# Example 101: canonical examples for overbinding w.r.t. order

Here, the second lower prooftree shows what huang would produce.

## Ex 101a

$$\dfrac{\dfrac{\overset{\Sigma}{P({\color{red}u},f({\color{red}u}))\vee Q({\color{red}u})} \qquad \overset{\Pi}{\neg Q(a)}}{P(a,f(a))}\; u\mapsto a \qquad \overset{\Pi}{\neg P(x,y)}}{\Box}\; x\mapsto a, y\mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{Q(a)}\; u\mapsto a \qquad \top}{P(a,f(a))\vee Q(a)}\; x\mapsto a, y\mapsto f(a) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 Q(x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}$$

Direct overbinding would not work without merging same variables!: $\forall x_1\forall x_2\exists x_3(P(x_2,x_3)\vee Q(x_1))$
counterexample: $Q\sim\{0\}, P\sim\{(1,0)\}$
Direct overbinding would work when considering original dependencies as highlighted above
arrow lemma:

$$\dfrac{\dfrac{\overset{\Sigma}{\Gamma\models\exists y_1(P({\color{red}u},y_1)\vee Q({\color{red}u})\vee\bot)} \qquad \overset{\Pi}{\Gamma\models\neg Q(x_1)\vee\top}}{\Gamma\models\exists y_1(P({\color{red}x_1},y_1)\vee Q({\color{red}x_1}))}\; u\mapsto a \qquad \overset{\Pi}{\Gamma\models\neg P(x,y)\vee\top}}{\Gamma\models(\forall x_1)\exists y_1(Q({\color{red}x_1})\vee P({\color{red}x_1},y_1))}\; x\mapsto a, y\mapsto f(a)$$

## Ex 101b – other resolution order

$$\dfrac{\dfrac{\overset{\Sigma}{P(u,f(u))\vee Q(u)} \qquad \overset{\Pi}{\neg P(x,y)}}{Q(u))}\; y\mapsto f(u), x\mapsto u \qquad \overset{\Pi}{\neg Q(a)}}{\Box}\; u\mapsto a$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(u,f(u))}\; x\mapsto f(u), x\mapsto u \qquad \top}{P(a,f(a))\vee Q(a)}\; u\mapsto a \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(u,x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}\; u\mapsto a$$

## Ex 101c – $\Pi$ and $\Sigma$ swapped

$$\dfrac{\dfrac{\overset{\Pi}{P(u,f(u))\vee Q(u)} \qquad \overset{\Sigma}{\neg P(x,y)}}{Q(u))}\; y\mapsto f(u), x\mapsto u \qquad \overset{\Sigma}{\neg Q(a)}}{\Box}\; u\mapsto a$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg P(u,f(u))}\; x\mapsto f(u), x\mapsto u \qquad \bot}{\neg P(a,f(a))\wedge\neg Q(a)}\; u\mapsto a \qquad\qquad \dfrac{\dfrac{\top \qquad \bot}{\forall x_2\neg P(u,x_2)} \qquad \bot}{\exists x_1\forall x_2(\neg P(x_1,x_2)\wedge\neg Q(x_1))}$$

arrow lemma:

$$\dfrac{\Gamma \models P(u,x_1) \overset{\Pi}{\vee} Q(u) \vee \top \qquad \Gamma \models \neg P(x,y) \overset{\Sigma}{\vee} \bot}{\Big(Q(u) \mid (\neg P(x,x_1)\wedge \top) \vee (P(u,/f(u))\wedge \bot)\Big)\sigma}\; y \mapsto f(u), x \mapsto u$$

$$Q(u) \mid (\neg P(u,x_1)\wedge \top) \vee (P(u,f(u))\wedge \bot)$$

employ $\sigma'$ !?!!?!?!?!??!?????!!!

$$\Gamma \models Q(u) \mid \neg P(u,x_1)$$

$$\dfrac{\Delta \models Q(u) \mid \exists x_1 P(u,x_1) \qquad\qquad \Gamma \models \exists y_1\, \overset{\Sigma}{\neg Q}(y_1)}{\text{both } u\text{'s on LHS need to become } a \text{ and then } y_1}\; u \mapsto a$$

$$\Gamma \models (\forall x_1)\exists y_1(\neg P(y_1,x_1) \vee \neg Q(y_1))$$

$$\Delta \models (\exists x_1)\forall y_1(P(y_1,x_1) \wedge Q(y_1))$$

### Ex 101d – $\Pi$ and $\Sigma$ swapped, other resolution order

$$\dfrac{\dfrac{P(u,f(u)) \overset{\Pi}{\vee} Q(u) \qquad \overset{\Sigma}{\neg Q(a)}}{P(a,f(a))}\; u \mapsto a \qquad \overset{\Sigma}{\neg P(x,y)}}{\square}\; x \mapsto a, y \mapsto f(a)$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg Q(a)}\; y \mapsto a \qquad \bot}{\neg Q(a) \wedge \neg P(a,f(a))}\; x \mapsto a, y \mapsto f(a)$$

$$\dfrac{\dfrac{\top \qquad \bot}{\exists x_1 \neg Q(x_1)} \qquad \bot}{\exists x_1 \forall x_2(\neg P(x_1,x_2) \wedge \neg Q(x_1))}$$

## 102 – similar to 101 but with intra-clause-set inferences

### Ex 102a

$$\dfrac{\dfrac{P(f(\textcolor{red}{x})) \overset{\Sigma}{\vee} Q(f(\textcolor{red}{x}),z) \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x),z)} \qquad \dfrac{\neg Q(x_1,\textcolor{blue}{y}) \overset{\Sigma}{\vee} R(\textcolor{blue}{y}) \qquad \overset{\Pi}{\neg R(g(z_1))}}{\neg Q(x_1,g(z_1))}\; y \mapsto g(z_1), x_1 \mapsto f(x)}{\square}\; x_1 \mapsto f(x), z \mapsto g(z_1)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(g(z_1))}\; y \mapsto g(z_1), x_1 \mapsto f(x)}{P(f(x)) \vee R(g(z_1))}\; x_1 \mapsto f(x), z \mapsto g(z_1)$$

