DYNAMIC POLYGONS

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1. Introduction

In this paper, we describe a transformation on polygons and analyze the properties of these polygons after this operation. Suppose that we have some polygon $\mathcal{P}_n$ with n vertices $p_1, p_2, ..., p_n$ in the two dimensional coordinate plane. Each point p_i is located at (x_i, y_i). Let us now describe a transformation on this polygon $\mathcal{P}_n$, which we define as the "flipping operation." We say that we flip some point p_i in our polygon $\mathcal{P}_n$ by reflecting p_i across the perpendicular bisector of the points p_{i-1} and p_{i+1}. We call the line over which p_i is flipped as the "flipping line." All of the other vertices in the transformed polygon remain unchanged. This operation can be seen in Figure 1 below.

In the following sections, we analyze how this flipping operation changes the structure of a polygon. Specifically, we want to understand how the structure of the polygon affects the set of points that the vertices can reach after any sequence of flips. We define this as the closure of polygons, and we analyze the shape of this set in the cases of triangles and quadrilaterals. We prove that a triangle's closure is contained in its circumscribed circle. For quadrilaterals, we prove the condition under which the closure is unbounded. This happens when the flipping lines of the quadrilateral are parallel but not equal. Otherwise, the closure lies entirely on two concentric circles centered at the intersection of the flipping lines.

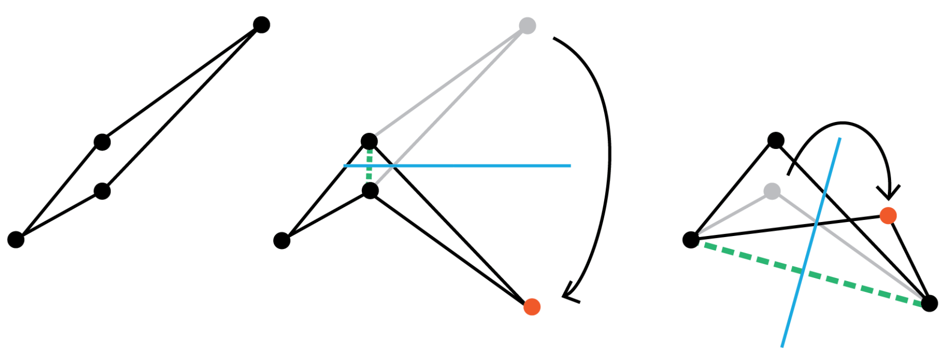

Figure 1. In this illustration of a polygon with four points, we show the transformation of the polygon after the flipping operation is applied twice. An example of a simple convex polygon can be seen in the first image. The polygon transforms into a concave polygon in the second image and finally becomes a complex polygon in the third image.

2. Problem Statement

A polygon $\mathcal{P}_n$ has n points, where the i^{th} point is denoted as p_i. The point p_i is located at position (x_i, y_i). There is an edge between the point p_i and point p_{i+1}, and there is also an edge between p_1 and p_n. Each side length can be indexed by i. A side length is denoted by a_i, and it is defined as the length of side that connects the point p_i to the point p_{i+1}. We define the set of all polygons with n vertices as $\mathbb{P}_n$. This includes both simple and complex polygons. A simple polygon is defined as a shape consisting of straight, non-intersecting line segments that are joined at specific points to form a closed path. However, if the sides intersect, then the polygon is complex. In Figure 1, we can see that under the flipping operation, a simple polygon can turn into a complex polygon.

We define the "flipping operation" on some polygon and some point as $f : \mathbb{P}_n \times [n] \to \mathbb{P}_n$, where $[n] = \{i \in \mathbb{N} \mid i \leq n\}$. This operation, $f(\mathcal{P}_n, i)$, takes some polygon $\mathcal{P}_n$ and flips the i^{th} vertex over the perpendicular bisector of p_{i-1} and p_{i+1}. We denote this line as l_i, and we refer to these l_i as the "flipping lines" of the polygon $\mathcal{P}_n$. This operation creates a new polygon denoted as $\mathcal{P}'_n$ where p_i becomes p'_i but all of the other points remain unchanged. In the edge case where p_{i+1} and p_{i-1} have the same coordinates, we define a flip $f(\mathcal{P}_n, i) = \mathcal{P}_n$, so the flipping operation does not change the polygon.

