Contents

0.1 referenced lemmas from previous sections

⟨lemma:lift_logic_commute⟩

Lemma 1 (Commutativity of lifting and logical operators)**.** *Let A and B be first-order formulas and s and t be terms. Then it holds that:*

1. $\ell^z_\Phi[\neg A] \Leftrightarrow \neg\ell^z_\Phi[A]$
2. $\ell^z_\Phi[A \circ B] \Leftrightarrow (\ell^z_\Phi[A] \circ \ell^z_\Phi[B])$ *for* $\circ \in \{\wedge, \vee\}$
3. $\ell^z_\Phi[s = t] \Leftrightarrow (\ell^z_\Phi[s] = \ell^z_\Phi[t])$

?⟨thmt@@lemmaCommutLiftSubst@data⟩? ?⟨thmt@@lemmaCommutLiftSubst⟩?

Lemma 2 (Commutativity of lifting and substitution)**.** *Let C be a clause and σ a substitution such that no lifting variable occurs in C or σ. Define σ' with $\mathrm{dom}(\sigma') = \mathrm{dom}(\sigma) \cup \{z_t \mid t\sigma \neq t\}$ such that for a variable z,*

$$x\sigma' = \begin{cases} z_{t\sigma} & \text{if } x = z_t \text{ and } t\sigma \neq t \\ \ell^z_\Phi[x\sigma] & \text{otherwise} \end{cases}$$

Then $\ell^z_\Phi[C\sigma] = \ell^z_\Phi[C]\sigma'$.

CHAPTER 1

Interpolant extraction from resolution proofs in one phase

While the previous chapter demonstrates that it is possible to extract propositional interpolants and lift them from the colored symbols later in order to obtain a proper interpolant, we now present a novel approach, which only operates with grey intermediary interpolants. This is established by lifting any term which is added to the interpolant.

By its nature, this approach requires an alternate strategy than the proof of the extraction in two phases as a commutation of substitution and lifting is no longer possible if lifting variables are present. Let us recall the corresponding lemma from the previous chapter:

Lemma 2 (Commutativity of lifting and substitution)**.** *Let C be a clause and σ a substitution such that no lifting variable occurs in C or σ. Define σ' with* $\mathrm{dom}(\sigma') = \mathrm{dom}(\sigma) \cup \{z_t \mid t\sigma \neq t\}$ *such that for a variable z,*

$$x\sigma' = \begin{cases} z_{t\sigma} & \textit{if } x = z_t \textit{ and } t\sigma \neq t \\ \ell^z_\Phi[x\sigma] & \textit{otherwise} \end{cases}$$

Then $\ell^z_\Phi[C\sigma] = \ell^z_\Phi[C]\sigma'$.

Consider the following illustration of a problem of the notion of applying this lemma to terms containing lifting variables:

Example 3. Let $\sigma = \{x \mapsto a\}$ and consider the terms $f(x)$ and $f(a)$, where f and a are colored symbols. Clearly $f(x)\sigma = f(a)$ and therefore necessarily $z_{f(x)}\sigma' = z_{f(a)}$.

But now consider $x_{f(x)}\sigma$. As $z_{f(x)}$ is a lifting variable, it is not affected by unifiers from resolution derivations and also not by σ. Hence $z_{f(x)}\sigma = z_{f(x)}$ and therefore $\ell[z_{f(x)}\sigma] = \ell[z_{f(x)}] = z_{f(x)}$, but $\ell[z_{f(x)}]\sigma' = z_{f(x)}\sigma' = z_{f(a)}$. So $\ell[z_{f(x)}\sigma] \neq \ell[z_{f(x)}]\sigma'$.

We see here that there are circumstances under which in order to commute lifting and substitution, the substitution σ' is required to conform to the equation $z_{f(x)}\sigma' = z_{f(a)}$, whereas in others, it must hold that $z_{f(x)}\sigma' = z_{f(x)}$. △

1.1 Definition of the extraction algorithm

The extracted interpolants are prenex formulas, where the quantifier block and the matrix of the formula are calculated separately in each step of the traversal of the resolution refutation.

1.1.1 Extraction of the interpolant formula matrix $\mathrm{AI}_{\mathrm{mat}}$ and calculation of $\mathrm{AI}_{\mathrm{cl}}$

$\mathrm{AI}_{\mathrm{mat}}$ is inspired by the propositional interpolants PI from Definition **??**. Its difference lies in the fact that the lifting occurs in every step of the extraction. This however necessitates applying these liftings to the clauses of the resolution refutation as well. For a clause C of the resolution refutation, we will denote the clause with the respective liftings applied by $\mathrm{AI}_{\mathrm{cl}}(C)$ (a formal definition will be given below), and for a term t at position p in C, we denote $\mathrm{AI}_{\mathrm{cl}}(C)|_p$ by t_{AIcl}.

Now we can define preliminary versions of $\mathrm{AI}^{\bullet}_{\mathrm{mat}}$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}$:

Definition 4 ($\mathrm{AI}^{\bullet}_{\mathrm{mat}}$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}$)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$.

For a clause C in π, $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C)$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C)$ are defined as follows:

Base case. If $C \in \Gamma$, $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \top$.

In any case, $\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[C]$.

Resolution. If the clause C is the result of a resolution step of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C)$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}$ are defined as follows:

$\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[(\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C_1)\backslash\{l_{\mathrm{AIcl}}\})\sigma] \ \vee\ \ell[(\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C_2)\backslash\{l'_{\mathrm{AIcl}}\})\sigma]$

1. If l is Γ-colored: $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma] \vee \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_2)\sigma]$
2. If l is Δ-colored: $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma] \wedge \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_2)\sigma]$
3. If l is grey: $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} (\neg\ell[l'_{\mathrm{AIcl}}\sigma] \wedge \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma]) \ \vee\ (\ell[l_{\mathrm{AIcl}}\sigma] \wedge \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_2)\sigma])$

Factorisation. If the clause C is the result of a factorisation of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma]$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[(\mathrm{AI}_{\mathrm{cl}}(C_1)\backslash\{l'_{\mathrm{AIcl}}\})\sigma]$. △

Note that in $\mathrm{AI}^{\bullet}_{\mathrm{mat}}$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}$, it is possible that there for a colored term t in C that $t_{\mathrm{AIcl}} \neq z_t$ as illustrated by the following examples:

Example 5. We consider a resolution refutation of the initial clause sets $\Gamma = \{R(c), \neg Q(v)\}$ and $\Delta = \{\neg R(u) \vee Q(g(u))\}$:

$$\dfrac{\dfrac{R(c) \qquad \neg R(u) \vee Q(g(u))}{Q(g(c))}\ \text{res}, y \mapsto c \qquad \neg Q(v)}{\Box}\ \text{res}, v \mapsto g(c)$$

We now replace every clause C by $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \mid \mathrm{AI}^{\bullet}_{\mathrm{cl}}(C)$ in order to visualize the steps of the algorithm:

$$\cfrac{\cfrac{\bot \mid R(y_c) \qquad \top \mid \neg R(u) \vee \neg Q(x_{g(u)})}{R(y_c) \mid Q(x_{g(u)})}\ \text{res}, y \mapsto c \qquad \bot \mid \neg Q(v)}{\neg Q(x_{g(c)}) \wedge R(y_c) \mid \Box}\ \text{res}, v \mapsto g(c)$$

By quantifying y_c existentially and $x_{g(c)}$ universally[1], we obtain an interpolant for $\Gamma \cup \Delta$: $\exists y_c \forall x_{g(c)}(\neg Q(x_{g_c}) \wedge R(y_c))$. Note however that $\ell[Q(g(c))] = Q(x_{g(c)})$, but $\mathrm{AI}_{\mathrm{mat}}(Q(g(c))) = Q(x_{g(u)})$. This example shows that this circumstance is not necessarily an obstacle for the correctness of this algorithm. △

⟨exa:2b⟩ **Example 6.** We consider a resolution refutation of the initial clause sets $\Gamma = \{R(c), P(c)\}$ and $\Delta = \{\neg R(u) \vee \neg Q(g(u)), \neg P(v) \vee Q(g(v))\}$:

$$\cfrac{\cfrac{\neg R(u) \vee \neg Q(g(u)) \qquad R(c)}{\neg Q(g(c))}\ \text{res}, u \mapsto c \qquad \cfrac{\neg P(v) \vee Q(g(v)) \qquad P(c)}{Q(g(c))}\ \text{res}, v \mapsto c}{\Box}\ \text{res}$$

