Example 101: canonical examples for overbinding w.r.t. order

Here, the second lower prooftree shows what huang would produce.

Ex 101a

$$\dfrac{\dfrac{\overset{\Sigma}{P(u,f(u))\vee Q(u)} \qquad \overset{\Pi}{\neg Q(a)}}{P(a,f(a))}\;u\mapsto a \qquad \overset{\Pi}{\neg P(x,y)}}{\square}\;x\mapsto a, y\mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{Q(a)}\;u\mapsto a \qquad \top}{P(a,f(a))\vee Q(a)}\;x\mapsto a, y\mapsto f(a) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 Q(x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}$$

Direct overbinding would not work without merging same variables!: $\forall x_1\forall x_2\exists x_3(P(x_2,x_3)\vee Q(x_1))$
counterexample: $Q\sim\{0\}, P\sim\{(1,0)\}$
Direct overbinding would work when considering original dependencies as highlighted above
arrow lemma:

$$\dfrac{\dfrac{\Gamma\models\overset{\Sigma}{\exists y_1(P(u,y_1)\vee Q(u)\vee\bot)} \qquad \Gamma\models\overset{\Pi}{\neg Q(x_1)\vee\top}}{\Gamma\models\exists y_1(P(x_1,y_1)\vee Q(x_1))}\;u\mapsto a \qquad \Gamma\models\overset{\Pi}{\neg P(x,y)\vee\top}}{\Gamma\models(\forall x_1)\exists y_1(Q(x_1)\vee P(x_1,y_1))}\;x\mapsto a, y\mapsto f(a)$$

Ex 101b – other resolution order

$$\dfrac{\dfrac{\overset{\Sigma}{P(u,f(u))\vee Q(u)} \qquad \overset{\Pi}{\neg P(x,y)}}{Q(u))}\;y\mapsto f(u), x\mapsto u \qquad \overset{\Pi}{\neg Q(a)}}{\square}\;u\mapsto a$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(u,f(u))}\;x\mapsto f(u), x\mapsto u \qquad \top}{P(a,f(a))\vee Q(a)}\;u\mapsto a \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(u,x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}\;u\mapsto a$$

Ex 101c – Π and Σ swapped

$$\dfrac{\dfrac{\overset{\Pi}{P(u,f(u))\vee Q(u)} \qquad \overset{\Sigma}{\neg P(x,y)}}{Q(u))}\;y\mapsto f(u), x\mapsto u \qquad \overset{\Sigma}{\neg Q(a)}}{\square}\;u\mapsto a$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg P(u,f(u))}\;x\mapsto f(u), x\mapsto u \qquad \bot}{\neg P(a,f(a))\wedge\neg Q(a)}\;u\mapsto a \qquad\qquad \dfrac{\dfrac{\top \qquad \bot}{\forall x_2\neg P(u,x_2)} \qquad \bot}{\exists x_1\forall x_2(\neg P(x_1,x_2)\wedge\neg Q(x_1))}$$

arrow lemma:

$$\dfrac{\Gamma \models \overset{\Pi}{P(u,x_1)} \lor Q(u) \lor \top \qquad \Gamma \models \neg \overset{\Sigma}{P(x,y)} \lor \bot}{\Big(Q(u) \mid (\neg P(x,x_1) \land \top) \lor (P(u,/f(u)) \land \bot)\Big)\sigma}\; y \mapsto f(u), x \mapsto u$$

$$Q(u) \mid (\neg P(u,x_1) \land \top) \lor (P(u,f(u)) \land \bot)$$

employ σ' !?!!?!?!?!??!?????!!!

$$\Gamma \models Q(u) \mid \neg P(u,x_1)$$

$$\dfrac{\Delta \models Q(u) \mid \exists x_1 P(u,x_1) \qquad \Gamma \models \exists y_1\, \neg \overset{\Sigma}{Q(y_1)}}{\text{both } u\text{'s on LHS need to become } a \text{ and then } y_1}\; u \mapsto a$$

$$\Gamma \models (\forall x_1)\exists y_1(\neg P(y_1,x_1) \lor \neg Q(y_1))$$

$$\Delta \models (\exists x_1)\forall y_1(P(y_1,x_1) \land Q(y_1))$$

Ex 101d – Π and Σ swapped, other resolution order

$$\dfrac{\dfrac{\overset{\Pi}{P(u,f(u))} \lor Q(u) \qquad \neg \overset{\Sigma}{Q(a)}}{P(a,f(a))}\; u \mapsto a \qquad \neg \overset{\Sigma}{P(x,y)}}{\square}\; x \mapsto a, y \mapsto f(a)$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg Q(a)}\; y \mapsto a \qquad \bot}{\neg Q(a) \land \neg P(a,f(a))}\; x \mapsto a, y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\top \qquad \bot}{\exists x_1 \neg Q(x_1)} \qquad \bot}{\exists x_1 \forall x_2(\neg P(x_1,x_2) \land \neg Q(x_1))}$$

102 – similar to 101 but with intra-clause-set inferences

Ex 102a

$$\dfrac{\dfrac{P(f({\color{red}x})) \lor \overset{\Sigma}{Q(f({\color{red}x}),z)} \qquad \neg \overset{\Pi}{P(y)}}{Q(f(x),z)} \qquad \dfrac{\neg \overset{\Sigma}{Q(x_1,{\color{blue}y})} \lor R({\color{blue}y}) \qquad \neg \overset{\Pi}{R(g(z_1))}}{\neg Q(x_1,g(z_1))}\; y \mapsto g(z_1), x_1 \mapsto f(x)}{\square}\; x_1 \mapsto f(x), z \mapsto g(z_1)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(g(z_1))}\; y \mapsto g(z_1), x_1 \mapsto f(x)}{P(f(x)) \lor R(g(z_1))}\; x_1 \mapsto f(x), z \mapsto g(z_1) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}}{\exists x_1 \forall x_2 (P(x_1) \lor R(x_2)) \quad \text{(order irrelevant!)}}$$

combined:

$$\dfrac{\dfrac{\bot \mid P(x_1) \lor Q(x_1,z) \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}}{\exists x_1 \forall x_2 (P(x_1) \lor R(x_2))}$$

Ex 102b

$$\dfrac{\dfrac{\overset{\Sigma}{P(f({\color{red}x})) \lor Q(f({\color{red}x}), z)} \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x), z)} \qquad \dfrac{\overset{\Sigma}{\neg Q(f({\color{blue}y}), z_1) \lor R({\color{blue}y})} \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a), z_1)}\; y \mapsto a}{\Box}\; x \mapsto a, z \mapsto z_1$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\; y \mapsto a}{P(f(a)) \lor R(a)}\; x \mapsto a, z \mapsto z_1 \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\; y \mapsto a}{\forall x_2 \exists x_1 (P(x_1) \lor R(x_2))}\; x \mapsto a, z \mapsto z_1$$

direct:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)}\; x_1 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\; x_2 \sim a}{\exists x_1 \forall x_2 (P(x_1) \lor R(x_2))}\; \text{order irrelevant!}$$