We also extend f to represent a sequence of transformations on some sequence of points, so $f(\mathcal{P}_n, (i, j)) = f(f(\mathcal{P}_n, i), j)$. For some sequence S of k indices $S \in [n]^k$, $f(\mathcal{P}_n, S)$ is the polygon computed after flipping each index in S sequentially.

2.1. Immediate Results.

In this section, we state some basic properties and invariants of polygons after the flipping operation is applied. They develop the intuition for what the flipping operation does to a polygon. Furthermore, they motivate the complexity of characterizing the polygon after a long sequence of flips.

Lemma 2.1. *Given some polygon $\mathcal{P}_n$, after flipping any vertex i to obtain $\mathcal{P}'_n = f(\mathcal{P}_n, i)$, all edges lengths a'_j in $\mathcal{P}'_n$ are equal to the corresponding a_j in $\mathcal{P}_n$ except for $j \in \{i-1, i\}$, in which case $a'_i = a_{i-1}$ and $a'_{i-1} = a_i$.*

Proof. By definition, all vertices $p_j = p'_j$ if $j \neq i$ by definition of f. Therefore, all edges between p_j, p_k where $j, k \neq i$ are unchanged, and the lengths of these edges, a_j, are also unchanged.

Now we consider the edges between (p'_{i-1}, p'_i) and (p'_i, p'_{i+1}). As discussed, $p'_{i-1} = p_{i-1}$ and $p'_{i+1} = p_{i+1}$. Also, without loss of generality, assume that p_{i-1} and p_{i+1} correspond to the two points on the x-axis $(0, 0)$ and $(d, 0)$, respectively. Let p_i have coordinates (x_i, y_i). The perpendicular bisector of p_{i-1} and p_{i+1} is the line $x = d/2$, so p'_i has $x'_i = d - x$ and $y'_i = y_i$.

When we compare the side lengths, we can see that before the operation $a_{i-1} = \sqrt{x_i^2 + y_i^2}$ and $a_i = \sqrt{(d - x_i)^2 + y_i^2}$, and after the operation $a'_{i-1} = \sqrt{(d - x_i)^2 + y_i^2}$ and $a'_i = \sqrt{x_i^2 + y_i^2}$. This completes the proof. $\square$

Proposition 2.2. *Let A be the set of side lengths of some polygon $\mathcal{P}_n$. The flipping operation conserves A.*

Proposition 2.3. *The flipping operation f conserves perimeter.*

Proposition 2.4. *If $\mathcal{P}_n$ is convex, it is not true that for all $i \in [n]$, $f(\mathcal{P}_n, i)$ is convex.*

Proof. Proof by counter-example. Consider the quadrilateral with vertices at the points $(0,0)$, $(1,0)$, $(3,2)$, $(1,1)$. This forms a convex quadrilateral. After flipping $(3,2)$ to $(3,-1.5)$, the shape of the quadrilateral is no longer convex. □

Proposition 2.5. *If $\mathcal{P}_n$ is a simple polygon, it is not true that for all $i \in [n]$, $f(\mathcal{P}_n, i)$ is a simple polygon.*

Proof. Proof by counter-example. Consider the quadrilateral with vertices at the points $(0,0)$, $(0.5,1)$, $(1.5,-0.75)$, $(0.5,2)$. This forms a simple polygon. After flipping $(0.5,1)$ to $(2,0.25)$, the quadrilateral is no longer simple. □

3. Closure

The main focus of our research is to understand the positions that the vertices of a polygon can occupy after some sequence of flips. We define this as the "closure" of the polygon under the flipping operation, or stated more simply, the closure of the polygon.

Our goal is to understand how the shape of the closure of some polygon $\mathcal{C}(\mathcal{P}_n)$ relates to the geometry of that polygon $\mathcal{P}_n$. We investigate whether the closure is bounded, or alternatively, whether a polygon can move arbitrarily far from its starting point given that enough flips have occurred.

