Lecture 8: Graph Algorithms (II)

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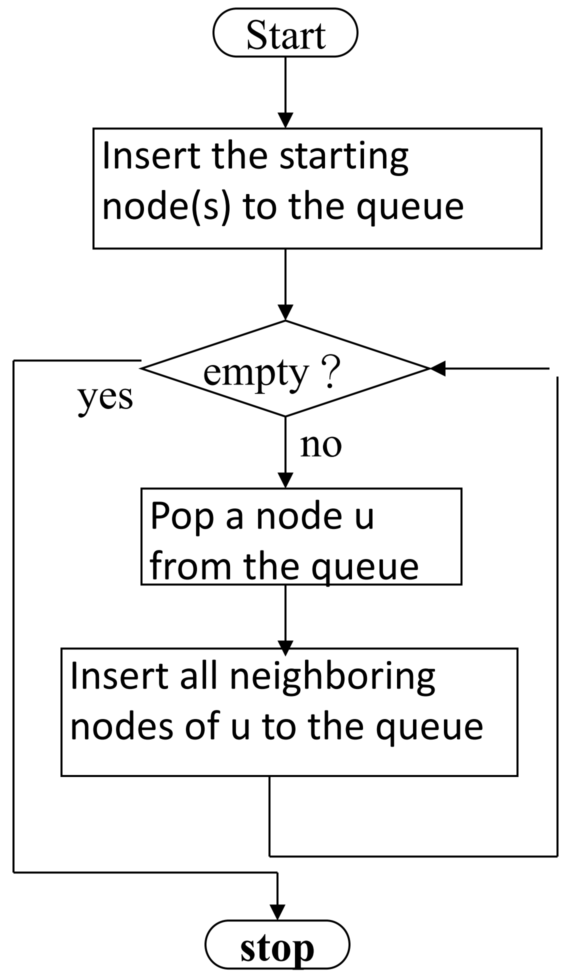
Graph Traversal

- **Depth First Search (DFS)**
  - Stack
  - The algorithm starts at the root node (selecting some arbitrary node as the root node in the case of a graph) and explores as far as possible along each branch before backtracking.
- **Breadth First Search (BFS)**
  - Queue
  - The algorithm starts at the root node, and explores all of the neighbor nodes at the present depth prior to moving on to the nodes at the next depth level.

DFS Algorithm

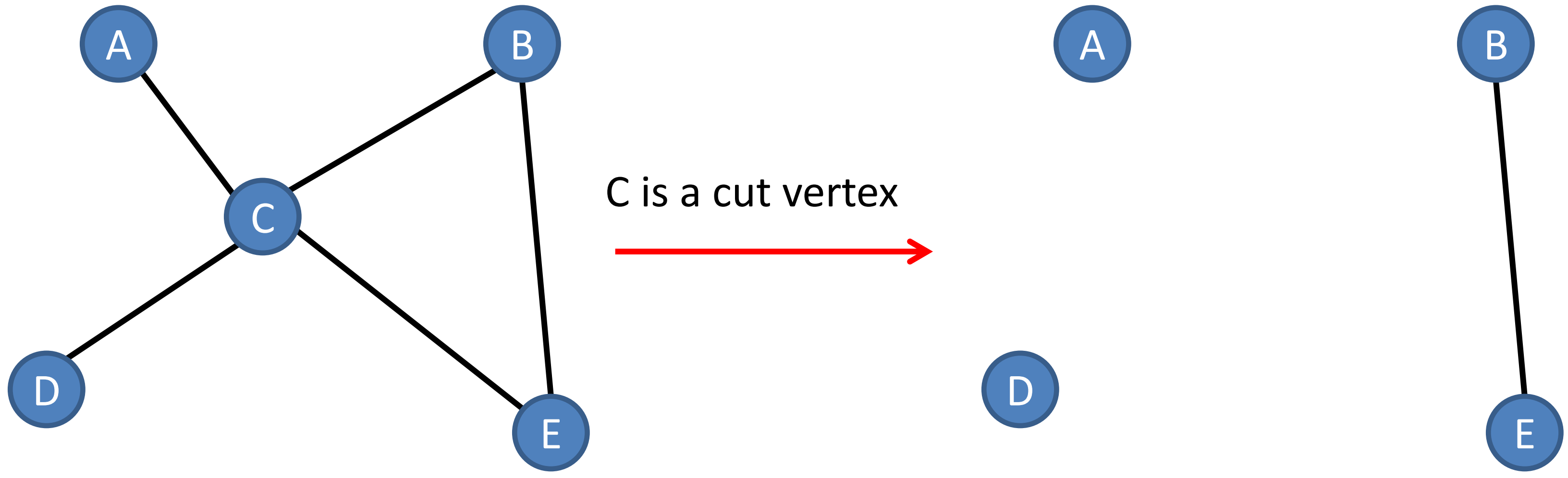
```
procedure DFS(G, v) is
    label v as discovered
    for all directed edges from v to w that are in G.adjacentEdges(v) do
        if vertex w is not labeled as discovered then
            recursively call DFS(G, w)
```

BFS Algorithm



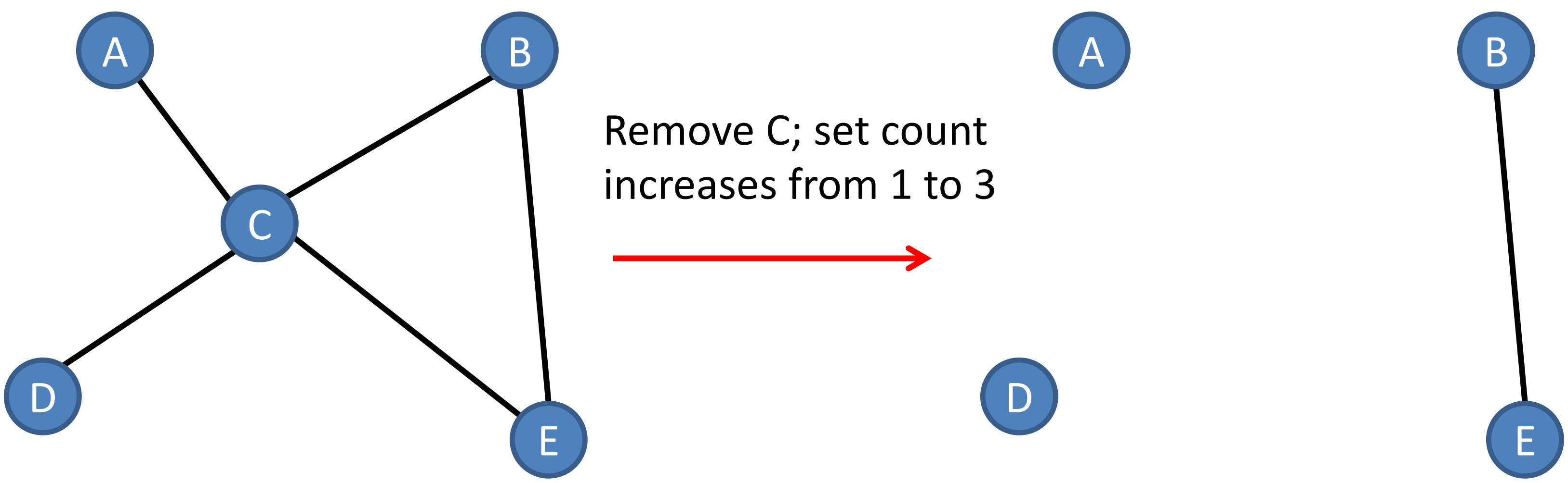
Cut Vertex (Articulation Point)

- **Formal definition**
  - A cut vertex or articulation point is a vertex in a graph such that removal of the vertex causes an increase in the number of connected components.
  - If the graph was connected before the removal of the vertex, it will be disconnected afterwards.



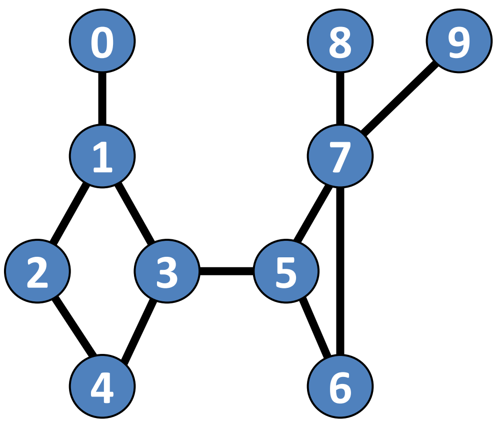
How to Find Cut Vertices?

