Well-Orderedness of the Bashicu Matrix System

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Abstract

The Bashicu Matrix System is a recursive system of ordinal notations created by the user BashicuHyudora of the japanese Googology Wiki. In this paper, we prove that the Bashicu Matrix System is well-ordered.

1 Introduction

The Bashicu Matrix System (BMS) is a recursive system of ordinal notations with a large order type created by the user BashicuHyudora of the japanese Googology Wiki [2]. Originally, it was defined informally in pseudocode based on the programming language BASIC, and the following is the agreed-upon formalization:

Definition 1.1. An array is a sequence of equal-length sequences of natural numbers, i.e. an element of $(\mathbb{N}^n)^m$ for some $n, m \in \mathbb{N}$. For every array $A \in (\mathbb{N}^n)^m$, the columns of A are its elements, and for each $n' < n$, the n'-th row of A is the sequence of length m such that for each $m' < m$, the m'-th element of the n'-th row is the n'-th element of the m'-th column. We will denote concatenation of sequences by $+$.

Let A be any array and n be any natural number. For every m smaller than the length of A's columns and every i smaller than the length of A, the m-parent of the i-th column is the last column before it whose m-th element is smaller than the m-th element of the i-th column, and which is an $(m-1)$-ancestor of the i-th column if $m > 0$, if such a column exists. If no such column exists, then the i-th column does not have an m-parent. The m-ancestors (also called strict m-ancestors) of a column are its m-parent and the m-ancestors of its parent. The non-strict m-ancestors of a column are the column itself and its m-ancestors.

If A is empty, then the expansion of A at n is $A[n] = A$. Otherwise let C be the last element of A and let m_0 be maximal such that C has an m_0-parent, if such an m_0 exists, otherwise m_0 is undefined. Let arrays $G, B_0, B_1, ..., B_n$ be such that:

- $A = G + B_0 + (C)$.
- The first element of B_0 is the m_0-parent of C if m_0 is defined and otherwise B_0 is empty.

- For each D in B_0 and $m < m_0$, if the first column in B_0 is D or an m-ancestor of D, then it the m-th element of D is said to ascend.
- B_i is a copy of B_0, but for each ascending element of each column in B_0, its copy in B_i is increased by $i \cdot ((m\text{-th element of } C) - (m\text{-th element of the first column in } B_0))$, where m is the index of the row in which that element is.

Then the expansion $A[n]$ of A at n is $G + B_0 + B_1 + ... + B_n$, with all rows of zeroes at the bottom removed.

BMS is the closure of $\{((\underbrace{0,0,...,0,0}_{n}),(\underbrace{1,1,...,1,1}_{n})) : n \in \mathbb{N}\}$ under expansion at each natural number, ordered by the $\subseteq$-minimal partial order such that $A[n] \leq A$ for each $n \in \mathbb{N}$ and $A \in BMS$. Here, a partial order $\leq$ is the set of pairs (x, y) such that $x \leq y$.

This is the fourth official version of the system, which is why it is also referred to as $BM4$. The previous versions $BM1$, $BM2$ and $BM3$ were not well-founded, but as we prove below, $BM4$ is well-founded. There are also unofficial versions, of which $BM2.3$ is strongly believed to be equivalent to $BM4$ [3], and $BM3.3$ is also notable for its similarity to $BM4$ and temporarily more predictable behavior. However, they are not the focus of this paper, so from now on, we will only refer to $BM4$.

The question of whether BMS is well-ordered has been an open problem for almost 8 years, and it was among the most significant open problems in googology. Although BMS is yet to be used outside of this field, its simplicity and large order type provide hope for future uses in proof theory and model theory. Before this paper, the research about BMS has brought the following results:

- BMS restricted to arrays with one row is also called the Primitive Sequence system (or $PrSS$), and has a simple isomorphism with the iterated base-ω Cantor normal form - intuitively, each column represents a single ω in the string, the element of the column is the "height" of the ω (the number of exponents it appears in), and distinct ωs with the same height are separated by a + at the same level, unless there is an ω between them with a lower height. ωs that do not have any ω in their exponent in the resulting string are exponentiated to 0. This isomorphism can be proven easily by transfinite induction on the Cantor normal form expression, thus the order type of $PrSS$ is ε_0.
- BMS restricted to arrays with two rows is also called the Pair Sequence System (or PSS), and was proven well-founded in 2018,[1] with its order type shown to be the proof-theoretic ordinal of $\Pi^1_1{-}CA_0$, i.e. the countable collapse of ω_ω using standard collapsing functions (such as Buchholz's function in this case).

