Chapter 1

Number of quantifier alternations in Huang and nested

1.1 Outline

Goal: try proof Huang and nested equal.

Method: proof for both:

Conjectured Proposition 1. *Let I be an interpolant created by \$algorithm. If I contains a term t such that t has n color changes, then I has at least n quantifier alternations.*

1.1.1 generally keep in mind

- Need to define all new terms here: color-changing, single-color, Φ-literal, substitutions from 0 to n
  - essentially same position: path from one position to other only contains grey symbol (this def allows for identical position as well)
- also note: literal is sometimes used for negated or not negated predicate with terms but in regular formulas with arbitrary connectives

1.2 Preliminaries

Quantifier alternations in I usually assumes the quantifier-alternation-minimizing arrangement of quantifiers in I

Definition 2 (Color alternation col-alt)**.** Colors Γ and Δ, term t:

$\text{col-alt}(t) \stackrel{\text{def}}{=} \text{col-alt}_{\perp}(t)$

Let $t = f(t_1, \dots, t_n)$ for constant, function and variable symbols (syntax abuse)

$$\text{col-alt}_{\Phi}(t) \stackrel{\text{def}}{=} \begin{cases} \max(\text{col-alt}_{\Phi}(t_1), \dots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is grey} \\ \max(\text{col-alt}_{\Phi}(t_1), \dots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_{\Psi}(t_1), \dots, \text{col-alt}_{\Psi}(t_n)) & f \text{ is of color } \Psi,\ \Phi \neq \Psi \end{cases}$$

$\triangle$

Definition 3. $\mathrm{PI}^{\circ}_{\mathrm{step}}$ is defined just like $\mathrm{PI}_{\mathrm{step}}$ but without applying any substitution. $\triangle$

Hence $\mathrm{PI}^{\circ}_{\mathrm{step}}(\cdot)\sigma = \mathrm{PI}_{\mathrm{step}}(\cdot)$. C° is somehow the same, i.e. if $C = D\sigma$, then $C^{\circ} = D$ where σ is derived from the context.

1.3 Random thoughts

- Quantifiers are introduced for lifting variables which actually occur in the interpolant
- If term t with $\text{col-alt}(t) = n$ enters I, we need subterm s of t with $\text{col-alt}(s) = n - 1$ to be in I (of course colors of t and s are exactly opposite)

1.3.1 Proof

- Induction over $\ell^{x}_{\Delta}[\mathrm{PI}(C) \vee C]$ and also about Γ-terms with Δ-lifting vars in that formula. Cf. `-final`
- NB: now somewhat described in the proper proof below describe proof method with $\sigma_{(0,\,i)}$: which PI?
  - Factorisation: easy: just apply σ_i for all i to $\mathrm{PI}(C) \vee C$. When done, a literal will be there twice and we can remove it without losing anything
  - Resolution: create propositional structure first.
    Ex.: $C_1 : D \vee l$, $C_2 : \neg l \vee E$:
    If we talk about properties for which it holds that if they hold for $\mathrm{PI}(C_i) \vee C_i$, $i \in \{1, 2\}$, then they also hold for $A \equiv \Big((l \wedge \mathrm{PI}(C_2)) \vee (\neg l \wedge \mathrm{PI}(C_1))\Big) \vee C^{\circ}$, then we can apply σ_i for all i to that formula.
    So if we can assume it for A and show it for all σ_i, we get that it holds for $\mathrm{PI}(C) \vee C$.

  Also: clauses are variable disjoint, so e.g. it's not possible that a color-changing var is created by $\mathrm{PI}_{\mathrm{step}}$

  Also: do it like a few lemmas further down, like $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \ldots, \mathrm{PI}(C_n)) \vee C^{\circ})\sigma_{(0,\,i)}$

1.4 Proof port attempt from `-final`

need to show that grey occurrences are in grey literals, all grey literals end up in the interpolant!

conj: if a Δ-term t occurs in a Γ-literal in a clause C, then t occurs in a grey literal in $\mathrm{PI}(C)$.

evidence:

- situation does not occur in Γ or Δ

- terms are only changed by unificators
- Δ- and Γ-terms are not unifiable, so one of the literals has to have a variable at a grey position when a Δ-term enters a Γ-literal
- that literal has to be grey
- QED?