Ex 102b' with Q grey

$$\dfrac{\dfrac{\overset{\Sigma}{P(f(x)) \lor Q(f(x), z)} \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x), z)} \qquad \dfrac{\overset{\Sigma}{\neg Q(f(y), z_1) \lor R(y)} \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a), z_1)}\; y \mapsto a}{\square}\; x \mapsto a, z_1 \mapsto z$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\; y \mapsto a}{(\neg Q(f(a), z) \land P(f(a))) \lor (Q(f(a), z) \land R(a))}\; x \mapsto a, z_1 \mapsto z$$

arrow lemma: (change of specification: $P(g(z_3))$ in clause in Σ instead of $P(f(x))$

$$\dfrac{\begin{array}{c}\dfrac{\overset{\Sigma}{\exists x_1(P(x_4) \lor Q(x_1, z))} \qquad \overset{\Pi}{\neg P(y)}}{\exists x_1(P(g(z_3)) \land \bot) \lor (P(x_4) \land \top) \mid Q(x_1, z)} \\ \exists x_1(P(x_4) \mid Q(x_1, z))\end{array} \qquad \begin{array}{c}\dfrac{\overset{\Sigma}{\exists x_2(\neg Q(x_2, z_1) \lor R(y))} \qquad \forall x_3\, \overset{\Pi}{\neg R(x_3)}}{\overset{\forall x_3}{\exists x_2}\, ((\neg R(x_3) \land \bot) \lor (R(a) \land \top) \mid \neg Q(x_2, z_1))}\; y \mapsto a \\ \overset{\forall x_3}{\exists x_2}\, ((\neg R(x_3) \land \bot) \lor (R(x_3) \land \top) \mid \neg Q(x_2, z_1))\end{array}}{\overset{\forall x_1}{\overset{\forall x_3}{\exists x_2}}\, \Big((\neg Q(x_2, z_1) \land P(x_4)) \lor (Q(x_1, z) \land R(x_3))\Big)}\; x \mapsto a, z_1 \mapsto z$$

arrow order: $x_3 < x_2$, x_2 same-block-as x_4: $\forall x_3 \exists x_2 \exists x_4 \forall x_1 \Big((\neg Q(x_2, z_1) \land \neg P(x_4)) \lor (Q(x_1, z) \land R(x_3))\Big)$

$\rightarrow$ bad example, plus some errors still in there

Huang:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)} \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\; y \mapsto a}{\forall x_1 \exists x_2 (\neg Q(x_2, z) \land P(x_2)) \lor (Q(x_2, z) \land R(x_1))}\; x \mapsto a, z_1 \mapsto z$$

direct:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)}\; x_2 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\; x_1 \sim a}{\forall x_1 \exists x_2 \exists x_3 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}\; x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3}{\text{OR:}\;\; \exists x_2 \forall x_1 \exists x_3 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}}{\text{OR:}\;\; \exists x_1 \exists x_3 \forall x_2 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}$$

Huang-Version passt weil man für existenzielle quantoren eh gleichen term einsetzt

direct w mixed, slightly different:

$$\dfrac{\dfrac{\dfrac{\bot \mid P(f(x)) \lor Q(x, z) \qquad \top \mid \neg P(y)}{\exists x_2 P(x_2) \mid Q(x, z)}\; x_2 \sim f(x) \qquad \dfrac{\bot \mid \overset{\Sigma}{\neg Q(f(y), z_1) \lor R(y)} \qquad \top \mid \overset{\Pi}{\neg R(a)}}{\forall x_1 R(x_1) \mid \neg Q(f(a), z_1)}\; x_1 \sim a}{\forall x_1 \exists x_3 \exists x_2 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}\; x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3}{(\neg Q(f(a), z) \land P(f(f(a)))) \lor (Q(f(a), z) \land R(a))}$$

last dependency not crucial because other arrow is a Σ-arrow as well, but just changing it to Π (and changing f for g should produce a quantifier alternation)

Example 103: variables in interpolant get unified and order might have to be changed

103a: length changes but old version is still valid, despite huang's algorithm doing a sort

$$\dfrac{\dfrac{\dfrac{Q(f(x)) \lor \overset{\Sigma}{P}(y) \lor R(x) \qquad \overset{\Pi}{\neg Q(y_1)}}{P(y) \lor R(x)}\,y_1 \mapsto f(x) \qquad \overset{\Pi}{\neg P(h(g(a)))}}{R(x)}\,y \mapsto h(g(a)) \qquad \overset{\Pi}{\neg R(g(g(a)))}}{\square}\,x \mapsto g(g(a))$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{Q(f(x))}\,y_1 \mapsto f(x) \qquad \top}{Q(f(x)) \lor P(h(g(a)))}\,y \mapsto h(g(a)) \qquad \top}{Q(f(g(g(a)))) \lor P(h(g(a))) \lor R(g(g(a)))}\,x \mapsto g(g(a)) \qquad\qquad \dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_1 \forall x_2 (Q(x_1) \lor P(x_2))} \qquad \top}{X}$$

X:

Huang's algo gives:

$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

Direct overbinding gives: $x_3 < x_1$, rest arbitrary, hence:

$\forall x_3 \exists x_1 \forall x_2 (Q(x_1) \lor P(x_2) \lor R(x_3))$ <- this you do not get with huang

$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

$\forall x_3 \forall x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

103b: length changes "uniformly"

$$\dfrac{\dfrac{\dfrac{Q(f(f(x))) \lor \overset{\Sigma}{P}(f(x)) \lor R(x) \qquad \overset{\Pi}{\neg Q(y_1)}}{P(f(x)) \lor R(x)}\,y_1 \mapsto f(f(x)) \qquad \overset{\Pi}{\neg P(y_2)}}{R(x)}\,y_2 \mapsto f(x) \qquad \overset{\Pi}{\neg R(g(a))}}{\square}\,x \mapsto g(a)$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{Q(f(f(x)))}\,y_1 \mapsto f(f(x)) \qquad \top}{Q(f(f(x))) \lor P(f(x))}\,y_2 \mapsto f(x) \qquad \top}{Q(f(f(g(a)))) \lor P(f(g(a))) \lor R(g(a))}\,x \mapsto g(a) \qquad\qquad \dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_2 \exists x_1 (Q(x_1) \lor P(x_2))} \qquad \top}{\forall x_3 \exists x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))}$$

Huang and direct overbinding somewhat coincide as $x_2 < x_1$ in both cases, and $x_3 < x_1$ and $x_3 < x_2$

new algo:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \mid Q(x_1) \lor P(x_2) \lor R(x) \qquad \top \mid \neg Q(y_1)}{Q(x_1) \mid P(x_2) \lor R(x)}\,y_1 \mapsto f(f(x)) \qquad \top \mid \neg P(y_2)}{Q(x_1) \lor P(x_2) \mid R(x)}\,y_2 \mapsto f(x) \qquad \top \mid R(x_3)}{Q(x_1) \lor P(x_2) \lor R(x_3)}\,x \mapsto g(a)}{Q(f(f(g(a)))) \lor P(f(g(a))) \lor R(g(a))}$$

