Chapter 1

trying to overbind mostly right away does not look promissing

Definition 1 (Q)**.** For a literal/clause φ, $Q(\varphi)$ denotes the quantifier block consisting of every lifting variable in φ with appropriate quantifier type. The order is yet to be defined $\triangle$

For $l \in C$ for $C \in \Gamma$: $Q(l) = \exists \bar{y}$
For $l \in C$ for $C \in \Delta$: $Q(l) = \forall \bar{x}$

basic axioms which should be fulfilled for a reasonable procedure

start

- $\Gamma \vDash \mathrm{LI}_{\mathrm{cl}}(C)$

  $\Gamma = \{P(f(x))\} \Rightarrow \mathrm{LI}_{\mathrm{cl}}(C) \overset{\text{must be}}{=} \exists x P(x)$

  $\Gamma = \{\neg P(f(x))\} \Rightarrow \mathrm{LI}_{\mathrm{cl}}(C) \overset{\text{must be}}{=} \exists x \neg P(x)$
- $\Delta \vDash ?$

inferences LI is always basically just Γ-part
always want: $\Gamma \vDash \mathrm{LI}$, $\Delta \vDash \neg\,\mathrm{LI}$

- $\Gamma : P(f(x)) \Rightarrow \exists x P(x)$

  $\Delta : \neg P(y) \Rightarrow \forall y \neg P(x)$
- $\Gamma : \neg P(f(x)) \Rightarrow \exists x \neg P(x)$

  $\Delta : P(y) \Rightarrow \forall y P(x)$
- $\Gamma : \neg P(x) \Rightarrow \forall x \neg P(x)$

  $\Delta : P(g(y)) \Rightarrow \exists y P(y)$
- $\Gamma : P(x) \Rightarrow \forall x P(x)$

  $\Delta : \neg P(g(y)) \Rightarrow \neg \exists y P(y)$

but must not tear apart $P(x) \vee \neg P(x)$ to $\forall x P(x) \vee \forall x \neg P(x)$

example for "var does not occur in clause any more-condition":

$$\frac{R(f(z)) \qquad \neg R(x) \vee P(x)}{\neg R(f(z)) \mid P(f(z))}$$

Note that $(\forall y_{f(x)} \neg R(y_{f(x)})) \vee P(x)$ is not valid!

Not sure what this example is supposed to demonstrate

attempt for a definition

Definition 2 (LI)**.**

Base case.

For $l \in C$ for $C \in \Gamma \cup \Delta$: $Q(l)\ell[C] \in \mathrm{LI}_{\mathrm{cl}}(C)$

LI as usual

Resolution.

Definition 3 (χ: lifting with quantification on literal level)**.**

$\chi(F \circ G) \overset{\text{def}}{=} \chi(F) \circ \chi(G)$

$\chi(\neg G) \overset{\text{def}}{=} \neg \chi(F)$

$\chi(Q(\lambda)\lambda) \overset{\text{def}}{=} Q(\lambda\sigma)\lambda\sigma$

where $Q(\lambda\sigma)$ is $Q(\lambda)$ with quantifiers and lifting variables for additional maximal colored terms introduced by σ into λ △

$\mathrm{LI}_{\mathrm{cl}}\, C \overset{\text{def}}{=} \chi(\mathrm{LI}_{\mathrm{cl}}(C_1) \backslash \{l_{\mathrm{LIcl}}\}) \vee \chi(\mathrm{LI}_{\mathrm{cl}}(C_2) \backslash \{l'_{\mathrm{LIcl}}\})$

1. If l is Γ-colored: $\mathrm{LI}(C) \overset{\text{def}}{=} \chi(\mathrm{LI}(C_1)) \vee \chi(\mathrm{LI}(C_2))$
2. If l is Δ-colored: $\mathrm{LI}(C) \overset{\text{def}}{=} \chi(\mathrm{LI}(C_1)) \wedge \chi(\mathrm{LI}(C_2))$
3. If l is grey: $\mathrm{LI}(C) \overset{\text{def}}{=} (l_{\mathrm{LIcl}}\tau \wedge \mathrm{LI}(C_2)\tau) \vee (\neg \ell[l'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_1)\tau])$

