1 small result

Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$.

1. if a Γ-lifting variable occurs multiple times in $\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$, then the terms at the corresponding positions in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C))$ are equal and
2. if a Δ-lifting variable occurs multiple times in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C))$, then the Δ-lifting variables at the corresponding positions in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C))$ are equal.

ad 1)
Suppose that a Γ-term t is introduced in a derivation step in $\mathrm{AI}_*(C)/\,\mathrm{AI}^{\Delta}_*(C)$ (or is "introduced initially") such that it does not occur in a Δ-term (then it actually is a lifting var in AI)

1. Suppose no variables occur in t. Then it will always have this form in the literal in $\mathrm{AI}^{\Delta}_*(C)$, and the lifting variable in $\mathrm{AI}_*(C)$ will always be y_t. So if multiple t's are introduced at any stage anywhere in the derivation, it will always be y_t.

2. Suppose a variable occurs in t.

   In $\mathrm{AI}_*(C)$, it's y_t; in $\mathrm{AI}^{\Delta}_*(C)$, it's $\ell_\Delta[t]$. Note that this is the case for any t introduced in this step, hence all these introduced ones correspond to $\ell_\Delta[t]$

   Suppose that y_t already occurs in $\mathrm{AI}_*(C)$.

   As it is now introduced by means of a unifier, no variable occurring in t has been substituted. (Lemma: if a variable is substituted for something, it does not occur at any later stage in the derivation (basically not in clauses, could also say not in unifiers) But if no variable of t has been altered, that old y_t still refers to $\ell_\Delta[t]$.

   Note that τ in this case has not changed y_t in $\mathrm{AI}_*(C)$ as τ only resets to the version in C, but also this has not changed.

   this establishes that if an y_t is introduced, it's fine with the old ones. so now that all are on the same level, note that any change in AI^{Δ}_* is performed for all occurrences of $\ell_\Delta[t]$ there.

   Also changes by means of τ affect all y_t in $\mathrm{AI}_*(C)$ the same way, and also, it does not unify lifting variables in a problematic way: If τ changes y_t, then y_t occurs in the resolved/factorised literal and y_t is replaced by say $\ell[s] = y_s$, if s is the term in C. but then every lifting variable which is a subterm of the position of s in $\mathrm{AI}^{\Delta}_*(C)$ is also treated by τ and reset to finally give $\ell_\Delta[s]$. So if in both the resolved/factorised literals we have y_t and $y_{t'}$, after the application of τ, both of them are y_s and also the ones in $\mathrm{AI}_*(C)$ are the same variables. [words about the IH]

Note that it is not the case that an occurrence of t or y_t can enter this thread of derivation as other clauses are variable disjoint and hence do not exhibit any of the variables in t.

2 serious stuff

Definition 1 (col change)**.** col change: a var x occurs in yet to specify location twice such that once in s.c. Γ-term and once in s.c. Δ-term. $\triangle$

Definition 2. $\sigma_{(i,j)} \overset{\text{def}}{=} \prod_{k=i}^{j} \sigma_k$. $\triangle$

⟨new_25⟩ **Lemma 3** (corresponds to lemma 25 in -final)**.** *Let* $\sigma = \text{mgu}(l, l') = \sigma_1 \cdots \sigma_n$*. Suppose a s.c.* Φ*-term* $s[y]$ *occs in* $l(')\sigma_{(0,i-1)}$ *where* $1 \leq i \leq n$ *and* $\sigma_0 = \text{id}$ *s.t.* $\text{dom}(\sigma_i) = \{y\}$ *and a var* x *occurs grey in* $y\sigma_i$*. At least one of the following statments holds:*

1. *x occurs grey in $l(')\sigma_{(0,i)}$ (and y in $l(')\sigma_{(0,i-1)}$)*
2. *x occur in s.c. Φ-term in $l(')\sigma_{(0,i-1)}$ ($\Rightarrow$ x occs in s.c. Φ-col term in $l(')\sigma_{(0,i)}$)*
3. *there is a col change where y is a col change var in $l(')\sigma_{(0,i-1)}$ (and x in $l(')\sigma_{(0,i)}$)*

Proof. If y occurs grey somewhere in $l(')\sigma_{(0,i-1)}$, we are done. ramp!
Suppose it only occurs colored in $l(')\sigma_{(0,i-1)}$. (1)
Suppose at least once in s.c. Ψ-term. Then in $l(')\sigma_{(0,i-1)}$, y is a col change variable (3)
Otw. it occs only in Φ-terms. There must exist an occurrence $\hat{y}$ of y in literal λ s.t. $\lambda'|_{\hat{y}}$ is $y\sigma_i$. But $\lambda|_{\hat{y}}$ and $\lambda'|_{\hat{y}}$ share the prefix, so $\lambda'|_{\hat{y}}$ is a s.c. Φ-term containing a grey occurrence of x. (2) $\square$

not BS:
Let $\sigma = \text{mgu}(l, l')$.
Suppose a variable y occs in $l(')\sigma_{(0,i-1)}$ where $1 \leq i \leq n$ and $\sigma_0 = \text{id}$ s.t. $\text{dom}(\sigma_i) = \{y\}$ and x occurs in a s.c. Φ-term in $y\sigma_i$.
THen in $l(')\sigma_{(0,i-1)}$, x occurs in a s.c. Φ-term.
BS:

Lemma 4 (corresponds to lemma 26 in -final)**.** *Let* $\sigma = \text{mgu}(l, l')$*. Suppose a variable* y *occs in* $l(')\sigma_{(0,i-1)}$ *where* $1 \leq i \leq n$ *and* $\sigma_0 = \text{id}$ *s.t.* $\text{dom}(\sigma_i) = \{y\}$ *and* x *occurs in a s.c.* Φ*-term in* $y\sigma_i$*. At least one of the following statments holds:*

1. *x occurs grey in $l(')\sigma_{(0,i)}$*
2. *x occurs grey in a s.c. Φ-term in $l(')\sigma_{(0,i)}$ (also in $l(')\sigma_{(0,i-1)}$)*
3. *there is a col change where x is the col change var in $l(')\sigma_{(0,i)}$*

Proof. Suppose that x does not occur grey in $l(')\sigma_{(0,i-1)}$ as otherwise we are done.
Suppose that x also does not occur grey in a s.c. Φ-term in $l(')\sigma_{(0,i-1)}$ as otherwise we are done.
So x only occurs in s.c. Ψ-terms in $l(')\sigma_{(0,i-1)}$.
Let $\hat{y}$ be the occ of y of the diff pair. Then $\lambda'|_{\hat{y}}$ contains an occ of x in a s.c. Φ-term. $\square$

⟨new_27⟩ **Lemma 5** (corresponds to lemma 27 in -final)**.** *Let* $\sigma = \mathrm{mgu}(l, l')$*,* C_1 *and* C_2 *var-disjoint and condition holds.*

NB: this means that it holds for all resolution refutations if we pretend to have extended it to factorisation by just applying induction on exactly this. perhaps we should do this.

