1 Overbinding in one step

Conjecture 1. *Suppose every variable occurs only once in $\Gamma \cup \Delta$. Then the order of the quantifiers for $\mathrm{PI}(\square)^*$ does not matter.*

Proposition 2. *Let $A(x_1, \ldots, x_n)$ be an atom in a relative interpolant. A variable occurs in one of the x_i if and only if there are atoms $A(y_1, \ldots, y_n)$ and $A(z_1, \ldots, z_n)$ in Γ and Δ respectively, where x_i can be unified with z_i and y_i such that there is still a variable at that location.*
This means that either the term structure above the variable is the same in the original clauses or there are some variables. Intended meaning: the original clauses prove at least the x_i, i.e. are at least as or more general.

Special case for outermost variables:
Let $A(x_1, \ldots, x_n)$ be an atom in a relative interpolant. An x_i is a variable if and only if there are atoms $A(y_1, \ldots, y_n)$ and $A(z_1, \ldots, z_n)$ in Γ and Δ respectively, where y_i and z_i are variables.

need more narrow version: clauses do appear in parent clauses in derivation.

Proposition 3. *Suppose in a partial interpolant, there are two maximal terms ζ_1 and ζ_2 such that w.l.o.g. ζ_1 is smaller (as defined in 5) than ζ_2. Then it the final interpolant, an overbinding can be defined where the variable corresponding to ζ_1 is quantified over before the variable corresponding to ζ_2 is.*

The subterm-relation is reflexive.

Definition 4. (OLD) Let s be a term that is in $\mathrm{PI}(C)$ but not in any predecessor $\mathrm{PI}(C_i)$, $i \in \{1, 2\}$. s is smaller than a term t in $\mathrm{PI}(C)$ if s is of strictly smaller length than t and there is a subterm in s which also occurs in t. $\triangle$

Definition 5. (NEW)
OUTDATED! DOES NOT WORK LIKE THIS
Let C be a clause.
A maximal term s of C is smaller than a maximal term t of C if s is a variable and occurs in t, but $s \neq t$.
OUTDATED! DOES NOT WORK LIKE THIS

$\triangle$

2 Half-baked approaches

Definition 6. Direct interpolation extraction.
This version of overline and star does NOT overbind variables! If they happen to be in the final interpolant, just overbind them somehow, but not earlier. This is ok as the interpolant only contains variables if both corresponding atoms in Γ and Δ do. Variables are the only terms in the interpolant that can "change their color", so we don't know a priori if there are constraints on the quantifier to overbind them with.

Convention w.r.t. a clause C which has been derived from C_1 and C_2: $\bar{Q}_n = Q_1 z_1 \ldots Q_n z_n$, such that the z_i correspond to the maximal terms ζ_i in $\mathrm{PI}(C)$. Same terms must be overbound by same variable, see 101a for counterexample to per-occurrence-overbinding. The z_i are ordered such that

1. the orderings in the Q_{n_1} and Q_{n_2} are respected (no circlular relations can occur in combination with merging as a term is only smaller than another term if it is smaller in length as well, which excludes cycles)

2. as well as ordering constraints of terms newly introduced in $\mathrm{PI}(C)$ (i.e. those that were not present in $\mathrm{PI}(C_1)$ and $\mathrm{PI}(C_2)$).

Basically, track dependencies and define actual order later.

Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \qquad \sigma = \mathrm{mgu}(l, l')$$

$\bar{Q}_{n_1} \, \mathrm{PI}(C_1)^*$

$\bar{Q}_{n_2} \, \mathrm{PI}(C_2)^*$

1. l and l' Γ-colored:

    $\mathrm{PI}(C) \equiv (\mathrm{PI}(C_1) \vee \mathrm{PI}(C_2))\sigma$

    $\mathrm{PI}(C)^* \equiv (\mathrm{PI}(C_1)^* \vee \mathrm{PI}(C_2)^*)\sigma$ (just replace maximal terms)

    intended meaning of σ: to change the free variables still in the $\mathrm{PI}(C_i)$

    TODO: basically do nothing here since no new atoms (revisit after mixed colored case has been dealt with)

    Let $\zeta_1, \ldots, \zeta_{n_1}$ be terms overbound in $\mathrm{PI}(C_1)$ and $s_1, \ldots, s_{n_2}$ terms overbound in $\mathrm{PI}(C_2)$.

    $\{z_1, \ldots, z_n\} = \{\zeta_1, \ldots, \zeta_{n_1}\}\sigma \cup \{s_1, \ldots, s_{n_2}\}\sigma$ // common terms are merged

    order relations as in C_1, C_2

    $\bar{Q}_n \, \mathrm{PI}(C)^* \equiv \bar{Q}_n(\mathrm{PI}(C_1)^* \vee \mathrm{PI}(C_2)^*)$

2. l and l' Δ-colored:

    similar to first case

3. l and l' grey: nothing here

$\triangle$

3 current proof attempts

Lemma 7. *If an atom A appears in the interpolant, it appeared in both original clause sets, once positively and once negatively.*

A is contained in some instance of the respective clauses in Γ and Δ.

Lemma 8. *Let $C \in \Phi$ for some initial clause set Φ.*

1. *Let x be an occurrence of a variable in C and x' another occurrence of the same variable in a different position but at the same term depth. Then $\Phi \models QyC[x/y][x'/y]$ for $Q \in \{\forall, \exists\}$.*

2. *Let x be an occurrence of a variable in C with the lowest depth and x' another occurrence of the same variable with a higher depth. Let t be the maximal colored term which contains x'. t is Φ-colored since it appears in Φ. Then $\Phi \models Qy\exists zC[x/y]\{t/z\}$ for $Q \in \{\forall, \exists\}$.*

Lemma 9. *Let t be a maximal colored term in C in Φ. It is Φ-colored. Let $x_1, \ldots, x_n$ be the variables which occur in t. Then $\Phi \models Q\bar{x}\exists yC\{t/y\}$ for $Q \in \{\forall, \exists\}$.*

actually

We have that $\Gamma \models \forall \bar{x} \ell_{\Delta,x}[\mathrm{PI}(C) \vee C]$.

