Chapter **1**

# Number of quantifier alternations of extracted interpolants

In this chapter, we derive lower bounds for the number of quantifier alternations for the extraction procedure in two phases as well as the one in one phase. We arrive at the conclusion that both procedures are equal with respect to this measure and that there is a tight connection between color alternations in terms and quantifier alternations in the interpolant.

## 1 Additional notation and definitions

A literal is called a $\Phi$-literal if its predicate symbol is $\Phi$-colored.

In a literal or term $\phi$ containing a subterm $t$, $t$ is said to occur *below* a $\Phi$-symbol $s$ if in the syntax tree representation of $\phi$, there is a node labeled $s$ on the path from the root to $t$. Note that the colored symbol may also be the predicate symbol. Moreover, $t$ is said to occur *directly below* the $\Phi$-symbol $s$ if it occurs below the $\Phi$-symbol $s$ and in the syntax tree representation of $\phi$ on the path from $s$ to $t$, no nodes with labels with colored symbol occur.

### 1.1 Unification

Let $\varphi$ and $\psi$ be two terms or literals. For $\sigma = \mathrm{mgu}(\varphi, \psi)$, we denote by $\sigma_i$ for $1 \leq i \leq |\operatorname{dom}(\sigma)|$ the $i$th substitution which is added to $\sigma$ by the unification algorithm. We define $\sigma_0 \stackrel{\text{def}}{=} \mathrm{id}$.

We furthermore denote the composition $\sigma_i \dots \sigma_j$ by $\sigma_{(i,j)}$. Hence $\sigma = \sigma_{(1,|\operatorname{dom}(\sigma)|)} = \sigma_{(0,|\operatorname{dom}(\sigma)|)}$.

### 1.2 Color and quantifier alternations

In the following, we assume that the maximum max of an empty sequence is defined to be 0 and constants are treated as function symbols of arity 0. Furthermore $\perp$ is used to denote a color which is not possessed by any symbol.

**Definition 1** (Color alternation col-alt)**.** Let $\Gamma$ and $\Delta$ be sets of formulas and $t$ be a term.

$\text{col-alt}(t) \stackrel{\text{def}}{=} \text{col-alt}_{\perp}(t)$

$$\text{col-alt}_{\Phi}(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } t \text{ is a variable} \\ \max(\text{col-alt}_{\Phi}(t_1), \ldots, \text{col-alt}_{\Phi}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ is gray} \\ \max(\text{col-alt}_{\Phi}(t_1), \ldots, \text{col-alt}_{\Phi}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_{\Psi}(t_1), \ldots, \text{col-alt}_{\Psi}(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ is of color } \Psi,\ \Phi \neq \Psi \end{cases}$$

$\triangle$

**Definition 2** (Quantifier alternation quant-alt)**.** Let $A$ be a formula.

$\text{quant-alt}(A) \stackrel{\text{def}}{=} \text{quant-alt}_{\perp}(A)$

$$\text{quant-alt}_{Q}(A) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } A \text{ is an atom} \\ \text{quant-alt}_{Q}(B) & \text{if } A \equiv \neg B \\ \max(\text{quant-alt}_{Q}(B), \\ \qquad \text{quant-alt}_{Q}(C)) & \text{if } A \equiv B \circ C,\ \circ \in \{\wedge, \vee, \supset\} \\ \text{quant-alt}_{Q}(B) & \text{if } A \equiv QxB \\ 1 + \text{quant-alt}_{Q'}(B) & \text{if } A \equiv Q'xB,\ Q \neq Q' \end{cases}$$

$\triangle$

Note that this definition of quantifier alternations handles formulas in prenex and non-prenex form.

## 2 Quantifier alternations in PI

**Definition 3** (PI*)**.** PI* is defined as PI with the difference that in PI*, all literals are considered to be gray. $\text{PI}^*_{\text{init}}$ and $\text{PI}^*_{\text{step}}$ are defined analogously. $\triangle$

Hence $\text{PI}^*_{\text{init}}$ coincides with $\text{PI}_{\text{init}}$. $\text{PI}^*_{\text{step}}$ coincides with $\text{PI}_{\text{step}}$ in case of factorization and paramodulation inferences. For resolution inferences, the first two cases in the definition of $\text{PI}_{\text{step}}$ do not occur for $\text{PI}^*_{\text{step}}$.

PI* enjoys the convenient property that it absorbs every literal which occurs in some clause:

op:every_lit_in_pi_star⟩ **Proposition 4.** *For every literal which occurs in a clause of a resolution refutation $\pi$, a respective successor occurs in* $\text{PI}^*(\pi)$*.*

*Proof.* By structural induction. □

Note that in PI*, we can conveniently reason about the occurrence of terms as no terms are lost throughout the extraction. However Lemma 5 allows us to transfer results about gray literals to PI:

y_lits_of_pi_star_in_pi⟩ **Lemma 5.** *For every clause $C$ of a resolution refutation, the literals and equalities of* $\mathrm{PI}(C)$ *are exactly the gray literals and equalities of* $\mathrm{PI}^*(C)$.

