

# 1 Proof of the correctness of Huang's algorithm without propositional refutations

Intuition of  $\sigma'$ :

If we pull a substitution out of a lifting which replaces  $\Delta$ -terms, we also have to replace the  $\Delta$ -terms in the “codomain” of the substitution. This is the second case in the definition of  $\sigma'$  below.

There is just a problem in the following case:  $\ell_{\Delta,x}[f(x)\sigma]$ , where  $x\sigma = a$  and  $f$  is a  $\Delta$ -symbol. Then  $\ell_{\Delta,x}[f(x)\sigma] = \ell_{\Delta,x}[f(a)] = x_i$ , but  $\ell_{\Delta,x}[f(x)]\sigma = x_j$  with  $i \neq j$ . The first case of the definition of  $x_j$  then fixes this by replacing  $x_j$  with  $x_i$ .

**Lemma 1.** *Let  $C$  be a clause and  $\sigma$  a substitution. Let  $t_1, \dots, t_n$  be all maximal  $\Delta$ -terms in this context, i.e. those that occur in  $C$  or  $C\sigma$ , and  $x_1, \dots, x_n$  the corresponding fresh variables to replace the  $t_i$ . Define  $\sigma'$  such that for a variable  $z$ ,*

$$z\sigma' = \begin{cases} x_l & \text{if } z = x_k \text{ and } t_k\sigma = t_l \\ \ell_{\Delta,x}[z\sigma] & \text{otherwise} \end{cases}$$

Then  $\ell_{\Delta,x}[C\sigma] = \ell_{\Delta,x}[C]\sigma'$ .

Note that the definition of  $\sigma'$  only depends on the  $x_i$  and  $t_i$ .

*Proof.* We prove this for an atom  $P(s_1, \dots, s_m)$  in  $C$ , which works since lifting and substitution commute over binary connectives and into an atom.

We show that  $\ell_{\Delta,x}[s_j\sigma] = \ell_{\Delta,x}[s_j]\sigma'$  for  $1 \leq j \leq m$ .

Note that anything in the term structure above a maximal  $\Delta$ -term is unaffected by both substitution and the lifting.

Let  $t_i$  be a maximal  $\Delta$ -term in  $s_i\sigma$ .

We show that  $\ell_{\Delta,x}[t_i\sigma] = \ell_{\Delta,x}[t_i]\sigma'$ , which proves the lemma.

Let  $t_i\sigma = t_j$ . Then  $\ell_{\Delta,x}[t_i\sigma] = \ell_{\Delta,x}[t_j] = x_j$ .

We show that  $x_j = \ell_{\Delta,x}[t_i]\sigma'$ .

Suppose that  $t_i = t_j$ , i.e.  $\sigma$  is trivial on  $t_i$ . Then  $i = j$  as the  $\Delta$ -terms have a unique number. Hence  $\ell_{\Delta,x}[t_i]\sigma' = x_i\sigma' = x_i = x_j$ .

Otherwise  $t_i \neq t_j$ . Then  $i \neq j$  and  $x_j \neq x_i$ .

$\ell_{\Delta,x}[t_i]\sigma' = x_i\sigma'$ . By the definition of  $\sigma'$ , as  $t_i\sigma = t_j$ ,  $x_i\sigma' = x_j$ . □

**Lemma 2** (corresponds to Lemma 4.8 in thesis and Lemma 11 in Huang). *Let  $A$  and  $B$  be first-order formulas and  $s$  and  $t$  be terms. Then it holds that:*

1.  $\ell_{\Phi,x}[\neg A] \Leftrightarrow \neg \ell_{\Phi,x}[A]$
2.  $\ell_{\Phi,x}[A \circ B] \Leftrightarrow (\ell_{\Phi,x}[A] \circ \ell_{\Phi,x}[B])$  for  $\circ \in \{\wedge, \vee\}$
3.  $\ell_{\Phi,x}[s = t] \Leftrightarrow (\ell_{\Phi,x}[s] = \ell_{\Phi,x}[t])$

**Lemma 3.** *Let  $s$  and  $t$  be terms such that no  $x_i$  occurs in them,  $\Phi$  a set of formulas and  $M$  a model. Then  $M \models \ell_{\Phi,x}[s] = \ell_{\Phi,x}[t]$  implies that  $M \models s = t$ .*

*Proof.* Suppose no  $\Delta$ -term occurs in  $s$  or  $t$ . Then  $\ell_{\Phi,x}[s] = s$  and  $\ell_{\Phi,x}[t] = t$ .

