Example 101: canonical examples for overbinding w.r.t. order

Here, the second lower prooftree shows what huang would produce.

Ex 101a

$$\cfrac{\cfrac{\overset{\Sigma}{P(u,f(u))\vee Q(u)} \qquad \overset{\Pi}{\neg Q(a)}}{P(a,f(a))}\ u\mapsto a \qquad \overset{\Pi}{\neg P(x,y)}}{\square}\ x\mapsto a, y\mapsto f(a)$$

$$\cfrac{\cfrac{\cfrac{\bot \qquad \top}{Q(a)}\ u\mapsto a \qquad \top}{P(a,f(a))\vee Q(a)}}{}\ x\mapsto a, y\mapsto f(a) \qquad\qquad \cfrac{\cfrac{\bot \qquad \top}{\forall x_1 Q(x_1)} \qquad \top}{\forall x_1 \exists x_2(P(x_1,x_2)\vee Q(x_1))}$$

Direct overbinding would not work without merging same variables!: $\forall x_1 \forall x_2 \exists x_3(P(x_2,x_3)\vee Q(x_1))$

counterexample: $Q\sim\{0\}, P\sim\{(1,0)\}$

Direct overbinding would work when considering original dependencies as highlighted above

arrow lemma:

$$\cfrac{\cfrac{\overset{\Sigma}{\Gamma\models\exists y_1(P(u,y_1)\vee Q(u)\vee\bot)} \qquad \overset{\Pi}{\Gamma\models\neg Q(x_1)\vee\top}}{\Gamma\models\exists y_1(P(x_1,y_1)\vee Q(x_1))}\ u\mapsto a \qquad \overset{\Pi}{\Gamma\models\neg P(x,y)\vee\top}}{\Gamma\models(\forall x_1)\exists y_1(Q(x_1)\vee P(x_1,y_1))}\ x\mapsto a, y\mapsto f(a)$$

Ex 101b − other resolution order

$$\cfrac{\cfrac{\overset{\Sigma}{P(u,f(u))\vee Q(u)} \qquad \overset{\Pi}{\neg P(x,y)}}{Q(u))}\ y\mapsto f(u), x\mapsto u \qquad \overset{\Pi}{\neg Q(a)}}{\square}\ u\mapsto a$$

$$\cfrac{\cfrac{\cfrac{\bot \qquad \top}{P(u,f(u))}\ x\mapsto f(u), x\mapsto u \qquad \top}{P(a,f(a))\vee Q(a)}}{}\ u\mapsto a \qquad\qquad \cfrac{\cfrac{\cfrac{\bot \qquad \top}{\exists x_1 P(u,x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}}{}\ u\mapsto a$$

Ex 101c − Π and Σ swapped

$$\cfrac{\cfrac{\overset{\Pi}{P(u,f(u))\vee Q(u)} \qquad \overset{\Sigma}{\neg P(x,y)}}{Q(u))}\ y\mapsto f(u), x\mapsto u \qquad \overset{\Sigma}{\neg Q(a)}}{\square}\ u\mapsto a$$

$$\cfrac{\cfrac{\cfrac{\top \qquad \bot}{\neg P(u,f(u))}\ x\mapsto f(u), x\mapsto u \qquad \bot}{\neg P(a,f(a))\wedge\neg Q(a)}}{}\ u\mapsto a \qquad\qquad \cfrac{\cfrac{\top \qquad \bot}{\forall x_2\neg P(u,x_2)} \qquad \bot}{\exists x_1\forall x_2(\neg P(x_1,x_2)\wedge\neg Q(x_1))}$$

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arrow lemma:

$$\frac{\Gamma \models \overset{\Pi}{P(u,x_1)} \vee Q(u) \vee \top \qquad \Gamma \models \overset{\Sigma}{\neg P(x,y)} \vee \bot}{\Big(Q(u) \mid (\neg P(x,x_1) \wedge \top) \vee (P(u,/f(u)) \wedge \bot) \Big)\sigma} \; y \mapsto f(u), x \mapsto u$$

$$Q(u) \mid (\neg P(u,x_1) \wedge \top) \vee (P(u,f(u)) \wedge \bot)$$

employ σ' !?!!?!?!?!??!?????!!!

$$\Gamma \models Q(u) \mid \neg P(u,x_1)$$

$$\frac{\Delta \models Q(u) \mid \exists x_1 P(u,x_1) \qquad\qquad\qquad\qquad \Gamma \models \exists y_1 \overset{\Sigma}{\neg Q(y_1)}}{\text{both } u\text{'s on LHS need to become } a \text{ and then } y_1} \; u \mapsto a$$

$$\Gamma \models (\forall x_1)\exists y_1(\neg P(y_1,x_1) \vee \neg Q(y_1))$$

$$\Delta \models (\exists x_1)\forall y_1(P(y_1,x_1) \wedge Q(y_1))$$

Ex 101d − Π and Σ swapped, other resolution order

$$\frac{\dfrac{\overset{\Pi}{P(u,f(u))} \vee Q(u) \qquad \overset{\Sigma}{\neg Q(a)}}{P(a,f(a))} \; u \mapsto a \qquad \overset{\Sigma}{\neg P(x,y)}}{\Box} \; x \mapsto a, y \mapsto f(a)$$

$$\frac{\dfrac{\top \qquad \bot}{\neg Q(a)} \; y \mapsto a \qquad \bot}{\neg Q(a) \wedge \neg P(a,f(a))} \; x \mapsto a, y \mapsto f(a) \qquad\qquad\qquad \frac{\dfrac{\top \qquad \bot}{\exists x_1 \neg Q(x_1)} \qquad \bot}{\exists x_1 \forall x_2(\neg P(x_1,x_2) \wedge \neg Q(x_1))}$$