*Proof.* Note that $\mathrm{PI}_{\mathrm{init}}$ and $\mathrm{PI}^*_{\mathrm{init}}$ coincide and $\mathrm{PI}_{\mathrm{step}}$ and $\mathrm{PI}^*_{\mathrm{step}}$ only differ for resolution inferences. More specifically, they only differ on resolution inferences, where the resolved literal is colored. Hence $\mathrm{PI}(C)$ and $\mathrm{PI}^*(C)$ contain the same grey literals and equalities. The colored resolved literals however are not added to $\mathrm{PI}(C)$ as desired. □

⟨lemma:0t8Gie7y⟩ **Lemma 6.** *Let $\iota$ be an inference of a resolution refutation using the clauses $C_1, \ldots, C_n$ which creates the clause $C$. If there is a gray literal $\lambda$ or an equality $s = t$ in $\mathrm{PI}(C_i) \vee C_i$ for $1 \leq i \leq n$, then a successor of $\lambda$ or $s = t$ respectively occurs in* $\mathrm{PI}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \ldots, \mathrm{PI}(C_n)) \vee C$.

*Proof.* Immediate by the definition of PI. □

y_lits_and_eq_all_in_PI⟩ **Corollary 7.** *If there is a gray literal $\lambda$ or an equality $s = t$ in $\mathrm{PI}(C) \vee C$ for a clause $C$ of a resolution refutation $\pi$, then a successor of $\lambda$ or $s = t$ respectively occurs in* $\mathrm{PI}(\pi)$.

*Proof.* This is a direct consequence of Lemma 6. □

We now make some considerations in the form of four lemmata about the construction of certain terms in the context of interpolant extraction. In order to do so, we frequently reason over the stepwise application of the respective unifiers, for which we employ the following definition:

**Definition 8.** We define $\tilde{\mathrm{PI}}^*_{\mathrm{step}}$ to coincide with $\mathrm{PI}^*_{\mathrm{step}}$ but without applying the substitution $\sigma$ in each of the cases. Furthermore, $\tilde{\mathrm{PI}}^*(C)$ is an abbreviation of $\tilde{\mathrm{PI}}^*_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_m))$ if $C$ is created by an inference $\iota$ from the clauses $C_1, \ldots, C_n$, and $\tilde{\mathrm{PI}}^*(C)$ coincides with $\mathrm{PI}^*(C)$ if $C \in \Gamma \cup \Delta$.

Analogously, if $C \equiv D\sigma$, we use $\tilde{C}$ to denote $D$. △

In the context of an inference $\iota$ using the clauses $C_1, \ldots, C_m$ to infer $C$, it holds that:

$$\begin{aligned}
\mathrm{PI}^*(C) \vee C &= \mathrm{PI}^*_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_m)) \vee C \\
&= \left(\tilde{\mathrm{PI}}^*_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_m)) \vee \tilde{C}\right)\sigma \\
&= \left(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C}\right)\sigma \\
&= \left(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C}\right)\sigma_{(0, |\operatorname{dom}(\sigma)|)}
\end{aligned}$$

Note that if we are able to show that the application of a substitution $\sigma_i$ to $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,i-1)}$ maintains an invariant and the invariant holds for $\tilde{\mathrm{PI}}^*(C) \vee \tilde{C}$, then it immediately follows that it holds for $\mathrm{PI}^*(C) \vee C$.

ma:var_below_phi_symbol⟩ **Lemma 9.** *Let $\iota$ be an inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable $u$ occurs directly below a $\Phi$-symbol in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,i)}$ for $i \geq 1$. Then at least one of the following statements holds:*

⟨15_1⟩

1. *The variable $u$ occurs directly below a $\Phi$-symbol in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,i-1)}$.*

⟨15_5⟩

2. *The variable $u$ occurs at a gray position in a gray literal or at a gray position in an equality in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,i)}$.*

⟨15_2⟩

3. *There is a variable $v$ such that*
   - *$u$ occurs gray in $v\sigma_i$ and*
   - *$v$ occurs in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,i-1)}$ directly below a $\Phi$-symbol as well as directly below a $\Psi$-symbol*

*Proof.* We consider all different situations under which the situation in question arises. Irrespective of the type of the inference $\iota$, one of these cases can apply:

- There is already a literal in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,i-1)}$ where $u$ occurs directly below a $\Phi$-symbol and $\sigma_i$ does not change this. Then clearly 1 is the case.
- There is a variable $v$ in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,i-1)}$ such that $v\sigma_i$ contains $u$ directly below a $\Phi$-symbol. As $v$ is unified with the term $v\sigma_i$, $v\sigma_i$ must occur in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,i-1)}$, which implies that 1 is the case.

