noteworthy thoughts

* if x occurs in $y\sigma$, then add an arrow from every grey occurrence of x to the network of y-occurrences. It should be possible to have this network reach every occurrence. not sure how wide-reaching this is as it does not cover any color-alternating terms.
* seem to not be able to construct $Q(f(h(x)), g(x))$ without arrow between arguments (either merge or a directed one)
* need some kind of backwards merging
** a possibly useful criterion: $z\sigma$ occurs in $y\sigma$ for $y, z \in C_1 \cup C_2$.
* what about label for arrows containing the variable, which is manipulated by the unifier?

basic facts which should be used in the algorithm

* variables only occur per clause and are only changed by unification. Hence need to establish conditions at beginning which are not violated by unification.
* without $x \leftrightarrow f(x)$ situation, no mixed-colored terms can occur
* other unifications transfer mixed-colored terms without producing them, but they also modify the mixed-colored terms

1 current version

TODO: find formulation of merge arrows: lemma about how terms of color are related when they share variables
TODO: check if old lemma 2 occurs anywhere else
TODO: basically check A, B and C for which new lemma 2 we need; consider longer NB comment in lemma 1
TODO: for rewriting, make sure to handle coloring correctly (lifting vars are not colored) and make sure to be clear on whether we talk about C, $\mathrm{AI}(C)$ or $\mathrm{AI}^{\Delta}(C)$.

from_grey_to_colored⟩ **Lemma 1.** *Let x be a variable in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ occurs in the maximal colored term $t[x]$. If it has a grey occurrence in some literal ($\mathrm{AI}(C)$ or or also in literals with colored predicates), then $x \rightsquigarrow t[x]$.*

Proof. Induction start by definition.
Induction step with usual notation.
We consider introductions of $t[x]$ by changing the variable y. Let $\hat{y}$ be the position of y which causes the variable to be changed by the unification algorithm. $\hat{y}$ is in a resolved literal, say l, so we denote it by $l|_{\hat{y}}$ and its counterpart in l' by $l'|_{\hat{y}}$.

Suppose x occurs grey in $y\sigma$ and a colored $t[y]$ occurs in C_1. Then $t[x]$ occurs in C and we have to show that $x \rightsquigarrow t[x]$ if x occurs grey in C.

- Suppose y has a grey occurrence $\dot{y}$ in C_1. Then by the induction hypothesis, $\dot{y} \rightsquigarrow t[y]$. As σ is applied in C, $\dot{y}\sigma[x] \rightsquigarrow t\sigma[x]$.
- Otherwise there are only colored occurrences of y, so also $l|_{\hat{y}}$ is a colored occurrence. Let it be contained in the maximal colored term $s[y]$.

  figure: $Q(\dots t[y] \dots) \vee P(\dots s[y]_p \dots)$ $\qquad \neg P(\dots s[x]_p)$

  - Suppose that x occurs grey in C_2. Then by the induction hypothesis, $x \rightsquigarrow l'|_{\hat{y}}$ and so $x \rightsquigarrow l|_{\hat{y}}$.

    As all other occurrences of y are contained in colored terms and x occurs in $y\sigma$, there is a unifier-induced arrow such that $x \rightsquigarrow y$. NB: don't need ind hyp anymore here

- Suppose that x does not occur grey in C_2. Suppose that x does occur grey in C as otherwise we are done.

  Then there exists a grey occurrence of a variable z in C_i such that x occurs grey in $z\sigma$.

  * Suppose $C_i = C_1$.

    figure: $Q(t[y]) \vee z \vee l[f(y), g(z)] \quad \neg l[f(h(x)), g(x)] \quad z\sigma = x; y\sigma = h(x)$

    By backwards merge special case 1', $z \rightsquigarrow t|_y$.

    NB: the backwards merge special case again does a lot; without it, we know that:

    · there is an arrow from z to $g(z)$ by the induction hypothesis
    · there should be some arrow at $f(h(x)), g(x))$, so after the resolution step, the same arrow applies to $f(y), g(z)$.
    · as y occurs colored, there should again be some arrow between $t[y]$ and $f(y)$.
    · these combined should yield $z \rightsquigarrow t(y)$, which is what we want to show

  * Suppose $C_i = C_2$.

    By backwards merge special case 1', $z \rightsquigarrow t|_y$

    figure: $Q(t[y]) \vee l[f(y), s[u], r[u]] \quad \neg l[f(h(x)), s'[z], r'[x]] \vee z$
    x grey in $u\sigma$, x grey in $z\sigma$,

    NB: without special case: z occurs grey in C_2, and also in the resolved literal, say at $\hat{z}$.