Suppose in $(C_1 \cup C_2)\sigma_{(0,i)}$ *where* $0 \leq i \leq n$ *and* $\sigma_0 = \mathrm{id}$ *there is a col change with var* x *of* Γ*-term* $s[x]$ *and* Δ*-term* $t[x]$*. Then* x *occs grey in* $(C_1 \cup C_2)\sigma_{(0,i)}$*.*

Proof. for σ_0, it holds.
suppose holds for σ_{i-1}.
3 possibilities for having a variable in a s.c. Φ-term :

1. was there in stage $i-1$ in $(C_1 \cup C_2)\sigma_{(0,i-1)}$
2. $(C_1 \cup C_2)\sigma_{(0,i-1)}$ contains term $t[y]$ with $\mathrm{dom}(\sigma_i) = \{y\}$ and x occs grey in $y\sigma_i$
3. $(C_1 \cup C_2)\sigma_{(0,i-1)}$ contains a variable z such that $\mathrm{dom}(\sigma_i) = \{z\}$ and x occs in a s.c. Φ-term in $z\sigma_i$.

apply this to both $s[x]$ and $t[x]$.
if both variables were present in both colors in s.c. terms, we are done by the IH.
So supp at least one introduced in stage i. this means at least for one of them situation 2 applies.
Hence lemma 3 applies, but not the case where x already appeared in a respectively single-colored term before.
but this means that for at least one of $s[x]$ or $t[x]$, x occs grey in stage $i-1$ (this is stage i in lemma 3), or there is a col change with x as var in $i-1$. In the first case, we are done right away (σ_i does not affect x as x still occurs after applying it), and in the second, we can use the IH. □

small version:

Lemma 6 (corresponds to lemma 27 in -final (but only for literal!))**.** *Let* $\sigma = \mathrm{mgu}(l, l')$*. Suppose in* $l(')\sigma_{(0,i)}$ *where* $0 \leq i \leq n$ *and* $\sigma_0 = \mathrm{id}$ *there is a col change with var* x *of* Γ*-term* $s[x]$ *and* Δ*-term* $t[x]$*. Then* x *occs grey in* $l(')\sigma_{(0,i)}$*.*

Proof. induction.
initially: $l\sigma_0$ and $l'\sigma_0$ var disjoint and condition holds for intra-vars. (so holds globally)
3 possibilities for having a variable in a s.c. Φ-term :

1. was there in stage $i-1$
2. $l(')\sigma_{(0,i-1)}$ contains term $t[y]$ with $\mathrm{dom}(\sigma_i) = \{y\}$ and x occs grey in $y\sigma_i$
3. $l(')\sigma_{(0,i-1)}$ contains a variable z such that $\mathrm{dom}(\sigma_i) = \{z\}$ and x occs in a s.c. Φ-term in $z\sigma_i$.

apply this to both $s[x]$ and $t[x]$.
continuing with slightly different train of thought after returning from lunch:
if both s.c. Γ and s.c. Δ were there in $i-1$, we are done by IH. this encompasses both 1 and 3, as by the non-BS lemma, it copies terms of a form.
So suppose at least one introduced by situation 2.
for both occs: either they were there in $i-1$, or we can apply lemma 3. in any case, we know that at least one of the three statments holds for both.
Note index shift, in lemma all indices are one too many.
If one of them has 1 (x occurs grey in $l(')\sigma_{(0,i-1)}$), we are done as σ_i does not affect x as x occurs in $l(')\sigma_{(0,i)}$.
If one of them has 3 (col change with x in $l(')\sigma_{(0,i-1)}$), then we apply the IH to it and get that x occs grey in $l(')\sigma_{(0,i-1)}$, so also in $l(')\sigma_{(0,i)}$.
Otw. both were there before, which we supposed not to be the case for both, so one of them has to hit one of the other cases. □

Conjectured Lemma 7 (corresponds to 29 in -final)**.** *If in* $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ *a* Γ*-term* $t[x_s]_p$ *contains a* Δ*-lifting variable* x_s*, then* x_s *occurs grey in* $\mathrm{AI}^{\Delta}_{*}(C)$*,*

Proof. induction; base case works.
supp resolution w/ usual notation.

1. Supp for some i $\sigma_i = \{u \mapsto s'\}$ s.t. s' contains a Δ-term, $s'\sigma = s$ and u occurs in a maximal colored Γ-term at a single-colored Γ-position (i.e., must be below Γ-symbol and must not contain any other colored symbol as otherwise it would be lifted).

   We basically perform an induction over all construction steps of σ. Base case works by outer induction hypothesis.

   ind step:

   As u is changed, it occurs in l or l', say in λ at $\hat{u}$.

   If u occs grey anywhere in $C_j\sigma_{(0,i-1)}$, in particular for example at $\lambda\sigma_{(0,i-1)}|_{\hat{u}}$, then done as $u\sigma_i = s'$, hence due to $s'\sigma = s$ we have that $u\sigma = s$.

   If u occs anywhere in $C_j\sigma_{(0,i-1)}$, in particular for example in $\lambda\sigma_{(0,i-1)}|_{\hat{u}}$, in a s.c. Δ-term, then by Lemma 5, u occs grey in $(C_1 \cup C_2)\sigma_{(0,i-1)}$ and we are done as above.

   So suppose u only occs in s.c. Γ-terms, in particular in $\lambda\sigma_{(0,i-1)}|_{\hat{u}}$. But as $\lambda'\sigma_{(0,i-1)}|_{\hat{u}}$ has the same prefix, but it is s', there is a Δ-term in a Γ-term, so by the induction hypothesis $x_{s'}$ occs grey in $\mathrm{AI}^{\Delta}_{*}(C_j)$ for some j.

   As Γ-terms are not lifted in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_j)$, $x_{s'}$ is not lifted there.

   As s' is in the range of the unifier, s' occurs in a resolved literal.

   By the definition of au, $\{x_{s'} \mapsto x_s\} \in \tau$ as s is the term at the position of $x_{s'}$ in $\lambda\sigma$ for λ the resolved literal where s' occurs.

   Hence there is a grey occurrence of x_s in $\mathrm{AI}^{\Delta}_{*}(C)$.

2. Suppose a variable u occurs in C_1 or C_2 grey or in a maximal colored single colored Γ-colored term such that $u\sigma$ contains a multi-colored Γ-term t

   Then $\lambda'\sigma_{(0,i-)}|_{\hat{u}}$ actually is $t \Rightarrow$ IH. ☐

TODO: ICI ICI ICI: this lemma should easily give the main result. extend to factorisation and write up nicely

3 old, incorrect version

s_somewhere_grey⟩ **Lemma 8.** *Let l and l' be variable disjoint literals and $\sigma = \mathrm{mgu}(l, l')$ such that for a variable x, t occurs grey in $x\sigma$.*

Then there is a sequence of variables $x_1, \dots, x_n$ with $x_1 = x$ such that for $1 \leq i \leq n-1$, t occurs grey in $x_i\sigma$ and $x_i \mapsto_{\mathrm{mgu}} r[x_{i+1}]$, where x_{i+1} occurs grey in $r[x_{i+1}]$. Furthermore, $x_n \mapsto_{\mathrm{mgu}} r_t$, where r_t contains the outermost symbol of t at a grey position.

TODO: prove here as well: if x_i occurs grey/in s.c. Φ-term, then x_{i+1} occs grey/in s.c. Φ-term due to literals same and term grey in unifier image.

Proof. TODO: accidentally proved below:

POSSIBLE BETTER STATEMENT: There is a sequence of variable $y_1, \dots, y_n$ such that $y_i\sigma$ contains x and $y_i \mapsto_{\mathrm{mgu}} r[y_{i+1}]$ for $1 \leq i \leq n-1$ where $r[y_{i+1}]$ is a grey term and $y_n \mapsto_{\mathrm{mgu}} r[x]$, where $r[x]$ is a grey term as well or a variable.

