1 version with merging of reachable components

TODO: CHECK UP ON THIS PROOF, then tackle next two.

TODO: Fix up major proof below with the three new lemmas

Conjectured Lemma 1. *Let x be a variable in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ which has a grey occurrence and a colored occurrence. Then for every grey occurrence $\hat{x}$ of x, there is an path from a term containing some grey occurrence to a term containing $\hat{x}$ using arrows from $\mathcal{A}(C)$.*

Proof. Proof by induction; base case obvious; suppose resolution with usual notation. Suppose the paths exist for C_1 and C_2.

We consider the different possibilities of introduction of colored occurrences of x and show that in each of them, there is a path from a term containing a grey occurrence to a term containing the colored occurrence.

Suppose x is introduced into a maximal colored term t by means of unification. So $t\sigma$ is a term containing x. Let y be a variable in t such that $y\sigma$ is a term containing x.

Let $\hat{y}$ be the position of y which causes the variable to be changed by the unification algorithm. $\hat{y}$ is in a resolved literal, say l, so we denote it by $l|_{\hat{y}}$ and its counterpart in l' by $l'|_{\hat{y}}$

- Suppose $l|_{\hat{y}}$ is a grey occurrence.

  figure: $P(f(y)) \vee Q(\hat{y}) \qquad \neg Q(\cdot)$

  Then by the induction hypothesis, there is an path from a term containing some grey occurrence of y to a term containing y in t. After applying σ, the path leads from $y\sigma[x]$ to $t\sigma[x]$. If $y\sigma[x]$ has a grey occurrence of x, we are done. Otherwise it has a colored occurrence of x. But as $l|_{\hat{y}}\sigma = l'|_{\hat{y}}\sigma$, there is a colored occurrence of x in C_2. If there also is a grey occurrence, then by the induction hypothesis, there is an arrow from some grey occurrence of x to $l'|_{\hat{y}}$ and hence there is a path from that grey occurrence to $t\sigma[x]$. If there is no grey occurrence of x in C_2, suppose that x originates from C_1 and there is a grey occurence of x in C_1, as otherwise we are done. As $l'|_{\hat{y}}\sigma$ contains x but $l|_{\hat{y}}\sigma$, x must occur in l, say at $\dot{x}$ and its corresponding term in l' is a variable, say z, such that $z\sigma = x$. z also occurs in $l'|_{\hat{y}}\sigma$.

  - Suppose $l|_{\dot{x}}$ is a grey occurrence. Then $l'|_{\dot{x}}$ is so as well and by the induction hypothesis, there is a path from a term containing a grey occurrence of z in C_1 to $l'|_{\hat{y}}\sigma$ and we are done.
  - Otherwise $l|_{\dot{x}}$ is a colored occurrence. Then so is $l'|_{\dot{x}}$ and by Lemma 2, there is a merge edge between a term containing $'l|_{\dot{x}}$ and a term containing $l'|_{\hat{y}}\sigma$.

As there is a grey occurrence of x in C_1, by the induction hypothesis, there is a path from a term containing a grey occurrence of x to $'l|_{\dot{x}}$ and we are done.

- Suppose $l|_{\hat{y}}$ is contained in a maximal colored term, say $s[y]$. Then in case $t[y] \neq s[y]$, by Lemma 2, there is a merge arrow between a term containing $t[y]$ and a term containing $s[y]$.

  As the arrows from terms containing $l|_{\hat{y}}$ and the corresponding terms containing $l'|_{\hat{y}'}$ are merged, we need to show that there is an arrow in the other clause. $l'|_{\hat{y}}$ is an abstraction from a term containing x.

  (⋇)

  - Suppose $l'|_{\hat{y}}$ contains x.

    If there is a grey occurrence of x in C_2, we are done as the arrows between $l|_{\hat{y}}$ and $l'|_{\hat{y}}$ are merged and by the induction hypothesis, $x \rightsquigarrow l'|_{\hat{y}}$.

    If there is a grey occurrence of y in C_1 and $y\sigma$ contains a grey occurrence of x, we are done as by the induction hypothesis, $y \rightsquigarrow t[y]$, and after applying σ, this path leads from a grey occurrence of x to $t[y]\sigma$.

    Otherwise there are no grey occurrences of x and there is nothing to prove.

  - Suppose $l'|_{\hat{y}}$ does not contain x. Then it contains a variable v such that $v\sigma$ is a term containing x.

    As $l'|_{\hat{y}}$ is by assumption contained in a colored term, where x is introduced, we know that there is an appropriate arrow by Remark $(*)$.

