

## 1 Outline

Goal: try proof Huang and nested equal.

Method: proof for both:

**Conjectured Proposition 1.** *Let  $I$  be an interpolant created by \$algorithm. If  $I$  contains a term  $t$  such that  $t$  has  $n$  color changes, then  $I$  has at least  $n$  quantifier alternations.*

### 1.1 generally keep in mind

- Need to define all new terms here: color-changing, single-color,  $\Phi$ -literal, substitutions from 0 to  $n$ 
  - essentially same position: path from one position to other only contains grey symbol (this def allows for identical position as well)
- also note: literal is sometimes used for negated or not negated predicate with terms but in regular formulas with arbitrary connectives

## 2 Preliminaries

Quantifier alternations in  $I$  usually assumes the quantifier-alternation-minimizing arrangement of quantifiers in  $I$

**Definition 2** (Color alternation col-alt). Colors  $\Gamma$  and  $\Delta$ , term  $t$ :

$$\text{col-alt}(t) \stackrel{\text{def}}{=} \text{col-alt}_{\perp}(t)$$

Let  $t = f(t_1, \dots, t_n)$  for constant, function and variable symbols (syntax abuse):

$$\text{col-alt}_{\Phi}(t) \stackrel{\text{def}}{=} \begin{cases} \max^1(\text{col-alt}_{\Phi}(t_1), \dots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is grey} \\ \max(\text{col-alt}_{\Phi}(t_1), \dots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_{\Psi}(t_1), \dots, \text{col-alt}_{\Psi}(t_n)) & f \text{ is of color } \Psi, \Phi \neq \Psi \end{cases}$$

$\triangle$

**Definition 3.**  $\text{PI}_{\text{step}}^{\circ}$  is defined just like  $\text{PI}_{\text{step}}$  but without applying any substitution.

$\triangle$

Hence  $\text{PI}_{\text{step}}^{\circ}(\cdot)\sigma = \text{PI}_{\text{step}}(\cdot)$ .  $C^{\circ}$  is somehow the same, i.e. if  $C = D\sigma$ , then  $C^{\circ} = D$  where  $\sigma$  is derived from the context.

## 3 Random thoughts

- Quantifiers are introduced for lifting variables which actually occur in the interpolant
- If term  $t$  with  $\text{col-alt}(t) = n$  enters  $I$ , we need subterm  $s$  of  $t$  with  $\text{col-alt}(s) = n - 1$  to be in  $I$  (of course colors of  $t$  and  $s$  are exactly opposite)

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<sup>1</sup>We assume that the maximum of an empty list of arguments is 0.

### 3.1 Proof

- Induction over  $\ell_\Delta^x[\text{PI}(C) \vee C]$  and also about  $\Gamma$ -terms with  $\Delta$ -lifting vars in that formula. Cf. `-final`
- **NB: now somewhat described in the proper proof below** describe proof method with  $\sigma_{(0,i)}$ : which PI?
  - Factorisation: easy: just apply  $\sigma_i$  for all  $i$  to  $\text{PI}(C) \vee C$ . When done, a literal will be there twice and we can remove it without losing anything
  - Resolution: create propositional structure first.  
 Ex.:  $C_1 : D \vee l, C_2 : \neg l \vee E$ :  
 If we talk about properties for which it holds that if they hold for  $\text{PI}(C_i) \vee C_i, i \in \{1, 2\}$ , then they also hold for  $A \equiv ((l \wedge \text{PI}(C_2)) \vee (\neg l \wedge \text{PI}(C_1))) \vee C^\circ$ , then we can apply  $\sigma_i$  for all  $i$  to that formula.  
 So if we can assume it for  $A$  and show it for all  $\sigma_i$ , we get that it holds for  $\text{PI}(C) \vee C$ .

