Example 101: canonical examples for overbinding w.r.t. order

Here, the second lower prooftree shows what huang would produce.

Ex 101a

$$\dfrac{\dfrac{\overset{\Sigma}{P({\color{red}u},f({\color{red}u}))\lor Q({\color{red}u})} \qquad \overset{\Pi}{\neg Q(a)}}{P(a,f(a))}\;u\mapsto a \qquad \overset{\Pi}{\neg P(x,y)}}{\square}\;x\mapsto a, y\mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{Q(a)}\;u\mapsto a \qquad \top}{P(a,f(a))\lor Q(a)}\;x\mapsto a, y\mapsto f(a) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 Q(x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\lor Q(x_1))}$$

Direct overbinding would not work without merging same variables!: $\forall x_1\forall x_2\exists x_3(P(x_2,x_3)\lor Q(x_1))$
counterexample: $Q\sim\{0\}, P\sim\{(1,0)\}$
Direct overbinding would work when considering original dependencies as highlighted above
arrow lemma:

$$\dfrac{\dfrac{\Gamma\models\exists y_1(\overset{\Sigma}{P({\color{red}u},y_1)}\lor Q({\color{red}u})\lor\bot) \qquad \Gamma\models\neg\overset{\Pi}{Q(x_1)}\lor\top}{\Gamma\models\exists y_1(P({\color{red}x_1},y_1)\lor Q({\color{red}x_1}))}\;u\mapsto a \qquad \Gamma\models\neg\overset{\Pi}{P(x,y)}\lor\top}{\Gamma\models(\forall x_1)\exists y_1(Q({\color{red}x_1})\lor P({\color{red}x_1},y_1))}\;x\mapsto a, y\mapsto f(a)$$

Ex 101b – other resolution order

$$\dfrac{\dfrac{\overset{\Sigma}{P(u,f(u))\lor Q(u)} \qquad \overset{\Pi}{\neg P(x,y)}}{Q(u))}\;y\mapsto f(u), x\mapsto u \qquad \overset{\Pi}{\neg Q(a)}}{\square}\;u\mapsto a$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(u,f(u))}\;x\mapsto f(u), x\mapsto u \qquad \top}{P(a,f(a))\lor Q(a)}\;u\mapsto a \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(u,x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\lor Q(x_1))}\;u\mapsto a$$

Ex 101c – Π and Σ swapped

$$\dfrac{\dfrac{\overset{\Pi}{P(u,f(u))\lor Q(u)} \qquad \overset{\Sigma}{\neg P(x,y)}}{Q(u))}\;y\mapsto f(u), x\mapsto u \qquad \overset{\Sigma}{\neg Q(a)}}{\square}\;u\mapsto a$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg P(u,f(u))}\;x\mapsto f(u), x\mapsto u \qquad \bot}{\neg P(a,f(a))\land\neg Q(a)}\;u\mapsto a \qquad\qquad \dfrac{\dfrac{\top \qquad \bot}{\forall x_2\neg P(u,x_2)} \qquad \bot}{\exists x_1\forall x_2(\neg P(x_1,x_2)\land\neg Q(x_1))}$$

arrow lemma:

$$\dfrac{\Gamma \models P(u, x_1) \overset{\Pi}{\vee} Q(u) \vee \top \qquad \Gamma \models \neg \overset{\Sigma}{P}(x, y) \vee \bot}{\Big(Q(u) \mid (\neg P(x, x_1) \wedge \top) \vee (P(u, /f(u)) \wedge \bot)\Big)\sigma} \; y \mapsto f(u), x \mapsto u$$

$$Q(u) \mid (\neg P(u, x_1) \wedge \top) \vee (P(u, f(u)) \wedge \bot)$$

employ σ' !?!!?!?!?!??!?????!!!

$$\Gamma \models Q(u) \mid \neg P(u, x_1)$$

$$\dfrac{\Delta \models Q(u) \mid \exists x_1 P(u, x_1) \qquad \Gamma \models \exists y_1 \, \neg \overset{\Sigma}{Q}(y_1)}{\text{both } u\text{'s on LHS need to become } a \text{ and then } y_1} \; u \mapsto a$$

$$\Gamma \models (\forall x_1)\exists y_1(\neg P(y_1, x_1) \vee \neg Q(y_1))$$

$$\Delta \models (\exists x_1)\forall y_1(P(y_1, x_1) \wedge Q(y_1))$$

Ex 101d – Π and Σ swapped, other resolution order

$$\dfrac{\dfrac{P(u, f(u)) \overset{\Pi}{\vee} Q(u) \qquad \neg \overset{\Sigma}{Q}(a)}{P(a, f(a))} \; u \mapsto a \qquad \neg \overset{\Sigma}{P}(x, y)}{\Box} \; x \mapsto a, y \mapsto f(a)$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg Q(a)} \; y \mapsto a \qquad \bot}{\neg Q(a) \wedge \neg P(a, f(a))} \; x \mapsto a, y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\top \qquad \bot}{\exists x_1 \neg Q(x_1)} \qquad \bot}{\exists x_1 \forall x_2 (\neg P(x_1, x_2) \wedge \neg Q(x_1))}$$

