1 Arrow-Algo

1. In the original clauses, find all occurrences of variables.

   Common case: If a variable appears as outermost symbol or only has grey ancestor-terms, add an arrow from it to all other occurrences.

   Uncommon case: if there is more than one occurrence of a variable under a Φ-colored term, add a *weak* dependency between them all (symmetric relation).

   NOTE: this creates double arrows for occurrences at same depth. This appears to be necessary for terms which are only variables, and doesn't hurt if the variable is contained in a term.

2. For each step in the derivation:

   a) Build propositional interpolant using $\mathrm{PI}(C_i)^*$, $i \in \{1, 2\}$, i.e. use ancestor PI without colored terms.

   b) If ancestors of atom added to $\mathrm{PI}(C)$ had arrows, merge them to atom in $\mathrm{PI}(C)$ (i.e. arrows starting in and leading to this atom).

   c) Replace colored terms in $\mathrm{PI}(C)$ (from new atom and unifier applied to $\mathrm{PI}(C_i)^*$) with fresh variables, except if a term has a double ended arrow to another overbinding variable, then use that variable.

      An arrow starts (ends) in one of the new variables if it starts (ends) somewhere in the term it replaced.

   d) Collect quantifiers: from $\mathrm{PI}(C_i)^*$, $i \in \{1, 2\}$ and ones from atom added to $\mathrm{PI}(C)$. Order such that arrows only point to variables to the right AND weakly connected variables appear in the same quantifier block.

      $\bar{Q}_n = \mathrm{sort}(Q_{n_1} \cup Q_{n_2} \cup \text{colored-terms}(l))$

1.1 algo more formally

Every literal in any initial clause set has a globally unique id/number
Ex: $P(y, a, f(z, g(y, b))) \lor Q(x)$
Term position:
0.2.1.0 means first literal, 3rd arg, 2nd arg, fst arg: y
0.1 is a
0.2.1 is $g(z, b)$
ρ calculates the position of a term or the term of a position, depending on the argument type.
for a position p_i, $\rho(p_i)$ denotes whatever p_i refers to in its respective clause.
for a term t, $\rho(t)$ denotes the position in t in its respective clause.
for a position p, $\rho_{\text{lit}}(p)$ denotes the position of the literal
for a position p, $\rho_{\text{term}}(p)$ denotes the position of the term in p_i
$\Rightarrow p = \rho_{\text{lit}}(p) . \rho_{\text{term}}(p)$
for a position p, $p \mod i$ denotes p with i least significant places cut off, $0.2.1.0 \mod 2 = 0.2$

1.2 Arrows:

$\mathcal{A}$ is a set of ordered pairs of term positions which point to positions in terms in literals
$\mathcal{W}$ is a set of unordered pairs of term positions which point to positions in terms in literals
w.r.t a refutation π of $\Gamma \cup \Delta$:

1. For each initial clause C in $\Gamma \cup \Delta$:

   Add to $\mathcal{A}$ all (p_1, p_2) in C such that p_1 contains only grey symbol and $\rho(p_1)$ is a variable and $\rho(p_1) = \rho(p_2)$ but $p_1 \neq p_2$.

   Add to $\mathcal{W}$ all $\{p_1, p_2\}$ such that there is a colored symbol in p_1 and a possible different one in p_2 and $\rho(p_1)$ is a var and $\rho(p_1) = \rho(p_2)$.

2. For each C resulting from a resolution step from $C_1 : D \lor l$ and $C_2 : E \lor \neg l$ to $C = D \lor E$ with prop interpolant $\mathrm{PI}(\cdot)$:

   Note: literals are added to the interpolant if they occur in both ancestors.

   Merge the respective ids of l and $\neg l$, i.e. their arrows. Their term structure will be the same, so all arrows point to valid positions.

1.3 algo

NOTE: for now, we assume that every colored-term has a globally unique id i and will be replaced by a variable with this index. This restriction is useful now and could potentially be lifted later, but it's not severe anyway.
Note: when a literal is added to the interpolant, the colored terms in one literal might have already been replaced with a certain variable before. we definitely have to use the same variable for both literals, and if one literal has other dependencies, we should stick with the variable we have.
PROBLEM: terms already replaced by variables still change! need to use same variable anyway, so note above not accurate!

