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0.1 from other pdf

`ma:lifting_order_not_relevant`⟩ **Lemma 1.** *Basically* $\ell^y_\Gamma[\ell^x_\Delta[\varphi]] = \ell^x_\Delta[\ell^y_\Gamma[\varphi]]$.

0.2 proof

Definition 2 ($\tau(\iota)$)**.** For an inference ι with $\sigma = \mathrm{mgu}(\iota)$, we define the infinite substitution $\tau(\iota)$ with $\mathrm{dom}(\tau(\iota)) = \mathrm{dom}(\sigma) \cup \{z_s \mid s\sigma \neq s\}$ as follows for a variable x:

define infinite substitutions properly and apply definition here

$$x\tau(\iota) = \begin{cases} x\sigma & x \text{ is a non-lifting variable} \\ z_{t\sigma} & x \text{ is a lifting variable } z_t \end{cases}$$

$\triangle$

Definition 3 (Incremental lifting)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$. We define $\mathrm{LI}(\pi)$ ($\mathrm{LI}_{\mathrm{cl}}(\pi)$) to be $\mathrm{LI}(\square)$ ($\mathrm{LI}_{\mathrm{cl}}(\square)$), where $\square$ is the empty clause derived in π.

Let C be a clause in π. For a literal λ in C, we denote the corresponding literal in $\mathrm{LI}_{\mathrm{cl}}(C)$ by λ_{LIcl}, which is exists by Proposition 4.

We define $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ as follows:

Base case. If $C \in \Gamma$, $\mathrm{LI}(C) \overset{\text{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{LI}(C) \overset{\text{def}}{=} \top$.

In any case, $\mathrm{LI}_{\mathrm{cl}}(C) \overset{\text{def}}{=} \ell[C]$.

Resolution. If the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then let $\tau = \tau(\iota)$ and define $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ as follows:

$\mathrm{LI}_{\mathrm{cl}}(C) \overset{\text{def}}{=} \ell[(\mathrm{LI}_{\mathrm{cl}}(C_1)\backslash\{l_{\mathrm{LIcl}}\})\tau] \ \vee\ \ell[(\mathrm{LI}_{\mathrm{cl}}(C_2)\backslash\{l'_{\mathrm{LIcl}}\})\tau]$

1. If l is Γ-colored: $\mathrm{LI}(C) \overset{\text{def}}{=} \ell[\mathrm{LI}(C_1)\tau] \vee \ell[\mathrm{LI}(C_2)\tau]$
2. If l is Δ-colored: $\mathrm{LI}(C) \overset{\text{def}}{=} \ell[\mathrm{LI}(C_1)\tau] \wedge \ell[\mathrm{LI}(C_2)\tau]$
3. If l is grey: $\mathrm{LI}(C) \overset{\text{def}}{=} (\ell[l_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_2)\tau]) \vee (\neg\ell[l'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_1)\tau])$

Factorisation. If the clause C is the result of a factorisation step ι of C_1 : $l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{LI}(C) \overset{\text{def}}{=} \ell[\mathrm{LI}(C_1)\tau(\iota)]$ and $\mathrm{LI}_{\mathrm{cl}}(C) \overset{\text{def}}{=} \ell[(\mathrm{LI}_{\mathrm{cl}}(C_1)\backslash\{l'_{\mathrm{LIcl}}\})\tau(\iota)]$. $\triangle$

⟨prop:corresponding_literal⟩ **Proposition 4.** *Every literal λ in C has a corresponding literal λ_{LIcl} in $\mathrm{LI}_{\mathrm{cl}}(C)$.*

Definition 5. $\mathrm{LI}^{\Delta}(C)$ ($\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$) for a clause C is defined as $\mathrm{LI}(C)$ ($\mathrm{LI}_{\mathrm{cl}}(C)$) with the difference that in its inductive definition, every lifting $\ell[\varphi]$ for a formula or term φ is replaced by a lifting of only the Δ-terms $\ell_{\Delta}[\varphi]$. $\triangle$

⟨lemma:gamma_proves_pide⟩ **Lemma 6.** *For a clause C of a resolution refutation of $\Gamma \cup \Delta$, $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$.*

Proof. Induction of the strengthening $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_{\Gamma})$

Base case. ✓

Resolution.

