Contents

0.1 Lemmas from other pdf

ma:lifting_order_not_relevant⟩ **Lemma 1.** $\ell_\Gamma[\ell_\Delta[\varphi]] = \ell_\Delta[\ell_\Gamma[\varphi]]$.

Chapter 1

Interpolant extraction from resolution proofs in one phase (subterm relation)

1.1 Incremental lifting and substitutions of lifting variables

Definition 2 (Substitution $\tau(\iota)$)**.** For an inference ι with $\sigma = \mathrm{mgu}(\iota)$, we define the infinite substitution $\tau(\iota)$ with $\mathrm{dom}(\tau(\iota)) = \mathrm{dom}(\sigma) \cup \{z_s \mid s\sigma \neq s\}$ as follows for a variable x:

define infinite substitutions properly and apply definition here

$$x\tau(\iota) = \begin{cases} x\sigma & x \text{ is a non-lifting variable} \\ z_{t\sigma} & x \text{ is a lifting variable } z_t \end{cases}$$

If the inference ι is clear from the context, we abbreviate $\tau(\iota)$ by τ. △

⟨lemma:lifting_tau_commute⟩ **Lemma 3.** *For a formula or term φ and an inference ι such that $\tau = \tau(\iota)$, $\ell[\ell[\varphi]\tau] = \ell[\varphi\tau]$.*

Proof. We proceed by induction.

- Suppose that t is a grey constant or function symbol of the form $f(t_1, \ldots, t_n)$. Then we can derive the following, where (IH) signifies a deduction by virtue of the induction hypothesis.

$$\begin{aligned}
\ell[\ell[t]\tau] &= \ell[\ell[f(t_1, \ldots, t_n)]\tau] \\
&= \ell[f(\ell[t_1]\tau, \ldots, \ell[t_n]\tau)] \\
&= f(\ell[\ell[t_1]\tau], \ldots, \ell[\ell[t_n]\tau]) \\
&\overset{\text{(IH)}}{=} f(\ell[t_1\tau], \ldots, \ell[t_n\tau]) \\
&= \ell[f(t_1, \ldots, t_n)\tau] \\
&= \ell[t\tau]
\end{aligned}$$

- Suppose that t is a colored constant or function symbol. Then:

$$\ell[\ell[t]\tau] = \ell[z_t\tau] = \ell[z_{t\sigma}] = z_{t\sigma} = \ell[t\sigma] = \ell[t\tau]$$

- Suppose that t is a variable x. Then:

$$\ell[\ell[t]\tau] = \ell[\ell[x]\tau] = \ell[x\tau] = \ell[t\tau]$$

- Suppose that t is a lifting variable z_t. Then:

$$\ell[\ell[z_t]\tau] = \ell[z_t\tau]$$ □

Definition 4 (Incrementally lifted interpolant)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$. We define $\mathrm{LI}(\pi)$ and $\mathrm{LI}_{\mathrm{cl}}(\pi)$ to be $\mathrm{LI}(\Box)$ and $\mathrm{LI}_{\mathrm{cl}}(\Box)$ respectively, where $\Box$ is the empty clause derived in π.

Let C be a clause in π. For a literal λ in C, we denote the corresponding literal in $\mathrm{LI}_{\mathrm{cl}}(C)$ by λ_{LIcl}, whose existence is ensured Lemma 5.

We define $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ as follows:

Base case. If $C \in \Gamma$, $\mathrm{LI}(C) \stackrel{\text{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{LI}(C) \stackrel{\text{def}}{=} \top$.

In any case, $\mathrm{LI}_{\mathrm{cl}}(C) \stackrel{\text{def}}{=} \ell[C]$.

