

# 1 Attempt without $P_P$

Intuition of  $\sigma'$ :

If we pull a substitution out of a lifting which replaced  $\Delta$ -terms, we also have to replace the  $\Delta$ -terms in the “domain” of the substitution. This is the lower case in the definition of  $\sigma'$  below.

There is just a problem in the following case:  $\ell_{\Delta,x}[f(x)\sigma]$ , where  $x\sigma = a$  and  $f$  is a  $\Delta$ -symbol. Then  $\ell_{\Delta,x}[f(x)\sigma] = \ell_{\Delta,x}[f(a)] = x_i$ , but  $\ell_{\Delta,x}[f(x)]\sigma = x_j$  with  $i \neq j$ . The first case of the definition of  $x_j$  then fixes this by replacing  $x_j$  with  $x_i$ .

**Lemma 1.** *Let  $C$  be a clause and  $t_1, \dots, t_n$  the set of maximal  $\Delta$ -terms in  $C$ ,  $x_1, \dots, x_n$  the corresponding fresh variables to replace the  $t_i$ , and  $\sigma$  be a substitution. Let  $\sigma'$  be defined such that*

$$z\sigma' = \begin{cases} x_l & \text{if } z = x_k \text{ and } t_k\sigma = t_l \\ \ell_{\Delta,x}[z\sigma] & \text{otherwise} \end{cases}$$

*Note that the definition of  $\sigma'$  only depends on the  $x_i$  and  $t_i$ .*

*Then  $\ell_{\Delta,x}[C\sigma] = \ell_{\Delta,x}[C]\sigma'$ .*

*Proof.* We prove this for an atom  $P(s_1, \dots, s_m)$  in  $C$ , which works since lifting and substitution commute over binary connectives and into an atom.

We show that  $\ell_{\Delta,x}[s_j\sigma] = \ell_{\Delta,x}[s_j]\sigma'$  for  $1 \leq j \leq m$ .

Note that anything in the term structure above a maximal  $\Delta$ -term is unaffected by both substitution and abstraction.

Let  $t_i$  be a maximal  $\Delta$ -term in  $s_i\sigma$ .

We show that  $\ell_{\Delta,x}[t_i\sigma] = \ell_{\Delta,x}[t_i]\sigma'$ , which proves the lemma.

Let  $t_i\sigma = t_j$ . Then  $\ell_{\Delta,x}[t_i\sigma] = \ell_{\Delta,x}[t_j] = x_j$ .

We show that  $x_j = \ell_{\Delta,x}[t_i]\sigma'$ .

Suppose that  $t_i = t_j$ , i.e.  $\sigma$  is trivial on  $t_i$ . Then  $i = j$  as the  $\Delta$ -terms have a unique number. Hence  $\ell_{\Delta,x}[t_i]\sigma' = x_i\sigma' = x_i = x_j$ .

Otherwise  $t_i \neq t_j$ . Then  $i \neq j$  and  $x_j \neq x_i$ .

$\ell_{\Delta,x}[t_i]\sigma' = x_i\sigma'$ . By the definition of  $\sigma'$ , as  $t_i\sigma = t_j$ ,  $x_i\sigma' = x_j$ . □

**Lemma 2** (currently 4.8 in thesis, Lemma 11 in Huang). *Let  $A$  and  $B$  be first-order formulas and  $s$  and  $t$  be terms. Then it holds that:*

1.  $\ell_{\Phi,x}[\neg A] \Leftrightarrow \neg \ell_{\Phi,x}[A]$
2.  $\ell_{\Phi,x}[A \circ B] \Leftrightarrow (\ell_{\Phi,x}[A] \circ \ell_{\Phi,x}[B])$  for  $\circ \in \{\wedge, \vee\}$
3.  $\ell_{\Phi,x}[s = t] \Leftrightarrow (\ell_{\Phi,x}[s] = \ell_{\Phi,x}[t])$

**Lemma 3.**  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C) \vee C]$ .

*Proof.* By induction on the resolution refutation of the strengthening:  $\Gamma \models \text{PI}(C) \vee C_\Gamma$

Base case: Either  $C \in \Gamma$ , then it does not contain  $\Delta$ -terms. Otherwise  $C \in \Delta$  and  $\text{PI}(C) = \top$ .

