Example 101: canonical examples for overbinding w.r.t. order

Here, the second lower prooftree shows what huang would produce.

Ex 101a

$$\cfrac{\cfrac{\overset{\Sigma}{P({\color{red}u},f({\color{red}u}))\vee Q({\color{red}u})} \qquad \overset{\Pi}{\neg Q(a)}}{P(a,f(a))}\;u\mapsto a \qquad \overset{\Pi}{\neg P(x,y)}}{\square}\;x\mapsto a, y\mapsto f(a)$$

$$\cfrac{\cfrac{\bot \qquad \top}{Q(a)}\;u\mapsto a \qquad \top}{P(a,f(a))\vee Q(a)}\;x\mapsto a, y\mapsto f(a) \qquad\qquad \cfrac{\cfrac{\bot \qquad \top}{\forall x_1 Q(x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}$$

Direct overbinding would not work without merging same variables!: $\forall x_1\forall x_2\exists x_3(P(x_2,x_3)\vee Q(x_1))$
counterexample: $Q\sim\{0\}, P\sim\{(1,0)\}$
Direct overbinding would work when considering original dependencies as highlighted above

Ex 101b – other resolution order

$$\cfrac{\cfrac{\overset{\Sigma}{P(u,f(u))\vee Q(u)} \qquad \overset{\Pi}{\neg P(x,y)}}{Q(u))}\;y\mapsto f(u), x\mapsto u \qquad \overset{\Pi}{\neg Q(a)}}{\square}\;u\mapsto a$$

$$\cfrac{\cfrac{\bot \qquad \top}{P(u,f(u))}\;x\mapsto f(u), x\mapsto u \qquad \top}{P(a,f(a))\vee Q(a)}\;u\mapsto a \qquad\qquad \cfrac{\cfrac{\bot \qquad \top}{\exists x_1 P(u,x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}\;u\mapsto a$$

Ex 101c – Π and Σ swapped

$$\cfrac{\cfrac{\overset{\Pi}{P(u,f(u))\vee Q(u)} \qquad \overset{\Sigma}{\neg P(x,y)}}{Q(u))}\;y\mapsto f(u), x\mapsto u \qquad \overset{\Sigma}{\neg Q(a)}}{\square}\;u\mapsto a$$

$$\cfrac{\cfrac{\top \qquad \bot}{\neg P(u,f(u))}\;x\mapsto f(u), x\mapsto u \qquad \bot}{\neg P(a,f(a))\wedge\neg Q(a)}\;u\mapsto a \qquad\qquad \cfrac{\cfrac{\top \qquad \bot}{\forall x_2\neg P(u,x_2)} \qquad \bot}{\exists x_1\forall x_2(\neg P(x_1,x_2)\wedge\neg Q(x_1))}$$

Ex 101d – Π and Σ swapped, other resolution order

$$\cfrac{\cfrac{P(u,f(\overset{\Pi}{u}))\vee Q(u) \qquad \neg\overset{\Sigma}{Q}(a)}{P(a,f(a))}\;u\mapsto a \qquad \neg\overset{\Sigma}{P}(x,y)}{\square}\;x\mapsto a, y\mapsto f(a)$$

$$\cfrac{\cfrac{\top \qquad \bot}{\neg Q(a)}\;y\mapsto a \qquad \bot}{\neg Q(a)\wedge\neg P(a,f(a))}\;x\mapsto a, y\mapsto f(a)$$

$$\cfrac{\cfrac{\top \qquad \bot}{\exists x_1\neg Q(x_1)} \qquad \bot}{\exists x_1\forall x_2(\neg P(x_1,x_2)\wedge\neg Q(x_1))}$$