Also: clauses are variable disjoint, so e.g. it's not possible that a color-changing var is created by  $\text{PI}_{\text{step}}$

Also: do it like a few lemmas further down, like  $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ) \sigma_{(0,i)}$

## 4 directly from old proof

### just for repetition:

`?{lemma:col_change}?` **Lemma 4.** *Resolution or factorisation step  $\iota$  from  $\bar{C}$ .*

*If  $u$  col-change var in  $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ) \sigma_{(0,i)}$ , then  $u$  also occurs grey in that formula.*

*Proof.* Abbreviation:  $F \equiv (\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)$

Induction over refutation and  $\sigma$ ; base case easy.

Step: Supp color change var  $u$  present in  $\chi \sigma_{(0,i)}$ . (could also say introduced, then proof would be somehow different)

Supp  $u$  not grey in  $\chi \sigma_{(0,i-1)}$  as otherwise done. As a first step, we show that if a (not necessarily color-changing) variable  $v$  occurs in a single-colored  $\Phi$ -term  $t[v]$  in  $\chi \sigma_{(0,i)}$ , then at least one of the following holds:

1.  $v$  occurs in some single-colored  $\Phi$ -term in  $\chi \sigma_{(0,i-1)}$
2. there is a color-changing variable  $w$  in  $\chi \sigma_{(0,i-1)}$  such that  $v$  occurs grey in  $w \sigma_i$ .

`(var_occ_2)` We consider unification process, and particularly the different cases which can introduce a variable  $v$  in a single-colored term  $\Phi$ : Either it has been there before, it was introduced in a s.c.  $\Phi$ -colored term, or a s.c.  $\Phi$ -term containing the var is in  $\text{ran}(\sigma)$ .

- Suppose a term  $t'[v]$  is present in  $\chi \sigma_{(0,i-1)}$  such that  $t'[v] \sigma_i = t[v]$ . Then 1 is the case.

- Suppose a variable  $w$  occurs in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$  such that  $v$  occurs grey in  $w\sigma_i$ . Suppose furthermore that 1 is not the case, i.e.  $v$  does not occur in a s.c.  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$ , as otherwise we would be done. We show that 2 is the case.

As  $v$  occurs neither grey nor in a s.c.  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$  but occurs in  $\text{ran}(\sigma_i)$ , it must occur in  $\chi\sigma_{(0,i-1)}$  and this can only be in a single-colored  $\Psi$ -term.

As by assumption  $v$  occurs grey in  $w\sigma_i$ , there must be an occurrence  $\hat{w}$  of  $w$  in a resolved or factorised literal, say  $\lambda\sigma_{(0,i-1)}$  such that for the other resolved literal  $\lambda'\sigma_{(0,i-1)}$ ,  $\lambda'\sigma_{(0,i-1)}|_{\hat{w}}$  is a subterm in which  $v$  occurs grey. But as the occurrence of  $v$  in  $\lambda'\sigma_{(0,i-1)}|_{\hat{w}}$  must be contained in a single-colored  $\Psi$ -term, so is  $\lambda\sigma_{(0,i-1)}|_{\hat{w}}$ , hence  $z$  occurs in a single-colored  $\Psi$ -term as well. Therefore 2 is the case.

- Suppose there is a variable  $z$  in  $\chi\sigma_{(0,i-1)}$  such that  $v$  occurs in a single-colored  $\Phi$ -term in  $z\sigma_i$ . Then  $z\sigma_i$  occurs in  $\chi\sigma_{(0,i-1)}$ , but this is a witness for 1.

