$\Sigma' = \{R(z) \vee \exists x P(f(x)), \neg Q(x), \}$
$\Pi' = \{\forall y\, g(y) = y, \forall y \neg P(g(y)) \vee Q(y), \neg R(d)\}$
$\Sigma = \mathrm{sk}(\Sigma') = \{R(z) \vee P(f(c)), \neg Q(y), \}$
$\Pi = \mathrm{sk}(\Pi') = \{g(u) = u, \neg P(g(v)) \vee Q(v), \neg R(d)\}$
$L(\Sigma) = \{R, P, Q, f, z, x, c\}$
$L(\Pi) = \{R, P, Q, g, u, v, d\}$
Refutation:

$$\dfrac{\dfrac{\overset{\Sigma}{R(z) \vee P(f(c))} \qquad \overset{\Pi}{\neg R(d)}}{P(f(c))}\; z \mapsto d \qquad \dfrac{\dfrac{\overset{\Pi}{\neg P(g(v)) \vee Q(v)} \qquad \overset{\Sigma}{\neg Q(y)}}{\neg P(g(y))}\; v \mapsto y \qquad \overset{\Pi}{g(u) = u}}{\neg P(u)}\; y \mapsto u}{\Box}\; u \mapsto f(c)$$

Interpolants:

$$\dfrac{\dfrac{\bot \qquad \top}{(\neg R(d) \wedge \bot) \vee (R(d) \wedge \top) \equiv R(d)}\; \theta_0 \qquad \dfrac{\dfrac{\top \qquad \bot}{(\neg Q(y) \wedge \top) \vee (Q(y) \wedge \top) \equiv \neg Q(y)}\; \theta_1 \qquad \top}{(\neg Q(u) \wedge g(u) = u) \vee (\top \wedge g(u) \neq u)}\; \theta_2}{(\neg P(f(c)) \wedge R(d)) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge g(f(c)) = f(c)) \quad \vee \quad g(f(c)) \neq f(c)))}\; \theta_3$$

Relative interpolant properties:

θ_0 :	$\Sigma \vdash R(d) \vee P(f(c))$	$\Pi \vdash \neg R(d) \vee P(f(c))$
θ_1 :	$\Sigma \vdash \neg Q(y) \vee \neg P(g(y))$	$\Pi \vdash Q(y) \vee \neg P(g(y))$
θ_2 :	$\Sigma \vdash (\neg Q(u) \wedge g(u) = u) \vee g(u) \neq u \quad \vee \quad \neg P(u)$	$\Pi \vdash \neg((\neg Q(u) \wedge g(u) = u) \vee g(u) \neq u) \quad \vee \quad \neg P(u)$ $\Pi \vdash ((Q(u) \vee g(u) \neq u) \wedge g(u) = u) \quad \vee \quad \neg P(u)$
θ_3 :	$\Sigma \vdash \theta_3$ Proof: Either $\neg P(f(c))$, then $R(d)$. Otw. either $g(f(c)) \neq f(c)$. Otw. also $\neg Q(f(c))$.	$\Pi \vdash \neg\theta_3$ Proof: $\neg(\neg P(fc) \wedge R(d)) \quad \vee \quad (P(fc) \wedge (\neg Q(fc) \wedge g(fc) = fc) \quad \vee \quad g(fc) \neq fc)$ $\equiv (P(fc) \vee \neg R(d)) \quad \wedge \quad (\neg P(fc) \vee (Q(fc) \vee g(fc) \neq fc) \quad \wedge \quad g(fc) = fc)$ Have $g(fc) = fc$ and $\neg R(d)$, so remaining: $\neg P(fc) \vee Q(fc)$. Get by axiom and unification with $g(u) = u$.