Inductive definition: Let $y_1 = y$. For each y_i, $y_i \mapsto_{\mathrm{mgu}} t$ for some t such that t is an abstraction of $y_i\sigma$, which is a term containing a grey occurrence of x. Hence either x occurs in t, then $i = n$. Otherwise x does not occur in t and there is a variable in t such that $v\sigma$ contains a grey occurrence of x. Let $y_{i+1} = v$. Note that as σ only changes a finite number of variables, a variable can only be added to the sequence finitely often and cycles are not possible by the nature of the unification algorithm. □

_contains_grey_x⟩ **Lemma 9.** *Let a single-colored Φ-term $s[y]$ occur in l or l' such that x occurs grey in $y\sigma$. Then at least one of the following statements holds:*

⟨27_z_grey⟩
1. *there is a variable z such that x occurs grey in $z\sigma$ and z occurs grey in l or l'*

⟨27_x_in_sc_phi⟩
2. *x occurs in a s.c. Φ-term*
3. *there is a variable z such that $z\sigma$ contains a grey occurrence of x and z occurs in either l or l' two times: once in s.c. Φ-term and once in s.c. Ψ-term.*

⟨27_mixed⟩
Proof. By Lemma 8, there is a sequence .... We distinguish on the coloring of y_n.

- Suppose that y_n occurs grey. Then we have established item 1 where $z = y_n$.
- Suppose that y_n occurs in a single-colored Φ-term. Then as $y_n \mapsto_{\mathrm{mgu}} r[x]$ where $r[x]$ contains a grey occurrence of x, x does so as well and we have established item 2.
- Suppose that y_n occurs in a single-colored Ψ-term for $\Psi \neq \Phi$. TODO: this is now proved in lemma 24, drop here As $y_1 = y$, y_1 occurs in a single-colored Φ-term. As for $1 \leq i \leq n-1$, $y_i \mapsto_{\mathrm{mgu}} r[y_{i+1}]$ where y_{i+1} occurs grey in $r[y_{i+1}]$, each successive variable occurs in the same coloring as the last one. As y_1 and y_n are contained in single-colored terms of different colors, there must be some j, $1 \leq j \leq n$, such that y_j occurs in a clause once in a single-colored Φ-term as well as in a single-colored Ψ-term, establishing item 3. □

ntains_colored_x⟩

Lemma 10. *Let a variable y occur in l or l' such that x occurs in a single-colored Φ-term in $y\sigma$. Then at least one of the following statements holds:*

1. *there is a variable z such that x occurs grey in $z\sigma$ and z occurs grey in l or l'*
2. *a single-colored Φ-term in l or l' contains x*

⟨29_grey_x⟩

3. *there is a variable z such that $z\sigma$ contains a grey occurrence of x and z occurs in either l or l' two times: once in s.c. Φ-term and once in s.c. Ψ-term.*

Proof. TODO: rewrite without the sequence; should be just like an algo and only an induction if i know how to do it properly

We attempt to build a sequence of variables $y_1, \dots, y_n$ such that $y_i \mapsto_{\text{mgu}} r[y_{i+1}]$, where $r[y_{i+1}]$ contains y_{i+1} and does not contain Ψ-terms. Furthermore for $1 \le i \le n-1$, $y_i\sigma$ contains a single-colored Φ-term containing x (and no Ψ-symbols) and $y_n\sigma$ contains a grey occurrence of x (and no Ψ-symbols).

Let $y_1 = y$. $y_i \mapsto_{\text{mgu}} t$.

- Suppose that t contains a single-colored Φ-term containing x. Then we have established item 2 and relinquish the partial sequence.
- Suppose that t contains a variable v such that x occurs grey in $v\sigma$ and v occurs in a single-colored Φ-term in t. Then by Lemma 9 gives the result.
- Suppose that t contains a variable v such that $v\sigma$ contains a single-colored Φ-term containing x and no Ψ-symbols. Then let $y_{i+1} = v$.

Note that since y_i contains a single-colored Φ-term containing x, one of the last two cases must be the case in case the first isn't.

□

olored_container⟩

Lemma 11. *Let a variable x occur in C once in a single-colored Γ-term and once in a single-colored Δ-term.*[1] *Then x occurs grey in $\mathrm{AI}_*(C)$.*

TODO: add formal details above and below if result works out

Proof. We proceed by induction on the resolution refutation:

Base case. Clauses contained in Γ do not contain Δ-terms and clauses contained in Δ do not contain Γ-terms.

Resolution/Factorisation. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ or of a factorisation step ι of $C_1 : l \vee l' \vee D$. Let $\sigma = \text{mgu}(\iota)$. TODO: avoid assigning C_1 twice here in final formulation

[1] Note that these terms may be subterms of other terms.

We consider an occurrence of a single-colored Φ-term containing x in C. There are three circumstances leading to this situation:

1. A single-colored Φ-term containing x occurs in a preceding clause.
2. A single-colored Φ-term $t[y]$ in a preceding clause contains a variable y such that x occurs grey in $y\sigma$.
⟨27_2⟩
3. A variable z occurs in a preceding clause such that $z\sigma$ contains a single-colored Φ-term containing x.

⟨27_3⟩

We apply Lemma 9 in the case of 2 and Lemma 10 in the case of 3 to obtain that in any of the cases, at least one of the following statements hold:

[copy formulation from lemma once it's finished there]

Now suppose that x occurs in a single-colored Γ-term and in a single-colored Δ-term in C. By applying the reasoning as just given, we know that one of the three statments holds for both occurrences.

If for any one z grey with $z\sigma$ contains grey x, then done

old way:

If IH-case, then: IH

otw both s.c. Γ and Δ-term respectively $\Rightarrow$ IH as well.

if for any one col change case, then col change var grey by IH, and this is unified to x.

otw both IH case, so one in s.c. Γ and one in s.c. Δ, but due variable disjointness in same clause, that's why IH works here. □

application of lemma below: Suppose such a term occurs in a clause. Then suppose that it occurs in same s.c. term in literal, otw grey (we are done) or other color (then IH). then lemma!

tains_delta_term⟩

Lemma 12. *Context: resolved literals. Suppose a single-colored Γ-term contains a variable u such that a Δ-term s occurs grey in $u\sigma$. Then one of the following statements holds:*

1. *there is a variable z such that s occurs grey in $z\sigma$ and z occurs grey in l or l' TODO: possibly change this everywhere to in $l\sigma$, s occurs grey*
2. *a single-colored Γ-term in l or l' contains s outermost symbol of s and variables such that in total, with the unifier we get s*

Proof. Suppose sequence with each unifying to next one, last one: $u_n \mapsto_{\mathrm{mgu}} r[s]$, where s occurs grey in r. also in lemma, successive variables in same coloring
u from lemma statement occs in $\bar{u}$.
Suppose u_i grey, then done as all $u_i\sigma$ contain grey s, hence case 1
Suppose one u_i occs s.c. Γ and s.c. Δ. by lemma 14, u_i occurs grey and s occs grey as above, hence case 1

Otw, all colored, and as successive vars same coloring, all same s.c. term. Start with Γ, hence all Γ. Hence case 2 (term contains var v at grey pos which has s in grey pos at $v\sigma$), hence s occs grey in Γ-term.

$\square$

Lemma 13. *If in* $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ *a* Γ*-term* $t[x_s]_p$ *contains a* Δ*-lifting variable* x_s*, then* $x_s \rightsquigarrow_{G_C} t[x_s]_p$*.*

Proof. We proceed by induction.

