0.1 referenced lemmas from previous sections

⟨lemma:logic_commute⟩?

Lemma 1 (Commutativity of lifting and logical operators)**.** *Let A and B be first-order formulas and s and t be terms. Then it holds that:*

1. $\ell^z_\Phi[\neg A] \Leftrightarrow \neg \ell^z_\Phi[A]$
2. $\ell^z_\Phi[A \circ B] \Leftrightarrow (\ell^z_\Phi[A] \circ \ell^z_\Phi[B])$ *for* $\circ \in \{\wedge, \vee\}$
3. $\ell^z_\Phi[s = t] \Leftrightarrow (\ell^z_\Phi[s] = \ell^z_\Phi[t])$

⟨lemmaCommutLiftSubst@data⟩? ⟨lemmaCommutLiftSubst⟩?

Lemma 2 (Commutativity of lifting and substitution)**.** *Let C be a clause and σ a substitution such that no lifting variable occurs in C or σ. Define σ' with $\mathrm{dom}(\sigma') = \mathrm{dom}(\sigma) \cup \{z_t \mid t\sigma \neq t\}$ such that for a variable z,*

$$x\sigma' = \begin{cases} z_{t\sigma} & \text{if } x = z_t \text{ and } t\sigma \neq t \\ \ell^z_\Phi[x\sigma] & \text{otherwise} \end{cases}$$

Then $\ell^z_\Phi[C\sigma] = \ell^z_\Phi[C]\sigma'$.

Chapter 1

Interpolant extraction from resolution proofs in one phase

While the previous chapter demonstrates that it is possible to extract propositional interpolants and lift them from the colored symbols later in order to obtain a proper interpolant, we now present a novel approach, which only operates with grey intermediary interpolants. This is established by lifting any term which is added to the interpolant.

By its nature, this approach requires an alternate strategy than the proof of the extraction in two phases as a commutation of substitution and lifting is no longer possible if lifting variables are present. Let us recall the corresponding lemma from the previous chapter:

Lemma 2 (Commutativity of lifting and substitution)**.** *Let C be a clause and σ a substitution such that no lifting variable occurs in C or σ. Define σ' with* $\mathrm{dom}(\sigma') = \mathrm{dom}(\sigma) \cup \{z_t \mid t\sigma \neq t\}$ *such that for a variable z,*

$$x\sigma' = \begin{cases} z_{t\sigma} & \text{if } x = z_t \text{ and } t\sigma \neq t \\ \ell^z_\Phi[x\sigma] & \text{otherwise} \end{cases}$$

Then $\ell^z_\Phi[C\sigma] = \ell^z_\Phi[C]\sigma'$.

Consider the following illustration of a problem of the notion of applying this lemma to terms containing lifting variables:

Example 3. Let $\sigma = \{x \mapsto a\}$ and consider the terms $f(x)$ and $f(a)$, where f and a are colored symbols. Clearly $f(x)\sigma = f(a)$ and therefore necessarily $z_{f(x)}\sigma' = z_{f(a)}$.

But now consider $x_{f(x)}\sigma$. As $z_{f(x)}$ is a lifting variable, it is not affected by unifiers from resolution derivations and also not by σ. Hence $z_{f(x)}\sigma = z_{f(x)}$ and therefore $\ell[z_{f(x)}\sigma] = \ell[z_{f(x)}] = z_{f(x)}$, but $\ell[z_{f(x)}]\sigma' = z_{f(x)}\sigma' = z_{f(a)}$. So $\ell[z_{f(x)}\sigma] \neq \ell[z_{f(x)}]\sigma'$.

We see here that there are circumstances under which in order to commute lifting and substitution, the substitution σ' is required to conform to the equation $z_{f(x)}\sigma' = z_{f(a)}$, whereas in others, it must hold that $z_{f(x)}\sigma' = z_{f(x)}$. $\triangle$

1.1 Definition of the extraction algorithm

The extracted interpolants are prenex formulas, where the quantifier block and the matrix of the formula are calculated separately in each step of the traversal of the resolution refutation.

