# Number of quantifier alternations of the extracted interpolants

In this chapter, we derive lower bounds for the number of quantifier alternations for the extraction procedure in two phases as well as the one in one phase. We arrive at the conclusion that both procedures are equal with respect to this measure and that there is a tight connection between color alternations in terms and quantifier alternations in the interpolant.

## 1 Additional notation and definitions

A literal $l$ is called a $\Phi$-literal if its predicate symbol is $\Phi$-colored.

In a literal or term $\phi$ containing a subterm $t$, $t$ is said to occur *below* a $\Phi$-symbol $s$ if in the syntax tree representation of $\phi$, there is a node labelled $s$ on the path from the root to $t$. Note that the colored symbol may also be the predicate symbol. Moreover, $t$ is said to occur *directly below* a $\Phi$-symbol if it occurs below the $\Phi$-symbol $s$ and in the syntax tree representation of $\phi$ on the path from $s$ to $t$, no nodes with labels with colored symbol occur.

### 1.1 Unification

Let $\varphi$ and $\psi$ be two terms or literals. For $\sigma = \mathrm{mgu}(\varphi, \psi)$, we denote by $\sigma_i$ for $1 \leq i \leq |\mathrm{dom}(\sigma)|$ the $i$th substitution which is added to $\sigma$ by the unification algorithm. We define $\sigma_0 \stackrel{\mathrm{def}}{=} \mathrm{id}$.

We furthermore denote the composition $\sigma_i \ldots \sigma_j$ by $\sigma_{(i,j)}$. Hence $\sigma = \sigma_{(1,\,|\mathrm{dom}(\sigma)|)} = \sigma_{(0,\,|\mathrm{dom}(\sigma)|)}$.

### 1.2 Color and quantifier alternations

In the following, we assume that the maximum max of an empty sequence is defined to be 0 and constants are treated as function symbols of arity 0. Furthermore $\perp$ is used to denote a color which is not possessed by any symbol.

**Definition 1** (Color alternation col-alt). Let $\Gamma$ and $\Delta$ be sets of formulas and $t$ be a term.

$$\mathrm{col\text{-}alt}(t) \stackrel{\mathrm{def}}{=} \mathrm{col\text{-}alt}_{\perp}(t)$$

$$\text{col-alt}_\Phi(t) \overset{\text{def}}{=} \begin{cases} 0 & t \text{ is a variable} \\ \max(\text{col-alt}_\Phi(t_1), \ldots, \text{col-alt}_\Phi(t_n)) & t = f(t_1, \ldots, t_n) \text{ is grey} \\ \max(\text{col-alt}_\Phi(t_1), \ldots, \text{col-alt}_\Phi(t_n)) & t = f(t_1, \ldots, t_n) \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_\Psi(t_1), \ldots, \text{col-alt}_\Psi(t_n)) & t = f(t_1, \ldots, t_n) \text{ is of color } \Psi, \\ & \Phi \neq \Psi \quad \triangle \end{cases}$$

**Definition 2** (Quantifier alternation quant-alt). Let $A$ be a formula.

$$\text{quant-alt}(A) \overset{\text{def}}{=} \text{quant-alt}_\perp(A)$$

$$\text{quant-alt}_Q(A) \overset{\text{def}}{=} \begin{cases} 0 & A \text{ is an atom} \\ \text{quant-alt}_Q(B) & A \equiv \neg B \\ \max(\text{quant-alt}_Q(B), & A \equiv B \circ C, \circ \in \{\wedge, \vee, \supset\} \\ \quad \text{quant-alt}_Q(C)) & \\ \text{quant-alt}_Q(B) & A \equiv QxB \\ 1 + \text{quant-alt}_{Q'}(B) & A \equiv Q'xB, Q \neq Q' \end{cases}$$

$$\triangle$$

Note that this definition of quantifier alternations handles formulas in prenex and non-prenex form.

## 2 Occurrence of terms in the interpolant

**Definition 3** (PI*). PI* is defined as PI with the difference that in PI*, all literals are considered to be grey. $\text{PI}^*_{\text{init}}$ and $\text{PI}^*_{\text{step}}$ are defined analogously. $\triangle$

Hence $\text{PI}^*_{\text{init}}$ coincides with $\text{PI}_{\text{init}}$. $\text{PI}^*_{\text{step}}$ coincides with $\text{PI}_{\text{step}}$ in case of factorisation and paramodulation inferences. For resolution inferences, the first two cases in the definition of $\text{PI}_{\text{step}}$ do not occur for $\text{PI}^*_{\text{step}}$.