Ind hyp gives $\Gamma \models \mathrm{LI}^{\Delta}(C_1) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(D) \vee l_{\mathrm{LIcl}^{\Delta}}$ and similar for C_2.

Don't really say $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(D)$ here, we only have $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$

$\Gamma \models \mathrm{LI}^{\Delta}(C_1) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(D) \vee l_{\mathrm{LIcl}^{\Delta}}$

+ `lemma:substitute_and_lift`

$\Gamma \models \ell^x_\Delta[\mathrm{LI}^{\Delta}(C_1)\tau] \vee \ell^x_\Delta[\mathrm{LI}^{\Delta}_{\mathrm{cl}}(D)\tau] \vee \ell^x_\Delta[l_{\mathrm{LIcl}^{\Delta}}\tau]$

have that $l\sigma = l'\sigma$, get also that $\ell_\Delta[l_{\mathrm{LIcl}^{\Delta}}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}^{\Delta}}\tau]$. Proof: Suppose not lifted, then same. Otw. lifting variables, but then for p pos of lft var z_t in $l_{\mathrm{LIcl}^{\Delta}}$, $l|_p$ is t after applying τ. Hence have z_t for both.

- supp Γ resolved literals not removed due to coloring. literals are equal, can do resolution. get everything in disjunction
- supp Δ literals removed. have: either one of the clauses, or else both interpolant pairs
- supp grey. as literals same, if l, then $\neg l$ not, so get rest there and vice versa

Factorisation.

Ind hyp gives $\Gamma \models \mathrm{LI}^{\Delta}(C_1) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(D) \vee l_{\mathrm{LIcl}^{\Delta}} \vee l'_{\mathrm{LIcl}^{\Delta}}$

Don't really say $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(D)$ here, we only have $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$

also have that $l\sigma = l'\sigma$ implies $\ell_\Delta[l_{\mathrm{LIcl}}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}}\tau]$.

+ `lemma:substitute_and_lift`:

$\Gamma \models \ell^x_\Delta[\mathrm{LI}^{\Delta}(C_1)\tau] \vee \ell^x_\Delta[\mathrm{LI}^{\Delta}_{\mathrm{cl}}(D)\tau] \vee \ell^x_\Delta[l_{\mathrm{LIcl}^{\Delta}}\tau] \vee \ell^x_\Delta[l'_{\mathrm{LIcl}^{\Delta}}\tau]$

hence can factorise here □

?⟨`def:arrow_quantifier_block`⟩? **Definition 7** (Quantifier block)**.** Let C be a clause in a resolution refutation π of $\Gamma \cup \Delta$ and $\bar{x}$ be the Δ-lifting variables and $\bar{y}$ the Γ-lifting variables occurring in $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$. $Q(C)$ denotes an arrangement of the elements of $\{\forall x_t \mid x_t \in \bar{x}\} \cup \{\exists y_t \mid y_t \in \bar{y}\}$ such that for two lifting variable z_s and z_r, if s is a subterm of r, then z_s is listed before z_r. We denote $Q(\Box)$ by $Q(\pi)$. △

Conjectured Lemma 8. $\ell[\ell[\varphi]\tau] = \ell[\varphi\tau]$.

Proof. proof by induction.
Supp constant: done.
Supp grey function: apply to children.
supp variable: $\ell[\ell[x]\tau] = \ell[x\tau]$
supp lft var: $\ell[\ell[z_t]\tau] = \ell[z_t\tau]$
supp col term t
$\ell[\ell[t]\tau] = \ell[z_t\tau] = \ell[z_{t\sigma}] = z_{t\sigma} = \ell[t\sigma] = \ell[t\tau]$

□

⟨`lemma:gamma_lifted_lide`⟩ **Lemma 9.** *For a clause C of a resolution refutation of $\Gamma \cup \Delta$, $\ell_\Gamma[\mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)] = \mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$.*

Proof. Base case.

