Example 101: canonical examples for overbinding w.r.t. order

Here, the second lower prooftree shows what huang would produce.

Ex 101a

$$\dfrac{\dfrac{\overset{\Sigma}{P(u,f(u))\vee Q(u)} \qquad \overset{\Pi}{\neg Q(a)}}{P(a,f(a))}\,u\mapsto a \qquad \overset{\Pi}{\neg P(x,y)}}{\square}\,x\mapsto a, y\mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{Q(a)}\,u\mapsto a \qquad \top}{P(a,f(a))\vee Q(a)}\,x\mapsto a, y\mapsto f(a) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 Q(x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}$$

Direct overbinding would not work without merging same variables!: $\forall x_1\forall x_2\exists x_3(P(x_2,x_3)\vee Q(x_1))$
counterexample: $Q\sim\{0\}, P\sim\{(1,0)\}$
Direct overbinding would work when considering original dependencies as highlighted above
arrow lemma:

$$\dfrac{\dfrac{\overset{\Sigma}{\Gamma\models\exists y_1(P(u,y_1)\vee Q(u)\vee\bot)} \qquad \overset{\Pi}{\Gamma\models\neg Q(x_1)\vee\top}}{\Gamma\models\exists y_1(P(x_1,y_1)\vee Q(x_1))}\,u\mapsto a \qquad \overset{\Pi}{\Gamma\models\neg P(x,y)\vee\top}}{\Gamma\models(\forall x_1)\exists y_1(Q(x_1)\vee P(x_1,y_1))}\,x\mapsto a, y\mapsto f(a)$$

Ex 101b – other resolution order

$$\dfrac{\dfrac{\overset{\Sigma}{P(u,f(u))\vee Q(u)} \qquad \overset{\Pi}{\neg P(x,y)}}{Q(u))}\,y\mapsto f(u), x\mapsto u \qquad \overset{\Pi}{\neg Q(a)}}{\square}\,u\mapsto a$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(u,f(u))}\,x\mapsto f(u), x\mapsto u \qquad \top}{P(a,f(a))\vee Q(a)}\,u\mapsto a \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(u,x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}\,u\mapsto a$$

Ex 101c – Π and Σ swapped

$$\dfrac{\dfrac{\overset{\Pi}{P(u,f(u))\vee Q(u)} \qquad \overset{\Sigma}{\neg P(x,y)}}{Q(u))}\,y\mapsto f(u), x\mapsto u \qquad \overset{\Sigma}{\neg Q(a)}}{\square}\,u\mapsto a$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg P(u,f(u))}\,x\mapsto f(u), x\mapsto u \qquad \bot}{\neg P(a,f(a))\wedge\neg Q(a)}\,u\mapsto a \qquad\qquad \dfrac{\dfrac{\top \qquad \bot}{\forall x_2\neg P(u,x_2)} \qquad \bot}{\exists x_1\forall x_2(\neg P(x_1,x_2)\wedge\neg Q(x_1))}$$

arrow lemma:

$$\dfrac{\Gamma \models P(u, \overset{\Pi}{x_1}) \vee Q(u) \vee \top \qquad \Gamma \models \neg P(\overset{\Sigma}{x}, y) \vee \bot}{\Big(Q(u) \mid (\neg P(x, x_1) \wedge \top) \vee (P(u, /f(u)) \wedge \bot)\Big)\sigma}\ y \mapsto f(u), x \mapsto u$$

$$Q(u) \mid (\neg P(u, x_1) \wedge \top) \vee (P(u, f(u)) \wedge \bot)$$

employ σ' !?!!?!?!?!??!?????!!!

$$\Gamma \models Q(u) \mid \neg P(u, x_1)$$

$$\dfrac{\Delta \models Q(u) \mid \exists x_1 P(u, x_1) \qquad \Gamma \models \exists y_1\, \neg \overset{\Sigma}{Q}(y_1)}{\text{both } u\text{'s on LHS need to become } a \text{ and then } y_1}\ u \mapsto a$$

$$\Gamma \models (\forall x_1)\exists y_1(\neg P(y_1, x_1) \vee \neg Q(y_1))$$

$$\Delta \models (\exists x_1)\forall y_1(P(y_1, x_1) \wedge Q(y_1))$$

Ex 101d – Π and Σ swapped, other resolution order

$$\dfrac{\dfrac{P(u, \overset{\Pi}{f(u)}) \vee Q(u) \qquad \neg \overset{\Sigma}{Q}(a)}{P(a, f(a))}\ u \mapsto a \qquad \neg P(\overset{\Sigma}{x}, y)}{\square}\ x \mapsto a, y \mapsto f(a)$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg Q(a)}\ y \mapsto a \qquad \bot}{\neg Q(a) \wedge \neg P(a, f(a))}\ x \mapsto a, y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\top \qquad \bot}{\exists x_1 \neg Q(x_1)} \qquad \bot}{\exists x_1 \forall x_2(\neg P(x_1, x_2) \wedge \neg Q(x_1))}$$

102 – similar to 101 but with intra-clause-set inferences

Ex 102a

$$\dfrac{\dfrac{P(f({\color{red}x})) \vee \overset{\Sigma}{Q}(f({\color{red}x}), z) \qquad \neg \overset{\Pi}{P}(y)}{Q(f(x), z)} \qquad \dfrac{\neg Q(\overset{\Sigma}{x_1}, {\color{blue}y}) \vee R({\color{blue}y}) \qquad \neg R(\overset{\Pi}{g}(z_1))}{\neg Q(x_1, g(z_1))}\ y \mapsto g(z_1), x_1 \mapsto f(x)}{\square}\ x_1 \mapsto f(x), z \mapsto g(z_1)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(g(z_1))}\ y \mapsto g(z_1), x_1 \mapsto f(x)}{P(f(x)) \vee R(g(z_1))}\ x_1 \mapsto f(x), z \mapsto g(z_1) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}}{\exists x_1 \forall x_2(P(x_1) \vee R(x_2)) \quad \text{(order irrelevant!)}}$$

