1 Outline

Goal: try proof Huang and nested equal.

Method: proof for both:

Conjectured Proposition 1. *Let I be an interpolant created by \$algorithm. If I contains a term t such that t has n color changes, then I has at least n quantifier alternations.*

1.1 generally keep in mind

- also note: literal is sometimes used for negated or not negated predicate with terms but in regular formulas with arbitrary connectives

2 Preliminaries

For $\sigma = \mathrm{mgu}(\varphi, \psi)$ for two terms or literals φ and ψ, we denote by σ_i for $1 \leq i \leq n$ the ith substitution which is added to σ by the unification algorithm, where $n = |\operatorname{dom}(\sigma)|$. We define $\sigma_0 \stackrel{\text{def}}{=} \mathrm{id}$.

We furthermore denote the composition of all σ_k for $i \leq k \leq j$ by $\sigma_{(i,\,j)}$. Hence $\sigma = \sigma_{(1,\,n)} = \sigma_{(0,\,n)}$.

A term t is *single-colored* if t is Φ-colored for some Φ and all colored symbols in t are Φ-colored.

A literal l is called a Φ-literal if its predicate symbol is Φ-colored.

Quantifier alternations in I usually assumes the quantifier-alternation-minimizing arrangement of quantifiers in I

Definition 2 (Color alternation col-alt)**.** Let Γ and Δ be sets of formulas and t be a term. We assume that the maximum max of an empty sequence is defined to be 0. We define:

$\text{col-alt}(t) \stackrel{\text{def}}{=} \text{col-alt}_\perp(t)$

$$\text{col-alt}_\Phi(t) \stackrel{\text{def}}{=} \begin{cases} 0 & t \text{ is a variable} \\ \max(\text{col-alt}_\Phi(t_1), \dots, \text{col-alt}_\Phi(t_n)) & t = f(t_1, \dots, t_n) \text{ is grey} \\ \max(\text{col-alt}_\Phi(t_1), \dots, \text{col-alt}_\Phi(t_n)) & t = f(t_1, \dots, t_n) \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_\Psi(t_1), \dots, \text{col-alt}_\Psi(t_n)) & t = f(t_1, \dots, t_n) \text{ is of color } \Psi, \\ & \Phi \neq \Psi \end{cases}$$

$\triangle$

Definition 3 (Quantifier alternation quant-alt)**.** Let A be a formula. We assume that the maximum max of an empty sequence is defined to be 0. We define:

$$\text{quant-alt}(A) \overset{\text{def}}{=} \text{quant-alt}_{\perp}(A)$$

$$\text{quant-alt}_Q(A) \overset{\text{def}}{=} \begin{cases} 0 & A \text{ is an atom} \\ \text{quant-alt}_Q(B) & A \equiv \neg B \\ \max(\text{quant-alt}_Q(B), \\ \qquad \text{quant-alt}_Q(C)) & A \equiv B \circ C,\ \circ \in \{\wedge, \vee, \supset\} \\ \text{quant-alt}_Q(B) & A \equiv Q'B,\ Q = Q' \\ 1 + \text{quant-alt}_{Q'}(B) & A \equiv Q'B,\ Q \neq Q' \end{cases}$$

△

Definition 4. We define $\mathrm{PI}^{\circ}_{\mathrm{step}}$ to coincide with $\mathrm{PI}_{\mathrm{step}}$ but without applying the substitution σ in each of the cases. Analogously, if $C \equiv D\sigma$, we use C° to denote D. △

Hence $\mathrm{PI}^{\circ}_{\mathrm{step}}(\cdot)\sigma = \mathrm{PI}_{\mathrm{step}}(\cdot)$.

3 Thursday prime

Definition 5 (PI^*)**.** PI^* is defined as PI with the difference that in PI^*, all literals are considered to be grey. △

Hence $\mathrm{PI}^*_{\mathrm{init}}$ coincides with $\mathrm{PI}_{\mathrm{init}}$. $\mathrm{PI}^*_{\mathrm{step}}$ coincides with $\mathrm{PI}_{\mathrm{step}}$ in case of factorisation and paramodulation inferences. For resolution inferences, the first two cases in the definition of $\mathrm{PI}_{\mathrm{step}}$ do not occur for $\mathrm{PI}^*_{\mathrm{step}}$.