We now again display $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \mid \mathrm{AI}^{\bullet}_{\mathrm{cl}}(C)$ for every clause C of the refutation:

$$\cfrac{\cfrac{\top \mid \neg R(u) \vee \neg Q(x_{g(u)}) \qquad \bot \mid R(y_c)}{R(y_c) \mid \neg Q(x_{g(u)})}\ \text{res}, u \mapsto c \qquad \cfrac{\top \mid \neg P(v) \vee Q(x_{g(v)}) \qquad \bot \mid P(y_c)}{P(y_c) \mid Q(x_{g(v)})}\ \text{res}, v \mapsto c}{(Q(x_{g(v)}) \wedge R(y_c)) \ \vee\ (\neg Q(x_{g(u)}) \wedge P(y_c)) \mid \Box}\ \text{res}$$

Note again that here, we have that $\ell[\neg Q(g(c))] = \neg Q(x_{g(c)}) \neq \mathrm{AI}^{\bullet}_{\mathrm{cl}}(\neg Q(g(c))) = \neg Q(x_{g(u)})$ and $\ell[Q(g(c))] = Q(x_{g(c)}) \neq \mathrm{AI}^{\bullet}_{\mathrm{cl}}(Q(g(c))) = Q(x_{g(v)})$. However in this instance, it is not possible to find quantifiers for the free variables of $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(\Box)$ such that by binding them, an interpolant is produced. For the naive approach, namely to use $\exists y_c \forall x_{g(v)} \forall x_{g(u)}$ as prefix, it holds that $\Gamma \models \exists y_c \forall x_{g(v)} \forall x_{g(u)}((Q(x_{g(v)}) \wedge R(y_c)) \ \vee\ (\neg Q(x_{g(u)}) \wedge P(y_c)))$. This failure is possible as intuitively, resolution deductions are valid by virtue of the resolved literals being equal. The interpolant extraction procedure exploits this property not directly on the clauses but on the lifted clause, i.e. on $\mathrm{AI}_{\mathrm{cl}}(C)$ for a clause C. Note that by ensuring that for resolved literals l and l', it holds that $l_{\mathrm{AIcl}} = l'_{\mathrm{AIcl}}$, we can obtain an interpolant, for instance: $\exists y_c \forall x^*((Q(x^*) \wedge R(y_c)) \ \vee\ (\neg Q(x^*) \wedge P(y_c)))$. △

In order to avoid the pitfall shown in Example 6 and to generalize the indicated solution, we define a function on resolved literals calculating a substitution, which ensures that the literals in the lifted clause, which correspond to the resolved literals, are equal.

Definition 7 (au)**.** Let ι be a resolution or factorisation rule application with l and l' as resolved or factorised literals and $\sigma = \mathrm{mgu}(\iota)$.

For terms s and t where $s = \ell[l_{\mathrm{AIcl}}\sigma]|_p$ and $t = \ell[l'_{\mathrm{AIcl}}\sigma]|_p$ for some position p, we define:

$$\mathrm{au}'(s,t) \overset{\mathrm{def}}{=} \begin{cases} \bigcup_{i=1}^{n} \mathrm{au}'(s_i, t_i) & \text{if } s \text{ is grey, } s = f_s(s_1, \ldots, s_n) \text{ and} \\ & t = f_t(t_1, \ldots, t_n)^2 \\ \{z_{s'} \mapsto z_r, z_{t'} \mapsto z_r\} & \text{if } s \text{ is a lifting variable } z_{s'},\ t = z_{t'}, \text{ and} \\ & z_r = \ell[l\sigma]|_p \end{cases}$$

[1] The procedure for calculating the quantifier block is defined in Definition 30

For $\ell[l_{\mathrm{AIcl}}\sigma] = P(s_1, \dots, s_n)$ and $\ell[l'_{\mathrm{AIcl}}\sigma] = P(t_1, \dots, t_n)$, we define:

$$\mathrm{au}'(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma]) = \mathrm{au}'(P(\bar{s}), P(\bar{t})) \stackrel{\mathrm{def}}{=} \bigcup_{i=1}^{n} \mathrm{au}'(s_i, t_i)$$

$$\mathrm{au}(\iota) \stackrel{\mathrm{def}}{=} \mathrm{au}'(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma]) \qquad \triangle$$

⟨prop:tau_dom_ran⟩ **Proposition 8.** *Let ι be a resolution or factorisation rule application with l and l' as resolved or factorised literals, $\sigma = \mathrm{mgu}(\iota)$. Then $\mathrm{dom}(\mathrm{au}(\iota))$ consists exactly of the lifting variables of $\ell[l_{\mathrm{AIcl}}\sigma]$ and $\ell[l'_{\mathrm{AIcl}}\sigma]$ and $\mathrm{ran}(\mathrm{au}(\iota))$ consists exactly of the lifting variables of $\ell[l\sigma]$.*

possibly argue here why au is well-defined (but it follows more or less directly from a later lemma)

Definition 9 ($\mathrm{AI}_{\mathrm{mat}}$ and $\mathrm{AI}_{\mathrm{cl}}$)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$. $\mathrm{AI}_{\mathrm{mat}}(\pi)$ is defined to be $\mathrm{AI}_{\mathrm{mat}}(\Box)$, where $\Box$ is the empty clause derived in π.

For a clause C in π, $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ are defined inductively as follows:

Base case. If $C \in \Gamma$, $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \top$.

In any case, $\mathrm{AI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[C]$.

Resolution. If the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then let $\tau = \mathrm{au}(\iota)$ and define $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ as follows:

$\mathrm{AI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[(\mathrm{AI}_{\mathrm{cl}}(C_1)\backslash\{l_{\mathrm{AIcl}}\})\sigma]\tau \;\vee\; \ell[(\mathrm{AI}_{\mathrm{cl}}(C_2)\backslash\{l'_{\mathrm{AIcl}}\})\sigma]\tau$

1. If l is Γ-colored: $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau$
2. If l is Δ-colored: $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau$
3. If l is grey: $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} (\neg\ell[l'_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau) \;\vee\; (\ell[l_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau)$

Factorisation. If the clause C is the result of a factorisation ι of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then let $\tau = \mathrm{au}(\iota)$ and define $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ as follows:

$\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau$

$\mathrm{AI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[(\mathrm{AI}_{\mathrm{cl}}(C_1)\backslash\{l'_{\mathrm{AIcl}}\})\sigma]\tau \qquad \triangle$

1.2 Lifting the Δ-terms

Definition 10. $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ ($\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$) for a clause C is defined as $\mathrm{AI}_{\mathrm{mat}}(C)$ ($\mathrm{AI}_{\mathrm{cl}}(C)$) with the difference that in its inductive definition, every lifting $\ell[\varphi]$ for a formula or term φ is replaced by a lifting of only the Δ-terms $\ell_{\Delta}[\varphi]$. $\triangle$

[2]Note that constants are treated as function symbols of arity zero.

Remark. Many results involving $\mathrm{AI}_{\mathrm{mat}}(C)$ ($\mathrm{AI}_{\mathrm{cl}}(C)$) are valid for $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ ($\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$) in a formulation which is adapted accordingly. This can easily be seen by the following proof idea:

Let $f_1, \dots, f_n$ be all Γ-colored function or constant symbols, c a fresh constant symbol and g be a fresh n-ary function symbol. Construct a formula $t = t$ such that $t = g(t_1, \dots, t_n)$, such that $t_i = f_i(c_1, \dots, c_m)$ for $1 \leq i \leq n$ where m is the arity of f_i and $c_j = c$ for $1 \leq j \leq m$. Let $\Delta' = \Delta$ and apply the desired result to the initial clause sets Γ and Δ'.

Under this construction, every originally Γ-colored symbol is now grey which implies that $\mathrm{AI}_{\mathrm{mat}}(C) = \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ as well as $\mathrm{AI}_{\mathrm{cl}}(C) = \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$. But $\Delta \models \varphi \Leftrightarrow \Delta' \models \varphi$ for any formula φ. $\triangle$

⟨lemma:no_colored_terms⟩ **Lemma 11.** *Let C be a clause of a resolution refutation π of $\Gamma \cup \Delta$. $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ do not contain colored symbols. $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ and $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ do not contain Δ-colored symbols.*

Proof. For $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$, consider the following: In the base case of the inductive definitions of $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$, no colored symbols occur. In the inductive steps, any colored symbol which is added by σ to intermediary formulas is lifted. By Proposition 8, $\mathrm{ran}(\mathrm{au}(\iota))$ for inferences ι in π only consists of lifting variables.

