om_grey_to_colored⟩

Lemma 1. *Let x be a variable in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$. Suppose there is a colored and a grey occurrence of x. Then for every colored occurrence p of x there is an arrow from some grey occurrence to p.* *// Should also hold for all of AI^{Δ}, but is currently not needed in the proof*

Proof. TODO: □

colored_to_colored⟩

Lemma 2. *Let x be a variable which occurs colored in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ and again colored somewhere else in $\mathrm{AI}^{\Delta}(C)$. Then there is a merge edge between the maximal colored terms containing the two occurrences.* *// This is exactly the case we need, possibly show something more general*

Proof. TODO: □

ion_in_arrow_proof⟩

Example 3. $\Gamma = \{Q(\gamma(x)) \lor P(x), \neg Q(\gamma(z)), R(\dots)\}$
$\Delta = \{\neg P(\delta(y)) \lor R(y), \neg R(a), Q(\dots)\}$
$a \sim x_k, \delta(y) \sim x_i, \delta(a) \sim x_j$

R only for coloring

Q only for coloring

$$\dfrac{\dfrac{\dfrac{\perp \mid Q(\gamma(x)) \lor P(x) \qquad \top \mid \neg P(x_i) \lor R(y)}{P(x_i) \mid Q(\gamma(x_i)) \lor R(y)} \qquad \top \mid \neg R(x_k)}{\dfrac{(\neg R(x_k) \land P(x_i)) \lor (R(x_k) \land \top) \mid Q(\gamma(x_i))}{P(x_i) \lor R(x_k) \mid Q(\gamma(x_i))}} \qquad \perp \mid \neg Q(\gamma(z))}{\dfrac{(\neg Q(x_j) \land (P(x_i) \lor R(x_k))) \lor (Q(x_j) \land \top) \mid \square}{\neg Q(x_j) \land (P(x_i) \lor R(x_k)) \mid \square}}$$

Gist: When $Q(\gamma(x_i))$ is the only symbol in $\mathrm{AI}^{\Delta}(\cdot)$, the lifting var means $\delta(x)$, but in the actual derivation, it's $\delta(a)$. however τ fixes this. So before Q is resolved, there is an arrow, but with the wrong lifting var (x_i instead of x_j) △

Remark (∗). Any substitution, in particular σ, only changes a finite number of variables. Furthermore a result of a run of the unification algorithm is acyclic in the sense that if a substitution $u \mapsto t$ is added to the resulting substitution, it is never the case that at a later stage $t \mapsto u$ is added. This can easily be seen by considering that at the point when $u \mapsto t$ is added to the resulting substitution, every occurrence of u is replaced by t, so u is not encountered by the algorithm at a later stage.
Therefore in order to show that a statement holds for every $u \mapsto t$ in a unifier σ, it suffices to show by an induction argument that for every substitution $v \mapsto s$ which is added to the resulting unifier by the unification algorithm that it holds for $v \mapsto s$ under

the assumption that it holds for every $w \mapsto r$ such that w occurs in s and $w \mapsto r$ is added to the resulting substitution at a later stage. $\triangle$

Conjecture 4. *Let C be a clause in a resolution refutation. Suppose that $\mathrm{AI}^{\Delta}(C)$ contains a maximal Γ-term $\gamma_j[z_i]$ which contains a lifting variable z_i. Then $z_i <_{\hat{\mathcal{A}}(C)} z_j$.*

Proof. We proceed by induction. For the base case, note that no multicolored terms occur in inital clauses, so no lifting term can occur inside of a Γ-term.

Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$. Furthermore suppose that for every lifting term inside a Γ-term in the clauses C_1 and C_2 of the refutation, for every term of the form $\gamma_j[z_i]$ we have that $z_i <_{\hat{\mathcal{A}}(C_1)} z_j$ or $z_i <_{\hat{\mathcal{A}}(C_2)} z_j$ respectively. Hence there is an arrow (p_1, p_2) in $\hat{\mathcal{A}}(C_1)$ or $\hat{\mathcal{A}}(C_2)$ such that z_i is contained in $\mathrm{P}(p_1)$ and z_j is contained in $\mathrm{P}(p_2)$. In $\mathrm{AI}^{\Delta}(C)$, $\mathrm{P}(p_1)$ contains $\ell[z_i\sigma]\tau = z_i\tau$ and $\mathrm{P}(p_2)$ contains $\ell[z_j\sigma]\tau = z_j\tau$. Hence the indicies of the lifting variables might change, but this renaming does not affect the relation of the objects as $\hat{\mathcal{A}}(C_1) \cup \hat{\mathcal{A}}(C_2) \subseteq \hat{\mathcal{A}}(C)$.