102 – similar to 101 but with intra-clause-set inferences

Ex 102a

$$\dfrac{\dfrac{P(f(x)) \overset{\Sigma}{\vee} Q(f(x), z) \qquad \neg \overset{\Pi}{P}(y)}{Q(f(x), z)} \qquad \dfrac{\neg Q(x_1, y) \overset{\Sigma}{\vee} R(y) \qquad \neg \overset{\Pi}{R}(g(z_1))}{\neg Q(x_1, g(z_1))} \; y \mapsto g(z_1), x_1 \mapsto f(x)}{\Box} \; x_1 \mapsto f(x), z \mapsto g(z_1)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(g(z_1))} \; y \mapsto g(z_1), x_1 \mapsto f(x)}{P(f(x)) \vee R(g(z_1))} \; x_1 \mapsto f(x), z \mapsto g(z_1) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}}{\exists x_1 \forall x_2 (P(x_1) \vee R(x_2)) \quad \text{(order irrelevant!)}}$$

Ex 102b

$$\dfrac{\dfrac{P(f(x)) \overset{\Sigma}{\vee} Q(f(x), z) \qquad \neg \overset{\Pi}{P}(y)}{Q(f(x), z)} \qquad \dfrac{\neg Q(f(y), z_1) \overset{\Sigma}{\vee} R(y) \qquad \neg \overset{\Pi}{R}(a)}{\neg Q(f(a), z_1)} \; y \mapsto a}{\Box} \; x \mapsto a, z \mapsto z_1$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\, y \mapsto a}{P(f(a)) \vee R(a)}\, x \mapsto a, z \mapsto z_1 \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\, y \mapsto a}{\forall x_2 \exists x_1 (P(x_1) \vee R(x_2))}\, x \mapsto a, z \mapsto z_1$$

direct:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)}\, x_1 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\, x_2 \sim a}{\exists x_1 \forall x_2 (P(x_1) \vee R(x_2))}\, \text{order irrelevant!}$$

Ex 102b' with Q grey

$$\dfrac{\dfrac{\overset{\Sigma}{P(f({\color{red}x})) \lor Q(f({\color{red}x}), z)} \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x), z)} \qquad \dfrac{\overset{\Sigma}{\neg Q(f({\color{blue}y}), z_1) \lor R({\color{blue}y})} \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a), z_1)}\; y \mapsto a}{\square}\; x \mapsto a, z_1 \mapsto z$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\; y \mapsto a}{(\neg Q(f(a), z) \land P(f(a))) \lor (Q(f(a), z) \land R(a))}\; x \mapsto a, z_1 \mapsto z$$

arrow lemma: (change of specification: $P(g(z_3))$ in clause in Σ instead of $P(f(x))$

$$\dfrac{\begin{array}{c}\dfrac{\overset{\Sigma}{\exists x_1(P(x_4) \lor Q(x_1, z))} \qquad \overset{\Pi}{\neg P(y)}}{\exists x_1(P(g(z_3)) \land \bot) \lor (P(x_4) \land \top) \mid Q(x_1, z)} \\ \exists x_1(P(x_4) \mid Q(x_1, z))\end{array} \qquad \begin{array}{c}\dfrac{\overset{\Sigma}{\exists x_2(\neg Q(x_2, z_1) \lor R(y))} \qquad \overset{\Pi}{\forall x_3\, \neg R(x_3)}}{\overset{\forall x_3}{\exists x_2}\; ((\neg R(x_3) \land \bot) \lor (R(a) \land \top) \mid \neg Q(x_2, z_1))}\; y \mapsto a \\ \overset{\forall x_3}{\exists x_2}\; ((\neg R(x_3) \land \bot) \lor (R(x_3) \land \top) \mid \neg Q(x_2, z_1))\end{array}}{\overset{\forall x_1}{\overset{\forall x_3}{\exists x_2}}\; \Big((\neg Q(x_2, z_1) \land P(x_4)) \lor (Q(x_1, z) \land R(x_3))\Big)}\; x \mapsto a, z_1 \mapsto z$$

arrow order: $x_3 < x_2$, x_2 same-block-as x_4: $\forall x_3 \exists x_2 \exists x_4 \forall x_1 \Big((\neg Q(x_2, z_1) \land \neg P(x_4)) \lor (Q(x_1, z) \land R(x_3))\Big)$