For $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ and $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$, a similar argument goes through by reading colored as Δ-colored. $\square$

⟨lemma:substitute_and_lift⟩ **Lemma 12.** *Let σ be a substitution and F a formula without Φ-colored terms such that for a set of formulas Ψ, $\Psi \models F$. Then $\Psi \models \ell_{\Phi}[F\sigma]$.*

Proof. $\ell^{z}_{\Phi}[F\sigma]$ is an instance of F: σ substitutes variables either for terms not containing Φ-colored symbols or by terms containing Φ-colored symbols. For the first kind, the lifting has no effect. For the latter, the lifting only replaces subterms of the terms introduced by the substitution by a lifting variable such that the original structure of F remains invariant as it by assumption does not contain colored terms. $\square$

emma:literals_clause_only_sim⟩ **Lemma 13.** *Let λ be a literal in a clause C occurring in a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{AI}_{\mathrm{cl}}(C)$ contains a literal λ_{AIcl} such that $\lambda_{\mathrm{AIcl}} \sim \ell[\lambda]$, where $\sim$ is defined as follows:*

$$\varphi \sim \varphi' \Leftrightarrow \begin{cases} P = P' \wedge \bigwedge_{i=1}^{n} s_i \sim s'_i & \text{if } \varphi = P(s_1, \dots, s_n) \text{ and } \varphi' = P'(s'_1, \dots, s'_n) \\ f = f' \wedge \bigwedge_{i=1}^{n} s_i \sim s'_i & \text{if } \varphi = f(s_1, \dots, s_n) \text{ and } \varphi' = f'(s'_1, \dots, s'_n) \\ x = x' & \text{if } \varphi, \varphi' \text{ are non-lifting variables, } \varphi = x \text{ and } \varphi' = x' \\ \top & \text{if } \varphi, \varphi' \text{ are lifting variables of the same color} \end{cases}$$

For resolved or factorised literals λ of an inference ι with $\tau = \mathrm{au}(\iota)$ we furthermore have that $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau \sim \ell[\lambda\sigma]$.

Proof. We proceed by induction on the resolution refutation.

Base case. If for a clause C either $C \in \Gamma$ or $C \in \Delta$ holds, then $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[C]$. Therefore for every literal l in C, there exists a literal l_{AIcl} in $\mathrm{AI}_{\mathrm{cl}}(C)$ such that $\ell[l] = l_{\mathrm{AIcl}}$, which implies $l_{\mathrm{AIcl}} \sim \ell[l]$.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$ such that $l\sigma = l'\sigma$ and $\tau = \mathrm{au}(\iota)$. Let λ be a literal in C_1 or C_2. Note that every literal in C is of the form $\lambda\sigma$ for some λ in C_1 or C_2. By the induction hypothesis, there is a literal in $\mathrm{AI}_{\mathrm{cl}}(C_1)$ or $\mathrm{AI}_{\mathrm{cl}}(C_2)$ respectively such that $\lambda_{\mathrm{AIcl}} \sim \ell[\lambda_{\mathrm{AIcl}}]$. If $\lambda \notin \{l, l'\}$, then $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau$ is contained in $\mathrm{AI}_{\mathrm{cl}}(C)$. Hence in any case, it remains to show that $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau \sim \ell[\lambda\sigma]$.

We perform an induction on the structure of λ_{AIcl} and λ by letting p be the position of the current term in the induction and $t_{\mathrm{AIcl}} = \lambda_{\mathrm{AIcl}}|_p$ as well as $t = \lambda|_p$.

- Suppose that t is a non-lifting variable. As by the induction hypothesis $\ell[t_{\mathrm{AIcl}}] \sim t$, t_{AIcl} is a non-lifting variable as well and $t = t_{\mathrm{AIcl}}$. But then $\ell[t_{\mathrm{AIcl}}\sigma] = \ell[t\sigma]$. Possibly $\ell[t_{\mathrm{AIcl}}\sigma]\tau \neq \ell[t_{\mathrm{AIcl}}\sigma]$, but as τ does not change the color of lifting variables, in any case $\ell[t_{\mathrm{AIcl}}\sigma]\tau \sim \ell[t\sigma]$[3]
- Suppose that t is colored term. Then $\ell[t]$ is a lifting variable and by the induction hypothesis, t_{AIcl} is one as well. Hence clearly $\ell[t_{\mathrm{AIcl}}\sigma]\tau \sim \ell[t\sigma]$.
- Suppose that t is a grey term of the form $f(s_1, \ldots, s_n)$. Then $\ell[t] = f(\ell[s_1], \ldots, \ell[s_n])$ and by the induction hypothesis, $t_{\mathrm{AIcl}} = f(r_1, \ldots, r_n)$ such that $\bigwedge_{i=1}^n r_i \sim \ell[s_i]$. By the induction hypothesis applied to the parameters of $\ell[t]$ and $\ell[t_{\mathrm{AIcl}}]$, we obtain that $\ell[r_i\sigma]\tau \sim \ell[s_i\sigma]$ for $1 \leq i \leq n$. Hence $f(\ell[r_1\sigma], \ldots, \ell[r_n\sigma]) \sim f(\ell[s_1\sigma], \ldots, \ell[s_n\sigma])$, which is nothing else than $\ell[t_{\mathrm{AIcl}}\sigma] \sim \ell[t\sigma]$. □

More elaborate but unnecessary lemma with faulty proof:

lemma:literals_clause_simgeq⟩? **Lemma 14.** *Let λ be a literal in a clause C occurring in a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{AI}_{\mathrm{cl}}(C)$ contains a literal λ_{AIcl} such that $\lambda_{\mathrm{AIcl}} \gtrsim \ell[\lambda]$, where $\gtrsim$ is defined as follows:*

$$\varphi \gtrsim \varphi' \Leftrightarrow \begin{cases} P = P' \wedge \bigwedge_{i=1}^n s_i \gtrsim s'_i & \text{if } \varphi = P(s_1, \ldots, s_n) \text{ and } \varphi' = P'(s'_1, \ldots, s'_n) \\ f = f' \wedge \bigwedge_{i=1}^n s_i \gtrsim s'_i & \text{if } \varphi = f(s_1, \ldots, s_n) \text{ and } \varphi' = f'(s'_1, \ldots, s'_n) \\ x = x' & \text{if } \varphi, \varphi' \text{ are non-lifting variables, } \varphi = x \text{ and } \varphi' = x' \\ s' \text{ is an instance of } s & \text{if } \varphi, \varphi' \text{ are lifting variables, } \varphi = z_s \text{ and } \varphi' = z_{s'} \end{cases}$$

For resolved or factorised literals λ of an inference ι with $\tau = \mathrm{au}(\iota)$ we furthermore have that $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau \gtrsim \ell[\lambda\sigma]$.

Proof. We proceed by induction on the resolution refutation.

Base case. If for a clause C either $C \in \Gamma$ or $C \in \Delta$ holds, then $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[C]$. Therefore for every literal l in C, there exists a literal l_{AIcl} in $\mathrm{AI}_{\mathrm{cl}}(C)$ such that $\ell[l] = l_{\mathrm{AIcl}}$, which implies $l_{\mathrm{AIcl}} \gtrsim \ell[l]$.

Resolution. If the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then let $\tau = \mathrm{au}(\iota)$. Let λ be a literal in C_1 or C_2. Note that every literal in C is of the form $\lambda\sigma$. By the induction hypothesis, there is a literal in $\mathrm{AI}_{\mathrm{cl}}(C_1)$ or $\mathrm{AI}_{\mathrm{cl}}(C_2)$ respectively such that $\lambda_{\mathrm{AIcl}} \gtrsim \ell[\lambda_{\mathrm{AIcl}}]$. If $\lambda \notin \{l, l'\}$, then $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau$ is contained in $\mathrm{AI}_{\mathrm{cl}}(C)$. Hence in any case, it remains to show that $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau \gtrsim \ell[\lambda\sigma]$.

We perform an induction on the structure of λ_{AIcl} and λ by letting p be the position of the current term in the induction and $t_{\mathrm{AIcl}} = \lambda_{\mathrm{AIcl}}|_p$ as well as $t = \lambda|_p$.

