

Number of quantifier alternations in Huang and nested

1.1 Outline

Goal: try proof Huang and nested equal.

Method: proof for both:

Conjectured Proposition 1. *Let I be an interpolant created by \$algorithm. If I contains a term t such that t has n color changes, then I has at least n quantifier alternations.*

1.1.1 generally keep in mind

- Need to define all new terms here: color-changing, single-color, Φ -literal, substitutions from 0 to n
 - essentially same position: path from one position to other only contains grey symbol (this def allows for identical position as well)
- also note: literal is sometimes used for negated or not negated predicate with terms but in regular formulas with arbitrary connectives

1.2 Preliminaries

Quantifier alternations in I usually assumes the quantifier-alternation-minimizing arrangement of quantifiers in I

Definition 2 (Color alternation col-alt). Colors Γ and Δ , term t :

$$\text{col-alt}(t) \stackrel{\text{def}}{=} \text{col-alt}_{\perp}(t)$$

Let $t = f(t_1, \dots, t_n)$ for constant, function and variable symbols (syntax abuse):

$$\text{col-alt}_{\Phi}(t) \stackrel{\text{def}}{=} \begin{cases} \max^1(\text{col-alt}_{\Phi}(t_1), \dots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is grey} \\ \max(\text{col-alt}_{\Phi}(t_1), \dots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_{\Psi}(t_1), \dots, \text{col-alt}_{\Psi}(t_n)) & f \text{ is of color } \Psi, \Phi \neq \Psi \end{cases}$$

\triangle

¹We assume that the maximum of an empty list of arguments is 0.

Definition 3. $\text{PI}_{\text{step}}^\circ$ is defined just like PI_{step} but without applying any substitution. \triangle

Hence $\text{PI}_{\text{step}}^\circ(\cdot)\sigma = \text{PI}_{\text{step}}(\cdot)$. C° is somehow the same, i.e. if $C = D\sigma$, then $C^\circ = D$ where σ is derived from the context.

1.3 Random thoughts

- Quantifiers are introduced for lifting variables which actually occur in the interpolant
- If term t with $\text{col-alt}(t) = n$ enters I , we need subterm s of t with $\text{col-alt}(s) = n - 1$ to be in I (of course colors of t and s are exactly opposite)

1.3.1 Proof

- Induction over $\ell_\Delta^x[\text{PI}(C) \vee C]$ and also about Γ -terms with Δ -lifting vars in that formula. Cf. **-final**
- **NB: now somewhat described in the proper proof below** describe proof method with $\sigma_{(0,i)}$: which PI?
 - Factorisation: easy: just apply σ_i for all i to $\text{PI}(C) \vee C$. When done, a literal will be there twice and we can remove it without losing anything
 - Resolution: create propositional structure first.
Ex.: $C_1 : D \vee l, C_2 : \neg l \vee E$:
If we talk about properties for which it holds that if they hold for $\text{PI}(C_i) \vee C_i, i \in \{1, 2\}$, then they also hold for $A \equiv ((l \wedge \text{PI}(C_2)) \vee (\neg l \wedge \text{PI}(C_1))) \vee C^\circ$, then we can apply σ_i for all i to that formula.
So if we can assume it for A and show it for all σ_i , we get that it holds for $\text{PI}(C) \vee C$.

Also: clauses are variable disjoint, so e.g. it's not possible that a color-changing var is created by PI_{step}

Also: do it like a few lemmas further down, like $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)\sigma_{(0,i)}$

1.4 directly from old proof

Lemma 4. *Resolution or factorisation step ι from \bar{C} .*

If u col-change var in $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)\sigma_{(0,i)}$, then u also occurs grey in that formula.

Proof. Abbreviation: $F \equiv (\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)$

Induction over refutation and σ ; base case easy.

Step: Supp color change var u present in $\chi\sigma_{(0,i)}$. (could also say introduced, then proof would be somehow different)

Supp u not grey in $\chi\sigma_{(0,i-1)}$ as otherwise done. As a first step, we show that if a (not necessarily color-changing) variable v occurs in a single-colored Φ -term $t[v]$ in $\chi\sigma_{(0,i)}$, then at least one of the following holds:

1. v occurs in some single-colored Φ -term in $\chi\sigma_{(0,i-1)}$
2. there is a color-changing variable w in $\chi\sigma_{(0,i-1)}$ such that v occurs grey in $w\sigma_i$.

