Contents

ng_order_not_relevant⟩? **Lemma 1.** $\ell_\Gamma[\ell_\Delta[\varphi]] = \ell_\Delta[\ell_\Gamma[\varphi]]$.

Chapter 1

Interpolant extraction from resolution proofs in one phase lifting terms whose quantifier position can be determined – nested

Definition 2 (Substitution $\tau(\iota)$)**.** For an inference ι with $\sigma = \mathrm{mgu}(\iota)$, we define the infinite substitution $\tau(\iota)$ with $\mathrm{dom}(\tau(\iota)) = \mathrm{dom}(\sigma) \cup \{z_s \mid s\sigma \neq s\}$ as follows for a variable x:

define infinite substitutions properly and apply definition here

$$x\tau(\iota) = \begin{cases} x\sigma & x \text{ is a non-lifting variable} \\ z_{t\sigma} & x \text{ is a lifting variable } z_t \end{cases}$$

If the inference ι is clear from the context, we abbreviate $\tau(\iota)$ by τ. △

⟨lemma:lifting_tau_commute⟩ **Lemma 3.** *For a formula or term φ and an inference ι such that $\tau = \tau(\iota)$, $\ell[\ell[\varphi]\tau] = \ell[\varphi\tau]$.*

Proof. We proceed by induction.

- Suppose that t is a grey constant or function symbol of the form $f(t_1, \dots, t_n)$. Then we can derive the following, where (IH) signifies a deduction by virtue of the induction hypothesis.

$$\begin{aligned}\ell[\ell[t]\tau] &= \ell[\ell[f(t_1,\ldots,t_n)]\tau] \\ &= \ell[f(\ell[t_1]\tau,\ldots,\ell[t_n]\tau)] \\ &= f(\ell[\ell[t_1]\tau],\ldots,\ell[\ell[t_n]\tau]) \\ &\overset{\text{(IH)}}{=} f(\ell[t_1\tau],\ldots,\ell[t_n\tau]) \\ &= \ell[f(t_1,\ldots,t_n)\tau] \\ &= \ell[t\tau]\end{aligned}$$

- Suppose that t is a colored constant or function symbol. Then:

$$\ell[\ell[t]\tau] = \ell[z_t\tau] = \ell[z_{t\sigma}] = z_{t\sigma} = \ell[t\sigma] = \ell[t\tau]$$

- Suppose that t is a variable x. Then:

$$\ell[\ell[t]\tau] = \ell[\ell[x]\tau] = \ell[x\tau] = \ell[t\tau]$$

- Suppose that t is a lifting variable z_t. Then:

$$\ell[\ell[z_t]\tau] = \ell[z_t\tau] \qquad \square$$

Definition 4 (Incrementally lifted interpolant)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$. We define $\mathrm{LI}(\pi)$ to be $\mathrm{LI}(\square)$, where $\square$ is the empty clause derived in π.

Let C be a clause in π.

We define the preliminary formula $\mathrm{LI}^\bullet(C)$ as follows:

Base case. If $C \in \Gamma$, $\mathrm{LI}(C) \overset{\text{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{LI}(C) \overset{\text{def}}{=} \top$.

Resolution.

If the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then define $\mathrm{LI}(C)$ as follows:

1. If l is Γ-colored: $\mathrm{LI}^\bullet(C) \overset{\text{def}}{=} \mathrm{LI}(C_1)\tau \vee \mathrm{LI}(C_2)\tau$
2. If l is Δ-colored: $\mathrm{LI}^\bullet(C) \overset{\text{def}}{=} \mathrm{LI}(C_1)\tau \wedge \mathrm{LI}(C_2)\tau$
3. If l is grey: $\mathrm{LI}^\bullet(C) \overset{\text{def}}{=} (l\tau \wedge \mathrm{LI}(C_2)\tau) \vee (\neg l'\tau \wedge \mathrm{LI}(C_1)\tau)$

Factorisation. If the clause C is the result of a factorisation step ι of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{LI}^\bullet(C) \overset{\text{def}}{=} \ell[\mathrm{LI}(C_1)\tau]$.

