

## Example 101: canonical examples for overbinding w.r.t. order

Here, the second lower prooftree shows what huang would produce.

### Ex 101a

$$\begin{array}{c}
 \frac{\frac{P(\textcolor{red}{u}, f(\textcolor{red}{u})) \vee Q(\textcolor{red}{u})}{P(a, f(a))} \quad \frac{\neg Q(a)}{u \mapsto a}}{\square} \quad \frac{\neg P(x, y)}{x \mapsto a, y \mapsto f(a)} \\
 \\
 \frac{\frac{\frac{\perp}{Q(a)} \quad \top}{u \mapsto a} \quad \top}{P(a, f(a)) \vee Q(a)} \quad \top \quad \frac{\frac{\frac{\perp}{\forall x_1 Q(x_1)} \quad \top}{\forall x_1 \exists x_2 (P(x_1, x_2) \vee Q(x_1))} \quad \top}{\forall x_1 \exists x_2 (P(x_1, x_2) \vee Q(x_1))}
 \end{array}$$

Direct overbinding would not work without merging same variables!:  $\forall x_1 \forall x_2 \exists x_3 (P(x_2, x_3) \vee Q(x_1))$

counterexample:  $Q \sim \{0\}, P \sim \{(1, 0)\}$

Direct overbinding would work when considering original dependencies as highlighted above

### Ex 101b – other resolution order

$$\begin{array}{c}
 \frac{\frac{P(u, f(u)) \vee Q(u)}{Q(u)} \quad \frac{\neg P(x, y)}{y \mapsto f(u), x \mapsto u}}{\square} \quad \frac{\neg Q(a)}{u \mapsto a} \\
 \\
 \frac{\frac{\frac{\perp}{P(u, f(u))} \quad \top}{x \mapsto f(u), x \mapsto u} \quad \top}{P(a, f(a)) \vee Q(a)} \quad \top \quad \frac{\frac{\frac{\perp}{\exists x_1 P(u, x_1)} \quad \top}{\forall x_1 \exists x_2 (P(x_1, x_2) \vee Q(x_1))} \quad \top}{\forall x_1 \exists x_2 (P(x_1, x_2) \vee Q(x_1))} \quad u \mapsto a
 \end{array}$$

### Ex 101c – $\Pi$ and $\Sigma$ swapped

$$\begin{array}{c}
 \frac{\frac{P(u, f(u)) \vee Q(u)}{Q(u)} \quad \frac{\neg P(x, y)}{y \mapsto f(u), x \mapsto u}}{\square} \quad \frac{\neg Q(a)}{u \mapsto a} \\
 \\
 \frac{\frac{\frac{\top}{\neg P(u, f(u))} \quad \perp}{x \mapsto f(u), x \mapsto u} \quad \perp}{\neg P(a, f(a)) \wedge \neg Q(a)} \quad \perp \quad \frac{\frac{\frac{\top}{\forall x_2 \neg P(u, x_2)} \quad \perp}{\exists x_1 \forall x_2 (\neg P(x_1, x_2) \wedge \neg Q(x_1))} \quad \perp}{\exists x_1 \forall x_2 (\neg P(x_1, x_2) \wedge \neg Q(x_1))}
 \end{array}$$

**Ex 101d –  $\Pi$  and  $\Sigma$  swapped, other resolution order**

$$\frac{\frac{P(u, f(u)) \vee Q(u) \quad \neg Q(a)}{P(a, f(a))} \quad \frac{\neg Q(a)}{u \mapsto a} \quad \frac{\neg P(x, y)}{x \mapsto a, y \mapsto f(a)}}{\square} \quad \Sigma$$

$$\frac{\frac{\top \quad \perp}{\neg Q(a)} \quad y \mapsto a}{\neg Q(a) \wedge \neg P(a, f(a))} \quad \frac{\perp}{x \mapsto a, y \mapsto f(a)}$$

$$\frac{\frac{\top \quad \perp}{\exists x_1 \neg Q(x_1)} \quad \perp}{\exists x_1 \forall x_2 (\neg P(x_1, x_2) \wedge \neg Q(x_1))}$$