Note that both are lifted.

We have that $\Gamma \models \forall \bar{x} \ell_{\Delta,x}[C] \vee C$.

Let t be a maximal Γ-term. It in general contains Γ-colored and grey terms, and also Δ-terms. The latter have entered it by unification.

If t contains no Δ terms, we can just overbind it existentially and give a witness.

Otherwise it contains Δ-terms. Then there is a variable in t at position say p which also occurs elsewhere in C, say at position q.

If q is the outermost term or if it has only grey term ancestors, then quantifying over whatever is in q before quantifying over whatever is in p is fine. Hence there is an arrow.

q can not be contained in a Δ term since dependencies cannot be introduced and must be there from the beginning, where no color mix is possible.

So otherwise q is contained in a maximal Γ-term s. For finding witnesses, we will put the same one for the variable at both q and t. As q introduces a Δ-term, at some point, there had had to be a unification with a formula from Δ (this then could have been passed on through "mirroring").

Conjecture: there are arrows along the path from the origin of the Δ-path to q.

Hence whatever is placed in q and p is quantified over earlier than the variables which replace t and s.

TODO: Proof or refute. . .

conjecture: put all terms that share variables and appear in the same clause and are all overbound with the same quantifier in the same quantifier block.

probably does not work when facing other dependencies, check that !

Notation:

p_1 is the position of s in t

p_2 is the elsewhere position of the shared var

A unification where a Γ-colored term s enters t happens when

- the other unified clause has a variable at the position of t (p_1)

- a variable is both in t (p_1) and elsewhere in the unified clause (p_2)

- p_2 is either in a grey term or as outermost or a Δ-colored term

  - if p_2 directly in grey term or as outermost, the ancestor of p_2 will not be overbound (only p_2). we need p_2 as witness for overbinding t, but not the other way.

  Hence quantifying over p_2 first is ok.

- if p_2 is in a Δ-colored term, say in maximal Δ-term s', Then s' and t are overbound with the same quantifier and order between them doesn't matter.

  for witness, we both need whatever the var is, and that we get by the inherited relation.

  ? there must be an inherited relation as since both s' and t are Γ and contain a Δ-term, the Δ term must have gotten into a Γ-colored term using aufschauckeln ?

4 structured proof

Lemma 10. $\Gamma \models PI(C) \vee C$ *for C in a prop proof.*

Proof. See Huang. □

Lemma 11. $\Gamma \models \forall x_1 \ldots \forall x_n \ell_{\Delta,x}[PI(C) \vee C]$ *for C in a prop proof.*

Proof. Still the same as in Huang. □

Lemma 12. $\mathrm{AI}_{\mathrm{mat}}(C) = \ell[PI(C)]$ *for $C \in \pi$.*

SUPPOSE NO VARIABLE OCCURS TWICE IN A COLORED TERM IN AN INITIAL CLAUSE SET

Lemma 13. *if there is a max Δ-term in a max Γ-term, there is an arrow from occurrence of Δ-term to the occurrence of Γ-term in* AI.

Proof. induction:
base case: no foreign terms
$C_1 : D \vee l$ and $C_2 : E \vee \neg l$
resolution, same color: induction hypothesis!!!
resolution, different color: l and l' unified. Disregard grey terms. Supp one of the unification locations, a term has a variable and all the prefix is grey/shared in l and l'. Then term from other literal enters, possibly foreign. Then by replacing all variables in C_1 or C_2, a Δ-term might enter a Γ-term. But in this case, we have an arrow.
Γ-terms and Δ-terms are different and hence not unifiable.
Suppose same prefix (i.e. same colored prefix), then different variables each. Supp variable at one end, foreign colored term at other. then arrows of literals are merged, and by induction hypothesis, the term with the foreign colored term has an arrow to the foreign colored term.
// for each unification where possibly foreign terms are introduced, there is an arrow.
Either resolution with same color: as long as just same color resolutions, no new literals in interpolant (but new terms by resolution?)
if after chain of same color resolution a grey literal is resolved, transitive edges kick in (chain of arrows)
resolution with grey literals: resolved literals share grey or aufgeschaukelter prefix before the variable.
RESTRICTION APPLIES HERE: no aufgeschaukelte prefixes

if the variable occurs elsewhere in one of the clauses, a foreign term might have been introduced in a colored term. in this case, there is an arrow $\square$

Lemma 14. $\Gamma \models \bar{Q}_n \ell_{\Gamma,y}[\ell_{\Delta,x}[PI(C) \vee C]]$ *for C in a prop proof.*

i.e.

$\Gamma \models \text{AI}(C)$ *for C in a prop proof.*

Proof. Show that the existential quantifiers in $\bar{Q}_n$ have witnesses.

If the clause C is the result of a resolution step of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$, then by induction hypothesis, we get that

supp l grey: $\text{AI}(C) = Q_1 u_i \ldots Q_m u_m (\neg \chi \wedge \text{AI}_{\text{mat}}(C_2)) \vee (l \wedge \text{AI}_{\text{mat}}(C_1))$

Know: $\Gamma \models \ell_{\Gamma,y}[PI(C) \vee C]$

By Lemma 12: Know: $\ell[PI(C) \vee C] = \text{AI}_{\text{mat}}(C) \vee \ell[C]$

Need: $\Gamma \models \text{AI}(C) \vee C$ // C in binding of AI

i.e. $\Gamma \models \{Q_{n_1} \cup Q_{n_2} \cup \text{new ones}\}(\ell[(l \wedge \ell[PI(C_2)]) \vee (\neg l \wedge \ell[PI(C_1)]) \vee D \vee E])$

existentially overbound variables in $\ell[PI(C_i)]$ will still work if relative order is maintained, which it is.

for new terms in l, we have lemma 11, which provide witnesses.

huang-style: show how a foreign term got into a colored term, and this is how it must have an arrow.