NB: in the last line, the terms corresponding to x_1 and x_2 change, but the interpolant stays the same

103c: Failed attempt: different variables, accidentally the same terms appear but no logical connection

$$\dfrac{\overset{\Sigma}{P(a,x)} \qquad \dfrac{\overset{\Sigma}{\neg Q(a)} \qquad \overset{\Pi}{\neg P(y, f(z)) \vee Q(z)}}{\neg P(y,f(a))}\; z\mapsto a}{\square}\; y\mapsto a, x\mapsto f(a)$$

Huang:

$$\dfrac{\perp \qquad \dfrac{\perp \qquad \top}{\neg Q(a)}\; z\mapsto a}{P(a,f(a))\wedge \neg Q(a)}\; y\mapsto a, x\mapsto f(a) \qquad\qquad \dfrac{\perp \qquad \dfrac{\perp\qquad\top}{\exists x_1 \neg Q(x_1)}}{\exists x_1\forall x_2(P(x_1,x_2)\wedge\neg Q(x_1))}$$

order required for Π

direct:

$$\dfrac{\dfrac{\perp \qquad \dfrac{\perp\qquad\top}{\exists x_1\neg Q(x_1)}\; x_1\sim a}{\exists x_1\exists x_2\forall x_3(P(x_2,x_3)\wedge\neg Q(x_1))}\; x_2\sim a, x_3\sim f(a); x_1<x_3}{\text{OR: } \exists x_1\forall x_3\exists x_2(P(x_2,x_3)\wedge\neg Q(x_1))}$$

invariant:

$$\dfrac{\dfrac{\exists x_2(P(x_2,x)\vee\perp) \qquad \dfrac{\exists x_1(Q(x_1)\vee\perp) \qquad \forall x_3((\neg P(y,x_3)\vee Q(z))\vee\top)}{\exists x_1\forall x_3\neg P(y,x_3)\vee\neg Q(x_1)}\; x_1\sim a}{\exists x_1\exists x_2\forall x_3(P(x_2,x_3)\wedge\neg Q(x_1))}\; x_2\sim a, x_3\sim f(a); x_1<x_3}{\text{OR: } \exists x_1\forall x_3\exists x_2(P(x_2,x_3)\wedge\neg Q(x_1))}$$

invariant in other resolution order

$$\dfrac{\dfrac{\perp \qquad \dfrac{\perp\qquad\top}{Q(z)\vee\exists x_2\forall x_3 P(x_2,x_3)}\; x_2\sim a, x_3\sim f(z)}{\exists x_1\exists x_2\forall x_3(P(x_2,x_3)\wedge\neg Q(x_1))}\; x_1\sim a; x_1<x_3}{\text{OR: } \exists x_1\forall x_3\exists x_2(P(x_2,x_3)\wedge\neg Q(x_1))}$$

invariant if Σ and Π swapped:

$$\dfrac{\dfrac{\perp \qquad \dfrac{\top\qquad\perp}{\neg P(y,f(x_1))\vee\forall x_1 Q(x_1)}\; x_1\sim a}{\forall x_1\forall x_2\exists x_3(\neg P(x_2,x_3)\vee Q(x_1))}\; x_2\sim a, x_3\sim f(a); x_1<x_3}{\text{OR: } \forall x_1\exists x_3\forall x_2(\neg P(x_2,x_3)\vee Q(x_1))}$$

SECOND ATTEMPT:

$$\dfrac{\overset{\Sigma}{P(a)} \qquad \dfrac{\overset{\Sigma}{Q(z)}\qquad \dfrac{\overset{\Sigma}{\neg S(a)}\qquad\overset{\Pi}{\neg P(y)\vee\neg Q(f(x))\vee S(x)}}{\neg P(y)\vee\neg Q(f(a))}\; x\mapsto a}{\neg P(y)}\; z\mapsto f(a)}{\square}\; y\mapsto a$$

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg S(a)}\, x \mapsto a}{\neg S(a) \wedge Q(f(a))}\, z \mapsto f(a)}{P(a) \wedge \neg S(a) \wedge Q(f(a))}\, y \mapsto a$$

Huang:

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 (\neg S(x_1) \wedge Q(x_2))}}{\exists x_1 \forall x_2 (P(x_1) \wedge \neg S(x_1) \wedge Q(x_2))}$$

$\forall x_1 \exists x_2 (\neg P(x_1) \vee S(x_1) \vee \neg Q(x_2))$

similar fail

$\Rightarrow$ anytime there is $P(a, f(a))$, either they have a dependency or they are not both differently colored (grey is uncolored)

for the record, direct method anyway:

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}\, x \sim a}{\exists x_1 \forall x_2 \neg S(x_1) \wedge Q(x_2)}\, z \sim f(a); x_1 < x_2}{\exists x_1 \forall x_2 \exists x_3 P(x_3) \wedge \neg S(x_1) \wedge Q(x_2)}\, x_3 \sim a; x_3 \text{ need not be merged w } x_1$$

Example: ordering on both ancestors where the merge forces a new ordering

202a – canonical

$$\dfrac{\dfrac{\overset{\Sigma}{P(a,x_{564})\vee R(y)} \qquad \overset{\Pi}{\neg P(x,fx)}}{R(y)}\;\begin{matrix}x_1\mapsto fa\\ x\mapsto a\end{matrix} \qquad \dfrac{\overset{\Sigma}{Q(x_2,gx_2)\vee\neg R(u)} \qquad \dfrac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg Q(fz,x_3)\vee S(z)}}{\neg Q(fa,x_3)}\;z\mapsto a}{\neg R(u)}\;\begin{matrix}x_2\mapsto fa,\\ x_3\mapsto gfa\end{matrix}}{\Box}$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,f(a))}\;\begin{matrix}x_1\mapsto f(a)\\ x\mapsto a\end{matrix} \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg S(a)}\;z\mapsto a}{Q(f(a),g(f(a)))\wedge\neg S(a)}\;\begin{matrix}x_2\mapsto f(a),\\ x_3\mapsto g(f(a))\end{matrix}}{P(a,f(a))\vee(Q(f(a),g(f(a)))\wedge\neg S(a))}$$

Huang

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1\forall x_2 P(x_1,x_2))} \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1\neg S(x_1)}}{\exists x_1\forall x_2\exists x_3 Q(x_2,x_3)\wedge\neg S(x_1)}}{\exists x_1\forall x_2\exists x_3 P(x_1,x_2)\vee(Q(x_2,x_3))\wedge\neg S(x_1))}$$

direct:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1\forall x_2 P(x_1,x_2))}\;\begin{matrix}x_1\sim a, x_2\sim fa\\ x_1<x_2\end{matrix} \qquad \begin{matrix}x_3\sim a, x_4\sim fa, x_5\sim gfa)\\ x_3<x_4, x_4<x_5\end{matrix}\; \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_3\neg S(x_3)}\;x_3\sim a}{\exists x_3\forall x_4\exists x_5 Q(x_4,x_5)\wedge\neg S(x_3)}}{\exists x_1\forall x_2\exists x_5 P(x_1,x_2)\vee(Q(x_2,x_5)\wedge\neg S(x_5))}\;\begin{matrix}x_3\mapsto x_1, x_4\mapsto x_2\\ x_1<x_2, x_2<x_5\end{matrix}$$

without merge in end: $(x_1<x_2, x_3<x_4, x_4<x_5)$

$\underline{\exists x_1\forall x_2}\;\underline{\exists x_3\forall x_4\exists x_5}\Big(P(x_1,x_2)\vee(Q(x_4,x_5)\wedge\neg S(x_3))\Big)$