Induction step:

Resolution.

$$\frac{C_1 : D \vee l \quad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad l\sigma = l'\sigma$$

By the induction hypothesis, we can assume that:

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee (D \vee l)_\Gamma] \text{ and } \Gamma \models \ell_{\Delta,x}[\text{PI}(C_2) \vee (E \vee \neg l')_\Gamma]$$

which by Lemma 2 implies that

$$\Gamma \stackrel{(*)}{\models} \ell_{\Delta,x}[\text{PI}(C_1)] \vee \ell_{\Delta,x}[D_\Gamma] \vee \ell_{\Delta,x}[l_\Gamma] \text{ and } \Gamma \stackrel{(\circ)}{\models} \ell_{\Delta,x}[\text{PI}(C_2)] \vee \ell_{\Delta,x}[E_\Gamma] \vee \neg \ell_{\Delta,x}[l'_\Gamma]$$

Let  $\sigma'$  be defined as in Lemma 1 with  $t_1, \dots, t_n$  all  $\Delta$ -terms in this context, i.e. from  $C_1$ ,  $C_2$ ,  $\text{PI}(C_1)$ ,  $\text{PI}(C_2)$  and  $\sigma$ .

1.  $l$  is  $\Gamma$ -colored. Then  $\text{PI}(C) = [\text{PI}(C_1) \vee \text{PI}(C_2)]\sigma$ .

We show that  $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2))\sigma \vee (D \vee E)_\Gamma\sigma]$ ,

i.e.  $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2) \vee D_\Gamma \vee E_\Gamma)\sigma]$ .

Hence by Lemma 1,  $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2) \vee D_\Gamma \vee E_\Gamma)]\sigma'$ .

Since  $l\sigma = l'\sigma$  (by resolution rule application),  $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l'\sigma]$ .

As by Lemma 1, with  $\sigma'$  as above,  $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l]\sigma'$  and  $\ell_{\Delta,x}[l'\sigma] = \ell_{\Delta,x}[l']\sigma'$ , we get  $\ell_{\Delta,x}[l]\sigma' = \ell_{\Delta,x}[l']\sigma'$ .

So by applying  $\sigma'$  to  $(*)$  and  $(\circ)$ , we can perform a resolution step on  $\ell_{\Delta,x}[l]\sigma'$  and get

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'.$$

and consequently  $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2) \vee D_\Gamma \vee E_\Gamma)]\sigma'$ .

So by Lemma 1,

$$\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2) \vee D_\Gamma \vee E_\Gamma)\sigma].$$

2.  $l$  is  $\Delta$ -colored. Then  $\text{PI}(C) = (\text{PI}(C_1) \wedge \text{PI}(C_2))\sigma$ .

We show that  $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \wedge \text{PI}(C_2))\sigma \vee (D_\Gamma \vee E_\Gamma)\sigma]$

which by Lemma 2 is equivalent to

$$\Gamma \models (\ell_{\Delta,x}[\text{PI}(C_1)\sigma] \wedge \ell_{\Delta,x}[\text{PI}(C_2)\sigma]) \vee \ell_{\Delta,x}[D_\Gamma\sigma] \vee \ell_{\Delta,x}[E_\Gamma\sigma]$$

and by Lemma 1 is equivalent to

$$\Gamma \stackrel{(\times)}{\models} (\ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma') \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$$

As  $l$  and  $l'$  are  $\Delta$ -colored, we can strengthen  $(*)$  and  $(\circ)$  as follows and apply  $\sigma'$ :

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \text{ and } \Gamma \models \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$$

These clearly imply  $(\times)$ .

3.  $l$  is grey. Then  $\text{PI}(C) = [(l \wedge \text{PI}(C_2)) \vee (\neg l' \wedge \text{PI}(C_2))]\sigma$ .

We show that  $\Gamma \models \ell_{\Delta,x}[(l \wedge \text{PI}(C_2)) \vee (\neg l' \wedge \text{PI}(C_2))]\sigma \vee (D_\Gamma \vee E_\Gamma)\sigma]$ , which by Lemma 2 and Lemma 1 is equivalent to

$\Gamma \models (\ell_{\Delta,x}[l]\sigma' \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma') \vee (\neg \ell_{\Delta,x}[l']\sigma' \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma') \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'.$   
 Suppose for a model  $M$  of  $\Gamma$  that  $M \not\models \ell_{\Delta,x}[D_\Gamma]\sigma'$  and  $M \not\models \ell_{\Delta,x}[E_\Gamma]\sigma'$  as otherwise we would be done. But then by  $(*)$  and  $(\circ)$ ,  $M \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[l]\sigma'$  and  $M \models \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \vee \neg \ell_{\Delta,x}[l']\sigma'.$

As observed in case 1,  $\ell_{\Delta,x}[l]\sigma' = \ell_{\Delta,x}[l']\sigma'.$  By a case distinction on the truth value of  $\ell_{\Delta,x}[l]\sigma',$  we obtain the result.

