$\Sigma' = \{R(z) \lor \exists x P(f(x)), \neg Q(x), \}$
$\Pi' = \{\forall y\, g(y) = y, \forall y \neg P(g(y)) \lor Q(y), \neg R(d)\}$
$\Sigma = \mathrm{sk}(\Sigma') = \{R(z) \lor P(f(c)), \neg Q(y), \}$
$\Pi = \mathrm{sk}(\Pi') = \{g(u) = u, \neg P(g(v)) \lor Q(v), \neg R(d)\}$
$L(\Sigma) = \{R, P, Q, f, z, x, c\}$
$L(\Pi) = \{R, P, Q, g, u, v, d\}$
Refutation:

$$\dfrac{\dfrac{R(z) \lor P(f(c))_\Sigma \qquad \neg R(d)_\Pi}{P(f(c))}\; z \mapsto d \qquad \dfrac{\dfrac{\neg P(g(v)) \lor Q(v)_\Pi \qquad \neg Q(y)_\Sigma}{\neg P(g(y))}\; v \mapsto y \qquad g(u) = u_\Pi}{\neg P(u)}\; y \mapsto u}{\square}\; u \mapsto f(c)$$

Interpolants:

$$\dfrac{\dfrac{\bot \qquad \top}{(\neg R(d) \land \bot) \lor (R(d) \land \top) \equiv R(d)}\; \theta_0 \qquad \dfrac{\dfrac{\top \qquad \bot}{(\neg Q(y) \land \top) \lor (Q(y) \land \top) \equiv \neg Q(y)}\; \theta_1 \qquad \top}{(\neg Q(u) \land g(u) = u) \lor (\top \land g(u) \neq u)}\; \theta_2}{(\neg P(f(c)) \land R(d)) \quad \lor \quad (P(f(c)) \land ((\neg Q(f(c)) \land g(f(c)) = f(c)) \quad \lor \quad g(f(c)) \neq f(c)))}\; \theta_3$$

Relative interpolant properties:

$\theta_0:$	$\Sigma \vdash R(d) \lor P(f(c))$	$\Pi \vdash \neg R(d) \lor P(f(c))$
$\theta_1:$	$\Sigma \vdash \neg Q(y) \lor \neg P(g(y))$	$\Pi \vdash Q(y) \lor \neg P(g(y))$
$\theta_2:$	$\Sigma \vdash (\neg Q(u) \land g(u) = u) \lor g(u) \neq u \quad \lor \quad \neg P(u)$	$\Pi \vdash \neg((\neg Q(u) \land g(u) = u) \lor g(u) \neq u) \quad \lor \quad \neg P(u)$ $\Pi \vdash ((Q(u) \lor g(u) \neq u) \land g(u) = u) \quad \lor \quad \neg P(u)$
$\theta_3:$	$\Sigma \vdash \theta_3$ Proof: Either $\neg P(f(c))$, then $R(d)$. Otw. either $g(f(c)) \neq f(c)$. Otw. also $\neg Q(f(c))$.	$\Pi \vdash \neg\theta_3$ Proof: $\neg(\neg P(fc) \land R(d)) \quad \lor \quad (P(fc) \land (\neg Q(fc) \land g(fc) = fc) \quad \lor \quad g(fc) \neq fc)$ $\equiv (P(fc) \lor \neg R(d)) \quad \land \quad (\neg P(fc) \lor (Q(fc) \lor g(fc) \neq fc) \quad \land \quad g(fc) = fc)$ Have $g(fc) = fc$ and $\neg R(d)$, so remaining: $\neg P(fc) \lor Q(fc)$. Get by axiom and unification with $g(u) = u$.

$\Sigma = \{R(z) \vee P(f(c)), \neg Q(y), \}$
$\Pi = \{g(u) = u, \neg P(g(v)) \vee Q(v), \neg R(d)\}$

Propositional refutation tree (no non-trivial unifiers):

$$\dfrac{\dfrac{R(d) \vee P(f(c))_\Sigma \qquad \neg R(d)_\Pi}{P(f(c))} \qquad \dfrac{\dfrac{\neg P(g(f(c))) \vee Q(f(c))_\Pi \qquad \neg Q(f(c))_\Sigma}{\neg P(g(f(c)))} \qquad g(f(c)) = f(c)_\Pi}{\neg P(f(c))}}{\Box}$$