TODO: show for restricted version: all variables occur once as maximal "colored" term. $\square$

5 proof attempt using $\mathrm{AI_{cl}}/\mathrm{AI_{mat}}$

Lemma 15. *Suppose no colored term occurs in* $\mathrm{PI}(C) \vee C$ *for* $C \in \pi$. *Then* $\Gamma \models \mathrm{AI_{mat}}(C) \vee \mathrm{AI_{cl}}(C)$
// implicit universal quantification

Proof. Proof by induction.

Base case:

For $C \in \Gamma$, $\mathrm{AI_{mat}}(C) = \bot$ and $\mathrm{AI_{cl}} = \ell[C] = \ell_{\Gamma,y}[C]$. By the restriction, $\ell_{\Gamma,y}[C] = C$ and $\Gamma \models C$.

For $C \in \Delta$, $\mathrm{AI_{mat}}(C) = \top$.

Induction step:

We know: $\Gamma \models \mathrm{AI_{mat}}(C_i) \vee \mathrm{AI_{cl}}(C_i)$, $i \in \{1, 2\}$.

- Suppose l and l' of opposite color.

  $$\mathrm{AI_{mat}}(C)' = \Big((\neg l \wedge \mathrm{AI_{mat}}(C_1)) \vee (l \wedge \mathrm{AI_{mat}}(C_2)) \Big) \sigma$$

  $$\mathrm{AI_{cl}}(C)' = \Big((\mathrm{AI_{cl}}(C_1) \setminus \{l\}) \vee (\mathrm{AI_{cl}}(C_2) \setminus \{\neg l'\}) \Big) \sigma \;\; // \;\; \text{setminus: remove clause with that ancestor}$$

  To show: $\Gamma \models \ell[\mathrm{AI_{mat}}(C)' \vee \mathrm{AI_{cl}}(C)']$, but as σ does not introduce a colored term, this is the same as $\Gamma \models \mathrm{AI_{mat}}(C)' \vee \mathrm{AI_{cl}}(C)'$.

  Suppose $\Gamma \not\models (\mathrm{AI_{cl}}(C_1) \setminus \{l\})\sigma$ and $\Gamma \not\models (\mathrm{AI_{cl}}(C_2) \setminus \{\neg l'\})\sigma$ as otherwise we would be done.

  Then $\Gamma \not\models (\mathrm{AI_{cl}}(C_1) \setminus \{l\})$ and $\Gamma \not\models (\mathrm{AI_{cl}}(C_1) \setminus \{l\})$.

  Hence the induction hypothesis reduces to $\Gamma \models \mathrm{AI_{mat}}(C_1) \vee l$ and $\Gamma \models \mathrm{AI_{mat}}(C_2) \vee \neg l'$.

  Therefore also $\Gamma \models (\mathrm{AI_{mat}}(C_1) \vee l)\sigma$ and $\Gamma \models (\mathrm{AI_{mat}}(C_2) \vee \neg l')\sigma$.

  Now as $l\sigma = l'\sigma$, their interpretation is linked, so we get the result in a similar way as in Huang's proof.

$\square$

Lemma 16. *Let* u *be a variable in a literal* l *being unified in a resolution step with* l' *using* σ. *Then* $\ell[u\sigma]\tau = \ell[u\sigma]$ *// possibly true and UNUSED*

Proof. Let $u\sigma = \zeta_j$ and hence $\ell[u\sigma] = z_j$.

NB: much weird stuff here, but last two paragraphs of first item seem to make sense

NB: possibly try to show that $u\sigma = \zeta_k$ in first item

as $l\sigma = l'\sigma$, in l' there is t with $u\sigma = t\sigma$, so $t\sigma = \zeta_j$.

- Suppose that $(z_j \mapsto z_k) \in \tau$ is of kind ①. Suppose that $k \neq j$ as otherwise we are done. Then by the definition of τ, we have a pair of corresponding terms $(a_{\mathrm{AIcl}}, b_{\mathrm{AIcl}})$ in l_{AIcl} and l'_{AIcl} respectively such that $a_{\mathrm{AIcl}} = z_j$ and b_{AIcl} is such that $\ell[b_{\mathrm{AIcl}}\sigma] = z_k$, hence $b_{\mathrm{AIcl}}\sigma = \zeta_k$. By Lemma **??** b_{AIcl} is a variable and $b = b_{\mathrm{AIcl}}$.

  By Lemma **??** $a_{\mathrm{AIcl}} \sim \ell[a]$. By Lemma **??**, $a\sigma = \zeta_k$ and there is a substitution ρ such that $\zeta_j \rho = \zeta_k$. Also $a = \zeta_j \rho'$ for some substituion ρ'.

Furthermore due to $l\sigma = l'\sigma$, $a\sigma = b\sigma$. Hence $\ell[b_{\mathrm{AIcl}}\sigma] = \ell[b\sigma] = \ell[a\sigma]$.

$a\sigma = b\sigma$, so $\zeta_j \rho'\sigma = b\sigma = \zeta_k$.

By Lemma 19, ζ_j contains a free variable. As C_1 and C_2 are used in a resolution step, they are variable disjoint. TODO: currently assume they are variable disjoint (tree derivation), try to generalise later. Hence ζ_j can only occur in one of them, w.l.o.g. let it occur in C_1.

As by assumption $\zeta_k \neq \zeta_j$ and $\zeta_j \rho = \zeta_k$, a free variable of ζ_j is substituted by .

TODO:

$u\sigma = t\sigma = \zeta_j$

we only have $(z_j \mapsto z_k)$ if the underlying term is unified, i.e. the variable is replaced. This variable only occurs in this clause (or related ones). But $t\sigma = \zeta_j$, i.e. either t still contains this variable or σ introduces it.

t cannot still contain it as $\zeta_j\sigma$ removes it, and σ cannot introduce it as C_1 and C_2 are variable disjoint, and it could only add it if a variable from the other clause is unified with it, but that variabel then cannot occur in t as it's from the other clause.