Factorisation.

$$\frac{C_1 : l \vee l' \vee D}{C : (l \vee D)\sigma} \quad \sigma = \text{mgu}(l, l')$$

Then  $\text{PI}(C) = \text{PI}(C_1)\sigma.$

The induction hypothesis gives that  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee l \vee l' \vee D].$  Let  $\sigma'$  be as in Lemma 1.

Then  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee l \vee l' \vee D]\sigma'$  and by Lemma 1,  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)\sigma \vee l\sigma \vee l'\sigma \vee D\sigma].$

By Lemma 2,  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)\sigma] \vee \ell_{\Delta,x}[l\sigma] \vee \ell_{\Delta,x}[l'\sigma] \vee \ell_{\Delta,x}[D\sigma].$

As  $\sigma = \text{mgu}(l, l'),$   $l\sigma$  and  $l'\sigma$  are syntactically equal, hence  $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l'\sigma].$

But then  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)\sigma] \vee \ell_{\Delta,x}[l\sigma] \vee \ell_{\Delta,x}[D\sigma]$  and consequently by Lemma 1,  $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)\sigma \vee l\sigma \vee D\sigma].$

Paramodulation.

$$\frac{C_1 : D \vee s = t \quad C_2 : E[r]_p}{C : (D \vee E[t]_p)\sigma} \quad \sigma = \text{mgu}(s, r)$$

By the induction hypothesis, we have:

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee (D \vee s = t)_\Gamma]$$

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_2) \vee (E[r]_p)_\Gamma]$$

By Lemma 2 and Lemma 1, these imply:

$$\stackrel{(\circ)}{\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[s]\sigma' = \ell_{\Delta,x}[t]\sigma'}$$

$$\stackrel{(*)}{\Gamma \models \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[(E[r]_p)_\Gamma]\sigma'}$$

We distinguish two cases:

1. Suppose  $s$  does not occur in a maximal  $\Delta$ -term  $h[s]$  in  $E[s]_p$  which occurs more than once in  $\text{PI}(E(s)) \vee E[s]_p.$

We show that  $\Gamma \models \ell_{\Delta,x}[(s = t \wedge \text{PI}(C_2)) \vee (s \neq t \wedge \text{PI}(C_1))]\sigma \vee ((D \vee E[t]_p)_\Gamma)\sigma,$  which subsumes the cases 2 and 3 of the definition of PI for paramodulation. By Lemma 2, we can pull the liftings inwards and by Lemma 1, we can commute substitution and lifting by employing  $\sigma'$  to arrive at

$$\Gamma \models ((\ell_{\Delta,x}[s]\sigma') = (\ell_{\Delta,x}[t]\sigma') \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma') \vee ((\ell_{\Delta,x}[s]\sigma') \neq (\ell_{\Delta,x}[t]\sigma') \wedge \ell_{\Delta,x}[\text{PI}(C_1)]\sigma') \vee (\ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[(E[t]_p)_\Gamma]\sigma')$$

syntactically  
equal?  
does  
"equal"  
suffice?  
see also  
 $s\sigma = r\sigma$   
below

Let  $M$  be a model of  $\Gamma$ . Let  $M \not\models \ell_{\Delta,x}[D\Gamma]\sigma' \vee \ell_{\Delta,x}[(E[t]_p)\Gamma]\sigma'$  as otherwise we would be done. We show that depending on the truth value of  $(\ell_{\Delta,x}[s]) = (\ell_{\Delta,x}[t])$  in  $M$ , either the first or second conjunct of the above formula holds.

Suppose that  $M \models (\ell_{\Delta,x}[s]) \neq (\ell_{\Delta,x}[t])$ . Then  $M \models (\ell_{\Delta,x}[s])\sigma' \neq (\ell_{\Delta,x}[t])\sigma'$ , so by  $(\circ)$ ,  $M \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma'$ .

On the other hand, suppose that  $M \models (\ell_{\Delta,x}[s]) = (\ell_{\Delta,x}[t])$ . The following two lemmas show that  $M \not\models \ell_{\Delta,x}[E[r]_p]\sigma'$ , so by  $(*)$ , we get that  $M \models \ell_{\Delta,x}[\text{PI}(C_2)]\sigma'$ .