Resolution. If the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then define $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ as follows:

$\mathrm{LI}_{\mathrm{cl}}(C) \stackrel{\text{def}}{=} \ell[(\mathrm{LI}_{\mathrm{cl}}(C_1) \backslash \{l_{\mathrm{LIcl}}\})\tau] \;\vee\; \ell[(\mathrm{LI}_{\mathrm{cl}}(C_2) \backslash \{l'_{\mathrm{LIcl}}\})\tau]$

1. If l is Γ-colored: $\mathrm{LI}(C) \stackrel{\text{def}}{=} \ell[\mathrm{LI}(C_1)\tau] \vee \ell[\mathrm{LI}(C_2)\tau]$
2. If l is Δ-colored: $\mathrm{LI}(C) \stackrel{\text{def}}{=} \ell[\mathrm{LI}(C_1)\tau] \wedge \ell[\mathrm{LI}(C_2)\tau]$
3. If l is grey: $\mathrm{LI}(C) \stackrel{\text{def}}{=} (\ell[l_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_2)\tau]) \vee (\neg\ell[l'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_1)\tau])$

Factorisation. If the clause C is the result of a factorisation step ι of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{LI}(C) \stackrel{\text{def}}{=} \ell[\mathrm{LI}(C_1)\tau]$ and $\mathrm{LI}_{\mathrm{cl}}(C) \stackrel{\text{def}}{=} \ell[(\mathrm{LI}_{\mathrm{cl}}(C_1) \backslash \{l'_{\mathrm{LIcl}}\})\tau]$. △

1.2 Properties of LI and $\mathrm{LI}_{\mathrm{cl}}$

vs_clause_plus_literals_equal⟩ **Lemma 5.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$.*

Then for every literal λ in C, there exists a literal λ_{LIcl} in $\mathrm{LI}_{\mathrm{cl}}(C)$ such that $\lambda_{\mathrm{LIcl}} = \ell[\lambda]$ and for resolved or factorised literals l and l' of a resolution or factorisation inference ι, we have that $\ell[l_{\mathrm{LIcl}}\tau] = \ell[l'_{\mathrm{LIcl}}\tau]$.

Proof. We proceed by induction.

Base case. For $C \in \Gamma \cup \Delta$, $\mathrm{LI}_{\mathrm{cl}}(C)$ is defined to be $\ell[C]$.

Resolution/Factorisation. Suppose the clause C is the result of a resolution or factorisation inference ι of the clauses $\bar{C}$ with $\sigma = \mathrm{mgu}(\iota)$.

Every literal in C is of the form $\lambda\sigma$ for a literal λ in $C_i \in \bar{C}$.

By the induction hypothesis, $\ell[\lambda]$ occurs in $\mathrm{LI}_{\mathrm{cl}}(C_i)$. By the construction of $\mathrm{LI}_{\mathrm{cl}}(C)$ and as λ is not a resolved or factorised literal, $\mathrm{LI}_{\mathrm{cl}}(C)$ contains a literal of the form $\ell[\ell[\lambda]\tau]$. But by Lemma 3, this is nothing else than

$\ell[\lambda\tau]$. As λ occurs in the resolution derivation, it does not contain lifting variables. Hence we get by the definition of τ that $\ell[\lambda\tau] = \ell[\lambda\sigma]$.

Let l and l' be the resolved or factorised literals of ι. In order to show that $\ell[l_{\mathrm{LIcl}}\tau] = \ell[l'_{\mathrm{LIcl}}\tau]$, consider that by the induction hypothesis, this is nothing else than $\ell[\ell[l]\tau] = \ell[\ell[l']\tau]$. But by applying a similar argument as above, this equation is equivalent to $\ell[l\sigma] = \ell[l'\sigma]$, which is implied by $l\sigma = l'\sigma$. □

⟨lemma:no_colored_terms⟩ **Lemma 6.** *Let C be a clause of a resolution refutation of $\Gamma \cup \Delta$. $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ do not contain colored symbols.*

Proof. For $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$, consider the following: In the base case of the inductive definitions of $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$, no colored symbols occur. In the inductive steps, any colored symbol which is added by τ to intermediary formulas is lifted. □

⟨lemma:substitute_and_lift⟩ **Lemma 7.** *Let σ be a substitution and F a formula without Φ-colored terms such that for a set of formulas Ψ which does not contain Φ-lifting variables, $\Psi \models F$. Then $\Psi \models \ell_\Phi[F\sigma]$.*