- **Brute force?**
  - Enumerate all vertices O(N)
    - Remove the vertex from the graph
    - Perform union-and-find algorithms to find the number of sets
    - If the number of disjoint sets increases, the vertex is a cut
  - Total time complexity is $O(N^2 \log N)$
    - Each union-and-find takes O(NlogN), needs N times



DFS can Do This for us More Efficiently

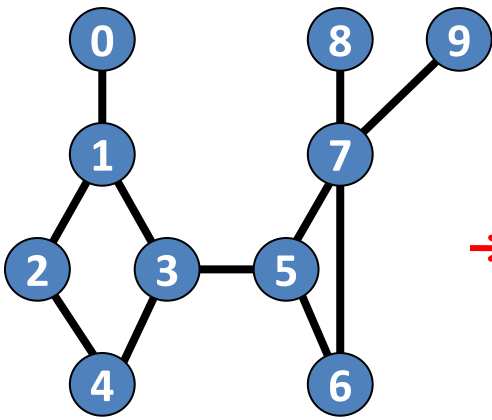
- **We have two edge types during DFS**
  - Forward edge u→v, v is not visited
  - Backward edge u→v, v is visited (except parent)



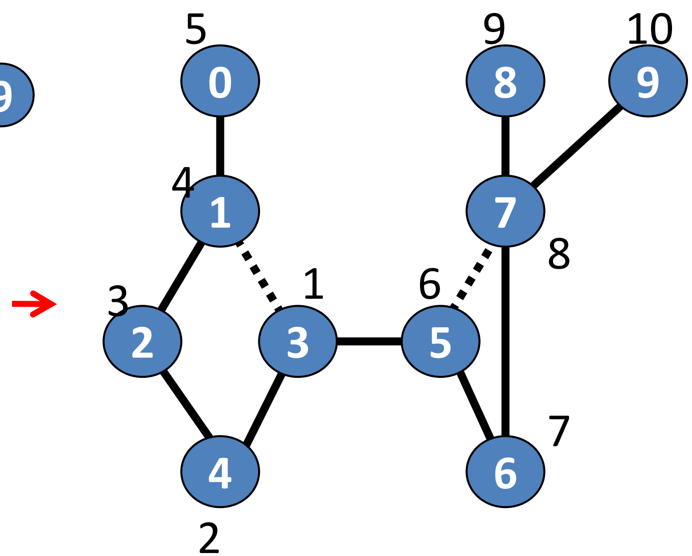
Let's start from 3

DFS can Do this for us

- ❑ **We have two edge type during DFS**
  - ❑ Forward edge u→v, v is not visited
  - ❑ Backward edge u→v, v is visited



Let's start from 3



Lebel the order of traversal
in a linear array "dfn"

DFS can Do this for us

- **We have two edge type during DFS**
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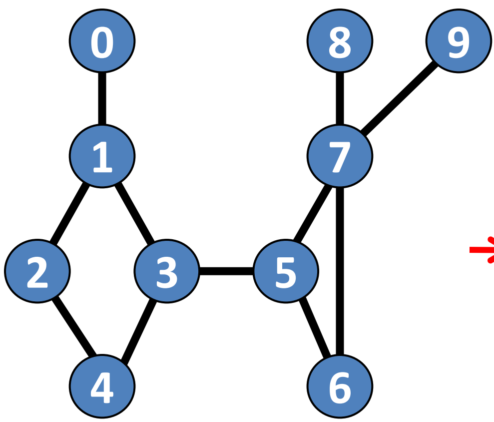
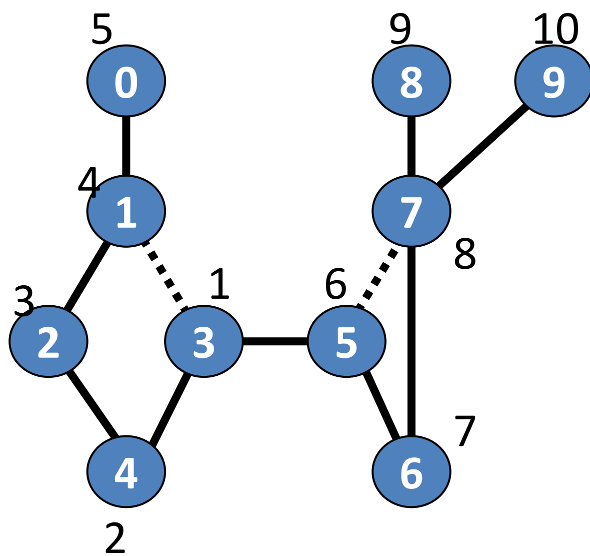
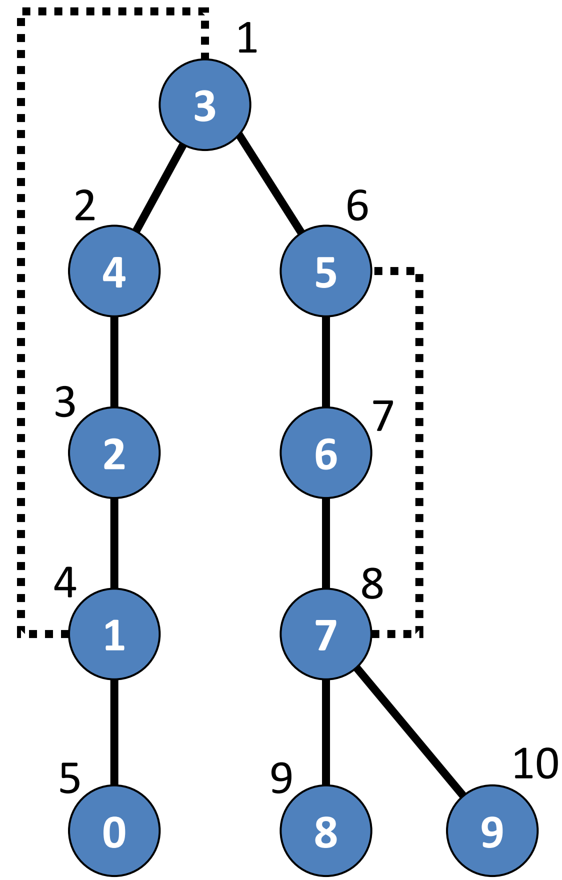


Let's start from 3



Lebel the order of traversal in a linear array "dfn"



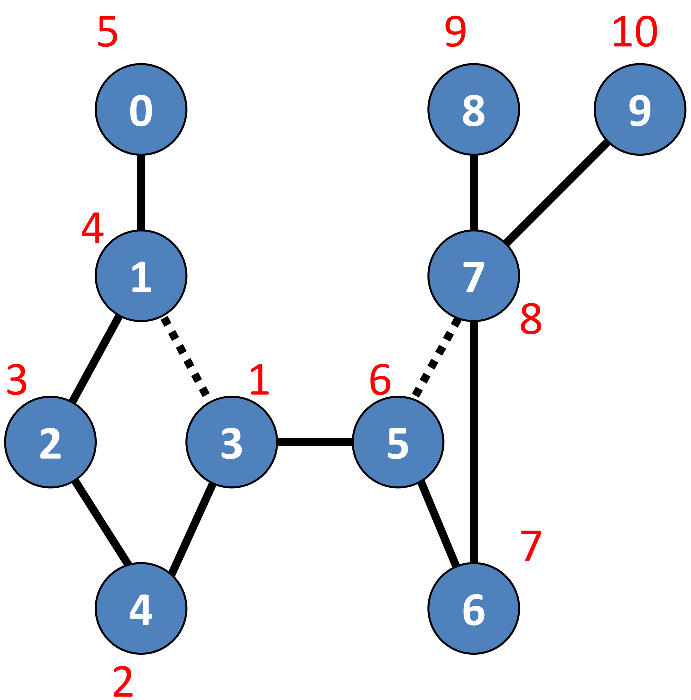
DFS gives us a spanning tree order of vertices

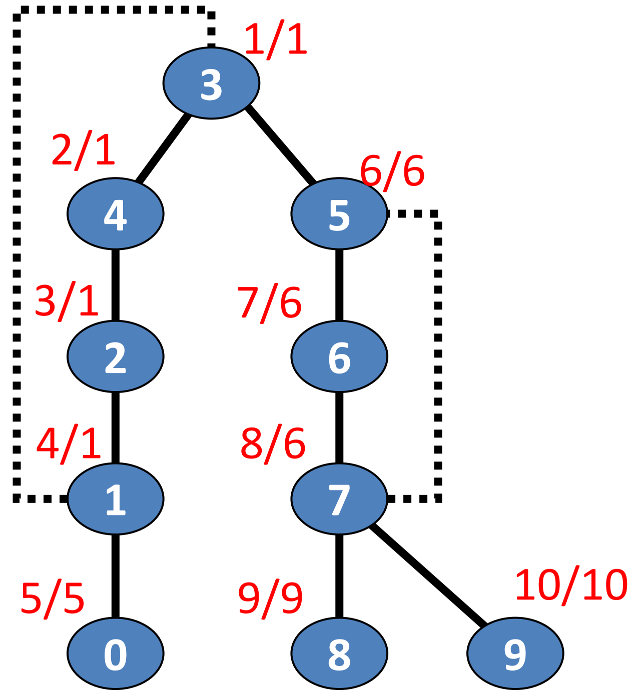
Cut Vertex Property

- **Observation**
  - If root has two children, => the root is a cut vertex
  - If a vertex u has a child v such that v can’t go back to u’s parent => u is a cut vertex
    - Assume no duplicate edges between vertices
- **Let’s quantify this**
  - low[u]: the minimum dfn value u can reach
    - min{ low[u], low[v] }, foreach edge u→v

Example of dfn[i] and low[i]

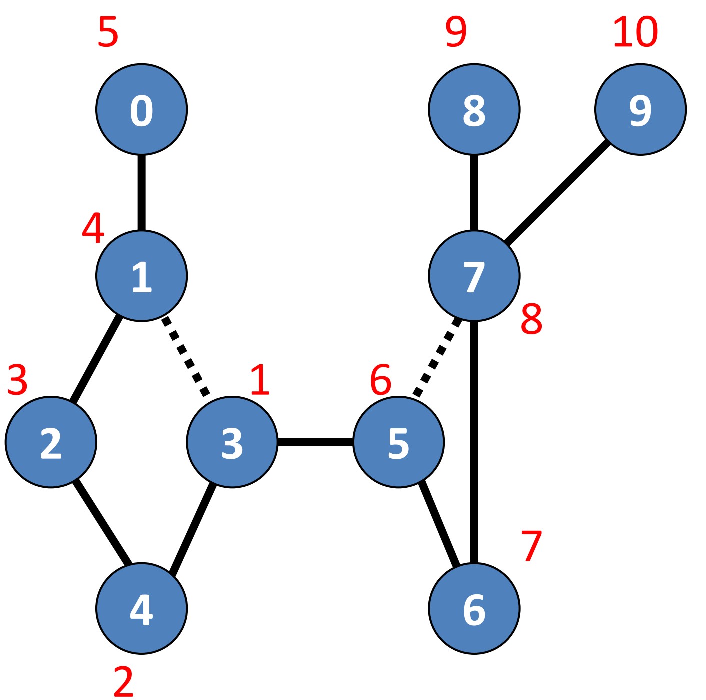
Vertex	0	1	2	3	4	5	6	7	8	9
dfn	5	4	3	1	2	6	7	8	9	10
low	5	1	1	1	1	6	6	6	9	10