- Suppose that t is a non-lifting variable. As by the induction hypothesis $\ell[t_{\mathrm{AIcl}}] \gtrsim t$, t_{AIcl} is a non-lifting variable as well and $t = t_{\mathrm{AIcl}}$. But then $\ell[t_{\mathrm{AIcl}}\sigma] = \ell[t\sigma]$. If τ is trivial on $\ell[t_{\mathrm{AIcl}}\sigma]$, we are done as then $\ell[t_{\mathrm{AIcl}}\sigma]\tau = \ell[t\sigma]$, so assume that it is not.
  But by the definition of au, the substitutions in τ only update lifting variables to correspond to the terms in the clause of the actual resolution derivation. More formally, $\ell[t_{\mathrm{AIcl}}\sigma]\tau = z_s$ for some term s implies that $\ell[\lambda\sigma]|_p = z_s$, but then $z_s = t$. this argument only holds for terms in the resolved literals, see remark in lemma statement
  outsource this thought to lemma after definition of au in case needed elsewhere

[3]Actually $\ell[t_{\mathrm{AIcl}}\sigma]\tau = \ell[t\sigma]$ does hold as τ "resets" lifting variables to the actual terms.

- Suppose that t is colored term. Then $\ell[t]$ is a lifting variable and by the induction hypothesis, t_{AIcl} is one as well such that $\ell[t]$ is an instance of t_{AIcl}. As lifting variables are not affected by the unifications occurring in resolution derivations, we only need to consider modifications by means of τ. But as we have seen in the previous case, if τ substitutes $\ell[t_{\mathrm{AIcl}}\sigma]$, then it does so by t.

  lemma

  Hence we obtain that $\ell[t_{\mathrm{AIcl}}\sigma]\tau \gtrsim \ell[t\sigma]$.

- Suppose that t is a grey term of the form $f(s_1, \dots, s_n)$. Then $\ell[t] = f(\ell[s_1], \dots, \ell[s_n])$ and by the induction hypothesis, $t_{\mathrm{AIcl}} = f(r_1, \dots, r_n)$ such that $\bigwedge_{i=1}^{n} r_i \gtrsim \ell[s_i]$. By the induction hypothesis applied to the parameters of $\ell[t]$ and $\ell[t_{\mathrm{AIcl}}]$, we obtain that $\ell[r_i\sigma]\tau \gtrsim \ell[s_i\sigma]$ for $1 \le i \le n$. Hence $f(\ell[r_1\sigma], \dots, \ell[r_n\sigma]) \gtrsim f(\ell[s_1\sigma], \dots, \ell[s_n\sigma])$, which however is nothing else than $\ell[t_{\mathrm{AIcl}}\sigma] \gtrsim \ell[t\sigma]$.

Factorisation. If the clause C is the result of a factorisation, then we can argue analogously as for resolution. □

d_literal_like_lifted_literal⟩ **Lemma 15.** *Let l be a resolved or factorised literal of a resolution derivation step ι employing the unifier σ such that $l\sigma = l'\sigma$ and let $\tau = \mathrm{au}(\iota)$. Then $\ell[l_{\mathrm{AIcl}}\sigma]\tau = \ell[l\sigma]$.*

Proof. By Lemma 13, we obtain that $\ell[l_{\mathrm{AIcl}}\sigma]\tau \sim \ell[l\sigma]$. Note that the $\sim$-relation guarantees that pairs of predicates and terms in this relation are equal up to the index of their lifting variables. Hence it only remains to show that the lifting variables of $\ell[l_{\mathrm{AIcl}}\sigma]\tau$ and $\ell[l\sigma]$ match. But the definition of au, τ substitutes any lifting variable at position p of $\ell[l_{\mathrm{AIcl}}\sigma]$ by the lifting variable $\ell[l\sigma]|_p$. □

lemma:resolved_literals_equal⟩ **Lemma 16.** *Let l and l' be the resolved or factorised literals of a resolution derivation step ι employing the unifier σ such that $l\sigma = l'\sigma$ and let $\tau = \mathrm{au}(\iota)$. Then $\ell[l_{\mathrm{AIcl}}\sigma]\tau = \ell[l'_{\mathrm{AIcl}}\sigma]\tau$.*

Proof. By Lemma 15, we obtain that $\ell[l_{\mathrm{AIcl}}\sigma]\tau = \ell[l\sigma]$ and $\ell[l'_{\mathrm{AIcl}}\sigma]\tau = \ell[l'\sigma]$. But due to $l\sigma \equiv l'\sigma$, it holds that $\ell[l\sigma] = \ell[l'\sigma]$. □

⟨lemma:gamma_entails_aide⟩ **Lemma 17.** *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then for clauses C in π, $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.*

Proof. We proceed by induction of the strengthening $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_\Gamma)$[4].

Base case. For $C \in \Gamma$, $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_\Gamma) = \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C) = \ell_\Delta[C] = C$, so $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_\Gamma)$. Otherwise $C \in \Delta$ and hence $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = \top$.

Resolution. Suppose the last rule application is an instance ι of resolution. Then it is of the following form:

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad l\sigma = l'\sigma$$

Let $\tau = \mathrm{au}(\iota)$. We introduce the following abbreviations:

$\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* = \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma) \backslash \{(l_{\mathrm{AIcl}^\Delta})_\Gamma\}$

$\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^* = \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma) \backslash \{\neg(l'_{\mathrm{AIcl}^\Delta})_\Gamma\}$

Note that $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C) = \ell_\Delta[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell_\Delta[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$.

[4] Recall that as in Lemma **??**, D_Φ denotes the clause created from the clause D by removing all literals which are not contained $\mathrm{L}(\Phi)$.

Employing these, the induction hypothesis yields $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* \vee (l_{\mathrm{AIcl}^\Delta})_\Gamma$ as well as $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^* \vee \neg(l'_{\mathrm{AIcl}^\Delta})_\Gamma$. By Lemma 11, $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_i)$ and $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_i)$ for $i \in \{1,2\}$ do not contain Δ-colored symbols. Hence by Lemma 12, pulling the lifting inwards using Lemma 1 and applying τ, we obtain:

$$\Gamma \overset{(\circ)}{\models} \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[(l_{\mathrm{AIcl}^\Delta})_\Gamma\sigma]\tau$$

$$\Gamma \overset{(*)}{\models} \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau \vee \neg\ell[(l'_{\mathrm{AIcl}^\Delta})_\Gamma\sigma]\tau$$

We continue by a case distinction on the color of l:

1. Suppose that l is Γ-colored. Then $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau$. As l is Γ-colored, $(l_{\mathrm{AIcl}^\Delta})_\Gamma = l_{\mathrm{AIcl}^\Delta}$ and as $l\sigma = l'\sigma$, also $(l'_{\mathrm{AIcl}^\Delta})_\Gamma = l'_{\mathrm{AIcl}^\Delta}$. By Lemma 16, $\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau = \ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau$. Hence we can perform a resolution step on $(\circ)$ and $(*)$ to arrive at $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$. This is however by Lemma 1 nothing else than $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.
2. Suppose that l is Δ-colored. Then $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau$. As l and l' are Δ-colored, $(\circ)$ and $(*)$ reduce to $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau$ and $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$ respectively. These however imply that $\Gamma \models (\ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau) \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$, which in turn is nothing else than $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.
3. Suppose that l is grey. Then $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = (\neg\ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau) \vee (\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau)$.
   Let M be a model of Γ. Suppose that $M \models \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ as otherwise we are done. Hence $M \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau$ and $M \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$ and $(\circ)$ and $(*)$ reduce to $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau$ and $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau$ respectively. As by Lemma 16 $\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau = \ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau$, a case distinction on the truth value of $\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau$ in M shows that $M \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$.

Factorisation. Suppose the last rule application is an instance of factorisation. Then it is of the following form:

$$\frac{C_1 : l \vee l' \vee D}{C : (l \vee D)\sigma} \qquad \sigma = \mathrm{mgu}(l, l')$$

Let $\tau = \mathrm{au}(\iota)$. We introduce the abbreviation $\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* \overset{\mathrm{def}}{=} \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma) \backslash \{(l_{\mathrm{AIcl}})_\Gamma, (l'_{\mathrm{AIcl}})_\Gamma\}$ and express the induction hypothesis as follows: $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* \vee (l_{\mathrm{AIcl}})_\Gamma \vee (l'_{\mathrm{AIcl}})_\Gamma$. By Lemma 11, Lemma 12 and Lemma 1 and after applying τ to the induction hypothesis, we obtain that $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[(l_{\mathrm{AIcl}})_\Gamma\sigma]\tau \vee \ell[(l'_{\mathrm{AIcl}})_\Gamma\sigma]\tau$.

