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g_order_not_relevant)? **Lemma 1.** $\ell_{\Gamma}[\ell_{\Delta}[\varphi]] = \ell_{\Delta}[\ell_{\Gamma}[\varphi]]$.

Interpolant extraction from resolution proofs in one phase lifting terms whose quantifier position can be determined – nested

Definition 2 (Substitution $\tau(\iota)$). For an inference ι with $\sigma = \text{mgu}(\iota)$, we define the infinite substitution $\tau(\iota)$ with $\text{dom}(\tau(\iota)) = \text{dom}(\sigma) \cup \{z_s \mid s\sigma \neq s\}$ as follows for a variable x :

$$x\tau(\iota) = \begin{cases} x\sigma & x \text{ is a non-lifting variable} \\ z_{t\sigma} & x \text{ is a lifting variable } z_t \end{cases}$$

define infinite substitutions properly and apply definition here

If the inference ι is clear from the context, we abbreviate $\tau(\iota)$ by τ . \triangle

:lifting_tau_commute) **Lemma 3.** For a formula or term φ and an inference ι such that $\tau = \tau(\iota)$, $\ell[\ell[\varphi]\tau] = \ell[\varphi\tau]$.

Proof. We proceed by induction.

- Suppose that t is a grey constant or function symbol of the form $f(t_1, \dots, t_n)$. Then we can derive the following, where (IH) signifies a deduction by virtue of

the induction hypothesis.

$$\begin{aligned}
 \ell[\ell[t]\tau] &= \ell[\ell[f(t_1, \dots, t_n)]\tau] \\
 &= \ell[f(\ell[t_1]\tau, \dots, \ell[t_n]\tau)] \\
 &= f(\ell[\ell[t_1]\tau], \dots, \ell[\ell[t_n]\tau]) \\
 &\stackrel{(IH)}{=} f(\ell[t_1\tau], \dots, \ell[t_n\tau]) \\
 &= \ell[f(t_1, \dots, t_n)\tau] \\
 &= \ell[t\tau]
 \end{aligned}$$

- Suppose that t is a colored constant or function symbol. Then:

$$\ell[\ell[t]\tau] = \ell[z_t\tau] = \ell[z_{t\sigma}] = z_{t\sigma} = \ell[t\sigma] = \ell[t\tau]$$

- Suppose that t is a variable x . Then:

$$\ell[\ell[t]\tau] = \ell[\ell[x]\tau] = \ell[x\tau] = \ell[t\tau]$$

- Suppose that t is a lifting variable z_t . Then:

$$\ell[\ell[z_t]\tau] = \ell[z_t\tau] \quad \square$$

Definition 4 (Incrementally lifted interpolant). Let π be a resolution refutation of $\Gamma \cup \Delta$. We define $\text{LI}(\pi)$ to be $\text{LI}(\square)$, where \square is the empty clause derived in π .

Let C be a clause in π .

We define the preliminary formula $\text{LI}^\bullet(C)$ as follows:

Base case. If $C \in \Gamma$, $\text{LI}(C) \stackrel{\text{def}}{=} \perp$. If otherwise $C \in \Delta$, $\text{LI}(C) \stackrel{\text{def}}{=} \top$.

Resolution.

If the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then define $\text{LI}(C)$ as follows:

1. If l is Γ -colored: $\text{LI}^\bullet(C) \stackrel{\text{def}}{=} \text{LI}(C_1)\tau \vee \text{LI}(C_2)\tau$
2. If l is Δ -colored: $\text{LI}^\bullet(C) \stackrel{\text{def}}{=} \text{LI}(C_1)\tau \wedge \text{LI}(C_2)\tau$
3. If l is grey: $\text{LI}^\bullet(C) \stackrel{\text{def}}{=} (l\tau \wedge \text{LI}(C_2)\tau) \vee (\neg l'\tau \wedge \text{LI}(C_1)\tau)$

Factorisation. If the clause C is the result of a factorisation step ι of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then $\text{LI}^\bullet(C) \stackrel{\text{def}}{=} \ell[\text{LI}(C_1)\tau]$.