**102 – similar to 101 but with intra-clause-set inferences**

**Ex 102a**

$$\frac{\frac{P(f(\textcolor{red}{x})) \vee Q(f(\textcolor{red}{x}), z) \quad \neg P(y)}{Q(f(x), z)} \quad \frac{\neg Q(x_1, \textcolor{blue}{y}) \vee R(\textcolor{blue}{y}) \quad \neg R(g(z_1))}{\neg Q(x_1, g(z_1))} \quad y \mapsto g(z_1), x_1 \mapsto f(x)}{x_1 \mapsto f(x), z \mapsto g(z_1)} \quad \Sigma$$

$$\frac{\frac{\perp \quad \top}{P(f(x))} \quad \frac{\perp \quad \top}{R(g(z_1))}}{P(f(x)) \vee R(g(z_1))} \quad y \mapsto g(z_1), x_1 \mapsto f(x)$$

$$\frac{\frac{\perp \quad \top}{\exists x_1 P(x_1)} \quad \frac{\perp \quad \top}{\forall x_2 R(x_2)}}{\exists x_1 \forall x_2 (P(x_1) \vee R(x_2))} \quad \text{(order irrelevant!)}$$

**Ex 102b**

$$\frac{\frac{P(f(\textcolor{red}{x})) \vee Q(f(\textcolor{red}{x}), z) \quad \neg P(y)}{Q(f(x), z)} \quad \frac{\neg Q(f(\textcolor{blue}{y}), z_1) \vee R(\textcolor{blue}{y}) \quad \neg R(a)}{\neg Q(f(a), z_1)} \quad y \mapsto a}{x \mapsto a, z \mapsto z_1} \quad \Sigma$$

$$\frac{\frac{\perp \quad \top}{P(f(x))} \quad \frac{\perp \quad \top}{R(a)}}{P(f(a)) \vee R(a)} \quad y \mapsto a$$

$$\frac{\frac{\perp \quad \top}{\exists x_1 P(x_1)} \quad \frac{\perp \quad \top}{\forall x_2 R(x_2)}}{\forall x_2 \exists x_1 (P(x_1) \vee R(x_2))} \quad y \mapsto a$$

direct:

$$\frac{\frac{\perp \quad \top}{\exists x_1 P(x_1)} \quad x_1 \sim f(x) \quad \frac{\perp \quad \top}{\forall x_2 R(x_2)} \quad x_2 \sim a}{\exists x_1 \forall x_2 (P(x_1) \vee R(x_2))} \quad \text{order irrelevant!}$$

Ex 102b' with  $Q$  grey

$$\frac{\frac{\frac{P(f(\textcolor{red}{x})) \vee Q(f(\textcolor{red}{x}), z)}{\Sigma} \quad \frac{\neg P(y)}{\Pi}}{Q(f(x), z)} \quad \frac{\frac{\neg Q(f(\textcolor{blue}{y}), z_1) \vee R(\textcolor{blue}{y})}{\Sigma} \quad \frac{\neg R(a)}{\Pi}}{\neg Q(f(a), z_1)} y \mapsto a}{\frac{\frac{\perp}{\bot} \quad \top}{P(f(x))} \quad \frac{\frac{\perp}{\bot} \quad \top}{R(a)} y \mapsto a} x \mapsto a, z_1 \mapsto z$$

$$\frac{}{(\neg Q(f(a), z) \wedge P(f(a))) \vee (Q(f(a), z) \wedge R(a))} x \mapsto a, z_1 \mapsto z$$

Huang:

$$\frac{\frac{\perp}{\bot} \quad \top}{\exists x_2 P(x_2)} \quad \frac{\perp}{\forall x_1 R(x_1)} y \mapsto a}{\forall x_1 \exists x_2 (\neg Q(x_2, z) \wedge P(x_2)) \vee (Q(x_2, z) \wedge R(x_1))} x \mapsto a, z_1 \mapsto z$$

direct:

$$\frac{\frac{\perp}{\bot} \quad \top}{\exists x_2 P(x_2)} x_2 \sim f(x) \quad \frac{\perp}{\forall x_1 R(x_1)} x_1 \sim a}{\forall x_1 \exists x_2 \exists x_3 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))} x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3$$

$$\frac{\text{OR: } \exists x_2 \forall x_1 \exists x_3 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}{\text{OR: } \exists x_1 \exists x_3 \forall x_2 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}$$