In the case that $\iota$ is a resolution or factorization inference, the following situations can apply:

- There is a variable $v$ which occurs directly below a $\Phi$-symbol such that $u$ occurs gray in $v\sigma_i$.

  Hence in the resolved or factorized literals $\lambda$ and $\lambda'$ in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,i-1)}$, there is a position $p$ such that without loss of generality $\lambda|_p = v$ and $u$ occurs gray in $\lambda'|_p$. Note that due to the definition of the unification algorithm, $\lambda$ and $\lambda'$ must coincide on the path to $p$.

  By Proposition 4, $\lambda$ and $\lambda'$ occur in $\tilde{\mathrm{PI}}^*(C) \vee \tilde{C}$ irrespective of their coloring.

  We distinguish cases based on the position $p$:

  - Suppose that $p$ occurs directly below a $\Phi$-symbol. Then as $u$ occurs gray in $\lambda'|_p$, $u$ occurs directly below a $\Phi$-symbol in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,i-1)}$ and 1 is the case.

   – Suppose that $p$ occurs directly below a $\Psi$-symbol. Then $v$ occurs directly below a $\Psi$-symbol in $\lambda|_p$ and 3 holds.
   – Suppose that $p$ does not occur directly below a colored symbol. Then $p$ does not occur below any colored symbol, hence $u$ is contained in a gray literal in a gray position in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$. As $\sigma_i$ is trivial on $u$, this occurrence of $u$ also is present in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$ and hence 2 is the case.

Now we consider the case that $\iota$ is a paramodulation inference of the clauses $C_1 : r_1 = r_2 \vee D$ and $C_2 : E[r]_p$ with $\sigma = \mathrm{mgu}(\iota) = \mathrm{mgu}(r_1, r)$ yielding $C : (D \vee E[r_2]_p)\sigma$. We again consider the different situations under which the situation in question arises:

- The variable $u$ occurs gray in $r_2$ and $p$ in $E$ is directly below a $\Phi$-symbol. But then $u$ occurs gray in an equality in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ and as $\sigma_i$ is trivial on $u$ also in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$, hence 2 holds.
- Suppose that some variable $v$ occurs directly below a $\Phi$-symbol in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ such that $u$ occurs gray in $v\sigma_i$. Then by the definition of the unification algorithm, there exists a position $q$ such that one of $r_1|_q$ and $r|_q$ is $v$ and the other one contains a gray occurrence of $u$.

  We distinguish cases based on the position $q$:

   – Suppose that $q$ occurs directly below a $\Phi$-symbol. Then clearly 1 is the case.
   – Suppose that $q$ occurs directly below a $\Psi$-symbol. Then as the variable $v$ also occurs directly below a $\Phi$-symbol and $u$ occurs gray in $v\sigma_i$, 3 is the case.
   – Suppose that $q$ is a gray position. Then 2 is the case: Either $u$ occurs gray in $r_1$ in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ and then also in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$, or otherwise $v$ occurs gray in $r_1$ in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$, but as $v\sigma_i$ contains $u$ gray, $u$ occurs gray in of $r_1\sigma_i$ in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$. □

⟨lemma:col_change⟩ **Lemma 10.** *Let $\iota$ be an inference of a resolution refutation of $\Gamma \cup \Delta$. Suppose that a variable $u$ occurs directly below a $\Phi$-symbol as well as directly below a $\Psi$-symbol in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$. Then $u$ occurs gray in a gray literal or gray in an equality in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$.*

*Proof.* We proceed by induction over the refutation. As the original clauses each contain symbols of at most one color, the base case is trivially true.

For the induction step, suppose that an inference makes use of the clauses $C_1, \ldots, C_n$ and that the lemma holds for $\mathrm{PI}^*(C_j) \vee C_j$ for $1 \leq j \leq n$.

Note that then, the lemma holds for $\tilde{\mathrm{PI}}^*_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_n)) \vee \tilde{C} = \tilde{\mathrm{PI}}^*(C) \vee \tilde{C}$. This is because as all clauses are variable-disjoint, if a variable occurs in $\tilde{\mathrm{PI}}^*(C) \vee \tilde{C}$ both directly below a $\Phi$-symbol as well as directly below a $\Psi$-symbol, then this must be the case also in $\mathrm{PI}^*(C_j) \vee C_j$ for some $j$, for which the lemma by assumption holds. Furthermore, by the definition of $\mathrm{PI}^*$, every literal which occurs in $\mathrm{PI}^*(C_j) \vee C_i$ for some $j$ occurs in $\tilde{\mathrm{PI}}^*(C) \vee \tilde{C}$.

