Interpolation in First-Order Logic with Equality

Master Thesis Presentation

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Agenda

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Craig Interpolation (1/2)

Theorem ([Craig, 1957])

Let Γ and Δ be finite sets of first-order formulas such that:

- $\Gamma \models \Delta$

Then there is a interpolant I such that:

- $\Gamma \models I$
- $I \models \Delta$

$$\begin{tikzcd} \Gamma \arrow[rr] \arrow[dr] & & \Delta \\ & I \arrow[ur] & \end{tikzcd}$$

Craig Interpolation (2/2)

Example

- Let $\Gamma = \{P(a))\}$ and $\Delta = \{\left(\forall x(P(x) \supset Q(x))\right) \supset \exists y Q(y)\}$.
- Interpolant: $\exists z\, P(z)$

Example

- Let $\Gamma = \{P(a), \neg P(b)\}$ and $\Delta = \{a \neq b\}$.
- Interpolant: $a \neq b$

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Applications and Motivation

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- Proof of Beth's Definability Theorem
- Model checking
- Reasoning with large knowledge bases
- ...

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Proof by Reduction due to Craig

Reduction to FOL without equality and function symbols:

$$\Big(P(c)\Big)^* \equiv \exists x(C(x) \wedge P(x))$$

$$\Big(P(f(c))\Big)^* \equiv \exists x(\exists y(C(y) \wedge F(y,x)) \wedge P(x))$$

$$\Big(s = t\Big)^* \equiv E(s,t)$$

$$\Big(\varphi\Big)^{\mathbf{*}} \equiv \Big(\mathsf{T}_E \wedge \bigwedge_{f \in \mathsf{FS}} \mathsf{T}_F\Big) \supset \varphi^*$$

Clearly φ and $\varphi^{\mathbf{*}}$ are equisatisfiable.

Proof in FOL without Equality and Function Symbols

- Use Maehara's Lemma for reduced logic
- [Baaz and Leitsch, 2011] presents a strengthening which handles function symbols
- **Open question:** Can it be extended to include equality?

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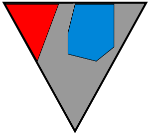
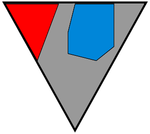
Interpolant extraction ([Huang, 1995])

Resolution proof:



$\Downarrow$ *Extract propositional interpolant structure from proof*

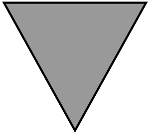
Propositional Interpolant:



$\ldots Q(f(c), c) \ldots$

$\Downarrow$ *Replace colored function and constant symbols*

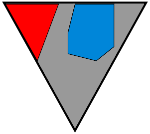
Prenex First-Order Interpolant:



$\exists x_3 \forall x_5 \ldots Q(x_5, x_3) \ldots$

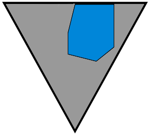
Interpolation Extraction in One Phase

Resolution proof:



$\Downarrow$ *Combined structure extraction and replacing of colored symbols*

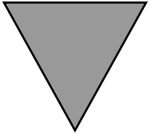
Interpolant modulo current clause:



$\forall x_5 \ldots Q(x_5, c) \ldots$

Recursively applied to all inferences of the proof results in:

Non-Prenex First-Order Interpolant:



$\exists x_3 \ldots \forall x_5 \ldots Q(x_5, x_3) \ldots$

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Semantic Proof

- Indirect proof of the interpolation theorem
- Builds a model non-constructively
- Equality does not require explicit handling

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Conclusion

- Craig's and Huang's proof based interpolant extraction from proofs
  $\Rightarrow$ but differ in practical applicability
- Craig shows that the interpolation theorem holds also in FOL with equality
- Huang shows that interpolants can efficiently be extracted in FOL with equality
  - Handling of equality does not require a different approach
- Huang's two-stage approach can be adapted to a one-stage approach yielding non-prenex interpolants
- Interpolation also allows for a model theoretic approach

References

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Trends in Logic. Springer.

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Linear Reasoning. A New Form of the Herbrand-Gentzen Theorem.
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