

$$\Sigma' = \{R(z) \vee \exists x P(f(x)), \neg Q(x), \}$$

$$\Pi' = \{\forall y g(y) = y, \forall y \neg P(g(y)) \vee Q(y), \neg R(d)\}$$

$$\Sigma = \text{sk}(\Sigma') = \{R(z) \vee P(f(c)), \neg Q(y), \}$$

$$\Pi = \text{sk}(\Pi') = \{g(u) = u, \neg P(g(v)) \vee Q(v), \neg R(d)\}$$

$$L(\Sigma) = \{R, P, Q, f, z, x, c\}$$

$$L(\Pi) = \{R, P, Q, g, u, v, d\}$$

Refutation:

$$\frac{\frac{R(z) \vee P(f(c))_{\Sigma} \quad \neg R(d)_{\Pi} \quad z \mapsto d}{P(f(c))} \quad \frac{\frac{\neg P(g(v)) \vee Q(v)_{\Pi} \quad \neg Q(y)_{\Sigma} \quad v \mapsto y}{\neg P(g(y))} \quad \frac{g(u) = u_{\Pi} \quad y \mapsto u}{\neg P(u)} \quad u \mapsto f(c)}{\square}$$

Interpolants:

$$\frac{\frac{\perp \quad \top}{(\neg R(d) \wedge \perp) \vee (R(d) \wedge \top) \equiv R(d)} \theta_0 \quad \frac{\frac{\top \quad \perp}{(\neg Q(y) \wedge \top) \vee (Q(y) \wedge \perp) \equiv \neg Q(y)} \theta_1 \quad \top \theta_2}{\frac{(\neg R(d) \wedge \perp) \vee (R(d) \wedge \top) \equiv R(d) \quad (\neg Q(u) \wedge g(u) = u) \vee (\top \wedge g(u) \neq u)}{(\neg P(f(c)) \wedge R(d)) \vee (P(f(c)) \wedge ((\neg Q(f(c)) \wedge g(f(c)) = f(c)) \vee g(f(c)) \neq f(c)))} \theta_3}$$

Relative interpolant properties:

| | | |
|--------------|---|---|
| $\theta_0 :$ | $\Sigma \vdash R(d) \vee P(f(c))$ | $\Pi \vdash \neg R(d) \vee P(f(c))$ |
| $\theta_1 :$ | $\Sigma \vdash \neg Q(y) \vee \neg P(g(y))$ | $\Pi \vdash Q(y) \vee \neg P(g(y))$ |
| $\theta_2 :$ | $\Sigma \vdash (\neg Q(u) \wedge g(u) = u) \vee g(u) \neq u \quad \vee \quad \neg P(u)$ | $\Pi \vdash \neg((\neg Q(u) \wedge g(u) = u) \vee g(u) \neq u) \quad \vee \quad \neg P(u)$ $\Pi \vdash ((Q(u) \vee g(u) \neq u) \wedge g(u) = u) \quad \vee \quad \neg P(u)$ |
| $\theta_3 :$ | $\Sigma \vdash \theta_3$ Proof: Either $\neg P(f(c))$, then $R(d)$. Otw. either $g(f(c)) \neq f(c)$. Otw. also $\neg Q(f(c))$. | $\Pi \vdash \neg \theta_3$ Proof: $\neg(\neg P(f(c)) \wedge R(d)) \quad \vee \quad (P(f(c)) \wedge (\neg Q(f(c)) \wedge g(f(c)) = f(c)) \quad \vee \quad g(f(c)) \neq f(c))$ $\equiv (P(f(c)) \vee \neg R(d)) \quad \wedge \quad (\neg P(f(c)) \vee (Q(f(c)) \vee g(f(c)) \neq f(c)) \quad \wedge \quad g(f(c)) = f(c))$ Have $g(f(c)) = f(c)$ and $\neg R(d)$, so remaining: $\neg P(f(c)) \vee Q(f(c))$. Get by axiom and unification with $g(u) = u$. |

