

# Part I - Interpolant extraction from resolution proofs in one phase

While the previous chapter demonstrates that it is possible to extract propositional interpolants and lift them from the colored symbols later in order to obtain a proper interpolant, we now present a novel approach, which only operates with grey intermediary interpolants.

## 1 Definition of the extraction algorithm

The extracted interpolants are prenex formulas, where the quantifier block and the matrix of the formula are calculated separately in each step of the traversal of the resolution refutation.

### 1.1 Extraction of the interpolant matrix $\text{AI}_{\text{mat}}$ and calculation of $\text{AI}_{\text{cl}}$

$\text{AI}_{\text{mat}}$  is inspired by the propositional interpolants PI from Definition ???. Its difference lies in the fact that the lifting occurs in every step of the extraction. This however necessitates applying these liftings to the clauses of the resolution refutation as well. For a clause  $C$  of the resolution refutation, we will denote the clause with the respective liftings applied by  $\text{AI}_{\text{cl}}(C)$  (a formal definition will be given below), and for a term  $t$  at position  $p$  in  $C$ , we denote  $\text{AI}_{\text{cl}}(C)|_p$  by  $t_{\text{AIcl}}$ .

Now we can define preliminary versions of  $\text{AI}_{\text{mat}}^\bullet$  and  $\text{AI}_{\text{cl}}^\bullet$ :

**Definition 1** ( $\text{AI}_{\text{mat}}^\bullet$  and  $\text{AI}_{\text{cl}}^\bullet$ ). Let  $\pi$  be a resolution refutation of  $\Gamma \cup \Delta$ .

For a clause  $C$  in  $\pi$ ,  $\text{AI}_{\text{mat}}^\bullet(C)$  and  $\text{AI}_{\text{cl}}^\bullet(C)$  are defined as follows:

Base case. If  $C \in \Gamma$ ,  $\text{AI}_{\text{mat}}^\bullet(C) \stackrel{\text{def}}{=} \perp$ . If otherwise  $C \in \Delta$ ,  $\text{AI}_{\text{mat}}^\bullet(C) \stackrel{\text{def}}{=} \top$ .

In any case,  $\text{AI}_{\text{cl}}^\bullet(C) \stackrel{\text{def}}{=} \ell[C]$ .

Resolution. If the clause  $C$  is the result of a resolution step of  $C_1 : D \vee l$  and  $C_2 : E \vee \neg l'$  using a unifier  $\sigma$  such that  $l\sigma = l'\sigma$ , then  $\text{AI}_{\text{mat}}^\bullet(C)$  and  $\text{AI}_{\text{cl}}^\bullet$  are defined as follows:

$$\text{AI}_{\text{cl}}^\bullet(C) \stackrel{\text{def}}{=} \ell[(\text{AI}_{\text{cl}}^\bullet(C_1) \setminus \{l_{\text{AIcl}}\})\sigma] \vee \ell[(\text{AI}_{\text{cl}}^\bullet(C_2) \setminus \{l'_{\text{AIcl}}\})\sigma]$$

1. If  $l$  is  $\Gamma$ -colored:  $\text{AI}_{\text{mat}}^\bullet(C) \stackrel{\text{def}}{=} \ell[\text{AI}_{\text{mat}}^\bullet(C_1)\sigma] \vee \ell[\text{AI}_{\text{mat}}^\bullet(C_2)\sigma]$
2. If  $l$  is  $\Delta$ -colored:  $\text{AI}_{\text{mat}}^\bullet(C) \stackrel{\text{def}}{=} \ell[\text{AI}_{\text{mat}}^\bullet(C_1)\sigma] \wedge \ell[\text{AI}_{\text{mat}}^\bullet(C_2)\sigma]$
3. If  $l$  is grey:  $\text{AI}_{\text{mat}}^\bullet(C) \stackrel{\text{def}}{=} (\neg \ell[l'_{\text{AIcl}}\sigma] \wedge \ell[\text{AI}_{\text{mat}}^\bullet(C_1)\sigma]) \vee (\ell[l_{\text{AIcl}}\sigma] \wedge \ell[\text{AI}_{\text{mat}}^\bullet(C_2)\sigma])$

Factorisation. If the clause  $C$  is the result of a factorisation of  $C_1 : l \vee l' \vee D$  using a unifier  $\sigma$  such that  $l\sigma = l'\sigma$ , then  $\text{AI}_{\text{mat}}^\bullet(C) \stackrel{\text{def}}{=} \ell[\text{AI}_{\text{mat}}^\bullet(C_1)\sigma]$  and  $\text{AI}_{\text{cl}}^\bullet(C) \stackrel{\text{def}}{=} \ell[(\text{AI}_{\text{cl}}^\bullet(C_1) \setminus \{l'_{\text{AIcl}}\})\sigma]$ .  
 $\triangle$

