1 Attempt without P_P

Intuition of σ':
If we pull a substitution out of a lifting which replaced Δ-terms, we also have to replace the Δ-terms in the "domain" of the substitution. This is the lower case in the definition of σ' below.
There is just a problem in the following case: $\ell_{\Delta,x}[f(x)\sigma]$, where $x\sigma = a$ and f is a Δ-symbol. Then $\ell_{\Delta,x}[f(x)\sigma] = \ell_{\Delta,x}[f(a)] = x_i$, but $\ell_{\Delta,x}[f(x)]\sigma = x_j$ with $i \neq j$. The first case of the definition of x_j then fixes this by replacing x_j with x_i.

Lemma 1. *Let C be a clause and $t_1, \ldots, t_n$ the set of maximal Δ-terms in C, $x_1, \ldots, x_n$ the corresponding fresh variables to replace the t_i, and σ be a substitution. Let σ' be defined such that*

$$z\sigma' = \begin{cases} x_l & \text{if } z = x_k \text{ and } t_k\sigma = t_l \\ \ell_{\Delta,x}[z\sigma] & \text{otherwise} \end{cases}$$

Note that the definition of σ' only depends on the x_i and t_i.
Then $\ell_{\Delta,x}[C\sigma] = \ell_{\Delta,x}[C]\sigma'$.

Proof. We prove this for an atom $P(s_1, \ldots, s_m)$ in C, which works since lifting and substitution commute over binary connectives and into an atom.
We show that $\ell_{\Delta,x}[s_j\sigma] = \ell_{\Delta,x}[s_j]\sigma'$ for $1 \leq j \leq m$.
Note that anything in the term structure above a maximal Δ-term is unaffected by both substitution and abstraction.
Let t_i be a maximal Δ-term in $s_i\sigma$.
We show that $\ell_{\Delta,x}[t_i\sigma] = \ell_{\Delta,x}[t_i]\sigma'$, which proves the lemma.
Let $t_i\sigma = t_j$. Then $\ell_{\Delta,x}[t_i\sigma] = \ell_{\Delta,x}[t_j] = x_j$.
We show that $x_j = \ell_{\Delta,x}[t_i]\sigma'$.
Suppose that $t_i = t_j$, i.e. σ is trivial on t_i. Then $i = j$ as the Δ-terms have a unique number. Hence $\ell_{\Delta,x}[t_i]\sigma' = x_i\sigma' = x_i = x_j$.
Otherwise $t_i \neq t_j$. Then $i \neq j$ and $x_j \neq x_i$.
$\ell_{\Delta,x}[t_i]\sigma' = x_i\sigma'$. By the definition of σ', as $t_i\sigma = t_j$, $x_i\sigma' = x_j$. $\square$

Lemma 2 (currently 4.8 in thesis, Lemma 11 in Huang)**.** *Let A and B be first-order formulas. Then it holds that:*

1. $\ell_{\Phi,x}[\neg A] \Leftrightarrow \neg \ell_{\Phi,x}[A]$
2. $\ell_{\Phi,x}[A \circ B] \Leftrightarrow (\ell_{\Phi,x}[A] \circ \ell_{\Phi,x}[B])$ *for* $\circ \in \{\wedge, \vee\}$

Lemma 3. $\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C) \vee C)]$.

Proof. By induction on the resolution refutation of the strengthening: $\Gamma \models \mathrm{PI}(C) \vee C_\Gamma$
Base case: Either $C \in \Gamma$, then it does not contain Δ-terms. Otherwise $C \in \Delta$ and $\mathrm{PI}(C) = \top$.
Induction step:

Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad l\sigma = l'\sigma$$

By the induction hypothesis, we can assume that:

$\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1) \vee (D \vee l)_\Gamma]$ and $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_2) \vee (E \vee \neg l')_\Gamma]$

which by Lemma 2 implies that

$\Gamma \overset{(*)}{\models} \ell_{\Delta,x}[\mathrm{PI}(C_1)] \vee \ell_{\Delta,x}[D_\Gamma] \vee \ell_{\Delta,x}[l_\Gamma]$ and $\Gamma \overset{(\circ)}{\models} \ell_{\Delta,x}[\mathrm{PI}(C_2)] \vee \ell_{\Delta,x}[E_\Gamma] \vee \neg \ell_{\Delta,x}[l'_\Gamma]$

Let σ' be defined as in Lemma 1 with $t_1, \ldots, t_n$ all Δ-terms in this context, i.e. from C_1, C_2, $\mathrm{PI}(C_1)$, $\mathrm{PI}(C_2)$ and σ.

