# Contents



## 0.1 Lemmas from other pdf

ma:lifting_order_not_relevant⟩ **Lemma 1.** $\ell_\Gamma[\ell_\Delta[\varphi]] = \ell_\Delta[\ell_\Gamma[\varphi]]$.

## 0.2 Proof

**Definition 2** (Substitution $\tau(\iota)$)**.** For an inference $\iota$ with $\sigma = \mathrm{mgu}(\iota)$, we define the infinite substitution $\tau(\iota)$ with $\mathrm{dom}(\tau(\iota)) = \mathrm{dom}(\sigma) \cup \{z_s \mid s\sigma \neq s\}$ as follows for a variable $x$:

define infinite substitutions properly and apply definition here

$$x\tau(\iota) = \begin{cases} x\sigma & x \text{ is a non-lifting variable} \\ z_{t\sigma} & x \text{ is a lifting variable } z_t \end{cases}$$

If the inference $\iota$ is clear from the context, we abbreviate $\tau(\iota)$ by $\tau$. △

⟨lemma:lifting_tau_commute⟩ **Lemma 3.** *For a formula or term $\varphi$ and an inference $\iota$ such that $\tau = \tau(\iota)$, $\ell[\ell[\varphi]\tau] = \ell[\varphi\tau]$.*

*Proof.* We proceed by induction.

- Suppose that $t$ is a grey constant or function symbol of the form $f(t_1, \ldots, t_n)$. Then we can derive the following, where (IH) signifies a deduction by virtue of the induction hypothesis.

$$\begin{aligned} \ell[\ell[t]\tau] &= \ell[\ell[f(t_1, \ldots, t_n)]\tau] \\ &= \ell[f(\ell[t_1]\tau, \ldots, \ell[t_n]\tau)] \\ &= f(\ell[\ell[t_1]\tau], \ldots, \ell[\ell[t_n]\tau]) \\ &\overset{\text{(IH)}}{=} f(\ell[t_1\tau], \ldots, \ell[t_n\tau]) \\ &= \ell[f(t_1, \ldots, t_n)\tau] \\ &= \ell[t\tau] \end{aligned}$$

- Suppose that $t$ is a colored constant or function symbol. Then:

$$\ell[\ell[t]\tau] = \ell[z_t\tau] = \ell[z_{t\sigma}] = z_{t\sigma} = \ell[t\sigma] = \ell[t\tau]$$

- Suppose that $t$ is a variable $x$. Then:

$$\ell[\ell[t]\tau] = \ell[\ell[x]\tau] = \ell[x\tau] = \ell[t\tau]$$

- Suppose that $t$ is a lifting variable $z_t$. Then:

$$\ell[\ell[z_t]\tau] = \ell[z_t\tau]$$ □

**Definition 4** (Incrementally lifted interpolant)**.** Let $\pi$ be a resolution refutation of $\Gamma \cup \Delta$. We define $\mathrm{LI}(\pi)$ and $\mathrm{LI}_{\mathrm{cl}}(\pi)$ to be $\mathrm{LI}(\Box)$ and $\mathrm{LI}_{\mathrm{cl}}(\Box)$ respectively, where $\Box$ is the empty clause derived in $\pi$.

Let $C$ be a clause in $\pi$. For a literal $\lambda$ in $C$, we denote the corresponding literal in $\mathrm{LI}_{\mathrm{cl}}(C)$ by $\lambda_{\mathrm{LIcl}}$, whose existence is ensured Lemma 5.

We define $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ as follows:

Base case. If $C \in \Gamma$, $\mathrm{LI}(C) \overset{\text{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{LI}(C) \overset{\text{def}}{=} \top$.

In any case, $\mathrm{LI}_{\mathrm{cl}}(C) \overset{\text{def}}{=} \ell[C]$.