Now recall that we have assumed  $u$  to be a color-changing variable in  $\chi\sigma_{(0,i)}$ . Hence it occurs in a single-colored  $\Gamma$ -term as well as in a single-colored  $\Delta$ -term. By the reasoning above, this leads to two case:

- In  $\chi\sigma_{(0,i-1)}$ ,  $u$  occurs both in some single-colored  $\Gamma$ -term as well as in some single-colored  $\Delta$ -term. Then we get the result by the induction hypothesis and the fact that  $u \notin \text{dom}(\sigma_i)$  as  $u$  does occur in  $\chi\sigma_{(0,i)}$ .
- Otherwise for some color  $\Phi$ ,  $u$  does not occur in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$ . Then case 2 above must hold and there is some color-changing variable  $w$  in  $\chi\sigma_{(0,i-1)}$  such that  $u$  occurs grey in  $w\sigma_{(0,i)}$ . But then by the induction hypothesis,  $w$  occurs grey in  $\chi\sigma_{(0,i-1)}$  and hence  $u$  occurs grey in  $\chi\sigma_{(0,i)}$ .  $\square$

## 5 Thursday

**WRONG: the resolved/factorised literal is in general not contained in  $\chi$**

In the following, we abbreviate  $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)$  by  $\chi$ .

(lemma:var\_grey\_col\_lit) **Lemma 5.** *Let  $\iota$  be an inference in a refutation of  $\Gamma \cup \Delta$ . Suppose that a variable  $u$  occurs grey in a  $\Phi$ -literal in  $\chi\sigma_{(0,i)}$ . Then at least one of the following statements holds:*

- (14\_1) 1. *The variable  $u$  occurs grey in a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$ .*
- (14\_5) 2. *The variable  $u$  occurs in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$ .*
- (14\_4) 3. *The variable  $u$  occurs at a grey position in a grey literal in  $\chi\sigma_{(0,i-1)}$ .*
- (14\_2) 4. *There is a variable  $v$  such that*
  - $u$  occurs grey in  $v\sigma_i$  and*
  - $v$  occurs in  $\chi\sigma_{(0,i-1)}$  grey in a  $\Phi$ -literal as well as grey in a  $\Psi$ -literal.*
- (14\_3) 5. *There is a variable  $v$  such that*
  - $u$  occurs grey in  $v\sigma_i^2$  and*
  - $v$  occurs in  $\chi\sigma_{(0,i-1)}$  either grey in a  $\Phi$ -literal as well as in a single-colored  $\Psi$ -term in any literal, or grey in a  $\Psi$ -literal as well as in a single-colored  $\Phi$ -term in any literal.*

*Proof.* We consider the unification process, and particularly the different cases which lead to the variable  $u$  in a grey position in a  $\Phi$ -literal in  $\chi\sigma_{(0,i)}$ :

- There already is a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$  which contains  $u$  at a grey position and  $\sigma_i$  does not change this. Then clearly 1 is the case.
- Otherwise there must be a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$ , which contains a variable  $v$  at a grey position such that  $u$  occurs grey in  $v\sigma_i$ . Hence in the resolved or factorised literals  $\lambda$  and  $\lambda'$ , there is a position  $p$  such that w.l.o.g.  $\lambda|_p = v$  and  $\lambda'|_p$  contains a grey occurrence of  $u$ , and  $\lambda$  and  $\lambda'$  coincide along the path to  $p$ . We distinguish based properties of the position  $p$ :
  - Suppose that  $p$  is contained in a single-colored  $\Phi$ -term. Then  $u$  occurs in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$  and 2 is the case.
  - Suppose that  $p$  is contained in a single-colored  $\Psi$ -term. Then  $u$  occurs grey in a  $\Phi$ -literal as well in a single-colored  $\Psi$ -term, which implies 5.
  - Otherwise  $p$  is a grey position. We distinguish further:
    - \* Suppose that the resolved or factorised literal is  $\Phi$ -colored. Then  $u$  occurs grey in a  $\Phi$ -literal and we have established item 5.

