Contents

_order_not_relevant⟩? **Lemma 1.** $\ell_\Gamma[\ell_\Delta[\varphi]] = \ell_\Delta[\ell_\Gamma[\varphi]]$.

CHAPTER 1

Interpolant extraction from resolution proofs in one phase lifting terms whose quantifier position can be determined – nested

now part of thesis, do not change here

Definition 2 (Substitution $\tau(\iota)$)**.** For an inference ι with $\sigma = \mathrm{mgu}(\iota)$, we define the infinite substitution $\tau(\iota)$ with $\mathrm{dom}(\tau(\iota)) = \mathrm{dom}(\sigma) \cup \{z_s \mid s\sigma \neq s\}$ as follows for a variable x:

$$x\tau(\iota) = \begin{cases} x\sigma & x \text{ is a non-lifting variable} \\ z_{t\sigma} & x \text{ is a lifting variable } z_t \end{cases}$$

If the inference ι is clear from the context, we abbreviate $\tau(\iota)$ by τ. △

:lifting_tau_commute⟩ **Lemma 3.** *For a formula or term φ and an inference ι such that $\tau = \tau(\iota)$, $\ell[\ell[\varphi]\tau] = \ell[\varphi\tau]$.*

Proof. We proceed by induction.

- Suppose that t is a grey constant or function symbol of the form $f(t_1, \ldots, t_n)$. Then we can derive the following, where (IH) signifies a deduction by virtue of the induction hypothesis.

$$\begin{aligned}\ell[\ell[t]\tau] &= \ell[\ell[f(t_1,\dots,t_n)]\tau]\\ &= \ell[f(\ell[t_1]\tau,\dots,\ell[t_n]\tau)]\\ &= f(\ell[\ell[t_1]\tau],\dots,\ell[\ell[t_n]\tau])\\ &\overset{(\mathrm{IH})}{=} f(\ell[t_1\tau],\dots,\ell[t_n\tau])\\ &= \ell[f(t_1,\dots,t_n)\tau]\\ &= \ell[t\tau]\end{aligned}$$

- Suppose that t is a colored constant or function symbol. Then:

$$\ell[\ell[t]\tau] = \ell[z_t\tau] = \ell[z_{t\sigma}] = z_{t\sigma} = \ell[t\sigma] = \ell[t\tau]$$

- Suppose that t is a variable x. Then:

$$\ell[\ell[t]\tau] = \ell[\ell[x]\tau] = \ell[x\tau] = \ell[t\tau]$$

- Suppose that t is a lifting variable z_t. Then:

$$\ell[\ell[z_t]\tau] = \ell[z_t\tau]$$ □

Definition 4 (Incrementally lifted interpolant)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$. We define $\mathrm{LI}(\pi)$ to be $\mathrm{LI}(\Box)$, where $\Box$ is the empty clause derived in π.

Let C be a clause in π.

We define the preliminary formula $\mathrm{LI}^\bullet(C)$ as follows:

Base case. If $C \in \Gamma$, $\mathrm{LI}(C) \overset{\mathrm{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{LI}(C) \overset{\mathrm{def}}{=} \top$.

Resolution.

If the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then define $\mathrm{LI}(C)$ as follows:

1. If l is Γ-colored: $\mathrm{LI}^\bullet(C) \overset{\mathrm{def}}{=} \mathrm{LI}(C_1)\tau \vee \mathrm{LI}(C_2)\tau$
2. If l is Δ-colored: $\mathrm{LI}^\bullet(C) \overset{\mathrm{def}}{=} \mathrm{LI}(C_1)\tau \wedge \mathrm{LI}(C_2)\tau$
3. If l is grey: $\mathrm{LI}^\bullet(C) \overset{\mathrm{def}}{=} (l\tau \wedge \mathrm{LI}(C_2)\tau) \vee (\neg l'\tau \wedge \mathrm{LI}(C_1)\tau)$

Factorisation. If the clause C is the result of a factorisation step ι of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{LI}^\bullet(C) \overset{\mathrm{def}}{=} \ell[\mathrm{LI}(C_1)\tau]$.