$$\Sigma = \{R(z) \vee P(f(c)), \neg Q(y), \}$$

$$\Pi = \{g(u) = u, \neg P(g(v)) \vee Q(v), \neg R(d)\}$$

Propositional refutation tree (no non-trivial unifiers):

$$\frac{\frac{R(d) \vee P(f(c))_{\Sigma} \quad \neg R(d)_{\Pi}}{P(f(c))} \quad \frac{\frac{\neg P(g(f(c))) \vee Q(f(c))_{\Pi} \quad \neg Q(f(c))_{\Sigma}}{\neg P(g(f(c)))} \quad \frac{g(f(c)) = f(c)_{\Pi}}{\neg P(f(c))}}{\square}$$

Lifting:

terms: $g(f(c)), f(c), d$

max Π -terms: $\{g(f(c)), d\} \sim \{x_1, x_2\}$

max Σ -terms: $\{f(c)\} \sim \{x_3\}$

$$\overline{(\neg P(f(c)) \wedge R(d)) \vee (P(f(c)) \wedge ((\neg Q(f(c)) \wedge g(f(c)) = f(c)) \vee g(f(c)) \neq f(c)))}(x_1, x_2)$$

$$\Leftrightarrow \neg P(f(c)) \wedge R(x_2) \vee (P(f(c)) \wedge ((\neg Q(f(c)) \wedge x_1 = f(c)) \vee x_1 \neq f(c)))$$

By Lemma 12, $\Sigma \models \overline{\theta_3}$ (proof from above still goes through).

$$\hat{\theta}(x_3) = (\neg P(x_3) \wedge R(x_2)) \vee (P(x_3) \wedge ((\neg Q(x_3) \wedge x_1 = x_3) \vee x_1 \neq x_3))$$

quantifiers according to order: $|d| < |f(c)| < |g(f(c))|$

$$\theta = \forall x_2 \exists x_3 \forall x_1 (\neg P(x_3) \wedge R(x_2)) \vee (P(x_3) \wedge (\neg Q(x_3) \vee x_1 \neq x_3))$$

$$\neg \theta = \exists x_2 \forall x_3 \exists x_1 (P(x_3) \vee \neg R(x_2)) \wedge (\neg P(x_3) \vee (Q(x_3) \wedge x_1 = x_3))$$

$$\Rightarrow \Sigma \vdash \theta; \Pi \vdash \neg \theta$$

Example 2:

$$\Sigma = \{P(c), \neg P(d)\}$$

$$\Pi = \{P(d) \vee g(u) = u, \neg P(g(x))\}$$

Refutation:

$$\frac{\frac{\frac{P(d) \vee g(u) = u}{\Pi} \quad \neg P(d)_{\Sigma}}{g(u) = u} \quad \frac{\neg P(g(x))_{\Pi} \quad u \mapsto x}{\neg P(x)} \quad \frac{P(c)_{\Sigma} \quad x \mapsto c}{\square}}$$

Relative interpolants:

$$\frac{\frac{\frac{\top \quad \perp}{(\neg P(d) \wedge \top) \vee (P(d) \wedge \perp) \equiv \neg P(d)}}{(g(x) = x \wedge \top) \vee (g(x) \neq x \wedge \neg P(d))} \quad \frac{\top \quad u \mapsto x}{\perp} \quad x \mapsto c}{(\neg P(c) \wedge \perp) \vee (P(c) \wedge (g(c) = c \vee (g(c) \neq c \wedge \neg P(d))))}$$

$$\theta = P(c) \wedge (g(c) = c \vee \neg P(d))$$

$$\neg \theta = \neg P(c) \vee (g(c) \neq c \wedge P(d))$$

terms: $g(c), c, d$

max Π -terms: $g(c)$

max Σ -terms: c

ordered by length ASCENDING: $\{c, g(c)\}$

$$\bar{\theta}(x_2) = P(c) \wedge (x_2 = c \vee \neg P(d))$$

$$\hat{\theta}(x_1) = P(x_1) \wedge (x_2 = x_1 \vee \neg P(d))$$

$$\Sigma \vdash \exists x_1 \forall x_2 P(x_1) \wedge (x_2 = x_1 \vee \neg P(d))$$

$$\Pi \vdash \neg \exists x_1 \forall x_2 P(x_1) \wedge (x_2 = x_1 \vee \neg P(d))$$

$$\Pi \vdash \forall x_1 \exists x_2 \neg P(x_1) \vee (x_2 \neq x_1 \wedge P(d))$$