However by Lemma 16, $\ell[(l_{\mathrm{AIcl}})_\Gamma\sigma]\tau = \ell[(l'_{\mathrm{AIcl}})_\Gamma\sigma]\tau$, hence we can perform a factorisation step to arrive at $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[(l_{\mathrm{AIcl}})_\Gamma\sigma]\tau$. This however is nothing else than $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$. □

As we have just seen, the formula $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ now satisfies one condition of interpolants. Using this, we are able to formulate a result on one-sided interpolants, which are defined as follows:

Definition 18. Let Γ and Δ be sets of first-order formulas. A *one-sided interpolant* of Γ and Δ is a first-order formula I such that

1. $\Gamma \models I$
2. $\mathrm{L}(I) \subseteq \mathrm{L}(\Gamma) \cap \mathrm{L}(\Delta)$ $\triangle$

Proposition 19. *Let Γ and Δ be sets of first-order formulas such that $\Gamma \cup \Delta$ is unsatisfiable. Then there is a one-sided interpolant of Γ and Δ which is a Π_1 formula.*

Proof. Let π be a resolution refutation of $\Gamma \cup \Delta$. By Lemma 17, $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(\pi) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(\pi)$, or in other words $\Gamma \models \forall x_1 \dots \forall x_n\, \mathrm{AI}^{\Delta}_{\mathrm{mat}}(\pi) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(\pi)$, where $x_1, \dots, x_n$ are the Δ-lifting variables occurring in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(\pi) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(\pi)$. By Lemma 11, the formula $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(\pi) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(\pi)$ does not contain Δ-colored symbols.

Let $y_1, \dots y_m$ be the Γ-lifting variables of $\ell^y_\Gamma[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(\pi) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(\pi)]$ and

$$I = \forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m \ell^y_\Gamma[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(\pi) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(\pi)].$$

Note that I does not contain any Γ-terms. As $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(\pi) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(\pi)$ contains witness terms for every existential quantifier in I with respect to Γ, $\Gamma \models I$. Hence I is a Π_1 formula which is a one-sided interpolant for $\Gamma \cup \Delta$. $\square$

1.3 Auxiliary sets

Definition 20 ($\mathrm{AI}_{\mathrm{col}}$)**.** The set of colored literals with respect to a clause C in a resolution derivation is defined as follows:

Base case. For $C \in \Gamma \cup \Delta$, $\mathrm{AI}_{\mathrm{col}}(C) \stackrel{\mathrm{def}}{=} \varnothing$.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$ and $\tau = \mathrm{au}(\iota)$. Then:

$\mathrm{AI}_{\mathrm{col}}(C) \stackrel{\mathrm{def}}{=} \{\ell[\varphi\sigma]\tau \mid \varphi \in \mathrm{AI}'_{\mathrm{col}}(C)\}$, where

$$\mathrm{AI}'_{\mathrm{col}}(C) \stackrel{\mathrm{def}}{=} \begin{cases} \mathrm{AI}_{\mathrm{col}}(C_1) \cup \mathrm{AI}_{\mathrm{col}}(C_2) \cup \{l_{\mathrm{AIcl}}, l'_{\mathrm{AIcl}}\} & \text{if } l \text{ is a colored literal} \\ \mathrm{AI}_{\mathrm{col}}(C_1) \cup \mathrm{AI}_{\mathrm{col}}(C_2) & \text{if } l \text{ is a grey literal} \end{cases}$$

Factorisation. If the clause C is the result of a factorisation of C_1, then $\mathrm{AI}_{\mathrm{col}}(C) \stackrel{\mathrm{def}}{=} \{\ell[\varphi\sigma]\tau \mid \varphi \in \mathrm{AI}_{\mathrm{col}}(C_1)\}$. $\triangle$

Definition 21 (AI_*)**.** For a clause C, $\mathrm{AI}_*(C)$ denotes $\mathrm{AI}_{\mathrm{mat}}(C) \cup \mathrm{AI}_{\mathrm{cl}}(C) \cup \mathrm{AI}_{\mathrm{col}}(C)$. $\triangle$

Remark. Let l be a literal in a clause in $\Gamma \cup \Delta$. Then for a clause C in a resolution refutation of $\Gamma \cup \Delta$, $\mathrm{AI}_*(C)$ contains at least one literal derived from l. Furthermore, no literal is removed from $\mathrm{AI}_*(C)$ during the derivation but only potentially moved from one of its sets to another one. $\triangle$

For a related definition for non-lifted literals, we introduce the following:

Definition 22. $\Lambda(C)$:

Base case: $C \in \Gamma \cup \Delta$: $\Lambda(C) \stackrel{\text{def}}{=} C$

Resolution: $\Lambda(C) \stackrel{\text{def}}{=} \{\ell[\varphi\sigma]\tau \mid \varphi \in \Lambda(C_1) \cup \Lambda(C_2)\}$

Factorisation: $\Lambda(C) \stackrel{\text{def}}{=} \{\ell[\varphi\sigma]\tau \mid \varphi \in \Lambda(C_1)\}$ // here, we could merge the factorised literals if we merge their arrows, but this is not necessary

// An alternate definition would be to collect all colored literal (like $\mathrm{AI}_{\mathrm{col}}$) and all of the ones in the interpolant (like $\mathrm{AI}_{\mathrm{mat}}$), then define $\Lambda(C) \stackrel{\text{def}}{=} \Lambda_{\mathrm{col}} \cup \Lambda_{\mathrm{mat}} \cup C$. △

⟨lemma:Lambda_vs_aiany⟩ **Lemma 23.** $\mathrm{AI}_*(C)$ *contains the same literals as* $\Lambda(C)$.

1.4 Definition of arrows

Remark. In the Definition 24 below, the set of occurrences of a certain color of a variable is required. In lifted clauses, it the structure of the term is in general not known.

However it is possible to algorithmically keep track of the required sets by complying to the following ideas:

- Build sets of occurrences in original clauses.
- If in a resolution or factorisation step with unifier σ, x occurs grey in $y\sigma$, then the contents of set of grey (colored) occurrences of y is added to the set grey (colored) occurrences of x. y does not occur in any further derivation step.
- If in a resolution or factorisation step with unifier σ, x occurs colored in $y\sigma$, then the contents of the set of grey and colored occurrences of y is added to the set of colored occurrences of x. △

Now we define the arrows which establish the required ordering on the lifting variables in the interpolant:

Definition 24. We define a directed graph G_C for every clause C of the derivation. The nodes are of the form $l.tp$, where l denotes a literal and tp a position of a term in l, which is not contained in a colored term. The node $l.tp$ in a graph G_C refers to any literal in $\mathrm{AI}_*(C)$, which is a descendant of l. The position will usually just be denoted by p or q as abbreviation of $l.tp$.

Note that term positions are well defined since arcs do not point into colored terms and are hence not removed by liftings and in the course of the derivation.

⟨def:arrows⟩

Base case. For $C \in \Gamma \cup \Delta$, we define G_C to be the empty graph.

Resolution. If the clause C is the result of a resolution or factorisation step ι of the clauses $\bar{C}$ with $\sigma = \mathrm{mgu}(\iota)$:

The first kind of arrows deals with introductions of colored terms into terms of other colors:

$\mathcal{A}_1 \stackrel{\text{def}}{=} \{(p, q) \mid$ maximal colored term t occurs in $x\sigma$ for some variable x, p grey occurrence of z_t in $\mathrm{AI}_*(C)$, q maximal colored term containing colored occurrence of x, where the color of x is different from the color of t in $\bar{C}$ }

this works in AI$^\Delta$, possibly not in AI

The second kind of arrows handles unifications, where mixed-colored terms occur in the range of the substitution:

$\mathcal{A}_2 \stackrel{\text{def}}{=} \{(p, q) \mid$ maximal Φ-term t occurs in maximal Ψ-term s in $x\sigma$ for some variable x, p grey occurrence of t in C, q grey occurrence of x or maximal colored term containing colored occurrence of x in $\bar{C}\}$

$G_C \stackrel{\text{def}}{=} G_{C_1} \cup G_{C_2} \cup \mathcal{A}_1 \cup \mathcal{A}_2$ $\triangle$

Definition 25 ($\rightsquigarrow$)**.** For terms s and t and a clause C, $s \rightsquigarrow_{G_C} t$ holds if there is some (p, q) in the edge set of G_C such that s is the term at p and t is the term at q. $\triangle$

1.5 Existence of arrows

⟨lemma:color_change⟩ **Lemma 26.** *Let π be a resolution refutation of $\Gamma \cup \Delta$ and $\bar{C}$ be the clauses used in a resolution or factorisation step ι with $\sigma = \mathrm{mgu}(\iota)$. Then if a variable x is a color-changing variable*[5] *in $\Lambda\sigma_{(0,i)}$, x also occurs grey in $\Lambda\sigma_{(0,i)}$.*

Proof. We proceed by induction. Note that in the initial clause sets, no foreign colored terms occur.

