CRITICAL BEHAVIOR IN STAIRCASE EROSION PROCESSES

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1. Introduction

The focus of this paper is a two-dimensional, discrete model of erosion. We consider an $N \times d$ rectangle of blocks that is bombarded by particles from the right. In this simple model, the rectangle of blocks decays over time and gradually takes on the form of a staircase. We can visualize this process as a rock in a desert, where a rock is gradually eroded by sand from one side. We analyze the rate of erosion of this system using numerical simulation.

In this random process, the $N \times d$ rectangle of blocks is eroded at every time step by a single incoming particle from the right side. An example of this process over a series of time steps can be seen in Figure 1. Particles are always fired at a height drawn uniformly over the range $\{1, ..., N\}$. At each time step, the system always has some rightmost "collision" boundary of blocks, as each block in this boundary has an equal probability of colliding with the incoming particle. If a block is hit, then it and all the blocks directly above it are removed from the system. The blocks to the left of each of these removed blocks form the collision boundary for the next time step. This produces a system with the staircase property that for each block at position (i, j) where $i, j > 1$, there must also be a block at position $(i-1, j)$ and position $(i, j-1)$. It is possible for the system's boundary to remain unchanged if an incoming particle is fired at a height where all the blocks have previously been removed.

Our primary research objective is to characterize the expected rate of decay, defined as the expected number of blocks removed from the system at each time step. We first introduce Monte Carlo methods, which allow us to efficiently explore the large number of possible realizations of the system. From there, we analyze the empirical results of the expected number of blocks remaining in the system in the case when $N = 10$ and $d = 60$. From these simulations, we discover that the rate of erosion has a critical point where the expected number of blocks remaining exhibits two different types of behaviors on either side of the critical point. Before the critical point, the number of blocks remaining decreases asymptotically to some nonzero constant. After the critical point, the asymptote suddenly changes the rate at which it decays and begins to approach zero. We motivate how the system can exhibit two modes of decay and why the transition between these two behaviors is rapid. Finally, we conclude with a few remarks about the future directions of this research.

2. Methods: Monte Carlo Simulation

We initially sought to model all of the possible realizations of the erosion process for some starting rectangle. A realization of the erosion process is defined as a sequence of states that the system takes when particles are fired over a series of

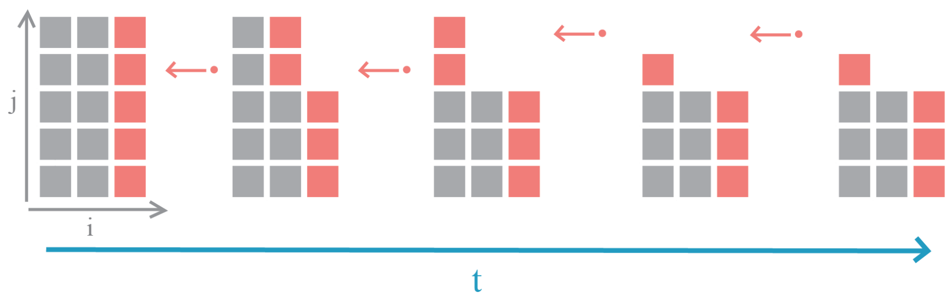


FIGURE 1. In this illustration of an $N = 5$ and $d = 3$ system, we show the erosion of a rectangle over five time steps. The red blocks correspond to blocks on the collision boundary. In each time step, a particle collides with a block on rightmost boundary and shifts the boundary in the next time step.

time steps. The number of realizations of a system grows exponentially in time, so we estimate desired distributions by random sampling to build an empirical distribution.

