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0.1 Lemmas other pdf

ma:lifting_order_not_relevant) **Lemma 1.** $\ell_\Gamma[\ell_\Delta[\varphi]] = \ell_\Delta[\ell_\Gamma[\varphi]]$.

0.2 Proof

Definition 2 ($\tau(\iota)$). For an inference ι with $\sigma = \text{mgu}(\iota)$, we define the infinite substitution $\tau(\iota)$ with $\text{dom}(\tau(\iota)) = \text{dom}(\sigma) \cup \{z_s \mid s\sigma \neq s\}$ as follows for a variable x :

$$x\tau(\iota) = \begin{cases} x\sigma & x \text{ is a non-lifting variable} \\ z_{t\sigma} & x \text{ is a lifting variable } z_t \end{cases}$$

If the inference ι is clear from the context, we abbreviate $\tau(\iota)$ by τ . \triangle

define infinite substitutions properly and apply definition here

(lemma:lifting_tau_commute) **Conjectured Lemma 3.** For a formula or term φ , $\ell[\ell[\varphi]\tau] = \ell[\varphi\tau]$.

Proof. We proceed by induction.

- Suppose that t is a grey constant or function symbol of the form $f(t_1, \dots, t_n)$. Then:

$$\begin{aligned} \ell[\ell[t]\tau] &= \ell[\ell[f(t_1, \dots, t_n)]\tau] \\ &= \ell[f(\ell[t_1]\tau, \dots, \ell[t_n]\tau)] \\ &= f(\ell[\ell[t_1]\tau], \dots, \ell[\ell[t_n]\tau]) \\ &= f(\ell[t_1\tau], \dots, \ell[t_n\tau]) \\ &= \ell[f(t_1, \dots, t_n)\tau] \\ &= \ell[t\tau] \end{aligned}$$

- Suppose that t is a colored constant or function symbol. Then:

$$\ell[\ell[t]\tau] = \ell[z_t\tau] = \ell[z_{t\sigma}] = z_{t\sigma} = \ell[t\sigma] = \ell[t\tau]$$

- Suppose that t is a variable x . Then:

$$\ell[\ell[t]\tau] = \ell[\ell[x]\tau] = \ell[x\tau]$$

- Suppose that t is a lifting variable z_t . Then:

$$\ell[\ell[z_t]\tau] = \ell[z_t\tau] \quad \square$$

Definition 4 (Incremental lifting). Let π be a resolution refutation of $\Gamma \cup \Delta$. We define $\text{LI}(\pi)$ ($\text{LI}_{\text{cl}}(\pi)$) to be $\text{LI}(\square)$ ($\text{LI}_{\text{cl}}(\square)$), where \square is the empty clause derived in π .

Let C be a clause in π . For a literal λ in C , we denote the corresponding literal in $\text{LI}_{\text{cl}}(C)$ by $\lambda_{\text{LI}_{\text{cl}}}$, which exists by lemma 5.

We define $\text{LI}(C)$ and $\text{LI}_{\text{cl}}(C)$ as follows:

Base case. If $C \in \Gamma$, $\text{LI}(C) \stackrel{\text{def}}{=} \perp$. If otherwise $C \in \Delta$, $\text{LI}(C) \stackrel{\text{def}}{=} \top$.

In any case, $\text{LI}_{\text{cl}}(C) \stackrel{\text{def}}{=} \ell[C]$.

Resolution. If the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then define $\text{LI}(C)$ and $\text{LI}_{\text{cl}}(C)$ as follows:

$$\text{LI}_{\text{cl}}(C) \stackrel{\text{def}}{=} \ell[(\text{LI}_{\text{cl}}(C_1) \setminus \{l_{\text{LI}_{\text{cl}}}\})\tau] \vee \ell[(\text{LI}_{\text{cl}}(C_2) \setminus \{l'_{\text{LI}_{\text{cl}}}\})\tau]$$

1. If l is Γ -colored: $\text{LI}(C) \stackrel{\text{def}}{=} \ell[\text{LI}(C_1)\tau] \vee \ell[\text{LI}(C_2)\tau]$
2. If l is Δ -colored: $\text{LI}(C) \stackrel{\text{def}}{=} \ell[\text{LI}(C_1)\tau] \wedge \ell[\text{LI}(C_2)\tau]$
3. If l is grey: $\text{LI}(C) \stackrel{\text{def}}{=} (\ell[l_{\text{LIcl}}\tau] \wedge \ell[\text{LI}(C_2)\tau]) \vee (\neg \ell[l'_{\text{LIcl}}\tau] \wedge \ell[\text{LI}(C_1)\tau])$