Refutated Lemma (this is wrong) 4. *If a Φ-term t occurs in a Ψ-literal in a clause C, then t occurs at a grey position in* $\mathrm{PI}(C)$.

Proof. As all grey literals of clauses involved in a refutation end up in the interpolant, it suffices to show that t occurs at a grey position in a grey literal.

Substitutions are applied to all variables, hence we only need to consider terms t which just enter a foreign colored literal.

TODO: propagation 1: Φ-terms vs Ψ-terms (in Ψ-literals)

TODO: propagation 2: Φ-terms vs other Φ-terms (in Ψ-literals)

Induction on refutation and σ; base case easy.

Resolution or factorisation inference ι. Let λ be a Γ-literal containing a variable u at position $\hat{u}$ such that $u\sigma_i$ contains a Δ-term t.

If the resolved or factorised literals are grey, they become part of $\mathrm{PI}(C)$ and if t occurs grey there, we are done.

- Suppose the resolved literals are Γ-colored. Then IH.
- Suppose the resolved literals are Δ-colored. TODO:
- Suppose the resolved literals are grey and t does not occur at a grey position in $\lambda\sigma = \lambda'\sigma$.

  TODO:

□

Conjectured Lemma 5. *If a Φ-term t occurs in a Ψ-literal in a clause C, then t occurs at a grey position in a grey literal in* $\mathrm{PI}(C) \vee C$.

there has to be a variable u in a Ψ-literal such that $u\sigma_i$ contains t.

Conjectured Lemma 6. *If a variable u occurs in a Φ-literal as well as in a Ψ-literal in a clause C, then t also occurs at a grey position in a grey literal in* $\mathrm{PI}(C)$.

Proof. Initially not the case.

Note that we can only resolve/factorise Γ-/Δ-/grey literals with other Γ-/Δ-/grey literals as clearly the predicate symbol must be the same for both literals. Hence if a variable occurs only in Γ- or only in Δ-literals, then it can never escape these. Hence u certainly is contained in a grey literal.

Now suppose that u only occurs colored in grey literals. Then it occurs in a Γ-(Δ-) term in the original Γ-(Δ-)clauses which contain it.

As shown before u must occur in some grey literal. Suppose it does not occur at a grey position in a grey literal as otherwise we are done. Then u only occurs in Γ-terms in grey literals as

TODO: it seems that now we have to deal with possible Γ-terms in Δ-literals and so on $\Rightarrow$ circular reasoning

The situation in question arises if some variable u occurs in a Γ-literal in some clause and some variable v occurs in a Δ-literal in some clause (possibly the same), such that in the unified literals, u and v both occur at the same position in the respective literals.

$\square$

1.5 directly from old proof

?⟨lemma:col_change⟩? **Lemma 7.** *Resolution or factorisation step ι from $\bar{C}$.*

If u col-change var in $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \ldots, \mathrm{PI}(C_n)) \vee C^{\circ})\sigma_{(0,\, i)}$, then u also occurs grey in that formula.

Proof. Abbreviation: $F \equiv (\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \ldots, \mathrm{PI}(C_n)) \vee C^{\circ})$

Induction over refutation and σ; base case easy.