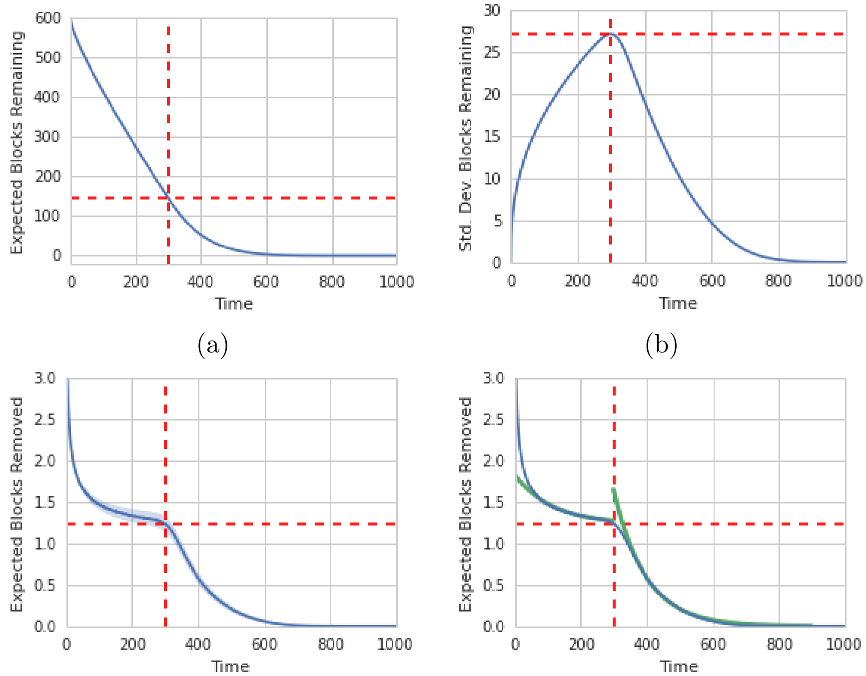
Monte Carlo methods are algorithms that rely on repeated random sampling to obtain a numerical result when considering every possible outcome is infeasible or inefficient. We randomly generated K possible realizations that the erosion process could take by firing particles at random heights at the rectangle for T time steps. For each realization, we calculated the number of blocks remaining in the system at each time step. By averaging over all of the counts at each time step, we obtain an empirical distribution over the number of blocks in the system over time. Furthermore, we are able to estimate the expected number of blocks remaining and the variance of the number of blocks remaining at each time step.

In our simulations, we use $K = 100,000$ random samples, a large number, in order to reduce the standard error in our procedure. In general, when estimating some random variable X with true mean μ and variance σ^2, a Monte Carlo estimate of the expected value of X with K random samples is distributed like $\mathcal{N}(\mu, \sigma^2/K)$. This means that the standard error of our estimate decreases proportional to $1/\sqrt{K}$.

3. Empirical Results

We use the Monte Carlo simulation described in Section 2 to understand the behavior of the rate of decay over time. Specifically, we analyze the expected number of blocks remaining in the system at every time step. In this section, we analyze the plots in Figure 2 when $N = 10$, $d = 60$, $K = 1 \times 10^5$, and $T = 1000$.

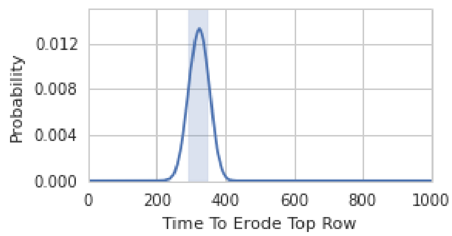
3.1. **Number of Blocks.** In Figure 2a, we plot the expected number of blocks remaining at every time step. Because the number of random samples is so large, the standard error is smaller than the thickness of the line in Figure 2a. This plot suggests that the type of decay changes at $t_{\text{crit}} = 299$, as indicated in all of the plots by the vertical dotted line. We defined t_{crit} by finding the maximum value of the standard deviation of the number of blocks remaining, shown in Figure 2b.



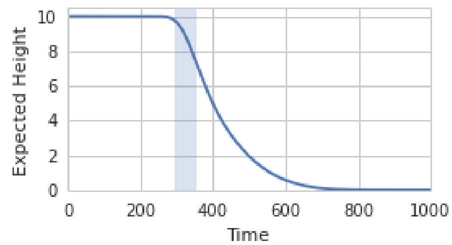
(a) (b) (c) (d)

FIGURE 2. Results of the Monte Carlo simulation with $N = 10$, $d = 60$, $K = 1 \times 10^5$, and $T = 1000$. In all plots, the vertical dotted line indicated $t_{crit} = 299$, and the horizontal dotted line indicates the value of the curve at t_{crit}. From left to right, top to bottom: (a) expected number of blocks remaining at each time step;(b) standard deviation of number of blocks remaining at each time step; (c) expected number of blocks removed at each time; (d) expected number of blocks removed at each time with exponential fits in green.