Definition 3.1. We define the closure of a polygon $\mathcal{P}_n$ as the set of all points (x,y) such that there exists a sequence S of flipping operations on $\mathcal{P}_n$, so that the polygon $f(\mathcal{P}_n, S)$ has a vertex with coordinates (x,y). The closure is denoted as $\mathcal{C}(\mathcal{P}_n)$. Formally,

$$\mathcal{C}(\mathcal{P}_n) = \{(x,y) \in \mathbb{R}^2 \mid \exists k \in \mathbb{N}, S \in [n]^k \text{ s.t. } f(\mathcal{P}_n, S) \text{ has a vertex at } (x,y)\}.$$

Definition 3.2. The closure of some polygon $\mathcal{C}(\mathcal{P}_n)$ is said to be bounded if there exists some ball $\mathcal{B} \subset \mathbb{R}^2$ such that $\mathcal{C}(\mathcal{P}_n) \subseteq \mathcal{B}$.

3.1. Example: Closure of Triangles.

In this section, we apply the properties of the flipping operation to triangles in order to characterize the closure of the triangle. First, we prove the property that all circumscribed polygons stay circumscribed under the flipping operation. We then use this fact to prove that all triangles, which are always circumscribed in some circle, lie on this same circle after a series of flips. This is the most simple class of polygons to describe, and the analysis below provides an example of the types of questions that we are interested in analyzing. In Section 3.2 we extend our analysis to include quadrilaterals.

Lemma 3.3. *If $\mathcal{P}_n$ is circumscribed by a circle C, then for all i, $f(\mathcal{P}_n, i)$ is also circumscribed by the same circle C.*

Proof. Consider some $\mathcal{P}_n$ circumscribed by a circle C, which implies that all vertices lie on C. Specifically, for all i, this means p_{i-1} and p_{i+1} are a chord on C, so their perpendicular bisector l_i goes through the center of C. Since C is symmetric by reflection across any diameter, the flipped vertex p_i' will lie on C. If all points of a complex polygon lie on some circle C, then they are circumscribed by it, so $\mathcal{P}_n'$ is circumscribed by the circle C. □

Theorem 3.4. *For all triangles $\mathcal{P}_3$, $\mathcal{C}(\mathcal{P}_3)$ is contained in some circle centered at the circumcenter of $\mathcal{P}_3$.*

Proof. All triangles $\mathcal{P}_3$ are circumscribed by some circle C, so by Lemma 3.3, after any sequence of flips S, $\mathcal{P}_3' = f(\mathcal{P}_3, S)$ will lie on the same circle C that circumscribes $\mathcal{P}_3$. We conclude $\mathcal{C}(\mathcal{P}_3) \subseteq C$. By definition, the center of C is the circumcenter of $\mathcal{P}_3$. □

3.2. **Closure of Quadrilaterals.** All of the properties of the closure of quadrilaterals under the flipping operation follow from the fact that there are two unique flipping lines, as proved in Lemma 3.5. Our main result is that the closure of a quadrilateral is unbounded if and only if the flipping lines of that quadrilateral are parallel but not equal. More specifically, if the flipping lines of a quadrilateral are parallel, then the closure of that quadrilateral lies on two parallel lines that are perpendicular to the flipping lines. If the flipping lines of a quadrilateral are not parallel, then the closure of that quadrilateral lies on two concentric circles centered at the intersection of the flipping lines.

3.2.1. *Flipping Lines of Quadrilaterals.* In the problem statement, there are n flipping lines for some complex polygon with n vertices. By this definition, a quadrilateral should have four flipping lines. However, for any quadrilateral, we will show that there are only two unique flipping lines.

Because of this, the quadrilateral case is much more restricted than generalizations to polygons with more vertices. We are able to exactly characterize the closure of a polygon by considering its flipping lines.

Lemma 3.5. *For all quadrilaterals $\mathcal{P}_4$, $\mathcal{P}_4$ has at most two unique flipping lines. Further, $l_1 = l_3$ and $l_2 = l_4$.*

Proof. The perpendicular bisector of the points (p_1, p_3) is the same as the perpendicular bisector of the points (p_3, p_1), because there can only be one perpendicular bisector for a pair of points. This also holds for the perpendicular bisector of the points (p_2, p_4) and (p_4, p_2). This means that there can only be two unique flipping lines for a quadrilateral because l_1 and l_3 are the same line, and the same holds for l_2 and l_4.