Extraction of the interpolant matrix $\mathrm{AI}_{\mathrm{mat}}$ and calculation of $\mathrm{AI}_{\mathrm{cl}}$

$\mathrm{AI}_{\mathrm{mat}}$ is inspired by the propositional interpolants PI from Definition **??**. Its difference lies in the fact that the lifting occurs in every step of the extraction. This however necessitates applying these liftings to the clauses of the resolution refutation as well. For a clause C of the resolution refutation, we will denote the clause with the respective liftings applied by $\mathrm{AI}_{\mathrm{cl}}(C)$ (a formal definition will be given below), and for a term t at position p in C, we denote $\mathrm{AI}_{\mathrm{cl}}(C)|_p$ by t_{AIcl}.

Now we can define preliminary versions of $\mathrm{AI}^\bullet_{\mathrm{mat}}$ and $\mathrm{AI}^\bullet_{\mathrm{cl}}$:

Definition 4 ($\mathrm{AI}^\bullet_{\mathrm{mat}}$ and $\mathrm{AI}^\bullet_{\mathrm{cl}}$)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$.

For a clause C in π, $\mathrm{AI}^\bullet_{\mathrm{mat}}(C)$ and $\mathrm{AI}^\bullet_{\mathrm{cl}}(C)$ are defined as follows:

Base case. If $C \in \Gamma$, $\mathrm{AI}^\bullet_{\mathrm{mat}}(C) \stackrel{\text{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{AI}^\bullet_{\mathrm{mat}}(C) \stackrel{\text{def}}{=} \top$.

In any case, $\mathrm{AI}^\bullet_{\mathrm{cl}}(C) \stackrel{\text{def}}{=} \ell[C]$.

Resolution. If the clause C is the result of a resolution step of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{AI}^\bullet_{\mathrm{mat}}(C)$ and $\mathrm{AI}^\bullet_{\mathrm{cl}}$ are defined as follows:

$\mathrm{AI}^\bullet_{\mathrm{cl}}(C) \stackrel{\text{def}}{=} \ell[(\mathrm{AI}^\bullet_{\mathrm{cl}}(C_1)\backslash\{l_{\mathrm{AIcl}}\})\sigma] \;\vee\; \ell[(\mathrm{AI}^\bullet_{\mathrm{cl}}(C_2)\backslash\{l'_{\mathrm{AIcl}}\})\sigma]$

1. If l is Γ-colored: $\mathrm{AI}^\bullet_{\mathrm{mat}}(C) \stackrel{\text{def}}{=} \ell[\mathrm{AI}^\bullet_{\mathrm{mat}}(C_1)\sigma] \vee \ell[\mathrm{AI}^\bullet_{\mathrm{mat}}(C_2)\sigma]$

2. If l is Δ-colored: $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma] \wedge \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_2)\sigma]$
3. If l is grey: $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} (\neg\ell[l'_{\mathrm{AIcl}}\sigma] \wedge \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma]) \vee (\ell[l_{\mathrm{AIcl}}\sigma] \wedge \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_2)\sigma])$

Factorisation. If the clause C is the result of a factorisation of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C_1)\sigma]$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[(\mathrm{AI}_{\mathrm{cl}}(C_1)\backslash\{l'_{\mathrm{AIcl}}\})\sigma]$. △

Note that in $\mathrm{AI}^{\bullet}_{\mathrm{mat}}$ and $\mathrm{AI}^{\bullet}_{\mathrm{cl}}$, it is possible that there for a colored term t in C that $t_{\mathrm{AIcl}} \neq z_t$ as illustrated by the following examples:

Example 5. We consider a resolution refutation of the initial clause sets $\Gamma = \{R(c), \neg Q(v)\}$ and $\Delta = \{\neg R(u) \vee Q(g(u))\}$:

$$\dfrac{\dfrac{R(c) \qquad \neg R(u) \vee Q(g(u))}{Q(g(c))}\,\mathrm{res}, y \mapsto c \qquad \neg Q(v)}{\square}\,\mathrm{res}, v \mapsto g(c)$$

