1 Outline

Goal: try proof Huang and nested equal.
Method: proof for both:

Conjectured Proposition 1. *Let I be an interpolant created by \$algorithm. If I contains a term t such that t has n color changes, then I has at least n quantifier alternations.*

1.1 generally keep in mind

- Need to define all new terms here: color-changing, single-color, Φ-literal, substitutions from 0 to n
  - essentially same position: path from one position to other only contains grey symbol (this def allows for identical position as well)
- also note: literal is sometimes used for negated or not negated predicate with terms but in regular formulas with arbitrary connectives

2 Preliminaries

Quantifier alternations in I usually assumes the quantifier-alternation-minimizing arrangement of quantifiers in I

Definition 2 (Color alternation col-alt)**.** Colors Γ and Δ, term t:
$\text{col-alt}(t) \overset{\text{def}}{=} \text{col-alt}_\perp(t)$
Let $t = f(t_1, \dots, t_n)$ for constant, function and variable symbols (syntax abuse):

$$\text{col-alt}_\Phi(t) \overset{\text{def}}{=} \begin{cases} \max{}^{1}(\text{col-alt}_\Phi(t_1), \dots, \text{col-alt}_\Phi(t_n)) & f \text{ is grey} \\ \max(\text{col-alt}_\Phi(t_1), \dots, \text{col-alt}_\Phi(t_n)) & f \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_\Psi(t_1), \dots, \text{col-alt}_\Psi(t_n)) & f \text{ is of color } \Psi,\ \Phi \neq \Psi \end{cases}$$

$\triangle$

Definition 3. $\text{PI}^\circ_{\text{step}}$ is defined just like PI_{step} but without applying any substitution. $\triangle$

Hence $\text{PI}^\circ_{\text{step}}(\cdot)\sigma = \text{PI}_{\text{step}}(\cdot)$. C° is somehow the same, i.e. if $C = D\sigma$, then $C^\circ = D$ where σ is derived from the context.

3 Random thoughts

- Quantifiers are introduced for lifting variables which actually occur in the interpolant
- If term t with $\text{col-alt}(t) = n$ enters I, we need subterm s of t with $\text{col-alt}(s) = n - 1$ to be in I (of course colors of t and s are exactly opposite)

[1] We assume that the maximum of an empty list of arguments is 0.

3.1 Proof

- Induction over $\ell^x_\Delta[\mathrm{PI}(C) \vee C]$ and also about Γ-terms with Δ-lifting vars in that formula. Cf. `-final`
- NB: now somewhat described in the proper proof below describe proof method with $\sigma_{(0,\,i)}$: which PI?
  - Factorisation: easy: just apply σ_i for all i to $\mathrm{PI}(C) \vee C$. When done, a literal will be there twice and we can remove it without losing anything
  - Resolution: create propositional structure first.
    Ex.: $C_1 : D \vee l$, $C_2 : \neg l \vee E$:
    If we talk about properties for which it holds that if they hold for $\mathrm{PI}(C_i) \vee C_i$, $i \in \{1, 2\}$, then they also hold for $A \equiv \Big((l \wedge \mathrm{PI}(C_2)) \vee (\neg l \wedge \mathrm{PI}(C_1))\Big) \vee C^\circ$, then we can apply σ_i for all i to that formula. So if we can assume it for A and show it for all σ_i, we get that it holds for $\mathrm{PI}(C) \vee C$.

  Also: clauses are variable disjoint, so e.g. it's not possible that a color-changing var is created by $\mathrm{PI}_{\mathrm{step}}$

  Also: do it like a few lemmas further down, like $(\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \ldots, \mathrm{PI}(C_n)) \vee C^\circ)\sigma_{(0,\,i)}$

4 directly from old proof

this may not be correct any more w.r.t. notation (χ)

just for repetition:

?⟨lemma:col_change⟩? **Lemma 4.** *Resolution or factorisation step ι from $\bar{C}$.*
If u col-change var in $(\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \ldots, \mathrm{PI}(C_n)) \vee C^\circ)\sigma_{(0,\,i)}$, then u also occurs grey in that formula.

Proof. Abbreviation: $F \equiv (\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \ldots, \mathrm{PI}(C_n)) \vee C^\circ)$

Induction over refutation and σ; base case easy.