PI^* enjoys the convenient property that it absorbs every literal which occurs some clause:

Proposition 6. *For every literal which occurs in a clause of a resolution refutation π, a respective successor occurs in $\mathrm{PI}^*(\pi)$.*

Proof. By structural induction. □

ef:grey_lits_of_pi_star_in_pi⟩ **Lemma 7.** *For every clause C of a resolution refutation, every grey literal, which occurs in $\mathrm{PI}^*(C)$, also occurs in $\mathrm{PI}(C)$.*

Proof. Note that $\mathrm{PI}_{\mathrm{init}}$ and $\mathrm{PI}^*_{\mathrm{init}}$ coincide and $\mathrm{PI}_{\mathrm{step}}$ and $\mathrm{PI}^*_{\mathrm{step}}$ only differ for resolution inferences. But more specifically, they only differ on resolution inferences, where the resolved literal is colored. However here, no grey literals are lost. □

Note that in PI^*, we can conveniently reason about the occurrence of terms as no terms are lost throughout the extraction. However Lemma 7 allows us to transfer results about grey literals to PI.

We now make some considerations in the form of four lemmata about the construction of terms of certain shapes in the context of interpolant extraction. In the following, we abbreviate $\mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \ldots, \mathrm{PI}^*(C_n)) \vee C^{\circ}$ by χ.

⟨lemma:var_grey_col_lit⟩ **Lemma 8.** *Let ι be a resolution or factorisation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

⟨14_1⟩
1. *The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨14_5⟩ *2. The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨14_4⟩ *3. The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨14_2⟩ *4. There is a variable v such that*

- *u occurs grey in $v\sigma_i$ and*
- *v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Φ-literal as well as grey in a Ψ-literal.*

⟨14_3⟩ *5. There is a variable v such that*

- *u occurs grey in $v\sigma_i$ and*
- *v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.*

Proof. We consider the unification process, and particularly the different cases which lead to the variable u in a grey position in a Φ-literal in $\chi\sigma_{(0,\,i)}$:

- There already is a Φ-literal in $\chi\sigma_{(0,\,i-1)}$ which contains u at a grey position and σ_i does not change this. Then clearly 1 is the case.
- Otherwise there must be a Φ-literal in $\chi\sigma_{(0,\,i-1)}$, which contains a variable v at a grey position such that u occurs grey in $v\sigma_i$. Hence in the resolved or factorised literals λ and λ', there is a position p such that w.l.o.g. $\lambda|_p = v$ and $\lambda'|_p$ contains a grey occurrence of u, and λ and λ coincide along the path to p.

  Note that λ and λ' are contained in χ as all literals are added to the interpolant since the definition of χ is based on PI*.

  We distinguish based properties of the position p:

  - Suppose that p is contained in a single-colored Φ-term. Then u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and 2 is the case.
  - Suppose that p is contained in a single-colored Ψ-term. Then u occurs grey in a Φ-literal as well in a single-colored Ψ-term, which implies 5.
  - Otherwise p is a grey position. We distinguish further:
    - ∗ Suppose that the resolved or factorised literal is Φ-colored. Then u occurs grey in a Φ-literal and we have established item 5.
    - ∗ Suppose that the resolved or factorised literal is Ψ-colored. Then the variable v occurs grey in a Φ-literal as well as grey in a Ψ-literal, hence 4 is the case.
      Otherwise the resolved or factorised literal is grey and u occurs grey in a grey literal, which is sufficient for 3. □

⟨lemma:var_in_sc_term⟩ **Lemma 9.** *Let ι be a resolution or factorisation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

⟨15_3⟩ *1. The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_1⟩ *2. The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_5⟩ *3. The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_2⟩ *4. There is a variable v such that*

- *u occurs grey in $v\sigma_i$ and*
- *v occurs in a single-colored Φ-term as well as in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_4⟩ *5. There is a variable v such that*

- *u occurs grey in $v\sigma_i$ and*
- *v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Ψ-literal as well as in a single-colored Φ-term.*

Proof. We consider the different cases of the unification process which lead to the variable u in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$:

- There is a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which contains u such that σ_i does not change this. Then 2 is the case.
- Suppose that there is a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which contains a variable v such that u occurs grey in $v\sigma_i$.

