# 1 Attempt without $P_P$

Intuition of $\sigma'$:
If we pull a substitution out of a lifting which replaced $\Delta$-terms, we also have to replace the $\Delta$-terms in the "domain" of the substitution. This is the lower case in the definition of $\sigma'$ below.
There is just a problem in the following case: $\ell_{\Delta,x}[f(x)\sigma]$, where $x\sigma = a$ and $f$ is a $\Delta$-symbol. Then $\ell_{\Delta,x}[f(x)\sigma] = \ell_{\Delta,x}[f(a)] = x_i$, but $\ell_{\Delta,x}[f(x)]\sigma = x_j$ with $i \neq j$. The first case of the definition of $x_j$ then fixes this by replacing $x_j$ with $x_i$.

**Lemma 1.** *Let $C$ be a clause and $t_1, \dots, t_n$ the set of maximal $\Delta$-terms in $C$, $x_1, \dots, x_n$ the corresponding fresh variables to replace the $t_i$, and $\sigma$ be a substitution. Let $\sigma'$ be defined such that*

$$z\sigma' = \begin{cases} x_l & \text{if } z = x_k \text{ and } t_k\sigma = t_l \\ \ell_{\Delta,x}[z\sigma] & \text{otherwise} \end{cases}$$

*Note that the definition of $\sigma'$ only depends on the $x_i$ and $t_i$.*
*Then $\ell_{\Delta,x}[C\sigma] = \ell_{\Delta,x}[C]\sigma'$.*

*Proof.* We prove this for an atom $P(s_1, \dots, s_m)$ in $C$, which works since lifting and substitution commute over binary connectives and into an atom.
We show that $\ell_{\Delta,x}[s_j\sigma] = \ell_{\Delta,x}[s_j]\sigma'$ for $1 \leq j \leq m$.
Note that anything in the term structure above a maximal $\Delta$-term is unaffected by both substitution and abstraction.
Let $t_i$ be a maximal $\Delta$-term in $s_i\sigma$.
We show that $\ell_{\Delta,x}[t_i\sigma] = \ell_{\Delta,x}[t_i]\sigma'$, which proves the lemma.
Let $t_i\sigma = t_j$. Then $\ell_{\Delta,x}[t_i\sigma] = \ell_{\Delta,x}[t_j] = x_j$.
We show that $x_j = \ell_{\Delta,x}[t_i]\sigma'$.
Suppose that $t_i = t_j$, i.e. $\sigma$ is trivial on $t_i$. Then $i = j$ as the $\Delta$-terms have a unique number. Hence $\ell_{\Delta,x}[t_i]\sigma' = x_i\sigma' = x_i = x_j$.
Otherwise $t_i \neq t_j$. Then $i \neq j$ and $x_j \neq x_i$.
$\ell_{\Delta,x}[t_i]\sigma' = x_i\sigma'$. By the definition of $\sigma'$, as $t_i\sigma = t_j$, $x_i\sigma' = x_j$. □

**Lemma 2** (currently 4.8 in thesis, Lemma 11 in Huang)**.** *Let $A$ and $B$ be first-order formulas and $s$ and $t$ be terms. Then it holds that:*

1. $\ell_{\Phi,x}[\neg A] \Leftrightarrow \neg \ell_{\Phi,x}[A]$
2. $\ell_{\Phi,x}[A \circ B] \Leftrightarrow (\ell_{\Phi,x}[A] \circ \ell_{\Phi,x}[B])$ *for* $\circ \in \{\wedge, \vee\}$
3. $\ell_{\Phi,x}[s = t] \Leftrightarrow (\ell_{\Phi,x}[s] = \ell_{\Phi,x}[t])$

**Lemma 3.** $\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C) \vee C)]$.

*Proof.* By induction on the resolution refutation of the strengthening: $\Gamma \models \mathrm{PI}(C) \vee C_\Gamma$
Base case: Either $C \in \Gamma$, then it does not contain $\Delta$-terms. Otherwise $C \in \Delta$ and $\mathrm{PI}(C) = \top$.
Induction step:

Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad l\sigma = l'\sigma$$

By the induction hypothesis, we can assume that:

$\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1) \vee (D \vee l)_\Gamma]$ and $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_2) \vee (E \vee \neg l')_\Gamma]$

which by Lemma 2 implies that

$\Gamma \overset{(*)}{\models} \ell_{\Delta,x}[\mathrm{PI}(C_1)] \vee \ell_{\Delta,x}[D_\Gamma] \vee \ell_{\Delta,x}[l_\Gamma]$ and $\Gamma \overset{(\circ)}{\models} \ell_{\Delta,x}[\mathrm{PI}(C_2)] \vee \ell_{\Delta,x}[E_\Gamma] \vee \neg\ell_{\Delta,x}[l'_\Gamma]$

Let $\sigma'$ be defined as in Lemma 1 with $t_1, \dots, t_n$ all $\Delta$-terms in this context, i.e. from $C_1$, $C_2$, $\mathrm{PI}(C_1)$, $\mathrm{PI}(C_2)$ and $\sigma$.