**Lemma 4.**  $M \models (\ell_{\Delta,x}[s]) = (\ell_{\Delta,x}[t])$  and  $M \not\models \ell_{\Delta,x}[E[t]_p]\sigma'$  imply that  $M \not\models \ell_{\Delta,x}[E[s]_p]\sigma'$ .

*Proof.*  $\ell_{\Delta,x}[E[t]_p]$  and  $\ell_{\Delta,x}[E[s]_p]$  only differ at position  $p$ , where at the first, there is  $\ell_{\Delta,x}[t]$ , and at the latter, there is  $\ell_{\Delta,x}[s]$ . But in  $M$ , they are interpreted the same way, hence  $M \models \ell_{\Delta,x}[E[t]_p] \Leftrightarrow \ell_{\Delta,x}[E[s]_p]$ , which implies the result.  $\square$

**Lemma 5.**  $\sigma = \text{mgu}(s, r)$  and  $M \not\models \ell_{\Delta,x}[E[s]_p]\sigma'$  imply that  $M \not\models \ell_{\Delta,x}[E[r]_p]\sigma'$ .

*Proof.* By Lemma 1,  $M \not\models \ell_{\Delta,x}[(E[s]_p)\sigma]$ .

Due to  $\sigma = \text{mgu}(s, r)$ , both  $s\sigma$  and  $r\sigma$  are syntactically equal. Suppose they are both not  $\Delta$ -colored. Then the lifting does not affect them and  $\ell_{\Delta,x}[(E[s]_p)\sigma] = \ell_{\Delta,x}[(E[r]_p)\sigma]$ . Otherwise the lifting will replace them with the same variable and we as well get that  $\ell_{\Delta,x}[(E[s]_p)\sigma] = \ell_{\Delta,x}[(E[r]_p)\sigma]$ .

By Lemma 1,  $\ell_{\Delta,x}[(E[s]_p)]\sigma' = \ell_{\Delta,x}[(E[r]_p)]\sigma'$ , which implies the result.  $\square$

2. Otherwise  $s$  occurs in a maximal  $\Delta$ -term  $h[s]_q$  in  $E[s]_p$  which occurs more than once in  $\text{PI}(E(s)) \vee E[s]_p$ .

Then we have to replace Lemma 4 by:

**Lemma 4'.**  $M \models (\ell_{\Delta,x}[s]) = (\ell_{\Delta,x}[t])$  and  $M \not\models \ell_{\Delta,x}[E[t]_p]\sigma'$  imply that  $M \not\models \ell_{\Delta,x}[E[s]_p]\sigma'$  or that  $\ell_{\Delta,x}[h[s]_q] \neq \ell_{\Delta,x}[h[t]_q]$ .

*Proof.* If  $\ell_{\Delta,x}[E[t]_p]$  and  $\ell_{\Delta,x}[E[s]_p]$  differ only at position  $p$ , then the proof of Lemma 4 applies.

Otherwise position  $p$  is in a maximal  $\Delta$ -term  $h[t]_q$ , such that  $h[t]_q$  and  $h[s]_q$  are replaced with distinct variables. But then clearly  $\ell_{\Delta,x}[h[s]_q] \neq \ell_{\Delta,x}[h[t]_q]$ .  $\square$

Hence the following holds:

$$\Gamma \models \left( (\ell_{\Delta,x}[s]\sigma') = (\ell_{\Delta,x}[t]\sigma') \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \right) \vee \left( (\ell_{\Delta,x}[s]\sigma') \neq (\ell_{\Delta,x}[t]\sigma') \wedge \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \right) \vee \left( (\ell_{\Delta,x}[s]\sigma') = (\ell_{\Delta,x}[t]\sigma') \wedge (\ell_{\Delta,x}[h[s]_q]) \neq (\ell_{\Delta,x}[h[t]_q]) \right) \vee \left( \ell_{\Delta,x}[D\Gamma]\sigma' \vee \ell_{\Delta,x}[(E[t]_p)\Gamma]\sigma' \right)$$

$\square$

Then the following from the thesis (also same in Huang) seem to go through:

Lemma 4.10: swap  $\Gamma$  and  $\Delta$  and obtain logical negation as interpolant

Corollary 4.11:  $\Delta \models \ell_{\Gamma,y}[\neg \text{PI}(C) \vee C]$

Lemma 4.12: not important if lifting delta or gamma terms first

Thm 4.13: ordering