Ex 102b

$$\dfrac{\dfrac{P(f({\color{red}x})) \vee \overset{\Sigma}{Q}(f({\color{red}x}), z) \qquad \neg \overset{\Pi}{P}(y)}{Q(f(x), z)} \qquad \dfrac{\neg Q(f(\overset{\Sigma}{{\color{blue}y}}), z_1) \vee R({\color{blue}y}) \qquad \neg \overset{\Pi}{R}(a)}{\neg Q(f(a), z_1)}\ y \mapsto a}{\square}\ x \mapsto a, z \mapsto z_1$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\, y \mapsto a}{P(f(a)) \vee R(a)}\, x \mapsto a, z \mapsto z_1 \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\, y \mapsto a}{\forall x_2 \exists x_1 (P(x_1) \vee R(x_2))}\, x \mapsto a, z \mapsto z_1$$

direct:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)}\, x_1 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\, x_2 \sim a}{\exists x_1 \forall x_2 (P(x_1) \vee R(x_2))}\,\text{order irrelevant!}$$

Ex 102b' with Q grey

$$\dfrac{\dfrac{\overset{\Sigma}{P(f(x)) \lor Q(f(x), z)} \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x), z)} \qquad \dfrac{\overset{\Sigma}{\neg Q(f(y), z_1) \lor R(y)} \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a), z_1)}\ y \mapsto a}{\square}\ x \mapsto a, z_1 \mapsto z$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\ y \mapsto a}{(\neg Q(f(a), z) \land P(f(a))) \lor (Q(f(a), z) \land R(a))}\ x \mapsto a, z_1 \mapsto z$$

arrow lemma: (change of specification: $P(g(z_3))$ in clause in Σ instead of $P(f(x))$

$$\dfrac{\begin{array}{c}\dfrac{\overset{\Sigma}{\exists x_1 (P(x_4) \lor Q(x_1, z))} \qquad \overset{\Pi}{\neg P(y)}}{\exists x_1 (P(g(z_3)) \land \bot) \lor (P(x_4) \land \top) \mid Q(x_1, z)} \\ \exists x_1 (P(x_4) \mid Q(x_1, z))\end{array} \qquad \begin{array}{c}\dfrac{\overset{\Sigma}{\exists x_2 (\neg Q(x_2, z_1) \lor R(y))} \qquad \overset{\Pi}{\forall x_3\ \neg R(x_3)}}{\overset{\forall x_3}{\exists x_2}\ ((\neg R(x_3) \land \bot) \lor (R(a) \land \top) \mid \neg Q(x_2, z_1))}\ y \mapsto a \\ \overset{\forall x_3}{\exists x_2}\ ((\neg R(x_3) \land \bot) \lor (R(x_3) \land \top) \mid \neg Q(x_2, z_1))\end{array}}{\overset{\forall x_1}{\overset{\forall x_3}{\exists x_2}} \Big((\neg Q(x_2, z_1) \land P(x_4)) \lor (Q(x_1, z) \land R(x_3)) \Big)}\ x \mapsto a, z_1 \mapsto z$$

arrow order: $x_3 < x_2$, x_2 same-block-as x_4: $\forall x_3 \exists x_2 \exists x_4 \forall x_1 \Big((\neg Q(x_2, z_1) \land \neg P(x_4)) \lor (Q(x_1, z) \land R(x_3)) \Big)$

$\rightarrow$ bad example, plus some errors still in there

Huang:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)} \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\ y \mapsto a}{\forall x_1 \exists x_2 (\neg Q(x_2, z) \land P(x_2)) \lor (Q(x_2, z) \land R(x_1))}\ x \mapsto a, z_1 \mapsto z$$

direct:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)}\ x_2 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\ x_1 \sim a}{\forall x_1 \exists x_2 \exists x_3 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}\ x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3}{\text{OR: } \exists x_2 \forall x_1 \exists x_3 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}}{\text{OR: } \exists x_1 \exists x_3 \forall x_2 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}$$

Huang-Version passt weil man für existenzielle quantoren eh gleichen term einsetzt

direct w mixed, slightly different:

$$\dfrac{\dfrac{\dfrac{\bot \mid P(f(x)) \lor Q(x, z) \qquad \top \mid \neg P(y)}{\exists x_2 P(x_2) \mid Q(x, z)}\ x_2 \sim f(x) \qquad \dfrac{\bot \mid \overset{\Sigma}{\neg Q(f(y), z_1) \lor R(y)} \qquad \top \mid \overset{\Pi}{\neg R(a)}}{\forall x_1 R(x_1) \mid \neg Q(f(a), z_1)}\ x_1 \sim a}{\forall x_1 \exists x_3 \exists x_2 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}\ x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3}{(\neg Q(f(a), z) \land P(f(f(a)))) \lor (Q(f(a), z) \land R(a))}$$

last dependency not crucial because other arrow is a Σ-arrow as well, but just changing it to Π (and changing f for g should produce a quantifier alternation)

Example 103: variables in interpolant get unified and order might have to be changed

103a: length changes but old version is still valid, despite huang's algorithm doing a sort

