

# Number of quantifier alternations in Huang and nested

## 1.1 Outline

Goal: try proof Huang and nested equal.

Method: proof for both:

**Conjectured Proposition 1.** *Let  $I$  be an interpolant created by  $\$algorithm$ . If  $I$  contains a term  $t$  such that  $t$  has  $n$  color changes, then  $I$  has at least  $n$  quantifier alternations.*

### 1.1.1 generally keep in mind

- Need to define all new terms here: color-changing, single-color,  $\Phi$ -literal, substitutions from 0 to  $n$ 
  - essentially same position: path from one position to other only contains grey symbol (this def allows for identical position as well)
- also note: literal is sometimes used for negated or not negated predicate with terms but in regular formulas with arbitrary connectives

## 1.2 Preliminaries

Quantifier alternations in  $I$  usually assumes the quantifier-alternation-minimising arrangement of quantifiers in  $I$

**Definition 2** (Color alternation col-alt). Colors  $\Gamma$  and  $\Delta$ , term  $t$ :

$$\text{col-alt}(t) \stackrel{\text{def}}{=} \text{col-alt}_{\perp}(t)$$

Let  $t = f(t_1, \dots, t_n)$  for constant, function and variable symbols (syntax abuse)

$$\text{col-alt}_{\Phi}(t) \stackrel{\text{def}}{=} \begin{cases} \max(\text{col-alt}_{\Phi}(t_1), \dots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is grey} \\ \max(\text{col-alt}_{\Phi}(t_1), \dots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_{\Psi}(t_1), \dots, \text{col-alt}_{\Psi}(t_n)) & f \text{ is of color } \Psi, \Phi \neq \Psi \end{cases}$$

$\triangle$

**Definition 3.**  $\text{PI}_{\text{step}}^\circ$  is defined just like  $\text{PI}_{\text{step}}$  but without applying any substitution.  $\triangle$

Hence  $\text{PI}_{\text{step}}^\circ(\cdot)\sigma = \text{PI}_{\text{step}}(\cdot)$ .  $C^\circ$  is somehow the same, i.e. if  $C = D\sigma$ , then  $C^\circ = D$  where  $\sigma$  is derived from the context.

### 1.3 Random thoughts

- Quantifiers are introduced for lifting variables which actually occur in the interpolant
- If term  $t$  with  $\text{col-alt}(t) = n$  enters  $I$ , we need subterm  $s$  of  $t$  with  $\text{col-alt}(s) = n - 1$  to be in  $I$  (of course colors of  $t$  and  $s$  are exactly opposite)

#### 1.3.1 Proof

- Induction over  $\ell_\Delta^x[\text{PI}(C) \vee C]$  and also about  $\Gamma$ -terms with  $\Delta$ -lifting vars in that formula. Cf. **-final**
- TODO: describe proof method with  $\sigma_{(0,i)}$ : which PI?
  - Factorisation: easy: just apply  $\sigma_i$  for all  $i$  to  $\text{PI}(C) \vee C$ . When done, a literal will be there twice and we can remove it without losing anything
  - Resolution: create propositional structure first.  
 Ex.:  $C_1 : D \vee l, C_2 : \neg l \vee E$ :  
 If we talk about properties for which it holds that if they hold for  $\text{PI}(C_i) \vee C_i, i \in \{1, 2\}$ , then they also hold for  $A \equiv ((l \wedge \text{PI}(C_2)) \vee (\neg l \wedge \text{PI}(C_1))) \vee C^\circ$ , then we can apply  $\sigma_i$  for all  $i$  to that formula.  
 So if we can assume it for  $A$  and show it for all  $\sigma_i$ , we get that it holds for  $\text{PI}(C) \vee C$ .

Also: clauses are variable disjoint, so e.g. it's not possible that a color-changing var is created by  $\text{PI}_{\text{step}}$

Also: do it like a few lemmas further down, like  $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)\sigma_{(0,i)}$

### 1.4 Proof port attempt from -final

**need to show that grey occurrences are in grey literals, all grey literals end up in the interpolant!**

conj: if a  $\Delta$ -term  $t$  occurs in a  $\Gamma$ -literal in a clause  $C$ , then  $t$  occurs in a grey literal in  $\text{PI}(C)$ .

evidence:

- situation does not occur in  $\Gamma$  or  $\Delta$
- terms are only changed by unificators

- $\Delta$ - and  $\Gamma$ -terms are not unifiable, so one of the literals has to have a variable at a grey position when a  $\Delta$ -term enters a  $\Gamma$ -literal
- that literal has to be grey
- QED?

