

Number of quantifier alternations in Huang and nested

1.1 Outline

Goal: try proof Huang and nested equal.

Method: proof for both:

Conjectured Proposition 1. *Let I be an interpolant created by $\$algorithm$. If I contains a term t such that t has n color changes, then I has at least n quantifier alternations.*

1.1.1 generally keep in mind

- Need to define all new terms here: color-changing, single-color, Φ -literal, substitutions from 0 to n
 - essentially same position: path from one position to other only contains grey symbol (this def allows for identical position as well)

1.2 Preliminaries

Quantifier alternations in I usually assumes the quantifier-alternation-minimising arrangement of quantifiers in I

Definition 2 (Color alternation col-alt). Colors Γ and Δ , term t :

$$\text{col-alt}(t) \stackrel{\text{def}}{=} \text{col-alt}_{\perp}(t)$$

Let $t = f(t_1, \dots, t_n)$ for constant, function and variable symbols (syntax abuse)

$$\text{col-alt}_{\Phi}(t) \stackrel{\text{def}}{=} \begin{cases} \max(\text{col-alt}_{\Phi}(t_1), \dots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is grey} \\ \max(\text{col-alt}_{\Phi}(t_1), \dots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_{\Psi}(t_1), \dots, \text{col-alt}_{\Psi}(t_n)) & f \text{ is of color } \Psi, \Phi \neq \Psi \end{cases}$$

\triangle

Definition 3. $\text{PI}_{\text{step}}^{\circ}$ is defined just like PI_{step} but without applying any substitution. \triangle

Hence $\text{PI}_{\text{step}}^\circ(\cdot)\sigma = \text{PI}_{\text{step}}(\cdot)$. C° is somehow the same, i.e. if $C = D\sigma$, then $C^\circ = D$ where σ is derived from the context.

1.3 Random thoughts

- Quantifiers are introduced for lifting variables which actually occur in the interpolant
- If term t with $\text{col-alt}(t) = n$ enters I , we need subterm s of t with $\text{col-alt}(s) = n - 1$ to be in I (of course colors of t and s are exactly opposite)

1.3.1 Proof

- Induction over $\ell_\Delta^x[\text{PI}(C) \vee C]$ and also about Γ -terms with Δ -lifting vars in that formula. Cf. `-final`
- TODO: describe proof method with $\sigma_{(0,i)}$: which PI?
 - Factorisation: easy: just apply σ_i for all i to $\text{PI}(C) \vee C$. When done, a literal will be there twice and we can remove it without losing anything
 - Resolution: create propositional structure first.
 Ex.: $C_1 : D \vee l, C_2 : \neg l \vee E$:
 If we talk about properties for which it holds that if they hold for $\text{PI}(C_i) \vee C_i, i \in \{1, 2\}$, then they also hold for $A \equiv ((l \wedge \text{PI}(C_2)) \vee (\neg l \wedge \text{PI}(C_1))) \vee C^\circ$, then we can apply σ_i for all i to that formula.
 So if we can assume it for A and show it for all σ_i , we get that it holds for $\text{PI}(C) \vee C$.

1.4 Proof port attempt from `-final`

need to show that grey occurrences are in grey literals, all grey literals end up in the interpolant!

conj: if a Δ -term t occurs in a Γ -literal in a clause C , then t occurs in a grey literal in $\text{PI}(C)$.

evidence:

- situation does not occur in Γ or Δ
- terms are only changed by unifiers
- Δ - and Γ -terms are not unifiable, so one of the literals has to have a variable at a grey position when a Δ -term enters a Γ -literal
- that literal has to be grey
- QED?

Refuted Lemma (this is wrong) 4. *If a Φ -term t occurs in a Ψ -literal in a clause C , then t occurs at a grey position in $\text{PI}(C)$.*

Proof. As all grey literals of clauses involved in a refutation end up in the interpolant, it suffices to show that t occurs at a grey position in a grey literal.

Substitutions are applied to all variables, hence we only need to consider terms t which just enter a foreign colored literal.

TODO: propagation 1: Φ -terms vs Ψ -terms (in Ψ -literals)

TODO: propagation 2: Φ -terms vs other Φ -terms (in Ψ -literals)

Induction on refutation and σ ; base case easy.

Resolution or factorisation inference ι . Let λ be a Γ -literal containing a variable u at position \hat{u} such that $u\sigma_i$ contains a Δ -term t .

If the resolved or factorised literals are grey, they become part of $\text{PI}(C)$ and if t occurs grey there, we are done.