If we abbreviate $\langle L_\alpha, \in\rangle \prec_{\Sigma_1} \langle L_\beta, \in\rangle$ as $\alpha <_0 \beta$, then informal estimates say that the order type of the set of arrays in BMS smaller than $((0,0,0),(1,1,1),(2,2,2))$ is most likely the supremum of, for each n, a recursive collapse (using standard collapsing functions) of the smallest ordinal α_0 for which there exist $\alpha_1, \alpha_2, ..., \alpha_n$ such that $\alpha_0 <_0 \alpha_1 <_0 \alpha_2 <_0 ... <_0 \alpha_n$.

The order type of the entirety of BMS has not been carefully estimated in terms of ordinal functions yet, but is expected to be the supremum of, for each n, a recursive collapse of the smallest ordinal α for which there exists β with $\langle L_\alpha, \in\rangle \prec_{\Sigma_n} \langle L_\beta, \in\rangle$, using collapsing functions that may be standard in the future.

Subjectively, BMS is a very elegant way to represent large recursive ordinals. With enough formalization effort, it could give rise to a system of recursively large ordinals. This system would be similar to stability in structure and, as far as we know, similar to stability in scale too, but perhaps easier to understand or easier to use for some purposes such as ordinal analysis.

We utilize this similarity to prove that BMS is well-ordered. Specifically, we first prove that BMS is totally ordered and the order is precisely the lexicographical order. We then prove that a certain reflection property holds for stable ordinals. We show that this property allows us to map elements of BMS to ordinals while preserving the order. Using this order-preserving function from BMS to Ord, any infinite descending sequence in BMS would be mapped to an infinite descending sequence in Ord, which cannot exist by definition, thus BMS is well-ordered.

2 The Proof

Given that a property holds for every element of a set X, and that if it holds for x then it holds for $f(x)$ for each f in some set F of functions, it is easy to see from the definition of closure that the property holds for all elements of the closure of X under the functions $f \in F$. We consider this fact trivial enough to be used implicitly.

It is clear that for $A, A' \in BMS$, $A' < A$ iff A is non-empty and $A' = A[n_0][n_1]...[n_m]$ for some $m, n_0, n_1, ..., n_m \in \mathbb{N}$.

Lemma 2.1. *For all $A \in BMS$ and $n \in \mathbb{N}$, $A[n]$ is lexicographically smaller than A (with the columns also compared lexicographically).*

Proof. Using the variable names from the definition of BMS, we have $A = G + B_0 + (C)$ and $A[n] = G + B_0 + B_1 + ... + B_n$. Then $A[n] <_{lex} A$ iff $B_1 + B_2 + ... + B_n <_{lex} (C)$, which is trivial if m_0 is undefined (the empty sequence is lexicographically smaller than all other sequences, including (C)), and otherwise equivalent to the first column in B_1 being lexicographically smaller than C.

Let R_i be the first column in B_i. Since R_0 is the m_0-parent of C, it is an m-ancestor of C for each $m \leq m_0$, thus the m-th element of R_0 is less than the m-th element of C. By definition, R_1 is a copy of R_0, but for each $m < m_0$, the m-th element is increased either by 0 or by the difference between itself and the m-th element of C. Then it is less than or equal to the m-th element of C, so the sequence of the first m_0 elements of R_1 is pointwise smaller than or equal to the sequence of the first m_0 elements of C (in fact, it is equal, but that is not necessary for this proof). However, the m_0-th element of R_1 is necessarily equal to the m_0-th element of R_0 since $m_0 < m_0$ is false, thus it is strictly smaller than the m_0-th element of C.