Huang-Version passt weil man für existenzielle quantoren eh gleichen term einsetzt

direct w mixed, slightly different:

$$\frac{\frac{\perp \mid P(f(x)) \vee Q(x, z)}{\exists x_2 P(x_2)} \quad \top \mid \neg P(y)}{x_2 \sim f(x)} \quad \frac{\frac{\perp \mid \neg Q(f(y), z_1) \vee R(y)}{\forall x_1 R(x_1)} \quad \top \mid \neg R(a)}{x_1 \sim a}$$

$$\frac{\frac{\frac{\frac{\perp \mid \neg Q(f(y), z_1) \vee R(y)}{\forall x_1 R(x_1)} \mid \neg Q(f(a), z_1)}{\exists x_2 P(x_2)} \mid Q(x, z)}{\forall x_1 \exists x_3 \exists x_2 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))} x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3$$

$$\frac{}{(\neg Q(f(a), z) \wedge P(f(f(a)))) \vee (Q(f(a), z) \wedge R(a))}$$

last dependency not crucial because other arrow is a  $\Sigma$ -arrow as well, but just changing it to  $\Pi$  (and changing  $f$  for  $g$  should produce a quantifier alternation)

## Example 103: variables in interpolant get unified and order might have to be changed

103a: length changes but old version is still valid, despite huang's algorithm doing a sort

$$\frac{\frac{\frac{\perp}{Q(f(\textcolor{red}{x}))} \top}{Q(f(x)) \vee P(y) \vee R(\textcolor{red}{x})} \neg Q(y_1)}{P(y) \vee R(x)} y_1 \mapsto f(x) \quad \frac{\neg P(h(g(a)))}{R(x)} y \mapsto h(g(a)) \quad \frac{\neg R(g(g(a)))}{\square} x \mapsto g(g(a))$$

$$\frac{\frac{\frac{\perp}{Q(f(x))} \top}{Q(f(x)) \vee P(h(g(a)))} \top}{Q(f(g(g(a)))) \vee P(h(g(a))) \vee R(g(g(a)))} y \mapsto h(g(a)) \quad \frac{\frac{\frac{\perp}{\exists x_1 Q(x_1)} \top}{\exists x_1 \forall x_2 (Q(x_1) \vee P(x_2))} \top}{X} x \mapsto g(g(a))$$

X:

Huang's algo gives:

$$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))$$

Direct overbinding gives:  $x_3 < x_1$ , rest arbitrary, hence:

$$\forall x_3 \exists x_1 \forall x_2 (Q(x_1) \vee P(x_2) \vee R(x_3)) \leftarrow \text{this you do not get with huang}$$

$$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))$$

$$\forall x_3 \forall x_2 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))$$

103b: length changes “uniformly”

$$\frac{\frac{\frac{\perp}{Q(f(f(\textcolor{red}{x})))} \top}{Q(f(f(x))) \vee P(f(\textcolor{red}{x})) \vee R(\textcolor{red}{x})} \neg Q(y_1)}{P(f(x)) \vee R(x)} y_1 \mapsto f(f(x)) \quad \frac{\neg P(y_2)}{R(x)} y_2 \mapsto f(x) \quad \frac{\neg R(g(a))}{\square} x \mapsto g(a)$$

$$\frac{\frac{\frac{\perp}{Q(f(f(x)))} \top}{Q(f(f(x))) \vee P(f(x))} \top}{Q(f(f(g(a)))) \vee P(f(g(a))) \vee R(g(a))} y_2 \mapsto f(x) \quad \frac{\frac{\frac{\perp}{\exists x_1 Q(x_1)} \top}{\exists x_2 \exists x_1 (Q(x_1) \vee P(x_2))} \top}{\forall x_3 \exists x_2 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))} x \mapsto g(a)$$

Huang and direct overbinding somewhat coincide as  $x_2 < x_1$  in both cases, and  $x_3 < x_1$  and  $x_3 < x_2$

103c: Failed attempt: different variables, accidentally the same terms appear but no logical connection

$$\frac{\frac{\frac{\perp}{P(a, x)} \top}{\neg Q(a)} \neg P(y, f(\textcolor{red}{z})) \vee Q(\textcolor{red}{z})}{\neg P(y, f(a))} z \mapsto a \quad \frac{\neg P(y, f(a))}{\square} y \mapsto a, x \mapsto f(a)$$

