1 Arrow-Algo

1. In the original clauses, find all occurrences of variables.

   Common case: If a variable appears as outermost symbol or only has grey ancestor-terms, add an arrow from it to all other occurrences.

   Uncommon case: if there is more than one occurrence of a variable under a Φ-colored term, add a *weak* dependency between them all (symmetric relation).

   NOTE: this creates double arrows for occurrences at same depth. This appears to be necessary for terms which are only variables, and doesn't hurt if the variable is contained in a term.

2. For each step in the derivation:

   a) Build propositional interpolant using $\mathrm{PI}(C_i)^*$, $i \in \{1, 2\}$, i.e. use ancestor PI without colored terms.

   b) If ancestors of atom added to $\mathrm{PI}(C)$ had arrows, merge them to atom in $\mathrm{PI}(C)$ (i.e. arrows starting in and leading to this atom).

   c) Replace colored terms in $\mathrm{PI}(C)$ (from new atom and unifier applied to $\mathrm{PI}(C_i)^*$) with fresh variables, except if a term has a double ended arrow to another overbinding variable, then use that variable.

      An arrow starts (ends) in one of the new variables if it starts (ends) somewhere in the term it replaced.

   d) Collect quantifiers: from $\mathrm{PI}(C_i)^*$, $i \in \{1, 2\}$ and ones from atom added to $\mathrm{PI}(C)$. Order such that arrows only point to variables to the right AND weakly connected variables appear in the same quantifier block.

      $\bar{Q}_n = \mathrm{sort}(Q_{n_1} \cup Q_{n_2} \cup \text{colored-terms}(l))$

1.1 algo more formally

Every literal in any initial clause set has a globally unique id/number
Ex: $P(y, a, f(z, g(y, b))) \lor Q(x)$
Term position:
0.2.1.0 means first literal, 3rd arg, 2nd arg, fst arg: y
0.1 is a
0.2.1 is $g(z, b)$
P calculates the position of a term or the term of a position, depending on the argument type.
for a position p_i, $\mathrm{P}(p_i)$ denotes whatever p_i refers to in its respective clause.
for a term t, $\mathrm{P}(t)$ denotes the position in t in its respective clause.
for a position p, $\mathrm{P}_{\mathrm{lit}}(p)$ denotes the position of the literal
for a position p, $\mathrm{P}_{\mathrm{term}}(p)$ denotes the position of the term in p_i
$\Rightarrow p = \mathrm{P}_{\mathrm{lit}}(p).\, \mathrm{P}_{\mathrm{term}}(p)$
for a position p, $p \mod i$ denotes p with i least significant places cut off, $0.2.1.0 \mod 2 = 0.2$

1.2 Arrows:

Definition 1 (Coloring of variable occurrences)**.** An occurrence of a variable x is called Φ-*colored* if it is contained in a maximal Φ-colored term. It is called *colored* if it is of any color and *grey* otherwise. △

$\mathcal{A}$ is a set of ordered pairs of term positions which point to positions in terms in literals
$\mathcal{W}$ is a set of unordered pairs of term positions which point to positions in terms in literals
Note that the "anchor point" for arrows are literals and not clauses. All literals occur in the initial clause sets. The literals in $\mathrm{AI}(C)$ are derived from these and the arrows apply to them.
Literals of colored predicates to not occur in $\mathrm{AI}(C)$ but can be relevant by transitivity.
w.r.t a refutation π of $\Gamma \cup \Delta$:

1. For each initial clause C in $\Gamma \cup \Delta$:

   For every variable x in C:

   Let Φ_x be the set of occurrences of a x in C. Let $\Phi_x^{\mathrm{col}} = \{p \in \Phi_x \mid p \text{ is contained in a colored term}\}$ and $\Phi_x^{\mathrm{grey}} = \Phi_x \setminus \Phi_x^{\mathrm{col}}$.

   Let $\mathrm{MaxCol}(\Phi) = \{p \mid p \text{ is a maximal colored term containing } q \text{ for } q \in \Phi\}$

   Add to $\mathcal{A}$ all (p_1, p_2) such that $p_1 \in \Phi_x^{\mathrm{grey}}$ and $p_2 \in \mathrm{MaxCol}(\Phi_x^{\mathrm{col}})$.

   Add to $\mathcal{W}$ all (p_1, p_2) such that $p_1 \in \mathrm{MaxCol}(\Phi_x^{\mathrm{col}})$ and $p_2 \in \mathrm{MaxCol}(\Phi_x^{\mathrm{col}})$ and $p_1 \neq p_2$.

2. For each C resulting from a resolution step from $C_1 : D \lor l$ and $C_2 : E \lor \neg l$ to $C = D \lor E$ with prop interpolant $\mathrm{PI}(\cdot)$:

The literals l and l' are unified and henceforth considered to be the same literal. Therefore the arrows of l and l' are merged.

1.3 algo

NOTE: for now, we assume that every colored-term has a globally unique id i and will be replaced by a variable with this index. This restriction is useful now and could potentially be lifted later, but it's not severe anyway.
Note: when a literal is added to the interpolant, the colored terms in one literal might have already been replaced with a certain variable before. we definitely have to use the same variable for both literals, and if one literal has other dependencies, we should stick with the variable we have.
PROBLEM: terms already replaced by variables still change! need to use same variable anyway, so note above not accurate!
POSSIBLE SOLUTION: for resolution, it is vital that the resolved literals are the unified, but nothing else is vital. We however ensure that the literals also have the same lifting variables. Furthermore, if variables are in the interpolant, then they have been present in both clauses, (e.g. $f(x)$ in C_1, $f(y)$ in C_2, then it's fine to universally quantify).

1.3.1 $\mathrm{AI}_{\mathrm{mat}}$ and $\mathrm{AI}_{\mathrm{cl}}$

Here, we define $\mathrm{AI}_{\mathrm{mat}}$, which represents the *matrix* of what will be the interpolant, and $\mathrm{AI}_{\mathrm{cl}}$, which represents the *clauses* in the refutation applied with the same unifications as $\mathrm{AI}_{\mathrm{mat}}$.

1. For each initial clause C, $\mathrm{AI}_{\mathrm{mat}}(C) = \mathrm{PI}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[C]$.

2. For each C resulting from a resolution step from $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ to $C = (D \vee E)\sigma$ with $l\sigma = l'\sigma$ with prop interpolant $\mathrm{PI}(\cdot)$:

   $l \dots$ literal in original clause

   $l_{\mathrm{AIcl}} \dots$ literal in $\mathrm{AI}_{\mathrm{cl}}$ (with unifications and liftings carried over, such that ind hyp goes through)

   $l\sigma = l'\sigma$, but l_{AIcl} and l'_{AIcl} might have been overbound with different variables. Still, they in a sense refer to the same ground literal, so we "can just" "unify" them.