    · merge arrow at $s[u], r[u]$, $f(h(x)), r'[x]$ and regular arrow from z to $s'[z]$
    · merge arrow at $t[y], f(y)$

Suppose x occurs colored in $y\sigma$ and y occurs in C_1 (colored or grey).

figure: $C_1 : Q(\ldots \dot{y} \ldots) \vee l[\hat{y}]_p \qquad C_2 : \neg l[\hat{y}']_p \quad$ ($\hat{y}'$ is abstraction of $t[x]$)

- Suppose $l'|_{\hat{y}}$ contains x.

  - Suppose that x occurs grey in C_2. Then by the induction hypothesis, $x \rightsquigarrow l'|_{\hat{y}}$ and hence $x \rightsquigarrow l|_{\hat{y}}$. Let $\dot{y}$ be an occurrence of y in C_1 different from $l|_{\hat{y}}$.

    If $l|_{\hat{y}}$ is a grey occurrence and $\dot{y}$ occurs colored in C_1, then by the induction hypothesis, $l|_{\hat{y}} \rightsquigarrow \dot{y}$. By combining the paths, we get that $x \rightsquigarrow \dot{y}$.

    If $l|_{\hat{y}}$ is a grey occurrence and $\dot{y}$ occurs grey in C_1, then by Lemma 2, there is a merge path between $l|_{\hat{y}}$ and $\dot{y}$ and hence $x \rightsquigarrow \dot{y}$.

    If $l|_{\hat{y}}$ is a colored occurrence and $\dot{y}$ occurs grey in C_1, then by the backwards merging case 3, $x \rightsquigarrow \dot{y}$.

    If $l|_{\hat{y}}$ is a colored occurrence and $\dot{y}$ occurs colored in C_1, <merge arrows>; Subsumed by: If $l|_{\hat{y}}$ is a colored occurrence, then $x \rightsquigarrow l|_{\hat{y}}$ since x occurs in $y\sigma$ by unifier-induced arrow 1.

  - Otherwise x occurs only colored in C_2.

    If x does not occur grey in C, we are done, so assume it does.

    Then there exists a grey occurrence of a variable z in C_i such that x occurs grey in $z\sigma$.

∗ Suppose $C_i = C_1$.

· Suppose y occurs grey in C_1. As x occurs grey in $z\sigma$ and x occurs colored in $y\sigma$, by the backwards merging 1 special case, we have an arrow from z to y. Full stop as in C, $x \rightsquigarrow y\sigma$.

· Otherwise y occurs colored in C_1. Then a similar reasoning goes through by backwards merging case 2.

need to see variable here

figure: $r[y] \vee z \vee l[f(y), g(z)] \quad \neg l[f(h(x)), g(x)] \quad z\sigma = x; y\sigma = h(x)$

NB: without special case: Then there is an occurrence of z in the resolved literal, sat at $l|_{\hat{z}}$ such that $l'|_{\hat{z}}$ is an occurrence of x. As x occurs grey in $z\sigma$ and x only occurs colored, both $l|_{\hat{z}}$ and $l'|_{\hat{z}}$ are colored occurrences.

· arrow $z, g(z)$, merge path $f(h(x)), g(x)$
· if $r[y]$ colored occurrence of y, then $f(y), r[y]$
· if $r[y]$ grey occurrence of y, really need some special case, but at least y is visible (probably special case 1)

∗ Suppose $C_i = C_2$.

figure: $r[y] \vee l[f(y), s[u], r[u]] \quad \neg l[f(h(x)), s'[z], r'[x]] \vee z$
x grey in $u\sigma$, x grey in $z\sigma$, x colored in $y\sigma$

We get $z \rightsquigarrow y$ by either backwards special case 1 or 2, depending whether y is a grey or colored occurrence.

NB: version without special case appears to be similar as above

- Suppose $l'|_{\hat{y}}$ does not contain x. Then it contains a variable u such that x occurs grey in $u\sigma$. So the situation repeats in C_2 as $l'|_{\hat{y}}$ is contained in a colored term and u is what y was now. Hence we obtain the result by Remark (∗).

□

⟨ow_from_grey_to_grey⟩ **Lemma 2.** *Let x be a variable in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$. Then there is a merge arrow between every pair of distinct grey occurrences of x. // possibly not really needed as same var is always lifted by same lifting var, they can never diverge syntactically. Still this is used in the proof. NB: possibly get rid of it by interpreting arrows as from term to term, not from occurrence to occurrence. This way, there is an implicit connection between every grey occurrence of a variable*

Proof. Induction start: by definition.
Suppose it holds for C_1 and C_2, usual notation.
Suppose for some grey variable occurrence x that y occurs grey in $x\sigma$ for some variable y which has a grey occurrence in C (so either it was there in C_i and $y\sigma = y$ or y occurs grey in $z\sigma$ for some z, but then some y occurs elsewhere).
Then there is a position $\hat{x}$ in a resolved literal, say w.l.o.g. l, such that $l|_{\hat{x}} = x$ and $l'|_{\hat{x}} = y$.