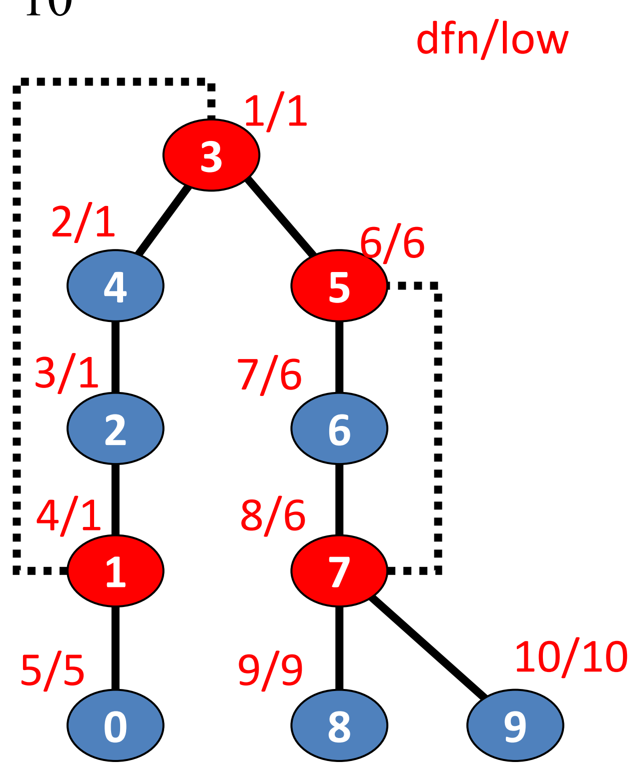




Cut Vertices Identified

Vertex	0	1	2	3	4	5	6	7	8	9
dfn	5	4	3	1	2	6	7	8	9	10
low	5	1	1	1	1	6	6	6	9	10





Algorithm

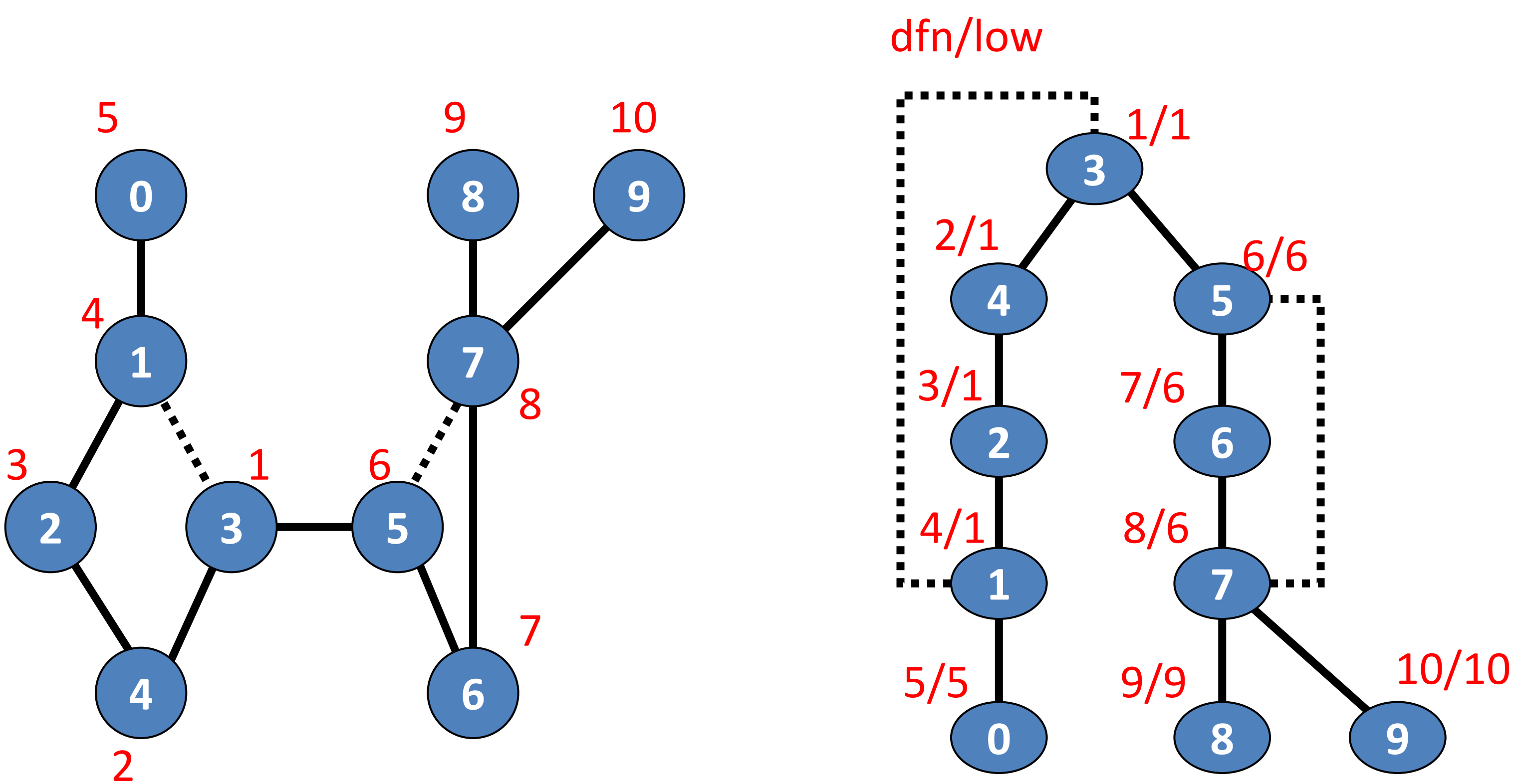
```
void dfs(int u)  {
  visited[u]=true;
  low[u]=dfn[u]=___________;
  int child=0;
  for(int i=0; i<adj[u].size(); i++) {
    int v=adj[u][i];
    if(visited[v]==false) {
      child++;
      parent[v]=u;
      dfs(v);
      low[u]=___________;
      if_______________________________________________________
        cut[u]=true;
    }
    else if(v!=parent[u]) {  // backward edge
      ____________________
    }
}
```

Initialization:
times=0, parent[i]=-1, cut[i]=0, visited[i]=0

- **Observation**
  - If root has two children, => the root is a cut vertex
  - If a vertex u has a child v such that v can't go back to u's parent => u is a cut vertex
    - Assume no duplicate edges between vertices
- **Let's quantify this**
  - low(u): the minimum dfn value u can reach
    - min{ low(u), low(v)}, foreach edge u→v

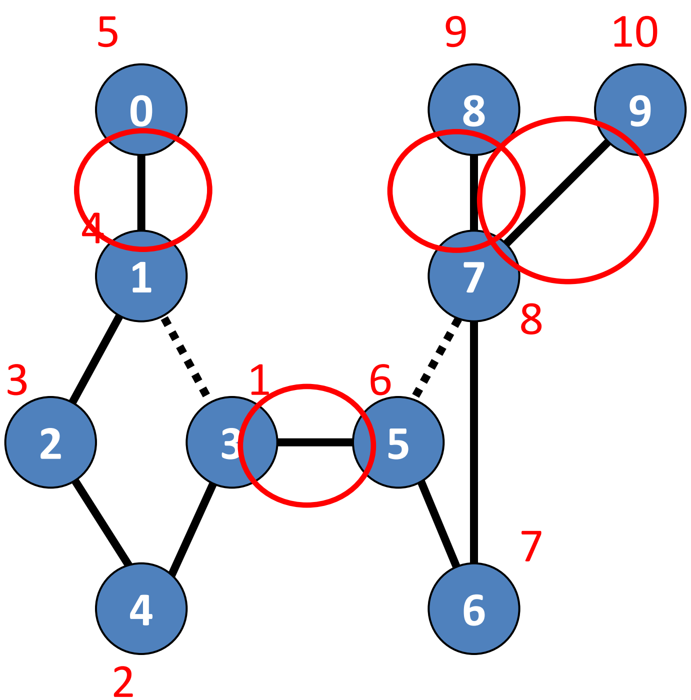
Cut Edge

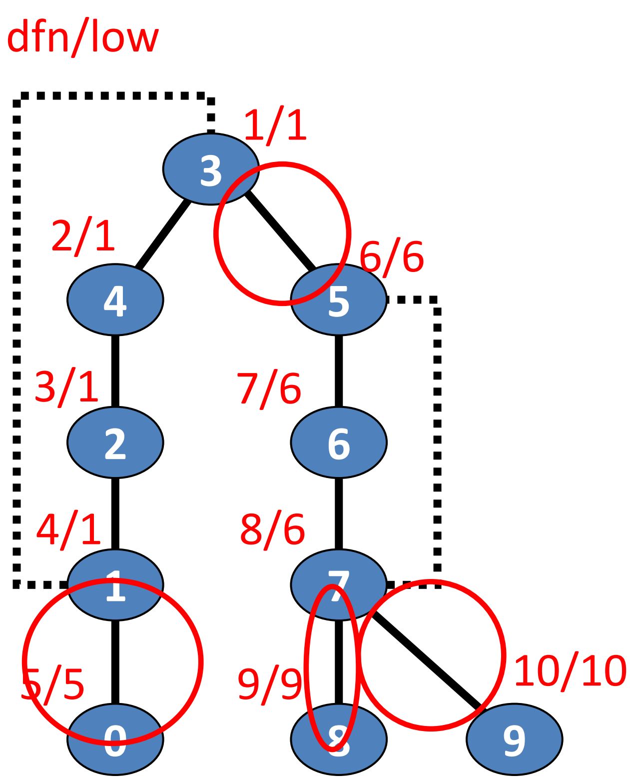
- ❑ **Observation**
  - ❑ For each edge u→v, if ___________ **=> cut edge**