   Prose explanation of formal definition below: Shape must be the same in the sense that grey terms are the same, otherwise there is Φ-replacing-var vs Φ-replacing-var (let arbitrary one win) or Φ-term and Φ-term (replace both with same var). (Φ-replacing-var vs Φ-term cannot happen as Φ-term is overbound as it has to be colored as otherwise the terms woudln't unify). Also apply these to substitutions in $\mathrm{AI}_{\mathrm{cl}}$ and $\mathrm{AI}_{\mathrm{mat}}$ here.

   au is defined on lifted terms of $\ell[l_{\mathrm{AIcl}}\sigma]$ and $\ell[l'_{\mathrm{AIcl}}\sigma]$, where the literals l_{AIcl} and l'_{AIcl} occur in $\mathrm{AI}_{\mathrm{cl}}(C_1)$, $\mathrm{AI}_{\mathrm{cl}}(C_2)$ such that for their corresponding l and l', $l\sigma = l'\sigma$. Note that if one of the arguments of au has assigned a color, the other one either has none or the same color. There cannot be a conflict as otherwise their original form would not be unifiable.

Note that $\mathrm{au}(a, b)$ is well-defined, i.e. never maps a variable to two different values as each occurrence of some x_j refers to a term with possible free variables, and since across the definition of au, always the same substitution σ is used as reference, every occurrence of x_j will be mapped to the same variable. (NOTE: this is what yet unproven conjectures in the other pdf are trying to formalize.)

l_{AIcl} and l'_{AIcl} are as they occur in $\mathrm{AI}_{\mathrm{cl}}\, C_1$ and $\mathrm{AI}_{\mathrm{cl}}\, C_2$. As the actual terms in the clause unify, we know that here, at least all terms have proper color after unification/lifting.

Let $\ell[l_{\mathrm{AIcl}}\sigma] = A(a_1, \dots, a_n)$, $\ell[l'_{\mathrm{AIcl}}\sigma] = A(b_1, \dots, b_n)$

$\mathrm{au}(A(a_1, \dots, a_n), A(b_1, \dots, b_n)) = \bigcup_{i=1}^{n} \mathrm{au}(a_i, b_i)$

$$\mathrm{au}(a_{\mathrm{AIcl}}, b_{\mathrm{AIcl}}) = \begin{cases} \bigcup_{j=1}^{n} \mathrm{au}(s_j, t_j) & \text{if } a_{\mathrm{AIcl}} = f_s(\bar{s}) \text{ grey and } b_{\mathrm{AIcl}} = f_t(\bar{t}) \text{ (includes } f_s \text{ being a con} \\ \{x_j \mapsto x_m, x_k \mapsto x_m\} & \text{if } a_{\mathrm{AIcl}} = x_j \text{ and } b_{\mathrm{AIcl}} = x_k\text{, both from lifting, and } x_m \text{ is} \\ & \text{the lifted term in the unified literal, i.e. } x_m = \ell[a] = \ell[b]. \\ & \text{More formally, } p = \mathrm{P}(a) = \mathrm{P}(b) \text{ and } \mathrm{P}(\mathrm{P}_{\mathrm{lit}}(\ell[l\sigma]).\, \mathrm{P}_{\mathrm{term}}(p)) = \\ & \mathrm{P}(\mathrm{P}_{\mathrm{lit}}(\ell[l'\sigma]).\, \mathrm{P}_{\mathrm{term}}(p)) = t_m \text{ and } \ell[t_m] = x_m. \\ \emptyset & \text{if } \ell[a\sigma] = \ell[b\sigma] = x_j \end{cases}$$

$\mathrm{AI}_{\mathrm{cl}}(C) = \ell[\Big((\mathrm{AI}_{\mathrm{cl}}(C_1) \setminus \{l_{\mathrm{AIcl}}\}) \vee (\mathrm{AI}_{\mathrm{cl}}(C_2) \setminus \{l'_{\mathrm{AIcl}}\})\Big)\sigma]\tau$

- if l and l' don't have the same color:

  $\mathrm{AI}_{\mathrm{mat}}(C) = (\neg\ell[l_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau) \vee (\ell[l_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma)]\tau$

- if l and l' are Γ-colored :

  $\mathrm{AI}_{\mathrm{mat}}(C) = \ell[\Big(\mathrm{AI}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}_{\mathrm{mat}}(C_2)\Big)\sigma]\tau$

- if l and l' are Δ-colored:

  $\mathrm{AI}_{\mathrm{mat}}(C) = \ell[\Big(\mathrm{AI}_{\mathrm{mat}}(C_1) \wedge \mathrm{AI}_{\mathrm{mat}}(C_2)\Big)\sigma]\tau$

1.3.2 AI

$\mathrm{AI}(C) = Q_1 u_1 \dots Q_m u_m \Big(\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)\Big)$
$u_1, \dots, u_m$ are comprised of all x_i and y_i PLUS all free variables in $\mathrm{AI}_{\mathrm{mat}}(C)$.
Q_i is $\exists$ if $u_i = y_i$ for some i, $\forall$ if $u_i = x_i$ for some i. For free variables, Q_i as in C_1/C_2. if $u_i\sigma \neq u_i$, u_i becomes one of x_j or y_j or grey term.
Combine dependencies of positions which are connected by weak arrows:
$\mathcal{A}' = \mathcal{A} \cup \{(a, c) \mid (a, b) \in \mathcal{A} \wedge \{b, c\} \in \mathcal{W}\} \cup \{(a, c) \mid (b, c) \in \mathcal{A} \wedge \{a, b\} \in \mathcal{W}\}$
$\mathcal{A}'' = \mathrm{TransitiveClosure}(\mathcal{A}')$
$(p_1, p_2) \in \mathcal{A}''$ implies that $u_i < u_j$ if $\mathrm{P}(p_1) = u_i$ and $\mathrm{P}(p_2) = u_j$ (positions in $\mathrm{AI}(C)$, not in C).

2 proof of propositional aspect of AI

The following lemma works in the other proof, but the remark below shows why it is not applicable here:

t_subst_commute⟩ **Lemma 2** (Restated from proof without propositional refutation, lemma 1). *Let C be a clause and σ a substitution. Let $\zeta_1, \dots, \zeta_n$ be all maximal Δ-terms in this context, i.e. those that occur in C or $C\sigma$, and $x_1, \dots, x_n$ the corresponding fresh variables to replace the ζ_i (i.e. none of the x_i occur in C). Define σ' such that for a variable z,*

$$z\sigma' = \begin{cases} x_l & \text{if } z = x_k \text{ and } \zeta_k\sigma = \zeta_l \\ \ell_{\Delta,x}[z\sigma] & \text{otherwise} \end{cases}$$

Then $\ell_{\Delta,x}[C\sigma] = \ell_{\Delta,x}[C]\sigma'$.

