Number of quantifier alternations in Huang and nested

1   Preliminaries

For $\sigma = \mathrm{mgu}(\varphi, \psi)$ for two terms or literals φ and ψ, we denote by σ_i for $1 \leq i \leq n$ the ith substitution which is added to σ by the unification algorithm, where $n = |\mathrm{dom}(\sigma)|$. We define $\sigma_0 \stackrel{\mathrm{def}}{=} \mathrm{id}$.

We furthermore denote the composition of all σ_k for $i \leq k \leq j$ by $\sigma_{(i,j)}$. Hence $\sigma = \sigma_{(1,n)} = \sigma_{(0,n)}$.

A term t is *single-colored* if t is Φ-colored for some Φ and all colored symbols in t are Φ-colored.

A literal l is called a Φ-literal if its predicate symbol is Φ-colored.

Quantifier alternations in I usually assumes the quantifier-alternation-minimizing arrangement of quantifiers in I. The lemma statements hence talk about the *minimal* number of quantifier alternations, which is indeed easily obtainable, i.e. it's a lower and upper bound at the same time.

In the following, we assume that the maximum max of an empty sequence is defined to be 0 and constants are treated as function symbols of arity 0. Furthermore $\perp$ is used to denote a color which is not possessed by any symbol.

Definition 1 (Color alternation col-alt)**.** Let Γ and Δ be sets of formulas and t be a term.

$$\text{col-alt}(t) \stackrel{\mathrm{def}}{=} \text{col-alt}_\perp(t)$$

$$\text{col-alt}_\Phi(t) \stackrel{\mathrm{def}}{=} \begin{cases} 0 & t \text{ is a variable} \\ \max(\text{col-alt}_\Phi(t_1), \dots, \text{col-alt}_\Phi(t_n)) & t = f(t_1, \dots, t_n) \text{ is grey} \\ \max(\text{col-alt}_\Phi(t_1), \dots, \text{col-alt}_\Phi(t_n)) & t = f(t_1, \dots, t_n) \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_\Psi(t_1), \dots, \text{col-alt}_\Psi(t_n)) & t = f(t_1, \dots, t_n) \text{ is of color } \Psi, \\ & \Phi \neq \Psi \end{cases}$$

$\triangle$

Definition 2 (Quantifier alternation quant-alt)**.** Let A be a formula.

quant-alt$(A) \stackrel{\text{def}}{=}$ quant-alt$_\perp(A)$

$$\text{quant-alt}_Q(A) \stackrel{\text{def}}{=} \begin{cases} 0 & A \text{ is an atom} \\ \text{quant-alt}_Q(B) & A \equiv \neg B \\ \max(\text{quant-alt}_Q(B), & A \equiv B \circ C, \circ \in \{\wedge, \vee, \supset\} \\ \quad \text{quant-alt}_Q(C)) & \\ \text{quant-alt}_Q(B) & A \equiv Q'B, Q = Q' \\ 1 + \text{quant-alt}_{Q'}(B) & A \equiv Q'B, Q \neq Q' \end{cases}$$

$\triangle$

Note that this definition of quantifier alternations handles formulas in prenex and non-prenex form.

Definition 3. We define $\text{PI}^\circ_{\text{step}}$ to coincide with PI_{step} but without applying the substitution σ in each of the cases. Analogously, if $C \equiv D\sigma$, we use C° to denote D. $\triangle$

Hence $\text{PI}^\circ_{\text{step}}(\cdot)\sigma = \text{PI}_{\text{step}}(\cdot)$.

2 Occurrence of terms in the interpolant

Definition 4 (PI*)**.** PI* is defined as PI with the difference that in PI*, all literals are considered to be grey. $\triangle$

Hence $\text{PI}^*_{\text{init}}$ coincides with PI_{init}. $\text{PI}^*_{\text{step}}$ coincides with PI_{step} in case of factorisation and paramodulation inferences. For resolution inferences, the first two cases in the definition of PI_{step} do not occur for $\text{PI}^*_{\text{step}}$.