Step: Supp color change var u present in $\chi\sigma_{(0,\, i)}$. (could also say introduced, then proof would be somehow different)

Supp u not grey in $\chi\sigma_{(0,\, i-1)}$ as otherwise done. As a first step, we show that if a (not necessarily color-changing) variable v occurs in a single-colored Φ-term $t[v]$ in $\chi\sigma_{(0,\, i)}$, then at least one of the following holds:

1. v occurs in some single-colored Φ-term in $\chi\sigma_{(0,\, i-1)}$

⟨var_occ_1⟩ 2. there is a color-changing variable w in $\chi\sigma_{(0,\, i-1)}$ such that v occurs grey in $w\sigma_i$.

⟨var_occ_2⟩ We consider the different cases which can introduce a variable v in a single-colored term Φ: Either it has been there before, it was introduced in a s.c. Φ-colored term, or a s.c. Φ-term containing the var is in $\mathrm{ran}(\sigma)$.

- Suppose a term $t'[v]$ is present in $\chi\sigma_{(0,\, i-1)}$ such that $t'[v]\sigma_i = t[v]$. Then 1 is the case.

- Suppose a variable w occurs in a single-colored Φ-term in $\chi\sigma_{(0,\, i-1)}$ such that v occurs grey in $w\sigma_i$. Suppose furthermore that 1 is not the case, i.e. v does not occur in a s.c. Φ-term in $\chi\sigma_{(0,\, i-1)}$, as otherwise we would be done. We show that 2 is the case.

  As v occurs neither grey nor in a s.c. Φ-term in $\chi\sigma_{(0,\, i-1)}$ but occurs in $\mathrm{ran}(\sigma_i)$, it must occur in $\chi\sigma_{(0,\, i-1)}$ and this can only be in a single-colored Ψ-term.

  As by assumption v occurs grey in $w\sigma_i$, there must be an occurrence $\hat{w}$ of w in a resolved or factorised literal, say $\lambda\sigma_{(0,\, i-1)}$ such that for the other resolved literal $\lambda'\sigma_{(0,\, i-1)}$, $\lambda'\sigma_{(0,\, i-1)}|_{\hat{w}}$ is a subterm in which v occurs grey. But as the occurrence of v in $\lambda'\sigma_{(0,\, i-1)}|_{\hat{w}}$ must be contained in a single-colored Ψ-term, so is $\lambda\sigma_{(0,\, i-1)}|_{\hat{w}}$, hence z occurs in a single-colored Ψ-term as well. Therefore 2 is the case.

- Suppose there is a variable z in $\chi\sigma_{(0,\, i-1)}$ such that v occurs in a single-colored Φ-term in $z\sigma_i$. Then $z\sigma_i$ occurs in $\chi\sigma_{(0,\, i-1)}$, but this is a witness for 1.

Now recall that we have assumed u to be a color-changing variable in $\chi\sigma_{(0,\, i)}$. Hence it occurs in a single-colored Γ-term as well as in a single-colored Δ-term. By the reasoning above, this leads to two case:

- In $\chi\sigma_{(0,\, i-1)}$, u occurs both in some single-colored Γ-term as well as in some single-colored Δ-term. Then we get the result by the induction hypothesis and the fact that $u \notin \mathrm{dom}(\sigma_i)$ as u does occur in $\chi\sigma_{(0,\, i)}$.

- Otherwise for some color Φ, u does not occur in a single-colored Φ-term in $\chi\sigma_{(0,\, i-1)}$. Then case 2 above must hold and there is some color-changing variable w in $\chi\sigma_{(0,\, i-1)}$ such that u occurs grey in $w\sigma_{(0,\, i)}$. But then by the induction hypothesis, w occurs grey in $\chi\sigma_{(0,\,i-1)}$ and hence u occurs grey in $\chi\sigma_{(0,\,i)}$. $\square$

1.6 Thursday:

In the following, we abbreviate $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \dots, \mathrm{PI}(C_n)) \vee C^{\circ})$ by χ.