Otherwise let  $t_i$  be a maximal  $\Delta$ -term in  $s$ . Suppose it occurs at position  $p$ . In  $\ell_{\Phi,x}[s]$ , it is replaced by  $x_i$ . But as  $M \models \ell_{\Phi,x}[s] = \ell_{\Phi,x}[t]$ , two situations can arise:

1.  $x_i$  occurs at  $p$  in  $\ell_{\Phi,x}[t]$ . As  $x_i$  does not occur in  $t$ , it is placed there by the lifting. But  $x_i$  is only employed in order to replace  $t_i$ , so at position  $p$  in  $t$ , we have  $t_i$ .
2. A term  $r$  occurs at  $p$  in  $\ell_{\Phi,x}[t]$  which does not influence the evaluation of  $\ell_{\Phi,x}[t]$  in  $M$ . This can be the case if  $r$  is contained in a subterm of  $u$  and in  $M$ , the function symbol of  $u$  is interpreted such that it does not depend on the argument that contains  $r$ .

But as the maximal  $\Delta$ -term  $t_i$  occurs in  $s$  at  $p$  and  $M \models \ell_{\Phi,x}[s] = \ell_{\Phi,x}[t]$ , there is a function symbol  $u'$  in  $\ell_{\Phi,x}[s]$  corresponding to  $u$  which also does not depend on this argument.

Hence even though  $s$  and  $t$  are not syntactically equal,  $M \models s = t$  in this case.  $\square$

**Lemma 4.** *Let  $E$  be a formula and  $s$  and  $t$  terms. Let  $h[t]$  be a maximal  $\Delta$ -colored term containing  $t$  at  $p$  in  $E[t]_p$ , if such a term exists. Then  $M \models (\ell_{\Delta,x}[s]) = (\ell_{\Delta,x}[t])$  and  $M \not\models \ell_{\Delta,x}[E[t]_p]$  imply that  $M \not\models \ell_{\Delta,x}[E[s]_p]$  or  $M \models s = t \wedge (\ell_{\Delta,x}[h[s]]) \neq (\ell_{\Delta,x}[h[t]])$ .*

*Proof.* Suppose that  $t$  at  $p$  in  $E[t]_p$  is not contained in a  $\Delta$ -colored term. Then  $\ell_{\Delta,x}[E[t]_p]$  and  $\ell_{\Delta,x}[E[s]_p]$  only differ at position  $p$ , where at the first, there is  $\ell_{\Delta,x}[t]$ , and at the latter, there is  $\ell_{\Delta,x}[s]$ . But in  $M$ , they are interpreted the same way, hence  $M \models \ell_{\Delta,x}[E[s]_p] \Leftrightarrow \ell_{\Delta,x}[E[t]_p]$ , which implies the result.

Otherwise  $t$  at  $p$  in  $E[t]_p$  is contained in the maximal  $\Delta$ -colored term  $h[t]$ . If  $s$  and  $t$  are syntactically equal, then  $(\ell_{\Delta,x}[h[s]]) = (\ell_{\Delta,x}[h[t]])$  and  $\ell_{\Delta,x}[E[s]_p] \Leftrightarrow \ell_{\Delta,x}[E[t]_p]$ . Otherwise, as by Lemma 3 it holds that  $M \models s = t$ , we get that  $M \models s = t \wedge (\ell_{\Delta,x}[h[s]]) \neq (\ell_{\Delta,x}[h[t]])$ .  $\square$

We use basically the same definition of PI as Huang with minor adaptations for paramodulation (deviations are marked):

**Definition 5** (Propositional interpolant extraction.). Let  $\pi$  be a resolution refutation of  $\Gamma \cup \Delta$ .  $\text{PI}(\pi)$  is defined to be  $\text{PI}(\square)$ , where  $\square$  is the empty clause derived in  $\pi$ .

