Contents

0.1 Lemmas from other pdf

:lifting_order_not_relevant⟩ **Lemma 1.** $\ell_\Gamma[\ell_\Delta[\varphi]] = \ell_\Delta[\ell_\Gamma[\varphi]]$.

CHAPTER 1

Interpolant extraction from resolution proofs in one phase lifting terms whose quantifier position can be determined – nested

1.1 Incremental lifting and substitutions of lifting variables

Definition 2 (Substitution $\tau(\iota)$)**.** For an inference ι with $\sigma = \mathrm{mgu}(\iota)$, we define the infinite substitution $\tau(\iota)$ with $\mathrm{dom}(\tau(\iota)) = \mathrm{dom}(\sigma) \cup \{z_s \mid s\sigma \neq s\}$ as follows for a variable x:

define infinite substitutions properly and apply definition here

$$x\tau(\iota) = \begin{cases} x\sigma & x \text{ is a non-lifting variable} \\ z_{t\sigma} & x \text{ is a lifting variable } z_t \end{cases}$$

If the inference ι is clear from the context, we abbreviate $\tau(\iota)$ by τ. △

⟨lemma:lifting_tau_commute⟩ **Lemma 3.** *For a formula or term φ and an inference ι such that $\tau = \tau(\iota)$, $\ell[\ell[\varphi]\tau] = \ell[\varphi\tau]$.*

Proof. We proceed by induction.

- Suppose that t is a grey constant or function symbol of the form $f(t_1, \dots, t_n)$. Then we can derive the following, where (IH) signifies a deduction by virtue of the induction hypothesis.

$$\begin{aligned}
\ell[\ell[t]\tau] &= \ell[\ell[f(t_1,\dots,t_n)]\tau] \\
&= \ell[f(\ell[t_1]\tau,\dots,\ell[t_n]\tau)] \\
&= f(\ell[\ell[t_1]\tau],\dots,\ell[\ell[t_n]\tau]) \\
&\overset{\text{(IH)}}{=} f(\ell[t_1\tau],\dots,\ell[t_n\tau]) \\
&= \ell[f(t_1,\dots,t_n)\tau] \\
&= \ell[t\tau]
\end{aligned}$$

- Suppose that t is a colored constant or function symbol. Then:
$$\ell[\ell[t]\tau] = \ell[z_t\tau] = \ell[z_{t\sigma}] = z_{t\sigma} = \ell[t\sigma] = \ell[t\tau]$$
- Suppose that t is a variable x. Then:
$$\ell[\ell[t]\tau] = \ell[\ell[x]\tau] = \ell[x\tau] = \ell[t\tau]$$
- Suppose that t is a lifting variable z_t. Then:
$$\ell[\ell[z_t]\tau] = \ell[z_t\tau]$$
□

Definition 4 (Incrementally lifted interpolant)**.** Let π be a resolution refutation of $\Gamma \cup \Delta$. We define $\mathrm{LI}(\pi)$ and $\mathrm{LI}_{\mathrm{cl}}(\pi)$ to be $\mathrm{LI}(\Box)$ and $\mathrm{LI}_{\mathrm{cl}}(\Box)$ respectively, where $\Box$ is the empty clause derived in π.

Let C be a clause in π.

We define $\mathrm{LI}_{\mathrm{cl}}(C) \stackrel{\mathrm{def}}{=} C$. TODO: if this version is final, drop $\mathrm{LI}_{\mathrm{cl}}(C)$ everywhere

We define the preliminary formula $\mathrm{LI}^\bullet(C)$ as follows:

Base case. If $C \in \Gamma$, $\mathrm{LI}(C) \stackrel{\mathrm{def}}{=} \bot$. If otherwise $C \in \Delta$, $\mathrm{LI}(C) \stackrel{\mathrm{def}}{=} \top$.

Resolution. If the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ using a unifier σ such that $l\sigma = l'\sigma$, then define $\mathrm{LI}(C)$ as follows:

1. If l is Γ-colored: $\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} \mathrm{LI}(C_1)\tau \vee \mathrm{LI}(C_2)\tau$
2. If l is Δ-colored: $\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} \mathrm{LI}(C_1)\tau \wedge \mathrm{LI}(C_2)\tau$
3. If l is grey: $\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} (l\tau \wedge \mathrm{LI}(C_2)\tau) \vee (\neg l'\tau \wedge \mathrm{LI}(C_1)\tau)$

Factorisation. If the clause C is the result of a factorisation step ι of $C_1 : l \vee l' \vee D$ using a unifier σ such that $l\sigma = l'\sigma$, then $\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} \ell[\mathrm{LI}(C_1)\tau]$.

Paramodulation. If the clause C is the result of a paramodulation step ι of $C_1 : s = t \vee D$ and $C_2 : E[r]$ with $\sigma = \mathrm{mgu}(\iota)$. Let $h[r]$ be the maximal colored term in which r occurs in $E[r]$. Then define $\mathrm{LI}(C)$ as follows:

1. If $h[r]$ is Δ-colored and $h[r]$ occurs more than once in $E[r] \vee \mathrm{LI}(E[r])$:
$\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} (s = t \wedge \mathrm{LI}(C_2))\tau \vee (s \neq t \wedge \mathrm{LI}(C_1))\tau \vee (s = t \wedge h[s] \neq h[t])\tau$
2. If $h[r]$ is Γ-colored and $h[r]$ occurs more than once in $E[r] \vee \mathrm{LI}(E[r])$:
$\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} [(s = t \wedge \mathrm{LI}(C_2))\tau \vee (s \neq t \wedge \mathrm{LI}(C_1))\tau] \wedge (s \neq t \vee h[s] = h[t])\tau$
3. If r does not occur in a colored term in $E[r]$ which occurs more than once in $E[r] \vee \mathrm{LI}(E[r])$:
$\mathrm{LI}^\bullet(C) \stackrel{\mathrm{def}}{=} (s = t \wedge \mathrm{LI}(C_2))\tau \vee (s \neq t \wedge \mathrm{LI}(C_1))\tau$

$\mathrm{LI}(C)$ is built from $\mathrm{LI}^\bullet(C)$ as follows:

1. Lift all maximal colored terms in $\mathrm{LI}^\bullet(C)$ which contains some variable which does not occur in C.
2. Let X (Y) be the Δ-(Γ-)lifting variables created in the previous step.
3. Prefix the resulting formula with an arrangement $Q(C)$ of the elements of $\{\forall x_t \mid x_t \in X\} \cup \{\exists y_t \mid y_t \in Y\}$ such that if s and r are terms such that s is a subterm of r, then z_s precedes z_r. △

1.2 Properties of LI and $\mathrm{LI}_{\mathrm{cl}}$

correct but useless here:

_clause_plus_literals_equal⟩ **Lemma 5.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$.*

Then for every literal λ in C, there exists a literal λ_{LIcl} in $\mathrm{LI}_{\mathrm{cl}}(C)$ such that $\lambda_{\mathrm{LIcl}} = \ell[\lambda]$ and for resolved or factorised literals l and l' of a resolution or factorisation inference ι, we have that $\ell[l_{\mathrm{LIcl}}\tau] = \ell[l'_{\mathrm{LIcl}}\tau]$.