Resolution. If the clause $C$ is the result of a resolution step $\iota$ of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier $\sigma$ such that $l\sigma = l'\sigma$, then define $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ as follows:

$$\mathrm{LI}_{\mathrm{cl}}(C) \overset{\mathrm{def}}{=} \ell[(\mathrm{LI}_{\mathrm{cl}}(C_1) \backslash \{l_{\mathrm{LIcl}}\})\tau] \ \vee \ \ell[(\mathrm{LI}_{\mathrm{cl}}(C_2) \backslash \{l'_{\mathrm{LIcl}}\})\tau]$$

1. If $l$ is $\Gamma$-colored: $\mathrm{LI}(C) \overset{\mathrm{def}}{=} \ell[\mathrm{LI}(C_1)\tau] \vee \ell[\mathrm{LI}(C_2)\tau]$
2. If $l$ is $\Delta$-colored: $\mathrm{LI}(C) \overset{\mathrm{def}}{=} \ell[\mathrm{LI}(C_1)\tau] \wedge \ell[\mathrm{LI}(C_2)\tau]$
3. If $l$ is grey: $\mathrm{LI}(C) \overset{\mathrm{def}}{=} (\ell[l_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_2)\tau]) \vee (\neg\ell[l'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_1)\tau])$

Factorisation. If the clause $C$ is the result of a factorisation step $\iota$ of $C_1 : l \vee l' \vee D$ using a unifier $\sigma$ such that $l\sigma = l'\sigma$, then $\mathrm{LI}(C) \overset{\mathrm{def}}{=} \ell[\mathrm{LI}(C_1)\tau]$ and $\mathrm{LI}_{\mathrm{cl}}(C) \overset{\mathrm{def}}{=} \ell[(\mathrm{LI}_{\mathrm{cl}}(C_1) \backslash \{l'_{\mathrm{LIcl}}\})\tau]$. △

ws_clause_plus_literals_equal⟩ **Lemma 5.** *Let $C$ be a clause in a resolution refutation of $\Gamma \cup \Delta$.*

*Then for every literal $\lambda$ in $C$, there exists a literal $\lambda_{\mathrm{LIcl}}$ in $\mathrm{LI}_{\mathrm{cl}}(C)$ such that $\lambda_{\mathrm{LIcl}} = \ell[\lambda]$ and for resolved or factorised literals $l$ and $l'$ of a resolution or factorisation inference $\iota$, we have that $\ell[l_{\mathrm{LIcl}}\tau] = \ell[l'_{\mathrm{LIcl}}\tau]$.*

*Proof.* We proceed by induction.

Base case. For $C \in \Gamma \cup \Delta$, $\mathrm{LI}_{\mathrm{cl}}(C)$ is defined to be $\ell[C]$.

Resolution/Factorisation. Suppose the clause $C$ is the result of a resolution or factorisation inference $\iota$ of the clauses $\bar{C}$ with $\sigma = \mathrm{mgu}(\iota)$.

Every literal in $C$ is of the form $\lambda\sigma$ for a literal $\lambda$ in $C_i \in \bar{C}$.

By the induction hypothesis, $\ell[\lambda]$ occurs in $\mathrm{LI}_{\mathrm{cl}}(C_i)$. By the construction of $\mathrm{LI}_{\mathrm{cl}}(C)$ and as $\lambda$ is not a resolved or factorised literal, $\mathrm{LI}_{\mathrm{cl}}(C)$ contains a literal of the form $\ell[\ell[\lambda]\tau]$. But by Lemma 3, this is nothing else than $\ell[\lambda\tau]$. As $\lambda$ occurs in the resolution derivation, it does not contain lifting variables. Hence we get by the definition of $\tau$ that $\ell[\lambda\tau] = \ell[\lambda\sigma]$.

Let $l$ and $l'$ be the resolved or factorised literals of $\iota$. In order to show that $\ell[l_{\mathrm{LIcl}}\tau] = \ell[l'_{\mathrm{LIcl}}\tau]$, consider that by the induction hypothesis, this is nothing else than $\ell[\ell[l]\tau] = \ell[\ell[l']\tau]$. But by applying a similar argument as above, this equation is equivalent to $\ell[l\sigma] = \ell[l'\sigma]$, which is implied by $l\sigma = l'\sigma$. □

**Definition 6.** $\mathrm{LI}^{\Delta}(C)$ and $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$ for a clause $C$ are defined as $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ respectively with the difference that in its inductive definition, every lifting $\ell[\varphi]$ for a formula or term $\varphi$ is replaced by a lifting of only the $\Delta$-terms $\ell_{\Delta}[\varphi]$. △