In Figure 2c, we plot the expected number of blocks removed at each time step. We define this as the rate of decay, and it also corresponds to the the derivative of number of blocks remaining shown in Figure 2a. The plot produced in Figure 2c clearly highlights that the the system has two distinct erosion behaviors on average, and that it transitions quickly from one behavior to the other at some critical point. We hoped to achieve a better characterization of the expected number of blocks in the system over time in Figure 2a by fitting the curves in Figure 2c and taking an integral. In Figure 2d, we made fitted exponential curves before and after the critical point for the line plotted in Figure 2c. We did so using standard methods. Because the rate of decay can be well-fitted by an exponential with a constant offset, we believe that the number of blocks remaining evolves as $Bt + C_1 e^{-\beta_1 t}$ before the critical point. After the critical point, we found a strong fit as an exponential $C_2 e^{-\beta_2 t}$, notably without any offset. After finding an exponential fit for both parts of the curve, shown in Figure 2d, we find that the time constants



(a)



(b)

FIGURE 3. Results of the Monte Carlo simulation with $N = 10$, $d = 60$, $K = 1 \times 10^5$, and $T = 1000$. From left to right: (a) the distribution over the first point in time when the top row is completely eroded with fitted mean $\hat{\mu} = 323.4$ and standard deviation $\hat{\sigma} = 29.6$; (b) the expected height of the system over time. In both plots, the light blue area indicates the interval $[\hat{\mu} - \hat{\sigma}, \hat{\mu} + \hat{\sigma}]$, which is $[293.8, 353.1]$.

are $\beta_1 = 9.00 \times 10^{-3} \pm 4.45 \times 10^{-5}$ before t_{crit} and $\beta_2 = 1.04 \times 10^{-2} \pm 2.76 \times 10^{-5}$ after t_{crit}.

3.2. **Height.** Around the critical time, on average the system undergoes a rapid transition in the rate of decay. To explain this, we pursue events of the system that are most likely to occur during a short window of time for most states. We find that one event like this is when the top row of the system erodes. This happens when the last block is removed from the top row.

The distribution over the time when the top row erodes is shown in Figure 3a. The distribution is narrow, especially considering the range of possible times when the last block in the top row can be removed. Using the pigeonhole principle, we find that the fastest and slowest times the last block in the top row can be removed are 60 and 591 respectively, but most of the probability mass sits between $[293.8, 353.1]$, about one tenth of the range.

The time interval where this transition has high probability includes t_{crit}. This leads us to believe that the erosion of the top row is one of the driving events behind for the critical transition in the expected rate of erosion.

Figure 3b shows how the expected height of the system where $N = 10$ and $d = 60$ decays over time. It shows the rate at which the system transitions to lesser heights from full height. The expected height stays at full height, meaning $H = N$, when there is low probability of eroding the top row. However, once the top row is gone, the next highest rows are quick to follow. This leads to some exponential decay of the height around this critical transition. We discuss how the height H is closely related to the rate of erosion in Section 4.1.

4. Discussion

In this section we develop intuition about the expected value of the rate of decay and number of blocks in the system. This intuition requires us to understand the expected number of blocks removed when the system is in a particular configuration, and to understand which configurations are most likely over time.