Remark (Restriction of Lemma 2). Lemma 2 does not hold in case x_i occurs in C. This can easily be seen using the following counterexample:
Let $\sigma = \{x \mapsto a\}$ and $\zeta_1 = f(x)$ and $\zeta_2 = f(a)$. Then clearly $\zeta_1\sigma = \zeta_2$ and therefore $x_1\sigma' = x_2$. But now consider $x_1\sigma$. As x_1 has its place in the domain of variables to replace colored terms, and σ is taken from a resolution refutation, they do not affect each other. Hence $x_1\sigma = x_1$ and therefore $\ell_{\Delta,x}[x_1\sigma] = x_1$, but $\ell_{\Delta,x}[x_1]\sigma' = \ell_{\Delta,x}[x_1]\sigma' = x_2$.
However such a situation arises naturally if we lift colored terms after every step of the interpolant extraction procedure, as there, the intermediate relative interpolants clearly contains variables to overbind terms, but we also need to treat terms that enter the interpolant by means of unification. △

t_logic_commute⟩ **Lemma 3** (corresponds to Lemma 4.8 in thesis and Lemma 11 in Huang). *Let A and B be first-order formulas and s and t be terms. Then it holds that:*

1. $\ell_{\Phi,x}[\neg A] \Leftrightarrow \neg\ell_{\Phi,x}[A]$
2. $\ell_{\Phi,x}[A \circ B] \Leftrightarrow (\ell_{\Phi,x}[A] \circ \ell_{\Phi,x}[B])$ *for* $\circ \in \{\wedge, \vee\}$
3. $\ell_{\Phi,x}[s = t] \Leftrightarrow (\ell_{\Phi,x}[s] = \ell_{\Phi,x}[t])$

o_colored_terms⟩ **Lemma 4.** *$\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ contain only grey terms and variables replacing colored terms. They do not contain colored terms.* // true and used

?⟨corr:lift_ai⟩? **Corollary 5.** *For a clause C in a resolution refutation π of $\Gamma \cup \Delta$:*

1. $\mathrm{AI}_{\mathrm{mat}}(C) = \ell[\mathrm{AI}_{\mathrm{mat}}(C)]$.
2. $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[\mathrm{AI}_{\mathrm{cl}}(C)]$.

// true and unused

g_vars_in_subst⟩ **Lemma 6.** *Lifting variables do not occur in any substitution of a resolution refutation. // true and used, also generally relevant*

titute_and_lift⟩ **Lemma 7.** *Let F be a formula without colored terms such that for a set of formulas Φ, $\Phi \models F$. Then $\Phi \models \ell[F\sigma]$ for a substitution σ.*

Proof. Note that substitutions only replace variables. Term positions, which are replaced by grey terms by σ are not affected by the lifting and hold due to being special cases of F.
Term positions, which are replaced by colored term by σ are again reduced to variables. All occurrences of a certain variable in F are substituted by the same term, so as the lifting replaces a certain term always be the same variables, all these occurrences of the variable are replaced by the same variable. □

_different_term⟩ **Example 8.** We illustrate that the given procedure, if a lifting variable x_k occurs in $\mathrm{AI}_{\mathrm{cl}}(C)$, it does not necessaily mean that ζ_k occurs in C:
$\Gamma = \{P(f(x)) \vee Q(x)\}$
$\Delta = \{\neg P(y), \neg Q(a)\}$

$$\dfrac{\dfrac{\bot \mid P(x_1) \vee Q(x) // P(f(x) \vee Q(x) \qquad \top \mid \neg P(y)}{P(x_1) \mid Q(x)} \qquad \top \mid \neg Q(y_1) // Q(a)}{Q(y_2) \vee P(x_1) \mid \Box}$$

Here, x_1 first refers to $f(x)$ and later to $f(a)$. This however is not essential for the correctness of the procedure, and it would be tedious to fix all such x_1 see also corresponding remark in case distinction in Lemma 10. △

Conjecture 9. au *is well defined: In a call of* $\mathrm{au}(a_{\mathrm{AIcl}}, b_{\mathrm{AIcl}})$, *if one of the arguments is a lifting variable of a certain color, then so is the other.*

_clause_similar⟩ **Lemma 10.** *If a literal l occurs in a clause C from a resolution refutation, then $\mathrm{AI}_{\mathrm{cl}}(C)$ contains a corresponding literal l_{AIcl} such that $l_{\mathrm{AIcl}} \sim \ell[l]$, where $\sim$ means equal up to the index of lifting variables. // true and used*

Proof. Base case: By Definition of $\mathrm{AI}_{\mathrm{cl}}$.
Let C be the result of a resolution step from $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ to $C = (D \vee E)\sigma$. Every literal of C is derived from a literal in C_1 or C_2. Let λ be a literal in C_1. The case for a literal in C_2 is analogous. Note that $\lambda \neq l$ as otherwise λ would not be contained in C.
By assumption $\lambda \in C_1$. Then by the resolution rule application, $\lambda\sigma \in C$.
By the induction hypothesis, there is a $\lambda_{\mathrm{AIcl}} \in \mathrm{AI}_{\mathrm{cl}}(C_1)$ such that $\lambda_{\mathrm{AIcl}} \sim \ell[\lambda]$. By the definition of $\mathrm{AI}_{\mathrm{cl}}$, $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau \in \mathrm{AI}_{\mathrm{cl}}(C)$ with $\tau = \mathrm{au}(l, l')$.
So we have to show that $\ell[\lambda\sigma] \sim \ell[\lambda_{\mathrm{AIcl}}\sigma]\tau$.

Remark on τ: τ only replaces lifting terms by other lifting by other lifting terms NB: this is where variable indices may not match.

We perform an induction on the depth of terms in λ (except the non-maximal colored terms). Note that as λ occurs in a clause of the refutation, it does not contain lifting variables.

- Suppose t is a term of size 1 in λ and it is a non-lifting variable, say u.

  As $\ell[u] \sim u_{\mathrm{AIcl}}$ and u is a variable, $u = u_{\mathrm{AIcl}}$. But then $u\sigma = u_{\mathrm{AIcl}}\sigma$ and also $\ell[u\sigma] = \ell[u_{\mathrm{AIcl}}\sigma]$, so clearly $\ell[u\sigma] \sim \ell[u_{\mathrm{AIcl}}\sigma]\tau$.

- Suppose t is a term of size 1 in λ and it is a constant.

  Suppose t is grey. Then it is unaffected by both liftings and substitutions, so we are done.

  Suppose t is colored. Then $\ell[t\sigma]$ is a lifting variable, but as $t_{\mathrm{AIcl}} = \ell[t]$, so is $\ell[t_{\mathrm{AIcl}}\sigma]\tau$.

  NB: From the point on where t was lifted, t_{AIcl} even always refers to exactly the lifting var $\ell[t] = x_k$ for some k (just the term in the refutation may change). Cf. Lemma **??**. Hence this case is no obstacle to showing the statement with $\ell[t] = t_{\mathrm{AIcl}}$ (and not just $\ell[t] \sim t_{\mathrm{AIcl}}$).