**Refuted Lemma (this is wrong) 4.** *If a  $\Phi$ -term  $t$  occurs in a  $\Psi$ -literal in a clause  $C$ , then  $t$  occurs at a grey position in  $\text{PI}(C)$ .*

*Proof.* As all grey literals of clauses involved in a refutation end up in the interpolant, it suffices to show that  $t$  occurs at a grey position in a grey literal.

Substitutions are applied to all variables, hence we only need to consider terms  $t$  which just enter a foreign colored literal.

TODO: propagation 1:  $\Phi$ -terms vs  $\Psi$ -terms (in  $\Psi$ -literals)

TODO: propagation 2:  $\Phi$ -terms vs other  $\Phi$ -terms (in  $\Psi$ -literals)

Induction on refutation and  $\sigma$ ; base case easy.

Resolution or factorisation inference  $\iota$ . Let  $\lambda$  be a  $\Gamma$ -literal containing a variable  $u$  at position  $\hat{u}$  such that  $u\sigma_i$  contains a  $\Delta$ -term  $t$ .

If the resolved or factorised literals are grey, they become part of  $\text{PI}(C)$  and if  $t$  occurs grey there, we are done.

- Suppose the resolved literals are  $\Gamma$ -colored. Then IH.
- Suppose the resolved literals are  $\Delta$ -colored. TODO:
- Suppose the resolved literals are grey and  $t$  does not occur at a grey position in  $\lambda\sigma = \lambda'\sigma$ .

TODO:

□

**Conjectured Lemma 5.** *If a  $\Phi$ -term  $t$  occurs in a  $\Psi$ -literal in a clause  $C$ , then  $t$  occurs at a grey position in a grey literal in  $\text{PI}(C) \vee C$ .*

there has to be a variable  $u$  in a  $\Psi$ -literal such that  $u\sigma_i$  contains  $t$ .

**Conjectured Lemma 6.** *If a variable  $u$  occurs in a  $\Phi$ -literal as well as in a  $\Psi$ -literal in a clause  $C$ , then  $t$  also occurs at a grey position in a grey literal in  $\text{PI}(C)$ .*

*Proof.* Initially not the case.

Note that we can only resolve/factorise  $\Gamma$ -/ $\Delta$ -/grey literals with other  $\Gamma$ -/ $\Delta$ -/grey literals as clearly the predicate symbol must be the same for both literals. Hence if a variable occurs only in  $\Gamma$ - or only in  $\Delta$ -literals, then it can never escape these. Hence  $u$  certainly is contained in a grey literal.

Now suppose that  $u$  only occurs colored in grey literals. Then it occurs in a  $\Gamma$ -( $\Delta$ -) term in the original  $\Gamma$ -( $\Delta$ -)clauses which contain it.

As shown before  $u$  must occur in some grey literal. Suppose it does not occur at a grey position in a grey literal as otherwise we are done. Then  $u$  only occurs in  $\Gamma$ -terms in grey literals as

TODO: it seems that now we have to deal with possible  $\Gamma$ -terms in  $\Delta$ -literals and so on  $\Rightarrow$  circular reasoning

The situation in question arises if some variable  $u$  occurs in a  $\Gamma$ -literal in some clause and some variable  $v$  occurs in a  $\Delta$ -literal in some clause (possibly the same), such that in the unified literals,  $u$  and  $v$  both occur at the same position in the respective literals.

□

## 1.5 directly from old proof

$\langle \text{lemma:col\_change} \rangle$  **Lemma 7.** *Resolution or factorisation step  $\iota$  from  $\bar{C}$ .*

*If  $u$  col-change var in  $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)\sigma_{(0,i)}$ , then  $u$  also occurs grey in that formula.*

*Proof.* Abbreviation:  $F \equiv (\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)$

Induction over refutation and  $\sigma$ ; base case easy.

Step: Supp color change var  $u$  present in  $F\sigma_{(0,i)}$ . (could also say introduced, then proof would be somehow different)

Supp  $u$  not grey in  $F\sigma_{(0,i-1)}$  as otherwise done. As a first step, we show that if a (not necessarily color-changing) variable  $v$  occurs in a single-colored  $\Phi$ -term  $t[v]$  in  $F\sigma_{(0,i)}$ , then at least one of the following holds:

1.  $v$  occurs in some single-colored  $\Phi$ -term in  $F\sigma_{(0,i-1)}$
2. there is a color-changing variable  $w$  in  $F\sigma_{(0,i-1)}$  such that  $v$  occurs grey in  $w\sigma_i$ .

$\langle \text{var\_occ\_2} \rangle$  We consider the different cases which can introduce a variable  $v$  in a single-colored term  $\Phi$ : Either it has been there before, it was introduced in a s.c.  $\Phi$ -colored term, or a s.c.  $\Phi$ -term containing the var is in  $\text{ran}(\sigma)$ .

