1) a ((x + a / x) / 2)
```

The function takes two arguments. The first is the number whose square root we want to calculate and the second is the number of iterations we want to do. We then call *go* with s, the number of iterations, a, the *Double* representation of n, and our first guess $a/2$.

In *go*, if we have reached $i = 0$, we yield the result x. Otherwise, we call *go* again with $i - 1$, a and the average of x and a/x.

Let us compute $\sqrt{2}$ following this approach for, say, five iterations. We first have

go 5 2 1 = *go* 4 2 ((1 + 2) / 2)
go 4 2 1.5 = *go* 3 2 ((1.5 + 2 / 1.5) / 2)
go 3 2 1.416666 = *go* 2 2 ((1.416666 + 2 / 1.416666) / 2)
go 2 2 1.414215 = *go* 1 2 ((1.414215 + 2 / 1.414215) / 2)
go 1 2 1.414213 = *go* 0 2 ((1.414213 + 2 / 1.414213) / 2)
go 1 2 1.414213 = 1.414213.

Note that we do not show the complete *Double* value, but only the first six digits. The results of the last two steps, therefore, are identical. They differ, in fact, at the twelfth digit: 1.4142135623746899 (*heron* 2 4) versus 1.414213562373095 (*heron* 2 5). Note that the result of five iterations has one digit less than that of four iterations. This is because that after the last digit, 5, the digit 0 follows and then the *precision* of the Double number type is exhausted. *Irrational* numbers, this is the designation of the type of numbers we are talking about, consist of infinitely many digits. Therefore, the last digit in the number presented above is in fact not the last digit of the number. With slightly higher precision, we would see that the number continues like 03 . . .

The result of (*heron* 2 5) ↑ 2 is fairly close to 2: 1.9999999999999996. It will not get any closer using *Double* representation.

7.2. Φ

Another interesting irrational number is τ or Φ, known as the *divine proportion* or *golden ratio*. The golden ratio is the relation of two quantities, such that the greater relates to the lesser as the sum of both to the greater. This may sound confusing, so here are two symbolic representations:

$$\frac{a}{b} = \frac{a+b}{a}. \tag{7.5}$$

In this equation, a is the greater number and b the lesser. The equation states that the relation of a to b equals the relation of $a + b$ to a. Alternatively we can say

$$\frac{b}{a} = \frac{a}{b-a}. \tag{7.6}$$

Here b is the sum of the two quantities and a is the greater one. The equation states that the relation of b to a is the same as the relation of a to $b - a$, which, of course, is then the smaller quantity.

The golden ratio is known since antiquity. The symbol Φ is an homage to ancient Greek sculptor and painter Phidias (480 – 430 BC). Its first heyday after antiquity was during Renaissance and then it was again extremely popular in the 18^{th} and 19^{th} centuries. Up

to our times, artists, writers and mathematicians have repeatedly called the golden ratio especially pleasing.

From equation 7.6 we can derive the numerical value of Φ in the form $\frac{\Phi}{1}$. We can then calculate b, for any value a, as $a\Phi$. The derivation uses some algebraic methods, which we will study more closely in the next part, especially *completing the square*. As such this section is also an algebra teaser.

We start by setting $\Phi = \frac{b}{a}$ with $a = 1$. We then have on the left side of the equation $\Phi = \frac{b}{1} = b$. On the right-hand side, we have $\frac{a=1}{\Phi-1}$:

$$\Phi = \frac{1}{\Phi - 1}. \tag{7.7}$$

Multiplying both sides by $\Phi - 1$, we get

$$\Phi^2 - \Phi = 1. \tag{7.8}$$

This is a quadratic equation and, with some phantasy, we even recognise the fragment of a binomial formula on the left side. A complete binomial formula would be

$$(a - b)^2 = a^2 - 2ab + b^2. \tag{7.9}$$

We try to pattern match the fragment above such that Φ^2 is a^2 and $-\Phi$ is $-2ab$. That would mean that the last term, b^2 would correspond to the square half of the number that is multiplied by Φ to yield $-\Phi$. $-\Phi$ can be seen as $-1 \times \Phi$. Half of that number is $-\frac{1}{2}$. That squared is $\frac{1}{4}$. So, if we add $\frac{1}{4}$ to both sides of the equation, we would end up with a complete binomial formula on the left side:

$$\Phi^2 - \Phi + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}. \tag{7.10}$$

We can apply the binomial theorem on the left side and get

$$\left(\Phi - \frac{1}{2}\right)^2 = \frac{5}{4}. \tag{7.11}$$

Now we take the square root:

$$\Phi - \frac{1}{2} = \frac{\pm\sqrt{5}}{2}. \tag{7.12}$$

Note that the $\pm$ in front of $\sqrt{5}$ reflects the fact that the square root of 5 (or any other number) may be positive or negative. Both solutions are possible. However, since we are looking for a positive relation, we only consider the positive solution and ignore the other one. We can therefore simplify to

$$\Phi - \frac{1}{2} = \frac{\sqrt{5}}{2}. \tag{7.13}$$

Finally, we add $\frac{1}{2}$ to both sides:

$$\Phi = \frac{1+\sqrt{5}}{2} \tag{7.14}$$

and voilá that is Φ. The numerical value is approximately 1.618 033 988 749 895. We can define it as a constant in Haskell as