102 – similar to 101 but with intra-clause-set inferences

Ex 102a

$$\cfrac{\cfrac{P(f({\color{red}x}))\vee \overset{\Sigma}{Q}(f({\color{red}x}),z) \qquad \neg\overset{\Pi}{P}(y)}{Q(f(x),z)} \qquad \cfrac{\neg\overset{\Sigma}{Q}(x_1,{\color{blue}y})\vee R({\color{blue}y}) \qquad \neg\overset{\Pi}{R}(g(z_1))}{\neg Q(x_1,g(z_1))}\;y\mapsto g(z_1), x_1\mapsto f(x)}{\square}\;x_1\mapsto f(x), z\mapsto g(z_1)$$

$$\cfrac{\cfrac{\bot \qquad \top}{P(f(x))} \qquad \cfrac{\bot \qquad \top}{R(g(z_1))}\;y\mapsto g(z_1), x_1\mapsto f(x)}{P(f(x))\vee R(g(z_1))}\;x_1\mapsto f(x), z\mapsto g(z_1)$$

$$\cfrac{\cfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \cfrac{\bot \qquad \top}{\forall x_2 R(x_2)}}{\exists x_1\forall x_2(P(x_1)\vee R(x_2)) \quad \text{(order irrelevant!)}}$$

Ex 102b

$$\cfrac{\cfrac{P(f({\color{red}x}))\vee \overset{\Sigma}{Q}(f({\color{red}x}),z) \qquad \neg\overset{\Pi}{P}(y)}{Q(f(x),z)} \qquad \cfrac{\neg\overset{\Sigma}{Q}(f({\color{blue}y}),z_1)\vee R({\color{blue}y}) \qquad \neg\overset{\Pi}{R}(a)}{\neg Q(f(a),z_1)}\;y\mapsto a}{\square}\;x\mapsto a, z\mapsto z_1$$

$$\cfrac{\cfrac{\bot \qquad \top}{P(f(x))} \qquad \cfrac{\bot \qquad \top}{R(a)}\;y\mapsto a}{P(f(a))\vee R(a)}\;x\mapsto a, z\mapsto z_1$$

$$\cfrac{\cfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \cfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\;y\mapsto a}{\forall x_2\exists x_1(P(x_1)\vee R(x_2))}\;x\mapsto a, z\mapsto z_1$$

direct:

$$\cfrac{\cfrac{\bot \qquad \top}{\exists x_1 P(x_1)}\;x_1\sim f(x) \qquad \cfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\;x_2\sim a}{\exists x_1\forall x_2(P(x_1)\vee R(x_2))}\;\text{order irrelevant!}$$

Ex 102b' with Q grey

$$\dfrac{\dfrac{\overset{\Sigma}{P(f({\color{red}x})) \vee Q(f({\color{red}x}), z)} \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x), z)} \qquad \dfrac{\overset{\Sigma}{\neg Q(f({\color{blue}y}), z_1) \vee R({\color{blue}y})} \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a), z_1)}\ y \mapsto a}{\Box}\ x \mapsto a, z_1 \mapsto z$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\ y \mapsto a}{(\neg Q(f(a), z) \wedge P(f(a))) \vee (Q(f(a), z) \wedge R(a))}\ x \mapsto a, z_1 \mapsto z$$

Huang:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)} \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\ y \mapsto a}{\forall x_1 \exists x_2 (\neg Q(x_2, z) \wedge P(x_2)) \vee (Q(x_2, z) \wedge R(x_1))}\ x \mapsto a, z_1 \mapsto z$$

direct:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)}\ x_2 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\ x_1 \sim a}{\forall x_1 \exists x_2 \exists x_3 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}\ x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3}{\text{OR: } \exists x_2 \forall x_1 \exists x_3 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}}{\text{OR: } \exists x_1 \exists x_3 \forall x_2 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}$$

Huang-Version passt weil man für existenzielle quantoren eh gleichen term einsetzt

Example 103: variables in interpolant get unified and order might have to be changed

103a: length changes but old version is still valid, despite huang's algorithm doing a sort