Base case. For $C \in \Gamma \cup \Delta$, consider that no mixed-colored terms occur in C and hence no Γ-term in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ can contain a Δ-lifting variable.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$ and $\tau = \mathrm{au}(\iota)$. There are two possible cases in which a Δ-lifting variable x_s can be subterm of a Γ-colored term $t[x_s]_p$ in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ such that this has not been the case in C_1 or C_2:

1. Suppose a maximal colored Γ-term in C_1 or C_2 contains a variable u such that s occurs grey in $u\sigma$.

⟨25_1⟩ Note that it suffices to show that x_s occurs grey in $\mathrm{AI}^{\Delta}_{*}(C)$, since if we suppose that it does so at position r, then $\mathcal{A}_1$ as defined in Definition **??** contains (r, q) such that $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)|_q$ is $t[x_s]_p$. As $\mathcal{A}_1 \subseteq G_C$, this implies $x_s \rightsquigarrow_{G_C} t[x_s]_p$.

We apply Lemma 12 as we can assume that this is also a s.c. Γ-term (otherwise it would contains a Δ-term and be lifted NB: afterthought, did not check global implications for this lemma).

in case 1, s occs grey.

in case 2, IH for that term, say s': $s' \rightsquigarrow_{G_{C_j}} \gamma'[s']$ s' is maximal Δ-term (else would be contained in r and we would talk about x_r). as Γ-terms not lifted, s' occurs "grey". As s is in range of subst, s occurs in literal being unified, by the definition of au, $\{x_s \mapsto x_r\} \in \tau$ as r is the term at the position of x_s in $\lambda\sigma$ for λ the resolved literal where s' occurs.

Hence there is a grey occurrence of x_s in $\mathrm{AI}^{\Delta}_{*}(C)$. TODO: check this

By Lemma **??**, there is a sequence of variable $u_1, \ldots, u_n$ such that $u_1 = u$ and s occurs grey in $u_i\sigma$ for $1 \leq i \leq n$. Note that if any variable u_i occurs grey in C_1 or C_2, then at the corresponding position in C, the term at this position is a grey occurrence of s and we are done. Therefore suppose that $u_1, \ldots, u_n$ occur only colored in C_1 and C_2.

Note that in the prefix of x_s in $t[x_s]_p$, no Δ-colored symbol occurs as otherwise x_s would not occur in this term. Hence the smallest colored term containing the occurrence of u in the predecessor of $t[x_s]$ is a Γ-term. this is the ramp!

Lemma **??** furthermore asserts that u_i occurs in a resolved literal l_i at $l_i|_{\hat{u}_i}$ such that in the respective opposite resolved literal l'_i, $l'_i|_{\hat{u}_i}$ contains u_{i+1} for $1 \leq i \leq n-1$ and $l'_n|_{\hat{u}_n}$ contains the outermost symbol of s. Note that for $1 \leq i \leq n$, u_i occurs at least twice in its respective clause. Note also that as $l_i\sigma = l'_i\sigma$, $l|_{\hat{u}_i}$ and $l'|_{\hat{u}_i}$ share the prefix of $\hat{u}_i$, so if $l|_{\hat{u}_i}$ is contained in a Φ-colored term, then so is the grey occurrence of u_{i+1} in $l'|_{\hat{u}_i}$.

If one of the u_i occurs in a clause twice such that for one occurrence, the smallest colored term containing it is Γ-colored and for the other one, the smallest colored term containing it is Δ-colored, then by Lemma 14, u_i occurs grey in $\mathrm{AI}_{*}(C)$ and we are done. Therefore assume that this situation does not arise for any u_i, $1 \leq i \leq n$.

Hence as the smallest colored term containing the occurrences of u_1 must be Γ-terms, the same holds for u_n. But as $l'_n|_{\hat{u}_n}$ contains the outermost symbol of s, which is a Δ-term, and $l_n\sigma = l'_n\sigma$ and the smallest colored term containing $l_n|_{\hat{u}_n}$ is a Γ-term, $l'_n|_{\hat{u}_n}$ is contained in a Γ-term. Let $r[x_\varphi]$ be hte maximal colored term containing $l'_n|_{\hat{u}_n}$ and x_φ be the lifting variable at the position of the outermost symbol of s in $l'_{n\,\mathrm{AIcl}}|_{\hat{u}_n}$. Let C_j be the clause containing l'_n.

2. Suppose a variable u occurs in C_1 or C_2 such that $u\sigma$ contains a multi-colored Γ-term t.

   Then by Lemma **??**, a variable u_n occurs in a resolved literal l at $l|_{\hat{u}_n}$ such that in the other resolved literal l', $l'|_{\hat{u}_n}$ contains the outermost symbol of t.

   If $l'|_{\hat{u}_n}$ is a multi-colored Γ-term, then by the induction hypothesis, dots

   Otherwise as the outermost symbol of t is Γ-colored, $l'|_{\hat{u}_n}$ contains a Γ-colored term which contains a variable v such that a Δ-term occurs grey in $v\sigma$, where case 1 gives the result, or a multi-colored Γ-term s occurs grey in v. But as s is strictly smaller than t, this case can only repeat finitely often before the other case is reached.

Factorisation. If the clause C is the result of a factorisation of C_1, then TODO: □

4 Attempts

olored_container⟩ **Conjectured Lemma 14.** *Let a variable x occur twice in C such that in one occ, the smallest colored term containing x is a Γ-term and for the other, the smallest colored term containing x is a Δ-term. Then x occurs grey in $\mathrm{AI}_*(C)$.*

Proof. **missing: variables don't have to occur grey in $y\sigma$, e.g. in $\gamma[y]$, $y\sigma$ might be $f(x)$ with f Γ-colored.**

- Suppose that in C_i, $\gamma[x]$ occurs and in C_j, we have $\delta[y]$ such that x occurs grey in $y\sigma$.

  Then y occurs in l at $l|_{\hat{y}}$ such that $l'|_{\hat{y}}$ is an abstraction of a term containing a grey occurrence of x.

  Suppose that $l|_{\hat{y}}$ (and therefore also $l|_{\hat{y}}$) is not a grey occurrence as otherwise we are done.

  As $l\sigma l'\sigma$, $l|_{\hat{y}}$ and $l|_{\hat{y}}$ share their prefix, so their color is the same.

  Then induction hypothesis.

- Suppose that in C_i, $\gamma[z]$ occurs and in C_j, $\delta[y]$ occurs such that x occurs grey in $y\sigma$ and in $z\sigma$.

  By Lemma **??**, exists $y_1, \dots, y_n$ and $z_1, \dots z_m$ such that x occurs grey in $y_i\sigma$ and in $z_i\sigma$ and term opposite of y_n and z_m actually contains x.

  If any y_i, z_j occurs grey, done, so assume all occur colored.

  z_m and y_n opposite of actual x, as x only in one clause, z_m and y_n in same clause. they do share prefix with the occurrences of x in the clause where x is.

  if they there are contained in smallest col terms of opposite color $\Rightarrow$ ind hyp

  otw of same smallest term color there.

  Note that every y_i, z_j occurs at least twice: once as opposite var of the last one, once to unify with the next one.

  as originally different colors and at meeting point at x same color, there has to be one alternation, where we use the ind hyp.

- Suppose that $\gamma[x]$ in C_i and $\delta[x]$ in $z\sigma$ such that z occurs grey in C_j.

  If $\delta[x]$ occurs in C_i (cannot occur in other clause), ind hyp.

  Suppose it does not occur. Then however exists $\delta[y]$ s.t. x occurs grey in $y\sigma \Rightarrow$ other case.

- Suppose that $\gamma[x]$ in $y\sigma$ such that y occurs grey in C_i and $\delta[x]$ in $z\sigma$ such that z occurs grey in C_j.

  If $\gamma[x]$ and $\delta[x]$ occur, ind hyp.

  If just one occurs, $\Rightarrow$ other case.