□

Lemma 3.6. *Let l_i be the flipping line corresponding to the i^{th} vertex of some polygon $\mathcal{P}_n$, and let the intersection of l_{i-1} and l_{i+1} with l_i be q_{i-1} and q_{i+1} respectively. After flipping vertex i to obtain $\mathcal{P}_n' = f(\mathcal{P}_n, i)$, the intersections of the new flipping lines are unchanged, meaning $q_{i-1} = q_{i-1}'$ and $q_{i+1} = q_{i+1}'$.*

Proof. Without loss of generality, let l_i be the x-axis, and p_i and p_{i+2} have coordinates (x_i, y_i) and (x_{i+2}, y_{i+2}) respectively. Note that l_{i+1} is the perpendicular bisector of these two points. The formula for l_{i+1} is $y - k = -(x-h)/m$, where the midpoint of p_i and p_{i+2} has coordinates (h, k) and m is the slope between p_i and p_{i+2}. We conclude that the intersection of this line and l_i, the x-axis, is $mk + h$. After flipping p_i to p_i', the new coordinates are $(x_i, -y_i)$. Plugging these in, we find:

$$
\begin{aligned}
m'k' + h' &= \left(\frac{y_{i+2} - y_i'}{x_{i+2} - x_i'}\right) \frac{y_{i+2} + y_i'}{2} + \frac{x_{i+2} + x_i'}{2} && \text{(definition of } m', h', k'\text{)} \\
&= \left(\frac{y_{i+2} + y_i}{x_{i+2} - x_i}\right) \frac{y_{i+2} - y_i}{2} + \frac{x_{i+2} + x_i}{2} && \text{(replace } y' \to -y, x' \to x\text{)} \\
&= \left(\frac{y_{i+2} - y_i}{x_{i+2} - x_i}\right) \frac{y_{i+2} + y_i}{2} + \frac{x_{i+2} + x_i}{2} && \text{(associative property)} \\
&= mk + h,
\end{aligned}
$$

so the intersection of the new flipping line l'_{i+1} with l_i is unchanged. The same analysis holds for l'_{i-1}, so the intersections of both lines with l_i are unchanged after a flip of vertex i. $\square$

Lemma 3.7. *If the flipping lines of the quadrilateral $\mathcal{P}_4$ are parallel, then for all i, the flipped quadrilateral $\mathcal{P}'_4 = f(\mathcal{P}_4, i)$ will also have parallel flipping lines.*

Proof. By Lemma 3.5, $l_1 = l_3$ and $l_2 = l_4$. Without loss of generality, assume l_1 and l_2 are vertical. Note that a flipping line l_i is vertical if and only if $y_{i-1} = y_{i+1}$. This means, that $y_1 = y_3$ and $y_2 = y_4$. Again, without loss of generality consider the flip $i = 2$, so $\mathcal{P}'_4 = f(\mathcal{P}_4, 2)$. All vertices besides p_2 are unchanged, and since l_2 is vertical, p'_2 is a horizontal reflection of p_2 across l_2. This means $y'_2 = y_2$. Further, $y'_2 = y_2 = y_4 = y'_4$, so the flipping line between points p'_2 and p'_4 is still vertical. The flipping line between points p'_1 and p'_3 is unchanged. We conclude that l'_1 and l'_2 are still parallel, and that this holds for all $\mathcal{P}_4$ and $f(\mathcal{P}_4, i)$. $\square$

3.2.2. *Characterizing Quadrilateral Closure.* Now we draw the connection from flipping lines to the closure of a quadrilateral under the flipping operation. We characterize the shape of the closure under three conditions on the flipping lines of the quadrilateral. From this analysis, we conclude with our main result that the closure of a quadrilateral is unbounded if and only if its flipping lines parallel but not equal.