Proof. $\ell_\Phi[F\sigma]$ is an instance of F: σ substitutes variables either for terms which do not contain Φ-colored symbols or by terms containing Φ-colored symbols. For the first kind, the lifting has no effect. For the latter, the lifting only replaces subterms of the terms introduced by the substitution by a lifting variable such that the original structure of F remains invariant as it by assumption does not contain colored terms. □

1.3 Lifting the Δ-terms

Definition 8. $\mathrm{LI}^\Delta(C)$ and $\mathrm{LI}^\Delta_{\mathrm{cl}}(C)$ for a clause C are defined as $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ respectively with the difference that in its inductive definition, every lifting $\ell[\varphi]$ for a formula or term φ is replaced by a lifting of only the Δ-terms $\ell_\Delta[\varphi]$. △

Remark. Many results involving $\mathrm{LI}(C)$ or $\mathrm{LI}_{\mathrm{cl}}(C)$ are valid for $\mathrm{LI}^\Delta(C)$ or $\mathrm{LI}^\Delta_{\mathrm{cl}}(C)$ in a formulation which is adapted accordingly. This can easily be seen by the following proof idea:

Let $f_1, \ldots, f_n$ be all Γ-colored function or constant symbols occurring in C, c a fresh constant symbol and g a fresh n-ary function symbol. Construct a formula $\varphi : g(t_1, \ldots, t_n) = g(t_1, \ldots, t_n)$, such that $t_i = f_i(c_1, \ldots, c_m)$ for $1 \leq i \leq n$ where m is the arity of f_i and $c_j = c$ for $1 \leq j \leq m$. Let $\Delta' = \Delta \cup \{\varphi\}$ and apply the desired result to the initial clause sets Γ and Δ'.

Under this construction, every originally Γ-colored symbol is now grey, which implies that $\mathrm{LI}(C) = \mathrm{LI}^\Delta(C)$ as well as $\mathrm{LI}_{\mathrm{cl}}(C) = \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$. But $\Delta \models \psi \Leftrightarrow \Delta' \models \psi$ for any formula ψ. △

⟨lemma:gamma_proves_pide⟩ **Lemma 9.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$.*

Proof. We proceed by induction of the strengthening $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C_\Gamma)$.

Base case. For $C \in \Gamma$, $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_\Gamma) = \ell_\Delta[C] = C$. Hence $\Gamma \models \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_\Gamma)$.

For $C \in \Delta$, $\mathrm{LI}^{\Delta}(C) = \top$, so $\Gamma \models \mathrm{LI}^{\Delta}(C)$.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

We define the following abbreviations:

$\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* = \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma \backslash \{l_{\mathrm{LIcl}^\Delta}\})$

$\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^* = \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma \backslash \{\neg l'_{\mathrm{LIcl}^\Delta}\})$

Hence the induction hypothesis can be stated as follows:

$\Gamma \models \mathrm{LI}^{\Delta}(C_1) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* \vee (l_{\mathrm{LIcl}^\Delta})_\Gamma$

$\Gamma \models \mathrm{LI}^{\Delta}(C_2) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^* \vee \neg(l'_{\mathrm{LIcl}^\Delta})_\Gamma$

By Lemma 6, $\mathrm{LI}^{\Delta}(C_i)$ and $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_i)$ for $i \in \{1, 2\}$ do not contain Δ-colored terms. Hence we are able to apply Lemma 7 in order to obtain

$\Gamma \overset{(\circ)}{\models} \ell_\Delta[\mathrm{LI}^{\Delta}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[(l_{\mathrm{LIcl}^\Delta})_\Gamma\tau]$

$\Gamma \overset{(*)}{\models} \ell_\Delta[\mathrm{LI}^{\Delta}(C_2)\tau] \vee \ell_\Delta[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\tau] \vee \neg\ell_\Delta[(l'_{\mathrm{LIcl}^\Delta})_\Gamma\tau]$

By Lemma 5, we obtain that $\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$.