$\Sigma = \{R(z) \vee P(f(c)), \neg Q(y), \}$
$\Pi = \{g(u) = u, \neg P(g(v)) \vee Q(v), \neg R(d)\}$

Propositional refutation tree (no non-trivial unifiers):

$$\dfrac{\dfrac{\overset{\Sigma}{R(d) \vee P(f(c))} \qquad \overset{\Pi}{\neg R(d)}}{P(f(c))} \qquad \dfrac{\dfrac{\overset{\Pi}{\neg P(g(f(c))) \vee Q(f(c))} \qquad \overset{\Sigma}{\neg Q(f(c))}}{\neg P(g(f(c)))} \qquad \overset{\Pi}{g(f(c)) = f(c)}}{\neg P(f(c))}}{\Box}$$

Lifting:

terms: $g(f(c)), f(c), d$

max Π-terms: $\{g(f(c)), d\} \sim \{x_1, x_2\}$

max Σ-terms: $\{f(c)\} \sim \{x_3\}$

$$\overline{(\neg P(f(c)) \wedge R(d)) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge g(f(c)) = f(c)) \quad \vee \quad g(f(c)) \neq f(c)))}(x_1, x_2)$$

$$\Leftrightarrow \neg P(f(c)) \wedge R(x_2) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge x_1 = f(c)) \quad \vee \quad x_1 \neq f(c)))$$

By Lemma 12, $\Sigma \models \overline{\theta_3}$ (proof from above still goes through).

$\hat{\theta}(x_3) = (\neg P(x_3) \wedge R(x_2)) \quad \vee \quad (P(x_3) \wedge ((\neg Q(x_3) \wedge x_1 = x_3) \quad \vee \quad x_1 \neq x_3))$

quantifiers according to order: $|d| < |f(c)| < |g(f(c))|$

$\theta = \forall x_2 \exists x_3 \forall x_1 (\neg P(x_3) \wedge R(x_2)) \quad \vee \quad (P(x_3) \wedge (\neg Q(x_3) \vee x_1 \neq x_3))$

$\neg\theta = \exists x_2 \forall x_3 \exists x_1 (P(x_3) \vee \neg R(x_2)) \quad \wedge \quad (\neg P(x_3) \vee (Q(x_3) \wedge x_1 = x_3))$

$\Rightarrow \Sigma \vdash \theta; \Pi \vdash \neg\theta$

Example 2:

$\Sigma = \{P(c), \neg P(d)\}$

$\Pi = \{P(d) \lor g(u) = u, \neg P(g(x))\}$

Refutation:

$$\cfrac{\cfrac{\cfrac{\overset{\Pi}{P(d) \lor g(u) = u} \qquad \overset{\Sigma}{\neg P(d)}}{g(u) = u} \qquad \overset{\Pi}{\neg P(g(x))}}{\neg P(x)}\, u \mapsto x \qquad \overset{\Sigma}{P(c)}}{\square}\, x \mapsto c$$

Relative interpolants:

$$\cfrac{\cfrac{\cfrac{\top \qquad \bot}{(\neg P(d) \land \top) \lor (P(d) \land \bot) \equiv \neg P(d)} \qquad \top}{(g(x) = x \land \top) \lor (g(x) \neq x \land \neg P(d))}\, u \mapsto x \qquad \bot}{(\neg P(c) \land \bot) \lor (P(c) \land (g(c) = c \lor (g(c) \neq c \land \neg P(d))))}\, x \mapsto c$$

$\theta = P(c) \land (g(c) = c \lor \neg P(d))$

$\neg\theta = \neg P(c) \lor (g(c) \neq c \land P(d))$

terms: $g(c), c, d$

max Π-terms: $g(c)$

max Σ-terms: c

ordered by length ASCENDING: $\{c, g(c)\}$

$\overline{\theta}(x_2) = P(c) \land (x_2 = c \lor \neg P(d))$

$\hat{\theta}(x_1) = P(x_1) \land (x_2 = x_1 \lor \neg P(d))$

$\Sigma \vdash \exists x_1 \forall x_2 P(x_1) \land (x_2 = x_1 \lor \neg P(d))$

$\Pi \vdash \neg\exists x_1 \forall x_2 P(x_1) \land (x_2 = x_1 \lor \neg P(d))$

$\Pi \vdash \forall x_1 \exists x_2 \neg P(x_1) \lor (x_2 \neq x_1 \land P(d))$

A possible interpolant: $\neg P(d) \land \exists x P(x)$

Example 2 (Craig translation):

