System Simulation Final Exam

Due: 03 May 2019

Problem 01

A set of floating magnets on a frictionless rod, with every other magnet having its poles reversed, makes an interesting demonstration of magnetic fields. The equations for nine such magnets are given by

$$\ddot{x}_m = -1 + \frac{1}{(x_m - x_{m-1})^2} - \frac{1}{(x_{m+1} - x_m)^2},$$

with $x_0 \equiv 0$ and $x_{10} \equiv 10$, for $1 \leq m \leq 9$. The initial conditions are $x_m(t) = m$ and $\dot{x}_m(t) = 0$ for all $t \leq 0$. It is assumed that there is no friction.

(A) Plot all nine magnet positions vs time (units in seconds) on the same plot using the Halijak double integrator, which approximates a double integrator as

$$\frac{1}{s^2} \leftrightharpoons H(z) = \frac{T^2 z}{z^2 - 2z + 1}.$$

(B) Generate the phase plot of x_1 versus x_2.

(C) Is this system chaotic? Why or why not?

Notes:Please do not use operational substitution! Note the all of the states should satisfy

$$0 \leq x_j(t) \leq 10.$$

To help understand the Halijak double integrator, draw a block diagram for $\frac{1}{s^2}$ where $x(t)$ is the output, and then the corresponding Halijak approximation where $x[k]$ is the output. A timestep T on the order of ms is sufficient to simulate this system.

Problem 02

The dynamic model for a certain chemical reaction is given by

$$\begin{aligned}
\dot{x}_1 &= s\left(x_2 - x_2 x_1 + x_1 - q x_1^2\right) \\
\dot{x}_2 &= s^{-1}\left(-x_2 - x_2 x_1 + \epsilon x_3\right) \\
\dot{x}_3 &= \omega(x_1 - x_3),
\end{aligned}$$

here $s = 77.27$, $\omega = 0.1610$, $q = 8.375 \times 10^{-6}$, and $\epsilon = 1$.

(A) Verify that the system has an equilibrium point when

$$x_1 = x_3 = 488.68$$

and

$$x_2 = 0.99796.$$

(B) Linearize the system about the four points

$$x_A = \begin{pmatrix} 1 & 10,000 & 1,000 \end{pmatrix}^T,$$

$$x_B = \begin{pmatrix} 31,623 & 10 & 100,000 \end{pmatrix}^T,$$

$$x_C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T,$$

and the equilibrium point. Calculate the eigenvalues and stiffness ratios of the linearizations.

(C) Simulate the system for at least 400 *seconds* for the initial conditions x_A, x_B, x_C, x_{SS+} and x_{SS-} where x_{SS+} is an initial condition near the equilibrium point for which x_1 increases and where x_{SS-} is an initial condition near the equilibrium point for which x_1 decreases. Provide the following plots:

(1) A plot with three subplots showing each $\log_{10}(x_j)$ plotted versus time for the x_C initial condition.

(2) A plot with $log_{10}(x_2)$ on the horizontal and $log_{10}(x_1)$ on the vertical for all five initial condition responses, each response in a different color.

(3) A plot with $log_{10}(x_2)$ on the horizontal and $log_{10}(x_3)$ on the vertical for all five initial condition responses, each response in a different color.

Note that all the simulations should be stable.

Problem 03

A dynamical model of a zombie outbreak was formulated by Munz et al.[1] This paper concluded that the only situation where both humans and zombies could coexist is when there was a treatment for "zombie-ism." This model for an outbreak consists of four groups, the suceptable group S of humans that could contract zombie-ism, the infected group I, that has contracted zombie-ism and will become zombies if the illness is left to take its course, the group of zombies Z, and finally, the individuals that have been removed, either humans who have died non-zombie deaths or zombies that have been eliminated. This model allows zombies that have been killed to be reanimated as zombies again. An additional group K of humans or zombies that are unable to be reanimated has been added. The modified model is given by

$$\begin{aligned}
\dot{S} &= \Pi S - \beta Z S - \delta S + cZ \\
\dot{I} &= \beta Z S - \rho I - \delta I \\
\dot{Z} &= \rho I + \zeta R - \alpha S Z - cZ \\
\dot{R} &= \delta S - \zeta R, \\
\dot{K} &= \delta I + \alpha S Z
\end{aligned}$$

where the parameters and possible values are given in Table 1. Note that state variables are in terms of thousands, and the timescale of t is in days. For example, $S(6) = 25$, indicates that the susceptable population is $25,000$ people at day six of the outbreak. Simulate, using any simulation technique that produces reasonable results, at least 180 days worth of data (more time would be needed of the simulation does not reach a steady state) for the four cities shown in Table . Create six plots: one plot for each city including the S, Z, I, and R populations, and a plot that includes the S populations for each city normalized to the largest value of S, and a plot that shows the total Z populations for each city. For your reference, a simulation has been performed for Middletown, OH (population 97,500) and is shown in Figure 1

[1] Munz, Philip, et al. "When zombies attack!: mathematical modelling of an outbreak of zombie infection." *Infectious Disease Modelling Research Progress 4* (2009): 133-150.

