Chapter 1

trying to overbind mostly right away does not look promissing

Definition 1 (Q)**.** For a literal/clause φ, $Q(\varphi)$ denotes the quantifier block consisting of every lifting variable in φ with appropriate quantifier type. The order is yet to be defined △

For $l \in C$ for $C \in \Gamma$: $Q(l) = \exists \bar{y}$
For $l \in C$ for $C \in \Delta$: $Q(l) = \forall \bar{x}$

basic axioms which should be fulfilled for a reasonable procedure

start

- $\Gamma \models \mathrm{LI}_{\mathrm{cl}}(C)$

  $\Gamma = \{P(f(x))\} \Rightarrow \mathrm{LI}_{\mathrm{cl}}(C) \overset{\text{must be}}{=} \exists x P(x)$

  $\Gamma = \{\neg P(f(x))\} \Rightarrow \mathrm{LI}_{\mathrm{cl}}(C) \overset{\text{must be}}{=} \exists x \neg P(x)$
- $\Delta \models ?$

inferences LI is always basically just Γ-part
always want: $\Gamma \models \mathrm{LI}$, $\Delta \models \neg\,\mathrm{LI}$

- $\Gamma : P(f(x)) \Rightarrow \exists x P(x)$
  $\Delta : \neg P(y) \Rightarrow \forall y \neg P(x)$
- $\Gamma : \neg P(f(x)) \Rightarrow \exists x \neg P(x)$
  $\Delta : P(y) \Rightarrow \forall y P(x)$
- $\Gamma : \neg P(x) \Rightarrow \forall x \neg P(x)$
  $\Delta : P(g(y)) \Rightarrow \exists y P(y)$
- $\Gamma : P(x) \Rightarrow \forall x P(x)$
  $\Delta : \neg P(g(y)) \Rightarrow \neg \exists y P(y)$

but must not tear apart $P(x) \vee \neg P(x)$ to $\forall x P(x) \vee \forall x \neg P(x)$

example for "var does not occur in clause any more-condition":

$$\frac{R(f(z)) \qquad \neg R(x) \vee P(x)}{\neg R(f(z)) \mid P(f(z))}$$

Note that $(\forall y_{f(x)} \neg R(y_{f(x)})) \vee P(x)$ is not valid!

Not sure what this example is supposed to demonstrate

attempt for a definition

Definition 2 (LI)**.**

Base case.

For $l \in C$ for $C \in \Gamma \cup \Delta$: $Q(l)\ell[C] \in \mathrm{LI}_{\mathrm{cl}}(C)$

LI as usual

Resolution.

Definition 3 (χ: lifting with quantification on literal level)**.**

$\chi(F \circ G) \stackrel{\text{def}}{=} \chi(F) \circ \chi(G)$

$\chi(\neg G) \stackrel{\text{def}}{=} \neg\chi(F)$

$\chi(Q(\lambda)\lambda) \stackrel{\text{def}}{=} Q(\lambda\sigma)\lambda\sigma$

where $Q(\lambda\sigma)$ is $Q(\lambda)$ with quantifiers and lifting variables for additional maximal colored terms introduced by σ into λ △

$\mathrm{LI}_{\mathrm{cl}}\, C \stackrel{\text{def}}{=} \chi(\mathrm{LI}_{\mathrm{cl}}(C_1) \backslash \{l_{\mathrm{LIcl}}\}) \vee \chi(\mathrm{LI}_{\mathrm{cl}}(C_2) \backslash \{l'_{\mathrm{LIcl}}\})$

1. If l is Γ-colored: $\mathrm{LI}(C) \stackrel{\text{def}}{=} \chi(\mathrm{LI}(C_1)) \vee \chi(\mathrm{LI}(C_2))$
2. If l is Δ-colored: $\mathrm{LI}(C) \stackrel{\text{def}}{=} \chi(\mathrm{LI}(C_1)) \wedge \chi(\mathrm{LI}(C_2))$
3. If l is grey: $\mathrm{LI}(C) \stackrel{\text{def}}{=} (l_{\mathrm{LIcl}}\tau \wedge \mathrm{LI}(C_2)\tau) \vee (\neg\ell[l'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_1)\tau])$