- Suppose that $(z_j \mapsto z_k) \in \tau$ (second kind). Suppose that $k \neq j$ as otherwise we are done.

  Similar reasoning: there is a variable in ζ_j occurring at least twice (as z_j). but then it must be unified to the same variable. so there, the same terms are present and the lifting variables are set accordingly by this crude method.

  So u is substituted for whatever happens on this other side, i.e. $u\sigma = \zeta_k$

$\square$

Lemma 17. *Let $\Phi_{\mathrm{AI_{cl}}(C)}$ be the set of occurrences of lifting variables x_j for some j in $\mathrm{AI_{cl}}(C)$ for C in a resolution refutation. Let Φ_C be the terms at the positions $\Phi_{\mathrm{AI_{cl}}(C)}$ in the respective corresponding literal in C. Then all terms at Φ_C are equal. NB: similar lemma with possibly better proof below*

Proof. Note that by Lemma **??**, incomparable clauses do not share lifting variables which replace terms without free variables. We proceed by induction.

Base case: $\mathrm{AI_{cl}}(C) = \ell[C]$ for C in some initial clause set. As x_j replaces a distinct term ζ_j, all occurrences of x_j in $\mathrm{AI_{cl}}(C)$ correspond to some ζ_j in C.

Induction step: Suppose the statement holds for C_1 and C_2, usual resolution step notation. Every $x_j \in \mathrm{AI_{cl}}(C)$ is derived from some $t \in \mathrm{AI_{cl}}(C_1)$ (w.l.o.g.).

- Suppose t is a lifting variable. Then $t = x_{j'}$ such that $\ell[x_{j'}\sigma]\tau = \ell[x_{j'}]\tau = x_{j'}\tau = x_j$.

  Suppose that t does not contain a free variable. Then by Lemma 19, τ is trivial on $x_{j'}$, so $x_{j'}\tau = x_{j'}$, but then $j = j'$. By the induction hypothesis, all terms at Φ_{C_1} are equal.

ind step for terms with variables

Let ϕ and ϕ' be contained in $\Phi_{\mathrm{AI_{cl}}(C)}$. We show that their corresponding positions ϕ_C and ϕ'_C in Φ_C refer to equal terms. Let ϕ and ϕ' refer to s_{AIcl} and r_{AIcl} respectively, so $s_{\mathrm{AIcl}} = r_{\mathrm{AIcl}} = x_j$.

- Suppose that s_{AIcl} and r_{AIcl} are both derived from terms in either C_1 or C_2. Then by the induction hypothesis, they both refer to some term t in C_1 or C_2 (TODO: be more precise). By the construction of $\mathrm{AI_{cl}}(C)$, they refer to $\ell[t\sigma]\tau$ in $\mathrm{AI_{cl}}(C)$.

- Suppose that w.l.o.g. s_{AIcl} is derived from a term s'_{AIcl} in C_1 and r_{AIcl} is derived from a term r'_{AIcl} in C_2.

  Suppose that s'_{AIcl} is a lifting variable and r'_{AIcl} is not. Then r'_{AIcl} is a variable as $\mathrm{AI_{cl}}(C_2)$ does not contain colored terms but $\ell[r'_{\mathrm{AIcl}}\sigma] = x_j$.

  $\ell[s_{\mathrm{AIcl}}] = \ell[r_{\mathrm{AIcl}}]$ which implies that $s_{\mathrm{AIcl}} = r_{\mathrm{AIcl}}$.

  $s_{\mathrm{AIcl}} = \ell[s'_{\mathrm{AIcl}}\sigma]\tau$ and $r_{\mathrm{AIcl}} = \ell[r'_{\mathrm{AIcl}}\sigma]\tau$.

  Hence $\ell[s'_{\mathrm{AIcl}}\sigma]\tau = \ell[r'_{\mathrm{AIcl}}\sigma]\tau$ TODO

  The opposite case is analogous.

  $\square$

Lemma 18. *Let s_{AIcl} and t_{AIcl} be two occurrences of the lifting variable x_i in $\mathrm{AI_{cl}}(C)$ for C in a resolution refutation. Then $s = t$.* // *not really used, is it really true? is it interesting?*
NB: similar lemma with possibly worse proof above

Proof. Base case: $\mathrm{AI_{cl}}(C) = \ell[C]$ for C in some initial clause set.
Suppose that ζ_i does not contain a variable. Then any occurrence of ζ_i is never altered throughout the derivation. If a substitution introduced ζ_i, it is lifted to x_i. By Lemma 19, τ does not alter x_i. Hence all occurrences to x_i refer to ζ_i at any given stage of the derivation.
Now we suppose that ζ_i does contain a variable. By Lemma **??**, s_{AIcl} and t_{AIcl} are both derived from one of the preceding clauses, say from C_1.
Induction step: $s = t$ in C_1. Then s and t as well as s_{AIcl} and t_{AIcl} are affected the same way by the algorithm as they are respectively equal. It remains to show that for newly introduced occurrences x_i in $\mathrm{AI_{cl}}(C)$, the condition holds. Let r be such a newly introduced occurrence. r_{AIcl} in $\mathrm{AI_{cl}}(C)$ is contained in $\ell[u_{\mathrm{AIcl}}\sigma]\tau$ for a term $u_{\mathrm{AIcl}} \in \mathrm{AI_{cl}}(C_1) \cup \mathrm{AI_{cl}}(C_2)$. As u_{AIcl} does not contain x_i, it was introduced either by τ or by σ introducing ζ_i which is then lifted to x_i.

- Suppose r_{AIcl} was introduced by σ and consecutive lifting. Then σ has introduced ζ_i.

  TODO: Note that with "strict variable renaming", each variable is renamed every time so ζ_i cannot be introduced here as the variable it would have to contain does not exist anymore.