Step: Supp color change var u present in $\chi\sigma_{(0,\,i)}$. (could also say introduced, then proof would be somehow different)

Supp u not grey in $\chi\sigma_{(0,\,i-1)}$ as otherwise done. As a first step, we show that if a (not necessarily color-changing) variable v occurs in a single-colored Φ-term $t[v]$ in $\chi\sigma_{(0,\,i)}$, then at least one of the following holds:

1. v occurs in some single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$
2. ⟨var_occ_1⟩ there is a color-changing variable w in $\chi\sigma_{(0,\,i-1)}$ such that v occurs grey in $w\sigma_i$.

⟨var_occ_2⟩ We consider unification process, and particularly the different cases which can introduce a variable v in a single-colored term Φ: Either it has been there before, it was introduced in a s.c. Φ-colored term, or a s.c. Φ-term containing the var is in $\mathrm{ran}(\sigma)$.

- Suppose a term $t'[v]$ is present in $\chi\sigma_{(0,\,i-1)}$ such that $t'[v]\sigma_i = t[v]$. Then 1 is the case.

- Suppose a variable w occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ such that v occurs grey in $w\sigma_i$. Suppose furthermore that 1 is not the case, i.e. v does not occur in a s.c. Φ-term in $\chi\sigma_{(0,\,i-1)}$, as otherwise we would be done. We show that 2 is the case.

  As v occurs neither grey nor in a s.c. Φ-term in $\chi\sigma_{(0,\,i-1)}$ but occurs in $\mathrm{ran}(\sigma_i)$, it must occur in $\chi\sigma_{(0,\,i-1)}$ and this can only be in a single-colored Ψ-term.

  As by assumption v occurs grey in $w\sigma_i$, there must be an occurrence $\hat{w}$ of w in a resolved or factorised literal, say $\lambda\sigma_{(0,\,i-1)}$ such that for the other resolved literal $\lambda'\sigma_{(0,\,i-1)}$, $\lambda'\sigma_{(0,\,i-1)}|_{\hat{w}}$ is a subterm in which v occurs grey. But as the occurrence of v in $\lambda'\sigma_{(0,\,i-1)}|_{\hat{w}}$ must be contained in a single-colored Ψ-term, so is $\lambda\sigma_{(0,\,i-1)}|_{\hat{w}}$, hence z occurs in a single-colored Ψ-term as well. Therefore 2 is the case.

- Suppose there is a variable z in $\chi\sigma_{(0,\,i-1)}$ such that v occurs in a single-colored Φ-term in $z\sigma_i$. Then $z\sigma_i$ occurs in $\chi\sigma_{(0,\,i-1)}$, but this is a witness for 1.

Now recall that we have assumed u to be a color-changing variable in $\chi\sigma_{(0,\,i)}$. Hence it occurs in a single-colored Γ-term as well as in a single-colored Δ-term. By the reasoning above, this leads to two case:

- In $\chi\sigma_{(0,\,i-1)}$, u occurs both in some single-colored Γ-term as well as in some single-colored Δ-term. Then we get the result by the induction hypothesis and the fact that $u \notin \mathrm{dom}(\sigma_i)$ as u does occur in $\chi\sigma_{(0,\,i)}$.

- Otherwise for some color Φ, u does not occur in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$. Then case 2 above must hold and there is some color-changing variable w in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $w\sigma_{(0,\,i)}$. But then by the induction hypothesis, w occurs grey in $\chi\sigma_{(0,\,i-1)}$ and hence u occurs grey in $\chi\sigma_{(0,\,i)}$. $\square$

5 Thursday prime

Definition 5 (PI*)**.** PI* is defined as PI with the difference that in PI*, all literals are considered to be grey. △

Hence $\mathrm{PI}^*_{\mathrm{init}}$ coincides with $\mathrm{PI}_{\mathrm{init}}$.

$\mathrm{PI}^*_{\mathrm{step}}$ coincides with $\mathrm{PI}_{\mathrm{step}}$ in case of factorisation and paramodulation inferences.

For resolution inferences, the first two cases in the definition of $\mathrm{PI}_{\mathrm{step}}$ do not occur for $\mathrm{PI}^*_{\mathrm{step}}$.

Proposition 6. *For every literal which occurs in a clause of a resolution refutation π, a respective successor occurs in* $\mathrm{PI}^*(\pi)$.