*Remark.* Many results involving $\mathrm{LI}(C)$ or $\mathrm{LI}_{\mathrm{cl}}(C)$ are valid for $\mathrm{LI}^{\Delta}(C)$ or $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$ in a formulation which is adapted accordingly. This can easily be seen by the following proof idea:

Let $f_1, \ldots, f_n$ be all $\Gamma$-colored function or constant symbols occurring in $C$, $c$ a fresh constant symbol and $g$ a fresh $n$-ary function symbol. Construct a formula $\varphi : g(t_1, \ldots, t_n) = g(t_1, \ldots, t_n)$, such that $t_i = f_i(c_1, \ldots, c_m)$ for $1 \leq i \leq n$ where $m$ is the arity of $f_i$ and $c_j = c$ for $1 \leq j \leq m$. Let $\Delta' = \Delta \cup \{\varphi\}$ and apply the desired result to the initial clause sets $\Gamma$ and $\Delta'$.

Under this construction, every originally $\Gamma$-colored symbol is now grey, which implies that $\mathrm{LI}(C) = \mathrm{LI}^{\Delta}(C)$ as well as $\mathrm{LI}_{\mathrm{cl}}(C) = \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$. But $\Delta \vDash \psi \Leftrightarrow \Delta' \vDash \psi$ for any formula $\psi$. $\triangle$

⟨lemma:no_colored_terms⟩ **Lemma 7.** *Let $C$ be a clause of a resolution refutation of $\Gamma \cup \Delta$. $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ do not contain colored symbols.*

*Proof.* For $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$, consider the following: In the base case of the inductive definitions of $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$, no colored symbols occur. In the inductive steps, any colored symbol which is added by $\tau$ to intermediary formulas is lifted. □

⟨lemma:substitute_and_lift⟩ **Lemma 8.** *Let $\sigma$ be a substitution and $F$ a formula without $\Phi$-colored terms such that for a set of formulas $\Psi$ which does not contain $\Phi$-lifting variables, $\Psi \vDash F$. Then $\Psi \vDash \ell_{\Phi}[F\sigma]$.*

*Proof.* $\ell_{\Phi}[F\sigma]$ is an instance of $F$: $\sigma$ substitutes variables either for terms which do not contain $\Phi$-colored symbols or by terms containing $\Phi$-colored symbols. For the first kind, the lifting has no effect. For the latter, the lifting only replaces subterms of the terms introduced by the substitution by a lifting variable such that the original structure of $F$ remains invariant as it by assumption does not contain colored terms. □

⟨lemma:gamma_proves_pide⟩ **Lemma 9.** *Let $C$ be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then $\Gamma \vDash \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$.*

*Proof.* We proceed by induction of the strengthening $\Gamma \vDash \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_{\Gamma})$.

Base case. For $C \in \Gamma$, $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_{\Gamma}) = \ell_{\Delta}[C] = C$. Hence $\Gamma \vDash \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_{\Gamma})$.

For $C \in \Delta$, $\mathrm{LI}^{\Delta}(C) = \top$, so $\Gamma \vDash \mathrm{LI}^{\Delta}(C)$.

Resolution. Suppose the clause $C$ is the result of a resolution step $\iota$ of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

We define the following abbreviations:

$\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_{\Gamma})^* = \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_{\Gamma} \backslash \{l_{\mathrm{LIcl}^{\Delta}}\})$

$\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_{\Gamma})^* = \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_{\Gamma} \backslash \{\neg l'_{\mathrm{LIcl}^{\Delta}}\})$

Hence the induction hypothesis can be stated as follows:

$\Gamma \vDash \mathrm{LI}^{\Delta}(C_1) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_{\Gamma})^* \vee (l_{\mathrm{LIcl}^{\Delta}})_{\Gamma}$

$\Gamma \vDash \mathrm{LI}^{\Delta}(C_2) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_{\Gamma})^* \vee \neg(l'_{\mathrm{LIcl}^{\Delta}})_{\Gamma}$

By Lemma 7, $\mathrm{LI}^{\Delta}(C_i)$ and $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_i)$ for $i \in \{1, 2\}$ do not contain $\Delta$-colored terms. Hence we are able to apply Lemma 8 in order to obtain