PI* enjoys the convenient property that it absorbs every literal which occurs some clause:

⟨prop:every_lit_in_pi_star⟩ **Proposition 4.** *For every literal which occurs in a clause of a resolution refutation $\pi$, a respective successor occurs in $\text{PI}^*(\pi)$.*

*Proof.* By structural induction. □

Note that in PI*, we can conveniently reason about the occurrence of terms as no terms are lost throughout the extraction. However Lemma 5 allows us to transfer results about grey literals to PI:

⟨ma:grey_lits_of_pi_star_in_pi⟩ **Lemma 5.** *For every clause $C$ of a resolution refutation, every grey literal, which occurs in $\text{PI}^*(C)$, also occurs in $\text{PI}(C)$.*

*Proof.* Note that $\text{PI}_{\text{init}}$ and $\text{PI}^*_{\text{init}}$ coincide and $\text{PI}_{\text{step}}$ and $\text{PI}^*_{\text{step}}$ only differ for resolution inferences. More specifically, they only differ on resolution inferences, where the resolved literal is colored. However here, no grey literals are removed but only colored ones. □

⟨lemma:grey_lits_all_in_PI⟩ **Lemma 6.** *If there is a grey literal $\lambda$ in $\text{PI}(C) \vee C$ for a clause $C$ of a resolution refutation $\pi$, then a successor of $\lambda$ occurs in $\text{PI}(\pi)$.*

*Proof.* Immediate by the definition of PI. □

⟨lemma:equalities_all_in_PI⟩ **Lemma 7.** *For every equality $s = t$ in $\mathrm{PI}(C) \vee C$ for a clause $C$ in a resolution refutation $\pi$, a successor of $s = t$ occurs in $\mathrm{PI}(\pi)$.*

*Proof.* Equalities in clauses are only removed by means of paramodulation and as $\pi$ derives the empty clause, all equalities are removed eventually. For any paramodulation inference $\iota$ using the equality $s = t$, $\mathrm{PI}_{\mathrm{step}}(\iota, I_1, I_2)$ contains $s = t$. □

We now make some considerations in the form of four lemmata about the construction of certain terms in the context of interpolant extraction. In order to do so, we frequently reason over the stepwise application of the respective unifiers, for which we employ the following definition:

**Definition 8.** We define $\mathrm{PI}^{\circ}_{\mathrm{step}}$ to coincide with $\mathrm{PI}_{\mathrm{step}}$ but without applying the substitution $\sigma$ in each of the cases. Analogously, if $C \equiv D\sigma$, we use $C^{\circ}$ to denote $D$. △

In the context of an inference $\iota$ using the clauses $C_1, \ldots, C_m$ to infer $C$, it holds that

$$
\begin{aligned}
\mathrm{PI}^*(C) &\vee C = \\
&\Big( \mathrm{PI}^*_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_m)) \Big) \vee C = \\
&\Big( \mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_m)) \vee C^{\circ} \Big)\sigma = \\
&\Big( \mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_m)) \vee C^{\circ} \Big)\sigma_{(0, \, |\,\mathrm{dom}(\sigma)|)}
\end{aligned}
$$

In the following, we abbreviate $\mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_m))$ by $\mathrm{PI}^{*\circ}(C) \vee C^{\circ}$. Note that if we are able to show that the application of a substitution $\sigma_i$ to $(\mathrm{PI}^{*\circ}(C) \vee C^{\circ})\sigma_{(0, \, i-1)}$ maintains an invariant and the invariant holds for $\mathrm{PI}^{*\circ}(C) \vee C^{\circ}$, then it immediately follows that it holds for $\mathrm{PI}^*(C) \vee C$.

⟨lemma:var_below_phi_symbol⟩ **Lemma 9.** *Let $\iota$ be a resolution or factorisation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable $u$ occurs directly below a $\Phi$-symbol in $(\mathrm{PI}^{*\circ}(C) \vee C^{\circ})\sigma_{(0, \, i)}$ for $i \geq 1$. Then at least one of the following statements holds:*

⟨14_1⟩
 *1. The variable $u$ occurs directly below a $\Phi$-symbol in $(\mathrm{PI}^{*\circ}(C) \vee C^{\circ})\sigma_{(0, \, i-1)}$.*

⟨14_5⟩
 *2. The variable $u$ occurs at a grey position in a grey literal or at a grey position in an equality in $(\mathrm{PI}^{*\circ}(C) \vee C^{\circ})\sigma_{(0, \, i)}$.*

⟨14_2⟩
 *3. There is a variable $v$ such that*

 – *$u$ occurs grey in $v\sigma_i$ and*

 – *$v$ occurs in $(\mathrm{PI}^{*\circ}(C) \vee C^{\circ})\sigma_{(0, \, i-1)}$ directly below a $\Phi$-symbol as well as directly below a $\Psi$-symbol*

*Proof.* We consider the different situations under which the situation in question is introduced by means of unification:

- There is already a literal in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ where $u$ occurs directly below a $\Phi$-symbol and $\sigma_i$ does not change this. Then clearly 1 is the case.

- There is a variable $v$ in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ such that $v\sigma_i$ contains $u$ directly below a $\Phi$-symbol. As then $v$ is unified with the term $v\sigma_i$, $v\sigma_i$ must occur in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$, which implies that 1 is the case.

- There is a variable $v$ which occurs directly below a $\Phi$-symbol such that $u$ occurs grey in $v\sigma_i$.