  Without strict renaming, we can show that then an arbitrary occurrence of x_i in $\mathrm{AI_{cl}}(C)$ different from r also refers to ζ_i. Note that C_1 and C_2 are variable disjoint. Hence the set of variables of ζ_i is a subset of the variables of C_1. TODO: clarify on the conditions that are ncessary for this. if some other ζ_i occurs in C_1, then no variable of ζ_i must occur in C_2. But if only $\zeta_i\rho$ occurs in C_1, C_2 might "reintroduce" a variable from ζ_i

Note furthermore that if any of the variables of ζ_i would not be present in C_1 anymore, than σ could not have produced ζ_i as unifications never introduce new variables.

And note that if a substitution is applied to a clause which is non-trivial on a variable, then this variable is not present in the clause afterwards.

Hence no variable of ζ_i has been unified in the derivation leading to C_1. But as term are only changed by means of unification, the ζ_i which were responsible for introducing the other occurrences of x_i in $\mathrm{AI}_{\mathrm{cl}}(C_1)$ are still present, and they are all equal to ζ_i.

- Suppose r_{AIcl} was introduced by τ. Then $(x_j \mapsto x_i) \in \tau$.

  By Lemma ??, $r = \zeta_i$ and either for a term $b_{\mathrm{AIcl}} \in \mathrm{AI}_{\mathrm{cl}}(C_1) \cup \mathrm{AI}_{\mathrm{cl}}(C_2)$ it holds that either, depending on the type of τ-substitution, $\ell[b_{\mathrm{AIcl}}\sigma] = x_i$ and $b = b_{\mathrm{AIcl}}$ or $a\sigma = b\sigma = x_i$. In both cases, the variables of ζ_i are present in $b\sigma$, so either they have been present before (i.e. in b) or were introduced by σ. In any case, by a similar reasoning as in the former case, the other occurrences of x_i in $\mathrm{AI}_{\mathrm{cl}} C_1$ refer to ζ_i as well. $\qquad\square$

Lemma 19. $(x_j \mapsto x_k) \in \mathrm{au}(l_{\mathrm{AIcl}}, l'_{\mathrm{AIcl}})$ with $j \neq k$ implies that ζ_j (corresponding to x_j) contains a free variable.
In other words: If ζ_j does not contain a free variable, then if it is lifted to x_j, τ will never change it to some x_k with $k \neq j$. // *true and somewhat used*

Proof. Let $\mathrm{au}(s_{\mathrm{AIcl}}, r_{\mathrm{AIcl}})$ introduce $\{x_j \mapsto x_k\}$. We perform a case distinction:

- Suppose s_{AIcl} is a lifting variable and but r_{AIcl} is not. Then by the definition of au, $s_{\mathrm{AIcl}} = x_j$ and $\ell[r_{\mathrm{AIcl}}\sigma] = x_k$. Suppose that ζ_j does not have a free variable. Then in the resolution derivation, from the point on as it is lifted by x_j the original term does not change, hence $s = \zeta_j$. As in consequence s does not contain free variables, $s\sigma = s$. As $s\sigma = r\sigma$ by the resolution rule application, we have that $s = r\sigma$. But as $\ell[s] = \ell[s_j] = x_j$, we must also have that $\ell[r\sigma] = x_j$. Hence $r\sigma = \zeta_j$.

  As r_{AIcl} occurs in $\mathrm{AI}_{\mathrm{cl}}(C)$ for some C, it is not a colored term, but as it is not a lifting variable and $r_{\mathrm{AIcl}}\sigma$ is a colored term, r_{AIcl} must be a variable. By Lemma ??, as $r \in C$, r_{AIcl} is equal to s up to the index of lifting variables, hence $r = r_{\mathrm{AIcl}}$.

  But then $r_{\mathrm{AIcl}}\sigma = r\sigma = \zeta_j$, so $\ell[r_{\mathrm{AIcl}}\sigma] = x_j$. But then $j = k$, a contradiction.

- Suppose r_{AIcl} is a lifting variable but s_{AIcl} is not. This case can be argued analogously.

- Suppose that both s_{AIcl} and r_{AIcl} are a lifting variables. Then by the definition of au, $s = r = \zeta_k$ such that $\ell[\zeta_k] = x_k$. Suppose that ζ_j does not contain a free variable. Then from the point on where s_{AIcl} has been lifted, ζ_j does not change. Therefore $s = \zeta_j$. But then $k = j$, a contradiction. $\qquad\square$

Lemma 20. *If a grey or maximal colored term t in a clause C does not contain a free variable, then for t_{AIcl} in $\mathrm{AI}_{\mathrm{cl}}$, we have that $\ell[t] = t_{\mathrm{AIcl}}$.* // *this is more of a comment, prove properly before actually using it*

Proof. Either t is there from beginning, then $t_{\text{AIcl}} = \ell[t]$. Otherwise it was introduced by a substitution, but then it was also introduced in $\text{AI}_{\text{cl}}(C)$ and lifted there.

Substitutions do not affect t_{AIcl} due to Lemma **??**. Hence also liftings do not affect t_{AIcl}. By Lemma 19, τ does not change t_{AIcl}.

$\Rightarrow t$ as well as t_{AIcl} remain invariant $\hfill\square$

Conjecture 21. *Let C_1 and C_2 be clauses of a resolution step such that a literal $l \in C_1$ and $l' \in C_2$ are resolved upon using σ such that $l\sigma = l'\sigma$. Let $l_{\text{AIcl}} \in \text{AI}_{\text{cl}}(C_1)$ correspond to l and $l'_{\text{AIcl}} \in \text{AI}_{\text{cl}}(C_2)$ correspond to l' (cf. Lemma **??**). Then $\text{au}(l_{\text{AIcl}}, l'_{\text{AIcl}})$ is well-defined, i.e. if it maps a variable x to another variable y, then y is unique.*

Proof. Suppose that $\{x_j \mapsto x_k\} \in \text{au}(l_{\text{AIcl}}, l'_{\text{AIcl}})$. Then either a (sub)term in l_{AIcl} or l'_{AIcl} is x_j and the corresponding (sub)term in the other literal is t such that $\ell[t\sigma] = x_k$, or ...