Paramodulation. If the clause C is the result of a paramodulation step ι of $C_1 : s = t \vee D$ and $C_2 : E[r]$ with $\sigma = \text{mgu}(\iota)$. Let $h[r]$ be the maximal colored term in which r occurs in $E[r]$ and define $\text{LI}(C)$ as follows:

1. If $h[r]$ is Δ -colored and $h[r]$ occurs more than once in $\text{LI}(C_2)\tau \vee E[r]\tau$:
 $\text{LI}^\bullet(C) \stackrel{\text{def}}{=} (s = t \wedge \text{LI}(C_2))\tau \vee (s \neq t \wedge \text{LI}(C_1))\tau \vee (s = t \wedge h[s] \neq h[t])\tau$
2. If $h[r]$ is Γ -colored and $h[r]$ occurs more than once in $\text{LI}(C_2)\tau \vee E[r]\tau$:
 $\text{LI}^\bullet(C) \stackrel{\text{def}}{=} [(s = t \wedge \text{LI}(C_2))\tau \vee (s \neq t \wedge \text{LI}(C_1))\tau] \wedge (s \neq t \vee h[s] = h[t])\tau$
3. If r does not occur in a colored term in $E[r]$ which occurs more than once in $\text{LI}(C_2)\tau \vee E[r]\tau$:
 $\text{LI}^\bullet(C) \stackrel{\text{def}}{=} (s = t \wedge \text{LI}(C_2))\tau \vee (s \neq t \wedge \text{LI}(C_1))\tau$

$\text{LI}(C)$ is built from $\text{LI}^\bullet(C)$ according to the following steps:

1. Lift all maximal colored terms t in $\text{LI}^\bullet(C)$ for at least which one of the following conditions applies:
 - The term t contains some variable x such that x does not occur in C .
 - The term t is ground and C does not contain t .
2. Let X (Y) be the Δ -(Γ)-lifting variables created in the previous step.
3. Prefix the resulting formula with an arrangement $Q(C)$ of the elements of $\{\forall x_t \mid x_t \in X\} \cup \{\exists y_t \mid y_t \in Y\}$ such that if s and r are terms such that s is a subterm of r , then z_s precedes z_r . \triangle

`:substitute_and_lift)` **Lemma 5.** Let σ be a substitution and F a formula without Φ -colored terms such that for a set of formulas Ψ which does not contain Φ -lifting variables, $\Psi \models F$. Then $\Psi \models \ell_\Phi[F\sigma]$.

Proof. $\ell_\Phi[F\sigma]$ is an instance of F : σ substitutes variables either for terms which do not contain Φ -colored symbols or by terms containing Φ -colored symbols. For the first kind, the lifting has no effect. For the latter, the lifting only replaces subterms of the terms introduced by the substitution by a lifting variable such that the original structure of F remains invariant as it by assumption does not contain colored terms. \square

Lemma 6. If a variable x occurs in $\text{LI}(C)$ but not in C , then x does not occur in the subsequent resolution derivation.

Lemma 7. If a ground term t occurs in $\text{LI}(C)$ but not in C , and a successor of C contains t , then “the second t has been derived from first principles in a different way”.

lta_lifted_invariant)

Lemma 8. *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then $\Gamma \models \ell_\Delta[\text{LI}(C)] \vee \ell_\Delta[C]$*

Proof. We show the strengthening $\Gamma \models \ell_\Delta[\text{LI}(C)] \vee \ell_\Delta[C_\Gamma]^1$.

As a first step, we prove by induction that $\Gamma \models \ell_\Delta[\text{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

Base case. If $C \in \Gamma$, then $\ell_\Delta[C] = C$ and clearly $\Gamma \models C$. If otherwise $C \in \Delta$, then $\text{LI}(C) = \top$.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \text{mgu}(\iota)$.