  Hence in the resolved or factorised literals $\lambda$ and $\lambda'$ in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$, there is a position $p$ such that without loss of generality $\lambda|_p = v$ and $u$ occurs grey in $\lambda'|_p$. Note that due to the definition of the unification algorithm, $\lambda$ and $\lambda'$ must coincide on the path to $p$.

  By Proposition 4, $\lambda$ and $\lambda'$ occur in $\text{PI}^{*\circ}(C) \vee C^\circ$ irrespective of their coloring.

  We distinguish cases based on the position $p$:

  - Suppose that $p$ occurs directly below a $\Phi$-symbol. Then as $u$ occurs grey in $\lambda'|_p$, $u$ occurs directly below a $\Phi$-symbol in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ and 1 is the case.

  - Suppose that $p$ occurs directly below a $\Psi$-symbol. Then $v$ occurs directly below a $\Psi$-symbol in $\lambda|_p$ and clearly 3 is the case.

  - Suppose that $p$ does not occur directly below a colored symbol. Then $p$ does not occur below any colored symbol, hence $u$ is contained in a grey literal in a grey position in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$. As $\sigma_i$ is trivial on $u$, this occurrence of $u$ also is present in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$ and hence 2 is the case. $\qquad\square$

⟨:var_below_phi_symbol_paramod⟩ **Lemma 10.** *Let $\iota$ be a paramodulation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable $u$ occurs directly below a $\Phi$-symbol in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$ for $i \geq 1$. Then at least one of the following statements holds:*

⟨15_1⟩
  1. *The variable $u$ occurs directly below a $\Phi$-symbol in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$.*

⟨15_5⟩
  2. *The variable $u$ occurs at a grey position in a grey literal or at a grey position in an equality in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$.*

⟨15_2⟩
  3. *There is a variable $v$ such that*

     - *$u$ occurs grey in $v\sigma_i$ and*
     - *$v$ occurs in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ directly below a $\Phi$-symbol as well as directly below a $\Psi$-symbol*

*Proof.* Let $\iota$ be a paramodulation inference of the clauses $C_1 : r_1 = r_2 \vee D$ and $C_2 : E[r]_p$ with $\sigma = \text{mgu}(\iota) = \text{mgu}(r_1, r)$ yielding $C : (D \vee E[r_2]_p)\sigma$. We consider the different situations under which the situation in question is introduced by means of unification:

- There is already a literal in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ where $u$ occurs directly below a $\Phi$-symbol and $\sigma_i$ does not change this. Then 1 is the case.

- There is a variable $v$ in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ such that $v\sigma_i$ contains $u$ directly below a $\Phi$-symbol. As then $v$ is unified with the term $v\sigma_i$, $v\sigma_i$ must occur in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$, which implies that 1 is the case.

- The variable $u$ occurs grey in $r_2$ and $p$ in $E$ is directly below a $\Phi$-symbol. But then $u$ occurs grey in an equality in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ and as $\sigma_i$ is trivial on $u$ also in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$, hence 2 holds.

- Suppose that some variable $v$ occurs directly below a $\Phi$-symbol in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ such that $u$ occurs grey in $v\sigma_i$. Then by the definition of the unification algorithm, there exists a position $q$ such that one of $r_1|_q$ and $r|_q$ is $v$ and the other one contains a grey occurrence of $u$.

  We distinguish cases based on the position $q$:

  - Suppose that $q$ occurs directly below a $\Phi$-symbol. Then clearly 1 is the case.

  - Suppose that $q$ occurs directly below a $\Psi$-symbol. Then as the variable $v$ also occurs directly below a $\Phi$-symbol and $u$ occurs grey in $v\sigma_i$, 3 is the case.

  - Suppose that $q$ is a grey position. Then 2 is the case: Either $u$ occurs grey in $r_1$ in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ and then also in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$, or otherwise $v$ occurs grey in $r_1$ in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$, but as $v\sigma_i$ contains $u$ grey, then $u$ occurs grey in the successor of $r_1$ in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$. $\square$

⟨lemma:col_change⟩ **Lemma 11.** *Let $\iota$ be an inference of a resolution refutation of $\Gamma \cup \Delta$. Suppose that a variable $u$ occurs directly below a $\Phi$-symbol as well as directly below a $\Psi$-symbol in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$. Then $u$ occurs grey in a grey literal or grey in an equality in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$.*

*Proof.* We proceed by induction over the refutation. As the original clauses each contain symbols of at most one color, the base case is trivially true.

For the induction step, suppose that an inference makes use of the clauses $C_1, \ldots, C_n$ and that the lemma holds for $\text{PI}^*(C_j) \vee C_j$ for $1 \le j \le n$.

Note that then, the lemma holds for $\text{PI}^{*\circ}(C) \vee C^\circ = \text{PI}^{*\circ}_{\text{step}}(\iota, \text{PI}^*(C_1), \ldots, \text{PI}^*(C_n)) \vee C^\circ)$. This is because as all clauses are variable-disjoint, no variable can occur $\text{PI}^{*\circ}(C) \vee C^\circ$ both directly below a $\Phi$-symbol as well as directly below a $\Psi$-symbol if this was not the case in $\text{PI}^*(C_j) \vee C_j$ for some $j$, for which the lemma by assumption holds. Furthermore, by the definition of $\text{PI}^*$, every literal which occurs in $\text{PI}^*(C_j) \vee C_i$ for some $j$ occurs in $\text{PI}^{*\circ}(C) \vee C^\circ$.