- Suppose the resolved literals are Γ -colored. Then IH.
- Suppose the resolved literals are Δ -colored. TODO:
- Suppose the resolved literals are grey and t does not occur at a grey position in $\lambda\sigma = \lambda'\sigma$.

TODO:

□

Conjectured Lemma 5. *If a Φ -term t occurs in a Ψ -literal in a clause C , then t occurs at a grey position in a grey literal in $\text{PI}(C) \vee C$.*

there has to be a variable u in a Ψ -literal such that $u\sigma_i$ contains t .

Conjectured Lemma 6. *If a variable u occurs in a Φ -literal as well as in a Ψ -literal in a clause C , then t also occurs at a grey position in a grey literal in $\text{PI}(C)$.*

Proof. Initially not the case.

Note that we can only resolve/factorise Γ -/ Δ -/grey literals with other Γ -/ Δ -/grey literals as clearly the predicate symbol must be the same for both literals. Hence if a variable occurs only in Γ - or only in Δ -literals, then it can never escape these. Hence u certainly is contained in a grey literal.

Now suppose that u only occurs colored in grey literals. Then it occurs in a Γ -(Δ -) term in the original Γ -(Δ -)clauses which contain it.

As shown before u must occur in some grey literal. Suppose it does not occur at a grey position in a grey literal as otherwise we are done. Then u only occurs in Γ -terms in grey literals as

TODO: it seems that now we have to deal with possible Γ -terms in Δ -literals and so on \Rightarrow circular reasoning

The situation in question arises if some variable u occurs in a Γ -literal in some clause and some variable v occurs in a Δ -literal in some clause (possibly the same), such that in the unified literals, u and v both occur at the same position in the respective literals.

□

1.5 directly from old proof

$\langle \text{lemma:col_change} \rangle$ **Lemma 7.** *Resolution or factorisation step ι from \bar{C} .*

If u col-change var in $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)\sigma_{(0,i)}$, then u also occurs grey in that formula.

Proof. Abbreviation: $F \equiv (\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)$

Induction over refutation and σ ; base case easy.

Step: Supp color change var u present in $F\sigma_{(0,i)}$. (could also say introduced, then proof would be somehow different)

Supp u not grey in $F\sigma_{(0,i-1)}$ as otherwise done. As a first step, we show that if a (not necessarily color-changing) variable v occurs in a single-colored Φ -term $t[v]$ in $F\sigma_{(0,i)}$, then at least one of the following holds:

1. v occurs in some single-colored Φ -term in $F\sigma_{(0,i-1)}$
2. there is a color-changing variable w in $F\sigma_{(0,i-1)}$ such that v occurs grey in $w\sigma_i$.

$\langle \text{var_occ_2} \rangle$ We consider the different cases which can introduce a variable v in a single-colored term Φ : Either it has been there before, it was introduced in a s.c. Φ -colored term, or a s.c. Φ -term containing the var is in $\text{ran}(\sigma)$.

- Suppose a term $t'[v]$ is present in $F\sigma_{(0,i-1)}$ such that $t'[v]\sigma_i = t[v]$. Then 1 is the case.
- Suppose a variable w occurs in a single-colored Φ -term in $F\sigma_{(0,i-1)}$ such that v occurs grey in $w\sigma_i$. Suppose furthermore that 1 is not the case, i.e. v does not occur in a s.c. Φ -term in $F\sigma_{(0,i-1)}$, as otherwise we would be done. We show that 2 is the case.

As v occurs neither grey nor in a s.c. Φ -term in $F\sigma_{(0,i-1)}$ but occurs in $\text{ran}(\sigma_i)$, it must occur in $F\sigma_{(0,i-1)}$ and this can only be in a single-colored Ψ -term.

As by assumption v occurs grey in $w\sigma_i$, there must be an occurrence \hat{w} of w in a resolved or factorised literal, say $\lambda\sigma_{(0,i-1)}$ such that for the other resolved literal $\lambda'\sigma_{(0,i-1)}$, $\lambda'\sigma_{(0,i-1)}|_{\hat{w}}$ is a subterm in which v occurs grey. But as the occurrence of v in $\lambda'\sigma_{(0,i-1)}|_{\hat{w}}$ must be contained in a single-colored Ψ -term, so is $\lambda\sigma_{(0,i-1)}|_{\hat{w}}$, hence z occurs in a single-colored Ψ -term as well. Therefore 2 is the case.

- Suppose there is a variable z in $F\sigma_{(0,i-1)}$ such that v occurs in a single-colored Φ -term in $z\sigma_i$. Then $z\sigma_i$ occurs in $F\sigma_{(0,i-1)}$, but this is a witness for 1.

