1 serious stuff

Definition 1 (col change)**.** col change: a var x occurs in yet to specify location twice such that once in s.c. Γ-term and once in s.c. Δ-term. $\triangle$

Definition 2. $\sigma_{i\to j} \overset{\text{def}}{=} \prod_{k=i}^{j} \sigma_k$. $\triangle$

⟨new_25⟩ **Lemma 3** (corresponds to lemma 25 in -final)**.** *Let $\sigma = \text{mgu}(l, l') = \sigma_1 \cdots \sigma_n$. Suppose a s.c. Φ-term $s[y]$ occs in $l(')\sigma_{0\to i-1}$ where $1 \le i \le n$ and $\sigma_0 = \text{id}$ s.t. $\text{dom}(\sigma_i) = \{y\}$ and a var x occurs grey in $y\sigma_i$. At least one of the following statments holds:*

1. *x occurs grey in $l(')\sigma_{0\to i}$ (and y in $l(')\sigma_{0\to i-1}$)*
2. *x occur in s.c. Φ-term in $l(')\sigma_{0\to i-1}$ ($\Rightarrow$ x occs in s.c. Φ-col term in $l(')\sigma_{0\to i}$)*
3. *there is a col change where y is a col change var in $l(')\sigma_{0\to i-1}$ (and x in $l(')\sigma_{0\to i}$)*

Proof. If y occurs grey somewhere in $l(')\sigma_{0\to i-1}$, we are done. ramp!
Suppose it only occurs colored in $l(')\sigma_{0\to i-1}$. (1)
Suppose at least once in s.c. Ψ-term. Then in $l(')\sigma_{0\to i-1}$, y is a col change variable (3)
Otw. it occs only in Φ-terms. There must exist an occurrence $\hat{y}$ of y in literal λ s.t. $\lambda'|_{\hat{y}}$ is $y\sigma_i$. But $\lambda|_{\hat{y}}$ and $\lambda'|_{\hat{y}}$ share the prefix, so $\lambda'|_{\hat{y}}$ is a s.c. Φ-term containing a grey occurrence of x. (2) $\square$

not BS:
Let $\sigma = \text{mgu}(l, l')$.
Suppose a variable y occs in $l(')\sigma_{0\to i-1}$ where $1 \le i \le n$ and $\sigma_0 = \text{id}$ s.t. $\text{dom}(\sigma_i) = \{y\}$ and x occurs in a s.c. Φ-term in $y\sigma_i$.
THen in $l(')\sigma_{0\to i-1}$, x occurs in a s.c. Φ-term.
BS:

Lemma 4 (corresponds to lemma 26 in -final)**.** *Let $\sigma = \text{mgu}(l, l')$. Suppose a variable y occs in $l(')\sigma_{0\to i-1}$ where $1 \le i \le n$ and $\sigma_0 = \text{id}$ s.t. $\text{dom}(\sigma_i) = \{y\}$ and x occurs in a s.c. Φ-term in $y\sigma_i$. At least one of the following statments holds:*

1. *x occurs grey in $l(')\sigma_{0\to i}$*
2. *x occurs grey in a s.c. Φ-term in $l(')\sigma_{0\to i}$ (also in $l(')\sigma_{0\to i-1}$)*
3. *there is a col change where x is the col change var in $l(')\sigma_{0\to i}$*

Proof. Suppose that x does not occur grey in $l(')\sigma_{0\to i-1}$ as otherwise we are done.
Suppose that x also does not occur grey in a s.c. Φ-term in $l(')\sigma_{0\to i-1}$ as otherwise we are done.
So x only occurs in s.c. Ψ-terms in $l(')\sigma_{0\to i-1}$.
Let $\hat{y}$ be the occ of y of the diff pair. Then $\lambda'|_{\hat{y}}$ contains an occ of x in a s.c. Φ-term. $\square$

⟨new_27⟩

Lemma 5 (corresponds to lemma 27 in -final). *Let $\sigma = \text{mgu}(l, l')$, C_1 and C_2 var-disjoint and condition holds.*

NB: this means that it holds for all resolution refutations if we pretend to have extended it to factorisation by just applying induction on exactly this. perhaps we should do this.

Suppose in $(C_1 \cup C_2)\sigma_{0\to i}$ where $0 \leq i \leq n$ and $\sigma_0 = \text{id}$ there is a col change with var x of Γ-term $s[x]$ and Δ-term $t[x]$. Then x occs grey in $(C_1 \cup C_2)\sigma_{0\to i}$.

Proof. for σ_0, it holds.

suppose holds for σ_{i-1}.