102 − similar to 101 but with intra-clause-set inferences

Ex 102a

$$\frac{\dfrac{\overset{\Sigma}{P(f(x))} \vee Q(f(x),z) \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x),z)} \qquad \dfrac{\overset{\Sigma}{\neg Q(x_1,y)} \vee R(y) \qquad \overset{\Pi}{\neg R(g(z_1))}}{\neg Q(x_1,g(z_1))} \; y \mapsto g(z_1), x_1 \mapsto f(x)}{\Box} \; x_1 \mapsto f(x), z \mapsto g(z_1)$$

$$\frac{\dfrac{\bot \quad \top}{P(f(x))} \quad \dfrac{\bot \quad \top}{R(g(z_1))} \; y \mapsto g(z_1), x_1 \mapsto f(x)}{P(f(x)) \vee R(g(z_1))} \; x_1 \mapsto f(x), z \mapsto g(z_1) \qquad\qquad \frac{\dfrac{\bot \quad \top}{\exists x_1 P(x_1)} \quad \dfrac{\bot \quad \top}{\forall x_2 R(x_2)}}{\exists x_1 \forall x_2(P(x_1) \vee R(x_2)) \quad \text{(order irrelevant!)}}$$

Ex 102b

$$\frac{\dfrac{\overset{\Sigma}{P(f(x))} \vee Q(f(x),z) \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x),z)} \qquad \dfrac{\overset{\Sigma}{\neg Q(f(y),z_1)} \vee R(y) \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a),z_1)} \; y \mapsto a}{\Box} \; x \mapsto a, z \mapsto z_1$$

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$$\frac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)} \; y \mapsto a}{P(f(a)) \lor R(a)} \; x \mapsto a, z \mapsto z_1 \qquad\qquad \frac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)} \; y \mapsto a}{\forall x_2 \exists x_1 (P(x_1) \lor R(x_2))} \; x \mapsto a, z \mapsto z_1$$

direct:

$$\frac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \; x_1 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)} \; x_2 \sim a}{\exists x_1 \forall x_2 (P(x_1) \lor R(x_2))} \text{ order irrelevant!}$$

Ex 102b' with Q grey

$$\frac{\overset{\Sigma}{P(f(x)) \vee Q(f(x), z)} \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x), z)} \qquad \frac{\overset{\Sigma}{\neg Q(f(y), z_1) \vee R(y)} \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a), z_1)} \, y \mapsto a}{\square} \, x \mapsto a, z_1 \mapsto z$$

$$\frac{\dfrac{\perp \qquad \top}{P(f(x))} \qquad \dfrac{\perp \qquad \top}{R(a)} \, y \mapsto a}{(\neg Q(f(a), z) \wedge P(f(a))) \vee (Q(f(a), z) \wedge R(a))} \, x \mapsto a, z_1 \mapsto z$$

arrow lemma: (change of specification: $P(g(z_3))$ in clause in Σ instead of $P(f(x))$)

$$\frac{\dfrac{\overset{\Sigma}{\exists x_1 (P(x_4) \vee Q(x_1, z))} \qquad \overset{\Pi}{\neg P(y)}}{\exists x_1 (P(g(z_3)) \wedge \perp) \vee (P(x_4) \wedge \top) \mid Q(x_1, z)}}{\exists x_1 (P(x_4) \mid Q(x_1, z))} \quad \frac{\dfrac{\overset{\Sigma}{\exists x_2 (\neg Q(x_2, z_1) \vee R(y))} \qquad \overset{\Pi}{\forall x_3 \, \neg R(x_3)}}{\overset{\forall x_3}{\exists x_2} ((\neg R(x_3) \wedge \perp) \vee (R(a) \wedge \top) \mid \neg Q(x_2, z_1))} \, y \mapsto a}{\overset{\forall x_3}{\exists x_2} ((\neg R(x_3) \wedge \perp) \vee (R(x_3) \wedge \top) \mid \neg Q(x_2, z_1))} \, x \mapsto a, z_1 \mapsto z}{\overset{\forall x_1}{\overset{\forall x_3}{\exists x_2}} \left((\neg Q(x_2, z_1) \wedge P(x_4)) \vee (Q(x_1, z) \wedge R(x_3)) \right)}$$

arrow order: $x_3 < x_2$, x_2 same-block-as x_4: $\forall x_3 \exists x_2 \exists x_4 \forall x_1 \left((\neg Q(x_2, z_1) \wedge \neg P(x_4)) \vee (Q(x_1, z) \wedge R(x_3)) \right)$

