CHAPTER 1

Number of quantifier alternations in Huang and nested

1 Preliminaries

For $\sigma = \mathrm{mgu}(\varphi, \psi)$ for two terms or literals φ and ψ, we denote by σ_i for $1 \leq i \leq n$ the ith substitution which is added to σ by the unification algorithm, where $n = |\operatorname{dom}(\sigma)|$. We define $\sigma_0 \overset{\text{def}}{=} \mathrm{id}$.

We furthermore denote the composition of all σ_k for $i \leq k \leq j$ by $\sigma_{(i,\,j)}$. Hence $\sigma = \sigma_{(1,\,n)} = \sigma_{(0,\,n)}$.

A literal l is called a Φ-literal if its predicate symbol is Φ-colored.

NB: The notion of single-colored is considered to be deprecated here.

In a literal or term ϕ containing a subterm t, t is said to occur *below* a Φ-symbol s if in the syntax tree representation of ϕ, there is a node labelled s on the path from the root to t. Note that the colored symbol may also be the predicate symbol. Moreover, t is said to occur *directly below* a Φ-symbol if it occurs below the Φ-symbol s and in the syntax tree representation of ϕ on the path from s to t, no nodes with labels with colored symbol occur.

Quantifier alternations in I usually assumes the quantifier-alternation-minimizing arrangement of quantifiers in I. The lemma statements hence talk about the *minimal* number of quantifier alternations, which is indeed easily obtainable, i.e. it's a lower and upper bound at the same time.

In the following, we assume that the maximum max of an empty sequence is defined to be 0 and constants are treated as function symbols of arity 0. Furthermore $\perp$ is used to denote a color which is not possessed by any symbol.

Definition 1 (Color alternation col-alt)**.** Let Γ and Δ be sets of formulas and t be a term.

$\text{col-alt}(t) \overset{\text{def}}{=} \text{col-alt}_{\perp}(t)$

$$\text{col-alt}_\Phi(t) \stackrel{\text{def}}{=} \begin{cases} 0 & t \text{ is a variable} \\ \max(\text{col-alt}_\Phi(t_1), \ldots, \text{col-alt}_\Phi(t_n)) & t = f(t_1, \ldots, t_n) \text{ is grey} \\ \max(\text{col-alt}_\Phi(t_1), \ldots, \text{col-alt}_\Phi(t_n)) & t = f(t_1, \ldots, t_n) \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_\Psi(t_1), \ldots, \text{col-alt}_\Psi(t_n)) & t = f(t_1, \ldots, t_n) \text{ is of color } \Psi, \\ & \Phi \neq \Psi \end{cases}$$

△

Definition 2 (Quantifier alternation quant-alt)**.** Let A be a formula.

$\text{quant-alt}(A) \stackrel{\text{def}}{=} \text{quant-alt}_\perp(A)$

$$\text{quant-alt}_Q(A) \stackrel{\text{def}}{=} \begin{cases} 0 & A \text{ is an atom} \\ \text{quant-alt}_Q(B) & A \equiv \neg B \\ \max(\text{quant-alt}_Q(B), & A \equiv B \circ C, \circ \in \{\wedge, \vee, \supset\} \\ \quad \text{quant-alt}_Q(C)) & \\ \text{quant-alt}_Q(B) & A \equiv Q'B, Q = Q' \\ 1 + \text{quant-alt}_{Q'}(B) & A \equiv Q'B, Q \neq Q' \end{cases}$$

△

Note that this definition of quantifier alternations handles formulas in prenex and non-prenex form.

Definition 3. We define $\text{PI}^\circ_{\text{step}}$ to coincide with PI_{step} but without applying the substitution σ in each of the cases. Analogously, if $C \equiv D\sigma$, we use C° to denote D. △

Hence $\text{PI}^\circ_{\text{step}}(\cdot)\sigma = \text{PI}_{\text{step}}(\cdot)$.

2 Occurrence of terms in the interpolant

Definition 4 (PI^*)**.** PI^* is defined as PI with the difference that in PI^*, all literals are considered to be grey. △

Hence $\text{PI}^*_{\text{init}}$ coincides with PI_{init}. $\text{PI}^*_{\text{step}}$ coincides with PI_{step} in case of factorisation and paramodulation inferences. For resolution inferences, the first two cases in the definition of PI_{step} do not occur for $\text{PI}^*_{\text{step}}$.

