Fourth Order Modified Laguerre's Method

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Abstract

We present a novel modification of Laguerre's method that results in a method for the concurrent approximation of all roots of a univariate polynomial. Our method has strong virtues including fourth-order convergence that is observed in practice and belonging to the class of embarrassingly parallel algorithms. A Fortran 90 implementation of our algorithm is available online and comparisons with several other software are provided.

The Algorithm

Let $p(\lambda) = a_0 + a_1\lambda + \cdots + a_m\lambda^m$ be a polynomial with $a_0 a_m \neq 0$ and denote by $(z_1, \ldots, z_m)$ the current approximations to the roots of $p(\lambda)$. The jth approximation is updated via

$$\hat{z}_j = z_j - \frac{m}{G_j \pm \sqrt{(m-1)(mH_j - G_j^2)}}, \tag{1}$$

where

$$G_j = \frac{p'(z_j)}{p(z_j)} - \sum_{\substack{i=1 \\ i\neq j}}^{m} \frac{1}{(z_j - z_i)} \quad \text{and} \quad H_j = -\left(\frac{p'(z_j)}{p(z_j)}\right)' - \sum_{\substack{i=1 \\ i\neq j}}^{m} \frac{1}{(z_j - z_i)^2}. \tag{2}$$

On each iteration, z_j is updated for $j = 1, \ldots, m$, unless it was accepted on a previous iteration.

Initial Estimates. In essence, we select complex numbers along circles of suitable radii. What constitutes suitable radii is formalized in [3] and can be computed via the upper envelope of the convex hull of the set $\{(i, \log|a_i|),\ i = 0, 1, \ldots, m\}$. We compute the convex hull via Andrew's Monotone Chain algorithm [1].

Backward Error. The backward error of an approximate root ξ is given by

$$\eta(\xi) = \frac{|p(\xi)|}{\alpha(\xi)}, \tag{3}$$

where $\alpha(\xi) = \sum_{i=0}^{m} |e_i||\xi|^i$ and e_i represent tolerances against which perturbations are measured. We accept a root approximation ξ if $\eta(\xi) < \mu$, where μ is machine precision.

Condition. The condition of a nonzero approximate root ξ is given by

$$\kappa(\xi) = \frac{\alpha(\xi)}{|\xi||p'(\xi)|}. \tag{4}$$

If the root approximation ξ is accepted, then we also return its condition.

Numerical Experiments

Comparisons between FPML, Polzeros [3], and the singleshift version of AMVW [2].

