Example 101: canonical examples for overbinding w.r.t. order

Here, the second lower prooftree shows what huang would produce.

Ex 101a

$$\dfrac{\dfrac{\overset{\Sigma}{P({\color{red}u},f({\color{red}u}))\vee Q({\color{red}u})} \qquad \overset{\Pi}{\neg Q(a)}}{P(a,f(a))}\,u\mapsto a \qquad \overset{\Pi}{\neg P(x,y)}}{\square}\,x\mapsto a, y\mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{Q(a)}\,u\mapsto a \qquad \top}{P(a,f(a))\vee Q(a)}\,x\mapsto a, y\mapsto f(a) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 Q(x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}$$

Direct overbinding would not work without merging same variables!: $\forall x_1\forall x_2\exists x_3(P(x_2,x_3)\vee Q(x_1))$
counterexample: $Q\sim\{0\}, P\sim\{(1,0)\}$
Direct overbinding would work when considering original dependencies as highlighted above

Ex 101b – other resolution order

$$\dfrac{\dfrac{\overset{\Sigma}{P(u,f(u))\vee Q(u)} \qquad \overset{\Pi}{\neg P(x,y)}}{Q(u))}\,y\mapsto f(u), x\mapsto u \qquad \overset{\Pi}{\neg Q(a)}}{\square}\,u\mapsto a$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(u,f(u))}\,x\mapsto f(u), x\mapsto u \qquad \top}{P(a,f(a))\vee Q(a)}\,u\mapsto a \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(u,x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}\,u\mapsto a$$

Ex 101c – Π and Σ swapped

$$\dfrac{\dfrac{\overset{\Pi}{P(u,f(u))\vee Q(u)} \qquad \overset{\Sigma}{\neg P(x,y)}}{Q(u))}\,y\mapsto f(u), x\mapsto u \qquad \overset{\Sigma}{\neg Q(a)}}{\square}\,u\mapsto a$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg P(u,f(u))}\,x\mapsto f(u), x\mapsto u \qquad \bot}{\neg P(a,f(a))\wedge\neg Q(a)}\,u\mapsto a \qquad\qquad \dfrac{\dfrac{\top \qquad \bot}{\forall x_2\neg P(u,x_2)} \qquad \bot}{\exists x_1\forall x_2(\neg P(x_1,x_2)\wedge\neg Q(x_1))}$$

Ex 101d – Π and Σ swapped, other resolution order

$$\dfrac{\dfrac{\overset{\Pi}{P(u,f(u))} \lor Q(u) \qquad \neg \overset{\Sigma}{Q}(a)}{P(a,f(a))}\ u \mapsto a \qquad \neg \overset{\Sigma}{P}(x,y)}{\square}\ x \mapsto a, y \mapsto f(a)$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg Q(a)}\ y \mapsto a \qquad \bot}{\neg Q(a) \land \neg P(a,f(a))}\ x \mapsto a, y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\top \qquad \bot}{\exists x_1 \neg Q(x_1)} \qquad \bot}{\exists x_1 \forall x_2 (\neg P(x_1,x_2) \land \neg Q(x_1))}$$

102 – similar to 101 but with intra-clause-set inferences

Ex 102a

$$\dfrac{\dfrac{P(f(x)) \lor \overset{\Sigma}{Q}(f(x),z) \qquad \neg \overset{\Pi}{P}(y)}{Q(f(x),z)} \qquad \dfrac{\neg \overset{\Sigma}{Q}(x_1,y) \lor R(y) \qquad \neg \overset{\Pi}{R}(g(z_1))}{\neg Q(x_1,g(z_1))}\ y \mapsto g(z_1), x_1 \mapsto f(x)}{\square}\ x_1 \mapsto f(x), z \mapsto g(z_1)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(g(z_1))}\ y \mapsto g(z_1), x_1 \mapsto f(x)}{P(f(x)) \lor R(g(z_1))}\ x_1 \mapsto f(x), z \mapsto g(z_1) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}}{\exists x_1 \forall x_2 (P(x_1) \lor R(x_2)) \quad \text{(order irrelevant!)}}$$