$\rightarrow$ bad example, plus some errors still in there

Huang:

$$\frac{\dfrac{\perp \qquad \top}{\exists x_2 P(x_2)} \qquad \dfrac{\perp \qquad \top}{\forall x_1 R(x_1)} \, y \mapsto a}{\forall x_1 \exists x_2 (\neg Q(x_2, z) \wedge P(x_2)) \vee (Q(x_2, z) \wedge R(x_1))} \, x \mapsto a, z_1 \mapsto z$$

direct:

$$\frac{\dfrac{\perp \qquad \top}{\exists x_2 P(x_2)} \, x_2 \sim f(x) \qquad \dfrac{\perp \qquad \top}{\forall x_1 R(x_1)} \, x_1 \sim a}{\forall x_1 \exists x_2 \exists x_3 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))} \, x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3$$
$$\text{OR: } \exists x_2 \forall x_1 \exists x_3 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))$$
$$\text{OR: } \exists x_1 \exists x_3 \forall x_2 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))$$

Huang-Version passt weil man für existenzielle quantoren eh gleichen term einsetzt

direct w mixed, slightly different:

$$\frac{\dfrac{\perp \mid P(f(x)) \vee Q(x, z) \qquad \top \mid \neg P(y)}{\exists x_2 P(x_2) \mid Q(x, z)} \, x_2 \sim f(x) \qquad \dfrac{\overset{\Sigma}{\perp \mid \neg Q(f(y), z_1) \vee R(y)} \qquad \overset{\Pi}{\top \mid \neg R(a)}}{\forall x_1 R(x_1) \mid \neg Q(f(a), z_1)} \, x_1 \sim a}{\dfrac{\forall x_1 \exists x_3 \exists x_2 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}{(\neg Q(f(a), z) \wedge P(f(f(a)))) \vee (Q(f(a), z) \wedge R(a))}} \, x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3$$

last dependency not crucial because other arrow is a Σ-arrow as well, but just changing it to Π (and changing f for g should produce a quantifier alternation)

Example 103: variables in interpolant get unified and order might have to be changed

103a: length changes but old version is still valid, despite huang's algorithm doing a sort

$$\cfrac{\cfrac{\cfrac{\overset{\Sigma}{Q(f(x)) \vee P(y) \vee R(x)} \quad \overset{\Pi}{\neg Q(y_1)}}{P(y) \vee R(x)} y_1 \mapsto f(x) \quad \overset{\Pi}{\neg P(h(g(a)))}}{R(x)} y \mapsto h(g(a)) \quad \overset{\Pi}{\neg R(g(g(a)))}}{\square} x \mapsto g(g(a))$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\bot \quad \top}{Q(f(x))} y_1 \mapsto f(x) \quad \top}{Q(f(x)) \vee P(h(g(a)))} y \mapsto h(g(a)) \quad \top}{Q(f(g(g(a)))) \vee P(h(g(a))) \vee R(g(g(a)))} x \mapsto g(g(a))}{}$$

$$\cfrac{\cfrac{\cfrac{\bot \quad \top}{\exists x_1 Q(x_1)} \quad \top}{\exists x_1 \forall x_2 (Q(x_1) \vee P(x_2))} \quad \top}{X}$$

X:

Huang's algo gives:

$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))$

Direct overbinding gives: $x_3 < x_1$, rest arbitrary, hence:

$\forall x_3 \exists x_1 \forall x_2 (Q(x_1) \vee P(x_2) \vee R(x_3))$ <- this you do not get with huang

$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))$

$\forall x_3 \forall x_2 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))$

103b: length changes "uniformly"

$$\cfrac{\cfrac{\cfrac{\overset{\Sigma}{Q(f(f(x))) \vee P(f(x)) \vee R(x)} \quad \overset{\Pi}{\neg Q(y_1)}}{P(f(x)) \vee R(x)} y_1 \mapsto f(f(x)) \quad \overset{\Pi}{\neg P(y_2)}}{R(x)} y_2 \mapsto f(x) \quad \overset{\Pi}{\neg R(g(a))}}{\square} x \mapsto g(a)$$

$$\cfrac{\cfrac{\cfrac{\bot \quad \top}{Q(f(f(x)))} y_1 \mapsto f(f(x)) \quad \top}{Q(f(f(x))) \vee P(f(x))} y_2 \mapsto f(x) \quad \top}{Q(f(f(g(a)))) \vee P(f(g(a))) \vee R(g(a))} x \mapsto g(a)$$

$$\cfrac{\cfrac{\cfrac{\bot \quad \top}{\exists x_1 Q(x_1)} \quad \top}{\exists x_2 \exists x_1 (Q(x_1) \vee P(x_2))} \quad \top}{\forall x_3 \exists x_2 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))}$$

Huang and direct overbinding somewhat coincide as $x_2 < x_1$ in both cases, and $x_3 < x_1$ and $x_3 < x_2$

103c: Failed attempt: different variables, accidentally the same terms appear but no logical connection

$$\cfrac{\overset{\Sigma}{P(a,x)} \quad \cfrac{\overset{\Sigma}{\neg Q(a)} \quad \overset{\Pi}{\neg P(y, f(z)) \vee Q(z)}}{\neg P(y, f(a))} z \mapsto a}{\square} y \mapsto a, x \mapsto f(a)$$

$$\cfrac{\cfrac{\cfrac{\bot \qquad \top}{\neg Q(a)}\; z \mapsto a}{\cfrac{\bot \qquad}{P(a, f(a)) \land \neg Q(a)}}}{} \; y \mapsto a, x \mapsto f(a)$$

$$\cfrac{\bot \qquad \cfrac{\bot \qquad \top}{\exists x_1 \neg Q(x_1)}}{\exists x_1 \forall x_2 (P(x_1, x_2) \land \neg Q(x_1))}$$

order required for Π

direct:

$$\cfrac{\cfrac{\bot \qquad \cfrac{\cfrac{\bot \qquad \top}{\exists x_1 \neg Q(x_1)}\; x_1 \sim a}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \land \neg Q(x_1))}\; x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{}}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \land \neg Q(x_1))}$$

invariant:

$$\cfrac{\cfrac{\exists x_2 (P(x_2, x) \lor \bot) \qquad \cfrac{\exists x_1(Q(x_1) \lor \bot) \qquad \cfrac{\forall x_3((\neg P(y, \textcolor{red}{x_3}) \lor Q(\textcolor{red}{z})) \lor \top)\; x_1 \sim a}{\exists x_1 \forall x_3 \neg P(y, \textcolor{red}{x_3}) \lor \neg Q(\textcolor{red}{x_1})}}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \land \neg Q(x_1))}\; x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{}}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \land \neg Q(x_1))}$$

invariant in other resolution order

$$\cfrac{\cfrac{\bot \qquad \cfrac{Q(\textcolor{red}{z}) \lor \exists x_2 \forall x_3 P(x_2, \textcolor{red}{x_3}) \qquad \cfrac{\bot \qquad \top}{}\; x_2 \sim a, x_3 \sim f(z)}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \land \neg Q(x_1))}\; x_1 \sim a; x_1 < x_3}{}}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \land \neg Q(x_1))}$$

invariant if Σ and Π swapped:

$$\cfrac{\cfrac{\bot \qquad \cfrac{\neg P(y, f(x_1)) \lor \forall x_1 Q(x_1) \qquad \cfrac{\top \qquad \bot}{}\; x_1 \sim a}{\forall x_1 \forall x_2 \exists x_3 (\neg P(x_2, x_3) \lor Q(x_1))}\; x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{}}{\text{OR: } \forall x_1 \exists x_3 \forall x_2 (\neg P(x_2, x_3) \lor Q(x_1))}$$

SECOND ATTEMPT:

$$\cfrac{\cfrac{\overset{\Sigma}{P(a)} \qquad \cfrac{\overset{\Sigma}{Q(z)} \qquad \cfrac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg P(y) \lor \neg Q(f(\textcolor{red}{x})) \lor S(\textcolor{red}{x})}\; x \mapsto a}{\neg P(y) \lor \neg Q(f(a))}\; z \mapsto f(a)}{\neg P(y)}\; y \mapsto a}{\Box}}{} \cdots$$

$$\cfrac{\cfrac{\bot \qquad \cfrac{\bot \qquad \cfrac{\bot \qquad \top}{\neg S(a)}\; x \mapsto a}{\neg S(a) \land Q(f(a))}\; z \mapsto f(a)}{P(a) \land \neg S(a) \land Q(f(a))}\; y \mapsto a}{}$$

Huang:

$$\cfrac{\cfrac{\bot \quad \cfrac{\bot \quad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 (\neg S(x_1) \wedge Q(x_2))}}{\exists x_1 \forall x_2 (P(x_1) \wedge \neg S(x_1) \wedge Q(x_2))}$$

$\forall x_1 \exists x_2 (\neg P(x_1) \vee S(x_1) \vee \neg Q(x_2))$

similar fail

$\Rightarrow$ anytime there is $P(a, f(a))$, either they have a dependency or they are not both differently colored (grey is uncolored)

for the record, direct method anyway:

$$\cfrac{\bot \quad \cfrac{\bot \quad \cfrac{\cfrac{\bot \quad \top}{\exists x_1 \neg S(x_1)} \quad x \sim a}{\exists x_1 \forall x_2 \neg S(x_1) \wedge Q(x_2)} \quad z \sim f(a); x_1 < x_2}{\exists x_1 \forall x_2 \exists x_3 P(x_3) \wedge \neg S(x_1) \wedge Q(x_2)} \quad x_3 \sim a; x_3 \text{ need not be merged w } x_1}$$

Example: ordering on both ancestors where the merge forces a new ordering

202a – canonical

$$\cfrac{\cfrac{\cfrac{\overset{\Sigma}{P(a,x_1) \vee R(y)} \quad \overset{\Pi}{\neg P(x,fx)}}{R(y)} \quad x_1 \mapsto fa \atop x \mapsto a \qquad \cfrac{\overset{\Sigma}{Q(x_2,gx_2) \vee \neg R(u)} \quad \cfrac{\overset{\Sigma}{\neg S(a)} \quad \overset{\Pi}{\neg Q(fz,x_3) \vee S(z)}}{\neg Q(fa,x_3)} \; z \mapsto a}{\neg R(u)} \; {x_2 \mapsto fa, \atop x_3 \mapsto gfa}}{\cfrac{\cfrac{\bot \quad \top}{P(a,f(a))} \; x_1 \mapsto f(a) \atop x \mapsto a \qquad \cfrac{\bot \quad \cfrac{\bot \quad \top}{\neg S(a)} z \mapsto a}{Q(f(a),g(f(a))) \wedge \neg S(a)} \; {x_2 \mapsto f(a), \atop x_3 \mapsto g(f(a))}}{}}}{P(a,f(a)) \vee (Q(f(a),g(f(a))) \wedge \neg S(a))}$$