⟨lemma:var_grey_col_lit⟩ **Lemma 8.** *Let ι be an inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

⟨14_1⟩
1. *The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨14_2⟩

2. *Some variable v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Φ-literal as well as grey in a Ψ-literal such that u occurs grey in $v\sigma_i$.*

⟨14_3⟩

3. *There is a variable v such that u occurs grey in $v\sigma_i$[1] and v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.*

⟨14_4⟩

4. *The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.*

Proof. We consider the different cases which lead to the variable u in a grey position in a Φ-literal in $\chi\sigma_{(0,\,i)}$:

- There already is a Φ-literal in $\chi\sigma_{(0,\,i-1)}$ which contains u at a grey position and σ_i does not change this. Then clearly 1 is the case.
- Otherwise there must be a Φ-literal in $\chi\sigma_{(0,\,i-1)}$, which contains a variable v at a grey position such that u occurs grey in $v\sigma_i$. Hence in the resolved or factorised literals λ and λ', there is a position p such that w.l.o.g. $\lambda|_p = v$ and $\lambda'|_p$ contains a grey occurrence of u, and λ and λ coincide along p. We distinguish based properties of the position p:
  - Suppose that p is contained in a single-colored Φ-term. Then v occurs grey in a Ψ-literal as well as in a single-colored Φ-term, which suffices for 3 as u occurs grey in $v\sigma_i$.
  - Suppose that p is contained in a single-colored Ψ-term. Then u occurs grey in a Φ-literal as well in a single-colored Ψ-term, which implies 3.
  - Otherwise p is a grey position. We distinguish further:
    * Suppose that the resolved or factorised literal is Φ-colored. Then u occurs grey in a Φ-literal and we have established item 3.
    * Suppose that the resolved or factorised literal is Ψ-colored. Then the variable v occurs grey in a Φ-literal as well as grey in a Ψ-literal, hence 2 is the case.

      Otherwise the resolved or factorised literal is grey and u occurs grey in a grey literal, which is sufficient for 4. ☐

⟨lemma:var_in_sc_term⟩ **Lemma 9.** *Let ι be an inference in a refutation of $\Gamma \cup \Delta$. Suppose that a variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$. Then at least one of the following statements holds:*

[1]Note that this includes the case that $v = u$ and σ_i is trivial on u.

⟨15_1⟩ 1. *The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_2⟩ 2. *There is a variable v such that u occurs grey in $v\sigma_i$ and v occurs in a single-colored Φ-term as well as in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_4⟩ 3. *There is a variable v such that u occurs grey in $v\sigma_i$ and v occurs in $\chi\sigma_{(0,\,i-1)}$ in a single-colored Φ-term as well as at a grey position in a Ψ-literal.*

⟨15_3⟩ 4. *The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.*

⟨15_5⟩ 5. *The variable u occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$.*

Proof. We consider the different cases which lead to the variable u in a single-colored Φ-term in $\chi\sigma_{(0,\,i)}$:

- There is a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which contains u such that σ_i does not change this. Then 1 is the case.

- Suppose that there is a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ which contains a variable v such that u occurs grey in $v\sigma_i$.

  Hence in the resolved or factorised literals λ and λ', there is a position p such that w.l.o.g. $\lambda|_p = v$ and $\lambda'|_p$ contains a grey occurrence of u, and λ and λ coincide along p. We distinguish based properties of the position p:

  - Suppose that p is contained in a single-colored Φ-term. Then u is contained in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$ and item 1 holds.
  - Suppose that p is contained in a single-colored Ψ-term. As then v is contained in a single-colored Φ-term as well as in a single-colored Ψ-term, 2 is the case.
  - Suppose that p is a grey position. We distinguish further:
    * Suppose that the resolved or factorised literal is Φ-colored. Then u occurs grey in a Φ-literal, which suffices for 4.
    * Suppose that the resolved or factorised literal is Ψ-colored. Then the variable v occurs in a single-colored Φ-term as well as grey in a Ψ-literal, which implies 3.
    * Otherwise the resolved or factorised literal is grey. But then u occurs grey in a grey literal and we have established item 5.
  - Suppose that a variable w occurs in $\chi\sigma_{(0,\,i-1)}$ such that u occurs in a single-colored Φ-term in $w\sigma_i$. This can only be the case if $w\sigma$ already occurs in $\chi\sigma_{(0,\,i-1)}$, which implies that 1 is the case. □