1.3.1 $\mathrm{AI}_{\mathrm{mat}}$ and $\mathrm{AI}_{\mathrm{cl}}$

Here, we define $\mathrm{AI}_{\mathrm{mat}}$, which represents the *matrix* of what will be the interpolant, and $\mathrm{AI}_{\mathrm{cl}}$, which represents the *clauses* in the refutation applied with the same unifications as $\mathrm{AI}_{\mathrm{mat}}$.

1. For each initial clause C, $\mathrm{AI}_{\mathrm{mat}}(C) = \mathrm{PI}(C)$ and $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[C]$.

2. For each C resulting from a resolution step from $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ to $C = (D \vee E)\sigma$ with $l\sigma = l'\sigma$ with prop interpolant $\mathrm{PI}(\cdot)$:

   $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[\Big((\mathrm{AI}_{\mathrm{cl}}(C_1) \setminus \{\ell[l]\}) \vee (\mathrm{AI}_{\mathrm{cl}}(C_2) \setminus \{\ell[\neg l']\})\Big)\sigma]$

   - if l and l' don't have the same color:

     $\mathrm{AI}_{\mathrm{mat}}(C) = \ell[\Big((\neg l \wedge \mathrm{AI}_{\mathrm{mat}}(C_1)) \vee (l \wedge \mathrm{AI}_{\mathrm{mat}}(C_2))\Big)\sigma]$
   - if l and l' are Γ-colored :

     as above but with $\mathrm{AI}_{\mathrm{mat}}(C)' = \Big(\mathrm{AI}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}_{\mathrm{mat}}(C_2)\Big)\sigma$
   - if l and l' are Δ-colored:

     as above but with $\mathrm{AI}_{\mathrm{mat}}(C)' = \Big(\mathrm{AI}_{\mathrm{mat}}(C_1) \wedge \mathrm{AI}_{\mathrm{mat}}(C_2)\Big)\sigma$

1.3.2 AI

$\mathrm{AI}(C) = Q_1 u_1 \dots Q_m u_m \, \mathrm{AI}_{\mathrm{mat}}(C)$
$u_1, \dots, u_m$ are comprised of all x_i and y_i PLUS all free variables in $\mathrm{AI}_{\mathrm{mat}}(C)$.
Q_i is $\exists$ if $u_i = y_i$ for some i, $\forall$ if $u_i = x_i$ for some i. For free variables, Q_i as in C_1/C_2. if $u_i\sigma \neq u_i$, u_i becomes one of x_j or y_j or grey term.
$(p_1, p_2) \in \mathrm{TransitiveClosure}(\mathcal{A})$ implies that $u_i < u_j$ if u_i replaces t_i and u_j replaces t_j and p_1 points into t_i and p_2 points into t_j.

2 proof of propositional aspect of AI

Lemma 1 (Restated from proof without propositional refutation, lemma 1)**.** *Let C be a clause and σ a substitution. Let $t_1, \dots, t_n$ be all maximal Δ-terms in this context, i.e. those that occur in C or $C\sigma$, and $x_1, \dots, x_n$ the corresponding fresh variables to replace the t_i (i.e. none of the x_i occur in C). Define σ' such that for a variable z,*

$$z\sigma' = \begin{cases} x_l & \text{if } z = x_k \text{ and } t_k\sigma = t_l \\ \ell_{\Delta,x}[z\sigma] & \text{otherwise} \end{cases}$$

Then $\ell_{\Delta,x}[C\sigma] = \ell_{\Delta,x}[C]\sigma'$.

Lemma 2 (corresponds to Lemma 4.8 in thesis and Lemma 11 in Huang)**.** *Let A and B be first-order formulas and s and t be terms. Then it holds that:*

1. $\ell_{\Phi,x}[\neg A] \Leftrightarrow \neg\ell_{\Phi,x}[A]$
2. $\ell_{\Phi,x}[A \circ B] \Leftrightarrow (\ell_{\Phi,x}[A] \circ \ell_{\Phi,x}[B])$ *for* $\circ \in \{\wedge, \vee\}$
3. $\ell_{\Phi,x}[s = t] \Leftrightarrow (\ell_{\Phi,x}[s] = \ell_{\Phi,x}[t])$

Lemma 3. $\mathrm{AI}_{\mathrm{mat}}(C)$ *and* $\mathrm{AI}_{\mathrm{cl}}(C)$ *do not contain colored terms. // true*

Corollary 4. *For a clause C in a resolution refutation π of $\Gamma \cup \Delta$:*

1. $\mathrm{AI}_{\mathrm{mat}}(C) = \ell[\mathrm{AI}_{\mathrm{mat}}(C)]$.
2. $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[\mathrm{AI}_{\mathrm{cl}}(C)]$.