Huang

$$\cfrac{\cfrac{\bot \quad \top}{\exists x_1 \forall x_2 P(x_1,x_2))} \qquad \cfrac{\bot \quad \cfrac{\bot \quad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 Q(x_2,x_3) \wedge \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 P(x_1,x_2) \vee (Q(x_2,x_3)) \wedge \neg S(x_1))}$$

direct:

$$\cfrac{\cfrac{\bot \quad \top}{\exists x_1 \forall x_2 P(x_1,x_2))} \; {x_1 \sim a, x_2 \sim fa \atop x_1 < x_2} \qquad {x_3 \sim a, x_4 \sim fa, x_5 \sim gfa) \atop x_3 < x_4, x_4 < x_5} \cfrac{\bot \quad \cfrac{\bot \quad \top}{\exists x_3 \neg S(x_3)} x_3 \sim a}{\exists x_3 \forall x_4 \exists x_5 Q(x_4,x_5) \wedge \neg S(x_3)} \; {x_3 \mapsto x_1, x_4 \mapsto x_2 \atop x_1 < x_2, x_2 < x_5}}{\exists x_1 \forall x_2 \exists x_5 P(x_1,x_2) \vee (Q(x_2,x_5) \wedge \neg S(x_5))}$$

without merge in end: $(x_1 < x_2, x_3 < x_4, x_4 < x_5)$

$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 P(x_1,x_2) \vee (Q(x_4,x_5) \wedge \neg S(x_3))$

$\exists x_3 \forall x_4 \exists x_5 \exists x_1 \forall x_2 P(x_1,x_2) \vee (Q(x_4,x_5) \wedge \neg S(x_3))$

(also interwoven ones appear to work)

combined presentation:

$$\cfrac{\cfrac{\bot \mid P(a,x_1) \vee R(y) \qquad \top \mid \neg P(x,f(x))}{P(a,f(a)) \mid R(y)} \begin{array}{l} x_1 \mapsto f(a) \\ x \mapsto a \end{array} \qquad \cfrac{\bot \mid Q(x_2,g(x_2)) \vee \neg R(u) \qquad \cfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(z),x_3) \vee S(z)}{\neg S(a) \mid \neg Q(f(a),x_3)} \begin{array}{l} z \mapsto a \\ x_2 \mapsto f(a), \\ x_3 \mapsto g(f(a)) \end{array}}{Q(f(a),g(f(a))) \wedge \neg S(a) \mid \neg R(u)}}{P(a,f(a)) \vee (Q(f(a),g(f(a))) \wedge \neg S(a)) \mid \Box}$$

combined presentation ground:

$$\cfrac{\cfrac{\bot \mid P(a,f(a)) \vee R(y) \qquad \top \mid \neg P(a,f(a))}{(P(a,f(a)) \wedge \top) \vee (\neg P(a,f(a)) \wedge \bot) \mid R(y)} \qquad \cfrac{\bot \mid Q(f(a),g(f(a))) \vee \neg R(u) \qquad \cfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(f(a),g(f(a))) \wedge \neg S(a) \mid \neg R(u)}}{P(a,f(a)) \vee (Q(f(a),g(f(a))) \wedge \neg S(a)) \mid \Box}$$

combined presentation ground with direct method but only Δ-terms removed :

$$\cfrac{\cfrac{\bot \mid P(a,f(a)) \vee R(y) \qquad \top \mid \neg P(a,f(a))}{(P(a,x_2) \wedge \top) \vee (\neg P(a,x_2) \wedge \bot) \mid R(y)} \qquad \cfrac{\bot \mid Q(f(a),g(f(a))) \vee \neg R(u) \qquad \cfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(x_4,g(x_4)) \wedge \neg S(a) \mid \neg R(u)}}{P(a,x_2) \vee (Q(x_4,g(x_4)) \wedge \neg S(a)) \mid \Box}$$

combined presentation ground with direct method:

$$\cfrac{\cfrac{\bot \mid P(a,f(a)) \vee R(y) \qquad \top \mid \neg P(a,f(a))}{\exists x_1 \forall x_2 (P(x_1,x_2) \wedge \top) \vee (\neg P(x_1,x_2) \wedge \bot) \mid R(y)} \qquad \cfrac{\bot \mid Q(f(a),g(f(a))) \vee \neg R(u) \qquad \cfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \vee S(a)}{\exists x_3 \neg S(x_3) \mid \neg Q(f(a),g(f(a)))}}{\exists x_3 \forall x_4 \exists x_5 (Q(x_4,x_5)) \wedge \neg S(x_3)) \mid \neg R(u)}}{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (P(x_1,x_2) \vee (Q(x_4,x_5)) \wedge \neg S(x_3)) \mid \Box}$$

203a – some alternations

$$\cfrac{\cfrac{\cfrac{\overset{\Sigma}{R(x) \vee \neg P(f(x))} \quad \overset{\Pi}{P(z) \vee Q(g(z))}}{R(x) \vee Q(g(f(x)))} \; z \mapsto f(x) \quad \overset{\Sigma}{\neg Q(y) \vee S(h(y))}}{R(x) \vee S(h(g(f(x))))} \; y \mapsto g(f(x))}{\cfrac{\overset{\Pi}{\neg R(a)} \qquad \qquad}{S(h(g(f(a))))} \; x \mapsto a}}{\cfrac{\overset{\Pi}{\neg S(x_1)} \qquad \qquad \qquad}{\square} \; x_1 \mapsto h(g(f(a)))}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\bot \quad \top}{\neg P(f(x))} \; z \mapsto f(x) \qquad \bot}{\neg Q(g(f(x))) \wedge \neg P(f(x))} \; y \mapsto g(f(x))}{\top \quad \neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a)} \; x \mapsto a}{\top \qquad \neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a) \vee S(h(g(f(a))))}}{\neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a) \vee S(h(g(f(a))))} \; x_1 \mapsto h(g(f(a)))$$