For a clause  $C$  in  $\pi$ ,  $\text{PI}(C)$  is defined as follows:

Base case. If  $C \in \Gamma$ ,  $\text{PI}(C) = \perp$ . If otherwise  $C \in \Delta$ ,  $\text{PI}(C) = \top$ .

Resolution. If the clause  $C$  is the result of a resolution step of  $C_1 : D \vee l$  and  $C_2 : E \vee \neg l'$  using a unifier  $\sigma$  such that  $l\sigma = l'\sigma$ , then  $\text{PI}(C)$  is defined as follows:

1. If  $l$  is  $\Gamma$ -colored:  $\text{PI}(C) = [\text{PI}(C_1) \vee \text{PI}(C_2)]\sigma$
2. If  $l$  is  $\Delta$ -colored:  $\text{PI}(C) = [\text{PI}(C_1) \wedge \text{PI}(C_2)]\sigma$
3. If  $l$  is grey:  $\text{PI}(C) = [(l \wedge \text{PI}(C_2)) \vee (\neg l' \wedge \text{PI}(C_1))]\sigma$

Factorisation. If the clause  $C$  is the result of a factorisation of  $C_1 : l \vee l' \vee D$  using a unifier  $\sigma$  such that  $l\sigma = l'\sigma$ , then  $\text{PI}(C) = \text{PI}(C_1)\sigma$ .

Paramodulation. If the clause  $C$  is the result of a paramodulation of  $C_1 : s = t \vee C$  and  $C_2 : D[r]$  using a unifier  $\sigma$  such that  $r\sigma = s\sigma$ , then  $\text{PI}(C)$  is defined according to the following case distinction:

1. If  $r$  occurs in a maximal  $\Delta$ -term  $h[r]$  in  $D[r]$ : // Huang's version has another restriction here  

$$\text{PI}(C) = [(s = t \wedge \text{PI}(C_2)) \vee (s \neq t \wedge \text{PI}(C_1))] \sigma \vee (s = t \wedge h[s] \neq h[t]) \sigma$$
2. If  $r$  occurs in a maximal  $\Gamma$ -term  $h[r]$  in  $D[r]$ : // Huang's version has another restriction here  

$$\text{PI}(C) = [(s = t \wedge \text{PI}(C_2)) \vee (s \neq t \wedge \text{PI}(C_1))] \sigma \wedge (s \neq t \vee h[s] = h[t]) \sigma$$
3. Otherwise:  

$$\text{PI}(C) = [(s = t \wedge (\text{PI}(C_2) \vee h[s] \neq h[t]) \vee (s \neq t \wedge \text{PI}(C_1))] \sigma \quad \Delta$$

Now we show the “main” lemma of Huang's proof without using a propositional deduction  $P_P$ . The remaining part of his proof after this lemma does not use the restriction to propositional deductions and hence goes through.

**Lemma 6** (corresponds to Lemma 12 in Huang and Lemma 4.9 in the thesis). *Let  $\pi$  be a resolution refutation of  $\Gamma \cup \Delta$ . Then for  $C \in \pi$ ,  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C) \vee C]$ .*

*Proof.* By induction on the resolution refutation of the strengthening:  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C) \vee C_\Gamma]$ , i.e. we only consider literals of  $C$  which are contained in  $L(\Gamma)$ .

Base case: Either  $C \in \Gamma$ , then it does not contain  $\Delta$ -terms. Otherwise  $C \in \Delta$  and  $\text{PI}(C) = \top$ .