$\underline{\exists x_3\forall x_4\exists x_5}\;\underline{\exists x_1\forall x_2}\Big(P(x_1,x_2)\vee(Q(x_4,x_5)\wedge\neg S(x_3))\Big)$

(also interwoven ones appear to work)

combined presentation of 202a:

$$\dfrac{\dfrac{\bot\mid P(a,x_1)\vee R(y) \qquad \top\mid\neg P(x,f(x))}{P(a,f(a))\mid R(y)}\;\begin{matrix}x_1\mapsto f(a)\\ x\mapsto a\end{matrix} \qquad \dfrac{\bot\mid Q(x_2,g(x_2))\vee\neg R(u) \qquad \dfrac{\bot\mid\neg S(a) \qquad \top\mid\neg Q(f(z),x_3)\vee S(z)}{\neg S(a)\mid\neg Q(f(a),x_3)}\;z\mapsto a}{Q(f(a),g(f(a)))\wedge\neg S(a)\mid\neg R(u)}\;\begin{matrix}x_2\mapsto f(a),\\ x_3\mapsto g(f(a))\end{matrix}}{P(a,f(a))\vee(Q(f(a),g(f(a)))\wedge\neg S(a))\mid\Box}$$

combined presentation ground:

$$\dfrac{\dfrac{\bot\mid P(a,f(a))\vee R(y) \qquad \top\mid\neg P(a,f(a))}{(P(a,f(a))\wedge\top)\vee(\neg P(a,f(a))\wedge\bot)\mid R(y)} \qquad \dfrac{\bot\mid Q(f(a),g(f(a)))\vee\neg R(u) \qquad \dfrac{\bot\mid\neg S(a) \qquad \top\mid\neg Q(f(a),g(f(a)))\vee S(a)}{\neg S(a)\mid\neg Q(f(a),g(f(a)))}}{Q(f(a),g(f(a)))\wedge\neg S(a)\mid\neg R(u)}}{P(a,f(a))\vee(Q(f(a),g(f(a)))\wedge\neg S(a))\mid\Box}$$

combined presentation ground with direct method but only Δ-terms removed :

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \vee R(y) \qquad \top \mid \neg P(a,f(a))}{(P(a,x_2) \wedge \top) \vee (\neg P(a,x_2) \wedge \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(x_4,g(x_4)) \wedge \neg S(a) \mid \neg R(u)}}{P(a,x_2) \vee (Q(x_4,g(x_4)) \wedge \neg S(a)) \mid \Box}$$

combined presentation ground with direct method:

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \vee R(y) \qquad \top \mid \neg P(a,f(a))}{\exists x_1 \forall x_2 (P(x_1,x_2) \wedge \top) \vee (\neg P(x_1,x_2) \wedge \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \vee S(a)}{\exists x_3 \neg S(x_3) \mid \neg Q(f(a),g(f(a)))}}{\exists x_3 \forall x_4 \exists x_5 (Q(x_4,x_5)) \wedge \neg S(x_3)) \mid \neg R(u)}}{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (P(x_1,x_2) \vee (Q(x_4,x_5)) \wedge \neg S(x_3)) \mid \Box}$$

203a – some alternations

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x)\vee \overset{\Sigma}{\neg P}(f(x)) \qquad P(z)\vee \overset{\Pi}{Q}(g(z))}{R(x)\vee Q(g(f(x)))}\; z\mapsto f(x) \qquad \neg Q(y)\vee \overset{\Sigma}{S}(h(y))}{R(x)\vee S(h(g(f(x))))}\; y\mapsto g(f(x))}{S(h(g(f(a))))}\; x\mapsto a}{\square}\; x_1\mapsto h(g(f(a)))$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\neg P(f(x))}\; z\mapsto f(x) \qquad \bot}{\neg Q(g(f(x)))\wedge\neg P(f(x))}\; y\mapsto g(f(x))}{\neg Q(g(f(a)))\wedge\neg P(f(a))\vee R(a)}\; x\mapsto a}{\neg Q(g(f(a)))\wedge\neg P(f(a))\vee R(a)\vee S(h(g(f(a))))}\; x_1\mapsto h(g(f(a)))$$

Huang:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1\neg P(x_1)} \qquad \bot}{\exists x_1\forall x_2(\neg Q(x_2)\wedge\neg P(x_1))}}{\forall x_0\exists x_1\forall x_2(\neg Q(x_2)\wedge\neg P(x_1)\vee R(x_0))}}{\forall x_0\exists x_1\forall x_2\exists x_3(\neg Q(x_2)\wedge\neg P(x_1)\vee R(x_0)\vee S(x_3))}$$

Direct:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1\neg P(x_1)}\; x_1\sim f(x) \qquad \bot}{\exists x_1\forall x_2(\neg Q(x_2)\wedge\neg P(f(x)))}\; x_2\sim g(f(x)); x_1<x_2}{\forall x_0\exists x_1\forall x_2(\neg Q(x_2)\wedge\neg P(x_1)\vee R(x_0))}\; x_0\sim a; x_0<x_1, x_0<x_2}{\forall x_0\exists x_1\forall x_2\exists x_3(\neg Q(x_2)\wedge\neg P(x_1)\vee R(x_0)\vee S(x_3))}\; x_3\sim h(g(f(a))); x_0<x_3, x_1<x_3, x_2<x_3$$

203b – many Σ-literals, coloring per occurrence

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x)\vee \overset{\Sigma}{\neg P}(f(x)) \qquad P(z)\vee \overset{\Sigma}{Q}(g(z))}{R(x)\vee Q(gfx)}\; z\mapsto fx \qquad \neg Q(y)\vee \overset{\Sigma}{S}(h(y))}{R(x)\vee S(hgfx)}\; y\mapsto gfx}{S(hgfa)}\; x\mapsto a}{\square}\; x_1\mapsto hgfa$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \bot}{\bot}\; z\mapsto fx \qquad \bot}{\bot}\; y\mapsto gfx}{R(a)}\; x\mapsto a}{S(hgfa)\vee R(a)}\; x_1\mapsto hgfa$$

$\rightarrow \forall x_1\exists x_2(R(x_1)\vee S(x_2))$

Example where variables are not the outermost symbol but order is still relevant

204a

$\Sigma = \{P(f(x), g(f(x)))\}$
$\Pi = \{P(f(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f(x_1), x_2)$

204b

$\Sigma = \{P(f^5(x), g(f(x)))\}$
$\Pi = \{P(f^5(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f^5(x_1), x_2)$

example with aufschaukelnde unification, such that direction of arrow isn't clear