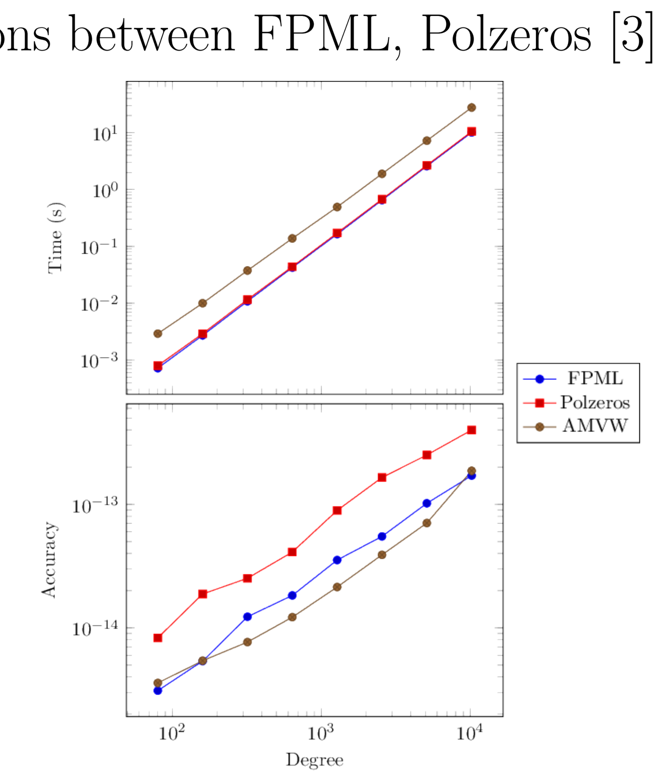


Fig. 1: Random Polynomials

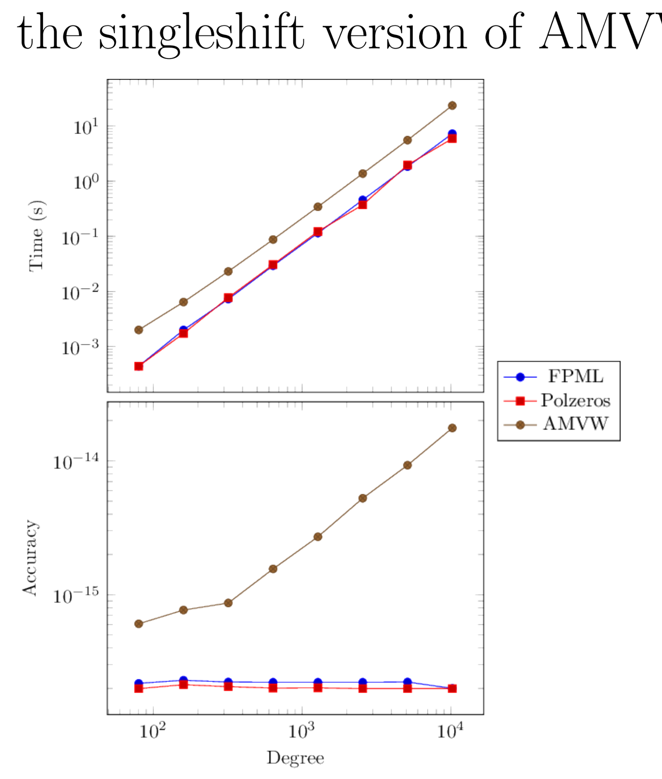


Fig. 2: Roots of Unity

Poly No.	Description	Deg.	Roots
1	Wilkinson polynomial	10	$1, \ldots, 10$
2	Wilkinson polynomial	15	$1, \ldots, 15$
3	Wilkinson polynomial	20	$1, \ldots, 20$
4	scale and shifted Wilkinson polynomial	20	$-2.1, -1.9, \ldots, 1.7$
5	reverse Wilkinson polynomial	10	$1, 1/2, \ldots, 1/10$
6	reverse Wilkinson polynomial	15	$1, 1/2, \ldots, 1/15$
7	reverse Wilkinson polynomial	20	$1, 1/2, \ldots, 1/20$
8	prescribed roots of varying scale	20	$2^{-10}, 2^{-9}, \ldots, 2^{9}$
9	prescribed roots of varying scale -3	20	$2^{-10}-3, 2^{-9}-3, \ldots, 2^{9}-3$
10	Chebyshev polynomial	20	$\cos(\frac{2j-1}{40}\pi)$
11	$z^{20} + z^{19} + \cdots + z + 1$	20	$e^{i\frac{2j}{21}\pi}$
12	C. Traverso	24	known
13	Mandelbrot	31	known
14	Mandelbrot	63	known

Poly No.	FPML	Polzeros	AMVW
1	$4.3 \cdot 10^{-11}$	$3.84 \cdot 10^{-10}$	$1.97 \cdot 10^{-7}$
2	$7.32 \cdot 10^{-7}$	$3.63 \cdot 10^{-6}$	$6.63 \cdot 10^{-2}$
3	$3.65 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	1
4	$7.1 \cdot 10^{-13}$	$1.89 \cdot 10^{-13}$	$3.3 \cdot 10^{-12}$
5	$2 \cdot 10^{-11}$	$1.07 \cdot 10^{-10}$	$7.13 \cdot 10^{-7}$
6	$1.35 \cdot 10^{-7}$	$2.85 \cdot 10^{-7}$	0.2
7	1.36	1.36	1.6
8	$8.55 \cdot 10^{-15}$	$2.21 \cdot 10^{-15}$	$1.98 \cdot 10^{-2}$
9	0.76	$2.2 \cdot 10^{-2}$	$4.78 \cdot 10^{-2}$
10	$5.21 \cdot 10^{-12}$	$3.72 \cdot 10^{-11}$	$3.9 \cdot 10^{-11}$
11	$3.7 \cdot 10^{-16}$	$2.65 \cdot 10^{-16}$	$5.63 \cdot 10^{-16}$
12	$1.99 \cdot 10^{-8}$	$3.83 \cdot 10^{-8}$	1
13	$4.9 \cdot 10^{-8}$	$3.94 \cdot 10^{-7}$	$8.71 \cdot 10^{-7}$
14	0.18	0.18	0.16