Paramodulation. If the clause C is the result of a paramodulation step ι of $C_1 : s = t \vee D$ and $C_2 : E[r]$ with $\sigma = \mathrm{mgu}(\iota)$. Let $h[r]$ be the maximal colored term in which r occurs in $E[r]$ and define $\mathrm{LI}(C)$ as follows:

1. If $h[r]$ is Δ-colored and $h[r]$ occurs more than once in $\mathrm{LI}(C_2)\tau \vee E[r]\tau$:
   $\mathrm{LI}^\bullet(C) \stackrel{\text{def}}{=} (s = t \wedge \mathrm{LI}(C_2))\tau \vee (s \neq t \wedge \mathrm{LI}(C_1))\tau \vee (s = t \wedge h[s] \neq h[t])\tau$
2. If $h[r]$ is Γ-colored and $h[r]$ occurs more than once in $\mathrm{LI}(C_2)\tau \vee E[r]\tau$:
   $\mathrm{LI}^\bullet(C) \stackrel{\text{def}}{=} [(s = t \wedge \mathrm{LI}(C_2))\tau \vee (s \neq t \wedge \mathrm{LI}(C_1))\tau] \wedge (s \neq t \vee h[s] = h[t])\tau$
3. If r does not occur in a colored term in $E[r]$ which occurs more than once in $\mathrm{LI}(C_2)\tau \vee E[r]\tau$:
   $\mathrm{LI}^\bullet(C) \stackrel{\text{def}}{=} (s = t \wedge \mathrm{LI}(C_2))\tau \vee (s \neq t \wedge \mathrm{LI}(C_1))\tau$

$\mathrm{LI}(C)$ is built from $\mathrm{LI}^\bullet(C)$ according to the following steps:

1. Lift all maximal colored terms in $\mathrm{LI}^\bullet(C)$ which contains some variable which does not occur in C.
2. Let X (Y) be the Δ-(Γ-)lifting variables created in the previous step.
3. Prefix the resulting formula with an arrangement $Q(C)$ of the elements of $\{\forall x_t \mid x_t \in X\} \cup \{\exists y_t \mid y_t \in Y\}$ such that if s and r are terms such that s is a subterm of r, then z_s precedes z_r. △

ma:substitute_and_lift⟩ **Lemma 5.** *Let σ be a substitution and F a formula without Φ-colored terms such that for a set of formulas Ψ which does not contain Φ-lifting variables, $\Psi \models F$. Then $\Psi \models \ell_\Phi[F\sigma]$.*

Proof. $\ell_\Phi[F\sigma]$ is an instance of F: σ substitutes variables either for terms which do not contain Φ-colored symbols or by terms containing Φ-colored symbols. For the first kind, the lifting has no effect. For the latter, the lifting only replaces subterms of the terms introduced by the substitution by a lifting variable such that the original structure of F remains invariant as it by assumption does not contain colored terms. □

delta_lifted_invariant⟩ **Lemma 6.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then $\Gamma \models \ell_\Delta[\mathrm{LI}(C)] \vee \ell_\Delta[C]$*

Proof. We show the strengthening $\Gamma \models \ell_\Delta[\mathrm{LI}(C)] \vee \ell_\Delta[C_\Gamma]$[1].

As a first step, we prove by induction that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

[1] Recall that D_Φ denotes the clause created from the clause D by removing all literals which are not contained $\mathrm{L}(\Phi)$.

Base case. If $C \in \Gamma$, then $\ell_\Delta[C] = C$ and clearly $\Gamma \models C$. If otherwise $C \in \Delta$, then $\mathrm{LI}(C) = \top$.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

By the induction hypothesis we obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)] \vee \ell_\Delta[D_\Gamma] \vee \ell_\Delta[l_\Gamma]$ as well as $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)] \vee \ell_\Delta[E_\Gamma] \vee \neg\ell_\Delta[l'_\Gamma]$. Hence by Lemma 5 and Lemma 3, we get:

$$\Gamma \overset{(\circ)}{\models} \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[D_\Gamma\tau] \vee \ell_\Delta[l_\Gamma\tau]$$

$$\Gamma \overset{(*)}{\models} \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[E_\Gamma\tau] \vee \neg\ell_\Delta[l'_\Gamma\tau]$$

As $l_\Gamma\sigma = l'_\Gamma\sigma$, it holds that $l_\Gamma\tau = l'_\Gamma\tau$ and consequently $\ell_\Delta[l_\Gamma\tau] = \ell_\Delta[l'_\Gamma\tau]$. We proceed by a case distinction on the color of the resolved literal to show that in each case, we have that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$:

- Suppose that l is Γ-colored. Then $l_\Gamma = l$ and $l'_\Gamma = l$, and we can perform a resolution step on $(\circ)$ and $(*)$ to obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[D_\Gamma\tau] \vee \ell_\Delta[E_\Gamma\tau]$. This however is nothing else than $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that l is Δ-colored. Then $(\circ)$ and $(*)$ reduce to $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[D_\Gamma\tau]$ and $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[E_\Gamma\tau]$ respectively, which clearly implies that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee (\ell_\Delta[D_\Gamma\tau] \wedge \ell_\Delta[E_\Gamma\tau])$. This is turn is however just the unfolding of $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that l is grey. Then $l_\Gamma = l$ and $l'_\Gamma = l$, and $(\circ)$ and $(*)$ imply that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee (\ell_\Delta[l_\Gamma\tau] \wedge \ell_\Delta[E_\Gamma\tau]) \vee (\neg\ell_\Delta[l'_\Gamma\tau] \wedge \ell_\Delta[D_\Gamma\tau])$. This however is equivalent to $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

Factorisation. Suppose the clause C is the result of a factorisation inference ι of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

The induction hypothesis gives $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)] \vee \ell_\Delta[l_\Gamma \vee l'_\Gamma \vee D_\Gamma]$. By Lemma 5, we obtain $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[l_\Gamma\tau \vee l'_\Gamma\tau \vee D_\Gamma\tau]$. As however $l\sigma \equiv l'\sigma$, also $l\tau \equiv l'\tau$, so we can apply a factorisation step and obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[l_\Gamma\tau \vee D_\Gamma\tau]$, which is nothing else than $\Gamma \models \mathrm{LI}^\bullet(C) \vee \ell_\Delta[C_\Gamma]$.

Paramodulation. Suppose the clause C is the result of a paramodulation inference ι of $C_1 : s = t \vee D$ and $C_2 : E[r]_p$ with $\sigma = \mathrm{mgu}(\iota)$.

By the induction hypothesis, we obtain the following:

$\Gamma \overset{(\circ)}{\models} \ell_\Delta[\mathrm{LI}(C_1)] \vee \ell_\Delta[D_\Gamma] \vee \ell_\Delta[s] = \ell_\Delta[t]$

$\Gamma \overset{(*)}{\models} \ell_\Delta[\mathrm{LI}(C_2)] \vee \ell_\Delta[(E[r]_p)_\Gamma]$

Suppose now that for a model M of Γ that $M \models \ell_\Delta[s] \neq \ell_\Delta[t]$. Then we get by $(\circ)$ that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)] \vee \ell_\Delta[D_\Gamma]$, which by Lemma 5 and Lemma 3 gives $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[D_\Gamma\tau]$. Note that $\Gamma \models \ell_\Delta[s\tau] \neq \ell_\Delta[t\tau] \wedge \ell_\Delta[\mathrm{LI}(C_1)\tau]$ suffices for $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)]$ and $\Gamma \models \ell_\Delta[D_\Gamma\tau]$ implies that $\Gamma \models \ell_\Delta[C_\Gamma]$. Therefore we obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

Now suppose to the contrary that for a model M of Γ that $M \models \ell_\Delta[s] = \ell_\Delta[t]$. By Lemma 5 and Lemma 3 we obtain from $(*)$ that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[(E[r]_p)_\Gamma\tau]$. As however $\sigma = \mathrm{mgu}(r, s)$, $r\tau \equiv s\tau$ and hence $\ell_\Delta[r\tau] \equiv \ell_\Delta[s\tau]$. Therefore we also have that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[(E[s]_p)_\Gamma\tau]$.

We proceed by a case distinction:

- Suppose that the position p in $E[s]_p$ is not contained in a Δ-term. Then $\ell_\Delta[(E[s]_p)_\Gamma\tau]$ and $\ell_\Delta[(E[t]_p)_\Gamma\tau]$ only differ at at position p. But as $\ell_\Delta[s] = \ell_\Delta[t]$, we can apply Lemma 5 and Lemma 3 to obtain that $\ell_\Delta[s\tau] = \ell_\Delta[t\tau]$. Thus $M \models \ell_\Delta[(E[s]_p)_\Gamma\tau] \Leftrightarrow \ell_\Delta[(E[t]_p)_\Gamma\tau]$ and consequently $M \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[(E[t]_p)_\Gamma\tau]$. As furthermore $\ell_\Delta[\tau s] = \ell_\Delta[t] \wedge \ell_\Delta[\mathrm{LI}(C_2)\tau]$ entails $\ell_\Delta[\mathrm{LI}^\bullet(C)]$ and $\ell_\Delta[(E[t]_p)_\Gamma\tau]$ is sufficient for $\ell_\Delta[C_\Gamma]$, we have that $M \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that the position p in $E[s]_p$ is contained in a maximal Δ-term $h[s]$. We distinguish further:
  - Suppose $h[s]$ occurs more than once in $\mathrm{LI}(C_2)\tau \vee E[s]_p\tau$. Assume that $M \models \ell_\Delta[h[s]] = \ell_\Delta[h[t]]$ as otherwise $M \models \ell_\Delta[s] = \ell_\Delta[t] \wedge \ell_\Delta[h[s]] \neq \ell_\Delta[h[t]]$, which implies that $M \models \ell_\Delta[\mathrm{LI}^\bullet(C)]$. But then we obtain that $\ell_\Delta[(E[s]_p)_\Gamma\tau]$ and $\ell_\Delta[(E[t]_p)_\Gamma\tau]$ differ solely at subterms which we can assume to be equal in the model, so by a similar line of argument as in the preceding case, we can deduce that $M \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C]$.
  - Suppose $h[s]$ occurs exactly once in $\mathrm{LI}(C_2)\tau \vee E[s]_p\tau$. Then the lifting variable $x_{h[s]}$ occurs exactly once in $\ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[E[s]_p\tau]$.