Ex 102b

$$\dfrac{\dfrac{P(f(x)) \lor \overset{\Sigma}{Q}(f(x),z) \qquad \neg \overset{\Pi}{P}(y)}{Q(f(x),z)} \qquad \dfrac{\neg \overset{\Sigma}{Q}(f(y),z_1) \lor R(y) \qquad \neg \overset{\Pi}{R}(a)}{\neg Q(f(a),z_1)}\ y \mapsto a}{\square}\ x \mapsto a, z \mapsto z_1$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\ y \mapsto a}{P(f(a)) \lor R(a)}\ x \mapsto a, z \mapsto z_1 \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\ y \mapsto a}{\forall x_2 \exists x_1 (P(x_1) \lor R(x_2))}\ x \mapsto a, z \mapsto z_1$$

direct:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)}\ x_1 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\ x_2 \sim a}{\exists x_1 \forall x_2 (P(x_1) \lor R(x_2))}\ \text{order irrelevant!}$$

Ex 102b' with Q grey

$$\dfrac{\dfrac{\overset{\Sigma}{P(f(x)) \vee Q(f(x), z)} \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x), z)} \qquad \dfrac{\overset{\Sigma}{\neg Q(f(y), z_1) \vee R(y)} \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a), z_1)}\ y \mapsto a}{\square}\ x \mapsto a, z_1 \mapsto z$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\ y \mapsto a}{(\neg Q(f(a), z) \wedge P(f(a))) \vee (Q(f(a), z) \wedge R(a))}\ x \mapsto a, z_1 \mapsto z$$

Huang:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)} \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\ y \mapsto a}{\forall x_1 \exists x_2 (\neg Q(x_2, z) \wedge P(x_2)) \vee (Q(x_2, z) \wedge R(x_1))}\ x \mapsto a, z_1 \mapsto z$$

direct:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)}\ x_2 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\ x_1 \sim a}{\forall x_1 \exists x_2 \exists x_3 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}\ x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3}{\text{OR: } \exists x_2 \forall x_1 \exists x_3 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}}{\text{OR: } \exists x_1 \exists x_3 \forall x_2 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}$$

Huang-Version passt weil man für existenzielle quantoren eh gleichen term einsetzt

Example 103: variables in interpolant get unified and order might have to be changed

103a: length changes but old version is still valid, despite huang's algorithm doing a sort

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{Q(f(\color{red}{x})) \lor P(y) \lor R(\color{red}{x})} \qquad \overset{\Pi}{\neg Q(y_1)}}{P(y) \lor R(x)}\, y_1 \mapsto f(x) \qquad \overset{\Pi}{\neg P(h(g(a)))}}{R(x)}\, y \mapsto h(g(a)) \qquad \overset{\Pi}{\neg R(g(g(a)))}}{\square}\, x \mapsto g(g(a))$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{Q(f(x))}\, y_1 \mapsto f(x) \qquad \top}{Q(f(x)) \lor P(h(g(a)))}\, y \mapsto h(g(a)) \qquad \top}{Q(f(g(g(a)))) \lor P(h(g(a))) \lor R(g(g(a)))}\, x \mapsto g(g(a))$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_1 \forall x_2 (Q(x_1) \lor P(x_2))} \qquad \top}{X}$$

X:

Huang's algo gives:

$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

Direct overbinding gives: $x_3 < x_1$, rest arbitrary, hence:

$\forall x_3 \exists x_1 \forall x_2 (Q(x_1) \lor P(x_2) \lor R(x_3))$ <- this you do not get with huang

$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

$\forall x_3 \forall x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

103b: length changes "uniformly"

