

Example 101: canonical examples for overbinding w.r.t. order

Here, the second lower proof tree shows what huang would produce.

Ex 101a

$$\begin{array}{c}
 \frac{\frac{P(\textcolor{red}{u}, f(\textcolor{red}{u})) \vee Q(\textcolor{red}{u})}{P(a, f(a))} \quad \frac{\neg Q(a)}{u \mapsto a} \quad \frac{\neg P(x, y)}{x \mapsto a, y \mapsto f(a)} \quad \frac{}{\Box} \\
 \frac{\frac{\frac{\perp}{Q(a)} \quad \top}{u \mapsto a} \quad \top}{P(a, f(a)) \vee Q(a)} \quad \top \quad \frac{\frac{\frac{\perp}{\forall x_1 Q(x_1)} \quad \top}{\forall x_1 \exists x_2 (P(x_1, x_2) \vee Q(x_1))} \quad \top
 \end{array}$$

Direct overbinding would not work without merging same variables!: $\forall x_1 \forall x_2 \exists x_3 (P(x_2, x_3) \vee Q(x_1))$

counterexample: $Q \sim \{0\}, P \sim \{(1, 0)\}$

Direct overbinding would work when considering original dependencies as highlighted above

arrow lemma:

$$\frac{\frac{\Gamma \models \exists y_1 (P(\textcolor{red}{u}, y_1) \vee Q(\textcolor{red}{u}) \vee \perp) \quad \Gamma \models \neg Q(x_1) \vee \top}{\Gamma \models \exists y_1 (P(\textcolor{red}{x}_1, y_1) \vee Q(\textcolor{red}{x}_1))} \quad \frac{\Gamma \models \neg P(x, y) \vee \top}{x \mapsto a, y \mapsto f(a)} \quad \frac{}{\Gamma \models (\forall x_1) \exists y_1 (Q(\textcolor{red}{x}_1) \vee P(\textcolor{red}{x}_1, y_1))}$$

Ex 101b – other resolution order

$$\begin{array}{c}
 \frac{\frac{P(u, f(u)) \vee Q(u)}{Q(u)} \quad \frac{\neg P(x, y)}{y \mapsto f(u), x \mapsto u} \quad \frac{\neg Q(a)}{u \mapsto a} \quad \frac{}{\Box} \\
 \frac{\frac{\frac{\perp}{P(u, f(u))} \quad \top}{x \mapsto f(u), x \mapsto u} \quad \top}{P(a, f(a)) \vee Q(a)} \quad \top \quad \frac{\frac{\frac{\perp}{\exists x_1 P(u, x_1)} \quad \top}{\forall x_1 \exists x_2 (P(x_1, x_2) \vee Q(x_1))} \quad \top}{u \mapsto a}
 \end{array}$$

Ex 101c – Π and Σ swapped

$$\begin{array}{c}
 \frac{\frac{P(u, f(u)) \vee Q(u)}{Q(u)} \quad \frac{\neg P(x, y)}{y \mapsto f(u), x \mapsto u} \quad \frac{\neg Q(a)}{u \mapsto a} \quad \frac{}{\Box} \\
 \frac{\frac{\top}{\neg P(u, f(u))} \quad \perp}{x \mapsto f(u), x \mapsto u} \quad \perp}{\neg P(a, f(a)) \wedge \neg Q(a)} \quad \frac{\frac{\top}{\forall x_2 \neg P(u, x_2)} \quad \perp}{\exists x_1 \forall x_2 (\neg P(x_1, x_2) \wedge \neg Q(x_1))}
 \end{array}$$

arrow lemma:

$$\begin{array}{c}
\frac{\Gamma \models P(u, x_1) \vee Q(u) \vee \top \quad \Gamma \models \neg P(x, y) \vee \perp}{\left(Q(u) \mid (\neg P(x, x_1) \wedge \top) \vee (P(u, f(u)) \wedge \perp) \right) \sigma} y \mapsto f(u), x \mapsto u \\
Q(u) \mid (\neg P(u, x_1) \wedge \top) \vee (P(u, f(u)) \wedge \perp) \\
\text{employ } \sigma' \text{ !?!?!?!?!?!?!?!?!} \\
\frac{\Gamma \models Q(u) \mid \neg P(u, x_1) \quad \Delta \models Q(u) \mid \exists x_1 P(u, x_1)}{\Gamma \models \exists y_1 \neg Q(y_1)} u \mapsto a \\
\text{both } u\text{'s on LHS need to become } a \text{ and then } y_1 \\
\Gamma \models (\forall x_1) \exists y_1 (\neg P(y_1, x_1) \vee \neg Q(y_1)) \\
\Delta \models (\exists x_1) \forall y_1 (P(y_1, x_1) \wedge Q(y_1))
\end{array}$$

Ex 101d – Π and Σ swapped, other resolution order

$$\frac{\frac{P(u, f(u)) \vee Q(u) \quad \neg Q(a)}{P(a, f(a))} u \mapsto a \quad \neg P(x, y)}{x \mapsto a, y \mapsto f(a)} \square$$

$$\frac{\frac{\top \quad \perp}{\neg Q(a)} y \mapsto a \quad \perp}{\neg Q(a) \wedge \neg P(a, f(a))} x \mapsto a, y \mapsto f(a) \quad \frac{\frac{\top \quad \perp}{\exists x_1 \neg Q(x_1)} \quad \perp}{\exists x_1 \forall x_2 (\neg P(x_1, x_2) \wedge \neg Q(x_1))}$$