    Note that from $(*)$ by applying Lemma 5 as well as Lemma 3, we obtain that $M \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[(E[s]_p)_\Gamma\tau]$. As $x_{h[s]}$ occurs only once and free in this formula, it is implicitly universally quantified and we can instantiate it arbitrarily, in particular by

$z_{h[t]}$. But thereby we get that $M \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[(E[t]_p)_\Gamma\tau]$, which implies that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

As we have now established that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$, we show that also $\Gamma \models \ell_\Delta[\mathrm{LI}(C)] \vee \ell_\Delta[C_\Gamma]$ holds.

The difference between $\ell_\Delta[\mathrm{LI}^\bullet(C)]$ and $\ell_\Delta[\mathrm{LI}(C)]$ lies only in certain maximal colored terms which are lifted in $\ell_\Delta[\mathrm{LI}(C)]$, hence it suffices to consider these. Let t be a term in $\mathrm{LI}^\bullet(C)$ at position p such that $\mathrm{LI}(C)|_p = \ell[t]$. Then t is a maximal colored term and contains a variable which does not occur in C.

If t is Δ-colored, then $\ell_\Delta[\mathrm{LI}^\bullet(C)]|_p = \mathrm{LI}(C)|_p = x_t$. Note that as t occurs at p in $\mathrm{LI}^\bullet(C)$, x_t occurs free at $\ell_\Delta[\mathrm{LI}^\bullet(C)]|_p$. Hence it is implicitly universally quantified and therefore entails that an explicit universal quantification in $\mathrm{LI}(C)$ is valid with an arbitrarily placed quantifier.

If otherwise t is a Γ-term, then $\ell_\Delta[\mathrm{LI}^\bullet(C)]|_p = \ell_\Delta[t]$. Hence $\ell_\Delta[t]$ represents a witness term for the existentially quantified lifting variable y_t at $\mathrm{LI}(C)|_p$. In general, $\ell_\Delta[t]$ however contains Δ-lifting variables, which require being lifted in the scope of the existential quantifier of y_t.

Let x_s be a Δ-lifting variable which occurs in $\ell_\Delta[t]$. It is essential to see that neither s nor a predecessor of s in the resolution derivation is lifted in a previous step of the interpolant extraction. Suppose to the contrary that this is the case in the inference creating the clause C'. Let s' and t' be the respective predecessors of s and t in C'. Then one of the following two contradictions eventuate:

- Suppose that s' is a subterm of the corresponding predecessor t'. Then due to the fact that s' is lifted, s' must contain a variable which does not occur in C'. But as t' contains s', t' contains this variable as well and would be lifted at this stage already.
- Otherwise t' does not contain s'. We have already established that s' contains a variable which does not occur in C'. As all clauses are variable-disjoint, no other clause contains this variable. But then it does not occur in any subsequent unifier, and in particular, it never enters t' by means of substitution, which implies that s' due to containing this variable does not become a subterm of a successor of t'.

Hence there are three possibilities for quantification of x_s. We show that in each of them, y_t is quantified in the scope of the quantifier of x_s.

1. Neither s nor a successor of s in the derivation occurs at a grey position. Then x_s is not quantified in the course of the interpolant extraction.

2. A variable which does not occur in C enters s by means of the current substitution σ or a variable is contained in s such that the only occurrences of it in C_1 and C_2 are in l and l'. Then x_s is lifted in the current step and as s is a subterm of t, x_s is quantified in $Q(C)$ prior to y_t.