  Hence in the resolved or factorised literals λ and λ' (which are both contained in χ), there is a position p such that w.l.o.g. $\lambda|_p = v$ and $\lambda'|_p$ contains a grey occurrence of u, and λ and λ' coincide along p. We distinguish based properties of the position p:

  - Suppose that p is contained in a single-colored Φ-term. Then u is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and 2 holds.
  - Suppose that p is contained in a single-colored Ψ-term. As then v is contained in a single-colored Φ-term as well as in a single-colored Ψ-term, 4 is the case.
  - Suppose that p is a grey position. We distinguish further:
    * Suppose that the resolved or factorised literal is Φ-colored. Then u occurs grey in a Φ-literal, which suffices for 1.
    * Suppose that the resolved or factorised literal is Ψ-colored. Then the variable v occurs in a single-colored Φ-term as well as grey in a Ψ-literal, which implies 5.
    * Otherwise the resolved or factorised literal is grey. But then u occurs grey in a grey literal and we have established item 3.
- Otherwise there is a variable w which occurs in $\chi\sigma_{(0,\,i-1)}$ such that u occurs in a single-colored Φ-term in $w\sigma_i$. This can only be the case if $w\sigma$ already occurs in $\chi\sigma_{(0,\,i-1)}$, which implies that 2 is the case. □

emma:var_grey_col_lit_paramod⟩ **Lemma 10.** *Let ι be a paramodulation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

⟨16_1⟩
1. *The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨16_5⟩
2. *The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨16_6⟩
3. *The variable u occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ or some variable v occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$.*

   NB: not subsumed by 1

⟨16_3⟩
4. *There is a variable v such that*
   - *u occurs grey in $v\sigma_i$ and*
   - *v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.*

Proof. Consider paramodulation: $s = t \vee D$ and $E[r]_p$ create $C : (D \vee E[t]_p)\sigma$ where $\sigma = \mathrm{mgu}(s, r)$.

A grey occurrence of variable u can be created in C by 2 means: either t contains a grey variable and p is a grey position or σ introduces a grey occurrence of a variable in a grey position

Let u be a variable in a grey position in a Φ-literal in $\chi\sigma_{(0,\,i)}$. We consider the cases which lead to this situation:

- The variable u is present in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$. Then clearly 1 is the case.
- Suppose that t contains a grey occurrence of u and p is a grey position in a Φ-literal. Then u occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ and 3 is the case.
- Suppose that σ_i introduces a grey occurrence of u in a Φ-literal (but not in p[1]). Then there exists a variable v which occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$. By that, we can derive that there exists a position q in s and r respectively such that 1 $s|_q = v$ and $r|_q$ contains u grey OR 2 $r|_q = v$ and $s|_q$ contains u grey. Furthermore, s and r agree on the path to q

  We distinguish:

  – Suppose that q is contained in a single-colored Φ-term.
    Then u is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which establishes 2.(1 + 2).
    (r might still be contained in other colored terms, but u remains in a single-colored Φ-term.

  – Suppose that q is contained in a single-colored Ψ-term. Then in $\chi\sigma_{(0,\,i-1)}$, v occurs grey in a Φ-literal and in a s.c. Ψ-term and $v\sigma_i$ contains u grey, thus establishing 4.

[1] not sure if this remark is really useful

– Suppose that q is a grey position.

NOTE: r might be in a colored term.

In case 2, u occurs at in a grey position in an equality in $\chi\sigma_{(0,\,i-1)}$, which shows that 3 is the case.