Now recall that we have assumed u to be a color-changing variable in $F\sigma_{(0,i)}$. Hence it occurs in a single-colored Γ -term as well as in a single-colored Δ -term. By the reasoning above, this leads to two case:

- In $F\sigma_{(0,i-1)}$, u occurs both in some single-colored Γ -term as well as in some single-colored Δ -term. Then we get the result by the induction hypothesis and the fact that $u \notin \text{dom}(\sigma_i)$ as u does occur in $F\sigma_{(0,i)}$.
- Otherwise for some color Φ , u does not occur in a single-colored Φ -term in $F\sigma_{(0,i-1)}$. Then case 2 above must hold and there is some color-changing variable w in $F\sigma_{(0,i-1)}$ such that u occurs grey in $w\sigma_{(0,i)}$. But then by

the induction hypothesis, w occurs grey in $F\sigma_{(0,i-1)}$ and hence u occurs grey in $F\sigma_{(0,i)}$. \square

NB: this is the heart of the proof:

Lemma 8. *If $\text{PI}(C) \vee C$ contains a maximal colored occurrence of a Γ -term $t[s]$ containing Δ -term s , then s occurs grey in $\text{PI}(C) \vee C$.*

works if todo's are handled in sublemmas

Proof. Note that it suffices to show that at the step where s is introduced as subterm of $t[s]$, s occurs grey in $\text{PI}(C) \vee C$ as any later modification by substitution is applied to both occurrences s , so they stay equal throughout the remaining derivation. **TODO: what if it's in $\text{PI}(C)$ and it disappears due to not being a colored literal?**

Induction over π and σ .

Base case: works

Step: Resolution or factorisation inference ι , $\text{mgu}(\iota) = \sigma = \sigma_1 \cdots \sigma_n$ The term $t[s]$ is created by one of the following two ways:

(we abbreviate $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ$ by F .)

- A variable u occurs in $F\sigma_{(0,i-1)}$ such that $u\sigma_i = t[s]$.

Then u occurs in a resolved or factorised literal $\lambda\sigma_{(0,i-1)}$ at \hat{u} such that at the other resolved or factorised literal $\lambda'\sigma_{(0,i-1)}$, $\lambda'\sigma_{(0,i-1)}|_{\hat{u}} = t[s]$. Then the condition is present at $F\sigma_{(0,i-1)}$ and we get the result by the induction hypothesis. **TODO: add grey here but get for free**

- Note that we only consider maximal colored terms.

Let $t[u]$ be a maximal colored Γ -term in $F\sigma_{(0,i-1)}$ such that in the tree-representation of $t[u]$, the path from the root to u does not contain a node labelled with a Δ -symbol, and $u\sigma_i$ contains a grey occurrence of s .

Suppose that u occurs grey in $F\sigma_{(0,i-1)}$. Then s occurs grey in $F\sigma_{(0,i)}$ and we are done.

Now suppose that u does not occur grey in $F\sigma_{(0,i-1)}$.

If u occurs in a single-colored Δ -term in $F\sigma_{(0,i-1)}$, then by Lemma 7, u also occurs grey in $F\sigma_{(0,i-1)}$ and s hence occurs grey in $F\sigma_{(0,i)}$. **TODO: must be in grey literals**

Otherwise u only occurs in single-colored Γ -terms in $F\sigma_{(0,i-1)}$. As $u\sigma_i$ contains a grey occurrence of s , there is a position p in the two resolved or factorised literals λ and λ' such that $\lambda|_p = u$ and $\lambda'|_p$ contains a grey occurrence of s . Furthermore, the prefix along the path to p is the same in both λ and λ' . But as by assumption u only occurs in single-colored Γ -terms, $\lambda'|_p$ does so as well, so s is contained in a single-colored Γ -term in $F\sigma_{(0,i-1)}$. Since s is a Δ -term, by the induction hypothesis, s occurs grey in $F\sigma_{(0,i-1)}$ and hence also in $F\sigma_i$. **TODO: add grey here but get for free** \square

are probably not same t and s as in lemma statement, which isn't technically wrong but confusing

1.6 random

Conjectured Lemma 9. *Supp u in s.c. Φ -term and that no variable occurs grey in a grey literal. Then u does not occur grey in a Ψ -literal.*

Proof. Holds initially.

Supp holds for first k deductions.

Supp that then there is some Ψ -literal l which contains v at a grey position such that $v\sigma_i$ contains a grey u . Then u and v face each other in the res/fac literals in at stage $k - 1$. By IH, there, u does not occur grey in a Ψ -literal (and of course not grey in a grey literal).