PI* enjoys the convenient property that it absorbs every literal which occurs some clause:

Proposition 5. *For every literal which occurs in a clause of a resolution refutation π, a respective successor occurs in $\text{PI}^*(\pi)$.*

Proof. By structural induction. $\square$

⟨ma:grey_lits_of_pi_star_in_pi⟩ **Lemma 6.** *For every clause C of a resolution refutation, every grey literal, which occurs in $\text{PI}^*(C)$, also occurs in $\text{PI}(C)$.*

Proof. Note that PI_{init} and $\text{PI}^*_{\text{init}}$ coincide and PI_{step} and $\text{PI}^*_{\text{step}}$ only differ for resolution inferences. But more specifically, they only differ on resolution inferences, where the resolved literal is colored. However here, no grey literals are lost. $\square$

Note that in PI*, we can conveniently reason about the occurrence of terms as no terms are lost throughout the extraction. However Lemma **??** allows us to transfer results about grey literals to PI. We can also give similar results about general literals and equalities occurring in the resolution refutation:

⟨lemma:grey_lits_all_in_PI⟩ **Lemma 7.** *If there is a grey literal λ in a clause C of a resolution refutation π, then a successor of λ occurs in $\text{PI}(\pi)$.*

Proof. Immediate by the definition of PI. □

⟨lemma:equalities_all_in_PI⟩ **Lemma 8.** *For every equality $s = t$ of a clause in a resolution refutation π, a successor of $s = t$ occurs in $\mathrm{PI}(\pi)$.*

Proof. Equalities in clauses are only removed by means of paramodulation and as π derives the empty clause, all equalities are removed eventually. For any paramodulation inference ι using the equality $s = t$, $\mathrm{PI}_{\mathrm{step}}(\iota, I_1, I_2)$ contains $s = t$. □

We now make some considerations in the form of four lemmata about the construction of terms of certain shapes in the context of interpolant extraction. In the following, we abbreviate $\mathrm{PI}_{\mathrm{step}}^{*\circ}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_n)) \vee C^\circ$ by χ.

⟨lemma:var_grey_col_lit⟩ **Lemma 9.** *Let ι be a resolution or factorisation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

⟨14_1⟩
1. The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.

⟨14_5⟩
2. The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨14_4⟩
3. The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.

⟨14_2⟩
4. There is a variable v such that

  – u occurs grey in $v\sigma_i$ and

  – v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Φ-literal as well as grey in a Ψ-literal.

⟨14_3⟩
5. There is a variable v such that

  – u occurs grey in $v\sigma_i$ and

  – v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.

Proof. We consider the unification process, and particularly the different cases which lead to the variable u in a grey position in a Φ-literal in $\chi\sigma_{(0,\,i)}$:

- There already is a Φ-literal in $\chi\sigma_{(0,\,i-1)}$ which contains u at a grey position and σ_i does not change this. Then clearly 1 is the case.

- Otherwise there must be a Φ-literal in $\chi\sigma_{(0,\,i-1)}$, which contains a variable v at a grey position such that u occurs grey in $v\sigma_i$. Hence in the resolved or factorised literals λ and λ', there is a position p such that w.l.o.g. $\lambda|_p = v$ and $\lambda'|_p$ contains a grey occurrence of u, and λ and λ coincide along the path to p.

  Note that λ and λ' are contained in χ as all literals are added to the interpolant since the definition of χ is based on PI^*.

  We distinguish based properties of the position p:

  – Suppose that p is contained in a single-colored Φ-term. Then u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and 2 is the case.

– Suppose that p is contained in a single-colored Ψ-term. Then u occurs grey in a Φ-literal as well in a single-colored Ψ-term, which implies 5.

– Otherwise p is a grey position. We distinguish further:

* Suppose that the resolved or factorised literal is Φ-colored. Then u occurs grey in a Φ-literal and we have established item 5.

* Suppose that the resolved or factorised literal is Ψ-colored. Then the variable v occurs grey in a Φ-literal as well as grey in a Ψ-literal, hence 4 is the case.
Otherwise the resolved or factorised literal is grey and u occurs grey in a grey literal, which is sufficient for 3. $\qquad\square$

⟨lemma:var_in_sc_term⟩ **Lemma 10.** *Let ι be a resolution or factorisation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

⟨15_3⟩
   1. The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.

⟨15_1⟩
   2. The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨15_5⟩
   3. The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.

⟨15_2⟩
   4. There is a variable v such that

     – u occurs grey in $v\sigma_i$ and

     – v occurs in a single-colored Φ-term as well as in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨15_4⟩
   5. There is a variable v such that

     – u occurs grey in $v\sigma_i$ and

     – v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Ψ-literal as well as in a single-colored Φ-term.

