Example 101: canonical examples for overbinding w.r.t. order

Here, the second lower prooftree shows what huang would produce.

Ex 101a

$$\cfrac{\cfrac{\overset{\Sigma}{P({\color{red}u},f({\color{red}u}))\vee Q({\color{red}u})} \qquad \overset{\Pi}{\neg Q(a)}}{P(a,f(a))}\;u\mapsto a \qquad \overset{\Pi}{\neg P(x,y)}}{\square}\;x\mapsto a, y\mapsto f(a)$$

$$\cfrac{\cfrac{\bot \qquad \top}{Q(a)}\;u\mapsto a \qquad \top}{P(a,f(a))\vee Q(a)}\;x\mapsto a, y\mapsto f(a)$$

$$\cfrac{\cfrac{\bot \qquad \top}{\forall x_1 Q(x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}$$

Direct overbinding would not work without merging same variables!: $\forall x_1\forall x_2\exists x_3(P(x_2,x_3)\vee Q(x_1))$
counterexample: $Q\sim\{0\}, P\sim\{(1,0)\}$
Direct overbinding would work when considering original dependencies as highlighted above

Ex 101b – other resolution order

$$\cfrac{\cfrac{\overset{\Sigma}{P(u,f(u))\vee Q(u)} \qquad \overset{\Pi}{\neg P(x,y)}}{Q(u))}\;y\mapsto f(u), x\mapsto u \qquad \overset{\Pi}{\neg Q(a)}}{\square}\;u\mapsto a$$

$$\cfrac{\cfrac{\bot \qquad \top}{P(u,f(u))}\;x\mapsto f(u), x\mapsto u \qquad \top}{P(a,f(a))\vee Q(a)}\;u\mapsto a$$

$$\cfrac{\cfrac{\bot \qquad \top}{\exists x_1 P(u,x_1)} \qquad \top}{\forall x_1\exists x_2(P(x_1,x_2)\vee Q(x_1))}\;u\mapsto a$$

Ex 101c – Π and Σ swapped

$$\cfrac{\cfrac{\overset{\Pi}{P(u,f(u))\vee Q(u)} \qquad \overset{\Sigma}{\neg P(x,y)}}{Q(u))}\;y\mapsto f(u), x\mapsto u \qquad \overset{\Sigma}{\neg Q(a)}}{\square}\;u\mapsto a$$

$$\cfrac{\cfrac{\top \qquad \bot}{\neg P(u,f(u))}\;x\mapsto f(u), x\mapsto u \qquad \bot}{\neg P(a,f(a))\wedge\neg Q(a)}\;u\mapsto a$$

$$\cfrac{\cfrac{\top \qquad \bot}{\forall x_2\neg P(u,x_2)} \qquad \bot}{\exists x_1\forall x_2(\neg P(x_1,x_2)\wedge\neg Q(x_1))}$$

Ex 101d – Π and Σ swapped, other resolution order

$$\cfrac{\cfrac{\overset{\Pi}{P(u,f(u))\vee Q(u)} \qquad \overset{\Sigma}{\neg Q(a)}}{P(a,f(a))}\;u\mapsto a \qquad \overset{\Sigma}{\neg P(x,y)}}{\square}\;x\mapsto a, y\mapsto f(a)$$

$$\dfrac{\dfrac{\top \qquad \bot}{\neg Q(a)}\; y \mapsto a \qquad\qquad \bot}{\neg Q(a) \wedge \neg P(a, f(a))}\; x \mapsto a, y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\top \qquad \bot}{\exists x_1 \neg Q(x_1)} \qquad \bot}{\exists x_1 \forall x_2 (\neg P(x_1, x_2) \wedge \neg Q(x_1))}$$