205a

situation not critical here

$$\dfrac{\overset{\Pi}{P(ffy,gy)} \qquad \dfrac{\overset{\Sigma}{\neg P(x,y)\lor Q(x)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffz)\lor Rz}}{\neg R(a) \mid \neg Q(ffa)}\ z\mapsto a}{\neg R(a)\land Q(ffa) \mid \neg P(ffa,y)}\ x\mapsto ffa}{(\neg R(a)\land Q(ffa))\lor \neg P(ffa,ga)}\ y\mapsto a$$

direct

$$\dfrac{\overset{\Pi}{P(ffy,gy)} \qquad \dfrac{\overset{\Sigma}{\neg P(x,y)\lor Q(x)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffz)\lor Rz}}{\exists x_1 \neg R(x_1) \mid \neg Q(ffa)}\ z\mapsto a}{\exists x_1\forall x_2(\neg R(x_1)\land Q(x_2)) \mid \neg P(ffa,u)}\ x\mapsto ffa}{\exists x_1\forall x_2\forall x_3((\neg R(x_1)\land Q(x_2))\lor \neg P(x_2,x_3))}\ y\mapsto a, u\mapsto ga$$

ground:

$$\dfrac{\overset{\Pi}{P(ffa,ga)} \qquad \dfrac{\overset{\Sigma}{\neg P(ffa,y)\lor Q(ffa)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffa)\lor Ra}}{\neg R(a) \mid \neg Q(ffa)}}{\neg R(a)\land Q(ffa) \mid \neg P(ffa,a)}}{(\neg R(a)\land Q(ffa))\lor \neg P(ffa,ga)}$$

arrow lemma: $x_1 \sim ffy$, $x_2 \sim gy$, $y_3 \sim a$ $x_4 \sim ffz$, $x_5 \sim ffa$, $y_6 \sim a$,

$$\dfrac{\overset{\Sigma}{\neg P(x,y)\lor Q(x) | \bot} \qquad \dfrac{\overset{\Sigma}{\exists y_3(\neg R(y_3) | \bot)} \qquad \overset{\Pi}{\forall x_4(\neg Q(x_4)\lor Rz | \bot)}}{\exists y_3\forall x_4((\neg R(y_3)\land\top)\lor \underline{(R(a)\land\bot)} \mid \neg Q(x_4))}\ z\mapsto a}{\exists y_3\forall x_4((\neg Q(x_4\land\bot))\lor(Q(x)\sigma\land\neg R(y_3)) \mid \neg P(x,y)\sigma)}\ x\mapsto ffa$$

$$\exists y_3\forall x_4((Q(ffa)\land\neg R(y_3)) \mid \neg P(ffa,y)) \quad (*)$$

$$\dfrac{\overset{\Pi}{\forall x_1\forall x_4(P(x_1,x_2) | \top)} \qquad \exists y_3\forall x_4\forall x_5((Q(x_5)\land\neg R(y_3)) \mid \neg P(x_5,y))}{(\neg P(x_5,y)\land\top)\lor(P(x_1,x_2)\land(Q(x_5)\land\neg R(y_3)))}\ y\mapsto a$$

$$(\neg P(x_5,a)\land\top)\lor(P(x_1,x_2)\land(Q(x_5)\land\neg R(y_3)))$$

$$\forall x_1\forall x_2\exists y_3\forall x_4\forall x_5\forall x_6(\neg P(x_5,y_6)\land\top)\lor(P(x_1,x_2)\land(Q(x_5)\land\neg R(y_3)))$$

$(*)$ "luckily", same overbinding for ffa, so this works

dashed underline: problem, but does not cause issues here.

205b $\sim$ 205a, but simpler

Suppose P occurs somewhere in Σ (result not that optimal in this setting, but correct)
not nice for proving, $\neg R(a)$ is a nice interpolant already

$$\dfrac{\overset{\Pi}{P(ffy,gy)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg P(ffz,x)\lor Rz}}{\neg R(a) \mid \neg P(ffa,x)}\ z\mapsto a}{\neg R(a)\lor\neg P(ffa,ga) \mid \Box}\ x\mapsto ga, y\mapsto a$$

$$\dfrac{\overset{\Pi}{\top \mid P(ffy,gy)} \qquad \dfrac{\overset{\Sigma}{\bot \mid \neg R(a)} \qquad \overset{\Pi}{\top \mid \neg P(ffz,x)\lor Rz}}{\exists x_1\neg R(x_1) \mid \neg P(ffa,x)}}{\exists x_1\forall x_2\forall x_3\neg R(x_1)\lor\neg P(x_2,x_3) \mid \Box}$$

$$\dfrac{\forall x_7\forall x_8\Big(\top \mid \overset{\Pi}{P}(x_7,x_8)\Big) \qquad \dfrac{\exists x_1\Big(\bot \mid \neg \overset{\Sigma}{R}(x_1)\Big) \qquad \forall x_5\Big(\top \mid \neg P(x_5,\overset{\Pi}{x})\vee Rz\Big)}{\exists x_1\forall x_5\Big(\neg R(x_1)\mid \neg P(x_5,x)\Big)}}{\exists x_1\forall x_2\forall x_3\neg R(x_1)\vee\neg P(x_2,x_3)\mid \Box}$$

in au, x_2 and x_3 win since these are the “actual” terms

last step in slow motion:

$\zeta_1 = a$ $\quad \zeta_2 = f(f(a))$ $\quad \zeta_3 = g(a)$ $\quad \zeta_5 = f(f(z))$ $\quad \zeta_7 = f(f(y))$ $\quad \zeta_8 = g(y)$

$\mathrm{AI}_{\mathrm{mat}}(\cdot) =$
$= \ell\Big[\Big(\Big(P(x_7,x_8)\wedge\neg R(x_1)\Big)\vee\Big(\neg P(x_5,x)\wedge\top\Big)\Big)\sigma\Big]\tau$
$= \Big(P(x_7,x_8)\wedge\neg R(x_1)\Big)\tau\vee\Big(\neg P(x_5,x_3)\wedge\top\Big)\Big)\tau$
$\mathrm{au}(x_7,x_5) = \{x_7\mapsto x_2, x_5\mapsto x_2\}$
$\mathrm{au}(x_8,x_3) = \{x_8\mapsto x_3, x_8\mapsto x_3\}$
$\exists x_1 R(x_1)$
$\exists x_1\forall x_2\forall x_3(R(x_1)\vee\neg P(x_2,x_3))$

example to demonstrate that literals being resolved upon have to be overbound with the same variable

206a

$$\dfrac{\dfrac{\overset{\Pi}{R(f(x))} \qquad \overset{\Sigma}{\neg R(y) \vee P(y)}}{(\neg R(x_3) \wedge \top) \vee (R(x_3) \wedge \bot) \mid P(x_3)} \qquad \dfrac{\overset{\Pi}{\neg P(f(z)) \vee S(z)} \qquad \overset{\Sigma}{\neg S(a)}}{(\neg S(y_2) \wedge \top) \vee (S(y_2) \wedge \bot) \mid \neg P(x_4)}}{(\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)}$$

Gist of this example: $P(f(x))$ is lifted on the left, but $P(f(a))$ on the right. So it's $P(x_3)$ vs $P(x_4)$, but both of them have to have the same variable.