△

Conjectured Lemma 4. $\Gamma \models \mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$

Proof. Start works.
Step:
resolved literals: have same coloring

IH:
$\Gamma \models \mathrm{LI}(C_1) \vee \mathrm{LI}_{\mathrm{cl}}(C_1^*) \vee l_{\mathrm{LIcl}}$
$\Gamma \models \mathrm{LI}(C_2) \vee \mathrm{LI}_{\mathrm{cl}}(C_2^*) \vee l'_{\mathrm{LIcl}}$ □

Chapter 2

overbind just within thight constraints

⟨A⟩ **Lemma 5.** *If a variable does occurs in $\bar{C}$ but does not in C, then it is not modified by any* mgu *of a subsequent inference.*

2.1 naive interpolant extraction based on 5

Definition 6 (LI with stepwise prenex interplants but globally non-prenex ones)**.**

Base case.

For $l \in C$ for $C \in \Gamma \cup \Delta$: $C \in \mathrm{LI}_{\mathrm{cl}}(C)$

LI as usual

Resolution.

$\mathrm{LI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \mathrm{LI}_{\mathrm{cl}}(C_1) \backslash \{l_{\mathrm{LIcl}}\}\sigma \vee \mathrm{LI}_{\mathrm{cl}}(C_2) \backslash \{l'_{\mathrm{LIcl}}\}\sigma$

$\Rightarrow \mathrm{LI}_{\mathrm{cl}}(C) = C$

$\chi(F)$: lift all maximal colored terms which contain some variable which does not occur in $\mathrm{LI}_{\mathrm{cl}}(C)$

TODO: not sure where we can quantify ground terms as they can be added arbitrarily (possibly lift every occurrence of a ground term t distinctly)

apropos ground term: imagine procedure which conceptually adds some variable as argument to every term. if then we can overbind ground terms, we should be able to have a convention to enable nested ground term lifting directly

TODO: need not be prenex here, can pull in as far as regular quantifier pull in rules allow

1. If l is Γ-colored: $\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} \mathrm{LI}(C_1)\tau \vee \mathrm{LI}(C_2)\tau$
2. If l is Δ-colored: $\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} \mathrm{LI}(C_1)\tau \wedge \mathrm{LI}(C_2)\tau$
3. If l is grey: $\mathrm{LI}(C)^\bullet \stackrel{\mathrm{def}}{=} (l_{\mathrm{LIcl}}\sigma \mathrm{LI}(C_2))\tau \vee (\neg l'_{\mathrm{LIcl}} \wedge \mathrm{LI}(C_1))\sigma$

$$\mathrm{LI}^*(C) \stackrel{\mathrm{def}}{=} \chi(\mathrm{LI}^\bullet(C))$$
$$\mathrm{LI}(C) \stackrel{\mathrm{def}}{=} Q_{\mathrm{LI}^*(C)}\, \mathrm{LI}^*(C)$$

△

$\Gamma \models \mathrm{LI}(C) \vee C$
$(\Delta \models \neg\, \mathrm{LI}(C) \vee C)$

2.2 lifting only Δ-terms in this way for now

does not really work like this because Γ-quantifiers are somewhat included, also nesting of quantifier is not treated in this “proof”

Conjectured Lemma 7. $\Gamma \models \mathrm{LI}^\Delta(C) \vee C$

Proof. THIS IS IMPLIED BY THE LEMMA FOR THE OTHER LIFTING STRATEGY, AS WE JUST LIFT *LESS* Δ-TERMS HERE, SO THIS IS ALWAYS AN INSTANCE OF THE OTHER LEMMA

induction on strenghtening, as always.