$\text{au}(l_{\text{AIcl}}, l'_{\text{AIcl}})$ is ill-defined if and only if there is another occurrence of x_j, at whose corresponding sub(term) in the other literal is s such that $\ell[s\sigma] = x_l$ with $k \neq l$. However as $l\sigma = l'\sigma$, we know that $t\sigma =$

BIG TODO

$\hfill\square$

6 old stuff, not sure if valuable

Lemma 22. *Let Φ be a set of formulas, t be a term and σ a substitution Then $\ell_{\Phi,x}[\ell_{\Phi,x}[t]\sigma] = \ell_{\Phi,x}[t\sigma]$.* // *WRONG*

Proof. With σ' as in Lemma **??**:

$\ell_{\Phi,x}[t\sigma] = \ell_{\Phi,x}[t]\sigma'$.

As σ just depends on the terms to replace and the variables to replace them with:

$\ell_{\Phi,x}[\ell_{\Phi,x}[t]\sigma] = \ell_{\Phi,x}[\ell_{\Phi,x}[t]]\sigma' = \ell_{\Phi,x}[t]\sigma'$. $\qquad\qquad\square$

OLD from first principles reasoning (complete this in case Lemma **??** is flawed):

(this was based on Δ-terms)

t contains Δ-terms ζ_i, grey terms g_i and free variables v_i. $\ell_{\Delta,x}[t]$ is t, where every maximal Δ-term ζ_i is replaced by free variable x_i.

A substitution σ occuring in a resolution refutation does not affect any of the x_i, as these symbols do not occur in the initial clause sets.

Hence $\ell_{\Delta,x}[t]\sigma$ differs from $\ell_{\Delta,x}[t]$ only in the v_i, which are potentially substituted. The v_i can be substituted to other free variables v'_i (not the same ones as factors of clauses are variable disjoint), grey terms g'_i or Δ-terms t'_i (these terms may contain grey or Δ-terms or free variables).

Assumption: $\ell_{\Delta,x}[]$ always replaces a certain Δ-term with the same variable, globally.

Goal: $\ell_{\Delta,x}[\ell_{\Delta,x}[t]\sigma] = \ell_{\Delta,x}[t\sigma]$.

Suppose σ does not affect t (and also not the x_i as it occurs in a resolution rule application). Then $\ell_{\Delta,x}[t\sigma] = \ell_{\Delta,x}[t] = \ell_{\Delta,x}[\ell_{\Delta,x}[t]]$ (

Otherwise σ changes a variable v_i at position p in t. $\ell_{\Delta,x}[t]$ has at position p v_i as well if the path to p does not contain colored symbol. So $\ell_{\Delta,x}[\ell_{\Delta,x}[t]\sigma]$ and $\ell_{\Delta,x}[t\sigma]$ coincide at p, irrespective of what σ introduces (var, Δ-/grey-term).

If p points into a maximal Δ-colored term, $p \mod k$ for some $k \geq 1$ is a Δ-colored term ζ_j. Hence in $\ell_{\Delta,x}[t]$, $\rho(p \mod k) = x_j$.

However in $\ell_{\Delta,x}[t\sigma]$, $\rho(p \mod k) = x_l$.

Lemma 23. *If $l \in C$, then $\ell[l] \in \mathrm{AI}_{\mathrm{cl}}(C)$.* // *WRONG*

Proof. By induction:

Base case by definition.

Let a literal λ be such that it hasn't been resolved upon in the deduction leading up to C.

Resolution $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $l\sigma = l'\sigma$ give $C : (D \vee E)\sigma$.

$\lambda \not\sim l$, as otherwise it would not be contained in C. W.l.o.g. $\lambda \in C_1$. Then $(\lambda\sigma) \in C$ by the resolution rule.

Induction hypothesis: $\ell[\lambda] \in \mathrm{AI}_{\mathrm{cl}}(C_1)$.

So have to show $\ell[\lambda\sigma] \in \mathrm{AI}_{\mathrm{cl}}(C)$ // as $(\lambda\sigma) \in C$

By Lemma 22, $\ell[\lambda\sigma] = \ell[\ell[\lambda]\sigma]$. We show that in all resolution cases, $\ell[\ell[\lambda]\sigma] \in \mathrm{AI}_{\mathrm{cl}}(C)$.

$\mathrm{AI}_{\mathrm{cl}}(C) = \ell[\big((\mathrm{AI}_{\mathrm{cl}}(C_1) \setminus \{\ell[l]\}) \vee (\mathrm{AI}_{\mathrm{cl}}(C_2) \setminus \{\neg\ell[l']\})\big)\sigma]$

As $\lambda \not\sim l$ and $\ell[\lambda] \in \mathrm{AI}_{\mathrm{cl}}(C_1)$, $\ell[\ell[\lambda]\sigma] \in \mathrm{AI}_{\mathrm{cl}}(C)$. $\qquad\qquad\square$

Proposition 24. $\Gamma \models Q_1 z_1 \ldots Q_n z_n \overline{\mathrm{PI}(C) \vee C}(z_1, \ldots, z_n)$, *quantifiers ordered as in 5, is a craig interpolant.*

Proof. Induction.

