0.1 referenced lemmas from previous sections

Lemma 1 (Commutativity of lifting and logical operators)**.** *Let A and B be first-order formulas and s and t be terms. Then it holds that:*

1. $\ell^z_\Phi[\neg A] \Leftrightarrow \neg \ell^z_\Phi[A]$
2. $\ell^z_\Phi[A \circ B] \Leftrightarrow (\ell^z_\Phi[A] \circ \ell^z_\Phi[B])$ *for* $\circ \in \{\wedge, \vee\}$
3. $\ell^z_\Phi[s = t] \Leftrightarrow (\ell^z_\Phi[s] = \ell^z_\Phi[t])$

Lemma 2 (Commutativity of lifting and substitution)**.** *Let C be a clause and σ a substitution such that no lifting variable occurs in C or σ. Define σ' with* $\mathrm{dom}(\sigma') = \mathrm{dom}(\sigma) \cup \{z_t \mid t\sigma \neq t\}$ *such that for a variable z,*

$$x\sigma' = \begin{cases} z_{t\sigma} & \text{if } x = z_t \text{ and } t\sigma \neq t \\ \ell^z_\Phi[x\sigma] & \text{otherwise} \end{cases}$$

Then $\ell^z_\Phi[C\sigma] = \ell^z_\Phi[C]\sigma'$.

Chapter 1

Interpolant extraction from resolution proofs in one phase

While the previous chapter demonstrates that it is possible to extract propositional interpolants and lift them from the colored symbols later in order to obtain a proper interpolant, we now present a novel approach, which only operates with grey intermediary interpolants. This is established by lifting any term which is added to the interpolant.

By its nature, this approach requires an alternate strategy than the proof of the extraction in two phases as a commutation of substitution and lifting is no longer possible if lifting variables are present. Let us recall the corresponding lemma from the previous chapter:

Lemma 2 (Commutativity of lifting and substitution)**.** *Let C be a clause and σ a substitution such that no lifting variable occurs in C or σ. Define σ' with* $\mathrm{dom}(\sigma') = \mathrm{dom}(\sigma) \cup \{z_t \mid t\sigma \neq t\}$ *such that for a variable z,*

$$x\sigma' = \begin{cases} z_{t\sigma} & \textit{if } x = z_t \textit{ and } t\sigma \neq t \\ \ell^z_\Phi[x\sigma] & \textit{otherwise} \end{cases}$$

Then $\ell^z_\Phi[C\sigma] = \ell^z_\Phi[C]\sigma'$.

Consider the following illustration of a problem of the notion of applying this lemma to terms containing lifting variables:

Example 3. Let $\sigma = \{x \mapsto a\}$ and consider the terms $f(x)$ and $f(a)$, where f and a are colored symbols. Clearly $f(x)\sigma = f(a)$ and therefore necessarily $z_{f(x)}\sigma' = z_{f(a)}$.

But now consider $x_{f(x)}\sigma$. As $z_{f(x)}$ is a lifting variable, it is not affected by unifiers from resolution derivations and also not by σ. Hence $z_{f(x)}\sigma = z_{f(x)}$ and therefore $\ell[z_{f(x)}\sigma] = \ell[z_{f(x)}] = z_{f(x)}$, but $\ell[z_{f(x)}]\sigma' = z_{f(x)}\sigma' = z_{f(a)}$. So $\ell[z_{f(x)}\sigma] \neq \ell[z_{f(x)}]\sigma'$.

We see here that there are circumstances under which in order to commute lifting and substitution, the substitution σ' is required to conform to the equation $z_{f(x)}\sigma' = z_{f(a)}$, whereas in others, it must hold that $z_{f(x)}\sigma' = z_{f(x)}$. △

1.1 Definition of the extraction algorithm

The extracted interpolants are prenex formulas, where the quantifier block and the matrix of the formula are calculated separately in each step of the traversal of the resolution refutation.