Huang:

$$\frac{\frac{\perp}{P(a, f(a)) \wedge \neg Q(a)} \quad \frac{\frac{\perp}{\neg Q(a)} \quad \top}{z \mapsto a}}{y \mapsto a, x \mapsto f(a)} \quad \frac{\frac{\perp}{\exists x_1 \neg Q(x_1)} \quad \top}{\exists x_1 \forall x_2 (P(x_1, x_2) \wedge \neg Q(x_1))}$$

order required for  $\Pi$

direct:

$$\frac{\frac{\frac{\perp}{\exists x_1 \neg Q(x_1)} \quad \top}{x_1 \sim a} \quad \frac{\perp}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}}{x_2 \sim a, x_3 \sim f(a); x_1 < x_3}$$

$$\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))$$

invariant:

$$\frac{\frac{\frac{\exists x_1 (Q(x_1) \vee \perp)}{\exists x_1 \forall x_3 \neg P(y, \mathbf{x}_3) \vee \neg Q(\mathbf{x}_1)} \quad \frac{\forall x_3 ((\neg P(y, \mathbf{x}_3) \vee Q(\mathbf{z})) \vee \top)}{x_1 \sim a}}{\exists x_2 (P(x_2, x) \vee \perp)} \quad \frac{\perp}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}}{x_2 \sim a, x_3 \sim f(a); x_1 < x_3}$$

$$\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))$$

invariant in other resolution order

$$\frac{\frac{\frac{\perp}{Q(\mathbf{z}) \vee \exists x_2 \forall x_3 P(x_2, \mathbf{x}_3)} \quad \top}{x_2 \sim a, x_3 \sim f(z)} \quad \frac{\perp}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}}{x_1 \sim a; x_1 < x_3}$$

$$\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))$$

invariant if  $\Sigma$  and  $\Pi$  swapped:

$$\frac{\frac{\frac{\top}{\neg P(y, f(x_1)) \vee \forall x_1 Q(x_1)} \quad \frac{\perp}{x_1 \sim a}}{\forall x_1 \forall x_2 \exists x_3 (\neg P(x_2, x_3) \vee Q(x_1))} \quad \frac{\perp}{\forall x_1 \exists x_3 \forall x_2 (\neg P(x_2, x_3) \vee Q(x_1))}}{x_2 \sim a, x_3 \sim f(a); x_1 < x_3}$$

$$\text{OR: } \forall x_1 \exists x_3 \forall x_2 (\neg P(x_2, x_3) \vee Q(x_1))$$

SECOND ATTEMPT:

$$\frac{\frac{\frac{\frac{\frac{\Sigma}{P(a)} \quad \frac{\Sigma}{Q(z)} \quad \frac{\frac{\Sigma}{\neg S(a)} \quad \frac{\Pi}{\neg P(y) \vee \neg Q(f(\mathbf{x})) \vee S(\mathbf{x})}}{\neg P(y) \vee \neg Q(f(a))} \quad x \mapsto a}}{z \mapsto f(a)} \quad \neg P(y)}{y \mapsto a} \quad \frac{\perp}{\square}}{x \mapsto a}$$

$$\frac{\frac{\perp}{\neg S(a)} \quad \top}{z \mapsto f(a)} \quad \frac{\perp}{P(a) \wedge \neg S(a) \wedge Q(f(a))} \quad y \mapsto a$$

Huang:

$$\frac{\perp \quad \frac{\perp \quad \frac{\top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 (\neg S(x_1) \wedge Q(x_2))}}{\exists x_1 \forall x_2 (P(x_1) \wedge \neg S(x_1) \wedge Q(x_2))}$$

$$\forall x_1 \exists x_2 (\neg P(x_1) \vee S(x_1) \vee \neg Q(x_2))$$

**similar fail**

$\Rightarrow$  anytime there is  $P(a, f(a))$ , either they have a dependency or they are not both differently colored (grey is uncolored)  
for the record, direct method anyway:

$$\frac{\perp \quad \frac{\perp \quad \frac{\top}{\exists x_1 \neg S(x_1)} \quad x \sim a}{\exists x_1 \forall x_2 \neg S(x_1) \wedge Q(x_2)} \quad z \sim f(a); x_1 < x_2}{\exists x_1 \forall x_2 \exists x_3 P(x_3) \wedge \neg S(x_1) \wedge Q(x_2)} \quad x_3 \sim a; x_3 \text{ need not be merged w } x_1$$

