Contents

0.1 random notes

- As long as every pair of literal is variable disjoint, the quantifier ordering is arbitrary (proof idea: establish that some ordering works, then pull quantifier inwards and back outwards in arbitrary order).
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  - lifted terms which contain variables are disjoint for different clauses, but ground lifted terms can be the same (which does not appear to be necessarily so!)
  - the resolved/factorised literal should be the same (else this kind of proof doesn't go through)
- $\forall x \exists y \varphi \Leftrightarrow \exists y \forall x$ does not hold for formula coding $f(0) = 1$, $f(1) = 0$: $(Z(y) \supset O(x)) \wedge (O(y) \supset Z(x)$, $\mathcal{U} = \{0, 1\}$, $Z/1$ and $O/1$ encode being 0 or 1 respectively.

⟨1⟩ **Lemma 1.** $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$.

⟨2⟩ **Lemma 2.** $\Gamma \models \forall \bar{x}\, \exists \bar{y}\, (\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C))$.

Proof. By 1, $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$.
Hence $\Gamma \models \forall \bar{x}\, (\mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C))$.
and also $\Gamma \models \forall \bar{x}\, \exists \bar{y}\, \ell_{\Gamma}[\mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)]$.
by some lemma then $\Gamma \models \forall \bar{x}\, \exists \bar{y}\, (\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C))$. □

but can't invert this idea:

Let $\hat{\Delta} = \Gamma$ and $\hat{\Gamma} = \Delta$.
Then with $\hat{\pi}$ and 2: $\hat{\Gamma} \models \forall \bar{\hat{x}}\, \exists \bar{\hat{y}}\, (\mathrm{LI}(\bar{\pi}))$
Hence (some lemma) $\Delta \models \forall \bar{y}\, \exists \bar{x}\, (\neg\, \mathrm{LI}(\pi))$.
Hence $\Delta \models \neg \exists \bar{y}\, \forall \bar{x}\, (\mathrm{LI}(\pi))$.
need some consistent ordering, so possibly just prove that all work, because we need to shuffle a lot anyway

example with same lifting var in two children of a connective:

601 – lifting vars interleaved so quantifier pull in/out trick doesn't work

$$\dfrac{\dfrac{\overset{\Sigma}{P(f(x)) \vee S(f(x))} \qquad \overset{\Pi}{\neg P(z) \vee Q(g(y)) \vee R(g(y))}}{P(f(x)) \mid S(f(x)) \vee Q(g(y)) \vee R(g(y))} \qquad \overset{\Sigma}{\neg Q(z)}}{\neg Q(g(y)) \wedge P(f(x)) \mid S(f(x)) \vee R(g(y))}$$

$\Sigma \models \forall u \exists v \big((\neg Q(u_{g(y)}) \wedge P(v_{f(x)})) \;\vee\; S(v_{f(x)}) \vee R(u_{g(y)})\big)$
$\Rightarrow$ not interesting as R is not mentioned, so it collapses.

$\Pi \models \exists u \forall v \big((Q(u_{g(y)}) \vee \neg P(v_{f(x)})) \;\vee\; S(v_{f(x)}) \vee R(u_{g(y)})\big)$

$$\dfrac{\overset{\text{(cont)}}{\neg Q(g(y)) \wedge P(f(x)) \mid S(f(x)) \vee R(g(y))} \qquad \overset{\Pi}{\neg S(x_7)}}{S(f(x)) \vee (\neg Q(g(y)) \wedge P(f(x))) \mid R(g(y))}$$

$\Sigma \models \forall u \exists v \Big(S(v) \vee (\neg Q(u) \vee P(v)) \vee R(u)\Big)$

$\Pi \models \exists u \forall v \Big((\neg S(v_{f(x)}) \wedge (Q(u_{g(y)}) \vee \neg P(v_{f(x)}))) \;\vee\; R(u_{g(y)})\Big)$

Can't see much of interest, but can not apply quantifier pulling in and out trick

same again with direct overbinding:

$$\dfrac{\overset{\Sigma}{\exists v(P(v) \vee S(v))} \qquad \overset{\Pi}{\forall u(\neg P(z) \vee Q(u) \vee R(u))}}{\overset{\exists v}{\forall u}\ (P(v) \mid S(v) \vee Q(u) \vee R(u))}$$

only Δ: $\forall u(P(f(x)) \mid S(f(x)) \vee Q(u) \vee R(u))$

no subterm relation anyway

602 – counterexample with alternating function

$$\dfrac{\dfrac{F(x) \lor \neg Z(\overset{\Sigma}{f}(x)) \lor O(\alpha) \qquad G(y) \lor \overset{\Pi}{\neg} O(g(y))}{O(g(y)) \mid F(x) \lor \neg Z(f(x)) \lor G(y)} \qquad Z(\alpha) \overset{\Pi}{\lor} M\beta}{O(g(y)) \lor \neg Z(f(x)) \mid F(x) \lor G(y) \lor M(\beta)}$$

$$\dfrac{\dfrac{F(x') \lor Z(\alpha) \overset{\Sigma}{\lor} \neg O(f(x')) \qquad G(y') \lor \overset{\Pi}{\neg} Z(g(y'))}{Z(g(y')) \mid F(x') \lor \neg O(f(x')) \lor G(y')} \qquad O(\alpha) \overset{\Pi}{\lor} \neg M(\beta)}{Z(g(y')) \lor \neg O(f(x')) \mid F(x') \lor G(y') \lor M(\beta)}$$

conbining:

$$\dfrac{(Z(g(y')) \lor \neg O(f(x'))) \quad \land \quad (O(g(y)) \lor \neg Z(f(x))) \quad \mid \quad F(x) \lor G(y) \lor F(x') \lor G(y')}{(Z(g(y)) \lor \neg O(f(x))) \quad \land \quad (O(g(y)) \lor \neg Z(f(x))) \quad \mid \quad F(x) \lor G(y)}$$

interpolant is lifted version:
$\forall y_g \exists y_f \Big((Z(y_g) \lor \neg O(y_f)) \quad \land \quad (O(y_g) \lor \neg Z(y_f)) \quad \mid \quad F(x) \lor G(y) \Big)$

602a: with constants

$$\dfrac{\dfrac{\neg Z(a) \overset{\Sigma}{\lor} O(\alpha) \qquad \neg \overset{\Pi}{O}(b)}{O(b) \mid \neg Z(a)} \qquad Z(\alpha) \overset{\Pi}{\lor} M\beta}{O(b) \lor \neg Z(a) \mid M(\beta)}$$

In such cases, we always have $O(\alpha)$, i.e. something universally quantified