Hence it remains to show that the lemma holds for $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma = (\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_0 \ldots \sigma_m$, which we do by induction over $i$ for $1 \leq i \leq m$. Suppose that the lemma holds for $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ and in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$, the variable $u$ occurs directly below a $\Phi$-symbol as well as directly below a $\Psi$-term.

Then by Lemma 9, we can deduce that one of the following statements holds for $\Phi = \Gamma$ as well as $\Phi = \Delta$. We denote case $j$ for $\Phi = \Gamma$ by $j^\Gamma$ and for $\Phi = \Delta$ by $j^\Delta$.

⟨16_1⟩
1. The variable $u$ occurs directly below a $\Phi$-symbol in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$.

⟨16_4⟩
2. The variable $u$ occurs at a gray position in a gray literal or at a gray position in an equality in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$.

⟨16_2⟩
3. There is a variable $v$ such that
   - $u$ occurs gray in $v\sigma_i$ and
   - $v$ occurs in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ directly below a $\Phi$-symbol as well as directly below a $\Psi$-symbol

If $2^\Gamma$ or $2^\Delta$ is the case, we clearly are done. On the other hand if $3^\Gamma$ or $3^\Delta$ is the case, then by the induction hypothesis, $v$ occurs gray in a gray literal or gray in an equality in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$. As $u$ occurs gray in $v\sigma_i$, we obtain that then, $u$ occurs gray in a gray literal or gray in an equality in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$.

Hence the only remaining possibility is that both $1^\Gamma$ and $1^\Delta$ hold. But then $u$ occurs directly below a $\Phi$-symbol as well as below a $\Psi$-symbol in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ and again by the induction hypothesis, we obtain that $u$ occurs gray in a gray literal or gray in an equality in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$, and as $\sigma_i$ is trivial on $u$, the same occurrence of $u$ is present in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$. $\square$

mma:subterm_in_gray_lit⟩ **Lemma 11.** *Let $C$ be a clause in a resolution refutation of $\Gamma \cup \Delta$. If $\mathrm{PI}^*(C) \vee C$ contains a maximal colored occurrence of a $\Phi$-term $t[s]$, which contains a maximal $\Psi$-colored term $s$, then $s$ occurs gray in a gray literal or gray in an equality in $\mathrm{PI}(C) \vee C$.*

*Proof.* Note that it suffices to show that the desired term occurs in a gray literal or equality in $\mathrm{PI}^*(C) \vee C$ since by Lemma 5, all gray literals and equalities of $\mathrm{PI}^*(C)$ also occur in $\mathrm{PI}(C)$. We do so by induction over the resolution refutation.

As the original clauses each contain symbols of at most one color, the base case is vacuously true.

The induction step is laid out similarly as in the proof of Lemma 10. We suppose that an inference makes use of the clauses $C_1, \dots, C_n$ and that the lemma holds for $\mathrm{PI}^*(C_j) \vee C_j$ for $1 \leq j \leq n$. Then the lemma holds for $\tilde{\mathrm{PI}}^*(C) \vee \tilde{C} = \tilde{\mathrm{PI}}^*_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \dots, \mathrm{PI}^*(C_n)) \vee \tilde{C})$ as no new terms are introduced in $\tilde{\mathrm{PI}}^*(C) \vee \tilde{C}$ and all literals from $\mathrm{PI}^*(C_j) \vee C_j)$ for $1 \leq j \leq n$ occur in $\tilde{\mathrm{PI}}^*(C) \vee \tilde{C}$.

It remains to show that the lemma holds for $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma = (\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_0 \dots \sigma_m$, which we do by induction over $i$ for $0 \leq i \leq m$. We distinguish based on the situation under which a unification leads to the term $t[s]$.

- Suppose for some variable $u$ that $u\sigma_i$ contains $t[s]$. Then $u$ is unified with a term which contains $t[s]$ and which occurs in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$. Hence by the induction hypothesis, $s$ occurs gray in a gray literal or gray in an equality in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ and, as $\sigma_i$ does not change this, also in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$.