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{Q(f({\color{red}x})) \lor P(y) \lor R({\color{red}x})} \qquad \overset{\Pi}{\neg Q(y_1)}}{P(y) \lor R(x)}\; y_1 \mapsto f(x) \qquad \overset{\Pi}{\neg P(h(g(a)))}}{R(x)}\; y \mapsto h(g(a)) \qquad \overset{\Pi}{\neg R(g(g(a)))}}{\Box}\; x \mapsto g(g(a))$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{Q(f(x))}\; y_1 \mapsto f(x) \qquad \top}{Q(f(x)) \lor P(h(g(a)))}\; y \mapsto h(g(a)) \qquad \top}{Q(f(g(g(a)))) \lor P(h(g(a))) \lor R(g(g(a)))}\; x \mapsto g(g(a))$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_1 \forall x_2 (Q(x_1) \lor P(x_2))} \qquad \top}{X}$$

X:

Huang's algo gives:

$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

Direct overbinding gives: $x_3 < x_1$, rest arbitrary, hence:

$\forall x_3 \exists x_1 \forall x_2 (Q(x_1) \lor P(x_2) \lor R(x_3))$ <- this you do not get with huang

$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

$\forall x_3 \forall x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

103b: length changes "uniformly"

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{Q(f(f({\color{red}x}))) \lor P(f({\color{red}x})) \lor R({\color{red}x})} \qquad \overset{\Pi}{\neg Q(y_1)}}{P(f(x)) \lor R(x)}\; y_1 \mapsto f(f(x)) \qquad \overset{\Pi}{\neg P(y_2)}}{R(x)}\; y_2 \mapsto f(x) \qquad \overset{\Pi}{\neg R(g(a))}}{\Box}\; x \mapsto g(a)$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{Q(f(f(x)))}\; y_1 \mapsto f(f(x)) \qquad \top}{Q(f(f(x))) \lor P(f(x))}\; y_2 \mapsto f(x) \qquad \top}{Q(f(f(g(a)))) \lor P(f(g(a))) \lor R(g(a))}\; x \mapsto g(a)$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_2 \exists x_1 (Q(x_1) \lor P(x_2))} \qquad \top}{\forall x_3 \exists x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))}$$

Huang and direct overbinding somewhat coincide as $x_2 < x_1$ in both cases, and $x_3 < x_1$ and $x_3 < x_2$

new algo:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \mid Q(x_1) \lor P(x_2) \lor R(x) \qquad \top \mid \neg Q(y_1)}{Q(x_1) \mid P(x_2) \lor R(x)}\; y_1 \mapsto f(f(x)) \qquad \top \mid \neg P(y_2)}{Q(x_1) \lor P(x_2) \mid R(x)}\; y_2 \mapsto f(x) \qquad \top \mid R(x_3)}{Q(x_1) \lor P(x_2) \lor R(x_3)}\; x \mapsto g(a)}{Q(f(f(g(a)))) \lor P(f(g(a))) \lor R(g(a))}$$

NB: in the last line, the terms corresponding to x_1 and x_2 change, but the interpolant stays the same

103c: Failed attempt: different variables, accidentally the same terms appear but no logical connection

$$\dfrac{\overset{\Sigma}{P(a,x)} \qquad \dfrac{\overset{\Sigma}{\neg Q(a)} \qquad \overset{\Pi}{\neg P(y, f({\color{red}z})) \vee Q({\color{red}z})}}{\neg P(y, f(a))}\ z \mapsto a}{\Box}\ y \mapsto a, x \mapsto f(a)$$

Huang:

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg Q(a)}\ z \mapsto a}{P(a, f(a)) \wedge \neg Q(a)}\ y \mapsto a, x \mapsto f(a) \qquad\qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg Q(x_1)}}{\exists x_1 \forall x_2 (P(x_1, x_2) \wedge \neg Q(x_1))}$$

order required for Π

direct:

$$\dfrac{\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg Q(x_1)}\ x_1 \sim a}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}\ x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))}$$

invariant:

$$\dfrac{\dfrac{\exists x_2 (P(x_2, x) \vee \bot) \qquad \dfrac{\exists x_1 (Q(x_1) \vee \bot) \qquad \forall x_3 ((\neg P(y, {\color{red}x_3}) \vee Q({\color{red}z})) \vee \top)}{\exists x_1 \forall x_3 \neg P(y, {\color{red}x_3}) \vee \neg Q({\color{red}x_1})}\ x_1 \sim a}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}\ x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))}$$

invariant in other resolution order

$$\dfrac{\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{Q({\color{red}z}) \vee \exists x_2 \forall x_3 P(x_2, {\color{red}x_3})}\ x_2 \sim a, x_3 \sim f(z)}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}\ x_1 \sim a; x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))}$$

invariant if Σ and Π swapped:

$$\dfrac{\dfrac{\bot \qquad \dfrac{\top \qquad \bot}{\neg P(y, f(x_1)) \vee \forall x_1 Q(x_1)}\ x_1 \sim a}{\forall x_1 \forall x_2 \exists x_3 (\neg P(x_2, x_3) \vee Q(x_1))}\ x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \forall x_1 \exists x_3 \forall x_2 (\neg P(x_2, x_3) \vee Q(x_1))}$$

SECOND ATTEMPT:

$$\dfrac{\overset{\Sigma}{P(a)} \qquad \dfrac{\overset{\Sigma}{Q(z)} \qquad \dfrac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg P(y) \vee \neg Q(f({\color{red}x})) \vee S({\color{red}x})}}{\neg P(y) \vee \neg Q(f(a))}\ x \mapsto a}{\neg P(y)}\ z \mapsto f(a)}{\Box}\ y \mapsto a$$

$$\dfrac{\perp \qquad \dfrac{\perp \qquad \dfrac{\perp \qquad \top}{\neg S(a)}\, x \mapsto a}{\neg S(a) \wedge Q(f(a))}\, z \mapsto f(a)}{P(a) \wedge \neg S(a) \wedge Q(f(a))}\, y \mapsto a$$