In case 1, v occurs at in a grey position in an equality in $\chi\sigma_{(0,\,i-1)}$ and $v\sigma_i$ contains u grey, which also that 3 is the case. □

⟨lemma:var_in_sc_term_paramod⟩ **Lemma 11.** *Let ι be a paramodulation inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$.*

Then at least one of the following statements holds:

⟨17_5⟩ 1. *The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨17_6⟩ 2. *The variable u occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ or some variable v occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$.*

⟨17_2⟩ 3. *There is a variable v such that*
   - *u occurs grey in $v\sigma_i$ and*
   - *v occurs in a single-colored Φ-term as well as in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$.*

Proof. Let u be a variable in a in single-colored Φ-term in $\chi\sigma_{(0,\,i)}$. We consider the cases which lead to this:

- The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and σ_i does not change this. Then 1 is the case.

- Suppose that t contains an occurrence of u a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$. Then 1 is the case.

  NB: the single-colored Φ-term which contains u occurs in an equality, but this is not relevant, i.e. we do not need a special equality case here like in the other lemma.

- Suppose that σ_i introduces a grey occurrence of u in a single-colored Φ-term. Then there exists a variable v which occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$.

  Hence in s and r, there exists a position q such that either $s|_q$ or $r|_q$ is v and the respective other subterm contains u grey. As s and r are unifiable, they agree on the path up to q.

  We distinguish:

  – Suppose that q is contained in a single-colored Φ-term. Then u is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and 1 is the case.

  – Suppose that q is contained in a single-colored Ψ-term. Then v occurs in $\chi\sigma_{(0,\,i-1)}$ in a single-colored Φ-term as well as in a single-colored Ψ-term and u occurs grey in $v\sigma_i$. Hence 3 holds.

  – Suppose that q is a grey position. Then $s|_q$ occurs grey in an equality. Hence either v or u occur grey in an equality in $\chi\sigma_{(0,\,i-1)}$. Both of these cases satisfy 2.

- Suppose that a single-colored Φ-term containing u is contained in $\mathrm{ran}(\sigma_i)$ and let $\mathrm{dom}(\sigma_i) = \{v\}$. Then v is unified with a term containing a single-colored Φ-term containing u, and $v\sigma_i$ occurs in $\chi\sigma_{(0,\,i-1)}$. This implies that 1 is the case. □

The preceding lemmata allow us to formulate a result which acts as the core machinery for the proof of existence of terms in grey literals or equalities. We first give it for PI^* but then generalise it to PI.

ange_and_grey_in_col_lit_star⟩ **Lemma 12.** *Let C be a clause in the resolution refutation π of $\Gamma \cup \Delta$ and u be a variable which occurs in $\mathrm{PI}^*(C) \vee C$ in some literal in a single-colored Φ-term or grey in a Φ-literal.*

Suppose that u also occurs in $\mathrm{PI}^*(C) \vee C$ in some literal in a single-colored Ψ-term or grey in a Ψ-literal.

Then u occurs grey in a grey literal or an equality.

Note that Φ and Ψ are to be read as any pair of distinct colors, i.e. Γ and Δ as well as Δ and Γ.

Proof. We proceed by induction over π and σ.

Note that initially, every pair of clauses is variable-disjoint and all symbols of a clause are either all grey or Φ-colored or all grey or Ψ-colored, hence the lemma is vacuously true.

For the induction step, we assume that the property holds for $\mathrm{PI}^*(C_i) \vee C_i$, $1 \leq i \leq n$, where $C_1, \dots, C_n$ are the clauses used in a resolution or factorisation inference ι. Note that then, the property also holds for χ, i.e. for $\mathrm{PI}^{*\circ}_{\mathrm{step}}(\iota, \mathrm{PI}^*(C_1), \dots, \mathrm{PI}^*(C_n)) \vee C^\circ$ as it contains all the literals present in $\mathrm{PI}^*(C_i) \vee C_i$ for any i (this is evident by the definition of $\mathrm{PI}^{*\circ}_{\mathrm{step}}$), and as clauses are pairwise variable-disjoint, the lemma condition can not become true for a variable for which it was not true in $\mathrm{PI}^*(C_i) \vee C$ for some i.