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}}{\exists x_1 \forall x_2(P(x_1) \vee R(x_2)) \quad \text{(order irrelevant!)}}$$

### Ex 102b

$$\dfrac{\dfrac{P(f(\textcolor{red}{x})) \overset{\Sigma}{\vee} Q(f(\textcolor{red}{x}),z) \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x),z)} \qquad \dfrac{\neg Q(f(\textcolor{blue}{y}),z_1) \overset{\Sigma}{\vee} R(\textcolor{blue}{y}) \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a),z_1)}\; y \mapsto a}{\square}\; x \mapsto a, z \mapsto z_1$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\, y \mapsto a}{P(f(a)) \vee R(a)}\, x \mapsto a, z \mapsto z_1 \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\, y \mapsto a}{\forall x_2 \exists x_1 (P(x_1) \vee R(x_2))}\, x \mapsto a, z \mapsto z_1$$

direct:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)}\, x_1 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\, x_2 \sim a}{\exists x_1 \forall x_2 (P(x_1) \vee R(x_2))}\, \text{order irrelevant!}$$

### Ex 102b' with $Q$ grey

$$\dfrac{\dfrac{\overset{\Sigma}{P(f({\color{red}x})) \lor Q(f({\color{red}x}),z)} \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x),z)} \qquad \dfrac{\overset{\Sigma}{\neg Q(f({\color{blue}y}),z_1) \lor R({\color{blue}y})} \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a),z_1)}\; y \mapsto a}{\square}\; x \mapsto a, z_1 \mapsto z$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\; y \mapsto a}{(\neg Q(f(a),z) \land P(f(a))) \lor (Q(f(a),z) \land R(a))}\; x \mapsto a, z_1 \mapsto z$$

arrow lemma: (change of specification: $P(g(z_3))$ in clause in $\Sigma$ instead of $P(f(x))$

$$\dfrac{\begin{array}{c}\dfrac{\overset{\Sigma}{\exists x_1(P(x_4) \lor Q(x_1,z))} \qquad \overset{\Pi}{\neg P(y)}}{\exists x_1(P(g(z_3)) \land \bot) \lor (P(x_4) \land \top) \mid Q(x_1,z)} \\ \exists x_1(P(x_4) \mid Q(x_1,z))\end{array} \qquad \begin{array}{c}\dfrac{\overset{\Sigma}{\exists x_2(\neg Q(x_2,z_1) \lor R(y))} \qquad \forall x_3\, \overset{\Pi}{\neg R(x_3)}}{\overset{\forall x_3}{\exists x_2}\,((\neg R(x_3) \land \bot) \lor (R(a) \land \top) \mid \neg Q(x_2,z_1))}\; y \mapsto a \\ \overset{\forall x_3}{\exists x_2}\,((\neg R(x_3) \land \bot) \lor (R(x_3) \land \top) \mid \neg Q(x_2,z_1))\end{array}}{\overset{\forall x_1}{\overset{\forall x_3}{\exists x_2}}\Big((\neg Q(x_2,z_1) \land P(x_4)) \lor (Q(x_1,z) \land R(x_3))\Big)}\; x \mapsto a, z_1 \mapsto z$$

arrow order: $x_3 < x_2$, $x_2$ same-block-as $x_4$: $\forall x_3 \exists x_2 \exists x_4 \forall x_1 \Big((\neg Q(x_2,z_1) \land \neg P(x_4)) \lor (Q(x_1,z) \land R(x_3))\Big)$

## $\longrightarrow$ bad example, plus some errors still in there

Huang:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)} \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\; y \mapsto a}{\forall x_1 \exists x_2(\neg Q(x_2,z) \land P(x_2)) \lor (Q(x_2,z) \land R(x_1))}\; x \mapsto a, z_1 \mapsto z$$

direct:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)}\; x_2 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\; x_1 \sim a}{\forall x_1 \exists x_2 \exists x_3(\neg Q(x_3,z) \land P(x_2)) \lor (Q(x_3,z) \land R(x_1))}\; x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3}{\text{OR: } \exists x_2 \forall x_1 \exists x_3(\neg Q(x_3,z) \land P(x_2)) \lor (Q(x_3,z) \land R(x_1))}}{\text{OR: } \exists x_1 \exists x_3 \forall x_2(\neg Q(x_3,z) \land P(x_2)) \lor (Q(x_3,z) \land R(x_1))}$$

Huang-Version passt weil man für existenzielle quantoren eh gleichen term einsetzt

direct w mixed, slightly different:

$$\dfrac{\dfrac{\dfrac{\bot \mid P(f(x)) \lor Q(x,z) \qquad \top \mid \neg P(y)}{\exists x_2 P(x_2) \mid Q(x,z)}\; x_2 \sim f(x) \qquad \dfrac{\bot \mid \overset{\Sigma}{\neg Q(f(y),z_1) \lor R(y)} \qquad \top \mid \overset{\Pi}{\neg R(a)}}{\forall x_1 R(x_1) \mid \neg Q(f(a),z_1)}\; x_1 \sim a}{\forall x_1 \exists x_3 \exists x_2(\neg Q(x_3,z) \land P(x_2)) \lor (Q(x_3,z) \land R(x_1))}\; x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3}{(\neg Q(f(a),z) \land P(f(f(a)))) \lor (Q(f(a),z) \land R(a))}$$

last dependency not crucial because other arrow is a $\Sigma$-arrow as well, but just changing it to $\Pi$ (and changing $f$ for $g$ should produce a quantifier alternation)