Cut Edge

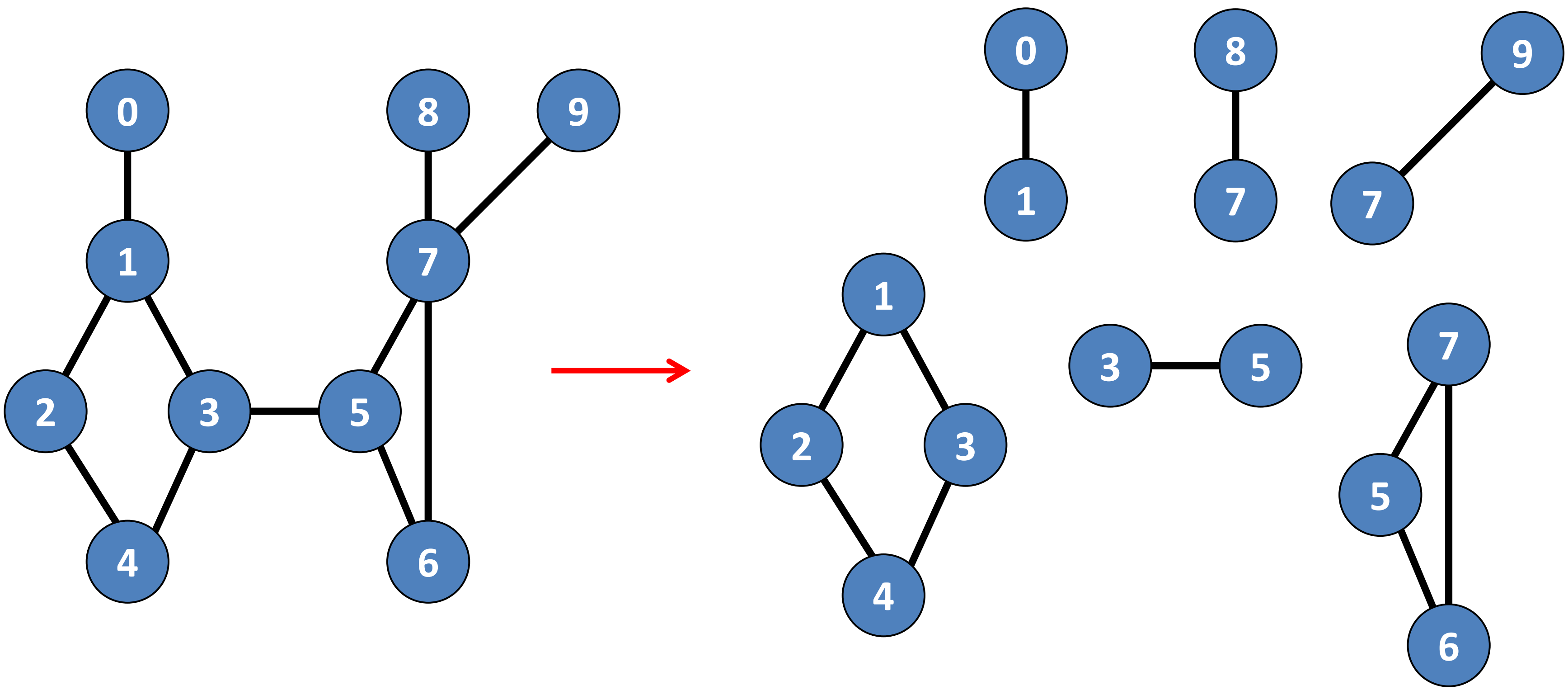
- We have four cut edges in this example
  - 0-1, 3-5, 8-7, 7-9





Bi-connected Components

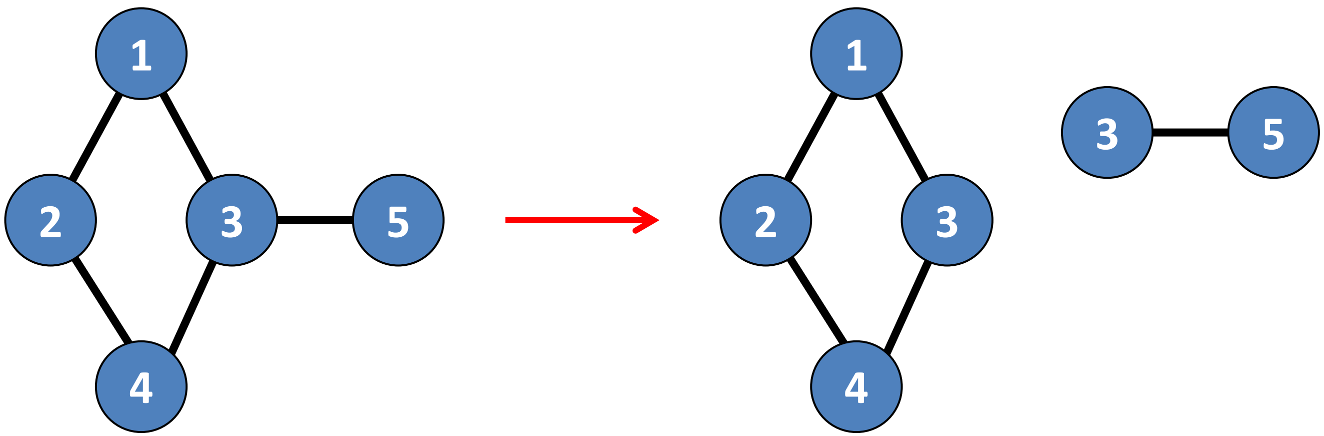
- ❑ A connected graph with NO cut vertices
- ❑ Find all biconnected <u>subgraphs</u> in a graph



six bi-connected components

Let’s Use a Simple Example first …

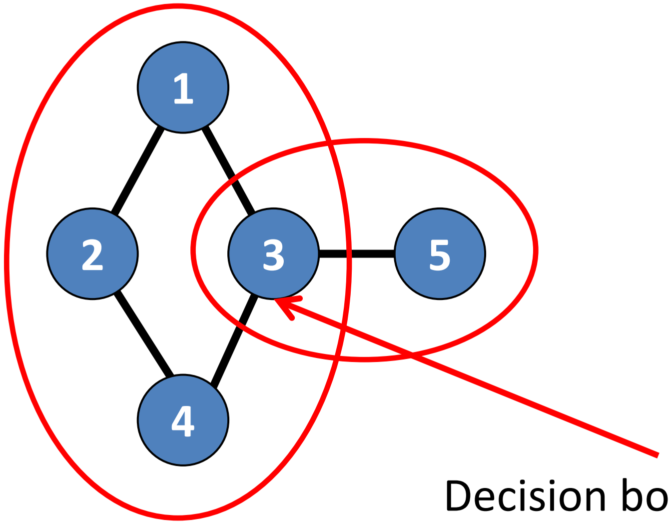
- **Apparently, we have two bi-connected components**
  - Cut vertices exist in the boundary between components
    - otherwise, it’s not connected by contradiction



two bi-connected components

Algorithms

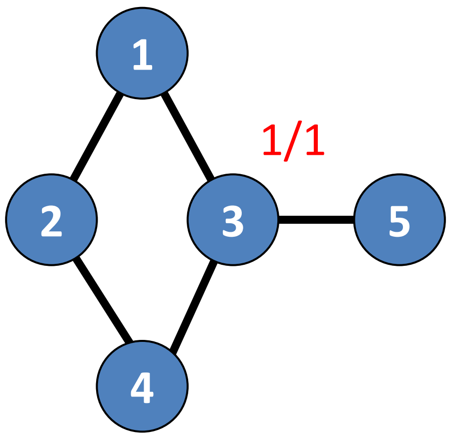
- **Maintain a stack of traversed edges so far**
  - This records the “trace” of our traversal
  - Edges between decision boundary form a component
- **When we find a cut vertex**
  - Pop all edges from the stack until the decision boundary