```
phi :: Double
phi = 0.5 * (1 + sqrt 5)
```

We can now define a function that, for any given a, yields b:

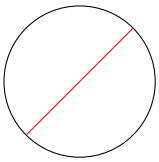
```
golden :: Double → Double
golden a = a * phi
```

golden 1, of course, is just Φ, *i.e.* 1.618 033 988 749 895. *golden* 2 is twice that value, namely 3.236 067 977 499 79. Furthermore, we can state that 2 / (*golden* 2 − 2), which is $\frac{a}{b-a}$, is again an approximation of Φ.

That is very nice. But there is more to come. Φ, in fact, is intimately connected to the Fibonacci sequence, as we will se in the next chapter.

7.3. π

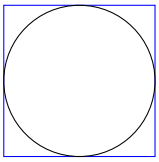
π is probably the most famous irrational number. It emerged in antique mathematics in studying the circle where it expresses the relation between the diameter (depicted in red in the image below) and the circumference (depicted in black):



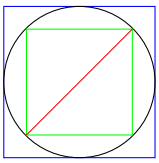
Since, often, the radius, which is half the diameter, is much more important in mathematics, it has been proposed to use $\tau = 2\pi$ where π is used today. But π has survived

through history and, even though slightly suboptimal in some situations, it is still in use today.

The reason why the perimeter instead of the radius was used to define the circle constant is probably because classic approaches to approximate π take the perimeter as basis. They start by drawing a square with side length 1 and inscribe a circle into the square with perimeter 1:



Since the square has side length 1, its perimeter, the sum of all its sides is $1+1+1+1 = 4$ and, as we can see clearly in the picture above, this perimeter is greater than that of the circle. 4, hence, is an upper bound for the circumference of the circle with perimeter 1. A lower bound would then be given by a square inscribed in the circle, such that the distance between its opposing corners (red) is 1, the perimeter of the circle:



We see two right triangles with two green sides on a red basis. The basis is the perimeter of the circle, of which we know that its length is 1. You certainly know the Pythagorean theorem, probably the most famous or notorious theorem of all mathematics, which states that, in a right triangle, one with a right angle, an angle of 90°, the sum of the squares of the sides to the left and right of that angle (the green sides) equals the square of the hypothenuse, the red side, which is opposite to the right angle. This can be stated as:

$$a^2 = b^2 + c^2, \tag{7.15}$$

where a is the red side, whose length we know, namely 1. We further know that the green sides are equal. We hence have:

$$1^2 = 2b^2. \tag{7.16}$$

and further derive

$$1 = \sqrt{2b^2}, \tag{7.17}$$

which is

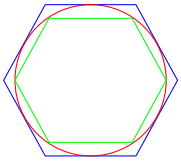
$$1 = \sqrt{2}b. \tag{7.18}$$

Dividing both sides by $\sqrt{2}$, we get

$$b = \frac{1}{\sqrt{2}}, \tag{7.19}$$

the side length of the green square, which is approximately 0.707. The perimeter of the inner square is thus 4×0.707, which is approximately 2.828. Thus π is some value between 2.828 and 4.

That result is not very satifactory, of course. There is room for a lot of numbers between 2.828 and 4. The method was therefore extended by choosing polygons with more than four sides to come closer to the real value of π. The ancient record holder for approximating π is Archimedes who started off with a hexagon, which is easy to construct with compass and ruler:



Then he subsequently doubled the number of sides of the polygon, so that he obtained polygons with 12, 24, 48 and, finally, 96 sides. With this approach he concluded that $\frac{223}{71} < \pi < \frac{22}{7}$, which translates to a number between 3.1408 and 3.1428 and is pretty close to the approximated value 3.14159.

In modern times, mathematicians started to search for approximations by other means than geometry, in particular by infinite series. One of the first series was discovered by Indian mathematician Nilakantha Somayaji (1444 – 1544). It goes like

$$\pi = 3 + \frac{4}{2 \times 3 \times 4} - \frac{4}{4 \times 5 \times 6} + \frac{4}{6 \times 7 \times 8} - \ldots \tag{7.20}$$

We can implement this in Haskell as

```
nilak :: Int → Double
nilak i    | even i     = nilak (i + 1)
           | otherwise = go i 2 3 4
  where go 0 _ _ _ = 3
        go n a b c = let k | even n = −4
                           | otherwise = 4
                     in (k / (a ∗ b ∗ c)) + go (n − 1) c (c + 1) (c + 2)
```

Here we use a negative term, whenever n, the counter for the step we are performing, is even. Since, with this approach, an even number of steps would produce a bad approximation, we perform, for i even, $i + 1$ and hence an odd number of steps. This way, the series converges to 3.14159 after about 35 steps, *i.e. nilak* 35 is some number that starts with 3.14159.

An even faster convergence is obtained by the beautiful series discovered by French mathematician François Viète (1540 – 1603) in 1593:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times \dots \tag{7.21}$$

In Haskell this gives rise to a very nice recursive function:

```
vietep :: Int → Double
vietep i = 2 / (go 0 (sqrt 2))
  where go n t | n ≡ i      = 1
               | otherwise = (t / 2) ∗ go (n + 1) (sqrt (2 + t))
```

The approximation 3.14159 is reached with *vietep* 10.

There are many other series, some focusing on early convergence, others on beauty. An exceptionally beautiful series is that of German polymath Gottfried Wilhelm Leibniz (1646 – 1716), who we will get to know more closely later on:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \tag{7.22}$$

In Haskell this is, for instance:

```
leipi :: Int → Double
leipi i = 4 ∗ go 0 1
  where go n d | n ≡ i      = 0
               | otherwise = let x | even n = 1
                                   | otherwise = −1
                             in x / d + go (n + 1) (d + 2)
```

This series converges really slowly. We reach 3.14159 only after about 400 000 steps.

π appears quite often in mathematics, particularly in geometry. But there are also some unexpected entries of this number. The inevitable Leonhard Euler solved a function, which today is called *Riemann zeta function*, for the special case $s = 2$:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{7.23}$$

Euler showed that, for the special case $s = 2$, $\zeta(s)$ converges to $\frac{\pi^2}{6}$; in fact, for any n, n a multiple of 2, $\zeta(n)$ converges to some fraction of a power of π, *e.g.* $\zeta(4)$ approaches $\frac{\pi^4}{90}$, $\zeta(6)$ approaches $\frac{\pi^6}{945}$ and so on.

This is surprising, because the zeta function is not related to circles, but to number theory. It appears for example, when calculating the probability of two numbers being coprime to each other. Two numbers are coprime if they do not share prime factors. The probability of a number being divisible by a given prime p is $\frac{1}{p}$, since every p^{th} number is divisible by p. For two independently chosen numbers, the probability that both are divisible by prime p is therefore $\frac{1}{p} \times \frac{1}{p} = \frac{1}{p^2}$. The reverse probability that both are not divisible by that prime, hence, is $1 - \frac{1}{p^2}$. The probablitiy that there is no prime at all that divides both is then

$$\prod_{p}^{\infty} 1 - \frac{1}{p^2}. \tag{7.24}$$

To cut a long story short, this equation can be transformed into the equation

$$\frac{1}{1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots} = \frac{1}{\zeta(2)} = \frac{1}{\frac{\pi^2}{6}} = \frac{6}{\pi^2} = 0.607 \approx 61\% \tag{7.25}$$

and with this π appears as a constant in number theory expressing the probability of two randomly chosen numbers being coprime to each other.

7.4. e

The Bernoullis were a family of Huguenots from Antwerp in the Spanish Netherlands from where they fled the repression by the Catholic Spanish authorities, first to Frankfurt am Main, later to Basel in Switzerland. Among the Bernoullis, there is a remarkable number of famous mathematicians who worked in calculus, probability theory, number theory and many areas of applied mathematics. One of the Basel Bernoullis was Johann Bernoulli (1667 – 1748) who worked mainly in calculus and tutored famous mathematicians like Guillaume L'Hôpital, but whose greatest contribution to the history of math

was perhaps to recognise the enormous talent of another of his pupils whose name was Leonhard Euler.

His brother Jacob Bernoulli (1655 – 1705), who worked, as his brother, in calculus, but most prominently in probability theory, is much better known today, partly perhaps because many of Johann's achievements were published under the name of L'Hôpital. Unfortunately, early modern mathematics and science in general was plagued with disputes over priorities in the authorship of contributions, a calamity that authors and authorities later tried to solve by introducing the *droite d'auteur*, better known in the English speaking world as *copyright*.

Among the many problems Jacob studied was the calculation of interests. He started off with a very simple problem. Suppose we have a certain amount of money and a certain interest credited after a given amount of time. To keep it simple, let the amount equal 1 (of any currency of your liking – currencies in Jacob's lifetime were extremely complicated, so we better ignore that detail). After one year 100% interest is paid. After that year, we hence have $1 + \frac{1*100}{100} = 2$ in our account. That is trivial. But what, if the interest is paid in shorter periods during the year? For instance, if the interest is paid twice a year, then the interest for that period would be 50%. After six months we would have $1 + \frac{1*50}{100} = 1.5$ in our account. After one year, the account would then be $1.5 + \frac{1.5*50}{100} = 1.5 + \frac{75}{100} = 1.5 + 0.75 = 2.25$.

Another way to see this is that the initial value is multiplied by 1.5 (the initial value plus the interest) twice: $1 \times 1.5 \times 1.5 = 1 \times 1.5^2 = 2.25$. When we reduce the period even further, say, to three months, then we had $1.25^4 \approx 2.4414$. On a monthly base, we would get $\left(1 + \frac{1}{12}\right)^{12} \approx 2.613$. On a daily basis, we would have $\left(1 + \frac{1}{365}\right)^{365} \approx 2.7145$. With hourly interests and the assumption that one year has $24 \times 365 = 8760$ hours, we would get $\left(1 + \frac{1}{8760}\right)^{8760} \approx 2.71812$. With interest paid per minute we would get $\left(1 + \frac{1}{525600}\right)^{525600} \approx 2.71827$ and on interest paid per second, we would get $\left(1 + \frac{1}{3156000}\right)^{3156000} \approx 2.71828$. In general, for interest on period n, we get:

$$\left(1 + \frac{1}{n}\right)^n.$$

You may have noticed in the examples above that this formula converges with greater and greater ns. For n approaching ∞, it converges to 2.71828, a number that is so beautiful that we should look at more than just the first 5 digits:

$$2.7\ 1828\ 1828\ 4590\ 4523\ldots$$

This is e. It is called Euler's number or, for the first written appearance of concepts related to it in 1618, Napier's number. It is a pity that its first mentioning was not in the year 1828. But who knows – perhaps in some rare Maya calendar the year 1618 actually is the year 1828.

An alternative way to approach e that converges much faster than the closed form above is the following:

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$$

or, in other words:

$$e = \sum_{n=1}^{\infty} \frac{1}{n!}. \tag{7.26}$$

We can implement this equation in Haskell as

```
e_ :: Integer → Double
e_ p = 1 + sum [1 / (dfac n) | n ← [1 .. p]]
  where dfac = fromInteger ∘ fac
```

After some experiments with this function, we see that it converges already after 17 recursions to a value that does not change with greater arguments at *Double* precision, such that $e_\ 17 \equiv e_\ 18 \equiv e_\ 19 \equiv \dots$ We could then implement e as

```
e :: Double
e = e_ 17
```

The fact that e is related to the factorial may led to the suspicion that it also appears directly in a formula dealing with factorials. There, indeed, is a formula derived by James Stirling who we already know for the Stirling numbers. This formula approximates the value of $n!$ without the need to go through all the steps of its recursive definition. Stirling's formula is as follows:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \tag{7.27}$$

This equation is nice already because of the fact that e and π appear together to compute the result of an important function. But how precise is the approximation? To answer this question, we first implement Stirling's formula:

```
stirfac :: Integer → Integer
stirfac i = ceiling $ (sqrt (2 * pi * n)) * (n / e) ↑ i
  where n = fromIntegral i
```

Note that we *ceil* the value, instead of rounding it just to the next integer value.

Then we define a function to compute the difference $difac\ n = fac\ n - stirfac\ n$. The result for the first 15 numbers is

$$0, 0, 0, 0, 1, 9, 59, 417, 3343, 30104, 301174, 3314113, 39781324, 517289459, 7243645800.$$

For the first numbers, the difference is 0. Indeed:

$$\begin{array}{lcrcr} 1! & = & stirfac(1) & = & 1 \\ 2! & = & stirfac(2) & = & 2 \\ 3! & = & stirfac(3) & = & 6 \\ 4! & = & stirfac(4) & = & 24 \end{array}$$

Then, the functions start to disagree, for instance $5! = 120 \neq stirfac(5) = 119$. The difference grows rapidly and reaches more than 3 million with 12!. But what is the deviation in relation to the real value? We define the function $100 * (\mathit{fromIntegral}\ \$\ \mathit{difac}\ n)\ /\ (\mathit{fromIntegral}\ \$\ \mathit{fac}\ n)$ to obtain the difference in terms of a percentage of the real value. We see starting from 5 (where the first difference occurs):

$$0.8333, 1.25, 1.1706, 1.0342, 0.9212, 0.8295, 0.7545, 0.6918, 0.6388, 0.5933, 0.5539, \ldots$$

For 5, the value jumps up from 0 to 0.8333%, climbs even higher to 1.25% and then starts to descrease slowly. At 42 the deviation falls below 0.2%. At 84, it falls below 0.1% and keeps falling. Even though the difference appears big in terms of absolute numbers, the percentage quickly shrinks and, for some problems, may even be neglible.

A completely different way to approximate e is by *continued fractions*. Continued fractions are infinite fractions, where each denominator is again a fraction. For instance:

$$e = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}}} \tag{7.28}$$

A more readable representation of continued fractions is by sequences of the denominator like:

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots] \tag{7.29}$$

where the first number is separated by a semicolon to highlight the fact that it is not a denominator, but an integral number added to the fraction that follows. We can capture this very nicely in Haskell, using just a list of integers. However, in some cases we might

have a fraction with numerators other than 1. An elegant way to represent this case is by using fractions instead of integers. We would then represent $\frac{2}{a+\ldots}$ as $\frac{1}{\frac{1}{2}a+\ldots}$. Here is an implementation:

```
contfrac :: [Quoz] → Double
contfrac [] = 1
contfrac [i] = fromQuoz i
contfrac (i : is) = n + 1 / (contfrac is)
  where n = fromQuoz i

fromQuoz :: Quoz → Double
fromQuoz i = case i of
  (Pos (Q nu d)) → fromIntegral nu / fromIntegral d
  (Neg (Q nu d)) → negate (fromIntegral nu / fromIntegral d)
```

For *contfrac* $[2, 1, 2, 1, 1, 4, 1, 1, 6]$ we get 2.7183, which is not bad, but not yet too close to e. With $[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10]$ we get 2.718 281 828, which is pretty close.

Examining the sequence a bit further, we see that is has a regular structure. We can generate it by means of the *Engel expansion* named for Friedrich Engel (1861 – 1941), a German mathematician who worked close with the great Norwegian algebraist Sophus Lie (1842 – 1899). The Engel expansion can be implemented as follows:

```
engelexp :: [Integer]
engelexp = 2 : 1 : go 1
  where go n = (2 * n) : 1 : 1 : go (n + 1)
```

The following fraction, however, conoverges must faster than the Engel expansion: $[1, \frac{1}{2}, 12, 5, 28, 9, 44, 13]$. Note that we take advantage of the datatype *Quoz* to represent a numerator that is not 1. This sequence can be generated by means of

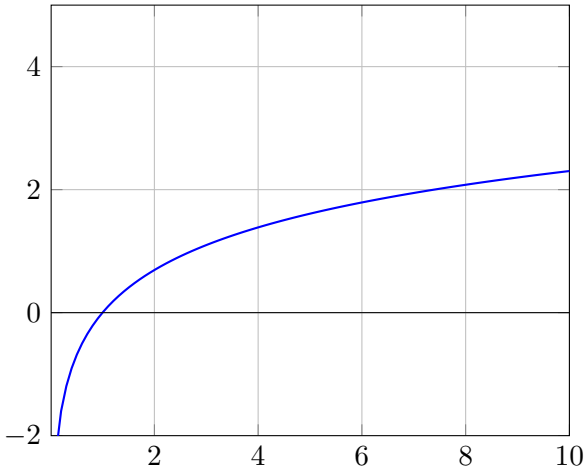
```
fastexp :: [Quoz]
fastexp = 1 : (Pos (1 % 2)) : go 1
  where go n = (16 * n − 4) : (4 * n + 1) : go (n + 1)
```

This fraction converges already after 7 steps to the value 2.718 281 828:

contfrac (*take* 7 *fastexp*).

The area of mathematics where e is really at home is analysis and its vast areas of application, which we will study in the third part of this series. The reason for the prominence of e in analysis stems from the *natural logarithm*, which we already introduced in the first chapter. The natural logarithm of a number n, usually denoted $\ln(n)$, is the exponent x, such that $e^x = n$.

The natural logarithm can be graphed as follows:



The curious fact that earned the natural logarithm its name is that, at $x = 1$, the curve has the slope 1. This might sound strange for the moment. We will investigate that later.

7.5. γ

The *harmonic series* is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \tag{7.30}$$

A harmonoic series with respect to a given number k, called the *harmonic number* H_k, then is

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \dots \frac{1}{k} \tag{7.31}$$

This is easily implemented in Haskell as