  If none of them occur, then occur $\delta[\alpha]$ s.t. x grey in $\alpha\sigma$ and similar for $\gamma[\beta] \Rightarrow$ other case.

$\square$

Conjectured Lemma 15. *Let σ unifier. exists unification order $\sigma = \sigma_1 \dots \sigma_n$ with $\sigma_i = \{x_i \mapsto r_i\}$ s.t. x_i does not occur in $\{r_i, r_{i+1}, \dots, r_n\}$.*

Proof. Suppose ordering does not exist, i.e. $l\sigma = l'\sigma$, but every x_i occurs in some r_j for j'i. But then last variable does not occur later.. □

Lemma 16. *Let σ unifier.*
At any stage in the run of the unification algo, exists var x as one part of a difference pair s.t. x does not occur in a function symbol in a difference pair.

Proof. Suppose no such var exists.
resolve all differences $x_i \sim r_i$ such that r_i does not contain a variable in a function symbol.
all variables, in particular the remaining x_i, occur in a function symbol in r_j for some j.
Iteratively resolve in some order: $x_i \mapsto r_i$, where every r_i contains at least one variable. Hence as every x_i occurs in some r_j, the variable in r_i then occurs in r_j.
so after a step, for the remaining difference pairs, it is still the case that every variable occurs in some r_j.
We do not get an occurs check error as by assumptions, the term are unifiable.
when we get to the point where there is only one subst left, it has to be of the form $x_i \mapsto r_i[x_i]$, so we do get an occurs check error, which contradicts the assumptions that the terms are unifiable.

□

Lemma 17. *Let σ unifier. At any stage in the run of the unification algorithm, there exists a variable as one part of a difference pair such that the other part does not contain a variable, which also occurs as one part of a difference pair, under a function symbol.*

Proof. Suppose to the contrary, that ....
Construct graph with vars as nodes and arrow from x, y if exists difference pair $(x, r[y])$ or the symmetric pair.
As every variable unifies to a term containg another variable, we have that $\forall x \exists y\, E(x, y)$.
Hence we can build a path of length $|V| + 1$, but this contains a cycle.

□

TODO ICI: does this mean that there is a variable which does not have a variable in a term at its RHS? (all difference pairs have a variable at some side, let's call it LHS and the other one RHS)
possibly: do induction along this order: take subst which has no var to the right, then this one occurs in the term. next term then does not actually exists necessarily, so need to show some induction property.
evil examples:
$P(z, z, \delta), \neg P(f(x), f(y), y)$
$P(z, f(z), f(f(\delta))), \neg P(f(x), y, y)$
$P(u, f(z), f(f(\delta))), \neg P(f(x), y, y)$

Conjectured Lemma 19. *Let* $\sigma = \mathrm{mgu}(l, l')$
Suppose Γ-term $s[y]$ in some unification pair, δ grey in $y\sigma$.

Conjectured Lemma 18. *Suppose* Γ*-term* $s(y)$ *in original diff pairs.*
Suppose $y\sigma = x$ *(simplification).*
Suppose no col change, i.e. no var x *occurs in a unified literal twice such that once in s.c.* Γ*-term and once in s.c.* Δ*-term.*
Suppose no x *grey in* $l\sigma\,(= l'\sigma)$.
Hence at some point have diff pair (y, v) *with* $v\sigma = x$.
by no col change and $s(y)$, y *does not occur in a s.c.* Δ*-term.*
As no x *grey in* $l\sigma$ *and* $y\sigma = x$, *no* y *grey.*
Hence y *only s.c.* Γ*-col.*
y *and* v *same prefix, so* v *s.c.* Γ*-col.*

Suppose no col change.

Suppose no δ *grey in* $l\sigma\,(= l'\sigma)$.

Then exists Γ*-term* $h[\delta']$ *in* l *or* l' *OR in earlier* mgu*-operation.*

Conjectured Lemma 20. *Suppose s.c. Γ-term containing Δ-term δ is created via unification of l and l'. Then at least one of the following statements holds:*

1. *In $l\sigma$ $(=l')$, δ occurs grey.*
2. *There is a variable x in l or l' such that it occurs once in an s.c. Γ-term and once in an s.c. Δ-term.*
3. *A δ-term occurs in a Γ-term in l or l' (TODO: be more precise on which term).*

Proof. We show that a term in question is created, then one of the statments holds, or a term in question has been created earlier during the run of the mgu.

1. Supp have $f(y)$ in some unification pair.

   Note y not grey somewhere as otherwise done.

   At some stage exists diff pair (y, t). note y, t same prefix, hence same color. t abstraction of $\varepsilon[\delta]$.

   - supp t contains outermost symbol of δ. as y, t same color, t is multi-col term either in l or l', or created earlier during unification algo.
   - otw t contains var v s.t. $v\sigma = \delta$ or $v\sigma = \varepsilon[\delta]$.

     Supp. v occurs grey in l or l'. then done.

     Note during unification procedure, coloring does not disappear, hence assume now all v colored.

     [hole: col change]

     hence can assume all occs of v are s.c. Γ-col.

     so have like $f(v)$, with $v\sigma = \delta$ or $v\sigma = \varepsilon[\delta]$. the corresponding diff pair is resolved earlier or later.

     possible argument: finitely often anyway?

     possible argument: after finitely many variable renamings, we hit an actual term, which then is strictly smaller, hence terminates?

2. var substituted for multi-colored term .

□

Conjectured Lemma 21. *Let $\sigma = \mathrm{mgu}(l, l')$. Let $\gamma[\delta]$ be a Γ-term containing a Δ-term δ in $l\sigma$. Then one of the following statments holds:*

1. *δ occurs at a grey position in $l\sigma$ TODO: argue about occurring $l\sigma$.*
2. *col change (where?)*
3. *in l or l', δ occurs in a Γ-term.*

Proof. Let $\sigma = \sigma_1 \cdots \sigma_n$, where σ_i stems from the ith substitution applied by the unification algo.
Let $l_j = l\sigma_1 \cdots \sigma_j$
Let σ_i be unifier $x \mapsto \delta$.
Suppose l_i contains a Δ-term in a Γ-term, where the respective predecessor of the Γ-term does not have a Δ-term at that position or does not exist in l_{i-1}.

1. Suppose a Γ-term $t[y]$ exists in l_{i-1}, such that it contains a grey occ of a variable y such that $y\sigma_i = \varepsilon[\delta]$ (where ε may be "empty" or else some grey term). The corresponding difference pair is $(y, \varepsilon[\delta])$, say at position $\hat{y}$

   So y occurs at say $\hat{y}$ in l or l', say λ. (y may occur in both, variable-disjointness might have already been broken).

   If it is a grey occurrence, we are done as δ occurs grey in $y\sigma$.

   So assume y occurs colored.

   $\lambda'_{i-1}|_{\hat{y}} = \varepsilon[\delta]$. Note that $\lambda_{i-1}|_{\hat{y}}$ and $\lambda'_{i-1}|_{\hat{y}}$ agree on the prefix (by virtue of being a difference pair).

   - Suppose $\lambda_{i-1}|_{\hat{y}}$ occurs in an s.c. Γ-term. Then $\lambda'_{i-1}|_{\hat{y}}$ is δ in a Γ-term in l or l' $\Rightarrow$ IH.
   - Suppose $\lambda_{i-1}|_{\hat{y}}$ occurs in an s.c. Δ-term. Then as y occurs in t in a Γ-term, we have a col change (but possibly distributed over l/l'). TODO: lemma for col change

   , say

2. Suppose y at $\hat{y}$ in λ_{i-1} s.t. $y\sigma_i$ is a Γ-term containing a Δ-term.

   Then $\lambda'_{i-1}|_{\hat{y}}$ actually is that term $\Rightarrow$ IH.