1. l is Γ-colored. Then $\mathrm{PI}(C) = [\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2)]\sigma$.

   We show that $\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2))\sigma \vee (D \vee E)_\Gamma \sigma]$,
   i.e. $\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2) \vee D_\Gamma \vee E_\Gamma)\sigma]$.

   Hence by Lemma 1, $\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2) \vee D_\Gamma \vee E_\Gamma)]\sigma'$.

   Since $l\sigma = l'\sigma$ (by resolution rule application), $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l'\sigma]$.

   As by Lemma 1, with σ' as above, $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l]\sigma'$ and $\ell_{\Delta,x}[l'\sigma] = \ell_{\Delta,x}[l']\sigma'$, we get $\ell_{\Delta,x}[l]\sigma' = \ell_{\Delta,x}[l']\sigma'$.

   So by applying σ' to $(*)$ and $(\circ)$, we can perform a resolution step on $\ell_{\Delta,x}[l]\sigma'$ and get

   $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$.

   and consequently $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2) \vee D_\Gamma \vee E_\Gamma]\sigma'$.

   So by Lemma 1,

   $\Gamma \models \ell_{\Delta,x}[\Big(\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2) \vee D_\Gamma \vee E_\Gamma\Big)\sigma]$.
2. l is Δ-colored. Then $\mathrm{PI}(C) = (\mathrm{PI}(C_1) \wedge \mathrm{PI}(C_2))\sigma$.

   We show that $\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C_1) \wedge \mathrm{PI}(C_2))\sigma \vee (D_\Gamma \vee E_\Gamma)\sigma]$

   which by Lemma 2 is equivalent to
   $\Gamma \models \Big(\ell_{\Delta,x}[\mathrm{PI}(C_1)\sigma] \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)\sigma]\Big) \vee \ell_{\Delta,x}[D_\Gamma\sigma] \vee \ell_{\Delta,x}[E_\Gamma\sigma]$

and by Lemma 1 is equivalent to

$\Gamma \overset{(\times)}{\models} \Big(\ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma'\Big) \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$

As l and l' are Δ-colored, we can strengthen $(*)$ and $(\circ)$ as follows and apply σ':

$\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma'$ and $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$

These clearly imply $(\times)$.

3. l is grey. Then $\mathrm{PI}(C) = [(l \wedge \mathrm{PI}(C_2)) \vee (\neg l' \wedge \mathrm{PI}(C_2))]\sigma$.

   We show that $\Gamma \models \ell_{\Delta,x}[[(l \wedge \mathrm{PI}(C_2)) \vee (\neg l' \wedge \mathrm{PI}(C_2))]\sigma \vee (D_\Gamma \vee E_\Gamma)\sigma]$, which by Lemma 2 and Lemma 1 is equivalent to

   $\Gamma \models \Big(\ell_{\Delta,x}[l]\sigma' \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma'\Big) \vee \Big(\neg\ell_{\Delta,x}[l']\sigma' \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma'\Big) \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$.

   Suppose for a model M of Γ that $M \not\models \ell_{\Delta,x}[D_\Gamma]\sigma'$ and $M \not\models \ell_{\Delta,x}[E_\Gamma]\sigma'$ as otherwise we would be done. But then by $(*)$ and $(\circ)$, $M \models \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[l]\sigma'$ and $M \models \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma' \vee \neg\ell_{\Delta,x}[l']\sigma'$.

   As observed in case 1, $\ell_{\Delta,x}[l]\sigma' = \ell_{\Delta,x}[l']\sigma'$. By a case distinction on the truth value of $\ell_{\Delta,x}[l]\sigma'$, we obtain the result.

Paramodulation.

$$\frac{C_1 : D \vee s = t \qquad C_2 : E[r]_p}{C : (D \vee E[t]_p)\sigma} \quad \sigma = \mathrm{mgu}(s, r)$$

By the induction hypothesis, we have:

$\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1) \vee (D \vee s = t)_\Gamma]$

$\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_2) \vee (E[r])_\Gamma]$

By Lemma 2 and Lemma 1, these imply:

$\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee (\ell_{\Delta,x}[s_\Gamma]\sigma') = (\ell_{\Delta,x}[t_\Gamma]\sigma')$