Proof. We consider the different cases of the unification process which lead to the variable u in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$:

• There is a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which contains u such that σ_i does not change this. Then 2 is the case.

• Suppose that there is a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which contains a variable v such that u occurs grey in $v\sigma_i$.

Hence in the resolved or factorised literals λ and λ' (which are both contained in χ), there is a position p such that w.l.o.g. $\lambda|_p = v$ and $\lambda'|_p$ contains a grey occurrence of u, and λ and λ' coincide along p. We distinguish based properties of the position p:

– Suppose that p is contained in a single-colored Φ-term. Then u is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and 2 holds.

– Suppose that p is contained in a single-colored Ψ-term. As then v is contained in a single-colored Φ-term as well as in a single-colored Ψ-term, 4 is the case.

– Suppose that p is a grey position. We distinguish further:

  * Suppose that the resolved or factorised literal is Φ-colored. Then u occurs grey in a Φ-literal, which suffices for 1.

  * Suppose that the resolved or factorised literal is Ψ-colored. Then the variable v occurs in a single-colored Φ-term as well as grey in a Ψ-literal, which implies 5.

  * Otherwise the resolved or factorised literal is grey. But then u occurs grey in a grey literal and we have established item 3.

- Otherwise there is a variable w which occurs in $\chi\sigma_{(0,\,i-1)}$ such that u occurs in a single-colored Φ-term in $w\sigma_i$. This can only be the case if $w\sigma$ already occurs in $\chi\sigma_{(0,\,i-1)}$, which implies that 2 is the case.   □

⟨lemma:var_grey_col_lit_paramod⟩ **Lemma 11.** *Let ι be a paramodulation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

⟨16_1⟩
  1. The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.

⟨16_5⟩
  2. The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨16_6⟩
  3. The variable u occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ or some variable v occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$.

  NB: not subsumed by 1

⟨16_3⟩
  4. There is a variable v such that

    – *u occurs grey in $v\sigma_i$ and*

    – *v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.*

Proof. Consider paramodulation: $s = t \lor D$ and $E[r]_p$ create $C : (D \lor E[t]_p)\sigma$ where $\sigma = \mathrm{mgu}(s, r)$.

A grey occurrence of variable u can be created in C by 2 means: either t contains a grey variable and p is a grey position or σ introduces a grey occurrence of a variable in a grey position

Let u be a variable in a grey position in a Φ-literal in $\chi\sigma_{(0,\,i)}$. We consider the cases which lead to this situation:

- The variable u is present in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$. Then clearly 1 is the case.

- Suppose that t contains a grey occurrence of u and p is a grey position in a Φ-literal. Then u occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ and 3 is the case.

- Suppose that σ_i introduces a grey occurrence of u in a Φ-literal (but not in p^1). Then there exists a variable v which occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$. By that, we can derive that

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[1] not sure if this remark is really useful

there exists a position q in s and r respectively such that ① $s|_q = v$ and $r|_q$ contains u grey OR ② $r|_q = v$ and $s|_q$ contains u grey. Furthermore, s and r agree on the path to q

We distinguish:

- Suppose that q is contained in a single-colored Φ-term.
  Then u is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which establishes 2.(① + ②).
  (r might still be contained in other colored terms, but u remains in a single-colored Φ-term.

- Suppose that q is contained in a single-colored Ψ-term. Then in $\chi\sigma_{(0,\,i-1)}$, v occurs grey in a Φ-literal and in a s.c. Ψ-term and $v\sigma_i$ contains u grey, thus establishing 4.

- Suppose that q is a grey position.
  NOTE: r might be in a colored term.
  In case ②, u occurs at in a grey position in an equality in $\chi\sigma_{(0,\,i-1)}$, which shows that 3 is the case.
  In case ①, v occurs at in a grey position in an equality in $\chi\sigma_{(0,\,i-1)}$ and $v\sigma_i$ contains u grey, which also that 3 is the case.   □

⟨`lemma:var_in_sc_term_paramod`⟩ **Lemma 12.** *Let ι be a paramodulation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$.*
Then at least one of the following statements holds:

⟨`17_5`⟩
1. *The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨`17_6`⟩
2. *The variable u occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ or some variable v occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$.*

⟨`17_2`⟩
3. *There is a variable v such that*

    - *u occurs grey in $v\sigma_i$ and*

    - *v occurs in a single-colored Φ-term as well as in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$.*

Proof. Let u be a variable in a in single-colored Φ-term in $\chi\sigma_{(0,\,i)}$. We consider the cases which lead to this:

- The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and σ_i does not change this. Then 1 is the case.