$\Sigma = \{P(c), \neg P(d)\}$

$\Pi = \{P(d) \vee g(u) = u, \neg P(g(x))\}$

$T(\Sigma) = \{\forall x\ x = x\} \cup \{\forall x \forall y\ x = y \supset P(x) \supset P(y)\} \cup \Sigma$

$$
\begin{aligned}
T(\Pi) = &\{\forall x\ x = x\} \cup \\
&\{\forall x \forall y\ x = y \supset P(x) \supset P(y), \forall x_1 \forall x_2 \forall y_1 \forall y_2\ x_1 = y_1 \supset x_2 = y_2 \supset x_1 = x_2 \supset y_1 = y_2, \forall x_1 \forall x_2 \forall y_1 \forall y_2\ x_1 = y_1 \supset x_2 = y_2 \supset G(x_1, x_2) \supset G(y_1, y_2)\} \cup \\
&\{P(d) \vee (\exists z G(u, z) \wedge (\forall y G(u, y) \supset z = y) \wedge z = u), \neg P(g(x))\}
\end{aligned}
$$

to continue seems to be not work the effort

Example 3 Bonacina/Johannson:

$\Sigma = \{A \vee B, \neg C\}$

$\Pi = \{\neg A \vee C, \neg B\}$

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{A \vee B} \quad \overset{\Pi}{\neg A \vee C}}{B \vee C} \quad \overset{\Sigma}{\neg C}}{B} \quad \overset{\Pi}{\neg B}}{\Box}$$

Bon/Joh:

$$\dfrac{\dfrac{\dfrac{\bot \quad \top}{(A \vee \bot) \wedge \top \equiv A} \quad \bot}{A \wedge (\neg C \vee \bot) \equiv A \wedge \neg C} \quad \top}{(B \vee (A \wedge \neg C)) \wedge \top}$$

Huang:

$$\dfrac{\dfrac{\dfrac{\bot \quad \top}{(\neg A \wedge \bot) \vee (A \wedge \top) \equiv A} \quad \bot}{(\neg C \wedge A) \vee (C \wedge \bot) \equiv \neg C \wedge A} \quad \top}{(\neg B \wedge (\neg C \wedge A)) \vee (B \wedge \top)}$$

-> logically equivalent

Example 3B Bonacina/Johannson:

$\Sigma = \{A \vee B, \neg C, \neg D\}$

$\Pi = \{\neg A \vee C, \neg B \vee D\}$

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{A \vee B} \quad \overset{\Pi}{\neg A \vee C}}{B \vee C} \quad \overset{\Sigma}{\neg C}}{B} \quad \dfrac{\overset{\Sigma}{\neg D} \quad \overset{\Pi}{\neg B \vee D}}{\neg B}}{\Box}$$

Bon/Joh:

$$\dfrac{\dfrac{\dfrac{\bot \quad \top}{(A \vee \bot) \wedge \top \equiv A} \quad \bot}{A \wedge (\neg C \vee \bot) \equiv A \wedge \neg C} \quad \dfrac{\bot \quad \top}{\top \wedge (\neg D \vee \bot) \equiv \neg D}}{(B \vee (A \wedge \neg C)) \wedge \neg D}$$

Huang:

$$\dfrac{\dfrac{\dfrac{\bot \quad \top}{(\neg A \wedge \bot) \vee (A \wedge \top) \equiv A} \quad \bot}{(\neg C \wedge A) \vee (C \wedge \bot) \equiv \neg C \wedge A} \quad \dfrac{\bot \quad \top}{(\neg D \wedge \top) \vee (D \wedge \bot) \equiv \neg D}}{(\neg B \wedge \neg C \wedge A) \vee (B \wedge \neg D)}$$