- Otherwise there is a variable $u$ which occurs directly below a $\Phi$-symbol and $v\sigma_i$ contains a gray occurrence of $s$. We distinguish based on the occurrences of $u$ in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$:

  - Suppose that $u$ occurs somewhere in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ gray in a gray literal or gray in an equality. Then clearly we are done.
  - Suppose that $u$ occurs somewhere in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ directly below a $\Psi$-symbol. Then by Lemma 10, $u$ occurs gray in a gray literal or gray in an equality in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$, whose successor in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i)}$ is an occurrence of $s$ of the same coloring. Hence we are done a well.
  - Suppose that $u$ occurs in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ only directly below a $\Psi$-symbol. Here, we differentiate between the types of inference of the current induction step:

    * Suppose that the inference of the current induction step is a resolution or a factorization inference. As $u$ occurs gray in $v\sigma_i$, there is a position $p$ such that for the resolved or factorized literals $\lambda$ and $\lambda'$ it holds without loss of generality that $\lambda|_p = u$ and $s$ occurs gray in $\lambda'|_p$. Note that $\lambda$ and $\lambda'$ agree on the path to $p$, including the predicate symbol..
      Now as by assumption $u$ only occurs directly below a $\Phi$-symbol, so must $s$. But then $s$ occurs directly below a $\Phi$-symbol in $(\tilde{\mathrm{PI}}^*(C) \vee \tilde{C})\sigma_{(0,\,i-1)}$ and we get the result by the induction hypothesis.

  - Suppose that the inference of the current induction step is a paramodulation inference. Assume it uses the the clauses $C_1 : r_1 = r_2 \vee D$ and $C_2 : E[r]_p$ with $\sigma = \mathrm{mgu}(\iota) = \mathrm{mgu}(r_1, r)$ to yield $C : (D \vee E[r_2]_p)\sigma$.
    As $u$ is affected by $\sigma_i$, it must occur in $r_1$ or $r$. Let $\hat{u}$ refer to this occurrence.
    - Suppose that $\hat{u}$ occurs directly below a $\Phi$-colored function symbol.
      If $\hat{u}$ is contained in $r_1$, then $s$ must be contained in $r$ directly below a $\Phi$-colored function symbol as $r_1$ and $r$ are unifiable. We then get the result by the induction hypothesis.
      If otherwise $\hat{u}$ is contained in $r$, then there are two possibilities for the occurrence of $s$ in $r_1$:
      Either $\hat{u}$ occurs in a $\Phi$-colored function symbol in $r$. Then $s$ occurs in a $\Phi$-colored function symbol in $r_1$ and we get the result by the induction hypothesis.
      Otherwise $\hat{u}$ occurs gray in $r$, but $r$ occurs directly below a $\Phi$-colored function symbol in $E$. Then however, as $r$ and $r_1$ are unifiable, $s$ must occur gray in $r_1$ and hence gray in an equality.
    - Suppose that $\hat{u}$ occurs directly below a $\Phi$-colored predicate symbol.
      Then as the equality predicate is not considered to be colored, $u$ must occur gray in $r$. But then as $r_1$ and $r$ are unifiable, $s$ must occur gray in $r_1$ and hence gray in an equality. □

gray_lit_then_quant_alt⟩ **Lemma 12.** *If a term with $n$ color alternations occurs in a gray literal or in an equality in $\mathrm{PI}^*(C) \vee C$ for a clause $C$, then the interpolant $I$ produced in Theorem 24 contains at least $n$ quantifier alternations.*

*Proof.* We perform an induction on $n$ and show the strengthening that the quantification of the lifting variable which replaces a term with $n$ color alternations is required to be in the scope of the quantification of $n-1$ alternating quantifiers.

For $n = 0$, no colored terms occur in $I$ and hence also no quantifiers. Moreover for $n = 1$, there are terms of one color which evidently require at least one quantifier.

Suppose that the statement holds for $n-1$ for $n > 1$ and that a term $t$ with $\text{col-alt}(t) = n$ occurs in $\mathrm{PI}^*(C) \vee C$. We assume without loss of generality that $t$ is a $\Phi$-term. Then $t$ contains some $\Psi$-colored term $s$ with $\text{col-alt}(s) = n-1$ and by Lemma 11, $s$ occurs gray in a gray literal or an equality in $\mathrm{PI}(C) \vee C$. By Lemma 7, a successor of $s$ occurs in $\mathrm{PI}(\pi)$. Note that as $s$ occurs in a gray position, any successor of $s$ also occurs in a gray position.

By the induction hypothesis, the quantification of the lifting variable for $s$ requires $n-1$ alternated quantifiers. As $s$ is a subterm of $t$ and $t$ is lifted, $t$ must be quantified in the scope of the quantification of $s$, and as $t$ and $s$ are of different color, their quantifier type is different. Hence the quantification of the lifting variable for $t$ requires $n$ quantifier alternations. $\square$

a:quant_alt_lower_bound⟩ **Lemma 13.** *If a term with $n$ color alternations occurs in* $\mathrm{PI}(C) \vee C$ *for a clause $C$, then the interpolant $I$ produced in Theorem 24 contains at least $n$ quantifier alternations.*

*Proof.* Let $t$ be a term in $\mathrm{PI}(C) \vee C$ with $\text{col-alt}(t) = n$. Then by Lemma 5, $t$ occurs grey in a grey literal or grey in an equality in $\mathrm{PI}^*(C)$. Lemma 12 gives the result. $\square$

We present an example which illustrates that the occurrence of a term with $n$ color alternations in $\mathrm{PI}(C) \vee C$ for a clause $C$ can lead to an interpolant with $n-1$ quantifier alternations (but no less as Lemma 13 shows).