$\rightarrow$ bad example, plus some errors still in there

Huang:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)} \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\; y \mapsto a}{\forall x_1 \exists x_2 (\neg Q(x_2, z) \land P(x_2)) \lor (Q(x_2, z) \land R(x_1))}\; x \mapsto a, z_1 \mapsto z$$

direct:

$$\begin{array}{c}\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)}\; x_2 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\; x_1 \sim a}{\forall x_1 \exists x_2 \exists x_3 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}\; x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3 \\ \hline \text{OR: } \exists x_2 \forall x_1 \exists x_3 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1)) \\ \hline \text{OR: } \exists x_1 \exists x_3 \forall x_2 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))\end{array}$$

Huang-Version passt weil man für existenzielle quantoren eh gleichen term einsetzt

direct w mixed, slightly different:

$$\dfrac{\dfrac{\dfrac{\bot \mid P(f(x)) \lor Q(x, z) \qquad \top \mid \neg P(y)}{\exists x_2 P(x_2) \mid Q(x, z)}\; x_2 \sim f(x) \qquad \dfrac{\overset{\Sigma}{\bot \mid \neg Q(f(y), z_1) \lor R(y)} \qquad \overset{\Pi}{\top \mid \neg R(a)}}{\forall x_1 R(x_1) \mid \neg Q(f(a), z_1)}\; x_1 \sim a}{\forall x_1 \exists\, x_3 \exists x_2 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}\; x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3}{(\neg Q(f(a), z) \land P(f(f(f(a))))) \lor (Q(f(a), z) \land R(a))}$$

last dependency not crucial because other arrow is a Σ-arrow as well, but just changing it to Π (and changing f for g should produce a quantifier alternation)

Example 103: variables in interpolant get unified and order might have to be changed

103a: length changes but old version is still valid, despite huang's algorithm doing a sort

$$\cfrac{\cfrac{\cfrac{Q(f(x)) \lor \overset{\Sigma}{P(y)} \lor R(x) \qquad \overset{\Pi}{\neg Q(y_1)}}{P(y) \lor R(x)}\; y_1 \mapsto f(x) \qquad \overset{\Pi}{\neg P(h(g(a)))}}{R(x)}\; y \mapsto h(g(a)) \qquad \overset{\Pi}{\neg R(g(g(a)))}}{\square}\; x \mapsto g(g(a))$$

$$\cfrac{\cfrac{\cfrac{\bot \qquad \top}{Q(f(x))}\; y_1 \mapsto f(x) \qquad \top}{Q(f(x)) \lor P(h(g(a)))}\; y \mapsto h(g(a)) \qquad \top}{Q(f(g(g(a)))) \lor P(h(g(a))) \lor R(g(g(a)))}\; x \mapsto g(g(a))$$

$$\cfrac{\cfrac{\cfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_1 \forall x_2 (Q(x_1) \lor P(x_2))} \qquad \top}{X}$$

X:

Huang's algo gives:

$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

Direct overbinding gives: $x_3 < x_1$, rest arbitrary, hence:

$\forall x_3 \exists x_1 \forall x_2 (Q(x_1) \lor P(x_2) \lor R(x_3))$ <- this you do not get with huang

$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

$\forall x_3 \forall x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

103b: length changes "uniformly"

$$\cfrac{\cfrac{\cfrac{Q(f(f(x))) \lor \overset{\Sigma}{P(f(x))} \lor R(x) \qquad \overset{\Pi}{\neg Q(y_1)}}{P(f(x)) \lor R(x)}\; y_1 \mapsto f(f(x)) \qquad \overset{\Pi}{\neg P(y_2)}}{R(x)}\; y_2 \mapsto f(x) \qquad \overset{\Pi}{\neg R(g(a))}}{\square}\; x \mapsto g(a)$$

$$\cfrac{\cfrac{\cfrac{\bot \qquad \top}{Q(f(f(x)))}\; y_1 \mapsto f(f(x)) \qquad \top}{Q(f(f(x))) \lor P(f(x))}\; y_2 \mapsto f(x) \qquad \top}{Q(f(f(g(a)))) \lor P(f(g(a))) \lor R(g(a))}\; x \mapsto g(a)$$

$$\cfrac{\cfrac{\cfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_2 \exists x_1 (Q(x_1) \lor P(x_2))} \qquad \top}{\forall x_3 \exists x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))}$$

Huang and direct overbinding somewhat coincide as $x_2 < x_1$ in both cases, and $x_3 < x_1$ and $x_3 < x_2$

103c: Failed attempt: different variables, accidentally the same terms appear but no logical connection

$$\cfrac{\overset{\Sigma}{P(a,x)} \qquad \cfrac{\overset{\Sigma}{\neg Q(a)} \qquad \overset{\Pi}{\neg P(y, f(z)) \lor Q(z)}}{\neg P(y, f(a))}\; z \mapsto a}{\square}\; y \mapsto a, x \mapsto f(a)$$