1. $l$ is $\Gamma$-colored. Then $\mathrm{PI}(C) = [\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2)]\sigma$.

   We show that $\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2))\sigma \vee (D \vee E)_\Gamma\sigma]$,
   i.e. $\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2) \vee D_\Gamma \vee E_\Gamma)\sigma]$.

   Hence by Lemma 1, $\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2) \vee D_\Gamma \vee E_\Gamma)]\sigma'$.

   Since $l\sigma = l'\sigma$ (by resolution rule application), $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l'\sigma]$.

   As by Lemma 1, with $\sigma'$ as above, $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l]\sigma'$ and $\ell_{\Delta,x}[l'\sigma] = \ell_{\Delta,x}[l']\sigma'$, we get $\ell_{\Delta,x}[l]\sigma' = \ell_{\Delta,x}[l']\sigma'$.

   So by applying $\sigma'$ to $(*)$ and $(\circ)$, we can perform a resolution step on $\ell_{\Delta,x}[l]\sigma'$ and get

   $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$.

   and consequently $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2) \vee D_\Gamma \vee E_\Gamma]\sigma'$.

   So by Lemma 1,

   $\Gamma \models \ell_{\Delta,x}[\Big(\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2) \vee D_\Gamma \vee E_\Gamma\Big)\sigma]$.

2. $l$ is $\Delta$-colored. Then $\mathrm{PI}(C) = (\mathrm{PI}(C_1) \wedge \mathrm{PI}(C_2))\sigma$.

   We show that $\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C_1) \wedge \mathrm{PI}(C_2))\sigma \vee (D_\Gamma \vee E_\Gamma)\sigma]$

   which by Lemma 2 is equivalent to
   $\Gamma \models \Big(\ell_{\Delta,x}[\mathrm{PI}(C_1)\sigma] \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)\sigma]\Big) \vee \ell_{\Delta,x}[D_\Gamma\sigma] \vee \ell_{\Delta,x}[E_\Gamma\sigma]$

   and by Lemma 1 is equivalent to
   $\Gamma \overset{(\times)}{\models} \Big(\ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma'\Big) \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$

   As $l$ and $l'$ are $\Delta$-colored, we can strengthen $(*)$ and $(\circ)$ as follows and apply $\sigma'$:

   $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma'$ and $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$

   These clearly imply $(\times)$.

3. $l$ is grey. Then $\mathrm{PI}(C) = [(l \wedge \mathrm{PI}(C_2)) \vee (\neg l' \wedge \mathrm{PI}(C_2))]\sigma$.

   We show that $\Gamma \models \ell_{\Delta,x}[[(l \wedge \mathrm{PI}(C_2)) \vee (\neg l' \wedge \mathrm{PI}(C_2))]\sigma \vee (D_\Gamma \vee E_\Gamma)\sigma]$, which by Lemma 2 and Lemma 1 is equivalent to

$\Gamma \models \Big(\ell_{\Delta,x}[l]\sigma' \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma'\Big) \vee \Big(\neg\ell_{\Delta,x}[l']\sigma' \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma'\Big) \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[E_\Gamma]\sigma'$.

Suppose for a model $M$ of $\Gamma$ that $M \not\models \ell_{\Delta,x}[D_\Gamma]\sigma'$ and $M \not\models \ell_{\Delta,x}[E_\Gamma]\sigma'$ as otherwise we would be done. But then by $(*)$ and $(\circ)$, $M \models \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[l]\sigma'$ and $M \models \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma' \vee \neg\ell_{\Delta,x}[l']\sigma'$.

As observed in case 1, $\ell_{\Delta,x}[l]\sigma' = \ell_{\Delta,x}[l']\sigma'$. By a case distinction on the truth value of $\ell_{\Delta,x}[l]\sigma'$, we obtain the result.

