Fourth Order Modified Laguerre's Method

Thomas R. Cameron
Davidson College

Abstract

We present a novel modification of Laguerre's method that results in a method for the concurrent approximation of all roots of a univariate polynomial. Our method has strong virtues including fourth-order convergence that is observed in practice and belonging to the class of embarrassingly parallel algorithms. A Fortran 90 implementation of our algorithm is available online and comparisons with several other software are provided to show the effectiveness of our approach.

The Algorithm

Let $p(\lambda) = a_0 + a_1\lambda + \cdots + a_m\lambda^m$ be a polynomial with $a_0 a_m \neq 0$ and denote by $(z_1, \ldots, z_m)$ the current approximations to the roots $r_1, \ldots, r_m$ of $p(\lambda)$. The jth approximation is updated via

$$\hat{z}_j = z_j - \frac{m}{G_j \pm \sqrt{(m-1)(mH_j - G_j^2)}}, \tag{1}$$

where

$$G_j = \frac{p'(z_j)}{p(z_j)} - \sum_{\substack{i=1\\ i\neq j}}^{m} \frac{1}{(z_j - z_i)} \quad \text{and} \quad H_j = -\left(\frac{p'(z_j)}{p(z_j)}\right)' - \sum_{\substack{i=1\\ i\neq j}}^{m} \frac{1}{(z_j - z_i)^2}. \tag{2}$$

On each iteration, z_j is updated for $j = 1, \ldots, m$, unless it was accepted on a previous iteration. In this sense, all roots of the polynomial are approximated concurrently, rather than sequentially.

Initial Estimates In essence, we select complex numbers along circles of suitable radii. What constitues suitable radii is formalized in [3] and can be computed via the upper envelope of the convex hull of the set $\{(i, \log|a_i|),\ i = 0, 1, \ldots, m\}$. We compute the convex hull via Andrew's Monotone Chain algorithm [1].

Backward Error The backward error of an approximate root ξ is given by

$$\eta(\xi) = \frac{|p(\xi)|}{\alpha(\xi)}, \tag{3}$$

where $\alpha(\xi) = \sum_{i=0}^{m} |e_i||\xi|^i$ and e_i are arbitrary and represent tolerances against which perturbations will be measured. We accept a root approximation ξ if $\eta(\xi) < \mu$, where μ is machine precision.

Condition The condition of a nonzero approximate root ξ is given by

$$\kappa(\xi) = \frac{\alpha(\xi)}{|\xi||p'(\xi)|}. \tag{4}$$

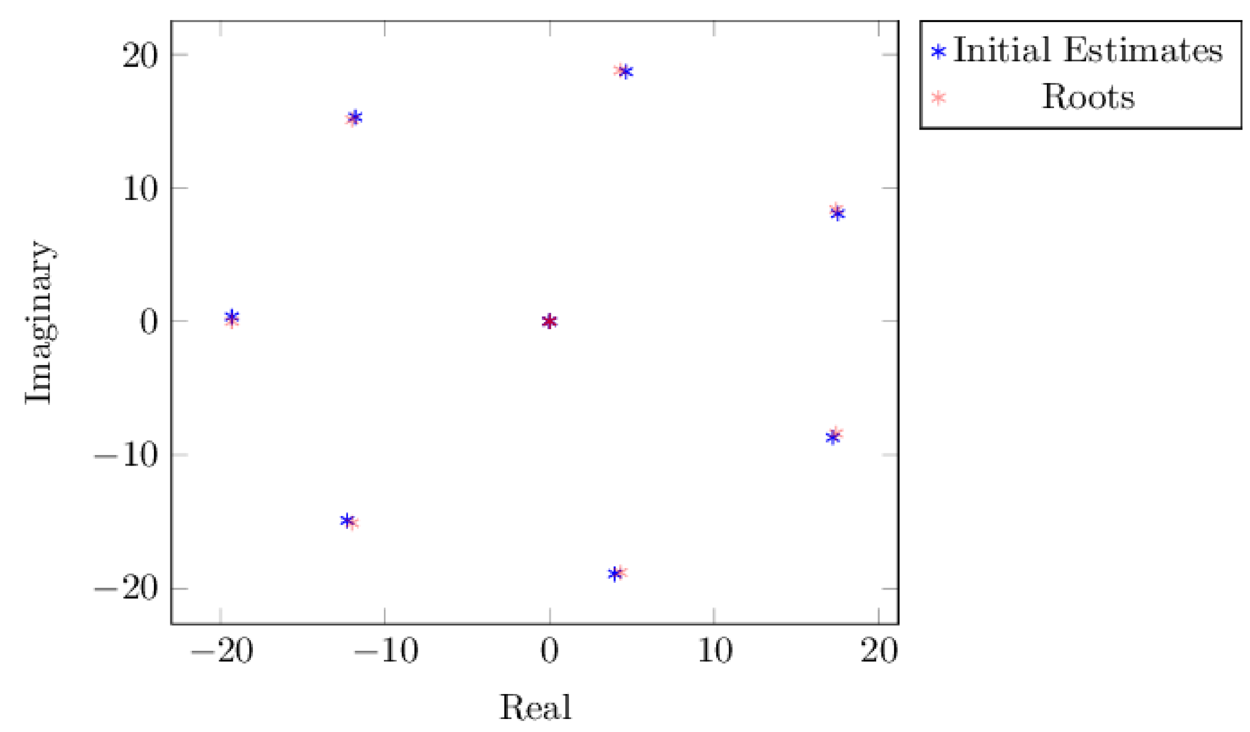
If the root approximation ξ is accepted, then we also return its condition. Thus, we are able to give the upper bound $\eta(\xi) \cdot \kappa(\xi)$ on the forward error in the root approximation ξ.