## Example: ordering on both ancestors where the merge forces a new ordering

202a – canonical

$$\begin{array}{c}
 \frac{\frac{P(a, x_1) \vee R(y)}{R(y)} \quad \frac{\neg P(\textcolor{violet}{x}, f\textcolor{violet}{x})}{x \mapsto fa} \quad \frac{Q(\textcolor{red}{x}_2, g\textcolor{red}{x}_2) \vee \neg R(u)}{\neg R(u)} \quad \frac{\frac{\neg S(a)}{\neg S(a)} \quad \frac{\neg Q(f\textcolor{blue}{z}, x_3) \vee S(\textcolor{blue}{z})}{\neg Q(fa, x_3)}}{z \mapsto a} \quad \frac{z \mapsto a}{x_3 \mapsto gfa}}{x_1 \mapsto fa, x_2 \mapsto fa, x_3 \mapsto gfa} \\
 \hline
 \frac{\frac{\frac{\perp}{\perp} \quad \top}{P(a, f(a))} \quad \frac{\frac{\perp}{\perp} \quad \top}{Q(f(a), g(f(a))) \wedge \neg S(a)} \quad \frac{\frac{\perp}{\perp} \quad \top}{z \mapsto a} \quad \frac{\frac{\perp}{\perp} \quad \top}{x_2 \mapsto f(a), x_3 \mapsto g(f(a))}}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a))} \quad \square
 \end{array}$$

Huang

$$\frac{\frac{\frac{\perp}{\perp} \quad \top}{\exists x_1 \forall x_2 P(x_1, x_2)} \quad \frac{\frac{\perp}{\perp} \quad \frac{\frac{\perp}{\perp} \quad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 Q(x_2, x_3) \wedge \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 P(x_1, x_2) \vee (Q(x_2, x_3) \wedge \neg S(x_1))}$$

direct:

$$\frac{\frac{\frac{\perp}{\perp} \quad \top}{\exists x_1 \forall x_2 P(x_1, x_2)} \quad \frac{x_1 \sim a, x_2 \sim fa \quad x_3 \sim a, x_4 \sim fa, x_5 \sim gfa)}{\textcolor{violet}{x}_1 < \textcolor{violet}{x}_2 \quad \textcolor{red}{x}_3 < \textcolor{red}{x}_4, \textcolor{blue}{x}_4 < \textcolor{blue}{x}_5} \quad \frac{\frac{\perp}{\perp} \quad \frac{\frac{\perp}{\perp} \quad \top}{\exists x_3 \neg S(x_3)}}{\exists x_3 \forall x_4 \exists x_5 Q(x_4, x_5) \wedge \neg S(x_3)} \quad \frac{x_3 \mapsto a}{x_3 \sim a}}{\exists x_1 \forall x_2 \exists x_5 P(x_1, x_2) \vee (Q(x_2, x_5) \wedge \neg S(x_5))} \quad \frac{x_3 \mapsto x_1, x_4 \mapsto x_2}{x_1 < x_2, x_2 < x_5}$$

without merge in end:  $(x_1 < x_2, x_3 < x_4, x_4 < x_5)$

$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))$

$\exists x_3 \forall x_4 \exists x_5 \exists x_1 \forall x_2 P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))$

(also interwoven ones appear to work)

combined presentation:

$$\begin{array}{c}
\frac{\frac{\frac{\perp \mid P(a, x_1) \vee R(y) \quad \top \mid \neg P(x, f(x))}{P(a, f(a)) \mid R(y)} \quad x_1 \mapsto f(a)}{x \mapsto a} \quad \frac{\frac{\perp \mid Q(x_2, g(x_2)) \vee \neg R(u)}{Q(f(a), g(f(a))) \wedge \neg S(a) \mid \neg R(u)} \quad \frac{\frac{\perp \mid \neg S(a) \quad \top \mid \neg Q(f(z), x_3) \vee S(z)}{\neg S(a) \mid \neg Q(f(a), x_3)} \quad \frac{z \mapsto a}{x_2 \mapsto f(a)},}{x_3 \mapsto g(f(a))} \\
\hline
P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a)) \mid \square
\end{array}$$

combined presentation ground:

$$\begin{array}{c}
\frac{\frac{\perp \mid P(a, f(a)) \vee R(y) \quad \top \mid \neg P(a, f(a))}{(P(a, f(a)) \wedge \top) \vee (\neg P(a, f(a)) \wedge \perp) \mid R(y)} \quad \frac{\frac{\perp \mid Q(f(a), g(f(a))) \vee \neg R(u)}{Q(f(a), g(f(a))) \wedge \neg S(a) \mid \neg R(u)} \quad \frac{\frac{\perp \mid \neg S(a) \quad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a), g(f(a)))}}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a)) \mid \square}
\end{array}$$

combined presentation ground with direct method but only  $\Delta$ -terms removed :

$$\begin{array}{c}
\frac{\frac{\perp \mid P(a, f(a)) \vee R(y) \quad \top \mid \neg P(a, f(a))}{(P(a, x_2) \wedge \top) \vee (\neg P(a, x_2) \wedge \perp) \mid R(y)} \quad \frac{\frac{\perp \mid Q(f(a), g(f(a))) \vee \neg R(u)}{Q(x_4, g(x_4)) \wedge \neg S(a) \mid \neg R(u)} \quad \frac{\frac{\perp \mid \neg S(a) \quad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a), g(f(a)))}}{P(a, x_2) \vee (Q(x_4, g(x_4)) \wedge \neg S(a)) \mid \square}
\end{array}$$