- Suppose t is of the form $f(t_1, \dots, t_n)$ in λ. Then by the induction hypothesis, $\ell[t_i\sigma] \sim \ell[(t_i)_{\mathrm{AIcl}}\sigma]\tau$ for $1 \leq i \leq n$.

  - Suppose f is grey. Then f is neither affected by substitutions nor by liftings.
  - Suppose f is colored. We only consider the case of occurrences of maximal colored terms as the other ones are discarded by the lifting. As $t_{\mathrm{AIcl}} \sim \ell[t]$, t_{AIcl} is a lifting variable. Hence also $\ell[t_{\mathrm{AIcl}}\sigma]\tau$ is a lifting variable. But so is $\ell[t\sigma]$. □

    NB: Note that even if it was the case that $\ell[t] = t_{\mathrm{AIcl}}$ (and not just $\ell[t] \sim t_{\mathrm{AIcl}}$), $\ell[t\sigma]$ might not be equal to $\ell[t_{\mathrm{AIcl}}\sigma]$, but only $\ell[t\sigma] \sim \ell[t_{\mathrm{AIcl}}\sigma]$.

    E.g. $t = f(x)$, $\ell[t] = x_1$, $t_{\mathrm{AIcl}} = x_1$, $\sigma = \{x \mapsto a\}$. Then $\ell[t\sigma] = \ell[f(a)] = x_2$, but $\ell[t_{\mathrm{AIcl}}\sigma] = x_1$. τ does not fix this, but could potentially if it is more careful than σ'. See also Example 8.

Example 11. TODO: example with terms in π vs AI, similar to 206a and last part of 208a:
$f(x)$ vs x_j
$f(g(y))$ vs x_j (actual term is changed but lifting variable stays the same
$f(g(h(z)))$ vs x_k (now x_j appears in resolution, either this occurrence or another occurrence of this var)
$f(g(h(a)))$ vs x_k (again actual term is changed without changing the lifting variable) △

raction_of_term⟩ **Lemma 12.** *Let $a_{\mathrm{AIcl}} = z_j$ a lifting variable for a term position a_{AIcl} in $\mathrm{AI}_{\mathrm{cl}}(C)$. Then $a = \zeta_j\rho$ for some substitution ρ. Even more, if a substitution $z_j \mapsto z_k$ for lifting variables z_j and z_k occurs, z_k refers to exactly ζ_k and there exists a substitution ρ' such that $\zeta_k = \zeta_j\rho'$.* // used

NB: this probably also hold in $\mathrm{AI}_{\mathrm{mat}}$ and for terms not occurring $\mathrm{AI}_{\mathrm{cl}}$ as well.

Proof. Base case: z_j is introduced to lift ζ_j, ρ is the identity function.
Induction step: Suppose z_j refers to $\zeta_j\rho$ for some ρ.
Suppose $\zeta_j\rho$ changes in the course of the resolution derivation. As it is a term in a resolution derivation, it changes only by means of unification, say by the unifier σ. Hence it changes to $\zeta_j\rho\sigma$ and z_j refers to $\zeta_j\rho\sigma$.
Suppose z_j changes. By the construction of $\mathrm{AI}_{\mathrm{cl}}/\mathrm{AI}_{\mathrm{mat}}$, lifting variables are not affected by the resolution unifications (cf. Lemma 6) or the liftings, but only by τ.
Suppose $(z_j \mapsto z_k) \in \tau$. Then by the definition of au, the term at position a is now ζ_k. As by the induction hypothesis $a = \zeta_j\rho$ in the preceding clause, $a\sigma = \zeta_k$. Hence $\zeta_j\rho\sigma = \zeta_k$. □

By Lemma 10, we have that $l_{\mathrm{AIcl}} \sim \ell[l]$. But we can also show that the terms in l only become more specialised, i.e. if a lifting variable z_j occurs in l_{AIcl}, the corresponding term in $\ell[l]$ is a specialisation of ζ_j,

ting_variables⟩? **Lemma 13.** *The set of lifting variables, which refer to terms which have free variables, is disjoint for every incomparable clause.* // *true but ok to have unused*

Proof. The free variables for every inital clause is disjoint. □

Apparently, τ establishes equality for the terms in the literals being resolved on (Lemma 14) and quasi-equality for other literals in the remaining clause (Lemma 10).

s_clauses_equal⟩ **Lemma 14.** *Let l_{AIcl}, l'_{AIcl} be the literal in $\mathrm{AI}_{\mathrm{cl}}(C_1)$ and $\mathrm{AI}_{\mathrm{cl}}(C_2)$ corresponding to l and l' where C is the result of a resolution step from $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ to $C = (D \vee E)\sigma$ (i.e. $l\sigma = l'\sigma$). Let $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$. Then $\ell[l_{\mathrm{AIcl}}\sigma]\tau = \ell[l'_{\mathrm{AIcl}}\sigma]\tau$.* // *true and used*

Proof. Let s_{AIcl} be a (sub-)term of a parameter of $l_{\mathrm{AIcl}}\sigma$ and t_{AIcl} the term at the same term position in $l'_{\mathrm{AIcl}}\sigma$. Let s and t be their corresponding (sub)-term at the same term position in $l\sigma$ and $l'\sigma$. We show that $\ell[s_{\mathrm{AIcl}}\sigma]\tau = \ell[t_{\mathrm{AIcl}}\sigma]\tau$ by induction on the structure of s_{AIcl} and t_{AIcl} respectively.
Note that by Lemma 4, s_{AIcl} and t_{AIcl} do not contain colored terms. This also implies that only grey terms can contain subterms.
By Lemma 10, $l_{\mathrm{AIcl}} \sim \ell[l]$ and $l'_{\mathrm{AIcl}} \sim \ell[l']$.

Lifting variables. Suppose that $s_{\mathrm{AIcl}} = z_i$ and/or $t_{\mathrm{AIcl}} = z_j$ for some i and j. Suppose that $s_{\mathrm{AIcl}} \neq t_{\mathrm{AIcl}}$ as otherwise we are done. By the resolution rule application $s\sigma = t\sigma$. Cases:

- $s_{\mathrm{AIcl}} = z_i$ and $t_{\mathrm{AIcl}} = z_j$ with $i \neq j$. As σ affects neither s_{AIcl} nor t_{AIcl}, $\ell[s_{\mathrm{AIcl}}\sigma] = s_{\mathrm{AIcl}}$ and $\ell[t_{\mathrm{AIcl}}\sigma] = t_{\mathrm{AIcl}}$. We show that $s_{\mathrm{AIcl}}\tau = t_{\mathrm{AIcl}}\tau$.
  Note that the function au visits all subterms and combines all mappings it encounters. Hence $\mathrm{au}(s_{\mathrm{AIcl}}, t_{\mathrm{AIcl}})$ is part of the final substitution τ. However due to the

just established circumstances, $\mathrm{au}(s_{\mathrm{AIcl}}, t_{\mathrm{AIcl}}) = \{z_i \mapsto z_m, z_j \mapsto z_m\}$ with m as in the definition of au, so $s_{\mathrm{AIcl}}\tau = t_{\mathrm{AIcl}}\tau$. NB: this is the somewhat crude step where all lifting variables in the resolved literal are just reset.