$\square$

Conjectured Lemma 22. *Let $\sigma = \mathrm{mgu}(l, l')$ such that in l and l', there are grey occs for col changes. Let $\gamma[x]$ be a s.c. Γ-term containing a variable x and $\delta[x]$ be a s.c. Δ-term containing the same variable x. Then x occurs at a grey position.*

Proof. probably revisit later when pre-lemmas are done

□

5 Structure (cases) of relevant unifications

Lemma 23. *For a difference pair or a not necessarily prefix-disjoint "unification pair" (s, t), s and t are both of same maximal and minimal color.*

Supp $f(x)$ occurs somewhere (original diff pairs or somewhere during run of algo) and $x\sigma = \varepsilon[\delta]$.

5.1 fst

Then $f(x) \sim t$, s.t. $t\sigma = f(\varepsilon[\delta])$.
(Suppose no col change.)

1. Supp $t = f(\varepsilon[\delta])$. ✓
2. Supp $t = f(y)$. $y\sigma = \varepsilon[\delta]$. Then IH (for some IH...).
3. Supp $t = f_{1/2}(y)$. $y\sigma = f_{1/2}(\varepsilon[\delta])$.
4. Supp $t = y$. $y\sigma = f(\varepsilon[\delta])$.
5. ? Supp $h(t) = y$.

5.2 snd

Then actually $x \sim t$, s.t. t possibly non-proper abstraction of $\varepsilon[\delta]$.

1. Supp $t = \varepsilon[\delta]$. ✓
2. Supp $t = y$. $y\sigma = \varepsilon[\delta]$.

5.3 random notes

suppose $z \sim f(x)$. then x is only changed if z is unified with something with an f-prefix.
look at terms where partial unification applies. the final state is just an extremely advanced applied partial unification.
outline of arrow part

5.4 Variable occurrences

Need for var x the set of colored occs and grey occs in initial clauses. lift clauses as usual s.t. to not see any of the colored structure, hence remember only in which max colored term the var is.

for resolution/factorisation, check unifier:

- if x occurs grey in $y\sigma$, then the set of occurrences of y is added to the ones of x, col to col and grey to grey
- if x occurs colored in $y\sigma$, then the set of occurrences of y is added to the ones of x, col and grey to col

Definition 24.

// (apparently not needed) arrows 1: if x occurs in $y\sigma$, add arrow from every *grey* occurrence of x in C to every colored occurrence of y in C_i.

arrows 2: if a maximal Φ-colored term t occurs grey in $x\sigma$, add arrow from every grey occurrence of t in C to every Ψ*-colored* occurrence of x in C_i.

arrows 3: if a maximal Ψ-colored term t occurs inside a maximal Ψ-colored term s in $x\sigma$, add an arrow from every grey occurrence of t in C to every occurrence of x in C_i. △

Lemma 25. *If in* $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ *a* Γ*-colored term* $t[x_s]$ *contains a* Δ*-lifting variable* x_s*, then* $x_s \rightsquigarrow t[x_s]$*.*

Proof.

Suppose term containing max colored term which is Δ-term is introduced into Γ-colored term.

Then Γ-colored occ of u in C_i s.t. δ_i grey in $u\sigma$ (δ_i is max col term). Hence by arrow 2, arrow from every grey δ_i to every colored u. TODO: as below, need existence

existence 1: If u occurs grey in C_i, then there, δ_i occurs grey in C (this is the necessary color change case $x, f(x)$) and hence the arrow actually exists.

existence 2 proper:

need to show that δ_i occurs grey given the assumptions.

unification algo produces a chain: $u \mapsto t$, $v \mapsto s$, ...

u only occurs colored in C_i. Hence also at $l|_{\hat{u}}$. Therefore $l'|_{\hat{u}}$ is a colored occurrence as well.

chain of colored variables:

if var occurs at some point grey s.t. Δ-term is still complete, then we are done.

if var occurs at some point at position we are unifying with, then we are done by the induction hypothesis.

AUX LEMMA: if a Δ-term enters a Γ-term, there is an arrow. Later, the terms always look the same as they are affected by the same unifications.

TODO: ICI; check example

NEW THING:

chain: either contain variables v s.t. $v\sigma$ contains Δ-term, or term contains Δ-term already (such that outermost symbol matches with the one we get in the end)

in both cases: if term occurs grey, we are done. in this case, we get exactly the lifting var we want.

if term occurs colored (can only be in Γ), then if we hit a Δ-symbol, we can use the ind hyp. Here, we get the lifting var which just is there. NOTE: different from whether both colors are lifted or just Δ-terms (see 212c).

NEW THING MORE FORMAL:

If for some u, δ_i grey in $u\sigma$ and u occurs in Γ-term, then δ_i occurs grey somewhere.

Prf. either u occurs grey, then we are done. Otw. u only occurs colored in Γ-terms. so $l'|_{\hat{u}}$ also colored.

Note: arguing along subst run.

If $l'|_{\hat{u}}$ contains outermost symbol of δ_i, then have Δ-term in Γ-term and ind hyp. Otw. $l'|_{\hat{u}}$ contains var v s.t. δ_i grey in $v\sigma$. Note that now, we can apply the same argument to v and this recursion terminates as mgu algo has terminated.

Suppose multi-colored Γ-term introduced.

Then u in C_i s.t. $\gamma[\delta_i]$ in $u\sigma$. Hence by arrow 3, arrow from every grey δ_i to every u. TODO: need make sure that grey δ_i exists (exactly δ_i? what if lifted)

existence: $l'|_{\hat{u}}$ is an abstraction of $u\sigma$ different from u. if contains multi-colored term $\Rightarrow$ ind hyp. Otw induction, Δ-term must come at some point. we either have other case, or some multi-colored term appears.

$\square$

6 Garbage

_unif_range_old⟩? **Lemma 26.** *Let l and l' be variable disjoint literals and $\sigma = \text{mgu}(l, l')$ and x and y be variables such that x occurs in a single-colored Δ-term in $y\sigma$.*
Then there is a sequence $y_1, \dots, y_n$ and some k such that $1 \le k \le n$, for $1 \le i \le k$, $y_i\sigma$ contains a single-colored Δ-term containing x and $y_i\sigma$ does not contain Γ-symbols, and for $k+1 \le i \le n$, $y_i\sigma$ contains a grey occurrence of x.
Furthermore, at least one of the following statements holds:

1. *some single-colored Δ-term containing x occurs in l or l'*

⟨25_delta_x⟩

2. *some single-colored Γ-term containing x occurs in l or l' and there is a color change: some y_i is contained in a single-col Δ-term and some y_{i+1} is contained in a single-col Γ-term*

⟨25_gamma_x⟩ *possible new text: y_i (and also y_{i+1} occurs grey, and they are unified to x as $i > k$*

3. *x occs grey.*

?⟨25_grey_x⟩? *additional conjecture: for the first y_i, but not y_1, the terms are contained in single-col Δ-terms. when the colored tiers are peeled off, the remaining y_i are grey occs of x. this is where color changes are possible.*