combined presentation ground with direct method:

$$\begin{array}{c}
\frac{\frac{\frac{\perp \mid P(a, f(a)) \vee R(y) \quad \top \mid \neg P(a, f(a))}{\exists x_1 \forall x_2 (P(x_1, x_2) \wedge \top) \vee (\neg P(x_1, x_2) \wedge \perp) \mid R(y)} \quad \frac{\frac{\perp \mid Q(f(a), g(f(a))) \vee \neg R(u)}{\exists x_3 \forall x_4 \exists x_5 (Q(x_4, x_5) \wedge \neg S(x_3)) \mid \neg R(u)} \quad \frac{\frac{\perp \mid \neg S(a) \quad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\exists x_3 \neg S(x_3) \mid \neg Q(f(a), g(f(a)))}}{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))) \mid \square}
\end{array}$$



## 203a – some alternations

$$\begin{array}{c}
\frac{\frac{\frac{\frac{\frac{}{R(x) \vee \neg P(f(x))}}{\Sigma} \quad \frac{\frac{}{P(z) \vee Q(g(z))}}{\Pi}}{R(x) \vee Q(g(f(x)))} \quad z \mapsto f(x)}{\Pi} \quad \frac{\frac{}{-Q(y) \vee S(h(y))}}{\Sigma}}{y \mapsto g(f(x))} \\
\frac{\frac{\frac{}{-R(a)}}{\Pi} \quad R(x) \vee S(h(g(f(x))))}{x \mapsto a} \\
\frac{\frac{}{\neg S(x_1)}}{\Pi} \quad S(h(g(f(a))))}{x_1 \mapsto h(g(f(a)))} \\
\hline
\Box
\end{array}$$

$$\frac{\frac{\frac{\frac{\perp}{\neg P(f(x))} z \mapsto f(x)}{\neg Q(g(f(x))) \wedge \neg P(f(x))} y \mapsto g(f(x))}{\neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a)} x \mapsto a}{\neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a) \vee S(h(g(f(a))))} x_1 \mapsto h(g(f(a)))$$

Huang:

$$\frac{\frac{\frac{\perp}{\exists x_1 \neg P(x_1)} \quad \top}{\exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1))} \quad \perp}{\top \quad \frac{\top}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0))}} \quad \frac{\top}{\forall x_0 \exists x_1 \forall x_2 \exists x_3 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0) \vee S(x_3))}$$

Direct:

$$\frac{\frac{\frac{\frac{\perp}{\exists x_1 \neg P(x_1)} \quad \top}{x_1 \sim f(x)} \quad \perp}{\exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(f(x)))} \quad x_2 \sim g(f(x)); x_1 < x_2}{\top \quad \frac{\top}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0))} \quad x_0 \sim a; x_0 < x_1, x_0 < x_2}{\top \quad \frac{\forall x_0 \exists x_1 \forall x_2 \forall x_3 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0) \vee S(x_3))}{x_3 \sim h(g(f(a))); x_0 < x_3, x_1 < x_3, x_2 < x_3}}$$

203b – many  $\Sigma$ -literals, coloring per occurrence

$$\begin{array}{c}
\frac{\frac{\frac{\frac{\frac{R(x) \vee \neg P(f(x))}{\Sigma} \quad \frac{P(z) \vee Q(g(z))}{\Sigma}}{R(x) \vee Q(gfx)} \quad z \mapsto fx}{\neg Q(y) \vee S(h(y))}{\Sigma}}{y \mapsto gfx} \\
\frac{\frac{\frac{\neg R(a)}{\Pi} \quad \frac{R(x) \vee S(hgfx)}{x \mapsto a}}{S(hgfa)} \quad x_1 \mapsto hgfa}{\perp} \quad \frac{\perp}{z \mapsto fx} \\
\frac{\frac{\frac{\frac{\frac{\perp}{\top} \quad \frac{\perp}{x \mapsto a}}{R(a)} \quad \frac{\perp}{y \mapsto gfx}}{\frac{\top}{S(hgfa) \vee R(a)} \quad x_1 \mapsto hgfa} \quad \frac{\perp}{\perp} \\
\hline
\frac{\neg S(x_1)}{\Pi} \quad \frac{\neg R(a)}{\Pi} \quad \frac{S(hgfa)}{\Pi} \quad \frac{x_1 \mapsto hgfa}{\Pi} \quad \frac{z \mapsto fx}{\Pi} \quad \frac{y \mapsto gfx}{\Pi} \\
\hline
\Box
\end{array}$$