**Example 14.** Let $\Gamma = \{\neg P(a)\}$ and $\Delta = \{P(x) \vee Q(f(x)), \neg Q(y)\}$. Consider the following refutation of $\Gamma \cup \Delta$:

$$\dfrac{\dfrac{\neg P(a) \mid \bot \qquad P(x) \vee Q(f(x)) \mid \top}{Q(f(a)) \mid \neg P(a)}\ \text{res}_{x \mapsto a} \qquad \neg Q(y) \mid \top}{\square \mid \neg P(a)}\ \text{res}_{y \mapsto f(a)}$$

In this example, Theorem 24 yields the interpolant $I \equiv \exists y_a \neg P(y_a)$ with $\text{quant-alt}(I) = 1$. The existence of the term $f(a)$ with $\text{col-alt}(f(a)) = 2$ in a clause of the refutation implies that $\text{quant-alt}(I) \geq 1$. $\triangle$

# 3 Upper bound

a:quant_alt_upper_bound⟩ **Lemma 15.** *Let $t$ be a term with the maximal number of color alternations in* $\mathrm{PI}(C) \vee C$ *for any clause $C$. Then there is an arrangement of the quantifier prefix in Theorem 24 which gives rise to an interpolant with at most* $\text{col-alt}(t)+1$ *quantifier alternations.*

*Proof.* Let $T_i^\Phi$ be the set of maximal $\Phi$-colored terms in $\mathrm{PI}(\pi)$ with $i$ color alternations for $1 \leq i \leq n$, where $n = \text{col-alt}(t)$. Note that every maximal colored term of $\mathrm{PI}(\pi)$ is contained in one of these sets. We use $\exists T_i^\Gamma$ ($\forall T_i^\Delta$) to denote $\exists y_{t_1} \dots \exists y_{t_m}$ ($\forall x_{t_1} \dots \forall x_{t_m}$) where $t_1, \dots, t_m$ is an arrangement of the elements of $T_i^\Gamma$ ($T_i^\Delta$) in ascending subterm order.

Now we construct the interpolant

$$I \equiv \forall T_1^\Delta \exists T_1^\Gamma \; \exists T_2^\Gamma \forall T_2^\Delta \; \forall T_3^\Delta \exists T_3^\Gamma \dots Q^\Phi T_n^\Phi Q^\Psi T_n^\Psi \; \ell_\Gamma^y[\ell_\Delta^x[\mathrm{PI}(\pi)]],$$

where $Q^\Phi T_n^\Phi Q^\Psi T_n^\Psi$ is $\forall T_n^\Delta \exists T_n^\Gamma$ if $n$ is odd and $\exists T_n^\Gamma \forall T_2^\Delta$ if $n$ is even. Clearly, $I$ has at most $n+1$ color alternations.

In order to show the result, it remains to show that $I$ is a valid interpolant with respect to Theorem 24. Note that the quantifier prefix binds all lifting variables occurring in $\ell_\Gamma^y[\ell_\Delta^x[\mathrm{PI}(\pi)]]$. We conclude by showing that the order of the quantifiers is admissible.

Let $t$ be a maximal colored term in $\ell_\Gamma^y[\ell_\Delta^x[\mathrm{PI}(\pi)]]$. We prove that the quantifier for the lifting variable of every subterm $s$ of $t$ precedes the quantifier for the lifting variable for $t$ in $I$. Suppose that col-alt$(t) = k$. Then we can deduce that col-alt$(s) \leq k$.

- If col-alt$(s) = k$, then $t$ and $s$ are of the same color and hence the quantifiers for their respective lifting variables are contained in the same block. However the quantifiers of each block are ordered as desired.