Proof. Let $y_1 = y$.
that for some single-colored Δ-term r, $y \mapsto_{\text{mgu}} r$. r furthermore contains x or a variable z such that $z\sigma$ does not contain a Γ-symbol and contains a grey occurrence of x or a single-colored Δ-term containing x.
We build the sequence inductively: By Lemma **??**, there is an occurrence of y_{i_n} of y_i such that $y_{i_n} \mapsto_{\text{mgu}} r$, where r shares the outermost symbol with $y_i\sigma$. As $y_i\sigma$ is a single-colored Δ-term containing x, r either contains x in which case $i = k = n$ and item 1 holds and we are done. Otherwise r contains a variable z such that $z\sigma$ contains a grey occurrence of x or $z\sigma$ does not contain Γ-terms and contains a single-colored Δ-term which contains x. Hence $y_{i+1} = z$ and in the first case, $k = i+1$. Note that the length of $z\sigma$ is a strictly smaller than the length of $y\sigma$, hence the second case can not occur infinitely often.
If we hit the first case and $k = i+1$, then we continue defining the sequence inductively. Let y_j be such that $y_j\sigma$ contains a grey occurrence of x. By Lemma **??**, there is an occurrence y_{j_n} of y_j such that $y_{j_n} \mapsto_{\text{mgu}} s[x]$, where $s[x]$ contains a grey occurrence of x. If $s[x]$ occurs grey or in a single-colored Δ-term, when we are done, so suppose it occurs in a single-colored Γ-term. Note that y_{j_n} is contained in a single-colored Φ-term if and only if $s[x]$ is. Note that y_k is contained in a single-colored Δ-term. As single-colored Δ-terms and single-colored Γ-terms are not unifiable, there is some i, $i < k \le n$such that y_i and y_{i+1} occur grey in either l or l', so 2 is the case.
TODO: check indices of i, k

∎

when we have finished peeling, there is at least one peeling step

varlt?

f_along_mgu_old⟩?

Lemma 27. *Let l and l' be variable disjoint literals and $\sigma = \mathrm{mgu}(l, l')$ such that for a variable x, $x\sigma$ contains a grey occurrence of a term t.*
old text: Then there is a sequence of variables $x_1, \dots, x_n$ with $x_1 = x$ such that for $1 \le i \le n$, t occurs grey in $x_i\sigma$ and x_i occurs in one of the literals, say l_i, at $l_i|_{\hat{x}_i}$ such that with l'_i being the respective other literal, $l'_i|_{\hat{x}_i}$ contains x_{i+1} for $1 \le i \le n-1$ and $l'_n|_{\hat{x}_n}$ contains the outermost symbol of t.
new text: Then there is a sequence of variables $x_1, \dots, x_n$ with $x_1 = x$ such that for $1 \le i \le n$, t occurs grey in $x_i\sigma$ and $x_i \mapsto_{\mathrm{mgu}} r[x_{i+1}]$ or $i = n \wedge x_n \mapsto_{\mathrm{mgu}} r_t$, where r_t contains the outermost symbol of t

Proof. Let $x_1 = x$ and note that t occurs in $x\sigma$ by assumption. We now consider the execution of the mgu algorithm as defined in **??** and show that for an x_i in the sequence, either we can find an element x_{i+1} which matches the requirement for the sequence or there is an occurrence of x_i which is unified with a term containing the outermost symbol of t. As the mgu algorithm produces a unifier which modifies x_i, x_i must occur in a literal, say in l_i at $l_i|_{\hat{x}_i}$, such that at the other literal l'_i, $l'_i|_{\hat{x}_i}$ is an abstraction of a term containing t which is different from x_i. We distinguish two cases:

- Suppose that $l'_i|_{\hat{x}_i}$ contains the outermost symbol of t. Then let $x_n = x_i$.
- Otherwise $l'_i|_{\hat{x}_i}$ contains a variable v such that t occurs grey in $v\sigma$. Let $x_{i+1} = v$. $\square$

tarts_somewhere⟩?

Lemma 28. *Let l and l' be variable disjoint literals and $\sigma = \mathrm{mgu}(l, l')$ such that for a variable x, $x\sigma$ contains a term t.*
new text: Then there is a sequence of variables $x_1, \dots, x_n$ with $x_1 = x$ such that for $1 \le i \le n$, t occurs in $x_i\sigma$ and $x_i \mapsto_{\mathrm{mgu}} r[x_{i+1}]$ or $i = n \wedge x_n \mapsto_{\mathrm{mgu}} r_t$, where r_t contains the outermost symbol of t

Proof. TODO: (but is virtually a subset of some lemma below) $\square$

comment
alternate version (unfinished)
Lemma **??** furthermore asserts that u_n occurs in a resolved literal λ at $\lambda|_{\hat{u}_n}$ such that $\lambda'|_{\hat{u}_n}$ contains the outermost symbol of the Δ-term s, where λ' is the respective other resolved literal. As u_n is a colored occurrence and $\lambda\sigma = \lambda'\sigma$, $\lambda'|_{\hat{u}_n}$ is a colored occurrence as well.

- Suppose $\lambda'|_{\hat{u}_n}$ is contained in a Γ-term. Let $r[x_\varphi]$ be the maximal colored term containing $\lambda'|_{\hat{u}_n}$ and x_φ be the lifting variable at the position of the outermost symbol of s in $\lambda'|_{\hat{u}_n}$ in $\mathrm{AI}_{\mathrm{cl}}(C_j)$ for $j = 1$ or $j = 2$. So by the induction hypothesis, $x_\varphi \rightsquigarrow_{G_{C_j}} r[x_\varphi]$, hence x_φ occurs grey in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_j)$, $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_j)$ or $\mathrm{AI}^{\Delta}_{\mathrm{col}}(C_j)$. As however x_φ occurs grey in λ'_{AIcl}, by the definition of au, $\{x_\varphi \mapsto x_s\} \in \tau$ as s is the term at the position of x_φ in $\lambda'\sigma$.

this is only guaranteed in AI^Δ, not in AI

Hence there is a grey occurrence of x_s in $\text{AI}^\Delta_{\text{mat}}(C)$, $\text{AI}^\Delta_{\text{cl}}(C)$ or $\text{AI}^\Delta_{\text{col}}(C)$ and we are done.

- Suppose that u_i for $1 \leq i \leq n$ is contained in a Δ-term which is contained in a Γ-term.

  TODO:

- Suppose $\lambda'|_{\hat{u}_n}$ is contained in a Δ-term. Due to $\lambda\sigma = \lambda'\sigma$, $\lambda|_{\hat{u}_n}$ is also contained in a Δ-term. As by assumption none of the u_i, $1 \leq i \leq n$ is a grey occurrence, there must be a clause which contains two occurrences of u_i such that one of them is a Γ-occurrence and one is a Δ-occurrence.

