Remark $(*)$. Any substitution, in particular σ, only changes a finite number of variables. Furthermore a result of a run of the unification algorithm is acyclic in the sense that if a substitution $u \mapsto t$ is added to the resulting substitution, it is never the case that at a later stage $t \mapsto u$ is added. This can easily be seen by considering that at the point when $u \mapsto t$ is added to the resulting substitution, every occurrence of u is replaced by t, so u is not encountered by the algorithm at a later stage.
Therefore in order to show that a statement holds for every $u \mapsto t$ in a unifier σ, it suffices to show by an induction argument that for every substitution $v \mapsto s$ which is added to the resulting unifier by the unification algorithm that it holds for $v \mapsto s$ under the assumption that it holds for every $w \mapsto r$ such that w occurs in s and $w \mapsto r$ is added to the resulting substitution at a later stage. △

Conjecture 1. *Let C be a clause in a resolution refutation. Suppose that $\mathrm{AI}^{\Delta}(C)$ contains a maximal Γ-term $\gamma_j[z_i]$ which contains a lifting variable z_i. Then $z_i <_{\hat{\mathcal{A}}(C)} y_j$.*

Proof. We proceed by induction. For the base case, note that no multicolored terms occur in inital clauses, so no lifting term can occur inside of a Γ-term.
Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$. Furthermore suppose that for every lifting term inside a Γ-term in the clauses C_1 and C_2 of the refutation, for every term of the form $\gamma_j[z_i]$ we have that $z_i <_{\hat{\mathcal{A}}(C_1)} y_j$ or $z_i <_{\hat{\mathcal{A}}(C_2)} y_j$ respectively. Hence there is an arrow (p_1, p_2) in $\hat{\mathcal{A}}(C_1)$ or $\hat{\mathcal{A}}(C_2)$ such that z_i is contained in $\mathrm{P}(p_1)$ and z_j is contained in $\mathrm{P}(p_2)$. In $\mathrm{AI}^{\Delta}(C)$, $\mathrm{P}(p_1)$ contains $\ell[z_i\sigma]\tau = z_i\tau$ and $\mathrm{P}(p_2)$ contains $\ell[z_j\sigma]\tau = z_j\tau$. Hence the indicies of the lifting variables might change, but this renaming does not affect the relation of the objects as $\hat{\mathcal{A}}(C_1) \cup \hat{\mathcal{A}}(C_2) \subseteq \hat{\mathcal{A}}(C)$.
We show that $z_i <_{\hat{\mathcal{A}}(C)} z_j$ holds true also for every new term of the form $\gamma_j[z_i]$ for some j, i in $\mathrm{AI}^{\Delta}(C)$. By "new", we mean terms which are not present in $\mathrm{AI}^{\Delta}(C_1)$ or $\mathrm{AI}^{\Delta}(C_2)$. Note that new terms in $\mathrm{AI}^{\Delta}(C)$ are of the form $\ell_{\Delta,x}[t\sigma]\tau$ for some $t \in \mathrm{AI}^{\Delta}(C_1) \cup \mathrm{AI}^{\Delta}(C_2)$. By Lemma **??**, σ does not introduce lifting variables. Hence a new term of the form $\gamma_j[z_i]$ is created either by introducing a Δ-term into a Γ-term or by introducing $\gamma_j[\delta_i]$ via σ, both followed by the lifting. Note that τ only substitutes lifting variables by other lifting variables and hence does not introduce lifting variables. Furthermore by Lemma **??**, τ only substitutes lifting variables for other lifting variables, whose corresponding term is more specialised. Hence if there exists an arrow from a lifting variable to $\gamma_j[z_i]$ according to this lemma, it is also an appropriate arrow if $\gamma_j[z_i]$ is replaced by $\gamma_j[z_i]\tau$.
We now distinguish the two cases under which a new term $\gamma_j[z_i]$ can occur in $\mathrm{AI}^{\Delta}(C)$:

Suppose for some Γ-term $\tilde{\gamma}_{j'}[u]$ in $\mathrm{AI}^{\Delta}(C_1)$ or $\mathrm{AI}^{\Delta}(C_2)$, $u\sigma$ contains a Δ-term.

Hence we have that $(\tilde{\gamma}_{j'}[u])\sigma = \gamma_j[\delta_i]$ for some i. Note that the position of u in $\tilde{\gamma}_{j'}[u]$ does not necessarily coincide with the position of δ_i in $\gamma_j[\delta_i]$ as u might be substituted by σ for a grey term containing δ_i.

We have that $\ell_\Delta[\tilde{\gamma}_{j'}[u]\sigma]\tau = \gamma_j[x_i]$.