Proof. We proceed by induction.

Base case. For $C \in \Gamma \cup \Delta$, $\mathrm{LI}_{\mathrm{cl}}(C)$ is defined to be $\ell[C]$.

Induction step. Suppose the clause C is the result of a resolution, factorisation or paramodulation inference ι of the clauses $\bar{C}$ with $\sigma = \mathrm{mgu}(\iota)$.

Every literal in C is of the form $\lambda\sigma$ for a literal λ in $C_i \in \bar{C}$.

By the induction hypothesis, $\ell[\lambda]$ occurs in $\mathrm{LI}_{\mathrm{cl}}(C_i)$. By the construction of $\mathrm{LI}_{\mathrm{cl}}(C)$ and as λ is not a resolved or factorised literal, $\mathrm{LI}_{\mathrm{cl}}(C)$ contains a literal of the form $\ell[\ell[\lambda]\tau]$. But by Lemma 3, this is nothing else than $\ell[\lambda\tau]$. As λ occurs in the resolution derivation, it does not contain lifting variables. Hence we get by the definition of τ that $\ell[\lambda\tau] = \ell[\lambda\sigma]$.

Let l and l' be the resolved or factorised literals of ι. In order to show that $\ell[l_{\mathrm{LIcl}}\tau] = \ell[l'_{\mathrm{LIcl}}\tau]$, consider that by the induction hypothesis, this is nothing else than $\ell[\ell[l]\tau] = \ell[\ell[l']\tau]$. But by applying a similar argument as above, this equation is equivalent to $\ell[l\sigma] = \ell[l'\sigma]$, which is implied by $l\sigma = l'\sigma$. □

⟨lemma:no_colored_terms⟩ **Lemma 6.** *Let C be a clause of a resolution refutation of $\Gamma \cup \Delta$. $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ do not contain colored symbols.*

Proof. For $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$, consider the following: In the base case of the inductive definitions of $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$, no colored symbols occur. In the inductive steps, any colored symbol which is added by τ to intermediary formulas is lifted. □

⟨lemma:substitute_and_lift⟩ **Lemma 7.** *Let σ be a substitution and F a formula without Φ-colored terms such that for a set of formulas Ψ which does not contain Φ-lifting variables, $\Psi \models F$. Then $\Psi \models \ell_{\Phi}[F\sigma]$.*

Proof. $\ell_{\Phi}[F\sigma]$ is an instance of F: σ substitutes variables either for terms which do not contain Φ-colored symbols or by terms containing Φ-colored symbols. For the first kind, the lifting has no effect. For the latter, the lifting only replaces subterms of the terms introduced by the substitution by a lifting variable such that the original structure of F remains invariant as it by assumption does not contain colored terms. □

ails_delta_lifted_invariant⟩ **Lemma 8.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then $\Gamma \models \ell_{\Delta}[\mathrm{LI}(C)] \vee \ell_{\Delta}[C]$*

Proof. We proceed by induction on the strengthening $\Gamma \models \ell_{\Delta}[\mathrm{LI}(C_{\Gamma})] \vee \ell_{\Delta}[C_{\Gamma}]$[1]

Base case. If $C \in \Gamma$, then $\ell_{\Delta}[C] = C$ and $\Gamma \models C$. If otherwise $C \in \Delta$, then $\mathrm{LI}(C) = \top$.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

By the induction hypothesis we obtain the following:

$\Gamma \models \ell_{\Delta}[\mathrm{LI}(C_1)] \vee \ell_{\Delta}[D_{\Gamma}] \vee \ell_{\Delta}[l_{\Gamma}]$

$\Gamma \models \ell_{\Delta}[\mathrm{LI}(C_2)] \vee \ell_{\Delta}[E_{\Gamma}] \vee \neg\ell_{\Delta}[l'_{\Gamma}]$

Hence by Lemma 7 and Lemma 3, we get:

$\Gamma \overset{(\circ)}{\models} \ell_{\Delta}[\mathrm{LI}(C_1)\tau] \vee \ell_{\Delta}[D_{\Gamma}\tau] \vee \ell_{\Delta}[l_{\Gamma}\tau]$

$\Gamma \overset{(*)}{\models} \ell_{\Delta}[\mathrm{LI}(C_2)\tau] \vee \ell_{\Delta}[E_{\Gamma}\tau] \vee \neg\ell_{\Delta}[l'_{\Gamma}\tau]$

[1] Recall that D_{Φ} denotes the clause created from the clause D by removing all literals which are not contained $\mathrm{L}(\Phi)$.

As $l_\Gamma\sigma = l'_\Gamma\sigma$, it holds that $l_\Gamma\tau = l'_\Gamma\tau$ and consequently $\ell_\Delta[l_\Gamma\tau] = \ell_\Delta[l'_\Gamma\tau]$. We proceed by a case distinction on the color of the resolved literal to show that in each case, we have that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$:

- Suppose that l is Γ-colored. Then $l_\Gamma = l$ and $l'_\Gamma = l$, and we can perform a resolution step on $(\circ)$ and $(*)$ to obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[D_\Gamma\tau] \vee \ell_\Delta[E_\Gamma\tau]$. This however is nothing else than $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that l is Δ-colored. Then $(\circ)$ and $(*)$ reduce to $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[D_\Gamma\tau]$ and $\Gamma \models \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee \ell_\Delta[E_\Gamma\tau]$ respectively, which clearly implies that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee (\ell_\Delta[D_\Gamma\tau] \wedge \ell_\Delta[E_\Gamma\tau])$. This is turn is however just the unfolding of $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.
- Suppose that l is grey. Then $(\circ)$ and $(*)$ imply that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}(C_2)\tau] \vee (\ell_\Delta[l_\Gamma\tau] \wedge \ell_\Delta[E_\Gamma\tau]) \vee (\neg\ell_\Delta[l'_\Gamma\tau] \wedge \ell_\Delta[D_\Gamma\tau])$. This however is equivalent to $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C_\Gamma]$.

