501 – like 502 but easier

$$\dfrac{\forall x_a \exists y_{f(a)} Q(x_a, y_{f(a)}) \mid A \qquad \forall x_a (S(x_a)) \mid \neg A}{\forall x_a (S(x_a)) \wedge \forall x_a \exists y_{f(a)} Q(x_a, y_{f(a)}) \mid \Box}$$

no first order operation in this last inference $\Rightarrow$ nothing to prove

502 – example with multiple, independent a's

derivation:

$$\dfrac{\dfrac{\dfrac{P(f(x),x) \overset{\Sigma}{\vee} Q(z) \vee R(z) \qquad \neg \overset{\Pi}{R}(a)}{R(a) \mid P(f(x),x) \vee Q(a)} \qquad \neg \overset{\Pi}{Q}(u)}{Q(a) \vee R(a) \mid P(f(x),x)} \qquad \neg \overset{\Pi}{P}(z,a)}{P(f(a),a) \vee Q(a) \vee R(a) \mid \Box}$$

invariant: $\ell_\Delta[\mathrm{LI}(C)] \mid \ell_\Delta[C]$

$$\dfrac{\dfrac{\dfrac{P(f(x),x) \overset{\Sigma}{\vee} Q(z) \vee R(z) \qquad \neg \overset{\Pi}{R}(x_a)}{R(x_a) \mid P(f(x),x) \vee Q(x_a)} \qquad \neg \overset{\Pi}{Q}(u)}{\forall x_a (Q(x_a) \vee R(x_a)) \mid P(f(x),x)} \qquad \neg \overset{\Pi}{P}(z,x_a)}{\forall x_a \exists y_{f(a)} \Big(P(y_{f(a)}, x_a) \vee \forall x_a (Q(x_a) \vee R(x_a)) \Big) \mid \Box}$$

Detailed derivation:
$\Gamma \vDash \forall x_a (Q(x_a) \vee R(x_a)) \mid P(f(x),x)$
$\Gamma \vDash P(z, x_a)$
hence
$\Gamma \vDash \forall x_a \Big(\forall x_a (Q(x_a) \vee R(x_a)) \mid P(f(x_a), x_a) \Big)$
$\Gamma \vDash \forall x_a P(f(x_a), x_a)$
hence
$\Gamma \vDash \forall x_a \Big(\forall x_a (Q(x_a) \vee R(x_a)) \mid P(f(x_a), x_a) \Big) \;\wedge\; \forall x_a P(f(x_a), x_a)$
hence
$\Gamma \vDash \forall x_a \Big(\forall x_a (Q(x_a) \vee R(x_a)) \mid P(f(x_a), x_a) \;\wedge\; P(f(x_a), x_a) \Big)$
simplified
$\Gamma \vDash \forall x_a \Big(\forall x_a (Q(x_a) \vee R(x_a)) \;\wedge\; P(f(x_a), x_a) \Big)$

lifting: $\mathrm{LI}(C) \mid C$

$$\begin{array}{c} \dfrac{\begin{array}{c} \dfrac{\dfrac{P(f(x),x) \overset{\Sigma}{\vee} Q(z) \vee R(z) \qquad \neg \overset{\Pi}{R}(a)}{R(a) \mid P(f(x),x) \vee Q(a)} \qquad \neg \overset{\Pi}{Q}(u)}{Q(a) \vee R(a) \mid P(f(x),x)} \\ \forall x_a (Q(x_a) \vee R(x_a)) \mid P(f(x),x) \end{array} \qquad \neg \overset{\Pi}{P}(z,a)}{P(f(a),a) \vee \forall x_a (Q(x_a) \vee R(x_a)) \mid \Box} \\ \forall x_a \exists y_{f(a)} \Big(P(y_{f(a)}, x_a) \vee \forall x_a (Q(x_a) \vee R(x_a)) \Big) \mid \Box \end{array}$$