Interpolation in First-Order Logic with Equality

Master Thesis Presentation

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Notes (cover this somewhere)

- interpolants can be extracted from resolution proofs since skolemisation and the cnf transformation doesn't change the set of interpolants

Agenda

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Introduction

- Want concrete algorithms for FOL/EQ
  $\Rightarrow$ Little attention so far
- Present different constructive proofs

Agenda

Craig Interpolation

Theorem (Craig). *Let Γ and Δ be sets of first-order formulas where*

- Γ *contains red and gray symbols and*
- Δ *contains blue and gray symbols*

such that:

- $\Gamma \models \Delta$

Then there is a interpolant I containing only gray symbols such that:

- $\Gamma \models I$
- $I \models \Delta$

$$\begin{tikzcd}
\Gamma \arrow[rr] \arrow[dr] & & \Delta \\
& I \arrow[ur] &
\end{tikzcd}$$

Interpolation and Equality

Example

- Let $\Gamma = \{P(a), \neg P(b)\}$ and $\Delta = \{a \neq b\}$.
- Clearly $\Gamma \models \Delta$.
- Only possible interpolant: $a \neq b$

Interpolation and Equality

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Applications

- Proof of Beth's Definability Theorem
- Model checking
- Detecting loop invariants

Agenda

Proof by reduction

Reduction to FOL without equality and function symbols:

Translate equality and function symbols:

$$\Big(P(c)\Big)^* \equiv \exists x(C(x) \wedge P(x))$$

$$\Big(P(f(c))\Big)^* \equiv \exists x(\exists y(C(y) \wedge F(y,x)) \wedge P(x))$$

$$\Big(s = t\Big)^* \equiv E(s,t)$$

Add axioms for equality and new predicate symbols:

$$\varphi \;\to\; \Big(\mathrm{T}_E \wedge \bigwedge_{f \in \mathrm{FS}} \mathrm{T}_f\Big) \supset \varphi^*$$

Clearly φ and φ^* are equisatisfiable.

Proof in FOL without equality and FS

Lemma (Maehara)

*Let Γ and Δ be sets of first-order formulas without equality and function symbols such that $\Gamma \vdash \Delta$ is provable in **sequent calculus**. Then for any partition $\langle(\Gamma_1; \Delta_1), (\Gamma_2; \Delta_2)\rangle$ there is an interpolant I such that*

1. $\Gamma_1 \vdash \Delta_1, I$ *is provable*
2. $\Gamma_2, I \vdash \Delta_2$ *is provable*
3. $\mathrm{L}(I) \subseteq \mathrm{L}(\Gamma_1, \Delta_1) \cap \mathrm{L}(\Gamma_2, \Delta_2)$

[Baaz and Leitsch, 2011] presents a strengthening which includes function symbols.

Open question: Can it be extended to include equality?

Proof in FOL without equality and FS

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1. Introduction
2. Craig Interpolation (10 min)
   - Interpolation and Equality (1 slide)
   - Applications (1 slide)
3. Proof by reduction (6 min)
4. Interpolant extraction from resolution proofs (10 min)
   - Introduction
   - Phase one (propositional, inductive)
   - Phase two (lifting, ordering)
   - Optional: Number of quantifier alternations
   - Proof with one phase: Can lift earlier.
5. Semantic Proof (6 min)
6. Conclusion
   - References

Interpolant extraction

Motivation

- Proof by reduction is impractical
- Goal: Compute interpolants from proof
- The following is based on [Huang, 1995]

Interpolant extraction from resolution proofs

- Skolemisation and clausal form transformation do no alter the set of interpolants
- Have to use "reverse" (but equivalent) formulation of interpolation

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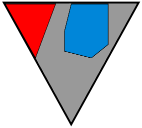
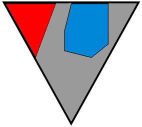
Huang's algorithm

Proof:



$\Downarrow$ *Extract propositional interpolant structure from proof*

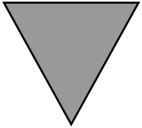
Propositional Interpolant:



$\ldots Q(f(c), c) \ldots$

$\Downarrow$ *Replace colored function and constant symbols*

Prenex First-Order Interpolant:



$\exists x_3 \forall x_5 \ldots Q(x_5, x_3) \ldots$

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Conclusion

- Craig's and Huang's proof based interpolant extraction from proofs
  $\Rightarrow$ differ in applicability
- Craig shows that the interpolation theorem holds also in FOL/EQ
- Huang shows that interpolants can efficiently be extracted in FOL/EQ
  - Does not require different methods
  - Little attention so far in research
- Interpolation also allows for a model theoretic approach

Baaz, M. and Leitsch, A. (2011).
Methods of Cut-Elimination.
Trends in Logic. Springer.

Huang, G. (1995).
Constructing Craig Interpolation Formulas.
In *Proceedings of the First Annual International Conference on Computing and Combinatorics*, COCOON '95, pages 181–190, London, UK, UK. Springer-Verlag.