lored_container⟩?

Conjectured Lemma 1. *Let a variable x occur twice in C such that in one occ, the smallest colored term containing x is a Γ-term and for the other, the smallest colored term containing x is a Δ-term. Then x occurs grey in $\mathrm{AI}_*(C)$.*

Proof. **missing: variables don't have to occur grey in $y\sigma$, e.g. in $\gamma[y]$, $y\sigma$ might be $f(x)$ with f Γ-colored.**

- Suppose that in C_i, $\gamma[x]$ occurs and in C_j, we have $\delta[y]$ such that x occurs grey in $y\sigma$.

  Then y occurs in l at $l|_{\hat{y}}$ such that $l'|_{\hat{y}}$ is an abstraction of a term containing a grey occurrence of x.

  Suppose that $l|_{\hat{y}}$ (and therefore also $l|_{\hat{y}}$) is not a grey occurrence as otherwise we are done.

  As $l\sigma l'\sigma$, $l|_{\hat{y}}$ and $l|_{\hat{y}}$ share their prefix, so their color is the same.

  Then induction hypothesis.

- Suppose that in C_i, $\gamma[z]$ occurs and in C_j, $\delta[y]$ occurs such that x occurs grey in $y\sigma$ and in $z\sigma$.

  By Lemma **??**, exists $y_1, \dots, y_n$ and $z_1, \dots z_m$ such that x occurs grey in $y_i\sigma$ and in $z_i\sigma$ and term opposite of y_n and z_m actually contains x.

  If any y_i, z_j occurs grey, done, so assume all occur colored.

  z_m and y_n opposite of actual x, as x only in one clause, z_m and y_n in same clause. they do share prefix with the occurrences of x in the clause where x is.

  if they there are contained in smallest col terms of opposite color $\Rightarrow$ ind hyp

  otw of same smallest term color there.

  Note that every y_i, z_j occurs at least twice: once as opposite var of the last one, once to unify with the next one.

  as originally different colors and at meeting point at x same color, there has to be one alternation, where we use the ind hyp.

- Suppose that $\gamma[x]$ in C_i and $\delta[x]$ in $z\sigma$ such that z occurs grey in C_j.

  If $\delta[x]$ occurs in C_i (cannot occur in other clause), ind hyp.

  Suppose it does not occur. Then however exists $\delta[y]$ s.t. x occurs grey in $y\sigma \Rightarrow$ other case.

- Suppose that $\gamma[x]$ in $y\sigma$ such that y occurs grey in C_i and $\delta[x]$ in $z\sigma$ such that z occurs grey in C_j.

  If $\gamma[x]$ and $\delta[x]$ occur, ind hyp.

  If just one occurs, $\Rightarrow$ other case.

  If none of them occur, then occur $\delta[\alpha]$ s.t. x grey in $\alpha\sigma$ and similar for $\gamma[\beta]$ $\Rightarrow$ other case.

$\square$

Conjectured Lemma 2. *Let σ unifier. exists unification order $\sigma = \sigma_1 \dots \sigma_n$ with $\sigma_i = \{x_i \mapsto r_i\}$ s.t. x_i does not occur in $\{r_i, r_{i+1}, \dots, r_n\}$.*

Proof. Suppose ordering does not exist, i.e. $l\sigma = l'\sigma$, but every x_i occurs in some r_j for j'i. But then last variable does not occur later.. □

Lemma 3. *Let σ unifier.*
At any stage in the run of the unification algo, exists var x as one part of a difference pair s.t. x does not occur in a function symbol in a difference pair.

Proof. Suppose no such var exists.
resolve all differences $x_i \sim r_i$ such that r_i does not contain a variable in a function symbol.
all variables, in particular the remaining x_i, occur in a function symbol in r_j for some j.
Iteratively resolve in some order: $x_i \mapsto r_i$, where every r_i contains at least one variable. Hence as every x_i occurs in some r_j, the variable in r_i then occurs in r_j.
so after a step, for the remaining difference pairs, it is still the case that every variable occurs in some r_j.
We do not get an occurs check error as by assumptions, the term are unifiable.
when we get to the point where there is only one subst left, it has to be of the form $x_i \mapsto r_i[x_i]$, so we do get an occurs check error, which contradicts the assumptions that the terms are unifiable.

□

Lemma 4. *Let σ unifier. At any stage in the run of the unification algorithm, there exists a variable as one part of a difference pair such that the other part does not contain a variable, which also occurs as one part of a difference pair, under a function symbol.*

Proof. Suppose to the contrary, that ....
Construct graph with vars as nodes and arrow from x, y if exists difference pair $(x, r[y])$ or the symmetric pair.
As every variable unifies to a term containg another variable, we have that $\forall x \exists y\, E(x, y)$. Hence we can build a path of length $|V| + 1$, but this contains a cycle.