Examples

Initial Estimates Let

$$p(\lambda) = 1 + 3 \cdot 10^3\lambda + 3 \cdot 10^6\lambda^2 + 1 \cdot 10^9\lambda^9 + \lambda^{10}.$$

The initial estimates and exact roots of p are below.

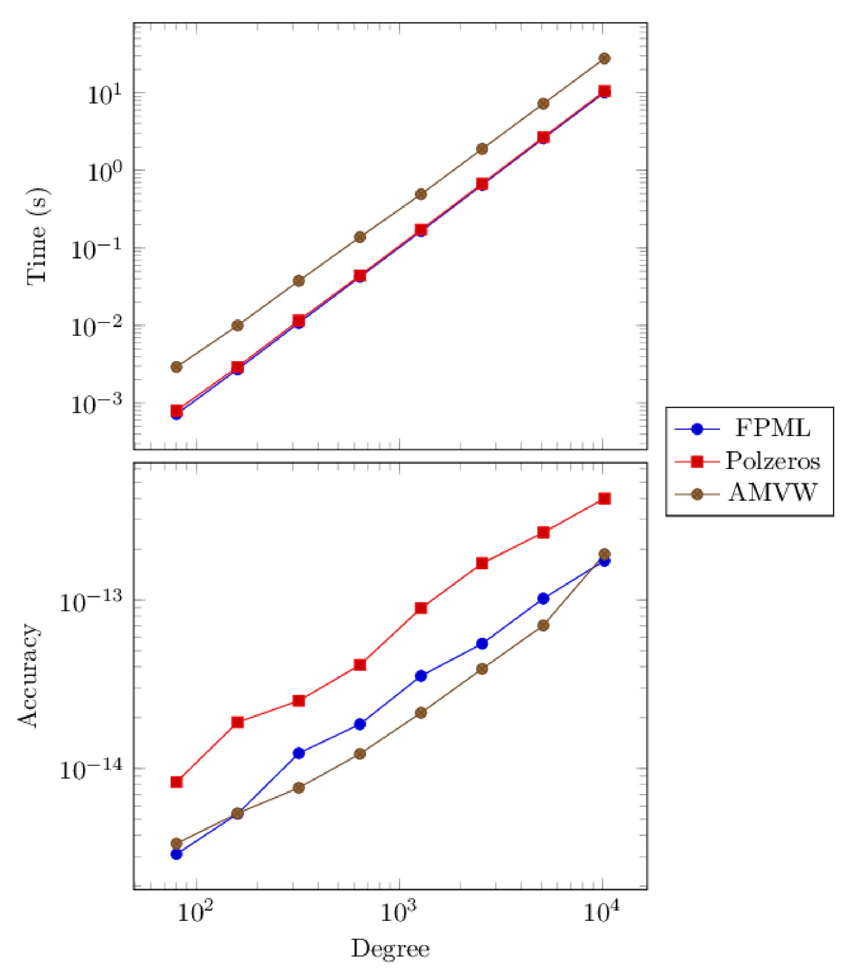


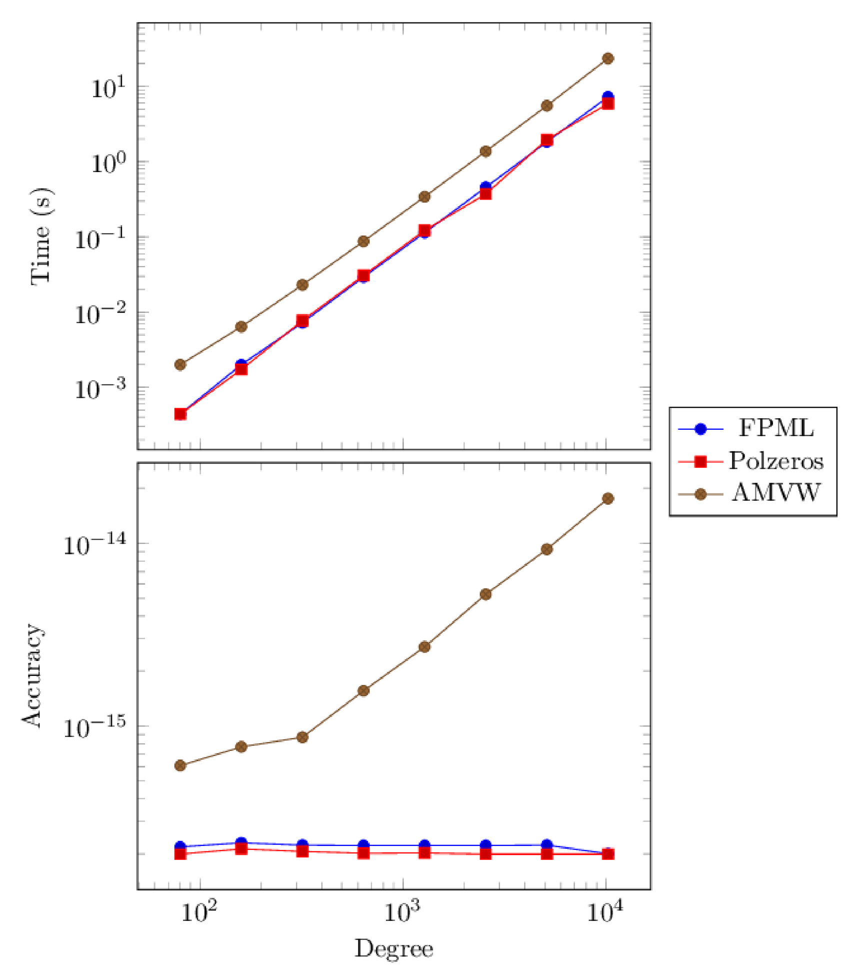
Convergence Here we test the convergence of the roots of three polynomials. The first polynomial is $z^5 - 1$, the second is the degree 10 Chebyshev polynomial, and the third is $z^{10} + \cdots + z + 1$. The error is measured as the maximum relative forward error. For each polynomial, the error after each iteration is recorded in the table below.

Iteration	Error-1	Error-2	Error-3
1	0.55	3.57	0.37
2	0.14	2.27	1.32
3	$1.91 \cdot 10^{-4}$	0.22	$2.55 \cdot 10^{-2}$
4	$3.33 \cdot 10^{-16}$	0.16	$5.93 \cdot 10^{-8}$
5	$3.33 \cdot 10^{-16}$	$1.49 \cdot 10^{-3}$	$1.96 \cdot 10^{-15}$
6	0	$2.39 \cdot 10^{-13}$	$1.96 \cdot 10^{-15}$
7	0	$1.02 \cdot 10^{-14}$	0
8	0	$1.02 \cdot 10^{-14}$	0
9	0	0	0
10	0	0	0

Numerical Experiments

Comparisons betweeen FPML, Polzeros [3], and the singleshift version of AMVW [2] are provided below.





Conclusion

References

[1] A. M. Andrew, *Another efficient algorithm for convex hulls in two dimensions*, Info. Proc. Letters **9** (1979), no. 15, 216–219.

[2] J. L. Aurentz, T. Mach, R. Vandebril, and D. S. Watkins, *Fast and backward stable computation of roots of polynomials*, SIAM J. Matrix Anal. Appl. **36** (2015), no. 3, 942–973.

[3] D. A. Bini, *Numerical computation of polynomial zeros by means of Aberth's method*, Numer. Algorithms **13** (1996), 179–200.