Factorisation. If the clause C is the result of a factorisation step ι of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then $\text{LI}(C) \stackrel{\text{def}}{=} \ell[\text{LI}(C_1)\tau]$ and $\text{LI}_{\text{cl}}(C) \stackrel{\text{def}}{=} \ell[(\text{LI}_{\text{cl}}(C_1) \setminus \{l'_{\text{LIcl}}\})\tau]$. \triangle

vs_clause_plus_literals_equal **Lemma 5.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$.*

Then for every literal λ in C , there exists a literal λ_{LIcl} in $\text{LI}_{\text{cl}}(C)$ such that $\lambda_{\text{LIcl}} = \ell[\lambda]$ and for resolved or factorised literals l and l' of a resolution or factorisation inference ι , we have that $\ell[l_{\text{LIcl}}\tau] = \ell[l'_{\text{LIcl}}\tau]$.

Proof. We proceed by induction.

Base case. For $C \in \Gamma \cup \Delta$, $\text{LI}_{\text{cl}}(C)$ is defined to be $\ell[C]$.

Resolution/Factorisation. Suppose the clause C is the result of a resolution or factorisation inference ι of the clauses \bar{C} with $\sigma = \text{mgu}(\iota)$.

For every literal in C , there exists a predecessor in a clause in \bar{C} . Let λ be a literal C_i with $C_i \in \bar{C}$, such that λ is not the predecessor of the literal being resolved or factorised in ι . Then $\lambda\sigma$ occurs in C .

By the induction hypothesis, $\ell[\lambda]$ occurs in $\text{LI}_{\text{cl}}(C_i)$. The successor of $\ell[\lambda]$ in $\text{LI}_{\text{cl}}(C)$ is of the form $\ell[\ell[\lambda]\tau]$. But by Lemma 3, this is nothing else than $\ell[\lambda\tau]$. As no lifting variables occur in λ , we get by the definition of τ that $\ell[\lambda\tau] = \ell[\lambda\sigma]$.

Let l and l' be the resolved or factorised literals. In order to show that $\ell[l_{\text{LIcl}}\tau] = \ell[l'_{\text{LIcl}}\tau]$, consider that by the induction hypothesis, this is nothing else than $\ell[\ell[l]\tau] = \ell[\ell[l']\tau]$. But by applying the same argument as above, this is equivalent to $\ell[l\sigma] = \ell[l'\sigma]$, which is implied by $l\sigma = l'\sigma$. \square

Definition 6. $\text{LI}^\Delta(C)$ ($\text{LI}_{\text{cl}}^\Delta(C)$) for a clause C is defined as $\text{LI}(C)$ ($\text{LI}_{\text{cl}}(C)$) with the difference that in its inductive definition, every lifting $\ell[\varphi]$ for a formula or term φ is replaced by a lifting of only the Δ -terms $\ell_\Delta[\varphi]$. \triangle

Remark. Many results involving $\text{LI}(C)$ ($\text{LI}_{\text{cl}}(C)$) are valid for $\text{LI}^\Delta(C)$ ($\text{LI}_{\text{cl}}^\Delta(C)$) in a formulation which is adapted accordingly. This can easily be seen by the following proof idea:

Let f_1, \dots, f_n be all Γ -colored function or constant symbols, c a fresh constant symbol and g be a fresh n -ary function symbol. Construct a formula $t = t$ such that $t = g(t_1, \dots, t_n)$, such that $t_i = f_i(c_1, \dots, c_m)$ for $1 \leq i \leq n$ where m is the arity of f_i and $c_j = c$ for $1 \leq j \leq m$. Let $\Delta' = \Delta$ and apply the desired result to the initial clause sets Γ and Δ' .

Under this construction, every originally Γ -colored symbol is now grey, which implies that $\text{LI}(C) = \text{LI}^\Delta(C)$ as well as $\text{LI}_{\text{cl}}(C) = \text{LI}_{\text{cl}}^\Delta(C)$. But $\Delta \models \varphi \Leftrightarrow \Delta' \models \varphi$ for any formula φ . \triangle

$\langle \text{lemma:no_colored_terms} \rangle$ **Lemma 7.** *Let C be a clause of a resolution refutation π of $\Gamma \cup \Delta$. $\text{LI}(C)$ and $\text{LI}_{\text{cl}}(C)$ do not contain colored symbols. $\text{LI}^\Delta(C)$ and $\text{LI}_{\text{cl}}^\Delta(C)$ do not contain Δ -colored symbols.*

Proof. For $\text{LI}(C)$ and $\text{LI}_{\text{cl}}(C)$, consider the following: In the base case of the inductive definitions of $\text{LI}(C)$ and $\text{LI}_{\text{cl}}(C)$, no colored symbols occur. In the inductive steps, any colored symbol which is added by τ to intermediary formulas is lifted.