Induction step:

Resolution.

$$\frac{C_1 : D \vee l \quad C_2 : E \vee \neg l'}{C : (D \vee E) \sigma} \quad l\sigma = l'\sigma$$

By the induction hypothesis, we can assume that:

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee (D \vee l)_\Gamma] \text{ and } \Gamma \models \ell_{\Delta,x}[\text{PI}(C_2) \vee (E \vee \neg l')_\Gamma]$$

which by Lemma 2 implies that

$$\stackrel{(*)}{\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)] \vee \ell_{\Delta,x}[D_\Gamma] \vee \ell_{\Delta,x}[l_\Gamma]} \text{ and } \stackrel{(\circ)}{\Gamma \models \ell_{\Delta,x}[\text{PI}(C_2)] \vee \ell_{\Delta,x}[E_\Gamma] \vee \neg \ell_{\Delta,x}[l'_\Gamma]}$$

Let  $\sigma'$  be defined as in Lemma 1 with  $t_1, \dots, t_n$  all  $\Delta$ -terms in this context (we need that every maximal  $\Delta$ -term has a distinct index, so take all occurring in  $C_1, C_2, \text{PI}(C_1), \text{PI}(C_2)$ , with and without  $\sigma$  applied to them).

Case distinction:

1.  $l$  is  $\Gamma$ -colored. Then  $\text{PI}(C) = [\text{PI}(C_1) \vee \text{PI}(C_2)] \sigma$ .

We show that  $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2)) \sigma \vee (D \vee E)_\Gamma \sigma]$ ,

i.e.  $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2) \vee D_\Gamma \vee E_\Gamma) \sigma]$ .

Hence by Lemma 1,  $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2) \vee D_\Gamma \vee E_\Gamma)] \sigma'$ .

Since  $\sigma = \text{mgu}(l, l')$ ,  $l\sigma$  and  $l'\sigma$  are syntactically equal and so  $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l'\sigma]$ .

As by Lemma 1  $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l]\sigma'$  and  $\ell_{\Delta,x}[l'\sigma] = \ell_{\Delta,x}[l']\sigma'$ , we get  $\ell_{\Delta,x}[l]\sigma' = \ell_{\Delta,x}[l']\sigma'$ .

So by applying  $\sigma'$  to  $(*)$  and  $(\circ)$  (note that  $l_\Gamma = l$  and  $l'_\Gamma = l'$  as they are  $\Gamma$ -colored), we can perform a resolution step on  $\ell_{\Delta,x}[l]\sigma'$  and get

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'.$$

and consequently  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee \text{PI}(C_2) \vee D_\Gamma \vee E_\Gamma]\sigma'$ .

So by Lemma 1,

$$\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2) \vee D_\Gamma \vee E_\Gamma)\sigma].$$

2.  $l$  is  $\Delta$ -colored. Then  $\text{PI}(C) = (\text{PI}(C_1) \wedge \text{PI}(C_2))\sigma$ .

We show that  $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \wedge \text{PI}(C_2))\sigma \vee (D_\Gamma \vee E_\Gamma)\sigma]$

which by Lemma 2 is equivalent to

$$\Gamma \models (\ell_{\Delta,x}[\text{PI}(C_1)\sigma] \wedge \ell_{\Delta,x}[\text{PI}(C_2)\sigma]) \vee \ell_{\Delta,x}[D_\Gamma\sigma] \vee \ell_{\Delta,x}[E_\Gamma\sigma]$$

and by Lemma 1 is equivalent to

$$\Gamma \stackrel{(*)}{\models} (\ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma') \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$$

As  $l$  and  $l'$  are  $\Delta$ -colored, we can simplify  $(*)$  and  $(\circ)$  as follows and apply  $\sigma'$ :

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \text{ and } \Gamma \models \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$$

These clearly imply  $(*)$ .

3.  $l$  is grey. Then  $\text{PI}(C) = [(l \wedge \text{PI}(C_2)) \vee (\neg l' \wedge \text{PI}(C_2))]\sigma$ .