Extraction of the interpolant matrix $\mathrm{AI}_{\mathrm{mat}}$ and calculation of $\mathrm{AI}_{\mathrm{cl}}$

$\mathrm{AI}_{\mathrm{mat}}$ is inspired by the propositional interpolants PI from Definition **??**. Its difference lies in the fact that the lifting occurs in every step of the extraction. This however necessitates applying these liftings to the clauses of the resolution refutation as well. For a clause C of the resolution refutation, we will denote the clause with the respective liftings applied by $\mathrm{AI}_{\mathrm{cl}}(C)$ (a formal definition will be given below), and for a term t at position p in C, we denote $\mathrm{AI}_{\mathrm{cl}}(C)|_p$ by t_{AIcl}.

Now we can define preliminary versions of $\mathrm{AI}^{\bullet}_{\mathrm{mat}}$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}$:

Definition 4 ($\mathrm{AI}^{\bullet}_{\mathrm{mat}}$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}$)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$.

For a clause C in π, $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C)$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C)$ are defined as follows:

Base case. If $C \in \Gamma$, $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \overset{\mathrm{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \overset{\mathrm{def}}{=} \top$.

In any case, $\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C) \overset{\mathrm{def}}{=} \ell[C]$.

Resolution. If the clause C is the result of a resolution step of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C)$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}$ are defined as follows:

$\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C) \overset{\mathrm{def}}{=} \ell[(\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C_1)\backslash\{l_{\mathrm{AIcl}}\})\sigma] \ \vee \ \ell[(\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C_2)\backslash\{l'_{\mathrm{AIcl}}\})\sigma]$

1. If l is Γ-colored: $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \overset{\mathrm{def}}{=} \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma] \vee \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_2)\sigma]$

2. If l is Δ-colored: $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \overset{\mathrm{def}}{=} \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma] \wedge \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_2)\sigma]$
3. If l is grey: $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \overset{\mathrm{def}}{=} (\neg\ell[l'_{\mathrm{AIcl}}\sigma] \wedge \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma]) \vee (\ell[l_{\mathrm{AIcl}}\sigma] \wedge \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_2)\sigma])$

Factorisation. If the clause C is the result of a factorisation of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \overset{\mathrm{def}}{=} \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma]$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C) \overset{\mathrm{def}}{=} \ell[(\mathrm{AI}_{\mathrm{cl}}(C_1)\backslash\{l'_{\mathrm{AIcl}}\})\sigma]$. △

Note that in $\mathrm{AI}^{\bullet}_{\mathrm{mat}}$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}$, it is possible that there for a colored term t in C that $t_{\mathrm{AIcl}} \neq z_t$ as illustrated by the following examples:

Example 5. We consider a resolution refutation of the initial clause sets $\Gamma = \{R(c), \neg Q(v)\}$ and $\Delta = \{\neg R(u) \vee Q(g(u))\}$:

$$\dfrac{\dfrac{R(c) \qquad \neg R(u) \vee Q(g(u))}{Q(g(c))}\ \mathrm{res}, y \mapsto c \qquad\qquad \neg Q(v)}{\square}\ \mathrm{res}, v \mapsto g(c)$$

We now replace every clause C by $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \mid \mathrm{AI}^{\bullet}_{\mathrm{cl}}(C)$ in order to visualise the steps of the algorithm:

$$\dfrac{\dfrac{\bot \mid R(y_c) \qquad \top \mid \neg R(u) \vee \neg Q(x_{g(u)})}{R(y_c) \mid Q(x_{g(u)})}\ \mathrm{res}, y \mapsto c \qquad\qquad \bot \mid \neg Q(v)}{\neg Q(x_{g(c)}) \wedge R(y_c) \mid \square}\ \mathrm{res}, v \mapsto g(c)$$

By quantifying y_c existentially and $x_{g(c)}$ universally[1], we obtain an interpolant for $\Gamma \cup \Delta$: $\exists y_c \forall x_{g(c)}(\neg Q(x_{g_c}) \wedge R(y_c))$. Note however that $\ell[Q(g(c))] = Q(x_{g(c)})$, but $\mathrm{AI}_{\mathrm{mat}}(Q(g(c))) = Q(x_{g(u)})$. This example shows that this circumstance is not necessarily an obstacle for the correctness of this algorithm. △