PI^* enjoys the convenient property that it absorbs every literal which occurs some clause:

⟨prop:every_lit_in_pi_star⟩ **Proposition 5.** *, For every literal which occurs in a clause of a resolution refutation π, a respective successor occurs in* $\text{PI}^*(\pi)$.

Proof. By structural induction. □

a:grey_lits_of_pi_star_in_pi⟩? **Lemma 6.** *For every clause C of a resolution refutation, every grey literal, which occurs in* $\text{PI}^*(C)$*, also occurs in* $\text{PI}(C)$.

Proof. Note that PI_{init} and $\text{PI}^*_{\text{init}}$ coincide and PI_{step} and $\text{PI}^*_{\text{step}}$ only differ for resolution inferences. But more specifically, they only differ on resolution inferences, where the resolved literal is colored. However here, no grey literals are lost. □

Note that in PI*, we can conveniently reason about the occurrence of terms as no terms are lost throughout the extraction. However Lemma **??** allows us to transfer results about grey literals to PI. We can also give similar results about general literals and equalities occurring in the resolution refutation:

?⟨lemma:grey_lits_all_in_PI⟩? **Lemma 7.** *If there is a grey literal λ in a clause C of a resolution refutation π, then a successor of λ occurs in* $\mathrm{PI}(\pi)$.

Proof. Immediate by the definition of PI. □

?⟨lemma:equalities_all_in_PI⟩? **Lemma 8.** *For every equality $s = t$ of a clause in a resolution refutation π, a successor of $s = t$ occurs in* $\mathrm{PI}(\pi)$.

Proof. Equalities in clauses are only removed by means of paramodulation and as π derives the empty clause, all equalities are removed eventually. For any paramodulation inference ι using the equality $s = t$, $\mathrm{PI}_{\mathrm{step}}(\iota, I_1, I_2)$ contains $s = t$. □

We now make some considerations in the form of four lemmata about the construction of terms of certain shapes in the context of interpolant extraction. In the following, we abbreviate $\mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_n)) \vee C^\circ$ by χ.

?⟨lemma:var_below_phi_symbol⟩? **Lemma 9.** *Let ι be a resolution or factorisation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs directly below a Φ-symbol in $\chi\sigma_{(0,\,i)}$ for $i \geq 1$. Then at least one of the following statements holds:*

⟨14_1⟩ 1. *The variable u occurs directly below a Φ-symbol in $\chi\sigma_{(0,\,i-1)}$.*

?⟨14_4⟩? 2. *The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨14_2⟩ 3. *There is a variable v such that*
   - *u occurs grey in $v\sigma_i$ and*
   - *v occurs in $\chi\sigma_{(0,\,i-1)}$ directly below a Φ-symbol as well as directly below a Ψ-symbol*

Proof. We consider the different situations under which the situation in question is introduced by means of unification:

- There is already a literal in $\chi\sigma_{(0,\,i-1)}$ where u occurs directly below a Φ-symbol and σ_i does not change this. Then clearly 1 is the case.
- There is a variable v in $\chi\sigma_{(0,\,i-1)}$ such that $v\sigma_i$ contains u directly below a Φ-symbol. As then v is unified with the term $v\sigma_i$, $v\sigma_i$ must occur in $\chi\sigma_{(0,\,i-1)}$, which implies that 1 is the case.
- There is a variable v which occurs directly below a Φ-symbol such that u occurs grey in $v\sigma_i$.

  Hence in the resolved or factorised literals λ and λ', there is a position p such that without loss of generality $\lambda|_p = v$ and u occurs grey in $\lambda'|_p$. Note that due to the definition of the unification algorithm, λ and λ' must coincide on the path to p.

  By Proposition 5, λ and λ' occur in χ irrespective of their coloring.

TODO: it must be λ and λ' with the appropriate amount of σ steps applied

We distinguish cases based on the position p:

- Suppose that p occurs directly below a Φ-symbol. Then as u occurs grey in $\lambda'|_p$, u occurs directly below a Φ-symbol in $\chi\sigma_{(0,\,i-1)}$ and 1 is the case.
- Suppose that p occurs directly below a Ψ-symbol. Then v occurs directly below a Ψ-symbol in $\lambda|_p$ and clearly 3 is the case.
- TODO: grey

$\square$