Chapter 1

The semantic perspective on interpolation

The proofs of the interpolation theorem in the preceding chapters have a common feature: They are proof-theoretic approaches.

TODO: finish this when the chapter is done content-wise

Theorem 1.1. *interpolation again*

Proof. Suppose that there is no interpolant for Γ and Δ We show that then, $\Gamma \cup \Delta$ is satisfiable by constructing a model.

Let $L_0 = \mathrm{L}(\Gamma) \cap \mathrm{L}(\Delta)$

Note that by assumption there is no formula I such that $\Gamma \vDash I$ and $\Delta \vDash \neg I$.

Let $\Phi = \varphi_1, \varphi_2, \ldots$ be an enumeration of all formulas in the language $\mathrm{L}(\Gamma)$ and $\Psi = \psi_1, \psi_2, \ldots$ be an enumeration of all formulas in the language $\mathrm{L}(\Delta)$.

Let $\Gamma_0 = \Gamma$ and $\Delta_0 = \Delta$. Let $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i\}$ if there is no interpolant for $\Gamma_i \cup \{\varphi_i\}$ and Δ_i, and $\Gamma_{i+1} = \Gamma_i$ otherwise. Let $\Delta_{i+1} = \Delta_i \cup \{\psi_i\}$ if there is no interpolant for $\Delta_i \cup \{\psi_i\}$ and Γ_i, and $\Delta_{i+1} = \Delta_i$ otherwise.

Let Γ_ω be the limit of the sequence of the Γ_i and Let Δ_ω be the limit of the sequence of the Δ_i.

Note that Γ_ω and Δ_ω are ??? (complete, maximal consistent). Suppose for a formula χ of language ??? that neither $\chi \in \Gamma_\omega$ nor $\neg\chi \in \Gamma_\omega$.

As then $\chi \notin \Gamma_\omega$, but the construction considers all formulas of appropriate language, it must hold that there is an interpolant $\Gamma_i \cup \{\chi\}$ and Δ_i for some i. Let I denote this interpolant.

Hence we obtain that $\Gamma_\omega \vDash \chi \supset I$ and $\Delta_\omega \vDash \neg I$.

As also $\neg\chi \notin \Gamma_\omega$, by a similar argument, we obtain that there must be an interpolant J for $\Gamma_j \cup \{\neg\chi\}$ and Δ_j for some j and consequently $\Gamma_\omega \vDash \neg\chi \supset J$ and $\Delta_\omega \vDash \neg J$.

But then $\Gamma_\omega \vDash (\chi \supset I) \wedge (\neg\chi \supset J)$, which implies $\Gamma_\omega \vDash I \vee J$, and also $\Delta \vDash \neg(I \vee J)$, thus $I \vee J$ is, in contradiction with the assumption, an interpolant for Γ_ω and Δ_ω.

By analogous argument, Δ_ω is ??? as well.

Note that for each formula α in $\mathrm{L}(\Gamma) \cap \mathrm{L}(\Delta)$, both Γ_ω and Δ_ω contain either α or $\neg\alpha$. Moreover, as there is no interpolant for Γ_ω and Δ_ω, they either both contain α or both contain $\neg\alpha$. Hence they agree on every formula in $\mathrm{L}(\Gamma) \cap \mathrm{L}(\Delta)$, and as both Γ_ω and Δ_ω are consistent, their reduct to this language is consistent.

But as $\Gamma \subseteq \Gamma_\omega$ and $\Delta \subseteq \Delta_\omega$, there is a model of $\Gamma \cup \Delta$.

$\square$

1.1 notes on shoenfield

$T[\Gamma]$:

Γ is a set of formulas in the theory T

$T[\Gamma]$: theory obtained from T with Γ as additional (nonlogical) axioms

Extension of theory: may contains additional symbols and new theorems, and every symbol/theorem of the old language/theory is a symbol/theorem of the new language/theory

elementary: from wikipedia: if all formulas of the language of the larger one with variables interpreted with elements of the smaller language hold in the smaller iff they hold in the larger.

Conservative extension of theory: T' is conservative extension of T if formulas in T which are theorems of T' are theorems in T.

Hence additional formulas may only be theorems if they involve new symbols.

Reduction theorem for consistency: if a set of formulas in a theory is inconsistent, then there is a disjunction of negated formulas of the set which is a theorem of T

Chain: a sequence of structure, such that every one is an extension of the former

a chain is elementary if every element is an elementary extension of the former

Tarksi's Lemma: the union of an elementary chain is an elementary extension of each member

Regular set of formulas: every $x = y$ or $x \neq y$ is contained, and for every formula $A[x_1, \ldots, x_n]$ is contained (does not say if x_i is var or term)