1 Overbinding in one step

Conjecture 1. *Suppose every variable occurs only once in* $\Gamma \cup \Delta$*. Then the order of the quantifiers for* $\mathrm{PI}(\square)^*$ *does not matter.*

Proposition 2. *Let* $A(x_1, \ldots, x_n)$ *be an atom in a relative interpolant. A variable occurs in one of the* x_i *if and only if there are atoms* $A(y_1, \ldots, y_n)$ *and* $A(z_1, \ldots, z_n)$ *in* Γ *and* Δ *respectively, where* x_i *can be unified with* z_i *and* y_i *such that there is still a variable at that location.*
This means that either the term structure above the variable is the same in the original clauses or there are some variables. Intended meaning: the original clauses prove at least the x_i*, i.e. are at least as or more general.*

Special case for outermost variables:
Let $A(x_1, \ldots, x_n)$ *be an atom in a relative interpolant. An* x_i *is a variable if and only if there are atoms* $A(y_1, \ldots, y_n)$ *and* $A(z_1, \ldots, z_n)$ *in* Γ *and* Δ *respectively, where* y_i *and* z_i *are variables.*

need more narrow version: clauses do appear in parent clauses in derivation.

Proposition 3. *Suppose in a partial interpolant, there are two maximal terms* t_1 *and* t_2 *such that w.l.o.g.* t_1 *is smaller (as defined in 5) than* t_2*. Then it the final interpolant, an overbinding can be defined where the variable corresponding to* t_1 *is quantified over before the variable corresponding to* t_2 *is.*

The subterm-relation is reflexive.

Definition 4. (OLD) Let s be a term that is in $\mathrm{PI}(C)$ but not in any predecessor $\mathrm{PI}(C_i)$, $i \in \{1, 2\}$. s is smaller than a term t in $\mathrm{PI}(C)$ if s is of strictly smaller length than t and there is a subterm in s which also occurs in t. △

Definition 5. (NEW)
OUTDATED! DOES NOT WORK LIKE THIS
Let C be a clause.
A maximal term s of C is smaller than a maximal term t of C if s is a variable and occurs in t, but $s \neq t$.
OUTDATED! DOES NOT WORK LIKE THIS
△

2 Half-baked approaches

Definition 6. Direct interpolation extraction.
This version of overline and star does NOT overbind variables! If they happen to be in the final interpolant, just overbind them somehow, but not earlier. This is ok as the interpolant only contains variables if both corresponding atoms in Γ and Δ do. Variables are the only terms in the interpolant that can "change their color", so we don't know a priori if there are constraints on the quantifier to overbind them with.

Convention w.r.t. a clause C which has been derived from C_1 and C_2: $\bar{Q}_n = Q_1 z_1 \dots Q_n z_n$, such that the z_i correspond to the maximal terms t_i in $\mathrm{PI}(C)$. Same terms must be overbound by same variable, see 101a for counterexample to per-occurrence-overbinding. The z_i are ordered such that

1. the orderings in the Q_{n_1} and Q_{n_2} are respected (no circlular relations can occur in combination with merging as a term is only smaller than another term if it is smaller in length as well, which excludes cycles)
2. as well as ordering constraints of terms newly introduced in $\mathrm{PI}(C)$ (i.e. those that were not present in $\mathrm{PI}(C_1)$ and $\mathrm{PI}(C_2)$).

Basically, track dependencies and define actual order later.

Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad \sigma = \mathrm{mgu}(l, l')$$

$\bar{Q}_{n_1}\, \mathrm{PI}(C_1)^*$

$\bar{Q}_{n_2}\, \mathrm{PI}(C_2)^*$

1. l and l' Γ-colored:

   $\mathrm{PI}(C) \equiv (\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2))\sigma$

   $\mathrm{PI}(C)^* \equiv (\mathrm{PI}(C_1)^* \vee \mathrm{PI}(C_2)^*)\sigma$ (just replace maximal terms)

   intended meaning of σ: to change the free variables still in the $\mathrm{PI}(C_i)$

   TODO: basically do nothing here since no new atoms (revisit after mixed colored case has been dealt with)

   Let $t_1, \dots, t_{n_1}$ be terms overbound in $\mathrm{PI}(C_1)$ and $s_1, \dots, s_{n_2}$ terms overbound in $\mathrm{PI}(C_2)$.