Now we distinguish cases based on the color of the resolved literal:

- Suppose that l is Γ-colored. Then as $\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$, we can perform a resolution step on $(\circ)$ and $(*)$, which gives that $\Gamma \models \ell_\Delta[\mathrm{LI}^{\Delta}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[\mathrm{LI}^{\Delta}(C_2)\tau] \vee \ell_\Delta[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\tau]$. This however is nothing else than $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$.
- Suppose that l is Δ-colored. Then $(\circ)$ and $(*)$ simply to the following:
  $\Gamma \models \ell_\Delta[\mathrm{LI}^{\Delta}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\tau]$
  $\Gamma \models \ell_\Delta[\mathrm{LI}^{\Delta}(C_2)\tau] \vee \ell_\Delta[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\tau]$
  These however imply that $\Gamma \models \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^* \vee (\ell_\Delta[\mathrm{LI}^{\Delta}(C_1)\tau] \wedge \ell_\Delta[\mathrm{LI}^{\Delta}(C_2)\tau])$, which is nothing else than $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$.
- Suppose that l is grey. Suppose that M is a model of Γ such that $M \not\models \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$, i.e. $M \not\models \ell_\Delta[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\tau]$. Then $M \models \ell_\Delta[\mathrm{LI}^{\Delta}(C_1)\tau] \vee \ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau]$ as well as $M \models \ell_\Delta[\mathrm{LI}^{\Delta}(C_2)\tau] \vee \neg\ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$. Due to $\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$, we obtain that $M \models (\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] \wedge \ell_\Delta[\mathrm{LI}^{\Delta}(C_2)\tau]) \vee (\neg\ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau] \wedge \ell_\Delta[\mathrm{LI}^{\Delta}(C_1)\tau])$, which is nothing else than $M \models \mathrm{LI}^{\Delta}(C)$.

Factorisation. Suppose the clause C is the result of a factorisation inference ι of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

We introduce the abbreviation $\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* = \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma \backslash \{l_{\mathrm{LIcl}^\Delta}, \neg l'_{\mathrm{LIcl}^\Delta}\})$ and express the induction hypothesis as follows:

$\Gamma \models \mathrm{LI}^{\Delta}(C_1) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* \vee (l_{\mathrm{LIcl}^\Delta})_\Gamma \vee \neg(l'_{\mathrm{LIcl}^\Delta})_\Gamma$

By Lemma 6, $\mathrm{LI}^{\Delta}(C_i)$ and $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_i)$ for $i \in \{1, 2\}$ do not contain Δ-colored terms. Hence we are able to apply Lemma 7 in order to obtain

$$\Gamma \overset{(\circledast)}{\models} \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[(l_{\mathrm{LIcl}^\Delta})_\Gamma\tau] \vee \neg\ell_\Delta[(l'_{\mathrm{LIcl}^\Delta})_\Gamma\tau]$$

As by Lemma 5 we get that $\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$, we can perform a factorisation step on $(\circledast)$ to obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[(l_{\mathrm{LIcl}^\Delta})_\Gamma\tau]$. But this is nothing else than $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C_\Gamma)$. □

⟨lemma:gamma_lifted_lide⟩ **Lemma 10.** *For a clause C of a resolution refutation of $\Gamma \cup \Delta$, $\ell_\Gamma[\mathrm{LI}^\Delta(C)] = \mathrm{LI}(C)$ and $\ell_\Gamma[\mathrm{LI}^\Delta_{\mathrm{cl}}(C)] = \mathrm{LI}_{\mathrm{cl}}(C)$.*

Proof. We proceed by induction.

Base case. For $C \in \Gamma \cup \Delta$, $\mathrm{LI}^\Delta_{\mathrm{cl}}(C) = \ell_\Delta[C]$. By Lemma 1, $\ell_\Gamma[\ell_\Delta[C]] = \ell[C]$, so $\ell_\Gamma[\mathrm{LI}^\Delta_{\mathrm{cl}}\, C] = \ell[C] = \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$.

$\mathrm{LI}^\Delta(C)$ does not contain colored symbols.

Inductions step. Suppose the clause C is the result of a resolution or factorisation inference ι of the clauses $\bar{C}$.

