Distributions

Discrete

`Combinatorial`	Finite space, equally likely outcomes $P(A) = card(A)/n$
`Bernoulli`	Success probability of bent coin $p_0 = q = (1-p)$ and $p_1 = p$ $\mu = p, \sigma^2 = pq$
`Binomial`	Probability of n heads in k tosses $p_k = \binom{n}{k} p^k (1-p)^{n-k}$ $\mu = np, \sigma^2 = npq$
`Geometric`	Failures before 1st success, $k \geq 0$ $p_k = (1-p)^k p$ $\mu = \frac{q}{p}, \sigma^2 = \frac{q}{p^2}$
`Negative Binomial`	Failures before rth success $k \geq 0$ $w_r(k;p) = \binom{k+r-1}{k} q^k p^r$ Note the negative binomial is sum of r **independent** geometric RVs. $\mu = \frac{rq}{p}, \sigma^2 = \frac{rq}{(p)^2}$
`Poisson`	Characterization of rare events $p_k = e^{-\lambda}\lambda^k/k!$ $\mu = \lambda, \sigma^2 = \lambda$

Continuous

`Uniform`	Agnostic about outcome on interval $f(x) = 1/(b-a)$ on $a < x < b$ and 0 $o.w.$ $\mu = \frac{1}{2}(a+b), \sigma^2 = \frac{1}{12}(b-a)$
`Gamma`	Note $\Gamma = (n-1)!$ $g_n(t;\alpha) = \alpha\frac{(\alpha t)^{n-1}}{(n-1)!}e^{-\alpha t}$ $\mu = \frac{1}{\alpha}, \sigma^2 = \frac{1}{\alpha^2}$
`Chi-squared`	Finite X_k $\mathcal{N}(0,1)$;$\mathrm{V}^2 = \sum_{k=1}^n X_k^2$ $g_{n/2}(t;1/2) = \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}t^{\frac{n}{2}-1}e^{-\frac{t}{2}}$ $\mu = n, \sigma^2 = 2n$
`Exponential`	A model for true randomness $f(x) = \alpha e^{-\alpha x}$ for $x > 0$ $\mu = \frac{1}{\alpha}, \sigma^2 = \frac{1}{\alpha^2}$
`Normal`	The one to rule them all... $f(x) = \frac{1}{\sqrt{(2\pi)}\sigma}e^{-(x-\mu)^2/2\sigma^2}$

Even function, strictly positive, dies monotonically...

`Multivariate Normal` Note that x and m are vectors

$$f(x) = \frac{1}{2\pi^{\frac{n}{2}} det(A)^{\frac{1}{2}}} exp(-\frac{1}{2}(x-m)A^{-1}(x-m)^T)$$

Negative Binomial Trick: $\binom{r+k-1}{k}q^k p^r = \binom{-r}{k}(-q)^k p^r$

Properties of Distributions

Shifting and Scaling *If two densities come from same type:* $\frac{1}{a}f(\frac{x-m}{a})$ *then* $\mu = a\mu + m$ $\sigma^2 = a^2\sigma^2$
Convolution *The convolution of any pair of distributions must necessarily be a distribution*
Stability of the Normal *Suppose* $X_1, X_2, ...X_n$ *are independent and for each k* X_k *is normal with mean* μ_k *and variance* σ_k*. Then* $S_n = \sum_{k=1}^n X_k$ *is normal with mean* $\sum_{k=1}^n \mu_k$ *and variance* $\sum_{k=1}^n \sigma_k$*, or in other words normal densities are stable under convolution. Also stable under* ***non-degenerate*** *linear transformation*
Translation Invariance: Uniform
Sum of Independent Poisson RVs *For* ***independent*** *random variables drawn from Poisson distribution is Poisson and the mean is* $\lambda_1 + \lambda_2$
Sum of Independent Binomials is also binomial.
Sum of Independent Gamma RVs *Given* $X_1 \sim g_{n_1}(x_1, \alpha)$ *and* $X_2 \sim g_{n_2}(x_2, \alpha)$ *then* $Z = X_1 + X_2 \sim g_{n_1+n_2}(z, \alpha)$*. Note the RV's must have the same mean* $\frac{1}{\alpha}$
The marginals of a multivariate normal are normal *Note: a system of variables with marginal normal densities need not have a jointly normal density, and, in fact a joint density need not even exist*
Sum of Exponentials *Sum of* ***independent*** *exponential random variables from a common exponential density* $\alpha e^{-\alpha x}$ *for* $x > 0$ *is* S_n*, the sum* $S_n = X_1 + X_2 + ... + X_n$ *has the* ***gamma density****.*
Memoryless Property ***ONLY*** *the exponential and geometric distributions are memoryless. The random variable X exhibits the memoryless property if* $P\{X > s+t | X > s\} = P\{X > t\}$ *or recast* $P\{X > s+t | X > s\} = P\{X > t\}$ *if and only if* $P\{X > s+t\} = P\{X > s\}P\{X > t\}$