Theorem 3.8. *If the flipping lines of quadrilateral $\mathcal{P}_4$ are parallel, then the closure of the quadrilateral $\mathcal{C}(\mathcal{P}_4)$ lies on two parallel lines.*

Proof. By Lemma 3.5, $l_1 = l_3$ and $l_2 = l_4$. Without loss of generality, assume l_1 and l_2 are vertical. Note that a flipping line l_i is vertical if and only if $y_{i-1} = y_{i+1}$. This means, that $y_1 = y_3$ and $y_2 = y_4$. Also, since the flipping lines are vertical, a flip only changes the x-coordinate of one of the vertices. By Lemma 3.7, the flipping lines will continue to stay parallel after any number of flips, and since at least one is vertical and unchanged after a flip, the flipping lines stay vertical after any number of flips. We conclude that after any sequence of flips, all of the points will fall on the lines $y = y_1$ or $y = y_2$. These are parallel, and this result holds in general. $\square$

Theorem 3.9. *If the flipping lines of a quadrilateral $\mathcal{P}_4$ are not parallel, then the closure $\mathcal{C}(\mathcal{P}_4)$ lies entirely on at most two concentric circles centered at the intersection of the flipping lines.*

Proof. By Lemma 3.5, there are at most two unique flipping lines l_1 and l_2. Since l_1 and l_2 are not parallel, then they intersect at exactly one point q. Since two points p_{i-1}, p_{i+1} are equidistant to every point on their perpendicular bisector, we know that p_1 and p_3 are equidistant to q with distance r_1, and p_2 and p_4 are equidistant to q with distance r_2. Thus, every quadrilateral has vertices that lie on at most two concentric circles with a shared center at q, the intersection of l_1 and l_2.

In the case when $r_1 = r_2$, then there is only one circle centered at q that contains all vertices, and this is the case of a circumscribed polygon as characterized in Lemma 3.3. In the rest of the proof, we consider $r_1 \neq r_2$.

Next, we show that after a flipping operation, that all of the vertices lie on the same two concentric circles, and that the new flipping lines intersect at the same point. Consider without loss of generality a flip of p_2, which lies on a circle with radius r_2 with center q. By definition of q, the flipping line l_2 goes through the center, so p_2' also lies on a circle of radius r_2 through q because circles have reflection symmetry across all diameters. Further, by Lemma 3.6, the intersection of the new flipping lines is still q.

This means all of the vertices of $f(\mathcal{P}_4, 2)$ lie on the same two concentric circles, and by induction, after any sequence of flips, all of the vertices lie on these two circles. □

Notice the parallel to the triangle case. In Theorem 3.4, we show that the closure of a triangle is a circle centered at the circumcenter of the triangle, which is the unique point where the flipping lines intersect. For quadrilaterals, we have shown that the center of the two concentric circles is also the unique intersection of the flipping lines. We believe larger polygons are harder to analyze because the flipping lines intersect in many points, so the center of the closure may not be a straightforward geometric property of the polygon.

Also, we can understand the two parallel lines described in Theorem 3.8 as being a continuous transformation of the concentric circles as the radii grow and the center moves infinitely farther away.

Theorem 3.10. *If all of the flipping lines of quadrilateral $\mathcal{P}_4$ are equal, then the closure of the quadrilateral $\mathcal{C}(\mathcal{P}_4)$ exactly equals the initial vertex locations.*

Proof. By Lemma 3.5, there are at most two unique flipping lines l_1 and l_2. Since the flipping lines are equal, $l_1 = l_2$. Without loss of generality say l_1 is the x-axis, and the vertices of $\mathcal{P}_4$ are $(x_1, y_1), (x_2, y_2), (x_1, -y_1), (x_2, -y_2)$. Either (x_i, y_i) is flipped to $(x_i, -y_i)$ or back. Note that when p_{i+1} has the same coordinates as p_{i-1}, then the flipping operation does not change the polygon, i.e. $f(\mathcal{P}_n, i) = \mathcal{P}_n$. Therefore, after any sequence of flips, the inital locations of the vertices are the only points in $\mathcal{C}(\mathcal{P}_4)$. □

The case described by Theorem 3.10 occurs when the flipping lines are parallel and intersect. The initial vertex locations both lie on two concentric circles and two parallel lines, so we can think of this case as the result of both Theorems 3.8 and 3.9 holding true.