Huang:

$$\dfrac{\perp \qquad \dfrac{\perp \qquad \top}{\neg Q(a)}\, z \mapsto a}{P(a, f(a)) \land \neg Q(a)}\, y \mapsto a, x \mapsto f(a) \qquad\qquad \dfrac{\perp \qquad \dfrac{\perp \qquad \top}{\exists x_1 \neg Q(x_1)}}{\exists x_1 \forall x_2 (P(x_1, x_2) \land \neg Q(x_1))}$$

order required for Π

direct:

$$\dfrac{\dfrac{\perp \qquad \dfrac{\perp \qquad \top}{\exists x_1 \neg Q(x_1)}\, x_1 \sim a}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \land \neg Q(x_1))}\, x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \land \neg Q(x_1))}$$

invariant:

$$\dfrac{\dfrac{\exists x_2 (P(x_2, x) \lor \perp) \qquad \dfrac{\exists x_1 (Q(x_1) \lor \perp) \qquad \forall x_3 ((\neg P(y, x_3) \lor Q(z)) \lor \top)}{\exists x_1 \forall x_3 \neg P(y, x_3) \lor \neg Q(x_1)}\, x_1 \sim a}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \land \neg Q(x_1))}\, x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \land \neg Q(x_1))}$$

invariant in other resolution order

$$\dfrac{\dfrac{\perp \qquad \dfrac{\perp \qquad \top}{Q(z) \lor \exists x_2 \forall x_3 P(x_2, x_3)}\, x_2 \sim a, x_3 \sim f(z)}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \land \neg Q(x_1))}\, x_1 \sim a; x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \land \neg Q(x_1))}$$

invariant if Σ and Π swapped:

$$\dfrac{\dfrac{\perp \qquad \dfrac{\top \qquad \perp}{\neg P(y, f(x_1)) \lor \forall x_1 Q(x_1)}\, x_1 \sim a}{\forall x_1 \forall x_2 \exists x_3 (\neg P(x_2, x_3) \lor Q(x_1))}\, x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \forall x_1 \exists x_3 \forall x_2 (\neg P(x_2, x_3) \lor Q(x_1))}$$

SECOND ATTEMPT:

$$\dfrac{\overset{\Sigma}{P(a)} \qquad \dfrac{\overset{\Sigma}{Q(z)} \qquad \dfrac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg P(y) \lor \neg Q(f(x)) \lor S(x)}}{\neg P(y) \lor \neg Q(f(a))}\, x \mapsto a}{\neg P(y)}\, z \mapsto f(a)}{\Box}\, y \mapsto a$$

$$\dfrac{\perp \qquad \dfrac{\perp \qquad \dfrac{\perp \qquad \top}{\neg S(a)}\, x \mapsto a}{\neg S(a) \land Q(f(a))}\, z \mapsto f(a)}{P(a) \land \neg S(a) \land Q(f(a))}\, y \mapsto a$$

Huang:

$$\cfrac{\bot \qquad \cfrac{\bot \qquad \cfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 (\neg S(x_1) \wedge Q(x_2))}}{\exists x_1 \forall x_2 (P(x_1) \wedge \neg S(x_1) \wedge Q(x_2))}$$

$\forall x_1 \exists x_2 (\neg P(x_1) \vee S(x_1) \vee \neg Q(x_2))$

similar fail

$\Rightarrow$ anytime there is $P(a, f(a))$, either they have a dependency or they are not both differently colored (grey is uncolored)

for the record, direct method anyway:

$$\cfrac{\bot \qquad \cfrac{\bot \qquad \cfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}\ x \sim a}{\exists x_1 \forall x_2 \neg S(x_1) \wedge Q(x_2)}\ z \sim f(a); x_1 < x_2}{\exists x_1 \forall x_2 \exists x_3 P(x_3) \wedge \neg S(x_1) \wedge Q(x_2)}\ x_3 \sim a; x_3 \text{ need not be merged w } x_1$$