102 – similar to 101 but with intra-clause-set inferences

Ex 102a

$$\dfrac{\dfrac{P(f({\color{red}x})) \overset{\Sigma}{\vee} Q(f({\color{red}x}), z) \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x), z)} \qquad \dfrac{\neg Q(x_1, {\color{blue}y}) \overset{\Sigma}{\vee} R({\color{blue}y}) \qquad \overset{\Pi}{\neg R(g(z_1))}}{\neg Q(x_1, g(z_1))}\; y \mapsto g(z_1), x_1 \mapsto f(x)}{\square}\; x_1 \mapsto f(x), z \mapsto g(z_1)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(g(z_1))}\; y \mapsto g(z_1), x_1 \mapsto f(x)}{P(f(x)) \vee R(g(z_1))}\; x_1 \mapsto f(x), z \mapsto g(z_1) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}}{\exists x_1 \forall x_2 (P(x_1) \vee R(x_2)) \quad \text{(order irrelevant!)}}$$

Ex 102b

$$\dfrac{\dfrac{P(f({\color{red}x})) \overset{\Sigma}{\vee} Q(f({\color{red}x}), z) \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x), z)} \qquad \dfrac{\neg Q(f({\color{blue}y}), z_1) \overset{\Sigma}{\vee} R({\color{blue}y}) \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a), z_1)}\; y \mapsto a}{\square}\; x \mapsto a, z \mapsto z_1$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\; y \mapsto a}{P(f(a)) \vee R(a)}\; x \mapsto a, z \mapsto z_1 \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)} \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\; y \mapsto a}{\forall x_2 \exists x_1 (P(x_1) \vee R(x_2))}\; x \mapsto a, z \mapsto z_1$$

direct:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_1 P(x_1)}\; x_1 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_2 R(x_2)}\; x_2 \sim a}{\exists x_1 \forall x_2 (P(x_1) \vee R(x_2))}\; \text{order irrelevant!}$$

Ex 102b' with Q grey

$$\dfrac{\dfrac{\overset{\Sigma}{P(f({\color{red}x})) \lor Q(f({\color{red}x}), z)} \qquad \overset{\Pi}{\neg P(y)}}{Q(f(x), z)} \qquad \dfrac{\overset{\Sigma}{\neg Q(f({\color{blue}y}), z_1) \lor R({\color{blue}y})} \qquad \overset{\Pi}{\neg R(a)}}{\neg Q(f(a), z_1)}\ y \mapsto a}{\square}\ x \mapsto a, z_1 \mapsto z$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(f(x))} \qquad \dfrac{\bot \qquad \top}{R(a)}\ y \mapsto a}{(\neg Q(f(a), z) \land P(f(a))) \lor (Q(f(a), z) \land R(a))}\ x \mapsto a, z_1 \mapsto z$$

Huang:

$$\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)} \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\ y \mapsto a}{\forall x_1 \exists x_2 (\neg Q(x_2, z) \land P(x_2)) \lor (Q(x_2, z) \land R(x_1))}\ x \mapsto a, z_1 \mapsto z$$

direct:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \qquad \top}{\exists x_2 P(x_2)}\ x_2 \sim f(x) \qquad \dfrac{\bot \qquad \top}{\forall x_1 R(x_1)}\ x_1 \sim a}{\forall x_1 \exists x_2 \exists x_3 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}\ x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3}{\text{OR: } \exists x_2 \forall x_1 \exists x_3 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}}{\text{OR: } \exists x_1 \exists x_3 \forall x_2 (\neg Q(x_3, z) \land P(x_2)) \lor (Q(x_3, z) \land R(x_1))}$$

Huang-Version passt weil man für existenzielle quantoren eh gleichen term einsetzt

Example 103: variables in interpolant get unified and order might have to be changed