Theorem 3.11. *The closure of a quadrilateral $\mathcal{C}(\mathcal{P}_4)$ is unbounded if and only if the flipping lines of $\mathcal{P}_4$ are parallel, but not equal.*

Proof. If the flipping lines are not parallel or equal, then either Theorem 3.9 or Theorem 3.10 holds. In both of these cases, where the closure of $\mathcal{P}_4$ lies on two concentric circles, it is clear that the closure is bounded.

To complete the proof, we show that if the flipping lines are parallel and not equal, then the closure is unbounded. By Lemma 3.5, there are at most two unique flipping lines l_1 and l_2. Without loss of generality say l_1 and l_2 are vertical. This means, the vertex with the lowest x-coordinate can always be flipped and moved to the right. By Lemma 3.7, this new $\mathcal{P}_4$ will also have parallel l_1' and l_2', which are still vertical, so the new leftmost vertex will be able to be moved right. This means for all x^*, there exists some n such that after n flips it is possible to flip a vertex to have x-coordinate greater than x^*, so the closure of $\mathcal{P}_4$ is unbounded in this case. □

4. Directions for Future Work

In this paper, we have analyzed how the closure of polygons, specifically in the cases of triangles and quadrilaterals, is directly related to its geometric properties. However, the general question of how to characterize the closure of higher order polygons is still open. We know that the closure of a polygon is invariant under flipping—this is trivial given the reversibility of flipping and the definition of closure—so in some sense a particular polygon encodes all of the information there is about its closure. However, the approach that we have taken with these simple cases does not seem to extend to more general cases. Let us consider some evidence for why our geometric approach might not be useful in understanding the closure of a general polygon.

For both triangles and quadrilaterals, the flipping lines are either identical or intersect at one point. In both cases, the intersection of the flipping lines is invariant under the flipping operation. While Lemma 3.6 hints at how these intersections are preserved in general, it is not true that the intersection of the flipping lines is preserved for polygons with more than four vertices.

Another observation is that in the cases of triangles and quadrilaterals, the closure of these polygons are either lines or circles. Either way, the vertices of the polygon always lie on the boundary of the closure for triangles and quadrilaterals. As we see in Figure 4, even for pentagons, the closure lies on an annulus. Furthermore, there is no guarantee that any of the vertices of a polygon lie on the boundary of the annulus that contains the closure of the polygon. This seemingly makes identifying the enclosing annulus harder just by looking at the polygon.

Lastly, the curious example in Figure 4 shows a particular 8-gon whose vertices trace out a non-trival shape after a particular sequence of flips. The sequence cycling around the integers 1 through 8 produces Figure 4. However, the sequence of flips cycling in the same manner and starting at point 2 produces the set of vertices shown in Figure 5. This reveals that for more complex polygons, the order of flips begins to matter more. So far, we have not developed proof methods which address this.

While the analysis of closure in general may be difficult, we are optimistic that it will yield interesting results. Figure 6 shows a nontrivial example of a pentagon whose closure is unbounded. A trivial example would be a pentagon whose vertices are all collinear. We suspect that for all n, there exists some non-trivial $\mathcal{P}_n$ such that $\mathcal{C}(\mathcal{P}_n)$ is unbounded. However, we do not believe that there is any $\mathcal{P}_n$ for any n whose closure is dense in the whole plane.

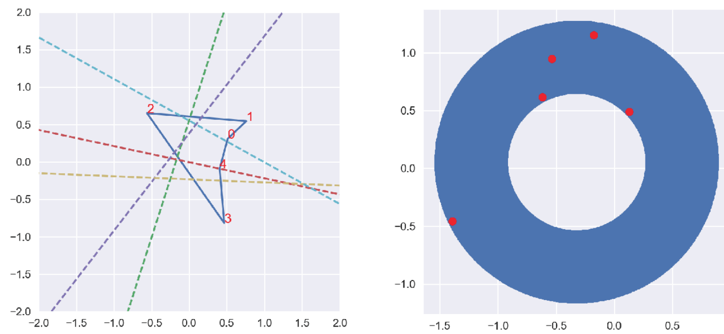

FIGURE 2. On the left we have a pentagon and its 5 flipping lines. We that the intersection of these lines is empty. On the right we have locations of the vertices of the polygon after flipping each vertex in a cycle for 1000 flips total.