$R(x_3) \in \mathrm{AI}_{\mathrm{mat}}(C_7)$

$P(x_3) \in \mathrm{AI}_{\mathrm{cl}}(C_7)$

$P(x_4) \in \mathrm{AI}_{\mathrm{cl}}(C_8)$

$\Sigma \models (\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)$

$\Sigma \models (\forall x_3)\forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee P(x_3)\Big)$

$\Sigma \not\models (\neg P(1) \wedge \neg R(0)) \vee P(0)$ // if $P \sim \{1\}$ and $R \sim \{0\}$

we know that for original clauses l and l' of $P(x_4)$ and $P(x_3)$, $l\sigma = l'\sigma$

hence same color, and can use different var as same value works.

inductive hypothesis:

$\Gamma \models \top \vee R(x_3)$

$\Gamma \models \bot \vee \neg R(y) \vee P(y)$

$\Gamma \models (\neg R(x_3) \wedge \top) \vee (R(x_3) \wedge \bot) \vee P(x_3) \quad \equiv \quad \neg R(x_3) \vee P(x_3)$

$\Gamma \models \top \vee \neg P(x_4) \vee S(z)$

$\Gamma \models \bot \vee \neg S(a)$

$\Gamma \models (\neg S(a) \wedge \top) \vee (S(a) \wedge \bot) \vee \neg P(x_4) \quad \equiv \quad \neg S(a) \vee \neg P(x_4)$

$\Gamma \models (\neg P(x_3) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(a))$

206b

WRONG: if a variable x_3 occurs, it always refers to $f(x)$, so it is always substituted to a particular value and cannot become $f(a)$ and $f(b)$ in the same clause as just the unifier σ is used.

$$\dfrac{\dfrac{\overset{\Pi}{R(f(x), f(x))} \qquad \overset{\Sigma}{\neg R(y,u) \vee P(y,u)}}{(\neg R(x_3,x_3) \wedge \top) \vee (R(x_3,x_3) \wedge \bot) \mid P(x_3,x_3)} \qquad \dfrac{\overset{\Pi}{\neg P(f(z), f(v)) \vee S(z,v)} \qquad \overset{\Sigma}{\neg S(a,b)}}{(\neg S(y_2,y_6) \wedge \top) \vee (S(y_2,y_6) \wedge \bot) \mid \neg P(x_4,x_7)}}{(\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)}$$

problems due to x_j not referring to actual term

208a

WRONG: variable x is used in two clauses

$$
\dfrac{
\dfrac{
\dfrac{P(f(x)) \vee \overset{\Pi}{Q}(x) \vee R(u) \qquad \overset{\Sigma}{\neg Q}(a)}{P(f(a)) \vee R(u)}
\qquad
\dfrac{P(f(x)) \vee \overset{\Pi}{Q}(x) \vee \neg R(u) \qquad \overset{\Sigma}{\neg Q}(b)}{P(f(b)) \vee \neg R(u)}
}{P(f(a)) \vee P(f(b))}
\qquad
\dfrac{\neg P(f\overset{\Pi}{(x)}) \vee S(x) \qquad \overset{\Sigma}{\neg S}(a)}{\neg P(f(a))}
}{P(f(b))}
$$

$$
\dfrac{
\dfrac{
\dfrac{\top \mid P(x_1) \vee Q(x) \vee R(u) \qquad \bot \mid \neg Q(y_2)}{\neg Q(y_2) \mid P(x_1) \vee R(u)}
\qquad
\dfrac{\top \mid P(x_1) \vee Q(x) \vee R(u) \qquad \bot \mid \neg Q(y_4)}{\neg Q(y_4) \mid P(x_1) \vee R(u)}
}{\neg Q(y_2) \wedge \neg Q(y_4) \mid P(x_1) \vee P(x_1)}
\qquad
\dfrac{\top \mid \neg P\overset{\Pi}{(x_1)} \vee S(x) \qquad \top \mid \overset{\Sigma}{\neg S}(y_2)}{\neg S(y_2) \mid \neg P(x_1)}
}{(\neg P(x_5) \wedge \neg Q(y_2) \wedge \neg Q(y_4)) \quad \vee \quad (P(x_5) \wedge \neg S(y_2)) \quad \mid \quad P(x_5)}
$$

NB: as the x_1 in the literal is actually $f(a)$, this way, all x_1 become x_5, but the other one is supposed to stand for $f(b)$

ACTUALLY:

$$
\dfrac{
\dfrac{
\dfrac{P(f(x)) \vee \overset{\Pi}{Q}(x) \vee R(u) \qquad \overset{\Sigma}{\neg Q}(a)}{P(f(a)) \vee R(u)}
\qquad
\dfrac{P(f(x')) \vee \overset{\Pi}{Q}(x') \vee \neg R(u') \qquad \overset{\Sigma}{\neg Q}(b)}{P(f(b)) \vee \neg R(u')}
}{P(f(a)) \vee P(f(b))}
\qquad
\dfrac{\neg P(f\overset{\Pi}{(x'')}) \vee S(x'') \qquad \overset{\Sigma}{\neg S}(a)}{\neg P(f(a))}
}{P(f(b))}
$$

$$
\dfrac{
\dfrac{
\dfrac{\top \mid P(x_1) \vee Q(x) \vee R(u) \qquad \bot \mid \neg Q(y_2)}{\neg Q(y_2) \mid P(x_1) \vee R(u)}
\qquad
\dfrac{\top \mid P(x_2) \vee Q(x) \vee R(u) \qquad \bot \mid \neg Q(y_4)}{\neg Q(y_4) \mid P(x_2) \vee R(u)}
}{\neg Q(y_2) \wedge \neg Q(y_4) \mid P(x_1) \vee P(x_2)}
\qquad
\dfrac{\top \mid \neg P\overset{\Pi}{(x_3)} \vee S(x) \qquad \top \mid \overset{\Sigma}{\neg S}(y_2)}{\neg S(y_2) \mid \neg P(x_3)}
}{(\neg P(x_5) \wedge \neg Q(y_2) \wedge \neg Q(y_4)) \quad \vee \quad (P(x_5) \wedge \neg S(y_2)) \quad \mid \quad P(x_2)}
$$

NB: $\mathrm{au}(P(x_1), P(x_3)) = \{x_1 \mapsto x_5, x_3 \mapsto x_5\}$

Hence a term with a free variable in a clause can never be lifted by the same variable as a term in another clause.
If two terms in the same clause are lifted with a certain variable, they are bound together in the derivation anyway.