   $\{z_1, \dots, z_n\} = \{t_1, \dots, t_{n_1}\}\sigma \cup \{s_1, \dots, s_{n_2}\}\sigma$ // common terms are merged

   order relations as in C_1, C_2

   $\bar{Q}_n\, \mathrm{PI}(C)^* \equiv \bar{Q}_n(\mathrm{PI}(C_1)^* \vee \mathrm{PI}(C_2)^*)$
2. l and l' Δ-colored:

   similar to first case
3. l and l' grey: nothing here

$\triangle$

3 Arrow-Algo

1. In the original clauses, find all occurrences of variables.

   Common case: If a variable appears as outermost symbol or only has grey ancestor-terms, add an arrow from it to all other occurrences.

   Uncommon case: if there is more than one occurrence of a variable under a Φ-colored term, add a *weak* dependency between them all (symmetric relation).

   NOTE: this creates double arrows for occurrences at same depth. This appears to be necessary for terms which are only variables, and doesn't hurt if the variable is contained in a term.

2. For each step in the derivation:

   a) Build propositional interpolant using $\mathrm{PI}(C_i)^*$, $i \in \{1, 2\}$, i.e. use ancestor PI without colored terms.

   b) If ancestors of atom added to $\mathrm{PI}(C)$ had arrows, merge them to atom in $\mathrm{PI}(C)$ (i.e. arrows starting in and leading to this atom).

   c) Replace colored terms in $\mathrm{PI}(C)$ (from new atom and unifier applied to $\mathrm{PI}(C_i)^*$) with fresh variables, except if a term has a double ended arrow to another overbinding variable, then use that variable.

      An arrow starts (ends) in one of the new variables if it starts (ends) somewhere in the term it replaced.

   d) Collect quantifiers: from $\mathrm{PI}(C_i)^*$, $i \in \{1, 2\}$ and ones from atom added to $\mathrm{PI}(C)$. Order such that arrows only point to variables to the right AND weakly connected variables appear in the same quantifier block.

      $\bar{Q}_n = \mathrm{sort}(Q_{n_1} \cup Q_{n_2} \cup \text{colored-terms}(l))$

3.1 algo more formally

Ex: $P(y, a, f(z, g(y, b))) \lor Q(x)$

Term position:

0.2.1.0 means first literal, 3rd arg, 2nd arg, fst arg: y

0.1 is a

0.2.1 is $g(z, b)$

for a position p, $p(p_i)$ denotes whatever p_i refers to in its respective clause.

for a term t, $p(t)$ denotes the position in t in its respective clause.

for a position p, $p_{\text{lit}}(p)$ denotes the position of the literal

for a position p, $p_{\text{term}}(p)$ denotes the position of the term in p_i

$\Rightarrow p = p_{\text{lit}}(p).p_{\text{term}}(p)$

3.2 Arrows:

$\mathcal{A}(C)$ for a clause C is a set of ordered pairs or term positions in C or $\text{PI}(C)$.

$\mathcal{W}(C)$ for a clause C is a set of unordered pairs or term positions in C or $\text{PI}(C)$.

w.r.t a refutation π of $\Gamma \cup \Delta$:

1. For each initial clause C in $\Gamma \cup \Delta$:

   Add to $\mathcal{A}(C)$ all (p_1, p_2) such that p_1 contains only grey symbol and $p(p_1) = p(p_2)$ but $p_1 \neq p_2$.

   Add to $\mathcal{W}(C)$ all $\{p_1, p_2\}$ such that there is a colored symbol in p_1 and a possible different one in p_2 and $p(p_1) = p(p_2)$.

2. For each C resulting from a resolution step from $C_1 : D \lor l$ and $C_2 : E \lor \neg l$ to $C = D \lor E$ with prop interpolant $\text{PI}(\cdot)$:

   Each literal in D (E) in C is said to come from the respective literal in C_1 (C_2).

   If l and/or l' is added to $\text{PI}(C)$, it comes from both l and l'.

   $(p_1, p_2) \in \mathcal{A}(C) \Leftrightarrow p_{\text{lit}}(p_i)$ comes from literal l_i in a clause D in an original clause set, $i \in \{1, 2\}$, (both from same D), and $(l_1.p_{\text{term}}(p_1), l_2.p_{\text{term}}(p_2)) \in \mathcal{A}(D)$

3.3 algo

1. For each initial clause C, $\text{AI}(C) = \text{PI}(C)$.

2. For each C resulting from a resolution step from $C_1 : D \lor l$ and $C_2 : E \lor \neg l$ to $C = D \lor E$ with prop interpolant $\text{PI}(\cdot)$:

   if l and l' don't have the same color:

   $\text{AI}_{\text{matrix}}(C) = (\neg \ell_{\Gamma,y}[\ell_{\Delta,x}[l]] \land \text{AI}_{\text{matrix}}(C_1)) \lor (\ell_{\Gamma,y}[\ell_{\Delta,x}[l]] \land \text{AI}_{\text{matrix}}(C_2))$

   $\text{AI}(C) = Q_1 u_1 \ldots Q_m u_m \, \text{AI}_{\text{matrix}}(C)$

   $u_1, \ldots, u_m$ are comprised of all x_i and y_i in $\text{AI}_{\text{matrix}}(C)$.