// true and used

Lemma 5. *The x_i, y_i and z_i do not occur in any substitution of a resolution refutation. // true but unused*

Lemma 6. *Let Φ be a set of formulas, t be a term and σ a substitution Then $\ell_{\Phi,x}[\ell_{\Phi,x}[t]\sigma] = \ell_{\Phi,x}[t\sigma]$. // true and used*

Proof. With σ' as in Lemma 1:
$\ell_{\Phi,x}[t\sigma] = \ell_{\Phi,x}[t]\sigma'$.
As σ just depends on the terms to replace and the variables to replace them with:
$\ell_{\Phi,x}[\ell_{\Phi,x}[t]\sigma] = \ell_{\Phi,x}[\ell_{\Phi,x}[t]]\sigma' = \ell_{\Phi,x}[t]\sigma'$. □

OLD from first principles reasoning (complete this in case Lemma 1 is flawed):
(this was based on Δ-terms)
t contains Δ-terms t_i, grey terms g_i and free variables v_i. $\ell_{\Delta,x}[t]$ is t, where every maximal Δ-term t_i is replaced by free variable x_i.
A substitution σ occuring in a resolution refutation does not affect any of the x_i, as these symbols do not occur in the initial clause sets.
Hence $\ell_{\Delta,x}[t]\sigma$ differs from $\ell_{\Delta,x}[t]$ only in the v_i, which are potentially substituted. The v_i can be substituted to other free variables v_i' (not the same ones as factors of clauses are variable disjoint), grey terms g_i' or Δ-terms t_i' (these terms may contain grey or Δ-terms or free variables). Assumption: $\ell_{\Delta,x}[]$ always replaces a certain Δ-term with the same variable, globally.
Goal: $\ell_{\Delta,x}[\ell_{\Delta,x}[t]\sigma] = \ell_{\Delta,x}[t\sigma]$.
Suppose σ does not affect t (and also not the x_i as it occurs in a resolution rule application). Then $\ell_{\Delta,x}[t\sigma] = \ell_{\Delta,x}[t] = \ell_{\Delta,x}[\ell_{\Delta,x}[t]]$ (
Otherwise σ changes a variable v_i at position p in t. $\ell_{\Delta,x}[t]$ has at position p v_i as well if the path to p does not contain colored symbol. So $\ell_{\Delta,x}[\ell_{\Delta,x}[t]\sigma]$ and $\ell_{\Delta,x}[t\sigma]$ coincide at p, irrespective of what σ introduces (var, Δ-/grey-term).
If p points into a maximal Δ-colored term, $p \mod k$ for some $k \geq 1$ is a Δ-colored term t_j. Hence in $\ell_{\Delta,x}[t]$, $\rho(p \mod k) = x_j$.

However in $\ell_{\Delta,x}[t\sigma]$, $\rho(p \mod k) = x_l$.

Lemma 7. *If $l \in C$, then $\ell[l] \in \mathrm{AI}_{\mathrm{cl}}(C)$. // true and used*

Proof. By induction:
Base case by definition.
Let a literal λ be such that it hasn't been resolved upon in the deduction leading up to C.

Resolution $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $l\sigma = l'\sigma$ give $C : (D \vee E)\sigma$.
$\lambda \not\sim l$, as otherwise it would not be contained in C. W.l.o.g. $\lambda \in C_1$. Then $(\lambda\sigma) \in C$ by the resolution rule.
Induction hypothesis: $\ell[\lambda] \in \mathrm{AI}_{\mathrm{cl}}(C_1)$.
So have to show $\ell[\lambda\sigma] \in \mathrm{AI}_{\mathrm{cl}}(C)$ // as $(\lambda\sigma) \in C$
By Lemma 6, $\ell[\lambda\sigma] = \ell[\ell[\lambda]\sigma]$. We show that in all resolution cases, $\ell[\ell[\lambda]\sigma] \in \mathrm{AI}_{\mathrm{cl}}(C)$.
$\mathrm{AI}_{\mathrm{cl}}(C) = \ell\Big[\Big((\mathrm{AI}_{\mathrm{cl}}(C_1) \setminus \{\ell[l]\}) \vee (\mathrm{AI}_{\mathrm{cl}}(C_2) \setminus \{\neg\ell[l']\})\Big)\sigma\Big]$
As $\lambda \not\sim l$ and $\ell[\lambda] \in \mathrm{AI}_{\mathrm{cl}}(C_1)$, $\ell[\ell[\lambda]\sigma] \in \mathrm{AI}_{\mathrm{cl}}(C)$. □