clause used multiple times

209a

$$
\dfrac{
  \dfrac{
    \overset{\Sigma}{P(a)}
    \qquad
    \dfrac{\overset{\Sigma}{\neg Q(a)} \qquad \overset{\Pi}{\neg P(x) \vee P(f(x)) \vee Q(y)}}{\neg P(x) \vee P(f(x))}
  }{
    \dfrac{P(f(a))}{P(f(f(a)))}
  }
  \qquad
  \overset{\Pi}{\neg P(f(f(z)))}
}{\Box}
$$

NB: we need to rename lifting variables, possibly rename all lifting variables which refer to a term which contains variables (an actual implementation might do this more efficiently, i.e. not always)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \bot \mid P(a)
      \qquad
      \dfrac{\bot \mid Q(a) \qquad \top \mid \neg P(x) \vee P(x_1) \vee Q(y)}{Q(a) \mid \neg P(x) \vee P(x_1)}
    }{P(a) \wedge Q(a) \mid P(x_1)}
    \qquad
    Q(a) \mid \neg P(x') \vee P(x'_1)
  }{(\neg P(x_1) \wedge P(a) \wedge Q(a)) \quad \vee \quad (P(x_1) \wedge Q(a)) \quad \mid \quad P(x'_1)}
  \qquad
  \top \mid \neg P(x_2)
}{\Big(\neg P(x_3) \wedge (\neg P(x_1) \wedge P(a) \wedge Q(a)) \quad \vee \quad (P(x_1) \wedge Q(a))\Big) \quad \vee \quad P(x_3) \mid \Box}
$$

NB: x_1 used to refer to $f(x)$, now: $f(a)$

$$\mathrm{au}(x'_1, x_2) = \{x'_1 \mapsto \ell[f(f(a))], x_2 \mapsto \ell[f(f(a))]\}$$

$\Big(\neg P(x_3) \wedge (\neg P(x_1) \wedge P(a) \wedge Q(a)) \quad \vee \quad (P(x_1) \wedge Q(a))\Big) \quad \vee \quad P(x_3)$

$\equiv \Big(Q(a) \wedge \Big((\neg P(x_1) \wedge P(a)) \quad \vee \quad P(x_1)\Big)\Big) \quad \vee \quad P(x_3)$

Σ✓

negated: $\Big(\neg Q(a) \vee \Big((P(x_1) \vee \neg P(a)) \quad \wedge \quad \neg P(x_1)\Big)\Big) \quad \wedge \quad \neg P(x_3)$

$\equiv \Big(\neg Q(a) \vee \Big(\neg P(a) \quad \wedge \quad \neg P(x_1)\Big)\Big) \quad \wedge \quad \neg P(x_3)$

Π✓

(none of the $P(f^n(x))$, $n \leq 2$, are allowed to be true in a model of Φ)

example with multicolored term without arrow

210a

$$\dfrac{\dfrac{P(f(x)) \overset{\Sigma}{\vee} Q(x) \qquad \neg Q(y) \overset{\Pi}{\vee} R(g(y))}{P(f(x)) \vee R(g(x))} \qquad \neg R(\overset{\Pi}{g}(a))}{P(f(a))}$$

AI$^{\Delta}$:

$$\dfrac{\dfrac{P(f(x)) \overset{\Sigma}{\vee} Q(x) \qquad \neg Q(y) \overset{\Pi}{\vee} R(x_2))}{P(f(x)) \vee R(x_2))} \qquad \neg \overset{\Pi}{R}(x_3)}{P(f(x_4))}$$

$$\dfrac{\dfrac{\exists x_1 \mid \bot \mid P(x_1) \overset{\Sigma}{\vee} Q(x) \qquad \forall x_2 \mid \top \mid \neg Q(y) \overset{\Pi}{\vee} R(x_2)}{\overset{\exists x_1}{\forall x_2} \mid Q(x) \mid P(x_1) \vee R(x_2)} \qquad \forall x_3 \mid \top \mid \neg \overset{\Pi}{R}(x_3)}{\forall x_4 \overset{\exists x_1}{\forall x_3} \mid R(x_3) \vee Q(x_4) \mid P(x_1)}$$

Gist of this example: a variable is contained once in a Γ- and once in a Δ-term in the same clause. Then we can unify basically any foreign term into the differently colored terms.
Here, the arrow actually exists since $Q(x)$ is in the interpolant.

210b – framework to get other (unrelated) terms into the interpolant

$$\dfrac{\dfrac{P(f(x)) \overset{\Sigma}{\vee} Q(x) \qquad \neg Q(y) \overset{\Pi}{\vee} R(g(y))}{P(f(x)) \vee R(g(x))} \qquad \dfrac{\neg R(g(z)) \overset{\Pi}{\vee} T(z) \qquad \dfrac{\neg S(z') \overset{\Sigma}{\vee} \neg T(z') \qquad \neg \overset{\Pi}{S}(a)}{\neg T(a)}}{\neg R(g(a))}}{P(f(a))}$$

AI$^{\Delta}$:

$$\dfrac{\dfrac{P(f(x)) \overset{\Sigma}{\vee} Q(x) \qquad \neg Q(y) \overset{\Pi}{\vee} R(x_2)}{P(f(x)) \vee R(x_2)} \qquad \dfrac{\neg R(x_5) \overset{\Pi}{\vee} T(z) \qquad \dfrac{\neg S(z') \overset{\Sigma}{\vee} \neg T(z') \qquad \neg \overset{\Pi}{S}(x_4)}{\neg T(x_4)}}{\neg R(x_5)}}{P(f(x_4))}$$

$$\dfrac{\dfrac{\bot \mid P(f(x)) \overset{\Sigma}{\vee} Q(x) \qquad \top \mid \neg Q(y) \overset{\Pi}{\vee} R(x_2)}{Q(x) \mid P(f(x)) \vee R(x_2)} \qquad \dfrac{\top \mid \neg R(x_5) \overset{\Pi}{\vee} T(z) \qquad \dfrac{\bot \mid \neg S(z') \overset{\Sigma}{\vee} \neg T(z') \qquad \top \mid \neg \overset{\Pi}{S}(x_4)}{\neg S(x_4) \mid \neg T(x_4)}}{\neg T(x_4) \vee \neg S(x_4) \mid \neg R(x_5)}}{\big(R(x_3) \wedge (\neg T(x_4) \vee \neg S(x_4))\big) \vee \big(\neg R(x_3) \wedge Q(x_4)\big) \quad \mid \quad P(f(x_4))}$$

again have arrow from $Q(x_4)$ to $P(f(x_4))$

210c – attempt to get it without any arrow

$$\dfrac{\dfrac{P(f(x)) \overset{\Sigma}{\vee} Q(x) \qquad \neg Q(y) \overset{\Sigma}{\vee} R(y)}{\bot \mid P(f(y)) \vee R(y)} \qquad \neg R(z) \overset{\Pi}{\vee} S(g(z)))}{R(y) \mid P(f(y)) \vee S(g(y))}$$

$\Rightarrow$ must port arrows on variable rename

? R possibly disappearing (Q disappearing) doesn't matter?)

