$\Sigma' = \{R(z) \vee \exists x P(f(x)), \neg Q(x), \}$

$\Pi' = \{\forall y\, g(y) = y, \forall y \neg P(g(y)) \vee Q(y), \neg R(d)\}$

$\Sigma = \mathrm{sk}(\Sigma') = \{R(z) \vee P(f(c)), \neg Q(y), \}$

$\Pi = \mathrm{sk}(\Pi') = \{g(u) = u, \neg P(g(v)) \vee Q(v), \neg R(d)\}$

$L(\Sigma) = \{R, P, Q, f, z, x, c\}$

$L(\Pi) = \{R, P, Q, g, u, v, d\}$

Refutation:

$$\dfrac{\dfrac{R(z) \vee P(f(c))_\Sigma \qquad \neg R(d)_\Pi}{P(f(c))}\; z \mapsto d \qquad \dfrac{\dfrac{\neg P(g(v)) \vee Q(v)_\Pi \qquad \neg Q(y)_\Sigma}{\neg P(g(y))}\; v \mapsto y \qquad g(u) = u_\Pi}{\neg P(u)}\; y \mapsto u}{\square}\; u \mapsto f(c)$$

Interpolants:

$$\dfrac{\dfrac{\bot \qquad \top}{(\neg R(d) \wedge \bot) \vee (R(d) \wedge \top) \equiv R(d)}\; \theta_0 \qquad \dfrac{\dfrac{\top \qquad \bot}{(\neg Q(y) \wedge \top) \vee (Q(y) \wedge \top) \equiv \neg Q(y)}\; \theta_1 \qquad \top}{(\neg Q(u) \wedge g(u) = u) \vee (\top \wedge g(u) \neq u)}\; \theta_2}{(\neg P(f(c)) \wedge R(d)) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge g(f(c)) = f(c)) \quad \vee \quad g(f(c)) \neq f(c)))}\; \theta_3$$

Relative interpolant properties:

θ_0 :	$\Sigma \vdash R(d) \vee P(f(c))$	$\Pi \vdash \neg R(d) \vee P(f(c))$
θ_1 :	$\Sigma \vdash \neg Q(y) \vee \neg P(g(y))$	$\Pi \vdash Q(y) \vee \neg P(g(y))$
θ_2 :	$\Sigma \vdash (\neg Q(u) \wedge g(u) = u) \vee g(u) \neq u \quad \vee \quad \neg P(u)$	$\Pi \vdash \neg((\neg Q(u) \wedge g(u) = u) \vee g(u) \neq u) \quad \vee \quad \neg P(u)$ $\Pi \vdash ((Q(u) \vee g(u) \neq u) \wedge g(u) = u) \quad \vee \quad \neg P(u)$
θ_3 :	$\Sigma \vdash \theta_3$ Proof: Either $\neg P(f(c))$, then $R(d)$. Otw. either $g(f(c)) \neq f(c)$. Otw. also $\neg Q(f(c))$.	$\Pi \vdash \neg\theta_3$ Proof: $\neg(\neg P(fc) \wedge R(d)) \quad \vee \quad (P(fc) \wedge (\neg Q(fc) \wedge g(fc) = fc) \quad \vee \quad g(fc) \neq fc)$ $\equiv (P(fc) \vee \neg R(d)) \quad \wedge \quad (\neg P(fc) \vee (Q(fc) \vee g(fc) \neq fc) \quad \wedge \quad g(fc) = fc)$ Have $g(fc) = fc$ and $\neg R(d)$, so remaining: $\neg P(fc) \vee Q(fc)$. Get by axiom and unification with $g(u) = u$.

$\Sigma = \{R(z) \vee P(f(c)), \neg Q(y), \}$
$\Pi = \{g(u) = u, \neg P(g(v)) \vee Q(v), \neg R(d)\}$

Propositional refutation tree (no non-trivial unifiers):

$$\dfrac{\dfrac{R(d) \vee P(f(c))_\Sigma \qquad \neg R(d)_\Pi}{P(f(c))} \qquad \dfrac{\dfrac{\neg P(g(f(c))) \vee Q(f(c))_\Pi \qquad \neg Q(f(c))_\Sigma}{\neg P(g(f(c)))} \qquad g(f(c)) = f(c)_\Pi}{\neg P(f(c))}}{\square}$$

Lifting:

terms: $g(f(c)), f(c), d$
max Π-terms: $\{g(f(c)), d\} \sim \{x_1, x_2\}$
max Σ-terms: $\{f(c)\} \sim \{x_3\}$