Therefore $R_1 <_{lex} C$, which implies $B_1 + B_2 + ... + B_n <_{lex} (C)$, and thus $A[n] <_{lex} A$. $\square$

Corollary 2.2. *For all $A, A' \in BMS$, $A' < A$ implies $A' <_{lex} A$.*

Lemma 2.3. *BMS is totally ordered.*

Proof. For every non-empty $A \in BMS$, $A[0]$ is simply A without the last column, as it is equal to $G + B_0$, and thus $A = A[0] + (C)$. Then it is trivial to prove by induction that for all $A, A' \in BMS$, if A' is a subsequence of A, then $A' = A\underbrace{[0][0]...[0][0]}_{n}$ for some $n \in \mathbb{N}$, and thus $A' \leq A$. Together with $A[n]$ being a subsequence of $A[n+1]$ for all $A \in BMS$ and $n \in \mathbb{N}$, this also means that for all $A, A' \in BMS$ and $n \in \mathbb{N}$, $A[n] \leq A[n+1]$, and if $A[n] < A' \leq A[n+1]$, then $A[n] \leq A'[0]$. This implies that if some subset X of BMS is totally ordered, then $X \cup \{A[n] : A \in X \wedge n \in \mathbb{N}\}$ is also totally ordered. By induction, it is clear that if $X \subseteq BMS$ is totally ordered, then $X \cup \{A[n_0] : A \in X \wedge n_0 \in \mathbb{N}\} \cup \{A[n_0][n_1] : A \in X \wedge n_0, n_1 \in \mathbb{N}\} \cup ... \cup \{A[n_0][n_1]...[n_m] : A \in X \wedge n_0, n_1, ..., n_m \in \mathbb{N}\}$ is totally ordered for each $m \in \mathbb{N}$. Let $X_0 = \{((\underbrace{0, 0, ..., 0, 0}_{n}), (\underbrace{1, 1, ..., 1, 1}_{n})) : n \in \mathbb{N}\}$. Since each $A \in BMS$ is in $\{A''[n_0][n_1]...[n_m] : A'' \in X_0 \wedge n_0, n_1, ..., n_m \in \mathbb{N}\}$ for some $m \in \mathbb{N}$, it is obvious that if X_0 is totally ordered, then for all $A, A' \in BMS$, there's some $m \in \mathbb{N}$ such that $A, A' \in X_0 \cup \{A''[n_0] : A'' \in X_0 \wedge n_0 \in \mathbb{N}\} \cup \{A''[n_0][n_1] : A'' \in X_0 \wedge n_0, n_1 \in \mathbb{N}\} \cup ... \cup \{A''[n_0][n_1]...[n_m] : A'' \in X_0 \wedge n_0, n_1, ..., n_m \in \mathbb{N}\}$, which is totally ordered, and therefore A, A' are comparable. So if X_0 is totally ordered, then BMS is totally ordered.

It is now sufficient to prove that X_0 is totally ordered. This is easy, since $((\underbrace{0, 0, ..., 0, 0}_{n+1}), (\underbrace{1, 1, ..., 1, 1}_{n+1}))[1]$ is trivially $((\underbrace{0, 0, ..., 0, 0}_{n}), (\underbrace{1, 1, ..., 1, 1}_{n}))$, and thus by induction, for each $n < m \in \mathbb{N}$, $((\underbrace{0, 0, ..., 0, 0}_{n}), (\underbrace{1, 1, ..., 1, 1}_{n})) = ((\underbrace{0, 0, ..., 0, 0}_{m}), (\underbrace{1, 1, ..., 1, 1}_{m}))\underbrace{[1][1]...[1][1]}_{m-n} < ((\underbrace{0, 0, ..., 0, 0}_{m}), (\underbrace{1, 1, ..., 1, 1}_{m}))$, and all elements of X_0 are of this form, so all elements of X_0 are pairwise comparable. $\square$

Corollary 2.4. *The ordering of BMS coincides with the lexicographical ordering with columns compared lexicographically.*