Proof. By structural induction. □

⟨ef:grey_lits_of_pi_star_in_pi⟩ **Lemma 7.** *For every clause C of a resolution refutation, every literal which is actually grey and occurs in* $\mathrm{PI}^*(C)$ *also occurs in* $\mathrm{PI}(C)$.

Proof. Note that $\mathrm{PI}_{\mathrm{init}}$ and $\mathrm{PI}^*_{\mathrm{init}}$ coincide and $\mathrm{PI}_{\mathrm{step}}$ and $\mathrm{PI}^*_{\mathrm{step}}$ only differ for resolution inferences. But more specifically, they only differ on resolution inferences, where the resolved literal is colored. However here, no grey literals are lost. □

Note that in PI*, we can conveniently reason about the occurrence of terms as no terms are lost throughout the extraction. However Lemma 7 allows us to transfer results about grey literals to PI.

Recall that by a postfix $\circ$, we denote the version where some context-dependent substitution σ is not applied.

In the following, we abbreviate $\mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_n)) \vee C^\circ$ by χ.

⟨lemma:var_grey_col_lit⟩ **Lemma 8.** *Let ι be an inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

⟨14_1⟩
1. *The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨14_5⟩
2. *The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨14_4⟩
3. *The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨14_2⟩
4. *There is a variable v such that*
   - *u occurs grey in $v\sigma_i$ and*
   - *v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Φ-literal as well as grey in a Ψ-literal.*

⟨14_3⟩
5. *There is a variable v such that*
   - *u occurs grey in $v\sigma_i$ and*
   - *v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.*

Proof. We consider the unification process, and particularly the different cases which lead to the variable u in a grey position in a Φ-literal in $\chi\sigma_{(0,\,i)}$:

- There already is a Φ-literal in $\chi\sigma_{(0,\,i-1)}$ which contains u at a grey position and σ_i does not change this. Then clearly 1 is the case.
- Otherwise there must be a Φ-literal in $\chi\sigma_{(0,\,i-1)}$, which contains a variable v at a grey position such that u occurs grey in $v\sigma_i$. Hence in the resolved or factorised literals λ and λ', there is a position p such that w.l.o.g. $\lambda|_p = v$ and $\lambda'|_p$ contains a grey occurrence of u, and λ and λ coincide along the path to p.

  Note that λ and λ' are contained in χ as all literals are added to the interpolant since the definition of χ is based on PI*.

  We distinguish based properties of the position p:

  – Suppose that p is contained in a single-colored Φ-term. Then u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and 2 is the case.

  – Suppose that p is contained in a single-colored Ψ-term. Then u occurs grey in a Φ-literal as well in a single-colored Ψ-term, which implies 5.

  – Otherwise p is a grey position. We distinguish further:

    ∗ Suppose that the resolved or factorised literal is Φ-colored. Then u occurs grey in a Φ-literal and we have established item 5.

    ∗ Suppose that the resolved or factorised literal is Ψ-colored. Then the variable v occurs grey in a Φ-literal as well as grey in a Ψ-literal, hence 4 is the case.
    Otherwise the resolved or factorised literal is grey and u occurs grey in a grey literal, which is sufficient for 3. □

⟨lemma:var_in_sc_term⟩ **Lemma 9.** *Let ι be an inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

⟨15_3⟩ 1. *The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_1⟩ 2. *The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_5⟩ 3. *The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_2⟩ 4. *There is a variable v such that u occurs grey in $v\sigma_i$ and*

   – *u occurs grey in $v\sigma_i$ and*

   – *v occurs in a single-colored Φ-term as well as in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_4⟩ 5. *There is a variable v such that*

   – *u occurs grey in $v\sigma_i$ and*

   – *v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Ψ-literal as well as in a single-colored Φ-term.*

Proof. We consider the different cases of the unification process which lead to the variable u in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$:

- There is a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which contains u such that σ_i does not change this. Then 2 is the case.
- Suppose that there is a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which contains a variable v such that u occurs grey in $v\sigma_i$.

  Hence in the resolved or factorised literals λ and λ' (which are both contained in χ), there is a position p such that w.l.o.g. $\lambda|_p = v$ and $\lambda'|_p$ contains a grey occurrence of u, and λ and λ' coincide along p. We distinguish based properties of the position p:

  – Suppose that p is contained in a single-colored Φ-term. Then u is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and 2 holds.

  – Suppose that p is contained in a single-colored Ψ-term. As then v is contained in a single-colored Φ-term as well as in a single-colored Ψ-term, 4 is the case.