Hence it remains to show that the lemma holds for $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma = (\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_0 \ldots \sigma_m$, which we do by induction over $i$ for $1 \le i \le m$. Suppose that the lemma holds for $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ and in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$, the variable $u$ occurs directly below a $\Phi$-symbol as well as directly below a $\Psi$-term.

Then by the lemmata 9 and 10, we can deduce that one of the following statements holds for $\Omega = \Phi$ as well as $\Omega = \Psi$. We denote case $j$ for $\Omega = \Phi$ by $j^\Phi$ and for $\Omega = \Psi$ by $j^\Psi$.

⟨16_1⟩ 1. The variable $u$ occurs directly below a $\Omega$-symbol in $(\text{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$.

⟨16_4⟩    2. The variable $u$ occurs at a grey position in a grey literal or at a grey position in an equality in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$.

⟨16_2⟩    3. There is a variable $v$ such that

- $u$ occurs grey in $v\sigma_i$ and
- $v$ occurs in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ directly below a $\Phi$-symbol as well as directly below a $\Psi$-symbol

If $2^\Phi$ or $2^\Psi$ is the case, we clearly are done. On the other hand $3^\Phi$ or $3^\Psi$ is the case, then by the induction hypothesis, $v$ occurs grey in a grey literal or grey in an equality in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$. As $u$ occurs grey in $v\sigma_i$, we obtain that then, $u$ occurs grey in a grey literal or grey in an equality in $\mathrm{PI}^{*\circ}(C) \vee C^\circ$.

Hence the only remaining possibility is that both $1^\Phi$ and $1^\Psi$ hold. But then $u$ occurs directly below a $\Phi$-symbol as well as below a $\Psi$-symbol in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ and again by the induction hypothesis, we obtain that $u$ occurs grey in a grey literal or grey in an equality in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ and as $\sigma_i$ is trivial on $u$, the same occurrence of $u$ is present in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$. □

⟨lemma:subterm_in_grey_lit_star⟩ **Lemma 12.** *If* $\mathrm{PI}^*(C) \vee C$ *for a clause* $C$ *in a resolution refutation of* $\Gamma \cup \Delta$ *contains a maximal colored occurrence of a* $\Phi$-*term* $t[s]$, *which contains a maximal* $\Psi$-*colored term* $s$, *then* $s$ *occurs grey in a grey literal or grey in an equality in* $\mathrm{PI}^*(C) \vee C$.

*Proof.* We proceed by induction over the resolution refutation. As the original clauses each contain symbols of at most one color, the base case is trivially true.

The induction step is laid out similarly as in the proof of Lemma 11. We suppose that an inference makes use of the clauses $C_1, \ldots, C_n$ and that the lemma holds for $\mathrm{PI}^*(C_j) \vee C_j$ for $1 \leq j \leq n$. Then the lemma holds for $\mathrm{PI}^{*\circ}(C) \vee C^\circ = \mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_n)) \vee C^\circ)$ as no new terms are introduced in $\mathrm{PI}^{*\circ}(C) \vee C^\circ$ and all literals from $\mathrm{PI}^*(C_j) \vee C_j$ for $1 \leq j \leq n$ occur in $\mathrm{PI}^{*\circ}(C) \vee C^\circ$.

It remains to show that the lemma holds for $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma = (\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_0 \ldots \sigma_m$, which we do by induction over $i$ for $0 \leq i \leq m$. We distinguish on the situation under which a unification leads to the term $t[s]$.

- Suppose for some variable $u$ that $u\sigma_i$ contains $t[s]$. Then $u$ is unified with a term which contains $t[s]$ and occurs in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$. Hence by the induction hypothesis, $s$ occurs grey in a grey literal or grey in an equality in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ and ,as $\sigma_i$ does not change this, also in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$.

- Otherwise there is a variable $u$ which occurs directly below a $\Phi$-symbol and $v\sigma_i$ contains a grey occurrence of $s$. We distinguish based on the occurrences of $u$ in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$:

    - Suppose that $u$ occurs in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ in grey a grey literal or grey in an equality. Then clearly we are done.

– Suppose that $u$ occurs in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ directly below a $\Psi$-symbol. Then by Lemma 11, $u$ occurs grey in a grey literal or grey in an equality in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$, whose successor in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i)}$ is an occurrence of $s$ of the same coloring.