Factorisation.

$$\frac{C_1 : l \vee l' \vee D}{C : (l \vee D)\sigma} \qquad \sigma = \mathrm{mgu}(l, l')$$

Then $\mathrm{PI}(C) = \mathrm{PI}(C_1)\sigma$.

The induction hypothesis gives that $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1) \vee l \vee l' \vee D]$. Let $\sigma'$ be as in Lemma 1.

Then $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1) \vee l \vee l' \vee D]\sigma'$ and by Lemma 1, $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1)\sigma \vee l\sigma \vee l'\sigma \vee D\sigma]$.

By Lemma 2, $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1)\sigma] \vee \ell_{\Delta,x}[l\sigma] \vee \ell_{\Delta,x}[l'\sigma] \vee \ell_{\Delta,x}[D\sigma]$.

As $\sigma = \mathrm{mgu}(l, l')$, $l\sigma$ and $l'\sigma$ are syntactically equal, hence $\ell_{\Delta,x}[l\sigma] = \ell_{\Delta,x}[l'\sigma]$.

syntactically equal? does "equal" suffice?. see also $s\sigma = r\sigma$ below

But then $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1)\sigma] \vee \ell_{\Delta,x}[l\sigma] \vee \ell_{\Delta,x}[D\sigma]$ and consequently by Lemma 1, $\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1)\sigma \vee l\sigma \vee D\sigma]$.

Paramodulation.

$$\frac{C_1 : D \vee s = t \qquad C_2 : E[r]_p}{C : (D \vee E[t]_p)\sigma} \qquad \sigma = \mathrm{mgu}(s, r)$$

By the induction hypothesis, we have:

$\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_1) \vee (D \vee s = t)_\Gamma]$

$\Gamma \models \ell_{\Delta,x}[\mathrm{PI}(C_2) \vee (E[r]_p)_\Gamma]$

By Lemma 2 and Lemma 1, these imply:

$\Gamma \overset{(\circ)}{\models} \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma' \vee \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[s]\sigma' = \ell_{\Delta,x}[t]\sigma'$

$\Gamma \overset{(*)}{\models} \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma' \vee \ell_{\Delta,x}[(E[r]_p)_\Gamma]\sigma'$

We distinguish two cases:

1. Suppose $s$ does not occur in a maximal $\Delta$-term $h[s]$ in $E[s]_p$ which occurs more than once in $\mathrm{PI}(E(s)) \vee E[s]_p$.

   We show that $\Gamma \models \ell_{\Delta,x}[\Big((s = t \wedge \mathrm{PI}(C_2)) \vee (s \neq t \wedge \mathrm{PI}(C_1))\Big)\sigma \vee \Big((D \vee E[t]_p)_\Gamma\Big)\sigma]$, which subsumes the cases **??** and **??** of Definition **??**. By Lemma 2, we can pull the liftings inwards and by Lemma 1, we can commute substitution and lifting by employing $\sigma'$ to arrive at

   $\Gamma \models \Big((\ell_{\Delta,x}[s]\sigma') = (\ell_{\Delta,x}[t]\sigma') \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma'\Big) \vee \Big((\ell_{\Delta,x}[s]\sigma') \neq (\ell_{\Delta,x}[t]\sigma') \wedge \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma'\Big) \vee \Big(\ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[(E[t]_p)_\Gamma]\sigma'\Big)$

Let $M$ be a model of $\Gamma$. Let $M \not\models \ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[(E[t]_p)_\Gamma]\sigma'$ as otherwise we would be done. We show that depending on the truth value of $(\ell_{\Delta,x}[s]) = (\ell_{\Delta,x}[t])$ in $M$, either the first or second conjunct of the above formula holds.

Suppose that $M \models (\ell_{\Delta,x}[s]) \neq (\ell_{\Delta,x}[t])$. Then $M \models (\ell_{\Delta,x}[s])\sigma' \neq (\ell_{\Delta,x}[t])\sigma'$, so by ($\circ$), $M \models \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma'$.

On the other hand, suppose that $M \models (\ell_{\Delta,x}[s]) = (\ell_{\Delta,x}[t])$. The following two lemmas show that $M \not\models \ell_{\Delta,x}[E[r]_p]\sigma'$, so by ($*$), we get that $M \models \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma'$.

