CHAPTER 1

Number of quantifier alternations in Huang and nested

1.1 Outline

Goal: try proof Huang and nested equal.
Method: proof for both:

Conjectured Proposition 1. *Let I be an interpolant created by $algorithm. If I contains a term t such that t has n color changes, then I has at least n quantifier alternations.*

1.2 Preliminaries

Quantifier alternations in I usually assumes the quantifier-alternation-minimising arrangement of quantifiers in I

Definition 2 (Color alternation col-alt)**.** Colors Γ and Δ, term t:
$\text{col-alt}(t) \stackrel{\text{def}}{=} \text{col-alt}_{\perp}(t)$
Let $t = f(t_1, \ldots, t_n)$ for constant, function and variable symbols (syntax abuse)

$$\text{col-alt}_{\Phi}(t) \stackrel{\text{def}}{=} \begin{cases} \max(\text{col-alt}_{\Phi}(t_1), \ldots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is grey} \\ \max(\text{col-alt}_{\Phi}(t_1), \ldots, \text{col-alt}_{\Phi}(t_n)) & f \text{ is of color } \Phi \\ 1 + \max(\text{col-alt}_{\Psi}(t_1), \ldots, \text{col-alt}_{\Psi}(t_n)) & f \text{ is of color } \Psi,\ \Phi \neq \Psi \end{cases}$$ △

Definition 3. $\text{PI}^{\circ}_{\text{step}}$ is the same as PI_{step} but no σ occurs in its definition. △

1.3 Random thoughts

- Quantifiers are introduced for lifting variables which actually occur in the interpolant
- If term t with $\text{col-alt}(t) = n$ enters I, we need subterm s of t with $\text{col-alt}(s) = n - 1$ to be in I (of course colors of t and s are exactly opposite)

1.3.1 Proof

- Induction over $\ell^x_\Delta[\mathrm{PI}(C) \vee C]$ and also about Γ-terms with Δ-lifting vars in that formula. Cf. `-final`
- TODO: describe proof method with $\sigma_{(0,i)}$: which PI?
  - Factorisation: easy: just apply σ_i for all i to $\mathrm{PI}(C) \vee C$. When done, a literal will be there twice and we can remove it without losing anything
  - Resolution: create propositional structure first.
    Ex.: $C_1 : D \vee l$, $C_2 : \neg l \vee E$:
    If we talk about properties for which it holds that if they hold for $\mathrm{PI}(C_i) \vee C_i$, $i \in \{1,2\}$, then they also hold for $A \equiv \Big((l \wedge \mathrm{PI}(C_2)) \vee (\neg l \wedge \mathrm{PI}(C_1))\Big) \vee C$, then we can apply σ_i for all i to that formula.
    So if we can assume it for A and show it for all σ_i, we get that it holds for $\mathrm{PI}(C) \vee C$.

1.4 Proof port attempt from `-final`

Conjectured Lemma 4. *Resolution or factorisation step ι from $\bar{C}$.*
If x col-change var (where?), then x also occurs grey (where?).

$\Rightarrow$ fill in blanks when known where we need this

Conjectured Lemma 5. *If $\mathrm{PI}(C) \vee C$ contains a maximal colored occurrence of a Γ-term $t[s]$ containing Δ-term s, then s occurs grey in $\mathrm{PI}(C) \vee C$.*

Proof. Note that it suffices to show that at the step where s is introduced as subterm of $t[s]$, s occurs grey in $\mathrm{PI}(C) \vee C$ as any later modification by substitution is applied to both occurrences s, so they stay equal throughout the remaining derivation. TODO: what if it's in $\mathrm{PI}(C)$ and it disappears due to not being a colored literal?

Induction over π and σ.

Base case: ✓

Step: Resolution or factorisation inference ι, $\mathrm{mgu}(\iota) = \sigma = \sigma_1 \cdots \sigma_n$ The term $t[s]$ is created by one of the following two ways:

- A variable u occurs in $(\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C)\sigma_{(0,i-1)}$ such that $u\sigma_i = t[s]$.

  Then u occurs in a resolved or factorised literal $\lambda\sigma_{(0,i-1)}$ at $\hat{u}$ such that at the other resolved or factorised literal $\lambda'\sigma_{(0,i-1)}$, $\lambda'\sigma_{(0,i-1)}|_{\hat{u}} = t[s]$. Then the condition is present at $(\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C)\sigma_{(0,i-1)}$ and we get the result by the induction hypothesis.

- Note that we only consider maximal colored terms. Let $t[u]$ be a maximal colored Γ-term in $(\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C)\sigma_{(0,i-1)}$ such that in the tree-representation of $t[u]$, the path from the root to u does not contain a node labelled with a Δ-symbol.

  Suppose that u occurs grey in $(\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C)\sigma_{(0,i-1)}$. Then s occurs grey in $(\mathrm{PI}^\circ_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C)\sigma_{(0,i)}$ and we are done.

is probably not same t as in lemma statement, which isn't technically wrong but confusing

Now suppose that u does not occur grey in $(\mathrm{PI}^{\circ}_{\mathrm{step}}(\iota, \mathrm{PI}(C_1), \mathrm{PI}(C_2)) \vee C)\sigma_{(0,i-1)}$.

TODO: need color changing variable lemma for $\mathrm{PI}(C) \vee C$, or actually the $\mathrm{PI}_{\mathrm{step}}$-representation

TODO: case with u in s.c. Γ-term

$\square$