– Suppose that $u$ occurs in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ only directly below a $\Psi$-symbol. Here, we differentiate between the types of inference of the current induction step:

  * Suppose that the inference of the current induction step is a resolution or a factorisation inference. As $u$ occurs grey in $v\sigma_i$, there is a position $p$ such that for the resolved or factorised literals $\lambda$ and $\lambda'$ it holds without loss of generality that $\lambda|_p = u$ and $s$ occurs grey in $\lambda'|_p$. Note that $\lambda$ and $\lambda'$ agree on the path to $p$, including the predicate symbol..
  Now as by assumption $u$ only occurs directly below a $\Phi$-symbol, so must $s$. But then $s$ occurs directly below a $\Phi$-symbol in $(\mathrm{PI}^{*\circ}(C) \vee C^\circ)\sigma_{(0,\,i-1)}$ and we get the result by the induction hypothesis.

  * Suppose that the inference of the current induction step is a paramodulation inference. Let it use the the clauses $C_1 : r_1 = r_2 \vee D$ and $C_2 : E[r]_p$ with $\sigma = \mathrm{mgu}(\iota) = \mathrm{mgu}(r_1, r)$ to yield $C : (D \vee E[r_2]_p)\sigma$.
  As $u$ is affected by $\sigma_i$, it must occur in $r_1$ or $r$. Let $\hat{u}$ refer to this occurrence.

    · Suppose that $\hat{u}$ occurs directly below a $\Phi$-colored function symbol.
    If $\hat{u}$ is contained in $r_1$, then $s$ must be contained in $r$ directly below a $\Phi$-symbol as $r_1$ and $r$ are unifiable. We then get the result by the induction hypothesis.
    If otherwise $\hat{u}$ is contained in $r$, then there are two possibilities for the occurrence of $s$ in $r_1$:
    Either $\hat{u}$ occurs in a $\Phi$-colored function symbol in $r$. Then $s$ occurs in a $\Phi$-colored function symbol in $r_1$ and we get the result by the induction hypothesis.
    Otherwise $\hat{u}$ occurs grey in $r$, but $r$ occurs directly below a $\Phi$-colored function symbol in $E$. Then however, as $r$ and $r_1$ are unifiable, $s$ must occur grey in $r_1$ and hence grey in an equality.
    · Suppose that $\hat{u}$ occurs directly below a $\Phi$-colored predicate symbol.
    Then as the equality predicate is not considered to be colored, $u$ must occur grey in $r$. But then as $r_1$ and $r$ are unifiable, $s$ must occur grey in $r_1$ and hence grey in an equality. $\qquad\square$

⟨lemma:subterm_in_grey_lit⟩ **Lemma 13.** *Same as 12 with PI in place of PI\*.*

*Proof.* Note that PI and PI\* only differ with respect to colored literals. Let $C$ be a clause of a resolution refutation of $\Gamma \cup \Delta$. As the literals occurring in $\mathrm{PI}(C)$ are a subset of the literals occurring in $\mathrm{PI}^*(C)$, the lemma prerequisites hold true only for terms for which they also hold in $\mathrm{PI}^*(C)$.

Therefore we can deduce that if a maximal colored $\Phi$-term $t[s]$ containing a maximal $\Psi$-colored term $s$ occurs in $\mathrm{PI}(C) \vee C$, then $t[s]$ also occurs in $\mathrm{PI}^*(C) \vee C$ and by Lemma 12, the term $s$ occurs grey in a grey literal or grey in an equality in $\mathrm{PI}^*(C) \vee C$. However by Lemma 5, all grey literals and equalities of $\mathrm{PI}^*(C)$ also occur in $\mathrm{PI}(C)$.                                      $\square$

⟨lt_in_grey_lit_then_quant_alt⟩ **Lemma 14.** *If a term with $n$ color alternations occurs in a grey literal or in an equality in $\mathrm{PI}(C) \vee C$ for a clause $C$, then the interpolant $I$ produced in Theorem* **??** *contains at least $n$ quantifier alternations.*

*Proof.* We perform an induction on $n$ and show the strengthening that the quantification of the lifting variable which replaces a term with $n$ color alternations is required to be in the scope of the quantification of $n - 1$ alternating quantifiers.

For $n = 0$, no colored terms occur in $I$ and hence also no quantifiers. Moreover for $n = 1$, there are terms of one color which evidently require at least one quantifier.

Suppose that the statement holds for $n - 1$ for $n > 1$ and that a term $t$ with col-alt$(t) = n$ occurs in $\mathrm{PI}(C)$. We assume without loss of generality that $t$ is a $\Phi$-term. Then $t$ contains some $\Psi$-colored term $s$ with col-alt$(s) = n - 1$ and by Lemma 13, $s$ occurs grey in a grey literal or an equality in $\mathrm{PI}(C) \vee C$. By Lemma 6 and Lemma 7, a successor of $s$ occurs in $\mathrm{PI}(\pi)$. Note that as $s$ occurs in a grey position, any successor of $s$ also occurs in a grey position.