$\langle \text{var_occ_2} \rangle$ We consider the different cases which can introduce a variable v in a single-colored term Φ : Either it has been there before, it was introduced in a s.c. Φ -colored term, or a s.c. Φ -term containing the var is in $\text{ran}(\sigma)$.

- Suppose a term $t'[v]$ is present in $\chi\sigma_{(0,i-1)}$ such that $t'[v]\sigma_i = t[v]$. Then 1 is the case.
- Suppose a variable w occurs in a single-colored Φ -term in $\chi\sigma_{(0,i-1)}$ such that v occurs grey in $w\sigma_i$. Suppose furthermore that 1 is not the case, i.e. v does not occur in a s.c. Φ -term in $\chi\sigma_{(0,i-1)}$, as otherwise we would be done. We show that 2 is the case.

As v occurs neither grey nor in a s.c. Φ -term in $\chi\sigma_{(0,i-1)}$ but occurs in $\text{ran}(\sigma_i)$, it must occur in $\chi\sigma_{(0,i-1)}$ and this can only be in a single-colored Ψ -term.

As by assumption v occurs grey in $w\sigma_i$, there must be an occurrence \hat{w} of w in a resolved or factorised literal, say $\lambda\sigma_{(0,i-1)}$ such that for the other resolved literal $\lambda'\sigma_{(0,i-1)}$, $\lambda'\sigma_{(0,i-1)}|_{\hat{w}}$ is a subterm in which v occurs grey. But as the occurrence of v in $\lambda'\sigma_{(0,i-1)}|_{\hat{w}}$ must be contained in a single-colored Ψ -term, so is $\lambda\sigma_{(0,i-1)}|_{\hat{w}}$, hence z occurs in a single-colored Ψ -term as well. Therefore 2 is the case.

- Suppose there is a variable z in $\chi\sigma_{(0,i-1)}$ such that v occurs in a single-colored Φ -term in $z\sigma_i$. Then $z\sigma_i$ occurs in $\chi\sigma_{(0,i-1)}$, but this is a witness for 1.

Now recall that we have assumed u to be a color-changing variable in $\chi\sigma_{(0,i)}$. Hence it occurs in a single-colored Γ -term as well as in a single-colored Δ -term. By the reasoning above, this leads to two case:

- In $\chi\sigma_{(0,i-1)}$, u occurs both in some single-colored Γ -term as well as in some single-colored Δ -term. Then we get the result by the induction hypothesis and the fact that $u \notin \text{dom}(\sigma_i)$ as u does occur in $\chi\sigma_{(0,i)}$.
- Otherwise for some color Φ , u does not occur in a single-colored Φ -term in $\chi\sigma_{(0,i-1)}$. Then case 2 above must hold and there is some color-changing variable w in $\chi\sigma_{(0,i-1)}$ such that u occurs grey in $w\sigma_{(0,i)}$. But then by the induction hypothesis, w occurs grey in $\chi\sigma_{(0,i-1)}$ and hence u occurs grey in $\chi\sigma_{(0,i)}$. \square

1.5 Thursday:

In the following, we abbreviate $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ)$ by χ .

(lemma:var_grey_col_lit) **Lemma 5.** *Let ι be an inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs grey in a Φ -literal in $\chi\sigma_{(0,i)}$. Then at least one of the following statements holds:*

- (14_1) 1. *The variable u occurs grey in a Φ -literal in $\chi\sigma_{(0,i-1)}$.*
- (14_2) 2. *Some variable v occurs in $\chi\sigma_{(0,i-1)}$ grey in a Φ -literal as well as grey in a Ψ -literal such that u occurs grey in $v\sigma_i$.*
- (14_3) 3. *There is a variable v such that u occurs grey in $v\sigma_i^2$ and v occurs in $\chi\sigma_{(0,i-1)}$ either grey in a Φ -literal as well as in a single-colored Ψ -term in any literal, or grey in a Ψ -literal as well as in a single-colored Φ -term in any literal.*
- (14_4) 4. *The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,i-1)}$.*

Proof. We consider the different cases which lead to the variable u in a grey position in a Φ -literal in $\chi\sigma_{(0,i)}$:

- There already is a Φ -literal in $\chi\sigma_{(0,i-1)}$ which contains u at a grey position and σ_i does not change this. Then clearly 1 is the case.
 - Otherwise there must be a Φ -literal in $\chi\sigma_{(0,i-1)}$, which contains a variable v at a grey position such that u occurs grey in $v\sigma_i$. Hence in the resolved or factorised literals λ and λ' , there is a position p such that w.l.o.g. $\lambda|_p = v$ and $\lambda'|_p$ contains a grey occurrence of u , and λ and λ' coincide along p . We distinguish based properties of the position p :
 - Suppose that p is contained in a single-colored Φ -term. Then v occurs grey in a Ψ -literal as well as in a single-colored Φ -term, which suffices for 3 as u occurs grey in $v\sigma_i$.
 - Suppose that p is contained in a single-colored Ψ -term. Then u occurs grey in a Φ -literal as well in a single-colored Ψ -term, which implies 3.
 - Otherwise p is a grey position. We distinguish further:
 - * Suppose that the resolved or factorised literal is Φ -colored. Then u occurs grey in a Φ -literal and we have established item 3.
 - * Suppose that the resolved or factorised literal is Ψ -colored. Then the variable v occurs grey in a Φ -literal as well as grey in a Ψ -literal, hence 2 is the case.
- Otherwise the resolved or factorised literal is grey and u occurs grey in a grey literal, which is sufficient for 4. \square

(lemma:var_in_sc_term) **Lemma 6.** *Let ι be an inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs in a single-colored Φ -term in $\chi\sigma_{(0,i)}$. Then at least one of the following statements holds:*

²Note that this includes the case that $v = u$ and σ_i is trivial on u .

- (15_1) 1. The variable u occurs in a single-colored Φ -term in $\chi\sigma_{(0,i-1)}$.
- (15_2) 2. There is a variable v such that u occurs grey in $v\sigma_i$ and v occurs in a single-colored Φ -term as well as in a single-colored Ψ -term in $\chi\sigma_{(0,i-1)}$.
- (15_4) 3. There is a variable v such that u occurs grey in $v\sigma_i$ and v occurs in $\chi\sigma_{(0,i-1)}$ in a single-colored Φ -term as well as at a grey position in a Ψ -literal.
- (15_3) 4. The variable u occurs grey in a Φ -literal in $\chi\sigma_{(0,i-1)}$.
- (15_5) 5. The variable u occurs grey in a grey literal in $\chi\sigma_{(0,i-1)}$.

Proof. We consider the different cases which lead to the variable u in a single-colored Φ -term in $\chi\sigma_{(0,i)}$:

- There is a single-colored Φ -term in $\chi\sigma_{(0,i-1)}$ which contains u such that σ_i does not change this. Then 1 is the case.
- Suppose that there is a single-colored Φ -term in $\chi\sigma_{(0,i-1)}$ which contains a variable v such that u occurs grey in $v\sigma_i$.

Hence in the resolved or factorised literals λ and λ' , there is a position p such that w.l.o.g. $\lambda|_p = v$ and $\lambda'|_p$ contains a grey occurrence of u , and λ and λ' coincide along p . We distinguish based properties of the position p :

- Suppose that p is contained in a single-colored Φ -term. Then u is contained in a single-colored Φ -term in $\chi\sigma_{(0,i-1)}$ and item 1 holds.
- Suppose that p is contained in a single-colored Ψ -term. As then v is contained in a single-colored Φ -term as well as in a single-colored Ψ -term, 2 is the case.
- Suppose that p is a grey position. We distinguish further:
 - * Suppose that the resolved or factorised literal is Φ -colored. Then u occurs grey in a Φ -literal, which suffices for 4.
 - * Suppose that the resolved or factorised literal is Ψ -colored. Then the variable v occurs in a single-colored Φ -term as well as grey in a Ψ -literal, which implies 3.
 - * Otherwise the resolved or factorised literal is grey. But then u occurs grey in a grey literal and we have established item 5.
- Suppose that a variable w occurs in $\chi\sigma_{(0,i-1)}$ such that u occurs in a single-colored Φ -term in $w\sigma_i$. This can only be the case if $w\sigma$ already occurs in $\chi\sigma_{(0,i-1)}$, which implies that 1 is the case. \square

ol_change_and_grey_in_col_lit) **Lemma 7.** Let C be a clause in the resolution refutation π of $\Gamma \cup \Delta$ and u be a variable which occurs in $\text{PI}(C) \vee C$ in some literal in a single-colored Φ -term or grey in a Φ -literal.