Note that in  $\text{AI}_{\text{mat}}^\bullet$  and  $\text{AI}_{\text{cl}}^\bullet$ , it is possible that there for a colored term  $t$  in  $C$  that  $t_{\text{AIcl}} \neq z_t$  as illustrated by the following examples:

**Example 2.A.** We consider a resolution refutation of the initial clause sets  $\Gamma = \{R(c), \neg Q(v)\}$  and  $\Delta = \{\neg R(u) \vee Q(g(u))\}$ :

$$\frac{\frac{R(c) \quad \neg R(u) \vee Q(g(u))}{Q(g(c))} \text{ res, } y \mapsto c \quad \neg Q(v)}{\square} \text{ res, } v \mapsto g(c)$$

We now replace every clause  $C$  by  $\text{AI}_{\text{mat}}^\bullet(C) \mid \text{AI}_{\text{cl}}^\bullet(C)$  in order to visualise the steps of the algorithm:

$$\frac{\frac{\perp \mid R(y_c) \quad \top \mid \neg R(u) \vee \neg Q(x_{g(u)})}{R(y_c) \mid Q(x_{g(u)})} \text{ res, } y \mapsto c \quad \perp \mid \neg Q(v)}{\neg Q(x_{g(c)}) \wedge R(y_c) \mid \square} \text{ res, } v \mapsto g(c)$$

By quantifying  $y_c$  existentially and  $x_{g(c)}$  universally<sup>1</sup>, we can obtain an interpolant for  $\Gamma \cup \Delta$ :  $\exists y_c \forall x_{g(c)} (\neg Q(x_{g(c)}) \wedge R(y_c))$ . Note however that  $\ell[Q(g(c))] = Q(x_{g(c)})$ , but  $\text{AI}_{\text{mat}}(Q(g(c))) = Q(x_{g(u)})$ . This example shows that this circumstance is not necessarily a problem for this algorithm.  $\triangle$

(**exa:2b**) **Example 2.B.** We consider a resolution refutation of the initial clause sets  $\Gamma = \{R(c), P(c)\}$  and  $\Delta = \{\neg R(u) \vee \neg Q(g(u)), \neg P(v) \vee Q(g(v))\}$ :

$$\frac{\frac{\neg R(u) \vee \neg Q(g(u)) \quad R(c)}{\neg Q(g(c))} \text{ res, } u \mapsto c \quad \frac{\neg P(v) \vee Q(g(v)) \quad P(c)}{Q(g(c))} \text{ res, } v \mapsto c}{\square} \text{ res}$$

We now again show  $\text{AI}_{\text{mat}}^\bullet(C) \mid \text{AI}_{\text{cl}}^\bullet(C)$  for every clause  $C$ :

$$\frac{\frac{\top \mid \neg R(u) \vee \neg Q(x_{g(u)}) \quad \perp \mid R(y_c)}{R(y_c) \mid \neg Q(x_{g(u)})} \text{ res, } u \mapsto c \quad \frac{\top \mid \neg P(v) \vee Q(x_{g(v)}) \quad \perp \mid P(y_c)}{P(y_c) \mid Q(x_{g(v)})} \text{ res, } v \mapsto c}{(Q(x_{g(v)}) \wedge R(y_c)) \vee (\neg Q(x_{g(u)}) \wedge P(y_c)) \mid \square} \text{ res}$$

Note again that  $\ell[\neg Q(g(c))] \neq \text{AI}_{\text{mat}}^\bullet(\neg Q(g(c))) = \neg Q(x_{g(u)})$  and  $\ell[Q(g(c))] \neq \text{AI}_{\text{mat}}^\bullet(Q(g(c))) = Q(x_{g(v)})$ . However in this instance, it is not possible to find quantifiers for the free variables of  $\text{AI}_{\text{mat}}^\bullet(\square)$  such that by binding them an interpolant is produced. For the naive approach, namely  $\exists y_c \forall x_{g(v)} \forall x_{g(u)} ((Q(x_{g(v)}) \wedge R(y_c)) \vee (\neg Q(x_{g(u)}) \wedge P(y_c)))$ , the problem lies in the fact that  $\Gamma \not\models \exists y_c \forall x_{g(v)} \forall x_{g(u)} ((Q(x_{g(v)}) \wedge R(y_c)) \vee (\neg Q(x_{g(u)}) \wedge P(y_c)))$ . This is possible as intuitively, resolution deductions are valid by virtue of the resolved literals being equal, which the interpolation extraction procedure in some sense exploit.  $\triangle$

<sup>1</sup>The procedure for calculating the quantifier block is defined in section 1.1

In order to avoid the pitfall shown in Example 2.B, we define a substitution which ensures that for literals  $l$  and  $l'$ , which are required to be equal by the calculus, it holds that  $l_{\text{AIcl}}$  and  $l'_{\text{AIcl}}$  are equal.