  – Suppose that one is only gamma and the other only delta

  – Suppose that mixed

comment

old proof of smallest colored container

We start by making an observation $(\divideontimes)$: If for two variables x and y it holds that x occurs grey in $y\sigma$, then by Lemma **??**, there exists a sequence $x_1, \ldots, x_n$ such that for $1 \leq i \leq n-1$, u_i occurs in $\lambda|_{\hat{u}_i}$ for a resolved literal λ such that the other resolved literal λ' has a grey occurrence of u_{i+1} at $\lambda'|_{\hat{u}_i}$. Hence if u_i occurs in a single-colored Φ-colored term in $\lambda|_{\hat{u}_i}$, then u_{i+1} does so too in $\lambda'|_{\hat{u}_i}$ as $\lambda\sigma = \lambda'\sigma$. As u_{i+1} also occurs in $\lambda'|_{\hat{u}_{i+1}}$ for $1 \leq 1 \leq n-1$, i.e. in the same clause as $\lambda'|_{\hat{u}_i}$, then if $\lambda'|_{\hat{u}_{i+1}}$ occurs in a single-colored term which is not Φ-colored, then by the induction hypothesis, u_{i+1} occurs grey in $\text{AI}_*(C_i)$ for $i \in \{1,2\}$ and as $u_{i+1}\sigma$ contains a grey occurrence of x, x occurs grey in $\text{AI}_*(C)$. Therefore we can assume that all variable of the sequence $x_1, \ldots, x_n$ occur only colored and each of the x_i, $1 \leq i \leq n$ is contained in some single-colored Φ-term, as otherwise we are done.
We make another observation $(*)$: If for two variables x an y it holds that $y\sigma = s[x]$ a single-colored Δ-term, then we can assume that x occurs grey or in some single-colored Δ-term in C_1 or C_2. Proof: We proceed by induction on the size of $s[x]$. By Lemma **??**, there is an occurrence of y_n of y in a resolved literal λ in say $\lambda[\hat{y}_n]$ such that $\lambda'[\hat{y}_n]$ contains the outermost symbol of $s[x]$.
Suppose for the induction start that $s[x]$ is of size 2. Note that this is the smallest size for a single-colored term containing a variable. Then $\lambda'|_{\hat{y}_n}$ either is $s[x]$, in which case we are done, or $\lambda'|_{\hat{y}_n}$ is $s[z]$ for a variable z such that $z\sigma = x$. Hence z occurs elsewhere in λ', say in $\lambda'|_{\hat{z}}$, such that $\lambda|_{\hat{z}}$ is x. So if $\lambda'|_{\hat{z}}$ is a grey occurrence or $\lambda'|_{\hat{z}}$ is contained in a single-colored Δ-term, then due to $\lambda\sigma = \lambda'\sigma$, $\lambda|_{\hat{z}}$ is a corresponding occurrence of x. Otherwise $\lambda'|_{\hat{z}}$ is contained in a single-colored Γ-term. meh
TODO: ICI: ind hyp should work for when z/x occur in a single-colored Γ-term, otw check what we need to have as lemma statement. all is in the resolved literal, so it's gone from the clause in the next step.
We distinguish between all four cases which produce a clause on which the lemma applies:

- Suppose that w.l.o.g. C_1 contains a single-colored Γ-term $s[x]$ which contains x and C_1 or C_2 contains a single-colored Δ-term containing a variable y such that x occurs grey or in a single-colored Δ-colored in $y\sigma$. Note that the case of an opposite assignment of colors can be argued in a symmetric manner.

  – Suppose that x occurs grey in $y\sigma$: Then by Lemma **??**, there is a variable y_n which occurs in a resolved literal λ at $\lambda|_{\hat{y}_n}$ such that $\lambda'|_{\hat{y}_n}$ contains a grey occurrence of x. By observation $(\divideontimes)$, $\lambda|_{\hat{y}_n}$ is contained in a single-colored Δ-term. But then so is $\lambda'|_{\hat{y}_n}$, and as clauses are variable disjoint, $s[x]$ also occurs in this clause. So by the induction hypothesis, there is a grey occurrence of x in $\text{AI}_*(C_j)$ where C_j is the clause containing $s[x]$, and as x is not affected by σ, x also occurs grey in $\text{AI}_*(C)$.

    clauses var-disjoint

  – Suppose that x occurs in a single-colored Δ-term $y\sigma$:

    Then by Lemma **??**, either x occurs grey, in which case we are done, or some y_i occurs grey in l or l' such that $y_i\sigma$ contains a grey occurrence of x, in which case we are done, or x occurs in a single-colored Δ-term $t[x]$. Then however as $s[x]$ occurs in C_1 and clauses are variable disjoint, $t[x]$ occurs in C_1 as well and x occurs grey in $\text{AI}_*(C_1)$ by the induction hypothesis.

    If a single-colored Δ-term $t[x]$ containing x occurs in C_1 or C_2, say in C_j, then as clauses are variable disjoint, it must be the same clause as $s[x]$. But then x occurs grey in $\text{AI}_*(C_j)$ by the induction hypothesis, so assume that no such $t[x]$ occurs in C_1 or C_2.

    But as a single-colored Δ-term containing x occurs in $y\sigma$, there must be a single-colored Δ-term in C_1 or C_2 which contains a variable z such that x occurs grey or in a single-colored Δ-term in $z\sigma$. Hence this case is repeated, but as $z\sigma$ is strictly smaller than $y\sigma$, this case can only repeat finitely often.

- Suppose that a single-colored Γ-term $s[y]$ occurs in C_i, $i \in \{1,2\}$ such that x occurs grey or in a single-colored Γ-term in $y\sigma$ and a single-colored Δ-term $t[z]$ occurs in C_j, $j \in \{1,2\}$ such that x occurs grey or in a single-colored Δ-term in $z\sigma$.
- 2 other items from arrow-final-conjectures.

old semi-main lemma reasoning:

- Suppose a single-colored Φ-term $s[x]$ in C_1 or C_2 contains a grey occurrence of x and a single-colored Ψ-term $t[x]$ is introduced in C. This is possible by two means:

  1. A single-colored Ψ-term $t[z]$ in C_1 or C_2 contains a variable z such that x occurs grey in $z\sigma$
  2. A variable u occurs in C_1 and C_2 such that $u\sigma$ contains a single-colored Ψ-term containing x

  We apply Lemma 10 in the first case and Lemma **??** Then by Lemma 10, at least one of the given three statments holds.

  (1) As there is a grey occurrence of z in C_1 or C_2, there is a grey occurrence of x in $\mathrm{AI}_*(C)$.

  (2) then this term occurs in the same clause as $s[x]$ as clauses are variable disjoint and x occurs grey by the induction hypothesis

  (3) then by IH, there is a grey occurrence of z in C_1 or C_2 and hence a grey occurrence of x in $\mathrm{AI}_*(C)$.

- Suppose a single-colored Φ-term $s[y]$ in C_1 or C_2 contains a variable y such that x occurs grey in $y\sigma$ and a single-colored Ψ-term $t[z]$ in C_1 or C_2 contains a variable z such that x occurs grey in $z\sigma$.

  Then we can apply Lemma 9 to both of $s[y]$ and $t[z]$.

  If any one yields case (1), we are done (as above).

  If any one yields case (3), we are done (IH, as above).

  Hence suppose that both yield case 2. Thus there is a single-colored Φ-term containing x and a single-colored Ψ-term containing x in C_1 or C_2. Note that as clauses are variable disjoint, both these terms must occur in the same clause, say in C_j. But then by the induction hypothesis, x occurs grey in $\mathrm{AI}_*(C_j)$ and so also in $\mathrm{AI}_*(C)$.

TODO: ICI; finish this proof

new distinction:

- Φ-col $s[x]$ in l/l', exists Ψ-col $t[z]$ with $z\sigma$ contains grey x

- exists Φ-col $s[y]$ with $y\sigma$ contains grey x and exists Ψ-col $t[z]$ with $z\sigma$ contains grey x

  by new 24 (for col occs of y), either

  - x occs grey
  - y_i grey in C_i OR y_i in once in s.c. Φ and once in s.c. Ψ-term
  - some Φ-term $r[x]$ in C_i

- Φ-col $s[x]$ in l/l', exists z in C_i s.t. $z\sigma$ contains s.c. Ψ-term containing x

- exists y in C_j s.t. $y\sigma$ contains s.c. Φ-term $s[x]$ and exists z in C_i s.t. $z\sigma$ contains s.c. Ψ-term $t[x]$

  by new 25, either:

  - some Φ-term $r[x]$ in C_i
  - y_i grey in C_i OR y_i in once in s.c. Φ and once in s.c. Ψ-term
  - x occs grey

  any of both case 2 or 3 $\Rightarrow$ done.

  otw both case 1, but then ind hyp