but additional strenghtening: lift all Δ-terms, just like in other lemma

$C_\Gamma = C^*_{1\,\Gamma} \vee C^*_{2\,\Gamma}$
IH:
$\Gamma \models \mathrm{LI}^\Delta(C_1) \vee C^*_{1\,\Gamma} \vee l_\Gamma$
$\Gamma \models \mathrm{LI}^\Delta(C_2) \vee C^*_{2\,\Gamma} \vee \neg l'_\Gamma$
Hence:
$\Gamma \models (\mathrm{LI}^\Delta(C_1) \vee C^*_{1\,\Gamma} \vee l_\Gamma)\sigma$
$\Gamma \models (\mathrm{LI}^\Delta(C_2) \vee C^*_{2\,\Gamma} \vee \neg l'_\Gamma)\sigma$
Supp grey:
$\Gamma \models (l \wedge \mathrm{LI}^\Delta(C_2))\sigma \vee (l' \wedge \mathrm{LI}^\Delta(C_1))\sigma \;\vee\; C_\Gamma$
$\Gamma \models \mathrm{LI}^\Delta(C) \;\vee\; C_\Gamma$
the literal is of course equal as by clearly C is not affected.

□

$X = \mathrm{LV}(\mathrm{LI}^\Delta(C)) \setminus \mathrm{LV}(\mathrm{LI}^\Delta_{\mathrm{cl}}(C_\Gamma))$

X': take from X those lifting variables, which contain variables which do not occur in C (this is safer than only $\mathrm{LI}^\Delta_{\mathrm{cl}}(C)$)

$Y = \mathrm{LV}(\ell_\Gamma[\mathrm{LI}^\Delta(C)])$

$Y' = \{z_t \in Y \mid t \text{ contains a variable which does not occur in } C\}$

From other pdf: $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$

Hence $\Gamma \models (Q(Y')\, \mathrm{LI}^\Delta(C)) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$

2.3 lifting with nesting

Conjectured Lemma 8. $\Gamma \models \ell_\Delta[\mathrm{LI}(C)] \vee \ell_\Delta[C]$

Proof. induction on C_Γ. take care of this properly when writing this up properly

Base case:

$C \in \Gamma$: Then $\ell_\Delta[C] = C$ and $\Gamma \models C$

$C \in \Delta$: Then $\mathrm{LI}(C) = \top$

Ind step:
Supp $\Gamma \models \ell_\Delta[\mathrm{LI}(C_i)] \vee \ell_\Delta[C_i]$
By lemma **??** $\Gamma \models \ell_\Delta[\ell_\Delta[\mathrm{LI}(C_i)]\tau] \vee \ell_\Delta[\ell_\Delta[C_i]\tau]$
By Lemma **??**,
$\Gamma \models \ell_\Delta[\mathrm{LI}(C_i)\tau] \vee \ell_\Delta[C_i\tau]$
formulate differently:
$(\circ)(*)$: $\Gamma \models \ell_\Delta[\mathrm{LI}(C_i)\tau] \vee \ell_\Delta[C_1^*\tau] \vee \ell_\Delta[l\tau]$
Clearly: $l\tau = l'\tau$
By Lemma TODO (not even needed): $\ell_\Delta[C] = \ell_\Delta[C_1^*\sigma] \vee \ell_\Delta[C_2^*\sigma] = \ell_\Delta[C_1]\tau \vee \ell_\Delta[C_2]\tau$

- Supp l grey.