Lemma 2.5. *Let A be a non-empty array and n be a natural number, let $G, B_0, B_1, ..., B_n, m_0$ be as in Definition 1.1, and let l_0, l_1 be the lengths of G, B_0.*
(i) For all $i < l_0$, $j < l_1$ and $k \in \mathbb{N}$, in $A[n]$, the i-th column in G is a k-ancestor of the j-th column in B_0 iff it is a k-ancestor of the j-th column in B_n.
(ii) For all $i, j < l_1$ and $k \in \mathbb{N}$, the i-th column in B_0 is a k-ancestor of the j-th column in B_0 iff the i-th column in B_n is a k-ancestor of the j-th column

in B_n.
(iii) If $n > 0$, then for all $i < l_1$ and $k < m_0$, in A, the i-th column in B_0 is a k-ancestor of the last column of A iff in $A[n]$, the i-th column in B_{n-1} is a k-ancestor of the first column in B_n.
(iv) For all $0 < i < l_1$ and $k \in \mathbb{N}$, in $A[n]$, the k-parent of the i-th column in B_n is either in B_n or in G.
(v) For all $i, j < l_1$ and $k \in \mathbb{N}$ and $n_0 < n_1 < n$, in $A[n]$, the i-th column in B_{n_0} is a k-ancestor of the j-th column in B_{n_1} iff it's a k-ancestor of the j-th column in B_{n_1+1}.

Proof. We can prove this by induction on k. The proof is relatively straightforward, but tedious. The author recommends drawing the mentioned ancestry relations in order to see what is happening.

Assume all 5 statements hold for all $k' < k$.

For (ii), fix i and j. If $j = 0$ then it is trivial, so we will only consider the case $j > 0$. From the assumption, it follows that for all $k' < k$ and $i' < l_0$, the i'-th column in B_0 is a k'-ancestor of the j-th column in B_0 iff the i'-th column in B_n is a k'-ancestor of the j-th column in B_n. Let I be the set of i' such that for all $k' < k$, the i'-th column in B_0 is a k'-ancestor of the j-th column in B_0.

Since for all $k' < k$, k'-ancestry is a total order on the columns with indices in I, the k-parent of each such column is simply the last such column before it with a smaller k-th element. The k-th element of the j-th column in B_0 ascends iff the first column in B_0 is in I and is a k-ancestor of the j-th column in B_0, which is equivalent to the k-th element of the first column in B_0 being smaller than the k-th element of all columns between it and the j-th column in B_0, so it is also a k-ancestor of all other columns with indices in I. This means that either the k-th elements of all columns in B_0 with indices in I ascend or the k-th element of the j-th column in B_0 doesn't ascend.

In the first case, the differences between the columns in B_n with indices in I are the same as in B_0, and since k'-ancestry relations between them are also the same as in B_0 for $k' < k$, k-ancestry must be the same too, because everything it depends on is the same. In the second case, since the j-th column doesn't ascend, in B_n, there trivially cannot be any k-ancestors of the j-th column that aren't copies of k-ancestors of the j-th column in B_0. Since this possibility requires that the first column in B_0 is not a k-ancestor of the j-th column, it is also not a k-ancestor of any k-ancestor of the j-th column, thus the k-th elements of the k-ancestors of the j-th column also don't ascend, and therefore the differences between them are the same, implying that the k-ancestry relations are preserved. Either way, (ii) holds for k.

The above can trivially be extended to include the next copy of the first column in B_0, and then since the k-th element of the first column in B_1 is easily seen to be the same as C as long as $k < m_0$, (iii) holds for k.

Then to prove (iv), we first observe that if for some $k' < k$, the k'-parent of the i-th column in B_n is in G, then all of its k'-ancestors are in G, and its k-parent must be one of its k'-ancestors so it is also in G. So we're left with the case that for all $k' < k$, the k'-parent of the i-th column in B_n is in B_n.

If the first column of B_n is a k'-ancestor of the i-th column in B_n for all $k' < k$, and yet its k-parent is not in B_n or G, then the first column in B_n is not a k-ancestor of the i-th column in B_n. Therefore from (ii) for k, which we have already proven, we get that the k-parent of the i-th column in B_0 is not in B_0 (therefore it is in G), which also implies that the k-th element of the i-th column in B_0 does not ascend in the expansion of A, so it is equal to the k-th element of the i-th column in B_n. But from (ii) for all $k' < k$ and the fact that the first column in B_n is a k'-ancestor of the i-th column in B_n for all $k' < k$, we get that the first column in B_0 is a k'-ancestor of the i-th column in B_0 for all $k' < k$.

