CHAPTER 1

Number of quantifier alternations in Huang and nested

1.1 Outline

Goal: try proof Huang and nested equal.
Method: proof for both:

Conjectured Proposition 1. *Let I be an interpolant created by \$algorithm. If I contains a term t such that t has n color changes, then I has at least n quantifier alternations.*

1.1.1 generally keep in mind

- Need to define all new terms here: color-changing, single-color, Φ-literal

1.2 Preliminaries

Quantifier alternations in I usually assumes the quantifier-alternation-minimising arrangement of quantifiers in I

Definition 2 (Color alternation col-alt)**.** Colors Γ and Δ, term t:
$\text{col-alt}(t) \stackrel{\text{def}}{=} \text{col-alt}_\perp(t)$
Let $t = f(t_1, \dots, t_n)$ for constant, function and variable symbols (syntax abuse)

$$\text{col-alt}_\Phi(t) \stackrel{\text{def}}{=} \begin{cases} \max(\text{col-alt}_\Phi(t_1), \dots, \text{col-alt}_\Phi(t_n)) & f \text{ is grey} \\ \max(\text{col-alt}_\Phi(t_1), \dots, \text{col-alt}_\Phi(t_n)) & f \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_\Psi(t_1), \dots, \text{col-alt}_\Psi(t_n)) & f \text{ is of color } \Psi,\ \Phi \neq \Psi \end{cases}$$ △

Definition 3. $\text{PI}^\circ_{\text{step}}$ is defined just like PI_{step} but without applying any substitution. △

Hence $\text{PI}^\circ_{\text{step}}(\cdot)\sigma = \text{PI}_{\text{step}}(\cdot)$. C° is somehow the same, i.e. if $C = D\sigma$, then $C^\circ = D$ where σ is derived from the context.

1.3 Random thoughts

- Quantifiers are introduced for lifting variables which actually occur in the interpolant
- If term t with col-alt$(t) = n$ enters I, we need subterm s of t with col-alt$(s) = n - 1$ to be in I (of course colors of t and s are exactly opposite)

1.3.1 Proof

- Induction over $\ell^x_\Delta[\mathrm{PI}(C) \vee C]$ and also about Γ-terms with Δ-lifting vars in that formula. Cf. `-final`
- TODO: describe proof method with $\sigma_{(0,i)}$: which PI?
  - Factorisation: easy: just apply σ_i for all i to $\mathrm{PI}(C) \vee C$. When done, a literal will be there twice and we can remove it without losing anything
  - Resolution: create propositional structure first.
    Ex.: $C_1 : D \vee l$, $C_2 : \neg l \vee E$:
    If we talk about properties for which it holds that if they hold for $\mathrm{PI}(C_i) \vee C_i$, $i \in \{1,2\}$, then they also hold for $A \equiv \Big((l \wedge \mathrm{PI}(C_2)) \vee (\neg l \wedge \mathrm{PI}(C_1))\Big) \vee C^\circ$, then we can apply σ_i for all i to that formula.
    So if we can assume it for A and show it for all σ_i, we get that it holds for $\mathrm{PI}(C) \vee C$.

1.4 Proof port attempt from `-final`

need to show that grey occurrences are in grey literals, all grey literals end up in the interpolant!

conj: if a Δ-term t occurs in a Γ-literal in a clause C, then t occurs in a grey literal in $\mathrm{PI}(C)$.
evidence:

- situation does not occur in Γ or Δ
- terms are only changed by unificators
- Δ- and Γ-terms are not unifiable, so one of the literals has to have a variable at a grey position when a Δ-term enters a Γ-literal
- that literal has to be grey
- QED?

Conjectured Lemma 4. *If a Φ-term t occurs in a Ψ-literal in a clause C, then t occurs at a grey position in* $\mathrm{PI}(C)$.

Proof. As all grey literals of clauses involved in a refutation end up in the interpolant, it suffices to show that t occurs at a grey position in a grey literal.

Substitutions are applied to all variables, hence we only need to consider terms t which just enter a foreign colored literal.

TODO: propagation 1: Φ-terms vs Ψ-terms (in Ψ-literals)
TODO: propagation 2: Φ-terms vs other Φ-terms (in Ψ-literals)

Induction on refutation and σ; base case easy.

Resolution or factorisation inference ι. Let λ be a Γ-literal containing a variable u at position $\hat{u}$ such that $u\sigma_i$ contains a Δ-term t.

If the resolved or factorised literals are grey, they become part of $\mathrm{PI}(C)$ and if t occurs grey there, we are done.

- Suppose the resolved literals are Γ-colored. Then IH.
- Suppose the resolved literals are Δ-colored. TODO:
- Suppose the resolved literals are grey and t does not occur at a grey position in $\lambda\sigma = \lambda'\sigma$. TODO:

Let λ' be the other resolved or factorised literal.

- Suppose λ' is Γ-colored
- Suppose λ' is grey.

$\square$

Conjectured Lemma 5. *Resolution or factorisation step ι from $\bar{C}$.*

If u col-change var in $(\mathrm{PI}^{\circ}_{\text{step}}(\iota, \mathrm{PI}(C_1), \dots, \mathrm{PI}(C_n)) \vee C^{\circ})\sigma_{(0,i)}$, then u also occurs grey in that formula.

Proof. Abbreviation: $F \equiv (\mathrm{PI}^{\circ}_{\text{step}}(\iota, \mathrm{PI}(C_1), \dots, \mathrm{PI}(C_n)) \vee C^{\circ})$

Induction over refutation and σ; base case easy.