By the induction hypothesis we obtain that $\Gamma \models \ell_\Delta[\text{LI}(C_1)] \vee \ell_\Delta[D_\Gamma] \vee \ell_\Delta[l_\Gamma]$ as well as $\Gamma \models \ell_\Delta[\text{LI}(C_2)] \vee \ell_\Delta[E_\Gamma] \vee \neg \ell_\Delta[l'_\Gamma]$. Hence by Lemma 5 and Lemma 3, we get:

$$\Gamma \stackrel{(\circ)}{\models} \ell_\Delta[\text{LI}(C_1)\tau] \vee \ell_\Delta[D_\Gamma\tau] \vee \ell_\Delta[l_\Gamma\tau]$$

$$\Gamma \stackrel{(*)}{\models} \ell_\Delta[\text{LI}(C_2)\tau] \vee \ell_\Delta[E_\Gamma\tau] \vee \neg \ell_\Delta[l'_\Gamma\tau]$$

As $l_\Gamma\sigma = l'_\Gamma\sigma$, it holds that $l_\Gamma\tau = l'_\Gamma\tau$ and consequently $\ell_\Delta[l_\Gamma\tau] = \ell_\Delta[l'_\Gamma\tau]$. We proceed by a case distinction on the color of the resolved literal to show that in each case, we have that $\Gamma \models \ell_\Delta[\text{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$:

- Suppose that l is Γ -colored. Then $l_\Gamma = l$ and $l'_\Gamma = l$, and we can perform a resolution step on (\circ) and $(*)$ to obtain that $\Gamma \models \ell_\Delta[\text{LI}(C_1)\tau] \vee \ell_\Delta[\text{LI}(C_2)\tau] \vee \ell_\Delta[D_\Gamma\tau] \vee \ell_\Delta[E_\Gamma\tau]$. This however is nothing else than $\Gamma \models \ell_\Delta[\text{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that l is Δ -colored. Then (\circ) and $(*)$ reduce to $\Gamma \models \ell_\Delta[\text{LI}(C_1)\tau] \vee \ell_\Delta[D_\Gamma\tau]$ and $\Gamma \models \ell_\Delta[\text{LI}(C_2)\tau] \vee \ell_\Delta[E_\Gamma\tau]$ respectively, which clearly implies that $\Gamma \models \ell_\Delta[\text{LI}(C_1)\tau] \vee \ell_\Delta[\text{LI}(C_2)\tau] \vee (\ell_\Delta[D_\Gamma\tau] \wedge \ell_\Delta[E_\Gamma\tau])$. This in turn is however just the unfolding of $\Gamma \models \ell_\Delta[\text{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that l is grey. Then $l_\Gamma = l$ and $l'_\Gamma = l$, and (\circ) and $(*)$ imply that $\Gamma \models \ell_\Delta[\text{LI}(C_1)\tau] \vee \ell_\Delta[\text{LI}(C_2)\tau] \vee (\ell_\Delta[l_\Gamma\tau] \wedge \ell_\Delta[E_\Gamma\tau]) \vee (\neg \ell_\Delta[l'_\Gamma\tau] \wedge \ell_\Delta[D_\Gamma\tau])$. This however is equivalent to $\Gamma \models \ell_\Delta[\text{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

Factorisation. Suppose the clause C is the result of a factorisation inference ι of $C_1 : l \vee l' \vee D$ with $\sigma = \text{mgu}(\iota)$.

¹Recall that D_Φ denotes the clause created from the clause D by removing all literals which are not contained in $L(\Phi)$.

The induction hypothesis gives $\Gamma \models \ell_\Delta[\text{LI}(C_1)] \vee \ell_\Delta[l_\Gamma \vee l'_\Gamma \vee D_\Gamma]$. By Lemma 5, we obtain $\Gamma \models \ell_\Delta[\text{LI}(C_1)\tau] \vee \ell_\Delta[l_\Gamma\tau \vee l'_\Gamma\tau \vee D_\Gamma\tau]$. As however $l\sigma \equiv l'\sigma$, also $l\tau \equiv l'\tau$, so we can apply a factorisation step and obtain that $\Gamma \models \ell_\Delta[\text{LI}(C_1)\tau] \vee \ell_\Delta[l_\Gamma\tau \vee D_\Gamma\tau]$, which is nothing else than $\Gamma \models \text{LI}^\bullet(C) \vee \ell_\Delta[C_\Gamma]$.