This, together with its k-parent being in G, means that for all columns in B_0 that are k'-ancestors of the i-th column in B_0 for all $k' < k$, their k-th element is at least as large as the k-th element of the i-th column in B_0, and therefore at least as large as the k-th element of the i-th column in B_n. This includes the first column in B_0, and since the k-th element of the first column in B_n is by definition at least as large as the k-th element of the first column in B_0, which is at least as large as the k-th element of the i-th column in B_n, which is by definition strictly larger than the k-th element of the k-parent of the i-th column in B_n, we get that the k-th element of the first column in B_n is strictly larger than the k-th element of the k-parent of the i-th column in B_n. With that, and due to the facts that k'-ancestry is a total order on the set of k'-ancestors of each column for each k', and that both the first column in B_n and the k-parent of the i-th column in B_n are k'-ancestors of the i-th column in B_n for every $k' < k$, and the latter is before the former, we get that the k-parent of the i-th column in B_n is also a k-ancestor of the first column in B_n.

If $k \geq m_0$ (using variable names from Definition 1.1), then this is already a contradiction, because the m_0-parent of the first column in B_n is easily seen to be in G. Otherwise, let $n' < n$ be the natural number such that the k-parent of the i-th column in B_n is in $B_{n'}$. From repeated applications of (iii) for k, which we have already proven, we get that the first column in $B_{n'}$ is a k-ancestor of the first column in B_n, and therefore by k-ancestry being a total order on the set of k-ancestors of the first column in B_n, we get that the first column in $B_{n'}$ is a k-ancestor of the k-parent of the i-th column in B_n, and thus is also a k-ancestor of the i-th column in B_n. This, however, by more repeated applications of (iii), implies that the first column in B_0 is a k-ancestor of the i-th column in B_n, which is in contradiction with the fact that the k-th element of the first column in B_0 is at least as large as the k-th element of the i-th column in B_n.

Now, for (iv), we're left with the case that for some $k' < k$, the first column in B_n is not a k'-ancestor of the i-th column in B_n. However, if we choose a specific such $k' < k$, then by (iv) for k' we get that the k'-parent of every k'-ancestor in B_n of the i-th column in B_n is either in B_n or in G, from which it follows that all k'-ancestors of the i-th column in B_n are either in B_n or in G, and that includes the k-parent of the i-th column in B_n, proving (iv) for k.

With (iv) proven for k, the proof of (ii) and (iii) for k can also be easily modified for relations between G and B_0 and between G and B_n, with all nontrivialities accounted for by (iv) for k: either the k-th element of the j-th column in B_0 ascends and the j-th column in B_n trivially has the first column in B_0 as a k-ancestor, thus the k-ancestors in G are simply the k-ancestors of that (by totality of k-ancestry on the set of k-ancestors of the j-th column in B_n), or the k-th element of the j-th column in B_0 doesn't ascend and B_n's copy C_n of the (j-th column in B_0)'s first non-strict k-ancestor C_0 in B_0 is easily seen to have the same k-parent as C_0, because the k-parents of C_0 and C_n are both in G, the k-th elements of C_0 and C_n are equal, and the sets of k'-ancestors of C_0 and of C_n are the same for every $k' < k$ by (i) for k', thus the k-ancestors in G of both C_0 and C_n are that k-parent and its k-ancestors. Therefore (i) also holds for k.