$\Gamma \overset{(\circ)}{\vDash} \ell_{\Delta}[\mathrm{LI}^{\Delta}(C_1)\tau] \vee \ell_{\Delta}[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_{\Gamma})^*\tau] \vee \ell_{\Delta}[(l_{\mathrm{LIcl}^{\Delta}})_{\Gamma}\tau]$

$\Gamma \overset{(*)}{\vDash} \ell_{\Delta}[\mathrm{LI}^{\Delta}(C_2)\tau] \vee \ell_{\Delta}[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_{\Gamma})^*\tau] \vee \neg\ell_{\Delta}[(l'_{\mathrm{LIcl}^{\Delta}})_{\Gamma}\tau]$

By Lemma 5, we obtain that $\ell_{\Delta}[l_{\mathrm{LIcl}^{\Delta}}\tau] = \ell_{\Delta}[l'_{\mathrm{LIcl}^{\Delta}}\tau]$.

Now we distinguish cases based on the color of the resolved literal:

- Suppose that $l$ is $\Gamma$-colored. Then as $\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$, we can perform a resolution step on ($\circ$) and ($*$), which gives that $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_2)_\Gamma)^*\tau]$. This however is nothing else than $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$.
- Suppose that $l$ is $\Delta$-colored. Then ($\circ$) and ($*$) simply to the following:
  $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau]$
  $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_2)_\Gamma)^*\tau]$
  These however imply that $\Gamma \models \mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^* \vee \mathrm{LI}^\Delta_{\mathrm{cl}}((C_2)_\Gamma)^* \vee (\ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \wedge \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau])$, which is nothing else than $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$.
- Suppose that $l$ is grey. Suppose that $M$ is a model of $\Gamma$ such that $M \not\models \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$, i.e. $M \not\models \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_2)_\Gamma)^*\tau]$. Then $M \models \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau]$ as well as $M \models \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau] \vee \neg\ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$. Due to $\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$, we obtain that $M \models (\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] \wedge \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau]) \vee (\neg\ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau] \wedge \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau])$, which is nothing else than $M \models \mathrm{LI}^\Delta(C)$.

Factorisation. Suppose the clause $C$ is the result of a factorisation inference $\iota$ of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

We introduce the abbreviation $\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^* = \mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma \backslash \{l_{\mathrm{LIcl}^\Delta}, \neg l'_{\mathrm{LIcl}^\Delta}\})$ and express the induction hypothesis as follows:

$\Gamma \models \mathrm{LI}^\Delta(C_1) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^* \vee (l_{\mathrm{LIcl}^\Delta})_\Gamma \vee \neg(l'_{\mathrm{LIcl}^\Delta})_\Gamma$

By Lemma 7, $\mathrm{LI}^\Delta(C_i)$ and $\mathrm{LI}^\Delta_{\mathrm{cl}}(C_i)$ for $i \in \{1,2\}$ do not contain $\Delta$-colored terms. Hence we are able to apply Lemma 8 in order to obtain

$\Gamma \overset{(\circledast)}{\models} \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[(l_{\mathrm{LIcl}^\Delta})_\Gamma\tau] \vee \neg\ell_\Delta[(l'_{\mathrm{LIcl}^\Delta})_\Gamma\tau]$

As by Lemma 5 we get that $\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$, we can perform a factorisation step on ($\circledast$) to obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[(l_{\mathrm{LIcl}^\Delta})_\Gamma\tau]$. But this is nothing else than $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C_\Gamma)$. □

?⟨def:arrow_quantifier_block⟩? **Definition 10** (Quantifier block)**.** Let $C$ be a clause in a resolution refutation $\pi$ of $\Gamma \cup \Delta$ and $\bar{x}$ the $\Delta$-lifting variables and $\bar{y}$ the $\Gamma$-lifting variables occurring in $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$. $Q(C)$ denotes an arrangement of the elements of $\{\forall x_t \mid x_t \in \bar{x}\} \cup \{\exists y_t \mid y_t \in \bar{y}\}$ such that for two lifting variable $z_s$ and $z_r$, if $s$ is a subterm of $r$, then $z_s$ is listed before $z_r$. We denote $Q(\Box)$ by $Q(\pi)$. △

⟨lemma:gamma_lifted_lide⟩ **Lemma 11.** *For a clause $C$ of a resolution refutation of $\Gamma \cup \Delta$, $\ell_\Gamma[\mathrm{LI}^\Delta(C)] = \mathrm{LI}(C)$ and $\ell_\Gamma[\mathrm{LI}^\Delta_{\mathrm{cl}}(C)] = \mathrm{LI}_{\mathrm{cl}}(C)$.*

*Proof.* We proceed by induction.

