

1 Attempt without P_P

Intuition of σ' :

If we pull a substitution out of a lifting which replaced Δ -terms, we also have to replace the Δ -terms in the “domain” of the substitution. This is the lower case.

There is just a problem in the following case: $\ell_{\Delta,x}[f(x)\sigma]$, where $x\sigma = a$ and f is a Δ -symbol. Then $\ell_{\Delta,x}[f(x)\sigma] = \ell_{\Delta,x}[f(a)] = x_i$, but $\ell_{\Delta,x}[f(x)]\sigma = x_j$ with $i \neq j$. The first case of the definition of x_j then fixes this by replacing x_j with x_i .

Lemma 1. *Let C be a clause and t_1, \dots, t_n the set of maximal Δ -terms in C and σ be a substitution. Let*

$$r\sigma' = \begin{cases} x_l & \text{if } r = x_k \text{ for some } k, l \neq k, \text{ and } t_k\sigma = t_l \\ \ell_{\Delta,x}[r\sigma] & \text{otherwise} \end{cases}$$

Then $\ell_{\Delta,x}[C\sigma] = \ell_{\Delta,x}[C]\sigma'$.

Proof. We prove this for an atom $P(s_1, \dots, s_m)$ in C , which works since lifting and substitution commute over binary connectives and into an atom.

We show that $\ell_{\Delta,x}[s_j\sigma] = \ell_{\Delta,x}[s_j]\sigma'$ for $1 \leq j \leq m$.

Note that anything in the term structure above a maximal Δ -term is unaffected by both substitution and abstraction.

Let t_i be a maximal Δ -term in $s_i\sigma$.

We show that $\ell_{\Delta,x}[t_i\sigma] = \ell_{\Delta,x}[t_i]\sigma'$, which proves the lemma.

Let $t_i\sigma = t_j$. Then $\ell_{\Delta,x}[t_i\sigma] = \ell_{\Delta,x}[t_j] = x_j$.

We show that $x_j = \ell_{\Delta,x}[t_i]\sigma'$.

Suppose that $t_i = t_j$, i.e. σ is trivial on t_i . Then $i = j$ as the Δ -terms have a unique number. Hence $\ell_{\Delta,x}[t_i]\sigma' = x_i\sigma' = x_i = x_j$.

Otherwise $t_i \neq t_j$. Then $i \neq j$ and $x_j \neq x_i$.

$\ell_{\Delta,x}[t_i]\sigma' = x_i\sigma'$. By the definition of σ' , as $t_i\sigma = t_j$, $x_i\sigma' = x_j$. □

Lemma 2. If $l\sigma = l'\sigma$, then $\ell_{\Delta,x}[l]\sigma' = \ell_{\Delta,x}[l']\sigma'$ for σ' defined as in lemma 1

Proof. $l\sigma = l'\sigma$

$\rightarrow \ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l'\sigma]$

by lemma 1, $\ell_{\Delta,x}[l]\sigma' = \ell_{\Delta,x}[l']\sigma'$

□

Lemma 3. // currently unused

$(\ell_{\Delta,x}[C](x_1, \dots, x_n))\sigma = (\ell_{\Delta,x}[C\sigma'](x_1, \dots, x_n))$ if σ does not change any of x_1, \dots, x_n or any of t_1, \dots, t_n .

it would work to fix substitutions of x_i by substituting t_i for that instead, as long as the result isn't another t_i , but this isn't actually relevant here.

Proposition 4. $\Gamma = \ell_{\Delta,x}[\Gamma](x_1, \dots, x_n)$.

Proof. By definition, Δ -terms only appear in Δ and not in Γ .

□

Lemma 5. $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C) \vee C)](x_1, \dots, x_n)$.

Proof. By induction on the resolution refutation.

Base case: Either $C \in \Gamma$, then it does not contain Δ -terms. Otherwise $C \in \Delta$ and $\text{PI}(C) = \top$.

Induction step:

Resolution.

$$\frac{C_1 : D \vee l \quad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad l\sigma = l'\sigma$$

By the induction hypothesis, we can assume that:

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee (D \vee l)](x_1, \dots, x_n)$$

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_2) \vee (E \vee \neg l')](x_1, \dots, x_n)$$

1. $\text{PS}(l) \in \text{L}(\Gamma) \setminus \text{L}(\Delta)$: Then $\text{PI}(C) = [\text{PI}(C_1) \vee \text{PI}(C_2)]\sigma$.

We show that $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2))\sigma \vee (D \vee E)\sigma]$,

i.e. $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2) \vee D \vee E)\sigma]$.

Let σ' be as in Lemma 1 with all Δ -terms occurring here (in formulas and substitutions) as t_1, \dots, t_n .

Hence by Lemma 1 $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2) \vee D \vee E)]\sigma'$.

By Lemma 11 (Huang) and the induction hypothesis,

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)] \vee \ell_{\Delta,x}[D] \vee \ell_{\Delta,x}[l] \quad (*)$$

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_2)] \vee \ell_{\Delta,x}[E] \vee \neg \ell_{\Delta,x}[l'] \quad (\circ)$$

Since $l\sigma = l'\sigma$ (by resolution rule application), $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l'\sigma]$.

As by Lemma 1 $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l]\sigma'$ and $\ell_{\Delta,x}[l'\sigma] = \ell_{\Delta,x}[l']\sigma'$, we get $\ell_{\Delta,x}[l']\sigma' = \ell_{\Delta,x}[l]\sigma'$.