Finally, (v) can be proven for k by simply letting $\{n_2, n_3\} = \{n_1, n_1 + 1\}$ (the two options together give the proofs of both directions of (v)), and noticing that if the j-th column in B_{n_2} has a k-ancestor in B_{n_0}, then the first column in B_{n_2} must also be its k-ancestor (similarly to the reasoning near the end of the previous paragraph - using (iv) for the last k-ancestor in B_{n_2} of the j-th column in B_{n_2}), and therefore by totality of k-ancestry on the set of k-ancestors of the j-th column in B_{n_2}, the i-th column in B_{n_0} is a k-ancestor of the first column in B_{n_2}. Then if $k \geq m_0$, we get a contradiction, because the k-ancestors of the j-th column in B_{n_2} are all in B_{n_2} or G, as we've already proven, so it must be that $k < m_0$. In that case, by application of (iii) and either (depending on $n_2 - n_1$) another application of the totality of k-ancestry on the set of k-ancestors of the j-th column in B_{n_2} or an application of transitivity of k-ancestry, we get that the i-th column in B_{n_0} is also a k-ancestor of the first column in B_{n_3}, and finally by an application of (ii) for k, we get that the first column in B_{n_3} is a k-ancestor of the j-th column in B_{n_3}, so by transitivity of k-ancestry, the i-th column in B_{n_0} is a k-ancestor of the j-th column in B_{n_3}, which concludes the proof of (v) for k.

By induction, all 5 statements in the lemma always hold. □

We will abbreviate $\langle L_\alpha, \in\rangle \preceq_{\Sigma_{n+1}} \langle L_\beta, \in\rangle$ as $\alpha \leq_n \beta$, and similarly for the strict versions of these relations. Here, L_α is the α-th level of the constructible hierarchy, and $M \preceq_{\Sigma_n} N$ means that M is a Σ_n-elementary substructure of N.

Let σ be the smallest ordinal α such that there exists an ordinal β with $\forall n \in \mathbb{N}(\alpha <_n \beta)$.

Lemma 2.6. *For all $\alpha, \beta \in \sigma$ and $n \in \mathbb{N}$, if $\omega < \alpha <_n \beta$, then for all finite $X, Y \subseteq Ord$ such that $\gamma < \alpha \leq \delta < \beta$ for all $\gamma \in X$ and $\delta \in Y$, there exists a finite $Y' \subseteq Ord$ and a bijection $f : Y \to Y'$ such that for all $\gamma \in X$, all $\delta_0, \delta_1 \in Y$, all $k \in \mathbb{N}$ and all $m < n$:*

- $\gamma < f(\delta_0) < \alpha$
- $\gamma <_k \delta_0 \Rightarrow \gamma <_k f(\delta_0)$
- $\delta_0 < \delta_1 \Rightarrow f(\delta_0) < f(\delta_1)$
- $\delta_0 <_k \delta_1 \Rightarrow f(\delta_0) <_k f(\delta_1)$
- $\delta_0 <_m \beta \Rightarrow f(\delta_0) <_m \alpha$

We can prove this by constructing a Σ_{n+1} formula that, when interpreted in L_β, asserts all the true instances of the statements on the left side of the implications, and when interpreted in L_α, asserts the corresponding instances of the statements on the right side of the implications. One small issue is the first assertion, which is unconditional. However, the $f(\delta_0) < \alpha$ part is simply asserting that $f(\delta_0)$ exists in L_α, which will be done by existentially quantifying the variable, and since $\gamma < \alpha \leq \delta_0$ is necessarily true, $\gamma < f(\delta_0)$ is equivalent to $\gamma < \delta_0 \Rightarrow \gamma < f(\delta_0)$, which is a conditional statement.

Proof. We construct a formula with parameters $\gamma_0, \gamma_1, ..., \gamma_{|X|-1}$, which are all the elements of X. Since they're ordinals smaller than α, they are in L_α, therefore we can use them as parameters in a formula that we want to reflect using the stability relation between α and β.

Let $\varphi_0(\eta, \xi)$ be a formula asserting $\eta < \xi$. Let $\varphi_1(\eta, \xi, k)$ be a formula asserting $\eta <_k \xi$. Let $\varphi_2(\eta, k)$ be a formula asserting $\eta <_k Ord$, i.e. $\langle L_\eta, \in \rangle \prec_{\Sigma_{k+1}} \langle L, \in \rangle$.