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{Q(f(f(\color{red}{x}))) \lor P(f(\color{red}{x})) \lor R(\color{red}{x})} \qquad \overset{\Pi}{\neg Q(y_1)}}{P(f(x)) \lor R(x)}\, y_1 \mapsto f(f(x)) \qquad \overset{\Pi}{\neg P(y_2)}}{R(x)}\, y_2 \mapsto f(x) \qquad \overset{\Pi}{\neg R(g(a))}}{\square}\, x \mapsto g(a)$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{Q(f(f(x)))}\, y_1 \mapsto f(f(x)) \qquad \top}{Q(f(f(x))) \lor P(f(x))}\, y_2 \mapsto f(x) \qquad \top}{Q(f(f(g(a)))) \lor P(f(g(a))) \lor R(g(a))}\, x \mapsto g(a)$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_2 \exists x_1 (Q(x_1) \lor P(x_2))} \qquad \top}{\forall x_3 \exists x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))}$$

Huang and direct overbinding somewhat coincide as $x_2 < x_1$ in both cases, and $x_3 < x_1$ and $x_3 < x_2$

103c: Failed attempt: different variables, accidentally the same terms appear but no logical connection

$$\dfrac{\overset{\Sigma}{P(a,x)} \qquad \dfrac{\overset{\Sigma}{\neg Q(a)} \qquad \overset{\Pi}{\neg P(y, f(\color{red}{z})) \vee Q(\color{red}{z})}}{\neg P(y, f(a))}\; z \mapsto a}{\Box}\; y \mapsto a, x \mapsto f(a)$$

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg Q(a)}\; z \mapsto a}{P(a, f(a)) \wedge \neg Q(a)}\; y \mapsto a, x \mapsto f(a) \qquad\qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg Q(x_1)}}{\exists x_1 \forall x_2 (P(x_1, x_2) \wedge \neg Q(x_1))}$$

order required for Π

direct:

$$\dfrac{\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg Q(x_1)}\; x_1 \sim a}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}\; x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))}$$

invariant:

$$\dfrac{\dfrac{\exists x_2 (P(x_2, x) \vee \bot) \qquad \dfrac{\exists x_1 (Q(x_1) \vee \bot) \qquad \forall x_3 ((\neg P(y, \color{red}{x_3}) \vee Q(\color{red}{z})) \vee \top)}{\exists x_1 \forall x_3 \neg P(y, \color{red}{x_3}) \vee \neg Q(\color{red}{x_1})}\; x_1 \sim a}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}\; x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))}$$

invariant in other resolution order

$$\dfrac{\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{Q(\color{red}{z}) \vee \exists x_2 \forall x_3 P(x_2, \color{red}{x_3})}\; x_2 \sim a, x_3 \sim f(z)}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}\; x_1 \sim a; x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))}$$

invariant if Σ and Π swapped:

$$\dfrac{\dfrac{\bot \qquad \dfrac{\top \qquad \bot}{\neg P(y, f(x_1)) \vee \forall x_1 Q(x_1)}\; x_1 \sim a}{\forall x_1 \forall x_2 \exists x_3 (\neg P(x_2, x_3) \vee Q(x_1))}\; x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \forall x_1 \exists x_3 \forall x_2 (\neg P(x_2, x_3) \vee Q(x_1))}$$

SECOND ATTEMPT:

$$\dfrac{\overset{\Sigma}{P(a)} \qquad \dfrac{\overset{\Sigma}{Q(z)} \qquad \dfrac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg P(y) \lor \neg Q(f({\color{red}x})) \lor S({\color{red}x})}}{\neg P(y) \lor \neg Q(f(a))}\; x \mapsto a}{\neg P(y)}\; z \mapsto f(a)}{\square}\; y \mapsto a$$

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg S(a)}\; x \mapsto a}{\neg S(a) \land Q(f(a))}\; z \mapsto f(a)}{P(a) \land \neg S(a) \land Q(f(a))}\; y \mapsto a$$

Huang:

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 (\neg S(x_1) \land Q(x_2))}}{\exists x_1 \forall x_2 (P(x_1) \land \neg S(x_1) \land Q(x_2))}$$