Paramodulation. Suppose the clause C is the result of a paramodulation inference ι of $C_1 : s = t \vee D$ and $C_2 : E[r]_p$ with $\sigma = \text{mgu}(\iota)$.

By the induction hypothesis, we obtain the following:

$$\Gamma \stackrel{(\circ)}{\models} \ell_\Delta[\text{LI}(C_1)] \vee \ell_\Delta[D_\Gamma] \vee \ell_\Delta[s] = \ell_\Delta[t]$$

$$\Gamma \stackrel{(*)}{\models} \ell_\Delta[\text{LI}(C_2)] \vee \ell_\Delta[(E[r]_p)_\Gamma]$$

Suppose now that for a model M of Γ and an assignment α of the free variables of $\ell_\Delta[s]$ and $\ell_\Delta[t]$ that $M_\alpha \models \ell_\Delta[s] \neq \ell_\Delta[t]$. Then we get by (\circ) that $M_\alpha \models \ell_\Delta[\text{LI}(C_1)] \vee \ell_\Delta[D_\Gamma]$, which by Lemma 5 and Lemma 3 gives $M_\alpha \models \ell_\Delta[\text{LI}(C_1)\tau] \vee \ell_\Delta[D_\Gamma\tau]$. Note that $M_\alpha \models \ell_\Delta[s\tau] \neq \ell_\Delta[t\tau] \wedge \ell_\Delta[\text{LI}(C_1)\tau]$ suffices for $M_\alpha \models \ell_\Delta[\text{LI}^\bullet(C)]$ and $M_\alpha \models \ell_\Delta[D_\Gamma\tau]$ implies that $M_\alpha \models \ell_\Delta[C_\Gamma]$. Therefore we obtain that $M_\alpha \models \ell_\Delta[\text{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

Now suppose to the contrary that for a model M of Γ that for any assignment of free variables $M \models \ell_\Delta[s] = \ell_\Delta[t]$.

By Lemma 5 and Lemma 3 we obtain from $(*)$ that $\Gamma \models \ell_\Delta[\text{LI}(C_2)\tau] \vee \ell_\Delta[(E[r]_p)_\Gamma\tau]$. As however $\sigma = \text{mgu}(r, s)$, $r\tau \equiv s\tau$ and hence $\ell_\Delta[r\tau] \equiv \ell_\Delta[s\tau]$. Therefore we also have that $\Gamma \models \ell_\Delta[\text{LI}(C_2)\tau] \vee \ell_\Delta[(E[s]_p)_\Gamma\tau]$.

We proceed by a case distinction:

- Suppose that the position p in $E[s]_p$ is not contained in a Δ -term. Then $\ell_\Delta[(E[s]_p)_\Gamma\tau]$ and $\ell_\Delta[(E[t]_p)_\Gamma\tau]$ only differ at at position p . As $M \models \ell_\Delta[s] = \ell_\Delta[t]$, we can apply Lemma 5 and Lemma 3 to obtain that $M \models \ell_\Delta[s\tau] = \ell_\Delta[t\tau]$. Thus $M \models \ell_\Delta[(E[s]_p)_\Gamma\tau] \Leftrightarrow \ell_\Delta[(E[t]_p)_\Gamma\tau]$ and consequently $M \models \ell_\Delta[\text{LI}(C_2)\tau] \vee \ell_\Delta[(E[t]_p)_\Gamma\tau]$. As furthermore $\ell_\Delta[\tau s] = \ell_\Delta[t] \wedge \ell_\Delta[\text{LI}(C_2)\tau]$ entails $\ell_\Delta[\text{LI}^\bullet(C)]$ and $\ell_\Delta[(E[t]_p)_\Gamma\tau]$ is sufficient for $\ell_\Delta[C_\Gamma]$, we have that $M \models \ell_\Delta[\text{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that the position p in $E[s]_p$ is contained in a maximal Δ -term $h[s]$. We distinguish further:
 - Suppose $h[s]$ occurs more than once in $\text{LI}(C_2)\tau \vee E[s]_p\tau$ and let α be an arbitrary assignment to the variables $\ell_\Delta[h[s]] = x_{h[s]}$ and $\ell_\Delta[h[t]] = x_{h[t]}$. If $M_\alpha \models \ell_\Delta[h[s]] \neq \ell_\Delta[h[t]]$, then we have that $M_\alpha \models \ell_\Delta[s] = \ell_\Delta[t] \wedge \ell_\Delta[h[s]] \neq \ell_\Delta[h[t]]$, which implies that $M_\alpha \models \ell_\Delta[\text{LI}^\bullet(C)]$.

Otherwise it holds that $M_\alpha \models \ell_\Delta[h[s]] = \ell_\Delta[h[t]]$. But then $\ell_\Delta[(E[s]_p)_\Gamma \tau]$ and $\ell_\Delta[(E[t]_p)_\Gamma \tau]$ differ in subterms which are equal in the given model and with the given interpretation of the free variables, so by a similar line of argument as in the preceding case, we can deduce that $M \models \ell_\Delta[\text{LI}^\bullet(C)] \vee \ell_\Delta[C]$.

- Suppose $h[s]$ occurs exactly once in $\text{LI}(C_2)\tau \vee E[s]_p \tau$. Then the lifting variable $x_{h[s]}$ occurs exactly once in $\ell_\Delta[\text{LI}(C_2)\tau] \vee \ell_\Delta[E[s]_p \tau]$. Note that from (*) by applying Lemma 5 as well as Lemma 3, we obtain that $M \models \ell_\Delta[\text{LI}(C_2)\tau] \vee \ell_\Delta[(E[s]_p)_\Gamma \tau]$. As $x_{h[s]}$ occurs only once and free in this formula, it is implicitly universally quantified and we can instantiate it arbitrarily, in particular by $z_{h[t]}$. But thereby we get that $M \models \ell_\Delta[\text{LI}(C_2)\tau] \vee \ell_\Delta[(E[t]_p)_\Gamma \tau]$, which implies that $\Gamma \models \ell_\Delta[\text{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

As we have now established that $\Gamma \models \ell_\Delta[\text{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$, we show that also $\Gamma \models \ell_\Delta[\text{LI}(C)] \vee \ell_\Delta[C_\Gamma]$ holds.

The difference between $\ell_\Delta[\text{LI}^\bullet(C)]$ and $\ell_\Delta[\text{LI}(C)]$ lies only in certain maximal colored terms which are lifted in $\ell_\Delta[\text{LI}(C)]$, hence it suffices to consider these. Let t be a colored term in $\text{LI}^\bullet(C)$ at position p such that $\text{LI}(C)|_p = \ell[t]$. Then t is a maximal colored term.

If t is Δ -colored, then $\ell_\Delta[\text{LI}^\bullet(C)]|_p = \text{LI}(C)|_p = x_t$. Note that as t occurs at p in $\text{LI}^\bullet(C)$, x_t occurs free at $\ell_\Delta[\text{LI}^\bullet(C)]|_p$. Hence it is implicitly universally quantified and therefore entails that an explicit universal quantification in $\text{LI}(C)$ is valid with an arbitrarily placed universal quantifier.

If otherwise t is a Γ -term, then $\ell_\Delta[\text{LI}^\bullet(C)]|_p = \ell_\Delta[t]$. Hence $\ell_\Delta[t]$ represents a witness term for the existentially quantified lifting variable y_t at $\text{LI}(C)|_p$. In general, $\ell_\Delta[t]$ however contains Δ -lifting variables, which require being lifted in the scope of the existential quantifier of y_t .