- Suppose a term  $t'[v]$  is present in  $F\sigma_{(0,i-1)}$  such that  $t'[v]\sigma_i = t[v]$ . Then 1 is the case.
- Suppose a variable  $w$  occurs in a single-colored  $\Phi$ -term in  $F\sigma_{(0,i-1)}$  such that  $v$  occurs grey in  $w\sigma_i$ . Suppose furthermore that 1 is not the case, i.e.  $v$  does not occur in a s.c.  $\Phi$ -term in  $F\sigma_{(0,i-1)}$ , as otherwise we would be done. We show that 2 is the case.

As  $v$  occurs neither grey nor in a s.c.  $\Phi$ -term in  $F\sigma_{(0,i-1)}$  but occurs in  $\text{ran}(\sigma_i)$ , it must occur in  $F\sigma_{(0,i-1)}$  and this can only be in a single-colored  $\Psi$ -term.

As by assumption  $v$  occurs grey in  $w\sigma_i$ , there must be an occurrence  $\hat{w}$  of  $w$  in a resolved or factorised literal, say  $\lambda\sigma_{(0,i-1)}$  such that for the other resolved literal  $\lambda'\sigma_{(0,i-1)}$ ,  $\lambda'\sigma_{(0,i-1)}|_{\hat{w}}$  is a subterm in which  $v$  occurs grey. But as the occurrence of  $v$  in  $\lambda'\sigma_{(0,i-1)}|_{\hat{w}}$  must be contained in a single-colored  $\Psi$ -term, so is  $\lambda\sigma_{(0,i-1)}|_{\hat{w}}$ , hence  $z$  occurs in a single-colored  $\Psi$ -term as well. Therefore 2 is the case.

- Suppose there is a variable  $z$  in  $F\sigma_{(0,i-1)}$  such that  $v$  occurs in a single-colored  $\Phi$ -term in  $z\sigma_i$ . Then  $z\sigma_i$  occurs in  $F\sigma_{(0,i-1)}$ , but this is a witness for 1.

Now recall that we have assumed  $u$  to be a color-changing variable in  $F\sigma_{(0,i)}$ . Hence it occurs in a single-colored  $\Gamma$ -term as well as in a single-colored  $\Delta$ -term. By the reasoning above, this leads to two case:

- In  $F\sigma_{(0,i-1)}$ ,  $u$  occurs both in some single-colored  $\Gamma$ -term as well as in some single-colored  $\Delta$ -term. Then we get the result by the induction hypothesis and the fact that  $u \notin \text{dom}(\sigma_i)$  as  $u$  does occur in  $F\sigma_{(0,i)}$ .
- Otherwise for some color  $\Phi$ ,  $u$  does not occur in a single-colored  $\Phi$ -term in  $F\sigma_{(0,i-1)}$ . Then case 2 above must hold and there is some color-changing variable  $w$  in  $F\sigma_{(0,i-1)}$  such that  $u$  occurs grey in  $w\sigma_{(0,i)}$ . But then by

the induction hypothesis,  $w$  occurs grey in  $F\sigma_{(0,i-1)}$  and hence  $u$  occurs grey in  $F\sigma_{(0,i)}$ .  $\square$

NB: this is the heart of the proof:

**Lemma 8.** *If  $\text{PI}(C) \vee C$  contains a maximal colored occurrence of a  $\Gamma$ -term  $t[s]$  containing  $\Delta$ -term  $s$ , then  $s$  occurs grey in  $\text{PI}(C) \vee C$ .*

works if todo's are handled in sublemmas

*Proof.* Note that it suffices to show that at the step where  $s$  is introduced as subterm of  $t[s]$ ,  $s$  occurs grey in  $\text{PI}(C) \vee C$  as any later modification by substitution is applied to both occurrences  $s$ , so they stay equal throughout the remaining derivation. **TODO: what if it's in  $\text{PI}(C)$  and it disappears due to not being a colored literal?**

Induction over  $\pi$  and  $\sigma$ .

Base case: works

Step: Resolution or factorisation inference  $\iota$ ,  $\text{mgu}(\iota) = \sigma = \sigma_1 \cdots \sigma_n$ . The term  $t[s]$  is created by one of the following two ways:

(we abbreviate  $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ$  by  $F$ .)

- A variable  $u$  occurs in  $F\sigma_{(0,i-1)}$  such that  $u\sigma_i = t[s]$ .

Then  $u$  occurs in a resolved or factorised literal  $\lambda\sigma_{(0,i-1)}$  at  $\hat{u}$  such that at the other resolved or factorised literal  $\lambda'\sigma_{(0,i-1)}$ ,  $\lambda'\sigma_{(0,i-1)}|_{\hat{u}} = t[s]$ . Then the condition is present at  $F\sigma_{(0,i-1)}$  and we get the result by the induction hypothesis. **TODO: add grey here but get for free**

- Note that we only consider maximal colored terms.