We now replace every clause C by $\mathrm{AI}^{\bullet}_{\mathrm{mat}}(C) \mid \mathrm{AI}^{\bullet}_{\mathrm{cl}}(C)$ in order to visualize the steps of the algorithm:

$$\dfrac{\dfrac{\bot \mid R(y_c) \qquad \top \mid \neg R(u) \vee \neg Q(x_{g(u)})}{R(y_c) \mid Q(x_{g(u)})}\,\mathrm{res}, y \mapsto c \qquad \bot \mid \neg Q(v)}{\neg Q(x_{g(c)}) \wedge R(y_c) \mid \square}\,\mathrm{res}, v \mapsto g(c)$$

By quantifying y_c existentially and $x_{g(c)}$ universally[1], we obtain an interpolant for $\Gamma \cup \Delta$: $\exists y_c \forall x_{g(c)}(\neg Q(x_{g_c}) \wedge R(y_c))$. Note however that $\ell[Q(g(c))] = Q(x_{g(c)})$, but $\mathrm{AI}_{\mathrm{mat}}(Q(g(c))) = Q(x_{g(u)})$. This example shows that this circumstance is not necessarily an obstacle for the correctness of this algorithm. △

⟨exa:2b⟩ **Example 6.** We consider a resolution refutation of the initial clause sets $\Gamma = \{R(c), P(c)\}$ and $\Delta = \{\neg R(u) \vee \neg Q(g(u)), \neg P(v) \vee Q(g(v))\}$:

$$\dfrac{\dfrac{\neg R(u) \vee \neg Q(g(u)) \qquad R(c)}{\neg Q(g(c))}\,\mathrm{res}, u \mapsto c \qquad \dfrac{\neg P(v) \vee Q(g(v)) \qquad P(c)}{Q(g(c))}\,\mathrm{res}, v \mapsto c}{\square}\,\mathrm{res}$$

[1]The procedure for calculating the quantifier block is defined in section 1.1

We now again display $\mathrm{AI}^\bullet_{\mathrm{mat}}(C) \mid \mathrm{AI}^\bullet_{\mathrm{cl}}(C)$ for every clause C of the refutation:

$$\dfrac{\dfrac{\top \mid \neg R(u) \vee \neg Q(x_{g(u)}) \qquad \bot \mid R(y_c)}{R(y_c) \mid \neg Q(x_{g(u)})}\ \text{res}, u \mapsto c \qquad \dfrac{\top \mid \neg P(v) \vee Q(x_{g(v)}) \qquad \bot \mid P(y_c)}{P(y_c) \mid Q(x_{g(v)})}\ \text{res}, v \mapsto c}{(Q(x_{g(v)}) \wedge R(y_c)) \ \vee\ (\neg Q(x_{g(u)}) \wedge P(y_c)) \mid \square}\ \text{res}$$

Note again that here, we have that $\ell[\neg Q(g(c))] = \neg Q(x_{g(c)}) \neq \mathrm{AI}^\bullet_{\mathrm{cl}}(\neg Q(g(c))) = \neg Q(x_{g(u)})$ and $\ell[Q(g(c))] = Q(x_{g(c)}) \neq \mathrm{AI}^\bullet_{\mathrm{cl}}(Q(g(c))) = Q(x_{g(v)})$. However in this instance, it is not possible to find quantifiers for the free variables of $\mathrm{AI}^\bullet_{\mathrm{mat}}(\square)$ such that by binding them, an interpolant is produced. For the naive approach, namely to use $\exists y_c \forall x_{g(v)} \forall x_{g(u)}$ as prefix, it holds that $\Gamma \not\models \exists y_c \forall x_{g(v)} \forall x_{g(u)}((Q(x_{g(v)}) \wedge R(y_c)) \ \vee\ (\neg Q(x_{g(u)}) \wedge P(y_c)))$. This failure is possible as intuitively, resolution deductions are valid by virtue of the resolved literals being equal. The interpolant extraction procedure exploits this property not directly on the clauses but on the lifted clause, i.e. on $\mathrm{AI}_{\mathrm{cl}}(C)$ for a clause C. Note that by ensuring that for resolved literals l and l', it holds that $l_{\mathrm{AIcl}} = l'_{\mathrm{AIcl}}$, we can obtain an interpolant, for instance: $\exists y_c \forall x^*((Q(x^*) \wedge R(y_c)) \ \vee\ (\neg Q(x^*) \wedge P(y_c)))$. △

In order to avoid the pitfall shown in Example 6 and to generalize the indicated solution, we define a function on resolved literals calculating a substitution, which ensures that the literals in the lifted clause, which correspond to the resolved literals, are equal.