φ_0 is clearly Σ_0, as it is simply the atomic formula $\eta \in \xi$. This means it is Σ_{n+1}. φ_1 only needs to assert the existence of L_ξ, the defining characteristics of it (specifically that it is a level of L, which is simply $V = L$ relativized to it, and that the ordinals in it are precisely the elements of ξ, which is trivially Σ_0), and then it needs to assert that $\varphi_2(\eta, k)$ relativized to L_ξ holds. The relativization of a first-order formula to a set is trivially always Σ_0. Assuming φ_2 is first-order, the only unbounded quantifier in φ_1 is the one existentially quantifying L_ξ. Then φ_1 is Σ_1, which means it's also Σ_{n+1}. Finally, $\varphi_2(\eta, k)$ is Π_{k+1}, as shown in [4] (Theorem 1.8), which means it is Σ_{k+2}, and therefore first-order. In all non-relativized uses of φ_2, we will require $k < n$, which means $k + 2 \leq n + 1$, thus it is Σ_{n+1}.

X and Y are finite, and all of their elements are smaller than σ so for each $\eta, \xi \in X \cup Y$, there are only finitely many k for which $\varphi_1(\eta, \xi, k)$ is true. Then there are finitely many instances of $\varphi_0(\gamma_i, \delta_j)$, $\varphi_1(\gamma_i, \delta_j, k)$, $\varphi_0(\delta_i, \delta_j)$, $\varphi_1(\delta_i, \delta_j, k)$ and $\varphi_2(\delta_i, m)$ with $k \in \mathbb{N}$ and $m < n$, which are true when each δ_i is interpreted as the i-th element of Y. So their conjunction φ is a conjunction of finitely many Σ_{n+1} formulae, therefore it is itself a Σ_{n+1} formula. Then we only need a Σ_{n+1} formula ψ asserting that all the δ_i are ordinals, which is trivial.

Now, the formula $\psi \wedge \varphi$ is Σ_{n+1}, therefore the formula $\exists \delta_0, \delta_1, ..., \delta_{|Y|-1}(\psi \wedge \varphi)$ is also Σ_{n+1}. In L_β, the witnesses of that existential quantifier are the elements of Y, therefore the formula is true in L_β. Then by $\alpha <_n \beta$, it must be true in L_α, and since it encodes all the relations between elements of X, elements of Y and β that need to be reflected to relations between elements of X, elements of Y' and α, the witnesses of that formula in L_α form a set Y' that, together with the unique order isomorphism $f : Y \to Y'$, satisfies the conditions in the lemma. □

Note that this reflection is similar to reflection in Patterns of Resemblance, and those could be used too. However, the author is not as experienced in working with Patterns of Resemblance, so it was easier to use stability.

Theorem 2.7. *BMS is well-ordered.*

Proof. We will define a function $o : BMS \to Ord$ in the following way. Consider an array A with length n. A stable representation of A is a function $f : n \to Ord$ such that for all $i, j < n$, $i < j \Rightarrow f(i) < f(j)$ and for all m, if the i-th column of A is an m-ancestor of the j-th column of A, then $f(i) <_m f(j)$. Let $o(A)$ be the minimal $\alpha \in Ord$ such that for some stable representation f of A, all outputs of f are smaller than α.

This proof is similar to the proof of Lemma 2.3 - we prove by induction on the number of expansions needed to reach an array, that o is defined and order-preserving on all of BMS, by starting from X_0 and proving that if it holds for some Z, then it holds for $Z \cup \{A[n] : A \in Z \wedge n \in \mathbb{N}\}$, and using the fact that every pair A, A' of arrays is reached after finitely many applications of this induction step.

Of course, $o(A)$ is defined for $A \in X_0 = \{((\underbrace{0, 0, ..., 0, 0}_{n}), (\underbrace{1, 1, ..., 1, 1}_{n})) : n \in \mathbb{N}\}$, and it is easy to see that $o(((\underbrace{0, 0, ..., 0, 0}_{n}), (\underbrace{1, 1, ..., 1, 1}_{n}))) < o(((\underbrace{0, 0, ..., 0, 0}_{m}), (\underbrace{1, 1, ..., 1, 1}_{m})))$ iff $n < m$, thus o is also order-preserving in this set.