We show that $z_i <_{\hat{\mathcal{A}}(C)} z_j$ holds true also for every new term of the form $\gamma_j[z_i]$ for some j, i in $\mathrm{AI}^{\Delta}(C)$. By "new", we mean terms which are not present in $\mathrm{AI}^{\Delta}(C_1)$ or $\mathrm{AI}^{\Delta}(C_2)$. Note that new terms in $\mathrm{AI}^{\Delta}(C)$ are of the form $\ell_{\Delta,x}[t\sigma]\tau$ for some $t \in \mathrm{AI}^{\Delta}(C_1) \cup \mathrm{AI}^{\Delta}(C_2)$. By Lemma **??**, σ does not introduce lifting variables. Hence a new term of the form $\gamma_j[z_i]$ is created either by introducing a Δ-term into a Γ-term or by introducing $\gamma_j[\delta_i]$ via σ, both followed by the lifting. Note that τ only substitutes lifting variables by other lifting variables and hence does not introduce lifting variables. Furthermore by Lemma **??**, τ only substitutes lifting variables for other lifting variables, whose corresponding term is more specialised. Hence if there exists an arrow from a lifting variable to $\gamma_j[z_i]$ according to this lemma, it is also an appropriate arrow if $\gamma_j[z_i]$ is replaced by $\gamma_j[z_i]\tau$.

We now distinguish the two cases under which a new term $\gamma_j[z_i]$ can occur in $\mathrm{AI}^{\Delta}(C)$:

Suppose for some Γ-term $\tilde{\gamma}_{j'}[u]$ in $\mathrm{AI}^{\Delta}(C_1)$ or $\mathrm{AI}^{\Delta}(C_2)$, $u\sigma$ contains a Δ-term.

Hence we have that $(\tilde{\gamma}_{j'}[u])\sigma = \gamma_j[\delta_i]$ for some i. Note that the position of u in $\tilde{\gamma}_{j'}[u]$ does not necessarily coincide with the position of δ_i in $\gamma_j[\delta_i]$ as u might be substituted by σ for a grey term containing δ_i.

We have that $\ell_{\Delta}[\tilde{\gamma}_{j'}[u]\sigma]\tau = \gamma_j[z_i]$.

At some well-defined point of application of the unification algorithm, u is substituted by an abstraction of a term which contains δ_i. This occurrence of u is in l and we denote it by $\hat{u}$. We furthermore denote the term at the corresponding position in l' by $t_{\hat{u}}$.

We distinguish cases based on the occurrences of $\hat{u}$ and $t_{\hat{u}}$.

- Suppose $\hat{u}$ is a grey occurrence.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\hat{u}) \qquad C_2 : \neg Q(t_{\hat{u}})}{C : P(\gamma_j[\delta_i])}$$

Figure 1: Example for this case

  Then by Lemma 1, there is an arrow from $\hat{u}$ to $\gamma_j[u]$ in $\hat{\mathcal{A}}(C)$. As $\hat{u}\sigma$ is a term containing the Δ-term δ_i, the term at the position of $\hat{u}$ in $\mathrm{AI}^{\Delta}(C)$ is $\ell[\hat{u}\sigma]\tau$, which by assumption contains z_i. But there is an arrow from this term containing z_i to $\gamma_j[z_i]$, so $z_i <_{\hat{\mathcal{A}}(C)} z_j$.

- Suppose $\hat{u}$ occurs in a maximal colored term which is a Γ-term.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\gamma_k[\hat{u}]_p) \qquad C_2 : \neg Q(\gamma_m[t_{\hat{u}}]_p)}{C : P(\gamma_j[\delta_i])}$$

$$\frac{C_1 : Q(\tilde{\gamma}_{j'}[\hat{u}]) \qquad C_2 : \neg Q(\gamma_m[t_{\hat{u}}])}{C : \square}$$

$//\gamma_j[\delta_i]$ occurs in the interpolant

Figure 2: Examples for this case

  Then either $\hat{u}$ is the occurrence of u in $\tilde{\gamma}_{j'}[\hat{u}]$ or it occurs in a different Γ-term $\gamma_j[\hat{u}]$. In the latter case, by Lemma 2, there is a merge edge between $\tilde{\gamma}_{j'}[\hat{u}]$ and $\gamma_j[\hat{u}]$. Hence in both cases, it suffices to show that there is an arrow from a term containing an occurrence of z_i to $t_{\hat{u}}$.