## Example 103: variables in interpolant get unified and order might have to be changed

### 103a: length changes but old version is still valid, despite huang's algorithm doing a sort

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{Q(f({\color{red}x}))\lor P(y)\lor R({\color{red}x})} \qquad \overset{\Pi}{\neg Q(y_1)}}{P(y)\lor R(x)}\; y_1\mapsto f(x) \qquad \overset{\Pi}{\neg P(h(g(a)))}}{R(x)}\; y\mapsto h(g(a)) \qquad \overset{\Pi}{\neg R(g(g(a)))}}{\square}\; x\mapsto g(g(a))$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{Q(f(x))}\; y_1\mapsto f(x) \qquad \top}{Q(f(x))\lor P(h(g(a)))}\; y\mapsto h(g(a)) \qquad \top}{Q(f(g(g(a))))\lor P(h(g(a)))\lor R(g(g(a)))}\; x\mapsto g(g(a))$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_1\forall x_2(Q(x_1)\lor P(x_2))} \qquad \top}{X}$$

$X$:

Huang's algo gives:

$\forall x_2\forall x_3\exists x_1(Q(x_1)\lor P(x_2)\lor R(x_3))$

Direct overbinding gives: $x_3 < x_1$, rest arbitrary, hence:

$\forall x_3\exists x_1\forall x_2(Q(x_1)\lor P(x_2)\lor R(x_3))$ <- this you do not get with huang

$\forall x_2\forall x_3\exists x_1(Q(x_1)\lor P(x_2)\lor R(x_3))$

$\forall x_3\forall x_2\exists x_1(Q(x_1)\lor P(x_2)\lor R(x_3))$

### 103b: length changes "uniformly"

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{Q(f(f({\color{red}x})))\lor P(f({\color{red}x}))\lor R({\color{red}x})} \qquad \overset{\Pi}{\neg Q(y_1)}}{P(f(x))\lor R(x)}\; y_1\mapsto f(f(x)) \qquad \overset{\Pi}{\neg P(y_2)}}{R(x)}\; y_2\mapsto f(x) \qquad \overset{\Pi}{\neg R(g(a))}}{\square}\; x\mapsto g(a)$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{Q(f(f(x)))}\; y_1\mapsto f(f(x)) \qquad \top}{Q(f(f(x)))\lor P(f(x))}\; y_2\mapsto f(x) \qquad \top}{Q(f(f(g(a))))\lor P(f(g(a)))\lor R(g(a))}\; x\mapsto g(a)$$

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_2\exists x_1(Q(x_1)\lor P(x_2))} \qquad \top}{\forall x_3\exists x_2\exists x_1(Q(x_1)\lor P(x_2)\lor R(x_3))}$$

Huang and direct overbinding somewhat coincide as $x_2 < x_1$ in both cases, and $x_3 < x_1$ and $x_3 < x_2$

new algo:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \mid Q(x_1)\lor P(x_2)\lor R(x) \qquad \top\mid\neg Q(y_1)}{Q(x_1)\mid P(x_2)\lor R(x)}\; y_1\mapsto f(f(x)) \qquad \top\mid\neg P(y_2)}{Q(x_1)\lor P(x_2)\mid R(x)}\; y_2\mapsto f(x) \qquad \top\mid R(x_3)}{Q(x_1)\lor P(x_2)\lor R(x_3)}\; x\mapsto g(a)}{Q(f(f(g(a))))\lor P(f(g(a)))\lor R(g(a))}$$

NB: in the last line, the terms corresponding to $x_1$ and $x_2$ change, but the interpolant stays the same

### 103c: Failed attempt: different variables, accidentally the same terms appear but no logical connection

$$\dfrac{\overset{\Sigma}{P(a,x)} \qquad \dfrac{\overset{\Sigma}{\neg Q(a)} \qquad \overset{\Pi}{\neg P(y,f(z)) \lor Q(z)}}{\neg P(y,f(a))}\; z\mapsto a}{\square}\; y\mapsto a, x\mapsto f(a)$$

Huang:

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg Q(a)}\; z\mapsto a}{P(a,f(a))\land\neg Q(a)}\; y\mapsto a, x\mapsto f(a)$$

$$\dfrac{\bot \qquad \dfrac{\bot\qquad\top}{\exists x_1\neg Q(x_1)}}{\exists x_1\forall x_2(P(x_1,x_2)\land\neg Q(x_1))}$$

order required for $\Pi$

direct:

$$\dfrac{\dfrac{\bot \qquad \dfrac{\bot\qquad\top}{\exists x_1\neg Q(x_1)}\; x_1\sim a}{\exists x_1\exists x_2\forall x_3(P(x_2,x_3)\land\neg Q(x_1))}\; x_2\sim a, x_3\sim f(a); x_1<x_3}{\text{OR: }\exists x_1\forall x_3\exists x_2(P(x_2,x_3)\land \neg Q(x_1))}$$

invariant:

$$\dfrac{\dfrac{\exists x_2(P(x_2,x)\lor\bot) \qquad \dfrac{\exists x_1(Q(x_1)\lor\bot)\qquad \forall x_3((\neg P(y,x_3)\lor Q(z))\lor\top)}{\exists x_1\forall x_3\neg P(y,x_3)\lor\neg Q(x_1)}\; x_1\sim a}{\exists x_1\exists x_2\forall x_3(P(x_2,x_3)\land\neg Q(x_1))}\; x_2\sim a, x_3\sim f(a); x_1<x_3}{\text{OR: }\exists x_1\forall x_3\exists x_2(P(x_2,x_3)\land \neg Q(x_1))}$$

invariant in other resolution order

$$\dfrac{\dfrac{\bot\qquad \dfrac{\bot\qquad\top}{Q(z)\lor\exists x_2\forall x_3 P(x_2,x_3)}\; x_2\sim a, x_3\sim f(z)}{\exists x_1\exists x_2\forall x_3(P(x_2,x_3)\land\neg Q(x_1))}\; x_1\sim a; x_1<x_3}{\text{OR: }\exists x_1\forall x_3\exists x_2(P(x_2,x_3)\land \neg Q(x_1))}$$

invariant if $\Sigma$ and $\Pi$ swapped:

$$\dfrac{\dfrac{\bot\qquad\dfrac{\top\qquad\bot}{\neg P(y,f(x_1))\lor\forall x_1 Q(x_1)}\; x_1\sim a}{\forall x_1\forall x_2\exists x_3(\neg P(x_2,x_3)\lor Q(x_1))}\; x_2\sim a, x_3\sim f(a); x_1<x_3}{\text{OR: }\forall x_1\exists x_3\forall x_2(\neg P(x_2,x_3)\lor Q(x_1))}$$