By the induction hypothesis, the quantification of the lifting variable for $s$ requires $n - 1$ alternated quantifiers. As $s$ is a subterm of $t$ and $t$ is lifted, $t$ must be quantified in the scope of the quantification of $s$, and as $t$ and $s$ are of different color, their quantifier type is different. Hence the quantification of the lifting variable for $t$ requires $n$ quantifier alternations.            $\square$

⟨prop:color_alt_eq_quant_alt⟩ **Proposition 15.** *If a term with $n$ color alternations occurs in $\mathrm{PI}(C) \vee C$ for a clause $C$, then the interpolant $I$ produced in Theorem* **??** *contains at least $n - 1$ quantifier alternations.*

*Proof.* By Lemma 13, a term with $n - 1$ color alternations occurs in a grey literal or an equality in $\mathrm{PI}(C) \vee C$. Lemma 14 gives the result.            $\square$

We present an example which illustrates that the occurrence of a term with $n$ color alternations in $\mathrm{PI}(C) \vee C$ for a clause $C$ can lead to an interpolant with $n - 1$ quantifier alternations (but no less as Proposition 15 shows).

**Example 16.** Let $\Gamma = \{\neg P(a)\}$ and $\Delta = \{P(x) \vee Q(f(x)), \neg Q(y)\}$. Consider the following refutation of $\Gamma \cup \Delta$:

$$
\cfrac{
  \cfrac{\neg P(a) \mid \bot \qquad P(x) \vee Q(f(x)) \mid \top}{Q(f(a)) \mid \neg P(a)}\ \text{res}_{\,x \mapsto a} \qquad \neg Q(y) \mid \top
}{
  \square \mid \neg P(a)
}\ \text{res}_{\,y \mapsto f(a)}
$$

In this example, Theorem **??** yields the interpolant $I \equiv \exists y_a \neg P(y_a)$ with quant-alt$(I) = 1$. The existence of the term $f(a)$ with col-alt$(f(a)) = 2$ in a clause of the refutation implies that quant-alt$(I) \geq 1$.            $\triangle$

## 3    LI

⟨`lemma:pi_li_1`⟩ **Conjectured Lemma 17.** *Let $C$ be a clause in a resolution refutation of $\Gamma \cup \Delta$. If a maximal colored term $t$ occurs in $\mathrm{LI}^{\bullet}(C)$ in a literal $\lambda$ at position $p$, then $t$ also occurs in a corresponding literal at position $p$ in $\mathrm{PI}(C)$.*

*Proof.* Note that PI and LI differ only in the lifting conditions of LI. As $t$ however is a colored term and hence not a lifting variable and it occurs in $\mathrm{LI}^{\bullet}(C)$, the lifting conditions have not become true during the derivation leading to $\mathrm{LI}^{\bullet}(C)$. □

⟨`lemma:pi_li_2`⟩ **Conjectured Lemma 18.** *Let $C$ be a clause in a resolution refutation of $\Gamma \cup \Delta$. If a maximal colored term $t$ occurs in $\mathrm{PI}(C)$ in a literal $\lambda$ at position $p$, then either $t$ or $z_t$ occurs in a corresponding literal at position $p$ in $\mathrm{LI}^{\bullet}(C)$.*

*Proof.* As pointed out previously, PI and LI differ only in the lifting conditions of LI. Suppose that the term in $\mathrm{LI}(C)$ which corresponds to $t$ or some respective predecessor fulfills a lifting condition. Then $z_t$ occurs at the position $p$ in $\mathrm{LI}^{\bullet}(C)$. Otherwise $t$ occurs at position $p$ in $\mathrm{LI}^{\bullet}(C)$. □

**Conjectured Lemma 19.** *TODO: merge with 5.3. of thesis (lemma:lifting_ conditions)*
⟨`lemma:pi_li_3`⟩ *Let $C$ be a clause of a resolution refutation such that $\mathrm{LI}^{\bullet}(C)$ contains a maximal colored $\Phi$-term $t$ which is lifted in $\mathrm{LI}(C)$. Suppose that $t$ contains a $\Psi$-colored subterm $s$. Then $s$ occurs as a subterm of $t$ in $\mathrm{LI}^{\bullet}(C)$.*

*Proof.* By the construction of LI, this lemma is only violated if the term $s$ or a respective predecessor is lifted due to fulfilling one of the lifting conditions.

For the sake of contradiction, suppose that this is the case in the inference creating the clause $C'$. Let $s'$ and $t'$ be the respective predecessors of $s$ and $t$ in $C'$.

- Suppose that $s'$ is lifted due to containing a variable which does not occur in $C'$. Then as $s'$ is a subterm of $t'$, $t'$ contains this variable as well and therefore is lifted in $\mathrm{LI}(C')$, contradicting the assumption.

- Suppose that $s'$ is lifted due to being a ground term which does not occur in $C$. Then $t'$ does not occur in $C'$ either as any occurrence of $t'$ would contain $s'$. Hence $t'$ is lifted in $\mathrm{LI}(C')$, contradicting the assumption.

□

**Definition 20.** Alternative, equivalent definition of LI.
    <insert def for base case>
    Let $\mathrm{LI}^{\bullet}_0(C) = \mathrm{LI}^{\bullet}(C)$ and $Z_0$ be the set of terms in $\mathrm{LI}^{\bullet}_i(C)$ for which some lifting conditions holds.
    We now define $\mathrm{LI}(C)$ and $\mathrm{LI}^{\bullet}_i(C)$ for $i \geq 1$:

1. Let $r \in Z_i$ be such that $r$ is not subterm of any other term in $Z_i$. If no such $r$ exists, let $\mathrm{LI}(C) = \mathrm{LI}^{\bullet}_i(C)$.