Conditional Probability

$P(A|H) = \frac{P(A \cap H)}{P(H)}$
$P_H : A \rightarrow P(A|H)$
$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1 | A_2 \cap ... \cap A_n)$
$\times P(A_2 | A_3 \cap ... \cap A_n) \times P(A_{n-1}|P(A_n) \times P(A_n)$

Bayes Rule

$P(A_k|H) = \frac{P(H|A_k)P(A_k)}{\sum_{j\geq 1} P(H|A_j)P(A_j)}$

Inclusion-Exclusion

Di Finneti's Theorem

Limit Laws

Weak Law of Large Numbers
Strong Law of Large Numbers
Central Limit Theorem
Suppose $X_1, X-2, ...$ *is a sequence of independent random variable drawn from a common distribution F with mean zero and variance one. For each n, let* $S_n* = (X_1 + ... + X_n)/\sqrt{n}$*. Then* S_n* *converges in distribution to the standard normal.*
$P\{a < S_n* < b\} \rightarrow \Phi(b) - \Phi(a) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}dx$ *If* $E(X_k) = \mu$ *and* $Var(X_k) = \sigma^2$ *then we can center and scale the variables such that* $S_n* = (S_n - n\mu/\sqrt{n}\sigma)$ *and the theorem holds for the new properly normalize* S_n**.*

Inequalities

Convexity The chord lies above the curve
$\Psi(\alpha x + (1-\alpha)y) \leq \alpha\Psi(x) + 1 - \alpha)\Psi(y)$
Jensen If Ψ is convex , X is integrable, and $\Psi(x)$ is integrable then: $\Psi(E(x)) \leq E(\Psi(x))$
Another of Jensen $E(X^2) \geq E(X)^2$
AM-GM $x_1^{p_1}x_2^{p_2}...x_n^{p_n} \leq p_1x_1 + p_2x_2 + ...p_nx_n$
Specialization of AM-GM $x^{1/p}y^{1/q} \leq (1/p)x + (1/q)y$ simplifies to: $xy \leq (1/p)x^p + (1/q)y^p$
Holder's $|E(XY)| \leq E(|XY|) \leq E(|X|)^{1/p}E(|Y|)^{1/q}$
Cauchy-Schwarz If $p = q = 2$ in Holder's $\rightarrow$ $|E(XY)^2| \leq E(X^2)E(Y^2)$
Minkowski $p \geq 1$ then $||x+y|_p \leq ||x||_p + ||y||_p$
Chebyshev$P\{|S_n - n\mu| \geq n\epsilon\} \leq \frac{n\sigma^2}{n^2\epsilon^2}$
Chernoff $P\{S_n \geq tSpecificationofa$

σ-algebras

Expectation

Variance $Var(X) = E(X^2) - E(X)^2$
Covariance $Cov(X,Y) = E(XY) - E(X)E(Y)$
Law of the Unconcious Statistician *If X is a discrete r.v. and g is a function from* $R \rightarrow R$ *then,* $E(g(X)) = \sum_x g(x)P(X = x)$
Independence *Independent variables are uncorrelated, but uncorrelated variables are not necessarily independent. However, this is a necessary and sufficient condition if we are talking about the marginals of a normal density* **Additivity** *Expectation is additive. Variance is additive if the summands are independent.*

Conditional Probability

$P(A|H) = \frac{P(A \cap H)}{P(H)}$
$P_H : A \rightarrow P(A|H)$
$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1 | A_2 \cap ... \cap A_n)$
$\times P(A_2 | A_3 \cap ... \cap A_n) \times P(A_{n-1}|P(A_n) \times P(A_n)$

Bayes Rule

$P(A_k|H) = \frac{P(H|A_k)P(A_k)}{\sum_{j\geq 1} P(H|A_j)P(A_j)}$

Inclusion-Exclusion

Di Finneti's Theorem

Limit Laws

Weak Law of Large Numbers
Strong Law of Large Numbers
Central Limit Theorem
Suppose $X_1, X-2, ...$ *is a sequence of independent random variable drawn from a common distribution F with mean zero and variance one. For each n, let* $S_n* = (X_1 + ... + X_n)/\sqrt{n}$*. Then* S_n* *converges in distribution to the standard normal.*
$P\{a < S_n* < b\} \rightarrow \Phi(b) - \Phi(a) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}dx$ *If* $E(X_k) = \mu$ *and* $Var(X_k) = \sigma^2$ *then we can center and scale the variables such that* $S_n* = (S_n - n\mu/\sqrt{n}\sigma)$ *and the theorem holds for the new properly normalize* S_n**.*