Suppose Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \qquad \sigma = \mathrm{mgu}(l, l')$$

$\Gamma \models \bar{Q}_{n_1} \overline{\mathrm{PI}(C_1) \vee D \vee l}$

$\Gamma \models \bar{Q}_{n_2} \overline{\mathrm{PI}(C_2) \vee E \vee \neg l'}$

to show:

$\Gamma \models \bar{Q}_n \overline{\mathrm{PI}(C) \vee (D \vee E)\sigma}$ // somewhat imprecise on $\bar{Q}_n$, but that's just useless quantifiers

$\Gamma \models (\bar{Q}_{n_1} \overline{PI(C_1)} \vee D \vee l)\sigma$

$\Gamma \models (\bar{Q}_{n_2} \overline{PI(C_2)} \vee E \vee \neg l')\sigma$

By resolution:

$\Gamma \models (\bar{Q}_{n_1} \overline{\mathrm{PI}(C_1)} \vee \bar{Q}_{n_2} \overline{\mathrm{PI}(C_2)})\sigma \vee (D \vee E)\sigma$

1. Suppose l, l' are from Γ alone: TODO

2. Suppose l and l' are colored with different colors and w.l.o.g l is Γ-colored and l' is Δ-colored.

   $\bar{Q}_n \overline{\mathrm{PI}(C)} \equiv \bar{Q}_n \overline{[(\neg l' \wedge \mathrm{PI}(C_1)^*) \vee (l \wedge \mathrm{PI}(C_2)^*)]\sigma}$

   $\equiv \bar{Q}_n (\overline{\neg l'\sigma} \wedge \overline{\overline{\mathrm{PI}(C_1)}\sigma}) \vee (\overline{l\sigma} \wedge \overline{\overline{\mathrm{PI}(C_2)}\sigma})$

   Adapt Huang proof to this, need to consider quantifiers:

   If $\Gamma \not\models D\sigma$ and $\Gamma \not\models E\sigma$ (else we are done), then

   $\Gamma \models [(\neg l' \wedge \bar{Q}_{n_1} \overline{PI(C_1)}) \vee (l \wedge \bar{Q}_{n_2} \overline{\mathrm{PI}(C_2)})]\sigma$

   As $\bar{Q}_{n_1}$ and $\bar{Q}_{n_2}$ disjoint and their variables do not appear in l or l',

   $\Gamma \models (\bar{Q}_{n_1} \bar{Q}_{n_2} [(\neg l' \wedge \overline{PI(C_1)}) \vee (l \wedge \overline{PI(C_2)})])\sigma$

   $\Gamma \models \bar{Q}_{n_1} \bar{Q}_{n_2} [(\neg l'\sigma \wedge \overline{\mathrm{PI}(C_1)}\sigma) \vee (l\sigma \wedge \overline{\mathrm{PI}(C_2)}\sigma)]$

   Consider the maximal terms of this expression which are Δ-colored.

   The $\mathrm{PI}(C_i)$, $i \in \{1, 2\}$ contain no colored terms. σ can introduce one by replacing a free variable x by a Δ-term t. But then overline replaces it with an universally quantified variable again, hence the formula is still entailed by Γ.

   $\Gamma \models \bar{Q}_{n_1} \bar{Q}_{n_2} [(\neg l'\sigma \wedge \overline{\overline{\mathrm{PI}(C_1)}\sigma}) \vee (l\sigma \wedge \overline{\overline{\mathrm{PI}(C_2)}\sigma})]$

   TODO: should work out similarily as huang if using P_P or it's the same as what i'm trying above.

$\square$

Proposition 25. $\Gamma \models Q_1 z_1 \ldots Q_n z_n \mathrm{PI}(C)^*(z_1, \ldots, z_n) \vee C$, *quantifiers ordered as in 5, is a craig interpolant.*

Proof. Induction.

Suppose Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \qquad \sigma = \mathrm{mgu}(l, l')$$

$\Gamma \models \bar{Q}_{n_1} \mathrm{PI}(C_1)^* \vee D \vee l$

$\Gamma \models \bar{Q}_{n_2} \mathrm{PI}(C_2)^* \vee E \vee \neg l'$

to show: $\Gamma \models \bar{Q}_n \mathrm{PI}(C)^* \vee (D \vee E)\sigma$

$\Gamma \models (\bar{Q}_{n_1} \mathrm{PI}(C_1)^* \vee D \vee l)\sigma$

$\Gamma \models (\bar{Q}_{n_2} \mathrm{PI}(C_2)^* \vee E \vee \neg l')\sigma$

By resolution:

$\Gamma \models (\bar{Q}_{n_1} \mathrm{PI}(C_1)^* \vee \bar{Q}_{n_2} \mathrm{PI}(C_2)^*)\sigma \vee (D \vee E)\sigma$

1. Suppose l, l' are from Γ alone: TODO

2. Suppose l and l' are colored with different colors and w.l.o.g l is Γ-colored and l' is Δ-colored.

   $\bar{Q}_n \mathrm{PI}(C)^* \equiv \bar{Q}_n([(\neg l' \wedge \mathrm{PI}(C_1)^*) \vee (l \wedge \mathrm{PI}(C_2)^*)]\sigma)^*$

   Adapt Huang proof to this, need to consider quantifiers:

   If $\Gamma \not\models D\sigma$ and $\Gamma \not\models E\sigma$ (else we are done), then

   $\Gamma \models [(\neg l' \wedge \bar{Q}_{n_1} \mathrm{PI}(C_1)^*) \vee (l \wedge \bar{Q}_{n_2} \mathrm{PI}(C_2)^*)]\sigma$

   As $\bar{Q}_{n_1}$ and $\bar{Q}_{n_2}$ disjoint and their variables do not appear in l or l',

   $\Gamma \models (\bar{Q}_{n_1}\bar{Q}_{n_2}[(\neg l' \wedge \mathrm{PI}(C_1)^*) \vee (l \wedge \mathrm{PI}(C_2)^*)])\sigma$

   The $\mathrm{PI}(C_i)$, $i \in \{1, 2\}$ contain no colored terms. σ can introduce one by replacing a free variable x.

   Consider the maximal terms of this expression which are Γ-colored.

   Either they only have grey subterms, then if they are existentially quantified, we can just use it as witness as the terms aren't replaced.

   Otherwise they contain at least a Γ- or a Δ-colored subterm.

Base case: simple.