102 – similar to 101 but with intra-clause-set inferences

Ex 102a

$$\frac{\frac{P(f(\textcolor{red}{x})) \vee Q(f(\textcolor{red}{x}), z) \quad \neg P(y)}{Q(f(x), z)} \quad \frac{\neg Q(x_1, \textcolor{blue}{y}) \vee R(\textcolor{blue}{y}) \quad \neg R(g(z_1))}{\neg Q(x_1, g(z_1))} y \mapsto g(z_1), x_1 \mapsto f(x)}{x_1 \mapsto f(x), z \mapsto g(z_1)} \square$$

$$\frac{\frac{\perp \quad \top}{P(f(x))} \quad \frac{\perp \quad \top}{R(g(z_1))} y \mapsto g(z_1), x_1 \mapsto f(x)}{P(f(x)) \vee R(g(z_1))} x_1 \mapsto f(x), z \mapsto g(z_1) \quad \frac{\frac{\perp \quad \top}{\exists x_1 P(x_1)} \quad \frac{\perp \quad \top}{\forall x_2 R(x_2)}}{\exists x_1 \forall x_2 (P(x_1) \vee R(x_2))} \text{ (order irrelevant!)}$$

Ex 102b

$$\frac{\frac{P(f(\textcolor{red}{x})) \vee Q(f(\textcolor{red}{x}), z) \quad \neg P(y)}{Q(f(x), z)} \quad \frac{\neg Q(f(\textcolor{blue}{y}), z_1) \vee R(\textcolor{blue}{y}) \quad \neg R(a)}{\neg Q(f(a), z_1)} y \mapsto a}{x \mapsto a, z \mapsto z_1} \square$$

$$\frac{\frac{\perp}{P(f(x))} \quad \top}{\frac{\perp}{R(a)} \quad y \mapsto a} \quad \frac{\perp}{P(f(a)) \vee R(a)} \quad x \mapsto a, z \mapsto z_1$$

$$\frac{\frac{\perp}{\exists x_1 P(x_1)} \quad \top}{\forall x_2 R(x_2)} \quad y \mapsto a \quad \frac{\perp}{\forall x_2 \exists x_1 (P(x_1) \vee R(x_2))} \quad x \mapsto a, z \mapsto z_1$$

direct:

$$\frac{\frac{\perp}{\exists x_1 P(x_1)} \quad \top \quad x_1 \sim f(x) \quad \frac{\perp}{\forall x_2 R(x_2)} \quad x_2 \sim a}{\exists x_1 \forall x_2 (P(x_1) \vee R(x_2))} \text{ order irrelevant!}$$

Ex 102b' with Q grey

$$\frac{\frac{P(f(\textcolor{red}{x})) \vee Q(f(\textcolor{red}{x}), z) \quad \neg P(y)}{Q(f(x), z)} \quad \frac{\neg Q(f(\textcolor{blue}{y}), z_1) \vee R(\textcolor{blue}{y}) \quad \neg R(a)}{\neg Q(f(a), z_1)} \quad y \mapsto a}{x \mapsto a, z_1 \mapsto z} \quad \square$$

$$\frac{\frac{\perp \quad \top}{P(f(x))} \quad \frac{\perp \quad \top}{R(a)} \quad y \mapsto a}{(\neg Q(f(a), z) \wedge P(f(a))) \vee (Q(f(a), z) \wedge R(a))} \quad x \mapsto a, z_1 \mapsto z$$

arrow lemma: (change of specification: $P(g(z_3))$ in clause in Σ instead of $P(f(x))$)

$$\frac{\frac{\exists x_1 (P(x_4) \vee Q(x_1, z)) \quad \neg P(y)}{\exists x_1 (P(g(z_3)) \wedge \perp) \vee (P(x_4) \wedge \top) \mid Q(x_1, z)} \quad \frac{\frac{\exists x_2 (\neg Q(x_2, z_1) \vee R(y)) \quad \forall x_3 \neg R(x_3)}{\forall x_3 \exists x_2 ((\neg R(x_3) \wedge \perp) \vee (R(a) \wedge \top) \mid \neg Q(x_2, z_1))} \quad y \mapsto a}{\frac{\exists x_1 (P(x_4) \mid Q(x_1, z)) \quad \forall x_3 \exists x_2 ((\neg R(x_3) \wedge \perp) \vee (R(x_3) \wedge \top) \mid \neg Q(x_2, z_1))}{\forall x_1 \forall x_3 \exists x_2 ((\neg Q(x_2, z_1) \wedge P(x_4)) \vee (Q(x_1, z) \wedge R(x_3)))} \quad x \mapsto a, z_1 \mapsto z$$

arrow order: $x_3 < x_2$, x_2 same-block-as x_4 : $\forall x_3 \exists x_2 \exists x_4 \forall x_1 ((\neg Q(x_2, z_1) \wedge \neg P(x_4)) \vee (Q(x_1, z) \wedge R(x_3)))$

\rightarrow bad example, plus some errors still in there

Huang:

$$\frac{\frac{\perp \quad \top}{\exists x_2 P(x_2)} \quad \frac{\perp \quad \top}{\forall x_1 R(x_1)} \quad y \mapsto a}{\forall x_1 \exists x_2 (\neg Q(x_2, z) \wedge P(x_2)) \vee (Q(x_2, z) \wedge R(x_1))} \quad x \mapsto a, z_1 \mapsto z$$

direct:

$$\frac{\frac{\perp \quad \top}{\exists x_2 P(x_2)} \quad x_2 \sim f(x) \quad \frac{\perp \quad \top}{\forall x_1 R(x_1)} \quad x_1 \sim a}{\forall x_1 \exists x_2 \exists x_3 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))} \quad x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3$$

$$\text{OR: } \frac{\exists x_2 \forall x_1 \exists x_3 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}{\text{OR: } \exists x_1 \exists x_3 \forall x_2 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}$$