3 possibilities for having a variable in a s.c. Φ-term :

1. was there in stage $i-1$ in $(C_1 \cup C_2)\sigma_{0\to i-1}$
2. $(C_1 \cup C_2)\sigma_{0\to i-1}$ contains term $t[y]$ with $\text{dom}(\sigma_i) = \{y\}$ and x occs grey in $y\sigma_i$
3. $(C_1 \cup C_2)\sigma_{0\to i-1}$ contains a variable z such that $\text{dom}(\sigma_i) = \{z\}$ and x occs in a s.c. Φ-term in $z\sigma_i$.

apply this to both $s[x]$ and $t[x]$.

if both variables were present in both colors in s.c. terms, we are done by the IH.

So supp at least one introduced in stage i. this means at least for one of them situation 2 applies.

Hence lemma 3 applies, but not the case where x already appeared in a respectively single-colored term before.

but this means that for at least one of $s[x]$ or $t[x]$, x occs grey in stage $i-1$ (this is stage i in lemma 3), or there is a col change with x as var in $i-1$. In the first case, we are done right away (σ_i does not affect x as x still occurs after applying it), and in the second, we can use the IH. □

small version:

Lemma 6 (corresponds to lemma 27 in -final (but only for literal!)). *Let $\sigma = \text{mgu}(l, l')$. Suppose in $l(')\sigma_{0\to i}$ where $0 \leq i \leq n$ and $\sigma_0 = \text{id}$ there is a col change with var x of Γ-term $s[x]$ and Δ-term $t[x]$. Then x occs grey in $l(')\sigma_{0\to i}$.*

Proof. induction.

initially: $l\sigma_0$ and $l'\sigma_0$ var disjoint and condition holds for intra-vars. (so holds globally)

3 possibilities for having a variable in a s.c. Φ-term :

1. was there in stage $i-1$
2. $l(')\sigma_{0\to i-1}$ contains term $t[y]$ with $\text{dom}(\sigma_i) = \{y\}$ and x occs grey in $y\sigma_i$
3. $l(')\sigma_{0\to i-1}$ contains a variable z such that $\text{dom}(\sigma_i) = \{z\}$ and x occs in a s.c. Φ-term in $z\sigma_i$.

apply this to both $s[x]$ and $t[x]$.

continuing with slightly different train of thought after returning from lunch:

if both s.c. Γ and s.c. Δ were there in $i-1$, we are done by IH. this encompasses both 1 and 3, as by the non-BS lemma, it copies terms of a form.

So suppose at least one introduced by situation 2.

for both occs: either they were there in $i-1$, or we can apply lemma 3. in any case, we know that at least one of the three statments holds for both.

Note index shift, in lemma all indices are one too many.

If one of them has 1 (x occurs grey in $l(')\sigma_{0\to i-1}$), we are done as σ_i does not affect x as x occurs in $l(')\sigma_{0\to i}$.

If one of them has 3 (col change with x in $l(')\sigma_{0\to i-1}$), then we apply the IH to it and get that x occs grey in $l(')\sigma_{0\to i-1}$, so also in $l(')\sigma_{0\to i}$.

Otw. both were there before, which we supposed not to be the case for both, so one of them has to hit one of the other cases. □

Conjectured Lemma 7 (corresponds to 29 in -final)**.** *If in* $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ *a* Γ*-term* $t[x_s]_p$ *contains a* Δ*-lifting variable* x_s*, then* x_s *occurs grey in* $\mathrm{AI}^{\Delta}_{*}(C)$*,*

Proof. induction; base case works.
supp resolution w/ usual notation.

1. Supp for some i $\sigma_i = \{u \mapsto s'\}$ s.t. s' contains a Δ-term, $s'\sigma = s$ and u occurs in a maximal colored Γ-term at a single-colored Γ-position (i.e., must be below Γ-symbol and must not contain any other colored symbol as otherwise it would be lifted).

   We basically perform an induction over all construction steps of σ. Base case works by outer induction hypothesis.

   ind step:

   As u is changed, it occurs in l or l', say in λ at $\hat{u}$.

   If u occs grey anywhere in $C_j\sigma_{0\to i-1}$, in particular for example at $\lambda\sigma_{0\to i-1}|_{\hat{u}}$, then done as $u\sigma_i = s'$, hence due to $s'\sigma = s$ we have that $u\sigma = s$.

   If u occs anywhere in $C_j\sigma_{0\to i-1}$, in particular for example in $\lambda\sigma_{0\to i-1}|_{\hat{u}}$, in a s.c. Δ-term, then by Lemma 5, u occs grey in $(C_1 \cup C_2)\sigma_{0\to i-1}$ and we are done as above.