For $\text{LI}^\Delta(C)$ and $\text{LI}_{\text{cl}}^\Delta(C)$, a similar argument goes through by reading colored as Δ -colored. \square

$\langle \text{lemma:substitute_and_lift} \rangle$ **Lemma 8.** *Let σ be a substitution and F a formula without Φ -colored terms such that for a set of formulas Ψ , $\Psi \models F$. Then $\Psi \models \ell_\Phi[F\sigma]$.*

Proof. $\ell_\Phi[F\sigma]$ is an instance of F : σ substitutes variables either for terms not containing Φ -colored symbols or by terms containing Φ -colored symbols. For the first kind, the lifting has no effect. For the latter, the lifting only replaces subterms of the terms introduced by the substitution by a lifting variable such that the original structure of F remains invariant as it by assumption does not contain colored terms. \square

$\langle \text{lemma:gamma_proves_pide} \rangle$ **Lemma 9.** *For a clause C in a resolution refutation of $\Gamma \cup \Delta$, $\Gamma \models \text{LI}^\Delta(C) \vee \text{LI}_{\text{cl}}^\Delta(C)$.*

Proof. We proceed by induction of the strengthening $\Gamma \models \text{LI}^\Delta(C) \vee \text{LI}_{\text{cl}}^\Delta(C_\Gamma)$.

Base case. For $C \in \Gamma$, $\text{LI}_{\text{cl}}^\Delta(C_\Gamma) = \ell_\Delta[C] = C$. Hence $\Gamma \models \text{LI}_{\text{cl}}^\Delta(C_\Gamma)$.

For $C \in \Delta$, $\text{LI}^\Delta(C) = \top$, so $\Gamma \models \text{LI}^\Delta(C)$.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \text{mgu}(\iota)$.

We define the following abbreviations:

$$\text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^* = \text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma \setminus \{l_{\text{LIcl}^\Delta}\})$$

$$\text{LI}_{\text{cl}}^\Delta((C_2)_\Gamma)^* = \text{LI}_{\text{cl}}^\Delta((C_2)_\Gamma \setminus \{\neg l'_{\text{LIcl}^\Delta}\})$$

Hence the induction hypothesis can be stated as follows:

$$\Gamma \models \text{LI}^\Delta(C_1) \vee \text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^* \vee (l_{\text{LIcl}^\Delta})_\Gamma$$

$$\Gamma \models \text{LI}^\Delta(C_2) \vee \text{LI}_{\text{cl}}^\Delta((C_2)_\Gamma)^* \vee \neg(l'_{\text{LIcl}^\Delta})_\Gamma$$

By Lemma 7, $\text{LI}^\Delta(C_i)$ and $\text{LI}_{\text{cl}}^\Delta(C_i)$ for $i \in \{1, 2\}$ do not contain Δ -colored terms. Hence we are able to apply Lemma 8 in order to obtain

$$\Gamma \stackrel{(\circ)}{\models} \ell_\Delta[\text{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[(l_{\text{LIcl}^\Delta})_\Gamma\tau]$$

$$\Gamma \stackrel{(*)}{\models} \ell_\Delta[\text{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[\text{LI}_{\text{cl}}^\Delta((C_2)_\Gamma)^*\tau] \vee \neg\ell_\Delta[(l'_{\text{LIcl}^\Delta})_\Gamma\tau]$$

By Lemma 5, we obtain that $\ell_\Delta[l_{\text{LIcl}^\Delta}\tau] = \ell_\Delta[l'_{\text{LIcl}^\Delta}\tau]$.