3. The lifting variable x_s or a respective successor is quantified at a later stage in the derivation. Then as the quantifier for y_t is contained in $\mathrm{LI}(C)$ and for any successor C' of C, $\mathrm{LI}(C')$ contains a successor $\mathrm{LI}(C)$, y_t is quantified in the scope of the quantifier for x_s. □

⟨lemma:li_symmetry⟩ **Lemma 7.** *Let π be a refutation of $\Gamma \cup \Delta$ and $\hat{\pi}$ be π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then for a clause C in π and its corresponding clause $\hat{C}$ in $\hat{\pi}$, $\mathrm{LI}(C) \Leftrightarrow \neg\, \mathrm{LI}(\hat{C})$.*

Proof. We proceed by induction to show that $\mathrm{LI}^\bullet(C) \Leftrightarrow \neg\, \mathrm{LI}^\bullet(\hat{C})$:

Base case. If $C \in \Gamma$, then $\mathrm{LI}(C) = \bot \Leftrightarrow \neg\top \Leftrightarrow \neg\, \mathrm{LI}(\hat{C})$ as $\hat{C} \in \Delta$. The case for $C \in \Delta$ can be argued analogously.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

As τ depends only on σ, τ is the same for both π and $\hat{\pi}$.

We now distinguish the following cases:

1. l is Γ-colored:

$$\begin{aligned}
\mathrm{LI}^\bullet(C) &= \mathrm{LI}(C_1)\tau \,\vee\, \mathrm{LI}(C_2)\tau \\
&\Leftrightarrow \neg(\neg\mathrm{LI}(C_1)\tau \,\wedge\, \neg\mathrm{LI}(C_2)\tau) \\
&\Leftrightarrow \neg(\mathrm{LI}(\hat{C}_1)\tau \,\wedge\, \mathrm{LI}(\hat{C}_2)\tau) \\
&= \neg\, \mathrm{LI}^\bullet(\hat{C})
\end{aligned}$$

2. l is Δ-colored: This case can be argued analogously.

3. l is grey: Note $l\tau \overset{(*)}{=} l'\tau$.

$$\begin{aligned}
\mathrm{LI}^\bullet(C) \;&=\; (\neg l'\tau \wedge \mathrm{LI}(C_1)\tau) \;\vee\; (l\tau \wedge \mathrm{LI}(C_2)\tau) \\
&\overset{(*)}{\Leftrightarrow}\; (l'\tau \vee \mathrm{LI}(C_1)\tau) \;\wedge\; (\neg l\tau \vee \mathrm{LI}(C_2)\tau) \\
&\Leftrightarrow \neg[(\neg l'\tau \wedge \neg\mathrm{LI}(C_1)\tau) \;\vee\; (l\tau \wedge \neg\mathrm{LI}(C_2)\tau)] \\
&= \neg[(\neg \hat{l}'\tau \wedge \mathrm{LI}(\hat{C}_1)\tau) \;\vee\; (\hat{l}\tau \wedge \mathrm{LI}(\hat{C}_2)\tau)] \\
&= \neg\, \mathrm{LI}^\bullet(\hat{C})
\end{aligned}$$

Factorisation. Suppose the clause C is the result of a factorisation ι of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

As the construction is not influenced by the coloring, the induction hypothesis $\mathrm{LI}^\bullet(C) = \mathrm{LI}(C_1)\tau$ suffices.

Then $\mathrm{LI}^\bullet(C) = \ell[\mathrm{LI}(C_1)\tau]$, so the construction is not influenced by the coloring and by the induction hypothesis, $\mathrm{LI}^\bullet(C) \Leftrightarrow \neg\,\mathrm{LI}^\bullet(\hat{C})$.

Paramodulation. Suppose the clause C is the result of a paramodulation inference ι of $C_1 : s = t \vee D$ and $C_2 : E[r]_p$ with $\sigma = \mathrm{mgu}(\iota)$.