We now conclude by showing that $\Gamma \models \ell_\Delta[\mathrm{LI}^\bullet(C)] \vee \ell_\Delta[C]$ implies that $\Gamma \models \ell_\Delta[\mathrm{LI}(C)] \vee \ell_\Delta[C]$.

The difference between $\ell_\Delta[\mathrm{LI}^\bullet(C)]$ and $\ell_\Delta[\mathrm{LI}(C)]$ lies only in certain maximal colored terms which are lifted in $\ell_\Delta[\mathrm{LI}(C)]$, hence it suffices to consider these. Let t be a term in $\mathrm{LI}^\bullet(C)$ at position p such that $\mathrm{LI}(C)|_p = \ell[t]$. Then t is a maximal colored term and contains a variable which does not occur in C.

If t is Δ-colored, then $\ell_\Delta[\mathrm{LI}^\bullet(C)]|_p = \mathrm{LI}(C)|_p = x_t$. Note that as t occurs at p in $\mathrm{LI}^\bullet(C)$, x_t occurs free at $\ell_\Delta[\mathrm{LI}^\bullet(C)]|_p$. Hence it is implicitly universally quantified and therefore entails that an explicit universal quantification in $\mathrm{LI}(C)$ is valid with an arbitrarily placed quantifier.

If otherwise t is a Γ-term, then $\ell_\Delta[\mathrm{LI}^\bullet(C)]|_p = \ell_\Delta[t]$. Hence $\ell_\Delta[t]$ represents a witness term for the existentially quantified lifting variable y_t at $\mathrm{LI}(C)|_p$. In general, $\ell_\Delta[t]$ however contains Δ-lifting variables, which require being lifted in the scope of the existential quantifier of y_t.

Let x_s be a Δ-lifting variable which occurs in $\ell_\Delta[t]$. It is essential to see that neither s nor a predecessor of s in the resolution derivation is lifted in a previous step of the interpolant extraction. Suppose to the contrary that this is the case in the inference creating the clause C'. Let s' and t' be the respective predecessors of s and t in C'. Then one of the following two contradictions eventuate:

- Suppose that s' is a subterm of the corresponding predecessor t'. Then due to the fact that s' is lifted, s' must contain a variable which does not occur in C'. But as t' contains s', t' contains this variable as well and would be lifted at this stage already.
- Otherwise t' does not contain s'. We have already established that s' contains a variable which does not occur in C'. As all clauses are variable-disjoint, no other clause contains this variable. But then it does not occur in any subsequent unifier, and in particular, it never enters t' by means of substitution, which implies that s' due to containing this variable does not become a subterm of a successor of t'.

Hence there are three possibilities for quantification of x_s:

1. Neither s nor a successor of s in the derivation occurs at a grey position. Then x_s is not quantified in the course of the interpolant extraction.
2. A variable which does not occur in C enters s by means of the current substitution σ or a variable is contained in s such that the only occurrences of it in C_1 and C_2 are in l and l'. Then x_s is lifted in the current step and as s is a subterm of t, x_s is quantified in $Q(C)$ prior to y_t.
3. The lifting variable x_s or a respective successor is quantified at a later stage in the derivation. Then as the quantifier for y_t is contained in $\mathrm{LI}(C)$ and for any successor C' of C, $\mathrm{LI}(C')$ contains a successor $\mathrm{LI}(C)$, y_t is quantified in the scope of the quantifier for x_s.

Factorisation. Suppose the clause C is the result of a factorisation inference ι of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

The induction hypothesis gives $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)] \vee \ell_\Delta[l \vee l' \vee D]$. By Lemma 7, we obtain $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[l\tau \vee l'\tau \vee D\tau]$. As however $l\sigma \equiv l'\sigma$, also $l\tau \equiv l'\tau$, so we can apply a factorisation step and obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}(C_1)\tau] \vee \ell_\Delta[l\tau \vee D\tau]$, which is nothing else than $\Gamma \models \mathrm{LI}^\bullet(C) \vee \ell_\Delta[C]$.

□

TODO: make sure this proof is valid; then define what happens at the end: lift all remaining terms (as in huang?). also check if symmetry works out. compare with other proofs.

⟨lemma:li_symmetry⟩ **Lemma 9.** *Let π be a refutation of $\Gamma \cup \Delta$ and $\hat{\pi}$ be π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then for a clause C in π and its corresponding clause $\hat{C}$ in $\hat{\pi}$, $\mathrm{LI}(C) \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$.*

Proof. We proceeded by induction to show that $\mathrm{LI}^\bullet(C) \Leftrightarrow \neg\,\mathrm{LI}^\bullet(\hat{C})$:

Base case. If $C \in \Gamma$, then $\mathrm{LI}(C) = \bot \Leftrightarrow \neg\top \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$ as $\hat{C} \in \Delta$. The case for $C \in \Delta$ can be argued analogously.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

As τ depends only on σ, τ is the same for both π and $\hat{\pi}$.

We now distinguish the following cases:

1. l is Γ-colored:

$$\begin{aligned}
\mathrm{LI}^\bullet(C) &= \mathrm{LI}(C_1)\tau \vee \mathrm{LI}(C_2)\tau \\
&\Leftrightarrow \neg(\neg\mathrm{LI}(C_1)\tau \wedge \neg\mathrm{LI}(C_2)\tau) \\
&\Leftrightarrow \neg(\mathrm{LI}(\hat{C}_1)\tau \wedge \mathrm{LI}(\hat{C}_2)\tau) \\
&= \neg\,\mathrm{LI}^\bullet(\hat{C})
\end{aligned}$$

2. l is Δ-colored: This case can be argued analogously.

3. l is grey: Note $l\tau \overset{(*)}{=} l'\tau$.

$$\begin{aligned}
\mathrm{LI}^\bullet(C) &= (\neg l'\tau \wedge \mathrm{LI}(C_1)\tau) \ \vee\ (l\tau \wedge \mathrm{LI}(C_2)\tau) \\
&\overset{(*)}{\Leftrightarrow} (l'\tau \vee \mathrm{LI}(C_1)\tau) \ \wedge\ (\neg l\tau \vee \mathrm{LI}(C_2)\tau) \\
&\Leftrightarrow \neg[(\neg l'\tau \wedge \neg\mathrm{LI}(C_1)\tau) \ \vee\ (l\tau \wedge \neg\mathrm{LI}(C_2)\tau)] \\
&= \neg[(\neg \hat{l}'\tau \wedge \mathrm{LI}(\hat{C}_1)\tau) \ \vee\ (\hat{l}\tau \wedge \mathrm{LI}(\hat{C}_2)\tau)] \\
&= \neg\,\mathrm{LI}^\bullet(\hat{C})
\end{aligned}$$

Factorisation. Suppose the clause C is the result of a factorisation ι of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

As the construction is not influenced by the coloring, the induction hypothesis $\mathrm{LI}^\bullet(C) = \mathrm{LI}(C_1)\tau$ suffices.