```
harmonic :: Natural → Double
harmonic n = sum [1 / d | d ← map fromIntegral [1 .. n]]
```

Some harmonic numbers are (*map harmonic* $[1 \,..\, 10]$):

$1, 1.5, 1.8\overline{3}, 2.08\overline{3}, 2.28\overline{3}, 2.44\overline{9}, 2.5928, 2.7178, 2.8289, 2.9289.$

The harmonic series is an interesting object of study in its own right. Here, however, we are interested in something else. Namely, the difference of the harmonic series and the

natural logarithm:

$$\gamma = \lim_{n\to\infty} H_n - \ln(n). \tag{7.32}$$

We can implement this equation as

```
harmonatural :: Natural → Double
harmonatural n = harmonic n − ln n
  where ln = log ∘ fromIntegral
```

Applied on the first numbers with *map harmonatural* $[1\,..\,10]$, the function does not show interesting results:

$1.0, 0.8068, 0.7347, 0.697, 0.6738, 0.6582, 0.6469, 0.6384, 0.6317, 0.6263, \ldots$

Applied to greater numbers, however, the results approach a constant value:

harmonatural $100 = 0.58220$
harmonatural $1000 = 0.57771$
harmonatural $10000 = 0.57726$
harmonatural $100000 = 0.57722$

With even greater numbers, the difference converges to 0.577 21. This number, γ, was first mentioned by – surprise – Leonhard Euler and some years later by Italian mathematician Lorenzo Mascheroni (1750 – 1800) and is therefore called the Euler-Mascheroni constant.

This mysterious number appears in different contexts and, apparently, quite often as a difference or average. An ingenious investigation was carried out by Belgian mathematician Charles Jean de la Vallée-Poussin (1866 – 1962) who is famous for his proof of the Prime number theorem. Vallée-Poussin studied the quotients of a number n and the primes up to that number. If n is not prime itself, then there are some prime numbers p, namely those of the prime factorisation of n, such that $\frac{n}{p}$ is an integer. For others, this quotient is a rational number, which falls short of the next natural number. For instance, there are four prime numbers less than 10: 2, 3, 5 and 7. The quotients are

$5, 3.\overline{3}, 2, 1.\overline{428571}$.

5 and 2, the quotients of 2 and 5, respectively, are integers and there, hence, is no difference. The quotient $\frac{10}{3} = 3.\overline{3}$, however, falls short of 4 by $0.\overline{6}$ and $\frac{10}{7} = 1.\overline{428571}$ falls short of 2 by $0.\overline{57142828}$.

Vallée-Poussin asked what the average of this difference is. For the example 10, the average is about $0.3\overline{095238}$. One might think that this average, computed for many numbers or for very big numbers, is about 0.5, so that the probability for the quotient of n and a random prime number to fall into the first or the second half of the rational numbers between two integers is equal, *i.e.* 50% for both cases. It turns out it is not. For

huge numbers, de la Vallée-Poussin's average converges to 0.577 21, the Euler-Mascheroni constant.

It converges quite slow, however. If we implement the prime quotient as

```
pquoz :: Natural → [Double]
pquoz n = [d / p | p ← ps]
  where ps = map fromIntegral (takeWhile (<n) allprimes)
        d  = fromIntegral n
```

and its average as

```
pquozavg :: Integer → Double
pquozavg n = (sum ds) / (fromIntegral $ length ds)
  where qs = pquoz n
        ns = map (fromIntegral ∘ ceiling) qs
        ds = [n − q | (n, q) ← zip ns qs]
```

we can experiment with some numbers like

pquozavg 10 =$0.3\overline{095238}$
pquozavg 100 = 0.548731
pquozavg 1000 = 0.5590468
pquozavg 10000 = 0.5666399
pquozavg 100000 = 0.5695143
...

With greater and greater numbers, this value approaches γ. Restricting n to prime numbers produces good approximations of γ much earlier. From 7 on, *pquozavg* with primes results in numbers of the form $0.5\ldots$ *pquozavg* 43 = 0.57416 is already very close to γ. It may be mentioned that 43 is suspiciously close to 42.

With de la Vallée-Pussin's result in mind, it is not too surprising that γ is related to divisors and Euler's totient number. A result of Gauss' immediate successor in Göttingen, Peter Gustav Lejeune-Dirichlet (1805 – 1859), is related to the average number of divisors of the numbers $1\ldots n$. We have already defined a function to generate the divisors of a number n, namely *divs*. Now we map this function on all numbers up to n:

```
divsupn :: Natural → [[Natural]]
divsupn n = map divs [1 .. n]
```

Applied to 10, this function yields:

$$[[1],[1,2],[1,3],[1,2,4],[1,5],[1,2,3,6],[1,7],[1,2,4,8],[1,3,9],[1,2,5,10]]$$

For modelling Lejeune-Dirichlet's result, we further need to count the numbers of divisors of each number:

```
ndivs :: Integer → [Int]
ndivs = map length ∘ divsupn
```

Applied again to 10, *ndivs* produces:

$[1, 2, 2, 3, 2, 4, 2, 4, 3, 4]$

Now we compute the average of this list using