Definition 7 (au)**.** For resolved or factorised literals l and l' of a resolution derivation step with a unifier σ such that $l\sigma = l'\sigma$, and $\ell[l_{\mathrm{AIcl}}\sigma] = P(s_1, \dots, s_n)$ and $\ell[l'_{\mathrm{AIcl}}\sigma] = P(t_1, \dots, t_n)$, we define:

$$\mathrm{au}(P(s_1, \dots, s_n), P(t_1, \dots, t_n)) \stackrel{\text{def}}{=} \bigcup_{i=1}^{n} \mathrm{au}(s_i, t_i)$$

For terms s and t where $s = \ell[l_{\mathrm{AIcl}}\sigma]|_p$ and $t = \ell[l'_{\mathrm{AIcl}}\sigma]|_p$ for some position p, we define:

$$\mathrm{au}(s,t) \stackrel{\text{def}}{=} \begin{cases} \bigcup_{i=1}^{n} \mathrm{au}(s_i, t_i) & \text{if } s \text{ is grey and } s = f_s(s_1, \dots, s_n),\ t = f_t(t_1, \dots, t_n)^2 \\ \{z_{s'} \mapsto z_r, z_{t'} \mapsto z_r\} & \text{if } s \text{ is a lifting variable } z_{s'},\ t = z_{t'}, \text{ and } z_r = \ell[l\sigma]|_p \end{cases} \quad \triangle$$

[2]Note that constants are treated as function symbols of arity zero.

⟨prop:tau_dom_ran⟩ **Proposition 8.** *Let l and l' be the resolved or factorised literals of a resolution derivation step and σ the unifier employed in this step. Furthermore let $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$. Then $\mathrm{dom}(\tau)$ consists exactly of the lifting variables of $\ell[l_{\mathrm{AIcl}}\sigma]$ and $\ell[l'_{\mathrm{AIcl}}\sigma]$ and $\mathrm{ran}(\tau)$ consists exactly of the lifting variables of $\ell[l\sigma]$.*

possibly argue here why au is well-defined (but it follows more or less directly from a later lemma)

Definition 9 ($\mathrm{AI}_{\mathrm{mat}}$ and $\mathrm{AI}_{\mathrm{cl}}$)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$. $\mathrm{AI}_{\mathrm{mat}}(\pi)$ is defined to be $\mathrm{AI}_{\mathrm{mat}}(\square)$, where $\square$ is the empty box derived in π.

For a clause C in π, $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ are defined inductively as follows:

Base case. If $C \in \Gamma$, $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \top$.

In any case, $\mathrm{AI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[C]$.

Resolution. If the clause C is the result of a resolution step of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then let $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$ and define $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ as follows:

$$\mathrm{AI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[(\mathrm{AI}_{\mathrm{cl}}(C_1) \backslash \{l_{\mathrm{AIcl}}\})\sigma]\tau \ \vee\ \ell[(\mathrm{AI}_{\mathrm{cl}}(C_2) \backslash \{l'_{\mathrm{AIcl}}\})\sigma]\tau$$

1. If l is Γ-colored: $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau$
2. If l is Δ-colored: $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau$
3. If l is grey: $\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} (\neg\ell[l'_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau) \ \vee\ (\ell[l_{\mathrm{AIcl}}\sigma]\tau \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)\sigma]\tau)$

Factorisation. If the clause C is the result of a factorisation of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then let $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$ and define $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ as follows:

$$\mathrm{AI}_{\mathrm{mat}}(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)\sigma]\tau$$

$$\mathrm{AI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} \ell[(\mathrm{AI}_{\mathrm{cl}}(C_1) \backslash \{l'_{\mathrm{AIcl}}\})\sigma]\tau$$