We show that  $\Gamma \models \ell_{\Delta,x}[(l \wedge \text{PI}(C_2)) \vee (\neg l' \wedge \text{PI}(C_2)) \vee D_\Gamma \vee E_\Gamma]\sigma$ , which by Lemma 2 and Lemma 1 is equivalent to

$$\Gamma \models (\ell_{\Delta,x}[l]\sigma' \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma') \vee (\neg \ell_{\Delta,x}[l']\sigma' \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma') \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'.$$

Suppose for a model  $M$  of  $\Gamma$  that  $M \not\models \ell_{\Delta,x}[D_\Gamma]\sigma'$  and  $M \not\models \ell_{\Delta,x}[E_\Gamma]\sigma'$  as otherwise we would be done. But then by  $(*)$  and  $(\circ)$ ,  $M \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[l]\sigma'$  and  $M \models \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \vee \neg \ell_{\Delta,x}[l']\sigma'$ .

As observed in case 1,  $\ell_{\Delta,x}[l]\sigma' = \ell_{\Delta,x}[l']\sigma'$ . By a case distinction on the truth value of  $\ell_{\Delta,x}[l]\sigma'$ , we obtain the result.

Factorisation.

$$\frac{C_1 : l \vee l' \vee D}{C : (l \vee D)\sigma} \quad \sigma = \text{mgu}(l, l')$$

Then  $\text{PI}(C) = \text{PI}(C_1)\sigma$ .

The induction hypothesis gives that  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee l \vee l' \vee D]$ . Let  $\sigma'$  be as in Lemma 1.

Then  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee l \vee l' \vee D]\sigma'$  and by Lemma 1,  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)\sigma \vee l\sigma \vee l'\sigma \vee D\sigma]$ .

By Lemma 2,  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)\sigma] \vee \ell_{\Delta,x}[l\sigma] \vee \ell_{\Delta,x}[l'\sigma] \vee \ell_{\Delta,x}[D\sigma]$ .

As  $\sigma = \text{mgu}(l, l')$ ,  $l\sigma$  and  $l'\sigma$  are syntactically equal, hence  $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l'\sigma]$ .

But then we can apply a factorisation step and get  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)\sigma] \vee \ell_{\Delta,x}[l\sigma] \vee \ell_{\Delta,x}[D\sigma]$  and by Lemma 1 and Lemma 2,  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)\sigma \vee l\sigma \vee D\sigma]$ .

Paramodulation.

$$\frac{C_1 : D \vee s = t \quad C_2 : E[r]_p}{C : (D \vee E[t]_p)\sigma} \quad \sigma = \text{mgu}(s, r)$$

By the induction hypothesis, we have:

$$\Gamma \models \ell_{\Delta, x}[\text{PI}(C_1) \vee (D \vee s = t)]_\Gamma$$

$$\Gamma \models \ell_{\Delta, x}[\text{PI}(C_2) \vee (E[r]_p)]_\Gamma$$

By Lemma 1 and Lemma 2, we get that:

$$\stackrel{(\circ)}{\Gamma \models \ell_{\Delta, x}[\text{PI}(C_1)] \vee \ell_{\Delta, x}[D_\Gamma] \vee \ell_{\Delta, x}[s] = \ell_{\Delta, x}[t]}$$

$$\stackrel{(*)}{\Gamma \models \ell_{\Delta, x}[\text{PI}(C_2)] \vee \ell_{\Delta, x}[(E[r]_p)]_\Gamma}$$

We distinguish two cases:

1. Suppose  $s$  does not occur in a maximal  $\Delta$ -term  $h[s]$  in  $E[s]_p$

We show that  $\Gamma \models \ell_{\Delta, x}[(s = t \wedge \text{PI}(C_2)) \vee (s \neq t \wedge \text{PI}(C_1))]_\sigma \vee ((D \vee E[t]_p)_\Gamma)_\sigma$ , which subsumes the cases 2 and 3 of the definition of PI for paramodulation. By Lemma 2, we can pull the liftings inwards and by Lemma 1, we can commute substitution and lifting by employing  $\sigma'$  to arrive at

$$\Gamma \models ((\ell_{\Delta, x}[s]\sigma') = (\ell_{\Delta, x}[t]\sigma') \wedge \ell_{\Delta, x}[\text{PI}(C_2)]\sigma') \vee ((\ell_{\Delta, x}[s]\sigma') \neq (\ell_{\Delta, x}[t]\sigma') \wedge \ell_{\Delta, x}[\text{PI}(C_1)]\sigma') \vee (\ell_{\Delta, x}[D_\Gamma]\sigma' \vee \ell_{\Delta, x}[(E[t]_p)_\Gamma]\sigma')$$