SECOND ATTEMPT:

$$\dfrac{\overset{\Sigma}{P(a)}\qquad \dfrac{\overset{\Sigma}{Q(z)}\qquad \dfrac{\overset{\Sigma}{\neg S(a)}\qquad\overset{\Pi}{\neg P(y)\lor\neg Q(f(x))\lor S(x)}}{\neg P(y)\lor\neg Q(f(a))}\; x\mapsto a}{\neg P(y)}\; z\mapsto f(a)}{\square}\; y\mapsto a$$

$$\cfrac{\bot \qquad \cfrac{\bot \qquad \cfrac{\bot \quad \top}{\neg S(a)}\, x \mapsto a}{\neg S(a) \wedge Q(f(a))}\, z \mapsto f(a)}{P(a) \wedge \neg S(a) \wedge Q(f(a))}\, y \mapsto a$$

Huang:

$$\cfrac{\bot \qquad \cfrac{\bot \qquad \cfrac{\bot \quad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 (\neg S(x_1) \wedge Q(x_2))}}{\exists x_1 \forall x_2 (P(x_1) \wedge \neg S(x_1) \wedge Q(x_2))}$$

$\forall x_1 \exists x_2 (\neg P(x_1) \vee S(x_1) \vee \neg Q(x_2))$

**similar fail**

$\Rightarrow$ anytime there is $P(a, f(a))$, either they have a dependency or they are not both differently colored (grey is uncolored)
for the record, direct method anyway:

$$\cfrac{\bot \qquad \cfrac{\bot \qquad \cfrac{\bot \quad \top}{\exists x_1 \neg S(x_1)}\, x \sim a}{\exists x_1 \forall x_2 \neg S(x_1) \wedge Q(x_2)}\, z \sim f(a); x_1 < x_2}{\exists x_1 \forall x_2 \exists x_3 P(x_3) \wedge \neg S(x_1) \wedge Q(x_2)}\, x_3 \sim a; x_3 \text{ need not be merged w } x_1$$

## Example: ordering on both ancestors where the merge forces a new ordering

### 202a – canonical

$$\dfrac{\dfrac{\overset{\Sigma}{P(a, x_{564}) \vee R(y)} \qquad \overset{\Pi}{\neg P(x, fx)}}{R(y)} \begin{array}{l} x_1 \mapsto fa \\ x \mapsto a \end{array} \qquad \dfrac{\overset{\Sigma}{Q(x_2, gx_2) \vee \neg R(u)} \qquad \dfrac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg Q(fz, x_3) \vee S(z)}}{\neg Q(fa, x_3)}\, z \mapsto a}{\neg R(u)} \begin{array}{l} x_2 \mapsto fa, \\ x_3 \mapsto gfa \end{array}}{\square}$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a, f(a))} \begin{array}{l} x_1 \mapsto f(a) \\ x \mapsto a \end{array} \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg S(a)}\, z \mapsto a}{Q(f(a), g(f(a))) \wedge \neg S(a)} \begin{array}{l} x_2 \mapsto f(a), \\ x_3 \mapsto g(f(a)) \end{array}}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a))}$$

Huang

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \forall x_2 P(x_1, x_2))} \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 Q(x_2, x_3) \wedge \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 P(x_1, x_2) \vee (Q(x_2, x_3)) \wedge \neg S(x_1))}$$

direct:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \forall x_2 P(x_1, x_2))} \begin{array}{c} x_1 \sim a, x_2 \sim fa \\ x_1 < x_2 \end{array} \qquad \begin{array}{c} x_3 \sim a, x_4 \sim fa, x_5 \sim gfa) \\ x_3 < x_4, x_4 < x_5 \end{array} \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_3 \neg S(x_3)}\, x_3 \sim a}{\exists x_3 \forall x_4 \exists x_5 Q(x_4, x_5) \wedge \neg S(x_3)}}{\exists x_1 \forall x_2 \exists x_5 P(x_1, x_2) \vee (Q(x_2, x_5) \wedge \neg S(x_5))} \begin{array}{l} x_3 \mapsto x_1, x_4 \mapsto x_2 \\ x_1 < x_2, x_2 < x_5 \end{array}$$

without merge in end: $(x_1 < x_2, x_3 < x_4, x_4 < x_5)$

$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \Big( P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3)) \Big)$

$\exists x_3 \forall x_4 \exists x_5 \exists x_1 \forall x_2 \Big( P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3)) \Big)$

(also interwoven ones appear to work)

combined presentation:

$$
\dfrac{\dfrac{\bot \mid P(a, x_1) \vee R(y) \qquad \top \mid \neg P(x, f(x))}{P(a, f(a)) \mid R(y)}\;\begin{matrix} x_1 \mapsto f(a) \\ x \mapsto a \end{matrix} \qquad \dfrac{\bot \mid Q(x_2, g(x_2)) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(z), x_3) \vee S(z)}{\neg S(a) \mid \neg Q(f(a), x_3)}\; z \mapsto a}{Q(f(a), g(f(a))) \wedge \neg S(a) \mid \neg R(u)}\;\begin{matrix} x_2 \mapsto f(a), \\ x_3 \mapsto g(f(a)) \end{matrix}}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a)) \mid \Box}
$$

combined presentation ground:

$$
\dfrac{\dfrac{\bot \mid P(a, f(a)) \vee R(y) \qquad \top \mid \neg P(a, f(a))}{(P(a, f(a)) \wedge \top) \vee (\neg P(a, f(a)) \wedge \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a), g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a), g(f(a)))}}{Q(f(a), g(f(a))) \wedge \neg S(a) \mid \neg R(u)}}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a)) \mid \Box}
$$