Inequalities

Convexity The chord lies above the curve
$\Psi(\alpha x + (1-\alpha)y) \leq \alpha\Psi(x) + 1 - \alpha)\Psi(y)$
Jensen If Ψ is convex , X is integrable, and $\Psi(x)$ is integrable then: $\Psi(E(x)) \leq E(\Psi(x))$
Another of Jensen $E(X^2) \geq E(X)^2$
AM-GM $x_1^{p_1}x_2^{p_2}...x_n^{p_n} \leq p_1x_1 + p_2x_2 + ...p_nx_n$
Specialization of AM-GM $x^{1/p}y^{1/q} \leq (1/p)x + (1/q)y$ simplifies to: $xy \leq (1/p)x^p + (1/q)y^p$
Holder's $|E(XY)| \leq E(|XY|) \leq E(|X|)^{1/p}E(|Y|)^{1/q}$
Cauchy-Schwarz If $p = q = 2$ in Holder's $\rightarrow$ $|E(XY)^2| \leq E(X^2)E(Y^2)$
Minkowski $p \geq 1$ then $||x+y|_p \leq ||x||_p + ||y||_p$
Chebyshev$P\{|S_n - n\mu| \geq n\epsilon\} \leq \frac{n\sigma^2}{n^2\epsilon^2}$
Chernoff $P\{S_n \geq t \leq (\inf_{\lambda \geq 0} e^{-\lambda t/n}M(\lambda))^n$ and
$P\{S_n \leq t\} \leq (\inf_{\lambda \geq 0} e^{-\lambda t/n}M(-\lambda))^n$

Where $M(\lambda) = E(e^{\lambda X_j}) = \int_{\mathcal{R}} e^{\lambda x} dF(x)$ is the moment generating function

Markov $P\{X \geq a\} \leq \frac{E(X)}{a}$

Normal Tail Bound p 317 $\frac{\phi(x)}{x}(1 - \frac{1}{x^2}) < 1 - \Phi(x) < \frac{\phi(x)}{x}$

Normal Tail Bound p 165 $\int_t^\infty \phi(x)dx \leq \frac{1}{2}e^{-t^2/2}$

Logarithmic Identities

$y = \log_b(x) \text{ iff} x = b^y$

$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

$\log_b(x) = \log_b(c)\log_c(x) = \frac{\log_c(x)}{\log_c(b)}$

$\log_b(x^n) = n\log_b(x)$

$\log_b(1) = 0$

$\log_b(b) = 1$

$\log_b(xy) = \log_b(x) + \log_b(y)$

$log(\infty) \to \infty$ $\quad$ $log(0) \to -\infty$ $\quad$ $log(1) = 0$

Exponential Identities

$x^a x^b = x^{(a+b)}$ $\quad$ $(x^a)^b = x^{(ab)}$

$x^a y^a = (xy)^a$ $\quad$ $x^{(a-b)} = \frac{x^a}{x^b}$

Common Infinite Series

Exponential

$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$

$\sum_{k=0}^{\infty} k\frac{z^k}{k!} = ze^z$ Mean of the Poisson

$\sum_{k=0}^{\infty} k^2\frac{z^k}{k!} = (z + z^2)e^z$ Second moment of Poisson

$\sum_{k=0}^{\infty} k^3\frac{z^k}{k!} = (z + 3z^2 + z^3)e^z$

$\sum_{k=0}^{\infty} k^4\frac{z^k}{k!} = (z + 7z^2 + 6z^3 + z^4)e^z$

Binomial

$\sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = (1+z)^\alpha, |z| < 1$

$\sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = (x+y)^n$

Integration

$\int x^n = \frac{1}{n+1}x^{n+1},\ n \neq -1$ $\quad$ $\int a^x = \frac{1}{\ln a}a^x$

$\int \frac{1}{x} = \ln|x|$ $\quad$ $\int \ln x = x\ln x - x$

$\int u\, v = uv - \int vdu$ $\quad$ $\int \sin x = -\cos x$

$\int e^x = e^x$ $\quad$ $\int \cos x = \sin x$