Huang:

$$\dfrac{\perp \qquad \dfrac{\perp \qquad \dfrac{\perp \qquad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 (\neg S(x_1) \wedge Q(x_2))}}{\exists x_1 \forall x_2 (P(x_1) \wedge \neg S(x_1) \wedge Q(x_2))}$$

$\forall x_1 \exists x_2 (\neg P(x_1) \vee S(x_1) \vee \neg Q(x_2))$

similar fail

$\Rightarrow$ anytime there is $P(a, f(a))$, either they have a dependency or they are not both differently colored (grey is uncolored)

for the record, direct method anyway:

$$\dfrac{\perp \qquad \dfrac{\perp \qquad \dfrac{\perp \qquad \top}{\exists x_1 \neg S(x_1)}\, x \sim a}{\exists x_1 \forall x_2 \neg S(x_1) \wedge Q(x_2)}\, z \sim f(a); x_1 < x_2}{\exists x_1 \forall x_2 \exists x_3 P(x_3) \wedge \neg S(x_1) \wedge Q(x_2)}\, x_3 \sim a; x_3 \text{ need not be merged w } x_1$$

Example: ordering on both ancestors where the merge forces a new ordering

202a – canonical

$$
\dfrac{\dfrac{\overset{\Sigma}{P(a, x_{564}) \vee R(y)} \qquad \overset{\Pi}{\neg P(x, fx)}}{R(y)}\;\begin{array}{c} x_1 \mapsto fa \\ x \mapsto a \end{array} \qquad \dfrac{\overset{\Sigma}{Q(x_2, gx_2) \vee \neg R(u)} \qquad \dfrac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg Q(fz, x_3) \vee S(z)}}{\neg Q(fa, x_3)}\; z \mapsto a}{\neg R(u)}\;\begin{array}{c} x_2 \mapsto fa, \\ x_3 \mapsto gfa \end{array}}{\Box}
$$

$$
\dfrac{\dfrac{\bot \qquad \top}{P(a, f(a))}\;\begin{array}{c} x_1 \mapsto f(a) \\ x \mapsto a \end{array} \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg S(a)}\; z \mapsto a}{Q(f(a), g(f(a))) \wedge \neg S(a)}\;\begin{array}{c} x_2 \mapsto f(a), \\ x_3 \mapsto g(f(a)) \end{array}}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a))}
$$

Huang

$$
\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \forall x_2 P(x_1, x_2))} \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 Q(x_2, x_3) \wedge \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 P(x_1, x_2) \vee (Q(x_2, x_3)) \wedge \neg S(x_1))}
$$

direct:

$$
\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \forall x_2 P(x_1, x_2))}\;\begin{array}{c} x_1 \sim a, x_2 \sim fa \\ x_1 < x_2 \end{array} \qquad \begin{array}{c} x_3 \sim a, x_4 \sim fa, x_5 \sim gfa) \\ x_3 < x_4, x_4 < x_5 \end{array} \; \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_3 \neg S(x_3)}\; x_3 \sim a}{\exists x_3 \forall x_4 \exists x_5 Q(x_4, x_5) \wedge \neg S(x_3)}}{\exists x_1 \forall x_2 \exists x_5 P(x_1, x_2) \vee (Q(x_2, x_5) \wedge \neg S(x_5))}\;\begin{array}{c} x_3 \mapsto x_1, x_4 \mapsto x_2 \\ x_1 < x_2, x_2 < x_5 \end{array}
$$

without merge in end: $(x_1 < x_2, x_3 < x_4, x_4 < x_5)$

$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))$

$\exists x_3 \forall x_4 \exists x_5 \exists x_1 \forall x_2 P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))$

(also interwoven ones appear to work)

combined presentation:

$$
\dfrac{
\dfrac{\bot \mid P(a, x_1) \vee R(y) \qquad \top \mid \neg P(x, f(x))}{P(a, f(a)) \mid R(y)}
\;\begin{array}{l} x_1 \mapsto f(a) \\ x \mapsto a \end{array}
\qquad
\dfrac{\bot \mid Q(x_2, g(x_2)) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(z), x_3) \vee S(z)}{\neg S(a) \mid \neg Q(f(a), x_3)}\; z \mapsto a}{Q(f(a), g(f(a))) \wedge \neg S(a) \mid \neg R(u)}
\;\begin{array}{l} x_2 \mapsto f(a), \\ x_3 \mapsto g(f(a)) \end{array}
}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a)) \mid \Box}
$$

combined presentation ground:

$$
\dfrac{
\dfrac{\bot \mid P(a, f(a)) \vee R(y) \qquad \top \mid \neg P(a, f(a))}{(P(a, f(a)) \wedge \top) \vee (\neg P(a, f(a)) \wedge \bot) \mid R(y)}
\qquad
\dfrac{\bot \mid Q(f(a), g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a), g(f(a)))}}{Q(f(a), g(f(a))) \wedge \neg S(a) \mid \neg R(u)}
}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a)) \mid \Box}
$$

combined presentation ground with direct method but only Δ-terms removed :

$$
\dfrac{
\dfrac{\bot \mid P(a, f(a)) \vee R(y) \qquad \top \mid \neg P(a, f(a))}{(P(a, x_2) \wedge \top) \vee (\neg P(a, x_2) \wedge \bot) \mid R(y)}
\qquad
\dfrac{\bot \mid Q(f(a), g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a), g(f(a)))}}{Q(x_4, g(x_4)) \wedge \neg S(a) \mid \neg R(u)}
}{P(a, x_2) \vee (Q(x_4, g(x_4)) \wedge \neg S(a)) \mid \Box}
$$

combined presentation ground with direct method:

$$
\dfrac{
\dfrac{\bot \mid P(a, f(a)) \vee R(y) \qquad \top \mid \neg P(a, f(a))}{\exists x_1 \forall x_2 (P(x_1, x_2) \wedge \top) \vee (\neg P(x_1, x_2) \wedge \bot) \mid R(y)}
\qquad
\dfrac{\bot \mid Q(f(a), g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\exists x_3 \neg S(x_3) \mid \neg Q(f(a), g(f(a)))}}{\exists x_3 \forall x_4 \exists x_5 (Q(x_4, x_5)) \wedge \neg S(x_3)) \mid \neg R(u)}
}{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (P(x_1, x_2) \vee (Q(x_4, x_5)) \wedge \neg S(x_3)) \mid \Box}
$$