Example: ordering on both ancestors where the merge forces a new ordering

202a – canonical

$$
\dfrac{
\dfrac{\overset{\Sigma}{P(a,x_1)\lor R(y)} \qquad \overset{\Pi}{\neg P(x,fx)}}{R(y)}\;\begin{array}{l}x_1\mapsto fa\\ x\mapsto a\end{array}
\qquad
\dfrac{\overset{\Sigma}{Q(x_2,gx_2)\lor\neg R(u)} \qquad \dfrac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg Q(fz,x_3)\lor S(z)}}{\neg Q(fa,x_3)}\; z\mapsto a}{\neg R(u)}\;\begin{array}{l}x_2\mapsto fa,\\ x_3\mapsto gfa\end{array}
}{\Box}
$$

$$
\dfrac{
\dfrac{\bot \qquad \top}{P(a,f(a))}\;\begin{array}{l}x_1\mapsto f(a)\\ x\mapsto a\end{array}
\qquad
\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg S(a)}\; z\mapsto a}{Q(f(a),g(f(a)))\land\neg S(a)}\;\begin{array}{l}x_2\mapsto f(a),\\ x_3\mapsto g(f(a))\end{array}
}{P(a,f(a))\lor(Q(f(a),g(f(a)))\land\neg S(a))}
$$

Huang

$$
\dfrac{
\dfrac{\bot \qquad \top}{\exists x_1\forall x_2 P(x_1,x_2))}
\qquad
\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1\neg S(x_1)}}{\exists x_1\forall x_2\exists x_3 Q(x_2,x_3)\land\neg S(x_1)}
}{\exists x_1\forall x_2\exists x_3 P(x_1,x_2)\lor(Q(x_2,x_3))\land\neg S(x_1))}
$$

direct:

$$
\dfrac{
\dfrac{\bot \qquad \top}{\exists x_1\forall x_2 P(x_1,x_2))}\;\begin{array}{c}x_1\sim a, x_2\sim fa\\ x_1<x_2\end{array}
\qquad
\begin{array}{c}x_3\sim a, x_4\sim fa, x_5\sim gfa)\\ x_3<x_4, x_4<x_5\end{array}
\;
\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_3\neg S(x_3)}\; x_3\sim a}{\exists x_3\forall x_4\exists x_5 Q(x_4,x_5)\land\neg S(x_3)}
}{\exists x_1\forall x_2\exists x_5 P(x_1,x_2)\lor(Q(x_2,x_5)\land\neg S(x_5))}\;\begin{array}{l}x_3\mapsto x_1, x_4\mapsto x_2\\ x_1<x_2, x_2<x_5\end{array}
$$

without merge in end: $(x_1<x_2, x_3<x_4, x_4<x_5)$
$\exists x_1\forall x_2\exists x_3\forall x_4\exists x_5 P(x_1,x_2)\lor(Q(x_4,x_5)\land\neg S(x_3))$
$\exists x_3\forall x_4\exists x_5\exists x_1\forall x_2 P(x_1,x_2)\lor(Q(x_4,x_5)\land\neg S(x_3))$
(also interwoven ones appear to work)

combined presentation:

$$\dfrac{\dfrac{\bot \mid P(a,x_1) \lor R(y) \qquad \top \mid \neg P(x,f(x))}{P(a,f(a)) \mid R(y)}\;\begin{matrix} x_1 \mapsto f(a) \\ x \mapsto a \end{matrix} \qquad \dfrac{\bot \mid Q(x_2,g(x_2)) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(z),x_3) \lor S(z)}{\neg S(a) \mid \neg Q(f(a),x_3)}\; z \mapsto a}{Q(f(a),g(f(a))) \land \neg S(a) \mid \neg R(u)}\;\begin{matrix} x_2 \mapsto f(a), \\ x_3 \mapsto g(f(a)) \end{matrix}}{P(a,f(a)) \lor (Q(f(a),g(f(a))) \land \neg S(a)) \mid \Box}$$

combined presentation ground:

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \lor R(y) \qquad \top \mid \neg P(a,f(a))}{(P(a,f(a)) \land \top) \lor (\neg P(a,f(a)) \land \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \lor S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(f(a),g(f(a))) \land \neg S(a) \mid \neg R(u)}}{P(a,f(a)) \lor (Q(f(a),g(f(a))) \land \neg S(a)) \mid \Box}$$

combined presentation ground with direct method but only Δ-terms removed :

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \lor R(y) \qquad \top \mid \neg P(a,f(a))}{(P(a,x_2) \land \top) \lor (\neg P(a,x_2) \land \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \lor S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(x_4,g(x_4)) \land \neg S(a) \mid \neg R(u)}}{P(a,x_2) \lor (Q(x_4,g(x_4)) \land \neg S(a)) \mid \Box}$$

combined presentation ground with direct method:

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \lor R(y) \qquad \top \mid \neg P(a,f(a))}{\exists x_1 \forall x_2 (P(x_1,x_2) \land \top) \lor (\neg P(x_1,x_2) \land \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \lor S(a)}{\exists x_3 \neg S(x_3) \mid \neg Q(f(a),g(f(a)))}}{\exists x_3 \forall x_4 \exists x_5 (Q(x_4,x_5)) \land \neg S(x_3)) \mid \neg R(u)}}{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (P(x_1,x_2) \lor (Q(x_4,x_5)) \land \neg S(x_3)) \mid \Box}$$