Let x_s be a Δ -lifting variable which occurs in $\ell_\Delta[t]$. It is essential to see that neither s nor a predecessor of s in the resolution derivation is lifted in a previous step of the interpolant extraction. Suppose to the contrary that this is the case in the inference creating the clause C' . Let s' and t' be the respective predecessors of s and t in C' .

- Suppose that t is lifted due to containing a variable which does not occur in C . Then as s' is a subterm of t' , t' contains this variable as well and would therefore be lifted in $\text{LI}(C')$.
- Suppose that t is lifted due to being a ground term which does not occur in C . Then s' does not occur in C' . But as t' contains s' , t' also does not occur in C' and would therefore be lifted in $\text{LI}(C')$.

check other paramod proofs

Note the the lifting conditions ensure that only terms are lifted which do not influence the subsequent resolution derivation: If a variable x occurs in $LI(C)$ but not in C , then as clauses are variable-disjoint, the variable x does not occur in any other clause. For ground terms r however which occur in $LI(C)$ but not in C , it is possible for them to cooccur in a subsequent clause. Let p be the occurrence of r in $LI(C)$ and q the occurrence of r in a successive clause of C . Then due to the fact that p is not used in any unification, q must have been created or originated from other occurrences of the same function and/or constant symbols. Note that the lifting conditions ensure that for these, the order of the quantifiers of their respective lifting variables is established in a fashion appropriate to ensuring the logical validity of the interpolant.

- Suppose that y_t is lifted due to containing a variable which does not occur in C . Then one of the following two contradictions eventuate:
 - Suppose that s' is a subterm of t' . Then due to the fact that s' is lifted, s' must contain a variable which does not occur in C' . But as t' contains s' , t' contains this variable as well and would be lifted at this stage already.
 - Otherwise t' does not contain s' . We have already established that s' contains a variable which does not occur in C' . As all clauses are variable-disjoint, no other clause contains this variable. But then it does not occur in any subsequent unifier, and in particular, it never enters t' by means of substitution, which implies that s' due to containing this variable does not become a subterm of a successor of t' .
- Suppose that y_t is lifted due to being a ground term which does not occur in C .
 - Suppose that s' is a subterm of t' . Then as s' occurs in t' , s' is not lifted.
 - Otherwise t' does not contain s' . Then if s' is lifted, s' is not contained in any term in C or $LI(C)$. Hence s' is not a predecessor of

what is lifted? all occs of some term? all maximal Φ -colored occs?

candidate for answer: effectively, we lift maximal colored terms, conceptually, every occurrence of these terms are lifted. with this version, we argue for correctness, and we should argue, that that's what we get

Supp s' subterm of t' : then s' is not lifted as it occurs in t' which occurs in $\text{LI}(C')$ or in C (well, exactly where s' occurs too)

Supp s' not subterm of t' . IDEA: then s' can be lifted, but if some t' contains s' , then this is a different occurrence of s' with a different “history”.

Hence there are three possibilities for quantification of x_s . We show that in each of them, y_t is quantified in the scope of the quantifier of x_s .

1. Neither s nor a successor of s in the derivation occurs at a grey position. Then x_s is not quantified in the course of the interpolant extraction.
2. A variable which does not occur in C enters s by means of the current substitution σ or a variable is contained in s such that the only occurrences of it in C_1 and C_2 are in l and l' . Then x_s is lifted in the current step and as s is a subterm of t , x_s is quantified in $Q(C)$ prior to y_t .
3. The lifting variable x_s or a respective successor is quantified at a later stage in the derivation. Then as the quantifier for y_t is contained in $\text{LI}(C)$ and for any successor C' of C , $\text{LI}(C')$ contains a successor $\text{LI}(C)$, y_t is quantified in the scope of the quantifier for x_s . \square