$\forall x_1 \exists x_2 (\neg P(x_1) \lor S(x_1) \lor \neg Q(x_2))$

similar fail

$\Rightarrow$ anytime there is $P(a, f(a))$, either they have a dependency or they are not both differently colored (grey is uncolored)

for the record, direct method anyway:

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}\; x \sim a}{\exists x_1 \forall x_2 \neg S(x_1) \land Q(x_2)}\; z \sim f(a); x_1 < x_2}{\exists x_1 \forall x_2 \exists x_3 P(x_3) \land \neg S(x_1) \land Q(x_2)}\; x_3 \sim a; x_3 \text{ need not be merged w } x_1$$

Example: ordering on both ancestors where the merge forces a new ordering

202a – canonical

$$
\dfrac{
\dfrac{\overset{\Sigma}{P(a,x_1)\vee R(y)} \qquad \overset{\Pi}{\neg P(x,fx)}}{R(y)}\;\begin{matrix}x_1\mapsto fa\\ x\mapsto a\end{matrix}
\qquad
\dfrac{\overset{\Sigma}{Q(x_2,gx_2)\vee\neg R(u)} \qquad \dfrac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg Q(fz,x_3)\vee S(z)}}{\neg Q(fa,x_3)}\; z\mapsto a}{\neg R(u)}\;\begin{matrix}x_2\mapsto fa,\\ x_3\mapsto gfa\end{matrix}
}{\Box}
$$

$$
\dfrac{
\dfrac{\bot \qquad \top}{P(a,f(a))}\;\begin{matrix}x_1\mapsto f(a)\\ x\mapsto a\end{matrix}
\qquad
\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg S(a)}\; z\mapsto a}{Q(f(a),g(f(a)))\wedge\neg S(a)}\;\begin{matrix}x_2\mapsto f(a),\\ x_3\mapsto g(f(a))\end{matrix}
}{P(a,f(a))\vee(Q(f(a),g(f(a)))\wedge\neg S(a))}
$$

Huang

$$
\dfrac{
\dfrac{\bot \qquad \top}{\exists x_1\forall x_2 P(x_1,x_2))}
\qquad
\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1\neg S(x_1)}}{\exists x_1\forall x_2\exists x_3 Q(x_2,x_3)\wedge\neg S(x_1)}
}{\exists x_1\forall x_2\exists x_3 P(x_1,x_2)\vee(Q(x_2,x_3))\wedge\neg S(x_1))}
$$

direct:

$$
\dfrac{
\dfrac{\bot \qquad \top}{\exists x_1\forall x_2 P(x_1,x_2))}\;\begin{matrix}x_1\sim a, x_2\sim fa\\ x_1<x_2\end{matrix}
\qquad
\begin{matrix}x_3\sim a, x_4\sim fa, x_5\sim gfa)\\ x_3<x_4, x_4<x_5\end{matrix}\;
\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_3\neg S(x_3)}\; x_3\sim a}{\exists x_3\forall x_4\exists x_5 Q(x_4,x_5)\wedge\neg S(x_3)}
}{\exists x_1\forall x_2\exists x_5 P(x_1,x_2)\vee(Q(x_2,x_5)\wedge\neg S(x_5))}\;\begin{matrix}x_3\mapsto x_1, x_4\mapsto x_2\\ x_1<x_2, x_2<x_5\end{matrix}
$$

without merge in end: $(x_1<x_2, x_3<x_4, x_4<x_5)$

$\exists x_1\forall x_2\exists x_3\forall x_4\exists x_5 P(x_1,x_2)\vee(Q(x_4,x_5)\wedge\neg S(x_3))$

$\exists x_3\forall x_4\exists x_5\exists x_1\forall x_2 P(x_1,x_2)\vee(Q(x_4,x_5)\wedge\neg S(x_3))$