⟨exa:2b⟩ **Example 6.** We consider a resolution refutation of the initial clause sets $\Gamma = \{R(c), P(c)\}$ and $\Delta = \{\neg R(u) \vee \neg Q(g(u)), \neg P(v) \vee Q(g(v))\}$:

$$\dfrac{\dfrac{\neg R(u) \vee \neg Q(g(u)) \qquad R(c)}{\neg Q(g(c))}\ \mathrm{res}, u \mapsto c \qquad \dfrac{\neg P(v) \vee Q(g(v)) \qquad P(c)}{Q(g(c))}\ \mathrm{res}, v \mapsto c}{\square}\ \mathrm{res}$$

We now again display $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \mid \mathrm{AI}^{\bullet}_{\mathrm{cl}}(C)$ for every clause C of the refutation:

[1] The procedure for calculating the quantifier block is defined in section 1.1

$$\dfrac{\dfrac{\top \mid \neg R(u) \lor \neg Q(x_{g(u)}) \qquad \bot \mid R(y_c)}{R(y_c) \mid \neg Q(x_{g(u)})}\,\text{res}, u \mapsto c \qquad \dfrac{\top \mid \neg P(v) \lor Q(x_{g(v)}) \qquad \bot \mid P(y_c)}{P(y_c) \mid Q(x_{g(v)})}\,\text{res}, v \mapsto c}{(Q(x_{g(v)}) \land R(y_c)) \;\lor\; (\neg Q(x_{g(u)}) \land P(y_c)) \mid \square}\,\text{res}$$

Note again that here, we have that $\ell[\neg Q(g(c))] = \neg Q(x_{g(c)}) \neq \mathrm{AI}^{\bullet}_{\mathrm{cl}}(\neg Q(g(c))) = \neg Q(x_{g(u)})$ and $\ell[Q(g(c))] = Q(x_{g(c)}) \neq \mathrm{AI}^{\bullet}_{\mathrm{cl}}(Q(g(c))) = Q(x_{g(v)})$. However in this instance, it is not possible to find quantifiers for the free variables of $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(\square)$ such that by binding them, an interpolant is produced. For the naive approach, namely to use $\exists y_c \forall x_{g(v)} \forall x_{g(u)}$ as prefix, it holds that $\Gamma \not\models \exists y_c \forall x_{g(v)} \forall x_{g(u)}((Q(x_{g(v)}) \land R(y_c)) \;\lor\; (\neg Q(x_{g(u)}) \land P(y_c)))$. This failure is possible as intuitively, resolution deductions are valid by virtue of the resolved literals being equal. The interpolant extraction procedure exploits this property not directly on the clauses but on the lifted clause, i.e. on $\mathrm{AI}_{\mathrm{cl}}(C)$ for a clause C. Note that by ensuring that for resolved literals l and l', it holds that $l_{\mathrm{AIcl}} = l'_{\mathrm{AIcl}}$, we can obtain an interpolant, for instance: $\exists y_c \forall x^*((Q(x^*) \land R(y_c)) \;\lor\; (\neg Q(x^*) \land P(y_c)))$. △

In order to avoid the pitfall shown in Example 6 and to generalise the indicated solution, we define a function on resolved literals calculating a substitution, which ensures that the literals in the lifted clause, which correspond to the resolved literals, are equal.

Definition 7 (au)**.** For resolved literals l and l' of a resolution derivation step with a unifier σ such that $l\sigma = l'\sigma$, and $\ell[l_{\mathrm{AIcl}}\sigma] = P(s_1, \dots, s_n)$ and $\ell[l'_{\mathrm{AIcl}}\sigma] = P(t_1, \dots, t_n)$, we define:

$$\mathrm{au}(P(s_1, \dots, s_n), P(t_1, \dots, t_n)) \stackrel{\text{def}}{=} \bigcup_{i=1}^{n} \mathrm{au}(s_i, t_i)$$