   So suppose u only occs in s.c. Γ-terms, in particular in $\lambda\sigma_{0\to i-1}|_{\hat{u}}$. But as $\lambda'\sigma_{0\to i-1}|_{\hat{u}}$ has the same prefix, but it is s', there is a Δ-term in a Γ-term, so by the induction hypothesis $x_{s'}$ occs grey in $\mathrm{AI}^{\Delta}_{*}(C_j)$ for some j.

   As Γ-terms are not lifted in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_j)$, $x_{s'}$ is not lifted there.

   As s' is in the range of the unifier, s' occurs in a resolved literal.

   By the definition of au, $\{x_{s'} \mapsto x_s\} \in \tau$ as s is the term at the position of $x_{s'}$ in $\lambda\sigma$ for λ the resolved literal where s' occurs.

   Hence there is a grey occurrence of x_s in $\mathrm{AI}^{\Delta}_{*}(C)$.

2. Suppose a variable u occurs in C_1 or C_2 grey or in a maximal colored single colored Γ-colored term such that $u\sigma$ contains a multi-colored Γ-term t

   Then $\lambda'\sigma_{0\to i-}|_{\hat{u}}$ actually is $t \Rightarrow$ IH. □

TODO: ICI ICI ICI: this lemma should easily give the main result. extend to factorisation and write up nicely

2 Attempts

lored_container⟩? **Conjectured Lemma 8.** *Let a variable x occur twice in C such that in one occ, the smallest colored term containing x is a Γ-term and for the other, the smallest colored term containing x is a Δ-term. Then x occurs grey in $\mathrm{AI}_*(C)$.*

Proof. **missing: variables don't have to occur grey in $y\sigma$, e.g. in $\gamma[y]$, $y\sigma$ might be $f(x)$ with f Γ-colored.**

- Suppose that in C_i, $\gamma[x]$ occurs and in C_j, we have $\delta[y]$ such that x occurs grey in $y\sigma$.

  Then y occurs in l at $l|_{\hat{y}}$ such that $l'|_{\hat{y}}$ is an abstraction of a term containing a grey occurrence of x.

  Suppose that $l|_{\hat{y}}$ (and therefore also $l|_{\hat{y}}$) is not a grey occurrence as otherwise we are done.

  As $l\sigma l'\sigma$, $l|_{\hat{y}}$ and $l|_{\hat{y}}$ share their prefix, so their color is the same.

  Then induction hypothesis.

- Suppose that in C_i, $\gamma[z]$ occurs and in C_j, $\delta[y]$ occurs such that x occurs grey in $y\sigma$ and in $z\sigma$.

  By Lemma **??**, exists $y_1, \ldots, y_n$ and $z_1, \ldots z_m$ such that x occurs grey in $y_i\sigma$ and in $z_i\sigma$ and term opposite of y_n and z_m actually contains x.

  If any y_i, z_j occurs grey, done, so assume all occur colored.

  z_m and y_n opposite of actual x, as x only in one clause, z_m and y_n in same clause. they do share prefix with the occurrences of x in the clause where x is.

  if they there are contained in smallest col terms of opposite color $\Rightarrow$ ind hyp

  otw of same smallest term color there.

  Note that every y_i, z_j occurs at least twice: once as opposite var of the last one, once to unify with the next one.

  as originally different colors and at meeting point at x same color, there has to be one alternation, where we use the ind hyp.

- Suppose that $\gamma[x]$ in C_i and $\delta[x]$ in $z\sigma$ such that z occurs grey in C_j.

  If $\delta[x]$ occurs in C_i (cannot occur in other clause), ind hyp.

  Suppose it does not occur. Then however exists $\delta[y]$ s.t. x occurs grey in $y\sigma \Rightarrow$ other case.

- Suppose that $\gamma[x]$ in $y\sigma$ such that y occurs grey in C_i and $\delta[x]$ in $z\sigma$ such that z occurs grey in C_j.

  If $\gamma[x]$ and $\delta[x]$ occur, ind hyp.

  If just one occurs, $\Rightarrow$ other case.

  If none of them occur, then occur $\delta[\alpha]$ s.t. x grey in $\alpha\sigma$ and similar for $\gamma[\beta] \Rightarrow$ other case.