We consider a resolution or factorisation step. We perform a nested induction over the construction steps of $\sigma = \sigma_1 \cdots \sigma_n$ with $\bar{C}$ as induction start.

Suppose that x does not occur grey in $\Lambda\sigma_{(0,i-1)}$ as otherwise we are done. We show that if a variable x occurs in a single-colored Φ-term in $\Lambda\sigma_{(0,i)}$, then (1) it does so in $\Lambda\sigma_{(0,i-1)}$ or (2) there is a color-changing variable y in $\Lambda\sigma_{(0,i-1)}$ such that x occurs grey in $y\sigma_i$. Consider the situations which produce a single-colored Φ-term containing a variable:

- Suppose a single-colored Φ-colored term containing x is present in $\Lambda\sigma_{(0,i-1)}$. Then it is as well in $\Lambda\sigma_{(0,i)}$.
- Suppose that a variable y occurs a single-colored Φ-term in $\Lambda\sigma_{(0,i-1)}$ such that x occurs grey in $y\sigma_i$. Suppose furthermore that x does not occur in a single-colored Φ-term in $\Lambda\sigma_{(0,i-1)}$ as otherwise we are done. As by assumption it does not occur grey in $\Lambda\sigma_{(0,i-1)}$, x only occurs in single-colored Ψ-terms in $\sigma_{(0,i-1)}$. But as x occurs grey in $y\sigma_i$, there must be an occurrence $\hat{y}$ of y in a resolved or factorised literal, say $l\sigma_{(0,i-1)}$, such that for the other resolved or factorised literal l', $l'\sigma_{(0,i-1)}|_{\hat{y}}$ is a subterm where x occurs grey. But as $l'\sigma_{(0,i-1)}|_{\hat{y}}$ is contained in a single-colored Ψ-term, so is $l\sigma_{(0,i-1)}$, hence y is a color-changing variable in $\Lambda\sigma_{(0,i-1)}$.
- Suppose that a variable y occurs in $\Lambda\sigma_{(0,i-1)}$ such that x occurs in a single-colored Φ-term in $y\sigma_i$. There must be an occurrence of $y\sigma_i$ in $\Lambda\sigma_{(0,i-1)}$, but this is nothing else than single-colored Φ-term containing x.

[5] Recall that a variable is a color-changing if it occurs both in a single-colored Γ-term and a single-colored Δ-term

Suppose now that x occurs in $\Lambda\sigma_{(0,i)}$ in a single-colored Φ-term as well as in a single-colored Ψ-term. If this is the case in $\Lambda\sigma_{(0,i-1)}$, then by the induction hypothesis, x occurs grey in $\Lambda\sigma_{(0,i-1)}$ and consequently also in $\Lambda\sigma_{(0,i)}$.

If otherwise x does not occur in a single-colored Φ- or Ψ-term in $\Lambda\sigma_{(0,i-1)}$, then by the reasoning given above, there is a color-changing variable y in $\Lambda\sigma_{(0,i-1)}$ such that x occurs grey in $y\sigma_i$. By the induction hypothesis, then y occurs grey in $\Lambda\sigma_{(0,i-1)}$, which directly implies that x occurs grey in $\Lambda\sigma_{(0,i)}$. □

⟨lemma:lft_var_occurs_grey⟩ **Lemma 27.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. If in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ a Γ-term $t[x_s]$ contains a Δ-lifting variable x_s, then x_s occurs grey in $\mathrm{AI}^{\Delta}_{*}(C)$.*

Proof. Note that it suffices to show that at the derivation step which introduces s as subterm of $t[s]$, x_s occurs grey in $\mathrm{AI}^{\Delta}_{*}(C)$ as any potential later modification of x_s is only performed by the substitution τ. However τ is applied globally in AI^{Δ}_{*}, so it affects each occurrence of x_s in the same manner.

Note that if a Γ-term containing a Δ-lifting variable occurs in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$, the corresponding non-lifted term in $\Lambda(C)$ is a Γ-term containing a Δ-term. Note also that if a term occurs grey in $\Lambda(C)$, a corresponding term occurs grey in $\mathrm{AI}^{\Delta}_{*}(C)$ (cf. 23).

We proceed by induction. Note that for $C \in \Gamma \cup \Delta$, no Δ-lifting variable occurs in a Γ-term in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.

For the induction step, suppose that the condition holds for the clauses $\bar{C}$ used in a resolution or factorisation step ι. Let $\sigma = \mathrm{mgu}(\iota)$. We continue by induction over the construction steps of $\sigma = \sigma_1 \cdots \sigma_n$ and consider the situations which produce Δ-terms in Γ-terms:

- Suppose a maximal colored single-colored Γ-term $t[u]$ in $\Lambda\sigma_{(0,i-1)}$ contains a variable u such that a Δ-term s' occurs grey in $u\sigma_i$ such that $s'\sigma_{(i+1,n)} = s$.

  We assume that u does not occur grey in $\Lambda\sigma_{(0,i-1)}$ as otherwise we are done. If u occurs in $\Lambda\sigma_{(0,i-1)}$ in a single-colored Δ-term, then by Lemma 26, x occurs grey in $\Lambda\sigma_{(0,i)}$ and we are done as well.

  Therefore suppose that u only occurs in single-colored Γ-terms in $\Lambda\sigma_{(0,i-1)}$. As $u \in \mathrm{dom}(\mathrm{mgu})$, u occurs in a resolved or factorised literal, say at $\hat{u}$ in $l\sigma_{(0,i-1)}$. The other resolved or factorised literal $l'\sigma_{(0,i-1)}$ contains a grey occurrence of s' at the subterm $l'\sigma_{(0,i-1)}$. But as $l\sigma_{(0,i-1)}|_{\hat{u}}$ and $l'\sigma_{(0,i-1)}|_{\hat{u}}$ agree on the prefix, s' occurs in a single-colored Γ-term in $l'\sigma_{(0,i-1)}$. So by the induction hypothesis, s' occurs grey in $\Lambda\sigma_{(0,i-1)}$. Note that if s' is introduced by $\sigma_{(0,i-1)}$, then due to $l\sigma|_{\hat{u}} = s$, σ introduces a grey occurrence of s, which in the corresponding literal in AI^{Δ}_{*} is lifted to yield x_s, in which case we are done.

  Otherwise s' has a predecessor s'' in C_1 or C_2 such that s'' is a Δ-term which is contained in a Γ-term and $s''\sigma_{(0,i-1)} = s'$. The lifting variable in $\mathrm{AI}^{\Delta}_{*}(C_1)$ or $\mathrm{AI}^{\Delta}_{*}(C_2)$ corresponding to s'' in general is of the form x_r with formulate a lemma about that this works
  $r \neq s$. But Lemma 15, we have that $\ell_{\Delta}[l_{\mathrm{AIcl}}\sigma]\tau = \ell_{\Delta}[l\sigma]$ for the resolved or factorised literal l with $\tau = \mathrm{au}(\iota)$. Since x_r occurs in l_{AIcl} and lifting variables are only modified by τ, it must be the case that $\{x_r \mapsto x_s\} \in \tau$. But then x_s occurs in $\ell_{\Delta}[l_{\mathrm{AIcl}}\sigma]\tau$, which is contained in $\mathrm{AI}^{\Delta}_{\mathrm{col}}(C)$ and hence in $\mathrm{AI}^{\Delta}_{*}(C)$.

- Suppose that a variable u occurs in C_1 or C_2 either grey or in a maximal colored single-colored Γ-term such that $u\sigma$ contains a multi-colored Γ-term t.