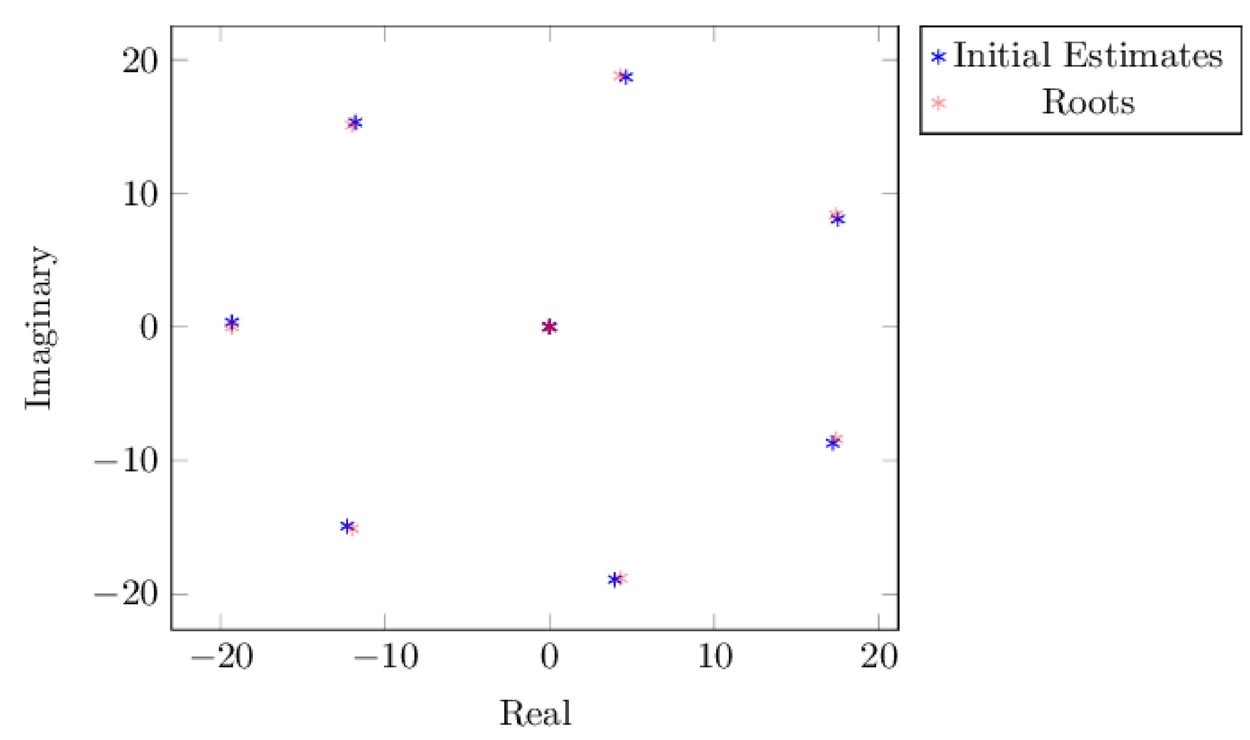
Fig. 3: Special Polynomials

Examples

Initial Estimates. Let

$$p(\lambda) = 1 + 3\cdot 10^3\lambda + 3\cdot 10^6\lambda^2 + 1\cdot 10^9\lambda^9 + \lambda^{10}.$$

The initial estimates and exact roots of p are below.



Convergence. Here we test the convergence of the roots of three polynomials. The first polynomial is $z^5 - 1$, the second is the degree 10 Chebyshev polynomial, and the third is $z^{10} + \cdots + z + 1$. The error is measured as the maximum relative forward error. For each polynomial, the error after each iteration is recorded in the table below.

Iteration	Error-1	Error-2	Error-3
1	0.55	3.57	0.37
2	0.14	2.27	1.32
3	$1.91 \cdot 10^{-4}$	0.22	$2.55 \cdot 10^{-2}$
4	$3.33 \cdot 10^{-16}$	0.16	$5.93 \cdot 10^{-8}$
5	$3.33 \cdot 10^{-16}$	$1.49 \cdot 10^{-3}$	$1.96 \cdot 10^{-15}$
6	0	$2.39 \cdot 10^{-13}$	$1.96 \cdot 10^{-15}$
7	0	$1.02 \cdot 10^{-14}$	0
8	0	$1.02 \cdot 10^{-14}$	0
9	0	0	0
10	0	0	0

Conclusion

Fortran 90 code along with installation instructions and additional experiment results and references are provided at https://github.com/trcameron/FPML.

References

[1] A. M. Andrew, *Another efficient algorithm for convex hulls in two dimensions*, Info. Proc. Letters **9** (1979), no. 15, 216–219.

[2] J. L. Aurentz, T. Mach, R. Vandebril, and D. S. Watkins, *Fast and backward stable computation of roots of polynomials*, SIAM J. Matrix Anal. Appl. **36** (2015), no. 3, 942–973.

[3] D. A. Bini, *Numerical computation of polynomial zeros by means of Aberth's method*, Numer. Algorithms **13** (1996), 179–200.