Suppose that u occurs in $\chi\sigma_{(0,\,i)}$ in a single-colored Φ-term or grey in a Φ-literal and that u also occurs in $\chi\sigma_{(0,\,i)}$ in a single-colored Ψ-term or grey in a Ψ-literal.

Then we can deduce by the Lemmata 8, 9, 10 and 11 that at least one of the following statements holds:

⟨oozoh70h1⟩
1. The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h5⟩
2. The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h4⟩
3. The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h2⟩
4. There is a variable v such that
   - u occurs grey in $v\sigma_i$ and
   - v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Φ-literal as well as grey in a Ψ-literal.

⟨oozoh70h6⟩
5. There is a variable v such that u occurs grey in $v\sigma_i$ and
   - u occurs grey in $v\sigma_i$ and
   - v occurs in a single-colored Φ-term as well as in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h3⟩

6. There is a variable v such that
   - u occurs grey in $v\sigma_i$ and
   - v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.

⟨oozoh70h7⟩

7. The variable u occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ or some variable v occurs grey in an equality in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $v\sigma_i$.

By the same lemmata, we get the same set of statements where Φ and Ψ are interchanged. We refer to them by the respective number followed by $^\triangle$.

Suppose that 3 is not the case as otherwise we are done since σ_i is trivial on u as u occurs in $\chi\sigma_{(0,\,i)}$. Furthermore, there are a number of cases which give the result by the induction hypothesis: For the cases 4, 5 and 6 we can infer that by the induction hypothesis, there is a grey occurrence of the variable v in a grey literal in $\chi\sigma_{(0,\,i-1)}$, and as u occurs grey in $v\sigma_i$, there is a grey occurrence of u in a grey literal in $\chi\sigma_{(0,\,i)}$.

If 7 or $7^\triangle$apply, then clearly u occurs grey in an equality in $\chi\sigma_{(0,\,i)}$ and we are done.

It remains to show that the lemma holds true in case the statements 1 or 2 as well as $1^\triangle$or $2^\triangle$hold. But note that in any combination of 1 or 2 and $1^\triangle$or $2^\triangle$in effect yields a situation under which the induction hypothesis again is applicable. Hence we may infer that u occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$ and since σ_i is trivial u as shown above, u occurs grey in a grey literal in $\chi\sigma_{(0,\,i)}$. □

ol_change_and_grey_in_col_lit⟩ **Lemma 13.** *Same as 12 with* PI *in place of* PI*.

Proof. First note that PI and PI* do not differ with respect to equalities. Therefore we only concern ourselfs with grey occurrences of variables in grey literals.

As PI(C) for any clause C is comprised of a subset of the literals of PI*(C), the lemma prerequesites hold true only for variables in PI(C) for which they also hold true in PI*(C). As by Lemma 12 the lemma holds for PI*(C), respective grey literals with grey occurrences of the variables in question exist in PI*(C). But by Lemma 15, these literals also occur in PI(C). □

4 Friday

⟨lemma:subterm_in_grey_lit⟩ **Lemma 14.** *If* PI(C) $\vee$ C *contains a maximal colored occurrence of a* Φ*-term* $t[s]$ *containing* Ψ*-term* s*, then* s *occurs grey in a grey literal or an equality in* PI(C) $\vee$ C.

Proof. Note that it suffices to show that at the step where s is introduced as subterm of $t[s]$, s occurs grey in PI(C) $\vee$ C as any later modification by substitution is applied to both occurrences s, so they stay equal throughout the remaining derivation.

Induction over π and σ. TODO: as in Lemma 13

Base case: vacuously true.

Step: Resolution or factorisation inference ι, $\mathrm{mgu}(\iota) = \sigma = \sigma_1 \cdots \sigma_n$ The term $t[s]$ is created by one of the following two ways:
(we abbreviate $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \dots, \mathrm{PI}(C_n)) \vee C^{\circ}$ by F.)