Paramodulation. If the clause C is the result of a paramodulation step ι of $C_1 : s = t \vee D$ and $C_2 : E[r]$ with $\sigma = \mathrm{mgu}(\iota)$. Let $h[r]$ be the maximal colored term in which r occurs in $E[r]$ and define $\mathrm{LI}(C)$ as follows:

1. If $h[r]$ is Δ-colored and $h[r]$ occurs more than once in $\mathrm{LI}(C_2)\tau \vee E[r]\tau$:
$\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} (s = t \wedge \mathrm{LI}(C_2))\tau \vee (s \neq t \wedge \mathrm{LI}(C_1))\tau \vee (s = t \wedge h[s] \neq h[t])\tau$
2. If $h[r]$ is Γ-colored and $h[r]$ occurs more than once in $\mathrm{LI}(C_2)\tau \vee E[r]\tau$:
$\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} [(s = t \wedge \mathrm{LI}(C_2))\tau \vee (s \neq t \wedge \mathrm{LI}(C_1))\tau] \wedge (s \neq t \vee h[s] = h[t])\tau$
3. If r does not occur in a colored term in $E[r]$ which occurs more than once in $\mathrm{LI}(C_2)\tau \vee E[r]\tau$:
$\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} (s = t \wedge \mathrm{LI}(C_2))\tau \vee (s \neq t \wedge \mathrm{LI}(C_1))\tau$

$\mathrm{LI}(C)$ is built from $\mathrm{LI}^\bullet(C)$ according to the following steps:

1. Lift all maximal colored occurrences of colored terms t in $\mathrm{LI}^\bullet(C)$ for which at least one of the following conditions, referred to as *lifting conditions*, applies:
   - The term t contains some variable x such that x does not occur in C.
   - The term t is ground and C does not contain t.
2. Let X (Y) be the Δ-(Γ-)lifting variables created in the previous step.
3. Prefix the resulting formula with an arrangement $Q(C)$ of the elements of $\{\forall x_t \mid x_t \in X\} \cup \{\exists y_t \mid y_t \in Y\}$ such that if s and r are terms such that s is a subterm of r, then z_s precedes z_r. $\triangle$

:substitute_and_lift⟩ **Lemma 5.** *Let σ be a substitution and F a formula without Φ-colored terms such that for a set of formulas Ψ which does not contain Φ-lifting variables, $\Psi \models F$. Then $\Psi \models \ell_\Phi[F\sigma]$.*

Proof. $\ell_\Phi[F\sigma]$ is an instance of F: σ substitutes variables either for terms which do not contain Φ-colored symbols or by terms containing Φ-colored symbols. For the first kind, the lifting has no effect. For the latter, the lifting only replaces subterms of the terms introduced by the substitution by a lifting variable such that the original structure of F remains invariant as it by assumption does not contain colored terms. $\square$

Lemma 6. *If a variable x occurs in $\mathrm{LI}(C)$ but not in C, then x does not occur in the subsequent resolution derivation.*

Lemma 7. *If a ground term t occurs in* $\mathrm{LI}(C)$ *but not in C, and a successor of C contains t, then "the second t has been derived from first principles in a different way".*

lta_lifted_invariant⟩ **Lemma 8.** *Let C be a clause in a resolution refutation of* $\Gamma \cup \Delta$. *Then* $\Gamma \models \ell_\Delta[\mathrm{LI}(C)] \vee \ell_\Delta[C]$

Proof. We show the strengthening $\Gamma \models \ell_\Delta[\mathrm{LI}(C)] \vee \ell_\Delta[C_\Gamma]$[1].