103a: length changes but old version is still valid, despite huang's algorithm doing a sort

$$\cfrac{\cfrac{\cfrac{Q(f(x)) \lor \overset{\Sigma}{P}(y) \lor R(x) \qquad \overset{\Pi}{\neg Q(y_1)}}{P(y) \lor R(x)}\; y_1 \mapsto f(x) \qquad \neg P(\overset{\Pi}{h}(g(a)))}{R(x)}\; y \mapsto h(g(a)) \qquad \neg R(\overset{\Pi}{g}(g(a)))}{\Box}\; x \mapsto g(g(a))$$

$$\cfrac{\cfrac{\cfrac{\bot \qquad \top}{Q(f(x))}\; y_1 \mapsto f(x) \qquad \top}{Q(f(x)) \lor P(h(g(a)))}\; y \mapsto h(g(a)) \qquad \top}{Q(f(g(g(a)))) \lor P(h(g(a))) \lor R(g(g(a)))}\; x \mapsto g(g(a)) \qquad\qquad \cfrac{\cfrac{\cfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_1 \forall x_2 (Q(x_1) \lor P(x_2))} \qquad \top}{X}$$

X:
Huang's algo gives:
$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$
Direct overbinding gives: $x_3 < x_1$, rest arbitrary, hence:
$\forall x_3 \exists x_1 \forall x_2 (Q(x_1) \lor P(x_2) \lor R(x_3))$ <- this you do not get with huang
$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$
$\forall x_3 \forall x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))$

103b: length changes "uniformly"

$$\cfrac{\cfrac{\cfrac{Q(f(f(x))) \lor \overset{\Sigma}{P}(f(x)) \lor R(x) \qquad \overset{\Pi}{\neg Q(y_1)}}{P(f(x)) \lor R(x)}\; y_1 \mapsto f(f(x)) \qquad \overset{\Pi}{\neg P(y_2)}}{R(x)}\; y_2 \mapsto f(x) \qquad \neg R(\overset{\Pi}{g}(a))}{\Box}\; x \mapsto g(a)$$

$$\cfrac{\cfrac{\cfrac{\bot \qquad \top}{Q(f(f(x)))}\; y_1 \mapsto f(f(x)) \qquad \top}{Q(f(f(x))) \lor P(f(x))}\; y_2 \mapsto f(x) \qquad \top}{Q(f(f(g(a)))) \lor P(f(g(a))) \lor R(g(a))}\; x \mapsto g(a) \qquad \cfrac{\cfrac{\cfrac{\bot \qquad \top}{\exists x_1 Q(x_1)} \qquad \top}{\exists x_2 \exists x_1 (Q(x_1) \lor P(x_2))} \qquad \top}{\forall x_3 \exists x_2 \exists x_1 (Q(x_1) \lor P(x_2) \lor R(x_3))}$$

Huang and direct overbinding somewhat coincide as $x_2 < x_1$ in both cases, and $x_3 < x_1$ and $x_3 < x_2$

103c: Failed attempt: different variables, accidentally the same terms appear but no logical connection

$$\cfrac{\overset{\Sigma}{P}(a, x) \qquad \cfrac{\overset{\Sigma}{\neg Q}(a) \qquad \neg P(y, \overset{\Pi}{f}(z)) \lor Q(z)}{\neg P(y, f(a))}\; z \mapsto a}{\Box}\; y \mapsto a, x \mapsto f(a)$$

$$\frac{\bot \qquad \frac{\bot \qquad \top}{\neg Q(a)}\, z \mapsto a}{P(a, f(a)) \wedge \neg Q(a)}\, y \mapsto a, x \mapsto f(a) \qquad\qquad \frac{\bot \qquad \frac{\bot \qquad \top}{\exists x_1 \neg Q(x_1)}}{\exists x_1 \forall x_2 (P(x_1, x_2) \wedge \neg Q(x_1))}$$

order required for Π

direct:

$$\frac{\frac{\bot \qquad \frac{\bot \qquad \top}{\exists x_1 \neg Q(x_1)}\, x_1 \sim a}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}\, x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))}$$

invariant:

$$\frac{\frac{\exists x_2 (P(x_2, x) \vee \bot) \qquad \frac{\exists x_1 (Q(x_1) \vee \bot) \qquad \forall x_3 ((\neg P(y, \textcolor{red}{x_3}) \vee Q(\textcolor{red}{z})) \vee \top)}{\exists x_1 \forall x_3 \neg P(y, \textcolor{red}{x_3}) \vee \neg Q(\textcolor{red}{x_1})}\, x_1 \sim a}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}\, x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))}$$

invariant in other resolution order

$$\frac{\frac{\bot \qquad \frac{\bot \qquad \top}{Q(\textcolor{red}{z}) \vee \exists x_2 \forall x_3 P(x_2, \textcolor{red}{x_3})}\, x_2 \sim a, x_3 \sim f(z)}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}\, x_1 \sim a; x_1 < x_3}{\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))}$$

invariant if Σ and Π swapped:

$$\frac{\frac{\bot \qquad \frac{\top \qquad \bot}{\neg P(y, f(x_1)) \vee \forall x_1 Q(x_1)}\, x_1 \sim a}{\forall x_1 \forall x_2 \exists x_3 (\neg P(x_2, x_3) \vee Q(x_1))}\, x_2 \sim a, x_3 \sim f(a); x_1 < x_3}{\text{OR: } \forall x_1 \exists x_3 \forall x_2 (\neg P(x_2, x_3) \vee Q(x_1))}$$

SECOND ATTEMPT:

$$\frac{\overset{\Sigma}{P(a)} \qquad \frac{\overset{\Sigma}{Q(z)} \qquad \frac{\overset{\Sigma}{\neg S(a)} \qquad \overset{\Pi}{\neg P(y) \vee \neg Q(f(\textcolor{red}{x})) \vee S(\textcolor{red}{x})}}{\neg P(y) \vee \neg Q(f(a))}\, x \mapsto a}{\neg P(y)}\, z \mapsto f(a)}{\square}\, y \mapsto a$$

$$\frac{\bot \qquad \frac{\bot \qquad \frac{\bot \qquad \top}{\neg S(a)}\, x \mapsto a}{\neg S(a) \wedge Q(f(a))}\, z \mapsto f(a)}{P(a) \wedge \neg S(a) \wedge Q(f(a))}\, y \mapsto a$$

Huang:

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 (\neg S(x_1) \wedge Q(x_2))}}{\exists x_1 \forall x_2 (P(x_1) \wedge \neg S(x_1) \wedge Q(x_2))}$$

$\forall x_1 \exists x_2 (\neg P(x_1) \vee S(x_1) \vee \neg Q(x_2))$

similar fail

$\Rightarrow$ anytime there is $P(a, f(a))$, either they have a dependency or they are not both differently colored (grey is uncolored)

for the record, direct method anyway:

$$\dfrac{\bot \qquad \dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}\, x \sim a}{\exists x_1 \forall x_2 \neg S(x_1) \wedge Q(x_2)}\, z \sim f(a); x_1 < x_2}{\exists x_1 \forall x_2 \exists x_3 P(x_3) \wedge \neg S(x_1) \wedge Q(x_2)}\, x_3 \sim a; x_3 \text{ need not be merged w } x_1$$

Example: ordering on both ancestors where the merge forces a new ordering

202a – canonical

$$
\dfrac{
\dfrac{P(a,\overset{\Sigma}{x_1}) \vee R(y) \qquad \neg P(\overset{\Pi}{x}, fx)}{R(y)}\;\begin{array}{c}x_1 \mapsto fa\\ x \mapsto a\end{array}
\qquad
\dfrac{Q(x_2, g\overset{\Sigma}{x_2}) \vee \neg R(u) \qquad \dfrac{\neg \overset{\Sigma}{S}(a) \qquad \neg Q(f\overset{\Pi}{z}, x_3) \vee S(z)}{\neg Q(fa, x_3)}\; z \mapsto a}{\neg R(u)}\;\begin{array}{l}x_2 \mapsto fa,\\ x_3 \mapsto gfa\end{array}
}{\Box}
$$

$$
\dfrac{
\dfrac{\bot \qquad \top}{P(a, f(a))}\;\begin{array}{c}x_1 \mapsto f(a)\\ x \mapsto a\end{array}
\qquad
\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\neg S(a)}\; z \mapsto a}{Q(f(a), g(f(a))) \wedge \neg S(a)}\;\begin{array}{l}x_2 \mapsto f(a),\\ x_3 \mapsto g(f(a))\end{array}
}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a))}
$$