(lemma:li_symmetry) **Lemma 9.** *Let π be a refutation of $\Gamma \cup \Delta$ and $\hat{\pi}$ be π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then for a clause C in π and its corresponding clause \hat{C} in $\hat{\pi}$, $\text{LI}(C) \Leftrightarrow \neg \text{LI}(\hat{C})$.*

Proof. We proceed by induction to show that $\text{LI}^\bullet(C) \Leftrightarrow \neg \text{LI}^\bullet(\hat{C})$:

Base case. If $C \in \Gamma$, then $\text{LI}(C) = \perp \Leftrightarrow \neg \top \Leftrightarrow \neg \text{LI}(\hat{C})$ as $\hat{C} \in \Delta$. The case for $C \in \Delta$ can be argued analogously.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \text{mgu}(\iota)$.

As τ depends only on σ , τ is the same for both π and $\hat{\pi}$.

We now distinguish the following cases:

1. l is Γ -colored:

$$\begin{aligned}
\text{LI}^\bullet(C) &= \text{LI}(C_1)\tau \vee \text{LI}(C_2)\tau \\
&\Leftrightarrow \neg(\neg\text{LI}(C_1)\tau \wedge \neg\text{LI}(C_2)\tau) \\
&\Leftrightarrow \neg(\text{LI}(\hat{C}_1)\tau \wedge \text{LI}(\hat{C}_2)\tau) \\
&= \neg\text{LI}^\bullet(\hat{C})
\end{aligned}$$

2. l is Δ -colored: This case can be argued analogously.

3. l is grey: Note $l\tau \stackrel{(*)}{=} l'\tau$.

$$\begin{aligned}
\text{LI}^\bullet(C) &= (\neg l'\tau \wedge \text{LI}(C_1)\tau) \vee (l\tau \wedge \text{LI}(C_2)\tau) \\
&\stackrel{(*)}{\Leftrightarrow} (l'\tau \vee \text{LI}(C_1)\tau) \wedge (\neg l\tau \vee \text{LI}(C_2)\tau) \\
&\Leftrightarrow \neg[(\neg l'\tau \wedge \neg\text{LI}(C_1)\tau) \vee (l\tau \wedge \neg\text{LI}(C_2)\tau)] \\
&= \neg[(\neg l'\tau \wedge \text{LI}(\hat{C}_1)\tau) \vee (l\tau \wedge \text{LI}(\hat{C}_2)\tau)] \\
&= \neg\text{LI}^\bullet(\hat{C})
\end{aligned}$$

Factorisation. Suppose the clause C is the result of a factorisation ι of $C_1 : l \vee l' \vee D$ with $\sigma = \text{mgu}(\iota)$.

As the construction is not influenced by the coloring, the induction hypothesis $\text{LI}^\bullet(C) = \text{LI}(C_1)\tau$ suffices.

Then $\text{LI}^\bullet(C) = \ell[\text{LI}(C_1)\tau]$, so the construction is not influenced by the coloring and by the induction hypothesis, $\text{LI}^\bullet(C) \Leftrightarrow \neg\text{LI}^\bullet(\hat{C})$.

Paramodulation. Suppose the clause C is the result of a paramodulation inference ι of $C_1 : s = t \vee D$ and $C_2 : E[r]_p$ with $\sigma = \text{mgu}(\iota)$.

We proceed by a case distinction:

- Suppose that p in $E[r]_p$ is contained in a maximal Δ -term $h[r]$, which occurs more than once in $E[r]_p \vee \text{LI}(E[r]_p)$. Then p in $\hat{E}[r]_p$ is contained in a maximal Γ -term $h[r]$, which occurs more than once in $\hat{E}[r]_p \vee \text{LI}(\hat{E}[r]_p)$.