As a first step, we prove by induction that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

Base case. If $C \in \Gamma$, then $\ell_\Delta[C] = C$ and clearly $\Gamma \models C$. If otherwise $C \in \Delta$, then $\mathrm{LI}(C) = \top$.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

By the induction hypothesis we obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)] \vee \ell_\Delta[D_\Gamma] \vee \ell_\Delta[l_\Gamma]$ as well as $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)] \vee \ell_\Delta[E_\Gamma] \vee \neg \ell_\Delta[l'_\Gamma]$. Hence by Lemma 5 and Lemma 3, we get:

$$\Gamma \overset{(\circ)}{\models} \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[D_\Gamma \tau] \vee \ell_\Delta[l_\Gamma \tau]$$

$$\Gamma \overset{(*)}{\models} \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[E_\Gamma \tau] \vee \neg \ell_\Delta[l'_\Gamma \tau]$$

As $l_\Gamma \sigma = l'_\Gamma \sigma$, it holds that $l_\Gamma \tau = l'_\Gamma \tau$ and consequently $\ell_\Delta[l_\Gamma \tau] = \ell_\Delta[l'_\Gamma \tau]$. We proceed by a case distinction on the color of the resolved literal to show that in each case, we have that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$:

- Suppose that l is Γ-colored. Then $l_\Gamma = l$ and $l'_\Gamma = l$, and we can perform a resolution step on $(\circ)$ and $(*)$ to obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[D_\Gamma \tau] \vee \ell_\Delta[E_\Gamma \tau]$. This however is nothing else than $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that l is Δ-colored. Then $(\circ)$ and $(*)$ reduce to $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[D_\Gamma \tau]$ and $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[E_\Gamma \tau]$ respectively, which clearly implies that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee (\ell_\Delta[D_\Gamma \tau] \wedge \ell_\Delta[E_\Gamma \tau])$. This is turn is however just the unfolding of $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that l is grey. Then $l_\Gamma = l$ and $l'_\Gamma = l$, and $(\circ)$ and $(*)$ imply that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee (\ell_\Delta[l_\Gamma \tau] \wedge \ell_\Delta[E_\Gamma \tau]) \vee (\neg \ell_\Delta[l'_\Gamma \tau] \wedge \ell_\Delta[D_\Gamma \tau])$. This however is equivalent to $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

[1] Recall that D_Φ denotes the clause created from the clause D by removing all literals which are not contained $\mathrm{L}(\Phi)$.

Factorisation. Suppose the clause C is the result of a factorisation inference ι of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

The induction hypothesis gives $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)] \vee \ell_\Delta[l_\Gamma \vee l'_\Gamma \vee D_\Gamma]$. By Lemma 5, we obtain $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[l_\Gamma\tau \vee l'_\Gamma\tau \vee D_\Gamma\tau]$. As however $l\sigma \equiv l'\sigma$, also $l\tau \equiv l'\tau$, so we can apply a factorisation step and obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[l_\Gamma\tau \vee D_\Gamma\tau]$, which is nothing else than $\Gamma \models \mathrm{LI}^\bullet(C) \vee \ell_\Delta[C_\Gamma]$.

Paramodulation. Suppose the clause C is the result of a paramodulation inference ι of $C_1 : s = t \vee D$ and $C_2 : E[r]_p$ with $\sigma = \mathrm{mgu}(\iota)$.

By the induction hypothesis, we obtain the following:

$$\Gamma \overset{(\circ)}{\models} \ell_\Delta[\mathrm{LI}(C_1)] \vee \ell_\Delta[D_\Gamma] \vee \ell_\Delta[s] = \ell_\Delta[t]$$

$$\Gamma \overset{(*)}{\models} \ell_\Delta[\mathrm{LI}(C_2)] \vee \ell_\Delta[(E[r]_p)_\Gamma]$$

Suppose now that for a model M of Γ and an assignment α of the free variables of $\ell_\Delta[s]$ and $\ell_\Delta[t]$ that $M_\alpha \models \ell_\Delta[s] \neq \ell_\Delta[t]$. Then we get by $(\circ)$ that $M_\alpha \models \ell_\Delta[\mathrm{LI}(C_1)] \vee \ell_\Delta[D_\Gamma]$, which by Lemma 5 and Lemma 3 gives $M_\alpha \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[D_\Gamma\tau]$. Note that $M_\alpha \models \ell_\Delta[s\tau] \neq \ell_\Delta[t\tau] \wedge \ell_\Delta[\mathrm{LI}(C_1)\tau]$ suffices for $M_\alpha \models \ell_\Delta[\mathrm{LI}^\bullet(C)]$ and $M_\alpha \models \ell_\Delta[D_\Gamma\tau]$ implies that $M_\alpha \models \ell_\Delta[C_\Gamma]$. Therefore we obtain that $M_\alpha \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

Now suppose to the contrary that for a model M of Γ that for any assignment of free variables $M \models \ell_\Delta[s] = \ell_\Delta[t]$.