$\overline{(\neg P(f(c)) \wedge R(d)) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge g(f(c)) = f(c)) \quad \vee \quad g(f(c)) \neq f(c)))}(x_1, x_2)$
$\Leftrightarrow \neg P(f(c)) \wedge R(x_2) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge x_1 = f(c)) \quad \vee \quad x_1 \neq f(c)))$

By Lemma 12, $\Sigma \models \overline{\theta_3}$ (proof from above still goes through).
$\hat{\theta}(x_3) = (\neg P(x_3) \wedge R(x_2)) \quad \vee \quad (P(x_3) \wedge ((\neg Q(x_3) \wedge x_1 = x_3) \quad \vee \quad x_1 \neq x_3))$
quantifiers according to order: $|d| < |f(c)| < |g(f(c))|$
$\theta = \forall x_2 \exists x_3 \forall x_1 (\neg P(x_3) \wedge R(x_2)) \quad \vee \quad (P(x_3) \wedge (\neg Q(x_3) \vee x_1 \neq x_3))$
$\neg\theta = \exists x_2 \forall x_3 \exists x_1 (P(x_3) \vee \neg R(x_2)) \quad \wedge \quad (\neg P(x_3) \vee (Q(x_3) \wedge x_1 = x_3))$
$\Rightarrow \Sigma \vdash \theta; \Pi \vdash \neg\theta$

Example 2:

$\Sigma = \{P(c), \neg P(d)\}$

$\Pi = \{P(d) \lor g(u) = u, \neg P(g(x))\}$

Refutation:

$$\dfrac{\dfrac{\dfrac{P(d) \lor g(u) = u_{\Pi} \qquad \neg P(d)_{\Sigma}}{g(u) = u} \qquad \neg P(g(x)_{\Pi}}{\neg P(x)}\; u \mapsto x \qquad P(c)_{\Sigma}}{\square}\; x \mapsto c$$

Relative interpolants:

$$\dfrac{\dfrac{\dfrac{\top \qquad \bot}{(\neg P(d) \land \top) \lor (P(d) \land \bot) \equiv \neg P(d)} \qquad \top}{(g(x) = x \land \top) \lor (g(x) \neq x \land \neg P(d))}\; u \mapsto x \qquad \bot}{(\neg P(c) \land \bot) \lor (P(c) \land (g(c) = c \lor (g(c) \neq c \land \neg P(d))))}\; x \mapsto c$$

$\theta = P(c) \land (g(c) = c \lor \neg P(d))$

$\neg\theta = \neg P(c) \lor (g(c) \neq c \land P(d))$

terms: $g(c), c, d$

max Π-terms: $g(c)$

max Σ-terms: c

ordered by length ASCENDING: $\{c, g(c)\}$

$\overline{\theta}(x_2) = P(c) \land (x_2 = c \lor \neg P(d))$

$\hat{\theta}(x_1) = P(x_1) \land (x_2 = x_1 \lor \neg P(d))$

$\Sigma \vdash \exists x_1 \forall x_2 P(x_1) \land (x_2 = x_1 \lor \neg P(d))$

$\Pi \vdash \neg\exists x_1 \forall x_2 P(x_1) \land (x_2 = x_1 \lor \neg P(d))$

$\Pi \vdash \forall x_1 \exists x_2 \neg P(x_1) \lor (x_2 \neq x_1 \land P(d))$

A possible interpolant: $\neg P(d) \land \exists x P(x)$

Example 2 (Craig translation):

$\Sigma = \{P(c), \neg P(d)\}$

$\Pi = \{P(d) \vee g(u) = u, \neg P(g(x))\}$

$T(\Sigma) = \{\forall x\; x = x\} \cup \{\forall x \forall y\; x = y \supset P(x) \supset P(y)\} \cup \Sigma$

$$
\begin{aligned}
T(\Pi) = & \{\forall x\; x = x\} \cup \\
& \{\forall x \forall y\; x = y \supset P(x) \supset P(y), \forall x_1 \forall x_2 \forall y_1 \forall y_2\; x_1 = y_1 \supset x_2 = y_2 \supset x_1 = x_2 \supset y_1 = y_2, \forall x_1 \forall x_2 \forall y_1 \forall y_2\; x_1 = y_1 \supset x_2 = y_2 \supset G(x_1, x_2) \supset G(y_1, y_2)\} \cup \\
& \{P(d) \vee (\exists z G(u, z) \wedge (\forall y G(u, y) \supset z = y) \wedge z = u), \neg P(g(x))\}
\end{aligned}
$$

to continue seems to be not work the effort

Example 3 Bonacina/Johannson:

$\Sigma = \{A \lor B, \neg C\}$

$\Pi = \{\neg A \lor C, \neg B\}$

$$\dfrac{\dfrac{\dfrac{A \lor B_\Sigma \qquad \neg A \lor C_\Pi}{B \lor C} \qquad \neg C_\Sigma}{B} \qquad \neg B_\Pi}{\Box}$$