Huang:

$$\cfrac{\cfrac{\cfrac{\cfrac{\bot \quad \top}{\exists x_1 \neg P(x_1)} \qquad \bot}{\top \quad \exists x_1 \forall x_2(\neg Q(x_2) \wedge \neg P(x_1))}}{\top \quad \forall x_0 \exists x_1 \forall x_2(\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0))}}{\forall x_0 \exists x_1 \forall x_2 \exists x_3(\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0) \vee S(x_3))}$$

Direct:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\bot \quad \top}{\exists x_1 \neg P(x_1)} \; x_1 \sim f(x) \qquad \bot}{\top \quad \exists x_1 \forall x_2(\neg Q(x_2) \wedge \neg P(f(x)))} \; x_2 \sim g(f(x)); x_1 < x_2}{\top \quad \forall x_0 \exists x_1 \forall x_2(\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0))} \; x_0 \sim a; x_0 < x_1, x_0 < x_2}{\forall x_0 \exists x_1 \forall x_2 \exists x_3(\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0) \vee S(x_3))} \; x_3 \sim h(g(f(a))); x_0 < x_3, x_1 < x_3, x_2 < x_3}$$

203b – many Σ-literals, coloring per occurrence

$$\cfrac{\cfrac{\cfrac{\overset{\Sigma}{R(x) \vee \neg P(f(x))} \quad \overset{\Sigma}{P(z) \vee Q(g(z))}}{R(x) \vee Q(gfx)} \; z \mapsto fx \quad \overset{\Sigma}{\neg Q(y) \vee S(h(y))}}{R(x) \vee S(hgfx)} \; y \mapsto gfx}{\cfrac{\overset{\Pi}{\neg R(a)} \qquad \qquad}{S(hgfa)} \; x \mapsto a}$$

$$\cfrac{\overset{\Pi}{\neg S(x_1)} \qquad \qquad}{\square} \quad x_1 \mapsto hgfa$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\bot \quad \bot}{\bot} \; z \mapsto fx \qquad \bot}{\top \quad R(a)} \; x \mapsto a}{\top \quad R(a)} \; x_1 \mapsto hgfa}{S(hgfa) \vee R(a)}$$

$\rightarrow \forall x_1 \exists x_2(R(x_1) \vee S(x_2))$

203b' – many Σ-literals, coloring per symbol, all predicates grey

$$\cfrac{\neg S(x_1)^{\Pi} \qquad \cfrac{\cfrac{\neg R(a)^{\Pi} \qquad R(x) \vee \neg P(f(x))^{\Sigma}}{R(a) \mid \neg P(fa)} \; x \mapsto a \qquad \cfrac{P(z) \vee Q(g(z))^{\Sigma}}{P(fa) \vee R(a) \mid Q(gfa)} \; z \mapsto fa \qquad \neg Q(y) \vee S(h(y))^{\Sigma}}{}}{}$$

TODO

Example where variables are not the outermost symbol but order is still relevant

204a

$\Sigma = \{P(f(x), g(f(x)))\}$

$\Pi = \{P(f(a), y)\}$

$\Rightarrow \forall x_1 \exists x_2 P(f(x_1), x_2)$

204b

$\Sigma = \{P(f^5(x), g(f(x)))\}$

$\Pi = \{P(f^5(a), y)\}$

$\Rightarrow \forall x_1 \exists x_2 P(f^5(x_1), x_2)$

example with aufschaukelnde unification, such that direction of arrow isn't clear

205a

$$\cfrac{\cfrac{}{P(ffy,gy)}\Pi \qquad \cfrac{\cfrac{}{\neg P(x,y)\vee Q(x)}\Sigma \qquad \cfrac{\cfrac{}{\neg R(a)}\Sigma \qquad \cfrac{}{\neg Q(ffz)\vee Rz}\Pi}{\neg R(a)\mid \neg Q(ffa)}\ z\mapsto a}{\cfrac{\neg R(a)\wedge Q(ffa)\mid \neg P(ffa,y)}{}\ x\mapsto ffa}}{(\neg R(a)\wedge Q(ffa))\vee \neg P(ffa,ga)}\ y\mapsto a$$

direct

$$\cfrac{\cfrac{}{P(ffy,gy)}\Pi \qquad \cfrac{\cfrac{}{\neg P(x,y)\vee Q(x)}\Sigma \qquad \cfrac{\cfrac{}{\neg R(a)}\Sigma \qquad \cfrac{}{\neg Q(ffz)\vee Rz}\Pi}{\exists x_1 \neg R(x_1)\mid \neg Q(ffa)}\ z\mapsto a}{\exists x_1\forall x_2(\neg R(x_1)\wedge Q(x_2))\mid \neg P(ffa,u)}\ x\mapsto ffa}{\exists x_1\forall x_2\forall x_3((\neg R(x_1)\wedge Q(x_2))\vee \neg P(x_2,x_3))}\ y\mapsto a, u\mapsto ga$$

ground:

$$\cfrac{\cfrac{}{P(ffa,ga)}\Pi \qquad \cfrac{\cfrac{}{\neg P(ffa,y)\vee Q(ffa)}\Sigma \qquad \cfrac{\cfrac{}{\neg R(a)}\Sigma \qquad \cfrac{}{\neg Q(ffa)\vee Ra}\Pi}{\neg R(a)\mid \neg Q(ffa)}}{\neg R(a)\wedge Q(ffa)\mid \neg P(ffa,a)}}{(\neg R(a)\wedge Q(ffa))\vee \neg P(ffa,ga)}$$

arrow lemma: $x_1\sim ffy, x_2\sim gy, y_3\sim a\ x_4\sim ffz, x_5\sim ffa, y_6\sim a,$

$$\cfrac{\cfrac{}{\forall x_1\forall x_4(P(x_1,x_2)\mid \top)}\Pi \qquad \cfrac{\cfrac{}{\neg P(x,y)\vee Q(x)\mid \bot}\Sigma \qquad \cfrac{\cfrac{}{\exists y_3(\neg R(y_3)\mid \bot)}\Sigma \qquad \cfrac{}{\forall x_4(\neg Q(x_4)\vee Rz\mid \bot)}\Pi}{\exists y_3\forall x_4((\neg R(y_3)\wedge \top)\vee (R(a)\wedge \bot)\mid \neg Q(x_4))}\ z\mapsto a}{\cfrac{\exists y_3\forall x_4((\neg Q(x_4\wedge \bot))\vee (Q(x)\sigma \wedge \neg R(y_3))\mid \neg P(x,y)\sigma)}{\exists y_3\forall x_4((Q(ffa)\wedge \neg R(y_3))\mid \neg P(ffa,y))\quad (*)}\ x\mapsto ffa}}{\cfrac{\exists y_3\forall x_4\forall x_5((Q(x_5)\wedge \neg R(y_3))\mid \neg P(x_5,y))}{(\neg P(x_5,y)\wedge \top)\vee (P(x_1,x_2)\wedge (Q(x_5)\wedge \neg R(y_3)))}\ y\mapsto a}}{(\neg P(x_5,a)\wedge \top)\vee (P(x_1,x_2)\wedge (Q(x_5)\wedge \neg R(y_3)))}$$

$$\forall x_1\forall x_2\exists y_3\forall x_4\forall x_5\forall x_6(\neg P(x_5,y_6)\wedge \top)\vee (P(x_1,x_2)\wedge (Q(x_5)\wedge \neg R(y_3)))$$

$(*)$ "luckily", same overbinding for ffa, so this works
dashed underline: problem, but does not cause issues here.

situations not critical here :'-(

205b $\sim$ 205a, but simpler

Suppose P occurs somewhere in Σ (result not that optimal in this setting, but correct)

not nice for proving, $\neg R(a)$ is a nice interpolant already

$$\cfrac{\cfrac{}{P(ffy,gy)}\Pi \qquad \cfrac{\cfrac{}{\neg R(a)}\Sigma \qquad \cfrac{}{\neg P(ffz,x)\vee Rz}\Pi}{\neg R(a)\mid \neg P(ffa,x)}\ z\mapsto a}{\neg R(a)\vee \neg P(ffa,ga)\mid \square}\ x\mapsto ga, y\mapsto a$$

$$\cfrac{\cfrac{}{\top \mid P(ffy,gy)}\Pi \qquad \cfrac{\cfrac{}{\bot \mid \neg R(a)}\Sigma \qquad \cfrac{}{\top \mid \neg P(ffz,x)\vee Rz}\Pi}{\exists x_1 \neg R(x_1)\mid \neg P(ffa,x)}\ z\mapsto a}{\exists x_1\forall x_2\forall x_3\neg R(x_1)\vee \neg P(x_2,x_3)\mid \square}\ x\mapsto ga, y\mapsto a$$

$\exists x_1 R(x_1)$

$\exists x_1\forall x_2\forall x_3(R(x_1)\vee \neg P(x_2,x_3))$

example to demonstrate that literals being resolved upon have to be overbound with the same variable

206a

$$\frac{\overset{\Pi}{R(f(x))} \quad \overset{\Sigma}{\neg R(y) \vee P(y)}}{(\neg R(x_3) \wedge \top) \vee (R(x_3) \wedge \bot) \mid P(x_3)} \quad \frac{\overset{\Pi}{\neg P(f(z)) \vee S(z)} \quad \overset{\Sigma}{\neg S(a)}}{(\neg S(y_2) \wedge \top) \vee (S(y_2) \wedge \bot) \mid \neg P(x_4)}$$

$$(\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2)) \Big)$$

Gist of this example: $P(f(x))$ is lifted to the left, but $P(f(a))$ to the right. So it's $P(x_3)$ vs $P(x_4)$, but both of them have to have the same variable.