(also interwoven ones appear to work)

combined presentation:

$$\dfrac{\dfrac{\bot \mid P(a,x_1) \lor R(y) \qquad \top \mid -}{P(a,f(a)) \mid R(y)}\;\begin{matrix} x_1 \mapsto f(a) \\ x \mapsto a \end{matrix} \qquad \dfrac{\bot \mid Q(x_2,g(x_2)) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(z),x_3) \lor S(z)}{\neg S(a) \mid \neg Q(f(a),x_3)}\; z \mapsto a}{Q(f(a),g(f(a))) \land \neg S(a) \mid \neg R(u)}\;\begin{matrix} x_2 \mapsto f(a), \\ x_3 \mapsto g(f(a)) \end{matrix}}{P(a,f(a)) \lor (Q(f(a),g(f(a))) \land \neg S(a)) \mid \Box}$$

combined presentation ground:

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \lor R(y) \qquad \top \mid \neg P(x,f(x))}{(P(a,f(a)) \land \top) \lor (\neg P(a,f(a)) \land \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \lor S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(f(a),g(f(a))) \land \neg S(a) \mid \neg R(u)}}{P(a,f(a)) \lor (Q(f(a),g(f(a))) \land \neg S(a)) \mid \Box}$$

combined presentation ground with direct method but only Δ-terms removed :

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \lor R(y) \qquad \top \mid \neg P(x,f(x))}{(P(a,x_2) \land \top) \lor (\neg P(a,x_2) \land \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \lor S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(x_4,g(x_4)) \land \neg S(a) \mid \neg R(u)}}{P(a,x_2) \lor (Q(x_4,g(x_4)) \land \neg S(a)) \mid \Box}$$

combined presentation ground with direct method:

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \lor R(y) \qquad \top \mid \neg P(x,f(x))}{\exists x_1 \forall x_2 (P(x_1,x_2) \land \top) \lor (\neg P(x_1,x_2) \land \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \lor \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \lor S(a)}{\exists x_3 \neg S(x_3) \mid \neg Q(f(a),g(f(a)))}}{\exists x_3 \forall x_4 \exists x_5 (Q(x_4,x_5)) \land \neg S(x_3)) \mid \neg R(u)}}{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (P(x_1,x_2) \lor (Q(x_4,x_5)) \land \neg S(x_3)) \mid \Box}$$

203a – some alternations

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x) \lor \overset{\Sigma}{\neg P}(f(x)) \qquad P(z) \lor \overset{\Pi}{Q}(g(z))}{R(x) \lor Q(g(f(x)))}\; z \mapsto f(x) \qquad \neg Q(y) \lor \overset{\Sigma}{S}(h(y))}{R(x) \lor S(h(g(f(x))))}\; y \mapsto g(f(x))}{S(h(g(f(a))))}\; x \mapsto a}{\square}\; x_1 \mapsto h(g(f(a)))$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\neg P(f(x))}\; z \mapsto f(x) \qquad \bot}{\neg Q(g(f(x))) \land \neg P(f(x))}\; y \mapsto g(f(x))}{\neg Q(g(f(a))) \land \neg P(f(a)) \lor R(a)}\; x \mapsto a}{\neg Q(g(f(a))) \land \neg P(f(a)) \lor R(a) \lor S(h(g(f(a))))}\; x_1 \mapsto h(g(f(a)))$$

Huang:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \neg P(x_1)} \qquad \bot}{\exists x_1 \forall x_2 (\neg Q(x_2) \land \neg P(x_1))}}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \land \neg P(x_1) \lor R(x_0))}}{\forall x_0 \exists x_1 \forall x_2 \exists x_3 (\neg Q(x_2) \land \neg P(x_1) \lor R(x_0) \lor S(x_3))}$$

Direct:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \neg P(x_1)}\; x_1 \sim f(x) \qquad \bot}{\exists x_1 \forall x_2 (\neg Q(x_2) \land \neg P(f(x)))}\; x_2 \sim g(f(x)); x_1 < x_2}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \land \neg P(x_1) \lor R(x_0))}\; x_0 \sim a; x_0 < x_1, x_0 < x_2}{\forall x_0 \exists x_1 \forall x_2 \exists x_3 (\neg Q(x_2) \land \neg P(x_1) \lor R(x_0) \lor S(x_3))}\; x_3 \sim h(g(f(a))); x_0 < x_3, x_1 < x_3, x_2 < x_3$$