For terms s and t where $s = \ell[l_{\mathrm{AIcl}}\sigma]|_p$ and $t = \ell[l'_{\mathrm{AIcl}}\sigma]|_p$ for some position p, we define:

$$\mathrm{au}(s,t) \stackrel{\text{def}}{=} \begin{cases} \bigcup_{i=1}^{n} \mathrm{au}(s_i, t_i) & \text{if } s \text{ is grey and } s = f_s(s_1, \dots, s_n),\ t = f_t(t_1, \dots, t_n)^2 \\ \{z_{s'} \mapsto z_r, z_{t'} \mapsto z_r\} & \text{if } s \text{ is a lifting variable } z_{s'},\ t = z_{t'}, \text{ and } r = \ell[l\sigma]|_p \end{cases} \quad \triangle$$

⟨prop:tau_dom_ran⟩ **Proposition 8.** *Let l and l' be the resolved literals of a resolution step and σ the unifier employed in this step. Furthermore let $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$. Then*

[2] Note that constants are treated as function symbols of arity zero.

$\mathrm{dom}(\tau)$ *consists exactly of the lifting variables of* $\ell[l_{\mathrm{AIcl}}\sigma]$ *and* $\ell[l'_{\mathrm{AIcl}}\sigma]$ *and* $\mathrm{ran}(\tau)$ *consists exactly of the lifting variables of* $\ell[l\sigma]$.

possibly argue here why au is well-defined (but it follows more or less directly from a later lemma)

Definition 9 ($\mathrm{AI}_{\mathrm{mat}}$ and $\mathrm{AI}_{\mathrm{cl}}$)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$. $\mathrm{AI}_{\mathrm{mat}}(\pi)$ is defined to be $\mathrm{AI}_{\mathrm{mat}}(\Box)$, where $\Box$ is the empty box derived in π.

For a clause C in π, $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ are defined inductively as follows:

Base case. If $C \in \Gamma$, $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \top$.

In any case, $\mathrm{AI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[C]$.

Resolution. If the clause C is the result of a resolution step of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then let $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$ and define $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ as follows:

$\mathrm{AI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[(\mathrm{AI}_{\mathrm{cl}}(C_1)\backslash\{l_{\mathrm{AIcl}}\})\sigma]\tau \ \vee\ \ell[(\mathrm{AI}_{\mathrm{cl}}(C_2)\backslash\{l'_{\mathrm{AIcl}}\})\sigma]\tau$

1. If l is Γ-colored: $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau$
2. If l is Δ-colored: $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau$
3. If l is grey: $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} (\neg\ell[l'_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau) \ \vee\ (\ell[l_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau)$

Factorisation. If the clause C is the result of a factorisation of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then:

$\tau \stackrel{\mathrm{def}}{=} \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$.

$\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau$

$\mathrm{AI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[(\mathrm{AI}_{\mathrm{cl}}(C_1)\backslash\{l'_{\mathrm{AIcl}}\})\sigma]\tau$ △

Definition 10. $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ ($\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$) for a clause C is defined as $\mathrm{AI}_{\mathrm{mat}}(C)$ ($\mathrm{AI}_{\mathrm{cl}}(C)$) with the difference that in its inductive definition, every lifting $\ell[\varphi]$ for a formula or term φ is replaced by a lifting of only the Δ-terms $\ell_\Delta[\varphi]$. △

ma:no_colored_terms⟩ **Lemma 11.** *Let C be a clause of a resolution refutation of* $\Gamma \cup \Delta$. $\mathrm{AI}_{\mathrm{mat}}(C)$ *and* $\mathrm{AI}_{\mathrm{cl}}(C)$ *do not contain colored symbols.* $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ *and* $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ *do not contain* Δ*-colored symbols.*

Proof. For $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$, consider the following: In the base case of the inductive definitions of $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$, no colored symbols occur. In the inductive steps, any colored symbol which is added by σ to intermediary formulas is lifted. By Proposition 8, $\mathrm{ran}(\tau)$ only consists of lifting variables.