By Lemma 5 and Lemma 3 we obtain from $(*)$ that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[(E[r]_p)_\Gamma\tau]$. As however $\sigma = \mathrm{mgu}(r,s)$, $r\tau \equiv s\tau$ and hence $\ell_\Delta[r\tau] \equiv \ell_\Delta[s\tau]$. Therefore we also have that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[(E[s]_p)_\Gamma\tau]$.

We proceed by a case distinction:

- Suppose that the position p in $E[s]_p$ is not contained in a Δ-term. Then $\ell_\Delta[(E[s]_p)_\Gamma\tau]$ and $\ell_\Delta[(E[t]_p)_\Gamma\tau]$ only differ at at position p. As $M \models \ell_\Delta[s] = \ell_\Delta[t]$, we can apply Lemma 5 and Lemma 3 to obtain that $M \models \ell_\Delta[s\tau] = \ell_\Delta[t\tau]$. Thus $M \models \ell_\Delta[(E[s]_p)_\Gamma\tau] \Leftrightarrow \ell_\Delta[(E[t]_p)_\Gamma\tau]$ and consequently $M \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[(E[t]_p)_\Gamma\tau]$. As furthermore $\ell_\Delta[\tau s] = \ell_\Delta[t] \wedge \ell_\Delta[\mathrm{LI}(C_2)\tau]$ entails $\ell_\Delta[\mathrm{LI}^\bullet(C)]$ and $\ell_\Delta[(E[t]_p)_\Gamma\tau]$ is sufficient for $\ell_\Delta[C_\Gamma]$, we have that $M \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that the position p in $E[s]_p$ is contained in a maximal Δ-term $h[s]$. We distinguish further:
  - Suppose $h[s]$ occurs more than once in $\mathrm{LI}(C_2)\tau \vee E[s]_p\tau$ and let α be an arbitrary assignment to the variables $\ell_\Delta[h[s]] = x_{h[s]}$ and $\ell_\Delta[h[t]] = x_{h[t]}$.

If $M_\alpha \models \ell_\Delta[h[s]] \neq \ell_\Delta[h[t]]$, then we have that $M_\alpha \models \ell_\Delta[s] = \ell_\Delta[t] \wedge \ell_\Delta[h[s]] \neq \ell_\Delta[h[t]]$, which implies that $M_\alpha \models \ell_\Delta[\mathrm{LI}^\bullet(C)]$.
Otherwise it holds that $M_\alpha \models \ell_\Delta[h[s]] = \ell_\Delta[h[t]]$. But then $\ell_\Delta[(E[s]_p)_\Gamma\tau]$ and $\ell_\Delta[(E[t]_p)_\Gamma\tau]$ differ in subterms which are equal in the given model and with the given interpretation of the free variables, so by a similar line of argument as in the preceding case, we can deduce that $M \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C]$.

– Suppose $h[s]$ occurs exactly once in $\mathrm{LI}(C_2)\tau \vee E[s]_p\tau$. Then the lifting variable $x_{h[s]}$ occurs exactly once in $\ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[E[s]_p\tau]$.
Note that from $(*)$ by applying Lemma 5 as well as Lemma 3, we obtain that $M \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[(E[s]_p)_\Gamma\tau]$. As $x_{h[s]}$ occurs only once and free in this formula, it is implicitly universally quantified and we can instantiate it arbitrarily, in particular by $z_{h[t]}$. But thereby we get that $M \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[(E[t]_p)_\Gamma\tau]$, which implies that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

As we have now established that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$, we show that also $\Gamma \models \ell_\Delta[\mathrm{LI}(C)] \vee \ell_\Delta[C_\Gamma]$ holds.

