$\Sigma' = \{R(z) \vee \exists x P(f(x)), \neg Q(x), \}$

$\Pi' = \{\forall y\, g(y) = y, \forall y \neg P(g(y)) \vee Q(y), \neg R(d)\}$

$\Sigma = \mathrm{sk}(\Sigma') = \{R(z) \vee P(f(c)), \neg Q(y), \}$

$\Pi = \mathrm{sk}(\Pi') = \{g(u) = u, \neg P(g(v)) \vee Q(v), \neg R(d)\}$

$L(\Sigma) = \{R, P, Q, f, z, x, c\}$

$L(\Pi) = \{R, P, Q, g, u, v, d\}$

Refutation:

$$\dfrac{\dfrac{R(z) \vee P(f(c))_\Sigma \qquad \neg R(d)_\Pi}{P(f(c))}\ z \mapsto d \qquad \dfrac{\dfrac{\neg P(g(v)) \vee Q(v)_\Pi \qquad \neg Q(y)_\Sigma}{\neg P(g(y))}\ v \mapsto y \qquad g(u) = u_\Pi}{\neg P(u)}\ y \mapsto u}{\square}\ u \mapsto f(c)$$

Interpolants:

$$\dfrac{\dfrac{\bot \qquad \top}{(\neg R(d) \wedge \bot) \vee (R(d) \wedge \top) \equiv R(d)}\ \theta_0 \qquad \dfrac{\dfrac{\top \qquad \bot}{(\neg Q(y) \wedge \top) \vee (Q(y) \wedge \top) \equiv \neg Q(y)}\ \theta_1 \qquad \top}{(\neg Q(u) \wedge g(u) = u) \vee (\top \wedge g(u) \neq u)}\ \theta_2}{(\neg P(f(c)) \wedge R(d)) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge g(f(c)) = f(c)) \quad \vee \quad g(f(c)) \neq f(c)))}\ \theta_3$$

Relative interpolant properties:

θ_0 :	$\Sigma \vdash R(d) \vee P(f(c))$	$\Pi \vdash \neg R(d) \vee P(f(c))$
θ_1 :	$\Sigma \vdash \neg Q(y) \vee \neg P(g(y))$	$\Pi \vdash Q(y) \vee \neg P(g(y))$
θ_2 :	$\Sigma \vdash (\neg Q(u) \wedge g(u) = u) \vee g(u) \neq u \quad \vee \quad \neg P(u)$	$\Pi \vdash \neg((\neg Q(u) \wedge g(u) = u) \vee g(u) \neq u) \quad \vee \quad \neg P(u)$ $\Pi \vdash ((Q(u) \vee g(u) \neq u) \wedge g(u) = u) \quad \vee \quad \neg P(u)$
θ_3 :	$\Sigma \vdash \theta_3$ Proof: Either $\neg P(f(c))$, then $R(d)$. Otw. either $g(f(c)) \neq f(c)$. Otw. also $\neg Q(f(c))$.	$\Pi \vdash \neg\theta_3$ Proof: $\neg(\neg P(fc) \wedge R(d)) \quad \vee \quad (P(fc) \wedge (\neg Q(fc) \wedge g(fc) = fc) \quad \vee \quad g(fc) \neq fc)$ $\equiv (P(fc) \vee \neg R(d)) \quad \wedge \quad (\neg P(fc) \vee (Q(fc) \vee g(fc) \neq fc) \quad \wedge \quad g(fc) = fc)$ Have $g(fc) = fc$ and $\neg R(d)$, so remaining: $\neg P(fc) \vee Q(fc)$. Get by axiom and unification with $g(u) = u$.