- Otherwise col-alt$(s) = l$ for some $l$ such that $l < k$. Then the lifting variable replacing $s$ is quantified in $\exists T_l^\Gamma$ or $\forall T_l^\Delta$. In any case, it precedes the quantifier for the lifting variable replacing $t$ which is contained in $\exists T_k^\Gamma$ or $\forall T_k^\Delta$. □

**Theorem 16.** *Let $n$ be the maximal number of color alternations of any term in $\mathrm{PI}(C) \vee C$ for any clause $C$ of a resolution refutation of $\Gamma \cup \Delta$. Then by arranging the quantifiers in a quantifier alternation minimizing fashion the interpolant of Theorem 24 has at least $n$ and at most $n+1$ quantifier alternations.*

*Proof.* Immediate by Lemma 13 and Lemma 15. □

# 4 Quantifier alternations in LI

We now show that the result of Lemma **??** holds in a very similar way also for LI, which we work out in detail in this section.

⟨lemma:pi_li_1⟩ **Conjectured Lemma 17.** *Let $C$ be a clause in a resolution refutation of $\Gamma \cup \Delta$. If a maximal colored term $t$ occurs in $\mathrm{LI}^\bullet(C)$ in a literal $\lambda$ at position $p$, then $t$ also occurs in a corresponding literal at position $p$ in $\mathrm{PI}(C)$.*

*Proof.* Note that PI and LI differ only in the lifting conditions of LI. As $t$ however is a colored term and not a lifting variable, and it also occurs in $\mathrm{LI}^\bullet(C)$, the lifting conditions are not true for any predecessor of $t$ in $\mathrm{LI}^\bullet(C)$. □

⟨lemma:pi_li_2⟩ **Conjectured Lemma 18.** *Let $C$ be a clause in a resolution refutation of $\Gamma \cup \Delta$. If a maximal colored term $t$ occurs in $\mathrm{PI}(C)$ in a literal $\lambda$ at position $p$, then either $t$ or $z_{t'}$, where $t'$ is an abstraction of $t$, occurs in a corresponding literal at position $p$ in $\mathrm{LI}^\bullet(C)$.*

*Proof.* As pointed out previously, PI and LI differ only in the lifting conditions of LI. Suppose that no predecessor of the term in $\mathrm{LI}(C)$ which corresponds to $t$ fulfills a lifting condition. Then $t$ occurs at position $p$ in $\mathrm{LI}^\bullet(C)$. Otherwise $z_{t'}$ occurs at the position $p$ in $\mathrm{LI}^\bullet(C)$, where $t'\tau = t$ for some substitution $\tau$. □

⟨lemma:pi_li_3⟩ **Conjectured Lemma 19.** *TODO: merge with 5.3. of thesis (lemma:lifting_ conditions)* *Let $C$ be a clause of a resolution refutation such that $\mathrm{LI}^\bullet(C)$ contains a maximal colored $\Phi$-term $t$ which is lifted in $\mathrm{LI}(C)$. Suppose that $t$ contains a $\Psi$-colored subterm $s$. Then $s$ occurs as a subterm of $t$ in $\mathrm{LI}^\bullet(C)$.*

*Proof.* By the construction of LI, this lemma is only violated if the term $s$ or a respective predecessor is lifted due to fulfilling one of the lifting conditions.

For the sake of contradiction, suppose that this is the case in the inference creating the clause $C'$. Let $s'$ and $t'$ be the respective predecessors of $s$ and $t$ in $C'$.

- Suppose that $s'$ is lifted due to containing a variable which does not occur in $C'$. Then as $s'$ is a subterm of $t'$ , $t'$ contains this variable as well and therefore is lifted in $\mathrm{LI}(C')$, contradicting the assumption.

- Suppose that $s'$ is lifted due to being a ground term which does not occur in $C$. Then $t'$ does not occur in $C'$ either as any occurrence of $t'$ would contain $s'$. Hence $t'$ is lifted in $\mathrm{LI}(C')$, contradicting the assumption. □

**Definition 20.** Alternative, equivalent definition of LI.

<insert def for base case>

Let $\mathrm{LI}^\bullet_0(C) = \mathrm{LI}^\bullet(C)$ and $Z_0$ be the set of terms in $\mathrm{LI}^\bullet_i(C)$ for which some lifting conditions holds.

We now define $\mathrm{LI}(C)$ and $\mathrm{LI}^\bullet_i(C)$ for $i \geq 1$:

1. Let $r \in Z_i$ be such that $r$ is not subterm of any other term in $Z_i$. If no such $r$ exists, let $\mathrm{LI}(C) = \mathrm{LI}^\bullet_i(C)$.