Lemma 8. *Suppose no* Γ*-term occurs in* Γ*. Then* $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$.

Proof. Proof by induction of the strenghtening: $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C_\Gamma)$.
Base case:
For $C \in \Gamma$, $\mathrm{AI}_{\mathrm{mat}}(C) = \bot$ and $\mathrm{AI}_{\mathrm{cl}}(C) = \ell[C_\Gamma] = \ell_{\Gamma,y}[C]$. By the restriction, $\ell_{\Gamma,y}[C] = C$ and $\Gamma \models C$.
For $C \in \Delta$, $\mathrm{AI}_{\mathrm{mat}}(C) = \top$.
Induction step:

restric used here

Resolution.

$$\frac{C_1 : D \vee l \qquad C_2 : E \vee \neg l'}{C : (D \vee E)\sigma} \quad \sigma = \mathrm{mgu}(l, l')$$

We introduce the following abbreviations, which are justified by Lemma 7 as $l \in C_1$ and $\neg l' \in C_2$:

$\mathrm{AI}_{\mathrm{cl}}(C_1)^* = \mathrm{AI}_{\mathrm{cl}}((C_1)_\Gamma) \setminus \{\ell[l_\Gamma]\}$

$\mathrm{AI}_{\mathrm{cl}}(C_2)^* = \mathrm{AI}_{\mathrm{cl}}((C_2)_\Gamma) \setminus \{\ell[\neg l'_\Gamma]\}$

Let σ' as in Lemma 1. Then by Lemma 1 and Lemma 2:

$$\begin{aligned}\mathrm{AI}_{\mathrm{cl}}(C_\Gamma) &= \ell\Big[\Big(\mathrm{AI}_{\mathrm{cl}}(C_1)^* \vee \mathrm{AI}_{\mathrm{cl}}(C_2)^*\Big)\sigma\Big] \\ &= \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)^*]\sigma' \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)^*]\sigma'\end{aligned}$$

By the induction hypothesis, $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C_i) \vee \mathrm{AI}_{\mathrm{cl}}(C_{i\Gamma})$, $i \in \{1, 2\}$, or expressed differently:

$\Gamma \overset{(\circ)}{\models} \mathrm{AI}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}_{\mathrm{cl}}(C_1)^* \vee \ell[l_\Gamma]$

$\Gamma \overset{(*)}{\models} \mathrm{AI}_{\mathrm{mat}}(C_2) \vee \mathrm{AI}_{\mathrm{cl}}(C_2)^* \vee \neg\ell[l'_\Gamma]$

Furthermore, $\sigma = \mathrm{mgu}(l, l')$, so $\ell[l\sigma] = \ell[l'\sigma]$ and $\ell[l]\sigma' = \ell[l']\sigma'$ by Lemma 1.

- Suppose that l and l' are Γ-colored.

$$\begin{aligned}\mathrm{AI}_{\mathrm{mat}}(C) &= \ell\Big[\Big((\mathrm{AI}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}_{\mathrm{mat}}(C_2))\Big)\sigma\Big] \\ &= \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)]\sigma' \vee \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)]\sigma'\end{aligned}$$

By Corollary 4 and applying σ' to $(\circ)$ and $(*)$, we obtain:

$\Gamma \models \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)]\sigma' \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)^*]\sigma' \vee \ell[l]\sigma'$

$\Gamma \models \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)]\sigma' \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)^*]\sigma' \neg\ell[l']\sigma'$

As $\ell[l]\sigma' = \ell[l']\sigma'$, we can perform a resolution step to arrive at:

$\Gamma \models \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)]\sigma' \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)^*]\sigma' \vee \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)]\sigma' \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)^*]\sigma'$

But this is nothing else than $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C_\Gamma)$.