2. Let $Z_{i+1} = Z_i \backslash \{r\}$.

3. Let $\mathrm{LI}^\bullet_{i+1}(C)$ be built from $\mathrm{LI}^\bullet{}_i(C)$ such that $r$ is lifted by a fresh lifting variable $z'_r$ and the formula is prefixed by $\forall x_r$ if $r$ is a $\Delta$-term and $\exists y_r$ if $r$ is a $\Gamma$ term. $\qquad\triangle$

⟨lemma:grey_lits_all_in_LI⟩ **Conjectured Lemma 21.** *Let $C$ be a clause in a resolution refutation $\pi$ of $\Gamma \cup \Delta$. Then for any grey literal $\lambda$ in $\mathrm{LI}^\bullet(C) \vee C$, a successor of $\lambda$ occurs in $\mathrm{LI}^\bullet(C') \vee C'$, where $C'$ is a successor of $C$.*

*Proof.* Consider that PI and LI coincide besides the lifting conditions of LI. Hence Lemma 6 gives the result. $\qquad\square$

⟨lemma:equalities_all_in_LI⟩ **Conjectured Lemma 22.** *Let $C$ be a clause in a resolution refutation $\pi$ of $\Gamma \cup \Delta$. Then for any equality $s = t$ in $\mathrm{LI}^\bullet(C) \vee C$, a successor of $s = t$ occurs in $\mathrm{LI}^\bullet(C') \vee C'$, where $C'$ is a successor of $C$.*

*Proof.* Consider that PI and LI coincide besides the lifting conditions of LI. Hence Lemma 7 gives the result. $\qquad\square$

**Conjectured Lemma 23.** *If a term $t$ with $\mathrm{col\text{-}alt}(t) = n$ occurs grey in a grey literal or grey in an equality $\mathrm{LI}^\bullet(C) \vee C$ for some $C$, then $\mathrm{LI}(\pi)$ has at least $n$ quantifier alternations.*

*Proof.* We proceed by induction over $n$.
<span style="color:red">TODO: base case</span>
We now turn to the induction step. By the Lemmata 21 and 22, a successor of $t$ occurs in $\mathrm{LI}^\bullet(C') \vee C'$ for any successor $C'$ of $C$. Let $C'$ be the successor of $C$ such that $\mathrm{LI}^\bullet(C')$ contains a successor of $t$ which is not a lifting variable and $\mathrm{LI}(C')$ contains a successor of $t$ which is a lifting variable. As all colored terms are lifted eventually, such a clause $C'$ must occur.

By Lemma 17, $t$ occurs at the same position in $\mathrm{PI}(C')$. We assume without loss of generality that $t$ is a $\Phi$-term. Then $t$ contains some $\Psi$-colored term $s$ with $\mathrm{col\text{-}alt}(s) = n - 1$. By Lemma 13, $s$ occurs grey in a grey literal or grey in an equality in $\mathrm{PI}(C') \vee C'$.

By Lemma 18, then either $s$ or $z_s$ occurs in $\mathrm{LI}^\bullet(C')$ at a corresponding literal and position. As $t$ is not lifted in $\mathrm{LI}^\bullet(C') \vee C'$ and $s$ is a subterm of $t$, by Lemma 19, $s$ is also not lifted in $\mathrm{LI}^\bullet(C') \vee C'$.

We show that if $t$ occurs in $\mathrm{LI}^\bullet(C)$, then the "successor" of $\mathrm{LI}^\bullet(C)$ which is contained in $\mathrm{LI}(\pi)$ is enclosed by $n$ alternating quantifiers.

Note that as $s$ is a subterm of $t$, $s$ is only lifted and quantified after $t$ (either in this stage or some subsequent one) such that the quantifier for $t$ is in the scope of the quantifier for $s$.

Let $i$ be the step where the lifted term in $\mathrm{LI}^\bullet_i(C)$ is $t$. In $\mathrm{LI}^\bullet_{i+1}(C)$, one quantifier is added in order to quantify the lifting variable for $t$. As $s$ can only be lifted after $t$, $s$ does occur in $\mathrm{LI}^\bullet_{i+1}(C)$ and by the induction hypothesis, $n-1$ alternated quantifiers are still added to $\mathrm{LI}^\bullet_{i+1}(C)$. As $t$ adds another one, we can infer that $t$ causes $\mathrm{LI}(\pi)$ to contain at least $n$ quantifier alternations. $\qquad\square$

## 4 cruft

### 4.1 Big picture

- if max col term $t$ in $\mathrm{LI}(C)$ not lifted, then $t$ also occurs in $\mathrm{PI}(C)$ ①.