Suppose a term containing $t[x]$ with t colored is in $\mathrm{ran}(\sigma)$. So $y\sigma$ contains $t[x]$ for some y, which occurs in w.l.o.g. C_1. Let $\dot{y}$ be an arbitrary occurrence of y.

There is an occurrence of y in l, say $l|_{\hat{y}}$, whose corresponding term $l'|_{\hat{y}}$ is an abstraction of $t[x]$. Note that the arrows of $l|_{\hat{y}}$ and $l'|_{\hat{y}}$ are merged.

figure: $C_1 : Q(\dots \dot{y} \dots) \vee l[\hat{y}]$ $\qquad$ $C_2 : \neg l[\hat{y}']$ $\quad$ ($\hat{y}'$ is abstraction of $t[x]$)

Now we can argue quite like starting at (⋇). This establishes that $x \rightsquigarrow l'|_{\hat{y}}$ and also $x \rightsquigarrow l|_{\hat{y}}$.

If y and $\hat{y}$ are grey occurrences, then by Lemma 3, there is a merge edge and we are done.

If $\hat{y}$ is grey and $\dot{y}$ is colored, then by the induction hypothesis, $y \rightsquigarrow \hat{y}$ NB: (which is fine if we order by symbols, not their occurrences).

If $\hat{y}$ is colored and y is grey, then by construction of $\mathcal{A}(C)$, in particular the special treatment of 210g' and as $y\sigma = t[x]$ there is an arrow from any grey occurrence of x to y (if there is one; if there are none, we are done anyway).

If $\hat{y}$ is colored and y is colored, then by Lemma 2, there is a merge edge between $\hat{y}$ and y and we are done. □

d_to_all_colored⟩ **Conjectured Lemma 2.** *Let x be a variable in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$. Then there is a merge path from every colored occurrence of x to every other colored occurrence of x in C. NB: in this version, we merge also terms of different color. TODO: check if this works out*

rom_grey_to_grey⟩ **Conjectured Lemma 3.** *Let x be a variable in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$. Then there is a merge path from every grey occurrence of x to every other grey occurrence of x in C.*

2 original proof

Ideas for simplification:

* Lemma for all cases about what is on the other side

grey_to_colored⟩?

Lemma 4. ***not true in this formulation, we can have x, $f(x)$ and $g(x)$ with arrows just from x to the two colored occurrences, even if f and g of same color.*** *Let x be a variable in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ which has a grey occurrence and a colored occurrence. Then there is an arrow in $\mathcal{A}(C)$ from a term containing a grey occurrence to a term containing a colored occurrence. // Should also hold for all of AI^{Δ}, but is currently not needed in the proof*

Proof. For clauses C in the initial clause set, $\mathcal{A}(C)$ is defined to contain an arrow from every grey occurrence to every colored occurrence for every variable occurring in the clause.
For the induction step, suppose the lemma holds for C_1 and C_2. Note that C_1 and C_2 are variable disjoint. TODO: how to continue without checking every single case?
Note that terms are only changed by means of substitution.
If a variable is substituted, it does not occur any further in the derivation.
If a variable is substituted by a term containing variables, this is fine because the original arrows still apply for the new terms. □

Lemma 5. ***(same as above) not true in this formulation, we can have x, $f(x)$ and $g(x)$ with arrows just***

from x to the two colored occurrences, even if f and g of same color.

ored_to_colored⟩? *Let x be a variable which occurs colored in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ and again colored in the same color somewhere else in $\mathrm{AI}^{\Delta}(C)$. Then there is a merge edge between the maximal colored terms containing the two occurrences.* *// This is exactly the case we need, possibly show something more general*

Proof. TODO: □

n_in_arrow_proof⟩ **Example 6.** $\Gamma = \{Q(\gamma(x)) \lor P(x), \neg Q(\gamma(z)), R(\dots)\}$
$\Delta = \{\neg P(\delta(y)) \lor R(y), \neg R(a), Q(\dots)\}$
$a \sim x_k, \delta(y) \sim x_i, \delta(a) \sim x_j$