- Suppose that t contains an occurrence of u a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$. Then 1 is the case.

  NB: the single-colored Φ-term which contains u occurs in an equality, but this is not relevant, i.e. we do not need a special equality case here like in the other lemma.

- Suppose that σ_i introduces a grey occurrence of u in a single-colored Φ-term. Then there exists a variable v which occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$.

  Hence in s and r, there exists a position q such that either $s|_q$ or $r|_q$ is v and the respective other subterm contains u grey. As s and r are unifiable, they agree on the path up to q.

  We distinguish:

  - Suppose that q is contained in a single-colored Φ-term. Then u is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and 1 is the case.

  - Suppose that q is contained in a single-colored Ψ-term. Then v occurs in $\chi\sigma_{(0,\,i-1)}$ in a single-colored Φ-term as well as in a single-colored Ψ-term and u occurs grey in $v\sigma_i$. Hence 3 holds.

  - Suppose that q is a grey position. Then $s|_q$ occurs grey in an equality. Hence either v or u occur grey in an equality in $\chi\sigma_{(0,\,i-1)}$. Both of these cases satisfy 2.

- Suppose that a single-colored Φ-term containing u is contained in $\mathrm{ran}(\sigma_i)$ and let $\mathrm{dom}(\sigma_i) = \{v\}$. Then v is unified with a term containing a single-colored Φ-term containing u, and $v\sigma_i$ occurs in $\chi\sigma_{(0,\,i-1)}$. This implies that 1 is the case. $\qquad\square$

The preceding lemmata allow us to formulate a result which acts as the core machinery for the proof of existence of terms in grey literals or equalities. We first give it for PI* but then generalise it to PI.

⟨ange_and_grey_in_col_lit_star⟩ **Lemma 13.** *Let C be a clause in the resolution refutation π of $\Gamma \cup \Delta$ and u be a variable which occurs in $\mathrm{PI}^*(C) \vee C$ in some literal in a single-colored Φ-term or grey in a Φ-literal.*

Suppose that u also occurs in $\mathrm{PI}^*(C) \vee C$ in some literal in a single-colored Ψ-term or grey in a Ψ-literal.

Then u occurs grey in a grey literal or an equality.

Note that Φ and Ψ are to be read as any pair of distinct colors, i.e. Γ and Δ as well as Δ and Γ.

Proof. We proceed by induction over π and σ.

Note that initially, every pair of clauses is variable-disjoint and all symbols of a clause are either all grey or Φ-colored or all grey or Ψ-colored, hence the lemma is vacuously true.

For the induction step, we assume that the property holds for $\mathrm{PI}^*(C_i) \vee C_i$, $1 \le i \le n$, where $C_1, \ldots, C_n$ are the clauses used in a resolution or factorisation inference ι. Note that then, the property also holds for χ, i.e. for $\mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_n)) \vee C^\circ$ as it contains all the literals present in $\mathrm{PI}^*(C_i) \vee C_i$ for any i (this is evident by the definition of $\mathrm{PI}^{*\circ}_{\mathrm{step}}$), and as clauses are pairwise variable-disjoint, the lemma condition can not become true for a variable for which it was not true in $\mathrm{PI}^*(C_i) \vee C$ for some i.

Suppose that u occurs in $\chi\sigma_{(0,\,i)}$ in a single-colored Φ-term or grey in a Φ-literal and that u also occurs in $\chi\sigma_{(0,\,i)}$ in a single-colored Ψ-term or grey in a Ψ-literal.

Then we can deduce by the Lemmata 9, 10, 11 and 12 that at least one of the following statements holds:

⟨oozoh70h1⟩
1. The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h5⟩
2. The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h4⟩
3. The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h2⟩
4. There is a variable v such that

   – u occurs grey in $v\sigma_i$ and
   – v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Φ-literal as well as grey in a Ψ-literal.

⟨oozoh70h6⟩
5. There is a variable v such that u occurs grey in $v\sigma_i$ and

   – u occurs grey in $v\sigma_i$ and
   – v occurs in a single-colored Φ-term as well as in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h3⟩
6. There is a variable v such that

   – u occurs grey in $v\sigma_i$ and
   – v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.

⟨oozoh70h7⟩
7. The variable u occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ or some variable v occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$.