The difference between $\ell_\Delta[\mathrm{LI}^\bullet(C)]$ and $\ell_\Delta[\mathrm{LI}(C)]$ lies only in certain maximal colored terms which are lifted and the resulting lifting variable bound in $\ell_\Delta[\mathrm{LI}(C)]$, hence it suffices to consider these. Let t be a colored term in $\mathrm{LI}^\bullet(C)$ at position p such that $\mathrm{LI}(C)|_p = \ell[t]$. Then t is a maximal colored term.

If t is Δ-colored, then $\ell_\Delta[\mathrm{LI}^\bullet(C)]|_p = \mathrm{LI}(C)|_p = x_t$. Note that as t occurs at p in $\mathrm{LI}^\bullet(C)$, x_t occurs free at $\ell_\Delta[\mathrm{LI}^\bullet(C)]|_p$. Hence it is implicitly universally quantified and therefore entails that an explicit universal quantification in $\mathrm{LI}(C)$ is valid with an arbitrarily placed universal quantifier.

If otherwise t is a Γ-term, then $\ell_\Delta[\mathrm{LI}^\bullet(C)]|_p = \ell_\Delta[t]$. Hence $\ell_\Delta[t]$ represents a witness term for the existentially quantified lifting variable y_t at $\mathrm{LI}(C)|_p$. In general, $\ell_\Delta[t]$ however contains Δ-lifting variables, hence for $\ell_\Delta[t]$ to be a valid witness term, these have to be bound such that the existential quantifier of y_t is in their scope. Note that occurrences of colored terms which are not maximal colored terms are not actually lifted in LI.

Let x_s be a Δ-lifting variable which occurs in $\ell_\Delta[t]$. We show that y_t is quantified in the scope of the quantification of x_s by discussing the different possibilities for quantification of x_s:

- Clearly if s or a respective successor is never bound due to not occurring at a maximal colored position, it is implicitly universally quantified.

- If s or a respective successor does occur at a maximal colored position but does not satisfy any of the lifting conditions up to the stage where t is lifted, it is bound at some later stage of the interpolant extraction, but as for any successor C' of C, $\mathrm{LI}(C)$ is contained in $\mathrm{LI}(C')$, the scope of its quantifier encompasses the quantifier for y_t.

- In the case that s and t are lifted at the same stage of the interpolant extraction, by the definition of the quantifier prefix, the quantification of x_s precedes the quantification for x_t as s is a subterm of t.
- It is furthermore essential to see that neither s nor a respective predecessor is lifted in a previous step of the interpolant extraction. Suppose to the contrary that this is the case in the inference creating the clause C'. Let s' and t' be the respective predecessors of s and t in C'.
  - Suppose that t is lifted due to containing a variable which does not occur in C. Then as s' is a subterm of t', t' contains this variable as well and would therefore be lifted in $\mathrm{LI}(C')$.
  - Suppose that t is lifted due to being a ground term which does not occur in C. Then s' does not occur in C'. But as t' contains s', t' also does not occur in C' and would therefore be lifted in $\mathrm{LI}(C')$. □

Note the the lifting conditions ensure that only terms are lifted, which do not exhibit a direct logical relation with any term in the remaining clause. More precisely, they do not influence the subsequent resolution derivation: If a variable x occurs in $\mathrm{LI}(C)$ but not in C, then as clauses are variable-disjoint, the variable x does not occur in any other clause. For ground terms r however which occur in $\mathrm{LI}(C)$ but not in C, it is possible for them to cooccur in a subsequent clause. Let p be the occurrence of r in $\mathrm{LI}(C)$ and q the occurrence of r in a successive clause of C. Then due to the fact that p is not used in any unification, q must have been created or originated from other occurrences of the same function and/or constant symbols. Note that the lifting conditions ensure that for these, the order of the quantifiers of their respective lifting variables is established in a fashion appropriate to ensuring the logical validity of the interpolant.

⟨lemma:li_symmetry⟩ **Lemma 9.** *Let π be a refutation of $\Gamma \cup \Delta$ and $\hat{\pi}$ be π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then for a clause C in π and its corresponding clause $\hat{C}$ in $\hat{\pi}$,* $\mathrm{LI}(C) \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$.

