Arrows only in literals

NB: it's possible to introduce a variable occurring grey and colored in a literal:
$P(f(x), y) \lor Q(x) \lor Q(y)$ and factorisation
OR
$P(f(x), y) \lor Q(g(x), h(y)) \qquad \neg Q(g(z), h(z))$ and resolution
However, Lemma 1 with full arrows implies that there are arrows from grey to colored variables in literals.
Suppose $\Gamma \models \mathrm{AI}^{\Delta}(C)$. We show:

- if in the original clause set, a variable occurs grey and colored in a literal, whatever is contained at the position of the grey occurrence is quantified before whatever is contained at the position of the colored literal.
- Quantifier order among other lifting variables is not relevant.

This shows that $\Gamma \models \mathrm{AI}(C)$.
Ad 1)

Conjectured Proposition 1 (Witness terms)**.**

Proof. cf. examples.
* for $P(u, f(u))$, to show $\forall u \exists x_1 P(u, x_1)$, u must be contained in the witness term. Hence we can not prove $\exists x_1 \forall u P(y, x_1)$ (this is not valid).
* for $P(f(u)) \lor Q(u)$, to show $\exists x_1 \forall u (P(x_1) \lor Q(u))$, u does not need to be (can not be?) contained in the witness term. $\forall y \exists x_1 (P(x_1) \lor Q(u))$ is easy to see (as above). $\square$

Thoughts on the proof:
Can't just do it as in Huang, if some existential quantifiers are before universal ones, we cannot argue that it's fine without talking about the formula structure, which can be quite arbitrary. But we can handle the formula structure by induction, possibly a proof that makes use of witness terms.
We know that $\Gamma \models Q_i \, \mathrm{AI}(C_i)$ for $i \in \{1, 2\}$ and $\Gamma \models \mathrm{AI}^{\Delta}(C)$.
Ind.Hyp.:
$\Gamma \models Q_1 \, \mathrm{AI}_{\mathrm{mat}}(C_1) \lor \mathrm{AI}_{\mathrm{cl}}(C_1)^* \lor (l_{\mathrm{AIcl}})_\Gamma$
$\Gamma \models Q_2 \, \mathrm{AI}_{\mathrm{mat}}(C_2) \lor \mathrm{AI}_{\mathrm{cl}}(C_2)^* \lor \neg(l'_{\mathrm{AIcl}})_\Gamma$
In these formulas, some variables are free and hence implicitly quantified. σ adds some terms, which is fine. The lifting lifts them again adding new variables, which is also fine (cf. corresponding Lemma).

Universally quantifying these new variables is generally fine as long as the universal quantifier isn't moved as far inwards that it overtakes an existential quantifier (because else it's just exchanging same quantifiers, which is fine). This is supposed to be exactly the known situation.

it's not easy for existential quantification. TODO: ici

$\Gamma \overset{(\circ)}{\models} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)^*\sigma]\tau \vee \ell[(l_{\mathrm{AIcl}})_\Gamma\sigma]\tau$

$\Gamma \overset{(*)}{\models} \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)^*\sigma]\tau \vee \neg\ell[(l'_{\mathrm{AIcl}})_\Gamma\sigma]\tau$