Now we distinguish cases based on the color of the resolved literal:

- Suppose that l is Γ -colored. Then as $\ell_\Delta[l_{\text{LIcl}\Delta}\tau] = \ell_\Delta[l'_{\text{LIcl}\Delta}\tau]$, we can perform a resolution step on (\circ) and $(*)$, which gives that $\Gamma \models \ell_\Delta[\text{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[\text{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[\text{LI}_{\text{cl}}^\Delta((C_2)_\Gamma)^*\tau]$. This however is nothing else than $\Gamma \models \text{LI}^\Delta(C) \vee \text{LI}_{\text{cl}}^\Delta(C)$.
- Suppose that l is Δ -colored. Then (\circ) and $(*)$ simply to the following:

$$\Gamma \models \ell_\Delta[\text{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^*\tau]$$

$$\Gamma \models \ell_\Delta[\text{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[\text{LI}_{\text{cl}}^\Delta((C_2)_\Gamma)^*\tau]$$
These however imply that $\Gamma \models \text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^* \vee \text{LI}_{\text{cl}}^\Delta((C_2)_\Gamma)^* \vee (\ell_\Delta[\text{LI}^\Delta(C_1)\tau] \wedge \ell_\Delta[\text{LI}^\Delta(C_2)\tau])$, which is nothing else than $\Gamma \models \text{LI}^\Delta(C) \vee \text{LI}_{\text{cl}}^\Delta(C)$.
- Suppose that l is grey. Suppose that M is a model of Γ such that $M \not\models \text{LI}_{\text{cl}}^\Delta(C)$, i.e. $M \not\models \ell_\Delta[\text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[\text{LI}_{\text{cl}}^\Delta((C_2)_\Gamma)^*\tau]$. Then $M \models \ell_\Delta[\text{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[l_{\text{LIcl}\Delta}\tau]$ as well as $M \models \ell_\Delta[\text{LI}^\Delta(C_2)\tau] \vee \neg \ell_\Delta[l'_{\text{LIcl}\Delta}\tau]$.
Due to $\ell_\Delta[l_{\text{LIcl}\Delta}\tau] = \ell_\Delta[l'_{\text{LIcl}\Delta}\tau]$, we obtain that

$$M \models (\ell_\Delta[l_{\text{LIcl}\Delta}\tau] \wedge \ell_\Delta[\text{LI}^\Delta(C_2)\tau]) \vee (\neg \ell_\Delta[l'_{\text{LIcl}\Delta}\tau] \wedge \ell_\Delta[\text{LI}^\Delta(C_1)\tau]),$$
which is nothing else than $M \models \text{LI}^\Delta(C)$.

Factorisation. Suppose the clause C is the result of a factorisation inference ι of $C_1 : l \vee l' \vee D$ with $\sigma = \text{mgu}(\iota)$.

We introduce the abbreviation $\text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^* = \text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma \setminus \{l_{\text{LIcl}\Delta}, \neg l'_{\text{LIcl}\Delta}\})$ and express the induction hypothesis as follows:

$$\Gamma \models \text{LI}^\Delta(C_1) \vee \text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^* \vee (l_{\text{LIcl}\Delta})_\Gamma \vee \neg (l'_{\text{LIcl}\Delta})_\Gamma$$

By Lemma 7, $\text{LI}^\Delta(C_i)$ and $\text{LI}_{\text{cl}}^\Delta(C_i)$ for $i \in \{1, 2\}$ do not contain Δ -colored terms. Hence we are able to apply Lemma 8 in order to obtain

$$\Gamma \stackrel{(*)}{\models} \ell_\Delta[\text{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[(l_{\text{LIcl}\Delta})_\Gamma\tau] \vee \neg \ell_\Delta[(l'_{\text{LIcl}\Delta})_\Gamma\tau]$$

As by Lemma 5 we get that $\ell_\Delta[l_{\text{LIcl}\Delta}\tau] = \ell_\Delta[l'_{\text{LIcl}\Delta}\tau]$, we can perform a factorisation step on $(*)$ to obtain that $\Gamma \models \ell_\Delta[\text{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\text{LI}_{\text{cl}}^\Delta((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[(l_{\text{LIcl}\Delta})_\Gamma\tau]$. But this is nothing else than $\Gamma \models \text{LI}^\Delta(C) \vee \text{LI}_{\text{cl}}^\Delta(C_\Gamma)$. \square

Definition 10 (Quantifier block). Let C be a clause in a resolution refutation π of $\Gamma \cup \Delta$ and \bar{x} be the Δ -lifting variables and \bar{y} the Γ -lifting variables occurring in $\text{LI}(C)$ and $\text{LI}_{\text{cl}}(C)$. $Q(C)$ denotes an arrangement of the elements of $\{\forall x_t \mid x_t \in \bar{x}\} \cup \{\exists y_t \mid y_t \in \bar{y}\}$ such that for two lifting variable z_s and z_r , if s is a subterm of r , then z_s is listed before z_r . We denote $Q(\square)$ by $Q(\pi)$. \triangle

Lemma 11. For a clause C of a resolution refutation of $\Gamma \cup \Delta$, $\ell_\Gamma[\text{LI}^\Delta(C)] = \text{LI}(C)$ and $\ell_\Gamma[\text{LI}_{\text{cl}}^\Delta(C)] = \text{LI}_{\text{cl}}(C)$.