By the same lemmata, we get the same set of statements where Φ and Ψ are interchanged. We refer to them by the respective number followed by $^{\triangle}$.

Suppose that 3 is not the case as otherwise we are done since σ_i is trivial on u as u occurs in $\chi\sigma_{(0,\,i)}$. Furthermore, there are a number of cases which give the result by the induction hypothesis: For the cases 4, 5 and 6 we can infer that by the induction hypothesis, there is a grey occurrence of the variable v in a grey literal in $\chi\sigma_{(0,\,i-1)}$, and as u occurs grey in $v\sigma_i$, there is a grey occurrence of u in a grey literal in $\chi\sigma_{(0,\,i)}$.

If 7 or $7^{\triangle}$apply, then clearly u occurs grey in an equality in $\chi\sigma_{(0,\,i)}$ and we are done.

It remains to show that the lemma holds true in case the statements 1 or 2 as well as $1^{\triangle}$or $2^{\triangle}$hold. But note that in any combination of 1 or 2 and $1^{\triangle}$or $2^{\triangle}$in effect yields a situation under which the induction hypothesis again is applicable. Hence we may infer that u occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$ and since σ_i is trivial u as shown above, u occurs grey in a grey literal in $\chi\sigma_{(0,\,i)}$. □

TODO: remove this and do same thing for lemma 15 because it's actually required for that

⟨l_change_and_grey_in_col_lit⟩? **Lemma 14.** *Same as 13 with PI in place of PI*.*

Proof. First note that PI and PI* do not differ with respect to equalities. Therefore we only concern ourselfs with grey occurrences of variables in grey literals.

As $\mathrm{PI}(C)$ for any clause C is comprised of a subset of the literals of $\mathrm{PI}^*(C)$, the lemma prerequesites hold true only for variables in $\mathrm{PI}(C)$ for which they also hold true in $\mathrm{PI}^*(C)$. As by Lemma 13 the lemma holds for $\mathrm{PI}^*(C)$, respective grey literals with grey occurrences of the variables in question exist in $\mathrm{PI}^*(C)$. But by Lemma 6, these literals also occur in $\mathrm{PI}(C)$. $\qquad\square$

3 Lifting of terms leads to quantifier alternations

⟨lemma:subterm_in_grey_lit⟩ **Lemma 15.** *If* $\mathrm{PI}^*(C) \vee C$ *contains a maximal colored occurrence of a* Φ*-term* $t[s]$ *containing* Ψ*-term* s*, then* s *occurs grey in a grey literal or an equality in* $\mathrm{PI}(C) \vee C$.

Proof. Note that it suffices to show that at the step where s is introduced as subterm of $t[s]$, s occurs grey in $\mathrm{PI}(C) \vee C$ as any later modification by substitution is applied to both occurrences s, so they remain equal throughout the remaining derivation.

We perform an induction over π and σ. For the base case, the lemma is vacuously true as all original clauses contain only symbols of one color.

For the induction step, we assume that the property holds for $\mathrm{PI}^*(C_i) \vee C_i$, $1 \le i \le n$ where $C_1, \ldots, C_n$ are the clauses used in ι. Then the property also holds for $\mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_n)) \vee C^\circ$ by the same argument as given in the proof of Lemma 13. This formula is however the unfolded definition of χ,

Hence it remains to show that the property holds for $\chi\sigma = \chi\sigma_0 \ldots \sigma_m = \mathrm{PI}^*(C) \vee C$. We do so by induction on m. Suppose the property holds for $\chi\sigma_{(0,i-1)}$. We consider all situations where a Φ-term $t[s]$ containing a Ψ-term s is introduced by means of unification.

- Suppose that a variable u occurs in $\chi\sigma_{(0,i-1)}$ such that $u\sigma_i$ contains $t[s]$. Then u is unified with a term in $\chi\sigma_{(0,i-1)}$ which is $t[s]$, hence this term occurs there and we obtain the result by the induction hypothesis.

- Suppose that a variable u occurs in $\chi\sigma_{(0,i-1)}$ in a maximal colored Φ-term t such that $u\sigma_i$ contains s at a grey position.