Decision boundary on
cut vertex

Example

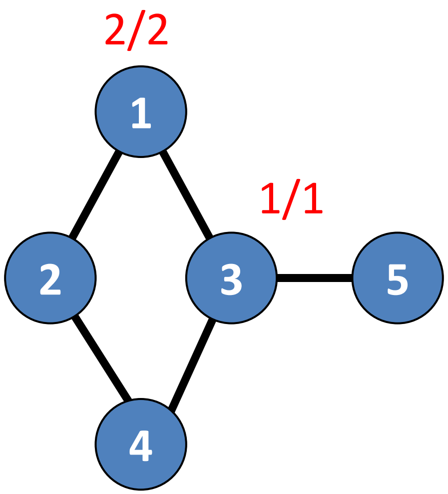
dfn/low



-1→3					

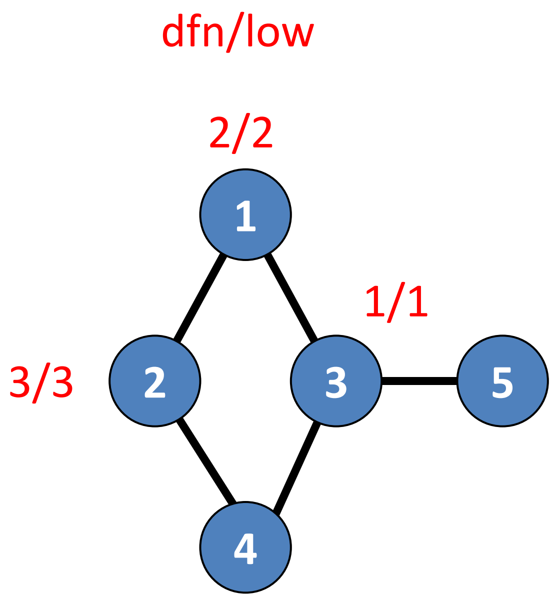
Example

dfn/low



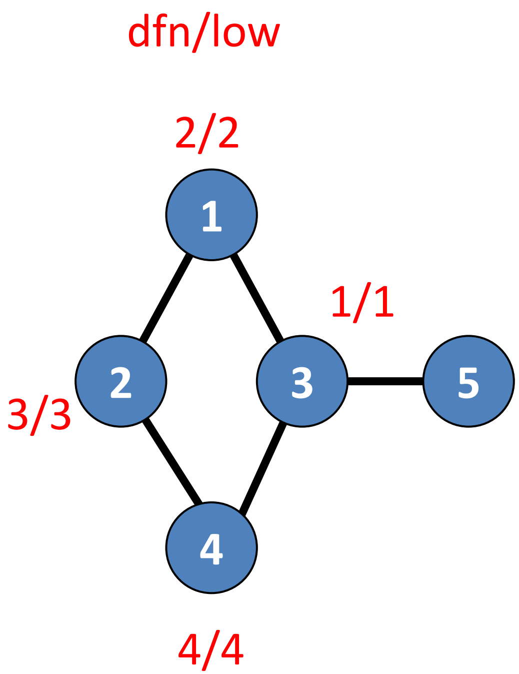
-1→3	3→1				

Example



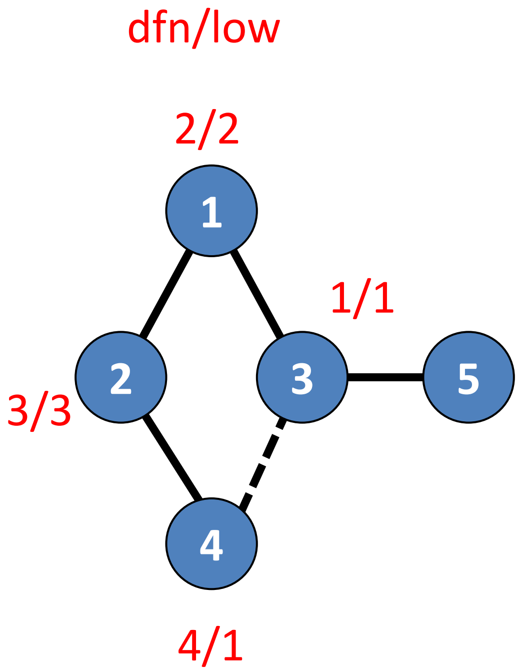
-1→3	3→1	1→2			

Example



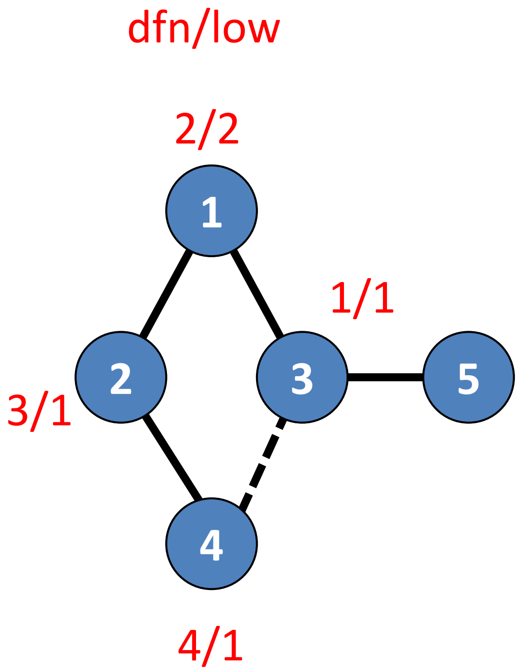
-1→3	3→1	1→2	2→4		

Example



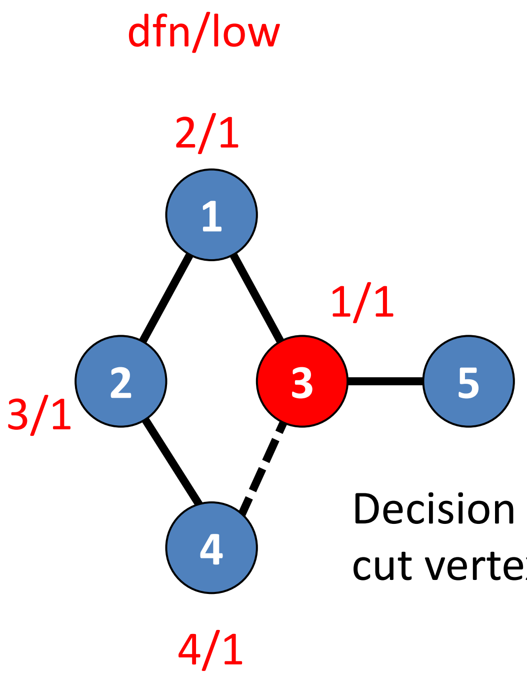
-1→3	3→1	1→2	2→4	4→3	

Example



-1→3	3→1	1→2	2→4	4→3	

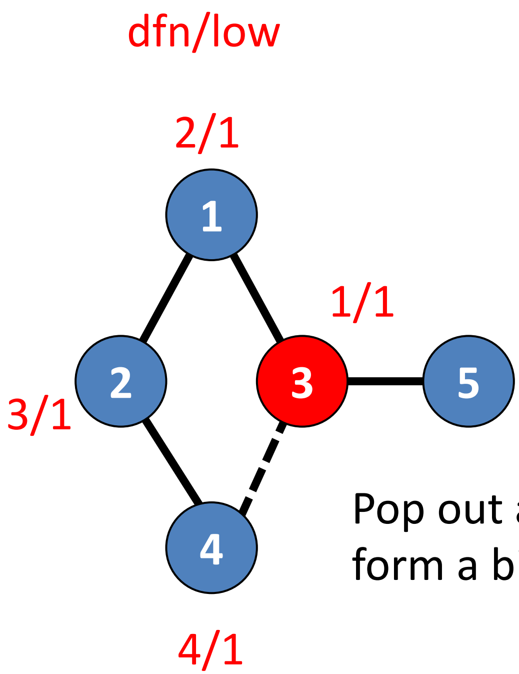
Example



-1→3	3→1	1→2	2→4	4→3	

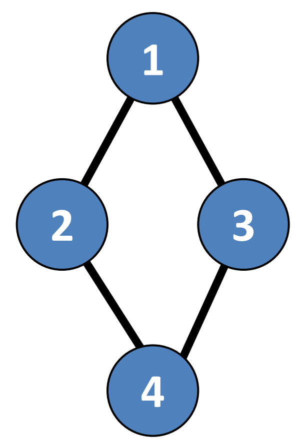
Decision boundary on cut vertex 3

Example



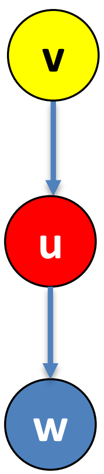
-1→3	3→1	1→2	2→4	4→3	

Pop out all edges until 3→1 to form a bi-connected component



Implementation Details

```
for (ptr = graph[u]; ptr; ptr = ptr→link) {
   w = ptr→vertex;
   if (v != w && dfn[w] < dfn[u])
     push(u,w); /* add edge to stack */
     if (dfn[w] <0) { /* w has not been visited */
        bicon(w,u);
        low[u] = MIN2(low[u],low[w]);
        if (low[w] >= dfn[u]) {
           printf("New biconnected component: ");
           do { /* delete edge from stack */
              pop(&x, &y);
              printf(" <%d,%d>",x,y);
           } while (!((x == u) && (y == w)));
           printf("\n");
        }
     }
     else if (w != v) low[u] = MIN2(low[u],dfn[w]);
}
```

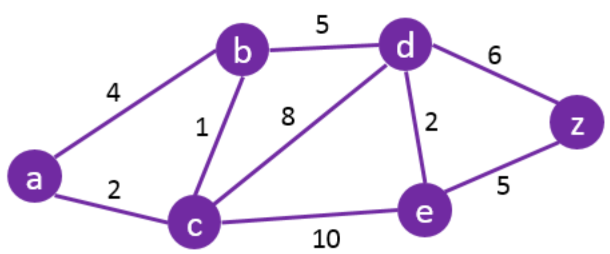


Cut vertex found!

Backward edge

Shortest Path Algorithms

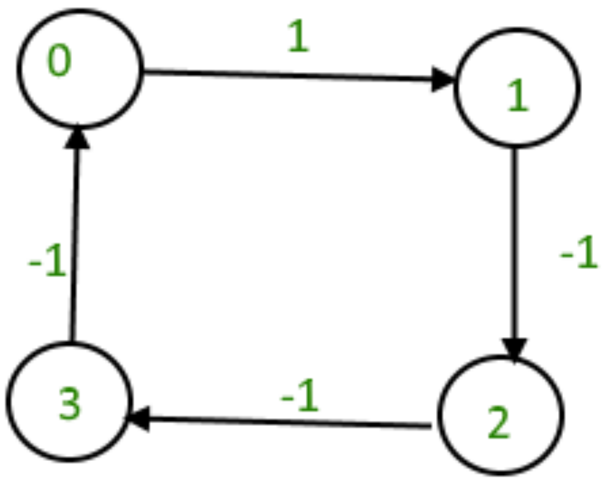
- **Single source all destinations**
  - Given a *weighted directed graph G*, find the minimum-weight path from a given source vertex s to all other vertices
  - "Shortest-path" = minimum summation weight
    - Weight of path is sum of edges
- **Tremendous applications**
  - Map: what is the shortest path from SLC to CA?
  - Circuit design: what is the minimum interconnect?