  We distinguish on the shape of $t_{\hat{u}}$:

  - $t_{\hat{u}}$ is a term which does not contain a Δ-term. Then it contains a variable that is be substituted by σ by a term which contains a Δ-term as $u\sigma = t_{\hat{u}}\sigma$ is a term containing a Δ-term. We denote by v the variable in $t_{\hat{u}}$ which is substituted by a term containing a Δ-term in case $t_{\hat{u}}$ is a grey term.

    In the course of the unification algorithm, there are further unifications of v since we know that $u\sigma = v\sigma$ is a term containing a Δ-term. Therefore by Remark ($*$), we can assume that there is an appropriate arrow to $t_{\hat{u}}$.

  - $t_{\hat{u}}$ is a term which contains a Δ-term. As $t_{\hat{u}}$ occurs in a Γ-term in C_1, say in $\gamma_m[t_{\hat{u}}]$, C_1 contains a multicolored Γ-term. Hence the corresponding term in $\mathrm{AI}^{\Delta}(C_1)$, is of the form $\gamma_m[z_{i'}]$ for some i'. Observe that i' in general is not equal to i as demonstrated in Example 3, even though we have that $t_{\hat{u}}\sigma = u\sigma$. This is because the lifting variables in $\mathrm{AI}(\cdot)$ represent abstractions of the terms in the clauses of the resolution derivation (cf. Lemma **??**). Therefore we only know by the induction hypothesis that $z_{i'} <_{\hat{\mathcal{A}}(C_1)} \ell[\gamma_m[z_{i'}]] = \ell[t_{\hat{u}}]$.

However by Lemma **??** and due to the fact that $\hat{u}$ and $t_{\hat{u}}$ respectively occur in the resolved literal, $\ell_\Delta[\hat{u}\sigma]\tau = \ell_\Delta[t_{\hat{u}}\sigma]\tau$. As $\ell_\Delta[\hat{u}\sigma]\tau = \ell_\Delta[\delta_i]\tau = z_i\tau$ as well as $\ell_\Delta[t_{\hat{u}}\sigma]\tau = \ell_\Delta[z_{i'}\sigma]\tau = z_{i'}\tau$, we must have that $z_i\tau = z_{i'}\tau$. As however $u\sigma = \delta_i$, by the definition of au, we have that $\{z_i \mapsto z_i\} \in \tau$, so $z_{i'}\tau = z_i$.

Since τ is applied to every literal in $\mathrm{AI}^\Delta(C)$ and in $\mathrm{AI}^\Delta(C_1)$ an arrow from a term containing $z_{i'}$ to $t_{\hat{u}}$ exists, the same arrow applied to $\mathrm{AI}^\Delta(C)$ points from a term containing $z_{i'}\tau = z_i$ to $t_{\hat{u}}$. Therefore $z_i <_{\hat{\mathcal{A}}(C)} z_j$.

- Suppose $\hat{u}$ occurs in a maximal colored term which is a Δ-term.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\delta_k[\hat{u}]_p) \qquad C_2 : \neg Q(\delta_m[t_{\hat{u}}]_p)}{C : P(\gamma_j[\delta_i])}$$

By Lemma **??**, TODO:

The substitution can also introduce a grey term containing a delta term, make sure to handle that!

The substitution can also introduce a gamma term containing a delta term, make sure to handle that!

Suppose for some variable v in $\mathrm{AI}^\Delta(C_1)$ or $\mathrm{AI}^\Delta(C_2)$, $v\sigma = \gamma_j[\delta_i]$ for some i.

As v is affected by the unifier, it occurs in the literal being unified, say w.l.o.g. in l in C_1. At some well-defined point in the unification algorithm, v is substituted by an abstraction of $\gamma_j[\delta_i]$. Let p be the position of the occurrence of v in l which causes this substitution. Furthermore, let p' be the position corresponding to p in l'.

Note that any arrow from or to p' also applies to p in $\hat{\mathcal{A}}(C)$ and hence to $\gamma_j[z_i]$ as they are merged due to occurring in the resolved literal. So it suffices to show that there is an arrow from an appropriate lifting variable to p'. We denote the term at p' by t.

Note that $t\sigma = \gamma_j[\delta_i]$. So t is either a Γ-term containing a Δ-term, in which case we know that there is an appropriate arrow by the induction hypothesis as t occurs in l' in C_2, or t is an abstraction of $\gamma_j[\delta_i]$, in which case we can assume the existence of an appropriate arrow by Remark $(*)$. □

recheck this paragraphs w.r.t. $<_{\hat{\mathcal{A}}(C)}$