LI^{Δ}: easy.

$\mathrm{LI}^{\Delta}_{\mathrm{cl}}$: By Lemma 1, $\ell_\Gamma[\ell_\Delta[C]] = \ell[C]$

Resolution.

IH:

$\ell_\Gamma[\mathrm{LI}^\Delta(C_1) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C_1)] = \mathrm{LI}(C_1) \vee \mathrm{LI}_{\mathrm{cl}}(C_1)$.

$\ell_\Gamma[\mathrm{LI}^\Delta(C_2) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C_2)] = \mathrm{LI}(C_2) \vee \mathrm{LI}_{\mathrm{cl}}(C_2)$.

$\mathrm{LI}^\Delta_{\mathrm{cl}}$:

$\ell_\Gamma[\mathrm{LI}^\Delta_{\mathrm{cl}}(C_1)] = \mathrm{LI}_{\mathrm{cl}}(C_1)$

$\ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}(C_1)\tau] \subseteq \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$

$\ell[\mathrm{LI}_{\mathrm{cl}}(C_1)\tau] \subseteq \mathrm{LI}_{\mathrm{cl}}(C)$

to show: $\ell^y_\Gamma[\mathrm{LI}^\Delta_{\mathrm{cl}}(C)] = \mathrm{LI}_{\mathrm{cl}}(C)$

$\ell[\ell_\Gamma[\mathrm{LI}^\Delta_{\mathrm{cl}}(C_1)]\tau] = \ell[\mathrm{LI}_{\mathrm{cl}}(C_1)\tau]$ IH + same op on both sides

new lemma above

$\ell[\ell_\Gamma[\mathrm{LI}^\Delta_{\mathrm{cl}}(C_1)]\tau] = \ell[\mathrm{LI}^\Delta_{\mathrm{cl}}(C_1)\tau]$

LI^Δ:

- Supp Γ:
  IH: $\ell_\Gamma[\mathrm{LI}^\Delta(C_1)] = \mathrm{LI}(C_1)$
  hence also: $\ell[\mathrm{LI}^\Delta(C_1)] = \mathrm{LI}(C_1)$ (by lemma: no Δ-terms in . . .)
  $+\ \tau$: $\ell[\mathrm{LI}^\Delta(C_1)]\tau = \mathrm{LI}(C_1)\tau$
  $+\ \ell$: $\ell[\ell[\mathrm{LI}^\Delta(C_1)]\tau] = \ell[\mathrm{LI}(C_1)\tau]$
  by new lemma $\ell[\mathrm{LI}^\Delta(C_1)\tau] = \ell[\mathrm{LI}(C_1)\tau]$
  hence by Lemma 1, $\ell_\Gamma[\ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau]] \subseteq \mathrm{LI}^\Delta(C)$
  hence $\ell_\Gamma[\mathrm{LI}^\Delta(C)] \subseteq \mathrm{LI}^\Delta(C)$

Factorisation.

□

Lemma 10. *For a clause C of a resolution refutation of $\Gamma \cup \Delta$, $\Gamma \models Q(C)(\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C))$.*

Proof. By Lemma 9 $\ell_\Gamma[\mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)] = \mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$.

By Lemma 6, $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$. Hence the terms in $\mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$ provide witness terms for the Γ-lifting variables in $\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$, which are existentially quantified in $Q(C)(\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C))$.

Furthermore, the ordering imposed on the quantifiers in $Q(C)$ implies that if a Δ-lifting variable x_s occurs in a witness term for a Γ-lifting variable y_r, y_r is quantified in the scope of the quantifier of x_s as s is a subterm of r. This however ensures that the witness terms are valid. □

?⟨lemma:li_symmetry⟩? **Lemma 11.** *symmetry: $Q(C)(\mathrm{LI}(C)) \Leftrightarrow Q(\hat{C})(\mathrm{LI}(\hat{C}))$.*

Proof. todo: copy from other pdf □

Theorem 12. *same as other pdf*