Every literal in $\mathrm{LI}(C)$ or $\mathrm{LI}_{\mathrm{cl}}(C)$ is of the form $\ell[\lambda\tau]$ for some λ in $\mathrm{LI}(C_i)$ or $\mathrm{LI}_{\mathrm{cl}}(C_i)$ for some $C_i \in \bar{C}$. Every literal in $\mathrm{LI}^\Delta(C)$ or $\mathrm{LI}^\Delta_{\mathrm{cl}}(C)$ is of the form $\ell_\Delta[\lambda\tau]$ for some λ in $\mathrm{LI}^\Delta(C_i)$ or $\mathrm{LI}^\Delta_{\mathrm{cl}}(C_i)$ for some $C_i \in \bar{C}$.

Hence it suffices to show that for a literal λ in $\mathrm{LI}^\Delta(C_i)$ or $\mathrm{LI}^\Delta_{\mathrm{cl}}(C_i)$ and its corresponding literal $\ell_\Gamma[\lambda]$ in $\mathrm{LI}(C_i)$ or $\mathrm{LI}_{\mathrm{cl}}(C_i)$ for some $C_i \in \bar{C}$ that $\ell_\Gamma[\ell_\Delta[\lambda\tau]] = \ell[\ell_\Gamma[\lambda]\tau]$.

By Lemma 6, no Δ-terms occur in λ. Hence $\ell[\lambda] = \ell_\Gamma[\lambda]$ and consequently $\ell[\ell[\lambda]\tau] = \ell[\ell_\Gamma[\lambda]\tau]$. By Lemma 3, $\ell[\ell[\lambda]\tau] = \ell[\lambda\tau]$ and by Lemma 1, $\ell[\lambda\tau] = \ell_\Gamma[\ell_\Delta[\lambda\tau]]$. Hence $\ell_\Gamma[\ell_\Delta[\lambda\tau]] = \ell[\ell_\Gamma[\lambda]\tau]$. □

1.4 Quantifying over the lifting variables

?⟨def:arrow_quantifier_block⟩? **Definition 11** (Quantifier block)**.** Let C be a clause in a resolution refutation π of $\Gamma \cup \Delta$ and $\bar{x}$ the Δ-lifting variables and $\bar{y}$ the Γ-lifting variables occurring in $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$. $Q(C)$ denotes an arrangement of the elements of $\{\forall x_t \mid x_t \in \bar{x}\} \cup \{\exists y_t \mid y_t \in \bar{y}\}$ such that for two lifting variable z_s and z_r, if s is a subterm of r, then z_s is listed before z_r. We denote $Q(\Box)$ by $Q(\pi)$. △

Note that at a certain stage of the interpolant extraction, the quantifier block possesses a certain partial ordering based on the subterm relation of the indices of the lifting variables. This implies that the ordering is monotonous in the sense that in the subsequent course of the extraction, this ordering is only extended but existing order-relations are not modified.

:gamma_proves_quantified_lide⟩ **Lemma 12.** *Let C be a clause of a resolution refutation of $\Gamma \cup \Delta$. Then $\Gamma \models Q(C)(\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C))$.*

Proof. By Lemma 10 $\ell_\Gamma[\mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)] = \mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$. By Lemma 9, $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$. Hence the terms in $\mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$ provide witness terms for the Γ-lifting variables in $\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$, which are existentially quantified in $Q(C)(\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C))$.

Furthermore, the ordering imposed on the quantifiers in $Q(C)$ implies that if a Δ-lifting variable x_s occurs in a witness term for a Γ-lifting variable y_r, y_r

is quantified in the scope of the quantifier of x_s as s is a subterm of r. This however ensures that the witness terms are valid. ☐

⟨lemma:li_symmetry⟩ **Lemma 13.** *Let π be a refutation of $\Gamma \cup \Delta$ and $\hat{\pi}$ be π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then for a clause C in π and its corresponding clause $\hat{C}$ in $\hat{\pi}$, $Q(C)(\mathrm{LI}(C)) \Leftrightarrow \neg Q(\hat{C})(\mathrm{LI}(\hat{C}))$.*

Proof. Consider furthermore that liftings variables of C and $\hat{C}$ only differ in the variable symbol, but not in the index, and that the quantifier type of any given lifting variable in C is dual to the corresponding one in $\hat{C}$. Hence for any formula ϕ, $Q(C)\neg\phi \Leftrightarrow \neg Q(\hat{C})\phi$.