Proof. We proceed by induction to show that $\mathrm{LI}^\bullet(C) \Leftrightarrow \neg\,\mathrm{LI}^\bullet(\hat{C})$:

Base case. If $C \in \Gamma$, then $\mathrm{LI}(C) = \bot \Leftrightarrow \neg\top \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$ as $\hat{C} \in \Delta$. The case for $C \in \Delta$ can be argued analogously.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

As τ depends only on σ, τ is the same for both π and $\hat{\pi}$.

We now distinguish the following cases:

1. l is Γ-colored:

$$\begin{aligned}\mathrm{LI}^\bullet(C) &= \mathrm{LI}(C_1)\tau \vee \mathrm{LI}(C_2)\tau \\ &\Leftrightarrow \neg(\neg\mathrm{LI}(C_1)\tau \wedge \neg\mathrm{LI}(C_2)\tau) \\ &\Leftrightarrow \neg(\mathrm{LI}(\hat{C}_1)\tau \wedge \mathrm{LI}(\hat{C}_2)\tau) \\ &= \neg\,\mathrm{LI}^\bullet(\hat{C})\end{aligned}$$

2. l is Δ-colored: This case can be argued analogously.
3. l is grey: Note $l\tau \overset{(*)}{=} l'\tau$.

$$\begin{aligned}\mathrm{LI}^\bullet(C) &= (\neg l'\tau \wedge \mathrm{LI}(C_1)\tau) \vee (l\tau \wedge \mathrm{LI}(C_2)\tau) \\ &\overset{(*)}{\Leftrightarrow} (l'\tau \vee \mathrm{LI}(C_1)\tau) \wedge (\neg l\tau \vee \mathrm{LI}(C_2)\tau) \\ &\Leftrightarrow \neg[(\neg l'\tau \wedge \neg\mathrm{LI}(C_1)\tau) \vee (l\tau \wedge \neg\mathrm{LI}(C_2)\tau)] \\ &= \neg[(\neg\hat{l}'\tau \wedge \mathrm{LI}(\hat{C}_1)\tau) \vee (\hat{l}\tau \wedge \mathrm{LI}(\hat{C}_2)\tau)] \\ &= \neg\,\mathrm{LI}^\bullet(\hat{C})\end{aligned}$$

Factorisation. Suppose the clause C is the result of a factorisation ι of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

As the construction is not influenced by the coloring, the induction hypothesis $\mathrm{LI}^\bullet(C) = \mathrm{LI}(C_1)\tau$ suffices.

Then $\mathrm{LI}^\bullet(C) = \ell[\mathrm{LI}(C_1)\tau]$, so the construction is not influenced by the coloring and by the induction hypothesis, $\mathrm{LI}^\bullet(C) \Leftrightarrow \neg\,\mathrm{LI}^\bullet(\hat{C})$.

Paramodulation. Suppose the clause C is the result of a paramodulation inference ι of $C_1 : s = t \vee D$ and $C_2 : E[r]_p$ with $\sigma = \mathrm{mgu}(\iota)$.

We proceed by a case distinction:

- Suppose that p in $E[r]_p$ is contained in a maximal Δ-term $h[r]$, which occurs more than once in $E[r]_p \vee \mathrm{LI}(E[r]_p)$. Then p in $\hat{E}[r]_p$ is contained in a maximal Γ-term $h[r]$, which occurs more than once in $\hat{E}[r]_p \vee \mathrm{LI}(\hat{E}[r]_p)$.

$$
\begin{aligned}
\mathrm{LI}^{\bullet}(C) &= (s\tau = t\tau \wedge \mathrm{LI}(C_2)\tau) \vee (s\tau \neq t\tau \wedge \mathrm{LI}(C_1)\tau) \vee (s\tau = t\tau \wedge h[s]\tau \neq h[t]\tau) \\
&\Leftrightarrow \neg[(s\tau \neq t\tau \vee \neg\mathrm{LI}(C_2)\tau) \wedge (s\tau = t\tau \vee \neg\mathrm{LI}(C_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&= \neg[(s\tau \neq t\tau \vee \mathrm{LI}(\hat{C}_2)\tau) \wedge (s\tau = t\tau \vee \mathrm{LI}(\hat{C}_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&\Leftrightarrow \neg[(s\tau = t\tau \wedge \mathrm{LI}(\hat{C}_2)\tau) \vee (s\tau \neq t\tau \wedge \mathrm{LI}(\hat{C}_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&= \neg\,\mathrm{LI}^{\bullet}(\hat{C})
\end{aligned}
$$