□

TODO ICI: does this mean that there is a variable which does not have a variable in a term at its RHS? (all difference pairs have a variable at some side, let's call it LHS and the other one RHS)
possibly: do induction along this order: take subst which has no var to the right, then this one occurs in the term. next term then does not actually exists necessarily, so need to show some induction property.
evil examples:
$P(z, z, \delta), \neg P(f(x), f(y), y)$
$P(z, f(z), f(f(\delta))), \neg P(f(x), y, y)$
$P(u, f(z), f(f(\delta))), \neg P(f(x), y, y)$

Definition 6 (col change)**.** col change: a var x occurs in yet to specify location twice such that once in s.c. Γ-term and once in s.c. Δ-term. △

Conjectured Lemma 5. *Suppose* Γ*-term* $s(y)$ *in original diff pairs.*
Suppose $y\sigma = x$ *(simplification).*
Suppose no col change, i.e. no var x *occurs in a unified literal twice such that once in s.c.* Γ*-term and once in s.c.* Δ*-term.*
Suppose no x *grey in* $l\sigma\,(= l'\sigma)$.
Hence at some point have diff pair (y, v) *with* $v\sigma = x$.
by no col change and $s(y)$, y *does not occur in a s.c.* Δ*-term.*
As no x *grey in* $l\sigma$ *and* $y\sigma = x$, *no* y *grey.*
Hence y *only s.c.* Γ*-col.*
y *and* v *same prefix, so* v *s.c.* Γ*-col.*

Conjectured Lemma 7. *Let* $\sigma = \mathrm{mgu}(l, l')$
Suppose Γ*-term* $s[y]$ *in some unification pair,* δ *grey in* $y\sigma$.
Suppose no col change.
Suppose no δ *grey in* $l\sigma\,(= l'\sigma)$.
Then exists Γ*-term* $h[\delta']$ *in* l *or* l' *OR in earlier* mgu*-operation.*

Conjectured Lemma 8. *Suppose s.c. Γ-term containing Δ-term δ is created via unification of l and l'. Then at least one of the following statements holds:*

1. *In $l\sigma$ $(=l')$, δ occurs grey.*
2. *There is a variable x in l or l' such that it occurs once in an s.c. Γ-term and once in an s.c. Δ-term.*
3. *A δ-term occurs in a Γ-term in l or l' (TODO: be more precise on which term).*

Proof. We show that a term in question is created, then one of the statments holds, or a term in question has been created earlier during the run of the mgu.

1. Supp have $f(y)$ in some unification pair.

   Note y not grey somewhere as otherwise done.

   At some stage exists diff pair (y, t). note y, t same prefix, hence same color. t abstraction of $\varepsilon[\delta]$.

   - supp t contains outermost symbol of δ. as y, t same color, t is multi-col term either in l or l', or created earlier during unification algo.
   - otw t contains var v s.t. $v\sigma = \delta$ or $v\sigma = \varepsilon[\delta]$.

     Supp. v occurs grey in l or l'. then done.

     Note during unification procedure, coloring does not disappear, hence assume now all v colored.

     [hole: col change]

     hence can assume all occs of v are s.c. Γ-col.

     so have like $f(v)$, with $v\sigma = \delta$ or $v\sigma = \varepsilon[\delta]$. the corresponding diff pair is resolved earlier or later.

     possible argument: finitely often anyway?

     possible argument: after finitely many variable renamings, we hit an actual term, which then is strictly smaller, hence terminates?

2. var substituted for multi-colored term .

□

Conjectured Lemma 9. *Let $\sigma = \mathrm{mgu}(l, l')$. Let $\gamma[\delta]$ be a Γ-term containing a Δ-term δ in $l\sigma$. Then one of the following statments holds:*

1. *δ occurs at a grey position in $l\sigma$ TODO: argue about occurring $l\sigma$.*
2. *col change (where?)*
3. *in l or l', δ occurs in a Γ-term.*

Proof. Let $\sigma = \sigma_1 \cdots \sigma_n$, where σ_i stems from the ith substitution applied by the unification algo.
Let $l_j = l\sigma_1 \cdots \sigma_j$
Let σ_i be unifier $x \mapsto \delta$.
Suppose l_i contains a Δ-term in a Γ-term, where the respective predecessor of the Γ-term does not have a Δ-term at that position or does not exist in l_{i-1}.

1. Suppose a Γ-term $t[y]$ exists in l_{i-1}, such that it contains a grey occ of a variable y such that $y\sigma_i = \varepsilon[\delta]$ (where ε may be "empty" or else some grey term). The corresponding difference pair is $(y, \varepsilon[\delta])$, say at position $\hat{y}$

   So y occurs at say $\hat{y}$ in l or l', say λ. (y may occur in both, variable-disjointness might have already been broken).

   If it is a grey occurrence, we are done as δ occurs grey in $y\sigma$.

   So assume y occurs colored.

   $\lambda'_{i-1}|_{\hat{y}} = \varepsilon[\delta]$. Note that $\lambda_{i-1}|_{\hat{y}}$ and $\lambda'_{i-1}|_{\hat{y}}$ agree on the prefix (by virtue of being a difference pair).