203b – many Σ-literals, coloring per occurrence

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x) \lor \overset{\Sigma}{\neg P}(f(x)) \qquad P(z) \lor \overset{\Sigma}{Q}(g(z))}{R(x) \lor Q(gfx)}\; z \mapsto fx \qquad \neg Q(y) \lor \overset{\Sigma}{S}(h(y))}{R(x) \lor S(hgfx)}\; y \mapsto gfx}{S(hgfa)}\; x \mapsto a}{\square}\; x_1 \mapsto hgfa$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \bot}{\bot}\; z \mapsto fx \qquad \bot}{\bot}\; y \mapsto gfx}{R(a)}\; x \mapsto a}{S(hgfa) \lor R(a)}\; x_1 \mapsto hgfa$$

$\to \forall x_1 \exists x_2 (R(x_1) \lor S(x_2))$

203b' – many Σ-literals, coloring per symbol, all predicates grey

$$\dfrac{\overset{\Pi}{\neg S}(x_1) \qquad \dfrac{\dfrac{\dfrac{\neg\overset{\Pi}{R}(a) \qquad R(x) \vee \overset{\Sigma}{\neg} P(f(x))}{R(a) \mid \neg P(fa)}\; x \mapsto a \qquad P(z) \vee \overset{\Sigma}{Q}(g(z))}{P(fa) \vee R(a) \mid Q(gfa)}\; z \mapsto fa \qquad \neg Q(y) \overset{\Sigma}{\vee} S(h(y))}{}}{}$$

TODO

Example where variables are not the outermost symbol but order is still relevant

204a

$\Sigma = \{P(f(x), g(f(x)))\}$
$\Pi = \{P(f(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f(x_1), x_2)$

204b

$\Sigma = \{P(f^5(x), g(f(x)))\}$
$\Pi = \{P(f^5(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f^5(x_1), x_2)$

example with aufschaukelnde unification, such that direction of arrow isn't clear

205a

$$\dfrac{\overset{\Pi}{P(ffy,gy)} \qquad \dfrac{\overset{\Sigma}{\neg P(x,y)\vee Q(x)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffz)\vee Rz}}{\neg R(a) \mid \neg Q(ffa)}\ z\mapsto a}{\neg R(a)\wedge Q(ffa) \mid \neg P(ffa,y)}\ x\mapsto ffa}{(\neg R(a)\wedge Q(ffa))\vee\neg P(ffa,ga)}\ y\mapsto a$$

direct

$$\dfrac{\overset{\Pi}{P(ffy,gy)} \qquad \dfrac{\overset{\Sigma}{\neg P(x,y)\vee Q(x)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffz)\vee Rz}}{\exists x_1 \neg R(x_1) \mid \neg Q(ffa)}\ z\mapsto a}{\exists x_1 \forall x_2(\neg R(x_1)\wedge Q(x_2)) \mid \neg P(ffa,u)}\ x\mapsto ffa}{\exists x_1 \forall x_2 \forall x_3((\neg R(x_1)\wedge Q(x_2))\vee\neg P(x_2,x_3))}\ y\mapsto a, u\mapsto ga$$

ground:

$$\dfrac{\overset{\Pi}{P(ffa,ga)} \qquad \dfrac{\overset{\Sigma}{\neg P(ffa,y)\vee Q(ffa)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffa)\vee Ra}}{\neg R(a) \mid \neg Q(ffa)}}{\neg R(a)\wedge Q(ffa) \mid \neg P(ffa,a)}}{(\neg R(a)\wedge Q(ffa))\vee\neg P(ffa,ga)}$$