**Lemma 4.** *$M \models (\ell_{\Delta,x}[s]) = (\ell_{\Delta,x}[t])$ and $M \not\models \ell_{\Delta,x}[E[t]_p]\sigma'$ imply that $M \not\models \ell_{\Delta,x}[E[s]_p]\sigma'$.*

*Proof.* $\ell_{\Delta,x}[E[t]_p]$ and $\ell_{\Delta,x}[E[s]_p]$ only differ at position $p$, where at the first, there is $\ell_{\Delta,x}[t]$, and at the latter, there is $\ell_{\Delta,x}[s]$. But in $M$, they are interpreted the same way, hence $M \models \ell_{\Delta,x}[E[t]_p] \Leftrightarrow \ell_{\Delta,x}[E[s]_p]$, which implies the result. □

**Lemma 5.** *$\sigma = \mathrm{mgu}(s, r)$ and $M \not\models \ell_{\Delta,x}[E[s]_p]\sigma'$ imply that $M \not\models \ell_{\Delta,x}[E[r]_p]\sigma'$.*

*Proof.* By Lemma 1, $M \not\models \ell_{\Delta,x}[(E[s]_p)\sigma]$.

Due to $\sigma = \mathrm{mgu}(s, r)$, both $s\sigma$ and $r\sigma$ are syntactically equal. Suppose they are both not $\Delta$-colored. Then the lifting does not affect them and $\ell_{\Delta,x}[(E[s]_p)\sigma] = \ell_{\Delta,x}[(E[r]_p)\sigma]$. Otherwise the lifting will replace them with the same variable and we as well get that $\ell_{\Delta,x}[(E[s]_p)\sigma] = \ell_{\Delta,x}[(E[r]_p)\sigma]$.

By Lemma 1, $\ell_{\Delta,x}[(E[s]_p)]\sigma' = \ell_{\Delta,x}[(E[r]_p)]\sigma'$, which implies the result. □

2. Otherwise $s$ occurs in a maximal $\Delta$-term $h[s]_q$ in $E[s]_p$ which occurs more than once in $\mathrm{PI}(E(s)) \vee E[s]_p$.

   Then we have to replace Lemma 4 by:
   **Lemma 4′.** *$M \models (\ell_{\Delta,x}[s]) = (\ell_{\Delta,x}[t])$ and $M \not\models \ell_{\Delta,x}[E[t]_p]\sigma'$ imply that $M \not\models \ell_{\Delta,x}[E[s]_p]\sigma'$ or that $\ell_{\Delta,x}[h[s]_q] \neq \ell_{\Delta,x}[h[t]_q]$.*

   *Proof.* If $\ell_{\Delta,x}[E[t]_p]$ and $\ell_{\Delta,x}[E[s]_p]$ differ only at position $p$, then the proof of Lemma 4 applies.

   Otherwise position $p$ is in a maximal $\Delta$-term $h[t]_q$, such that $h[t]_q$ and $h[s]_q$ are replaced with distinct variables. But then clearly $\ell_{\Delta,x}[h[s]_q] \neq \ell_{\Delta,x}[h[t]_q]$. □

   Hence the following holds:

$$\Gamma \models \Big((\ell_{\Delta,x}[s]\sigma') = (\ell_{\Delta,x}[t]\sigma') \wedge \ell_{\Delta,x}[\mathrm{PI}(C_2)]\sigma'\Big) \vee \Big((\ell_{\Delta,x}[s]\sigma') \neq (\ell_{\Delta,x}[t]\sigma') \wedge \ell_{\Delta,x}[\mathrm{PI}(C_1)]\sigma'\Big) \vee \Big((\ell_{\Delta,x}[s]\sigma') = (\ell_{\Delta,x}[t]\sigma') \wedge (\ell_{\Delta,x}[h[s]_q]) \neq (\ell_{\Delta,x}[h[t]_q])\Big) \vee \Big(\ell_{\Delta,x}[D_\Gamma]\sigma' \vee \ell_{\Delta,x}[(E[t]_p)_\Gamma]\sigma'\Big)$$

$\square$

General layout of this proof:

$\Gamma \models \ell_{\Delta,x}[(\mathrm{PI}(C) \vee C)]$

Lemma 4.10: swap $\Gamma$ and $\Delta$ and obtain logical negationas interpolant <– seems to go through

Corollary 4.11: $\Delta \models \ell_{\Gamma,y}[\neg\, \mathrm{PI}(C) \vee C]$ <– just a corollary

Lemma 4.12: not important if lifting delta or gamma terms first <– seems to go through

Thm 4.13: ordering <— also?