- W.l.o.g. $s_{\mathrm{AIcl}} = z_i$ and t_{AIcl} is not a lifting variable. As $t_{\mathrm{AIcl}} \sim \ell[t]$, t is not a colored term. But due to $s_{\mathrm{AIcl}} \sim \ell[s]$, s is a colored term. As $s\sigma = t\sigma$, t must be a variable and $t\sigma$ a colored term. So $\ell[t\sigma] = z_k$ for some k. Note that the function au visits all subterms and combines all mappings it encounters. By the construction of au, at $\mathrm{au}(s_{\mathrm{AIcl}}, t_{\mathrm{AIcl}})$, $\{z_i \mapsto z_k\}$ is added. Therefore $\ell[s_{\mathrm{AIcl}}\sigma]\tau = \ell[z_i\sigma]\tau = \ell[z_i]\tau = z_i\tau = z_k$.
  Due to $t_{\mathrm{AIcl}} \sim \ell[t]$ and as t is a variable, $t = t_{\mathrm{AIcl}}$. Then $\ell[t_{\mathrm{AIcl}}\sigma]\tau = \ell[t\sigma]\tau = z_k\tau$. It remains to show that $z_k\tau = z_k$.
  As t is a variable and due to $t\sigma = \zeta_k$ and as σ is the most general unifier, it is necessary to substitute ζ_k in order to unify the literals.
  We continue with a proof by constradiction and suppose that $(z_k \mapsto z_l) \in \tau$. Let $k \neq l$ as otherwise we are done. By the definition of au, z_k must occur in either $\mathrm{AI}_{\mathrm{cl}}(C_1)$ or $\mathrm{AI}_{\mathrm{cl}}(C_2)$. Furthermore, at least one of the ancestors of C_1 or C_2, or C_1 or C_2 themselves, contains ζ_k, as only this term is lifted using z_k.
  By Lemma 12, z_l refers to precisely ζ_l. As z_l replaces z_k, and z_k used to refer to ζ_k, some sequence of substitutions occurred which changed ζ_k to ζ_l. This sequence of substitutions has substituted at least one variable of ζ_k as $\zeta_k \neq \zeta_l$. As the set of clauses is unique for a clause, this variable does not occur in the subsequent derivation.
  As however σ introduces ζ_k and therefore all of its variables as subterms, ζ_k has never been updated to ζ_l, but then $(z_k \mapsto z_l) \notin \tau$.

Grey terms. Suppose that at least one of s_{AIcl} and t_{AIcl} is a grey term.

- Suppose that both s_{AIcl} and t_{AIcl} are grey terms: By $s_{\mathrm{AIcl}} = \ell[s]$ and $t_{\mathrm{AIcl}} = \ell[t]$, and as $s\sigma = t\sigma$, their outermost symbol is the same in all these terms. The equality of the parameters is established by the induction hypothesis. Note that grey constants can be treated as grey functions without parameters.

- Suppose that exactly one of s_{AIcl} and t_{AIcl} is a grey terms. W.l.o.g. let s_{AIcl} be a grey term. Then as $s\sigma = t\sigma$, $s_{\mathrm{AIcl}} = \ell[s]$ and $t_{\mathrm{AIcl}} = \ell[t]$, t_{AIcl} is a variable and $t = t_{\mathrm{AIcl}}$. Furthermore, $t_{\mathrm{AIcl}}\sigma$ is a grey term. Due to $s\sigma = t\sigma$, the outermost symbol of s_{AIcl} and t_{AIcl} is the same. Equality of potential parameters in s_{AIcl} is established by the induction hypothesis.

Variables. Suppose that both s_{AIcl} and t_{AIcl} are variables. Suppose that σ is non-trivial on at least s_{AIcl} or t_{AIcl}, as otherwise we are done. Due to $s_{\mathrm{AIcl}} = \ell[s]$ and $t_{\mathrm{AIcl}} = \ell[t]$, $s = s_{\mathrm{AIcl}}$ and $t = t_{\mathrm{AIcl}}$. As $s\sigma = t\sigma$, the outermost symbol of both $s_{\mathrm{AIcl}}\sigma$ is the same as the one of $t_{\mathrm{AIcl}}\sigma$. As the equality of potential parameters of $s_{\mathrm{AIcl}}\sigma$ and $t_{\mathrm{AIcl}}\sigma$ is established by the induction hypothesis, we are done. $\square$

ta_terms_lifted⟩ **Lemma 15.** *Let every* Γ*-term be grey. (To establish valid conditions, for each* Γ*-term* t*, add* $P(t)$ *to* Δ *where* P *is a fresh predicate symbol. Then the resolution refutation is unaffected). Then* $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$.

Proof. Proof by induction of the strenghtening: $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C_\Gamma)$.
Base case:
For $C \in \Gamma$, $\mathrm{AI}_{\mathrm{mat}}(C) = \bot$ and $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[C_\Gamma] = \ell_{\Gamma,y}[C]$. As Γ-terms are not lifted, $\ell_{\Gamma,y}[C] = C$ and $\Gamma \models C$.
For $C \in \Delta$, $\mathrm{AI}_{\mathrm{mat}}(C) = \top$.
Induction step:

Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad \sigma = \mathrm{mgu}(l, l')$$

We introduce the following abbreviations:

$\mathrm{AI}_{\mathrm{cl}}(C_1)^* = \mathrm{AI}_{\mathrm{cl}}((C_1)_\Gamma) \setminus \{\ell[(l_{\mathrm{AIcl}})_\Gamma]\}$

$\mathrm{AI}_{\mathrm{cl}}(C_2)^* = \mathrm{AI}_{\mathrm{cl}}((C_2)_\Gamma) \setminus \{\ell[\neg(l'_{\mathrm{AIcl}})_\Gamma]\}$

$\tau = \mathrm{au}((l_{\mathrm{AIcl}})_\Gamma, (l'_{\mathrm{AIcl}})_\Gamma)$

$\mathrm{AI}_{\mathrm{cl}}(C_\Gamma) = \ell[\Big(\mathrm{AI}_{\mathrm{cl}}(C_1)^* \vee \mathrm{AI}_{\mathrm{cl}}(C_2)^* \Big)\sigma]\tau$.

By Lemma 3, $\mathrm{AI}_{\mathrm{cl}}(C_\Gamma) = \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)^*\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)^*\sigma]\tau$.