  Then u occurs in a resolved or factorised literal $\lambda\sigma_{(0,i-1)}$ at $\hat{u}$ such that at the other resolved or factorised literal $\lambda'\sigma_{(0,i-1)}$, $\lambda'\sigma_{(0,i-1)}|_{\hat{u}} = t$. But then by the induction hypothesis, $\mathrm{AI}^{\Delta}_{*}(C)$ contains grey occurrences for every lifting variable in t and as t occurs in the resolved or factorised literal, but a similar reasoning as given in the other case, τ substitutes these lifting variables to exactly the ones occurring in $t\sigma$. □

Example 28. $R(h(y)) \vee P(f(y))$; $\neg P(f(x_{g(x)})) \vee Q(x_{g(x)})$ such that in the actual clause, it is $g(a)$ and not $g(x)$ any more. Then $\{x_{g(x)} \mapsto x_{g(a)}\} \in \tau$ as desired. △

⟨lemma:arrow_for_lft_var⟩ **Lemma 29.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. If in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ a maximal colored Γ-term $t[x_s]$ contains a Δ-lifting variable x_s, then $x_s \rightsquigarrow_{G_C} t[x_s]$.*

Proof. We proceed by induction. Note that for $C \in \Gamma \cup \Delta$, no Δ-lifting variable occurs in a Γ-term in $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.

For the induction step, suppose that the condition holds for C_1 and C_2 which are used in a resolution or factorisation step ι. By Lemma 27, x_s occurs grey in $\mathrm{AI}^{\Delta}_{*}(C)$. Let r be the position of x_s in $\mathrm{AI}^{\Delta}_{*}(C)$. We consider the two situations which produce Δ-terms in Γ-terms:

- Suppose a maximal colored single-colored Γ-term $t[u]$ in Λ contains a variable u such that a Δ-term s occurs grey in $u\sigma$. Then $\mathcal{A}_1$ as defined in Definition 24 contains (r, q) such that $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)|_q$ is $t[x_s]$.

- Suppose that a variable u occurs in C_1 or C_2 either grey or in a maximal colored single-colored Γ-term such that $u\sigma$ contains a multi-colored Γ-term t. Then $\mathcal{A}_2$ as defined in Definition 24 contains (r, q) such that $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)|_q$ is the grey occurrence of u or the maximal colored term containing u respectively. □

1.6 Combining the results

⟨def:arrow_quantifier_block⟩ **Definition 30** (Quantifier block)**.** Let C be a clause in a resolution refutation π of $\Gamma \cup \Delta$ and $\bar{x}$ be the Δ-lifting variables and $\bar{y}$ the Γ-lifting variables occurring in $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$. $Q(C)$ denotes an arrangement of the elements of $\{\forall x_t \mid x_t \in \bar{x}\} \cup \{\exists y_t \mid y_t \in \bar{y}\}$ such that for two lifting variable z_s and z_r $z_s \rightsquigarrow_{G_C} z_r$ implies that z_s is listed before z_r. We denote $Q(\square)$ by $Q(\pi)$. △

⟨lemma:ai_vs_aide_1⟩ **Lemma 31.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. If a Γ-lifting variable occurs multiple times in $\mathrm{AI}_{*}(C)$, then the terms at the corresponding positions in $\mathrm{AI}^{\Delta}_{*}(C)$) are equal.*

Proof. We proceed by induction over the resolution refutation of $\Gamma \cup \Delta$.

Base case. For $C \in \Gamma \cup \Delta$, if $\mathrm{AI}_{*}(C)$ contains a Γ-lifting variable y_t at position p, then $\mathrm{AI}^{\Delta}_{*}(C)|_p = \ell_{\Delta}[t]$.

Induction step. Suppose the clause C is the result of a resolution or factorisation step ι of the clauses $\bar{C}$. Let $\sigma = \mathrm{mgu}(\iota)$ and $\tau = \mathrm{au}(\iota)$.

We first show that the property holds for newly introduced lifting variables, and then that the property is not violated in the course of the derivation.

- Suppose a Γ-lifting variable y_t is introduced in $\mathrm{AI}_*(C)$ in ι. Then σ introduces a Γ-term t at a grey position, so the corresponding term in $\mathrm{AI}_*^\Delta(C)$ is t.

  Let $\hat{y}_t$ be another occurrence of y_t in $\mathrm{AI}_*(C)$. We show that $\mathrm{AI}_*^\Delta(C)|_{\hat{y}_t} = t$ by contradiction, i.e. suppose that $\mathrm{AI}_*^\Delta(C)|_{\hat{y}_t} = s$ with $s \neq t$. Then from the inference in the derivation where y_t is introduced created up to ι, at least one variable contained in t, say u, has been substituted. But then u does not occur in the subsequent derivation, but $\mathrm{ran}(\sigma)$ contains u since it contains t. Therefore no variable occurring in t is changed from the point of its introduction up to ι.

  This however implies that if for a position p, $\mathrm{AI}_*(C)|_p = y_t$, then $\mathrm{AI}_*^\Delta(C)|_p = t$.

- Let $\hat{y}_s$ and $\dot{y}_s$ be two occurrences of a lifting variable y_s in $\mathrm{AI}_*(C)$.

  Note that if s does not contain a variable, then in the course of the derivation, the term as well as the lifting variable is not modified, in particular also not by τ. Hence suppose in the following that s does contain a variable.

  Suppose that there exist ancestors of $\hat{y}_s$ and $\dot{y}_s$ in $\mathrm{AI}_*(D)$ for $D \in \bar{C}$ which are both occurrences of y_s. Then by the induction hypothesis, $\mathrm{AI}_*^\Delta(D)|_{\hat{y}_s} = \mathrm{AI}_*^\Delta(D)|_{\dot{y}_s}$. $\hat{y}_s$ and $\dot{y}_s$ can only be modified by τ, but this affects both of them the same way. $\mathrm{AI}_*^\Delta(D)|_{\hat{y}_s}$ and $\mathrm{AI}_*^\Delta(D)|_{\dot{y}_s}$ on the other hand are modified by σ, but this again affects both terms the same way.

  Suppose that otherwise at both ancestors of $\hat{y}_s$ and $\dot{y}_s$ in $\mathrm{AI}_*(C)$ are lifting variables but at least one ancestor is y_r such that $r \neq s$. Then it is modified by τ and hence occurs in a resolved or factorised literal λ, say at p. However τ modifies it such $\mathrm{AI}_*(C)|_p = y_s$. Also all lifting variable contained in $\mathrm{AI}_*^\Delta|_p$ are contained in the resolved or factorised literal and therefore τ resets them to reflect the term in the non-lifted literal.

  Hence of both $\hat{y}_s$ and $\dot{y}_s$ are not occurrences of y_s in the clauses preceding C, in $\mathrm{AI}_*^\Delta(C)$, they both are occurrences of $\ell_\Delta^x[s]$.

  If however one of $\hat{y}_s$ and $\dot{y}_s$ is an occurrence of y_s in a preceding clause, then by a similar argument as in the other case, as s is present in C, no variable contained in s has been affected by a substitution, hence $\hat{y}_s$ and $\dot{y}_s$ are both occurrences of $\ell_\Delta^x[s]$ in $\mathrm{AI}_*^\Delta(C)$. □

⟨lemma:ai_vs_aide_2⟩ **Lemma 32.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. If a Δ-lifting variable occurs multiple times in $\mathrm{AI}_*^\Delta(C)$, then the terms at the corresponding positions in $\mathrm{AI}_*(C)$) are equal.*

Proof. (Sketch.) Note that if a Δ-lifting variable in $\mathrm{AI}_*^\Delta(C)$ is contained in a Γ-term, there is no corresponding term in $\mathrm{AI}_*(C)$. Therefore we only consider Δ-lifting variables in $\mathrm{AI}_*^\Delta(C)$ at grey positions.

We proceed by induction.

For $C \in \Gamma \cup \Delta$, the condition clearly holds.

Induction step:

Similar as in the proof of Lemma 32, if a colored term t is introduced, then none of its variables has been substituted, so all present occurrences of x_t correspond to t.

If a lifting variable contains no variables, then it does not change, hence this case is fine.