  – Suppose that p is a grey position. We distinguish further:

    ∗ Suppose that the resolved or factorised literal is Φ-colored. Then u occurs grey in a Φ-literal, which suffices for 1.

    ∗ Suppose that the resolved or factorised literal is Ψ-colored. Then the variable v occurs in a single-colored Φ-term as well as grey in a Ψ-literal, which implies 5.

    ∗ Otherwise the resolved or factorised literal is grey. But then u occurs grey in a grey literal and we have established item 3.

- Otherwise there is a variable w which occurs in $\chi\sigma_{(0,\,i-1)}$ such that u occurs in a single-colored Φ-term in $w\sigma_i$. This can only be the case if $w\sigma$ already occurs in $\chi\sigma_{(0,\,i-1)}$, which implies that 2 is the case. □

ange_and_grey_in_col_lit_star⟩ **Lemma 10.** *Let C be a clause in the resolution refutation π of $\Gamma \cup \Delta$ and u be a variable which occurs in $\mathrm{PI}^*(C) \vee C$ in some literal in a single-colored Φ-term or grey in a Φ-literal.*

Suppose that u also occurs in $\mathrm{PI}^*(C) \vee C$ in some literal in a single-colored Ψ-term or grey in a Ψ-literal.

Then u occurs grey in a grey literal.

Note that Φ and Ψ are to be read as any pair of distinct colors, i.e. Γ and Δ as well as Δ and Γ.

Proof. We proceed by induction over π and σ.

Note that initially, every pair of clauses is variable-disjoint and all symbols of a clause are either all grey or Φ-colored or all grey or Ψ-colored, hence the lemma is vacuously true.

For the induction step, we assume that the property holds for $\mathrm{PI}^*(C_i) \vee C_i$, $1 \leq i \leq n$, where $C_1, \dots, C_n$ are the clauses used in a resolution or factorisation inference ι. Note that then, the property also holds for χ, i.e. for $\mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \dots, \mathrm{PI}^*(C_n)) \vee C^\circ$ as it contains all the literals present in $\mathrm{PI}^*(C_i) \vee C_i$ for any i (this is evident by the definition of $\mathrm{PI}^{*\circ}_{\mathrm{step}}$), and as clauses are pairwise variable-disjoint, the lemma condition can not become true for a variable for which it was not true in $\mathrm{PI}^*(C_i) \vee C$ for some i.

Suppose that u occurs in $\chi\sigma_{(0,\,i)}$ in a single-colored Φ-term or grey in a Φ-literal and that u also occurs in $\chi\sigma_{(0,\,i)}$ in a single-colored Ψ-term or grey in a Ψ-literal.

Then we can deduce by Lemma 8 and Lemma 9 that at least one of the following statements holds:

⟨oozoh70h1⟩
1. The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h5⟩
2. The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h4⟩
3. The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h2⟩
4. There is a variable v such that
   - u occurs grey in $v\sigma_i$ and
   - v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Φ-literal as well as grey in a Ψ-literal.

⟨oozoh70h6⟩
5. There is a variable v such that u occurs grey in $v\sigma_i$ and
   - u occurs grey in $v\sigma_i$ and
   - v occurs in a single-colored Φ-term as well as in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h3⟩
6. There is a variable v such that
   - u occurs grey in $v\sigma_i$ and
   - v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.

By the same lemmata, we get the same set of statements where Φ and Ψ are interchanged. We refer to them by the respective number followed by $^{\triangle}$.

Suppose that 3 is not the case as otherwise we are done since σ_i is trivial on u as u occurs in $\chi\sigma_{(0,\,i)}$. Furthermore, there are a number of cases which give the result by the induction hypothesis: For the cases 4, 5 and 6 we can infer that by the induction hypothesis, there is a grey occurrence of the variable v in a grey literal in $\chi\sigma_{(0,\,i-1)}$, and as u occurs grey in $v\sigma_i$, there is a grey occurrence of u in a grey literal in $\chi\sigma_{(0,\,i)}$.