   Q_i is $\exists$ if $u_i = y_i$ for some i, $\forall$ otherwise.

$(p_1, p_2) \in \mathcal{A}(C)$ implies that $u_i < u_j$ if u_i replaces t_i and u_j replaces t_j and p_1 points into t_i and p_2 points into t_j.

4 current proof attempts

Lemma 7. *If an atom A appears in the interpolant, it appeared in both original clause sets, once positively and once negatively.*
A is contained in some instance of the respective clauses in Γ and Δ.

Lemma 8. *Let $C \in \Phi$ for some initial clause set Φ.*

1. *Let x be an occurrence of a variable in C and x' another occurrence of the same variable in a different position but at the same term depth. Then $\Phi \models QyC[x/y][x'/y]$ for $Q \in \{\forall, \exists\}$.*

2. *Let x be an occurrence of a variable in C with the lowest depth and x' another occurrence of the same variable with a higher depth. Let t be the maximal colored term which contains x'. t is Φ-colored since it appears in Φ. Then $\Phi \models Qy\exists zC[x/y]\{t/z\}$ for $Q \in \{\forall, \exists\}$.*

Lemma 9. *Let t be a maximal colored term in C in Φ. It is Φ-colored. Let $x_1, \dots, x_n$ be the variables which occur in t. Then $\Phi \models Q\bar{x}\exists yC\{t/y\}$ for $Q \in \{\forall, \exists\}$.*

We have that $\Gamma \models \forall \bar{x} \ell_{\Delta,x}[C] \vee C$.

Let t be a maximal Γ-term. It in general contains Γ-colored and grey terms, and also Δ-terms. The latter have entered it by unification.
If t contains no Δ terms, we can just overbind it existentially and give a witness.
Otherwise it contains Δ-terms. Then there is a variable in t at position say p which also occurs elsewhere in C, say at position q.
If q is the outermost term or if it has only grey term ancestors, then quantifying over whatever is in q before quantifying over whatever is in p is fine. Hence there is an arrow.
q can not be contained in a Δ term since dependencies cannot be introduced and must be there from the beginning, where no color mix is possible.
So otherwise q is contained in a maximal Γ-term s. For finding witnesses, we will put the same one for the variable at both q and t. As q introduces a Δ-term, at some point, there had had to be a unification with a formula from Δ (this then could have been passed on through "mirroring").
Conjecture: there are arrows along the path from the origin of the Δ-path to q.
Hence whatever is placed in q and p is quantified over earlier than the variables which replace t and s.
TODO: Proof or refute...

conjecture: put all terms that share variables and appear in the same clause and are all overbound with the same quantifier in the same quantifier block.
probably does not work when facing other dependencies, check that !

Notation:
p_1 is the position of s in t
p_2 is the elsewhere position of the shared var
A unification where a Γ-colored term s enters t happens when

- the other unified clause has a variable at the position of t (p_1)
- a variable is both in t (p_1) and elsewhere in the unified clause (p_2)
- p_2 is either in a grey term or as outermost or a Δ-colored term
  - if p_2 directly in grey term or as outermost, the ancestor of p_2 will not be overbound (only p_2). we need p_2 as witness for overbinding t, but not the other way.

    Hence quantifying over p_2 first is ok.

- if p_2 is in a Δ-colored term, say in maximal Δ-term s', Then s' and t are overbound with the same quantifier and order between them doesn't matter.

  for witness, we both need whatever the var is, and that we get by the inherited relation.

  ? there must be an inherited relation as since both s' and t are Γ and contain a Δ-term, the Δ term must have gotten into a Γ-colored term using aufschauckeln ?