  Need to show for $\mathrm{LI}^\bullet(C)$:

  $\Gamma \models \ell_\Delta[C] \;\vee\; (\ell_\Delta[l]\tau \vee \ell_\Delta[\mathrm{LI}(C_2)]\tau) \;\vee\; (\neg\ell_\Delta[l']\tau \vee \ell_\Delta[\mathrm{LI}(C_1)]\tau)$

  By $(\circ)(*)$:

  $\Gamma \models \ell_\Delta[\mathrm{LI}_{\mathrm{cl}}(C_1)\tau] \vee \ell_\Delta[C_1^*\tau] \vee \ell_\Delta[l\tau]$

  $\Gamma \models \ell_\Delta[\mathrm{LI}_{\mathrm{cl}}(C_2)\tau] \vee \ell_\Delta[C_2^*\tau] \vee \neg\ell_\Delta[l'\tau]$

  Hence:

  $\Gamma \models \ell_\Delta[\mathrm{LI}_{\mathrm{cl}}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}_{\mathrm{cl}}(C_2)\tau] \;\vee\; (\ell_\Delta[l\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau]) \;\vee\; (\neg\ell_\Delta[l'\tau] \vee \ell_\Delta[\mathrm{LI}(C_1)\tau])$

  is nothing else than:

  $\Gamma \models \ell_\Delta[C] \vee \ell_\Delta[\mathrm{LI}^\bullet(C)]$

- Supp l Γ-colored

  Then resolve on l in $(\circ)(*)$ and obtain the same thing as in the other cases

- Supp l Δ-colored

  Then $(\circ)(*)$ collapse to:

  $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[C_1^*\tau]$

  $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[C_2^*\tau]$

  Clearly $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[C_1^*\tau] \wedge \ell_\Delta[C_2^*\tau]$

  is nothing else than:

  $\Gamma \models \ell_\Delta[C] \vee \ell_\Delta[\mathrm{LI}^\bullet(C)]$

Let t be a term in $\mathrm{LI}^\bullet(C)$ at p such that t is maximal colored, contains a variable which does not occur in C.

IDEA: "do not occur in $\ell_\Delta[C]$

- Suppose t is Δ-colored:

  $\mathrm{LI}(C)|_p$ is $\ell_\Delta^x[t] = x_t$

  However $\ell_\Delta[\mathrm{LI}^\bullet]|_p = x_t$

- Suppose t is Γ-colored:

  Then $\mathrm{LI}^*(C)|_p = y_t$

  and $\exists y_t$ is contained in $Q_{\mathrm{LI}*(C)}$

  But $\mathrm{LI}^\bullet(C)|_p = t$ (t by assumption has not been lifted before)

  also $\ell_\Delta[\mathrm{LI}^\bullet(C)]|_p = \ell_\Delta[t]$

  Hence here we have the witness, and it contains quantified Δ-lifting vars.

Also more Δ-colored terms in $\ell_\Delta[\mathrm{LI}^\bullet(C)]$ are lifted, but due to τ, they correspond exactly to the non-lifted terms. Hence with respect to the Δ-terms, $\mathrm{LI}(C)$ is an instance of $\ell_\Delta[\mathrm{LI}^\bullet(C)]$.

As t contains a variable which does not occur elsewhere, t is not a subterm of a term which does not occur here.

$Q_{\mathrm{LI}*(C)}$ sorts according to subterm relation. Hence if t is a Γ-term and its witness term in $\ell_\Delta[\mathrm{LI}^\bullet(C)]$ contains a Δ-lifting variable, say x_s, then it is quantified in $Q_{\mathrm{LI}*(C)}$ before y_t is.

if s contains the important variable, then t and s are both lifted at this inference.

if s does not, then t contains the important variable. Then s might be lifted later if it occurs elsewhere, or possibly not at all.

irrelevant remark: it is quantified at some point if it contains a variable as the last inference creates the empty clause, which does not contain variables. □

2.4 random ideas

– we can pull apart existentially quantified variables: $\exists x(P(x) \vee Q(x))$ implies $\exists x P(x) \vee \exists y P(y)$. this does not work with universally quantified variables $(P(f(x)) \vee \neg P(f(x))$

but interpolants are somewhat symmetric, if it's existential for Γ, it's universal for Δ.

– suppose we lift all ground terms in the interpolant if no maximal colored term in the current clause is a subterm.