ol_change_and_grey_in_col_lit⟩ **Lemma 10.** *Let C be a clause in the resolution refutation π of $\Gamma \cup \Delta$ and u be a variable which occurs in $\mathrm{PI}(C) \vee C$ in some literal in a single-colored Φ-term or grey in a Φ-literal.*

Suppose that u also occurs in $\mathrm{PI}(C) \vee C$ in some literal in a single-colored Ψ-term or grey in a Ψ-literal.

Then u occurs grey in a grey literal.

Note that Φ and Ψ are to be read as any pair of different colors, i.e. Γ and Δ as well as Δ and Γ.

Proof. We proceed by induction over π and σ.

Note that initially, every pair of clauses is variable-disjoint and all symbols of a clause are either all grey or Φ-colored or all grey or Ψ-colored, hence the lemma is vacuously true.

For the induction step, we assume that the property holds for $\mathrm{PI}(C_i) \vee C_i$, $1 \leq i \leq n$, where $C_1, \dots, C_n$ are the clauses used in a resolution or factorisation inference ι. Note that then, the property also holds for χ, i.e. for $\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \dots, \mathrm{PI}(C_n)) \vee C^\circ$ as it contains all the grey literals present in $\mathrm{PI}(C_i) \vee C_i$ for any i (this is evident by the definition of $\mathrm{PI}^\circ_{\mathrm{step}}$), and as clauses are pairwise variable-disjoint, the lemma condition can not become true for a variable for which it was not true in $\mathrm{PI}(C_i) \vee C$ for some i.

Suppose that u occurs in $\chi\sigma_{(0,\,i)}$ in a single-colored Φ-term or grey in a Φ-literal and that u also occurs in $\chi\sigma_{(0,\,i)}$ in a single-colored Ψ-term or grey in a Ψ-literal.

Then we can deduce by Lemma 8 and Lemma 9 that at least one of the following statements holds:

⟨oozoh70h1⟩
1. The variable u occurs grey in a Φ-literal in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h2⟩
2. Some variable v occurs in $\chi\sigma_{(0,\,i-1)}$ grey in a Φ-literal as well as grey in a Ψ-literal such that u occurs grey in $v\sigma_i$.

⟨oozoh70h3⟩
3. There is a variable v such that u occurs grey in $v\sigma_i$ and v occurs in $\chi\sigma_{(0,\,i-1)}$ either grey in a Φ-literal as well as in a single-colored Ψ-term in any literal, or grey in a Ψ-literal as well as in a single-colored Φ-term in any literal.

⟨oozoh70h4⟩
4. The variable u occurs at a grey position in a grey literal in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h5⟩
5. The variable u occurs in a single-colored Φ-term in $\chi\sigma_{(0,\,i-1)}$.

⟨oozoh70h6⟩
6. There is a variable v such that u occurs grey in $v\sigma_i$ and v occurs in a single-colored Φ-term as well as in a single-colored Ψ-term in $\chi\sigma_{(0,\,i-1)}$.

By the same lemmata, we get the same set of statements where Φ and Ψ are interchanged. We refer to them by the respective number followed by *.

Suppose that 4 is not the case as otherwise we are done since σ_i is trivial on u as u occurs in $\chi\sigma_{(0,\,i)}$. Furthermore, there are a number of cases which give the result by the induction hypothesis: For the cases 2, 3 and 6, we can infer that by the induction hypothesis, there is a grey occurrence of the variable v in a grey literal in $\chi\sigma_{(0,\,i-1)}$, and as u occurs grey in $v\sigma_i$, there is a grey occurrence of u in a grey literal in $\chi\sigma_{(0,\,i)}$.