Bon/Joh:

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{(A \lor \bot) \land \top \equiv A} \qquad \bot}{A \land (\neg C \lor \bot) \equiv A \land \neg C} \qquad \top}{(B \lor (A \land \neg C)) \land \top}$$

Huang:

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{(\neg A \land \bot) \lor (A \land \top) \equiv A} \qquad \bot}{(\neg C \land A) \lor (C \land \bot) \equiv \neg C \land A} \qquad \top}{(\neg B \land (\neg C \land A)) \lor (B \land \top)}$$

-> logically equivalent

Example 3B Bonacina/Johannson:

$\Sigma = \{A \lor B, \neg C, \neg D\}$

$\Pi = \{\neg A \lor C, \neg B \lor D\}$

$$\dfrac{\dfrac{\dfrac{A \lor B_\Sigma \qquad \neg A \lor C_\Pi}{B \lor C} \qquad \neg C_\Sigma}{B} \qquad \dfrac{\neg D_\Sigma \qquad \neg B \lor D_\Pi}{\neg B}}{\Box}$$

Bon/Joh:

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{(A \lor \bot) \land \top \equiv A} \qquad \bot}{A \land (\neg C \lor \bot) \equiv A \land \neg C} \qquad \dfrac{\bot \qquad \top}{\top \land (\neg D \lor \bot) \equiv \neg D}}{(B \lor (A \land \neg C)) \land \neg D}$$

Huang:

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{(\neg A \land \bot) \lor (A \land \top) \equiv A} \qquad \bot}{(\neg C \land A) \lor (C \land \bot) \equiv \neg C \land A} \qquad \dfrac{\bot \qquad \top}{(\neg D \land \top) \lor (D \land \bot) \equiv \neg D}}{(\neg B \land \neg C \land A) \lor (B \land \neg D)}$$

-> not logically equivalent

Example 4: Paramodulation special case in Huang

$$\cfrac{\cfrac{\cfrac{P(x) \vee \neg Q(x)_\Sigma \qquad Q(h(r))_\Pi}{P(h(r))} \qquad s = t_\Pi}{P(h(s))}\, r \mapsto s \qquad \neg P(h(s))_\Pi}{\Box}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\bot \qquad \top}{\neg Q(h(r))} \qquad \top}{(s = t \wedge \neg Q(h(t))) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t))} \qquad \top}{(\neg P(h(s)) \wedge (s = t \wedge \neg Q(h(t))) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t))) \vee P(h(s))}$$

$\Sigma = \{P(x) \vee \neg Q(x)\}$

$\Pi = \{\neg P(h(s)), Q(h(r)), s = t\}$

$((s = t \wedge \neg Q(h(t))) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t))) \vee P(h(s))$

$\theta = \neg Q(h(t)) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t)) \vee P(h(s))$

$\neg\theta = Q(h(t)) \wedge (s = t) \wedge (s \neq t \vee h(s) = h(t)) \wedge \neg P(h(s))$

$\theta^* = \forall x_1 \forall x_2 \forall x_3 \forall x_4 \neg Q(x_2) \vee (x_3 \neq x_4) \vee (x_3 = x_4 \wedge x_1 \neq x_2) \vee P(x_1)$

$\neg\theta^* = \exists x_1 \exists x_2 \exists x_3 \exists x_4 Q(x_2) \wedge (x_3 = x_4) \wedge (x_3 \neq x_4 \vee x_1 = x_2) \wedge \neg P(x_1)$

$\Rightarrow$ special case not needed here

Example 4b: Paramodulation special case in Huang

$$\frac{\dfrac{\dfrac{P(x) \lor \neg Q(x)_{\Sigma} \qquad Q(h(t))_{\Pi}}{P(h(t))} \qquad s = t_{\Sigma}}{P(h(s))} \qquad \dfrac{\neg P(x) \lor Q(x)_{\Sigma} \qquad \neg Q(h(s))_{\Pi}}{\neg P(h(s))}}{\Box}$$

$$\frac{\dfrac{\dfrac{\bot \qquad \top}{\neg Q(h(t))} \qquad \bot}{((s = t) \land \neg Q(h(t))) \lor (s = t \land h(s) \neq h(t)} \qquad \dfrac{\bot \qquad \top}{Q(h(s))}}{(\neg P(h(s)) \land ((s = t) \land \neg Q(h(t))) \lor (s = t \land h(s) \neq h(t))) \lor (P(h(s) \land Q(h(s))))}$$

$\Sigma = \{P(x) \lor \neg Q(x), \neg P(x) \lor Q(x), s = t\}$ $\quad // \; P(x) \leftrightarrow Q(x)$