Huang

$$
\dfrac{
\dfrac{\bot \qquad \top}{\exists x_1 \forall x_2 P(x_1, x_2))}
\qquad
\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 Q(x_2, x_3) \wedge \neg S(x_1)}
}{\exists x_1 \forall x_2 \exists x_3 P(x_1, x_2) \vee (Q(x_2, x_3)) \wedge \neg S(x_1))}
$$

direct:

$$
\dfrac{
\dfrac{\bot \qquad \top}{\exists x_1 \forall x_2 P(x_1, x_2))}\;\begin{array}{c}x_1 \sim a, x_2 \sim fa\\ x_1 < x_2\end{array}
\qquad
\begin{array}{c}x_3 \sim a, x_4 \sim fa, x_5 \sim gfa)\\ x_3 < x_4, x_4 < x_5\end{array}
\dfrac{\bot \qquad \dfrac{\bot \qquad \top}{\exists x_3 \neg S(x_3)}\; x_3 \sim a}{\exists x_3 \forall x_4 \exists x_5 Q(x_4, x_5) \wedge \neg S(x_3)}
}{\exists x_1 \forall x_2 \exists x_5 P(x_1, x_2) \vee (Q(x_2, x_5) \wedge \neg S(x_5))}\;\begin{array}{l}x_3 \mapsto x_1, x_4 \mapsto x_2\\ x_1 < x_2, x_2 < x_5\end{array}
$$

without merge in end: $(x_1 < x_2, x_3 < x_4, x_4 < x_5)$
$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))$
$\exists x_3 \forall x_4 \exists x_5 \exists x_1 \forall x_2 P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))$
(also interwoven ones appear to work)

combined presentation:

$$\dfrac{\dfrac{\bot \mid P(a,x_1) \vee R(y) \qquad \top \mid -}{P(a,f(a)) \mid R(y)}\;{\scriptstyle \begin{array}{c} x_1 \mapsto f(a) \\ x \mapsto a \end{array}} \qquad \dfrac{\bot \mid Q(x_2,g(x_2)) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(z),x_3) \vee S(z)}{\neg S(a) \mid \neg Q(f(a),x_3)}\;{\scriptstyle z \mapsto a}}{Q(f(a),g(f(a))) \wedge \neg S(a) \mid \neg R(u)}\;{\scriptstyle \begin{array}{c} x_2 \mapsto f(a), \\ x_3 \mapsto g(f(a)) \end{array}}}{P(a,f(a)) \vee (Q(f(a),g(f(a))) \wedge \neg S(a)) \mid \Box}$$

combined presentation ground:

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \vee R(y) \qquad \top \mid \neg P(x,f(x))}{(P(a,f(a)) \wedge \top) \vee (\neg P(a,f(a)) \wedge \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(f(a),g(f(a))) \wedge \neg S(a) \mid \neg R(u)}}{P(a,f(a)) \vee (Q(f(a),g(f(a))) \wedge \neg S(a)) \mid \Box}$$

combined presentation ground with direct method but only Δ-terms removed :