$\Sigma = \{R(z) \vee P(f(c)), \neg Q(y), \}$
$\Pi = \{g(u) = u, \neg P(g(v)) \vee Q(v), \neg R(d)\}$

Propositional refutation tree (no non-trivial unifiers):

$$\dfrac{\dfrac{R(d) \vee P(f(c))_\Sigma \qquad \neg R(d)_\Pi}{P(f(c))} \qquad \dfrac{\dfrac{\neg P(g(f(c))) \vee Q(f(c))_\Pi \qquad \neg Q(f(c))_\Sigma}{\neg P(g(f(c)))} \qquad g(f(c)) = f(c)_\Pi}{\neg P(f(c))}}{\square}$$

Lifting:

terms: $g(f(c)), f(c), d$
max Π-terms: $\{g(f(c)), d\} \sim \{x_1, x_2\}$
max Σ-terms: $\{f(c)\} \sim \{x_3\}$

$\overline{(\neg P(f(c)) \wedge R(d)) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge g(f(c)) = f(c)) \quad \vee \quad g(f(c)) \neq f(c)))}(x_1, x_2)$
$\Leftrightarrow \neg P(f(c)) \wedge R(x_2) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge x_1 = f(c)) \quad \vee \quad x_1 \neq f(c)))$

By Lemma 12, $\Sigma \models \overline{\theta_3}$ (proof from above still goes through).
$\hat{\theta}(x_3) = (\neg P(x_3) \wedge R(x_2)) \quad \vee \quad (P(x_3) \wedge ((\neg Q(x_3) \wedge x_1 = x_3) \quad \vee \quad x_1 \neq x_3))$
quantifiers according to order: $|d| < |f(c)| < |g(f(c))|$
$\theta = \forall x_2 \exists x_3 \forall x_1 (\neg P(x_3) \wedge R(x_2)) \quad \vee \quad (P(x_3) \wedge (\neg Q(x_3) \vee x_1 \neq x_3))$
$\neg\theta = \exists x_2 \forall x_3 \exists x_1 (P(x_3) \vee \neg R(x_2)) \quad \wedge \quad (\neg P(x_3) \vee (Q(x_3) \wedge x_1 = x_3))$
$\Rightarrow \Sigma \vdash \theta; \Pi \vdash \neg\theta$

Example 2:

$\Sigma = \{P(c), \neg P(d)\}$

$\Pi = \{P(d) \lor g(u) = u, \neg P(g(x))\}$

Refutation:

$$\dfrac{\dfrac{\dfrac{P(d) \lor g(u) = u_{\Pi} \qquad \neg P(d)_{\Sigma}}{g(u) = u} \qquad \neg P(g(x)_{\Pi}}{\neg P(x)}\; u \mapsto x \qquad P(c)_{\Sigma}}{\square}\; x \mapsto c$$

Relative interpolants:

$$\dfrac{\dfrac{\dfrac{\top \qquad \bot}{(\neg P(d) \land \top) \lor (P(d) \land \bot) \equiv \neg P(d)} \qquad \top}{(g(x) = x \land \top) \lor (g(x) \neq x \land \neg P(d))}\; u \mapsto x \qquad \bot}{(\neg P(c) \land \bot) \lor P(c) \land (g(c) = c \lor (g(c) \neq c \land \neg P(d)))}\; x \mapsto c$$

$\theta = P(c) \land (g(c) = c \lor \neg P(d))$

$\neg\theta = \neg P(c) \lor (g(c) \neq c \land P(d))$

terms: $g(c), c, d$

max Π-terms: $g(c)$

max Σ-terms: c

ordered by length ASCENDING: $\{c, g(c)\}$

$\overline{\theta}(x_2) = P(c) \land (x_2 = c \lor \neg P(d))$

$\hat{\theta}(x_1) = P(x_1) \land (x_2 = x_1 \lor \neg P(d))$

$\Sigma \vdash \exists x_1 \forall x_2 P(x_1) \land (x_2 = x_1 \lor \neg P(d))$

$\Pi \vdash \neg\exists x_1 \forall x_2 P(x_1) \land (x_2 = x_1 \lor \neg P(d))$

$\Pi \vdash \forall x_1 \exists x_2 \neg P(x_1) \lor (x_2 \neq x_1 \land P(d))$

A possible interpolant: $\neg P(d) \land \exists x P(x)$