$R(x_3) \in \mathrm{AI}_{\mathrm{mat}}(C_7)$

$P(x_3) \in \mathrm{AI}_{\mathrm{cl}}(C_7)$

$P(x_4) \in \mathrm{AI}_{\mathrm{cl}}(C_8)$

$\Sigma \models (\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2)) \Big)$

$\Sigma \models (\forall x_3)\forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee P(x_3) \Big)$

$\Sigma \not\models (\neg P(1) \wedge \neg R(0)) \vee P(0)$ // if $P \sim \{1\}$ and $R \sim \{0\}$

we know that for original clauses l and l' of $P(x_4)$ and $P(x_3)$, $l\sigma = l'\sigma$

hence same color, and can use different var as same value works.

inductive hypothesis:

$\Gamma \models \top \vee R(x_3)$

$\Gamma \models \bot \vee \neg R(y) \vee P(y)$

$\Gamma \models (\neg R(x_3) \wedge \top) \vee (R(x_3) \wedge \bot) \vee P(x_3) \quad \equiv \quad \neg R(x_3) \vee P(x_3)$

$\Gamma \models \top \vee \neg P(x_4) \vee S(z)$

$\Gamma \models \bot \vee \neg S(a)$

$\Gamma \models (\neg S(a) \wedge \top) \vee (S(a) \wedge \bot) \vee \neg P(x_4) \quad \equiv \quad \neg S(a) \vee \neg P(x_4)$

$\Gamma \models (\neg P(x_3) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(a))$

206b

WRONG: if a variable x_3 occurs, it always refers to $f(x)$, so it is always substituted to a particular value and cannot become $f(a)$ and $f(b)$ in the same clause as just the unifier σ is used.

$$\frac{\overset{\Pi}{R(f(x), f(x))} \quad \overset{\Sigma}{\neg R(y, u) \vee P(y, u)}}{(\neg R(x_3, x_3) \wedge \top) \vee (R(x_3, x_3) \wedge \bot) \mid P(x_3, x_3)} \quad \frac{\overset{\Pi}{\neg P(f(z), f(v)) \vee S(z, v)} \quad \overset{\Sigma}{\neg S(a, b)}}{(\neg S(y_2, y_6) \wedge \top) \vee (S(y_2, y_6) \wedge \bot) \mid \neg P(x_4, x_7)}$$

$$(\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2)) \Big)$$

$f(x) \vee g(x)$ with f, g different colors

$$\cfrac{\cfrac{}{\neg P(z)}^{\Pi} \quad \cfrac{\cfrac{}{\neg R(g(a))}^{\Pi} \quad \cfrac{\cfrac{}{P(f(x)) \vee Q(x)}^{\Sigma} \quad \cfrac{}{R(g(y)) \vee \neg Q(y)}^{\Pi}}{\forall x_1 Q(x_1) \mid P(f(x)) \vee R(g(x))} \; y \mapsto x}{\forall x_1 Q(x_1) \mid P(f(a)) // R(g(a))} \; x \mapsto a}{\forall x_1 \exists x_2 \big(Q(x_1) \vee P(x_2)\big) // R(g(a))} \; z \mapsto f(a)$$

$\Rightarrow$ free vars in the interpolant have to be overbound (if there are arrows, but we can just always do so)

misc examples

201a

$$\cfrac{\cfrac{\overset{\Sigma}{P(x,y)} \vee \neg Q(y) \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(y)} \; x \mapsto a \qquad \cfrac{\overset{\Sigma}{Q(f(z))} \vee R(z) \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a))} \; z \mapsto a}{\square} \; y \mapsto f(a)$$

$$\cfrac{\cfrac{\bot \qquad \top}{P(a,y)} \; x \mapsto a \qquad \cfrac{\bot \qquad \top}{R(a)} \; z \mapsto a}{P(a,f(a)) \vee R(a)} \; y \mapsto f(a) \qquad\qquad \cfrac{\cfrac{\bot \qquad \top}{\forall x_1 P(x_1,y)} \; x \mapsto a \qquad \cfrac{\bot \qquad \top}{\forall x_3 R(x_3)} \; z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 (P(x_1,x_2) \vee R(x_3))} \; y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

201b

$$\cfrac{\cfrac{\overset{\Sigma}{P(x,f(y))} \vee \neg Q(f(y)) \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(f(y))} \; x \mapsto a \qquad \cfrac{\overset{\Sigma}{Q(f(z))} \vee R(z) \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a))} \; z \mapsto a}{\square} \; y \mapsto f(a)$$

$$\cfrac{\cfrac{\bot \qquad \top}{P(a,f(y))} \; x \mapsto a \qquad \cfrac{\bot \qquad \top}{R(a)} \; z \mapsto a}{P(a,f(a)) \vee R(a)} \; y \mapsto a \qquad\qquad \cfrac{\cfrac{\bot \qquad \top}{\forall x_1 \exists x_2 P(x_1,x_2)} \; x \mapsto a \qquad \cfrac{\bot \qquad \top}{\forall x_3 R(x_3)} \; z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_3)} \; y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

16

arrow in element which is not in interpolant or resolution clause

206

$$\cfrac{\cfrac{\overset{\Sigma}{P(x) \vee \neg Q(f(x))} \qquad \overset{\Pi}{\neg P(a)}}{\forall x_1 P(x_1) \mid \neg Q(f(a))} \; x \mapsto a \qquad \cfrac{\overset{\Sigma}{Q(y) \vee R(g(y))} \qquad \overset{\Pi}{\neg R(z)}}{\exists x_2 R(x_2) \mid Q(y)} \; z \mapsto g(y)}{\forall x_1 \exists x_2 (P(x_1) \vee R(x_2)) \mid \Box} \; y \mapsto f(a)$$

$$P(a) \vee R(g(f(a)))$$

for first interpolant, $\Sigma \not\models \ell_{\Delta,x}[\mathrm{PI}(C)] \vee C$

$\Rightarrow$ need to overbind clause as well