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \vee R(y) \qquad \top \mid \neg P(x,f(x))}{(P(a,x_2) \wedge \top) \vee (\neg P(a,x_2) \wedge \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a),g(f(a)))}}{Q(x_4,g(x_4)) \wedge \neg S(a) \mid \neg R(u)}}{P(a,x_2) \vee (Q(x_4,g(x_4)) \wedge \neg S(a)) \mid \Box}$$

combined presentation ground with direct method:

$$\dfrac{\dfrac{\bot \mid P(a,f(a)) \vee R(y) \qquad \top \mid \neg P(x,f(x))}{\exists x_1 \forall x_2 (P(x_1,x_2) \wedge \top) \vee (\neg P(x_1,x_2) \wedge \bot) \mid R(y)} \qquad \dfrac{\bot \mid Q(f(a),g(f(a))) \vee \neg R(u) \qquad \dfrac{\bot \mid \neg S(a) \qquad \top \mid \neg Q(f(a),g(f(a))) \vee S(a)}{\exists x_3 \neg S(x_3) \mid \neg Q(f(a),g(f(a)))}}{\exists x_3 \forall x_4 \exists x_5 (Q(x_4,x_5)) \wedge \neg S(x_3)) \mid \neg R(u)}}{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (P(x_1,x_2) \vee (Q(x_4,x_5)) \wedge \neg S(x_3)) \mid \Box}$$

203a

$$\dfrac{\overset{\Pi}{\neg S(x_1)} \qquad \dfrac{\overset{\Pi}{\neg R(a)} \qquad \dfrac{\dfrac{R(x) \lor \overset{\Sigma}{\neg P(f(x))} \qquad P(z) \lor \overset{\Pi}{Q(g(z))}}{R(x) \lor Q(g(f(x)))}\ z \mapsto f(x) \qquad \neg Q(y) \lor \overset{\Sigma}{S(h(y))}}{R(x) \lor S(h(g(f(x))))}\ y \mapsto g(f(x))}{S(h(g(f(a))))}\ x \mapsto a}{\square}\ x_1 \mapsto h(g(f(a)))$$

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\neg P(f(x))}\ z \mapsto f(x) \qquad \bot}{\neg Q(g(f(x))) \land \neg P(f(x))}\ y \mapsto g(f(x))}{\neg Q(g(f(a))) \land \neg P(f(a)) \lor R(a)}\ x \mapsto a}{\neg Q(g(f(a))) \land \neg P(f(a)) \lor R(a) \lor S(h(g(f(a))))}\ x_1 \mapsto h(g(f(a)))$$

Huang:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \neg P(x_1)} \qquad \bot}{\exists x_1 \forall x_2 (\neg Q(x_2) \land \neg P(x_1))}}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \land \neg P(x_1) \lor R(x_0))}}{\forall x_0 \exists x_1 \forall x_2 \exists x_3 (\neg Q(x_2) \land \neg P(x_1) \lor R(x_0) \lor S(x_3))}$$

Direct:

$$\dfrac{\top \qquad \dfrac{\top \qquad \dfrac{\dfrac{\bot \qquad \top}{\exists x_1 \neg P(x_1)}\ x_1 \sim f(x) \qquad \bot}{\exists x_1 \forall x_2 (\neg Q(x_2) \land \neg P(f(x)))}\ x_2 \sim g(f(x)); x_1 < x_2}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \land \neg P(x_1) \lor R(x_0))}\ x_0 \sim a; x_0 < x_1, x_0 < x_2}{\forall x_0 \exists x_1 \forall x_2 \exists x_3 (\neg Q(x_2) \land \neg P(x_1) \lor R(x_0) \lor S(x_3))}\ x_3 \sim h(g(f(a))); x_0 < x_3, x_1 < x_3, x_2 < x_3$$

Example where variables are not the outermost symbol but order is still relevant

204a

$\Sigma = \{P(f(x), g(f(x)))\}$
$\Pi = \{P(f(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f(x_1), x_2)$

204b

$\Sigma = \{P(f^5(x), g(f(x)))\}$
$\Pi = \{P(f^5(a), y)\}$
$\Rightarrow \forall x_1 \exists x_2 P(f^5(x_1), x_2)$

example with aufschaukelnde unification, such that direction of arrow isn't clear