$\Pi = \{\neg Q(h(s)), Q(h(t))\}$

$\theta = (\neg P(h(s)) \land ((s = t) \land \neg Q(h(t))) \lor (s = t \land h(s) \neq h(t))) \lor (P(h(s) \land Q(h(s))))$

$\neg\theta = (P(h(s)) \lor ((s \neq t) \lor Q(h(t))) \land (s \neq t \lor h(s) = h(t))) \land (\neg P(h(s) \lor \neg Q(h(s))))$

$\theta^* = \forall x_1 \forall x_2 (\neg P(x_1) \land ((s = t) \land \neg Q(x_2)) \lor (s = t \land x_1 \neq x_2)) \lor (P(x_1) \land Q(x_1))$

$\neg\theta^* = \exists x_1 \exists x_2 (P(x_1) \lor ((s \neq t) \lor Q(x_2)) \land (s \neq t \lor x_1 = x_2)) \land (\neg P(x_1) \lor \neg Q(x_2))$

$\Rightarrow$ special case IS needed

Example 5: cases for one pass overbinding algo

want to have step in between where only one of the "critical" terms appears in the interpolant and a decision on the order is forced

$$\dfrac{P(y_1, y_2)_\Sigma \qquad \dfrac{Q(\alpha)_\Sigma \qquad \neg Q(z) \vee \neg P(z, \beta)_\Pi}{\neg P(\alpha, \beta)}\ z \mapsto \alpha}{\square}$$

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{Q(\alpha)^\circ}\ z \mapsto \alpha}{P(\alpha, \beta) \wedge Q(\alpha)}$$

$\Rightarrow$ need to overbind α first, no matter which order would be assigned later

NOTE: b might be $f(a)$, i.e. we don't know a priori at which level it is and how many smaller or larger terms will be added.

Let $\alpha = b$, $\beta = g(z)$.

$\theta^* = \exists x_1 \forall x_2 P(x_1, x_2) \wedge Q(x_1)$

$\neg\theta^* = \forall x_1 \exists x_2 \neg P(x_1, x_2) \vee \neg Q(x_1)$

$\theta^{\circ *} = \exists x_1 Q(x_1)$

Let $\alpha = g(x)$, $\beta = b$.

$\theta^* = \exists x_1 \forall x_2 P(x_1, x_2) \wedge Q(x_1)$

$\neg\theta^* = \forall x_1 \exists x_2 \neg P(x_1, x_2) \vee \neg Q(x_1)$

$\theta^{\circ *} = \exists x_1 Q(x_1)$

$\Rightarrow$ works (need not change quantifier order like this, but here, no predicate has parameters which depend on each other)

Example 5b: no equality, but quantifier order still matters

$$\frac{P(u,g(u))_\Sigma \qquad \neg P(a,x)_\Pi}{\square}\ u \mapsto a, x \mapsto g(a)$$

Prop Interpolant: $P(a, g(a)$

Interpolant: $\forall x_1 \exists x_2 P(x_1, x_2)$

Example 5b': order matters, construction in multiple steps:

$$\frac{\dfrac{P(u,v,f(u,v)) \vee Q(u)_\Sigma \qquad \neg Q(a)_\Pi}{P(a,v,f(a,v))}\ u \mapsto a \qquad \neg P(x,b,y)_\Pi}{\square}\ x \mapsto a, v \mapsto b, y \mapsto f(a,b)$$

$$\frac{\dfrac{\bot \qquad \top}{Q(a)}\ u \mapsto a \qquad \top}{P(a,b,f(a,b)) \vee (\neg P(a,b,f(a,b)) \wedge Q(a))}\ x \mapsto a, v \mapsto b, y \mapsto f(a,b)$$

Hence: $P(a, b, f(a, b)) \vee Q(a)$

Interpolant: $\forall x_1 \forall x_2 \exists x_3 P(x_1, x_2, x_3) \vee Q(x_1)$

Example 5c: overbinding with equality

for arbitrary a, b:

$\Sigma = \{P(a), a = b\}$

$\Pi = \{\neg P(b)\}$

$$\dfrac{\dfrac{P(a)_\Sigma \qquad a = b_\Sigma}{P(b)} \qquad \neg P(b)_\Pi}{\Box}$$

$$\dfrac{\dfrac{\bot \qquad \bot}{\bot} \qquad \top}{P(b)}$$

OR

$$\dfrac{\dfrac{a = b_\Sigma \qquad \neg P(b)_\Pi}{\neg P(a)} \qquad P(a)_\Sigma}{\Box}$$

$$\dfrac{\dfrac{\bot \qquad \top}{a = b} \qquad \bot}{(P(a) \wedge a = b)}$$