Let  $t[u]$  be a maximal colored  $\Gamma$ -term in  $F\sigma_{(0,i-1)}$  such that in the tree-representation of  $t[u]$ , the path from the root to  $u$  does not contain a node labelled with a  $\Delta$ -symbol, and  $u\sigma_i$  contains a grey occurrence of  $s$ .

Suppose that  $u$  occurs grey in  $F\sigma_{(0,i-1)}$ . Then  $s$  occurs grey in  $F\sigma_{(0,i)}$  and we are done.

Now suppose that  $u$  does not occur grey in  $F\sigma_{(0,i-1)}$ .

If  $u$  occurs in a single-colored  $\Delta$ -term in  $F\sigma_{(0,i-1)}$ , then by Lemma 7,  $u$  also occurs grey in  $F\sigma_{(0,i-1)}$  and  $s$  hence occurs grey in  $F\sigma_{(0,i)}$ . **TODO: must be in grey literals**

Otherwise  $u$  only occurs in single-colored  $\Gamma$ -terms in  $F\sigma_{(0,i-1)}$ . As  $u\sigma_i$  contains a grey occurrence of  $s$ , there is a position  $p$  in the two resolved or factorised literals  $\lambda$  and  $\lambda'$  such that  $\lambda|_p = u$  and  $\lambda'|_p$  contains a grey occurrence of  $s$ . Furthermore, the prefix along the path to  $p$  is the same in both  $\lambda$  and  $\lambda'$ . But as by assumption  $u$  only occurs in single-colored  $\Gamma$ -terms,  $\lambda'|_p$  does so as well, so  $s$  is contained in a single-colored  $\Gamma$ -term in  $F\sigma_{(0,i-1)}$ . Since  $s$  is a  $\Delta$ -term, by the induction hypothesis,  $s$  occurs grey in  $F\sigma_{(0,i-1)}$  and hence also in  $F\sigma_i$ . **TODO: add grey here but get for free**  $\square$

are probably not same  $t$  and  $s$  as in lemma statement, which isn't technically wrong but confusing

## 1.6 random

**Conjectured Lemma 9.** *Supp  $u$  in s.c.  $\Phi$ -term and that no variable occurs grey in a grey literal. Then  $u$  does not occur grey in a  $\Psi$ -literal.*

*Proof.* Holds initially.

Supp holds for first  $k$  deductions.

Supp that then there is some  $\Psi$ -literal  $l$  which contains  $v$  at a grey position such that  $v\sigma_i$  contains a grey  $u$ . Then  $u$  and  $v$  face each other in the res/fac literals in at stage  $k - 1$ . By IH, there,  $u$  does not occur grey in a  $\Psi$ -literal (and of course not grey in a grey literal).

As  $v$  occurs grey in a  $\Psi$ -literal, by contraposition,  $v$  does not occur in a s.c.  $\Phi$ -term.

- Supp res/fac lits are  $\Psi$ -colored. Then  $u$  does not occur grey here.

**TODO:**

- Supp res/fac lits are grey. Then  $u$  and  $v$  do not occur grey here. **TODO:**
- Supp res/fac lits are  $\Phi$ -colored. **TODO:**

□

**Conjectured Lemma 10.** *Every variable, which does not occur grey in a grey literal either does not occur in a single-colored  $\Phi$ -( $\Psi$ )-term or does not occur grey in a  $\Psi$ -( $\Phi$ )-literal.*

*Proof.* Supp  $u$  occurs grey in  $\Phi$ -literal. We show that  $u$  does not occur in a s.c.  $\Psi$ -term given that it does not occur grey in a grey literal.

Supp s.c.  $\Psi$ -term  $t[v]$  s.t.  $v\sigma_i$  contains grey  $u$ . By IH,  $v$  does not occur grey in a  $\Phi$ -literal (given it does not occur grey in a grey literal).

By IH, there is no s.c.  $\Psi$ -term which contains  $u$ . Hence  $t[v]$  is not directly unified with some  $t[u]$ .

Supp  $v$  occurs grey where it is unified. then  $u$  occurs grey there as well. Then it can't be a grey literal, and it can't be a  $\Phi$ -literal. So it is a  $\Psi$ -literal.