- Suppose that $l|_{\hat{x}}$ is a grey occurrence. Then so is $l'|_{\hat{x}}$. By the induction hypothesis, both occurrences have merge edges to all other occurrences of the variable, and these are merged. Note that C_1 and C_2 are variable disjoint, so x does not occur in C_2 and y does not occur in C_1.

- Otherwise suppose that $l|_{\hat{x}}$ is a colored occurrence. There are merge edges between all grey occurrences of x in C_1 and y in C_2 by the induction hypothesis. As y occurs grey in $x\sigma$, by backwards merge case 4, there is a merge edge between every grey occurrence of x and y. □

2 proof for AI^{Δ}

variables_connected⟩? **Proposition 3.** *Consider the edges of $\mathcal{M}(C)$ and $\mathcal{A}(C)$ as edge-set of an undirected graph G. Then all occurrences of a variable x are connected in G.*

Proof. Initially, this clearly holds.
Usual induction step:
By the induction hypothesis, variable occurrences in the resolving clauses are connected.
Only variables are changed which occur in the resolved literal. They are unified with the term at the corresponding literal in the other clause. This is the only way that new variables can be introduced, but the arrows of the corresponding terms in the resolved literals are merged. □

Lemma 4. *(this is actually what the procedure should establish. Make sure to have some kind of network of all colored occurrences of a variable.)*
:colored_merge_ai_de⟩ *NB: it seems to be more useful to just write "unifier arrow 1/2"*

- *Let a variable x occur colored in $s[x]$ in $\mathrm{AI}^{\Delta}(C)$. If x also occurs in a colored term in the resolved literal which is unified with a term t of the other resolved literal such that for a variable y, it holds that $y \rightsquigarrow t$, then add an arrow from every grey occurrence of y in C to $s[x]$. // for the upper lemmas*
- *Let $s[x]$ be a maximal colored term containing an occurrence of x or the occurrence of x itself in case it is grey. If x also occurs in a colored term in the resolved literal which is unified with a term t of the other resolved literal such that for a lifting variable x_i, it holds that $x_i \rightsquigarrow t$, then add an arrow from every grey occurrence of x_i in C to $s[x]$. // for AI^{Δ} lemma (probably needs to be applied to AI^{Γ} separately)*

NB: intuition of difference: grey variable changes are handled elsewhere

Proof. By unifier arrows 1 and 2. □

Conjecture 5. *Let C be a clause in a resolution refutation. Suppose that $\mathrm{AI}^{\Delta}(C)$ contains a maximal Γ-term $\gamma_j[z_i]$ which contains a lifting variable z_i. Then $z_i \rightsquigarrow \gamma_j[z_i]$.*

Proof. Induction, usual notation, induction step:
(Notation: $\ell_\Delta[\gamma_{j'}[u]\sigma]\tau = \gamma_j[z_i]$; i.e. $\ell_\Delta[u\sigma]\tau$ contains z_i)

Suppose for some $\gamma_{j'}[u]$ in (w.l.o.g.) $\mathrm{AI}^{\Delta}(C_1)$ that a maximal colored term which is a Δ-term occurs in $u\sigma$.

NB: By unifier-induced arrows 2, we have that $z_i \rightsquigarrow \gamma_{j'}[u]$ straight away in case there is a grey occurrence of z_i. The induction hypothesis seems to make sure that it actually exists.

Then there is an occurrence of u in the resolved literal, say at $l|_{\hat{u}}$, such that the unification algorithm modifies u .

- Suppose u occurs grey in C_1. Then by Lemma 1, $u \rightsquigarrow \gamma_{j'}[u]$. But $\ell_\Delta[u\sigma]\tau$ contains z_i, the same lifting variable as $\ell_\Delta[\gamma_j[u]\sigma]\tau$. So $z_i \rightsquigarrow \gamma_j[z_i]$.

  Remark: To create a mixed-colored term, a variable must appear grey in a resolved literal (and elsewhere occur colored). This is this case and it necessarily yields a grey occurrence of the foreign colored term.