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<sup>2</sup>Note that this includes the case that  $v = u$  and  $\sigma_i$  is trivial on  $u$ . **TODO: this really necessary? what about case 1, doesn't that one subsume this?**

- \* Suppose that the resolved or factorised literal is  $\Psi$ -colored. Then the variable  $v$  occurs grey in a  $\Phi$ -literal as well as grey in a  $\Psi$ -literal, hence 4 is the case.
- Otherwise the resolved or factorised literal is grey and  $u$  occurs grey in a grey literal, which is sufficient for 3.  $\square$

(lemma:var\_in\_sc\_term) **Lemma 6.** *Let  $\iota$  be an inference in a refutation of  $\Gamma \cup \Delta$ . Suppose that a variable  $u$  occurs in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i)}$ . Then at least one of the following statements holds:*

- (15\_1) 1. *The variable  $u$  occurs in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$ .*
- (15\_2) 2. *There is a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$  and  $v$  occurs in a single-colored  $\Phi$ -term as well as in a single-colored  $\Psi$ -term in  $\chi\sigma_{(0,i-1)}$ .*
- (15\_4) 3. *There is a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$  and  $v$  occurs in  $\chi\sigma_{(0,i-1)}$  in a single-colored  $\Phi$ -term as well as at a grey position in a  $\Psi$ -literal.*
- (15\_3) 4. *The variable  $u$  occurs grey in a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$ .*
- (15\_5) 5. *The variable  $u$  occurs grey in a grey literal in  $\chi\sigma_{(0,i-1)}$ .*

*Proof.* We consider the different cases which lead to the variable  $u$  in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i)}$ :

- There is a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$  which contains  $u$  such that  $\sigma_i$  does not change this. Then 1 is the case.
- Suppose that there is a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$  which contains a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$ .

Hence in the resolved or factorised literals  $\lambda$  and  $\lambda'$ , there is a position  $p$  such that w.l.o.g.  $\lambda|_p = v$  and  $\lambda'|_p$  contains a grey occurrence of  $u$ , and  $\lambda$  and  $\lambda'$  coincide along  $p$ . We distinguish based properties of the position  $p$ :

- Suppose that  $p$  is contained in a single-colored  $\Phi$ -term. Then  $u$  is contained in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$  and item 1 holds.
- Suppose that  $p$  is contained in a single-colored  $\Psi$ -term. As then  $v$  is contained in a single-colored  $\Phi$ -term as well as in a single-colored  $\Psi$ -term, 2 is the case.
- Suppose that  $p$  is a grey position. We distinguish further:
  - \* Suppose that the resolved or factorised literal is  $\Phi$ -colored. Then  $u$  occurs grey in a  $\Phi$ -literal, which suffices for 4.
  - \* Suppose that the resolved or factorised literal is  $\Psi$ -colored. Then the variable  $v$  occurs in a single-colored  $\Phi$ -term as well as grey in a  $\Psi$ -literal, which implies 3.
  - \* Otherwise the resolved or factorised literal is grey. But then  $u$  occurs grey in a grey literal and we have established item 5.
- Suppose that a variable  $w$  occurs in  $\chi\sigma_{(0,i-1)}$  such that  $u$  occurs in a single-colored  $\Phi$ -term in  $w\sigma_i$ . This can only be the case if  $w\sigma$  already occurs in  $\chi\sigma_{(0,i-1)}$ , which implies that 1 is the case.  $\square$

`_change_and_grey_in_col_lit`) **Lemma 7.** *Let  $C$  be a clause in the resolution refutation  $\pi$  of  $\Gamma \cup \Delta$  and  $u$  be a variable which occurs in  $\text{PI}(C) \vee C$  in some literal in a single-colored  $\Phi$ -term or grey in a  $\Phi$ -literal.*

*Suppose that  $u$  also occurs in  $\text{PI}(C) \vee C$  in some literal in a single-colored  $\Psi$ -term or grey in a  $\Psi$ -literal.*

*Then  $u$  occurs grey in a grey literal.*

Note that  $\Phi$  and  $\Psi$  are to be read as any pair of different colors, i.e.  $\Gamma$  and  $\Delta$  as well as  $\Delta$  and  $\Gamma$ .

*Proof.* We proceed by induction over  $\pi$  and  $\sigma$ .