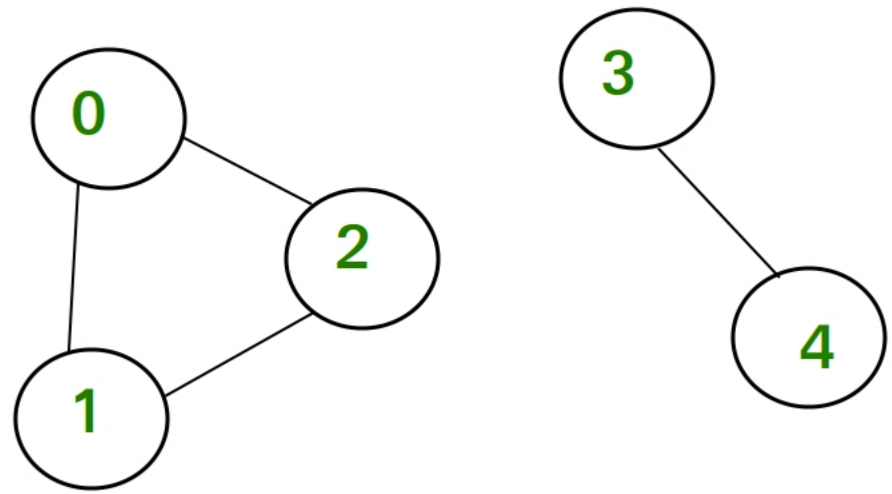
What is the shortest path from a to z?

Shortest Paths may Not Exist

- If graph contains non-reachable targets
- If graph contains negative cycles



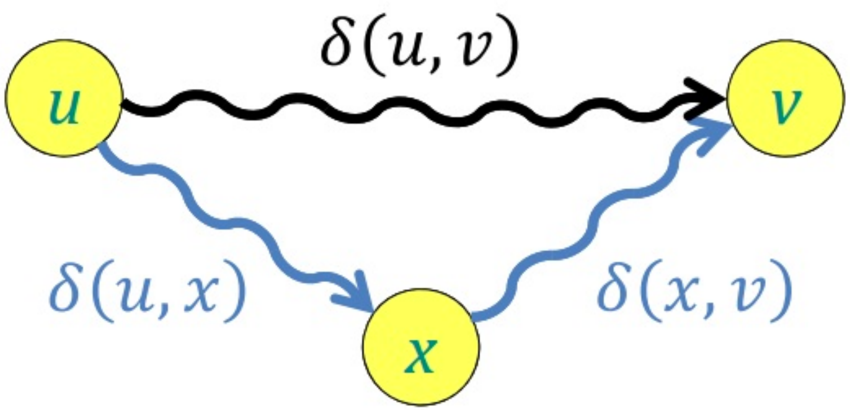
Negative cycle



No route from vertex 0 to vertex 4

Shortest Path Property

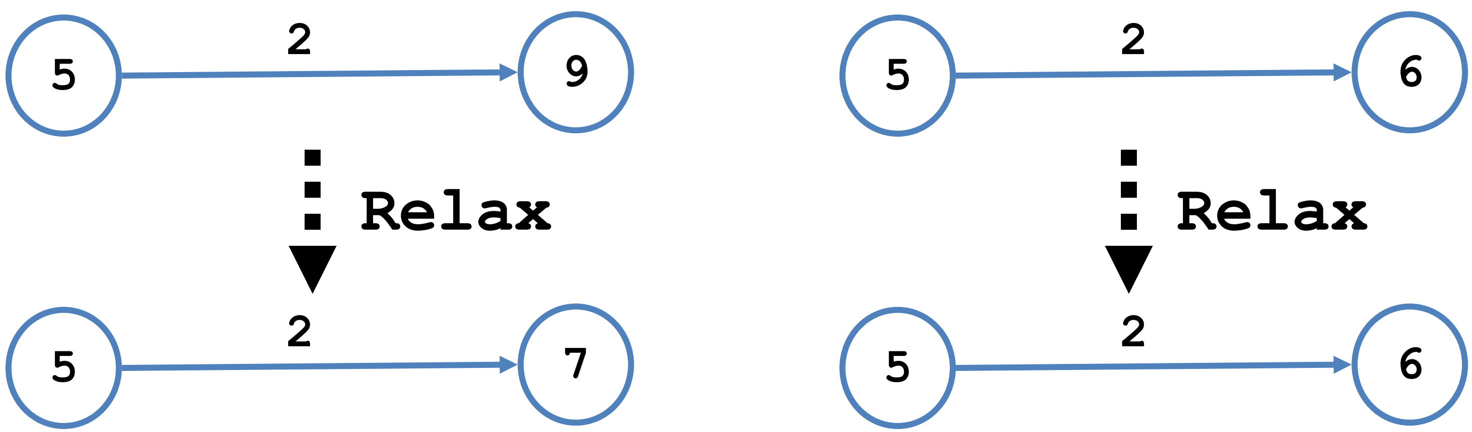
- ***Optimal substructure***
  - The shortest path consists of shortest subpaths
  - Easy to prove by contradiction
- **Let δ(u,v) be the the shortest path from u to v:**
  - Shortest paths satisfy the *triangle inequality*
    - $\delta(u,v) \leq \delta(u,x) + \delta(x,v)$



Algorithm

- ❑ **Key technique: *relaxation***
  - ❑ Maintain upper bound d[*v*] on δ(s,v):

```
Relax(u,v,w) {
   if (d[v] > d[u]+w) then d[v]=d[u]+w;
}
```



Bellman Ford

```
BellmanFord()
   for each v ∈ V
      d[v] = ∞;
   d[s] = 0;
   for i=1 to |V|-1
      for each edge (u,v) ∈ E
         Relax(u,v, w(u,v));
```

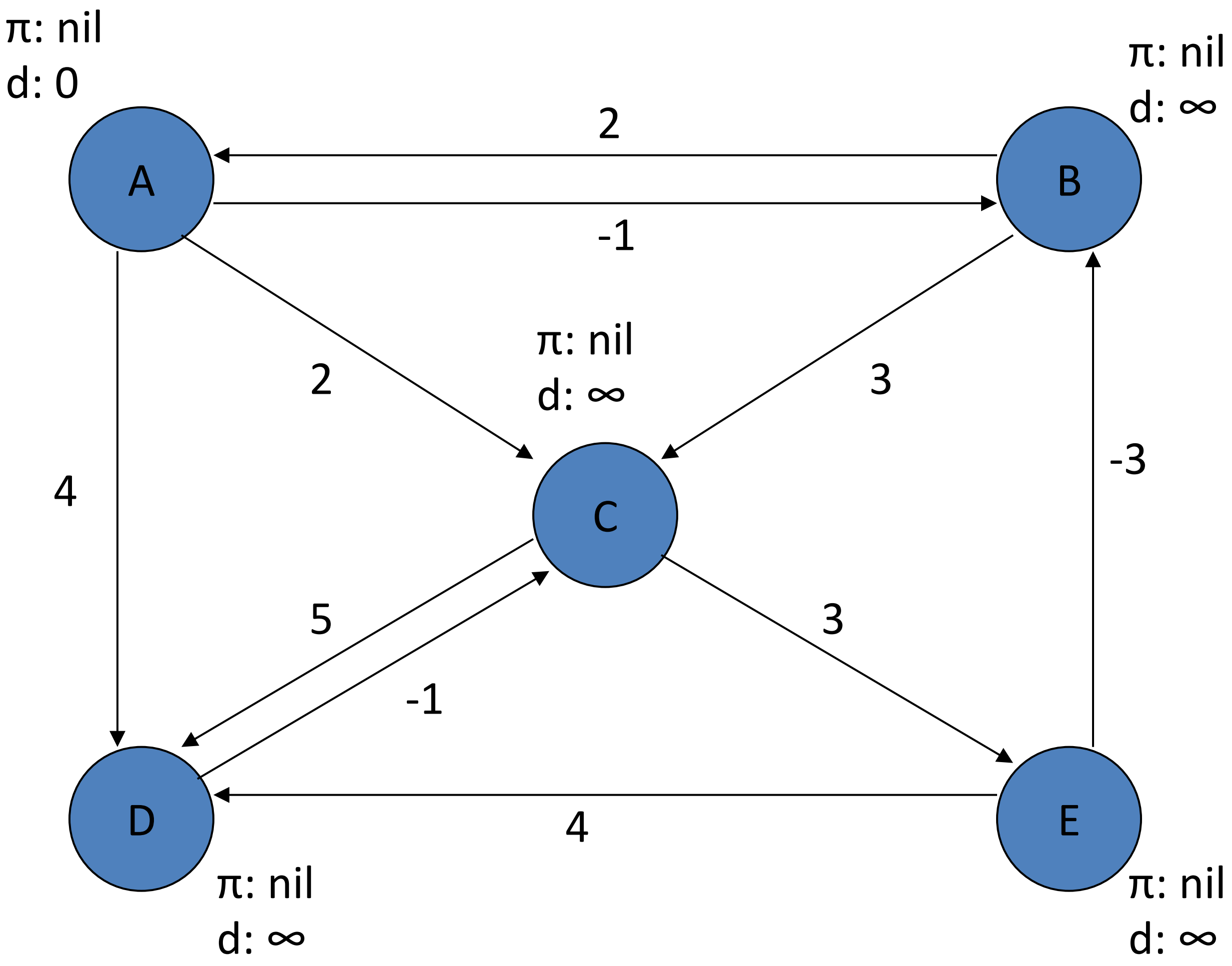
Initialize d[], which will converge to shortest-path value δ