Suppose that u also occurs in $\text{PI}(C) \vee C$ in some literal in a single-colored Ψ -term or grey in a Ψ -literal.

Then u occurs grey in a grey literal.

Note that Φ and Ψ are to be read as any pair of different colors, i.e. Γ and Δ as well as Δ and Γ .

Proof. We proceed by induction over π and σ .

Note that initially, every pair of clauses is variable-disjoint and all symbols of a clause are either all grey or Φ -colored or all grey or Ψ -colored, hence the lemma is vacuously true.

For the induction step, we assume that the property holds for $\text{PI}(C_i) \vee C_i$, $1 \leq i \leq n$, where C_1, \dots, C_n are the clauses used in a resolution or factorisation inference ι . Note that then, the property also holds for χ , i.e. for $\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ$ as it contains all the grey literals present in $\text{PI}(C_i) \vee C_i$ for any i (this is evident by the definition of $\text{PI}_{\text{step}}^\circ$), and as clauses are pairwise variable-disjoint, the lemma condition can not become true for a variable for which it was not true in $\text{PI}(C_i) \vee C$ for some i .

Suppose that u occurs in $\chi\sigma_{(0,i)}$ in a single-colored Φ -term or grey in a Φ -literal and that u also occurs in $\chi\sigma_{(0,i)}$ in a single-colored Ψ -term or grey in a Ψ -literal.

Then we can deduce by Lemma 5 and Lemma 6 that at least one of the following statements holds:

- $\langle \text{oozoh70h1} \rangle$ 1. The variable u occurs grey in a Φ -literal in $\chi\sigma_{(0,i-1)}$.
- $\langle \text{oozoh70h2} \rangle$ 2. Some variable v occurs in $\chi\sigma_{(0,i-1)}$ grey in a Φ -literal as well as grey in a Ψ -literal such that u occurs grey in $v\sigma_i$.
- $\langle \text{oozoh70h3} \rangle$ 3. There is a variable v such that u occurs grey in $v\sigma_i$ and v occurs in $\chi\sigma_{(0,i-1)}$ either grey in a Φ -literal as well as in a single-colored Ψ -term in any literal, or grey in a Ψ -literal as well as in a single-colored Φ -term in any literal.
- $\langle \text{oozoh70h4} \rangle$ 4. The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,i-1)}$.
- $\langle \text{oozoh70h5} \rangle$ 5. The variable u occurs in a single-colored Φ -term in $\chi\sigma_{(0,i-1)}$.
- $\langle \text{oozoh70h6} \rangle$ 6. There is a variable v such that u occurs grey in $v\sigma_i$ and v occurs in a single-colored Φ -term as well as in a single-colored Ψ -term in $\chi\sigma_{(0,i-1)}$.

By the same lemmata, we get the same set of statements where Φ and Ψ are interchanged. We refer to them by the respective number followed by $*$.

Suppose that 4 is not the case as otherwise we are done since σ_i is trivial on u as u occurs in $\chi\sigma_{(0,i)}$. Furthermore, there are a number of cases which give the result by the induction hypothesis: For the cases 2, 3 and 6, we can infer that by the induction hypothesis, there is a grey occurrence of the variable v in a grey literal in $\chi\sigma_{(0,i-1)}$, and as u occurs grey in $v\sigma_i$, there is a grey occurrence of u in a grey literal in $\chi\sigma_{(0,i)}$.

It remains to show that the lemma holds true in case the statements 1 or 5 as well as 1* or 5* hold. But note that in any combination of 1 or 5 and 1* or 5* in effect yields a situation under which the induction hypothesis again is applicable. Hence we may infer that u occurs grey in a grey literal in $\chi\sigma_{(0,i-1)}$ and since σ_i is trivial on u as shown above, u occurs grey in a grey literal in $\chi\sigma_{(0,i)}$. \square

1.6 Friday

Lemma 8. *If $\text{PI}(C) \vee C$ contains a maximal colored occurrence of a Φ -term $t[s]$ containing Ψ -term s , then s occurs grey in a grey literal in $\text{PI}(C) \vee C$.*

Proof. Note that it suffices to show that at the step where s is introduced as subterm of $t[s]$, s occurs grey in $\text{PI}(C) \vee C$ as any later modification by substitution is applied to both occurrences s , so they stay equal throughout the remaining derivation.

Induction over π and σ . **TODO:** as in Lemma 7

Base case: vacuously true.