Then $\mathrm{LI}^\bullet(C) = \ell[\mathrm{LI}(C_1)\tau]$, so the construction is not influenced by the coloring and by the induction hypothesis, $\mathrm{LI}^\bullet(C) \Leftrightarrow \neg\,\mathrm{LI}^\bullet(\hat{C})$.

Paramodulation. Suppose the clause C is the result of a paramodulation inference ι of $C_1 : s = t \vee D$ and $C_2 : E[r]_p$ with $\sigma = \mathrm{mgu}(\iota)$.

We proceed by a case distinction:

- Suppose that p in $E[r]_p$ is contained in a maximal Δ-term $h[r]$, which occurs more than once in $E[r]_p \vee \mathrm{LI}(E[r]_p)$. Then p in $\hat{E}[r]_p$ is contained in a maximal Γ-term $h[r]$, which occurs more than once in $\hat{E}[r]_p \vee \mathrm{LI}(\hat{E}[r]_p)$.

$$\begin{aligned}
\mathrm{LI}^\bullet(C) &= (s\tau = t\tau \wedge \mathrm{LI}(C_2)\tau) \vee (s\tau \neq t\tau \wedge \mathrm{LI}(C_1)\tau) \vee (s\tau = t\tau \wedge h[s]\tau \neq h[t]\tau) \\
&\Leftrightarrow \neg[(s\tau \neq t\tau \vee \neg\mathrm{LI}(C_2)\tau) \wedge (s\tau = t\tau \vee \neg\mathrm{LI}(C_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&= \neg[(s\tau \neq t\tau \vee \mathrm{LI}(\hat{C}_2)\tau) \wedge (s\tau = t\tau \vee \mathrm{LI}(\hat{C}_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&\Leftrightarrow \neg[(s\tau = t\tau \wedge \mathrm{LI}(\hat{C}_2)\tau) \vee (s\tau \neq t\tau \wedge \mathrm{LI}(\hat{C}_1)\tau) \wedge (s\tau \neq t\tau \vee h[s]\tau = h[t]\tau)] \\
&= \neg\,\mathrm{LI}^\bullet(\hat{C})
\end{aligned}$$

- Suppose that p in $E[r]_p$ is contained in a maximal Γ-term $h[r]$, which occurs more than once in $E[r]_p \vee \mathrm{LI}(E[r]_p)$. This case can be argued analogously.
- Otherwise:

$$\begin{aligned}
\mathrm{LI}^\bullet(C) &= (s\tau = t\tau \wedge \mathrm{LI}(C_2)\tau) \ \vee\ (s\tau \neq t\tau \wedge \mathrm{LI}(C_1)\tau) \\
&\Leftrightarrow \neg[(s\tau \neq t\tau \vee \neg\mathrm{LI}(C_2)\tau) \ \wedge\ (s\tau = t\tau \vee \neg\mathrm{LI}(C_1)\tau)] \\
&= \neg[(s\tau \neq t\tau \vee \mathrm{LI}(\hat{C}_2)\tau) \ \wedge\ (s\tau = t\tau \vee \mathrm{LI}(\hat{C}_1)\tau)] \\
&\Leftrightarrow \neg[(s\tau = t\tau \wedge \mathrm{LI}(\hat{C}_2)\tau) \ \vee\ (s\tau \neq t\tau \wedge \mathrm{LI}(\hat{C}_1)\tau)] \\
&= \neg\,\mathrm{LI}^\bullet(\hat{C})
\end{aligned}$$

We conclude by showing that $\mathrm{LI}^\bullet(C) \Leftrightarrow \neg\,\mathrm{LI}^\bullet(\hat{C})$ entails that $\mathrm{LI}(C) \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$: Clearly the terms to be lifted in $\mathrm{LI}^\bullet(C)$ and $\mathrm{LI}^\bullet(\hat{C})$ are the same and differ only in their color. Even though this results in different lifting variables, that is of no relevance as all lifted variables are instantly bound. Additionally, the quantifier type of any given lifting variable in $Q(C)$ is dual to the respective one in $Q(\hat{C})$. Furthermore note that the subterm-relation is not affected by the coloring, so the ordering of the quantifiers in $Q(C)$ and $Q(\hat{C})$ is identical. Hence $\mathrm{LI}(C) \Leftrightarrow \neg\, \mathrm{LI}(\hat{C})$. □

⟨lemma:delta_entails_li⟩ **Lemma 10.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then* $\Delta \models \neg \ell_\Gamma[\mathrm{LI}(C)] \vee \ell_\Gamma[C]$.

Proof. Construct $\hat{\pi}$ with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then by Lemma 8, $\hat{\Gamma} \models \ell_{\hat{\Delta}}[\mathrm{LI}(\hat{C})] \vee \ell_{\hat{\Delta}}[\hat{C}]$, which by Lemma 9 is nothing else than $\Delta \models \neg \ell_\Gamma[\mathrm{LI}(C)] \vee \ell_\Gamma[C]$. □

Theorem 11. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{LI}(\pi)$ is an interpolant of Γ and Δ.*

Proof. 8 10 plus lifting of ground terms (todo)

□

old stuff

1.3 Lifting the Δ-terms

Definition 12. $\mathrm{LI}^{\Delta}(C)$ and $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$ for a clause C are defined as $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$ respectively with the difference that in its inductive definition, every lifting $\ell[\varphi]$ for a formula or term φ is replaced by a lifting of only the Δ-terms $\ell_{\Delta}[\varphi]$. $\triangle$

Remark. Many results involving $\mathrm{LI}(C)$ or $\mathrm{LI}_{\mathrm{cl}}(C)$ are valid for $\mathrm{LI}^{\Delta}(C)$ or $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$ in a formulation which is adapted accordingly. This can easily be seen by the following proof idea:

Let $f_1, \dots, f_n$ be all Γ-colored function or constant symbols occurring in C, c a fresh constant symbol and g a fresh n-ary function symbol. Construct a formula $\varphi : g(t_1, \dots, t_n) = g(t_1, \dots, t_n)$, such that $t_i = f_i(c_1, \dots, c_m)$ for $1 \leq i \leq n$ where m is the arity of f_i and $c_j = c$ for $1 \leq j \leq m$. Let $\Delta' = \Delta \cup \{\varphi\}$ and apply the desired result to the initial clause sets Γ and Δ'.