△

Definition 10. $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ ($\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$) for a clause C is defined as $\mathrm{AI}_{\mathrm{mat}}(C)$ ($\mathrm{AI}_{\mathrm{cl}}(C)$) with the difference that in its inductive definition, every lifting $\ell[\varphi]$ for a formula or term φ is replaced by a lifting of only the Δ-terms $\ell_{\Delta}[\varphi]$. △

⟨lemma:no_colored_terms⟩ **Lemma 11.** *Let C be a clause of a resolution refutation of $\Gamma \cup \Delta$. $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$ do not contain colored symbols. $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ and $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ do not contain Δ-colored symbols.*

Proof. For $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$, consider the following: In the base case of the inductive definitions of $\mathrm{AI}_{\mathrm{mat}}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C)$, no colored symbols occur. In the inductive steps, any colored symbol which is added by σ to intermediary formulas is lifted. By Proposition 8, $\mathrm{ran}(\tau)$ only consists of lifting variables.

For $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$ and $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$, a similar argument goes through by reading colored as Δ-colored. □

⟨lemma:substitute_and_lift⟩ **Lemma 12.** *Let σ be a substitution and F a formula without Φ-colored terms such that for a set of formulas Ψ, $\Psi \models F$. Then $\Psi \models \ell^{z}_{\Phi}[F\sigma]$.*

Proof. $\ell^{z}_{\Phi}[F\sigma]$ is an instance of F: σ substitutes variables either for terms not containing Φ-colored symbols or by terms containing Φ-colored symbols. For the first kind, the lifting has no effect. For the latter, the lifting only replaces subterms of the terms introduced by the substitution by a lifting variable such that the original structure of F remains invariant as it by assumption does not contain colored terms. □

Lemma 13. *Let l and l' be resolved or factorised literals in a resolution derivation step creating a clause C and $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$. For any substitution $(z_s \mapsto z_t) \in \tau$,*

TODO: check which statment we actually need (resolved literal, clause?)

make sure that it works for positions in the resolved literals as well as in the clause

Lemma 14. ***either reduce to "equal up to index of lifting variables" or use elaborate version as given below with additional lemma about how every x_s refers to the same term PLUS variable renaming convention***

⟨lemma:literals_clause_simgeq⟩

Let λ be a literal in a clause C occurring in a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{AI}_{\mathrm{cl}}(C)$ contains a literal λ_{AIcl} such that $\lambda_{\mathrm{AIcl}} \gtrsim \ell[\lambda]$, where $\gtrsim$ is defined as follows:

$$\varphi \gtrsim \varphi' \Leftrightarrow \begin{cases} P = P' \wedge \bigwedge_{i=1}^{n} s_i \gtrsim s'_i & \text{if } \varphi = P(s_1, \dots, s_n) \text{ and } \varphi' = P'(s'_1, \dots, s'_n) \\ f = f' \wedge \bigwedge_{i=1}^{n} s_i \gtrsim s'_i & \text{if } \varphi = f(s_1, \dots, s_n) \text{ and } \varphi' = f'(s'_1, \dots, s'_n) \\ x = x' & \text{if } \varphi, \varphi' \text{ are non-lifting variables, } \varphi = x \text{ and } \varphi' = x' \\ s' \text{ is a specialisation of } s & \text{if } \varphi, \varphi' \text{ are lifting variables, } \varphi = z_s \text{ and } \varphi' = z_{s'} \end{cases}$$

For resolved or factorised literals λ we furthermore have that $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau \gtrsim \ell[\lambda\sigma]$.

introduce definition for characterising the relation between C and $\mathrm{AI}_{\mathrm{cl}}(C)$

Proof. We proceed by induction on the resolution refutation.

Base case. If for a clause C either $C \in \Gamma$ or $C \in \Delta$ holds, then $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[C]$. Therefore for every literal l in C, there exists a literal l_{AIcl} in $\mathrm{AI}_{\mathrm{cl}}(C)$ such that $\ell[l] = l_{\mathrm{AIcl}}$, which implies $l_{\mathrm{AIcl}} \gtrsim \ell[l]$.