Huang-Version passt weil man für existenzielle quantoren eh gleichen term einsetzt

direct w mixed, slightly different:

$$\frac{\frac{\perp \mid P(f(x)) \vee Q(x, z) \quad \top \mid \neg P(y)}{\exists x_2 P(x_2) \mid Q(x, z)} \quad x_2 \sim f(x) \quad \frac{\frac{\perp \mid \neg Q(f(y), z_1) \vee R(y) \quad \top \mid \neg R(a)}{\forall x_1 R(x_1) \mid \neg Q(f(a), z_1)} \quad x_1 \sim a}{\frac{\forall x_1 \exists x_3 \exists x_2 (\neg Q(x_3, z) \wedge P(x_2)) \vee (Q(x_3, z) \wedge R(x_1))}{(\neg Q(f(a), z) \wedge P(f(f(a)))) \vee (Q(f(a), z) \wedge R(a))} \quad x_3 \sim f(a); x_2 \parallel x_3, x_1 < x_3$$

last dependency not crucial because other arrow is a Σ -arrow as well, but just changing it to Π (and changing f for g should produce a quantifier alternation)

Example 103: variables in interpolant get unified and order might have to be changed

103a: length changes but old version is still valid, despite huang's algorithm doing a sort

$$\frac{\frac{\frac{\perp}{Q(f(\textcolor{red}{x}))} \top}{Q(f(x)) \vee P(y) \vee R(\textcolor{red}{x})} \neg Q(y_1)}{P(y) \vee R(x)} y_1 \mapsto f(x) \quad \frac{\neg P(h(g(a)))}{R(x)} y \mapsto h(g(a)) \quad \frac{\neg R(g(g(a)))}{\square} x \mapsto g(g(a))$$

$$\frac{\frac{\frac{\perp}{Q(f(x))} \top}{Q(f(x)) \vee P(h(g(a)))} \top}{Q(f(g(g(a)))) \vee P(h(g(a))) \vee R(g(g(a)))} y \mapsto h(g(a)) \quad \frac{\frac{\frac{\perp}{\exists x_1 Q(x_1)} \top}{\exists x_1 \forall x_2 (Q(x_1) \vee P(x_2))} \top}{X} x \mapsto g(g(a))$$

X:

Huang's algo gives:

$$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))$$

Direct overbinding gives: $x_3 < x_1$, rest arbitrary, hence:

$$\forall x_3 \exists x_1 \forall x_2 (Q(x_1) \vee P(x_2) \vee R(x_3)) \leftarrow \text{this you do not get with huang}$$

$$\forall x_2 \forall x_3 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))$$

$$\forall x_3 \forall x_2 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))$$

103b: length changes “uniformly”

$$\frac{\frac{\frac{\perp}{Q(f(f(\textcolor{red}{x})))} \top}{Q(f(f(x))) \vee P(f(x))} \neg Q(y_1)}{P(f(x)) \vee R(x)} y_1 \mapsto f(f(x)) \quad \frac{\neg P(y_2)}{R(x)} y_2 \mapsto f(x) \quad \frac{\neg R(g(a))}{\square} x \mapsto g(a)$$

$$\frac{\frac{\frac{\perp}{Q(f(f(x)))} \top}{Q(f(f(x))) \vee P(f(x))} \top}{Q(f(f(g(a)))) \vee P(f(g(a))) \vee R(g(a))} y_2 \mapsto f(x) \quad \frac{\frac{\frac{\perp}{\exists x_1 Q(x_1)} \top}{\exists x_2 \exists x_1 (Q(x_1) \vee P(x_2))} \top}{\forall x_3 \exists x_2 \exists x_1 (Q(x_1) \vee P(x_2) \vee R(x_3))} x \mapsto g(a)$$

Huang and direct overbinding somewhat coincide as $x_2 < x_1$ in both cases, and $x_3 < x_1$ and $x_3 < x_2$

103c: Failed attempt: different variables, accidentally the same terms appear but no logical connection

$$\frac{\frac{\perp}{P(a, x)} \top}{\square} \quad \frac{\frac{\neg Q(a)}{\neg P(y, f(a))} \neg P(y, f(\textcolor{red}{z})) \vee Q(\textcolor{red}{z})}{y \mapsto a, x \mapsto f(a)} z \mapsto a$$