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{Q(f(x)) \lor P(y) \lor R(x)} \qquad \overset{\Pi}{\neg Q(y_1)}}{P(y) \lor R(x)}\; y_1 \mapsto f(x) \qquad \overset{\Pi}{\neg P(h(g(a)))}}{R(x)}\; y \mapsto h(g(a)) \qquad \overset{\Pi}{\neg R(g(g(a)))}}{\square}\; x \mapsto g(g(a))$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{Q(f(x))}\; y_1 \mapsto f(x) \qquad \top}{Q(f(x)) \lor P(h(g(a)))}\; y \mapsto h(g(a)) \qquad \top}{Q(f(g(g(a)))) \lor P(h(g(a))) \lor R(g(g(a)))}\; x \mapsto g(g(a))$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_1 \forall x_2 (Q(x_1) \lor P(x_2))} \qquad \top}{X}$$

X:
Huang's algo gives:
$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$
Direct overbinding gives: $x_3 < x_1$, rest arbitrary, hence:
$\forall x_3 \exists x_1 \forall x_2 (Q(x_1) \lor P(x_2) \lor R(x_3))$ <- this you do not get with huang
$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$
$\forall x_3 \forall x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

103b: length changes "uniformly"

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{Q(f(f(x))) \lor P(f(x)) \lor R(x)} \qquad \overset{\Pi}{\neg Q(y_1)}}{P(f(x)) \lor R(x)}\; y_1 \mapsto f(f(x)) \qquad \overset{\Pi}{\neg P(y_2)}}{R(x)}\; y_2 \mapsto f(x) \qquad \overset{\Pi}{\neg R(g(a))}}{\square}\; x \mapsto g(a)$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{Q(f(f(x)))}\; y_1 \mapsto f(f(x)) \qquad \top}{Q(f(f(x))) \lor P(f(x))}\; y_2 \mapsto f(x) \qquad \top}{Q(f(f(g(a)))) \lor P(f(g(a))) \lor R(g(a))}\; x \mapsto g(a)$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_2 \exists x_1 (Q(x_1) \lor P(x_2))} \qquad \top}{\forall x_3 \exists x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))}$$

Huang and direct overbinding somewhat coincide as $x_2 < x_1$ in both cases, and $x_3 < x_1$ and $x_3 < x_2$

103c: Failed attempt: different variables, accidentally the same terms appear but no logical connection

$$\dfrac{\overset{\Sigma}{P(a,x)} \qquad \dfrac{\overset{\Sigma}{\neg Q(a)} \qquad \overset{\Pi}{\neg P(y, f(\textcolor{red}{z})) \vee Q(\textcolor{red}{z})}}{\neg P(y,f(a))}\, z \mapsto a}{\Box}\, y\mapsto a, x\mapsto f(a)$$

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg Q(a)}\, z\mapsto a}{P(a,f(a)) \wedge \neg Q(a)}\, y\mapsto a, x\mapsto f(a) \qquad\qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg Q(x_1)}}{\exists x_1\forall x_2 (P(x_1,x_2)\wedge\neg Q(x_1))}$$

order required for Π

direct:

$$\dfrac{\dfrac{\bot \qquad \dfrac{\bot\qquad\top}{\exists x_1\neg Q(x_1)}\, x_1\sim a}{\exists x_1\exists x_2\forall x_3(P(x_2,x_3)\wedge\neg Q(x_1))}\, x_2\sim a, x_3\sim f(a); x_1<x_3}{\text{OR: } \exists x_1\forall x_3\exists x_2(P(x_2,x_3)\wedge \neg Q(x_1))}$$

invariant:

$$\dfrac{\dfrac{\exists x_2(P(x_2,x)\vee\bot) \qquad \dfrac{\exists x_1(Q(x_1)\vee\bot) \qquad \forall x_3((\neg P(y,\textcolor{red}{x_3})\vee Q(\textcolor{red}{z}))\vee\top)}{\exists x_1\forall x_3\neg P(y,\textcolor{red}{x_3})\vee\neg Q(\textcolor{red}{x_1})}\, x_1\sim a}{\exists x_1\exists x_2\forall x_3(P(x_2,x_3)\wedge\neg Q(x_1))}\, x_2\sim a, x_3\sim f(a); x_1<x_3}{\text{OR: } \exists x_1\forall x_3\exists x_2(P(x_2,x_3)\wedge \neg Q(x_1))}$$