Note that initially, every pair of clauses is variable-disjoint and all symbols of a clause are either all grey or  $\Phi$ -colored or all grey or  $\Psi$ -colored, hence the lemma is vacuously true.

For the induction step, we assume that the property holds for  $\text{PI}(C_i) \vee C_i$ ,  $1 \leq i \leq n$ , where  $C_1, \dots, C_n$  are the clauses used in a resolution or factorisation inference  $\iota$ . Note that then, the property also holds for  $\chi$ , i.e. for  $\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ$  as it contains all the grey literals present in  $\text{PI}(C_i) \vee C_i$  for any  $i$  (this is evident by the definition of  $\text{PI}_{\text{step}}^\circ$ ), and as clauses are pairwise variable-disjoint, the lemma condition can not become true for a variable for which it was not true in  $\text{PI}(C_i) \vee C$  for some  $i$ .

Suppose that  $u$  occurs in  $\chi\sigma_{(0,i)}$  in a single-colored  $\Phi$ -term or grey in a  $\Phi$ -literal and that  $u$  also occurs in  $\chi\sigma_{(0,i)}$  in a single-colored  $\Psi$ -term or grey in a  $\Psi$ -literal.

Then we can deduce by Lemma 5 and Lemma 6 that at least one of the following statements holds:

- (oozoh70h1) 1. The variable  $u$  occurs grey in a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$ .
- (oozoh70h2) 2. Some variable  $v$  occurs in  $\chi\sigma_{(0,i-1)}$  grey in a  $\Phi$ -literal as well as grey in a  $\Psi$ -literal such that  $u$  occurs grey in  $v\sigma_i$ .
- (oozoh70h3) 3. There is a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$  and  $v$  occurs in  $\chi\sigma_{(0,i-1)}$  either grey in a  $\Phi$ -literal as well as in a single-colored  $\Psi$ -term in any literal, or grey in a  $\Psi$ -literal as well as in a single-colored  $\Phi$ -term in any literal.
- (oozoh70h4) 4. The variable  $u$  occurs at a grey position in a grey literal in  $\chi\sigma_{(0,i-1)}$ .
- (oozoh70h5) 5. The variable  $u$  occurs in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$ .
- (oozoh70h6) 6. There is a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i$  and  $v$  occurs in a single-colored  $\Phi$ -term as well as in a single-colored  $\Psi$ -term in  $\chi\sigma_{(0,i-1)}$ .

By the same lemmata, we get the same set of statements where  $\Phi$  and  $\Psi$  are interchanged. We refer to them by the respective number followed by  $*$ .

Suppose that 4 is not the case as otherwise we are done since  $\sigma_i$  is trivial on  $u$  as  $u$  occurs in  $\chi\sigma_{(0,i)}$ . Furthermore, there are a number of cases which give the result by the induction hypothesis: For the cases 2, 3 and 6, we can infer that by the induction hypothesis, there is a grey occurrence of the variable  $v$  in a grey literal in  $\chi\sigma_{(0,i-1)}$ , and as  $u$  occurs grey in  $v\sigma_i$ , there is a grey occurrence of  $u$  in a grey literal in  $\chi\sigma_{(0,i)}$ .

It remains to show that the lemma holds true in case the statements 1 or 5 as well as 1\* or 5\* hold. But note that in any combination of 1 or 5 and 1\* or 5\* in

effect yields a situation under which the induction hypothesis again is applicable. Hence we may infer that  $u$  occurs grey in a grey literal in  $\chi\sigma_{(0,i-1)}$  and since  $\sigma_i$  is trivial  $u$  as shown above,  $u$  occurs grey in a grey literal in  $\chi\sigma_{(0,i)}$ .  $\square$

## 6 Friday

**Lemma 8.** *If  $\text{PI}(C) \vee C$  contains a maximal colored occurrence of a  $\Phi$ -term  $t[s]$  containing  $\Psi$ -term  $s$ , then  $s$  occurs grey in a grey literal in  $\text{PI}(C) \vee C$ .*

*Proof.* Note that it suffices to show that at the step where  $s$  is introduced as subterm of  $t[s]$ ,  $s$  occurs grey in  $\text{PI}(C) \vee C$  as any later modification by substitution is applied to both occurrences  $s$ , so they stay equal throughout the remaining derivation.