Under this construction, every originally Γ-colored symbol is now grey, which implies that $\mathrm{LI}(C) = \mathrm{LI}^{\Delta}(C)$ as well as $\mathrm{LI}_{\mathrm{cl}}(C) = \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$. But $\Delta \models \psi \Leftrightarrow \Delta' \models \psi$ for any formula ψ. $\triangle$

d:lemma:gamma_entails_lide⟩? **Lemma 13.** *Let C be a clause in a resolution refutation of $\Gamma \cup \Delta$. Then $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$.*

Proof. We proceed by induction of the strengthening $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_{\Gamma})$.

Base case. For $C \in \Gamma$, $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_{\Gamma}) = \ell_{\Delta}[C] = C$. Hence $\Gamma \models \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_{\Gamma})$.

For $C \in \Delta$, $\mathrm{LI}^{\Delta}(C) = \top$, so $\Gamma \models \mathrm{LI}^{\Delta}(C)$.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

We define the following abbreviations:

$\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_{\Gamma})^* = \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_{\Gamma} \backslash \{l_{\mathrm{LIcl}^{\Delta}}\})$

$\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_{\Gamma})^* = \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_{\Gamma} \backslash \{\neg l'_{\mathrm{LIcl}^{\Delta}}\})$

Hence the induction hypothesis can be stated as follows:

$\Gamma \models \mathrm{LI}^{\Delta}(C_1) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_{\Gamma})^* \vee (l_{\mathrm{LIcl}^{\Delta}})_{\Gamma}$

$\Gamma \models \mathrm{LI}^{\Delta}(C_2) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_{\Gamma})^* \vee \neg(l'_{\mathrm{LIcl}^{\Delta}})_{\Gamma}$

By Lemma 6, $\mathrm{LI}^{\Delta}(C_i)$ and $\mathrm{LI}^{\Delta}_{\mathrm{cl}}(C_i)$ for $i \in \{1,2\}$ do not contain Δ-colored terms. Hence we are able to apply Lemma 7 in order to obtain

$$\Gamma \overset{(\circ)}{\models} \ell_{\Delta}[\mathrm{LI}^{\Delta}(C_1)\tau] \vee \ell_{\Delta}[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_{\Gamma})^*\tau] \vee \ell_{\Delta}[(l_{\mathrm{LIcl}^{\Delta}})_{\Gamma}\tau]$$

$$\Gamma \overset{(*)}{\models} \ell_{\Delta}[\mathrm{LI}^{\Delta}(C_2)\tau] \vee \ell_{\Delta}[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_{\Gamma})^*\tau] \vee \neg\ell_{\Delta}[(l'_{\mathrm{LIcl}^{\Delta}})_{\Gamma}\tau]$$

By Lemma 5, we obtain that $\ell_{\Delta}[l_{\mathrm{LIcl}^{\Delta}}\tau] = \ell_{\Delta}[l'_{\mathrm{LIcl}^{\Delta}}\tau]$.

Now we distinguish cases based on the color of the resolved literal:

- Suppose that l is Γ-colored. Then as $\ell_{\Delta}[l_{\mathrm{LIcl}^{\Delta}}\tau] = \ell_{\Delta}[l'_{\mathrm{LIcl}^{\Delta}}\tau]$, we can perform a resolution step on $(\circ)$ and $(*)$, which gives that $\Gamma \models \ell_{\Delta}[\mathrm{LI}^{\Delta}(C_1)\tau] \vee \ell_{\Delta}[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_1)_{\Gamma})^*\tau] \vee \ell_{\Delta}[\mathrm{LI}^{\Delta}(C_2)\tau] \vee \ell_{\Delta}[\mathrm{LI}^{\Delta}_{\mathrm{cl}}((C_2)_{\Gamma})^*\tau]$. This however is nothing else than $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$.

- Suppose that l is Δ-colored. Then ($\circ$) and ($*$) simply to the following:
  $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau]$
  $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_2)_\Gamma)^*\tau]$
  These however imply that $\Gamma \models \mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^* \vee \mathrm{LI}^\Delta_{\mathrm{cl}}((C_2)_\Gamma)^* \vee (\ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \wedge \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau])$, which is nothing else than $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$.
- Suppose that l is grey. Suppose that M is a model of Γ such that $M \not\models \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$, i.e. $M \not\models \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_2)_\Gamma)^*\tau]$. Then $M \models \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau]$ as well as $M \models \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau] \vee \neg\ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$. Due to $\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$, we obtain that $M \models (\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] \wedge \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau]) \vee (\neg\ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau] \wedge \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau])$, which is nothing else than $M \models \mathrm{LI}^\Delta(C)$.

Factorisation. Suppose the clause C is the result of a factorisation inference ι of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

We introduce the abbreviation $\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^* = \mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma \backslash \{l_{\mathrm{LIcl}^\Delta}, \neg l'_{\mathrm{LIcl}^\Delta}\})$ and express the induction hypothesis as follows:

$\Gamma \models \mathrm{LI}^\Delta(C_1) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^* \vee (l_{\mathrm{LIcl}^\Delta})_\Gamma \vee \neg(l'_{\mathrm{LIcl}^\Delta})_\Gamma$

By Lemma 6, $\mathrm{LI}^\Delta(C_i)$ and $\mathrm{LI}^\Delta_{\mathrm{cl}}(C_i)$ for $i \in \{1,2\}$ do not contain Δ-colored terms. Hence we are able to apply Lemma 7 in order to obtain

$\Gamma \overset{(\circledast)}{\models} \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[(l_{\mathrm{LIcl}^\Delta})_\Gamma\tau] \vee \neg\ell_\Delta[(l'_{\mathrm{LIcl}^\Delta})_\Gamma\tau]$

As by Lemma 5 we get that $\ell_\Delta[l_{\mathrm{LIcl}^\Delta}\tau] = \ell_\Delta[l'_{\mathrm{LIcl}^\Delta}\tau]$, we can perform a factorisation step on ($\circledast$) to obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^*\tau] \vee \ell_\Delta[(l_{\mathrm{LIcl}^\Delta})_\Gamma\tau]$. But this is nothing else than $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C_\Gamma)$. $\square$

Paramodulation. Suppose the clause C is the result of a paramodulation inference ι of $C_1 : s = t \vee D$ and $C_2 : E[r]_p$ with $\sigma = \mathrm{mgu}(\iota)$.