It remains to show that the lemma holds true in case the statements 1 or 2 as well as $1^{\triangle}$or $2^{\triangle}$hold. But note that in any combination of 1 or 2 and $1^{\triangle}$or $2^{\triangle}$in effect yields a situation under which the induction hypothesis again is applicable. Hence we may infer that u occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$ and since σ_i is trivial u as shown above, u occurs grey in a grey literal in $\chi\sigma_{(0,\,i)}$. □

ol_change_and_grey_in_col_lit⟩ **Lemma 11.** *Same as 10 with* PI *in place of* PI*.

Proof. As PI(C) for any clause C is comprised of a subset of the literals of PI$^*(C)$, the lemma prerequesites hold true only for variables in PI(C) for which they also hold true in PI$^*(C)$. As by Lemma 10 the lemma holds for PI$^*(C)$, respective grey literals with grey occurrences of the variables in question exist in PI$^*(C)$. But by Lemma 13, these literals also occur in PI(C). □

6 Friday

Lemma 12. *If* PI$(C) \vee C$ *contains a maximal colored occurrence of a* Φ*-term* $t[s]$ *containing* Ψ*-term* s*, then* s *occurs grey in a grey literal in* PI$(C) \vee C$*.*

Proof. Note that it suffices to show that at the step where s is introduced as subterm of $t[s]$, s occurs grey in PI$(C) \vee C$ as any later modification by substitution is applied to both occurrences s, so they stay equal throughout the remaining derivation.

Induction over π and σ. TODO: as in Lemma 11

Base case: vacuously true.

Step: Resolution or factorisation inference ι, $\mathrm{mgu}(\iota) = \sigma = \sigma_1 \cdots \sigma_n$ The term $t[s]$ is created by one of the following two ways:
(we abbreviate $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \dots, \mathrm{PI}(C_n)) \vee C^{\circ}$ by F.)

- A variable u occurs in $\chi\sigma_{(0,\,i-1)}$ such that $u\sigma_i = t[s]$.

  Then u occurs in a resolved or factorised literal $\lambda\sigma_{(0,\,i-1)}$ at $\hat{u}$ such that at the other resolved or factorised literal $\lambda'\sigma_{(0,\,i-1)}$, $\lambda'\sigma_{(0,\,i-1)}|_{\hat{u}} = t[s]$. Then the condition is present at $\chi\sigma_{(0,\,i-1)}$ and we get the result by the induction hypothesis.

- Note that we only consider maximal colored terms.

  Let $t[u]$ be a maximal colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ such that in the tree-representation of $t[u]$, the path from the root to u does not contain a node labelled with a Ψ-symbol, and $u\sigma_i$ contains a grey occurrence of s.

  are probably not same t and s as in lemma statement, which isn't technically wrong but confusing

  Suppose that u occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$. Then s occurs grey in a grey literal in $\chi\sigma_{(0,\,i)}$ as σ_i does not affect u since u occurs in $\chi\sigma_{(0,\,i)}$ and we are done.

  If u occurs grey in a Ψ-literal or if u occurs in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$, then by Lemma 11, u also occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$ and s hence occurs grey in a grey literal in $\chi\sigma_{(0,\,i)}$.

  Now suppose that u does not occur grey in a grey literal $\chi\sigma_{(0,\,i-1)}$ as otherwise clearly we are done.

  Hence as all other cases are excluded, u can only occur in $\chi\sigma_{(0,\,i-1)}$ in a single-colored Φ-term or grey in a Φ-colored literal. But then, since $u\sigma_i$ contains a grey occurrence of s, there is a position p in the two resolved or factorised literals λ and λ' such that $\lambda|_p = u$ and $\lambda'|_p$ contains a grey occurrence of s. Furthermore, the prefix along the path to p is the same in both λ and λ'. As u only occurs in single-colored Φ-terms, $\lambda'|_p$ does so as well, so s is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$. Since s is a Ψ-term, by the induction hypothesis, s occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$ and hence also in $\chi\sigma_i$. □

⟨lemma:grey_lits_all_in_PI⟩ **Lemma 13.** *If there is a grey literal* λ *in a clause* C *of a resolution refutation* π*, then a successor of* λ *occurs in* PI(π)*.*

Proof. Immediate by the definition of PI. □

TODO: define quantifier alternations as col-alt, 0 == no quants, 1 == one quant, 2 is Π_2 or Σ_2

Proposition 14. *If a term with n color alternations occurs in* $\mathrm{PI}(C) \vee C$ *for a clause C, then the interpolant I produced in Theorem* **??** *contains at least n quantifier alternations.*

Proof. We perform an induction on n and show the strenghtening that the quantification of the lifting variable corresponding to a term with n color alternations is required to be in the scope of the quantification of $n-1$ alternating quantifiers.