slow motion:

$$\dfrac{P(f(x)) \overset{\Sigma}{\vee} Q(x) \qquad \neg Q(y) \overset{\Sigma}{\vee} R(y)}{\begin{array}{c}\sigma = \{x \mapsto y\} \\ \text{Merge arrows of } Q(x) \text{ and } Q(y) \\ \bot \mid P(f(y)) \vee R(y)\end{array}}$$

$f(x) \vee g(x)$ with f, g different colors

207a

ORIGINAL VERSION:

$$
\cfrac{\overset{\Pi}{\neg P(z)} \qquad \cfrac{\overset{\Pi}{\neg R(g(a))} \qquad \cfrac{\overset{\Sigma}{P(f(x)) \vee Q(x)} \qquad \overset{\Pi}{R(g(y)) \vee \neg Q(y)}}{\forall x_1 Q(x_1) \mid P(f(x)) \vee R(g(x))}\; y \mapsto x}{\forall x_1 Q(x_1) \mid P(f(a)) // R(g(a))}\; x \mapsto a}{\forall x_1 \exists x_2 \big(Q(x_1) \vee P(x_2)\big) // R(g(a))}\; z \mapsto f(a)
$$

$\Rightarrow$ free vars in the interpolant have to be overbound (if there are arrows, but we can just always do so)

VERSION WITH "CURRENT" ALGO:

$x_4 = g(x)$ $\qquad$ $x_5 = g(a)$ $\qquad$ $x_6 = g(y)$ $\qquad$ $x_7 = f(x)$ $\qquad$ $x_8 = f(a)$

$$
\cfrac{\overset{\Pi}{\neg P(z)} \qquad \cfrac{\overset{\Pi}{\neg R(x_5)} \qquad \cfrac{\overset{\Sigma}{\exists x_7 \mid \bot \mid P(x_7) \vee Q(x)} \qquad \overset{\Pi}{\forall x_6 \mid \top \mid R(x_6) \vee \neg Q(y)}}{\forall x_6 \exists x_7 \mid (Q(x) \wedge \top) \vee (\neg Q(x) \wedge \bot) \mid P(x_7) \vee R(x_6)}\; y \mapsto x}{\forall x_1 \forall x_6 \exists x_7 \mid Q(x_1) \mid P(x_7) // R(x_6)}\; x \mapsto a}{\forall x_1 \exists x_8 \mid \big(Q(x_1) \vee P(x_8)\big) // R(x_6)}\; z \mapsto f(a)
$$

$\Rightarrow$ a free variable can be left as is as it is implicitly universally bound at highest level, which is also the case in the preceding clauses (else it would have been unified and changed)

misc examples

201a

$$\dfrac{\dfrac{P(x,y) \overset{\Sigma}{\vee} \neg Q(y) \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(y)}\; x \mapsto a \qquad \dfrac{Q(f(z)) \overset{\Sigma}{\vee} R(z) \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a))}\; z \mapsto a}{\Box}\; y \mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,y)}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{R(a)}\; z \mapsto a}{P(a,f(a)) \vee R(a)}\; y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 P(x_1,y)}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{\forall x_3 R(x_3)}\; z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 (P(x_1,x_2) \vee R(x_3))}\; y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

201b

$$\dfrac{\dfrac{P(x,f(y)) \overset{\Sigma}{\vee} \neg Q(f(y)) \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(f(y))}\; x \mapsto a \qquad \dfrac{Q(f(z)) \overset{\Sigma}{\vee} R(z) \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a))}\; z \mapsto a}{\Box}\; y \mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,f(y))}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{R(a)}\; z \mapsto a}{P(a,f(a)) \vee R(a)}\; y \mapsto a \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 \exists x_2 P(x_1,x_2)}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{\forall x_3 R(x_3)}\; z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_3)}\; y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

arrow in element which is not in interpolant or resolution clause

206

$$\dfrac{\dfrac{\overset{\Sigma}{P(x) \lor \neg Q(f(x))} \qquad \overset{\Pi}{\neg P(a)}}{\forall x_1 P(x_1) \mid \neg Q(f(a))}\, x \mapsto a \qquad \dfrac{\overset{\Sigma}{Q(y) \lor R(g(y))} \qquad \overset{\Pi}{\neg R(z)}}{\exists x_2 R(x_2) \mid Q(y)}\, z \mapsto g(y)}{\forall x_1 \exists x_2 (P(x_1) \lor R(x_2)) \mid \Box}\, y \mapsto f(a)$$

$$P(a) \lor R(g(f(a)))$$

for first interpolant, $\Sigma \not\models \ell_{\Delta,x}[\mathrm{PI}(C)] \lor C$

=> need to overbind clause as well

border cases: arrows not within supposedly connected components

211a

$$\frac{Q(x) \lor P(f(x,a)) \qquad \neg Q(y) \lor R(f(y,b))}{Q(x) \mid P(f(x,a)) \lor R(f(x,b))}$$

$\Rightarrow$ no arr between P and R

211a'

$$\cfrac{\cfrac{Q(x) \lor \overset{\Sigma}{P}(f(x,a)) \qquad \neg Q(y) \lor \overset{\Sigma}{R}(f(y,b))}{Q(x) \mid P(f(x,a)) \lor R(f(x,b))} \qquad \cfrac{\overset{\Sigma}{\neg P(f(u,z)) \lor S(u))} \qquad \overset{\Pi}{\neg S}(c)}{S(c) \mid \neg P(f(c,z))}}{(P(f(c,a) \land S(c)) \lor (\neg P(f(c,a)) \land Q(c))) \mid R(f(c,b))}$$

$c \sim x_1$ $\qquad$ $f(c,a) \sim y_2$ $\qquad$ $f(c,b) \sim y_3$

$(P(y_2) \land S(x_1)) \lor (\neg P(y_2) \land Q(x_1)) \mid R(y_3)$

$\forall x_1 \exists y_2 \exists y_3$

this is not valid per se as the left hand side only contains Σ-formulas, but it probably could be fixed by adding some Π-inferences

Lesson is: no extra arrows needed, if a term enters, it does so via x, but there is a variable from the grey x to both colored x.

211b

$$\frac{Q(x) \lor P(f(x)) \qquad R(y) \lor \neg P(f(y))}{P(f(x)) \mid Q(x) \lor R(x)}$$

$\Rightarrow$ no arr between Q and R

211b'

$$\cfrac{\cfrac{Q(x) \lor \overset{\Sigma}{P}(f(x)) \qquad R(y) \lor \overset{\Sigma}{\neg P}(f(y))}{P(f(x)) \mid Q(x) \lor R(x)} \qquad \overset{\Pi}{\neg Q}(a)}{P(f(a)) \lor Q(a) \mid R(a)}$$

conjecture: Q and R do not need arrows as they are lifted by the same variable anyway, so constraints on Q do the work

Example: no nice arrow start/end points

212a
$$\frac{\bot \mid P(g(f(y))) \vee Q(y) \qquad \top \mid \neg Q(g(x)) \vee R(h(x))}{Q(g(x)) \mid P(g(f(g(x)))) \vee R(h(x))}$$

how to handle with components? problem with lifting vars? is this ok due to "contains lifting term"-semantics?

merge all terms which are contained in each other?