Base case. For $C \in \Gamma \cup \Delta$, $\mathrm{LI}^\Delta_{\mathrm{cl}}(C) = \ell_\Delta[C]$. By Lemma 1, $\ell_\Gamma[\ell_\Delta[C]] = \ell[C]$, so $\ell_\Gamma[\mathrm{LI}^\Delta_{\mathrm{cl}}\, C] = \ell[C] = \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$.

$\mathrm{LI}^\Delta(C)$ does not contain colored symbols.

Inductions step. Suppose the clause $C$ is the result of a resolution or factorisation inference $\iota$ of the clauses $\bar{C}$.

Every literal in $\mathrm{LI}(C)$ or $\mathrm{LI}_{\mathrm{cl}}(C)$ is of the form $\ell[\lambda\tau]$ for some $\lambda$ in $\mathrm{LI}(C_i)$ or $\mathrm{LI}_{\mathrm{cl}}(C_i)$ for some $C_i \in \bar{C}$.

Every literal in $\mathrm{LI}^{\Delta}(C)$ or $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$ is of the form $\ell_{\Delta}[\lambda\tau]$ for some $\lambda$ in $\mathrm{LI}^{\Delta}(C_i)$ or $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_i)$ for some $C_i \in \bar{C}$.

Hence it suffices to show that for a literal $\lambda$ in $\mathrm{LI}^{\Delta}(C_i)$ or $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_i)$ and its corresponding literal $\kappa$ in $\mathrm{LI}(C_i)$ or $\mathrm{LI}_{\mathrm{cl}}(C_i)$ for some $C_i \in \bar{C}$ that $\ell_{\Gamma}[\ell_{\Delta}[\lambda\tau]] = \ell[\kappa\tau]$.

By the induction hypothesis, $\ell_{\Gamma}[\lambda] = \kappa$. By Lemma 7, no $\Delta$-terms occur in $\lambda$. Hence $\ell[\lambda] = \kappa$ and also $\ell[\ell[\lambda]\tau] = \ell[\kappa\tau]$. By Lemma 3, $\ell[\lambda\tau] = \ell[\kappa\tau]$, which by Lemma 1 is nothing else than $\ell_{\Gamma}[\ell_{\Delta}[\lambda\tau]] = \ell[\kappa\tau]$. ☐

:gamma_proves_quantified_lide⟩ **Lemma 12.** *For a clause $C$ of a resolution refutation of $\Gamma \cup \Delta$, $\Gamma \models Q(C)(\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C))$.*

*Proof.* By Lemma 11 $\ell_{\Gamma}[\mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)] = \mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$.

By Lemma 9, $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$. Hence the terms in $\mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$ provide witness terms for the $\Gamma$-lifting variables in $\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$, which are existentially quantified in $Q(C)(\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C))$.

Furthermore, the ordering imposed on the quantifiers in $Q(C)$ implies that if a $\Delta$-lifting variable $x_s$ occurs in a witness term for a $\Gamma$-lifting variable $y_r$, $y_r$ is quantified in the scope of the quantifier of $x_s$ as $s$ is a subterm of $r$. This however ensures that the witness terms are valid. ☐

⟨lemma:li_symmetry⟩ **Lemma 13.** *Let $\pi$ be a refutation of $\Gamma \cup \Delta$ and $\hat{\pi}$ be $\pi$ with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then for a clause $C$ in $\pi$ and its corresponding clause $\hat{C}$ in $\hat{\pi}$, $Q(C)(\mathrm{LI}(C)) \Leftrightarrow Q(\hat{C})(\mathrm{LI}(\hat{C}))$.*

*Proof.* Note that $\mathrm{LI}_{\mathrm{cl}}$ is defined irrespective of the coloring, so $\mathrm{LI}_{\mathrm{cl}}(C) = \mathrm{LI}_{\mathrm{cl}}(\hat{C})$.