If a lifting variable is changed by τ, then it is contained in the resolved or factorised literal, so it is changed in $\mathrm{AI}_*(C)$ as well as in $\mathrm{AI}_*^\Delta(C)$ and in both instances reset to the correct term. This includes the case where in C_1 and C_2 there are lifting variables such that in C, they are the same lifting variable. □

a:gamma_entails_quantified_ai⟩ **Lemma 33.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then* $\Gamma \models Q(C)(\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C))$.

Proof. By Lemma 17, we obtain that $\Gamma \models \forall x_1 \cdots \forall x_n (\mathrm{AI}_{\mathrm{mat}}^\Delta(C) \vee \mathrm{AI}_{\mathrm{cl}}^\Delta(C))$, where $x_1, \dots, x_n$ are the Δ-lifting variables occurring in $\mathrm{AI}_{\mathrm{mat}}^\Delta(C) \vee \mathrm{AI}_{\mathrm{cl}}^\Delta(C)$. Note that $\mathrm{AI}_{\mathrm{mat}}^\Delta(C) \vee \mathrm{AI}_{\mathrm{cl}}^\Delta(C)$ and $\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$ are structurally equal in the sense that for a position p in the latter, the first has a related term: For a grey term, it is the same grey term. For a Δ-lifting variable, it is also a Δ-lifting variable but possibly a different one and for Γ-lifting variables, the first has a maximal colored Γ-term.

Lemma 31 and Lemma 32 give the following:

1. If a Γ-lifting variable occurs multiple times in $\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$, then the terms at the corresponding positions in $\mathrm{AI}_{\mathrm{mat}}^\Delta(C) \vee \mathrm{AI}_{\mathrm{cl}}^\Delta(C))$ are equal.
2. If a Δ-lifting variable occurs multiple times in $\mathrm{AI}_{\mathrm{mat}}^\Delta(C) \vee \mathrm{AI}_{\mathrm{cl}}^\Delta(C))$, then the Δ-lifting variables at the corresponding positions in $\mathrm{AI}_{\mathrm{mat}}^\Delta(C) \vee \mathrm{AI}_{\mathrm{cl}}^\Delta(C))$ are equal.

These conditions allow for inserting the respective terms of $\mathrm{AI}_{\mathrm{mat}}^\Delta(C) \vee \mathrm{AI}_{\mathrm{cl}}^\Delta(C))$ as witness terms for the Γ-lifting variables in $\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$ to yield a formula where the universal quantifiers are either equally or less restrictive than in $\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$, but otherwise the formula coincides.

In the witness terms for the Γ-lifting variables extracted from $\mathrm{AI}_{\mathrm{mat}}^\Delta(C) \vee \mathrm{AI}_{\mathrm{cl}}^\Delta(C)$, in general Δ-lifting variables occur. By Lemma 29, for every lifting variable x_s in $t[x_s]$, we have that $x_s \rightsquigarrow_{G_C} t[x_s]$, hence by Definition 30, the lifting variables are quantified in some order such that the lifting variable for x_s is quantified before the lifting variable for $t[x_s]$ is. This shows that $\Gamma \models Q(C)(\mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C))$. □

⟨lemma:ai_symmetry⟩ **Lemma 34.** *Let π be a refutation of $\Gamma \cup \Delta$ and $\hat{\pi}$ be π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then* $Q(\pi)\,\mathrm{AI}_{\mathrm{mat}}(\pi) \Leftrightarrow \neg Q(\hat{\pi})\,\mathrm{AI}_{\mathrm{mat}}(\hat{\pi})$.

Proof. Let C be a clause in π and $\hat{C}$ the corresponding clause in $\hat{\pi}$. Note that $\mathrm{AI}_{\mathrm{cl}}$ is defined irrespective of the coloring, so $\mathrm{AI}_{\mathrm{cl}}(C) \equiv \mathrm{AI}_{\mathrm{cl}}(\hat{C})$.

Consider furthermore that liftings variables of C and $\hat{C}$ only differ in the variable symbol, but not in the index, and that the quantifier type of any given lifting variable in C is exactly contrary to the corresponding one in $\hat{C}$. Hence for any formula ϕ, $Q(C)\neg\phi \Leftrightarrow \neg Q(\hat{C})\phi$.

It remains to show that $\mathrm{AI}_{\mathrm{mat}}(C) \Leftrightarrow \neg \mathrm{AI}_{\mathrm{mat}}(\hat{C})$, which we do be induction:

Base case. If $C \in \Gamma$, then $\mathrm{AI}_{\mathrm{mat}}(C) = \bot \Leftrightarrow \neg\top \Leftrightarrow \neg \mathrm{AI}_{\mathrm{mat}}(\hat{C})$ as $\hat{C} \in \Delta$. The case for $C \in \Delta$ can be argued analogously.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$ and $\tau = \mathrm{au}(\iota)$.

As $\mathrm{au}(\iota)$ depends only on l_{AIcl}, l'_{AIcl} and σ and as $\mathrm{AI}_{\mathrm{cl}}(C) \equiv \mathrm{AI}_{\mathrm{cl}}(\hat{C})$, τ is the same for both π and $\hat{\pi}$.

We now distinguish the following cases:

1. l is Γ-colored:

$$\begin{aligned}
\mathrm{AI}_{\mathrm{mat}}(C) &= \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau \\
&\Leftrightarrow \neg(\neg\ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \neg\ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau) \\
&\Leftrightarrow \neg(\ell[(\neg \mathrm{AI}_{\mathrm{mat}}(C_1))\sigma]\tau \wedge \ell[(\neg \mathrm{AI}_{\mathrm{mat}}(C_2))\sigma]\tau) \\
&\Leftrightarrow \neg(\ell[\mathrm{AI}_{\mathrm{mat}}(\hat{C}_1)\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(\hat{C}_2)\sigma]\tau) \\
&= \neg \mathrm{AI}_{\mathrm{mat}}(\hat{C})
\end{aligned}$$

2. l is Δ-colored: This case can be argued analogously
3. l is grey: Note that by Lemma 16, $\ell[l_{\mathrm{AIcl}}\sigma] = \ell[l'_{\mathrm{AIcl}}\sigma]$ $(*)$.

$$\begin{aligned}
\mathrm{AI}_{\mathrm{mat}}(C) &= (\neg\ell[l'_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau) \ \vee\ (\ell[l_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau) \\
&\overset{(*)}{\Leftrightarrow} (\ell[l'_{\mathrm{AIcl}}\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau) \ \wedge\ (\neg\ell[l_{\mathrm{AIcl}}\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau) \\
&\Leftrightarrow \neg\Big((\neg\ell[l'_{\mathrm{AIcl}}\sigma]\tau \wedge \neg\ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau) \ \vee\ (\ell[l_{\mathrm{AIcl}}\sigma]\tau \wedge \neg\ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau)\Big) \\
&= \neg\Big((\neg\ell[\hat{l}'_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(\hat{C}_1)\sigma]\tau) \ \vee\ (\ell[\hat{l}_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(\hat{C}_2)\sigma]\tau)\Big) \\
&= \mathrm{AI}_{\mathrm{mat}}(\hat{C})
\end{aligned}$$

Factorisation. Suppose the clause C is the result of a factorisation ι of C_1 : $l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$ and $\tau = \mathrm{au}(\iota)$.

Then $\mathrm{AI}_{\mathrm{mat}}(C) = \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau$, so the construction is not influenced by the coloring and the induction hypothesis gives the result. □

Theorem 35. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{AI}_{\mathrm{mat}}(\pi)$ is an interpolant.*

Proof. By Lemma 33, $\Gamma \models Q(\pi)(\mathrm{AI}_{\mathrm{mat}}(\pi) \vee \mathrm{AI}_{\mathrm{cl}}(\pi))$. But as $\mathrm{AI}_{\mathrm{cl}}(\pi) = \square$, this simplifies to $\Gamma \models Q(\pi)\,\mathrm{AI}_{\mathrm{mat}}(\pi)$.

By constructing a proof $\hat{\pi}$ from π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$, we obtain by Lemma 33 that $\hat{\Gamma} \models Q(\hat{\pi})\,\mathrm{AI}_{\mathrm{mat}}(\hat{\pi})$. By Lemma 34, this however is nothing else than $\Delta \models \neg Q(\pi)\,\mathrm{AI}_{\mathrm{mat}}(\pi)$.

As furthermore by construction no colored symbols occur in $Q(\pi)\,\mathrm{AI}_{\mathrm{mat}}(\pi)$, this formula is an interpolant for $\Gamma \cup \Delta$. □