Suppose Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \qquad \sigma = \mathrm{mgu}(l, l')$$

$\Gamma \models \bar{Q}_{n_1} \mathrm{PI}(C_1)^* \vee D \vee l$

$\Gamma \models \bar{Q}_{n_2} \mathrm{PI}(C_2)^* \vee E \vee \neg l'$

to show: $\Gamma \models \bar{Q}_n \mathrm{PI}(C)^*\sigma \vee (D \vee E)\sigma$

Note that a term newly introduced in $\mathrm{PI}(C)$ occurs in either l or l', but not in both.

Let t be a colored term in $\mathrm{PI}(C)$, which has just been added W.l.o.g. let it occur in l, i.e. in C_1.

Case distinction:

1. Suppose l, l' are from Γ alone:

   By induction hypothesis:

   $\Gamma \models (\bar{Q}_{n_1} \operatorname{PI}(C_1)^* \vee D \vee l)\sigma$

   $\Gamma \models (\bar{Q}_{n_2} \operatorname{PI}(C_2)^* \vee E \vee \neg l')\sigma$

   By resolution:

   $\Gamma \models (\bar{Q}_{n_1} \operatorname{PI}(C_1)^* \vee \bar{Q}_{n_2} \operatorname{PI}(C_2)^*)\sigma \vee (D \vee E)\sigma$

   Suppose t is Γ-colored.

   Then it will be replaced by x_i and existentially quantified. It appears in either $\operatorname{PI}(C_1)$ or $\operatorname{PI}(C_2)$.

   t is a witness for x_i because it contains subterms $\zeta_1, \ldots, \zeta_n$. If they are overbound as well, they are so before t and are available here.

   TODO: derive properties using examples 103 or so

OTHER TRY:

Then σ replaces variables $y_1, \ldots, y_k$ in $E \vee \neg l'$ with terms that contain t.

By the induction hypothesis, $\Gamma \models Q_1 z_1 \ldots Q_{n_2} z_{n_2} \operatorname{PI}(C_2)^*(z_1, \ldots, z_{n_2}) \vee E \vee \neg l'$.

Hence $\Gamma \models (Q_1 z_1 \ldots Q_{n_2} z_{n_2} \operatorname{PI}(C_2)^*(z_1, \ldots, z_{n_2}) \vee E \vee \neg l')\sigma$.

Also $\Gamma \models Q_1 z_1 \ldots Q_{n_2} z_{n_2} (\operatorname{PI}(C_2)^*(z_1, \ldots, z_{n_2})\sigma) \vee E\sigma \vee \neg l'\sigma$.

Similarily, $\Gamma \models Q_1 z_1 \ldots Q_{n_1} z_{n_1} (\operatorname{PI}(C_1)^*(z_1, \ldots, z_{n_1})\sigma) \vee D\sigma \vee l\sigma$

$\Gamma \models Q_1 z_1 \ldots Q_n z_n ((\neg l \wedge \operatorname{PI}(C_2)) \vee (l \wedge \operatorname{PI}(C_1)))^*(z_1, \ldots, z_n)\sigma) \vee D\sigma \vee l\sigma$

l basically is the only new thing ($l\sigma = l'\sigma$).

Either l does not contain any subterms of other terms, then it does not depend on anything and l serves as witness for itself.

Otherwise it does depend on other terms and we have to make sure that that term is available. Depending on another term means that it uses information that is only available from another term, i.e. it contains a subterm of another term. but then that subterm is quantified over before the variable that replaces t is, so it works out.

t is Δ-colored. Then it is replaced by a universally quantified variable. But it "was already universally quantified" in the induction hypothesis. There, it was some free variable, because that's the only thing that can be substituted, but even with this free var, it worked out.

$\square$

Conjecture 26. $\Gamma \cup \Delta$ *unsat*, π *propositional resolution refutation. Then* $\Gamma \models \bar{Q}_n \operatorname{PI}(C)^* \vee C$ *and* $\Delta \models \neg\bar{Q}_n \operatorname{PI}(C)^* \vee C$ *for all C in π.*

Proof. Base case as in Huang.

Induction.

Suppose Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l}{C : D \vee E}$$

$\Gamma \models \bar{Q}_{n_1} \text{PI}(C_1)^* \vee D \vee l$

$\Gamma \models \bar{Q}_{n_2} \text{PI}(C_2)^* \vee E \vee \neg l$

to show: $\Gamma \models \bar{Q}_n \text{PI}(C)^* \vee D \vee E$, i.e.

$\Gamma \models \text{sort}(Q_{n_1} \cup Q_{n_2} \cup \text{colored-terms}(l))((\neg l^* \wedge PI(C_1)^*) \vee (l^* \wedge PI(C_2)^*)) \vee D \vee E$

If $\Gamma \not\models D$ and $\Gamma \not\models E$ (else we are done), then

$\Gamma \models (\neg l \wedge \bar{Q}_{n_1} PI(C_1)^*) \vee (l \wedge \bar{Q}_{n_2} PI(C_2)^*)$

As $\bar{Q}_{n_1}$ and $\bar{Q}_{n_2}$ disjoint and their variables do not appear in l or l,

$\Gamma \models \bar{Q}_{n_1} \bar{Q}_{n_2} [(\neg l \wedge PI(C_1)^*) \vee (l \wedge PI(C_2)^*)]$

Since we've pushed the variables outside, no colored terms appear in $\text{PI}(C_i)^*$.

Suppose l does not contain colored terms. Then $l = l^*$ and we are done.

Otherwise let t be a maximal colored term in l.

By lemma 7, l appears in Γ with a certain polarity, say in clause E. l is an instance of E.

In fact, l is contained in $C\sigma$ where σ is the composition of unifiers applied in the deriviation up to the current point.

Hence $\Gamma \models C\sigma$.

1. Suppose t is Γ-colored. $\Gamma \models l$ implies that $\Gamma \models \exists y \, l\{t/y\}$

2. Suppose t is Δ-colored. $\Gamma \models \forall y \, l\{t/y\}$ because:

$\square$