By the induction hypothesis, $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C_i) \vee \mathrm{AI}_{\mathrm{cl}}(C_{i\Gamma})$, $i \in \{1, 2\}$, or expressed differently:

$\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}_{\mathrm{cl}}(C_1)^* \vee (l_{\mathrm{AIcl}})_\Gamma$

$\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C_2) \vee \mathrm{AI}_{\mathrm{cl}}(C_2)^* \vee \neg(l'_{\mathrm{AIcl}})_\Gamma$

By Lemma 4, $\mathrm{AI}_{\mathrm{mat}}(C_1)$ and $\mathrm{AI}_{\mathrm{cl}}(C_1)$ as well as $\mathrm{AI}_{\mathrm{mat}}(C_2)$ and $\mathrm{AI}_{\mathrm{cl}}(C_2)$ do not contain colored terms. Hence by Lemma 7, Lemma 3 and applying τ, we get that

$\Gamma \overset{(\circ)}{\models} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)^*\sigma]\tau \vee \ell[(l_{\mathrm{AIcl}})_\Gamma\sigma]\tau$

$\Gamma \overset{(*)}{\models} \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)^*\sigma]\tau \vee \neg\ell[(l'_{\mathrm{AIcl}})_\Gamma\sigma]\tau$

By Lemma 14, $\ell[(l_{\mathrm{AIcl}})_\Gamma\sigma]\tau = \ell[(l'_{\mathrm{AIcl}})_\Gamma\sigma]\tau$.

- If l and l' grey:

  $\mathrm{AI}_{\mathrm{mat}}(C) = (\neg\ell[l_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau) \vee (\ell[l_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma)]\tau$

Suppose for a model M of Γ that $M \not\models \mathrm{AI}_{\mathrm{cl}}(C)$, i.e. $M \not\models \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)\sigma]\tau$ and $M \not\models \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)\sigma]\tau$ as otherwise we would be done. Then by ($\circ$) and ($*$):

$M \models \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[l_{\mathrm{AIcl}}\sigma]\tau$

$M \models \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \neg\ell[l'_{\mathrm{AIcl}}\sigma]\tau$

By Lemma 14, $\ell[l_{\mathrm{AIcl}}\sigma]\tau = \ell[l'_{\mathrm{AIcl}}\sigma]\tau$. By a case distinction on the truth value of $\ell[l_{\mathrm{AIcl}}\sigma]\tau$ in M, we obtain that $M \models \mathrm{AI}_{\mathrm{mat}}(C)$.

- If l and l' are Γ-colored: $\mathrm{AI}_{\mathrm{mat}}(C) = \ell[\Big(\mathrm{AI}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}_{\mathrm{mat}}(C_2)\Big)\sigma]\tau$

  By Lemma 14, we can do a resolution step on $\ell[l_{\mathrm{AIcl}}\sigma]\tau$ of ($\circ$) and ($*$) to arrive at

  $\Gamma \models \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \;\vee\; \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)^*\sigma]\tau \;\vee\; \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau \;\vee\; \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)^*\sigma]\tau$

  This however is by Lemma 3 nothing else than $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$

- If l and l' are Δ-colored: $\mathrm{AI}_{\mathrm{mat}}(C) = \ell[\Big(\mathrm{AI}_{\mathrm{mat}}(C_1) \wedge \mathrm{AI}_{\mathrm{mat}}(C_2)\Big)\sigma]\tau$

  As l is Δ-colored, ($\circ$) and ($*$) reduce to:

  $\Gamma \models \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)^*\sigma]\tau$

  $\Gamma \models \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)^*\sigma]\tau$

  But this implies that

  $\Gamma \models \Big(\ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau\Big) \;\vee\; \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)^*\sigma]\tau \;\vee\; \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)^*\sigma]\tau$

  This however is by Lemma 3 nothing else than $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$. □

Lemma 16. *The previous lemma basically calculates* AI^{Δ}*, where in* $\mathrm{AI}_{\mathrm{cl}}$ *and* $\mathrm{AI}_{\mathrm{mat}}$ *instead of* $\ell[\cdot]$ $\ell_{\Delta,x}[\cdot]$ *is employed. TODO: NEEDS A PROOF*

Conjecture 17. $\Gamma \models \mathrm{AI}(C)$. *(Recall that* $\mathrm{AI}(C) = Q_1u_1 \dots Q_mu_m\Big(\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)\Big)$*.)*

Proof. By Lemma 15, by considering Γ-terms to be grey, there is a "witness formula" $\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$ which contains all Γ-terms and all Δ-terms are lifted and implicitly universally quantified.
If we now again consider Γ-terms to be Γ-terms, then in the "witness formula", the Γ-terms are lifted and existentially quantified. As a Γ-term ζ_i in general contains lifting variables which lift Δ-terms, we have to ensure that these are quantified before y_i is. Note that this implies that in C, a Δ-term is contained in a Γ-term.
As the quantifier prefix is ordered according to the arrows, Lemma 23 gives the result. TODO: verify this after the proof of the lemma □

Definition 18 (Unification algorithm)**.** Let id denote the identity function and **fail** be returned by mgu in case the arguments are not unifiable. Let s and t denote terms and x a variable.
The most general unifier mgu of two literals $l = A(s_1, \dots, s_n)$ and $l' = A(t_1, \dots, t_n)$ is defined to be $\mathrm{mgu}(\{(s_1, t_1), \dots, (s_n, t_n)\})$.

The mgu for a set of pairs of terms T is defined as follows:
$\mathrm{mgu}(\emptyset) = \mathrm{id}$

$$\mathrm{mgu}(\{t\} \cup T) = \begin{cases} \textbf{fail} & \text{if } t = (x, s) \text{ or } t = (s, x) \text{ and } x \text{ occurs} \\ & \text{in } s \text{ but } x \neq s \\ \mathrm{mgu}(T[x/s])[x/s] \cup \{x \mapsto s\} & \text{if } t = (x, s) \text{ or } t = (s, x) \text{ and } x \text{ does} \\ & \text{not occur in } s \text{ or } x = x \\ \textbf{fail} & \text{if } t = (f(s_1, \ldots, s_n), g(s_1, \ldots, s_n)) \\ & \text{with } f \neq g \\ \mathrm{mgu}(T \cup \{(s_1, t_1), \ldots, (t_n, s_n)\}) & \text{if } t = (f(s_1, \ldots, s_n), f(t_1, \ldots, t_n)) \\ \mathrm{mgu}(T) & \text{if } t = (s, s) \end{cases} \quad \triangle$$

Definition 19. A term is called *multicolored* if it contains both Γ- and Δ-colored subterms. △

Note that a multicolored Φ-term consequently is a term whose outermost symbol is Φ-colored and contains a colored but not Φ-colored subterm.

n_introduction⟩?