- Suppose u does not occur grey in C_1. Then $l|_{\hat{u}}$ is a colored occurrence of u. As there are no Δ-terms, u is a Γ-colored occurrence. $l'|_{\hat{u}}$ is an abstraction of a term which contains a Γ-colored term containing z_i at the position $\hat{u}$.

  only mention of "no Δ-terms" here, looks somewhat generalisable

  - Suppose that $l'|_{\hat{u}}$ does not contain a lifting variable. Let $s[u]$ be the maximal colored term containing $l'|_{\hat{u}}$. Then $l'|_{\hat{u}}$ contains a variable v such that $v\sigma$ contains a lifting variable. This is handled by another case (Remark $(*)$) and we assume that $z_i \rightsquigarrow s[u]$.
  - Suppose that $l'|_{\hat{u}}$ contains a lifting variable. As $l|_{\hat{u}}$ is a Γ-colored occurrence of u and $l|_{\hat{u}}\sigma = l'|_{\hat{u}}\sigma$, $l'|_{\hat{u}}$ is contained in a Γ-term and contains a Δ-term, so by the induction hypothesis, $z_i \rightsquigarrow s[u]$.

  By Lemma 4, we also have that $z_i \rightsquigarrow \gamma_{j'}[u]$ and hence $z_i \rightsquigarrow \gamma_j[z_i]$.

Suppose for some variable u in (w.l.o.g.) $\mathrm{AI}^{\Delta}(C_1)$ that $u\sigma$ contains a mixed-colored Γ-term containing z_i. Then u occurs in the resolved literal, say at $\hat{u}$, and $l'|_{\hat{u}}$ is an abstraction of $u\sigma$ but $l'|_{\hat{u}} \neq u$. Note that $u\sigma = l'|_{\hat{u}}\sigma$

NB: unifier-induced arrows seem to replace all this as in the other case

- Suppose $l'|_{\hat{u}}$ contains a mixed-colored term. Then by the induction hypothesis, $z_i \rightsquigarrow l'|_{\hat{u}}$.
- Suppose $l'|_{\hat{u}}$ does not contain a mixed-colored term. As $l'|_{\hat{u}}\sigma$ contains a mixed-colored term, $l'|_{\hat{u}}$ contains a variable v such that either $v\sigma$ contains a mixed-colored term, or v occurs in a Γ-colored term in $l'|_{\hat{u}}$ and $v\sigma$ is a Δ-colored term $\Rightarrow$ Remark $(*)$.

  So this shows that $z_i \rightsquigarrow l'|_{\hat{u}}$. TODO: recheck when remark is made more formal, but it probably can't imply less than the other case does.

As the arrows of $l|_{\hat{u}}\sigma$ and $l'|_{\hat{u}}\sigma$ are merged, we get that $z_i \rightsquigarrow l|_{\hat{u}}$ if z_i occurs in $\mathrm{AI}^{\Delta}(C_2)$. It remains to show that the arrow also applies to other occurrences of u. Let $\dot{u}$ be an occurrence of u in $\mathrm{AI}^{\Delta}(C_1)$ different from $l|_{\hat{u}}$.

- Suppose $\dot{u}$ is a grey occurrence and $\hat{u}$ is a grey occurrence. Then by Lemma 2, $\hat{u} \rightsquigarrow_= \dot{u}$.
- Suppose $\dot{u}$ is a colored occurrence and $\hat{u}$ is a grey occurrence. Then by Lemma 1, $\hat{u} \rightsquigarrow \dot{u}$.
- Suppose $\dot{u}$ is a grey occurrence (irrespective of the $\hat{u}$ occurrence). Then by unifier-induced arrows 2b, $z_i \rightsquigarrow \dot{u}$. $\square$

3 original proof

Ideas for simplification:
* Lemma for all cases about what is on the other side

Remark $(*)$. Any substitution, in particular σ, only changes a finite number of variables. Furthermore a result of a run of the unification algorithm is acyclic in the sense that if a substitution $u \mapsto t$ is added to the resulting substitution, it is never the case that at a later stage $t \mapsto u$ is added. This can easily be seen by considering that at the point when $u \mapsto t$ is added to the resulting substitution, every occurrence of u is replaced by t, so u is not encountered by the algorithm at a later stage.
Therefore in order to show that a statement holds for every $u \mapsto t$ in a unifier σ, it suffices to show by an induction argument that for every substitution $v \mapsto s$ which is added to the resulting unifier by the unification algorithm that it holds for $v \mapsto s$

under the assumption that it holds for every $w \mapsto r$ such that w occurs in s and $w \mapsto r$ is added to the resulting substitution at a later stage. $\triangle$

Conjecture 6. *Let C be a clause in a resolution refutation. Suppose that* $\mathrm{AI}^{\Delta}(C)$ *contains a maximal* Γ*-term* $\gamma_j[z_i]$ *which contains a lifting variable* z_i*. Then* $z_i <_{\hat{\mathcal{A}}(C)} z_j$*.*