Relaxation: Make |V|-1 passes, relax each edge

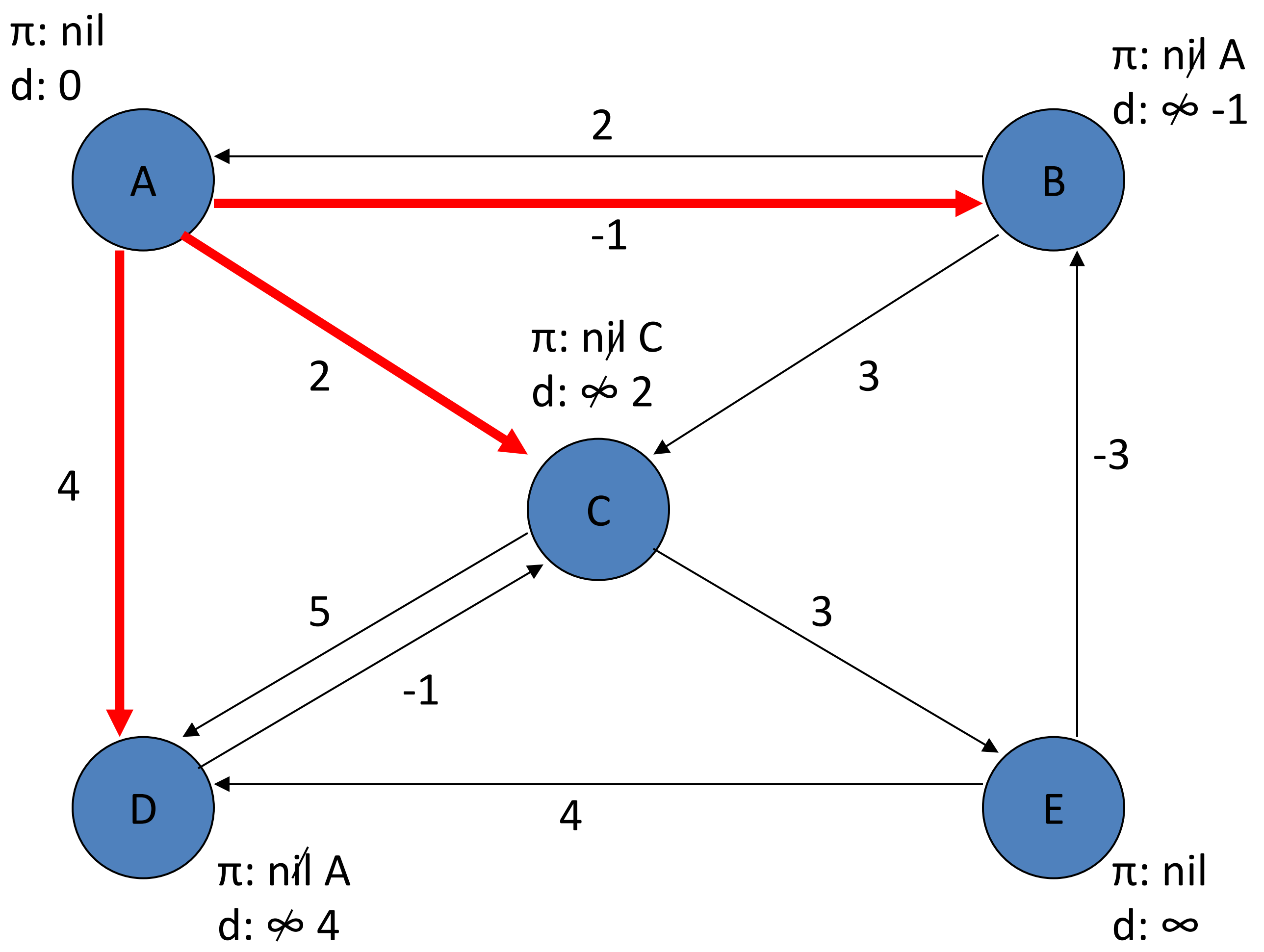
```
Relax(u,v,w): if (d[v] > d[u]+w) then d[v]=d[u]+w
```

- The simplest shortest path algorithm

Bellman Ford



Bellman Ford



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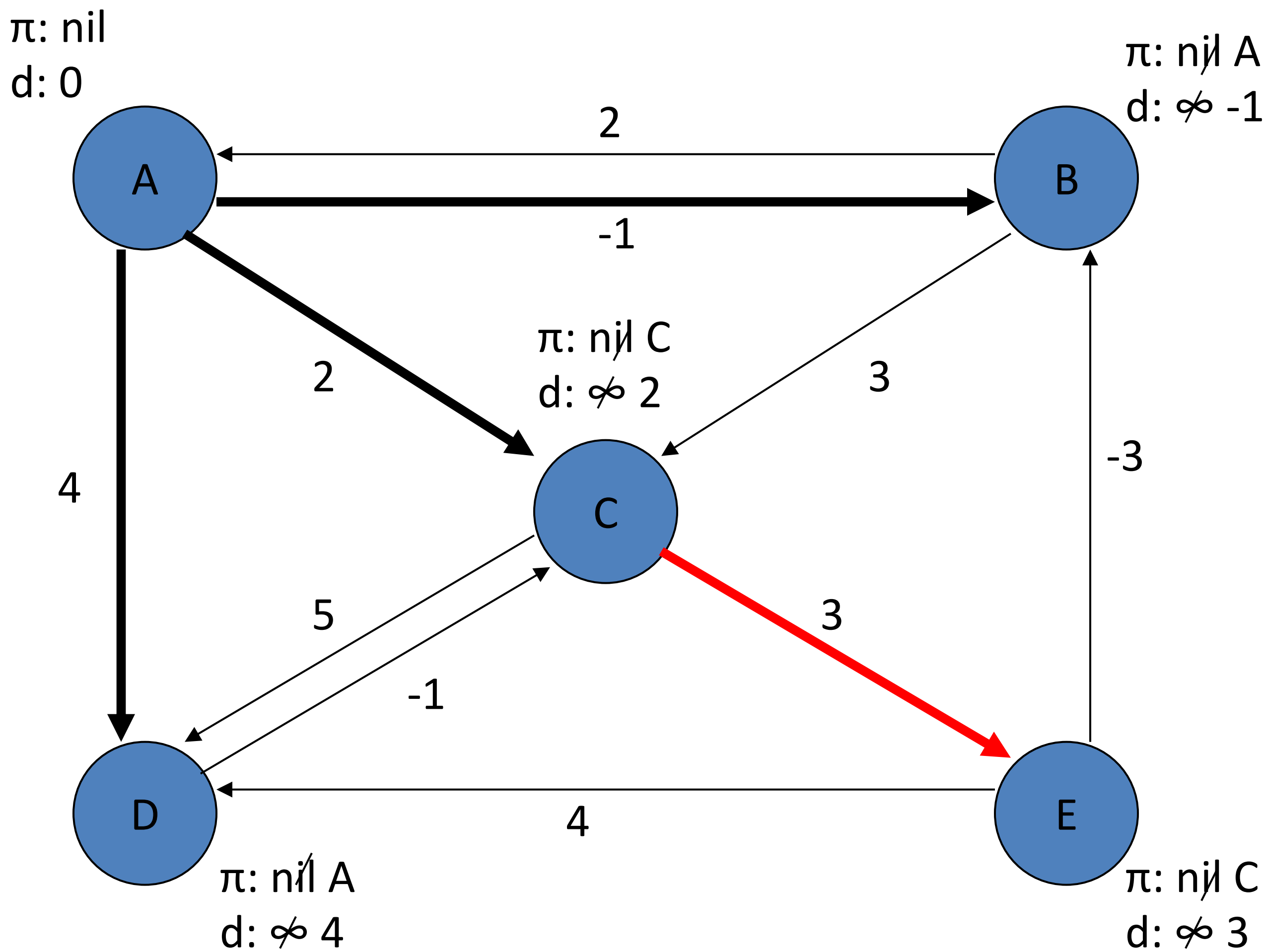
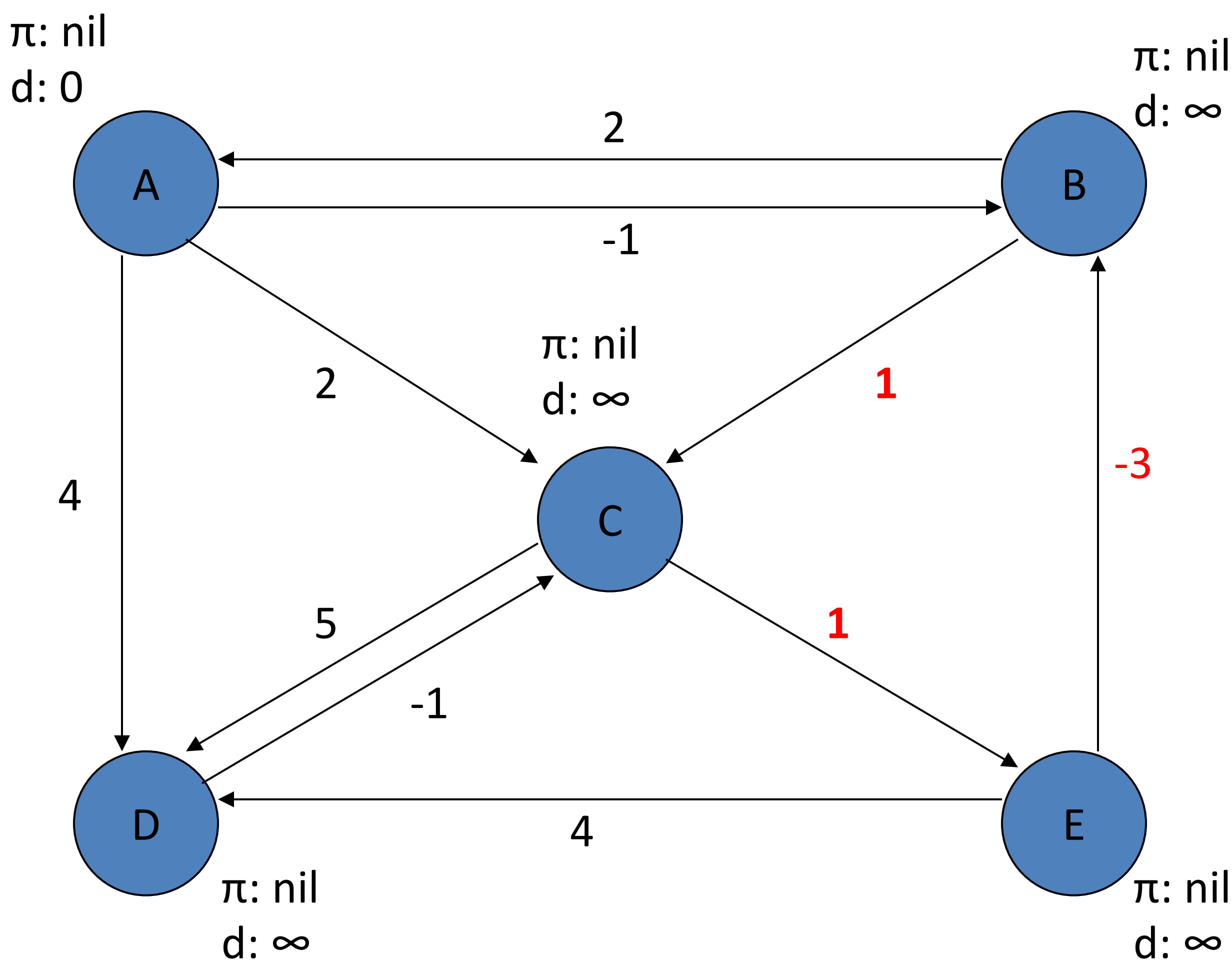


