Need I_1, I_2, I_1', I_2', l, l', σ such that there is no $\varphi_{\text{res}}(X_1, X_2, l, l', \sigma)$ with

- $\varphi_{\text{res}}(I_1, I_2, l, l', \sigma)$ is interpolant for π
- $\varphi_{\text{res}}(I_1', I_2', l, l', \sigma)$ is interpolant for π'

I_i° is interpolant for π_i°
φ_{res} is "formula scheme", i.e. a rigid formula with occurrences of arguments.
it exists in prop, but not in fol.

π:

$$\dfrac{\begin{matrix}[\pi_1] & & [\pi_2] \\ C \lor l & & D \lor \neg l'\end{matrix}}{(C \lor D)\sigma}$$

π':

$$\dfrac{\begin{matrix}[\pi_1'] & & [\pi_2'] \\ E \lor l & & F \lor \neg l'\end{matrix}}{(E \lor F)\sigma}$$

Attempt 1

$\Sigma = \{P(u, f(u) \lor Q(u)), C(c)\}$
$\Pi = \{\neg Q(a), \neg P(a, y) \lor \neg C(c)\}$
π:

$$\dfrac{\dfrac{\overset{\Sigma}{P(u, f(u) \lor Q(u)} \qquad \overset{\Pi}{\neg Q(a)}}{\forall x_1 Q(x_1) \parallel P(a, f(a))} \qquad \dfrac{\overset{\Sigma}{C(c)} \qquad \overset{\Pi}{\neg P(a, y) \lor \neg C(c)}}{C(c) \parallel \neg P(a, y)}}{(P(a, f(a)) \land C(c)) \lor (\neg P(a, f(a)) \land \forall x_1 Q(x_1)) \parallel \forall \mathbf{x_1} \exists \mathbf{x_2} (\mathbf{P}(\mathbf{x_1}, \mathbf{x_2}) \land \mathbf{C}(\mathbf{c})) \lor (\neg \mathbf{P}(\mathbf{x_1}, \mathbf{x_2}) \land \forall \mathbf{x_1} \mathbf{Q}(\mathbf{x_1})) \parallel \Box}$$

$\Sigma' = \{Q(a), P(a, y) \lor C(c)\}$
$\Pi' = \{\neg P(u, f(u) \lor \neg Q(f(u))), \neg C(c)\}$
π':

$$\dfrac{\dfrac{\overset{\Pi}{\neg P(u, f(u) \lor \neg Q(f(u)))} \qquad \overset{\Sigma}{Q(x)}}{\forall x_1 Q(x_1) \parallel \neg P(u, f(u))} \qquad \dfrac{\overset{\Pi}{\neg C(c)} \qquad \overset{\Sigma}{P(a, y) \lor C(c)}}{C(c) \parallel P(a, y)}}{(P(a, f(a)) \land \forall x_1 Q(x_1)) \lor (\neg P(a, f(a)) \land C(c))}$$

Then we get:
$\varphi_{\text{res}}(\forall x_1 Q(x_1), C(c), P(a, f(a)), \neg P(a, y), \sigma) = \forall x_1 \exists x_2 (P(x_1, x_2) \land C(c)) \lor (\neg P(x_1, x_2) \land \forall x_1 Q(x_1))$
$\varphi_{\text{res}}(\forall x_1 Q(x_1), \bot, P(a, f(a)), \neg P(a, y), \sigma') = \exists x_2 \forall x_1 (Q(x_1) \lor P(x_1, x_2))$
// NOTE: we merged the x_1 here which we can't do with a homomorphism
// TODO: example with f(x) instead of f(u) in P with free x