Proof. We proceed by induction. For the base case, note that no multicolored terms occur in initial clauses, so no lifting term can occur inside of a Γ-term.
Suppose a clause C is the result of a resolution of $C_1 : D \vee l$ and $C_2 : E \vee \neg l$ with $l\sigma = l'\sigma$. Furthermore suppose that for every lifting term inside a Γ-term in the clauses C_1 and C_2 of the refutation, for every term of the form $\gamma_j[z_i]$ we have that $z_i <_{\hat{\mathcal{A}}(C_1)} z_j$ or $z_i <_{\hat{\mathcal{A}}(C_2)} z_j$ respectively. Hence there is an arrow (p_1, p_2) in $\hat{\mathcal{A}}(C_1)$ or $\hat{\mathcal{A}}(C_2)$ such that z_i is contained in $\mathrm{P}(p_1)$ and z_j is contained in $\mathrm{P}(p_2)$. In $\mathrm{AI}^{\Delta}(C)$, $\mathrm{P}(p_1)$ contains $\ell[z_i\sigma]\tau = z_i\tau$ and $\mathrm{P}(p_2)$ contains $\ell[z_j\sigma]\tau = z_j\tau$. Hence the indices of the lifting variables might change, but this renaming does not affect the relation of the objects as $\hat{\mathcal{A}}(C_1) \cup \hat{\mathcal{A}}(C_2) \subseteq \hat{\mathcal{A}}(C)$.
We show that $z_i <_{\hat{\mathcal{A}}(C)} z_j$ holds true also for every new term of the form $\gamma_j[z_i]$ for some j, i in $\mathrm{AI}^{\Delta}(C)$. By "new", we mean terms which are not present in $\mathrm{AI}^{\Delta}(C_1)$ or $\mathrm{AI}^{\Delta}(C_2)$. Note that new terms in $\mathrm{AI}^{\Delta}(C)$ are of the form $\ell^x_{\Delta}[t\sigma]\tau$ for some $t \in \mathrm{AI}^{\Delta}(C_1) \cup \mathrm{AI}^{\Delta}(C_2)$. By Lemma **??**, σ does not introduce lifting variables. Hence a new term of the form $\gamma_j[z_i]$ is created either by introducing a Δ-term into a Γ-term or by introducing $\gamma_j[\delta_i]$ via σ, both followed by the lifting. Note that τ only substitutes lifting variables by other lifting variables and hence does not introduce lifting variables. Furthermore by Lemma **??**, τ only substitutes lifting variables for other lifting variables, whose corresponding term is more specialised. Hence if there exists an arrow from a lifting variable to $\gamma_j[z_i]$ according to this lemma, it is also an appropriate arrow if $\gamma_j[z_i]$ is replaced by $\gamma_j[z_i]\tau$.
We now distinguish the two cases under which a new term $\gamma_j[z_i]$ can occur in $\mathrm{AI}^{\Delta}(C)$:

Suppose for some Γ-term $\tilde{\gamma}_{j'}[u]$ in $\mathrm{AI}^{\Delta}(C_1)$ or $\mathrm{AI}^{\Delta}(C_2)$, $u\sigma$ contains a Δ-term.

Hence we have that $(\tilde{\gamma}_{j'}[u])\sigma = \gamma_j[\delta_i]$ for some i. Note that the position of u in $\tilde{\gamma}_{j'}[u]$ does not necessarily coincide with the position of δ_i in $\gamma_j[\delta_i]$ as u might be substituted by σ for a grey term containing δ_i.

We have that $\ell_{\Delta}[\tilde{\gamma}_{j'}[u]\sigma]\tau = \gamma_j[z_i]$.

At some well-defined point of application of the unification algorithm, u is substituted by an abstraction of a term which contains δ_i. This occurrence of u is in l and we denote it by $\hat{u}$. We furthermore denote the term at the corresponding position in l' by $t_{\hat{u}}$.

We distinguish cases based on the occurrences of $\hat{u}$ and $t_{\hat{u}}$.

- Suppose $\hat{u}$ is a grey occurrence.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\hat{u}) \qquad C_2 : \neg Q(t_{\hat{u}})}{C : P(\gamma_j[\delta_i])}$$

Figure 1: Example for this case

Then by Lemma 1, there is an arrow from a term containing u to a term containing $\gamma_j[u]$ in $\hat{\mathcal{A}}(C)$. As $\hat{u}\sigma$ is a term containing the Δ-term δ_i, the term at the position of $\hat{u}$ in $\mathrm{AI}^{\Delta}(C)$ is $\ell[\hat{u}\sigma]\tau$, which by assumption contains z_i. But there is an arrow from this term containing z_i to $\gamma_j[z_i]$, so $z_i <_{\hat{\mathcal{A}}(C)} z_j$.