- A variable u occurs in $\chi\sigma_{(0,\,i-1)}$ such that $u\sigma_i = t[s]$.

  Then u occurs in a resolved or factorised literal $\lambda\sigma_{(0,\,i-1)}$ at $\hat{u}$ such that at the other resolved or factorised literal $\lambda'\sigma_{(0,\,i-1)}$, $\lambda'\sigma_{(0,\,i-1)}|_{\hat{u}} = t[s]$. Then the condition is present at $\chi\sigma_{(0,\,i-1)}$ and we get the result by the induction hypothesis.

- Note that we only consider maximal colored terms.

  Let $t[u]$ be a maximal colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ such that in the tree-representation of $t[u]$, the path from the root to u does not contain a node labelled with a Ψ-symbol, and $u\sigma_i$ contains a grey occurrence of s.

  are probably not same t and s as in lemma statement, which isn't technically wrong but confusing

  Suppose that u occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$. Then s occurs grey in a grey literal in $\chi\sigma_{(0,\,i)}$ as σ_i does not affect u since u occurs in $\chi\sigma_{(0,\,i)}$ and we are done.

  If u occurs grey in a Ψ-literal or if u occurs in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$, then by Lemma 13, u also occurs grey in a grey literal or an equality in $\chi\sigma_{(0,\,i-1)}$ and s hence occurs grey in a grey literal or an equality in $\chi\sigma_{(0,\,i)}$.

  Now suppose that u does not occur grey in a grey literal or an equality in $\chi\sigma_{(0,\,i-1)}$ as otherwise clearly we are done.

  Hence as all other cases are excluded, u can only occur in $\chi\sigma_{(0,\,i-1)}$ in a single-colored Φ-term or grey in a Φ-colored literal. But then, since $u\sigma_i$ contains a grey occurrence of s, there is a position p in the two resolved or factorised literals λ and λ' such that $\lambda|_p = u$ and $\lambda'|_p$ contains a grey occurrence of s. Furthermore, the prefix along the path to p is the same in both λ and λ'. As u only occurs in single-colored Φ-terms, $\lambda'|_p$ does so as well, so s is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$. Since s is a Ψ-term, by the induction hypothesis, s occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$ and hence also in $\chi\sigma_i$. □

⟨lemma:grey_lits_all_in_PI⟩ **Lemma 15.** *If there is a grey literal λ in a clause C of a resolution refutation π, then a successor of λ occurs in* $\mathrm{PI}(\pi)$.

Proof. Immediate by the definition of PI. □

⟨lemma:equalities_all_in_PI⟩ **Lemma 16.** *For every equality $s = t$ of a clause in a resolution refutation π, a successor of $s = t$ occurs in* $\mathrm{PI}(\pi)>$

Proof. Equalities in clauses are only removed by means of paramodulation and as π derives the empty clause, all equalities are removed eventually. For any paramodulation inference ι, $\mathrm{PI}_{\mathrm{step}}(\iota, I_1, I_2)$ contains $s = t$. □

We present an example which illustrates that the occurrence of a term with n color alternations in $\mathrm{PI}(C) \vee C$ for a clause C can lead to an interpolant with $n-1$ quantifier alternations (but no less as Proposition 19 shows).

Example 17. Let $\Gamma = \{\neg P(a)\}$ and $\Delta = \{P(x) \vee Q(f(x)), \neg Q(y)\}$. Consider the following refutation of $\Gamma \cup \Delta$:

$$\dfrac{\dfrac{\neg P(a) \mid \bot \qquad P(x) \vee Q(f(x)) \mid \top}{Q(f(a)) \mid \neg P(a)}\ \mathrm{res}_{x \mapsto a} \qquad \neg Q(y) \mid \top}{\Box \mid \neg P(a)}\ \mathrm{res}_{y \mapsto f(a)}$$