We introduce the abbreviation $\mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma)^* = \mathrm{LI}^\Delta_{\mathrm{cl}}((C_1)_\Gamma \backslash \{(s = t)_{\mathrm{LIcl}}\})$ and express the induction hypothesis as follows:

$\Gamma \overset{(\circ)}{\models} \mathrm{LI}^\Delta(C_1) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C_1)^* \vee (s = t)_{\mathrm{LIcl}}$

$\Gamma \overset{(*)}{\models} \mathrm{LI}^\Delta(C_2) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C_2)$

Suppose now that for a model M of Γ that $M \models \ell_\Delta[s] \neq \ell_\Delta[t]$. Then by Lemma 5, $M \models s_{\mathrm{LIcl}} \neq t_{\mathrm{LIcl}}$. Hence we get by ($\circ$) that $\Gamma \models \mathrm{LI}^\Delta(C_1) \vee \mathrm{LI}_{\mathrm{cl}}(C_1)^*$ and consequently by Lemma 6 and Lemma 7 that $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_1)\tau] \vee \ell_\Delta[\mathrm{LI}_{\mathrm{cl}}(C_1)^*\tau]$. But this however implies that $\Gamma \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$.

Now suppose to the contrary that for a model M of Γ that $M \models \ell_\Delta[s] = \ell_\Delta[t]$. Note that by Lemma 5, $\ell_\Delta[(E[r]_p)_\Gamma] = \mathrm{LI}^\Delta_{\mathrm{cl}}(C_2)$. Hence ($*$) is nothing else than $\Gamma \models \mathrm{LI}^\Delta(C_2) \vee \ell_\Delta[(E[r]_p)_\Gamma]$. From this, it also follows by Lemma 6 and Lemma 7 that $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[\ell_\Delta[(E[r]_p)_\Gamma]\tau]$, which by Lemma 3 simplifies to $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[(E[r]_p)_\Gamma\tau]$,

Due to $\sigma = \mathrm{mgu}(r,s)$, $r\tau \equiv s\tau$ and consequently $\ell_\Delta[r\tau] = \ell_\Delta[s\tau]$. Hence $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[(E[s]_p)_\Gamma\tau]$,

We proceed by a case distinction:

- Suppose that p in $E[s]_p$ is not contained in a Δ-term. Then $\ell_\Delta[(E[s]_p)_\Gamma\tau]$ and $\ell_\Delta[(E[t]_p)_\Gamma\tau]$ only differ at position p. But as $M \models \ell_\Delta[s] = \ell_\Delta[t]$, by Lemma 7 and Lemma 3 we derive that $M \models \ell_\Delta[s\tau] = \ell_\Delta[t\tau]$. Then however $M \models \ell_\Delta[(E[s]_p)_\Gamma\tau] \Leftrightarrow \ell_\Delta[(E[t]_p)_\Gamma\tau]$ and thus $M \models \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[(E[t]_p)_\Gamma\tau]$, which is sufficient for $M \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$.
- Suppose that p in $E[r]_p$ is contained in a maximal Δ-term $h[r]$, which occurs more than once in $E[r]_p \vee \mathrm{LI}^\Delta(E[r]_p)$. Suppose furthermore that $M \models \ell_\Delta[h[s]] = \ell_\Delta[h[t]]$ as otherwise $M \models s = t \wedge \ell_\Delta[h[s]] = \ell_\Delta[h[t]]$, which implies that $M \models \mathrm{LI}^\Delta(C)$.

  But then we again obtain that $\ell_\Delta[(E[s]_p)_\Gamma\tau]$ and $\ell_\Delta[(E[t]_p)_\Gamma\tau]$ only differ at position p and by a similar line of argument as in the former case, we can deduce that $M \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$.
- Suppose that p in $E[s]_p$ is contained in a maximal Δ-term $h[s]$, which occurs exactly once in $E[s]_p \vee \mathrm{LI}^\Delta(E[r]_p)$. Then the lifting variable $z_{h[s]}$ occurs exactly once in $(*)$, where it is implicitly universally quantified. Therefore we can instantiate this variable by any term, in particular by $z_{h[t]}$, so we obtain that $\Gamma \models \ell_\Delta[\mathrm{LI}^\Delta(C_2)\tau] \vee \ell_\Delta[(E[t]_p)_\Gamma\tau]$. which again is sufficient for $M \models \mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$.

ld:lemma:gamma_lifted_lide⟩? **Lemma 14.** *For a clause C of a resolution refutation of $\Gamma \cup \Delta$, $\ell_\Gamma[\mathrm{LI}^\Delta(C)] = \mathrm{LI}(C)$ and $\ell_\Gamma[\mathrm{LI}^\Delta_{\mathrm{cl}}(C)] = \mathrm{LI}_{\mathrm{cl}}(C)$.*

Proof. We proceed by induction.

Base case. For $C \in \Gamma \cup \Delta$, $\mathrm{LI}^\Delta_{\mathrm{cl}}(C) = \ell_\Delta[C]$. By Lemma 1, $\ell_\Gamma[\ell_\Delta[C]] = \ell[C]$, so $\ell_\Gamma[\mathrm{LI}^\Delta_{\mathrm{cl}}\, C] = \ell[C] = \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$.

$\mathrm{LI}^\Delta(C)$ does not contain colored symbols.

Inductions step. Suppose the clause C is the result of a resolution, factorisation or paramodulation inference ι of the clauses $\bar{C}$.

As liftings do not affect the predicate, we do not consider them further. Note that every term in $\mathrm{LI}(C)$ or $\mathrm{LI}_{\mathrm{cl}}(C)$ is of the form $\ell[t\tau]$ for some term t in $\mathrm{LI}(C_i)$ or $\mathrm{LI}_{\mathrm{cl}}(C_i)$ for some $C_i \in \bar{C}$. Furthermore, every term in $\mathrm{LI}^\Delta(C)$ or $\mathrm{LI}^\Delta_{\mathrm{cl}}(C)$ is of the form $\ell_\Delta[t\tau]$ for some term t in $\mathrm{LI}^\Delta(C_i)$ or $\mathrm{LI}^\Delta_{\mathrm{cl}}(C_i)$ for some $C_i \in \bar{C}$.

Hence it suffices to show that for a term t in $\mathrm{LI}^\Delta(C_i)$ or $\mathrm{LI}^\Delta_{\mathrm{cl}}(C_i)$ and its corresponding term $\ell_\Gamma[t]$ in $\mathrm{LI}(C_i)$ or $\mathrm{LI}_{\mathrm{cl}}(C_i)$ for some $C_i \in \bar{C}$ that $\ell_\Gamma[\ell_\Delta[t\tau]] = \ell[\ell_\Gamma[t]\tau]$.