Resolution. If the clause C is the result of a resolution step of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then let $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$. Let λ be a literal in C_1 or C_2. Note that every literal in C is of the form $\lambda\sigma$. By the induction hypothesis, there is a literal in $\mathrm{AI}_{\mathrm{cl}}(C_1)$ or $\mathrm{AI}_{\mathrm{cl}}(C_2)$ respectively such that $\lambda_{\mathrm{AIcl}} \gtrsim \ell[\lambda_{\mathrm{AIcl}}]$. If $\lambda \notin \{l, l'\}$, then $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau$ is contained in $\mathrm{AI}_{\mathrm{cl}}(C)$. Hence in any case, it remains to show that $\ell[\lambda_{\mathrm{AIcl}}\sigma]\tau \gtrsim \ell[\lambda\sigma]$.

We perform an induction on the structure of λ_{AIcl} and λ by letting p be the position of the current term in the induction and $t_{\mathrm{AIcl}} = \lambda_{\mathrm{AIcl}}|_p$ as well as $t = \lambda|_p$.

- Suppose that t is a non-lifting variable. As by the induction hypothesis $\ell[t_{\mathrm{AIcl}}] \gtrsim t$, t_{AIcl} is a non-lifting variable as well and $t = t_{\mathrm{AIcl}}$. But then $\ell[t_{\mathrm{AIcl}}\sigma] = \ell[t\sigma]$. If τ is trivial on $\ell[t_{\mathrm{AIcl}}\sigma]$, we are done as then $\ell[t_{\mathrm{AIcl}}\sigma]\tau = \ell[t\sigma]$, so assume that it is not.

  But by the definition of au, the substitutions in τ only update lifting variables to correspond to the terms in the clause of the actual resolution derivation. More formally, $\ell[t_{\mathrm{AIcl}}\sigma]\tau = z_s$ for some term s implies that $\ell[\lambda\sigma]|_p = z_s$, but then $z_s = t$.

  outsource this thought to lemma after definition of au in case needed elsewhere

- Suppose that t is colored term. Then $\ell[t]$ is a lifting variable and by the induction hypothesis, t_{AIcl} is one as well such that $\ell[t]$ is a specialisation of t_{AIcl}. As lifting variables are not affected by the unifications occurring in resolution derivations, we only need to consider modifications by means of τ. But as we have seen in the previous case, if τ substitutes $\ell[t_{\mathrm{AIcl}}\sigma]$, then it does so by t.

  lemma

  Hence we obtain that $\ell[t_{\mathrm{AIcl}}\sigma]\tau \gtrsim \ell[t\sigma]$.

- Suppose that t is a grey term of the form $f(s_1, \ldots, s_n)$. Then $\ell[t] = f(\ell[s_1], \ldots, \ell[s_n])$ and by the induction hypothesis, $t_{\mathrm{AIcl}} = f(r_1, \ldots, r_n)$ such that $\bigwedge_{i=1}^{n} r_i \gtrsim \ell[s_i]$. By the induction hypothesis applied to the parameters of $\ell[t]$ and $\ell[t_{\mathrm{AIcl}}]$, we obtain that $\ell[r_i\sigma]\tau \gtrsim \ell[s_i\sigma]$ for $1 \leqslant i \leqslant n$. Hence $f(\ell[r_1\sigma], \ldots, \ell[r_n\sigma]) \gtrsim f(\ell[s_1\sigma], \ldots, \ell[s_n\sigma])$, which however is nothing else than $\ell[t_{\mathrm{AIcl}}\sigma] \gtrsim \ell[t\sigma]$.

Factorisation. If the clause C is the result of a factorisation, then we can argue analoguously as for resolution. □

ma:lifted_literal_equal⟩ **Lemma 15.** *Let l and l' be the resolved or factorised literals of a resolution derivation step employing the unifier σ such that $l\sigma = l'\sigma$. Furthermore let $\tau = \mathrm{au}(\ell[l_{\mathrm{AIcl}}\sigma], \ell[l'_{\mathrm{AIcl}}\sigma])$. Then $\ell[l_{\mathrm{AIcl}}\sigma]\tau = \ell[l'_{\mathrm{AIcl}}\sigma]\tau$.*