Proof. We proceed by induction.

Base case. For $C \in \Gamma \cup \Delta$, $\text{LI}_{\text{cl}}^\Delta(C) = \ell_\Delta[C]$. By Lemma 1, $\ell_\Gamma[\ell_\Delta[C]] = \ell[C]$, so $\ell_\Gamma[\text{LI}_{\text{cl}}^\Delta(C)] = \ell[C] = \text{LI}_{\text{cl}}^\Delta(C)$.

$\text{LI}^\Delta(C)$ does not contain colored symbols.

Inductions step. Suppose the clause C is the result of a resolution or factorisation inference ι of the clauses \bar{C} .

Every literal in $\text{LI}(C)$ or $\text{LI}_{\text{cl}}(C)$ is of the form $\ell[\lambda\tau]$ for some λ in $\text{LI}(C_i)$ or $\text{LI}_{\text{cl}}(C_i)$ for some $C_i \in \bar{C}$.

Every literal in $\text{LI}^\Delta(C)$ or $\text{LI}_{\text{cl}}^\Delta(C)$ is of the form $\ell_\Delta[\lambda\tau]$ for some λ in $\text{LI}^\Delta(C_i)$ or $\text{LI}_{\text{cl}}^\Delta(C_i)$ for some $C_i \in \bar{C}$.

Hence it suffices to show that for a literal λ in $\text{LI}^\Delta(C_i)$ or $\text{LI}_{\text{cl}}^\Delta(C_i)$ and its corresponding literal κ in $\text{LI}(C_i)$ or $\text{LI}_{\text{cl}}(C_i)$ for some $C_i \in \bar{C}$ that $\ell_\Gamma[\ell_\Delta[\lambda\tau]] = \ell[\kappa\tau]$.

By the induction hypothesis, $\ell_\Gamma[\lambda] = \kappa$. By Lemma 7, no Δ -terms occur in λ . Hence $\ell[\lambda] = \kappa$ and also $\ell[\ell[\lambda]\tau] = \ell[\kappa\tau]$. By Lemma 3, $\ell[\lambda\tau] = \ell[\kappa\tau]$, which by Lemma 1 is nothing else than $\ell_\Gamma[\ell_\Delta[\lambda\tau]] = \ell[\kappa\tau]$. \square

(gamma_proves_quantified_lide) **Lemma 12.** *For a clause C of a resolution refutation of $\Gamma \cup \Delta$, $\Gamma \models Q(C)(\text{LI}(C) \vee \text{LI}_{\text{cl}}(C))$.*

Proof. By Lemma 11 $\ell_\Gamma[\text{LI}^\Delta(C) \vee \text{LI}_{\text{cl}}^\Delta(C)] = \text{LI}(C) \vee \text{LI}_{\text{cl}}(C)$.

By Lemma 9, $\Gamma \models \text{LI}^\Delta(C) \vee \text{LI}_{\text{cl}}^\Delta(C)$. Hence the terms in $\text{LI}^\Delta(C) \vee \text{LI}_{\text{cl}}^\Delta(C)$ provide witness terms for the Γ -lifting variables in $\text{LI}(C) \vee \text{LI}_{\text{cl}}(C)$, which are existentially quantified in $Q(C)(\text{LI}(C) \vee \text{LI}_{\text{cl}}(C))$.

Furthermore, the ordering imposed on the quantifiers in $Q(C)$ implies that if a Δ -lifting variable x_s occurs in a witness term for a Γ -lifting variable y_r , y_r is quantified in the scope of the quantifier of x_s as s is a subterm of r . This however ensures that the witness terms are valid. \square

(lemma:li_symmetry) **Lemma 13.** *Let π be a refutation of $\Gamma \cup \Delta$ and $\hat{\pi}$ be π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then for a clause C in π and its corresponding clause \hat{C} in $\hat{\pi}$, $Q(C)(\text{LI}(C)) \Leftrightarrow Q(\hat{C})(\text{LI}(\hat{C}))$.*

Proof. Note that LI_{cl} is defined irrespective of the coloring, so $\text{LI}_{\text{cl}}(C) = \text{LI}_{\text{cl}}(\hat{C})$.