- Suppose $\hat{u}$ occurs in a maximal colored term which is a Γ-term.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\gamma_k[\hat{u}]_p) \qquad C_2 : \neg Q(\gamma_m[t_{\hat{u}}]_p)}{C : P(\gamma_j[\delta_i])}$$

$$\frac{C_1 : Q(\tilde{\gamma}_{j'}[\hat{u}]) \qquad C_2 : \neg Q(\gamma_m[t_{\hat{u}}])}{C : \square}$$

$//\gamma_j[\delta_i]$ occurs in the interpolant

Figure 2: Examples for this case

Then either $\hat{u}$ is the occurrence of u in $\tilde{\gamma}_{j'}[\hat{u}]$ or it occurs in a different Γ-term $\gamma_j[\hat{u}]$. In the latter case, by Lemma **??**, there is a merge edge between $\tilde{\gamma}_{j'}[\hat{u}]$ and $\gamma_j[\hat{u}]$. TODO: or no direct connection but via other term Hence in both cases, it suffices to show that there is an arrow from a term containing an occurrence of z_i to $t_{\hat{u}}$.

A (does not work like this)

We distinguish on the shape of $t_{\hat{u}}$:

  - $t_{\hat{u}}$ is a term which does not contain a Δ-term. Then it contains a variable that is be substituted by σ by a term which contains a Δ-term as $u\sigma = t_{\hat{u}}\sigma$ is a term containing a Δ-term. We denote by v the variable in $t_{\hat{u}}$ which is substituted by a term containing a Δ-term in case $t_{\hat{u}}$ is a grey term.

    In the course of the unification algorithm, there are further unifications of v since we know that $u\sigma = v\sigma$ is a term containing a Δ-term. Therefore by Remark $(*)$, we can assume that there is an appropriate arrow to $t_{\hat{u}}$.

  - $t_{\hat{u}}$ is a term which contains a Δ-term. As $t_{\hat{u}}$ occurs in a Γ-term in C_1, say in $\gamma_m[t_{\hat{u}}]$, C_1 contains a multicolored Γ-term. Hence the corresponding term in $\mathrm{AI}^\Delta(C_1)$, is of the form $\gamma_m[z_{i'}]$ for some i'. Observe that i' in general is not equal to i as demonstrated in Example 7, even though we have that $t_{\hat{u}}\sigma = u\sigma$. This is because the lifting variables in $\mathrm{AI}(\cdot)$ represent abstractions of the terms in the clauses of the resolution derivation (cf. Lemma **??**). Therefore we only know by the induction hypothesis that $z_{i'} <_{\hat{\mathcal{A}}(C_1)} \ell[\gamma_m[z_{i'}]] = \ell[t_{\hat{u}}]$.

    However by Lemma **??** and due to the fact that $\hat{u}$ and $t_{\hat{u}}$ respectively occur in the resolved literal, $\ell_\Delta[\hat{u}\sigma]\tau = \ell_\Delta[t_{\hat{u}}\sigma]\tau$. As $\ell_\Delta[\hat{u}\sigma]\tau = \ell_\Delta[\delta_i]\tau = z_i\tau$ as well as $\ell_\Delta[t_{\hat{u}}\sigma]\tau = \ell_\Delta[z_{i'}\sigma]\tau = z_{i'}\tau$, we must have that $z_i\tau = z_{i'}\tau$. As however $u\sigma = \delta_i$, by the definition of au, we have that $\{z_i \mapsto z_i\} \in \tau$, so $z_{i'}\tau = z_i$.

    Since τ is applied to every literal in $\mathrm{AI}^\Delta(C)$ and in $\mathrm{AI}^\Delta(C_1)$ an arrow from a term containing $z_{i'}$ to $t_{\hat{u}}$ exists, the same arrow applied to $\mathrm{AI}^\Delta(C)$ points from a term containing $z_{i'}\tau = z_i$ to $t_{\hat{u}}$. Therefore $z_i <_{\hat{\mathcal{A}}(C)} z_j$.

- Suppose $\hat{u}$ occurs in a maximal colored term which is a Δ-term.

no Δ-terms in $\mathrm{AI}^\Delta(C)$.

$$\frac{C_1 : P(\tilde{\gamma}_{j'}[u]) \vee Q(\delta_k[\hat{u}]_p) \qquad C_2 : \neg Q(\delta_m[t_{\hat{u}}]_p)}{C : P(\gamma_j[\delta_i])}$$

By Lemma ??, TODO:

B (check how we need colored term arrows)

Suppose for some variable v in $\mathrm{AI}^{\Delta}(C_1)$ or $\mathrm{AI}^{\Delta}(C_2)$, $v\sigma = \gamma_j[\delta_i]$ for some i. As v is affected by the unifier, it occurs in the literal being unified, say w.l.o.g. in l in C_1. At some well-defined point in the unification algorithm, v is substituted by an abstraction of $\gamma_j[\delta_i]$. Let p be the position of the occurrence of v in l which causes this substitution. Furthermore, let p' be the position corresponding to p in l'.