combined presentation ground with direct method but only $\Delta$-terms removed :

$$
\dfrac{\dfrac{\bot \mid P(a, f(a)) \vee R(y) \qquad \top \mid \neg P(a, f(a))}{(P(a, x_2) \wedge \top) \vee (\neg P(a, x_2) \wedge \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a), g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a), g(f(a)))}}{Q(x_4, g(x_4)) \wedge \neg S(a) \mid \neg R(u)}}{P(a, x_2) \vee (Q(x_4, g(x_4)) \wedge \neg S(a)) \mid \Box}
$$

combined presentation ground with direct method:

$$
\dfrac{\dfrac{\bot \mid P(a, f(a)) \vee R(y) \qquad \top \mid \neg P(a, f(a))}{\exists x_1 \forall x_2 (P(x_1, x_2) \wedge \top) \vee (\neg P(x_1, x_2) \wedge \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a), g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\exists x_3 \neg S(x_3) \mid \neg Q(f(a), g(f(a)))}}{\exists x_3 \forall x_4 \exists x_5 (Q(x_4, x_5)) \wedge \neg S(x_3)) \mid \neg R(u)}}{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (P(x_1, x_2) \vee (Q(x_4, x_5)) \wedge \neg S(x_3)) \mid \Box}
$$

## 203a – some alternations

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x) \vee \overset{\Sigma}{\neg P(f(x))} \qquad P(z) \vee \overset{\Pi}{Q(g(z))}}{R(x) \vee Q(g(f(x)))}\, z \mapsto f(x) \qquad \neg Q(y) \vee \overset{\Sigma}{S(h(y))}}{R(x) \vee S(h(g(f(x))))}\, y \mapsto g(f(x))}{S(h(g(f(a))))}\, x \mapsto a}{\square}\, x_1 \mapsto h(g(f(a)))$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\neg P(f(x))}\, z \mapsto f(x) \qquad \bot}{\neg Q(g(f(x))) \wedge \neg P(f(x))}\, y \mapsto g(f(x))}{\neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a)}\, x \mapsto a}{\neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a) \vee S(h(g(f(a))))}\, x_1 \mapsto h(g(f(a)))$$

Huang:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \neg P(x_1)} \qquad \bot}{\exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1))}}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0))}}{\forall x_0 \exists x_1 \forall x_2 \exists x_3 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0) \vee S(x_3))}$$

Direct:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \neg P(x_1)}\, x_1 \sim f(x) \qquad \bot}{\exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(f(x)))}\, x_2 \sim g(f(x)); x_1 < x_2}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0))}\, x_0 \sim a; x_0 < x_1, x_0 < x_2}{\forall x_0 \exists x_1 \forall x_2 \exists x_3 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0) \vee S(x_3))}\, x_3 \sim h(g(f(a))); x_0 < x_3, x_1 < x_3, x_2 < x_3$$

## 203b – many $\Sigma$-literals, coloring per occurrence

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x) \vee \overset{\Sigma}{\neg P(f(x))} \qquad P(z) \vee \overset{\Sigma}{Q(g(z))}}{R(x) \vee Q(gfx)}\, z \mapsto fx \qquad \neg Q(y) \vee \overset{\Sigma}{S(h(y))}}{R(x) \vee S(hgfx)}\, y \mapsto gfx}{S(hgfa)}\, x \mapsto a}{\square}\, x_1 \mapsto hgfa$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \bot}{\bot}\, z \mapsto fx \qquad \bot}{\bot}\, y \mapsto gfx}{R(a)}\, x \mapsto a}{S(hgfa) \vee R(a)}\, x_1 \mapsto hgfa$$

$\to \forall x_1 \exists x_2 (R(x_1) \vee S(x_2))$

### 203b' – many $\Sigma$-literals, coloring per symbol, all predicates grey

$$\dfrac{\neg \overset{\Pi}{S}(x_1) \qquad \dfrac{\dfrac{\dfrac{\neg \overset{\Pi}{R}(a) \qquad R(x) \vee \overset{\Sigma}{\neg} P(f(x))}{R(a) \mid \neg P(fa)}\; x \mapsto a \qquad P(z) \vee \overset{\Sigma}{Q}(g(z))}{P(fa) \vee R(a) \mid Q(gfa)}\; z \mapsto fa \qquad \neg Q(y) \vee \overset{\Sigma}{S}(h(y))}{}}{}$$

TODO

## Example where variables are not the outermost symbol but order is still relevant

### 204a

$\Sigma = \{P(f(x), g(f(x)))\}$
$\Pi = \{P(f(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f(x_1), x_2)$

### 204b

$\Sigma = \{P(f^5(x), g(f(x)))\}$
$\Pi = \{P(f^5(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f^5(x_1), x_2)$

## example with aufschaukelnde unification, such that direction of arrow isn't clear

### 205a

$$
\dfrac{\overset{\Pi}{P(ffy, gy)} \qquad \dfrac{\overset{\Sigma}{\neg P(x, y) \vee Q(x)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffz) \vee Rz}}{\neg R(a) \mid \neg Q(ffa)}\; z \mapsto a}{\neg R(a) \wedge Q(ffa) \mid \neg P(ffa, y)}\; x \mapsto ffa}{(\neg R(a) \wedge Q(ffa)) \vee \neg P(ffa, ga)}\; y \mapsto a
$$

direct

$$
\dfrac{\overset{\Pi}{P(ffy, gy)} \qquad \dfrac{\overset{\Sigma}{\neg P(x, y) \vee Q(x)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffz) \vee Rz}}{\exists x_1 \neg R(x_1) \mid \neg Q(ffa)}\; z \mapsto a}{\exists x_1 \forall x_2 (\neg R(x_1) \wedge Q(x_2)) \mid \neg P(ffa, u)}\; x \mapsto ffa}{\exists x_1 \forall x_2 \forall x_3 ((\neg R(x_1) \wedge Q(x_2)) \vee \neg P(x_2, x_3))}\; y \mapsto a, u \mapsto ga
$$