A possible interpolant: $\neg P(d) \wedge \exists x P(x)$

Example 2 (Craig translation):

$$\Sigma = \{P(c), \neg P(d)\}$$

$$\Pi = \{P(d) \vee g(u) = u, \neg P(g(x))\}$$

$$T(\Sigma) = \{\forall x \, x = x\} \cup \{\forall x \forall y \, x = y \supset P(x) \supset P(y)\} \cup \Sigma$$

$$T(\Pi) = \{\forall x \, x = x\} \cup$$

$$\{\forall x \forall y \, x = y \supset P(x) \supset P(y), \forall x_1 \forall x_2 \forall y_1 \forall y_2 \, x_1 = y_1 \supset x_2 = y_2 \supset x_1 = x_2 \supset y_1 = y_2, \forall x_1 \forall x_2 \forall y_1 \forall y_2 \, x_1 = y_1 \supset x_2 = y_2 \supset G(x_1, x_2) \supset G(y_1, y_2)\} \cup$$

$$\{P(d) \vee (\exists z G(u, z) \wedge (\forall y G(u, y) \supset z = y) \wedge z = u), \neg P(g(x))\}$$

Example 3 Bonacina/Johannson:

$$\Sigma = \{A \vee B, \neg C\}$$

$$\Pi = \{\neg A \vee C, \neg B\}$$

$$\frac{\frac{A \vee B_{\Sigma} \quad \neg A \vee C_{\Pi}}{B \vee C} \quad \neg C_{\Sigma}}{B} \quad \neg B_{\Pi} \quad \square$$

Bon/Joh:

$$\frac{\frac{\frac{\perp \quad \top}{(A \vee \perp) \wedge \top \equiv A} \quad \perp}{A \wedge (\neg C \vee \perp) \equiv A \wedge \neg C} \quad \top}{(B \vee (A \wedge \neg C)) \wedge \top}$$

Huang:

$$\frac{\frac{\frac{\perp \quad \top}{(\neg A \wedge \perp) \vee (A \wedge \top) \equiv A} \quad \perp}{(\neg C \wedge A) \vee (C \wedge \perp) \equiv \neg C \wedge A} \quad \top}{(\neg B \wedge (\neg C \wedge A)) \vee (B \wedge \top)}$$

-> logically equivalent

Example 3B Bonacina/Johannson:

$$\Sigma = \{A \vee B, \neg C, \neg D\}$$

$$\Pi = \{\neg A \vee C, \neg B \vee D\}$$

$$\frac{\frac{A \vee B_{\Sigma} \quad \neg A \vee C_{\Pi}}{B \vee C} \quad \neg C_{\Sigma} \quad \frac{\neg D_{\Sigma} \quad \neg B \vee D_{\Pi}}{\neg B}}{B} \quad \square$$

Bon/Joh:

$$\frac{\frac{\frac{\perp \quad \top}{(A \vee \perp) \wedge \top \equiv A} \quad \perp}{A \wedge (\neg C \vee \perp) \equiv A \wedge \neg C} \quad \frac{\frac{\perp \quad \top}{\top \wedge (\neg D \vee \perp) \equiv \neg D}}{(B \vee (A \wedge \neg C)) \wedge \neg D}$$