-> not logically equivalent

Example 4: Paramodulation special case in Huang failed, see next page

$$
\dfrac{\dfrac{\dfrac{\overset{\Sigma}{P(x) \vee \neg Q(x)} \qquad \overset{\Pi}{Q(h(r))}}{P(h(r))} \qquad \overset{\Pi}{s = t}}{P(h(s))}\, r \mapsto s \qquad \overset{\Pi}{\neg P(h(s))}}{\Box}
$$

$$
\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\neg Q(h(r))} \qquad \top}{(s = t \wedge \neg Q(h(t))) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t))} \qquad \top}{(\neg P(h(s)) \wedge (s = t \wedge \neg Q(h(t))) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t))) \vee P(h(s))}
$$

$\Sigma = \{P(x) \vee \neg Q(x)\}$

$\Pi = \{\neg P(h(s)), Q(h(r)), s = t\}$

$((s = t \wedge \neg Q(h(t))) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t))) \vee P(h(s))$

$\theta = \neg Q(h(t)) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t)) \vee P(h(s))$

$\neg\theta = Q(h(t)) \wedge (s = t) \wedge (s \neq t \vee h(s) = h(t)) \wedge \neg P(h(s))$

$\theta^* = \forall x_1 \forall x_2 \forall x_3 \forall x_4 \neg Q(x_2) \vee (x_3 \neq x_4) \vee (x_3 = x_4 \wedge x_1 \neq x_2) \vee P(x_1)$

$\neg\theta^* = \exists x_1 \exists x_2 \exists x_3 \exists x_4 Q(x_2) \wedge (x_3 = x_4) \wedge (x_3 \neq x_4 \vee x_1 = x_2) \wedge \neg P(x_1)$

$\Rightarrow$ special case not needed here

Example 4b: Paramodulation special case in Huang

$\Sigma = \{P(x) \vee \neg Q(x), \neg P(x) \vee Q(x), s = t\}$ // $P(x) \leftrightarrow Q(x)$ $\quad\Pi = \{\neg Q(h(s)), Q(h(t))\}$

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{P(x) \vee \neg Q(x)} \qquad \overset{\Pi}{Q(h(t))}}{P(h(t))} \qquad \overset{\Sigma}{s = t}}{P(h(s))} \qquad \dfrac{\overset{\Sigma}{\neg P(x) \vee Q(x)} \qquad \overset{\Pi}{\neg Q(h(s))}}{\neg P(h(s))}}{\square}$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\neg Q(h(t))} \qquad \bot}{((s = t) \wedge \neg Q(h(t))) \vee (s = t \wedge h(s) \neq h(t))} \qquad \dfrac{\bot \qquad \top}{Q(h(s))}}{(((s = t) \wedge \neg Q(h(t))) \vee (s = t \wedge h(s) \neq h(t))) \vee Q(h(s))}$$

$$\dfrac{\dfrac{\dfrac{\Sigma \models \bot \mid P(x) \vee \neg Q(x) \qquad \Sigma \models \top \mid Q(x_2)}{\Sigma \models \neg Q(x_2) \mid P(x_2)} \qquad \Sigma \models \bot \mid s = t}{\Sigma \models ((s = t) \wedge \neg Q(x_2)) \vee (s = t \wedge x_1 \neq x_2) \mid P(x_1)} \qquad \dfrac{//\ \overset{\Sigma}{\neg P(x) \vee Q(x)} \qquad //\ \overset{\Pi}{\neg Q(h(s))}}{//\neg P(h(s))}}{\square}$$

$\theta = (((s = t) \wedge \neg Q(h(t))) \vee (s = t \wedge h(s) \neq h(t))) \vee Q(h(s))$

$\neg\theta = (((s \neq t) \vee Q(h(t))) \wedge (s \neq t \vee h(s) = h(t))) \wedge \neg Q(h(s))$

$\theta^* = \forall x_1 \forall x_2 (((s = t) \wedge \neg Q(x_2)) \vee (s = t \wedge x_1 \neq x_2)) \vee Q(x_1)$