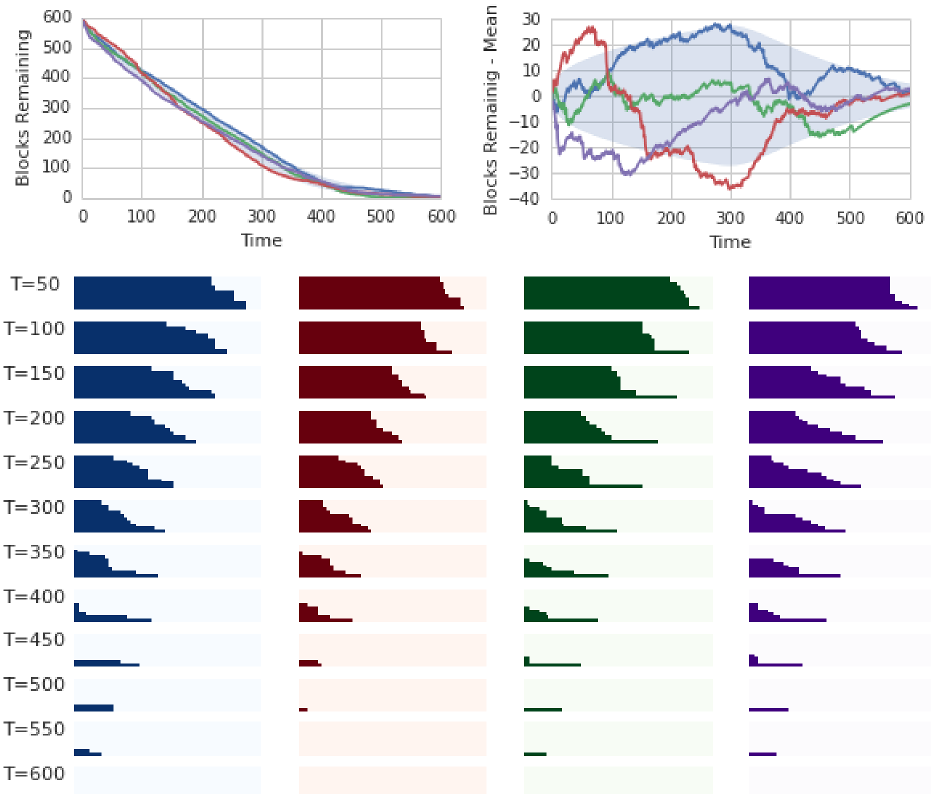


FIGURE 4. Four realizations of the erosion process with $N = 10$ and $d = 60$. In the top left we plot the number of blocks over time for each realization, or path. In the top left, we normalize by subtracting the mean from Figure 2a better understand how these paths vary around the mean. One standard deviation above and below is illustrated by the blue area. Below, we illustrate the state of each path at regular intervals.

In Figure 4 we illustrate four realizations of the erosion process with $N = 10$ and $d = 60$ as before. These examples serve both to clarify how much the state changes over time, and to understand what variation in the system looks like at a given time. Our results in Section 3 suggest that there is some critical transition point, but this shift in behavior only exists as a weighted average over all states. The criticality phenomenon is surprising because for any given realization of the process, it is impossible to identify a critical point, but taken as a whole, the rates of erosion of all realizations change over a short period.

The other insight we take from Figure 4 is that as time goes on, the system begins to look more "diagonal," meaning there are fewer rows with the same width. This change in shape reduces the effect of removing multiple blocks by hitting the bottom of a stack. This property, like many of the statements to follow in this

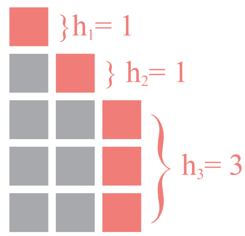


FIGURE 5. Illustration of vertical faces indexed by i

section, do not hold for every realization, but on average across the likely states at a given time.

The outline of our analysis is as follows: first, there is a simple mathematical relationship between the rate of erosion for a particular state given its height and how "diagonal" it is—to be formalized in the next section. Next, we argue that systems become more diagonal on average and how diagonality causes the height of the system to decay rapidly once the first layer of blocks has been eroded away. The last piece of intuition to develop is why the distribution over times when the first layer erodes is so narrow, but we do not discuss this in our paper.

4.1. **Expected Rate of Erosion.** The expected number of blocks at each time step can be written concisely as a function of the height of the vertical faces of the system, which we define as vertical segments of the collision boundary at the right. For example, in Figure 4.1, there are 3 vertical faces with heights 1, 1 and 3. We index these faces from left to right. Let h_i be height of the i^{th} vertical face, and H is total height of the system. Note that $\sum_{i=1}^{M} h_i = H$, where M is the total number of vertical faces.

If we are given $\mathcal{H} = \{h_1, \ldots, h_M\}$, we know the probability of hitting each vertical face is h_i/N. Given the incoming particle hits the i^{th} vertical face, we know the probability that $j \in \{1, \ldots, h_i\}$ blocks are removed is $1/h_i$. Thus we can calculate the expected number of blocks removed from a system, ΔB, in a system with initial height N in Equation 4.1, where C is a random variable indicating which vertical face the particle collides with.

$$
\begin{aligned}
\mathbb{E}[\ \Delta B \mid \mathcal{H}\ ] &= \sum_{i=1}^{M} \mathbb{E}[\ \Delta B \mid C = i, \mathcal{H}\ ]\ \mathbb{P}[\ C = i \mid \mathcal{H}\ ] \\
&= \sum_{i=1}^{M} \sum_{j=1}^{h_i} \frac{j}{h_i} \frac{h_i}{N} \\
&= \sum_{i=1}^{M} \frac{h_i(1+h_i)}{2N} \\
&= \frac{H}{2N} + \sum_{i=1}^{M} \frac{h_i^2}{2N} \qquad (4.1)
\end{aligned}
$$

To do a full analysis, we need to know the distribution over $\mathcal{H}$ at each time. By developing intuition about this distribution over time we can explain our results in Figure 2c.