Huang:

$$\frac{\frac{\perp}{P(a, f(a)) \wedge \neg Q(a)} \quad \frac{\frac{\perp}{\neg Q(a)} \quad \top}{z \mapsto a}}{y \mapsto a, x \mapsto f(a)} \quad \frac{\frac{\perp}{\exists x_1 \neg Q(x_1)} \quad \top}{\exists x_1 \forall x_2 (P(x_1, x_2) \wedge \neg Q(x_1))}$$

order required for Π

direct:

$$\frac{\frac{\frac{\perp}{\exists x_1 \neg Q(x_1)} \quad \top}{x_1 \sim a} \quad \frac{\perp}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}}{x_2 \sim a, x_3 \sim f(a); x_1 < x_3}$$

$$\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))$$

invariant:

$$\frac{\frac{\frac{\exists x_1 (Q(x_1) \vee \perp)}{\exists x_1 \forall x_3 \neg P(y, \mathbf{x}_3) \vee \neg Q(\mathbf{x}_1)} \quad \frac{\forall x_3 ((\neg P(y, \mathbf{x}_3) \vee Q(\mathbf{z})) \vee \top)}{x_1 \sim a}}{\exists x_2 (P(x_2, x) \vee \perp)} \quad \frac{\perp}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}}{x_2 \sim a, x_3 \sim f(a); x_1 < x_3}$$

$$\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))$$

invariant in other resolution order

$$\frac{\frac{\frac{\perp}{Q(\mathbf{z}) \vee \exists x_2 \forall x_3 P(x_2, \mathbf{x}_3)} \quad \top}{x_2 \sim a, x_3 \sim f(z)} \quad \frac{\perp}{\exists x_1 \exists x_2 \forall x_3 (P(x_2, x_3) \wedge \neg Q(x_1))}}{x_1 \sim a; x_1 < x_3}$$

$$\text{OR: } \exists x_1 \forall x_3 \exists x_2 (P(x_2, x_3) \wedge \neg Q(x_1))$$

invariant if Σ and Π swapped:

$$\frac{\frac{\frac{\top}{\neg P(y, f(x_1)) \vee \forall x_1 Q(x_1)} \quad \frac{\perp}{x_1 \sim a}}{\forall x_1 \forall x_2 \exists x_3 (\neg P(x_2, x_3) \vee Q(x_1))} \quad \frac{\perp}{\forall x_1 \exists x_3 \forall x_2 (\neg P(x_2, x_3) \vee Q(x_1))}}{x_2 \sim a, x_3 \sim f(a); x_1 < x_3}$$

$$\text{OR: } \forall x_1 \exists x_3 \forall x_2 (\neg P(x_2, x_3) \vee Q(x_1))$$

SECOND ATTEMPT:

$$\frac{\frac{\frac{\frac{\frac{\frac{\Sigma}{P(a)} \quad \frac{\Sigma}{Q(z)}}{\neg P(y)} \quad \frac{\frac{\frac{\Sigma}{\neg S(a)} \quad \frac{\Pi}{\neg P(y) \vee \neg Q(f(\mathbf{x})) \vee S(\mathbf{x})}}{\neg P(y) \vee \neg Q(f(a))}}{x \mapsto a}}{z \mapsto f(a)}}{y \mapsto a} \quad \frac{\perp}{\neg P(y)}}{\square} \quad \frac{\frac{\frac{\perp}{\neg S(a)} \quad \top}{x \mapsto a} \quad \frac{\perp}{\neg S(a) \wedge Q(f(a))}}{z \mapsto f(a)}}{P(a) \wedge \neg S(a) \wedge Q(f(a))} \quad y \mapsto a$$

Huang:

$$\frac{\perp \quad \frac{\perp \quad \frac{\top}{\exists x_1 \neg S(x_1)}}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 (\neg S(x_1) \wedge Q(x_2))} \quad \perp}{\exists x_1 \forall x_2 (P(x_1) \wedge \neg S(x_1) \wedge Q(x_2))}$$

$$\forall x_1 \exists x_2 (\neg P(x_1) \vee S(x_1) \vee \neg Q(x_2))$$

similar fail

\Rightarrow anytime there is $P(a, f(a))$, either they have a dependency or they are not both differently colored (grey is uncolored)
for the record, direct method anyway:

$$\frac{\perp \quad \frac{\perp \quad \frac{\top}{\exists x_1 \neg S(x_1)} \quad x \sim a}{\exists x_1 \neg S(x_1)} \quad z \sim f(a); x_1 < x_2}{\exists x_1 \forall x_2 \neg S(x_1) \wedge Q(x_2)} \quad \perp}{\exists x_1 \forall x_2 \exists x_3 P(x_3) \wedge \neg S(x_1) \wedge Q(x_2)} \quad x_3 \sim a; x_3 \text{ need not be merged w } x_1$$

Example: ordering on both ancestors where the merge forces a new ordering

202a – canonical

$$\begin{array}{c}
 \frac{\frac{P(a, x_1) \vee R(y)}{R(y)} \quad \frac{\neg P(\textcolor{violet}{x}, f\textcolor{violet}{x})}{x \mapsto fa} \quad \frac{Q(\textcolor{red}{x}_2, g\textcolor{red}{x}_2) \vee \neg R(u)}{\neg R(u)} \quad \frac{\frac{\neg S(a)}{\neg S(a)} \quad \frac{\neg Q(f\textcolor{blue}{z}, x_3) \vee S(\textcolor{blue}{z})}{\neg Q(fa, x_3)}}{z \mapsto a} \quad \frac{x_2 \mapsto fa, \quad x_3 \mapsto gfa}{x_3 \mapsto gfa} \\
 \hline
 \frac{\frac{\frac{\perp}{P(a, f(a))} \quad \frac{\top}{x \mapsto a}}{x \mapsto f(a)} \quad \frac{\frac{\perp}{Q(f(a), g(f(a))) \wedge \neg S(a)} \quad \frac{\frac{\perp}{\neg S(a)} \quad \frac{\top}{z \mapsto a}}{x_2 \mapsto f(a), \quad x_3 \mapsto g(f(a))}}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a))} \\
 \hline
 \square
 \end{array}$$