Proof. As $l\sigma = l'\sigma$, it also holds that $\ell[l\sigma] = \ell[l'\sigma]$. By Lemma 14, we obtain that $\ell[l_{\mathrm{AIcl}}\sigma]\tau \gtrsim \ell[l\sigma]$ and $\ell[l'_{\mathrm{AIcl}}\sigma]\tau \gtrsim \ell[l'\sigma]$. Furthermore note that the $\gtrsim$-relation guarantees that pairs of predicates and terms in this relation are equal up to the index of their lifting variables. Hence it only remains to show that the lifting variables of $\ell[l_{\mathrm{AIcl}}\sigma]\tau$ and $\ell[l'_{\mathrm{AIcl}}\sigma]\tau$ match. But by the definition of au, τ substitutes any lifting variable at position p of $\ell[l_{\mathrm{AIcl}}\sigma]$ and $\ell[l'_{\mathrm{AIcl}}\sigma]$ by the lifting variable $\ell[l\sigma]|_p$, thus making them equal. □

Lemma 16. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then for clauses C in π, $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.*

Proof. We proceed by induction of the strengthening $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_\Gamma)$[3].

Base case. For $C \in \Gamma$, $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_\Gamma) = \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C) = \ell_\Delta[C] = C$, so $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_\Gamma)$. Otherwise $C \in \Delta$ and hence $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = \top$.

[3]Recall that as in Lemma **??**, D_Φ denotes the clause created from the clause D by removing all literals which are not contained $\mathrm{L}(\Phi)$.

Resolution. Suppose the last rule application is an instance of resolution. Then it is of the following form:

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad l\sigma = l'\sigma$$

We introduce the following abbreviations:

$\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* = \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma) \backslash \{(l_{\mathrm{AIcl}^\Delta})_\Gamma\}$

$\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^* = \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma) \backslash \{\neg(l'_{\mathrm{AIcl}^\Delta})_\Gamma\}$

Note that $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C) = \ell_\Delta[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell_\Delta[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$, where τ is defined as in the interpolant extraction procedure as $\mathrm{au}(\ell[l_{\mathrm{AIcl}^\Delta}\sigma], \ell[l'_{\mathrm{AIcl}^\Delta}\sigma])$.

Employing these, the induction hypothesis yields $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* \vee (l_{\mathrm{AIcl}^\Delta})_\Gamma$ as well as $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^* \vee \neg(l'_{\mathrm{AIcl}^\Delta})_\Gamma$. By Lemma 11, $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_i)$ and $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C_i)$ for $i \in \{1, 2\}$ do not contain Δ-colored symbols. Hence by Lemma 12, pulling the lifting inwards using Lemma 1 and applying τ, we obtain:

$$\Gamma \overset{(\circ)}{\models} \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[(l_{\mathrm{AIcl}^\Delta})_\Gamma\sigma]\tau$$

$$\Gamma \overset{(*)}{\models} \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau \vee \neg\ell[(l'_{\mathrm{AIcl}^\Delta})_\Gamma\sigma]\tau$$

We continue by a case distinction on the color of l:

1. Suppose that l is Γ-colored. Then $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau$. As l is Γ-colored, $(l_{\mathrm{AIcl}^\Delta})_\Gamma = l_{\mathrm{AIcl}^\Delta}$ and as $l\sigma = l'\sigma$, also $(l'_{\mathrm{AIcl}^\Delta})_\Gamma = l'_{\mathrm{AIcl}^\Delta}$. By Lemma 15, $\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau = \ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau$. Hence we can perform a resolution step on ($\circ$) and ($*$) to arrive at $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$. This is however by Lemma 1 nothing else than $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.
2. Suppose that l is Δ-colored. Then $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau$. As l and l' are Δ-colored, ($\circ$) and ($*$) reduce to $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau$ and $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$ respectively. These however imply that $\Gamma \models (\ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau) \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$, which in turn is nothing else than $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$.
3. Suppose that l is grey. Then $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) = (\neg\ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau) \vee (\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau \wedge \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau)$.