   - Suppose $\lambda_{i-1}|_{\hat{y}}$ occurs in an s.c. Γ-term. Then $\lambda'_{i-1}|_{\hat{y}}$ is δ in a Γ-term in l or l' $\Rightarrow$ IH.
   - Suppose $\lambda_{i-1}|_{\hat{y}}$ occurs in an s.c. Δ-term. Then as y occurs in t in a Γ-term, we have a col change (but possibly distributed over l/l'). TODO: lemma for col change

   , say

2. Suppose y at $\hat{y}$ in λ_{i-1} s.t. $y\sigma_i$ is a Γ-term containing a Δ-term.

   Then $\lambda'_{i-1}|_{\hat{y}}$ actually is that term $\Rightarrow$ IH.

□

Definition 10. $\sigma_{i \to j} \stackrel{\text{def}}{=} \prod_{k=i}^{j} \sigma_k$. △

Lemma 11 (corresponds to lemma 25 in -final). *Let $\sigma = \mathrm{mgu}(l, l') = \sigma_1 \cdots \sigma_n$. Suppose a s.c. Φ-term $s[y]$ occs in $l\sigma_{0\to i}$ or $l'\sigma_{0\mapsto i}$ where $0 \le i \le n$ and $\sigma_0 = \mathrm{id}$ s.t. a var x occurs grey in $y\sigma$. At least one of the following statments holds:*

1. *x occurs grey in $l\sigma$*
2. *var z s.t. x occs grey in $z\sigma$ OR the var x itself occur in s.c. Φ-term in $l\sigma_{0\to i-1}$*
3. *there is a col change where x is the col change var in $l\sigma$*

Proof. At some point, say i, the is diff pair (y, t) is resolved, where t is an abstraction of $\varepsilon[x]$, say $t = \varepsilon[z]$
So $\sigma_i = \{y \mapsto \varepsilon[z]\}$, where either $z = x$ or x occurs grey in $z\sigma$ (in total: $\varepsilon[z]\sigma = \varepsilon[x]$).
If y occs grey, we are done as this then is a grey occ of x in $l\sigma$. (1)
If y occs in a s.c. Ψ-term, then y is a col change var, where x then is a col change var in $l\sigma$. (3)
So supp it only occurs in Φ-terms (note that clauses aren't var disjoint anymore here in general).
y occs at $\hat{y}$ in λ and $\lambda'|_{\hat{y}}$ is exactly $\varepsilon[z]$. But $\lambda|_{\hat{y}}$ and $\lambda'|_{\hat{y}}$ share the prefix, so $\varepsilon[z]$ is contained in a Φ-term, and in $l\sigma$, that's a grey occurrence of x (this occurrence of $\varepsilon[z]$ exists at least in stage $i-1$, but possibly not in l). (2) TODO: □

Lemma 12 (corresponds to lemma 26 in -final). *Let $\sigma = \mathrm{mgu}(l, l')$. Suppose a variable y occs in $l\sigma_{0\to i}$ or $l'\sigma_{0\mapsto i}$ where $0 \le i \le n$ and $\sigma_0 = \mathrm{id}$ s.t. x occurs in a s.c. Φ-term in $y\sigma$. At least one of the following statments holds:*

1. *x occurs grey in $l\sigma$*
2. *x occurs in a s.c. Φ-term in ???? TODO:*
3. *there is a col change where x is the col change var in $l\sigma$ TODO:*

Conjectured Lemma 13. *Let $\sigma = \mathrm{mgu}(l, l')$ such that in l and l', there are grey occs for col changes. Let $\gamma[x]$ be a s.c. Γ-term containing a variable x and $\delta[x]$ be a s.c. Δ-term containing the same variable x. Then x occurs at a grey position.*

Proof. probably revisit later when pre-lemmas are done

□

TODO: express idea: all of same color or color change
TODO: possibly assume: something (var, Δ-term) does not occur grey $\Rightarrow$ then other special case

1 Structure (cases) of relevant unifications

Lemma 14. *For a difference pair or a not necessarily prefix-disjoint "unification pair"* (s, t)*, s and t are both of same maximal and minimal color.*

Supp $f(x)$ occurs somewhere (original diff pairs or somewhere during run of algo) and $x\sigma = \varepsilon[\delta]$.

1.1 fst

Then $f(x) \sim t$, s.t. $t\sigma = f(\varepsilon[\delta])$.
(Suppose no col change.)

1. Supp $t = f(\varepsilon[\delta])$. ✓
2. Supp $t = f(y)$. $y\sigma = \varepsilon[\delta]$. Then IH (for some IH...).
3. Supp $t = f_{1/2}(y)$. $y\sigma = f_{1/2}(\varepsilon[\delta])$.
4. Supp $t = y$. $y\sigma = f(\varepsilon[\delta])$.
5. ? Supp $h(t) = y$.

1.2 snd

Then actually $x \sim t$, s.t. t possibly non-proper abstraction of $\varepsilon[\delta]$.

1. Supp $t = \varepsilon[\delta]$. ✓
2. Supp $t = y$. $y\sigma = \varepsilon[\delta]$.

1.3 random notes

suppose $z \sim f(x)$. then x is only changed if z is unified with something with an f-prefix.
look at terms where partial unification applies. the final state is just an extremely advanced applied partial unification.