$\neg\theta^* = \exists x_1 \exists x_2 (((s \neq t) \vee Q(x_2)) \wedge (s \neq t \vee x_1 = x_2)) \wedge \neg Q(x_1)$

special case relevant for Σ as it does not know about the relation of x_1 and x_2

$\Sigma \models \forall x_1 \forall x_2 (((s = t) \wedge \neg Q(x_2)) \vee (s = t \wedge x_1 \neq x_2)) \vee Q(x_1)$

$\Sigma \models \forall x_1 \forall x_2 ((s = t) \wedge (\neg Q(x_2) \vee x_1 \neq x_2)) \vee Q(x_1)$

$\Sigma \models ((s = t) \wedge (\neg Q(b) \vee a \neq b)) \vee Q(a)$

Intuition:

Get $s = t$ for free, but else not relevant

$\Sigma \models \neg Q(b) \vee a \neq b \vee Q(a)$

$\Sigma \models Q(b) \supset a = b \supset Q(a)$

$\Rightarrow$ special case IS needed

Example 4c: Paramodulation special case in Huang, term contained in both Γ- and Δ-term

$\Sigma = \{P(x) \vee \neg Q(x), \neg P(y) \vee Q(y), s = t, \neg R_1(g(t)), \neg R_2(g(s))\}$ // $P(x) \leftrightarrow Q(x)$ $\qquad \Pi = \{R_2(x_3) \vee \neg Q(h(x_3)), R_1(x_2) \vee Q(h(x_2))\}$

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{\neg R_1(g(t))} \qquad \overset{\Pi}{R_1(x_2) \vee Q(h(x_2))}}{\neg R_1(g(t)) \mid Q(h(g(t)))}\; x_2 \mapsto g(t) \qquad \overset{\Sigma}{P(x) \vee \neg Q(x)}}{\underbrace{\neg R_1(g(t)) \wedge \neg Q(h(g(t)))}_{\alpha} \mid P(h(g(t)))}\; x \mapsto h(g(t)) \qquad \overset{\Sigma}{s = t}}{\nu \mid P(h(g(s)))}$$

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{\neg R_2(g(s))} \qquad \overset{\Pi}{R_2(x_3) \vee \neg Q(h(x_3))}}{\neg R_2(g(s)) \mid \neg Q(h(g(s)))}\; x_3 \mapsto g(s) \qquad \overset{\Sigma}{\neg P(y) \vee Q(y)}}{\underbrace{\neg R_2(g(s)) \wedge Q(h(g(s)))}_{\beta} \mid \neg P(h(g(s)))}\; y \mapsto h(g(s))}{\square \text{ // by resolution with upper tree}}$$

$$\nu_0 = \Big((s = t \wedge \alpha) \vee (s \neq t \wedge \bot)\Big)\sigma \quad \equiv \quad (s = t \wedge \alpha)\sigma \quad \equiv \quad s = t \wedge \neg R_1(g(t)) \wedge \neg Q(h(g(t)))$$