Huang

$$\frac{\frac{\frac{\perp}{\exists x_1 \forall x_2 P(x_1, x_2)}}{\exists x_1 \forall x_2 \exists x_3 P(x_1, x_2) \vee (Q(x_2, x_3) \wedge \neg S(x_1))} \quad \frac{\frac{\frac{\perp}{\exists x_1 \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 Q(x_2, x_3) \wedge \neg S(x_1)}}{\exists x_1 \forall x_2 \exists x_3 P(x_1, x_2) \vee (Q(x_2, x_3) \wedge \neg S(x_1))}$$

direct:

$$\frac{\frac{\frac{\perp}{\exists x_1 \forall x_2 P(x_1, x_2)}}{\exists x_1 \forall x_2 \exists x_5 P(x_1, x_2) \vee (Q(x_2, x_5) \wedge \neg S(x_5))} \quad \frac{\frac{\frac{\frac{\perp}{\exists x_3 \neg S(x_3)}}{\exists x_3 \forall x_4 \exists x_5 Q(x_4, x_5) \wedge \neg S(x_3)}}{\exists x_3 \forall x_4 \exists x_5 Q(x_4, x_5) \wedge \neg S(x_3)}}{x_3 \mapsto x_1, x_4 \mapsto x_2} \quad \frac{x_1 \sim a, x_2 \sim fa \quad x_3 \sim a, x_4 \sim fa, x_5 \sim gfa}{x_1 < x_2, x_2 < x_5}$$

without merge in end: $(x_1 < x_2, x_3 < x_4, x_4 < x_5)$

$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))$

$\exists x_3 \forall x_4 \exists x_5 \exists x_1 \forall x_2 P(x_1, x_2) \vee (Q(x_4, x_5) \wedge \neg S(x_3))$

(also interwoven ones appear to work)

combined presentation:

$$\begin{array}{c}
\frac{\frac{\frac{\perp \mid P(a, x_1) \vee R(y) \quad \top \mid \neg P(x, f(x))}{P(a, f(a)) \mid R(y)} \quad x_1 \mapsto f(a)}{x \mapsto a} \quad \frac{\frac{\perp \mid Q(x_2, g(x_2)) \vee \neg R(u)}{Q(f(a), g(f(a))) \wedge \neg S(a) \mid \neg R(u)} \quad \frac{\frac{\perp \mid \neg S(a) \quad \top \mid \neg Q(f(z), x_3) \vee S(z)}{\neg S(a) \mid \neg Q(f(a), x_3)} \quad \frac{z \mapsto a}{x_2 \mapsto f(a)},}{x_3 \mapsto g(f(a))} \\
\hline
P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a)) \mid \square
\end{array}$$

combined presentation ground:

$$\begin{array}{c}
\frac{\frac{\perp \mid P(a, f(a)) \vee R(y) \quad \top \mid \neg P(a, f(a))}{(P(a, f(a)) \wedge \top) \vee (\neg P(a, f(a)) \wedge \perp) \mid R(y)} \quad \frac{\frac{\perp \mid Q(f(a), g(f(a))) \vee \neg R(u)}{Q(f(a), g(f(a))) \wedge \neg S(a) \mid \neg R(u)} \quad \frac{\frac{\perp \mid \neg S(a) \quad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a), g(f(a)))}}{P(a, f(a)) \vee (Q(f(a), g(f(a))) \wedge \neg S(a)) \mid \square}
\end{array}$$

combined presentation ground with direct method but only Δ -terms removed :

$$\begin{array}{c}
\frac{\frac{\perp \mid P(a, f(a)) \vee R(y) \quad \top \mid \neg P(a, f(a))}{(P(a, x_2) \wedge \top) \vee (\neg P(a, x_2) \wedge \perp) \mid R(y)} \quad \frac{\frac{\perp \mid Q(f(a), g(f(a))) \vee \neg R(u)}{Q(x_4, g(x_4)) \wedge \neg S(a) \mid \neg R(u)} \quad \frac{\frac{\perp \mid \neg S(a) \quad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\neg S(a) \mid \neg Q(f(a), g(f(a)))}}{P(a, x_2) \vee (Q(x_4, g(x_4)) \wedge \neg S(a)) \mid \square}
\end{array}$$

combined presentation ground with direct method:

$$\begin{array}{c}
\frac{\frac{\frac{\perp \mid P(a, f(a)) \vee R(y) \quad \top \mid \neg P(a, f(a))}{\exists x_1 \forall x_2 (P(x_1, x_2) \wedge \top) \vee (\neg P(x_1, x_2) \wedge \perp) \mid R(y)} \quad \frac{\frac{\perp \mid Q(f(a), g(f(a))) \vee \neg R(u)}{\exists x_3 \forall x_4 \exists x_5 (Q(x_4, x_5)) \wedge \neg S(x_3) \mid \neg R(u)} \quad \frac{\frac{\perp \mid \neg S(a) \quad \top \mid \neg Q(f(a), g(f(a))) \vee S(a)}{\exists x_3 \neg S(x_3) \mid \neg Q(f(a), g(f(a)))}}{\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 (P(x_1, x_2) \vee (Q(x_4, x_5)) \wedge \neg S(x_3)) \mid \square}
\end{array}$$