invariant in other resolution order

$$\dfrac{\dfrac{\bot \qquad \dfrac{\bot\qquad\top}{Q(\textcolor{red}{z})\vee\exists x_2\forall x_3 P(x_2,\textcolor{red}{x_3})}\, x_2\sim a, x_3\sim f(z)}{\exists x_1\exists x_2\forall x_3(P(x_2,x_3)\wedge\neg Q(x_1))}\, x_1\sim a; x_1<x_3}{\text{OR: } \exists x_1\forall x_3\exists x_2(P(x_2,x_3)\wedge \neg Q(x_1))}$$

invariant if Σ and Π swapped:

$$\dfrac{\dfrac{\bot \qquad \dfrac{\top\qquad\bot}{\neg P(y,f(x_1))\vee\forall x_1 Q(x_1)}\, x_1\sim a}{\forall x_1\forall x_2\exists x_3(\neg P(x_2,x_3)\vee Q(x_1))}\, x_2\sim a, x_3\sim f(a); x_1<x_3}{\text{OR: }\forall x_1\exists x_3\forall x_2(\neg P(x_2,x_3)\vee Q(x_1))}$$

SECOND ATTEMPT:

$$
\dfrac{\overset{\Sigma}{P(a)} \qquad \dfrac{\overset{\Sigma}{Q(z)} \qquad \dfrac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg P(y) \vee \neg Q(f(\textcolor{red}{x})) \vee S(\textcolor{red}{x})}}{\neg P(y) \vee \neg Q(f(a))}\; x \mapsto a}{\neg P(y)}\; z \mapsto f(a)}{\Box}\; y \mapsto a
$$

$$
\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg S(a)}\; x \mapsto a}{\neg S(a) \wedge Q(f(a))}\; z \mapsto f(a)}{P(a) \wedge \neg S(a) \wedge Q(f(a))}\; y \mapsto a
$$

Huang:

$$
\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 (\neg S(x_1) \wedge Q(x_2))}}{\exists x_1 \forall x_2 (P(x_1) \wedge \neg S(x_1) \wedge Q(x_2))}
$$

$\forall x_1 \exists x_2 (\neg P(x_1) \vee S(x_1) \vee \neg Q(x_2))$

similar fail

$\Rightarrow$ anytime there is $P(a, f(a))$, either they have a dependency or they are not both differently colored (grey is uncolored)

for the record, direct method anyway:

$$
\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}\; x \sim a}{\exists x_1 \forall x_2 \neg S(x_1) \wedge Q(x_2)}\; z \sim f(a); x_1 < x_2}{\exists x_1 \forall x_2 \exists x_3 P(x_3) \wedge \neg S(x_1) \wedge Q(x_2)}\; x_3 \sim a; x_3 \text{ need not be merged w } x_1
$$

Example: ordering on both ancestors where the merge forces a new ordering

202a – canonical

$$\dfrac{\dfrac{P(a,\overset{\Sigma}{x_1}) \vee R(y) \qquad \neg P(\overset{\Pi}{x}, fx)}{R(y)}\;\begin{array}{l} x_1 \mapsto fa \\ x \mapsto a \end{array} \qquad \dfrac{Q(x_2, g\overset{\Sigma}{x_2}) \vee \neg R(u) \qquad \dfrac{\overset{\Sigma}{\neg S(a)} \qquad \neg Q(f\overset{\Pi}{z}, x_3) \vee S(z)}{\neg Q(fa, x_3)}\; z \mapsto a}{\neg R(u)}\;\begin{array}{l} x_2 \mapsto fa, \\ x_3 \mapsto gfa \end{array}}{\square}$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a, f(a))}\;\begin{array}{l} x_1 \mapsto f(a) \\ x \mapsto a \end{array} \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg S(a)}\; z \mapsto a}{Q(f(a), g(f(a))) \wedge \neg S(a)}\;\begin{array}{l} x_2 \mapsto f(a), \\ x_3 \mapsto g(f(a)) \end{array}}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a))}$$