Step: Supp color change var u present in $F\sigma_{(0,i)}$. (could also say introduced, then proof would be somehow different)

Supp u not grey in $F\sigma_{(0,i-1)}$ as otherwise done. As a first step, we show that if a (not necessarily color-changing) variable v occurs in a single-colored Φ-term $t[v]$ in $F\sigma_{(0,i)}$, then at least one of the following holds:

1. v occurs in some single-colored Φ-term in $F\sigma_{(0,i-1)}$ ⟨var_occ_1⟩
2. there is a color-changing variable w in $F\sigma_{(0,i-1)}$ such that v occurs grey in $w\sigma_i$. ⟨var_occ_2⟩

We consider the different cases which can introduce a variable v in a single-colored term Φ: Either it has been there before, it was introduced in a s.c. Φ-colored term, or a s.c. Φ-term containing the var is in $\mathrm{ran}(\sigma)$.

- Suppose a term $t'[v]$ is present in $F\sigma_{(0,i-1)}$ such that $t'[v]\sigma_i = t[v]$. Then 1 is the case.
- Suppose a variable w occurs in a single-colored Φ-term in $F\sigma_{(0,i-1)}$ such that v occurs grey in $w\sigma_i$. Suppose furthermore that 1 is not the case, i.e. v does not occur in a s.c. Φ-term in $F\sigma_{(0,i-1)}$, as otherwise we would be done. We show that 2 is the case.

  As v occurs neither grey nor in a s.c. Φ-term in $F\sigma_{(0,i-1)}$ but occurs in $\mathrm{ran}(\sigma_i)$, it must occur in $F\sigma_{(0,i-1)}$ and this can only be in a single-colored Ψ-term.

As by assumption v occurs grey in $w\sigma_i$, there must be an occurrence $\bar{w}$ of w in a resolved or factorised literal, say $\lambda\sigma_{(0,i-1)}$ such that for the other resolved literal $\lambda'\sigma_{(0,i-1)}$, $\lambda'\sigma_{(0,i-1)}|_{\bar{w}}$ is a subterm in which v occurs grey. But as the occurrence of v in $\lambda'\sigma_{(0,i-1)}|_{\bar{w}}$ must be contained in a single-colored Ψ-term, so is $\lambda\sigma_{(0,i-1)}|_{\bar{w}}$, hence z occurs in a single-colored Ψ-term as well. Therefore 2 is the case.

- Suppose there is a variable z in $F\sigma_{(0,i-1)}$ such that v occurs in a single-colored Φ-term in $z\sigma_i$. Then $z\sigma_i$ occurs in $F\sigma_{(0,i-1)}$, but this is a witness for 1.

Now recall that we have assumed u to be a color-changing variable in $F\sigma_{(0,i)}$. Hence it occurs in a single-colored Γ-term as well as in a single-colored Δ-term. By the reasoning above, this leads to two case:

- In $F\sigma_{(0,i-1)}$, u occurs both in some single-colored Γ-term as well as in some single-colored Δ-term. Then we get the result by the induction hypothesis and the fact that $u \notin \mathrm{dom}(\sigma_i)$ as u does occur in $F\sigma_{(0,i)}$.
- Otherwise for some color Φ, u does not occur in a single-colored Φ-term in $F\sigma_{(0,i-1)}$. Then case 2 above must hold and there is some color-changing variable w in $F\sigma_{(0,i-1)}$ such that u occurs grey in $w\sigma_{(0,i)}$. But then by the induction hypothesis, w occurs grey in $F\sigma_{(0,i-1)}$ and hence u occurs grey in $F\sigma_{(0,i)}$. □

Conjectured Lemma 6. *If* $\mathrm{PI}(C) \vee C$ *contains a maximal colored occurrence of a* Γ*-term* $t[s]$ *containing* Δ*-term* s*, then* s *occurs grey in* $\mathrm{PI}(C) \vee C$*.*

Proof. Note that it suffices to show that at the step where s is introduced as subterm of $t[s]$, s occurs grey in $\mathrm{PI}(C) \vee C$ as any later modification by substitution is applied to both occurrences s, so they stay equal throughout the remaining derivation. TODO: what if it's in $\mathrm{PI}(C)$ and it disappears due to not being a colored literal?

Induction over π and σ.

Base case: ✓

Step: Resolution or factorisation inference ι, $\mathrm{mgu}(\iota) = \sigma = \sigma_1 \cdots \sigma_n$ The term $t[s]$ is created by one of the following two ways:

- A variable u occurs in $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C^{\circ})\sigma_{(0,i-1)}$ such that $u\sigma_i = t[s]$.

  Then u occurs in a resolved or factorised literal $\lambda\sigma_{(0,i-1)}$ at $\hat{u}$ such that at the other resolved or factorised literal $\lambda'\sigma_{(0,i-1)}$, $\lambda'\sigma_{(0,i-1)}|_{\hat{u}} = t[s]$. Then the condition is present at $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C^{\circ})\sigma_{(0,i-1)}$ and we get the result by the induction hypothesis.

- Note that we only consider maximal colored terms. Let $t[u]$ be a maximal colored Γ-term in $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C^{\circ})\sigma_{(0,i-1)}$ such that in the tree-representation of $t[u]$, the path from the root to u does not contain a node labelled with a Δ-symbol.

  is probably not same t as in lemma statement, which isn't technically wrong but confusing

  Suppose that u occurs grey in $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C^{\circ})\sigma_{(0,i-1)}$. Then s occurs grey in $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C^{\circ})\sigma_{(0,i)}$ and we are done.

  Now suppose that u does not occur grey in $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C^{\circ})\sigma_{(0,i-1)}$.

  TODO: need color changing variable lemma for $\mathrm{PI}(C) \vee C$, or actually the $\mathrm{PI}_{\mathrm{step}}$-representation

  TODO: case with u in s.c. Γ-term

□