Let M be a model of Γ. Suppose that $M \models \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ as otherwise we are done. Hence $M \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau$ and $M \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_2)_\Gamma)^*\sigma]\tau$ and $(\circ)$ and $(*)$ reduce to $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau$ and $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_2)\sigma]\tau \vee \ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau$ respectively. As by Lemma 15 $\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau = \ell[l'_{\mathrm{AIcl}^\Delta}\sigma]\tau$, a case distinction on the truth value of $\ell[l_{\mathrm{AIcl}^\Delta}\sigma]\tau$ in M shows that $M \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C)$.

Factorisation. Suppose the last rule application is an instance of factorisation. Then it is of the following form:

$$\frac{C_1 : l \vee l' \vee D}{C : (l \vee D)\sigma} \quad \sigma = \mathrm{mgu}(l, l')$$

We introduce the abbreviation $\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* \overset{\mathrm{def}}{=} \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)\backslash\{(l_{\mathrm{AIcl}})_\Gamma, (l'_{\mathrm{AIcl}})_\Gamma\}$ and express the induction hypothesis as follows: $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^* \vee (l_{\mathrm{AIcl}})_\Gamma \vee (l'_{\mathrm{AIcl}})_\Gamma$. By Lemma 11, Lemma 12 and Lemma 1 and after applying τ to the induction hypothesis, we obtain that $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[(l_{\mathrm{AIcl}})_\Gamma\sigma]\tau \vee \ell[(l'_{\mathrm{AIcl}})_\Gamma\sigma]\tau$.

However by Lemma 15, $\ell[(l_{\mathrm{AIcl}})_\Gamma\sigma]\tau = \ell[(l'_{\mathrm{AIcl}})_\Gamma\sigma]\tau$, hence we can perform a factorisation step to arrive at $\Gamma \models \ell[\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C_1)\sigma]\tau \vee \ell[\mathrm{AI}^{\Delta}_{\mathrm{cl}}((C_1)_\Gamma)^*\sigma]\tau \vee \ell[(l_{\mathrm{AIcl}})_\Gamma\sigma]\tau$. This however is nothing else than $\Gamma \models \mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$. □

Lemma 17. *Let $\overline{x}$ be the Δ-lifting variables and $\overline{y}$ be the Γ-lifting variables of $\mathrm{AI}(C)$. Let $\overline{x'}$ be the Δ-lifting vairables of $\mathrm{AI}^{\Delta}(C)$.*

$\Gamma \models \overline{\forall x}\, \mathrm{AI}^{\Delta}(C)$ *implies* $\Gamma \models \overline{\forall x | \exists y}\, \mathrm{AI}(C)$.

Proof. wie auf zettel

□

Theorem 18. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{AI}_{\mathrm{mat}}(\pi)$ is an interpolant.*

Proof. This needs too many things I don't yet know how to make precise, so let's start with $\Gamma \models \ldots$

□

c:arrow_quantifier_block⟩

Chapter 2

outline of arrow part

Definition 19. arrows 1: if x occurs in $y\sigma$, add arrow from every *grey* occurrence of x in C to every colored occurrence of y in C_i.

arrows 2: if a maximal colored term t occurs grey in $x\sigma$, add arrow from every grey occurrence of t in (?C or C_i?) to every *colored* occurrence of x in C_i.

arrows 3: if a maximal Φ-colored term t occurs inside a maximal Ψ-colored term s in $x\sigma$, add an arrow from every grey occurrence of t in (?C or C_i?) to every occurrence of x in C_i. $\triangle$

Lemma 20. *If in* $\mathrm{AI}^{\Delta}_{\mathrm{mat}}(C) \vee \mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ *a* Γ*-colored term* $t[x_s]$ *contains a* Δ*-lifting variable* x_s*, then* $x_s \rightsquigarrow t[x_s]$.

Proof. Suppose term containing max colored term which is Δ-term is introduced into Γ-colored term.

Then colored u in C_i s.t. δ_i in $u\sigma$. Hence by arrow 2, arrow from every grey δ_i to every colored u. TODO: as below, need existence

Suppose multicolored Γ-term introduced.

Then u in C_i s.t. $\gamma[\delta_i]$ in $u\sigma$. Hence by arrow 3, arrow from every grey δ_i to every u. TODO: need make sure that grey δ_i exists (exaclty δ_i? what if lifted) $\square$