Lifting:

terms: $g(f(c)), f(c), d$
max Π-terms: $\{g(f(c)), d\} \sim \{x_1, x_2\}$
max Σ-terms: $\{f(c)\} \sim \{x_3\}$

$\overline{(\neg P(f(c)) \wedge R(d)) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge g(f(c)) = f(c)) \quad \vee \quad g(f(c)) \neq f(c)))}(x_1, x_2)$
$\Leftrightarrow \neg P(f(c)) \wedge R(x_2) \quad \vee \quad (P(f(c)) \wedge ((\neg Q(f(c)) \wedge x_1 = f(c)) \quad \vee \quad x_1 \neq f(c)))$

By Lemma 12, $\Sigma \models \overline{\theta_3}$ (proof from above still goes through).
$\hat{\theta}(x_3) = (\neg P(x_3) \wedge R(x_2)) \quad \vee \quad (P(x_3) \wedge ((\neg Q(x_3) \wedge x_1 = x_3) \quad \vee \quad x_1 \neq x_3))$
quantifiers according to order: $|d| < |f(c)| < |g(f(c))|$
$\theta = \forall x_2 \exists x_3 \forall x_1 (\neg P(x_3) \wedge R(x_2)) \quad \vee \quad (P(x_3) \wedge (\neg Q(x_3) \vee x_1 \neq x_3))$
$\neg\theta = \exists x_2 \forall x_3 \exists x_1 (P(x_3) \vee \neg R(x_2)) \quad \wedge \quad (\neg P(x_3) \vee (Q(x_3) \wedge x_1 = x_3))$
$\Rightarrow \Sigma \vdash \theta; \Pi \vdash \neg\theta$

Example 2:

$\Sigma = \{P(c), \neg P(d)\}$

$\Pi = \{P(d) \vee g(u) = u, \neg P(g(x))\}$

Refutation:

$$\dfrac{\dfrac{\dfrac{P(d) \vee g(u) = u_{\Pi} \qquad \neg P(d)_{\Sigma}}{g(u) = u} \qquad \neg P(g(x)_{\Pi}}{\neg P(x)}\; u \mapsto x \qquad P(c)_{\Sigma}}{\Box}\; x \mapsto c$$

Relative interpolants:

$$\dfrac{\dfrac{\dfrac{\top \qquad \bot}{(\neg P(d) \wedge \top) \vee (P(d) \wedge \bot) \equiv \neg P(d)} \qquad \top}{(g(x) = x \wedge \top) \vee (g(x) \neq x \wedge \neg P(d))}\; u \mapsto x \qquad \bot}{(\neg P(c) \wedge \bot) \vee P(c) \wedge (g(c) = c \vee (g(c) \neq c \wedge \neg P(d)))}\; x \mapsto c$$

$\theta = P(c) \wedge (g(c) = c \vee \neg P(d))$

$\neg\theta = \neg P(c) \vee (g(c) \neq c \wedge P(d))$

terms: $g(c), c, d$

max Π-terms: $g(c)$

max Σ-terms: c

ordered by length ASCENDING: $\{c, g(c)\}$

$\overline{\theta}(x_2) = P(c) \wedge (x_2 = c \vee \neg P(d))$

$\hat{\theta}(x_1) = P(x_1) \wedge (x_2 = x_1 \vee \neg P(d))$

$\Sigma \vdash \exists x_1 \forall x_2 P(x_1) \wedge (x_2 = x_1 \vee \neg P(d))$

$\Pi \vdash \neg\exists x_1 \forall x_2 P(x_1) \wedge (x_2 = x_1 \vee \neg P(d))$

$\Pi \vdash \forall x_1 \exists x_2 \neg P(x_1) \vee (x_2 \neq x_1 \wedge P(d))$

A possible interpolant: $\neg P(d) \wedge \exists x P(x)$

Example 2 (Craig translation):

$\Sigma = \{P(c), \neg P(d)\}$

$\Pi = \{P(d) \vee g(u) = u, \neg P(g(x))\}$

$T(\Sigma) = \{\forall x\ x = x\} \cup \{\forall x \forall y\ x = y \supset P(x) \supset P(y)\} \cup \Sigma$

$$\begin{aligned} T(\Pi) = &\{\forall x\ x = x\} \cup \\ &\{\forall x \forall y\ x = y \supset P(x) \supset P(y), \forall x_1 \forall x_2 \forall y_1 \forall y_2\ x_1 = y_1 \supset x_2 = y_2 \supset x_1 = x_2 \supset y_1 = y_2, \forall x_1 \forall x_2 \forall y_1 \forall y_2\ x_1 = y_1 \supset x_2 = y_2 \supset G(x_1, x_2) \supset G(y_1, y_2)\} \cup \\ &\{P(d) \vee (\exists z G(u, z) \wedge (\forall y G(u, y) \supset z = y) \wedge z = u), \neg P(g(x))\} \end{aligned}$$