As v occurs grey in a Ψ -literal, by contraposition, v does not occur in a s.c. Φ -term.

- Supp res/fac lits are Ψ -colored. Then u does not occur grey here.

TODO:

- Supp res/fac lits are grey. Then u and v do not occur grey here. **TODO:**
- Supp res/fac lits are Φ -colored. **TODO:**

□

Conjectured Lemma 10. *Every variable, which does not occur grey in a grey literal either does not occur in a single-colored Φ -(Ψ)-term or does not occur grey in a Ψ -(Φ)-literal.*

Proof. Supp u occurs grey in Φ -literal. We show that u does not occur in a s.c. Ψ -term given that it does not occur grey in a grey literal.

Supp s.c. Ψ -term $t[v]$ s.t. $v\sigma_i$ contains grey u . By IH, v does not occur grey in a Φ -literal (given it does not occur grey in a grey literal).

By IH, there is no s.c. Ψ -term which contains u . Hence $t[v]$ is not directly unified with some $t[u]$.

Supp v occurs grey where it is unified. then u occurs grey there as well. Then it can't be a grey literal, and it can't be a Φ -literal. So it is a Ψ -literal.

□

basically also grey occ of v with $t[u]$ conceivable, but that should be IH

Conjectured Lemma 11. *If a var occurs in Γ but not grey in a grey literal, it does not occur grey in Δ .*

1.7 Thursday:

(last_lemma) **Conjectured Lemma 12.** *Suppose a variable u occurs in $\text{PI}(C) \vee C$ in a s.c. Φ -term OR grey in a Φ -literal. If u occurs in $\text{PI}(C) \vee C$ in a grey position in a Ψ -literal OR in s.c. Ψ -term in any literal, then it also occurs at a grey position in a grey literal in $\text{PI}(C) \vee C$.*

Proof. ind step; supp u as in statement ubiquitously

Supp Ψ -literal with grey v such that u grey in $v\sigma_i$. Other occs of v :

If v also occurs in Φ -term, then it occurs in a Φ -term and in a grey position in a Ψ -literal, then by IH, it occurs grey in a grey literal in $F\sigma_{(0,i-1)}$. By σ_i , in $F\sigma_i$, u occurs at a grey position in a grey literal in $\text{PI}(C) \vee C$.

If v also occurs grey in a Φ -literal, then similar argument.
done?

Supp s.c. Ψ -term contains w such that u grey in $w\sigma_i$. Supp u not grey in grey literal as otw done. other occs of w :

Supp w in s.c. Φ -term: then IH.

Supp u in Ψ -term: then IH

Supp grey in res/fac lit:

Supp grey: then both occ grey in grey, done.

Supp Φ -lit: then IH on w

Supp Ψ :- then IH on u

□

Conjectured Lemma 13. *12 nochmal gscheid:*

Let u be a variable which occurs in $\text{PI}(C) \vee C$ in some literal in a single-colored Φ -term or grey in a Φ -literal.

Suppose that u also occurs in $\text{PI}(C) \vee C$ in some literal in a single-colored Ψ -term or grey in a Ψ -literal.

Then u occurs grey in a grey literal.

Proof. Note that initially, every pair of clauses is variable-disjoint and all symbols of a clause are either all grey or Φ -colored or all grey or Ψ -colored, hence the lemma is vacuously true.

Induction of π and σ .

Let u occur in $F\sigma_{(0,i)}$ in a s.c. Φ -term. **Suppose u also occurs in $F\sigma_{(0,i)}$ in a s.c. Φ -term.** Supppose that u occurs in $F\sigma_{(0,i)}$ in a s.c. Ψ -term but not in $F\sigma_{(0,i-1)}$ as otherwise we are done by the induction hypothesis. Then one of the following cases applies:

- There is a s.c. Ψ -term in $F\sigma_{(0,i-1)}$ containing a variable v such that u occurs grey in $v\sigma_i$.

In the res/fac lits in λ and λ' respectively, u and v occur at the essentially same position.

If that position is in an s.c. Φ -term, then we are done by the IH on v .

If that position is in an s.c. Ψ -term, then we are done by the IH on u .

Hence suppose that position is grey: If it's in a Φ -literal, then we are done by the IH on v . If it's in a Ψ -literal, then we are done by the IH on u . Otherwise the occurrence is grey in a grey literal, which we wanted to show.

- There is a variable v in $F\sigma_{(0,i-1)}$ such that $v\sigma_i$ contains a s.c. Ψ -term which contains u . Then $v\sigma_i$ occurs in $F\sigma_{(0,i-1)}$ and we are done by the induction hypothesis.

□