So by applying σ' to $(*)$ and (\circ) , we can resolve on $\ell_{\Delta,x}[l']$ and get

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D]\sigma' \vee \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[E]\sigma'.$$

and

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee \text{PI}(C_2) \vee D \vee E]\sigma'.$$

So by Lemma 1,

$$\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \vee \text{PI}(C_2) \vee D \vee E)\sigma].$$

2. $\text{PS}(l) \in \text{L}(\Delta) \setminus \text{L}(\Gamma)$: Then $\text{PI}(C) = [\text{PI}(C_1) \wedge \text{PI}(C_2)]\sigma$.

We show that $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \wedge \text{PI}(C_2)) \vee D \vee E]\sigma(x_1, \dots, x_n)$. By lemma 1 with σ' as in the lemma, $\Gamma \models \ell_{\Delta,x}[(\text{PI}(C_1) \wedge \text{PI}(C_2)) \vee D \vee E](x_1, \dots, x_n)\sigma'$.

TODO

Paramodulation.

$$\frac{C_1 : D \vee s = t \quad C_2 : E[r]}{C : (D \vee E[t])\sigma} \quad \sigma = \text{mgu}(s, r)$$

By the induction hypothesis, we have:

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_1) \vee (D \vee s = t)]$$

$$\Gamma \models \ell_{\Delta,x}[\text{PI}(C_2) \vee (E[r])]$$

easy case: $\text{PI}(C) = [(s = t \wedge \text{PI}(C_2)) \vee (s \neq t \wedge \text{PI}(C_1))]\sigma$

to show: $\Gamma \models \ell_{\Delta,x}[\ell_{\Delta,x}[(s = t \wedge \text{PI}(C_2)) \vee (s \neq t \wedge \text{PI}(C_1))] \vee (D \vee E[t])]\sigma$

proof idea: either $s = t$, then also $\text{PI}(C_2)$, or else $s \neq t$, but then also $\text{PI}(C_1)$

by lemma 1 for σ' as in lemma, $\Gamma \models \ell_{\Delta,x}[(s = t \wedge \text{PI}(C_2)) \vee (s \neq t \wedge \text{PI}(C_1))] \vee (D \vee E[t])\sigma'$

by lemma 11 (huang) $\Gamma \models [(\ell_{\Delta,x}[s] = \ell_{\Delta,x}[t] \wedge \ell_{\Delta,x}[\text{PI}(C_2)]) \vee (\ell_{\Delta,x}[s \neq t] \wedge \ell_{\Delta,x}[\text{PI}(C_1)])] \vee (\ell_{\Delta,x}[D] \vee \ell_{\Delta,x}[E[t]])\sigma'$

reformulate: $\Gamma \models ((\ell_{\Delta,x}[s]\sigma' = \ell_{\Delta,x}[t]\sigma' \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma') \vee (\ell_{\Delta,x}[s]\sigma' \neq \ell_{\Delta,x}[t]\sigma' \wedge \ell_{\Delta,x}[\text{PI}(C_1)]\sigma')) \vee (\ell_{\Delta,x}[D]\sigma' \vee \ell_{\Delta,x}[E[t]]\sigma')$

By the rule: $s\sigma = r\sigma$, hence also $\ell_{\Delta,x}[s\sigma] = \ell_{\Delta,x}[r\sigma]$ and $\ell_{\Delta,x}[s]\sigma' = \ell_{\Delta,x}[r]\sigma'$ REALLY TRUE? – think so...

Suppose $M \models \Gamma$ and $M \not\models (\ell_{\Delta,x}[D]\sigma' \vee \ell_{\Delta,x}[E[t]]\sigma')$.

Suppose $M \models \ell_{\Delta,x}[s]\sigma' = \ell_{\Delta,x}[t]\sigma'$.

By induction hypothesis (and lemma 11 (huang) and adding the substitution σ'), $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[(E[r])]\sigma'$.

However by assumption $\Gamma \not\models \ell_{\Delta,x}[E[t]]\sigma'$.

Hence $\Gamma \not\models \ell_{\Delta,x}[E[s]]\sigma'$, and $\Gamma \not\models \ell_{\Delta,x}[E[r]]\sigma'$. Therefore $\Gamma \models \ell_{\Delta,x}[\text{PI}(C_2)]\sigma'$.

Suppose on the other hand $M \models \ell_{\Delta,x}[s]\sigma' \neq \ell_{\Delta,x}[t]\sigma'$.

By the induction hypothesis, $M \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma' \vee (\ell_{\Delta,x}[D]\sigma' \vee (\ell_{\Delta,x}[s] = \ell_{\Delta,x}[t])\sigma')$, hence then $M \models \ell_{\Delta,x}[\text{PI}(C_1)]\sigma'$.

Consequently, $M \models (\ell_{\Delta,x}[s]\sigma' \neq \ell_{\Delta,x}[t]\sigma' \wedge \ell_{\Delta,x}[\text{PI}(C_1)]\sigma') \vee (\ell_{\Delta,x}[s]\sigma' = \ell_{\Delta,x}[t]\sigma' \wedge \ell_{\Delta,x}[\text{PI}(C_2)]\sigma')$.

By lemma 11 (huang), $M \models \ell_{\Delta,x}[(s \neq t \wedge \text{PI}(C_1)) \vee (s = t \wedge \text{PI}(C_2))]\sigma'$.

Hence $\Gamma \models \ell_{\Delta,x}[(s \neq t \wedge \text{PI}(C_1)) \vee (s = t \wedge \text{PI}(C_2))]\sigma' \vee (\ell_{\Delta,x}[D] \vee \ell_{\Delta,x}[E[t]])\sigma'$.

IS THIS REALLY WHAT I NEED TO SHOW?

□