Step: Resolution or factorisation inference ι , $\text{mgu}(\iota) = \sigma = \sigma_1 \cdots \sigma_n$. The term $t[s]$ is created by one of the following two ways:

(we abbreviate $(\text{PI}_{\text{step}}^\circ(\iota, \text{PI}(C_1), \dots, \text{PI}(C_n)) \vee C^\circ$ by F .)

- A variable u occurs in $\chi\sigma_{(0,i-1)}$ such that $u\sigma_i = t[s]$.

Then u occurs in a resolved or factorised literal $\lambda\sigma_{(0,i-1)}$ at \hat{u} such that at the other resolved or factorised literal $\lambda'\sigma_{(0,i-1)}$, $\lambda'\sigma_{(0,i-1)}|_{\hat{u}} = t[s]$. Then the condition is present at $\chi\sigma_{(0,i-1)}$ and we get the result by the induction hypothesis.

- Note that we only consider maximal colored terms.

Let $t[u]$ be a maximal colored Φ -term in $\chi\sigma_{(0,i-1)}$ such that in the tree-representation of $t[u]$, the path from the root to u does not contain a node labelled with a Ψ -symbol, and $u\sigma_i$ contains a grey occurrence of s .

Suppose that u occurs grey in a grey literal in $\chi\sigma_{(0,i-1)}$. Then s occurs grey in a grey literal in $\chi\sigma_{(0,i)}$ as σ_i does not affect u since u occurs in $\chi\sigma_{(0,i)}$ and we are done.

If u occurs grey in a Ψ -literal or if u occurs in a single-colored Ψ -term in $\chi\sigma_{(0,i-1)}$, then by Lemma 7, u also occurs grey in a grey literal in $\chi\sigma_{(0,i-1)}$ and s hence occurs grey in a grey literal in $\chi\sigma_{(0,i)}$.

Now suppose that u does not occur grey in a grey literal $\chi\sigma_{(0,i-1)}$ as otherwise clearly we are done.

Hence as all other cases are excluded, u can only occur in $\chi\sigma_{(0,i-1)}$ in a single-colored Φ -term or grey in a Φ -colored literal. But then, since $u\sigma_i$ contains a grey occurrence of s , there is a position p in the two resolved or factorised literals λ and λ' such that $\lambda|_p = u$ and $\lambda'|_p$ contains a grey occurrence of s . Furthermore, the prefix along the path to p is the same in both λ and λ' . As u only occurs in single-colored Φ -terms, $\lambda'|_p$ does so as well, so s is contained in a single-colored Φ -term in $\chi\sigma_{(0,i-1)}$. Since s is a Ψ -term, by the induction hypothesis, s occurs grey in a grey literal in $\chi\sigma_{(0,i-1)}$ and hence also in $\chi\sigma_i$. \square

are probably not same t and s as in lemma statement, which isn't technically wrong but confusing

(lemma:grey_lits_all_in_PI) **Lemma 9.** *If there is a grey literal λ in a clause C of a resolution refutation π , then a successor of λ occurs in $\text{PI}(\pi)$.*

Proof. Immediate by the definition of PI . \square

TODO: define quantifier alternations as col-alt, 0 == no quants, 1 == one quant, 2 is Π_2 or Σ_2

Proposition 10. *If a term with n color alternations occurs in $\text{PI}(C) \vee C$ for a clause C , then the interpolant I produced in Theorem ?? contains at least n quantifier alternations.*

Proof. We perform an induction on n and show the strengthening that the quantification of the lifting variable corresponding to a term with n color alternations is required to be in the scope of the quantification of $n-1$ alternating quantifiers.

For $n = 0$, no colored terms occur in I and hence by construction no quantifiers and for $n = 1$, there are only single-colored terms

Suppose the statement holds for $n-1$ for $n > 1$ and a term t with $\text{col-alt}(t) = n$ occurs in $\text{PI}(C)$. We assume that t is a Φ -term. Then t contains a Ψ -colored term s and by Lemma 7, s occurs grey in a grey literal in $\text{PI}(C) \vee C$. By Lemma 9, a successor of s occurs in $\text{PI}(\pi)$. By the induction hypothesis, the quantification of the lifting variable for s requires $n-1$ alternated quantifiers. As s is a subterm of t and t is lifted, t must be quantified in the scope of the quantification of s , and as t and s are of different color, their quantifier type is different. Hence the quantification of the lifting variable for t requires n quantifier alternations. \square