

Remark ()*. Any substitution, in particular σ , only changes a finite number of variables. Furthermore a result of a run of the unification algorithm is acyclic in the sense that if a substitution $u \mapsto t$ is added to the resulting substitution, it is never the case that at a later stage $t \mapsto u$ is added. This can easily be seen by considering that at the point when $u \mapsto t$ is added to the resulting substitution, every occurrence of u is replaced by t , so u is not encountered by the algorithm at a later stage.

Therefore in order to show that a statement holds for every $u \mapsto t$ in a unifier σ , it suffices to show by an induction argument that for every substitution $v \mapsto s$ which is added to the resulting unifier by the unification algorithm that it holds for $v \mapsto s$ under the assumption that it holds for every $w \mapsto r$ such that w occurs in s and $w \mapsto r$ is added to the resulting substitution at a later stage. \triangle

Conjecture 1. *Let C be a clause in a resolution refutation. Suppose that $\text{AI}^\Delta(C)$ contains a maximal Γ -term $\gamma_j[z_i]$ which contains a lifting variable z_i . Then $z_i <_{\hat{\mathcal{A}}(C)} z_j$.*

Proof. We proceed by induction. For the base case, note that no multicolored terms occur in initial clauses, so no lifting term can occur inside of a Γ -term.

Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$. Furthermore suppose that for every lifting term inside a Γ -term in the clauses C_1 and C_2 of the refutation, for every term of the form $\gamma_j[z_i]$ we have that $z_i <_{\hat{\mathcal{A}}(C_1)} z_j$ or $z_i <_{\hat{\mathcal{A}}(C_2)} z_j$ respectively. Hence there is an arrow (p_1, p_2) in $\hat{\mathcal{A}}(C_1)$ or $\hat{\mathcal{A}}(C_2)$ such that z_i is contained in $P(p_1)$ and z_j is contained in $P(p_2)$. In $\text{AI}^\Delta(C)$, $P(p_1)$ contains $\ell[z_i\sigma]\tau = z_i\tau$ and $P(p_2)$ contains $\ell[z_j\sigma]\tau = z_j\tau$. Hence the indices of the lifting variables might change, but this renaming does not affect the relation of the objects as $\hat{\mathcal{A}}(C_1) \cup \hat{\mathcal{A}}(C_2) \subseteq \hat{\mathcal{A}}(C)$.

We show that $z_i <_{\hat{\mathcal{A}}(C)} z_j$ holds true also for every new term of the form $\gamma_j[z_i]$ for some j, i in $\text{AI}^\Delta(C)$. By “new”, we mean terms which are not present in $\text{AI}^\Delta(C_1)$ or $\text{AI}^\Delta(C_2)$. Note that new terms in $\text{AI}^\Delta(C)$ are of the form $\ell_{\Delta, x}[t\sigma]\tau$ for some $t \in \text{AI}^\Delta(C_1) \cup \text{AI}^\Delta(C_2)$. By Lemma ??, σ does not introduce lifting variables. Hence a new term of the form $\gamma_j[z_i]$ is created either by introducing a Δ -term into a Γ -term or by introducing $\gamma_j[\delta_i]$ via σ , both followed by the lifting. Note that τ only substitutes lifting variables by other lifting variables and hence does not introduce lifting variables. Furthermore by Lemma ??, τ only substitutes lifting variables for other lifting variables, whose corresponding term is more specialised. Hence if there exists an arrow from a lifting variable to $\gamma_j[z_i]$ according to this lemma, it is also an appropriate arrow if $\gamma_j[z_i]$ is replaced by $\gamma_j[z_i]\tau$.

We now distinguish the two cases under which a new term $\gamma_j[z_i]$ can occur in $\text{AI}^\Delta(C)$:

Suppose for some Γ -term $\tilde{\gamma}_{j'}[u]$ in $\text{AI}^\Delta(C_1)$ or $\text{AI}^\Delta(C_2)$, $u\sigma$ contains a Δ -term.

Hence we have that $(\tilde{\gamma}_{j'}[u])\sigma = \gamma_j[\delta_i]$ for some i . Note that the position of u in $\tilde{\gamma}_{j'}[u]$ does not necessarily coincide with the position of δ_i in $\gamma_j[\delta_i]$ as u might be substituted by σ for a grey term containing δ_i .

We have that $\ell_\Delta[\tilde{\gamma}_{j'}[u]\sigma]\tau = \gamma_j[z_i]$.

At some well-defined point of application of the unification algorithm, u is substituted by an abstraction of a term which contains δ_i . This occurrence of u is in l and we denote it by \hat{u} . We furthermore denote the term at the corresponding position in l' by $t_{\hat{u}}$.

We distinguish cases based on the occurrences of \hat{u} and $t_{\hat{u}}$.

- Suppose \hat{u} is a grey occurrence.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\hat{u}) \quad C_2 : \neg Q(t_{\hat{u}})}{C : P(\gamma_j[\delta_i])}$$

Figure 1: Example for this case

Then by Lemma ??, there is an arrow from \hat{u} to $\gamma_j[u]$ in $\hat{\mathcal{A}}(C)$. As $\hat{u}\sigma$ is a term containing the Δ -term δ_i , the term at the position of \hat{u} in $\text{AI}^\Delta(C)$ is $\ell[\hat{u}\sigma]\tau$, which by assumption contains z_i . But there is an arrow from this term containing z_i to $\gamma_j[z_i]$, so $z_i <_{\hat{\mathcal{A}}(C)} z_j$.