Induction over  $\pi$  and  $\sigma$ . **TODO: as in Lemma 7**

Base case: vacuously true.

Step: Resolution or factorisation inference  $\iota$ ,  $\text{mgu}(\iota) = \sigma = \sigma_1 \cdots \sigma_n$ . The term  $t[s]$  is created by one of the following two ways:

(we abbreviate  $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)$  by  $F$ .)

- A variable  $u$  occurs in  $\chi\sigma_{(0,i-1)}$  such that  $u\sigma_i = t[s]$ .

Then  $u$  occurs in a resolved or factorised literal  $\lambda\sigma_{(0,i-1)}$  at  $\hat{u}$  such that at the other resolved or factorised literal  $\lambda'\sigma_{(0,i-1)}$ ,  $\lambda'\sigma_{(0,i-1)}|_{\hat{u}} = t[s]$ . Then the condition is present at  $\chi\sigma_{(0,i-1)}$  and we get the result by the induction hypothesis.

- Note that we only consider maximal colored terms.

Let  $t[u]$  be a maximal colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$  such that in the tree-representation of  $t[u]$ , the path from the root to  $u$  does not contain a node labelled with a  $\Psi$ -symbol, and  $u\sigma_i$  contains a grey occurrence of  $s$ .

Suppose that  $u$  occurs grey in a grey literal in  $\chi\sigma_{(0,i-1)}$ . Then  $s$  occurs grey in a grey literal in  $\chi\sigma_{(0,i)}$  as  $\sigma_i$  does not affect  $u$  since  $u$  occurs in  $\chi\sigma_{(0,i)}$  and we are done.

If  $u$  occurs grey in a  $\Psi$ -literal or if  $u$  occurs in a single-colored  $\Psi$ -term in  $\chi\sigma_{(0,i-1)}$ , then by Lemma 7,  $u$  also occurs grey in a grey literal in  $\chi\sigma_{(0,i-1)}$  and  $s$  hence occurs grey in a grey literal in  $\chi\sigma_{(0,i)}$ .

Now suppose that  $u$  does not occur grey in a grey literal  $\chi\sigma_{(0,i-1)}$  as otherwise clearly we are done.

Hence as all other cases are excluded,  $u$  can only occur in  $\chi\sigma_{(0,i-1)}$  in a single-colored  $\Phi$ -term or grey in a  $\Phi$ -colored literal. But then, since  $u\sigma_i$  contains a grey occurrence of  $s$ , there is a position  $p$  in the two resolved or factorised literals  $\lambda$  and  $\lambda'$  such that  $\lambda|_p = u$  and  $\lambda'|_p$  contains a grey occurrence of  $s$ . Furthermore, the prefix along the path to  $p$  is the same in both  $\lambda$  and  $\lambda'$ . As  $u$  only occurs in single-colored  $\Phi$ -terms,  $\lambda'|_p$  does so as well, so  $s$  is contained in a single-colored  $\Phi$ -term in  $\chi\sigma_{(0,i-1)}$ . Since  $s$  is a  $\Psi$ -term, by the induction hypothesis,  $s$  occurs grey in a grey literal in  $\chi\sigma_{(0,i-1)}$  and hence also in  $\chi\sigma_i$ .  $\square$

are probably not same  $t$  and  $s$  as in lemma statement, which isn't technically wrong but confusing

(lemma:grey\_lits\_all\_in\_PI) **Lemma 9.** *If there is a grey literal  $\lambda$  in a clause  $C$  of a resolution refutation  $\pi$ , then a successor of  $\lambda$  occurs in  $\text{PI}(\pi)$ .*