205b ∼ 205a, but simpler

Suppose P occurs somewhere in Σ (result not that optimal in this setting, but correct)
not nice for proving, $\neg R(a)$ is a nice interpolant already

$$\dfrac{\overset{\Pi}{P(ffy,gy)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg P(ffz,x)\vee Rz}}{\neg R(a) \mid \neg P(ffa,x)}\ z\mapsto a}{\neg R(a)\vee\neg P(ffa,ga) \mid \Box}\ x\mapsto ga, y\mapsto a$$

$$\dfrac{\top \mid \overset{\Pi}{P(ffy,gy)} \qquad \dfrac{\bot \mid \overset{\Sigma}{\neg R(a)} \qquad \top \mid \overset{\Pi}{\neg P(ffz,x)\vee Rz}}{\exists x_1 \neg R(x_1) \mid \neg P(ffa,x)}\ z\mapsto a}{\exists x_1 \forall x_2 \forall x_3 \neg R(x_1)\vee\neg P(x_2,x_3) \mid \Box}\ x\mapsto ga, y\mapsto a$$

$\exists x_1 R(x_1)$

$\exists x_1 \forall x_2 \forall x_3 (R(x_1)\vee\neg P(x_2,x_3))$

misc examples

201a

$$\dfrac{\dfrac{P(x,y) \overset{\Sigma}{\vee} \neg Q(y) \qquad \neg P(\overset{\Pi}{a}, y_2)}{\neg Q(y)}\, x \mapsto a \qquad \dfrac{Q(f(z)\overset{\Sigma}{)}) \vee R(z) \qquad \neg \overset{\Pi}{R}(a)}{Q(f(a)}\, z \mapsto a}{\Box}\, y \mapsto f(a)$$

$$\dfrac{\dfrac{\perp \qquad \top}{P(a,y)}\, x \mapsto a \qquad \dfrac{\perp \qquad \top}{R(a)}\, z \mapsto a}{P(a, f(a)) \vee R(a)}\, y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\perp \qquad \top}{\forall x_1 P(x_1, y)}\, x \mapsto a \qquad \dfrac{\perp \qquad \top}{\forall x_3 R(x_3)}\, z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 (P(x_1, x_2) \vee R(x_3))}\, y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1, x_2) \vee R(x_1)$

201b

$$\dfrac{\dfrac{P(x, f(y)) \overset{\Sigma}{\vee} \neg Q(f(y)) \qquad \neg P(\overset{\Pi}{a}, y_2)}{\neg Q(f(y))}\, x \mapsto a \qquad \dfrac{Q(f(z)\overset{\Sigma}{)}) \vee R(z) \qquad \neg \overset{\Pi}{R}(a)}{Q(f(a)}\, z \mapsto a}{\Box}\, y \mapsto f(a)$$

$$\dfrac{\dfrac{\perp \qquad \top}{P(a, f(y))}\, x \mapsto a \qquad \dfrac{\perp \qquad \top}{R(a)}\, z \mapsto a}{P(a, f(a)) \vee R(a)}\, y \mapsto a \qquad\qquad \dfrac{\dfrac{\perp \qquad \top}{\forall x_1 \exists x_2 P(x_1, x_2)}\, x \mapsto a \qquad \dfrac{\perp \qquad \top}{\forall x_3 R(x_3)}\, z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 P(x_1, x_2) \vee R(x_3)}\, y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1, x_2) \vee R(x_1)$

arrow in element which is not in interpolant or resolution clause

206

$$\cfrac{\cfrac{\overset{\Sigma}{P(x) \vee \neg Q(f(x))} \qquad \overset{\Pi}{\neg P(a)}}{\forall x_1 P(x_1) \mid \neg Q(f(a))}\; x \mapsto a \qquad \cfrac{\overset{\Sigma}{Q(y) \vee R(g(y))} \qquad \overset{\Pi}{\neg R(z)}}{\exists x_2 R(x_2) \mid Q(y)}\; z \mapsto g(y)}{\forall x_1 \exists x_2 (P(x_1) \vee R(x_2)) \mid \Box}\; y \mapsto f(a)$$

$$P(a) \vee R(g(f(a)))$$

for first interpolant, $\Sigma \not\models \ell_{\Delta,x}[\mathrm{PI}(C)] \vee C$