- Suppose that l and l' are Δ-colored.

  $$\begin{aligned}\mathrm{AI}_{\mathrm{mat}}(C) &= \ell[\Big((\mathrm{AI}_{\mathrm{mat}}(C_1) \wedge \mathrm{AI}_{\mathrm{mat}}(C_2))\Big)\sigma] \\ &= \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)]\sigma' \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)]\sigma'\end{aligned}$$

  $(\circ)$ and $(*)$ then reduce to:

  $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C_1) \vee \mathrm{AI}_{\mathrm{cl}}(C_1)^*$

  $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C_2) \vee \mathrm{AI}_{\mathrm{cl}}(C_2)^*$

  But this implies that $\Gamma \models (\mathrm{AI}_{\mathrm{mat}}(C_1) \wedge \mathrm{AI}_{\mathrm{mat}}(C_2)) \vee \mathrm{AI}_{\mathrm{cl}}(C_1)^* \vee \mathrm{AI}_{\mathrm{cl}}(C_2)^*$, which by Corollary 4 and applying σ' gives $\Gamma \models (\ell[\mathrm{AI}_{\mathrm{mat}}(C_1)]\sigma' \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)]\sigma') \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_1)^*]\sigma' \vee \ell[\mathrm{AI}_{\mathrm{cl}}(C_2)^*]\sigma'$. This however is nothing else than $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C) \vee \mathrm{AI}_{\mathrm{cl}}(C)$.

- Suppose that l and l' are grey:

  $$\begin{aligned}\mathrm{AI}_{\mathrm{mat}}(C) &= \ell[\Big((\neg l \wedge \mathrm{AI}_{\mathrm{mat}}(C_1)) \vee (l \wedge \mathrm{AI}_{\mathrm{mat}}(C_2))\Big)\sigma] \\ &= (\ell[\neg l]\sigma' \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)]\sigma') \vee (\ell[l]\sigma' \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)]\sigma')\end{aligned}$$

  Note that $l_\Gamma = l$ and $l'_\Gamma = l'$.

  Let M be a model of Γ. Suppose that $M \not\models \mathrm{AI}_{\mathrm{cl}}(C)$ as otherwise we would be done.

  Then by $(\circ)$ and $(*)$:

  $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C_1) \vee \ell[l]$

  $\Gamma \models \mathrm{AI}_{\mathrm{mat}}(C_2) \vee \neg\ell[l']$

  By Corollary 4 and applying σ':

  $\Gamma \models \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)]\sigma' \vee \ell[l]\sigma'$

  $\Gamma \models \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)]\sigma' \vee \neg\ell[l']\sigma'$

  As $\ell[l]\sigma' = \ell[l']\sigma'$, we can deduce that $M \models \Big(\neg\ell[l]\sigma' \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_1)]\sigma'\Big) \vee \Big(\ell[l]\sigma' \wedge \ell[\mathrm{AI}_{\mathrm{mat}}(C_2)]\sigma'\Big)$, i.e. $M \models \mathrm{AI}_{\mathrm{mat}}(C)$. □

3 arrow proof

Lemma 9. *Whenever the same variable appears multiple times in* $\mathrm{PI}(C) \vee C$ *for* $C \in \pi$*, there are arrows.*

- *if both variables are contained only in grey terms, there is a double arrow // they unify to exactly the same*
- *if only one variable is only contained in grey terms, there is an arrow from it to the other one // either unify the one in grey term, then other one must be overbound later. if otherwise var in the colored term is unified, we can still overbind the grey one first.*
- *otherwise there are weak arrows between them // have same quantifier, so order does not matter, but want to keep dependencies on both the same*

Proof. By induction. Note: As required by resolution, all initial clauses are variable disjoint.
Base case: In the initial clause sets, consider for a clause C two different positions p_1 and p_2 pointing to the same variable. Then either:

- p_1 and p_2 contain only grey symbols. Then $(p_1, p_2) \in \mathcal{A}$.
- Only p_i, $i \in \{1, 2\}$ contains only grey symbols. Then $(p_i, p_{(i \mod 2)+1}) \in \mathcal{A}$.
- There are not only grey symbols in both p_1 and p_2, i.e. both contain at least a colored symbol. Then $\{p_1, p_2\} \in \mathcal{W}$.