203a – some alternations

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x)\lor \overset{\Sigma}{\neg} P(f(x)) \qquad P(z)\lor \overset{\Pi}{Q}(g(z))}{R(x)\lor Q(g(f(x)))}\; z\mapsto f(x) \qquad \neg Q(y) \overset{\Sigma}{\lor} S(h(y))}{R(x)\lor S(h(g(f(x))))}\; y\mapsto g(f(x))}{S(h(g(f(a))))}\; x\mapsto a}{\Box}\; x_1\mapsto h(g(f(a)))$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\neg P(f(x))}\; z\mapsto f(x) \qquad \bot}{\neg Q(g(f(x)))\land\neg P(f(x))}\; y\mapsto g(f(x))}{\neg Q(g(f(a)))\land\neg P(f(a))\lor R(a)}\; x\mapsto a}{\neg Q(g(f(a)))\land\neg P(f(a))\lor R(a)\lor S(h(g(f(a))))}\; x_1\mapsto h(g(f(a)))$$

Huang:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \neg P(x_1)} \qquad \bot}{\exists x_1 \forall x_2(\neg Q(x_2)\land\neg P(x_1))}}{\forall x_0\exists x_1\forall x_2(\neg Q(x_2)\land\neg P(x_1)\lor R(x_0))}}{\forall x_0\exists x_1\forall x_2\exists x_3(\neg Q(x_2)\land\neg P(x_1)\lor R(x_0)\lor S(x_3))}$$

Direct:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \neg P(x_1)}\; x_1\sim f(x) \qquad \bot}{\exists x_1 \forall x_2(\neg Q(x_2)\land\neg P(f(x)))}\; x_2\sim g(f(x)); x_1<x_2}{\forall x_0\exists x_1\forall x_2(\neg Q(x_2)\land\neg P(x_1)\lor R(x_0))}\; x_0\sim a; x_0<x_1, x_0<x_2}{\forall x_0\exists x_1\forall x_2\exists x_3(\neg Q(x_2)\land\neg P(x_1)\lor R(x_0)\lor S(x_3))}\; x_3\sim h(g(f(a))); x_0<x_3, x_1<x_3, x_2<x_3$$

203b – many Σ-literals, coloring per occurrence

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x)\lor \overset{\Sigma}{\neg} P(f(x)) \qquad P(z)\lor \overset{\Sigma}{Q}(g(z))}{R(x)\lor Q(gfx)}\; z\mapsto fx \qquad \neg Q(y) \overset{\Sigma}{\lor} S(h(y))}{R(x)\lor S(hgfx)}\; y\mapsto gfx}{S(hgfa)}\; x\mapsto a}{\Box}\; x_1\mapsto hgfa$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \bot}{\bot}\; z\mapsto fx \qquad \bot}{\bot}\; y\mapsto gfx}{R(a)}\; x\mapsto a}{S(hgfa)\lor R(a)}\; x_1\mapsto hgfa$$

$\to \forall x_1\exists x_2(R(x_1)\lor S(x_2))$

203b' – many Σ-literals, coloring per symbol, all predicates grey

$$\dfrac{\neg \overset{\Pi}{S}(x_1) \qquad \dfrac{\dfrac{\dfrac{\neg \overset{\Pi}{R}(a) \qquad R(x) \vee \overset{\Sigma}{\neg} P(f(x))}{R(a) \mid \neg P(fa)}\; x \mapsto a \qquad P(z) \vee \overset{\Sigma}{Q}(g(z))}{P(fa) \vee R(a) \mid Q(gfa)}\; z \mapsto fa \qquad \neg Q(y) \vee \overset{\Sigma}{S}(h(y))}{\quad}}{\quad}$$

TODO

Example where variables are not the outermost symbol but order is still relevant

204a

$\Sigma = \{P(f(x), g(f(x)))\}$
$\Pi = \{P(f(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f(x_1), x_2)$

204b

$\Sigma = \{P(f^5(x), g(f(x)))\}$
$\Pi = \{P(f^5(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f^5(x_1), x_2)$

example with aufschaukelnde unification, such that direction of arrow isn't clear