Using the maximal colored symbol, $h(g(t))$ 1. $\nu = \nu_0 \vee (s = t \wedge h(g(s)) \neq h(g(t)))\sigma$	Using the maximal Σ symbol, $g(t)$ 2. $\nu = \nu_0 \wedge (s \neq t \wedge g(s) = g(t))\sigma$
$\Big(s = t \wedge \neg R_1(g(t)) \wedge \neg Q(h(g(t)))\Big) \vee (s = t \wedge h(g(s)) \neq h(g(t))) \vee P(h(g(s)))$ $\Sigma \models \exists y_1 \forall x_2 \forall x_3 (\Big(s = t \wedge \neg R_1(y_1) \wedge \neg Q(x_2)\Big) \vee (s = t \wedge x_3 \neq x_2) \vee P(x_3))$ $\Rightarrow \Sigma \models \exists y_1 \forall x_2 \forall x_3 (\Big(\neg R_1(y_1) \wedge \neg Q(x_2)\Big) \vee (x_3 \neq x_2) \vee P(x_3))$ $\Rightarrow \Sigma \models \forall x_2 \forall x_3 (\neg Q(x_2) \vee (x_3 \neq x_2) \vee P(x_3))$ special case saves the day! ✓ $\Pi \models \forall y_1 \exists x_2 \exists x_3 (\Big(s \neq t \vee R_1(y_1) \vee Q(x_2)\Big) \wedge (s \neq t \vee x_3 = x_2) \vee \neg P(x_3))$ $\Rightarrow \Pi \models \forall y_1 \exists x_2 \Big(R_1(y_1) \vee Q(x_2)\Big) \wedge (x_3 = x_2)$ ✓ Final interpolant: $\mu = \nu \vee \beta = \Big(s = t \wedge \neg R_1(g(t)) \wedge \neg Q(h(g(t)))\Big) \vee (s = t \wedge h(g(s)) \neq h(g(t))) \vee \neg R_2(g(s)) \wedge Q(h(g(s)))$ $\Rightarrow$ looks good	$\Big(s = t \wedge \neg R_1(g(t)) \wedge \neg Q(h(g(t)))\Big) \wedge (s \neq t \vee g(s) = g(t)) \vee P(h(g(s)))$ $\Sigma \models \exists y_1 \exists y_2 \forall x_3 \forall x_4 (\Big(s = t \wedge \neg R_1(y_1) \wedge \neg Q(x_3)\Big) \wedge (s \neq t \vee y_2 = y_1)) \vee P(x_4)$ $\Rightarrow \Sigma \models \exists y_1 \exists y_2 \forall x_3 \forall x_4 (\Big(\neg R_1(g(t)) \wedge \neg Q(x_3)\Big)) \vee P(x_4)$ $\Rightarrow \Sigma \models \forall x_3 \forall x_4 (\neg Q(x_3) \vee P(x_4))$ ☒ This makes sense: ν_0 needs some help, but the extra case here becomes effective for Π only

$\Rightarrow$ need to use outermost colored symbol, not just maximal Φ-term!

Example 5: cases for one pass overbinding algo

want to have step in between where only one of the "critical" terms appears in the interpolant and a decision on the order is forced

$$\frac{\overset{\Sigma}{P(y_1, y_2)} \qquad \dfrac{\overset{\Sigma}{Q(\alpha)} \qquad \overset{\Pi}{\neg Q(z) \vee \neg P(z, \beta)}}{\neg P(\alpha, \beta)}\; z \mapsto \alpha}{\Box}$$

$$\frac{\bot \qquad \dfrac{\bot \qquad \top}{Q(\alpha)^{\circ}}\; z \mapsto \alpha}{P(\alpha, \beta) \wedge Q(\alpha)}$$

$\Rightarrow$ need to overbind α first, no matter which order would be assigned later

NOTE: b might be $f(a)$, i.e. we don't know a priori at which level it is and how many smaller or larger terms will be added.

Let $\alpha = b$, $\beta = g(z)$.

$\theta^* = \exists x_1 \forall x_2 P(x_1, x_2) \wedge Q(x_1)$

$\neg\theta^* = \forall x_1 \exists x_2 \neg P(x_1, x_2) \vee \neg Q(x_1)$

$\theta^{\circ *} = \exists x_1 Q(x_1)$

Let $\alpha = g(x)$, $\beta = b$.