**Definition 3** (au).  $\text{au}(a_{\text{AIcl}}, b_{\text{AIcl}}) \stackrel{\text{def}}{=} \begin{cases} \bigcup_{i=1}^n \text{au}(s_i, t_i) & \text{if } a_{\text{AIcl}} = f_s(\bar{s}) \text{ grey and } b_{\text{AIcl}} = f_t(\bar{t}) \text{ (includes } f_s \text{ being a constant)} \\ \{x_j \mapsto x_m, x_k \mapsto x_m\} & \text{if } a_{\text{AIcl}} = x_j \text{ and } b_{\text{AIcl}} = x_k, \text{ both lifting variables, and } x_m \text{ is the} \\ & \text{corresponding lifted term in the unified literal, i.e. } x_m = \ell[a\sigma] = \ell[b\sigma]. \text{ This means to just take the term from the literal in the} \\ & \text{actual clause.} \\ \text{id} & \text{if } \ell[a\sigma] = \ell[b\sigma] = x_j \text{ (this basically is a special case of the upper one)} \end{cases}$

TODO: fix up def

△

**Definition 4** ( $\text{AI}_{\text{mat}}$  and  $\text{AI}_{\text{cl}}$ ). Let  $\pi$  be a resolution refutation of  $\Gamma \cup \Delta$ .

For a clause  $C$  in  $\pi$ ,  $\text{AI}_{\text{mat}}(C)$  and  $\text{AI}_{\text{cl}}(C)$  are defined as follows:

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$$\begin{aligned} \text{AI}_{\text{cl}}(C) &\stackrel{\text{def}}{=} \ell[(\text{AI}_{\text{cl}}(C_1) \setminus \{l_{\text{AIcl}}\})\sigma]\tau \vee \ell[(\text{AI}_{\text{cl}}(C_2) \setminus \{l'_{\text{AIcl}}\})\sigma]\tau \\ \tau &\stackrel{\text{def}}{=} \text{au}(\ell[l_{\text{AIcl}}\sigma], \ell[l'_{\text{AIcl}}\sigma]) \end{aligned}$$

$$1. \text{ If } l \text{ is } \Gamma\text{-colored: } \text{AI}_{\text{mat}}(C) \stackrel{\text{def}}{=} \ell[\text{AI}_{\text{mat}}(C_1)\sigma]\tau \vee \ell[\text{AI}_{\text{mat}}(C_2)\sigma]\tau$$

$$2. \text{ If } l \text{ is } \Delta\text{-colored: } \text{AI}_{\text{mat}}(C) \stackrel{\text{def}}{=} \ell[\text{AI}_{\text{mat}}(C_1)\sigma]\tau \wedge \ell[\text{AI}_{\text{mat}}(C_2)\sigma]\tau$$

$$3. \text{ If } l \text{ is grey: } \text{AI}_{\text{mat}}(C) \stackrel{\text{def}}{=} (\neg \ell[l'_{\text{AIcl}}\sigma]\tau \wedge \ell[\text{AI}_{\text{mat}}(C_1)\sigma]\tau) \vee (\ell[l_{\text{AIcl}}\sigma]\tau \wedge \ell[\text{AI}_{\text{mat}}(C_2)\sigma]\tau)$$

Factorisation. If the clause  $C$  is the result of a factorisation of  $C_1 : l \vee l' \vee D$  using a unifier  $\sigma$  such that  $l\sigma = l'\sigma$ , then:

$$\tau \stackrel{\text{def}}{=} \text{au}(\ell[l_{\text{AIcl}}\sigma], \ell[l'_{\text{AIcl}}\sigma]).$$

$$\text{AI}_{\text{mat}}(C) \stackrel{\text{def}}{=} \ell[\text{AI}_{\text{mat}}(C_1)\sigma]\tau$$

$$\text{AI}_{\text{cl}}(C) \stackrel{\text{def}}{=} \ell[(\text{AI}_{\text{cl}}(C_1) \setminus \{l'_{\text{AIcl}}\})\sigma]\tau$$

△

w\_quantifier\_block>