```
dirichlet :: Integer → Double
dirichlet n   = s / l
  where ds = ndivs n
        l  = fromIntegral $ length ds
        s  = fromIntegral $ sum ds
```

For *dirichlet* 10 we see 2.7. This does not appear too spectacular. Greater numbers show:

dirichlet $100 = 4.759$
dirichlet $250 = 5.684$
dirichlet $500 = 6.38$
dirichelt $1000 = 7.069$

As we can see, the number is slowly increasing resembling a log function or, more specifically, the natural log. When we compare the natural log, we indeed see that the results are close:

$\ln 100 = 4.605$
$\ln 250 = 5.521$
$\ln 500 = 6.214$
$\ln 1000 = 6.907$

For greater and greater numbers, the difference of the *dirichlet* function and the natural logarithm approaches

$$0.154435 \approx 2\gamma - 1. \tag{7.33}$$

For the five examples above, the difference is still significantly away from that number:

$\Delta 100 = 0.2148$
$\Delta 250 = 0.1625$
$\Delta 500 = 0.1635$
$\Delta 1000 = 0.1612,$

but already $\Delta 2000 = 0.158$ comes close and $\Delta 4000 = 0.1572$ approaches the value even further.

An important constant derived from γ is e^{γ}, which is a limit often seen in number theory. One instance is the lower bound of the totient function. There is a clear upper bound, namely $n - 1$. Indeed, $\varphi(n)$ can never yield a value greater $n - 1$ and this upper bound is reached exclusively by prime numbers. There is no such linear lower bound. That is,

$\varphi(n)$ can assume values that are much smaller than the value seen for $n-1$ or other numbers less than n. But there is a lower bound that slowly grows with n. This lower bound is often given as $\frac{n}{\ln\ln n}$. This lower bound, however, is too big. There are some values that are still below that border. $\varphi(40)$, for instance, is 16. $\frac{40}{\ln\ln 40}$, however, is around 30. A better, even still not perfect approximation, is

$$\frac{n}{e^\gamma \ln\ln n}.$$

For the $n = 40$ again, $\frac{40}{e^\gamma \ln\ln 40}$ is around 17 and, hence, very close to the real value.

We see that γ is really a quite mysterious number that appears in different contexts, sometimes in quite a subtle manner. The greatest mystery, however, is that it is not so clear that this number belongs here in the first place. Indeed, it has not yet been shown that γ is irrational. In the approximations, we have studied in this section, we actually have not seen the typical techniques to create irrational numbers like roots, continuous fractions and infinite series. If γ is indeed rational, then it must be the fraction of two really large numbers. In 2003, it has been shown that the denominator of such a fraction must be greater than 10^{242080}. A number big enough, for my taste, to speak of *irrational*.

7.6. Representation of Real Numbers

The set of real numbers $\mathbb{R}$ is the union of the rational and the irrational numbers. When we write real numbers on paper, we use the decimal notation. A number in decimal notation corresponds to an ordinary integer terminated by a dot called the decimal point; this integer corresponds to the part of the real number greater 1 or 0 of course. After the dot a stream of digits follows, which is not necessarily a number in the common sense, since it may start with zeros, *e.g.* 0.0001. In fact, one could say that the part after the dot corresponds to a a reversed integer, since the zero following this number have no impact on the value of the whole expression, *i.e.* $0.10 = 0.1$.

Any rational number can be expressed in this system. An integer corresponds just to the part before the dot: $1.0 = 1$. A fraction like $\frac{1}{2}$ is written as 0.5. We will later look at how this is computed concretely. Rationals in decimal notation can be easily identified: all numbers in decimal notation with a finite part after the dot are rational: 0.25 is $\frac{1}{4}$, 0.75 is $\frac{3}{4}$, 0.2 is $\frac{1}{5}$ and so on.

There are some rational numbers that are infinite. For example, $\frac{1}{3}$ is $0.333333\ldots$, which we encode as $0.\overline{3}$. Such periodic decimals are easy to convert to fractions. We just have to multiply them by a power of 10, such that there is a part greater 0 before the decimal point and that the first number of the the repeating period is aligned to it. For $0.\overline{3}$, this is just $10 \times 0.\overline{3} = 3.\overline{3}$. For $0.1\overline{6}$, it would be $10 \times 0.1\overline{6} = 1.\overline{6}$. For $0.\overline{09}$, it would be $10^2 \times 0.\overline{09} = 9.\overline{09}$. We then subtract the original number from the result. If the original

number is x, we now have $10^n x - x = (10^n - 1)x$. For $x = 0.\overline{3}$ this is $9x$; for $x = 0.1\overline{6}$ this, too, is $9x$ and for $x = 0.\overline{09}$ this is $99x$. The results are 3, 1.5 and 9 respectively. We now build a fraction of this result as numerator and the factor (9 or 99) in the denominator. Hence, $0.\overline{3} = \frac{3}{9} = \frac{1}{3}$, $0.1\overline{6} = \frac{1.5}{9} = \frac{3}{18} = \frac{1}{6}$ and $0.\overline{09} = \frac{9}{99} = \frac{1}{11}$.

A curiosity resulting from this calculations is that $9 \times 1/3 = 3$ and $9 \times 0.\overline{3} = 2.\overline{9}$ are actually the same! A correct implementation must take this into account. Try it with Haskell, you will see that $9 * 0.3333333333333333$ is indeed 3. It will not work if you forget a three. The precision of the *Double* number type is 16 digits after the decimal point. With only 15 threes, the result will be 2.9999999999999997.

Irrational numbers in the decimal notation have infinite many digits after the decimal point. With this said, it is obvious that we cannot represent irrational numbers in this system. We can of course represent any number by some kind of formula like $\sqrt{5}$ and do some math in this way such as $\frac{1+\sqrt{5}}{2}$, *etc.* But often, when we are dealing with applied mathematics, such formulas are not very useful. We need an explicit number. But, unfortunately or not, we have only limited resources in paper, brainpower and time. That is, at some point we have to abandon the calculations and work with what can be achieved with the limited resources we have at our disposal.

The point in time at which we take the decision that we now have calculated enough is the measure for the precision of the real number type in question. On paper, we would hence say that we write only a limited number of digits after the decimal point. In most day-to-day situations where real numbers play a role, like in dealing with money, cooking, medication or travelling distances, we calculate up to one or two decimal places. Prices, for instances are often given as 4.99 or something, but hardly 4.998. Recipes would tell that we need 2.5 pounds or whatever of something, using one decimal place. One would say that it is about 1.5km to somewhere, but hardly that it is 1.49km. In other areas, especially in science much more precision is needed. We therefore need a flexible datatype.

A nice and clean format to represent real numbers uses two integers or, as the following definition, two natural numbers:

data *RealN* = *R Natural Natural*

The first number represents the integral part. You will remember that a number is a list of digits where every digit is multiplied by a power of ten according to the place of the digit. The digit, counted from the right, is multiplied by $10^0 = 1$. The second is multiplied by 10^1, the third by 10^2 and so on. The digit multiplied by 10^0, is the last digit before the decimal point. If we wanted to push it to the right of the decimal point, we would need to reduce the exponent. So, we multiplied it not by 10^0, but by 10^{-1} to push it to the first decimal place. This is the function of the second number in the datatype above. It represents the value of the least significant bit in terms of the exponent to which we have to raise 10 to obtain the number represented by this datatype. Since our datatype uses a natural number, we have to negate it to find the

exponent we need.

For instance, the number $R\ 25\ 2$ corresponds to 25×10^{-2}, which we can reduce stepwise to 2.5×10^{-1} and 0.25×10^{0}. A meaningful way to show this datatype would therefore be:

```
instance Show RealN where
  show (R a e) = show a ++ "*10^(-" ++ show e ++ ")"
```

There are obviously many ways to represent the same number with this number type. 1, for instance, can be represented as

```
one = R 1 0
one = R 10 1
one = R 100 2
one = R 1000 3
...
```

To keap numbers as concise as possible, we define a function to simplify numbers with redundant zeros:

```
simplify :: RealN → RealN
simplify (R 0 _) = R 0 0
simplify (R a e) | e > 0 ∧
  a `rem` 10 ≡ 0 = simplify (R (a `div` 10) (e − 1))
  | otherwise = R a e
```

As long as the exponent is greater 0 and the base a is divisible by 10, we reduce the exponent by one and divide a by 10. In other words, we remove unnecessary zeros. The following constructor uses *simplify* to create clean real numbers:

```
real :: Natural → Natural → RealN
real i e = simplify (R i e)
```

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We now define how to check two real numbers for equality:

```
instance Eq RealN where
  r1@(R a e1) ≡ r2@(R b e2) | e1 ≡ e2 = a ≡ b
                            | e1 > e2 = r1 ≡ blowup e1 r2
                            | e1 < e2 = blowup e2 r1 ≡ r2
```

If the exponents are equal, then we trivially compare the coefficients. Otherwise, we first expand the number with the smaller exponent using *blowup*:

```
blowup :: Natural → RealN → RealN
blowup i (R r e) | i ⩽ e       = R r e
                 | otherwise = R (r * 10↑(i − e)) i
```

That is simple! If the target i is greater than the current exponent of the number, we just multiply the coefficient by 10 raised to the difference of the target exponent and the current exponent and make the target the new exponent. Otherwise, nothing changes.

We continue with comparison, which follows exactly the same logic:

```
instance Ord RealN where
  compare r1@(R a e1) r2@(R b e2) | e1 ≡ e2 = compare a b
                                  | e1 > e2 = compare r1 (blowup e1 r2)
                                  | e1 < e2 = compare (blowup e2 r1) r2
```

Now we make *RealN* instance of *Num*:

```
instance Num RealN where
  (R a e1) + (R b e2) | e1 ≡ e2   = simplify $ R (a + b) e1
                      | e1 > e2   = simplify $ R (a + b * 10↑(e1 − e2)) e1
                      | otherwise = simplify $ R (a * 10↑(e2 − e1) + b) e2
  (R a e1) − (R b e2) | e1 ≡ e2 ∧
                        a ⩾ b     = simplify $ R (a − b) e1
                      | e1 ≡ e2   = error "subtraction beyond zero!"
                      | e1 > e2   = simplify $ (R a e1) − (R (b * 10↑(e1 − e2)) e1)
                      | otherwise = simplify $ (R (a * 10↑(e2 − e1)) e2) − (R b e2)
  (R a e1) * (R b e2) = real (a * b) (e1 + e2)
  negate r      = r  -- we cannot negate natural numbers
  abs    r      = r
  signum r      = r
  fromInteger i = R (fromIntegral i) 0
```

Addition is again the same logic. For two numbers with equal exponents, we just add the coefficients. If the exponents differ, we first convert the smaller number to the greater exponent.

For subtraction, note that we define *RealN* like numbers before without negatives. To consider signedness, we still have to use the datatype *Signed RealN*. Consequently, we have to rule out the case where the first number is smaller than the second one.

Multiplication is interesting. We multiply two real numbers by multiplying the coefficients and adding the exponents. We have already seen this logic, when defining the natural number type. Some simple examples may convince you that this is the right way to go. 1×0.1, for instance, is 0.1. In terms of our *RealN* type, this corresponds to $(R\ 1\ 0) * (R\ 1\ 1) \equiv (R\ (1 * 1)\ (0 + 1))$.

The next task is to make *RealN* instance of *Fractional*:

```
instance Fractional RealN where
  (/) = rdiv 17
  fromRational r = (R (fromIntegral $ numerator r) 0) /
                   (R (fromIntegral $ denominator r) 0)
```

The method *fromRational* is quite simple. We just create two real numbers, the numerator of the original fraction and its denominator, and the we divide them. What we need to do this, of course, is division. Division, as usual, is a bit more complicated than the other arithmetic operations. We define it as follows:

```
rdiv :: Natural → RealN → RealN → RealN
rdiv n r1@(R a e1) r2@(R b e2) | e1 < e2 =
                                 rdiv n (blowup e2 r1) r2
                               | a < b ∧ e1 ≡ e2 =
                                 rdiv n (blowup (e2 + 1) r1) r2
                               | otherwise =
                                 simplify (R (go n a b) (e1 − e2 + n))
  where go i x y | i ⩽ 0 = 0
                 | otherwise =
                   case x `quotRem` y of
                     (q, 0) → 10 ↑ i ∗ q
                     (q, r) → let (r′, e)  = borrow r y
                                  q′       = 10 ↑ i ∗ q
                              in if e > i then q′
                                          else q′ + go (i − e) r′ y
```

rdiv has one more argument than the arithmetic operations seen before. This additional argument, n, defines the precision of the result. This is necessary, because, as we will see, the number of iterations the function has to perform depends on the precision the result is expected to have.

If the first number is smaller than the second, either because its exponent or its coefficient is smaller, we blow it up so that it is at least the same size. Then we calculate the new coefficient by means of *go* and the new exponent as the difference of the first and the second exponent plus the expected precision. Note that division has the inverse effect on the size of the exponents as multiplication. When we look again at the example 1 and 0.1, we have $1/0.1 = 10$, which translates to $(R\ 1\ 0)\ /\ (R\ 1\ 1) = R\ (1\ /\ 1)\ (0 - 1)$, which of course is the same as $R\ 10\ 0$.

The inner function *go* proceeds until i, which initially is n, becomes 0 or smaller. In each step, we divide x, initially the coefficient of the first number, and y, the coefficient of the second number. If the result leaves no remainder, we are done. We just raise q to the power of the step in question. Otherwise, we continue dividing the remainder r by y. But before we continue, we borrow from y, that is, we increase r until it is at least y. i is then decremented by the number of zeros we borrowed this way. If we run out of i, so to speak, that is if $e > i$, then we terminate with q raised to the current step.

borrow is just the same as *blowup* applied to two natural numbers:

```
borrow :: Natural → Natural → (Natural, Natural)
borrow a b | a ⩾ b      = (a, 0)
           | otherwise = let (x, e) = borrow (10 * a) b in (x, e + 1)
```

We now make *RealN* instance of *Real*. We need to define just one method, namely how to convert *RealN* to *Rational*, which we do just by creating a fraction with the coefficient of the real number in the numerator and 10 raised to the exponent in the denominator. The rest is done by the *Rational* number type:

```
instance Real RealN where
  toRational (R r e) = i % (10 ↑ x)
    where i = fromIntegral r :: Integer
          x = fromIntegral e :: Integer
```

We further add a function to convert real numbers to our *Ratio* type:

```
r2R :: RealN → Ratio
r2R (R a e) = ratio a (10 ↑ e)
```

We also add a function to convert our real number type to the standard *Double* type:

```
r2d :: RealN → Double
r2d r@(R a e) | e > 16      = r2d (roundr 16 r)
              | otherwise = (fromIntegral a) / 10 ↑ e
```

An inconvenience is that we have to round a number given in our number type so it fits into a *Double*. For this, we assume that the *Double* has a precision of 16 decimal digits. This is not quite true. The *Double* type has room for 16 digits. But if the first digits after the decimal point are zeros, the *Double* type will present this as a number raised to a negative exponent, just as we do with our real type. In this case, the *Double* type may have a much higher precision than 16. For our purpose, however, this is not too relevant.

So, here is how we round:

```
roundr :: Natural → RealN → RealN
roundr n (R a e) | n ⩾ e      = R a e
                 | otherwise = let b = a `div` 10
                                   l = a − 10 * b
                                   d | l < 5 = 0
                                     | otherwise = 1
                               in roundr n (R (b + d) (e − 1))
```

That is, we first get the least significant digit l as $l = a - 10 * (div\ a\ 10)$, where, if *div* is the Euclidian division, the last digit of a remains. If the last digit is less than 5, we just drop it off. Otherwise, we add 1 to the whole number. If we now define

$one = R\ 1\ 0$
$three = R\ 3\ 0$
$third = one\ /\ three$,

then *roundr* 16 (*three* $*$ *third*) yields 1, as desired.

Finally, we define

type *SReal* = *Signed RealN*

and have a fulfledged real number datatype.

7.8. The Stern-Brocot Tree

7.9. Real Factorials

7.10. Real Binomial Coefficients

7.11. The Continuum

7.12. Review of the Number Zoo

Part II.

Algebra and Geometry

8. Polynomials

8.1. Numeral Systems

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8.3. Factoring Polynomials

8.4. Polynomials and Binomial Coefficients

8.5. The Method of partial Fractions

8.6. The Difference Engine

8.7. Generationfunctionology 1

8.8. The closed Form of the Fibonacci Sequence

$$G(x) = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \dots \tag{8.1}$$

$$G(x) = 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \dots \tag{8.2}$$

$$xG(x) = F_0 x + F_1 x^2 + F_2 x^3 + F_3 x^4 + \dots \tag{8.3}$$

$$x^2 G(x) = F_0 x^2 + F_1 x^3 + F_2 x^4 + F_3 x^5 + \dots \tag{8.4}$$

$$G(x) - xG(x) - x^2 G(x) = (1 - x - x^2)G(x). \tag{8.5}$$

$$\begin{array}{lllllll}(1-x-x^2)G(x) = & (F_0 & +F_1x & +F_2x^2 & +F_3x^3 & +\dots) & - \\ & (& F_0x & +F_1x^2 & +F_2x^3 & +\dots) & - \\ & (& & +F_0x^2 & +F_1x^3 & +\dots) & \end{array}$$

$$\begin{aligned}(1-x-x^2)G(x) = \quad & F_0 + (F_1 - F_0)x \\ & + (F_2 - F_1 - F_0)x^2 \\ & + (F_3 - F_2 - F_1)x^3 \\ & + \dots\end{aligned}$$

$$(1-x-x^2)G(x) = x. \tag{8.6}$$

$$G(x) = \frac{x}{1-x-x^2}. \tag{8.7}$$

9. Relations, Functions and the Cartesian Plane

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