ground:

$$
\dfrac{\overset{\Pi}{P(ffa, ga)} \qquad \dfrac{\overset{\Sigma}{\neg P(ffa, y) \vee Q(ffa)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffa) \vee Ra}}{\neg R(a) \mid \neg Q(ffa)}}{\neg R(a) \wedge Q(ff\,a) \mid \neg P(ffa, a)}}{(\neg R(a) \wedge Q(ffa)) \vee \neg P(ff\,a, ga)}
$$

arrow lemma: $x_1 \sim ffy$, $x_2 \sim gy$, $y_3 \sim a$ $x_4 \sim ffz$, $x_5 \sim ffa$, $y_6 \sim a$,

$$
\dfrac{\overset{\Sigma}{\neg P(x, y) \vee Q(x) \mid \bot} \qquad \dfrac{\overset{\Sigma}{\exists y_3 (\neg R(y_3) \mid \bot)} \qquad \overset{\Pi}{\forall x_4 (\neg Q(x_4) \vee Rz \mid \bot)}}{\exists y_3 \forall x_4 ((\neg R(y_3) \wedge \top) \vee (R(a) \wedge \bot) \mid \neg Q(x_4))}\; z \mapsto a}{\exists y_3 \forall x_4 ((\neg Q(x_4 \wedge \bot)) \vee (Q(x)\sigma \wedge \neg R(y_3)) \mid \neg P(x, y)\sigma)}\; x \mapsto ffa
$$

$$
\exists y_3 \forall x_4 ((Q(ffa) \wedge \neg R(y_3)) \mid \neg P(ffa, y)) \quad (*)
$$

$$
\dfrac{\overset{\Pi}{\forall x_1 \forall x_4 (P(x_1, x_2) \mid \top)} \qquad \exists y_3 \forall x_4 \forall x_5 ((Q(x_5) \wedge \neg R(y_3)) \mid \neg P(x_5, y))}{(\neg P(x_5, y) \wedge \top) \vee (P(x_1, x_2) \wedge (Q(x_5) \wedge \neg R(y_3)))}\; y \mapsto a
$$

$$
(\neg P(x_5, a) \wedge \top) \vee (P(x_1, x_2) \wedge (Q(x_5) \wedge \neg R(y_3)))
$$

$$
\forall x_1 \forall x_2 \exists y_3 \forall x_4 \forall x_5 \forall x_6 (\neg P(x_5, y_6) \wedge \top) \vee (P(x_1, x_2) \wedge (Q(x_5) \wedge \neg R(y_3)))
$$

$(*)$ "luckily", same overbinding for $ffa$, so this works
dashed underline: problem, but does not cause issues here.

situations not critical here :'-(

### 205b $\sim$ 205a, but simpler

Suppose $P$ occurs somewhere in $\Sigma$ (result not that optimal in this setting, but correct)
not nice for proving, $\neg R(a)$ is a nice interpolant already

$$
\dfrac{\overset{\Pi}{P(ffy, gy)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg P(ffz, x) \vee Rz}}{\neg R(a) \mid \neg P(ffa, x)}\; z \mapsto a}{\neg R(a) \vee \neg P(ffa, ga) \mid \Box}\; x \mapsto ga, y \mapsto a
$$

$$
\dfrac{\overset{\Pi}{\top \mid P(ffy, gy)} \qquad \dfrac{\overset{\Sigma}{\bot \mid \neg R(a)} \qquad \overset{\Pi}{\top \mid \neg P(ffz, x) \vee Rz}}{\exists x_1 \neg R(x_1) \mid \neg P(ffa, x)}\; z \mapsto a}{\exists x_1 \forall x_2 \forall x_3 \neg R(x_1) \vee \neg P(x_2, x_3) \mid \Box}\; x \mapsto ga, y \mapsto a
$$

$\exists x_1 R(x_1)$

$\exists x_1 \forall x_2 \forall x_3 (R(x_1) \vee \neg P(x_2, x_3))$

## example to demonstrate that literals being resolved upon have to be overbound with the same variable

### 206a

$$\dfrac{\dfrac{\overset{\Pi}{R(f(x))} \qquad \overset{\Sigma}{\neg R(y) \vee P(y)}}{(\neg R(x_3) \wedge \top) \vee (R(x_3) \wedge \bot) \mid P(x_3)} \qquad \dfrac{\overset{\Pi}{\neg P(f(z)) \vee S(z)} \qquad \overset{\Sigma}{\neg S(a)}}{(\neg S(y_2) \wedge \top) \vee (S(y_2) \wedge \bot) \mid \neg P(x_4)}}{(\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)}$$

Gist of this example: $P(f(x))$ is lifted on the left, but $P(f(a))$ on the right. So it's $P(x_3)$ vs $P(x_4)$, but both of them have to have the same variable.
$R(x_3) \in \mathrm{AI}_{\mathrm{mat}}(C_7)$
$P(x_3) \in \mathrm{AI}_{\mathrm{cl}}(C_7)$
$P(x_4) \in \mathrm{AI}_{\mathrm{cl}}(C_8)$
$\Sigma \models (\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)$
$\Sigma \models (\forall x_3)\forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee P(x_3)\Big)$
$\Sigma \not\models (\neg P(1) \wedge \neg R(0)) \vee P(0)$ // if $P \sim \{1\}$ and $R \sim \{0\}$
we know that for original clauses $l$ and $l'$ of $P(x_4)$ and $P(x_3)$, $l\sigma = l'\sigma$
hence same color, and can use different var as same value works.

inductive hypothesis:
$\Gamma \models \top \vee R(x_3)$
$\Gamma \models \bot \vee \neg R(y) \vee P(y)$
$\Gamma \models (\neg R(x_3) \wedge \top) \vee (R(x_3) \wedge \bot) \vee P(x_3) \quad \equiv \quad \neg R(x_3) \vee P(x_3)$