By Lemma 6, no Δ-terms occur in t. Hence $\ell[t] = \ell_\Gamma[t]$ and consequently $\ell[\ell[t]\tau] = \ell[\ell_\Gamma[t]\tau]$. By Lemma 3, $\ell[\ell[t]\tau] = \ell[t\tau]$ and by Lemma 1, $\ell[t\tau] = \ell_\Gamma[\ell_\Delta[t\tau]]$. Hence $\ell_\Gamma[\ell_\Delta[t\tau]] = \ell[\ell_\Gamma[t]\tau]$. ☐

1.4 One-sided interpolants

As we have just seen, the formula $\mathrm{LI}^\Delta(C) \vee \mathrm{LI}^\Delta_{\mathrm{cl}}(C)$ now satisfies one condition of interpolants. Using this, we are able to formulate a result on one-sided interpolants, which are defined as follows:

Definition 15. Let Γ and Δ be sets of first-order formulas. A *one-sided interpolant* of Γ and Δ is a first-order formula I such that

1. $\Gamma \models I$
2. $\mathrm{L}(I) \subseteq \mathrm{L}(\Gamma) \cap \mathrm{L}(\Delta)$ $\triangle$

Proposition 16. *Let Γ and Δ be sets of first-order formulas such that $\Gamma \cup \Delta$ is unsatisfiable. Then there is a one-sided interpolant of Γ and Δ which is a Π_1 formula.*

Proof. Let π be a resolution refutation of $\Gamma \cup \Delta$. By Lemma **??**, $\Gamma \models \mathrm{LI}^{\Delta}(\pi) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(\pi)$, or in other words $\Gamma \models \forall x_1 \ldots \forall x_n \mathrm{LI}^{\Delta}(\pi) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(\pi)$, where $x_1, \ldots, x_n$ are the Δ-lifting variables occurring in $\mathrm{LI}^{\Delta}(\pi) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(\pi)$. By Lemma 6, the formula $\mathrm{LI}^{\Delta}(\pi) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(\pi)$ does not contain Δ-colored symbols.

Let $y_1, \ldots y_m$ be the Γ-lifting variables of $\ell^{y}_{\Gamma}[\mathrm{LI}^{\Delta}(\pi) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(\pi)]$ and

$$I = \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m \ell^{y}_{\Gamma}[\mathrm{LI}^{\Delta}(\pi) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(\pi)].$$

Note that I does not contain any Γ-terms. As $\mathrm{LI}^{\Delta}(\pi) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(\pi)$ contains witness terms for every existential quantifier in I with respect to Γ, $\Gamma \models I$. Hence I is a Π_1 formula which is a one-sided interpolant for $\Gamma \cup \Delta$. $\square$

1.5 Quantifying over the lifting variables

def:arrow_quantifier_block⟩? **Definition 17** (Quantifier block)**.** Let C be a clause in a resolution refutation π of $\Gamma \cup \Delta$ and $\bar{x}$ the Δ-lifting variables and $\bar{y}$ the Γ-lifting variables occurring in $\mathrm{LI}(C)$ and $\mathrm{LI}_{\mathrm{cl}}(C)$. $Q(C)$ denotes an arrangement of the elements of $\{\forall x_t \mid x_t \in \bar{x}\} \cup \{\exists y_t \mid y_t \in \bar{y}\}$ such that for two lifting variable z_s and z_r, if s is a subterm of r, then z_s precedes z_r. We denote $Q(\square)$ by $Q(\pi)$. $\triangle$

Note that at a certain stage of the interpolant extraction, the quantifier block possesses a certain partial ordering based on the subterm relation of the indices of the lifting variables. This implies that the ordering is monotonous in the sense that in the subsequent course of the extraction, this ordering is only extended but existing order-relations are not modified, even though the indices of the lifting variables are altered by means of substitution.

ma_entails_quantified_lide⟩? **Lemma 18.** *Let C be a clause of a resolution refutation of $\Gamma \cup \Delta$. Then $\Gamma \models Q(C)(\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C))$.*

Proof. By Lemma **??**, $\Gamma \models \mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$ and by Lemma **??** $\ell_{\Gamma}[\mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)] = \mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$. Hence the terms in $\mathrm{LI}^{\Delta}(C) \vee \mathrm{LI}^{\Delta}_{\mathrm{cl}}(C)$ provide witness terms for the Γ-lifting variables in $\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C)$, which are existentially quantified in $Q(C)(\mathrm{LI}(C) \vee \mathrm{LI}_{\mathrm{cl}}(C))$.

Furthermore, the ordering imposed on the quantifiers in $Q(C)$ implies that if a Δ-lifting variable x_s occurs in a witness term for a Γ-lifting variable y_r, y_r is quantified in the scope of the quantifier of x_s as s is a subterm of r. This however ensures that the witness terms are valid. $\square$

?⟨old:lemma:li_symmetry⟩? **Lemma 19.** *Let π be a refutation of $\Gamma \cup \Delta$ and $\hat{\pi}$ be π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$. Then for a clause C in π and its corresponding clause $\hat{C}$ in $\hat{\pi}$, $Q(C)(\mathrm{LI}(C)) \Leftrightarrow \neg Q(\hat{C})(\mathrm{LI}(\hat{C}))$.*

Proof. Consider furthermore that liftings variables of C and $\hat{C}$ only differ in the variable symbol, but not in the index, and that the quantifier type of any given lifting variable in C is dual to the corresponding one in $\hat{C}$. Hence for any formula ϕ, $Q(C)\neg\phi \Leftrightarrow \neg Q(\hat{C})\phi$.

It remains to show that $\mathrm{LI}(C) \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$, which we establish by induction:

Base case. If $C \in \Gamma$, then $\mathrm{LI}(C) = \bot \Leftrightarrow \neg\top \Leftrightarrow \neg\,\mathrm{LI}(\hat{C})$ as $\hat{C} \in \Delta$. The case for $C \in \Delta$ can be argued analogously.

Resolution. Suppose the clause C is the result of a resolution step ι of $C_1 : D \vee l$ and $C_2 : E \vee \neg l'$ with $\sigma = \mathrm{mgu}(\iota)$.

As τ depends only on σ, τ is the same for both π and $\hat{\pi}$.