$\square$

Conjectured Lemma 9. *Let σ unifier. exists unification order $\sigma = \sigma_1 \ldots \sigma_n$ with $\sigma_i = \{x_i \mapsto r_i\}$ s.t. x_i does not occur in $\{r_i, r_{i+1}, \ldots, r_n\}$.*

Proof. Suppose ordering does not exist, i.e. $l\sigma = l'\sigma$, but every x_i occurs in some r_j for j‘i. But then last variable does not occur later.. □

Lemma 10. *Let σ unifier.*
At any stage in the run of the unification algo, exists var x as one part of a difference pair s.t. x does not occur in a function symbol in a difference pair.

Proof. Suppose no such var exists.
resolve all differences $x_i \sim r_i$ such that r_i does not contain a variable in a function symbol.
all variables, in particular the remaining x_i, occur in a function symbol in r_j for some j.
Iteratively resolve in some order: $x_i \mapsto r_i$, where every r_i contains at least one variable. Hence as every x_i occurs in some r_j, the variable in r_i then occurs in r_j.
so after a step, for the remaining difference pairs, it is still the case that every variable occurs in some r_j.
We do not get an occurs check error as by assumptions, the term are unifiable.
when we get to the point where there is only one subst left, it has to be of the form $x_i \mapsto r_i[x_i]$, so we do get an occurs check error, which contradicts the assumptions that the terms are unifiable.

□

Lemma 11. *Let σ unifier. At any stage in the run of the unification algorithm, there exists a variable as one part of a difference pair such that the other part does not contain a variable, which also occurs as one part of a difference pair, under a function symbol.*

Proof. Suppose to the contrary, that ....
Construct graph with vars as nodes and arrow from x, y if exists difference pair $(x, r[y])$ or the symmetric pair.
As every variable unifies to a term containg another variable, we have that $\forall x \exists y \, E(x, y)$. Hence we can build a path of length $|V| + 1$, but this contains a cycle.

□

TODO ICI: does this mean that there is a variable which does not have a variable in a term at its RHS? (all difference pairs have a variable at some side, let's call it LHS and the other one RHS)
possibly: do induction along this order: take subst which has no var to the right, then this one occurs in the term. next term then does not actually exists necessarily, so need to show some induction property.
evil examples:
$P(z, z, \delta), \neg P(f(x), f(y), y)$
$P(z, f(z), f(f(\delta))), \neg P(f(x), y, y)$
$P(u, f(z), f(f(\delta))), \neg P(f(x), y, y)$

Conjectured Lemma 13. *Let $\sigma = \mathrm{mgu}(l, l')$*
Suppose Γ-term $s[y]$ in some unification pair, δ grey in $y\sigma$.

Conjectured Lemma 12. *Suppose* Γ*-term* $s(y)$ *in original diff pairs.*
Suppose $y\sigma = x$ *(simplification).*
Suppose no col change, i.e. no var x *occurs in a unified literal twice such that once in s.c.* Γ*-term and once in s.c.* Δ*-term.*
Suppose no x *grey in* $l\sigma\,(= l'\sigma)$.
Hence at some point have diff pair (y, v) *with* $v\sigma = x$.
by no col change and $s(y)$, y *does not occur in a s.c.* Δ*-term.*
As no x *grey in* $l\sigma$ *and* $y\sigma = x$, *no* y *grey.*
Hence y *only s.c.* Γ*-col.*
y *and* v *same prefix, so* v *s.c.* Γ*-col.*

Suppose no col change.
Suppose no δ *grey in* $l\sigma\,(= l'\sigma)$.
Then exists Γ*-term* $h[\delta']$ *in* l *or* l' *OR in earlier* mgu*-operation.*

Conjectured Lemma 14. *Suppose s.c. Γ-term containing Δ-term δ is created via unification of l and l'. Then at least one of the following statements holds:*

1. *In $l\sigma$ ($=l'$), δ occurs grey.*
2. *There is a variable x in l or l' such that it occurs once in an s.c. Γ-term and once in an s.c. Δ-term.*
3. *A δ-term occurs in a Γ-term in l or l' (TODO: be more precise on which term).*

Proof. We show that a term in question is created, then one of the statments holds, or a term in question has been created earlier during the run of the mgu.

1. Supp have $f(y)$ in some unification pair.

   Note y not grey somewhere as otherwise done.

   At some stage exists diff pair (y, t). note y, t same prefix, hence same color. t abstraction of $\varepsilon[\delta]$.

   - supp t contains outermost symbol of δ. as y, t same color, t is multi-col term either in l or l', or created earlier during unification algo.
   - otw t contains var v s.t. $v\sigma = \delta$ or $v\sigma = \varepsilon[\delta]$.

     Supp. v occurs grey in l or l'. then done.

     Note during unification procedure, coloring does not disappear, hence assume now all v colored.