---

$\Gamma \models \top \vee \neg P(x_4) \vee S(z)$
$\Gamma \models \bot \vee \neg S(a)$
$\Gamma \models (\neg S(a) \wedge \top) \vee (S(a) \wedge \bot) \vee \neg P(x_4) \quad \equiv \quad \neg S(a) \vee \neg P(x_4)$

---

$\Gamma \models (\neg P(x_3) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(a))$

### 206b

WRONG: if a variable $x_3$ occurs, it always refers to $f(x)$, so it is always substituted to a particular value and cannot become $f(a)$ and $f(b)$ in the same clause as just the unifier $\sigma$ is used.

$$\dfrac{\dfrac{\overset{\Pi}{R(f(x), f(x))} \qquad \overset{\Sigma}{\neg R(y,u) \vee P(y,u)}}{(\neg R(x_3,x_3) \wedge \top) \vee (R(x_3,x_3) \wedge \bot) \mid P(x_3,x_3)} \qquad \dfrac{\overset{\Pi}{\neg P(f(z), f(v)) \vee S(z,v)} \qquad \overset{\Sigma}{\neg S(a,b)}}{(\neg S(y_2,y_6) \wedge \top) \vee (S(y_2,y_6) \wedge \bot) \mid \neg P(x_4,x_7)}}{(\forall x_3)\exists y_2 \forall x_4 (\forall x_3)\Big((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2))\Big)}$$

**problems due to $x_j$ not referring to actual term**

**208a**

# WRONG: variable $x$ is used in two clauses

$$\dfrac{\dfrac{\dfrac{P(f(x)) \lor \overset{\Pi}{Q}(x) \lor R(u) \qquad \neg \overset{\Sigma}{Q}(a)}{P(f(a)) \lor R(u)} \qquad \dfrac{P(f(x)) \lor \overset{\Pi}{Q}(x) \lor \neg R(u) \qquad \neg \overset{\Sigma}{Q}(b)}{P(f(b)) \lor \neg R(u)}}{P(f(a)) \lor P(f(b))} \qquad \dfrac{\neg \overset{\Pi}{P(f(x))} \lor S(x) \qquad \neg \overset{\Sigma}{S}(a)}{\neg P(f(a))}}{P(f(b))}$$

$$\dfrac{\dfrac{\dfrac{\top \mid P(x_1) \lor Q(x) \lor R(u) \qquad \bot \mid \neg Q(y_2)}{\neg Q(y_2) \mid P(x_1) \lor R(u)} \qquad \dfrac{\top \mid P(x_1) \lor Q(x) \lor R(u) \qquad \bot \mid \neg Q(y_4)}{\neg Q(y_4) \mid P(x_1) \lor R(u)}}{\neg Q(y_2) \land \neg Q(y_4) \mid P(x_1) \lor P(x_1)} \qquad \dfrac{\top \mid \neg \overset{\Pi}{P(x_1)} \lor S(x) \qquad \top \mid \neg \overset{\Sigma}{S}(y_2)}{\neg S(y_2) \mid \neg P(x_1)}}{(\neg P(x_5) \land \neg Q(y_2) \land \neg Q(y_4)) \quad \lor \quad (P(x_5) \land \neg S(y_2)) \quad \mid \quad P(x_5)}$$

NB: as the $x_1$ in the literal is actually $f(a)$, this way, all $x_1$ become $x_5$, but the other one is supposed to stand for $f(b)$

# ACTUALLY:

$$\dfrac{\dfrac{\dfrac{P(f(x)) \lor \overset{\Pi}{Q}(x) \lor R(u) \qquad \neg \overset{\Sigma}{Q}(a)}{P(f(a)) \lor R(u)} \qquad \dfrac{P(f(x')) \lor \overset{\Pi}{Q}(x') \lor \neg R(u') \qquad \neg \overset{\Sigma}{Q}(b)}{P(f(b)) \lor \neg R(u')}}{P(f(a)) \lor P(f(b))} \qquad \dfrac{\neg \overset{\Pi}{P(f(x''))} \lor S(x'') \qquad \neg \overset{\Sigma}{S}(a)}{\neg P(f(a))}}{P(f(b))}$$

$$\dfrac{\dfrac{\dfrac{\top \mid P(x_1) \lor Q(x) \lor R(u) \qquad \bot \mid \neg Q(y_2)}{\neg Q(y_2) \mid P(x_1) \lor R(u)} \qquad \dfrac{\top \mid P(x_2) \lor Q(x) \lor R(u) \qquad \bot \mid \neg Q(y_4)}{\neg Q(y_4) \mid P(x_2) \lor R(u)}}{\neg Q(y_2) \land \neg Q(y_4) \mid P(x_1) \lor P(x_2)} \qquad \dfrac{\top \mid \neg \overset{\Pi}{P(x_3)} \lor S(x) \qquad \top \mid \neg \overset{\Sigma}{S}(y_2)}{\neg S(y_2) \mid \neg P(x_3)}}{(\neg P(x_5) \land \neg Q(y_2) \land \neg Q(y_4)) \quad \lor \quad (P(x_5) \land \neg S(y_2)) \quad \mid \quad P(x_2)}$$

NB: $\mathrm{au}(P(x_1), P(x_3)) = \{x_1 \mapsto x_5, x_3 \mapsto x_5\}$

Hence a term with a free variable in a clause can never be lifted by the same variable as a term in another clause.
If two terms in the same clause are lifted with a certain variable, they are bound together in the derivation anyway.