Note that any arrow from or to p' also applies to p in $\hat{\mathcal{A}}(C)$ and hence to $\gamma_j[z_i]$ as they are merged due to occurring in the resolved literal. So it suffices to show that there is an arrow from an appropriate lifting variable to p'. We denote the term at p' by t.

Note that $t\sigma = \gamma_j[\delta_i]$. So t is either a Γ-term containing a Δ-term, in which case we know that there is an appropriate arrow by the induction hypothesis as t occurs in l' in C_2, or t is an abstraction of $\gamma_j[\delta_i]$, in which case we can assume the existence of an appropriate arrow by Remark $(*)$.WRONG: probably last half sentence, this is usually not the situation where remark $(*)$is applicable

□

C (might need some lemma as well)

ction_in_arrow_proof⟩ **Example 7.** $\Gamma = \{Q(\gamma(x)) \vee P(x), \neg Q(\gamma(z)), R(\dots)\}$

$\Delta = \{\neg P(\delta(y)) \vee R(y), \neg R(a), Q(\dots)\}$

$a \sim x_k, \delta(y) \sim x_i, \delta(a) \sim x_j$

R only for coloring

Q only for coloring

$$
\dfrac{\dfrac{\dfrac{\dfrac{\perp \mid Q(\gamma(x)) \vee P(x) \qquad \top \mid \neg P(x_i) \vee R(y)}{P(x_i) \mid Q(\gamma(x_i)) \vee R(y)} \qquad \top \mid \neg R(x_k)}{(\neg R(x_k) \wedge P(x_i)) \vee (R(x_k) \wedge \top) \mid Q(\gamma(x_i))}}{P(x_i) \vee R(x_k) \mid Q(\gamma(x_i))} \qquad \perp \mid \neg Q(\gamma(z))}{\dfrac{(\neg Q(x_j) \wedge (P(x_i) \vee R(x_k))) \vee (Q(x_j) \wedge \top) \mid \square}{\neg Q(x_j) \wedge (P(x_i) \vee R(x_k)) \mid \square}}
$$

Gist: When $Q(\gamma(x_i))$ is the only symbol in $\mathrm{AI}^{\Delta}(\cdot)$, the lifting var means $\delta(x)$, but in the actual derivation, it's $\delta(a)$. however τ fixes this. So before Q is resolved, there is an arrow, but with the wrong lifting var (x_i instead of x_j) △

something about when i started with connected components

unification is for resolved literals.
connections between resolved literals and the rest of the clauses is covered by arrows.
if a term enters, merge arrows ensure that everything is propagated.
the special thing about colored occurrences is the fact that they can create multicolored terms in cooperation with grey occurrences..
a variable only occurs in a clause if it was never substituted by anything. Hence in particular all grey occurrences are "original" (TODO: renamings of variables)
Let u be a grey occurrence. let $f(u)$ be a colored occurrence. either it is original, then we are fine by arrow propagation. otherwise it has been introduced, but then it has used the network of another variable.
more precisely: a variable v occurs in a related literal in a related position in another clause as u in $f(u)$. so the variable is substituted by a term containing u, say $t[u]$ the arrows at the entry points are merged.

effect: $t[u]$ occurs at every grey occurrence of v. all arrows mentioning them are merged with the ones mentioning the entry point. this is justified as the terms there appear "as they are", i.e. as they are produced at the entry point.
however a colored occurrence cannot be produced from a grey occurrence ($\mathrm{mgu}(x, f(u))$) but only if a grey occ is in the literal and a colored occ is elsewhere in the clause (the network of the other var). but then there are (directed) arrows.
Every variable has a connected network in a clause.
there is a barrier between colored terms.

4 misc results

ta_term_in_ai_delta⟩? **Proposition 8.** *In $\mathrm{AI}^{\Delta}(C)$, all terms are either variables, grey, Γ-terms or Δ-lifting variables. In particular, no Γ-term is contained in a Δ-term and there is at most one color alternation.*
In other words, the coloring of all terms follows this grammar: (grey | gamma) [delta]*
// not sure how this is really useful in the end of the proof where we have to switch colors and show that it also works from the Δ-side

5 results in spe

Conjectured Lemma 9. *there is a merge path between all occurrences of a variable x in all colored terms of the same "stage" (and directed arrows between stages).NB: not sure where this is going and if it's true*

Proof. a stage means the color alternation level: only Γ, $1\Gamma + 1\Delta$, and so on.
more formally: on the prefix to x in a maximal colored term t, iterate in order and increase counter whenever the current symbol has a different color than the previous encountered color. the counter is increased for the first colored symbol. this number plus the color of t define the stage.
induction start: by def.
induction step, usual notation.
Suppose a term $t[y]$ changes its stage. So it contains a variable y h

□

6 missteps

ored_to_all_colored⟩? **Conjectured Lemma 10.** *WRONG: does not work out Let x be a variable in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$. Then there is a merge path from every colored occurrence of x to every other colored occurrence of x in C.*

Proof. Induction start: by definition.
Suppose holds for C_1 and C_2, usual notation.
We consider introductions of colored occurrences of x.