△

Conjectured Lemma 4. $\Gamma \models \mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$

Proof. Start works.
Step:
resolved literals: have same coloring

IH:
$\Gamma \models \mathrm{LI}(C_1) \vee \mathrm{LI}_{\mathrm{cl}}(C_1^*) \vee l_{\mathrm{LIcl}}$
$\Gamma \models \mathrm{LI}(C_2) \vee \mathrm{LI}_{\mathrm{cl}}(C_2^*) \vee l'_{\mathrm{LIcl}}$ □

Chapter 2

overbind just within thight constraints

⟨A⟩ **Lemma 5.** *If a variable does occurs in $\bar{C}$ but does not in C, then it is not modified by any* mgu *of a subsequent inference.*

2.1 naive interpolant extraction based on 5

Definition 6 (LI with stepwise prenex interplants but globally non-prenex ones)**.**

Base case.

For $l \in C$ for $C \in \Gamma \cup \Delta$: $C \in \mathrm{LI}_{\mathrm{cl}}(C)$

LI as usual

Resolution.

$\mathrm{LI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \mathrm{LI}_{\mathrm{cl}}(C_1)\backslash\{l_{\mathrm{LIcl}}\}\sigma \vee \mathrm{LI}_{\mathrm{cl}}(C_2)\backslash\{l'_{\mathrm{LIcl}}\}\sigma$

$\Rightarrow \mathrm{LI}_{\mathrm{cl}}(C) = C$

$\chi(F)$: lift all terms which do contain only variables which do not occur in $\mathrm{LI}_{\mathrm{cl}}(C)$ AND at least one such, and quantify prenex Does it suffice to have one, and other variables which are not final? proof idea: this term will never be a subterm of another term, as this variable cannot enter another term. other terms might become subterms of this term, but then they only need to be quantified before this one, which is implicitly given as it is lifted here in a nested way.

TODO: not sure where we can quantify ground terms as they can be added arbitrarily (possibly lift every occurrence of a ground term t distinctly)

apropos ground term: imagine procedure which conceptually adds some variable as argument to every term. if then we can overbind ground terms, we should be able to have a convention to enable nested ground term lifting directly

TODO: need not be prenex here, can pull in as far as regular quantifier pull in rules allow

1. If l is Γ-colored: $\mathrm{LI}^\bullet(C) \stackrel{\text{def}}{=} \mathrm{LI}(C_1) \vee \mathrm{LI}(C_2)\sigma$
2. If l is Δ-colored: $\mathrm{LI}(C)^\bullet \stackrel{\text{def}}{=} \mathrm{LI}(C_1) \wedge \mathrm{LI}(C_2)\sigma$
3. If l is grey: $\mathrm{LI}(C)^\bullet \stackrel{\text{def}}{=} (l_{\mathrm{LIcl}}\sigma \mathrm{LI}(C_2))\tau \vee (\neg l'_{\mathrm{LIcl}} \wedge \mathrm{LI}(C_1))\sigma$

$$\mathrm{LI}^*(C) \stackrel{\text{def}}{=} \chi(\mathrm{LI}^\bullet(C))$$

$$\mathrm{LI}(C) \stackrel{\text{def}}{=} Q_{\mathrm{LI}^*(C)}\, \mathrm{LI}^*(C)$$

△

$\Gamma \models \mathrm{LI}(C) \vee C$
$(\Delta \models \neg\, \mathrm{LI}(C) \vee C)$

2.2 lifting only Δ-terms in this way for now

does not really work like this because Γ-quantifiers are somewhat included, also nesting of quantifier is not treated in this "proof"

Conjectured Lemma 7. $\Gamma \models \mathrm{LI}^\Delta(C) \vee C$

Proof. THIS IS IMPLIED BY THE LEMMA FOR THE OTHER LIFTING STRATEGY, AS WE JUST LIFT *LESS* Δ-TERMS HERE, SO THIS IS ALWAYS AN INSTANCE OF THE OTHER LEMMA

induction on strenghtening, as always.

