Interpolation in First-Order Logic with Equality

Master Thesis Presentation

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Agenda

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Craig Interpolation (1/2)

Theorem ([Craig, 1957]). *Let Γ and Δ be sets of first-order formulas where*

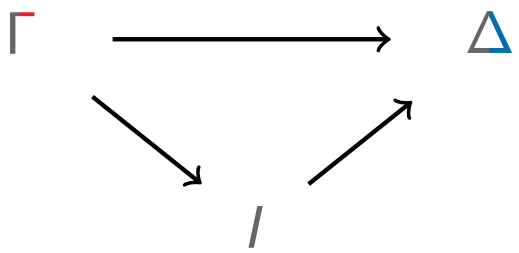
- *Γ contains red and gray symbols and*
- *Δ contains blue and gray symbols*

such that:

- $\Gamma \models \Delta$

Then there is a interpolant I containing only gray symbols such that:

- $\Gamma \models I$
- $I \models \Delta$



Craig Interpolation (2/2)

Example

- Let $\Gamma = \{P(a))\}$ and $\Delta = \{\forall x(P(x) \supset Q(x)), \exists y\, Q(y)\}$.
- Interpolant: $\exists z\, P(z)$

Example

- Let $\Gamma = \{P(a), \neg P(b)\}$ and $\Delta = \{a \neq b\}$.
- Only possible interpolant: $a \neq b$

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- Let $\Gamma = \{P(a), \neg P(a)\}$, $\Delta = \emptyset$.
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Applications and Motivation

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- Proof of Beth's Definability Theorem
- Model checking
- Detecting loop invariants
- Reasoning with large knowledge bases

Motivation

- Craig interpolation in full first-order logic with equality has received little attention so far
- Interest for constructive proofs giving rise to interpolant extraction algorithms

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Proof by Reduction

Reduction to FOL without equality and function symbols:

Translate equality and function symbols:

$$\Big(P(c)\Big)^* \equiv \exists x(C(x) \wedge P(x))$$
$$\Big(P(f(c))\Big)^* \equiv \exists x(\exists y(C(y) \wedge F(y,x)) \wedge P(x))$$
$$\Big(s = t\Big)^* \equiv E(s,t)$$

Add axioms for equality and new predicate symbols:

$$\Big(\varphi\Big)^* \equiv \Big(\mathsf{T}_E \wedge \bigwedge_{f \in \mathsf{FS}} \mathsf{T}_F\Big) \supset \varphi^*$$

Clearly φ and φ^* are equisatisfiable.

Proof in FOL without Equality and Function Symbols

Lemma (Maehara)

Let Γ and Δ be sets of first-order formulas without equality and function symbols such that $\Gamma \vdash \Delta$ is provable in ***sequent calculus****. Then for any partition $\langle(\Gamma_1; \Delta_1), (\Gamma_2; \Delta_2)\rangle$ there is an interpolant I such that*

1. $\Gamma_1 \vdash \Delta_1, I$ *is provable*
2. $\Gamma_2, I \vdash \Delta_2$ *is provable*
3. $\mathsf{L}(I) \subseteq \mathsf{L}(\Gamma_1, \Delta_1) \cap \mathsf{L}(\Gamma_2, \Delta_2)$

[Baaz and Leitsch, 2011] presents a strengthening which includes function symbols.

Open question: Can it be extended to include equality?

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Interpolant Extraction

Motivation

- Proof by reduction is impractical
- Goal: Compute interpolants from proof
- The following is based on [Huang, 1995]

Interpolant extraction from resolution proofs

- Skolemisation and clausal form transformation do no alter the set of interpolants
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Interpolant extraction from resolution proofs

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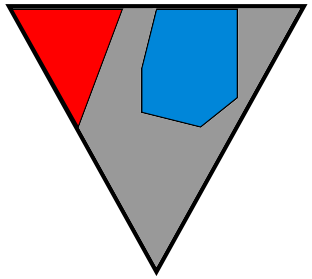
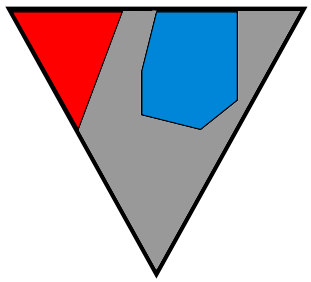
Huang's Algorithm (1/2) (6 min)

Proof:



⇓ *Extract propositional interpolant structure from proof*

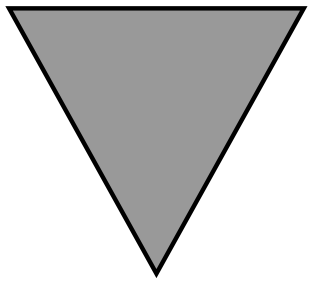
Propositional Interpolant:



$\ldots Q(f(c), c) \ldots$

⇓ *Replace colored function and constant symbols*

Prenex First-Order Interpolant:



$\exists x_3 \forall x_5 \ldots Q(x_5, x_3) \ldots$

Huang's Algorithm (2/2)

- Propositional interpolant is interpolant modulo function and constant symbols
- For the lifting phase, the ordering of the lifting variables is crucial
- The type of the quantifier is determined by the coloring of the symbol

Theorem

The number of quantifier alternations in the resulting interpolant directly corresponds to the number of color alternations of terms in the resolution proof.

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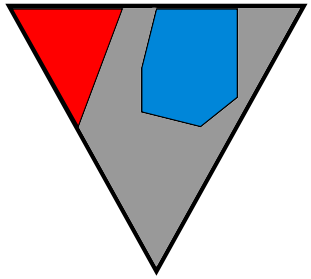
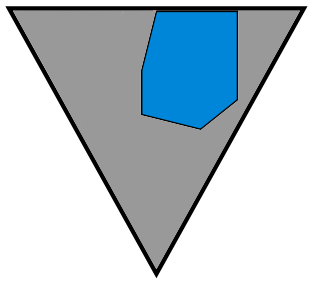
Interpolation Extraction in one Phase

Proof:



⇓ *Combined structure extraction and replacing of colored symbols*

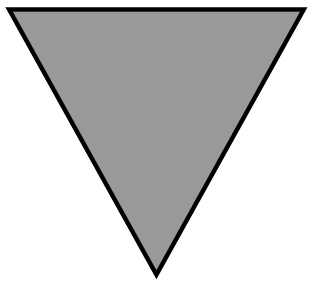
Interpolant modulo current clause:



$\forall x_5 \ldots Q(x_5, c) \ldots$

Recursively applied to all inferences of the proof results in:

Non-Prenex First-Order Interpolant:



$\exists x_3 \ldots \forall x_5 \ldots Q(x_5, x_3) \ldots$

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Semantic Proof

TODO

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Conclusion

- Craig's and Huang's proof based interpolant extraction from proofs
  $\Rightarrow$ differ in applicability
- Craig shows that the interpolation theorem holds also in FOL/EQ
- Huang shows that interpolants can efficiently be extracted in FOL/EQ
  - Does not require different methods
  - Little attention so far in research
- Interpolation also allows for a model theoretic approach


Baaz, M. and Leitsch, A. (2011).
Methods of Cut-Elimination.
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Linear Reasoning. A New Form of the Herbrand-Gentzen Theorem.
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