Lemma 20. *A resolution derivation can only contain multicolored* Γ*-term if there is a variable which has a* Γ*-colored occurrence and a grey occurrence in a clause.*

Proof. Suppose no such variable exists in Γ and Δ. By the definition of the colors, no Γ-colored term initially contains a Δ-colored term. We show that no resolution rule application can introduce one.
As terms in clauses are only changed by means of unification, we have to show that no most general unifier σ exists for any clauses derived from Γ and Δ which introduces a Δ-term in an existing Γ-term or a new Γ-term, which contains a Δ-term.

⟨a5hsefdgsy6⟩

1. Suppose a unifier σ of two literals l and l' of C_1 and C_2 respectively introduces a Δ-term t in a Γ-term s in C_1. Then a variable x occurs in s such that $x\sigma = t$. By the unification algorithm, this implies that x is directly unified with a term t' which has the same outermost symbol as t. Note that clauses involved in a resolution step are variable disjoint, so x only occurs in C_1 and its preceding clauses, but not in C_2. Since x has a Γ-colored occurrence, it does by assumption not have a grey occurrence. Hence all occurrences of x are of the form $r[x]$, where the outermost symbol of r is colored. As x occurs in a Γ-term, it must originate from a clause in Γ. As by Lemma 22 variables are never added in a resolution derivation, all $r[x]$ must be Γ-colored terms.

   Γ- and Δ-colored terms are not unifiable, so a unifier mapping x to a Δ-colored term must be created from a unification of a term of the form $r[x]$ and a Γ-term. But by assumption, no Γ-colored term contains a Δ-colored term.

   Hence there cannot be an mgu of l and l' which maps x to a Δ-colored term.

2. Suppose a unifier σ of two literals l and l' of C_1 and C_2 respectively introduces a Γ-term s, which contains a Δ-term t. As by assumption no such term s exists in C_1 or C_2, the unification algorithm does not encounter it as term to unify a variable with directly.

   This however does not exclude the case that a variable y is unified first with a Γ-term containing a variable x, where later x is unified with a Δ-term. However the argumentation in case 1 excludes precisely the case that a variable, which is contained in a Γ-term, is unified with a Δ-term. □

Conjecture 21. *Let C be a clause which contains a multicolored Γ-term.*

ble_never_added⟩ **Conjecture 22.** *Variables are never added in a resolution derivation. Hence if they are present in a clause C at some point in the derivation, their position in the original clause has been a variable or has been contained in a variable in case a varaiable has been added inside of a term.*

r_lifting_terms⟩ **Conjecture 23.** *In $\mathrm{AI}^{\Delta}(C)$, if a lifting variable x_i (lifting a Δ-term) occurs inside of a maximal Γ-term t, then there is an arrow from an occurrence of x_i to t in $\mathrm{AI}^{\Delta}(C)$ in $\mathcal{A}''$.*

Proof. For a term to occur in $\mathrm{AI}(C)$ means to occur in $\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$.
Base case: No foreign terms occur in the inital clauses.
Induction step: Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$. By the induction hypothesis, all t which occur in $\mathrm{AI}(C_1)$ and $\mathrm{AI}(C_2)$ satisfy the condition.
Note that every x_i from $\mathrm{AI}(C_1)$ or $\mathrm{AI}(C_2)$ is carried over to $\mathrm{AI}(C)$. τ might rename x_i to another lifting variable, but as this applies to every occurrence of x_i, the statement of this lemma is not violated.
Hence we consider all introductions of new t^* in $\mathrm{AI}(C)$, i.e. those that have not been present in $\mathrm{AI}(C_1)$ or $\mathrm{AI}(C_2)$. Let t be the term in w.l.o.g. $\mathrm{AI}(C_1)$ where t^* originates from, i.e. $t\sigma = t^*$
$\mathrm{AI}(C) = Q_1 u_1 \dots Q_m u_m \Big(\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C) \Big)$
All terms of $\mathrm{AI}_{\mathrm{cl}}(C)$ are contained in $\ell[(\mathrm{AI}_{\mathrm{cl}}(C_1) \setminus \{l_{\mathrm{AIcl}}\})\sigma]\tau$ or $\ell[(\mathrm{AI}_{\mathrm{cl}}(C_2) \setminus \{l'_{\mathrm{AIcl}}\})\sigma]\tau$ and all terms of $\mathrm{AI}_{\mathrm{mat}}(C)$ are contained in $\ell[l_{\mathrm{AIcl}}\sigma]\tau$, $\ell[l'_{\mathrm{AIcl}}\sigma]\tau$, $\ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau$ or $\ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma)]\tau$.
As by Lemma 6 σ does not introduce lifting variables, σ does not introduce t. There are 2 cases, we show that in both of them, there is an arrow from an occurrence of x_i to t in $\mathcal{A}''$

- σ introduces the Δ-colored term ζ_i in t^*. Hence t contains a variable u such that $u\sigma = \zeta_i$.
  - Suppose t does not occur in the literal being unified. Then u is only set due to occurring in the literal being unified, say at position p.

    The occurrence at p is either grey or colored. As variables are never added (cf. 22) and arrows never removed, the original arrows still apply. u in t is a colored occurrence.

Suppose p is a grey occurrence. Then there is an arrow from p to u in t.

Otherwise suppose p is a colored occurrence. Then there is a weak connection between p and u in t. But as then the literal p is unified to does not introduce the Δ-term into a Γ-term, there has to be some kind of induction hypothesis which gives the result here.

– Suppose t does occur in the literal being unified, say in l. Let t' be the term in l' that t is unified with.

($\circ$) Suppose t' is a variable TODO: blablub

($*$) Suppose t' is not variable.

We know that t' is the lifting of an abstraction of ζ_i. We assume it is not a variable. Hence it is a maximal Γ-term. If it contains a lifting var just like t^*, we get the result by the induction hypothesis as there must be an arrow towards t' and the arrows of the literals are being merged.

Otherwise t' does not contain a lifting variable. As t does not either, but $t\sigma = t'\sigma = t^*$ does, variables in t or t' or both are substituted for terms containing Δ-terms. These variables occur elsewhere in the respective clauses.

They occur either in colored terms as well. Then this is the same situation as this case (if not one of the fast bailouts above applies, in which case we are done). However there can only be finitely many such cases as formulas are finite and there cannot be a circle as the vars have proper substitutions, which must have originated somewhere.

Otherwise these variables occur as outermost symbol. Then case ($\circ$) applies.