For $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ and $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$, a similar argument goes through by reading colored as Δ-colored. □

emma:substitute_and_lift⟩ **Lemma 12.** *Let σ be a substitution and F a formula without Φ-colored terms such that for a set of formulas Ψ, $\Psi \models F$. Then $\Psi \models \ell^z_\Phi[F\sigma]$.*

Proof. $\ell^z_\Phi[F\sigma]$ is an instance of F: σ substitutes variables either for terms not containing Φ-colored symbols or by terms containing Φ-colored symbols. For the first kind, the lifting has no effect. For the latter, the lifting only replaces subterms of the terms introduced by the substitution by a lifting variable such that the original structure of F remains invariant as it by assumption does not contain colored terms. □

Lemma 13. *Let λ be a literal in a clause C occurring in a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{AI}_{\mathrm{cl}}(C)$ contains a literal λ_{AIcl} such that $\lambda_{\mathrm{AIcl}} \gtrsim \ell[\lambda]$, where $\gtrsim$ is defined as follows:*

$$\varphi \gtrsim \varphi' \Leftrightarrow \begin{cases} P = P' \wedge \bigwedge_{i=1}^{n} s_i \gtrsim s'_i & \textit{if } \varphi = P(s_1,\dots,s_n) \textit{ and } \varphi' = P'(s'_1,\dots,s'_n) \\ f = f' \wedge \bigwedge_{i=1}^{n} s_i \gtrsim s'_i & \textit{if } \varphi = f(s_1,\dots,s_n) \textit{ and } \varphi' = f'(s'_1,\dots,s'_n) \\ x = x' & \textit{if } \varphi, \varphi' \textit{ are non-lifting variables, } \varphi = x \textit{ and } \varphi' = x' \\ s' \textit{ is a specialisation of } s & \textit{if } \varphi, \varphi' \textit{ are lifting variables, } \varphi = z_s \textit{ and } \varphi' = z_{s'} \end{cases}$$

Proof. We proceed by induction on the resolution refutation.

Base case. If for a clause C either $C \in \Gamma$ or $C \in \Delta$ holds, then $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[(]C)$. Therefore for every literal l in C, there exists a literal l_{AIcl} in $\mathrm{AI}_{\mathrm{cl}}(C)$ such that $\ell[l] = l_{\mathrm{AIcl}}$, which implies $l_{\mathrm{AIcl}} \gtrsim \ell[l]$.

Resolution. If the clause C is the result of a resolution step of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then let $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$. Every literal in C is of the form $\lambda\sigma$ for some $\lambda \in C_1$ or $\lambda \in C_2$. Without loss of generality let $\lambda \in C_1$. By the inductilon hypothesis, there is a literal in $\mathrm{AI}_{\mathrm{cl}}(C_1)$ such that $\lambda_{\mathrm{AIcl}} \gtrsim \ell[\lambda_{\mathrm{AIcl}}]$. As $\lambda \neq l$, by the definition of $\mathrm{AI}_{\mathrm{cl}}$, $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau$ is contained in $\mathrm{AI}_{\mathrm{cl}}(C)$. It remains to show that $\ell[l_{\mathrm{AIcl}}\sigma]\tau \gtrsim \ell[\lambda\sigma]$.

Factorisation. If the clause C is the result of a factorisation of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then let $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$. The rest of this proof resembles the case of resolution: Every literal in C is of the form $\lambda\sigma$ for some $\lambda \in C_1$. By the induction hypothesis, there is a literal in $\mathrm{AI}_{\mathrm{cl}}(C_1)$ such that $\lambda_{\mathrm{AIcl}} \gtrsim \ell[\lambda]$. As $\lambda \neq l$, by the definition of $\mathrm{AI}_{\mathrm{cl}}$, $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau$ is contained in $\mathrm{AI}_{\mathrm{cl}}(C)$. It remains to show that $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau \gtrsim \ell[\lambda\sigma]$.

TODO: ICI: it seems that we need something special, move this to the resolution case and claim at factorisation that something similar goes through

We perform an induction on the structure of $\ell[t]$ for a term t to show that $\ell[t'] \gtrsim t$ implies that $\ell[t'\sigma]\tau \gtrsim \ell[t\sigma]$.

- Suppose t is a non-lifting variable. Then $\ell[t] = t$, which in conjunction with $\ell[t'] \gtrsim t$ implies that $t' = t$. But then $\ell[t'\sigma] = \ell[t\sigma]$. If τ is trivial on $\ell[t'\sigma]$, we are done as then $\ell[t'\sigma]\tau = \ell[t\sigma]$. Hence assume that $\ell[t'\sigma]\tau \neq \ell[t'\sigma]$. By the definition of au, this entails that $\ell[t'\sigma]\tau = z_s$ for some term s such that $s =$????