For $n = 0$, no colored terms occur in I and hence by construction no quantifiers and for $n = 1$, there are only single-colored terms

Suppose the statement holds for $n-1$ for $n > 1$ and a term t with $\text{col-alt}(t) = n$ occurs in $\mathrm{PI}(C)$. We assume that t is a Φ-term. Then t contains a Ψ-colored term s and by Lemma 11, s occurs grey in a grey literal in $\mathrm{PI}(C) \vee C$. By Lemma 13, a successor of s occurs in $\mathrm{PI}(\pi)$. By the induction hypothesis, the quantification of the lifting variable for s requires $n-1$ alternated quantifiers. As s is a subterm of t and t is lifted, t must be quantified in the scope of the quantification of s, and as t and s are of different color, their quantifier type is different. Hence the quantification of the lifting variable for t requires n quantifier alternations. □

7 Monday: Paramodulation

7.1 Notes

1. Every equality which is used ends up in the interpolant, i.e. it's a grey literal (binary)
2. Every equality is used eventually

7.2 Proof

Extension of Lemma 8

ar_grey_col_lit_paramod_star⟩? **Lemma 15.** *Let ι be a* ***paramodulation*** *inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

⟨16_1⟩ 1. *The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨16_5⟩ 2. *The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨16_6⟩ 3. *The variable u occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$.*

subsumed by case above

⟨16_3⟩ 4. *There is a variable v such that*

   - *u occurs grey in $v\sigma_i$ and*

– *v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.*

Proof. Consider paramodulation: $s = t \vee D$ and $E[r]_p$ create $C : (D \vee E[t]_p)\sigma$ where $\sigma = \mathrm{mgu}(s, r)$.

A grey occurrence of variable u can be created in C by 2 means: either t contains a grey variable and p is a grey position or σ introduces a grey occurrence of a variable in a grey position

Let u be a variable in a grey position in a Φ-literal in $\chi\sigma_{(0,\,i)}$. We consider the cases which lead to this situation:

- The variable u is present in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$. Then clearly 1 is the case.
- Suppose that t contains a grey occurrence of u and p is a grey position in a Φ-literal. Then u occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ and 3 is the case.
- Suppose that σ_i introduces a grey occurrence of u in a Φ-literal (but not in p[2]). Then there exists a variable v which occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$. By that, we can derive that there exists a position q in s and r respectively such that 1 $s|_q = v$ and $r|_q$ contains u grey OR 2 $r|_q = v$ and $s|_q$ contains u grey. Furthermore, s and r agree on the path to q

  We distinguish:

  – Suppose that q is contained in a single-colored Φ-term.
    Then u is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which establishes 2.(1 + 2).
    (r might still be contained in other colored terms, but u remains in a single-colored Φ-term.

  – Suppose that q is contained in a single-colored Ψ-term. Then v occurs grey in a Φ-literal and in a s.c. Ψ-term and $v\sigma_i$ contains u grey, thus establishing 4.

  – Suppose that q is a grey position.
    NOTE: r might be in a colored term.
    Then u occurs at in a grey position in an equality which shows that 3 is the case.

$\square$

lemma:var_in_sc_term_paramod⟩? **Lemma 16.** *Let ι be a* ***paramodulation*** *inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$.*

Then at least one of the following statements holds:

?⟨17_1⟩? *1. The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.*

[2] not sure if this remark is really useful

⟨17_5⟩ *2. The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨17_6⟩ *3. The variable u occurs in a single-colored Φ-term an equality in $\chi\sigma_{(0,\,i-1)}$.*

subsumed by case above

?⟨17_3⟩? *4. There is a variable v such that*

- *u occurs grey in $v\sigma_i$ and*
- *v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.*

Proof. Let u be a variable in a in single-colored Φ-term in $\chi\sigma_{(0,\,i)}$. We consider the cases which lead to this:

- u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$. Then 2 is the case.
- Suppose that t contains an occurrence of u a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$. Then 3 is the case.
- Suppose that σ_i introduces a grey occurrence of u in a single-colored Φ-term. Then there exists a variable v which occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$ such that u occurs grey in $v\sigma_i$ TODO:
- direct introduction in $v\sigma_i$ TODO:

□