5 structured proof

Lemma 10. *$\Gamma \models PI(C) \vee C$ for C in a prop proof.*

Proof. See Huang. □

Lemma 11. *$\Gamma \models \forall x_1 \ldots \forall x_n(\ell_{\Delta,x}[PI(C)]) \vee C$ for C in a prop proof.*

Proof. Still the same as in Huang. □

Lemma 12. *$\Gamma \models \bar{Q}_n(\ell_{\Gamma,y}[\ell_{\Delta,x}[PI(C)]]) \vee C$ for C in a prop proof.*

Proof. By 11, $\Gamma \models \forall x_1 \ldots \forall x_n(\ell_{\Delta,x}[PI(C)]) \vee C$. Show that the existential quantifiers in $\bar{Q}_n$ have witnesses. □

6 old stuff, not sure if valuable

Proposition 13. $\Gamma \models Q_1 z_1 \dots Q_n z_n \overline{\mathrm{PI}(C) \vee C}(z_1, \dots, z_n)$ *, quantifiers ordered as in 5, is a craig interpolant.*

Proof. Induction.
Suppose Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \qquad \sigma = \mathrm{mgu}(l, l')$$

$\Gamma \models \bar{Q}_{n_1} \overline{\mathrm{PI}(C_1) \vee D \vee l}$
$\Gamma \models \bar{Q}_{n_2} \overline{\mathrm{PI}(C_2) \vee E \vee \neg l'}$
to show:
$\Gamma \models \bar{Q}_n \overline{\mathrm{PI}(C) \vee (D \vee E)\sigma}$ // somewhat imprecise on $\bar{Q}_n$, but that's just useless quantifiers
$\Gamma \models (\bar{Q}_{n_1} \overline{PI(C_1)} \vee D \vee l)\sigma$
$\Gamma \models (\bar{Q}_{n_2} \overline{PI(C_2)} \vee E \vee \neg l')\sigma$
By resolution:
$\Gamma \models (\bar{Q}_{n_1} \overline{\mathrm{PI}(C_1)} \vee \bar{Q}_{n_2} \overline{\mathrm{PI}(C_2)})\sigma \vee (D \vee E)\sigma$

1. Suppose l, l' are from Γ alone: TODO

2. Suppose l and l' are colored with different colors and w.l.o.g l is Γ-colored and l' is Δ-colored.

   $\bar{Q}_n \overline{\mathrm{PI}(C)} \equiv \bar{Q}_n \overline{[(\neg l' \wedge \mathrm{PI}(C_1)^*) \vee (l \wedge \mathrm{PI}(C_2)^*)]\sigma}$
   $\equiv \bar{Q}_n (\overline{\neg l' \sigma} \wedge \overline{\overline{\mathrm{PI}(C_1)}\sigma}) \vee (\overline{l\sigma} \wedge \overline{\overline{\mathrm{PI}(C_2)}\sigma})$

   Adapt Huang proof to this, need to consider quantifiers:

   If $\Gamma \not\models D\sigma$ and $\Gamma \not\models E\sigma$ (else we are done), then

   $\Gamma \models [(\neg l' \wedge \bar{Q}_{n_1} \overline{PI(C_1)}) \vee (l \wedge \bar{Q}_{n_2} \overline{\mathrm{PI}(C_2)})]\sigma$

   As $\bar{Q}_{n_1}$ and $\bar{Q}_{n_2}$ disjoint and their variables do not appear in l or l',

   $\Gamma \models (\bar{Q}_{n_1} \bar{Q}_{n_2} [(\neg l' \wedge \overline{PI(C_1)}) \vee (l \wedge \overline{PI(C_2)})])\sigma$

   $\Gamma \models \bar{Q}_{n_1} \bar{Q}_{n_2} [(\neg l' \sigma \wedge \overline{\mathrm{PI}(C_1)}\sigma) \vee (l\sigma \wedge \overline{\mathrm{PI}(C_2)}\sigma)]$

   Consider the maximal terms of this expression which are Δ-colored.

   The $\mathrm{PI}(C_i)$, $i \in \{1, 2\}$ contain no colored terms. σ can introduce one by replacing a free variable x by a Δ-term t. But then overline replaces it with an universally quantified variable again, hence the formula is still entailed by Γ.

   $\Gamma \models \bar{Q}_{n_1} \bar{Q}_{n_2} [(\neg l' \sigma \wedge \overline{\overline{\mathrm{PI}(C_1)}\sigma}) \vee (l\sigma \wedge \overline{\overline{\mathrm{PI}(C_2)}\sigma})]$

   TODO: should work out similarily as huang if using P_P or it's the same as what i'm trying above.