- Suppose that p in $E[r]_p$ is contained in a maximal Γ-term $h[r]$, which occurs more than once in $E[r]_p \vee \mathrm{LI}(E[r]_p)$. This case can be argued analogously.
- Otherwise:

$$
\begin{aligned}
\mathrm{LI}^{\bullet}(C) &= (s\tau = t\tau \wedge \mathrm{LI}(C_2)\tau) \ \vee\ (s\tau \neq t\tau \wedge \mathrm{LI}(C_1)\tau) \\
&\Leftrightarrow \neg[(s\tau \neq t\tau \vee \neg\mathrm{LI}(C_2)\tau) \ \wedge\ (s\tau = t\tau \vee \neg\mathrm{LI}(C_1)\tau)] \\
&= \neg[(s\tau \neq t\tau \vee \mathrm{LI}(\hat{C}_2)\tau) \ \wedge\ (s\tau = t\tau \vee \mathrm{LI}(\hat{C}_1)\tau)] \\
&\Leftrightarrow \neg[(s\tau = t\tau \wedge \mathrm{LI}(\hat{C}_2)\tau) \ \vee\ (s\tau \neq t\tau \wedge \mathrm{LI}(\hat{C}_1)\tau)] \\
&= \neg\,\mathrm{LI}^{\bullet}(\hat{C})
\end{aligned}
$$

We conclude by showing that $\mathrm{LI}^{\bullet}(C) \Leftrightarrow \neg\,\mathrm{LI}^{\bullet}(\hat{C})$ entails that $\mathrm{LI}(C) \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$: Clearly the terms to be lifted in $\mathrm{LI}^{\bullet}(C)$ and $\mathrm{LI}^{\bullet}(\hat{C})$ are the same and differ only in their color. Even though this results in different lifting variables, that is of no relevance as all lifted variables are instantly bound. Additionally, the quantifier type of any given lifting variable in $Q(C)$ is dual to the respective one in $Q(\hat{C})$. Furthermore note that the subterm-relation is not affected by the coloring, so the ordering of the quantifiers in $Q(C)$ and $Q(\hat{C})$ is identical. Hence $\mathrm{LI}(C) \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$. $\square$

mma:delta_entails_li⟩ **Lemma 10.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then $\Delta \models \neg\ell_{\Gamma}[\mathrm{LI}(C)] \vee \ell_{\Gamma}[C]$.*

Proof. Construct $\hat{\pi}$ with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then by Lemma 8, $\hat{\Gamma} \models \ell_{\hat{\Delta}}[\mathrm{LI}(\hat{C})] \vee \ell_{\hat{\Delta}}[\hat{C}]$, which by Lemma 9 is nothing else than $\Delta \models \neg\ell_{\Gamma}[\mathrm{LI}(C)] \vee \ell_{\Gamma}[C]$. $\square$

Theorem 11. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{LI}(\pi)$ is an interpolant of Γ and Δ.*

Proof. We obtain by Lemma 8 that $\Gamma \models \ell_{\Delta}[\mathrm{LI}(\pi)]$ and by Lemma 10 that $\Delta \models \neg\ell_{\Gamma}[\mathrm{LI}(\pi)]$. As the empty clause derived in π trivially contains neither variables nor ground terms, all colored terms are lifted in $\mathrm{LI}(\pi)$. Therefore $\ell_{\Delta}[\mathrm{LI}(\pi)] = \mathrm{LI}(\pi)$ and $\ell_{\Gamma}[\mathrm{LI}(\pi)] = \mathrm{LI}(\pi)$. $\square$