2. Let $Z_{i+1} = Z_i \backslash \{r\}$.

3. Let $\mathrm{LI}^\bullet_{i+1}(C)$ be built from $\mathrm{LI}^\bullet{}_i(C)$ such that $r$ is lifted by a fresh lifting variable $z'_r$ and the formula is prefixed by $\forall x_r$ if $r$ is a $\Delta$-term and $\exists y_r$ if $r$ is a $\Gamma$ term. △

⟨…ubterm_of_multicol_term⟩ **Lemma 21.** *Let $t$ be a term with $\text{col-alt}(t) = n$ which occurs gray in a gray literal or gray in an equality $\mathrm{LI}^\bullet(C) \vee C$ for some clause $C$ and let $s$ be a subterm of $t$ with $\text{col-alt}(s) = n - 1$. Then there exists a clause $C'$ which is a successor of $C$ such that respective successors of $t$ and $s$, which are not lifting variables, occur in $\mathrm{LI}^\bullet(C') \vee C'$, and the successor of $t$ in $\mathrm{LI}(C) \vee C$ is a lifting variable.*

*Proof.* By Lemma **??**, a successor of $t$ occurs in $\mathrm{LI}^{\bullet}(C') \vee C'$ for any successor $C'$ of $C$. Let $C'$ be the successor of $C$ such that $\mathrm{LI}^{\bullet}(C')$ contains a successor of $t$ which is not a lifting variable and $\mathrm{LI}(C')$ contains a successor of $t$ which is a lifting variable. As all colored terms are lifted eventually, such a clause $C'$ must occur.

By Lemma 17, $t$ occurs at the same position in $\mathrm{PI}(C')$. We assume without loss of generality that $t$ is a $\Phi$-term. Then $t$ contains some $\Psi$-colored term $s$ with $\text{col-alt}(s) = n - 1$. By Lemma 11, $s$ occurs gray in a gray literal or gray in an equality in $\mathrm{PI}(C') \vee C'$.

By Lemma 18, then either $s$ or $z_{s'}$, where $s'$ is an abstraction of $s$, occurs in $\mathrm{LI}^{\bullet}(C')$ at a corresponding literal and position. As $t$ is not lifted in $\mathrm{LI}^{\bullet}(C') \vee C'$ and $s$ is a subterm of $t$, Lemma 19 proves that $s$ is also not lifted in $\mathrm{LI}^{\bullet}(C') \vee C'$. Hence the case of $z_{s'}$ is excluded. □

**Conjectured Lemma 22.** *If a term $t$ with* $\text{col-alt}(t) = n$ *occurs gray in a gray literal or gray in an equality* $\mathrm{LI}^{\bullet}(C) \vee C$ *for some $C$, then* $\mathrm{LI}(\pi)$ *has at least $n$ quantifier alternations.*

*Proof.* We proceed by induction over $n$ to show that the lifting variable, which replace the successor of $t$ in $\mathrm{LI}(\pi)$, is enclosed by $n$ alternating quantifiers.

For $n = 0$, no colored terms occur and hence no lifting variables which induce quantifiers. Moreover, for $n = 1$, the interpolant clearly must have at least one quantifier.

We now turn to the induction step and suppose that the lemma holds for $n - 1$ for $n > 1$. By Lemma 21, there exists a clause $C'$ which is a successor of $C$ such that respective successors of $t$ and $s$, which are not lifting variables, occur in $\mathrm{LI}^{\bullet}(C') \vee C'$, and the successor of $t$ in $\mathrm{LI}(C) \vee C$ is a lifting variable.

Note that as $s$ is a subterm of $t$, $s$ is only quantified after $t$ (either in this stage or some subsequent one) such that the quantifier for $t$ is in the scope of the quantifier for $s$.

By the induction hypothesis, the quantifier for the lifting variable, which is the successor of $s$ in $\mathrm{LI}(\pi)$, is enclosed by $n - 1$ alternating quantifiers in $\mathrm{LI}(\pi)$. As $s$ and $t$ are of different colors, their quantifier type is different. Since the quantifier for $t$ occurs in the scope of the quantifier for $s$, we get that the lifting variable replacing the successor of $t$ in $\mathrm{LI}(\pi)$ is enclosed by $n$ alternating quantifiers. □

TODO: conclude by explaining two main results ; describe how to apply result (term $t$ occs in ??)

**Conjectured Lemma 23.** *need term which exist initially, and with all subsets applied has max color alternations*

*note that there is no term in PI, LI, or anywhere, with more alternations this term does occur in* PI* *hence a term with one less col alt occs gray in* PI*, *hence also in* PI. *(then the usual)*

*however this term may not actually occur in* LI *(cf. 703)*

*POSSIBLY: do apply subsets to lft vars. then show that you can jump back and forth from li to pi (basically not* $z_{s'}$ *but* $z_s$ *in that lemma above).*

*Then lemmas for* PI *could suffice*

*possible lemma statement:*

*IF* $t$ *WITH* $\text{col-alt}(t) = n$ *OR* $z_t$ *OCCURS, THEN* $n$ *QUANT ALT*

**Theorem 24.** *<Huang's thm> (this is here only to make the references to it in this document work)*

⟨`thm:two_phases`⟩