It remains to show that $\mathrm{LI}(C) \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$, which we establish by induction:

Base case. If $C \in \Gamma$, then $\mathrm{LI}(C) = \bot \Leftrightarrow \neg\top \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$ as $\hat{C} \in \Delta$. The case for $C \in \Delta$ can be argued analogously.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

As τ depends only on σ, τ is the same for both π and $\hat{\pi}$.

We now distinguish the following cases:

1. l is Γ-colored:

$$\begin{aligned}\mathrm{LI}(C) &= \ell[\mathrm{LI}(C_1)\tau] \vee \ell[\mathrm{LI}(C_2)\tau] \\ &\Leftrightarrow \neg(\neg\ell[\mathrm{LI}(C_1)\tau] \wedge \neg\ell[\mathrm{LI}(C_2)\tau]) \\ &\Leftrightarrow \neg(\ell[\mathrm{LI}(\hat{C}_1)\tau] \wedge \ell[\mathrm{LI}(\hat{C}_2)\tau]) \\ &= \neg\,\mathrm{LI}(\hat{C})\end{aligned}$$

2. l is Δ-colored: This case can be argued analogously.
3. l is grey: Note that by Lemma 5, $\ell[l_{\mathrm{LIcl}}\tau] = \ell[l'_{\mathrm{LIcl}}\tau]$ $(*)$.

$$\begin{aligned}\mathrm{LI}(C) &= (\neg\ell[l'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_1)\tau]) \;\vee\; (\ell[l_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_2)\tau]) \\ &\overset{(*)}{\Leftrightarrow} (\ell[l'_{\mathrm{LIcl}}\tau] \vee \ell[\mathrm{LI}(C_1)\tau]) \;\wedge\; (\neg\ell[l_{\mathrm{LIcl}}\tau] \vee \ell[\mathrm{LI}(C_2)\tau]) \\ &\Leftrightarrow \neg\Big((\neg\ell[l'_{\mathrm{LIcl}}\tau] \wedge \neg\ell[\mathrm{LI}(C_1)\tau]) \;\vee\; (\ell[l_{\mathrm{LIcl}}\tau] \wedge \neg\ell[\mathrm{LI}(C_2)\tau])\Big) \\ &= \neg\Big((\neg\ell[\hat{l}'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(\hat{C}_1)\tau]) \;\vee\; (\ell[\hat{l}_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(\hat{C}_2)\tau])\Big) \\ &= \neg\,\mathrm{LI}(\hat{C})\end{aligned}$$

Factorisation. Suppose the clause C is the result of a factorisation ι of C_1 : $l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

Then $\mathrm{LI}(C) = \ell[\mathrm{LI}(C_1)\tau]$, so the construction is not influenced by the coloring and the induction hypothesis gives the result. ☐

Theorem 14. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{LI}(\pi)$ is an interpolant.*

Proof. By Lemma 12 $\Gamma \models Q(\pi)(\mathrm{LI}(\pi) \vee \mathrm{LI}_{\mathrm{cl}}(\pi))$. But as $\mathrm{LI}_{\mathrm{cl}}(\pi) = \Box$, this simplifies to $\Gamma \models Q(\pi)\,\mathrm{LI}(\pi)$.

By constructing a proof $\hat{\pi}$ from π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$, we obtain by Lemma 12 that $\hat{\Gamma} \models Q(\hat{\pi})\,\mathrm{LI}(\hat{\pi})$. By Lemma 13, this however is nothing else than $\Delta \models \neg Q(\pi)\,\mathrm{LI}(\pi)$.

As furthermore by construction no colored symbols occur in $Q(\pi)\,\mathrm{LI}(\pi)$, this formula is an interpolant for $\Gamma \cup \Delta$. $\Box$