203a – some alternations

$$\begin{array}{c}
\frac{\frac{\frac{\frac{\frac{}{R(x) \vee \neg P(f(x))}}{\Sigma} \quad \frac{\frac{}{P(z) \vee Q(g(z))}}{\Pi}}{R(x) \vee Q(g(f(x)))} \quad z \mapsto f(x)}{\Pi} \quad \frac{\frac{}{-Q(y) \vee S(h(y))}}{\Sigma}}{y \mapsto g(f(x))} \\
\frac{\frac{\frac{}{-R(a)}}{\Pi} \quad R(x) \vee S(h(g(f(x))))}{x \mapsto a} \\
\frac{\frac{}{\neg S(x_1)}}{\Pi} \quad S(h(g(f(a))))}{x_1 \mapsto h(g(f(a)))} \\
\boxed{\square}
\end{array}$$

$$\frac{\frac{\frac{\frac{\perp}{\neg P(f(x))} z \mapsto f(x)}{\neg Q(g(f(x))) \wedge \neg P(f(x))} y \mapsto g(f(x))}{\neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a)} x \mapsto a}{\neg Q(g(f(a))) \wedge \neg P(f(a)) \vee R(a) \vee S(h(g(f(a))))} x_1 \mapsto h(g(f(a)))$$

Huang:

$$\frac{\frac{\frac{\perp}{\exists x_1 \neg P(x_1)} \quad \top}{\exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1))} \quad \perp}{\top \quad \frac{\top}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0))}} \frac{}{\forall x_0 \exists x_1 \forall x_2 \exists x_3 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0) \vee S(x_3))}$$

Direct:

$$\frac{\frac{\frac{\frac{\perp}{\exists x_1 \neg P(x_1)} \quad \top}{x_1 \sim f(x)} \quad \perp}{\exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(f(x)))} \quad x_2 \sim g(f(x)); x_1 < x_2}{\top \quad \frac{\top}{\forall x_0 \exists x_1 \forall x_2 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0))} \quad x_0 \sim a; x_0 < x_1, x_0 < x_2}{\top \quad \frac{\forall x_0 \exists x_1 \forall x_2 \forall x_3 (\neg Q(x_2) \wedge \neg P(x_1) \vee R(x_0) \vee S(x_3))}{x_3 \sim h(g(f(a))); x_0 < x_3, x_1 < x_3, x_2 < x_3}}$$

203b – many Σ -literals, coloring per occurrence

$$\begin{array}{c}
\frac{\frac{\frac{\frac{}{R(x) \vee \neg P(f(x))}}{\Sigma} \quad \frac{\frac{}{P(z) \vee Q(g(z))}}{\Sigma}}{R(x) \vee Q(gfx)} \quad z \mapsto fx \quad \frac{\frac{}{\neg Q(y) \vee S(h(y))}}{\Sigma}}{y \mapsto gfx}}{R(x) \vee S(hgfx)} \quad x \mapsto a \\
\frac{\frac{\frac{}{\neg R(a)}}{\Pi} \quad \frac{}{\neg S(x_1)}}{\Pi}}{S(hgfa)} \quad x_1 \mapsto hgfa \\
\frac{}{\Box} \quad \frac{\frac{\perp}{\bot} \quad \frac{\perp}{\bot} \quad z \mapsto fx}{\perp} \quad \frac{\perp}{\top} \quad y \mapsto gfx \\
\frac{\frac{\top}{\top} \quad \frac{\perp}{\bot} \quad x \mapsto a}{R(a)} \quad \frac{\top}{\top} \quad \frac{}{S(hgfa) \vee R(a)} \quad x_1 \mapsto hgfa
\end{array}$$

$$\rightarrow \forall x_1 \exists x_2 (R(x_1) \vee S(x_2))$$

203b' – many Σ -literals, coloring per symbol, all predicates grey

$$\frac{\frac{\frac{\neg R(a)}{\Pi} \quad \frac{R(x) \vee \neg P(f(x))}{\Sigma} \quad x \mapsto a}{R(a) \mid \neg P(fa)} \quad \frac{P(z) \vee Q(g(z))}{\Sigma} \quad z \mapsto fa}{\frac{\neg S(x_1)}{\Pi} \quad \frac{P(fa) \vee R(a) \mid Q(gfa)}{\Sigma} \quad \neg Q(y) \vee S(h(y))}$$

TODO

Example where variables are not the outermost symbol but order is still relevant

204a

$$\Sigma = \{P(f(x), g(f(x)))\}$$

$$\Pi = \{P(f(a), y)\}$$

$$\Rightarrow \forall x_1 \exists x_2 P(f(x_1), x_2)$$

204b

$$\Sigma = \{P(f^5(x), g(f(x)))\}$$

$$\Pi = \{P(f^5(a), y)\}$$

$$\Rightarrow \forall x_1 \exists x_2 P(f^5(x_1), x_2)$$

example with aufschaukelnde unification, such that direction of arrow isn't clear