$$\rightarrow \forall x_1 \exists x_2 (R(x_1) \vee S(x_2))$$

203b' – many  $\Sigma$ -literals, coloring per symbol, all predicates grey

$$\begin{array}{c}
 \frac{\frac{\frac{\neg R(a)}{\Pi} \quad \frac{R(x) \vee \neg P(f(x))}{\Sigma}}{R(a) \mid \neg P(fa)} \quad x \mapsto a \quad \frac{P(z) \vee Q(g(z))}{\Sigma} \quad z \mapsto fa}{\frac{\neg S(x_1)}{\Pi} \quad \frac{P(fa) \vee R(a) \mid Q(gfa)}{\Sigma} \quad \neg Q(y) \vee S(h(y))}{\Sigma}
 \end{array}$$

TODO

## Example where variables are not the outermost symbol but order is still relevant

204a

$$\Sigma = \{P(f(x), g(f(x)))\}$$

$$\Pi = \{P(f(a), y)\}$$

$$\Rightarrow \forall x_1 \exists x_2 P(f(x_1), x_2)$$

204b

$$\Sigma = \{P(f^5(x), g(f(x)))\}$$

$$\Pi = \{P(f^5(a), y)\}$$

$$\Rightarrow \forall x_1 \exists x_2 P(f^5(x_1), x_2)$$

example with aufschaukelnde unification, such that direction of arrow isn't clear

205a

$$\frac{P(ff\textcolor{blue}{y}, g\textcolor{blue}{y}) \quad \frac{\neg P(\textcolor{red}{x}, y) \vee Q(\textcolor{red}{x}) \quad \frac{\neg R(a) \quad \neg Q(ff\textcolor{violet}{z}) \vee R\textcolor{violet}{z}}{\neg R(a) \mid \neg Q(ffa)} z \mapsto a}{\neg R(a) \mid \neg P(ffa, y)} x \mapsto ffa}{(\neg R(a) \wedge Q(ffa)) \vee \neg P(ffa, ga)} y \mapsto a$$

direct

$$\frac{P(ff\textcolor{blue}{y}, g\textcolor{blue}{y}) \quad \frac{\neg P(\textcolor{red}{x}, y) \vee Q(\textcolor{red}{x}) \quad \frac{\neg R(a) \quad \neg Q(ff\textcolor{violet}{z}) \vee R\textcolor{violet}{z}}{\exists x_1 \neg R(x_1) \mid \neg Q(ffa)} z \mapsto a}{\exists x_1 \forall x_2 (\neg R(x_1) \wedge Q(x_2)) \mid \neg P(ffa, u)} x \mapsto ffa}{\exists x_1 \forall x_2 \forall x_3 ((\neg R(x_1) \wedge Q(x_2)) \vee \neg P(x_2, x_3))} y \mapsto a, u \mapsto ga$$

ground:

$$\frac{P(ff\textcolor{blue}{a}, ga) \quad \frac{\neg P(f\textcolor{red}{f}a, y) \vee Q(f\textcolor{red}{f}a) \quad \frac{\neg R(a) \quad \neg Q(ff\textcolor{violet}{a}) \vee R\textcolor{violet}{a}}{\neg R(a) \mid \neg Q(ffa)} }{\neg R(a) \wedge Q(ffa) \mid \neg P(ffa, a)} }{(\neg R(a) \wedge Q(ffa)) \vee \neg P(ffa, ga)}$$

205b ~ 205a, but simpler

Suppose  $P$  occurs somewhere in  $\Sigma$  (result not that optimal in this setting, but correct)

not nice for proving,  $\neg R(a)$  is a nice interpolant already

$$\frac{P(ff\textcolor{blue}{y}, g\textcolor{blue}{y}) \quad \frac{\neg R(a) \quad \neg P(ff\textcolor{violet}{z}, x) \vee R\textcolor{violet}{z}}{\neg R(a) \mid \neg P(ffa, x)} z \mapsto a}{\neg R(a) \mid \neg P(ffa, ga)} x \mapsto ga, y \mapsto a$$

$$\frac{\top \mid P(ff\textcolor{blue}{y}, g\textcolor{blue}{y}) \quad \frac{\perp \mid \neg R(a) \quad \top \mid \neg P(ff\textcolor{violet}{z}, x) \vee R\textcolor{violet}{z}}{\exists x_1 \neg R(x_1) \mid \neg P(ffa, x)} z \mapsto a}{\exists x_1 \forall x_2 \forall x_3 \neg R(x_1) \vee \neg P(x_2, x_3) \mid \square} x \mapsto ga, y \mapsto a$$