205a

$$\dfrac{\overset{\Pi}{P(ff{\color{blue}y},g{\color{blue}y})} \qquad \dfrac{\overset{\Sigma}{\neg P({\color{red}x},y)\lor Q({\color{red}x})} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ff{\color{purple}z})\lor R{\color{purple}z}}}{\neg R(a) \mid \neg Q(ffa)}\; z\mapsto a}{\neg R(a)\land Q(ffa) \mid \neg P(ffa,y)}\; x\mapsto ffa}{(\neg R(a)\land Q(ffa))\lor\neg P(ffa,ga)}\; y\mapsto a$$

direct

$$\dfrac{\overset{\Pi}{P(ff{\color{blue}y},g{\color{blue}y})} \qquad \dfrac{\overset{\Sigma}{\neg P({\color{red}x},y)\lor Q({\color{red}x})} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ff{\color{purple}z})\lor R{\color{purple}z}}}{\exists x_1 \neg R(x_1) \mid \neg Q(ffa)}\; z\mapsto a}{\exists x_1\forall x_2(\neg R(x_1)\land Q(x_2)) \mid \neg P(ffa,u)}\; x\mapsto ffa}{\exists x_1\forall x_2\forall x_3((\neg R(x_1)\land Q(x_2))\lor\neg P(x_2,x_3))}\; y\mapsto a, u\mapsto ga$$

ground:

$$\dfrac{\overset{\Pi}{P(ff{\color{blue}a},g{\color{blue}a})} \qquad \dfrac{\overset{\Sigma}{\neg P({\color{red}ffa},y)\lor Q({\color{red}ffa})} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ff{\color{purple}a})\lor R{\color{purple}a}}}{\neg R(a) \mid \neg Q(ffa)}}{\neg R(a)\land Q(ff\,a) \mid \neg P(ffa,a)}}{(\neg R(a)\land Q(ffa))\lor\neg P(ff\,a,ga)}$$

arrow lemma: $x_1 \sim ffy$, $x_2 \sim gy$, $y_3 \sim a$ $x_4 \sim ffz$, $x_5 \sim ffa$, $y_6 \sim a$,

$$\dfrac{\overset{\Sigma}{\neg P({\color{red}x},y)\lor Q({\color{red}x}) \mid \bot} \qquad \dfrac{\overset{\Sigma}{\exists y_3(\neg R(y_3) \mid \bot)} \qquad \overset{\Pi}{\forall x_4(\neg Q(x_4)\lor R{\color{purple}z} \mid \bot)}}{\exists y_3\forall x_4((\neg R(y_3)\land\top)\lor\underline{(R(a)\land\bot)} \mid \neg Q(x_4))}\; z\mapsto a}{\exists y_3\forall x_4((\neg Q(x_4\land\bot))\lor(Q(x)\sigma\land\neg R(y_3)) \mid \neg P(x,y)\sigma)}\; x\mapsto ffa$$

$$\exists y_3\forall x_4((Q(ffa)\land\neg R(y_3)) \mid \neg P(ffa,y)) \quad (*)$$

$$\dfrac{\overset{\Pi}{\forall x_1\forall x_4(P(x_1,x_2) \mid \top)} \qquad \exists y_3\forall x_4\forall x_5((Q(x_5)\land\neg R(y_3)) \mid \neg P(x_5,y))}{(\neg P(x_5,y)\land\top)\lor(P(x_1,x_2)\land(Q(x_5)\land\neg R(y_3)))}\; y\mapsto a$$

$$(\neg P(x_5,a)\land\top)\lor(P(x_1,x_2)\land(Q(x_5)\land\neg R(y_3)))$$

$$\forall x_1\forall x_2\exists y_3\forall x_4\forall x_5\forall x_6(\neg P(x_5,y_6)\land\top)\lor(P(x_1,x_2)\land(Q(x_5)\land\neg R(y_3)))$$

$(*)$ "luckily", same overbinding for ffa, so this works
dashed underline: problem, but does not cause issues here.

situations not critical here :'-(

205b $\sim$ 205a, but simpler

Suppose P occurs somewhere in Σ (result not that optimal in this setting, but correct)
not nice for proving, $\neg R(a)$ is a nice interpolant already

$$\dfrac{\overset{\Pi}{P(ff{\color{blue}y},g{\color{blue}y})} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg P(ff{\color{purple}z},x)\lor R{\color{purple}z}}}{\neg R(a) \mid \neg P(ffa,x)}\; z\mapsto a}{\neg R(a)\lor\neg P(ffa,ga) \mid \Box}\; x\mapsto ga, y\mapsto a$$

$$\dfrac{\overset{\Pi}{\top \mid P(ff{\color{blue}y},g{\color{blue}y})} \qquad \dfrac{\overset{\Sigma}{\bot \mid \neg R(a)} \qquad \overset{\Pi}{\top \mid \neg P(ff{\color{purple}z},x)\lor R{\color{purple}z}}}{\exists x_1\neg R(x_1) \mid \neg P(ffa,x)}\; z\mapsto a}{\exists x_1\forall x_2\forall x_3\neg R(x_1)\lor\neg P(x_2,x_3) \mid \Box}\; x\mapsto ga, y\mapsto a$$

$\exists x_1 R(x_1)$

$\exists x_1\forall x_2\forall x_3(R(x_1)\lor\neg P(x_2,x_3))$

example to demonstrate that literals being resolved upon have to be overbound with the same variable