Consider furthermore that liftings variables of $C$ and $\hat{C}$ only differ in the variable symbol, but not in the index, and that the quantifier type of any given lifting variable in $C$ is exactly contrary to the corresponding one in $\hat{C}$. Hence for any formula $\phi$, $Q(C)\neg\phi \Leftrightarrow \neg Q(\hat{C})\phi$.

It remains to show that $\mathrm{LI}(C) \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$, which we do be induction:

Base case. If $C \in \Gamma$, then $\mathrm{LI}(C) = \bot \Leftrightarrow \neg\top \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$ as $\hat{C} \in \Delta$. The case for $C \in \Delta$ can be argued analogously.

Resolution. Suppose the clause $C$ is the result of a resolution step $\iota$ of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

As $\tau$ depends only on $\sigma$, $\tau$ is the same for both $\pi$ and $\hat{\pi}$.

We now distinguish the following cases:

1. $l$ is $\Gamma$-colored:

$$\begin{aligned}\mathrm{LI}(C) &= \ell[\mathrm{LI}(C_1)\tau] \vee \ell[\mathrm{LI}(C_2)\tau]\\ &\Leftrightarrow \neg(\neg\ell[\mathrm{LI}(C_1)\tau] \wedge \neg\ell[\mathrm{LI}(C_2)\tau])\\ &\Leftrightarrow \neg(\ell[\mathrm{LI}(\hat{C}_1)\tau] \wedge \ell[\mathrm{LI}(\hat{C}_2)\tau])\\ &= \neg\,\mathrm{LI}(\hat{C})\end{aligned}$$

2. $l$ is $\Delta$-colored: This case can be argued analogously
3. $l$ is grey: Note that by Lemma 5, $\ell[l_{\mathrm{LIcl}}\tau] = \ell[l'_{\mathrm{LIcl}}\tau]$ $(*)$.

$$\begin{aligned}\mathrm{LI}(C) &= (\neg\ell[l'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_1)\tau]) \;\vee\; (\ell[l_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_2)\tau])\\ &\overset{(*)}{\Leftrightarrow} (\ell[l'_{\mathrm{LIcl}}\tau] \vee \ell[\mathrm{LI}(C_1)\tau]) \;\wedge\; (\neg\ell[l_{\mathrm{LIcl}}\tau] \vee \ell[\mathrm{LI}(C_2)\tau])\\ &\Leftrightarrow \neg\Big((\neg\ell[l'_{\mathrm{LIcl}}\tau] \wedge \neg\ell[\mathrm{LI}(C_1)\tau]) \;\vee\; (\ell[l_{\mathrm{LIcl}}\tau] \wedge \neg\ell[\mathrm{LI}(C_2)\tau])\Big)\\ &= \neg\Big((\neg\ell[\hat{l}'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(\hat{C}_1)\tau]) \;\vee\; (\ell[\hat{l}_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(\hat{C}_2)\tau])\Big)\\ &= \mathrm{LI}(\hat{C})\end{aligned}$$

Factorisation. Suppose the clause $C$ is the result of a factorisation $\iota$ of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

Then $\mathrm{LI}(C) = \ell[\mathrm{LI}(C_1)\tau]$, so the construction is not influenced by the coloring and the induction hypothesis gives the result. □

**Theorem 14.** *Let $\pi$ be a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{LI}(\pi)$ is an interpolant.*

*Proof.* By Lemma 12 $\Gamma \models Q(\pi)(\mathrm{LI}(\pi) \vee \mathrm{LI}_{\mathrm{cl}}(\pi))$. But as $\mathrm{LI}_{\mathrm{cl}}(\pi) = \Box$, this simplifies to $\Gamma \models Q(\pi)\,\mathrm{LI}(\pi)$.

By constructing a proof $\hat{\pi}$ from $\pi$ with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$, we obtain by Lemma 12 that $\hat{\Gamma} \models Q(\hat{\pi})\,\mathrm{LI}(\hat{\pi})$. By Lemma 13, this however is nothing else than $\Delta \models \neg Q(\pi)\,\mathrm{LI}(\pi)$.

As furthermore by construction no colored symbols occur in $Q(\pi)\,\mathrm{LI}(\pi)$, this formula is an interpolant for $\Gamma \cup \Delta$. □