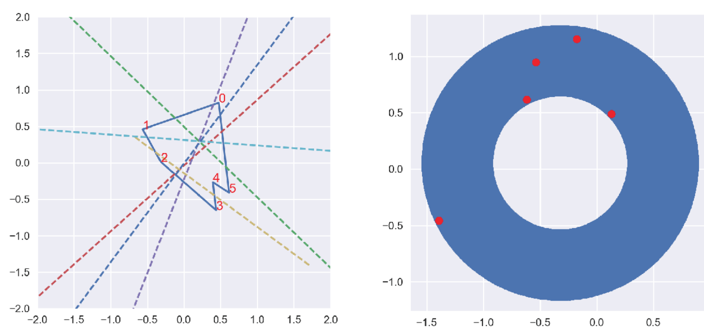

FIGURE 3. As in figure 2, here we have a hexagon and its closure, which also looks similar to a washer.

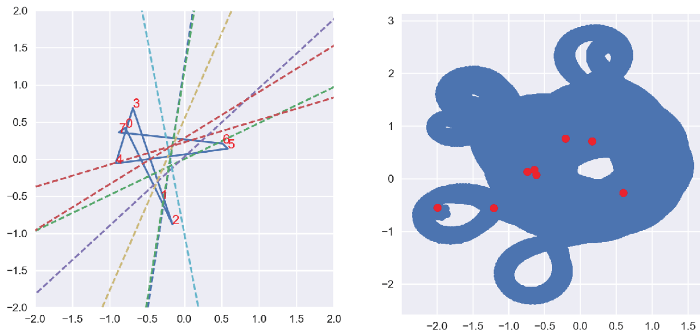

FIGURE 4. On the left we have a octagon and its 8 flipping lines. On the right we have locations of the vertices of the polygon after flipping each vertex in a cycle for 50,000 flips total. This particular shape occurs when cycle through the vertices in order and start at 1. However, if we start at 2 we obtain a different shape as shown in Figure 5.

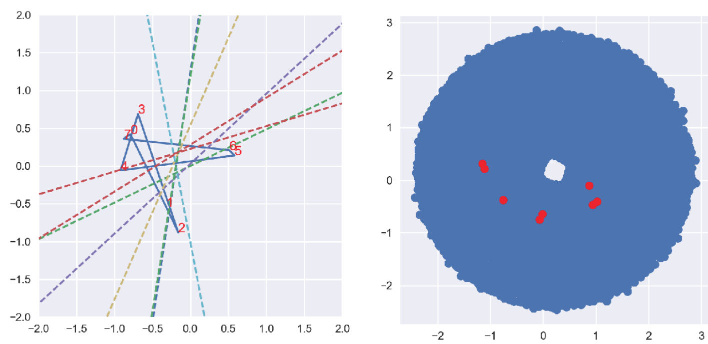

FIGURE 5. Here we have the same octagon as Figure 4. On the right we have locations of the vertices of the polygon after flipping each vertex in a cycle for 50,000 flips total. This particular shape occurs when cycle through the vertices in order and start at 2. However, if we start at 1 we obtain a different shape as shown in Figure 4.

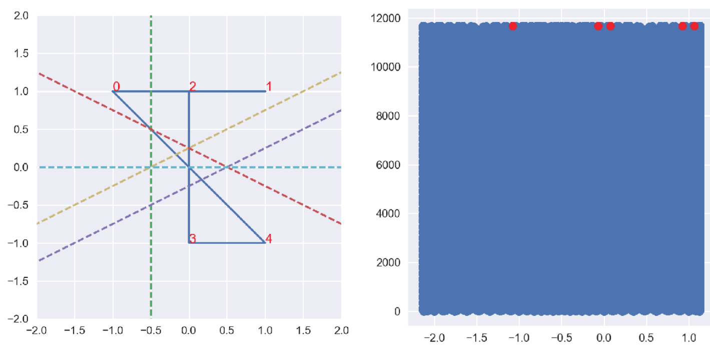

FIGURE 6. Here we have a pentagon that exhibits an unbounded closure. On the right we have locations of the vertices of the polygon after flipping each vertex in a cycle for 10,000 flips total. We can see the vertices are marching up, leading us to believe that its closure is an unbounded strip.