  We distinguish based on the occurrence of u in $\chi\sigma_{(0,i-1)}$:

  - Suppose that u occurs grey in a grey literal or grey in an equality in $\chi\sigma_{(0,i-1)}$. Then we are done.
  - Suppose that u occurs grey in a Ψ-literal or in a single-colored Ψ-term in $\chi\sigma_{(0,i-1)}$. Then by Lemma 13, u also occurs grey in a grey literal or an equality in $\chi\sigma_{(0,i-1)}$ and we are done.
  - If none of the above cases apply, then u occurs only grey in Φ-literals and in single-colored Φ-terms in $\chi\sigma_{(0,i-1)}$.

    We distinguish further based on the type of the inference:

    * Suppose that ι is a resolution or factorisation inference. As $u \in \mathrm{dom}(\sigma_i)$, u occurs in on of the resolved or factorised literals λ and λ'. Let it occur in λ at position p. Then $\lambda'|_p = u\sigma_i$.

Due to the definition of the unification algorithm, λ and λ' must agree on the path to p. Hence as u only occurs grey in a Φ-colored literal or in a single-colored Φ-term, so does $u\sigma_i$. However $u\sigma_i$ contains s at a grey position. So if $\lambda'|_p$ is contained in a single-colored Φ-term, then s occurs here in a Φ-term and we obtain the result by the induction hypothesis.
Otherwise $\lambda'|_p$ is a grey position in a Φ-literal. TODO:

* Suppose that ι is a paramodulation inference of $r_1 = r_2 \vee D$ and $E[r]$ which results in $(D \vee E[r_2])\sigma$. As $\sigma = \mathrm{mgu}(r_1, r)$ and $u \in \mathrm{dom}(\sigma_i)$, u occurs either in r_1 or r at position p such that the respective other term contains s grey at p.
TODO:

GO THROUGH THIS AGAIN AND CONSIDER PARAMODULATION

For the induction step, Step: Resolution or factorisation inference ι, $\mathrm{mgu}(\iota) = \sigma = \sigma_1 \cdots \sigma_n$ The term $t[s]$ is created by one of the following two ways:

- A variable u occurs in $\chi\sigma_{(0,\,i-1)}$ such that $u\sigma_i = t[s]$.

  Then u occurs in a resolved or factorised literal $\lambda\sigma_{(0,\,i-1)}$ at $\hat{u}$ such that at the other resolved or factorised literal $\lambda'\sigma_{(0,\,i-1)}$, $\lambda'\sigma_{(0,\,i-1)}|_{\hat{u}} = t[s]$. Then the condition is present at $\chi\sigma_{(0,\,i-1)}$ and we get the result by the induction hypothesis.

- Note that we only consider maximal colored terms.

  Let $t[u]$ be a maximal colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ such that in the tree-representation of $t[u]$, the path from the root to u does not contain a node labelled with a Ψ-symbol, and $u\sigma_i$ contains a grey occurrence of s.

  Suppose that u occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$. Then s occurs grey in a grey literal in $\chi\sigma_{(0,\,i)}$ as σ_i does not affect u since u occurs in $\chi\sigma_{(0,\,i)}$ and we are done. wtf, probably wrong

  If u occurs grey in a Ψ-literal or if u occurs in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$, then by Lemma 13, u also occurs grey in a grey literal or an equality in $\chi\sigma_{(0,\,i-1)}$ and s hence occurs grey in a grey literal or an equality in $\chi\sigma_{(0,\,i)}$.

  Now suppose that u does not occur grey in a grey literal or an equality in $\chi\sigma_{(0,\,i-1)}$ as otherwise we are done.

  Hence as all other cases are excluded, u can only occur in $\chi\sigma_{(0,\,i-1)}$ in a single-colored Φ-term or grey in a Φ-colored literal. But then, since $u\sigma_i$ contains a grey occurrence of s, there is a position p in the two resolved or factorised literals λ and λ' such that $\lambda|_p = u$ and $\lambda'|_p$ contains a grey occurrence of s. Furthermore, the prefix along the path to p is the same in both λ and λ'. As u only occurs in single-colored Φ-terms, $\lambda'|_p$ does so

are probably not same t and s as in lemma statement, which isn't technically wrong but confusing

as well, so s is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$. Since s is a Ψ-term, by the induction hypothesis, s occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$ and hence also in $\chi\sigma_i$. INCOMPLETE: does not consider s grey in Φ-lits □

We present an example which illustrates that the occurrence of a term with n color alternations in $\mathrm{PI}(C) \vee C$ for a clause C can lead to an interpolant with $n-1$ quantifier alternations (but no less as Proposition 18 shows).