□

basically also grey occ of  $v$  with  $t[u]$  conceivable, but that should be IH

**Conjectured Lemma 11.** *If a var occurs in  $\Gamma$  but not grey in a grey literal, it does not occur grey in  $\Delta$ .*

## 1.7 Thursday:

(last\_lemma) **Conjectured Lemma 12.** *Suppose a variable  $u$  occurs in  $\text{PI}(C) \vee C$  in a s.c.  $\Phi$ -term OR grey in a  $\Phi$ -literal. If  $u$  occurs in  $\text{PI}(C) \vee C$  in a grey position in a  $\Psi$ -literal OR in s.c.  $\Psi$ -term in any literal, then it also occurs at a grey position in a grey literal in  $\text{PI}(C) \vee C$ .*

*Proof.* ind step; supp  $u$  as in statement ubiquitously

Supp  $\Psi$ -literal with grey  $v$  such that  $u$  grey in  $v\sigma_i$ . Other occs of  $v$ :

If  $v$  also occurs in  $\Phi$ -term, then it occurs in a  $\Phi$ -term and in a grey position in a  $\Psi$ -literal, then by IH, it occurs grey in a grey literal in  $F\sigma_{(0, i-1)}$ . By  $\sigma_i$ , in  $F\sigma_i$ ,  $u$  occurs at a grey position in a grey literal in  $\text{PI}(C) \vee C$ .

If  $v$  also occurs grey in a  $\Phi$ -literal, then similar argument.  
done?

Supp s.c.  $\Psi$ -term contains  $w$  such that  $u$  grey in  $w\sigma_i$ . Supp  $u$  not grey in grey literal as otw done. other occs of  $w$ :

Supp  $w$  in s.c.  $\Phi$ -term: then IH.

Supp  $u$  in  $\Psi$ -term: then IH

Supp grey in res/fac lit:

Supp grey: then both occ grey in grey, done.

Supp  $\Phi$ -lit: then IH on  $w$

Supp  $\Psi$ :- then IH on  $u$

□



**Conjectured Lemma 13.** *12 nochmal gscheid:*

*Let  $u$  be a variable which occurs in  $\text{PI}(C) \vee C$  in some literal in a single-colored  $\Phi$ -term or grey in a  $\Phi$ -literal.*

*Suppose that  $u$  also occurs in  $\text{PI}(C) \vee C$  in some literal in a single-colored  $\Psi$ -term or grey in a  $\Psi$ -literal.*

*Then  $u$  occurs grey in a grey literal.*

Note that  $\Phi$  and  $\Psi$  are to be read as any pair of different colors, i.e.  $\Gamma$  and  $\Delta$  as well as  $\Delta$  and  $\Gamma$ .

*Proof.* Note that initially, every pair of clauses is variable-disjoint and all symbols of a clause are either all grey or  $\Phi$ -colored or all grey or  $\Psi$ -colored, hence the lemma is vacuously true.

Induction of  $\pi$  and  $\sigma$ .

Let  $u$  occur in  $F\sigma_{(0,i)}$  in a s.c.  $\Phi$ -term. **Suppose  $u$  also occurs in  $F\sigma_{(0,i)}$  in a s.c.  $\Phi$ -term.** Suppose that  $u$  occurs in  $F\sigma_{(0,i)}$  in a s.c.  $\Psi$ -term but not in  $F\sigma_{(0,i-1)}$  as otherwise we are done by the induction hypothesis ( **$u$  in s.c.  $\Psi$ -term before**). Then one of the following cases applies (positions of  $v$  in res/fac lit):

- There is a s.c.  $\Psi$ -term in  $F\sigma_{(0,i-1)}$  containing a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$ .

In the res/fac lits in  $\lambda$  and  $\lambda'$  respectively,  $u$  and  $v$  occur at the essentially same position.

- If that position is in an s.c.  $\Phi$ -term, then we are done by the IH on  $v$ .  **$v$  col change**
- If that position is in an s.c.  $\Psi$ -term, then we are done by the IH on  $u$ .  **$u$  col change NB: also:  $u$  is in  $\Phi$ -term before**
- Hence suppose that position is grey:  
If it's in a  $\Phi$ -literal, then we are done by the IH on  $v$ .  **$\Phi$ -lit + s.c.  $\Psi$ -term**  
If it's in a  $\Psi$ -literal, then we are done by the IH on  $u$ .  **$\Psi$ -lit + s.c.  $\Phi$ -term** (basically somewhat of propagation of preexisting circumstances)  
Otherwise the occurrence is grey in a grey literal, which we wanted to show. **?**

- There is a variable  $v$  in  $F\sigma_{(0,i-1)}$  such that  $v\sigma_i$  contains a s.c.  $\Psi$ -term which contains  $u$ . Then  $v\sigma_i$  occurs in  $F\sigma_{(0,i-1)}$  and we are done by the induction hypothesis.  **$u$  in s.c.  $\Psi$ -term before**

**TODO: formulate as other proof, this part more or less must work out**

**2nd part:**

Let  $u$  occur in  $F\sigma_{(0,i)}$  in a s.c.  $\Phi$ -term.