Consider furthermore that liftings variables of C and \hat{C} only differ in the variable symbol, but not in the index, and that the quantifier type of any given lifting variable in C is exactly contrary to the corresponding one in \hat{C} . Hence for any formula ϕ , $Q(C)\neg\phi \Leftrightarrow \neg Q(\hat{C})\phi$.

It remains to show that $\text{LI}(C) \Leftrightarrow \neg \text{LI}(\hat{C})$, which we do by induction:

Base case. If $C \in \Gamma$, then $\text{LI}(C) = \perp \Leftrightarrow \neg \top \Leftrightarrow \neg \text{LI}(\hat{C})$ as $\hat{C} \in \Delta$. The case for $C \in \Delta$ can be argued analogously.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \text{mgu}(\iota)$.

As τ depends only on σ , τ is the same for both π and $\hat{\pi}$.

We now distinguish the following cases:

1. l is Γ -colored:

$$\begin{aligned} \text{LI}(C) &= \ell[\text{LI}(C_1)\tau] \vee \ell[\text{LI}(C_2)\tau] \\ &\Leftrightarrow \neg(\neg\ell[\text{LI}(C_1)\tau] \wedge \neg\ell[\text{LI}(C_2)\tau]) \\ &\Leftrightarrow \neg(\ell[\text{LI}(\hat{C}_1)\tau] \wedge \ell[\text{LI}(\hat{C}_2)\tau]) \\ &= \neg\text{LI}(\hat{C}) \end{aligned}$$

2. l is Δ -colored: This case can be argued analogously

3. l is grey: Note that by Lemma 5, $\ell[l_{\text{LIcl}}\tau] = \ell[l'_{\text{LIcl}}\tau]$ (*).

$$\begin{aligned} \text{LI}(C) &= (\neg\ell[l'_{\text{LIcl}}\tau] \wedge \ell[\text{LI}(C_1)\tau]) \vee (\ell[l_{\text{LIcl}}\tau] \wedge \ell[\text{LI}(C_2)\tau]) \\ &\stackrel{(*)}{\Leftrightarrow} (\ell[l'_{\text{LIcl}}\tau] \vee \ell[\text{LI}(C_1)\tau]) \wedge (\neg\ell[l_{\text{LIcl}}\tau] \vee \ell[\text{LI}(C_2)\tau]) \\ &\Leftrightarrow \neg\left((\neg\ell[l'_{\text{LIcl}}\tau] \wedge \neg\ell[\text{LI}(C_1)\tau]) \vee (\ell[l_{\text{LIcl}}\tau] \wedge \neg\ell[\text{LI}(C_2)\tau])\right) \\ &= \neg\left((\neg\ell[\hat{l}_{\text{LIcl}}\tau] \wedge \ell[\text{LI}(\hat{C}_1)\tau]) \vee (\ell[\hat{l}_{\text{LIcl}}\tau] \wedge \ell[\text{LI}(\hat{C}_2)\tau])\right) \\ &= \text{LI}(\hat{C}) \end{aligned}$$

Factorisation. Suppose the clause C is the result of a factorisation ι of C_1 :
 $l \vee l' \vee D$ with $\sigma = \text{mgu}(\iota)$.

Then $\text{LI}(C) = \ell[\text{LI}(C_1)\tau]$, so the construction is not influenced by the coloring and the induction hypothesis gives the result. \square

Theorem 14. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then $\text{LI}(\pi)$ is an interpolant.*

Proof. By Lemma 12 $\Gamma \models Q(\pi)(\text{LI}(\pi) \vee \text{LI}_{\text{cl}}(\pi))$. But as $\text{LI}_{\text{cl}}(\pi) = \square$, this simplifies to $\Gamma \models Q(\pi)\text{LI}(\pi)$.

By constructing a proof $\hat{\pi}$ from π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$, we obtain by Lemma 12 that $\hat{\Gamma} \models Q(\hat{\pi})\text{LI}(\hat{\pi})$. By Lemma 13, this however is nothing else than $\Delta \models \neg Q(\pi)\text{LI}(\pi)$.

As furthermore by construction no colored symbols occur in $Q(\pi)\text{LI}(\pi)$, this formula is an interpolant for $\Gamma \cup \Delta$. \square