Example 16. Let $\Gamma = \{\neg P(a)\}$ and $\Delta = \{P(x) \vee Q(f(x)), \neg Q(y)\}$. Consider the following refutation of $\Gamma \cup \Delta$:

$$\cfrac{\cfrac{\neg P(a) \mid \bot \qquad P(x) \vee Q(f(x)) \mid \top}{Q(f(a)) \mid \neg P(a)}\ \underset{x \mapsto a}{\mathrm{res}} \qquad \neg Q(y) \mid \top}{\square \mid \neg P(a)}\ \underset{y \mapsto f(a)}{\mathrm{res}}$$

In this example, Theorem **??** yields the interpolant $I \equiv \exists y_a \neg P(y_a)$ with quant-alt$(I) = 1$. The existence of the term $f(a)$ with col-alt$(f(a)) = 2$ in a clause of the refutation implies that quant-alt$(I) \geq 1$. △

⟨lt_in_grey_lit_then_quant_alt⟩ **Lemma 17.** *If a term with n color alternations occurs in a grey literal or an equality $\mathrm{PI}(C) \vee C$ for a clause C, then the interpolant I produced in Theorem **??** contains at least n quantifier alternations.*

Proof. We perform an induction on n and show the strenghtening that the quantification of the lifting variable corresponding to a term with n color alternations is required to be in the scope of the quantification of $n-1$ alternating quantifiers.

For $n = 0$, no colored terms occur in I and hence by construction no quantifiers and for $n = 1$, there are only single-colored terms.

Suppose that the statement holds for $n-1$ for $n > 1$ and that a term t with col-alt$(t) = n$ occurs in $\mathrm{PI}(C)$. We assume without loss of generality that t is a Φ-term. Then t contains a Ψ-colored term s and by Lemma 15, s occurs grey in a grey literal or an equality in $\mathrm{PI}(C) \vee C$. By Lemma 7 and Lemma 8, a successor of s occurs in $\mathrm{PI}(\pi)$. Note that as s occurs in a grey position, any successor of s also occurs in a grey position.

By the induction hypothesis, the quantification of the lifting variable for s requires $n-1$ alternated quantifiers. As s is a subterm of t and t is lifted, t must be quantified in the scope of the quantification of s, and as t and s are of different color, their quantifier type is different. Hence the quantification of the lifting variable for t requires n quantifier alternations. □

⟨prop:color_alt_eq_quant_alt⟩ **Proposition 18.** *If a term with n color alternations occurs in $\mathrm{PI}(C) \vee C$ for a clause C, then the interpolant I produced in Theorem **??** contains at least $n-1$ quantifier alternations.*

Proof. By Lemma 15, a term with $n-1$ color alternations occurs in a grey literal or an equality in $\mathrm{PI}(C) \vee C$. Lemma 17 gives the result. □

4   Monday: Paramodulation

4.1   Notes

1. Every equality which is used ends up in the interpolant, i.e. it's a grey literal (binary)

2. Every equality is used eventually

4.2   Proof

Extension of Lemma 9

5   Monday prime: LI

1. Agrees with PI except if lifting conditions apply.

Lemma 19. *If a*

6   directly from old proof

this may not be correct any more w.r.t. notation (χ)

just for repetition:

?⟨lemma:col_change⟩? **Lemma 20.** *Resolution or factorisation step ι from $\bar{C}$.*

If u col-change var in $(\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \dots, \mathrm{PI}(C_n)) \vee C^\circ)\sigma_{(0,i)}$, then u also occurs grey in that formula.

Proof. Abbreviation: $F \equiv (\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \dots, \mathrm{PI}(C_n)) \vee C^\circ)$

Induction over refutation and σ; base case easy.

Step: Supp color change var u present in $\chi\sigma_{(0,i)}$. (could also say introduced, then proof would be somehow different)

Supp u not grey in $\chi\sigma_{(0,i-1)}$ as otherwise done. As a first step, we show that if a (not necessarily color-changing) variable v occurs in a single-colored Φ-term $t[v]$ in $\chi\sigma_{(0,i)}$, then at least one of the following holds:

1. v occurs in some single-colored Φ-term in $\chi\sigma_{(0,i-1)}$

⟨var_occ_1⟩ 2. there is a color-changing variable w in $\chi\sigma_{(0,i-1)}$ such that v occurs grey in $w\sigma_i$.