$\theta^* = \exists x_1 \forall x_2 P(x_1, x_2) \wedge Q(x_1)$

$\neg\theta^* = \forall x_1 \exists x_2 \neg P(x_1, x_2) \vee \neg Q(x_1)$

$\theta^{\circ *} = \exists x_1 Q(x_1)$

$\Rightarrow$ works (need not change quantifier order like this, but here, no predicate has parameters which depend on each other)

Example 5b: no equality, but quantifier order still matters

$$\frac{\overset{\Sigma}{P(u,g(u))} \qquad \overset{\Pi}{\neg P(a,x)}}{\square}\; u \mapsto a, x \mapsto g(a)$$

Prop Interpolant: $P(a,g(a))$
Interpolant: $\forall x_1 \exists x_2 P(x_1,x_2)$

Example 5b': order matters, construction in multiple steps:

$$\frac{\dfrac{\overset{\Sigma}{P(u,v,f(u,v)) \vee Q(u)} \qquad \overset{\Pi}{\neg Q(a)}}{P(a,v,f(a,v))}\; u \mapsto a \qquad \overset{\Pi}{\neg P(x,b,y)}}{\square}\; x \mapsto a, v \mapsto b, y \mapsto f(a,b)$$

$$\frac{\dfrac{\bot \qquad \top}{Q(a)}\; u \mapsto a \qquad \top}{P(a,b,f(a,b)) \vee (\neg P(a,b,f(a,b)) \wedge Q(a))}\; x \mapsto a, v \mapsto b, y \mapsto f(a,b)$$

Non-trivial interpolants:
$\forall x_1 Q(x_1)$
$\forall x_1 \forall x_2 \exists x_3 P(x_1,x_2,x_3) \vee Q(x_1)$

Example 5b": 5b' with different resolution order

$$\frac{\dfrac{\overset{\Sigma}{P(u,v,f(u,v)) \vee Q(u)} \qquad \overset{\Pi}{\neg P(x,b,y)}}{Q(u)}\; x \mapsto u, v \mapsto b, y \mapsto f(u,b) \qquad \overset{\Pi}{\neg Q(a)}}{\square}\; u \mapsto a$$

$$\frac{\dfrac{\bot \qquad \top}{P(u,b,f(u,b))}\; x \mapsto u, v \mapsto b, y \mapsto f(u,b) \qquad \top}{P(a,b,f(a,b)) \vee Q(a)}\; u \mapsto a$$

Non-trivial interpolants:
$\forall x_2 \exists x_3 P(u,x_2,x_3)$
$\forall x_1 \forall x_2 \exists x_3 (P(x_1,x_2,x_3) \vee Q(x_1))$

Example 5c: example with $\exists\forall$ necessarily in interpolant

$\Rightarrow$ as shown in Huang, swap Σ and Π from 5b'

$$\dfrac{\dfrac{\overset{\Pi}{P(u,v,f(u,v)) \lor Q(u)} \qquad \overset{\Sigma}{\neg Q(a)}}{P(a,v,f(a,v))}\; u \mapsto a \qquad \overset{\Sigma}{\neg P(x,b,y)}}{\Box}\; x \mapsto a, v \mapsto b, y \mapsto f(a,b)$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg Q(a)}\; u \mapsto a \qquad \bot}{\neg P(a,b,f(a,b)) \land \neg Q(a)}\; x \mapsto a, v \mapsto b, y \mapsto f(a,b)$$

Non-trivial interpolants:

$\exists x_1 Q(x_1)$

$\exists x_1 \exists x_2 \forall x_3 (\neg P(x_1, x_2, x_3) \land \neg Q(x_1))$

$\Rightarrow$ similar for 5b"

$$\dfrac{\dfrac{\overset{\Pi}{P(u,v,f(u,v)) \lor Q(u)} \qquad \overset{\Sigma}{\neg P(x,b,y)}}{Q(u)}\; x \mapsto u, v \mapsto b, y \mapsto f(u,b) \qquad \overset{\Sigma}{\neg Q(a)}}{\Box}\; u \mapsto a$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg P(u,b,f(u,b))}\; x \mapsto u, v \mapsto b, y \mapsto f(u,b) \qquad \bot}{\neg Q(a) \land \neg P(a,b,f(a,b))}\; u \mapsto a$$

Non-trival interpolants:

$\exists x_2 \forall x_3 \neg(P(u, x_2, x_3))$

$\exists x_1 \exists x_2 \forall x_3 (\neg Q(x_1) \land \neg(P(x_2, x_3)))$