but additional strenghtening: lift all Δ-terms, just like in other lemma

$C_\Gamma = C^*_{1\,\Gamma} \vee C^*_{2\,\Gamma}$
IH:
$\Gamma \models \mathrm{LI}^\Delta(C_1) \vee C^*_{1\,\Gamma} \vee l_\Gamma$
$\Gamma \models \mathrm{LI}^\Delta(C_2) \vee C^*_{2\,\Gamma} \vee \neg l'_\Gamma$
Hence:
$\Gamma \models (\mathrm{LI}^\Delta(C_1) \vee C^*_{1\,\Gamma} \vee l_\Gamma)\sigma$
$\Gamma \models (\mathrm{LI}^\Delta(C_2) \vee C^*_{2\,\Gamma} \vee \neg l'_\Gamma)\sigma$
Supp grey:
$\Gamma \models (l \wedge \mathrm{LI}^\Delta(C_2))\sigma \vee (l' \wedge \mathrm{LI}^\Delta(C_1))\sigma \;\vee\; C_\Gamma$
$\Gamma \models \mathrm{LI}^\Delta(C) \;\vee\; C_\Gamma$
the literal is of course equal as by clearly C is not affected.

□

$X = \mathrm{LV}(\mathrm{LI}^\Delta(C)) \setminus \mathrm{LV}(\mathrm{LI}^\Delta_{\mathrm{cl}}(C_\Gamma))$

X': take from X those lifting variables, which contain variables which do not occur in C (this is safer than only $\mathrm{LI}^\Delta_{\mathrm{cl}}(C)$)

$Y = \mathrm{LV}(\ell_\Gamma[\mathrm{LI}^\Delta(C)])$

$Y' = \{z_t \in Y \mid t \text{ contains a variable which does not occur in } C\}$

From other pdf: $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$

Hence $\Gamma \models (Q(Y')\, \mathrm{LI}^\Delta(C)) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$

2.3 lifting with nesting

Conjectured Lemma 8. $\Gamma \models \ell_\Delta[\mathrm{LI}(C)] \vee \ell_\Delta[C]$

Proof. induction on C_Γ.

Base case:
$C \in \Gamma$: Then $\ell_\Delta[C] = C$ and $\Gamma \models C$
$C \in \Delta$: Then $\mathrm{LI}(C) = \top$

Ind step:
Supp $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)] \vee \ell_\Delta[C_1]$ and $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)] \vee \ell_\Delta[C_2]$.
By Lemma TODO: $\ell_\Delta[C] = \ell_\Delta[C_1^*\sigma] \vee \ell_\Delta[C_2^*\sigma] = \ell_\Delta[C_1]\tau \vee \ell_\Delta[C_2]\tau$
Then also
$\Gamma \models \mathrm{LI}(C_1) \vee C_1^* \vee l$
$\Gamma \models \mathrm{LI}(C_2) \vee C_2^* \vee l'$.
also:
$\Gamma \models \mathrm{LI}(C_1)\sigma \vee C_1^*\sigma \vee l\sigma$
$\Gamma \models \mathrm{LI}(C_2)\sigma \vee C_2^*\sigma \vee \neg l'\sigma$.

- Supp l grey. Note $l\sigma = l'\sigma$. then $\Gamma \models C_1^*\sigma \vee C_2^*\sigma \vee (l\sigma \wedge \mathrm{LI}(C_2)) \vee (\neg l'\sigma \wedge \mathrm{LI}(C_1)\sigma$.

  Variables in $\mathrm{LI}(C)$ can only not occur in C anymore if it was contained in l/l' and not anywhere else in C_1/C_2. variables might disappear due to σa removing them, but then they also disappear in $\mathrm{LI}(C)$.

  here we might be able to proof this w.r.t ground terms

  So suppose l contains a term t with variables such that they do not occur in C.

∎

2.4 random ideas

– we can pull apart existentially quantified variables: $\exists x(P(x) \vee Q(x))$ implies $\exists x P(x) \vee \exists y P(y)$. this does not work with universally quantified variables $(P(f(x)) \vee \neg P(f(x))$

but interpolants are somewhat symmetric, if it's existential for Γ, it's universal for Δ.