Huang

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \forall x_2 P(x_1, x_2))} \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 Q(x_2, x_3) \wedge \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 P(x_1, x_2) \vee (Q(x_2, x_3)) \wedge \neg S(x_1))}$$

direct:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \forall x_2 P(x_1, x_2))}\;\begin{array}{c} x_1 \sim a, x_2 \sim fa \\ x_1 < x_2 \end{array} \qquad \begin{array}{c} x_3 \sim a, x_4 \sim fa, x_5 \sim gfa) \\ x_3 < x_4, x_4 < x_5 \end{array} \; \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_3 \neg S(x_3)}\; x_3 \sim a}{\exists x_3 \forall x_4 \exists x_5 Q(x_4, x_5) \wedge \neg S(x_3)}}{\exists x_1 \forall x_2 \exists x_5 P(x_1, x_2) \vee (Q(x_2, x_5) \wedge \neg S(x_5))}\;\begin{array}{l} x_3 \mapsto x_1, x_4 \mapsto x_2 \\ x_1 < x_2, x_2 < x_5 \end{array}$$

without merge in end: $(x_1 < x_2, x_3 < x_4, x_4 < x_5)$

$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))$

$\exists x_3 \forall x_4 \exists x_5 \exists x_1 \forall x_2 P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))$

(also interwoven ones appear to work)

combined presentation:

$$\dfrac{\dfrac{\bot \mid P(a,x_1) \lor R(y) \qquad \top \mid -}{P(a,f(a)) \mid R(y)}\ \begin{array}{c} x_1 \mapsto f(a) \\ x \mapsto a \end{array} \qquad \dfrac{\bot \mid Q(x_2,g(x_2)) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(z),x_3) \lor S(z)}{\neg S(a) \mid \neg Q(f(a),x_3)}\ z \mapsto a}{Q(f(a),g(f(a))) \land \neg S(a) \mid \neg R(u)}\ \begin{array}{c} x_2 \mapsto f(a), \\ x_3 \mapsto g(f(a)) \end{array}}{P(a,f(a)) \lor (Q(f(a),g(f(a))) \land \neg S(a)) \mid \Box}$$

combined presentation ground:

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \lor R(y) \qquad \top \mid \neg P(x,f(x))}{(P(a,f(a)) \land \top) \lor (\neg P(a,f(a)) \land \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \lor S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(f(a),g(f(a))) \land \neg S(a) \mid \neg R(u)}}{P(a,f(a)) \lor (Q(f(a),g(f(a))) \land \neg S(a)) \mid \Box}$$

combined presentation ground with direct method but only Δ-terms removed :

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \lor R(y) \qquad \top \mid \neg P(x,f(x))}{(P(a,x_2) \land \top) \lor (\neg P(a,x_2) \land \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \lor S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(x_4,g(x_4)) \land \neg S(a) \mid \neg R(u)}}{P(a,x_2) \lor (Q(x_4,g(x_4)) \land \neg S(a)) \mid \Box}$$

combined presentation ground with direct method:

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \lor R(y) \qquad \top \mid \neg P(x,f(x))}{\exists x_1 \forall x_2 (P(x_1,x_2) \land \top) \lor (\neg P(x_1,x_2) \land \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \lor S(a)}{\exists x_3 \neg S(x_3) \mid \neg Q(f(a),g(f(a)))}}{\exists x_3 \forall x_4 \exists x_5 (Q(x_4,x_5)) \land \neg S(x_3)) \mid \neg R(u)}}{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (P(x_1,x_2) \lor (Q(x_4,x_5)) \land \neg S(x_3)) \mid \Box}$$