$$\begin{aligned}
\text{LI}^\bullet(C) &= (s\tau = t\tau \wedge \text{LI}(C_2)\tau) \vee (s\tau \neq t\tau \wedge \text{LI}(C_1)\tau) \vee (s\tau = t\tau \wedge h[s]\tau \neq h[t]\tau) \\
&\Leftrightarrow \neg[(s\tau \neq t\tau \vee \neg\text{LI}(C_2)\tau) \wedge (s\tau = t\tau \vee \neg\text{LI}(C_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&= \neg[(s\tau \neq t\tau \vee \text{LI}(\hat{C}_2)\tau) \wedge (s\tau = t\tau \vee \text{LI}(\hat{C}_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&\Leftrightarrow \neg[(s\tau = t\tau \wedge \text{LI}(\hat{C}_2)\tau) \vee (s\tau \neq t\tau \wedge \text{LI}(\hat{C}_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&= \neg\text{LI}^\bullet(\hat{C})
\end{aligned}$$

- Suppose that p in $E[r]_p$ is contained in a maximal Γ -term $h[r]$, which occurs more than once in $E[r]_p \vee \text{LI}(E[r]_p)$. This case can be argued analogously.
- Otherwise:

$$\begin{aligned}
\text{LI}^\bullet(C) &= (s\tau = t\tau \wedge \text{LI}(C_2)\tau) \vee (s\tau \neq t\tau \wedge \text{LI}(C_1)\tau) \\
&\Leftrightarrow \neg[(s\tau \neq t\tau \vee \neg\text{LI}(C_2)\tau) \wedge (s\tau = t\tau \vee \neg\text{LI}(C_1)\tau)] \\
&= \neg[(s\tau \neq t\tau \vee \text{LI}(\hat{C}_2)\tau) \wedge (s\tau = t\tau \vee \text{LI}(\hat{C}_1)\tau)] \\
&\Leftrightarrow \neg[(s\tau = t\tau \wedge \text{LI}(\hat{C}_2)\tau) \vee (s\tau \neq t\tau \wedge \text{LI}(\hat{C}_1)\tau)] \\
&= \neg\text{LI}^\bullet(\hat{C})
\end{aligned}$$

We conclude by showing that $\text{LI}^\bullet(C) \Leftrightarrow \neg\text{LI}^\bullet(\hat{C})$ entails that $\text{LI}(C) \Leftrightarrow \neg\text{LI}(\hat{C})$: Clearly the terms to be lifted in $\text{LI}^\bullet(C)$ and $\text{LI}^\bullet(\hat{C})$ are the same and differ only in their color. Even though this results in different lifting variables, that is of no relevance as all lifted variables are instantly bound. Additionally, the quantifier type of any given lifting variable in $Q(C)$ is dual to the respective one in $Q(\hat{C})$. Furthermore note that the subterm-relation is not affected by the coloring, so the ordering of the quantifiers in $Q(C)$ and $Q(\hat{C})$ is identical. Hence $\text{LI}(C) \Leftrightarrow \neg\text{LI}(\hat{C})$. \square

$\text{mma:delta_entails_li}$ **Lemma 10.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then $\Delta \models \neg\ell_\Gamma[\text{LI}(C)] \vee \ell_\Gamma[C]$.*

Proof. Construct $\hat{\pi}$ with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then by Lemma 8, $\hat{\Gamma} \models \ell_{\hat{\Delta}}[\text{LI}(\hat{C})] \vee \ell_{\hat{\Delta}}[\hat{C}]$, which by Lemma 9 is nothing else than $\Delta \models \neg\ell_\Gamma[\text{LI}(C)] \vee \ell_\Gamma[C]$. \square

Theorem 11. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then $\text{LI}(\pi)$ is an interpolant of Γ and Δ .*

Proof. We obtain by Lemma 8 that $\Gamma \models \ell_\Delta[\text{LI}(\pi)]$ and by Lemma 10 that $\Delta \models \neg\ell_\Gamma[\text{LI}(\pi)]$. As the empty clause derived in π trivially contains neither variables nor ground terms, all colored terms are lifted in $\text{LI}(\pi)$. Therefore $\ell_\Delta[\text{LI}(\pi)] = \text{LI}(\pi)$ and $\ell_\Gamma[\text{LI}(\pi)] = \text{LI}(\pi)$. \square