- for every max col term $t$ in $\mathrm{PI}(C)$, either $t$ or $\ell[t]$ occurs in $\mathrm{LI}(C)$ ②

- if max col term $t$ in $\mathrm{LI}(C)$ not lifted, then all subterms $s$ of $t$ are not lifted as well. ③

- Hence:

  Supp max col $t$ in $\mathrm{LI}(C)$ is lifted in $\mathrm{LI}(C)$ but not in $\mathrm{LI}^\bullet(C)$.
  Then $t$ also occs in $\mathrm{PI}(C)$ by ①.
  Then by that lemma, any subterm $s$ of $t$ occs in $\mathrm{PI}(C)$ grey in grey lit or grey in equality. Hence it occs also in $\mathrm{LI}(C)$, either lifted or not lifted ②.
  By ③, it is not lifted.

  It however will eventually be lifted (cf. TODO), so the quantification of $t$ will be in the scope of the quantifier for $s$.

As $t$ is lifted in this step, one quantifier is added. Note that even if $s$ is also lifted in this step, then it is lifted after $t$, or otherwise it is lifted at a later stage.

Let $\mathrm{LI}^\bullet(C)'$ be the "formula right after $t$ has been lifted and quantified, but not the remaining ones of this stage".

Then $s$ occurs in $\mathrm{LI}^\bullet(C)' \vee C$. Hence by the induction hypothesis, the successor of $\mathrm{LI}^\bullet(C)'$ in $\mathrm{LI}(\pi)$ is enclosed by $n-1$ quantifier alternations. As $s$ and $t$ are of opposite colors, their quantifier type is different and the lifting and quantification of $t$ adds one more quantifier alternation to $\mathrm{LI}^\bullet(C)$. Hence the successor of $\mathrm{LI}^\bullet(C)$ in $\mathrm{LI}(\pi)$ is enclosed by $n$ quantifier alternations.

*Proof.* Supp $t$ not lifted in $\mathrm{LI}^\bullet(C)$ but in $\mathrm{LI}(C)$ and col-alt$(t) = 1$. Then $t$ is colored and $\mathrm{LI}(C)$ (and $\mathrm{LI}(\pi)$) has one color alternation

Supp $t$ not lifted in $\mathrm{LI}^\bullet(C)$ but in $\mathrm{LI}(C)$ and col-alt$(t) = n$. Then by Lemma 19, no subterm of $t$ is lifted. By TODO, the subterm $s$ of $t$ occs in $\mathrm{LI}^\bullet(C)$ and by the argument above is not lifted. By the induction hypothesis, $n-1$ quantifier alternations are added to the "current lifted interpolant".

as $s$ and $t$ of different color, different quantifier. hence the lifting of $t$ adds one quantifier alternation, plus the $n-1$ which come later, so we have at least $n$ in the end. □

**Conjectured Lemma 24.** *Let $\pi$ be a refutation of $\Gamma \cup \Delta$ and $C$ a clause in $\pi$. Let $t$ be a term with $n$ color alternations such that $t$ is not a lifting variable. If $t$ occurs in a grey literal or in an equality in $\mathrm{LI}^\bullet(C) \vee C$, then $\mathrm{LI}(\pi)$ contains at least $n$ quantifier alternations.*

*Proof.* ONLY IND STEP FOR NOW:

By the Lemmata 21 and 22, a successor of $t$ occurs in $\mathrm{LI}^\bullet(C') \vee C'$ for any successor $C'$ of $C$. Let $C'$ be the successor of $C$ such that $\mathrm{LI}^\bullet(C')$ contains a successor of $t$ which is not a lifting variable and $\mathrm{LI}(C')$ contains a successor of

$t$ which is a lifting variable. As all colored terms are lifted eventually, such a clause $C'$ must occur.

By Lemma 17, $t$ occurs at the same position in $\mathrm{PI}(C')$. We assume without loss of generality that $t$ is a $\Phi$-term. Then $t$ contains some $\Psi$-colored term $s$ with col-alt$(s) = n - 1$. By Lemma 13, $s$ occurs grey in a grey literal or grey in an equality in $\mathrm{PI}(C') \vee C'$.

By Lemma 18, then either $s$ or $z_s$ occurs in $\mathrm{LI}(C')$ TODO:  OR $\mathrm{LI}^\bullet(C')$  at a corresponding literal and position. As $t$ is not lifted in $\mathrm{LI}^\bullet(C') \vee C'$ and $s$ is a subterm of $t$, by Lemma 19, $s$ is also not lifted in $\mathrm{LI}^\bullet(C') \vee C'$.

As however $s$ is a colored term which occurs in a grey literal or an equality in $\mathrm{LI}^\bullet(C') \vee C'$, by the induction hypothesis, $\mathrm{LI}(\pi)$ contains at least $n - 1$ quantifier alternations. TODO: weak!

## OTHER STATEMENT: $n$ col alt $\Rightarrow$ $n$ nested liftings still to come

$\square$

TODO:  show that terms in $\mathrm{PI}(C)$ actually occur in $\mathrm{LI}(C)$, hence $\mathrm{PI}(C)$ and $\mathrm{LI}(C)$ have same quant alts