Suppose x is introduced in t by means of unification. $t\sigma$ contains x, hence there is a variable y in t such that $y\sigma = s[x]$.

Let $\hat{y}$ be the position of y which causes the variable to be changed by the unification algorithm. $\hat{y}$ is in a resolved literal, say l, so we denote it by $l|_{\hat{y}}$ and its counterpart in l' by $l'|_{\hat{y}}$ // copied

- Suppose $l|_{\hat{y}}$ is a grey occurrence. $l'|_{\hat{y}}$ is an abstraction of $s[x]$. TODO:
- Suppose $l|_{\hat{y}}$ is a colored occurrence. Then by the induction hypothesis, $l|_{\hat{y}} \rightsquigarrow_{=} \dot{y}$ for every other colored occurrence $\dot{y}$ of y. As the substitution $s[x]$ for y, there are merge edges between all these occurrences.

  For the grey occurrences of y TODO:

Suppose a colored term $t[x]$ containing x is in $\mathrm{ran}(\sigma)$.

∎

olor_to_other_color⟩? **Conjectured Lemma 11.** *WRONG: $Q(f(x)) \vee R(x); \neg R(g(y))$ If a variable x occurs in a maximal colored term $s[x]$ which is a Γ-term as well as in a maximal colored term $t[x]$ which is a Δ-term in C, then $x \rightsquigarrow s[x]$ and $x \rightsquigarrow t[x]$.*

Proof. This situation does not occur in the induction start.
Induction step, usual notation.
We consider resolution steps which create this situation. Clauses are variable disjoint, so if a variable occurs in a term of a color, it can only also occur in a term of another color by entering the other term via unification.
TODO: case distinction: var x introduced into t or $y\sigma$ gives such a term ? ∎

?⟨wrong:same_color2⟩? _same_colored_terms⟩? **Conjectured Lemma 12.** *WRONG: probably wrong for same reason as 13 Let x be a variable $s[x]$ and $t[x]$ be terms containing x such that $x \not\rightsquigarrow s[x]$ and $x \not\rightsquigarrow t[x]$. Then $s[x] \rightsquigarrow_{=} t[x]$.*

⟨wrong:same_color⟩ **Conjectured Lemma 13.** *WRONG: consider: $f(x)$, $g(f(x))$, $h(g(f(x)))$, $f, h : \Gamma, g : \Delta$ Let $s[x]$ and $t[x]$ be maximal colored terms of the same color containing a variable x. Then $s[x] \rightsquigarrow_{=} t[x]$*

rom_grey_to_colored⟩? **Lemma 14.** ***not true in this formulation, we can have x, $f(x)$ and $g(x)$ with arrows just from x to the two colored occurrences, even if f and g of same color.***
Let x be a variable in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ which has a grey occurrence and a colored occurrence. Then there is an arrow in $\mathcal{A}(C)$ from a term containing a grey occurrence to a term containing a colored occurrence. // Should also hold for all of AI^{Δ}, but is currently not needed in the proof

Proof. For clauses C in the initial clause set, $\mathcal{A}(C)$ is defined to contain an arrow from every grey occurrence to every colored occurrence for every variable occurring in the clause.
For the induction step, suppose the lemma holds for C_1 and C_2. Note that C_1 and C_2 are variable disjoint. TODO: how to continue without checking every single case?
Note that terms are only changed by means of substitution.
If a variable is substituted, it does not occur any further in the derivation.
If a variable is substituted by a term containing variables, this is fine because the original arrows still apply for the new terms. ∎

Lemma 15. ***(same as above) not true in this formulation, we can have x, $f(x)$ and $g(x)$ with***

arrows just from x to the two colored occurrences, even if f and g of same color.

_colored_to_colored⟩? *Let x be a variable which occurs colored in $\mathrm{AI}^{\Delta}_{\mathrm{cl}}(C)$ and again colored in the same color somewhere else in $\mathrm{AI}^{\Delta}(C)$. Then there is a merge edge between the maximal colored terms containing the two occurrences.* *// This is exactly the case we need, possibly show something more general*

Proof. TODO: □