R only for coloring

Q only for coloring

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \mid Q(\gamma(x)) \lor P(x) \qquad \top \mid \neg P(x_i) \lor R(y)}{P(x_i) \mid Q(\gamma(x_i)) \lor R(y)} \qquad \top \mid \neg R(x_k)}{(\neg R(x_k) \land P(x_i)) \lor (R(x_k) \land \top) \mid Q(\gamma(x_i))}}{P(x_i) \lor R(x_k) \mid Q(\gamma(x_i))} \qquad \bot \mid \neg Q(\gamma(z))}{\dfrac{(\neg Q(x_j) \land (P(x_i) \lor R(x_k))) \lor (Q(x_j) \land \top) \mid \square}{\neg Q(x_j) \land (P(x_i) \lor R(x_k)) \mid \square}}$$

Gist: When $Q(\gamma(x_i))$ is the only symbol in $\mathrm{AI}^{\Delta}(\cdot)$, the lifting var means $\delta(x)$, but in the actual derivation, it's $\delta(a)$. however τ fixes this. So before Q is resolved, there is an arrow, but with the wrong lifting var (x_i instead of x_j) △

Remark (∗). Any substitution, in particular σ, only changes a finite number of variables. Furthermore a result of a run of the unification algorithm is acyclic in the sense that if a substitution $u \mapsto t$ is added to the resulting substitution, it is never the case that at a later stage $t \mapsto u$ is added. This can easily be seen by considering that at the point when $u \mapsto t$ is added to the resulting substitution, every occurrence of u is replaced by t, so u is not encountered by the algorithm at a later stage.
Therefore in order to show that a statement holds for every $u \mapsto t$ in a unifier σ, it suffices to show by an induction argument that for every substitution $v \mapsto s$ which is added to the resulting unifier by the unification algorithm that it holds for $v \mapsto s$ under the assumption that it holds for every $w \mapsto r$ such that w occurs in s and $w \mapsto r$ is added to the resulting substitution at a later stage. △

Conjecture 7. *Let C be a clause in a resolution refutation. Suppose that $\mathrm{AI}^{\Delta}(C)$ contains a maximal Γ-term $\gamma_j[z_i]$ which contains a lifting variable z_i. Then $z_i <_{\hat{A}(C)} z_j$.*

Proof. We proceed by induction. For the base case, note that no multicolored terms occur in initial clauses, so no lifting term can occur inside of a Γ-term.

Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$. Furthermore suppose that for every lifting term inside a Γ-term in the clauses C_1 and C_2 of the refutation, for every term of the form $\gamma_j[z_i]$ we have that $z_i <_{\hat{\mathcal{A}}(C_1)} z_j$ or $z_i <_{\hat{\mathcal{A}}(C_2)} z_j$ respectively. Hence there is an arrow (p_1, p_2) in $\hat{\mathcal{A}}(C_1)$ or $\hat{\mathcal{A}}(C_2)$ such that z_i is contained in $\mathrm{P}(p_1)$ and z_j is contained in $\mathrm{P}(p_2)$. In $\mathrm{AI}^\Delta(C)$, $\mathrm{P}(p_1)$ contains $\ell[z_i\sigma]\tau = z_i\tau$ and $\mathrm{P}(p_2)$ contains $\ell[z_j\sigma]\tau = z_j\tau$. Hence the indices of the lifting variables might change, but this renaming does not affect the relation of the objects as $\hat{\mathcal{A}}(C_1) \cup \hat{\mathcal{A}}(C_2) \subseteq \hat{\mathcal{A}}(C)$.

We show that $z_i <_{\hat{\mathcal{A}}(C)} z_j$ holds true also for every new term of the form $\gamma_j[z_i]$ for some j, i in $\mathrm{AI}^\Delta(C)$. By "new", we mean terms which are not present in $\mathrm{AI}^\Delta(C_1)$ or $\mathrm{AI}^\Delta(C_2)$. Note that new terms in $\mathrm{AI}^\Delta(C)$ are of the form $\ell^x_\Delta[t\sigma]\tau$ for some $t \in \mathrm{AI}^\Delta(C_1) \cup \mathrm{AI}^\Delta(C_2)$. By Lemma **??**, σ does not introduce lifting variables. Hence a new term of the form $\gamma_j[z_i]$ is created either by introducing a Δ-term into a Γ-term or by introducing $\gamma_j[\delta_i]$ via σ, both followed by the lifting. Note that τ only substitutes lifting variables by other lifting variables and hence does not introduce lifting variables. Furthermore by Lemma **??**, τ only substitutes lifting variables for other lifting variables, whose corresponding term is more specialised. Hence if there exists an arrow from a lifting variable to $\gamma_j[z_i]$ according to this lemma, it is also an appropriate arrow if $\gamma_j[z_i]$ is replaced by $\gamma_j[z_i]\tau$.

