

Remark ()*. Any substitution, in particular σ , only changes a finite number of variables. Furthermore a result of a run of the unification algorithm is acyclic in the sense that if a substitution $u \mapsto t$ is added to the resulting substitution, it is never the case that at a later stage $t \mapsto u$ is added. This can easily be seen by considering that at the point when $u \mapsto t$ is added to the resulting substitution, every occurrence of u is replaced by t , so u is not encountered by the algorithm at a later stage.

Therefore in order to show that a statement holds for every $u \mapsto t$ in a unifier σ , it suffices to show by an induction argument that for every substitution $v \mapsto s$ which is added to the resulting unifier by the unification algorithm that it holds for $v \mapsto s$ under the assumption that it holds for every $w \mapsto r$ such that w occurs in s and $w \mapsto r$ is added to the resulting substitution at a later stage. \triangle

Conjecture 1. *Let C be a clause in a resolution refutation. Suppose that $\text{AI}^\Delta(C)$ contains a maximal Γ -term $\gamma_j[z_i]$ which contains a lifting variable z_i . Then $z_i <_{\hat{\mathcal{A}}(C)} y_j$.*

Proof. We proceed by induction. For the base case, note that no multicolored terms occur in initial clauses, so no lifting term can occur inside of a Γ -term.

Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$. Furthermore suppose that for every lifting term inside a Γ -term in the clauses C_1 and C_2 of the refutation, for every term of the form $\gamma_j[z_i]$ we have that $z_i <_{\hat{\mathcal{A}}(C_1)} y_j$ or $z_i <_{\hat{\mathcal{A}}(C_2)} y_j$ respectively. Hence there is an arrow (p_1, p_2) in $\hat{\mathcal{A}}(C_1)$ or $\hat{\mathcal{A}}(C_2)$ such that z_i is contained in $P(p_1)$ and z_j is contained in $P(p_2)$. In $\text{AI}^\Delta(C)$, $P(p_1)$ contains $\ell[z_i\sigma]\tau = z_i\tau$ and $P(p_2)$ contains $\ell[z_j\sigma]\tau = z_j\tau$. Hence the indices of the lifting variables might change, but this renaming does not affect the relation of the objects as $\hat{\mathcal{A}}(C_1) \cup \hat{\mathcal{A}}(C_2) \subseteq \hat{\mathcal{A}}(C)$.

We show that $z_i <_{\hat{\mathcal{A}}(C)} z_j$ holds true also for every new term of the form $\gamma_j[z_i]$ for some j, i in $\text{AI}^\Delta(C)$. By “new”, we mean terms which are not present in $\text{AI}^\Delta(C_1)$ or $\text{AI}^\Delta(C_2)$. Note that new terms in $\text{AI}^\Delta(C)$ are of the form $\ell_{\Delta, x}[t\sigma]\tau$ for some $t \in \text{AI}^\Delta(C_1) \cup \text{AI}^\Delta(C_2)$. By Lemma ??, σ does not introduce lifting variables. Hence a new term of the form $\gamma_j[z_i]$ is created either by introducing a Δ -term into a Γ -term or by introducing $\gamma_j[\delta_i]$ via σ , both followed by the lifting. Note that τ only substitutes lifting variables by other lifting variables and hence does not introduce lifting variables. Furthermore by Lemma ??, τ only substitutes lifting variables for other lifting variables, whose corresponding term is more specialised. Hence if there exists an arrow from a lifting variable to $\gamma_j[z_i]$ according to this lemma, it is also an appropriate arrow if $\gamma_j[z_i]$ is replaced by $\gamma_j[z_i]\tau$.

We now distinguish the two cases under which a new term $\gamma_j[z_i]$ can occur in $\text{AI}^\Delta(C)$:

Suppose for some Γ -term $\tilde{\gamma}_j[u]$ in $\text{AI}^\Delta(C_1)$ or $\text{AI}^\Delta(C_2)$, $(\tilde{\gamma}_j[u])\sigma = \gamma_j[\delta_i]$ for some i .

Note that the position of u in $\tilde{\gamma}_j[u]$ does not necessarily coincide with the position of δ_i in $\gamma_j[\delta_i]$ as u might be substituted for a grey term containing δ_i , but we assume that u is the variable which is substituted for a term containing δ_i .

We have that $\ell[\tilde{\gamma}_j[u]\sigma]\tau = \gamma_j[x_i]$.

At some well-defined point of application of the unification algorithm, u is substituted by an abstraction of a term which contains δ_i . This occurrence of u is in l and we denote it by \hat{u} . We furthermore denote the term at the corresponding position in l' by \hat{u}' .

We distinguish cases based on the occurrences of \hat{u} and \hat{u}' .

- Suppose \hat{u} is a grey occurrence. Then by Lemma ??, there is an arrow from \hat{u} to $\gamma_j[u]$ in $\hat{\mathcal{A}}(C)$. As $\hat{u}\sigma$ is a term containing the Δ -term δ_i , the term at the position of \hat{u} in $\text{AI}^\Delta(C)$ is $\ell[\hat{u}\sigma]\tau$, which by assumption contains x_i . But there is an arrow from this term containing x_i to $\gamma_j[x_i]$, so $z_j <_{\hat{\mathcal{A}}(C)} x_i$.
- Suppose \hat{u} occurs in a maximal colored term which is a Γ -term.

TODO:

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\Rightarrow what if it is overbound?

Suppose for some variable v in $\text{AI}^\Delta(C_1)$ or $\text{AI}^\Delta(C_2)$, $v\sigma = \gamma_j[\delta_i]$ for some i .

As v is affected by the unifier, it occurs in the literal being unified, say w.l.o.g. in l in C_1 . At some well-defined point in the unification algorithm, v is substituted by an abstraction of $\gamma_j[\delta_i]$. Let p be the position of the occurrence of v in l which causes this substitution. Furthermore, let p' be the position corresponding to p in l' .

Note that any arrow from or to p' also applies to p in $\hat{\mathcal{A}}(C)$ and hence to $\gamma_j[z_i]$ as they are merged due to occurring in the resolved literal. So it suffices to show that

there is an arrow from an appropriate lifting variable to p' . We denote the term at p' by t .

Note that $t\sigma = \gamma_j[\delta_i]$. So t is either a Γ -term containing a Δ -term, in which case we know that there is an appropriate arrow by the induction hypothesis as t occurs in l' in C_2 , or t is an abstraction of $\gamma_j[\delta_i]$, in which case we can assume the existence of an appropriate arrow by Remark (*). □

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graphs
w.r.t. $<_{\hat{\mathcal{A}}(C)}$