203a – some alternations

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{\overset{\Sigma}{R(x)\lor\neg P(f(x))} \qquad \overset{\Pi}{P(z)\lor Q(g(z))}}{R(x)\lor Q(g(f(x)))}\ z\mapsto f(x) \qquad \overset{\Sigma}{\neg Q(y)\lor S(h(y))}}{R(x)\lor S(h(g(f(x))))}\ y\mapsto g(f(x))}{S(h(g(f(a))))}\ x\mapsto a}{\square}\ x_1\mapsto h(g(f(a)))$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\neg P(f(x))}\ z\mapsto f(x) \qquad \bot}{\neg Q(g(f(x)))\land\neg P(f(x))}\ y\mapsto g(f(x))}{\neg Q(g(f(a)))\land\neg P(f(a))\lor R(a)}\ x\mapsto a}{\neg Q(g(f(a)))\land\neg P(f(a))\lor R(a)\lor S(h(g(f(a))))}\ x_1\mapsto h(g(f(a)))$$

Huang:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1\neg P(x_1)} \qquad \bot}{\exists x_1\forall x_2(\neg Q(x_2)\land\neg P(x_1))}}{\forall x_0\exists x_1\forall x_2(\neg Q(x_2)\land\neg P(x_1)\lor R(x_0))}}{\forall x_0\exists x_1\forall x_2\exists x_3(\neg Q(x_2)\land\neg P(x_1)\lor R(x_0)\lor S(x_3))}$$

Direct:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1\neg P(x_1)}\ x_1\sim f(x) \qquad \bot}{\exists x_1\forall x_2(\neg Q(x_2)\land\neg P(f(x)))}\ x_2\sim g(f(x)); x_1<x_2}{\forall x_0\exists x_1\forall x_2(\neg Q(x_2)\land\neg P(x_1)\lor R(x_0))}\ x_0\sim a; x_0<x_1, x_0<x_2}{\forall x_0\exists x_1\forall x_2\exists x_3(\neg Q(x_2)\land\neg P(x_1)\lor R(x_0)\lor S(x_3))}\ x_3\sim h(g(f(a))); x_0<x_3, x_1<x_3, x_2<x_3$$

203b – many Σ-literals, coloring per occurrence

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{\overset{\Sigma}{R(x)\lor\neg P(f(x))} \qquad \overset{\Sigma}{P(z)\lor Q(g(z))}}{R(x)\lor Q(gfx)}\ z\mapsto fx \qquad \overset{\Sigma}{\neg Q(y)\lor S(h(y))}}{R(x)\lor S(hgfx)}\ y\mapsto gfx}{S(hgfa)}\ x\mapsto a}{\square}\ x_1\mapsto hgfa$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \bot}{\bot}\ z\mapsto fx \qquad \bot}{\bot}\ y\mapsto gfx}{R(a)}\ x\mapsto a}{S(hgfa)\lor R(a)}\ x_1\mapsto hgfa$$

$\to \forall x_1\exists x_2(R(x_1)\lor S(x_2))$

203b' – many Σ-literals, coloring per symbol, all predicates grey

$$\dfrac{\overset{\Pi}{\neg S}(x_1) \qquad \dfrac{\dfrac{\dfrac{\neg \overset{\Pi}{R}(a) \qquad R(x) \vee \overset{\Sigma}{\neg} P(f(x))}{R(a) \mid \neg P(fa)}\ x \mapsto a \qquad P(z) \vee \overset{\Sigma}{Q}(g(z))}{P(fa) \vee R(a) \mid Q(gfa)}\ z \mapsto fa \qquad \neg Q(y) \overset{\Sigma}{\vee} S(h(y))}{\quad}}{\quad}$$

TODO

Example where variables are not the outermost symbol but order is still relevant

204a

$\Sigma = \{P(f(x), g(f(x)))\}$
$\Pi = \{P(f(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f(x_1), x_2)$

204b

$\Sigma = \{P(f^5(x), g(f(x)))\}$
$\Pi = \{P(f^5(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f^5(x_1), x_2)$

example with aufschaukelnde unification, such that direction of arrow isn't clear