We proceed by a case distinction:

- Suppose that p in $E[r]_p$ is contained in a maximal Δ-term $h[r]$, which occurs more than once in $E[r]_p \vee \mathrm{LI}(E[r]_p)$. Then p in $\hat{E}[r]_p$ is contained in a maximal Γ-term $h[r]$, which occurs more than once in $\hat{E}[r]_p \vee \mathrm{LI}(\hat{E}[r]_p)$.

$$\begin{aligned}
\mathrm{LI}^\bullet(C) &= (s\tau = t\tau \wedge \mathrm{LI}(C_2)\tau) \vee (s\tau \neq t\tau \wedge \mathrm{LI}(C_1)\tau) \vee (s\tau = t\tau \wedge h[s]\tau \neq h[t]\tau) \\
&\Leftrightarrow \neg[(s\tau \neq t\tau \vee \neg\mathrm{LI}(C_2)\tau) \wedge (s\tau = t\tau \vee \neg\mathrm{LI}(C_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&= \neg[(s\tau \neq t\tau \vee \mathrm{LI}(\hat{C}_2)\tau) \wedge (s\tau = t\tau \vee \mathrm{LI}(\hat{C}_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&\Leftrightarrow \neg[(s\tau = t\tau \wedge \mathrm{LI}(\hat{C}_2)\tau) \vee (s\tau \neq t\tau \wedge \mathrm{LI}(\hat{C}_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&= \neg\,\mathrm{LI}^\bullet(\hat{C})
\end{aligned}$$

- Suppose that p in $E[r]_p$ is contained in a maximal Γ-term $h[r]$, which occurs more than once in $E[r]_p \vee \mathrm{LI}(E[r]_p)$. This case can be argued analogously.
- Otherwise:

$$\begin{aligned}
\mathrm{LI}^\bullet(C) &= (s\tau = t\tau \wedge \mathrm{LI}(C_2)\tau) \ \vee\ (s\tau \neq t\tau \wedge \mathrm{LI}(C_1)\tau) \\
&\Leftrightarrow \neg[(s\tau \neq t\tau \vee \neg\mathrm{LI}(C_2)\tau) \ \wedge\ (s\tau = t\tau \vee \neg\mathrm{LI}(C_1)\tau)] \\
&= \neg[(s\tau \neq t\tau \vee \mathrm{LI}(\hat{C}_2)\tau) \ \wedge\ (s\tau = t\tau \vee \mathrm{LI}(\hat{C}_1)\tau)] \\
&\Leftrightarrow \neg[(s\tau = t\tau \wedge \mathrm{LI}(\hat{C}_2)\tau) \ \vee\ (s\tau \neq t\tau \wedge \mathrm{LI}(\hat{C}_1)\tau)] \\
&= \neg\,\mathrm{LI}^\bullet(\hat{C})
\end{aligned}$$

We conclude by showing that $\mathrm{LI}^\bullet(C) \Leftrightarrow \neg\,\mathrm{LI}^\bullet(\hat{C})$ entails that $\mathrm{LI}(C) \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$: Clearly the terms to be lifted in $\mathrm{LI}^\bullet(C)$ and $\mathrm{LI}^\bullet(\hat{C})$ are the same and differ only in their color. Even though this results in different lifting variables, that is of no relevance as all lifted variables are instantly bound. Additionally, the quantifier type of any given lifting variable in $Q(C)$ is dual to the respective one in $Q(\hat{C})$. Furthermore note that the subterm-relation is not affected by the coloring, so the ordering of the quantifiers in $Q(C)$ and $Q(\hat{C})$ is identical. Hence $\mathrm{LI}(C) \Leftrightarrow \neg\, \mathrm{LI}(\hat{C})$. $\square$

⟨lemma:delta_entails_li⟩ **Lemma 8.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then $\Delta \models \neg \ell_\Gamma[\mathrm{LI}(C)] \vee \ell_\Gamma[C]$.*

Proof. Construct $\hat{\pi}$ with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then by Lemma 6, $\hat{\Gamma} \models \ell_{\hat{\Delta}}[\mathrm{LI}(\hat{C})] \vee \ell_{\hat{\Delta}}[\hat{C}]$, which by Lemma 7 is nothing else than $\Delta \models \neg \ell_\Gamma[\mathrm{LI}(C)] \vee \ell_\Gamma[C]$. $\square$

Theorem 9. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{LI}(\pi)$ is an interpolant of Γ and Δ.*

Proof. 6 8 plus lifting of ground terms (todo)

$\square$

proof idea for ground terms:

- idea 1: double all arities of function symbols. exchange parameter t by t, x, where x is a fresh variable. change interpretation such that even second parameter is ignored. Then there are no ground terms.
- idea 2:

  replace every constant by a variable
- idea 3:

  every ground term contains a constant.

  Let f_c be fresh function symbols for each constant c

  replace every constant a by $f_a(x)$, where x is a fresh variable symbol. then unifiers where a is mentioned are to be replaced by $f_a(x)$ with the respective x, which is universally unifiable.