In this example, Theorem **??** yields the interpolant $I \equiv \exists y_a \neg P(y_a)$ with quant-alt$(I) = 1$. The existence of the term $f(a)$ with col-alt$(f(a)) = 2$ in a clause of the refutation implies that quant-alt$(I) \geq 1$. △

⟨lt_in_grey_lit_then_quant_alt⟩ **Lemma 18.** *If a term with n color alternations occurs in a grey literal or an equality* $\mathrm{PI}(C) \vee C$ *for a clause C, then the interpolant I produced in Theorem* **??** *contains at least n quantifier alternations.*

Proof. We perform an induction on n and show the strenghtening that the quantification of the lifting variable corresponding to a term with n color alternations is required to be in the scope of the quantification of $n-1$ alternating quantifiers.

For $n = 0$, no colored terms occur in I and hence by construction no quantifiers and for $n = 1$, there are only single-colored terms.

Suppose that the statement holds for $n-1$ for $n > 1$ and that a term t with col-alt$(t) = n$ occurs in $\mathrm{PI}(C)$. We assume without loss of generality that t is a Φ-term. Then t contains a Ψ-colored term s and by Lemma 14, s occurs grey in a grey literal or an equality in $\mathrm{PI}(C) \vee C$. By Lemma 15 and Lemma 16, a successor of s occurs in $\mathrm{PI}(\pi)$. Note that as s occurs in a grey position, any successor of s also occurs in a grey position.

By the induction hypothesis, the quantification of the lifting variable for s requires $n-1$ alternated quantifiers. As s is a subterm of t and t is lifted, t must be quantified in the scope of the quantification of s, and as t and s are of different color, their quantifier type is different. Hence the quantification of the lifting variable for t requires n quantifier alternations. □

⟨prop:color_alt_eq_quant_alt⟩ **Proposition 19.** *If a term with n color alternations occurs in* $\mathrm{PI}(C) \vee C$ *for a clause C, then the interpolant I produced in Theorem* **??** *contains at least $n-1$ quantifier alternations.*

Proof. By Lemma 14, a term with $n-1$ color alternations occurs in a grey literal or an equality in $\mathrm{PI}(C) \vee C$. Lemma 18 gives the result. □

5 Monday: Paramodulation

5.1 Notes

1. Every equality which is used ends up in the interpolant, i.e. it's a grey literal (binary)
2. Every equality is used eventually

5.2 Proof

Extension of Lemma 8

6 Monday prime: LI

1. Agrees with PI except if lifting conditions apply.

Lemma 20. *If a*

7 directly from old proof

this may not be correct any more w.r.t. notation (χ)

just for repetition:

?⟨lemma:col_change⟩? **Lemma 21.** *Resolution or factorisation step ι from $\bar{C}$.*

If u col-change var in $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \ldots, \mathrm{PI}(C_n)) \vee C^{\circ})\sigma_{(0,\,i)}$, then u also occurs grey in that formula.

Proof. Abbreviation: $F \equiv (\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \ldots, \mathrm{PI}(C_n)) \vee C^{\circ})$

Induction over refutation and σ; base case easy.

Step: Supp color change var u present in $\chi\sigma_{(0,\,i)}$. (could also say introduced, then proof would be somehow different)

Supp u not grey in $\chi\sigma_{(0,\,i-1)}$ as otherwise done. As a first step, we show that if a (not necessarily color-changing) variable v occurs in a single-colored Φ-term $t[v]$ in $\chi\sigma_{(0,\,i)}$, then at least one of the following holds:

⟨var_occ_1⟩
1. v occurs in some single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$
2. there is a color-changing variable w in $\chi\sigma_{(0,\,i-1)}$ such that v occurs grey in $w\sigma_i$.

⟨var_occ_2⟩ We consider unification process, and particularly the different cases which can introduce a variable v in a single-colored term Φ: Either it has been there before, it was introduced in a s.c. Φ-colored term, or a s.c. Φ-term containing the var is in $\mathrm{ran}(\sigma)$.

- Suppose a term $t'[v]$ is present in $\chi\sigma_{(0,\,i-1)}$ such that $t'[v]\sigma_i = t[v]$. Then 1 is the case.