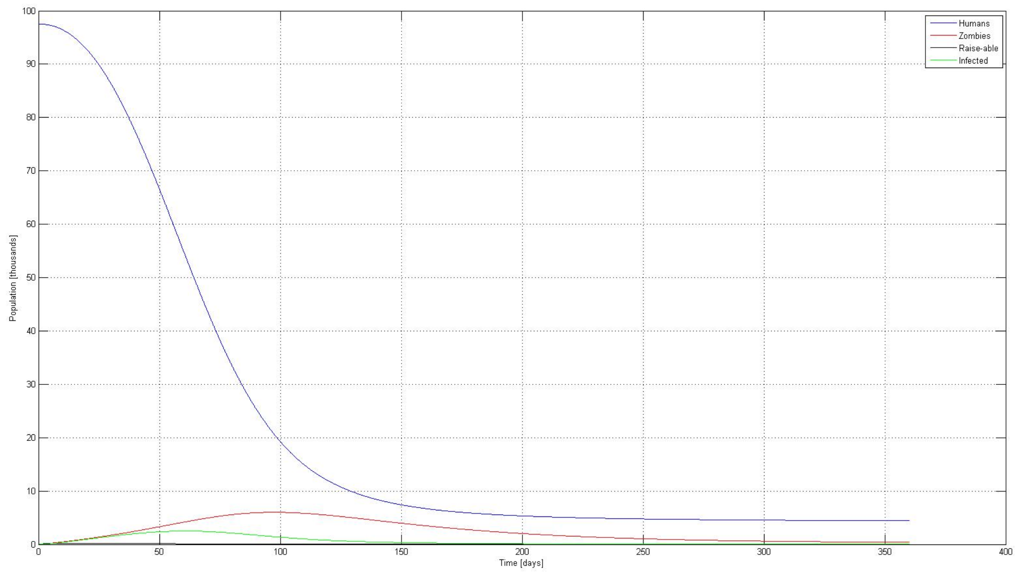

Figure 1: Zombie Simulation for Middletown, OH (Population 97,500).

Paramter	Description	Value
α	Rate of defeat of zombies	0.005
β	Bite rate of the zombies	0.0055
ζ	Raising rate from the removed class	0.5
δ	Death rate due to non-zombie causes	0.001
ρ	Rate of infected indivuals zombie-fying	0.5
c	Rate of curing zombies	0.002
Π	Birth rate in humans	0.001

Table 1: Parameters for Zombie Outbreak Model.

Problem 04

Sprott[2] considers several dynamical systems to model happiness. Using a fourth-order Runge-Kutta method, simulate the model given by Equation (4) in the paper using the parameters shown below Figure 7 in the paper. Use small initial conditions. Submit plots of R vs t, H vs. t, and H vs R.

Problem 05

A dynamic model of love was formulated by Rinaldi et al[3] and applied to the Disney film *Beauty and the Beast*[4] as follows. The state variables are the feelings at the time t of the individuals for

[2]Sprott, J. C. "Dynamical models of happiness." Nonlinear Dynamics, Psychology, and Life Sciences 9.1 (2005): 23-36.

[3]Rinaldi et all, "Love and appeal in standard couples." *International Journal of Bifurcation and Chaos*, 2000.

[4]Rinaldiet al, "Small discoveries can have great consequences in love affairs: the case of Beauty and the Beast." *International Journal of Bifurcation and Chaos*, 2013.

City	Population (thousands)
Akron	197.859
Canal Fulton	5.479
Cleveland	389.521
Columbus	835.957

Table 2: Cities for the Zombie Simulation.

their partners. The dynamic model is

$$\begin{aligned}\dot{x}_1(t) &= -\alpha_1 x_1(t) + R_1(x_2) + A_2(t)\\ \dot{x}_2(t) &= -\alpha_2 x_2(t) + R_2(x_1) + A_1(t)\end{aligned} \tag{1}$$

where

$$\alpha_1 = 0.1,$$

$$\alpha_1 = 0.3,$$

$$R_1(x_2) = \frac{e^{x_2} - e^{-x_2}}{e^{x_2} + e^{-x_2}},$$

and The next equation was modified.

$$R_2(x_1) = \frac{2e^{x_1} - 2e^{-x_1}}{e^{x_1} + 2e^{-x_1}}.$$

The α_1 and α_2 parameters have to do with placing higher significance on more recent events. The $R_1(x_2)$ and $R_2(x_1)$ are descriptions of how one person reacts to the feelings of the other person. The final pair of parameters $A_1(t)$ and $A_2(t)$ represent the appeal of one person to the other. In *Beauty and the Beast* appeal of Belle is $A_1(t)$ high and a constant, let $A_1(t) = 1.2$. The appeal of the Beast will change with time, and make the love story possible. Note that $x_1(t)$ represents the feelings that Belle has for the Beast and that $x_2(t)$ represents the feelings that the Beast has for Belle. Assume that the two start with indifferent feelings toward each other, that is $x_1(0) = x_2(0) = 0$.

(A) Initially, the appeal of the Beast is very negative because of his, well, bestial appearance and behavior. For the first simulation, keep the appeal of the Beast constant at $A_2 = -1.9$, and simulate the resulting system until a steady state is reached.

(B) In the story, the Beast becomes less unappealing, i.e. $A_2(t)$ increases, through his actions of saving Belle from the wolves, allowing Belle into his library, etc. We will represent this scenario by setting

$$A_2(t) = \begin{cases} 0.02t - 1.9, & 0 \le t \le 35 \\ -1, & 35 < t \end{cases}.$$

Though the Beast becomes more appealing, in this scenario Belle is still frightened by the Beast's bestial appearance. Simulate this scenario until some steady state has been reached.

(C) Repeat part (B), except use

$$A_2(t) = \begin{cases} 0.02t - 1.9, & 0 \le t \le 80 \\ -0.3, & 80 < t \end{cases}.$$

In this scenario, Belle is more able to look past the Beast's appearance, but there is still some revulsion to it.

(D) Briefly discuss the three scenarios, and suggest which of the scenarios fits the story best. *Hint*: In the end, Belle and the Beast fall in love and live happily ever after.