203a – some alternations

$$
\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x) \vee \overset{\Sigma}{\neg P}(f(x)) \qquad P(z) \vee \overset{\Pi}{Q}(g(z))}{R(x) \vee Q(g(f(x)))}\ z \mapsto f(x) \qquad \neg Q(y) \vee \overset{\Sigma}{S}(h(y))}{R(x) \vee S(h(g(f(x))))}\ y \mapsto g(f(x))}{S(h(g(f(a))))}\ x \mapsto a}{\Box}\ x_1 \mapsto h(g(f(a)))
$$

$$
\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\neg P(f(x))}\ z \mapsto f(x) \qquad \bot}{\neg Q(g(f(x))) \wedge \neg P(f(x))}\ y \mapsto g(f(x))}{\neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a)}\ x \mapsto a}{\neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a) \vee S(h(g(f(a))))}\ x_1 \mapsto h(g(f(a)))
$$

Huang:

$$
\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \neg P(x_1)} \qquad \bot}{\exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1))}}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0))}}{\forall x_0 \exists x_1 \forall x_2 \exists x_3 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0) \vee S(x_3))}
$$

Direct:

$$
\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \neg P(x_1)}\ x_1 \sim f(x) \qquad \bot}{\exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(f(x)))}\ x_2 \sim g(f(x)); x_1 < x_2}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0))}\ x_0 \sim a; x_0 < x_1, x_0 < x_2}{\forall x_0 \exists x_1 \forall x_2 \exists x_3 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0) \vee S(x_3))}\ x_3 \sim h(g(f(a))); x_0 < x_3, x_1 < x_3, x_2 < x_3
$$

203b – many Σ-literals, coloring per occurrence

$$
\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x) \vee \overset{\Sigma}{\neg P}(f(x)) \qquad P(z) \vee \overset{\Sigma}{Q}(g(z))}{R(x) \vee Q(gfx)}\ z \mapsto fx \qquad \neg Q(y) \vee \overset{\Sigma}{S}(h(y))}{R(x) \vee S(hgfx)}\ y \mapsto gfx}{S(hgfa)}\ x \mapsto a}{\Box}\ x_1 \mapsto hgfa
$$

$$
\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \bot}{\bot}\ z \mapsto fx \qquad \bot}{\bot}\ y \mapsto gfx}{R(a)}\ x \mapsto a}{S(hgfa) \vee R(a)}\ x_1 \mapsto hgfa
$$

$\to \forall x_1 \exists x_2 (R(x_1) \vee S(x_2))$

203b' – many Σ-literals, coloring per symbol, all predicates grey

$$
\dfrac{\overset{\Pi}{\neg S}(x_1) \qquad \dfrac{\dfrac{\dfrac{\overset{\Pi}{\neg R}(a) \qquad R(x) \vee \overset{\Sigma}{\neg P}(f(x))}{R(a) \mid \neg P(fa)}\, x \mapsto a \qquad P(z) \vee \overset{\Sigma}{Q}(g(z))}{P(fa) \wedge R(a) \mid Q(gfa)}\, z \mapsto fa \qquad \neg Q(y) \vee \overset{\Sigma}{S}(h(y))}{\neg Q(gfa) \wedge P(fa) \wedge R(a) \mid S(hgfa)}\, y \mapsto gfa}{S(hgfa) \vee (\neg Q(gfa) \wedge P(fa) \wedge R(a))}
$$

direct:

$$
\dfrac{\top \qquad \dfrac{\dfrac{\dfrac{\top \qquad \bot}{\forall x_1 R(x_1) \mid \neg P(fa)}\, x \mapsto a \qquad \bot}{\forall x_1 \exists x_2 (P(x_2) \wedge R(x_1)) \mid Q(gfa)}\, z \mapsto fa \qquad \bot}{\forall x_1 \exists x_2 \exists x_3 (\neg Q(x_3) \wedge P(x_2) \wedge R(x_1)) \mid S(hgfa)}\, y \mapsto gfa}{\forall x_1 \exists x_2 \exists x_3 \exists x_4 (S(x_4) \wedge (\neg Q(x_3) \wedge P(x_2) \wedge R(x_1)))}
$$

Example where variables are not the outermost symbol but order is still relevant

204a

$\Sigma = \{P(f(x), g(f(x)))\}$
$\Pi = \{P(f(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f(x_1), x_2)$