We now distinguish the following cases:

1. l is Γ-colored:

$$\begin{aligned}
\mathrm{LI}(C) &= \ell[\mathrm{LI}(C_1)\tau] \vee \ell[\mathrm{LI}(C_2)\tau] \\
&\Leftrightarrow \neg(\neg\ell[\mathrm{LI}(C_1)\tau] \wedge \neg\ell[\mathrm{LI}(C_2)\tau]) \\
&\Leftrightarrow \neg(\ell[\mathrm{LI}(\hat{C}_1)\tau] \wedge \ell[\mathrm{LI}(\hat{C}_2)\tau]) \\
&= \neg\,\mathrm{LI}(\hat{C})
\end{aligned}$$

2. l is Δ-colored: This case can be argued analogously.
3. l is grey: Note that by Lemma 5, $\ell[l_{\mathrm{LIcl}}\tau] = \ell[l'_{\mathrm{LIcl}}\tau]$ $(*)$.

$$\begin{aligned}
\mathrm{LI}(C) \;&=\; (\neg\ell[l'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_1)\tau]) \;\vee\; (\ell[l_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(C_2)\tau]) \\
&\overset{(*)}{\Leftrightarrow} (\ell[l'_{\mathrm{LIcl}}\tau] \vee \ell[\mathrm{LI}(C_1)\tau]) \;\wedge\; (\neg\ell[l_{\mathrm{LIcl}}\tau] \vee \ell[\mathrm{LI}(C_2)\tau]) \\
&\Leftrightarrow \neg\Big((\neg\ell[l'_{\mathrm{LIcl}}\tau] \wedge \neg\ell[\mathrm{LI}(C_1)\tau]) \;\vee\; (\ell[l_{\mathrm{LIcl}}\tau] \wedge \neg\ell[\mathrm{LI}(C_2)\tau])\Big) \\
&= \neg\Big((\neg\ell[\hat{l}'_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(\hat{C}_1)\tau]) \;\vee\; (\ell[\hat{l}_{\mathrm{LIcl}}\tau] \wedge \ell[\mathrm{LI}(\hat{C}_2)\tau])\Big) \\
&= \neg\,\mathrm{LI}(\hat{C})
\end{aligned}$$

Factorisation. Suppose the clause C is the result of a factorisation ι of $C_1 : l \vee l' \vee D$ with $\sigma = \mathrm{mgu}(\iota)$.

Then $\mathrm{LI}(C) = \ell[\mathrm{LI}(C_1)\tau]$, so the construction is not influenced by the coloring and the induction hypothesis gives the result.

Paramodulation. Suppose the clause C is the result of a paramodulation inference ι of $C_1 : s = t \vee D$ and $C_2 : E[r]_p$ with $\sigma = \mathrm{mgu}(\iota)$.

We proceed by a case distinction:

- Suppose that p in $E[r]_p$ is contained in a maximal Δ-term $h[r]$, which occurs more than once in $E[r]_p \vee \mathrm{LI}(E[r]_p)$. Then p in $\hat{E}[r]_p$ is contained in a maximal Γ-term $h[r]$, which occurs more than once in $\hat{E}[r]_p \vee \mathrm{LI}(\hat{E}[r]_p)$.

$$
\begin{aligned}
&\mathrm{LI}(C) \\
&= (\ell[s\tau = t\tau] \wedge \ell[\mathrm{LI}(C_2)\tau]) \vee (\ell[s\tau \neq t\tau] \wedge \ell[\mathrm{LI}(C_1)\tau]) \vee (\ell[s\tau = t\tau] \wedge \ell[h[s]\tau \neq h[t]\tau]) \\
&\Leftrightarrow \neg[(\ell[s\tau \neq t\tau] \vee \neg\ell[\mathrm{LI}(C_2)\tau]) \wedge (\ell[s\tau = t\tau] \vee \neg\ell[\mathrm{LI}(C_1)\tau]) \wedge (\ell[s\tau \neq t\tau] \vee \ell[h[s]\tau = h[t]\tau])] \\
&= \neg[(\ell[s\tau \neq t\tau] \vee \ell[\mathrm{LI}(\hat{C}_2)\tau]) \wedge (\ell[s\tau = t\tau] \vee \ell[\mathrm{LI}(\hat{C}_1)\tau]) \wedge (\ell[s\tau \neq t\tau] \vee \ell[h[s]\tau = h[t]\tau])] \\
&\Leftrightarrow \neg[(\ell[s\tau = t\tau] \wedge \ell[\mathrm{LI}(\hat{C}_2)\tau]) \vee (\ell[s\tau \neq t\tau] \wedge \ell[\mathrm{LI}(\hat{C}_1)\tau]) \wedge (\ell[s\tau \neq t\tau] \vee \ell[h[s]\tau = h[t]\tau])] \\
&= \neg\,\mathrm{LI}(\hat{C})
\end{aligned}
$$

- Suppose that p in $E[r]_p$ is contained in a maximal Γ-term $h[r]$, which occurs more than once in $E[r]_p \vee \mathrm{LI}(E[r]_p)$. This case can be argued analogously.
- Otherwise:

$$
\begin{aligned}
&\mathrm{LI}(C) \\
&= (\ell[s\tau = t\tau] \wedge \ell[\mathrm{LI}(C_2)\tau]) \ \vee\ (\ell[s\tau \neq t\tau] \wedge \ell[\mathrm{LI}(C_1)\tau]) \\
&\Leftrightarrow \neg[(\ell[s\tau \neq t\tau] \vee \neg\ell[\mathrm{LI}(C_2)\tau]) \ \wedge\ (\ell[s\tau = t\tau] \vee \neg\ell[\mathrm{LI}(C_1)\tau])] \\
&= \neg[(\ell[s\tau \neq t\tau] \vee \ell[\mathrm{LI}(\hat{C}_2)\tau]) \ \wedge\ (\ell[s\tau = t\tau] \vee \ell[\mathrm{LI}(\hat{C}_1)\tau])] \\
&\Leftrightarrow \neg[(\ell[s\tau = t\tau] \wedge \ell[\mathrm{LI}(\hat{C}_2)\tau]) \ \vee\ (\ell[s\tau \neq t\tau] \wedge \ell[\mathrm{LI}(\hat{C}_1)\tau])] \\
&= \neg\,\mathrm{LI}(\hat{C})
\end{aligned}
$$

□

Theorem 20. *Let π be a resolution refutation of $\Gamma \cup \Delta$. Then $\mathrm{LI}(\pi)$ is an interpolant.*

Proof. By Lemma **??** $\Gamma \models Q(\pi)(\mathrm{LI}(\pi) \vee \mathrm{LI}_{\mathrm{cl}}(\pi))$. But as $\mathrm{LI}_{\mathrm{cl}}(\pi) = \Box$, this simplifies to $\Gamma \models Q(\pi)\,\mathrm{LI}(\pi)$.

By constructing a proof $\hat{\pi}$ from π with $\hat{\Gamma} = \Delta$ and $\hat{\Delta} = \Gamma$, we obtain by Lemma **??** that $\hat{\Gamma} \models Q(\hat{\pi})\,\mathrm{LI}(\hat{\pi})$. By Lemma 9, this however is nothing else than $\Delta \models \neg Q(\pi)\,\mathrm{LI}(\pi)$.

As furthermore by construction no colored symbols occur in $Q(\pi)\,\mathrm{LI}(\pi)$, this formula is an interpolant for $\Gamma \cup \Delta$. □