We now distinguish the two cases under which a new term $\gamma_j[z_i]$ can occur in $\mathrm{AI}^\Delta(C)$:

Suppose for some Γ-term $\tilde{\gamma}_{j'}[u]$ in $\mathrm{AI}^\Delta(C_1)$ or $\mathrm{AI}^\Delta(C_2)$, $u\sigma$ contains a Δ-term.

Hence we have that $(\tilde{\gamma}_{j'}[u])\sigma = \gamma_j[\delta_i]$ for some i. Note that the position of u in $\tilde{\gamma}_{j'}[u]$ does not necessarily coincide with the position of δ_i in $\gamma_j[\delta_i]$ as u might be substituted by σ for a grey term containing δ_i.

We have that $\ell_\Delta[\tilde{\gamma}_{j'}[u]\sigma]\tau = \gamma_j[z_i]$.

At some well-defined point of application of the unification algorithm, u is substituted by an abstraction of a term which contains δ_i. This occurrence of u is in l and we denote it by $\hat{u}$. We furthermore denote the term at the corresponding position in l' by $t_{\hat{u}}$.

We distinguish cases based on the occurrences of $\hat{u}$ and $t_{\hat{u}}$.

- Suppose $\hat{u}$ is a grey occurrence.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\hat{u}) \qquad C_2 : \neg Q(t_{\hat{u}})}{C : P(\gamma_j[\delta_i])}$$

Figure 1: Example for this case

Then by Lemma **??**, there is an arrow from a term containing u to a term containing $\gamma_j[u]$ in $\hat{\mathcal{A}}(C)$. As $\hat{u}\sigma$ is a term containing the Δ-term δ_i, the term at the position of $\hat{u}$ in $\mathrm{AI}^{\Delta}(C)$ is $\ell[\hat{u}\sigma]\tau$, which by assumption contains z_i. But there is an arrow from this term containing z_i to $\gamma_j[z_i]$, so $z_i <_{\hat{\mathcal{A}}(C)} z_j$.

- Suppose $\hat{u}$ occurs in a maximal colored term which is a Γ-term.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\gamma_k[\hat{u}]_p) \qquad C_2 : \neg Q(\gamma_m[t_{\hat{u}}]_p)}{C : P(\gamma_j[\delta_i])}$$

$$\frac{C_1 : Q(\tilde{\gamma}_{j'}[\hat{u}]) \qquad C_2 : \neg Q(\gamma_m[t_{\hat{u}}])}{C : \Box}$$

$//\gamma_j[\delta_i]$ occurs in the interpolant

Figure 2: Examples for this case

Then either $\hat{u}$ is the occurrence of u in $\tilde{\gamma}_{j'}[\hat{u}]$ or it occurs in a different Γ-term $\gamma_j[\hat{u}]$. In the latter case, by Lemma **??**, there is a merge edge between $\tilde{\gamma}_{j'}[\hat{u}]$ and $\gamma_j[\hat{u}]$. TODO: or that other combination, which is fine as well Hence in both cases, it suffices to show that there is an arrow from a term containing an occurrence of z_i to $t_{\hat{u}}$.

We distinguish on the shape of $t_{\hat{u}}$:

  - $t_{\hat{u}}$ is a term which does not contain a Δ-term. Then it contains a variable that is be substituted by σ by a term which contains a Δ-term as $u\sigma = t_{\hat{u}}\sigma$ is a term containing a Δ-term. We denote by v the variable in $t_{\hat{u}}$ which is substituted by a term containing a Δ-term in case $t_{\hat{u}}$ is a grey term.

    In the course of the unification algorithm, there are further unifications of v since we know that $u\sigma = v\sigma$ is a term containing a Δ-term. Therefore by Remark $(*)$, we can assume that there is an appropriate arrow to $t_{\hat{u}}$.

  - $t_{\hat{u}}$ is a term which contains a Δ-term. As $t_{\hat{u}}$ occurs in a Γ-term in C_1, say in $\gamma_m[t_{\hat{u}}]$, C_1 contains a multicolored Γ-term. Hence the corresponding term in $\mathrm{AI}^{\Delta}(C_1)$, is of the form $\gamma_m[z_{i'}]$ for some i'. Observe that i' in general is not equal to i as demonstrated in Example 6, even though we have that $t_{\hat{u}}\sigma = u\sigma$. This is because the lifting variables in $\mathrm{AI}(\cdot)$ represent abstractions of the terms in the clauses of the resolution derivation (cf. Lemma **??**). Therefore we only know by the induction hypothesis that $z_{i'} <_{\hat{\mathcal{A}}(C_1)} \ell[\gamma_m[z_{i'}]] = \ell[t_{\hat{u}}]$.