205a

$$\cfrac{\overset{\Pi}{P(ffy,gy)} \qquad \cfrac{\overset{\Sigma}{\neg P(x,y)\vee Q(x)} \qquad \cfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffz)\vee Rz}}{\neg R(a) \mid \neg Q(ffa)}\; z\mapsto a}{\neg R(a)\wedge Q(ffa)\mid \neg P(ffa,y)}\; x\mapsto ffa}{(\neg R(a)\wedge Q(ffa))\vee\neg P(ffa,ga)}\; y\mapsto a$$

direct

$$\cfrac{\overset{\Pi}{P(ffy,gy)} \qquad \cfrac{\overset{\Sigma}{\neg P(x,y)\vee Q(x)} \qquad \cfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffz)\vee Rz}}{\exists x_1 \neg R(x_1) \mid \neg Q(ffa)}\; z\mapsto a}{\exists x_1\forall x_2(\neg R(x_1)\wedge Q(x_2))\mid \neg P(ffa,u)}\; x\mapsto ffa}{\exists x_1\forall x_2\forall x_3((\neg R(x_1)\wedge Q(x_2))\vee\neg P(x_2,x_3))}\; y\mapsto a, u\mapsto ga$$

ground:

$$\cfrac{\overset{\Pi}{P(ffa,ga)} \qquad \cfrac{\overset{\Sigma}{\neg P(ffa,y)\vee Q(ffa)} \qquad \cfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffa)\vee Ra}}{\neg R(a) \mid \neg Q(ffa)}}{\neg R(a)\wedge Q(ffa)\mid \neg P(ffa,a)}}{(\neg R(a)\wedge Q(ffa))\vee\neg P(ffa,ga)}$$

arrow lemma: $x_1 \sim ffy$, $x_2 \sim gy$, $y_3 \sim a$ $x_4 \sim ffz$, $x_5 \sim ffa$, $y_6 \sim a$,

$$\cfrac{\overset{\Sigma}{\neg P(x,y)\vee Q(x) \mid \bot} \qquad \cfrac{\overset{\Sigma}{\exists y_3(\neg R(y_3)\mid \bot)} \qquad \overset{\Pi}{\forall x_4(\neg Q(x_4)\vee Rz\mid \bot)}}{\exists y_3\forall x_4((\neg R(y_3)\wedge\top)\vee(R(a)\wedge\bot)\mid\neg Q(x_4))}\; z\mapsto a}{\exists y_3\forall x_4((\neg Q(x_4\wedge\bot))\vee(Q(x)\sigma\wedge\neg R(y_3))\mid\neg P(x,y)\sigma)}\; x\mapsto ffa$$

$$\exists y_3\forall x_4((Q(ffa)\wedge\neg R(y_3))\mid\neg P(ffa,y)) \quad (*)$$

$$\cfrac{\overset{\Pi}{\forall x_1\forall x_4(P(x_1,x_2)\mid\top)} \qquad \exists y_3\forall x_4\forall x_5((Q(x_5)\wedge\neg R(y_3))\mid\neg P(x_5,y))}{(\neg P(x_5,y)\wedge\top)\vee(P(x_1,x_2)\wedge(Q(x_5)\wedge\neg R(y_3)))}\; y\mapsto a$$

$$(\neg P(x_5,a)\wedge\top)\vee(P(x_1,x_2)\wedge(Q(x_5)\wedge\neg R(y_3)))$$

$$\forall x_1\forall x_2\exists y_3\forall x_4\forall x_5\forall x_6(\neg P(x_5,y_6)\wedge\top)\vee(P(x_1,x_2)\wedge(Q(x_5)\wedge\neg R(y_3)))$$

$(*)$ "luckily", same overbinding for ffa, so this works
dashed underline: problem, but does not cause issues here.

situations not critical here :'-(

205b $\sim$ 205a, but simpler

Suppose P occurs somewhere in Σ (result not that optimal in this setting, but correct)
not nice for proving, $\neg R(a)$ is a nice interpolant already

$$\cfrac{\overset{\Pi}{P(ffy,gy)} \qquad \cfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg P(ffz,x)\vee Rz}}{\neg R(a)\mid\neg P(ffa,x)}\; z\mapsto a}{\neg R(a)\vee\neg P(ffa,ga)\mid\Box}\; x\mapsto ga, y\mapsto a$$

$$\cfrac{\overset{\Pi}{\top\mid P(ffy,gy)} \qquad \cfrac{\overset{\Sigma}{\bot\mid\neg R(a)} \qquad \overset{\Pi}{\top\mid\neg P(ffz,x)\vee Rz}}{\exists x_1\neg R(x_1)\mid\neg P(ffa,x)}\; z\mapsto a}{\exists x_1\forall x_2\forall x_3\neg R(x_1)\vee\neg P(x_2,x_3)\mid\Box}\; x\mapsto ga, y\mapsto a$$

$\exists x_1 R(x_1)$

$\exists x_1\forall x_2\forall x_3(R(x_1)\vee\neg P(x_2,x_3))$

example to demonstrate that literals being resolved upon have to be overbound with the same variable

206a

$$\dfrac{\dfrac{\overset{\Pi}{R(f(x))} \qquad \overset{\Sigma}{\neg R(y) \vee P(y)}}{(\neg R(x_3) \wedge \top) \vee (R(x_3) \wedge \bot) \mid P(x_3)} \qquad \dfrac{\overset{\Pi}{\neg P(f(z)) \vee S(z)} \qquad \overset{\Sigma}{\neg S(a)}}{(\neg S(y_2) \wedge \top) \vee (S(y_2) \wedge \bot) \mid \neg P(x_4)}}{(\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)}$$

Gist of this example: $P(f(x))$ is lifted to the left, but $P(f(a))$ to the right. So it's $P(x_3)$ vs $P(x_4)$, but both of them have to have the same variable.