□

Proposition 14. $\Gamma \models Q_1 z_1 \dots Q_n z_n \, \mathrm{PI}(C)^*(z_1, \dots, z_n) \vee C$*, quantifiers ordered as in 5, is a craig interpolant.*

Proof. Induction.
Suppose Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad \sigma = \mathrm{mgu}(l, l')$$

$\Gamma \models \bar{Q}_{n_1}\,\mathrm{PI}(C_1)^* \vee D \vee l$
$\Gamma \models \bar{Q}_{n_2}\,\mathrm{PI}(C_2)^* \vee E \vee \neg l'$
to show: $\Gamma \models \bar{Q}_n\,\mathrm{PI}(C)^* \vee (D \vee E)\sigma$
$\Gamma \models (\bar{Q}_{n_1}\,\mathrm{PI}(C_1)^* \vee D \vee l)\sigma$
$\Gamma \models (\bar{Q}_{n_2}\,\mathrm{PI}(C_2)^* \vee E \vee \neg l')\sigma$
By resolution:
$\Gamma \models (\bar{Q}_{n_1}\,\mathrm{PI}(C_1)^* \vee \bar{Q}_{n_2}\,\mathrm{PI}(C_2)^*)\sigma \vee (D \vee E)\sigma$

1. Suppose l, l' are from Γ alone: TODO

2. Suppose l and l' are colored with different colors and w.l.o.g l is Γ-colored and l' is Δ-colored.

   $\bar{Q}_n\,\mathrm{PI}(C)^* \equiv \bar{Q}_n([(\neg l' \wedge \mathrm{PI}(C_1)^*) \vee (l \wedge \mathrm{PI}(C_2)^*)]\sigma)^*$

   Adapt Huang proof to this, need to consider quantifiers:

   If $\Gamma \not\models D\sigma$ and $\Gamma \not\models E\sigma$ (else we are done), then

   $\Gamma \models [(\neg l' \wedge \bar{Q}_{n_1}\,\mathrm{PI}(C_1)^*) \vee (l \wedge \bar{Q}_{n_2}\,\mathrm{PI}(C_2)^*)]\sigma$

   As $\bar{Q}_{n_1}$ and $\bar{Q}_{n_2}$ disjoint and their variables do not appear in l or l',

   $\Gamma \models (\bar{Q}_{n_1}\bar{Q}_{n_2}[(\neg l' \wedge \mathrm{PI}(C_1)^*) \vee (l \wedge \mathrm{PI}(C_2)^*)])\sigma$

   The $\mathrm{PI}(C_i)$, $i \in \{1, 2\}$ contain no colored terms. σ can introduce one by replacing a free variable x.

   Consider the maximal terms of this expression which are Γ-colored.

   Either they only have grey subterms, then if they are existentially quantified, we can just use it as witness as the terms aren't replaced.

   Otherwise they contain at least a Γ- or a Δ-colored subterm.

Base case: simple.
Suppose Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad \sigma = \mathrm{mgu}(l, l')$$

$\Gamma \models \bar{Q}_{n_1}\,\mathrm{PI}(C_1)^* \vee D \vee l$
$\Gamma \models \bar{Q}_{n_2}\,\mathrm{PI}(C_2)^* \vee E \vee \neg l'$
to show: $\Gamma \models \bar{Q}_n\,\mathrm{PI}(C)^*\sigma \vee (D \vee E)\sigma$
Note that a term newly introduced in $\mathrm{PI}(C)$ occurs in either l or l', but not in both.
Let t be a colored term in $\mathrm{PI}(C)$, which has just been added W.l.o.g. let it occur in l, i.e. in C_1.
Case distinction:

1. Suppose l, l' are from Γ alone:

   By induction hypothesis:

   $\Gamma \models (\bar{Q}_{n_1} \operatorname{PI}(C_1)^* \vee D \vee l)\sigma$

   $\Gamma \models (\bar{Q}_{n_2} \operatorname{PI}(C_2)^* \vee E \vee \neg l')\sigma$

   By resolution:

   $\Gamma \models (\bar{Q}_{n_1} \operatorname{PI}(C_1)^* \vee \bar{Q}_{n_2} \operatorname{PI}(C_2)^*)\sigma \vee (D \vee E)\sigma$

   Suppose t is Γ-colored.

   Then it will be replaced by x_i and existentially quantified. It appears in either $\operatorname{PI}(C_1)$ or $\operatorname{PI}(C_2)$.

   t is a witness for x_i because it contains subterms $t_1, \dots, t_n$. If they are overbound as well, they are so before t and are available here.