Induction step: Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$. $\mathrm{PI}(C)$ is $[\mathrm{PI}(C_1) \circ \mathrm{PI}(C)]\sigma$ or $[(l \wedge \mathrm{PI}(C_2)) \vee (\neg l \wedge \mathrm{PI}(C_1))]\sigma$.
Assumption: C_1 and C_2 are variable disjoint, i.e. variables are renamed in case C_1 and C_2 are derived from some common original clause and share variables.
By the induction hypothesis, there are appropriate arrows in both $\mathrm{PI}(C_i) \vee C_i$, $i \in \{1, 2\}$.
If the variables were present in C_1 or C_2, the arrow is still there, either in $\mathrm{PI}(C)$ (in the case of l or l'), C (in case of D and E) or in currently not shown literal (in case l and l' have the same color).
Otherwise, it was introduced by unification in $l\sigma$ or $\mathrm{PI}(C_i)\sigma$. In this case, there is some term position q in with $\rho(l).q$ a variable and $\rho(l').q$ a variable or a term containing variables (or other way around). Hence unification maps a variable to a variable or a term containing variables. The variable being unified is in $\mathrm{PI}(C_i) \vee C_i$ for some $i \in \{1, 2\}$. But by the induction hypothesis, all occurrences of each variable does already have appropriate arrows, which are still present. □

Lemma 10. *In* $\mathrm{PI}(C) \vee C$ *for* $C \in \pi$*, if there is a* Δ*-colored term* s *in a* Γ*-term* t*, then there is an arrow from* p_1 *to* p_2 *such that* $\rho(p_1) = s$ *and* $\rho(p_2) = s$ *and for some* i*,* $\rho(p_2 \mod i) = t$*.*

Note: p_1 might be in some clause, the prop interpolant or none of both.

Proof. By induction.
Base case: There are no foreign terms in the initial clause sets, so no arrows necessary.
Induction step:

Resolution. Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$.

1. Suppose l is colored. This case is similar to the grey one, with the exception that the cases applying to l in PI do not apply.

2. Suppose l is grey. Then $\mathrm{PI}(C) = [(l \wedge \mathrm{PI}(C_2)) \vee (\neg l \wedge \mathrm{PI}(C_1))]\sigma$

   By the induction hypothesis, there are appropriate arrows in $\mathrm{PI}(C_1) \vee C_1$ and $\mathrm{PI}(C_2) \vee C_2$.

   We show that for all maximal Γ-terms in $\mathrm{PI}(C) \vee C$ with Δ-terms in them which were not present in $\mathrm{PI}(C_i) \vee C_i$, $i \in \{1, 2\}$, there is an arrow.

   Γ-terms and Δ-terms are not unifiable. Hence all pairs of terms (t_1, t_2) in the same positions in l and l' (if both positions exist) either point to the same symbol or (w.l.o.g.) t_1 is a variable and t_2 is a term. If there are Δ-terms in Γ-terms in the prefix, they are present in both ancestors and handled by the induction hypothesis.

   The only way a Δ-colored term may enter a Γ-colored term is in the situation where t_1 is a variable and t_2 a colored term. But then $\mathrm{mgu}(t_1, t_2)$ applied to t_1 yields t_2, i.e. "the parts of σ concerned with unifying t_1 and t_2" do not introduce new Δ-terms in Γ-terms. In other words, all such situation have been present in $\mathrm{PI}(C_i) \vee C_i$ for $i \in \{1, 2\}$ and since the arrows for l and l' are merged, they are present for $l\sigma$ in $\mathrm{PI}(C)$.

   This handles the case where terms t_1 and t_2 are unified. But unification also affects all other occurrences of variables, this means "the parts of σ not concerned with unifying t_1 and t_2". The relevant case for this lemma is when a Γ-term contains a variable, that is substituted by a term containing Δ-terms. But in this case, by Lemma 9, there is an arrow from the other occurrence of the variable to the one in the Γ-term: either double arrow in $\mathcal{A}$ if both prefixes are grey, one in $\mathcal{A}$ if one of the prefixes is grey or one in $\mathcal{W}$ if both prefixes contain a colored symbol. □