206a

$$\dfrac{\dfrac{\overset{\Pi}{R(f(x))} \qquad \overset{\Sigma}{\neg R(y) \vee P(y)}}{(\neg R(x_3) \wedge \top) \vee (R(x_3) \wedge \bot) \mid P(x_3)} \qquad \dfrac{\overset{\Pi}{\neg P(f(z)) \vee S(z)} \qquad \overset{\Sigma}{\neg S(a)}}{(\neg S(y_2) \wedge \top) \vee (S(y_2) \wedge \bot) \mid \neg P(x_4)}}{(\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)}$$

Gist of this example: $P(f(x))$ is lifted to the left, but $P(f(a))$ to the right. So it's $P(x_3)$ vs $P(x_4)$, but both of them have to have the same variable.

$R(x_3) \in \mathrm{AI}_{\mathrm{mat}}(C_7)$
$P(x_3) \in \mathrm{AI}_{\mathrm{cl}}(C_7)$
$P(x_4) \in \mathrm{AI}_{\mathrm{cl}}(C_8)$
$\Sigma \models (\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)$
$\Sigma \models (\forall x_3)\forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee P(x_3)\Big)$
$\Sigma \not\models (\neg P(1) \wedge \neg R(0)) \vee P(0)$ // if $P \sim \{1\}$ and $R \sim \{0\}$
we know that for original clauses l and l' of $P(x_4)$ and $P(x_3)$, $l\sigma = l'\sigma$
hence same color, and can use different var as same value works.

$f(x) \lor g(x)$ with f, g different colors

$$\dfrac{\overset{\Pi}{\neg P(z)} \qquad \dfrac{\overset{\Pi}{\neg R(g(a))} \qquad \dfrac{\overset{\Sigma}{P(f(x)) \lor Q(x)} \qquad \overset{\Pi}{R(g(y)) \lor \neg Q(y)}}{\forall x_1 Q(x_1) \mid P(f(x)) \lor R(g(x))}\; y \mapsto x}{\forall x_1 Q(x_1) \mid P(f(a)) // R(g(a))}\; x \mapsto a}{\forall x_1 \exists x_2 \big(Q(x_1) \lor P(x_2)\big) // R(g(a))}\; z \mapsto f(a)$$

$\Rightarrow$ free vars in the interpolant have to be overbound (if there are arrows, but we can just always do so)

misc examples

201a

$$\dfrac{\dfrac{\overset{\Sigma}{P(x,y) \vee \neg Q(y)} \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(y)}\, x \mapsto a \qquad \dfrac{\overset{\Sigma}{Q(f(z)) \vee R(z)} \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a))}\, z \mapsto a}{\Box}\, y \mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,y)}\, x \mapsto a \qquad \dfrac{\bot \qquad \top}{R(a)}\, z \mapsto a}{P(a,f(a)) \vee R(a)}\, y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 P(x_1,y)}\, x \mapsto a \qquad \dfrac{\bot \qquad \top}{\forall x_3 R(x_3)}\, z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 (P(x_1,x_2) \vee R(x_3))}\, y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

201b

$$\dfrac{\dfrac{\overset{\Sigma}{P(x,f(y)) \vee \neg Q(f(y))} \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(f(y))}\, x \mapsto a \qquad \dfrac{\overset{\Sigma}{Q(f(z)) \vee R(z)} \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a))}\, z \mapsto a}{\Box}\, y \mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,f(y))}\, x \mapsto a \qquad \dfrac{\bot \qquad \top}{R(a)}\, z \mapsto a}{P(a,f(a)) \vee R(a)}\, y \mapsto a \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 \exists x_2 P(x_1,x_2)}\, x \mapsto a \qquad \dfrac{\bot \qquad \top}{\forall x_3 R(x_3)}\, z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_3)}\, y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

arrow in element which is not in interpolant or resolution clause

206

$$
\dfrac{\dfrac{\overset{\Sigma}{P(x) \vee \neg Q(f(x))} \qquad \overset{\Pi}{\neg P(a)}}{\forall x_1 P(x_1) \mid \neg Q(f(a))}\; x \mapsto a \qquad \dfrac{\overset{\Sigma}{Q(y) \vee R(g(y))} \qquad \overset{\Pi}{\neg R(z)}}{\exists x_2 R(x_2) \mid Q(y)}\; z \mapsto g(y)}{\forall x_1 \exists x_2 (P(x_1) \vee R(x_2)) \mid \Box}\; y \mapsto f(a)
$$

$$P(a) \vee R(g(f(a)))$$

for first interpolant, $\Sigma \not\models \ell_{\Delta,x}[\mathrm{PI}(C)] \vee C$

=> need to overbind clause as well