Let  $M$  be a model of  $\Gamma$ . Let  $M \not\models \ell_{\Delta, x}[D_\Gamma]\sigma' \vee \ell_{\Delta, x}[(E[t]_p)_\Gamma]\sigma'$  as otherwise we would be done. We show that depending on the truth value of  $(\ell_{\Delta, x}[s]) = (\ell_{\Delta, x}[t])$  in  $M$ , either the first or second conjunct of the above formula holds.

Suppose that  $M \models (\ell_{\Delta, x}[s]) \neq (\ell_{\Delta, x}[t])$ . Then by  $(\circ)$ ,  $M \models \ell_{\Delta, x}[\text{PI}(C_1)]$  and hence  $M \models \ell_{\Delta, x}[\text{PI}(C_1)]\sigma'$ .

On the other hand, suppose that  $M \models (\ell_{\Delta, x}[s]) = (\ell_{\Delta, x}[t])$ . Then by Lemma 4, as  $s$  at  $p$  in  $E[s]_p$  does not occur in a maximal  $\Delta$ -term,  $M \not\models \ell_{\Delta, x}[E[s]_p]$ .

Hence also  $M \not\models \ell_{\Delta, x}[E[s]_p]\sigma'$  and by Lemma 1,  $M \not\models \ell_{\Delta, x}[(E[s]_p)\sigma]$ .

Due to  $\sigma = \text{mgu}(s, r)$ , both  $s\sigma$  and  $r\sigma$  are syntactically equal. Suppose they are both not  $\Delta$ -colored. Then the lifting does not affect them and  $\ell_{\Delta, x}[(E[s]_p)\sigma] = \ell_{\Delta, x}[(E[r]_p)\sigma]$ . Otherwise the lifting will replace them with the same variable and we as well get that  $\ell_{\Delta, x}[(E[s]_p)\sigma] = \ell_{\Delta, x}[(E[r]_p)\sigma]$ .

By Lemma 1,  $\ell_{\Delta, x}[(E[s]_p)]\sigma' = \ell_{\Delta, x}[(E[r]_p)]\sigma'$ , so from  $M \not\models \ell_{\Delta, x}[E[s]_p]\sigma'$ , it follows that  $M \not\models \ell_{\Delta, x}[(E[r]_p)]\sigma'$

Then by  $(*)$ , we arrive at  $M \models \ell_{\Delta, x}[\text{PI}(C_2)]\sigma'$

2. Otherwise  $s$  occurs in a maximal  $\Delta$ -term  $h[s]_q$  in  $E[s]_p$ .

Then a similar line of argument can be employed, with the difference that the application of Lemma 4 yields the extra case that  $M \models s = t \wedge (\ell_{\Delta, x}[h[s]]) \neq (\ell_{\Delta, x}[h[t]])$ .

Hence the following holds:

$$\Gamma \models \left( (\ell_{\Delta,x}[s]\sigma') = (\ell_{\Delta,x}[t]\sigma') \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \right) \vee \left( (\ell_{\Delta,x}[s]\sigma') \neq (\ell_{\Delta,x}[t]\sigma') \wedge \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \right) \vee \left( (\ell_{\Delta,x}[s]\sigma') = (\ell_{\Delta,x}[t]\sigma') \wedge (\ell_{\Delta,x}[h[s]_q]) \neq (\ell_{\Delta,x}[h[t]_q]) \right) \vee \left( \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[(E[t]_p)_\Gamma]\sigma' \right)$$

□

Then the following from the thesis (also same in Huang) seem to go through:

Lemma 4.10: swap  $\Gamma$  and  $\Delta$  and obtain logical negation as interpolant

Corollary 4.11:  $\Delta \models \ell_{\Gamma,y}[\neg \text{PI}(C) \vee C]$

Lemma 4.12: not important if lifting delta or gamma terms first

Thm 4.13: ordering