Example 3 Bonacina/Johannson:

$\Sigma = \{A \lor B, \neg C\}$

$\Pi = \{\neg A \lor C, \neg B\}$

$$\dfrac{\dfrac{\dfrac{A \lor B_\Sigma \qquad \neg A \lor C_\Pi}{B \lor C} \qquad \neg C_\Sigma}{B} \qquad \neg B_\Pi}{\Box}$$

Bon/Joh:

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{(A \lor \bot) \land \top \equiv A} \qquad \bot}{A \land (\neg C \lor \bot) \equiv A \land \neg C} \qquad \top}{(B \lor (A \land \neg C)) \land \top}$$

Huang:

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{(\neg A \land \bot) \lor (A \land \top) \equiv A} \qquad \bot}{(\neg C \land A) \lor (C \land \bot) \equiv \neg C \land A} \qquad \top}{(\neg B \land (\neg C \land A)) \lor (B \land \top)}$$

-> logically equivalent

Example 3B Bonacina/Johannson:

$\Sigma = \{A \lor B, \neg C, \neg D\}$

$\Pi = \{\neg A \lor C, \neg B \lor D\}$

$$\dfrac{\dfrac{\dfrac{A \lor B_\Sigma \qquad \neg A \lor C_\Pi}{B \lor C} \qquad \neg C_\Sigma}{B} \qquad \dfrac{\neg D_\Sigma \qquad \neg B \lor D_\Pi}{\neg B}}{\Box}$$

Bon/Joh:

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{(A \lor \bot) \land \top \equiv A} \qquad \bot}{A \land (\neg C \lor \bot) \equiv A \land \neg C} \qquad \dfrac{\bot \qquad \top}{\top \land (\neg D \lor \bot) \equiv \neg D}}{(B \lor (A \land \neg C)) \land \neg D}$$

Huang:

$$\dfrac{\dfrac{\dfrac{\bot \qquad \top}{(\neg A \land \bot) \lor (A \land \top) \equiv A} \qquad \bot}{(\neg C \land A) \lor (C \land \bot) \equiv \neg C \land A} \qquad \dfrac{\bot \qquad \top}{(\neg D \land \top) \lor (D \land \bot) \equiv \neg D}}{(\neg B \land \neg C \land A) \lor (B \land \neg D)}$$

-> not logically equivalent

Example 4: Paramodulation special case in Huang

$\Sigma = \{c = d, \neg C(x)\}$

$\Pi = \{P(f(c)) \lor Q(f(c)) \lor C(f(c), \neg P(f(d), \neg Q(f(d))\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(f(c)) \lor Q(f(c)) \lor C(f(c))_{\Pi} \qquad \neg C(x)_{\Sigma}}{P(f(c)) \lor Q(f(c)) \qquad\qquad c = d_{\Sigma}}}{P(f(d) \lor Q(f(d)) \qquad\qquad \neg P(f(d))_{\Pi}}}{Q(f(d)) \qquad\qquad \neg Q(f(d))_{\Pi}}}{\Box}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\top \qquad \bot}{\neg C(f(c)) \qquad \bot}}{(c = d \land \neg C(f(c))) \lor (c \neq d \land \bot) \lor (c = d \land f(c) \neq f(d) \equiv (c = d \land \neg C(f(c))) \lor (c = d \land f(c) \neq f(d)) \qquad \top}}{(c = d \land \neg C(f(c)) \lor (c = d \land f(c) \neq f(d)) \land \top \qquad \top}}{(c = d \land \neg C) \lor (c = d \land f(c) \neq f(d)) \land \top}$$

$(c = d \land \neg C(f(c))) \lor (c = d \land f(c) \neq f(d))$

max Π-terms: $f(c), f(d)$

$\theta^* = \forall x_1 \forall x_2 (c = d \land \neg C(x_1)) \lor (c = d \land x_1 \neq x_2)$

$\Rightarrow \Sigma \vdash \theta^*; \Pi \vdash \neg\theta^*$