Let Z be a set of arrays, on which o is order-preserving and defined for all of Z's elements. Let $A \in Z$. If A is empty, then trivially for all $n \in \mathbb{N}$, $o(A[n]) = o(A) \leq o(A)$, and thus $o(A[n])$ is also defined and order is preserved. Otherwise, let f be a stable representation of A whose outputs are all smaller than $o(A)$. We can then recursively define stable representations of $A[n]$ in the following way.

Let l_n be the length of $A[n]$ for all $n \in \mathbb{N}$. A stable representation f_0 of $A[0]$ is simply f restricted to l_0. Using variable names from the definition of BMS, this representation trivially maps indices (in $A[0]$) of columns in B_0 to the ordinals to which f maps indices (in A) of columns in B_0.

Let f_n be a stable representation of $A[n]$ that maps indices (in $A[n]$) of columns in B_n to the ordinals to which f maps indices (in A) of columns in B_0. Then using the reflection property from Lemma 2.6 with α being the ordinal to which f_n maps the index of the first column in B_n, β being the ordinal to which f maps the last column of A, X being the set of ordinals to which f_n maps indices of columns before B_n, and Y being the set of ordinals to which f_n maps indices of columns in B_n (or to which f maps indices of columns in B_0), we get a set Y' of ordinals due to $\alpha <_{m_0} \beta$. We can then define f_{n+1} by making it the same as f_n for indices of columns before B_n, mapping indices of columns in B_n to the elements of Y', and mapping indices (in $A[n+1]$) of columns in B_{n+1} to the elements of Y.

It follows from Lemma 2.5 that f_{n+1} is a stable representation of $A[n+1]$. Then it's trivially a stable representation of $A[n+1]$ that maps indices of columns in B_{n+1} to the ordinals to which f maps indices of columns in B_0, therefore by induction, for all $m \in \mathbb{N}$, there is a stable representation of $A[m]$ that maps indices of columns in B_m to the ordinals to which f maps indices of columns in B_0. Since all these ordinals are smaller than β, $o(A[m])$ is defined and is at most

β, and since β is an output of f, it is smaller than $o(A)$, so $o(A[m]) < o(A)$, which means o is defined and order-preserving (due to the order being originally defined only by comparing an array with its expansions) on $Z \cup \{A[m] : A \in Z \wedge m \in \mathbb{N}\}$.

Now, similarly to the proof of Lemma 2.3, with $X_0 = \{((\underbrace{0, 0, ..., 0, 0}_{n}), (\underbrace{1, 1, ..., 1, 1}_{n})) : n \in \mathbb{N}\}$, we conclude that o is defined and order-preserving on $X_0 \cup \{A[n_0] : A \in X_0 \wedge n_0 \in \mathbb{N}\} \cup \{A[n_0][n_1] : A \in X_0 \wedge n_0, n_1 \in \mathbb{N}\} \cup ... \cup \{A[n_0][n_1]...[n_m] : A \in X_0 \wedge n_0, n_1, ..., n_m \in \mathbb{N}\}$ for each $m \in \mathbb{N}$, and since all $A, A' \in BMS$ are also in this set for some m, o is defined for them their order is preserved by o, so o is defined and order-preserving on all of BMS.

Then if BMS was not well-ordered, there would be an infinite descending sequence in BMS, which would get mapped to an infinite descending sequence of ordinals by o, and that cannot exist by the definition of ordinals. Therefore BMS is well-ordered. □

3 Future research

We hope to use BMS in ordinal analysis, first using it to rewrite analyses of theories that have already been analyzed by other means, and then analyzing even stronger theories, ideally up to full second-order arithmetic if the order type of BMS is large enough for that.

Once this approach proves viable, we also plan to continue proving the well-orderedness of similar notation systems with larger order types, such as Y sequence [6] and its extension $\omega - Y$ sequence [5].

Another challenge that is relevant is the task to find a "self-contained" proof of well-orderedness of BMS (that is, a proof using only concepts that are directly related to BMS, which excludes stability and ordinal collapsing functions), as this would simplify the translation of the proof to theories that only deal with basic structures, such as third-order arithmetics.

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