205a

$$\frac{\frac{\Pi}{P(ff\textcolor{blue}{y},gy)} \quad \frac{\Sigma}{\neg P(\textcolor{red}{x},y) \vee Q(\textcolor{red}{x})} \quad \frac{\frac{\Sigma}{\neg R(a)} \quad \frac{\Pi}{\neg Q(ff\textcolor{violet}{z}) \vee R\textcolor{violet}{z}}}{\neg R(a) \mid \neg Q(ffa)} z \mapsto a}{\frac{\neg R(a) \wedge Q(ffa) \mid \neg P(ffa,y)}{y \mapsto a} \quad \frac{\Pi}{P(ff\textcolor{blue}{y},gy)}} x \mapsto ffa$$

direct

$$\frac{\frac{\Pi}{P(ff\textcolor{blue}{y},gy)} \quad \frac{\Sigma}{\neg P(\textcolor{red}{x},y) \vee Q(\textcolor{red}{x})} \quad \frac{\frac{\Sigma}{\neg R(a)} \quad \frac{\Pi}{\neg Q(ff\textcolor{violet}{z}) \vee R\textcolor{violet}{z}}}{\exists x_1 \neg R(x_1) \mid \neg Q(ffa)} z \mapsto a}{\frac{\exists x_1 \forall x_2 (\neg R(x_1) \wedge Q(x_2)) \mid \neg P(ffa,u)}{y \mapsto a, u \mapsto ga} \quad \frac{\Pi}{P(ff\textcolor{blue}{y},gy)}} x \mapsto ffa$$

ground:

$$\frac{\frac{\Pi}{P(ff\textcolor{blue}{a},ga)} \quad \frac{\Sigma}{\neg P(\textcolor{red}{ffa},y) \vee Q(\textcolor{red}{ffa})} \quad \frac{\frac{\Sigma}{\neg R(a)} \quad \frac{\Pi}{\neg Q(ff\textcolor{violet}{a}) \vee R\textcolor{violet}{a}}}{\neg R(a) \mid \neg Q(ffa)} z \mapsto a}{\frac{\neg R(a) \wedge Q(ffa) \mid \neg P(ffa,a)}{y \mapsto a} \quad \frac{\Pi}{P(ff\textcolor{blue}{a},ga)}} x \mapsto ffa$$

arrow lemma: $x_1 \sim ffy, x_2 \sim gy, y_3 \sim a, x_4 \sim ffz, x_5 \sim ffa, y_6 \sim a$

$$\frac{\frac{\Sigma}{\neg P(x,y) \vee Q(x)} \mid \perp \quad \frac{\frac{\Sigma}{\exists y_3 (\neg R(y_3) \mid \perp)} \quad \frac{\Pi}{\forall x_4 (\neg Q(x_4) \vee Rz \mid \perp)}}{\exists y_3 \forall x_4 ((\neg R(y_3) \wedge \top) \vee (R(a) \wedge \perp) \mid \neg Q(x_4))} z \mapsto a}{\frac{\exists y_3 \forall x_4 ((\neg Q(x_4) \wedge \perp) \vee (Q(x) \sigma \wedge \neg R(y_3))) \mid \neg P(x,y) \sigma}{\exists y_3 \forall x_4 ((Q(ffa) \wedge \neg R(y_3)) \mid \neg P(ffa,y)) \quad (*)} x \mapsto ffa}$$

$$\frac{\frac{\Pi}{\forall x_1 \forall x_4 (P(x_1, x_2) \mid \top)} \quad \frac{\exists y_3 \forall x_4 ((Q(x_5) \wedge \neg R(y_3)) \mid \neg P(x_5, y))}{(\neg P(x_5, y) \wedge \top) \vee (P(x_1, x_2) \wedge (Q(x_5) \wedge \neg R(y_3)))} y \mapsto a}{\frac{(\neg P(x_5, a) \wedge \top) \vee (P(x_1, x_2) \wedge (Q(x_5) \wedge \neg R(y_3)))}{\forall x_1 \forall x_2 \exists y_3 \forall x_4 \forall x_5 \forall x_6 (\neg P(x_5, y_6) \wedge \top) \vee (P(x_1, x_2) \wedge (Q(x_5) \wedge \neg R(y_3)))}}$$

(*) "luckily", same overbinding for ffa , so this works
dashed underline: problem, but does not cause issues here.

situations not critical here :-)

205b \sim 205a, but simpler

Suppose P occurs somewhere in Σ (result not that optimal in this setting, but correct)

not nice for proving, $\neg R(a)$ is a nice interpolant already

$$\frac{\frac{\Pi}{P(ff\textcolor{blue}{y},gy)} \quad \frac{\Sigma}{\neg R(a)} \quad \frac{\Pi}{\neg P(ff\textcolor{violet}{z},x) \vee R\textcolor{violet}{z}}}{\neg R(a) \mid \neg P(ffa,x)} z \mapsto a$$

$$\frac{\frac{\Pi}{P(ff\textcolor{blue}{y},gy)} \quad \frac{\Sigma}{\neg R(a)} \quad \frac{\Pi}{\neg P(ff\textcolor{violet}{z},x) \vee R\textcolor{violet}{z}}}{\neg R(a) \vee \neg P(ffa,ga) \mid \Box} x \mapsto ga, y \mapsto a$$

$$\frac{\frac{\Pi}{\top \mid P(ff\textcolor{blue}{y},gy)} \quad \frac{\Sigma}{\perp \mid \neg R(a)} \quad \frac{\Pi}{\top \mid \neg P(ff\textcolor{violet}{z},x) \vee R\textcolor{violet}{z}}}{\exists x_1 \neg R(x_1) \mid \neg P(ffa,x)} z \mapsto a$$

$$\frac{\frac{\Pi}{\top \mid P(ff\textcolor{blue}{y},gy)} \quad \frac{\Sigma}{\perp \mid \neg R(a)} \quad \frac{\Pi}{\top \mid \neg P(ff\textcolor{violet}{z},x) \vee R\textcolor{violet}{z}}}{\exists x_1 \forall x_2 \forall x_3 \neg R(x_1) \vee \neg P(x_2, x_3) \mid \Box} x \mapsto ga, y \mapsto a$$