□

⟨lifted_literal_equal⟩ **Lemma 14.** *lifted literal equal*

Lemma 15. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then for clauses C in π, $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.*

Proof. We proceed by induction of the strengthening $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_\Gamma)$[3].

Base case. For $C \in \Gamma$, $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_\Gamma) = \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C) = \ell_\Delta[C] = C$, so $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_\Gamma)$. Otherwise $C \in \Delta$ and hence $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = \top$.

Resolution. Suppose the last rule application is an instance of resolution. Then it is of the following form:

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad l\sigma = l'\sigma$$

[3] Recall that as in Lemma **??**, D_Φ denotes the clause created from the clause D by removing all literals which are not contained $\mathrm{L}(\Phi)$.

We introduce the following abbreviations:

$\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* = \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)\backslash\{\ell[(l_{\mathrm{AIcl}^\Delta})_\Gamma]\}$

$\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^* = \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)\backslash\{\ell[\neg(l'_{\mathrm{AIcl}^\Delta})_\Gamma]\}$

Note that $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C) = \ell_\Delta[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell_\Delta[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$, where τ is defined as in the interpolant extraction procedure as $\mathrm{au}(\ell[l_{\mathrm{AIcl}^\Delta}\sigma], \ell[l'_{\mathrm{AIcl}^\Delta}\sigma])$.

Employing these, the induction hypothesis yields $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* \vee (l_{\mathrm{AIcl}^\Delta})_\Gamma$ as well as $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^* \vee \neg(l'_{\mathrm{AIcl}^\Delta})_\Gamma$. By Lemma 11, $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_i)$ and $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_i)$ for $i \in \{1,2\}$ do not contain Δ-colored symbols. Hence by Lemma 12, pulling the lifting inwards using Lemma 1 and applying τ, we obtain:

$$\Gamma \overset{(\circ)}{\models} \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[(l_{\mathrm{AIcl}^\Delta})_\Gamma\sigma]\tau$$

$$\Gamma \overset{(*)}{\models} \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau \vee \neg\ell[(l'_{\mathrm{AIcl}^\Delta})_\Gamma\sigma]\tau$$

We continue by a case distinction on the color of l:

1. Suppose that l is Γ-colored. Then $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau$. As l is Γ-colored, $(l_{\mathrm{AIcl}^\Delta})_\Gamma = l_{\mathrm{AIcl}^\Delta}$ and as $l\sigma = l'\sigma$, also $(l'_{\mathrm{AIcl}^\Delta})_\Gamma = l'_{\mathrm{AIcl}^\Delta}$. By Lemma 14, $\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau = \ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau$. Hence we can perform a resolution step on $(\circ)$ and $(*)$ to arrive at $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$. This is however by Lemma 1 nothing else than $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.
2. Suppose that l is Δ-colored. Then $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau$. As l and l' are Δ-colored, $(\circ)$ and $(*)$ reduce to $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau$ and $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$ respectively. These however imply that $\Gamma \models (\ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau) \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$, which in turn is nothing else than $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.
3. Suppose that l is grey. Then $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = (\neg\ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau) \vee (\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau)$.
   Let M be a model of Γ. Suppose that $M \models \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ as otherwise we are done. Hence $M \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau$ and $M \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$ and $(\circ)$ and $(*)$ reduce to $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau$ and $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau$ respectively. As by Lemma 14 $\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau = \ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau$, a case distinction on the truth value of $\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau$ in M shows that $M \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$.

Factorisation. Suppose the last rule application is an instance of factorisation. Then it is of the following form:

$$\frac{C_1 : l \vee l' \vee D}{C : (l \vee D)\sigma} \quad \sigma = \mathrm{mgu}(l, l')$$

TODO: continue after literal equal lemma has been formalised

□

row_quantifier_block⟩

Theorem 16. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{AI}_{\mathrm{mat}}(\pi)$ is an interpolant.*

Proof. This needs to many things I don't yet know how to make precise, so let's start with $\Gamma \models \ldots$

□