Supp  $u$  occurs grey in  $\Psi$ -literal in  $F\sigma_{(0,i)}$ . Supp not in  $F\sigma_{(0,i-1)}$  as then IH would take effect.  **$u$  grey in  $\Psi$ -lit**

Then exists  $v$  grey in  $\Psi$ -literal in  $F\sigma_{(0,i-1)}$  s.t.  $u$  grey in  $v\sigma_i$ . Distinguish based on occurrence of  $u$ ,  $v$  pair in res/fac lit:

- supp in s.c.  $\Psi$ -term. then IH on  $u$ .  **$u$  col change**
- supp in s.c.  $\Phi$ -term. then IH on  $v$ .  **$v$  grey in  $\Psi$ -lit and in s.c.  $\Phi$ -term**
- supp grey.
  - supp in  $\Phi$ -lit: then IH on  $v$   **$\Psi$ -lit +  $\Phi$ -lit**
  - supp in  $\Psi$ -lit: IH on  $u$   **$\Psi$ -lit + s.c.  $\Phi$ -term**, but basically same as before
  - supp in grey lit: done

**TODO: formulate similarly, in any case the IH or the desired statement is the case**

**3rd/4th part:**

Let  $u$  occur in  $F\sigma_{(0,i)}$  in grey in a  $\Phi$ -lit. Either it was there before, or:

There exists a  $\Phi$ -lit with a variable  $v$  at a grey position such that  $v\sigma_i$  contains a grey  $u$ . Occurrences of position of  $u$  and  $v$  in res/fac lit:

- Supp in s.c.  $\Psi$ -term. Then IH on  $v$ .  **$v$  in s.c.  $\Psi$ -term and grey in  $\Phi$ -lit**
- Supp in s.c.  $\Phi$ -term.

□

Reformulation sketch:

If some var  $u$  occurs in  $F\sigma_{(0,i)}$  in a s.c.  $\Phi$ -term, then:

- was there before
- exists col change var which is substituted for  $u$  in  $F\sigma_{(0,i)}$
- exists var grey in  $\Phi$ -lit and (usually elsewhere) in s.c.  $\Psi$ -term (or opposite coloring)

Sublemma: If  $u$  occurs grey in a  $\Phi$ -literal in  $F\sigma_{(0,i)}$ , then at least one of the following statements hold:

- $\langle 14_1 \rangle$
1.  $u$  occurs grey in a  $\Phi$ -literal in  $F\sigma_{(0,i)}$ .
- $\langle 14_2 \rangle$
2. A variable  $v$  occurs grey in a  $\Phi$ -lit and grey in a  $\Psi$ -lit such that  $v\sigma_i$  contains a grey occ of  $u$ .
- $\langle 14_3 \rangle$
3. A variable occurs grey in a  $\Phi$ -lit and in an s.c.  $\Psi$ -term, OR  
a variable occurs grey in a  $\Psi$ -lit and in an s.c.  $\Phi$ -term.  
Additionally, either this variable is  $u$  or it is some variable  $v$  such that  $v\sigma_i$  contains a grey  $u$ .
- $\langle 14_4 \rangle$
4.  $u$  occurs at a grey position in a grey literal.

Subproof: Suppose the lemma precondition. we consider the formation cases for this:

- $u$  occurs grey in a  $\Phi$ -literal in  $F\sigma_{(0,i)}$  and this fact is not changed by  $\sigma_i$ . Then we clearly have 1.
- Some  $\Phi$ -literal  $\lambda$  in  $F\sigma_{(0,i-1)}$  contains a variable  $v$  at a grey position such that  $v\sigma_i$  contains a grey occ of  $u$ . We distinguish further based on the position  $p$  in the literal where  $v$  and  $u$  face each other.
  - Suppose  $p$  is in a s.c.  $\Phi$ -term. Then we have 3 on  $v$ .
  - Suppose  $p$  is in a s.c.  $\Psi$ -term. Then we have 3 on  $u$ .
  - Suppose the res/fac lit is  $\Psi$ -colored, and the position is grey. Then we have 2 on  $v$ .
  - Suppose the res/fac lit is  $\Phi$ -colored, and the position is grey. Then we have 1
  - Suppose the res/fac lit is grey, and the position is grey. Then we have 4

In the following, we abbreviate  $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)$  by  $\chi$ .