⟨var_occ_2⟩ We consider unification process, and particularly the different cases which can introduce a variable v in a single-colored term Φ: Either it has been there before, it was introduced in a s.c. Φ-colored term, or a s.c. Φ-term containing the var is in $\mathrm{ran}(\sigma)$.

- Suppose a term $t'[v]$ is present in $\chi\sigma_{(0,i-1)}$ such that $t'[v]\sigma_i = t[v]$. Then 1 is the case.

- Suppose a variable w occurs in a single-colored Φ-term in $\chi\sigma_{(0,i-1)}$ such that v occurs grey in $w\sigma_i$. Suppose furthermore that 1 is not the case, i.e.

v does not occur in a s.c. Φ-term in $\chi\sigma_{(0,\,i-1)}$, as otherwise we would be done. We show that 2 is the case.

As v occurs neither grey nor in a s.c. Φ-term in $\chi\sigma_{(0,\,i-1)}$ but occurs in $\mathrm{ran}(\sigma_i)$, it must occur in $\chi\sigma_{(0,\,i-1)}$ and this can only be in a single-colored Ψ-term.

As by assumption v occurs grey in $w\sigma_i$, there must be an occurrence $\hat{w}$ of w in a resolved or factorised literal, say $\lambda\sigma_{(0,\,i-1)}$ such that for the other resolved literal $\lambda'\sigma_{(0,\,i-1)}$, $\lambda'\sigma_{(0,\,i-1)}|_{\hat{w}}$ is a subterm in which v occurs grey. But as the occurrence of v in $\lambda'\sigma_{(0,\,i-1)}|_{\hat{w}}$ must be contained in a single-colored Ψ-term, so is $\lambda\sigma_{(0,\,i-1)}|_{\hat{w}}$, hence z occurs in a single-colored Ψ-term as well. Therefore 2 is the case.

- Suppose there is a variable z in $\chi\sigma_{(0,\,i-1)}$ such that v occurs in a single-colored Φ-term in $z\sigma_i$. Then $z\sigma_i$ occurs in $\chi\sigma_{(0,\,i-1)}$, but this is a witness for 1.

Now recall that we have assumed u to be a color-changing variable in $\chi\sigma_{(0,\,i)}$. Hence it occurs in a single-colored Γ-term as well as in a single-colored Δ-term. By the reasoning above, this leads to two case:

- In $\chi\sigma_{(0,\,i-1)}$, u occurs both in some single-colored Γ-term as well as in some single-colored Δ-term. Then we get the result by the induction hypothesis and the fact that $u \notin \mathrm{dom}(\sigma_i)$ as u does occur in $\chi\sigma_{(0,\,i)}$.

- Otherwise for some color Φ, u does not occur in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$. Then case 2 above must hold and there is some color-changing variable w in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $w\sigma_{(0,\,i)}$. But then by the induction hypothesis, w occurs grey in $\chi\sigma_{(0,\,i-1)}$ and hence u occurs grey in $\chi\sigma_{(0,\,i)}$. $\qquad\square$

6.1 Proof

- Induction over $\ell^x_\Delta[\mathrm{PI}(C) \vee C]$ and also about Γ-terms with Δ-lifting vars in that formula. Cf. -final

- NB: now somewhat described in the proper proof below describe proof method with $\sigma_{(0,\,i)}$: which PI?

  - Factorisation: easy: just apply σ_i for all i to $\mathrm{PI}(C) \vee C$. When done, a literal will be there twice and we can remove it without losing anything

  - Resolution: create propositional structure first.
    Ex.: $C_1 : D \vee l$, $C_2 : \neg l \vee E$:
    If we talk about properties for which it holds that if they hold for $\mathrm{PI}(C_i) \vee C_i$, $i \in \{1, 2\}$, then they also hold for $A \equiv \big((l \wedge \mathrm{PI}(C_2)) \vee (\neg l \wedge \mathrm{PI}(C_1))\big) \vee C^\circ$, then we can apply σ_i for all i to that formula. So if we can assume it for A and show it for all σ_i, we get that it holds for $\mathrm{PI}(C) \vee C$.

Also: clauses are variable disjoint, so e.g. it's not possible that a color-changing var is created by $\mathrm{PI_{step}}$

Also: do it like a few lemmas further down, like $(\mathrm{PI}_{\mathrm{step}}^{\circ}(\iota, \mathrm{PI}(C_1), \dots, \mathrm{PI}(C_n)) \vee C^{\circ})\sigma_{(0,\,i)}$