It remains to show that the lemma holds true in case the statements 1 or 5 as well as 1*or 5*hold. But note that in any combination of 1 or 5 and 1*or 5*in effect yields a situation under which the induction hypothesis again is applicable. Hence we may infer that u occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$ and since σ_i is trivial u as shown above, u occurs grey in a grey literal in $\chi\sigma_{(0,\,i)}$. □

1.7 Friday

NB: this is the heart of the proof:

Lemma 11. *If* $\mathrm{PI}(C) \vee C$ *contains a maximal colored occurrence of a* Γ*-term* $t[s]$ *containing* Δ*-term* s*, then* s *occurs grey in a grey literal in* $\mathrm{PI}(C) \vee C$*.*

Proof. Note that it suffices to show that at the step where s is introduced as subterm of $t[s]$, s occurs grey in $\mathrm{PI}(C) \vee C$ as any later modification by substitution is applied to both occurrences s, so they stay equal throughout the remaining derivation.

Induction over π and σ. TODO: as in Lemma 10

Base case: vacuously true.

Step: Resolution or factorisation inference ι, $\mathrm{mgu}(\iota) = \sigma = \sigma_1 \cdots \sigma_n$ The term $t[s]$ is created by one of the following two ways:
(we abbreviate $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \ldots, \mathrm{PI}(C_n)) \vee C^{\circ}$ by F.)

- A variable u occurs in $\chi\sigma_{(0,\,i-1)}$ such that $u\sigma_i = t[s]$.

  Then u occurs in a resolved or factorised literal $\lambda\sigma_{(0,\,i-1)}$ at $\hat{u}$ such that at the other resolved or factorised literal $\lambda'\sigma_{(0,\,i-1)}$, $\lambda'\sigma_{(0,\,i-1)}|_{\hat{u}} = t[s]$. Then the condition is present at $\chi\sigma_{(0,\,i-1)}$ and we get the result by the induction hypothesis.

- Note that we only consider maximal colored terms.

  Let $t[u]$ be a maximal colored Γ-term in $\chi\sigma_{(0,\,i-1)}$ such that in the tree-representation of $t[u]$, the path from the root to u does not contain a node labelled with a Δ-symbol, and $u\sigma_i$ contains a grey occurrence of s.

  are probably not same t and s as in lemma statement, which isn't technically wrong but confusing

  Suppose that u occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$. Then s occurs grey in a grey literal in $\chi\sigma_{(0,\,i)}$ as σ_i does not affect u since u occurs in $\chi\sigma_{(0,\,i)}$ and we are done.

  If u occurs grey in a Δ-literal or if u occurs in a single-colored Δ-term in $\chi\sigma_{(0,\,i-1)}$, then by Lemma 10, u also occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$ and s hence occurs grey in a grey literal in $\chi\sigma_{(0,\,i)}$.

  Now suppose that u does not occur grey in a grey literal $\chi\sigma_{(0,\,i-1)}$ as otherwise clearly we are done.

  Hence as all other cases are excluded, u can only occur in $\chi\sigma_{(0,\,i-1)}$ in a single-colored Γ-term or grey in a Γ-colored literal. But then, since $u\sigma_i$ contains a grey occurrence of s, there is a position p in the two resolved or factorised literals λ and λ' such that $\lambda|_p = u$ and $\lambda'|_p$ contains a grey occurrence of s. Furthermore, the prefix along the path to p is the same in both λ and λ'. As u only occurs in single-colored Γ-terms, $\lambda'|_p$ does so as well, so s is contained in a single-colored Γ-term in $\chi\sigma_{(0,\,i-1)}$. Since s is a Δ-term, by the induction hypothesis, s occurs grey in a grey literal in $\chi\sigma_{(0,\,i-1)}$ and hence also in $\chi\sigma_i$. □