   TODO: derive properties using examples 103 or so

OTHER TRY:
Then σ replaces variables $y_1, \dots, y_k$ in $E \vee \neg l'$ with terms that contain t.
By the induction hypothesis, $\Gamma \models Q_1 z_1 \dots Q_{n_2} z_{n_2} \operatorname{PI}(C_2)^*(z_1, \dots, z_{n_2}) \vee E \vee \neg l'$.
Hence $\Gamma \models (Q_1 z_1 \dots Q_{n_2} z_{n_2} \operatorname{PI}(C_2)^*(z_1, \dots, z_{n_2}) \vee E \vee \neg l')\sigma$.
Also $\Gamma \models Q_1 z_1 \dots Q_{n_2} z_{n_2} (\operatorname{PI}(C_2)^*(z_1, \dots, z_{n_2})\sigma) \vee E\sigma \vee \neg l'\sigma$.
Similarly, $\Gamma \models Q_1 z_1 \dots Q_{n_1} z_{n_1} (\operatorname{PI}(C_1)^*(z_1, \dots, z_{n_1})\sigma) \vee D\sigma \vee l\sigma$
$\Gamma \models Q_1 z_1 \dots Q_n z_n ((\neg l \wedge \operatorname{PI}(C_2)) \vee (l \wedge \operatorname{PI}(C_1)))^*(z_1, \dots, z_n)\sigma) \vee D\sigma \vee l\sigma$
l basically is the only new thing ($l\sigma = l'\sigma$).
Either l does not contain any subterms of other terms, then it does not depend on anything and l serves as witness for itself.
Otherwise it does depend on other terms and we have to make sure that that term is available. Depending on another term means that it uses information that is only available from another term, i.e. it contains a subterm of another term. but then that subterm is quantified over before the variable that replaces t is, so it works out.

t is Δ-colored. Then it is replaced by a universally quantified variable. But it "was already universally quantified" in the induction hypothesis. There, it was some free variable, because that's the only thing that can be substituted, but even with this free var, it worked out.

□

Conjecture 15. *$\Gamma \cup \Delta$ unsat, π propositional resolution refutation. Then $\Gamma \models \bar{Q}_n \operatorname{PI}(C)^* \vee C$ and $\Delta \models \neg \bar{Q}_n \operatorname{PI}(C)^* \vee C$ for all C in π.*

Proof. Base case as in Huang.
Induction.
Suppose Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l}{C : D \vee E}$$

$\Gamma \models \bar{Q}_{n_1}\,\mathrm{PI}(C_1)^* \vee D \vee l$
$\Gamma \models \bar{Q}_{n_2}\,\mathrm{PI}(C_2)^* \vee E \vee \neg l$
to show: $\Gamma \models \bar{Q}_n\,\mathrm{PI}(C)^* \vee D \vee E$, i.e.
$\Gamma \models \mathrm{sort}(Q_{n_1} \cup Q_{n_2} \cup \text{colored-terms}(l))((\neg l^* \wedge PI(C_1)^*) \vee (l^* \wedge PI(C_2)^*)) \vee D \vee E$
If $\Gamma \not\models D$ and $\Gamma \not\models E$ (else we are done), then
$\Gamma \models (\neg l \wedge \bar{Q}_{n_1} PI(C_1)^*) \vee (l \wedge \bar{Q}_{n_2}\,\mathrm{PI}(C_2)^*)$
As $\bar{Q}_{n_1}$ and $\bar{Q}_{n_2}$ disjoint and their variables do not appear in l or l,
$\Gamma \models \bar{Q}_{n_1}\bar{Q}_{n_2}[(\neg l \wedge PI(C_1)^*) \vee (l \wedge PI(C_2)^*)]$
Since we've pushed the variables outside, no colored terms appear in $\mathrm{PI}(C_i)^*$.
Suppose l does not contain colored terms. Then $l = l^*$ and we are done.
Otherwise let t be a maximal colored term in l.
By lemma 7, l appears in Γ with a certain polarity, say in clause E. l is an instance of E.
In fact, l is contained in $C\sigma$ where σ is the composition of unifiers applied in the deriviation up to the current point.
Hence $\Gamma \models C\sigma$.

1. Suppose t is Γ-colored. $\Gamma \models l$ implies that $\Gamma \models \exists y\, l\{t/y\}$
2. Suppose t is Δ-colored. $\Gamma \models \forall y\, l\{t/y\}$ because:

$\square$