Huang:

$$\frac{\frac{\frac{\perp \quad \top}{(\neg A \wedge \perp) \vee (A \wedge \top) \equiv A} \quad \perp}{(\neg C \wedge A) \vee (C \wedge \perp) \equiv \neg C \wedge A} \quad \frac{\frac{\perp \quad \top}{(\neg D \wedge \top) \vee (D \wedge \perp) \equiv \neg D}}{(\neg B \wedge \neg C \wedge A) \vee (B \wedge \neg D)}$$

-> not logically equivalent

Example 4: Paramodulation special case in Huang

$$\begin{array}{c}
\frac{P(x) \vee \neg Q(x)_{\Sigma} \quad Q(h(r))_{\Pi}}{P(h(r))} \quad \frac{s = t_{\Pi} \quad r \mapsto s}{P(h(s))} \quad \neg P(h(s))_{\Pi} \\
\hline
\Box \\
\\
\frac{\frac{\perp \quad \top}{\neg Q(h(r))} \quad \top}{(s = t \wedge \neg Q(h(t))) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t))} \quad \top \\
\hline
(\neg P(h(s)) \wedge (s = t \wedge \neg Q(h(t)))) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t)) \vee P(h(s))
\end{array}$$

$$\Sigma = \{P(x) \vee \neg Q(x)\}$$

$$\Pi = \{\neg P(h(s)), Q(h(r)), s = t\}$$

$$((s = t \wedge \neg Q(h(t))) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t))) \vee P(h(s))$$

$$\theta = \neg Q(h(t)) \vee (s \neq t) \vee (s = t \wedge h(s) \neq h(t)) \vee P(h(s))$$

$$\neg \theta = Q(h(t)) \wedge (s = t) \wedge (s \neq t \vee h(s) = h(t)) \wedge \neg P(h(s))$$

$$\theta^* = \forall x_1 \forall x_2 \forall x_3 \forall x_4 \neg Q(x_2) \vee (x_3 \neq x_4) \vee (x_3 = x_4 \wedge x_1 \neq x_2) \vee P(x_1)$$

$$\neg \theta^* = \exists x_1 \exists x_2 \exists x_3 \exists x_4 Q(x_2) \wedge (x_3 = x_4) \wedge (x_3 \neq x_4 \vee x_1 = x_2) \wedge \neg P(x_1)$$

Example 4b: Paramodulation special case in Huang

$$\frac{\frac{P(x) \vee \neg Q(x)_{\Sigma} \quad Q(h(t))_{\Pi}}{P(h(t))} \quad s = t_{\Sigma} \quad \frac{\neg P(x) \vee Q(x)_{\Sigma} \quad \neg Q(h(s))_{\Pi}}{\neg P(h(s))}}{P(h(s))} \quad \square$$

$$\frac{\frac{\frac{\perp \quad \top}{\neg Q(h(t))} \quad \perp}{((s = t) \wedge \neg Q(h(t))) \vee (s = t \wedge h(s) \neq h(t))} \quad \frac{\perp \quad \top}{Q(h(s))}}{(\neg P(h(s)) \wedge ((s = t) \wedge \neg Q(h(t))) \vee (s = t \wedge h(s) \neq h(t))) \vee (P(h(s) \wedge Q(h(s))))}$$

$$\Sigma = \{P(x) \vee \neg Q(x), \neg P(x) \vee Q(x), s = t\} \quad // \quad P(x) \leftrightarrow Q(x)$$

$$\Pi = \{\neg Q(h(s)), Q(h(t))\}$$

$$\theta = (\neg P(h(s)) \wedge ((s = t) \wedge \neg Q(h(t))) \vee (s = t \wedge h(s) \neq h(t))) \vee (P(h(s) \wedge Q(h(s))))$$

$$\neg \theta = (P(h(s)) \vee ((s \neq t) \vee Q(h(t))) \wedge (s \neq t \vee h(s) = h(t))) \wedge (\neg P(h(s) \vee \neg Q(h(s))))$$

$$\theta^* = \forall x_1 \forall x_2 (\neg P(x_1) \wedge ((s = t) \wedge \neg Q(x_2)) \vee (s = t \wedge x_1 \neq x_2)) \vee (P(x_1) \wedge Q(x_1))$$

$$\neg \theta^* = \exists x_1 \exists x_2 (P(x_1) \vee ((s \neq t) \vee Q(x_2)) \wedge (s \neq t \vee x_1 = x_2)) \wedge (\neg P(x_1) \vee \neg Q(x_2))$$