- Suppose \hat{u} occurs in a maximal colored term which is a Γ -term.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\gamma_k[\hat{u}]_p) \quad C_2 : \neg Q(\gamma_m[t_{\hat{u}}]_p)}{C : P(\gamma_j[\delta_i])}$$

$$\frac{C_1 : Q(\tilde{\gamma}_{j'}[\hat{u}]) \quad C_2 : \neg Q(\gamma_m[t_{\hat{u}}])}{C : \square}$$

// $\gamma_j[\delta_i]$ occurs in the interpolant

Figure 2: Examples for this case

Then either \hat{u} is the occurrence of u in $\tilde{\gamma}_{j'}[\hat{u}]$ or it occurs in a different Γ -term $\gamma_j[\hat{u}]$. In the latter case, by Lemma ??, there is a merge edge between $\tilde{\gamma}_{j'}[\hat{u}]$ and $\gamma_j[\hat{u}]$. Hence in both cases, it suffices to show that there is an arrow from a term containing an occurrence of z_i to $t_{\hat{u}}$.

We distinguish on the shape of $t_{\hat{u}}$:

- $t_{\hat{u}}$ is a variable or a grey term. If $t_{\hat{u}}$ is a grey term, then it contains a variable that is substituted by σ by a term which contains a Δ -term as $u\sigma = t_{\hat{u}}\sigma$ is a term containing a Δ -term. However this is true for the case where $t_{\hat{u}}$ is a variable. We denote by v either $t_{\hat{u}}$ if it is a variable, or the variable in $t_{\hat{u}}$ which is substituted by a term containing a Δ -term in case $t_{\hat{u}}$ is a grey term.

In the course of the unification algorithm, there are further unifications of v since we know that $u\sigma = v\sigma$ is a term containing a Δ -term. Therefore by Remark (*), we can assume that there is an appropriate arrow to $t_{\hat{u}}$.

- $t_{\hat{u}}$ is a Δ -term. As $t_{\hat{u}}$ occurs in a Γ -term in C_1 , say in $\gamma_m[t_{\hat{u}}]$, C_1 contains a multicolored Γ -term. Hence the corresponding term in $\text{AI}^\Delta(C_1)$, is of the form $\gamma_m[z_{i'}]$ for some i' . Observe that i' in general is not equal to i as demonstrated in Example ??, even though we have that $t_{\hat{u}}\sigma = u\sigma$. This is because the lifting variables in $\text{AI}(\cdot)$ represent abstractions of the terms in the clauses of the resolution derivation (cf. Lemma ??). Therefore we only know by the induction hypothesis that $z_{i'} <_{\hat{\mathcal{A}}(C_1)} \ell[\gamma_m[z_{i'}]] = \ell[t_{\hat{u}}]$.

However by Lemma ?? and due to the fact that \hat{u} and $t_{\hat{u}}$ respectively occur in the resolved literal, $\ell_\Delta[\hat{u}\sigma]\tau = \ell_\Delta[t_{\hat{u}}\sigma]\tau$. As $\ell_\Delta[\hat{u}\sigma]\tau = \ell_\Delta[\delta_i]\tau = z_i\tau$ as well as $\ell_\Delta[t_{\hat{u}}\sigma]\tau = \ell_\Delta[z_{i'}\sigma]\tau = z_{i'}\tau$, we must have that $z_i\tau = z_{i'}\tau$. As however $u\sigma = \delta_i$, by the definition of au , we have that $\{z_i \mapsto z_{i'}\} \in \tau$, so $z_{i'}\tau = z_i$.

Since τ is applied to every literal in $\text{AI}^\Delta(C)$ and in $\text{AI}^\Delta(C_1)$ an arrow from a term containing $z_{i'}$ to $t_{\hat{u}}$ exists, the same arrow applied to $\text{AI}^\Delta(C)$ points from a term containing $z_{i'}\tau = z_i$ to $t_{\hat{u}}$. Therefore $z_i <_{\hat{\mathcal{A}}(C)} z_j$

- Suppose \hat{u} occurs in a maximal colored term which is a Δ -term.

TODO:

The substitution can also introduce a grey term containing a delta term, make sure to handle that!

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\Rightarrow what if it is overbound?

Suppose for some variable v in $\text{AI}^\Delta(C_1)$ or $\text{AI}^\Delta(C_2)$, $v\sigma = \gamma_j[\delta_i]$ for some i .

As v is affected by the unifier, it occurs in the literal being unified, say w.l.o.g. in

l in C_1 . At some well-defined point in the unification algorithm, v is substituted by an abstraction of $\gamma_j[\delta_i]$. Let p be the position of the occurrence of v in l which causes this substitution. Furthermore, let p' be the position corresponding to p in l' .

Note that any arrow from or to p' also applies to p in $\hat{\mathcal{A}}(C)$ and hence to $\gamma_j[z_i]$ as they are merged due to occurring in the resolved literal. So it suffices to show that there is an arrow from an appropriate lifting variable to p' . We denote the term at p' by t .

Note that $t\sigma = \gamma_j[\delta_i]$. So t is either a Γ -term containing a Δ -term, in which case we know that there is an appropriate arrow by the induction hypothesis as t occurs in l' in C_2 , or t is an abstraction of $\gamma_j[\delta_i]$, in which case we can assume the existence of an appropriate arrow by Remark (*). □

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graphs
w.r.t. $<_{\hat{\mathcal{A}}(C)}$