$\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[(E[r]_p)_\Gamma]\sigma'$

$\mathrm{PI}(C) =$

We show that $\Gamma \models$

easy case: $\mathrm{PI}(C) = [(s = t \wedge \mathrm{PI}(C_2)) \vee (s \neq t \wedge \mathrm{PI}(C_1))]\sigma$

to show: $\Gamma \models \ell_{\Delta,x}[[((s = t \wedge \mathrm{PI}(C_2)) \vee (s \neq t \wedge \mathrm{PI}(C_1))) \vee (D \vee E[t])]\sigma]$

proof idea: either $s = t$, then also $\mathrm{PI}(C_2)$, or else $s \neq t$, but then also $\mathrm{PI}(C_1)$

by lemma 1 for σ' as in lemma, $\Gamma \models \ell_{\Delta,x}[((s = t \wedge \mathrm{PI}(C_2)) \vee (s \neq t \wedge \mathrm{PI}(C_1))) \vee (D \vee E[t])]\sigma'$

by lemma 11 (huang) $\Gamma \models [((\ell_{\Delta,x}[s] = \ell_{\Delta,x}[t] \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]) \vee (\ell_{\Delta,x}[s \neq t] \wedge \ell_{\Delta,x}[\mathrm{PI}(C_1)])) \vee (\ell_{\Delta,x}[D] \vee \ell_{\Delta,x}[E[t]])]\sigma'$

reformulate: $\Gamma \models ((\ell_{\Delta,x}[s]\sigma' = \ell_{\Delta,x}[t]\sigma' \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma') \vee (\ell_{\Delta,x}[s]\sigma' \neq \ell_{\Delta,x}[t]\sigma' \wedge \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma')) \vee (\ell_{\Delta,x}[D]\sigma' \vee \ell_{\Delta,x}[E[t]]\sigma')$

By the rule: $s\sigma = r\sigma$, hence also $\ell_{\Delta,x}[s\sigma] = \ell_{\Delta,x}[r\sigma]$ and $\ell_{\Delta,x}[s]\sigma' = \ell_{\Delta,x}[r]\sigma'$ REALLY TRUE? – think so…

Suppose $M \models \Gamma$ and $M \not\models (\ell_{\Delta,x}[D]\sigma' \vee \ell_{\Delta,x}[E[t]]\sigma')$.

Suppose $M \models \ell_{\Delta,x}[s]\sigma' = \ell_{\Delta,x}[t]\sigma'$.

By induction hypothesis (and lemma 11 (huang) and adding the substitution σ'), $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[(E[r])]\sigma'$.

However by assumption $\Gamma \not\models \ell_{\Delta,x}[E[t]]\sigma'$.

Hence $\Gamma \not\models \ell_{\Delta,x}[E[s]]\sigma'$, and $\Gamma \not\models \ell_{\Delta,x}[E[r]]\sigma'$. Therefore $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma'$.

Suppose on the other hand $M \models \ell_{\Delta,x}[s]\sigma' \neq \ell_{\Delta,x}[t]\sigma'$.

By the induction hypothesis, $M \models \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \vee (\ell_{\Delta,x}[D]\sigma' \vee (\ell_{\Delta,x}[s] = \ell_{\Delta,x}[t])\sigma')$, hence then $M \models \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma'$.

Consequently, $M \models (\ell_{\Delta,x}[s]\sigma' \neq \ell_{\Delta,x}[t]\sigma' \wedge \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma') \vee (\ell_{\Delta,x}[s]\sigma' = \ell_{\Delta,x}[t]\sigma' \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma')$.

By lemma 11 (huang), $M \models \ell_{\Delta,x}[s \neq t \wedge \mathrm{PI}(C_1) \vee (s = t \wedge \mathrm{PI}(C_2))]\sigma'$.

Hence $\Gamma \models \ell_{\Delta,x}[(s \neq t \wedge \mathrm{PI}(C_1) \vee (s = t \wedge \mathrm{PI}(C_2))]\sigma' \vee (\ell_{\Delta,x}[D] \vee \ell_{\Delta,x}[E[t]])\sigma')$.

is this really what i need to show?

$\square$

General layout of this proof:
$\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C) \vee C)]$
Lemma 4.10: swap Γ and Δ and obtain logical negationas interpolant
Lemma 4.11: $\Delta \models \ell_{\Gamma,y}[\neg \mathrm{PI}(C) \vee C]$
$\Gamma \models \bar{Q}\ell_{\Gamma\cup\Delta,z}[\mathrm{PI}(\pi)]$; $\Delta \models \neg\bar{Q}\ell_{\Gamma\cup\Delta,z}[\mathrm{PI}(\pi)]$;