$\exists x_1 R(x_1)$

$\exists x_1 \forall x_2 \forall x_3 (R(x_1) \vee \neg P(x_2, x_3))$

$f(x) \vee g(x)$  **with**  $f, g$  **different colors**

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{\Pi}{\neg P(z)} \quad \frac{\Pi}{\neg R(g(a))} \quad \frac{\frac{\Sigma}{P(f(x)) \vee Q(x)}{\forall x_1 Q(x_1) \mid P(f(x)) \vee R(g(x))} \quad \frac{\Pi}{R(g(y)) \vee \neg Q(y)}}{\forall x_1 Q(x_1) \mid P(f(a)) \vee R(g(a))} \quad y \mapsto x}{\forall x_1 Q(x_1) \mid P(f(a)) \vee R(g(a))} \quad x \mapsto a}{\forall x_1 \exists x_2 (Q(x_1) \vee P(x_2)) \vee R(g(a))} \quad z \mapsto f(a)
 \end{array}$$

$\Rightarrow$  free vars in the interpolant have to be overbound (if there are arrows, but we can just always do so)

## misc examples

201a

$$\frac{\frac{P(x, y) \vee \neg Q(y) \quad \neg P(a, y_2)}{\neg Q(y)} \quad x \mapsto a \quad \frac{Q(f(z)) \vee R(z) \quad \neg R(a)}{Q(f(a))} \quad z \mapsto a}{\square} \quad y \mapsto f(a)$$

$$\frac{\frac{\perp \quad \top}{P(a, y)} \quad x \mapsto a \quad \frac{\perp \quad \top}{R(a)} \quad z \mapsto a}{P(a, f(a)) \vee R(a)} \quad y \mapsto f(a)$$

$$\frac{\frac{\perp \quad \top}{\forall x_1 P(x_1, y)} \quad x \mapsto a \quad \frac{\perp \quad \top}{\forall x_3 R(x_3)} \quad z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 (P(x_1, x_2) \vee R(x_3))} \quad y \mapsto f(a)$$

Huang would produce:  $\forall x_1 \exists x_2 P(x_1, x_2) \vee R(x_1)$

201b

$$\frac{\frac{P(x, f(y)) \vee \neg Q(f(y)) \quad \neg P(a, y_2)}{\neg Q(f(y))} \quad x \mapsto a \quad \frac{Q(f(z)) \vee R(z) \quad \neg R(a)}{Q(f(a))} \quad z \mapsto a}{\square} \quad y \mapsto f(a)$$

$$\frac{\frac{\perp \quad \top}{P(a, f(y))} \quad x \mapsto a \quad \frac{\perp \quad \top}{R(a)} \quad z \mapsto a}{P(a, f(a)) \vee R(a)} \quad y \mapsto a$$

$$\frac{\frac{\perp \quad \top}{\forall x_1 \exists x_2 P(x_1, x_2)} \quad x \mapsto a \quad \frac{\perp \quad \top}{\forall x_3 R(x_3)} \quad z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 P(x_1, x_2) \vee R(x_3)} \quad y \mapsto f(a)$$

Huang would produce:  $\forall x_1 \exists x_2 P(x_1, x_2) \vee R(x_1)$

arrow in element which is not in interpolant or resolution clause

206

$$\begin{array}{c}
 \frac{P(x) \vee \neg Q(f(x)) \quad \neg P(a)}{\forall x_1 P(x_1) \mid \neg Q(f(a))} x \mapsto a \quad \frac{Q(y) \vee R(g(y)) \quad \neg R(z)}{\exists x_2 R(x_2) \mid Q(y)} z \mapsto g(y) \\
 \hline
 \frac{\quad}{\forall x_1 \exists x_2 (P(x_1) \vee R(x_2)) \mid \square} y \mapsto f(a) \\
 \hline
 P(a) \vee R(g(f(a)))
 \end{array}$$

for first interpolant,  $\Sigma \not\models \ell_{\Delta,x}[\text{PI}(C)] \vee C$

$\Rightarrow$  need to overbind clause as well