$\exists x_1 R(x_1)$

$\exists x_1 \forall x_2 \forall x_3 (R(x_1) \vee \neg P(x_2, x_3))$

example to demonstrate that literals being resolved upon have to be overbound with the same variable

$$\frac{\frac{R(f(x))^\Pi \quad \neg R(y) \vee P(y)^\Sigma}{(\neg R(x_3) \wedge \top) \vee (R(x_3) \wedge \perp) \mid P(x_3)} \quad \frac{\neg P(f(z)) \vee S(z)^\Pi \quad \neg S(a)^\Sigma}{(\neg S(y_2) \wedge \top) \vee (S(y_2) \wedge \perp) \mid \neg P(x_4)}}{(\forall x_3) \exists y_2 \forall x_4 (\forall x_3) \left((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2)) \right)}$$

$$\Sigma \models (\forall x_3) \exists y_2 \forall x_4 (\forall x_3) \left((\neg P(x_4) \wedge \neg R(x_3)) \vee (P(x_3) \wedge S(y_2)) \right)$$

$$\Sigma \models (\forall x_3) \forall x_4 (\forall x_3) \left((\neg P(x_4) \wedge \neg R(x_3)) \vee P(x_3) \right)$$

$$\Sigma \not\models (\neg P(1) \wedge \neg R(0)) \vee P(0) \quad // \text{ if } P \sim \{1\} \text{ and } R \sim \{0\}$$

we know that for original clauses l and l' of $P(x_4)$ and $P(x_3)$, $l\sigma = l'\sigma$

hence same color, and can use different var as same value works.

$f(x) \vee g(x)$ **with** f, g **different colors**

$$\begin{array}{c}
 \frac{\frac{\frac{\Pi}{\neg P(z)} \quad \frac{\Pi}{\neg R(g(a))} \quad \frac{\frac{\Sigma}{P(f(x)) \vee Q(x)}{\forall x_1 Q(x_1) \mid P(f(x)) \vee R(g(x))} \quad \frac{\Pi}{R(g(y)) \vee \neg Q(y)}{y \mapsto x}}{x \mapsto a} \quad \frac{\Pi}{\forall x_1 Q(x_1) \mid P(f(a)) \vee R(g(a))}}{z \mapsto f(a)} \\
 \frac{\forall x_1 \exists x_2 (Q(x_1) \vee P(x_2)) \vee R(g(a))}{\forall x_1 \exists x_2 (Q(x_1) \vee P(x_2)) \vee R(g(a))}
 \end{array}$$

\Rightarrow free vars in the interpolant have to be overbound (if there are arrows, but we can just always do so)

misc examples

201a

$$\frac{\frac{P(x, y) \vee \neg Q(y) \quad \neg P(a, y_2)}{\neg Q(y)} \quad x \mapsto a \quad \frac{Q(f(z)) \vee R(z) \quad \neg R(a)}{Q(f(a))} \quad z \mapsto a}{\square} \quad y \mapsto f(a)$$

$$\frac{\frac{\perp \quad \top}{P(a, y)} \quad x \mapsto a \quad \frac{\perp \quad \top}{R(a)} \quad z \mapsto a}{P(a, f(a)) \vee R(a)} \quad y \mapsto f(a)$$

$$\frac{\frac{\perp \quad \top}{\forall x_1 P(x_1, y)} \quad x \mapsto a \quad \frac{\perp \quad \top}{\forall x_3 R(x_3)} \quad z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 (P(x_1, x_2) \vee R(x_3))} \quad y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1, x_2) \vee R(x_1)$

201b

$$\frac{\frac{P(x, f(y)) \vee \neg Q(f(y)) \quad \neg P(a, y_2)}{\neg Q(f(y))} \quad x \mapsto a \quad \frac{Q(f(z)) \vee R(z) \quad \neg R(a)}{Q(f(a))} \quad z \mapsto a}{\square} \quad y \mapsto f(a)$$

$$\frac{\frac{\perp \quad \top}{P(a, f(y))} \quad x \mapsto a \quad \frac{\perp \quad \top}{R(a)} \quad z \mapsto a}{P(a, f(a)) \vee R(a)} \quad y \mapsto a$$

$$\frac{\frac{\perp \quad \top}{\forall x_1 \exists x_2 P(x_1, x_2)} \quad x \mapsto a \quad \frac{\perp \quad \top}{\forall x_3 R(x_3)} \quad z \mapsto a}{\forall x_3 \forall x_1 \exists x_2 P(x_1, x_2) \vee R(x_3)} \quad y \mapsto f(a)$$

Huang would produce: $\forall x_1 \exists x_2 P(x_1, x_2) \vee R(x_1)$

arrow in element which is not in interpolant or resolution clause

206

$$\begin{array}{c}
 \frac{P(x) \vee \neg Q(f(x)) \quad \neg P(a)}{\forall x_1 P(x_1) \mid \neg Q(f(a))} \quad x \mapsto a \quad \frac{Q(y) \vee R(g(y)) \quad \neg R(z)}{\exists x_2 R(x_2) \mid Q(y)} \quad z \mapsto g(y) \\
 \hline
 \frac{\quad}{\forall x_1 \exists x_2 (P(x_1) \vee R(x_2)) \mid \square} \quad y \mapsto f(a) \\
 \hline
 P(a) \vee R(g(f(a)))
 \end{array}$$

for first interpolant, $\Sigma \not\models \ell_{\Delta,x}[\text{PI}(C)] \vee C$

\Rightarrow need to overbind clause as well