     [hole: col change]

     hence can assume all occs of v are s.c. Γ-col.

     so have like $f(v)$, with $v\sigma = \delta$ or $v\sigma = \varepsilon[\delta]$. the corresponding diff pair is resolved earlier or later.

     possible argument: finitely often anyway?

     possible argument: after finitely many variable renamings, we hit an actual term, which then is strictly smaller, hence terminates?

2. var substituted for multi-colored term .

□

Conjectured Lemma 15. *Let $\sigma = \mathrm{mgu}(l, l')$. Let $\gamma[\delta]$ be a Γ-term containing a Δ-term δ in $l\sigma$. Then one of the following statments holds:*

1. *δ occurs at a grey position in $l\sigma$ TODO: argue about occurring $l\sigma$.*
2. *col change (where?)*
3. *in l or l', δ occurs in a Γ-term.*

Proof. Let $\sigma = \sigma_1 \cdots \sigma_n$, where σ_i stems from the ith substitution applied by the unification algo.
Let $l_j = l\sigma_1 \cdots \sigma_j$
Let σ_i be unifier $x \mapsto \delta$.
Suppose l_i contains a Δ-term in a Γ-term, where the respective predecessor of the Γ-term does not have a Δ-term at that position or does not exist in l_{i-1}.

1. Suppose a Γ-term $t[y]$ exists in l_{i-1}, such that it contains a grey occ of a variable y such that $y\sigma_i = \varepsilon[\delta]$ (where ε may be "empty" or else some grey term). The corresponding difference pair is $(y, \varepsilon[\delta])$, say at position $\hat{y}$

   So y occurs at say $\hat{y}$ in l or l', say λ. (y may occur in both, variable-disjointness might have already been broken).

   If it is a grey occurrence, we are done as δ occurs grey in $y\sigma$.

   So assume y occurs colored.

   $\lambda'_{i-1}|_{\hat{y}} = \varepsilon[\delta]$. Note that $\lambda_{i-1}|_{\hat{y}}$ and $\lambda'_{i-1}|_{\hat{y}}$ agree on the prefix (by virtue of being a difference pair).

   - Suppose $\lambda_{i-1}|_{\hat{y}}$ occurs in an s.c. Γ-term. Then $\lambda'_{i-1}|_{\hat{y}}$ is δ in a Γ-term in l or l' $\Rightarrow$ IH.
   - Suppose $\lambda_{i-1}|_{\hat{y}}$ occurs in an s.c. Δ-term. Then as y occurs in t in a Γ-term, we have a col change (but possibly distributed over l/l'). TODO: lemma for col change

   , say

2. Suppose y at $\hat{y}$ in λ_{i-1} s.t. $y\sigma_i$ is a Γ-term containing a Δ-term.

   Then $\lambda'_{i-1}|_{\hat{y}}$ actually is that term $\Rightarrow$ IH.

$\square$

Conjectured Lemma 16. *Let $\sigma = \mathrm{mgu}(l, l')$ such that in l and l', there are grey occs for col changes. Let $\gamma[x]$ be a s.c. Γ-term containing a variable x and $\delta[x]$ be a s.c. Δ-term containing the same variable x. Then x occurs at a grey position.*

Proof. probably revisit later when pre-lemmas are done

□

3 Structure (cases) of relevant unifications

Lemma 17. *For a difference pair or a not necessarily prefix-disjoint "unification pair" (s, t), s and t are both of same maximal and minimal color.*

Supp $f(x)$ occurs somewhere (original diff pairs or somewhere during run of algo) and $x\sigma = \varepsilon[\delta]$.

3.1 fst

Then $f(x) \sim t$, s.t. $t\sigma = f(\varepsilon[\delta])$.
(Suppose no col change.)

1. Supp $t = f(\varepsilon[\delta])$. ✓
2. Supp $t = f(y)$. $y\sigma = \varepsilon[\delta]$. Then IH (for some IH...).
3. Supp $t = f_{1/2}(y)$. $y\sigma = f_{1/2}(\varepsilon[\delta])$.
4. Supp $t = y$. $y\sigma = f(\varepsilon[\delta])$.
5. ? Supp $h(t) = y$.

3.2 snd

Then actually $x \sim t$, s.t. t possibly non-proper abstraction of $\varepsilon[\delta]$.

1. Supp $t = \varepsilon[\delta]$. ✓
2. Supp $t = y$. $y\sigma = \varepsilon[\delta]$.

3.3 random notes

suppose $z \sim f(x)$. then x is only changed if z is unified with something with an f-prefix.
look at terms where partial unification applies. the final state is just an extremely advanced applied partial unification.