4.2. **Diagonality and Full Heightedness.** Now let us use Equation 4.1 and the difference in shapes of the system to explain the differences in decay behavior. We define diagonality as the property that the staircase does not have tall vertical faces. Equivalently, not many rows have the same width, and all $h_i \in \mathcal{H}$ are small. Initially the system is not diagonal, since $\mathcal{H} = \{N\}$ in the initial state. However, over time the system becomes more diagonal on average. An important consequence of a system being highly diagonal is that at the time when the first layer is eroded off, the width of the next topmost layers is small, presumably much less than the initial width d. We see this by example in Figure 4, and it is possible to make this statement more formal, but we do not do it here.

Since the height cannot begin to decrease until the first layer is removed, for a long period of time the system is at full height, meaning $H = N$. During this time, the system is becoming more diagonal on average. We can see this in Figure 4 when $t \leq 250$. This causes the second term in Equation 4.1 to decrease on average since states where $h_i = 1$ become more likely over time. These are completely diagonal states. In these completely diagonal states, the expected number of blocks removed simplifies to H/N, so that the expected rate of erosion removed is proportional to the height.

Taken all together, there is some period of time where the system is full height, and the expected rate of erosion decays because more diagonal states become more likely, but always at least one block is removed at each time. At some later time, when it is more likely that the height has decreased from N, it is also more likely to be diagonal, so the expected rate of erosion is approximately H/N. When the height begins decreasing, it decreases quickly at first because of the diagonal shape, and slows down. This explains the shapes of the plots in Figures 2c and 3b.

If we also know that the time when the first layer is eroded off falls in a short window of time, then we can also explain why this system exhibits criticality: its because most realizations of the system stop being full height around the same time. We see this empirically in Figure 3a, but we do not have an analytical explanation of this.

5. Future work

While we were able to explore the average behavior of this system, this paper fell short of a full analytical exploration of the problem. We believe that all the features of the rate of erosion over time can be explained by careful analysis of the geometry of the Markov graph over states. Preliminary investigation into this is promising, and suggests that this complex state transition graph can be approximated by a much simpler one with less node degree. Higher connectivity graphs tend to be more dispersive, and if the system behaved like a more sparse graph on fewer states then it might explain why so many states erode their top row around the same time. This explanation, along with the analysis we presented in Section 4 would completely characterize how the rate of erosion changes over time.

References

1. M. Bona, *A Walk through combinatorics* World Scientific (2011), pp 98–160.

Appendix

For this problem, we also developed an analytical approach from a different perspective. Instead of considering how the particles erode the system, we instead think about how the many different staircases we can construct given n blocks. This directly corresponds to the integer partition number $p(n)$, where $p(n)$ is the number of ways to partition n into non-empty sets. For example, $n = 4$ can be partitioned into $\{4\}, \{3,1\}, \{2,2\}, \{2,1,1\}, \{1,1,1,1\}$, so $p(n) = 5$. Thus, there are 5 possible staircases attainable given 4 blocks.

Following this train of thought, the staircases bear very close resemblance to Ferrers shapes$_{[1,99]}$. Ferrers shapes have interesting properties regarding symmetry that could provide insight to our problem. For example, if we reflect a Ferrers shape diagonally (switch rows and columns) we obtain another Ferrers shape, which we call its conjugate. A Ferrers shape whose conjugate is itself is called self-conjugate. The number of self-conjugate partitions is the same as the number of partitions with distinct odd parts$_{[1,100]}$.

Furthermore, as n increases, $p(n)$ increases at a rate between exponential and polynomial time$_{[1,98]}$. To be more precise, the rate is approximated by

$$\frac{1}{4\sqrt{3}} exp(\pi\sqrt{\frac{2n}{3}}) \tag{5.1}$$

This means the number of staircases we can obtain given n blocks increases at a rate between exponential and polynomial time.

In addition to this, the following generating function$_{[1,159]}$ can be used to describe the behavior of $p(n)$:

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} \tag{5.2}$$

In the future, we could tie this in to our numerical simulation for a more thorough explanation of the causes of certain types of behavior found in the numerical simulation.