205a

$$\dfrac{\overset{\Pi}{P(ffy, gy)} \qquad \dfrac{\overset{\Sigma}{\neg P(x, y) \vee Q(x)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffz) \vee Rz}}{\neg R(a) \mid \neg Q(ffa)}\, z \mapsto a}{\neg R(a) \wedge Q(ffa) \mid \neg P(ffa, y)}\, x \mapsto ffa}{(\neg R(a) \wedge Q(ffa)) \vee \neg P(ffa, ga)}\, y \mapsto a$$

direct

$$\dfrac{\overset{\Pi}{P(ffy, gy)} \qquad \dfrac{\overset{\Sigma}{\neg P(x, y) \vee Q(x)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg Q(ffz) \vee Rz}}{\exists x_1 \neg R(x_1) \mid \neg Q(ffa)}\, z \mapsto a}{\exists x_1 \forall x_2 (\neg R(x_1) \wedge Q(x_2)) \mid \neg P(ffa, u)}\, x \mapsto ffa}{\exists x_1 \forall x_2 ((\neg R(x_1) \wedge Q(x_2)) \vee \neg P(x_2, ga))}\, y \mapsto a, u \mapsto ga$$

$\exists x_1 \forall x_2 \forall x_3 ((\neg R(x_1) \wedge Q(x_2)) \vee \neg P(x_2, x_3))$

arrow between x_1 and x_3 isn't really there, only weak connection between x_2 and x_3

205b $\sim$ 205a, but simpler

Suppose P occurs somewhere in Σ (result not that optimal in this setting, but correct)

$$\dfrac{\overset{\Pi}{P(ffy, gy)} \qquad \dfrac{\overset{\Sigma}{\neg R(a)} \qquad \overset{\Pi}{\neg P(ffz, x) \vee Rz}}{\neg R(a) \mid \neg P(ffa, x)}\, z \mapsto a}{\neg R(a) \vee \neg P(ffa, ga) \mid \Box}\, x \mapsto ga, y \mapsto a$$

$\exists x_1 R(x_1)$

$\exists x_1 \forall x_2 \forall x_3 (R(x_1) \vee \neg P(x_2, x_3))$

misc examples

201a

$$\dfrac{\dfrac{P(x,y) \overset{\Sigma}{\vee} \neg Q(y) \qquad \neg P(\overset{\Pi}{a}, y_2)}{\neg Q(y)}\; x \mapsto a \qquad \dfrac{Q(f(z\overset{\Sigma}{)}) \vee R(z) \qquad \neg \overset{\Pi}{R}(a)}{Q(f(a))}\; z \mapsto a}{\Box}\; y \mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,y)}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{R(a)}\; z \mapsto a}{P(a,f(a)) \vee R(a)}\; y \mapsto f(a) \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 P(x_1,y)}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{\forall x_3 R(x_3)}\; z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 (P(x_1,x_2) \vee R(x_3))}\; y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$

201b

$$\dfrac{\dfrac{P(x,f(y)) \overset{\Sigma}{\vee} \neg Q(f(y)) \qquad \neg P(\overset{\Pi}{a}, y_2)}{\neg Q(f(y))}\; x \mapsto a \qquad \dfrac{Q(f(z\overset{\Sigma}{)}) \vee R(z) \qquad \neg \overset{\Pi}{R}(a)}{Q(f(a))}\; z \mapsto a}{\Box}\; y \mapsto f(a)$$

$$\dfrac{\dfrac{\bot \qquad \top}{P(a,f(y))}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{R(a)}\; z \mapsto a}{P(a,f(a)) \vee R(a)}\; y \mapsto a \qquad\qquad \dfrac{\dfrac{\bot \qquad \top}{\forall x_1 \exists x_2 P(x_1,x_2)}\; x \mapsto a \qquad \dfrac{\bot \qquad \top}{\forall x_3 R(x_3)}\; z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_3)}\; y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1,x_2) \vee R(x_1)$