(lemma:var\_grey\_col\_lit) **Lemma 14.** *Let  $\iota$  be an inference in a refutation of  $\Gamma \cup \Delta$ . Suppose that a variable  $u$  occurs grey in a  $\Phi$ -literal in  $\chi\sigma_{(0,i)}$ . Then at least one of the following statements holds:*

- (14\_1) 1. *The variable  $u$  occurs grey in a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$ .*
- (14\_2) 2. *Some variable  $v$  occurs in  $\chi\sigma_{(0,i-1)}$  grey in a  $\Phi$ -literal as well as grey in a  $\Psi$ -literal such that  $u$  occurs grey in  $v\sigma_i$ .*
- (14\_3) 3. *There is a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i^1$  and  $v$  occurs in  $\chi\sigma_{(0,i-1)}$  either grey in a  $\Phi$ -literal as well as in a single-colored  $\Psi$ -term in any literal, or grey in a  $\Psi$ -literal as well as in a single-colored  $\Phi$ -term in any literal.*
- (14\_4) 4. *The variable  $u$  occurs at a grey position in a grey literal in  $\chi\sigma_{(0,i-1)}$ .*

*Proof.* We consider the different cases which lead to the variable  $u$  in a grey position in a  $\Phi$ -literal in  $\chi\sigma_{(0,i)}$ :

- There already is a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$  which contains  $u$  at a grey position and  $\sigma_i$  does not change this. Then clearly 1 is the case.
  - Otherwise there must be a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$ , which contains a variable  $v$  at a grey position such that  $u$  occurs grey in  $v\sigma_i$ . Hence in the resolved or factorised literals  $\lambda$  and  $\lambda'$ , there is a position  $p$  such that w.l.o.g.  $\lambda|_p = v$  and  $\lambda'|_p$  contains a grey occurrence of  $u$ , and  $\lambda$  and  $\lambda'$  coincide along  $p$ . We distinguish based properties of the position  $p$ :
    - Suppose that  $p$  is contained in a single-colored  $\Phi$ -term. Then  $v$  occurs grey in a  $\Psi$ -literal as well as in a single-colored  $\Phi$ -term, which suffices for 3 as  $u$  occurs grey in  $v\sigma_i$ .
    - Suppose that  $p$  is contained in a single-colored  $\Psi$ -term. Then  $u$  occurs grey in a  $\Phi$ -literal as well in a single-colored  $\Psi$ -term, which implies 3.
    - Otherwise  $p$  is a grey position. We distinguish further:
      - \* Suppose that the resolved or factorised literal is  $\Phi$ -colored. Then  $u$  occurs grey in a  $\Phi$ -literal and we have established item 3.
      - \* Suppose that the resolved or factorised literal is  $\Psi$ -colored. Then the variable  $v$  occurs grey in a  $\Phi$ -literal as well as grey in a  $\Psi$ -literal, hence 2 is the case.
- Otherwise the resolved or factorised literal is grey and  $u$  occurs grey in a grey literal, which is sufficient for 4.  $\square$

(lemma:var\_in\_sc\_term) **Lemma 15.** *Let  $\iota$  be an inference in a refutation of  $\Gamma \cup \Delta$ . Suppose that a variable  $u$  occurs in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i)}$ . Then at least one of the following statements holds:*

- (15\_1) 1. *The variable  $u$  occurs in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$ .*

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<sup>1</sup>Note that this includes the case that  $v = u$  and  $\sigma_i$  is trivial on  $u$ .

- (15\_2) 2. There is a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$  and  $v$  occurs in a single-colored  $\Phi$ -term as well as in a single-colored  $\Psi$ -term in  $\chi\sigma_{(0,i-1)}$ .
- (15\_4) 3. There is a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$  and  $v$  occurs in  $\chi\sigma_{(0,i-1)}$  in a single-colored  $\Phi$ -term as well as at a grey position in a  $\Psi$ -literal.
- (15\_3) 4. The variable  $u$  occurs grey in a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$ .
- (15\_5) 5. The variable  $u$  occurs grey in a grey literal in  $\chi\sigma_{(0,i-1)}$ .

*Proof.* We consider the different cases which lead to the variable  $u$  in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i)}$ :

- There is a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$  which contains  $u$  such that  $\sigma_i$  does not change this. Then 1 is the case.
- Suppose that there is a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$  which contains a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$ .

Hence in the resolved or factorised literals  $\lambda$  and  $\lambda'$ , there is a position  $p$  such that w.l.o.g.  $\lambda|_p = v$  and  $\lambda'|_p$  contains a grey occurrence of  $u$ , and  $\lambda$  and  $\lambda'$  coincide along  $p$ . We distinguish based properties of the position  $p$ :