At some well-defined point of application of the unification algorithm, u is substituted by an abstraction of a term which contains δ_i. This occurrence of u is in l and we denote it by $\hat{u}$. We furthermore denote the term at the corresponding position in l' by $t_{\hat{u}}$.

We distinguish cases based on the occurrences of $\hat{u}$ and $t_{\hat{u}}$.

- Suppose $\hat{u}$ is a grey occurrence.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\hat{u}) \qquad C_2 : \neg Q(t_{\hat{u}})}{C : P(\gamma_j[\delta_i])}$$

Figure 1: Example for this case

Then by Lemma **??**, there is an arrow from $\hat{u}$ to $\gamma_j[u]$ in $\hat{\mathcal{A}}(C)$. As $\hat{u}\sigma$ is a term containing the Δ-term δ_i, the term at the position of $\hat{u}$ in $\mathrm{AI}^{\Delta}(C)$ is $\ell[\hat{u}\sigma]\tau$, which by assumption contains x_i. But there is an arrow from this term containing x_i to $\gamma_j[x_i]$, so $z_j <_{\hat{\mathcal{A}}(C)} x_i$.

- Suppose $\hat{u}$ occurs in a maximal colored term which is a Γ-term.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\gamma_k[\hat{u}]_p) \qquad C_2 : \neg Q(\gamma_m[t_{\hat{u}}]_p)}{C : P(\gamma_j[\delta_i])}$$

$$\frac{C_1 : Q(\tilde{\gamma}_{j'}[\hat{u}]) \qquad C_2 : \neg Q(\gamma_k[t_{\hat{u}}])}{C : \Box}$$

$//\gamma_j[\delta_i]$ occurs in the interpolant

Figure 2: Examples for this case

Then either $\hat{u}$ is the occurrence of u in $\tilde{\gamma}_{j'}[\hat{u}]$ or it occurs in a different Γ-term $\gamma_j[\hat{u}]$. In the latter case, by Lemma **??**, there is a merge edge between $\tilde{\gamma}_{j'}[\hat{u}]$ and $\gamma_j[\hat{u}]$. Hence in both cases, it suffices to show that there is an arrow from a term containing an occurrence of z_i to $t_{\hat{u}}$.

We distinguish on the shape of $t_{\hat{u}}$:

- $t_{\hat{u}}$ is a variable or a grey term. If $t_{\hat{u}}$ is a grey term, then it contains a variable that is be substituted by σ by a term which contains a Δ-term as $u\sigma = t_{\hat{u}}\sigma$ is a term containing a Δ-term. However this is true for the case where $t_{\hat{u}}$ is a variable. We denote by v either $t_{\hat{u}}$ if it is a variable, or the variable in $t_{\hat{u}}$ which is substituted by a term containing a Δ-term in case $t_{\hat{u}}$ is a grey term.

  In the course of the unification algorithm, there are further unifications of v since we know that $u\sigma = v\sigma$ is a term containing a Δ-term. Therefore by Remark $(*)$, we can assume that there is an appropriate arrow to $t_{\hat{u}}$.
- $t_{\hat{u}}$ is a grey term.

• Suppose $\hat{u}$ occurs in a maximal colored term which is a Δ-term.

TODO:

The substitution can also introduce a grey term containing a delta term, make sure to handle that!

The substitution can also introduce a gamma term containing a delta term, make sure to handle that!

$\Rightarrow$ what if it is overbound?

Suppose for some variable v in $\mathrm{AI}^{\Delta}(C_1)$ or $\mathrm{AI}^{\Delta}(C_2)$, $v\sigma = \gamma_j[\delta_i]$ for some i.

As v is affected by the unifier, it occurs in the literal being unified, say w.l.o.g. in l in C_1. At some well-defined point in the unification algorithm, v is substituted by an abstraction of $\gamma_j[\delta_i]$. Let p be the position of the occurrence of v in l which causes this substitution. Furthermore, let p' be the position corresponding to p in l'.

Note that any arrow from or to p' also applies to p in $\hat{\mathcal{A}}(C)$ and hence to $\gamma_j[z_i]$ as they are merged due to occurring in the resolved literal. So it suffices to show that there is an arrow from an appropriate lifting variable to p'. We denote the term at p' by t.

Note that $t\sigma = \gamma_j[\delta_i]$. So t is either a Γ-term containing a Δ-term, in which case we know that there is an appropriate arrow by the induction hypothesis as t occurs in l' in C_2, or t is an abstraction of $\gamma_j[\delta_i]$, in which case we can assume the existence of an appropriate arrow by Remark $(*)$. □

recheck this paragraphs w.r.t. $<_{\hat{\mathcal{A}}(C)}$