*Proof.* Immediate by the definition of PI.  $\square$

**TODO:** define quantifier alternations as col-alt, 0 == no quants, 1 == one quant, 2 is  $\Pi_2$  or  $\Sigma_2$

**Proposition 10.** *If a term with  $n$  color alternations occurs in  $\text{PI}(C) \vee C$  for a clause  $C$ , then the interpolant  $I$  produced in Theorem ?? contains at least  $n$  quantifier alternations.*

*Proof.* We perform an induction on  $n$  and show the strengthening that the quantification of the lifting variable corresponding to a term with  $n$  color alternations is required to be in the scope of the quantification of  $n - 1$  alternating quantifiers.

For  $n = 0$ , no colored terms occur in  $I$  and hence by construction no quantifiers and for  $n = 1$ , there are only single-colored terms

Suppose the statement holds for  $n - 1$  for  $n > 1$  and a term  $t$  with  $\text{col-alt}(t) = n$  occurs in  $\text{PI}(C)$ . We assume that  $t$  is a  $\Phi$ -term. Then  $t$  contains a  $\Psi$ -colored term  $s$  and by Lemma 7,  $s$  occurs grey in a grey literal in  $\text{PI}(C) \vee C$ . By Lemma 9, a successor of  $s$  occurs in  $\text{PI}(\pi)$ . By the induction hypothesis, the quantification of the lifting variable for  $s$  requires  $n - 1$  alternated quantifiers. As  $s$  is a subterm of  $t$  and  $t$  is lifted,  $t$  must be quantified in the scope of the quantification of  $s$ , and as  $t$  and  $s$  are of different color, their quantifier type is different. Hence the quantification of the lifting variable for  $t$  requires  $n$  quantifier alternations.  $\square$

## 7 Monday: Paramodulation

### 7.1 Notes

1. Every equality which is used ends up in the interpolant, i.e. it's a grey literal (binary)
2. Every equality is used eventually

### 7.2 Proof

Extension of Lemma 5

a:var\_grey\_col\_lit\_paramod)?

**Lemma 11.** *Let  $\iota$  be a **paramodulation** inference in a refutation of  $\Gamma \cup \Delta$ . Suppose that a variable  $u$  occurs grey in a  $\Phi$ -literal in  $\chi\sigma_{(0,i)}$ . Then at least one of the following statements holds:*

- ?(11\_1)? 1. *The variable  $u$  occurs grey in a  $\Phi$ -literal in  $\chi\sigma_{(0,i-1)}$ .*
- ?(11\_2)? 2. *Some variable  $v$  occurs in  $\chi\sigma_{(0,i-1)}$  grey in a  $\Phi$ -literal as well as grey in a  $\Psi$ -literal such that  $u$  occurs grey in  $v\sigma_i$ .*
- ?(11\_3)? 3. *There is a variable  $v$  such that  $u$  occurs grey in  $v\sigma_i^3$  and  $v$  occurs in  $\chi\sigma_{(0,i-1)}$  either grey in a  $\Phi$ -literal as well as in a single-colored  $\Psi$ -term in any literal, or grey in a  $\Psi$ -literal as well as in a single-colored  $\Phi$ -term in any literal.*
- ?(11\_4)? 4. *The variable  $u$  occurs at a grey position in a grey literal in  $\chi\sigma_{(0,i-1)}$ .*



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*Proof.* Consider paramodulation:  $s = t \vee D$  and  $E[r]_p$  create  $C : (D \vee E[t]_p)\sigma$  where  $\sigma = \text{mgu}(s, r)$ .

A grey occurrence of variable can be created in  $C$  by 2 means: either  $t$  contains a grey variable and  $p$  is a grey position or  $\sigma$  introduces a grey occurrence of a variable in a grey position

- case 1: everything which is grey in an equality predicate ends up grey in the interpolant, so this case is easy

□