## clause used multiple times

**209a**

$$\dfrac{\dfrac{\dfrac{\overset{\Sigma}{P(a)} \qquad \dfrac{\overset{\Sigma}{\neg Q(a)} \qquad \overset{\Pi}{\neg P(x) \vee P(f(x)) \vee Q(y)}}{\neg P(x) \vee P(f(x))}}{P(f(a))}}{P(f(f(a)))} \qquad \overset{\Pi}{\neg P(f(f(z)))}}{\Box}$$

NB: we need to rename lifting variables, possibly rename all lifting variables which refer to a term which contains variables (an actual implementation might do this more efficiently, i.e. not always)

$$\dfrac{\dfrac{\dfrac{\bot \mid P(a) \qquad \dfrac{\bot \mid Q(a) \qquad \top \mid \neg P(x) \vee P(x_1) \vee Q(y)}{Q(a) \mid \neg P(x) \vee P(x_1)}}{P(a) \wedge Q(a) \mid P(x_1)} \qquad Q(a) \mid \neg P(x') \vee P(x_1')}{(\neg P(x_1) \wedge P(a) \wedge Q(a)) \;\vee\; (P(x_1) \wedge Q(a)) \;\mid\; P(x_1')} \qquad \top \mid \neg P(x_2)}{\Big(\neg P(x_3) \wedge (\neg P(x_1) \wedge P(a) \wedge Q(a)) \;\vee\; (P(x_1) \wedge Q(a))\Big) \;\vee\; P(x_3) \mid \Box}$$

NB: $x_1$ used to refer to $f(x)$, now: $f(a)$

$$\mathrm{au}(x_1', x_2) = \{x_1' \mapsto \ell[f(f(a))], x_2 \mapsto \ell[f(f(a))]\}$$

$\Big(\neg P(x_3) \wedge (\neg P(x_1) \wedge P(a) \wedge Q(a)) \;\vee\; (P(x_1) \wedge Q(a))\Big) \;\vee\; P(x_3)$

$\equiv \Big(Q(a) \wedge \Big((\neg P(x_1) \wedge P(a)) \;\vee\; P(x_1)\Big)\Big) \;\vee\; P(x_3)$

$\Sigma$✓

negated: $\Big(\neg Q(a) \vee \Big((P(x_1) \vee \neg P(a)) \;\wedge\; \neg P(x_1)\Big)\Big) \;\wedge\; \neg P(x_3)$

$\equiv \Big(\neg Q(a) \vee \Big(\neg P(a) \;\wedge\; \neg P(x_1)\Big)\Big) \;\wedge\; \neg P(x_3)$

$\Pi$✓

(none of the $P(f^n(x))$, $n \leq 2$, are allowed to be true in a model of $\Phi$)

## $f(x) \vee g(x)$ with $f$, $g$ different colors

### 207a

$$\dfrac{\overset{\Pi}{\neg P(z)} \qquad \dfrac{\overset{\Pi}{\neg R(g(a))} \qquad \dfrac{\overset{\Sigma}{P(f(x)) \vee Q(x)} \qquad \overset{\Pi}{R(g(y)) \vee \neg Q(y)}}{\forall x_1 Q(x_1) \mid P(f(x)) \vee R(g(x))}\ y \mapsto x}{\forall x_1 Q(x_1) \mid P(f(a)) // R(g(a))}\ x \mapsto a}{\forall x_1 \exists x_2 \big(Q(x_1) \vee P(x_2)\big) // R(g(a))}\ z \mapsto f(a)$$

$\Rightarrow$ free vars in the interpolant have to be overbound (if there are arrows, but we can just always do so)

## misc examples

### 201a

$$\dfrac{\dfrac{P(x,y) \overset{\Sigma}{\vee} \neg Q(y) \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(y)}\; x \mapsto a \qquad \dfrac{Q(f(z)) \overset{\Sigma}{\vee} R(z) \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a))}\; z \mapsto a}{\Box}\; y \mapsto f(a)$$

$$\dfrac{\dfrac{\perp \qquad \top}{P(a,y)}\; x \mapsto a \qquad \dfrac{\perp \qquad \top}{R(a)}\; z \mapsto a}{P(a,f(a)) \vee R(a)}\; y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\perp \qquad \top}{\forall x_1 P(x_1,y)}\; x \mapsto a \qquad \dfrac{\perp \qquad \top}{\forall x_3 R(x_3)}\; z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 (P(x_1,x_2) \vee R(x_3))}\; y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

### 201b

$$\dfrac{\dfrac{P(x,f(y)) \overset{\Sigma}{\vee} \neg Q(f(y)) \qquad \overset{\Pi}{\neg P(a,y_2)}}{\neg Q(f(y))}\; x \mapsto a \qquad \dfrac{Q(f(z)) \overset{\Sigma}{\vee} R(z) \qquad \overset{\Pi}{\neg R(a)}}{Q(f(a))}\; z \mapsto a}{\Box}\; y \mapsto f(a)$$

$$\dfrac{\dfrac{\perp \qquad \top}{P(a,f(y))}\; x \mapsto a \qquad \dfrac{\perp \qquad \top}{R(a)}\; z \mapsto a}{P(a,f(a)) \vee R(a)}\; y \mapsto a \qquad\qquad \dfrac{\dfrac{\perp \qquad \top}{\forall x_1 \exists x_2 P(x_1,x_2)}\; x \mapsto a \qquad \dfrac{\perp \qquad \top}{\forall x_3 R(x_3)}\; z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_3)}\; y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

**arrow in element which is not in interpolant or resolution clause**

**206**

$$\dfrac{\dfrac{\overset{\Sigma}{P(x) \lor \neg Q(f(x))} \qquad \overset{\Pi}{\neg P(a)}}{\forall x_1 P(x_1) \mid \neg Q(f(a))}\; x \mapsto a \qquad \dfrac{\overset{\Sigma}{Q(y) \lor R(g(y))} \qquad \overset{\Pi}{\neg R(z)}}{\exists x_2 R(x_2) \mid Q(y)}\; z \mapsto g(y)}{\forall x_1 \exists x_2 (P(x_1) \lor R(x_2)) \mid \Box}\; y \mapsto f(a)$$

$$P(a) \lor R(g(f(a)))$$

for first interpolant, $\Sigma \not\models \ell_{\Delta,x}[\mathrm{PI}(C)] \lor C$

=> need to overbind clause as well