$R(x_3) \in \mathrm{AI}_{\mathrm{mat}}(C_7)$

$P(x_3) \in \mathrm{AI}_{\mathrm{cl}}(C_7)$

$P(x_4) \in \mathrm{AI}_{\mathrm{cl}}(C_8)$

$\Sigma \models (\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)$

$\Sigma \models (\forall x_3)\forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee P(x_3)\Big)$

$\Sigma \not\models (\neg P(1) \wedge \neg R(0)) \vee P(0)$ // if $P \sim \{1\}$ and $R \sim \{0\}$

we know that for original clauses l and l' of $P(x_4)$ and $P(x_3)$, $l\sigma = l'\sigma$

hence same color, and can use different var as same value works.

inductive hypothesis:

$\Gamma \models \top \vee R(x_3)$

$\Gamma \models \bot \vee \neg R(y) \vee P(y)$

$\Gamma \models (\neg R(x_3) \wedge \top) \vee (R(x_3) \wedge \bot) \vee P(x_3) \quad \equiv \quad \neg R(x_3) \vee P(x_3)$

$\Gamma \models \top \vee \neg P(x_4) \vee S(z)$

$\Gamma \models \bot \vee \neg S(a)$

$\Gamma \models (\neg S(a) \wedge \top) \vee (S(a) \wedge \bot) \vee \neg P(x_4) \quad \equiv \quad \neg S(a) \vee \neg P(x_4)$

$\Gamma \models (\neg P(x_3) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(a))$

206b

WRONG: if a variable x_3 occurs, it always refers to $f(x)$, so it is always substituted to a particular value and cannot become $f(a)$ and $f(b)$ in the same clause as just the unifier σ is used.

$$\dfrac{\dfrac{\overset{\Pi}{R(f(x), f(x))} \qquad \overset{\Sigma}{\neg R(y,u) \vee P(y,u)}}{(\neg R(x_3,x_3) \wedge \top) \vee (R(x_3,x_3) \wedge \bot) \mid P(x_3,x_3)} \qquad \dfrac{\overset{\Pi}{\neg P(f(z), f(v)) \vee S(z,v)} \qquad \overset{\Sigma}{\neg S(a,b)}}{(\neg S(y_2,y_6) \wedge \top) \vee (S(y_2,y_6) \wedge \bot) \mid \neg P(x_4,x_7)}}{(\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)}$$

problems due to x_j not referring to actual term

208a

WRONG: variable x is used in two clauses

$$\dfrac{\dfrac{\dfrac{P(f(x)) \lor \overset{\Pi}{Q}(x) \lor R(u) \qquad \neg \overset{\Sigma}{Q}(a)}{P(f(a)) \lor R(u)} \qquad \dfrac{P(f(x)) \lor \overset{\Pi}{Q}(x) \lor \neg R(u) \qquad \neg \overset{\Sigma}{Q}(b)}{P(f(b)) \lor \neg R(u)}}{P(f(a)) \lor P(f(b))} \qquad \dfrac{\neg \overset{\Pi}{P(f(x))} \lor S(x) \qquad \neg \overset{\Sigma}{S}(a)}{\neg P(f(a))}}{P(f(b))}$$

$$\dfrac{\dfrac{\dfrac{\top \mid P(x_1) \lor Q(x) \lor R(u) \qquad \bot \mid \neg Q(y_2)}{\neg Q(y_2) \mid P(x_1) \lor R(u)} \qquad \dfrac{\top \mid P(x_1) \lor Q(x) \lor R(u) \qquad \bot \mid \neg Q(y_4)}{\neg Q(y_4) \mid P(x_1) \lor R(u)}}{\neg Q(y_2) \land \neg Q(y_4) \mid P(x_1) \lor P(x_1)} \qquad \dfrac{\top \mid \neg \overset{\Pi}{P(x_1)} \lor S(x) \qquad \top \mid \neg \overset{\Sigma}{S}(y_2)}{\neg S(y_2) \mid \neg P(x_1)}}{(\neg P(x_5) \land \neg Q(y_2) \land \neg Q(y_4)) \quad \lor \quad (P(x_5) \land \neg S(y_2)) \quad \mid \quad P(x_5)}$$

NB: as the x_1 in the literal is actually $f(a)$, this way, all x_1 become x_5, but the other one is supposed to stand for $f(b)$

ACTUALLY:

$$\dfrac{\dfrac{\dfrac{P(f(x)) \lor \overset{\Pi}{Q}(x) \lor R(u) \qquad \neg \overset{\Sigma}{Q}(a)}{P(f(a)) \lor R(u)} \qquad \dfrac{P(f(x')) \lor \overset{\Pi}{Q}(x') \lor \neg R(u') \qquad \neg \overset{\Sigma}{Q}(b)}{P(f(b)) \lor \neg R(u')}}{P(f(a)) \lor P(f(b))} \qquad \dfrac{\neg \overset{\Pi}{P(f(x''))} \lor S(x'') \qquad \neg \overset{\Sigma}{S}(a)}{\neg P(f(a))}}{P(f(b))}$$

$$\dfrac{\dfrac{\dfrac{\top \mid P(x_1) \lor Q(x) \lor R(u) \qquad \bot \mid \neg Q(y_2)}{\neg Q(y_2) \mid P(x_1) \lor R(u)} \qquad \dfrac{\top \mid P(x_2) \lor Q(x) \lor R(u) \qquad \bot \mid \neg Q(y_4)}{\neg Q(y_4) \mid P(x_2) \lor R(u)}}{\neg Q(y_2) \land \neg Q(y_4) \mid P(x_1) \lor P(x_2)} \qquad \dfrac{\top \mid \neg \overset{\Pi}{P(x_3)} \lor S(x) \qquad \top \mid \neg \overset{\Sigma}{S}(y_2)}{\neg S(y_2) \mid \neg P(x_3)}}{(\neg P(x_5) \land \neg Q(y_2) \land \neg Q(y_4)) \quad \lor \quad (P(x_5) \land \neg S(y_2)) \quad \mid \quad P(x_2)}$$

NB: $\mathrm{au}(P(x_1), P(x_3)) = \{x_1 \mapsto x_5, x_3 \mapsto x_5\}$

Hence a term with a free variable in a clause can never be lifted by the same variable as a term in another clause.
If two terms in the same clause are lifted with a certain variable, they are bound together in the derivation anyway.