    However by Lemma **??** and due to the fact that $\hat{u}$ and $t_{\hat{u}}$ respectively occur in the resolved literal, $\ell_\Delta[\hat{u}\sigma]\tau = \ell_\Delta[t_{\hat{u}}\sigma]\tau$. As $\ell_\Delta[\hat{u}\sigma]\tau = \ell_\Delta[\delta_i]\tau = z_i\tau$ as well as $\ell_\Delta[t_{\hat{u}}\sigma]\tau = \ell_\Delta[z_{i'}\sigma]\tau = z_{i'}\tau$, we must have that $z_i\tau = z_{i'}\tau$. As however $u\sigma = \delta_i$, by the definition of au, we have that $\{z_i \mapsto z_i\} \in \tau$, so $z_{i'}\tau = z_i$.

Since τ is applied to every literal in $\mathrm{AI}^{\Delta}(C)$ and in $\mathrm{AI}^{\Delta}(C_1)$ an arrow from a term containing $z_{i'}$ to $t_{\hat{u}}$ exists, the same arrow applied to $\mathrm{AI}^{\Delta}(C)$ points from a term containing $z_{i'}\tau = z_i$ to $t_{\hat{u}}$. Therefore $z_i <_{\hat{\mathcal{A}}(C)} z_j$.

- Suppose $\hat{u}$ occurs in a maximal colored term which is a Δ-term.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\delta_k[\hat{u}]_p) \qquad C_2 : \neg Q(\delta_m[t_{\hat{u}}]_p)}{C : P(\gamma_j[\delta_i])}$$

By Lemma **??**, TODO:

The substitution can also introduce a grey term containing a delta term, make sure to handle that!

The substitution can also introduce a gamma term containing a delta term, make sure to handle that!

Suppose for some variable v in $\mathrm{AI}^{\Delta}(C_1)$ or $\mathrm{AI}^{\Delta}(C_2)$, $v\sigma = \gamma_j[\delta_i]$ for some i.

As v is affected by the unifier, it occurs in the literal being unified, say w.l.o.g. in l in C_1. At some well-defined point in the unification algorithm, v is substituted by an abstraction of $\gamma_j[\delta_i]$. Let p be the position of the occurrence of v in l which causes this substitution. Furthermore, let p' be the position corresponding to p in l'.

Note that any arrow from or to p' also applies to p in $\hat{\mathcal{A}}(C)$ and hence to $\gamma_j[z_i]$ as they are merged due to occurring in the resolved literal. So it suffices to show that there is an arrow from an appropriate lifting variable to p'. We denote the term at p' by t.

Note that $t\sigma = \gamma_j[\delta_i]$. So t is either a Γ-term containing a Δ-term, in which case we know that there is an appropriate arrow by the induction hypothesis as t occurs in l' in C_2, or t is an abstraction of $\gamma_j[\delta_i]$, in which case we can assume the existence of an appropriate arrow by Remark ($*$). $\square$

recheck this paragraphs w.r.t. $<_{\hat{\mathcal{A}}(C)}$

something about when i started with connected components

unification is for resolved literals.
connections between resolved literals and the rest of the clauses is covered by arrows.
if a term enters, merge arrows ensure that everything is propagated.
the special thing about colored occurrences is the fact that they can create multicolored terms in cooperation with grey occurrences..
a variable only occurs in a clause if it was never substituted by anything. Hence in particular all grey occurrences are "original" (TODO: renamings of variables)
Let u be a grey occurrence. let $f(u)$ be a colored occurrence. either it is original, then we are fine by arrow propagation. otherwise it has been introduced, but then it has used the network of another variable.

more precisely: a variable v occurs in a related literal in a related position in another clause as u in $f(u)$. so the variable is substituted by a term containing u, say $t[u]$ the arrows at the entry points are merged.
effect: $t[u]$ occurs at every grey occurrence of v. all arrows mentioning them are merged with the ones mentioning the entry point. this is justified as the terms there appear "as they are", i.e. as they are produced at the entry point.
however a colored occurrence cannot be produced from a grey occurrence (mgu$(x, f(u))$) but only if a grey occ is in the literal and a colored occ is elsewhere in the clause (the network of the other var). but then there are (directed) arrows.
Every variable has a connected network in a clause.
there is a barrier between colored terms.