Example 5: cases for one pass overbinding algo

want to have step in between where only one of the “critical” terms appears in the interpolant and a decision on the order is forced

$$\frac{P(y_1, y_2)_\Sigma \quad \frac{Q(\alpha)_\Sigma \quad \neg Q(z) \vee \neg P(z, \beta)_\Pi}{\neg P(\alpha, \beta)} z \mapsto \alpha}{\square}$$

$$\frac{\perp \quad \frac{\perp \quad \top}{Q(\alpha)^\circ} z \mapsto \alpha}{P(\alpha, \beta) \wedge Q(\alpha)}$$

\Rightarrow need to overbind α first, no matter which order would be assigned later

NOTE: b might be $f(a)$, i.e. we don't know a priori at which level it is and how many smaller or larger terms will be added.

Let $\alpha = b, \beta = g(z)$.

$$\theta^* = \exists x_1 \forall x_2 P(x_1, x_2) \wedge Q(x_1)$$

$$\neg \theta^* = \forall x_1 \exists x_2 \neg P(x_1, x_2) \vee \neg Q(x_1)$$

$$\theta^{\circ*} = \exists x_1 Q(x_1)$$

Let $\alpha = g(x), \beta = b$.

$$\theta^* = \exists x_1 \forall x_2 P(x_1, x_2) \wedge Q(x_1)$$

$$\neg \theta^* = \forall x_1 \exists x_2 \neg P(x_1, x_2) \vee \neg Q(x_1)$$

$$\theta^{\circ*} = \exists x_1 Q(x_1)$$

\Rightarrow works (need not change quantifier order like this, but here, no predicate has parameters which depend on each other)

Example 5b: no equality, but quantifier order still matters

$$\frac{P(u, g(u))_{\Sigma} \quad \neg P(a, x)_{\Pi}}{\square} u \mapsto a, x \mapsto g(a)$$

Prop Interpolant: $P(a, g(a))$

Interpolant: $\forall x_1 \exists x_2 P(x_1, x_2)$

Example 5b': order matters, construction in multiple steps:

$$\frac{\frac{P(u, v, f(u, v)) \vee Q(u)_{\Sigma} \quad \neg Q(a)_{\Pi}}{P(a, v, f(a, v))} u \mapsto a \quad \neg P(x, b, y)_{\Pi}}{\square} x \mapsto a, v \mapsto b, y \mapsto f(a, b)$$

$$\frac{\frac{\perp \quad \top}{Q(a)} u \mapsto a \quad \top}{P(a, b, f(a, b)) \vee (\neg P(a, b, f(a, b)) \wedge Q(a))} x \mapsto a, v \mapsto b, y \mapsto f(a, b)$$

Hence: $P(a, b, f(a, b)) \vee Q(a)$

Interpolant: $\forall x_1 \forall x_2 \exists x_3 P(x_1, x_2, x_3) \vee Q(x_1)$

Example 5c: overbinding with equality

for arbitrary a, b:

$$\Sigma = \{P(a), a = b\}$$

$$\Pi = \{\neg P(b)\}$$

$$\frac{\frac{P(a)_{\Sigma} \quad a = b_{\Sigma}}{P(b)} \quad \neg P(b)_{\Pi}}{\square}$$

$$\frac{\frac{\perp \quad \perp}{\perp} \quad \top}{P(b)}$$

OR

$$\frac{\frac{a = b_{\Sigma} \quad \neg P(b)_{\Pi}}{\neg P(a)} \quad P(a)_{\Sigma}}{\square}$$

$$\frac{\frac{\perp \quad \top}{a = b} \quad \perp}{(P(a) \wedge a = b)}$$