- σ introduces the mixed-colored term t'. TODO:

□

Example 24. TODO: example showing that if lifting vars occur in both $\mathrm{AI}_{\mathrm{cl}}(C_1)$ and $\mathrm{AI}_{\mathrm{cl}}(C_2)$, combining them with the same quantifier is fine as they both have the same dependencies. $\forall x F(x)$ and $\exists y G(y)$ combines nicely to $\forall x \exists y (F(x) \vee G(y))$ anyway as the bound variables are different. △

Conjecture 25. *Let ζ_i be contained in some literal in Γ and a term t derived from ζ_i occurs in $\mathrm{AI}(C)$ for some C as in Lemma 15. Suppose t_{AI} contains some Δ-colored ζ_j. $t = \zeta_i \rho$ for some ρ. Then as Δ-terms are lifted in t, it contains z_j for some j.*
There is an arrow from some occurrence of z_j in $\mathrm{AI}(C)$ to the position of ζ_i (where there's actually some kind of lifting variable)
NB: this is what the proof above needs

Conjecture 26. *Let ζ_i be a term of some color and ζ_j be a term of a different color, which contains ζ_i as subterm. If the corresponding lifting variables z_i and z_j occur in $\mathrm{AI}(C)$ for some C, then there is an arrow from an occurrence of z_i to an occurrence of z_j (or for different i, j, as the z's can become more specialised).*
NB: this seems to be provable, but check what it actually implies/expresses

3 arrow proof

Conjecture 27. *Let x be a variable in $\mathrm{AI}_{\mathrm{cl}}(C)$. Suppose there is a colored and a grey occurrence of x. Then for every colored occurrence p of x there is an arrow from some grey occurrence to p.*

Conjecture 28. *If there is a term in C which contains a subterm of a different color, then there is variable in an ancestor of C which has a grey and a colored occurrence.*

If a variable occurs twice in colored terms, foreign terms can be propagated. If it's once as grey and once as colored occurrence, foreign terms can be introduced.

_same_variables⟩ **Lemma 29.** *Whenever the same variable appears multiple times in $\mathrm{PI}(C) \vee C$ for $C \in \pi$, there are arrows.*

- *If both variables are contained only in grey terms, there is a double arrow // they unify to exactly the same*
- *If only one variable is only contained in grey terms, there is an arrow from it to the other one // either unify the one in grey term, then other one must be overbound later. if otherwise var in the colored term is unified, we can still overbind the grey one first.*
- *otherwise there are weak arrows between them // have same quantifier, so order does not matter, but want to keep dependencies on both the same*

Proof. By induction. Note: As required by resolution, all initial clauses are variable disjoint. Base case: In the initial clause sets, consider for a clause C two different positions p_1 and p_2 pointing to the same variable. Then either:

- p_1 and p_2 contain only grey symbols. Then $(p_1, p_2) \in \mathcal{A}$.
- Only p_i, $i \in \{1, 2\}$ contains only grey symbols. Then $(p_i, p_{(i \bmod 2)+1}) \in \mathcal{A}$.
- There are not only grey symbols in both p_1 and p_2, i.e. both contain at least a colored symbol. Then $\{p_1, p_2\} \in \mathcal{W}$.

Induction step: Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$. $\mathrm{PI}(C)$ is $[\mathrm{PI}(C_1) \circ \mathrm{PI}(C)]\sigma$ or $[(l \wedge \mathrm{PI}(C_2)) \vee (\neg l \wedge \mathrm{PI}(C_1))]\sigma$.
Assumption: C_1 and C_2 are variable disjoint, i.e. variables are renamed in case C_1 and C_2 are derived from some common original clause and share variables.
By the induction hypothesis, there are appropriate arrows in both $\mathrm{PI}(C_i) \vee C_i$, $i \in \{1, 2\}$.
If the variables were present in C_1 or C_2, the arrow is still there, either in $\mathrm{PI}(C)$ (in the case of l or l'), C (in case of D and E) or in currently not shown literal (in case l and l' have the same color).
Otherwise, it was introduced by unification in $l\sigma$ or $\mathrm{PI}(C_i)\sigma$. In this case, there is some term position q in with $\mathrm{P}(l).q$ a variable and $\mathrm{P}(l').q$ a variable or a term containing variables (or other way around). Hence unification maps a variable to a variable or a term containing variables. The variable being unified is in $\mathrm{PI}(C_i) \vee C_i$ for some $i \in \{1, 2\}$. But by the induction hypothesis, all occurrences of each variable does already have appropriate arrows, which are still present. □

Lemma 30. *In $\mathrm{PI}(C) \vee C$ for $C \in \pi$, if there is a Δ-colored term s in a Γ-term t, then there is an arrow from p_1 to p_2 such that $\mathrm{P}(p_1) = s$ and $\mathrm{P}(p_2) = s$ and for some i, $\mathrm{P}(p_2 \mod i) = t$.*

Note: p_1 might be in some clause, the prop interpolant or none of both.

Proof. By induction.
Base case: There are no foreign terms in the initial clause sets, so no arrows necessary.
Induction step:

Resolution. Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$.

1. Suppose l is colored. This case is similar to the grey one, with the exception that the cases applying to l in PI do not apply.
2. Suppose l is grey. Then $\mathrm{PI}(C) = [(l \wedge \mathrm{PI}(C_2)) \vee (\neg l \wedge \mathrm{PI}(C_1))]\sigma$

   By the induction hypothesis, there are appropriate arrows in $\mathrm{PI}(C_1) \vee C_1$ and $\mathrm{PI}(C_2) \vee C_2$.

   We show that for all maximal Γ-terms in $\mathrm{PI}(C) \vee C$ with Δ-terms in them which were not present in $\mathrm{PI}(C_i) \vee C_i$, $i \in \{1, 2\}$, there is an arrow.

   Γ-terms and Δ-terms are not unifiable. Hence all pairs of terms (ζ_1, ζ_2) in the same positions in l and l' (if both positions exist) either point to the same symbol or (w.l.o.g.) ζ_1 is a variable and ζ_2 is a term. TODO: or the outermost symbol is the same and contains variables. If there are Δ-terms in Γ-terms in the prefix, they are present in both ancestors and handled by the induction hypothesis.

The only way a Δ-colored term may enter a Γ-colored term is in the situation where ζ_1 is a variable and ζ_2 a colored term. But then $\mathrm{mgu}(\zeta_1, \zeta_2)$ applied to ζ_1 yields ζ_2, i.e. "the parts of σ concerned with unifying ζ_1 and ζ_2" do not introduce new Δ-terms in Γ-terms. In other words, all such situation have been present in $\mathrm{PI}(C_i) \vee C_i$ for $i \in \{1, 2\}$ and since the arrows for l and l' are merged, they are present for $l\sigma$ in $\mathrm{PI}(C)$.

This handles the case where terms ζ_1 and ζ_2 are unified. But unification also affects all other occurrences of variables, this means "the parts of σ not concerned with unifying ζ_1 and ζ_2". The relevant case for this lemma is when a Γ-term contains a variable, that is substituted by a term containing Δ-terms. But in this case, by Lemma 29, there is an arrow from the other occurrence of the variable to the one in the Γ-term: either double arrow in $\mathcal{A}$ if both prefixes are grey, one in $\mathcal{A}$ if one of the prefixes is grey or one in $\mathcal{W}$ if both prefixes contain a colored symbol. $\square$

4 thoughts

Conjecture 31. *Double ended arrows are not important as terms are overbound with same variable anyway as always same unifier applies.*