- Suppose a variable w occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ such that v occurs grey in $w\sigma_i$. Suppose furthermore that 1 is not the case, i.e. v does not occur in a s.c. Φ-term in $\chi\sigma_{(0,\,i-1)}$, as otherwise we would be done. We show that 2 is the case.

  As v occurs neither grey nor in a s.c. Φ-term in $\chi\sigma_{(0,\,i-1)}$ but occurs in $\mathrm{ran}(\sigma_i)$, it must occur in $\chi\sigma_{(0,\,i-1)}$ and this can only be in a single-colored Ψ-term.

  As by assumption v occurs grey in $w\sigma_i$, there must be an occurrence $\hat{w}$ of w in a resolved or factorised literal, say $\lambda\sigma_{(0,\,i-1)}$ such that for the other resolved literal $\lambda'\sigma_{(0,\,i-1)}$, $\lambda'\sigma_{(0,\,i-1)}|_{\hat{w}}$ is a subterm in which v occurs grey. But as the occurrence of v in $\lambda'\sigma_{(0,\,i-1)}|_{\hat{w}}$ must be contained in a single-colored Ψ-term, so is $\lambda\sigma_{(0,\,i-1)}|_{\hat{w}}$, hence z occurs in a single-colored Ψ-term as well. Therefore 2 is the case.

- Suppose there is a variable z in $\chi\sigma_{(0,\,i-1)}$ such that v occurs in a single-colored Φ-term in $z\sigma_i$. Then $z\sigma_i$ occurs in $\chi\sigma_{(0,\,i-1)}$, but this is a witness for 1.

Now recall that we have assumed u to be a color-changing variable in $\chi\sigma_{(0,\,i)}$. Hence it occurs in a single-colored Γ-term as well as in a single-colored Δ-term. By the reasoning above, this leads to two case:

- In $\chi\sigma_{(0,\,i-1)}$, u occurs both in some single-colored Γ-term as well as in some single-colored Δ-term. Then we get the result by the induction hypothesis and the fact that $u \notin \mathrm{dom}(\sigma_i)$ as u does occur in $\chi\sigma_{(0,\,i)}$.
- Otherwise for some color Φ, u does not occur in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$. Then case 2 above must hold and there is some color-changing variable w in $\chi\sigma_{(0,\,i-1)}$ such that u occurs grey in $w\sigma_{(0,\,i)}$. But then by the induction hypothesis, w occurs grey in $\chi\sigma_{(0,\,i-1)}$ and hence u occurs grey in $\chi\sigma_{(0,\,i)}$. □

7.1 Proof

- Induction over $\ell^x_\Delta[\mathrm{PI}(C) \vee C]$ and also about Γ-terms with Δ-lifting vars in that formula. Cf. `-final`
- NB: now somewhat described in the proper proof below describe proof method with $\sigma_{(0,\,i)}$: which PI?
  - Factorisation: easy: just apply σ_i for all i to $\mathrm{PI}(C) \vee C$. When done, a literal will be there twice and we can remove it without losing anything
  - Resolution: create propositional structure first.
    Ex.: $C_1 : D \vee l$, $C_2 : \neg l \vee E$:
    If we talk about properties for which it holds that if they hold for $\mathrm{PI}(C_i) \vee C_i$, $i \in \{1, 2\}$, then they also hold for $A \equiv \Big((l \wedge \mathrm{PI}(C_2)) \vee (\neg l \wedge \mathrm{PI}(C_1))\Big) \vee C^\circ$, then we can apply σ_i for all i to that formula.
    So if we can assume it for A and show it for all σ_i, we get that it holds for $\mathrm{PI}(C) \vee C$.

  Also: clauses are variable disjoint, so e.g. it's not possible that a color-changing var is created by $\mathrm{PI}_{\mathrm{step}}$

  Also: do it like a few lemmas further down, like $(\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \dots, \mathrm{PI}(C_n)) \vee C^\circ)\sigma_{(0,\,i)}$