clause used multiple times

209a

$$\cfrac{\cfrac{\cfrac{\overset{\Sigma}{P(a)} \qquad \cfrac{\overset{\Sigma}{\neg Q(a)} \qquad \overset{\Pi}{\neg P(x) \lor P(f(x)) \lor Q(y)}}{\neg P(x) \lor P(f(x))}}{P(f(a))}}{P(f(f(a)))} \qquad \overset{\Pi}{\neg P(f(f(z)))}}{\square}$$

NB: we need to rename lifting variables, possibly rename all lifting variables which refer to a term which contains variables (an actual implementation might do this more efficiently, i.e. not always)

$$\cfrac{\cfrac{\cfrac{\bot \mid P(a) \qquad \cfrac{\bot \mid Q(a) \qquad \top \mid \neg P(x) \lor P(x_1) \lor Q(y)}{Q(a) \mid \neg P(x) \lor P(x_1)}}{P(a) \land Q(a) \mid P(x_1)} \qquad Q(a) \mid \neg P(x') \lor P(x_1')}{(\neg P(x_1) \land P(a) \land Q(a)) \quad \lor \quad (P(x_1) \land Q(a)) \quad \mid \quad P(x_1')} \qquad \top \mid \neg P(x_2)}{\Big(\neg P(x_3) \land (\neg P(x_1) \land P(a) \land Q(a)) \quad \lor \quad (P(x_1) \land Q(a))\Big) \quad \lor \quad P(x_3) \mid \square}$$

NB: x_1 used to refer to $f(x)$, now: $f(a)$

$$\mathrm{au}(x_1', x_2) = \{x_1' \mapsto \ell[f(f(a))], x_2 \mapsto \ell[f(f(a))]\}$$

$\Big(\neg P(x_3) \land (\neg P(x_1) \land P(a) \land Q(a)) \quad \lor \quad (P(x_1) \land Q(a))\Big) \quad \lor \quad P(x_3)$
$\equiv \Big(Q(a) \land \Big((\neg P(x_1) \land P(a)) \quad \lor \quad P(x_1)\Big)\Big) \quad \lor \quad P(x_3)$
Σ✓
negated: $\Big(\neg Q(a) \lor \Big((P(x_1) \lor \neg P(a)) \quad \land \quad \neg P(x_1)\Big)\Big) \quad \land \quad \neg P(x_3)$
$\equiv \Big(\neg Q(a) \lor \Big(\neg P(a) \quad \land \quad \neg P(x_1)\Big)\Big) \quad \land \quad \neg P(x_3)$
Π✓
(none of the $P(f^n(x))$, $n \leq 2$ are allowed to be true in a model of Φ)

$f(x) \vee g(x)$ with f, g different colors

207a

$$
\dfrac{\overset{\Pi}{\neg P(z)} \qquad \dfrac{\overset{\Pi}{\neg R(g(a))} \qquad \dfrac{\overset{\Sigma}{P(f(x)) \vee Q(x)} \qquad \overset{\Pi}{R(g(y)) \vee \neg Q(y)}}{\forall x_1 Q(x_1) \mid P(f(x)) \vee R(g(x))}\ y \mapsto x}{\forall x_1 Q(x_1) \mid P(f(a)) // R(g(a))}\ x \mapsto a}{\forall x_1 \exists x_2 \big(Q(x_1) \vee P(x_2)\big) // R(g(a))}\ z \mapsto f(a)
$$

$\Rightarrow$ free vars in the interpolant have to be overbound (if there are arrows, but we can just always do so)

misc examples

201a

$$\dfrac{\dfrac{P(x,y) \overset{\Sigma}{\vee} \neg Q(y) \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(y)}\; x \mapsto a \qquad \dfrac{Q(f(z)) \overset{\Sigma}{\vee} R(z) \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a))}\; z \mapsto a}{\square}\; y \mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,y)}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{R(a)}\; z \mapsto a}{P(a,f(a)) \vee R(a)}\; y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 P(x_1,y)}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{\forall x_3 R(x_3)}\; z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 (P(x_1,x_2) \vee R(x_3))}\; y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

201b

$$\dfrac{\dfrac{P(x,f(y)) \overset{\Sigma}{\vee} \neg Q(f(y)) \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(f(y))}\; x \mapsto a \qquad \dfrac{Q(f(z)) \overset{\Sigma}{\vee} R(z) \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a))}\; z \mapsto a}{\square}\; y \mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,f(y))}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{R(a)}\; z \mapsto a}{P(a,f(a)) \vee R(a)}\; y \mapsto a \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 \exists x_2 P(x_1,x_2)}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{\forall x_3 R(x_3)}\; z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_3)}\; y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

arrow in element which is not in interpolant or resolution clause

206

$$\dfrac{\dfrac{\overset{\Sigma}{P(x) \lor \neg Q(f(x))} \qquad \overset{\Pi}{\neg P(a)}}{\forall x_1 P(x_1) \mid \neg Q(f(a))}\; x \mapsto a \qquad \dfrac{\overset{\Sigma}{Q(y) \lor R(g(y))} \qquad \overset{\Pi}{\neg R(z)}}{\exists x_2 R(x_2) \mid Q(y)}\; z \mapsto g(y)}{\forall x_1 \exists x_2 (P(x_1) \lor R(x_2)) \mid \square}\; y \mapsto f(a)$$

$$P(a) \lor R(g(f(a)))$$

for first interpolant, $\Sigma \not\models \ell_{\Delta,x}[\mathrm{PI}(C)] \lor C$

=> need to overbind clause as well