204b

$\Sigma = \{P(f^5(x), g(f(x)))\}$
$\Pi = \{P(f^5(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f^5(x_1), x_2)$

example with aufschaukelnde unification, such that direction of arrow isn't clear

205a

$$\dfrac{\overset{\Pi}{P(ff{\color{blue}y},g{\color{blue}y})} \qquad \dfrac{\overset{\Sigma}{\neg P({\color{red}x},y) \vee Q({\color{red}x})} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ff{\color{purple}z}) \vee R{\color{purple}z}}}{\neg R(a) \mid \neg Q(ffa)}\; z \mapsto a}{\neg R(a) \wedge Q(ffa) \mid \neg P(ffa,y)}\; x \mapsto ffa}{(\neg R(a) \wedge Q(ffa)) \vee \neg P(ffa,ga)}\; y \mapsto a$$

direct

$$\dfrac{\overset{\Pi}{P(ff{\color{blue}y},g{\color{blue}y})} \qquad \dfrac{\overset{\Sigma}{\neg P({\color{red}x},y) \vee Q({\color{red}x})} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ff{\color{purple}z}) \vee R{\color{purple}z}}}{\exists x_1 \neg R(x_1) \mid \neg Q(ffa)}\; z \mapsto a}{\exists x_1 \forall x_2 (\neg R(x_1) \wedge Q(x_2)) \mid \neg P(ffa,u)}\; x \mapsto ffa}{\exists x_1 \forall x_2 ((\neg R(x_1) \wedge Q(x_2)) \vee \neg P(x_2,ga))}\; y \mapsto a, u \mapsto ga$$

$\exists x_1 \forall x_2 \forall x_3 ((\neg R(x_1) \wedge Q(x_2)) \vee \neg P(x_2, x_3))$
arrow between x_1 and x_3 isn't really there, only weak connection between x_2 and x_3

205b $\sim$ 205a, but simpler

Suppose P occurs somewhere in Σ (result not that optimal in this setting, but correct)

$$\dfrac{\overset{\Pi}{P(ff{\color{blue}y},g{\color{blue}y})} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg P(ff{\color{purple}z},x) \vee R{\color{purple}z}}}{\neg R(a) \mid \neg P(ffa,x)}\; z \mapsto a}{\neg R(a) \vee \neg P(ffa,ga) \mid \Box}\; x \mapsto ga, y \mapsto a$$

$\exists x_1 R(x_1)$
$\exists x_1 \forall x_2 \forall x_3 (R(x_1) \vee \neg P(x_2, x_3))$

misc examples

201a

$$\dfrac{\dfrac{P(x,y) \overset{\Sigma}{\vee} \neg Q(y) \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(y)}\,x \mapsto a \qquad \dfrac{Q(f(z)) \overset{\Sigma}{\vee} R(z) \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a)}\,z \mapsto a}{\Box}\,y \mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,y)}\,x \mapsto a \qquad \dfrac{\bot \qquad \top}{R(a)}\,z \mapsto a}{P(a,f(a)) \vee R(a)}\,y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 P(x_1,y)}\,x \mapsto a \qquad \dfrac{\bot \qquad \top}{\forall x_3 R(x_3)}\,z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 (P(x_1,x_2) \vee R(x_3))}\,y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

201b

$$\dfrac{\dfrac{P(x,f(y)) \overset{\Sigma}{\vee} \neg Q(f(y)) \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(f(y))}\,x \mapsto a \qquad \dfrac{Q(f(z)) \overset{\Sigma}{\vee} R(z) \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a)}\,z \mapsto a}{\Box}\,y \mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,f(y))}\,x \mapsto a \qquad \dfrac{\bot \qquad \top}{R(a)}\,z \mapsto a}{P(a,f(a)) \vee R(a)}\,y \mapsto a \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 \exists x_2 P(x_1,x_2)}\,x \mapsto a \qquad \dfrac{\bot \qquad \top}{\forall x_3 R(x_3)}\,z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_3)}\,y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$