- Suppose that  $p$  is contained in a single-colored  $\Phi$ -term. Then  $u$  is contained in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$  and item 1 holds.
- Suppose that  $p$  is contained in a single-colored  $\Psi$ -term. As then  $v$  is contained in a single-colored  $\Phi$ -term as well as in a single-colored  $\Psi$ -term, 2 is the case.
- Suppose that  $p$  is a grey position. We distinguish further:
  - \* Suppose that the resolved or factorised literal is  $\Phi$ -colored. Then  $u$  occurs grey in a  $\Phi$ -literal, which suffices for 4.
  - \* Suppose that the resolved or factorised literal is  $\Psi$ -colored. Then the variable  $v$  occurs in a single-colored  $\Phi$ -term as well as grey in a  $\Psi$ -literal, which implies 3.
  - \* Otherwise the resolved or factorised literal is grey. But then  $u$  occurs grey in a grey literal and we have established item 5.
- Suppose that a variable  $w$  occurs in  $\chi\sigma_{(0,i-1)}$  such that  $u$  occurs in a single-colored  $\Phi$ -term in  $w\sigma_i$ . This can only be the case if  $w\sigma$  already occurs in  $\chi\sigma_{(0,i-1)}$ , which implies that 1 is the case.  $\square$

**Lemma 16.** Let  $C$  be a clause in the resolution refutation  $\pi$  of  $\Gamma \cup \Delta$  and  $u$  be a variable which occurs in  $\text{PI}(C) \vee C$  in some literal in a single-colored  $\Phi$ -term or grey in a  $\Phi$ -literal.

Suppose that  $u$  also occurs in  $\text{PI}(C) \vee C$  in some literal in a single-colored  $\Psi$ -term or grey in a  $\Psi$ -literal.

Then  $u$  occurs grey in a grey literal.

Note that  $\Phi$  and  $\Psi$  are to be read as any pair of different colors, i.e.  $\Gamma$  and  $\Delta$  as well as  $\Delta$  and  $\Gamma$ .

*Proof.* We proceed by induction over  $\pi$ . **TODO: also this proof strategy as outlined above**

Note that initially, every pair of clauses is variable-disjoint and all symbols of a clause are either all grey or  $\Phi$ -colored or all grey or  $\Psi$ -colored, hence the lemma is vacuously true.

For the induction step, suppose that  $u$  occurs in  $\chi\sigma_{(0,i)}$  in a single-colored  $\Phi$ -term or grey in a  $\Phi$ -literal and that  $u$  also occurs in  $\chi\sigma_{(0,i)}$  in a single-colored  $\Psi$ -term or grey in a  $\Psi$ -literal.

Then we can deduce by Lemma 14 and Lemma 15 that at least one of the following statements holds:

- $\langle \text{oozoh70h1} \rangle$  1. The variable  $u$  occurs grey in a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$ .
- $\langle \text{oozoh70h2} \rangle$  2. Some variable  $v$  occurs in  $\chi\sigma_{(0,i-1)}$  grey in a  $\Phi$ -literal as well as grey in a  $\Psi$ -literal such that  $u$  occurs grey in  $v\sigma_i$ .
- $\langle \text{oozoh70h3} \rangle$  3. There is a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$  and  $v$  occurs in  $\chi\sigma_{(0,i-1)}$  either grey in a  $\Phi$ -literal as well as in a single-colored  $\Psi$ -term in any literal, or grey in a  $\Psi$ -literal as well as in a single-colored  $\Phi$ -term in any literal.
- $\langle \text{oozoh70h4} \rangle$  4. The variable  $u$  occurs at a grey position in a grey literal in  $\chi\sigma_{(0,i-1)}$ .
- $\langle \text{oozoh70h5} \rangle$  5. The variable  $u$  occurs in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$ .
- $\langle \text{oozoh70h6} \rangle$  6. There is a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$  and  $v$  occurs in a single-colored  $\Phi$ -term as well as in a single-colored  $\Psi$ -term in  $\chi\sigma_{(0,i-1)}$ .

By the same lemmata, we get the same set of statements where  $\Phi$  and  $\Psi$  are interchanged. We refer to them by the respective number followed by  $*$ .

Suppose that 4 is not the case as otherwise we are done since  $\sigma_i$  is trivial on  $u$  as  $u$  occurs in  $\chi\sigma_{(0,i)}$ . Furthermore, there are a number of cases which give the result by the induction hypothesis: For the cases 2, 3 and 6, we can infer that by the induction hypothesis, there is a grey occurrence of the variable  $v$  in a grey literal in  $\chi\sigma_{(0,i-1)}$ , and as  $u$  occurs grey in  $v\sigma_i$ , there is a grey occurrence of  $u$  in a grey literal in  $\chi\sigma_{(0,i)}$ .

It remains to show that the lemma holds true in case the statements 1 or 5 as well as 1\* or 5\* hold. But note that in any combination of 1 or 5 and 1\* or 5\* in effect yields a situation under which the induction hypothesis again is applicable. Hence we may infer that  $u$  occurs grey in a grey literal in  $\chi\sigma_{(0,i-1)}$  and since  $\sigma_i$  is trivial  $u$  as shown above,  $u$  occurs grey in a grey literal in  $\chi\sigma_{(0,i)}$ .  $\square$