Complexity

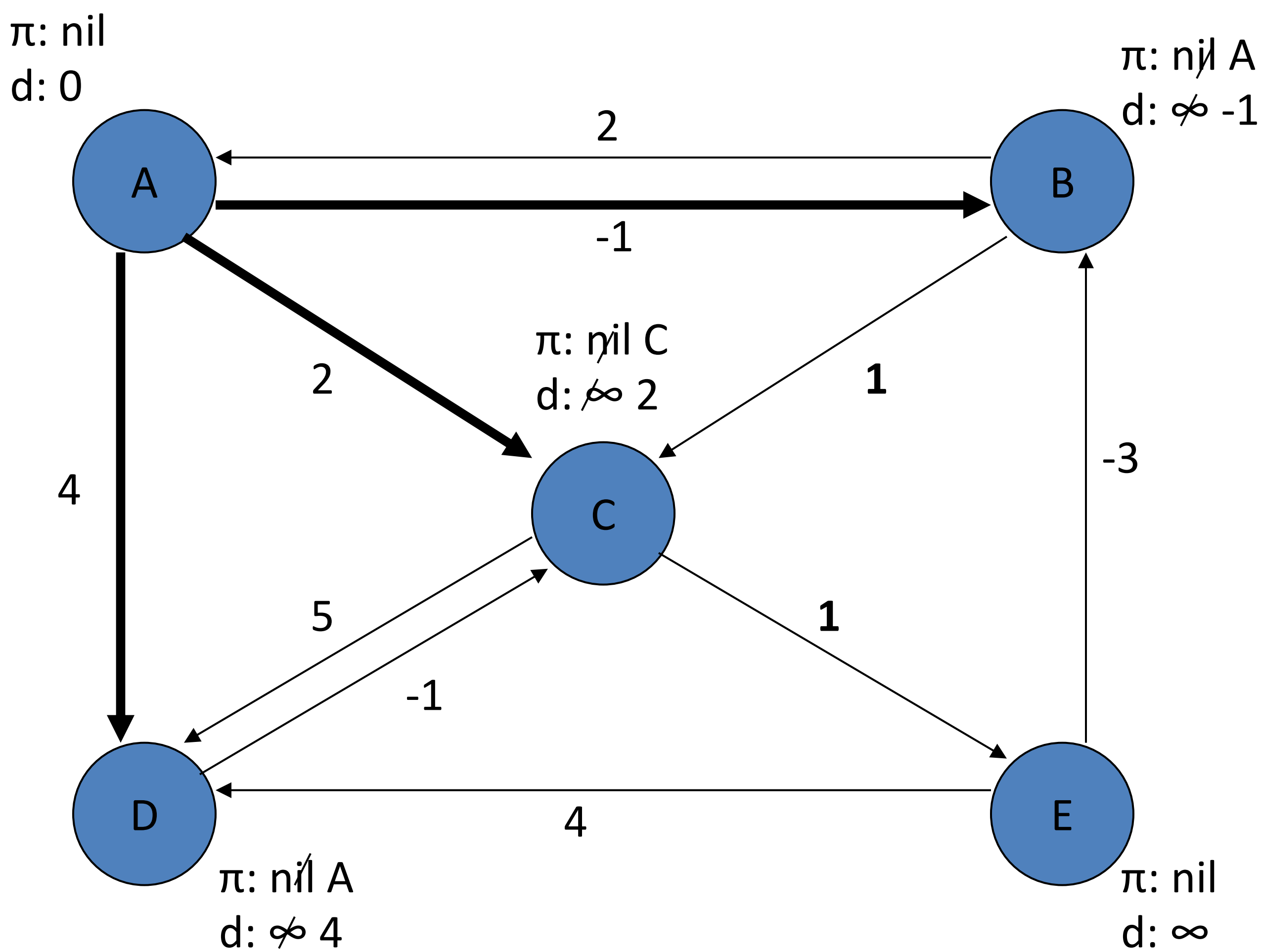
- **Running time: O(VE)**
  - Not so good for large dense graphs
  - But a very practical algorithm in many ways

- **What about graph with negative cycles …?**

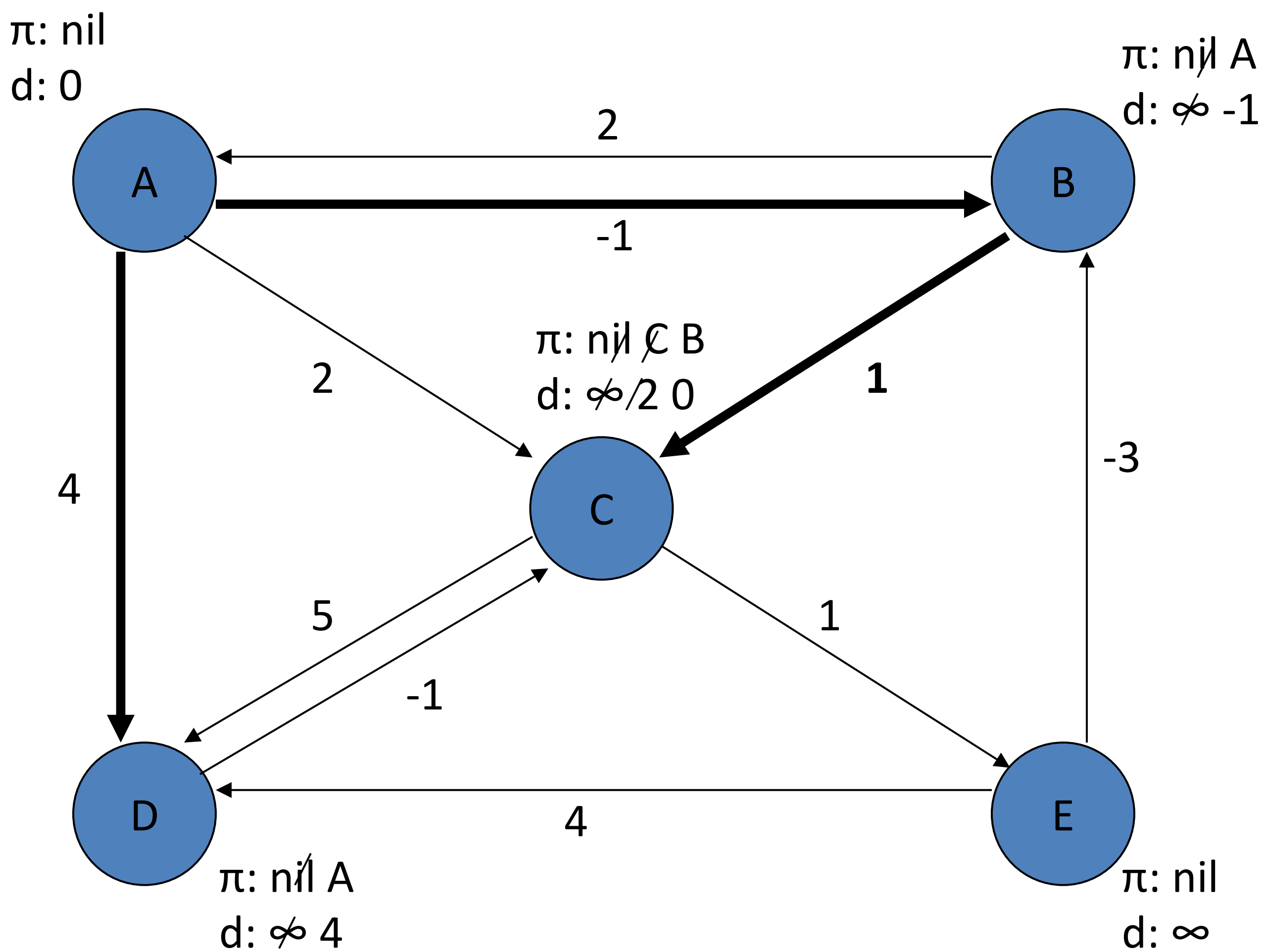
Bellman Ford



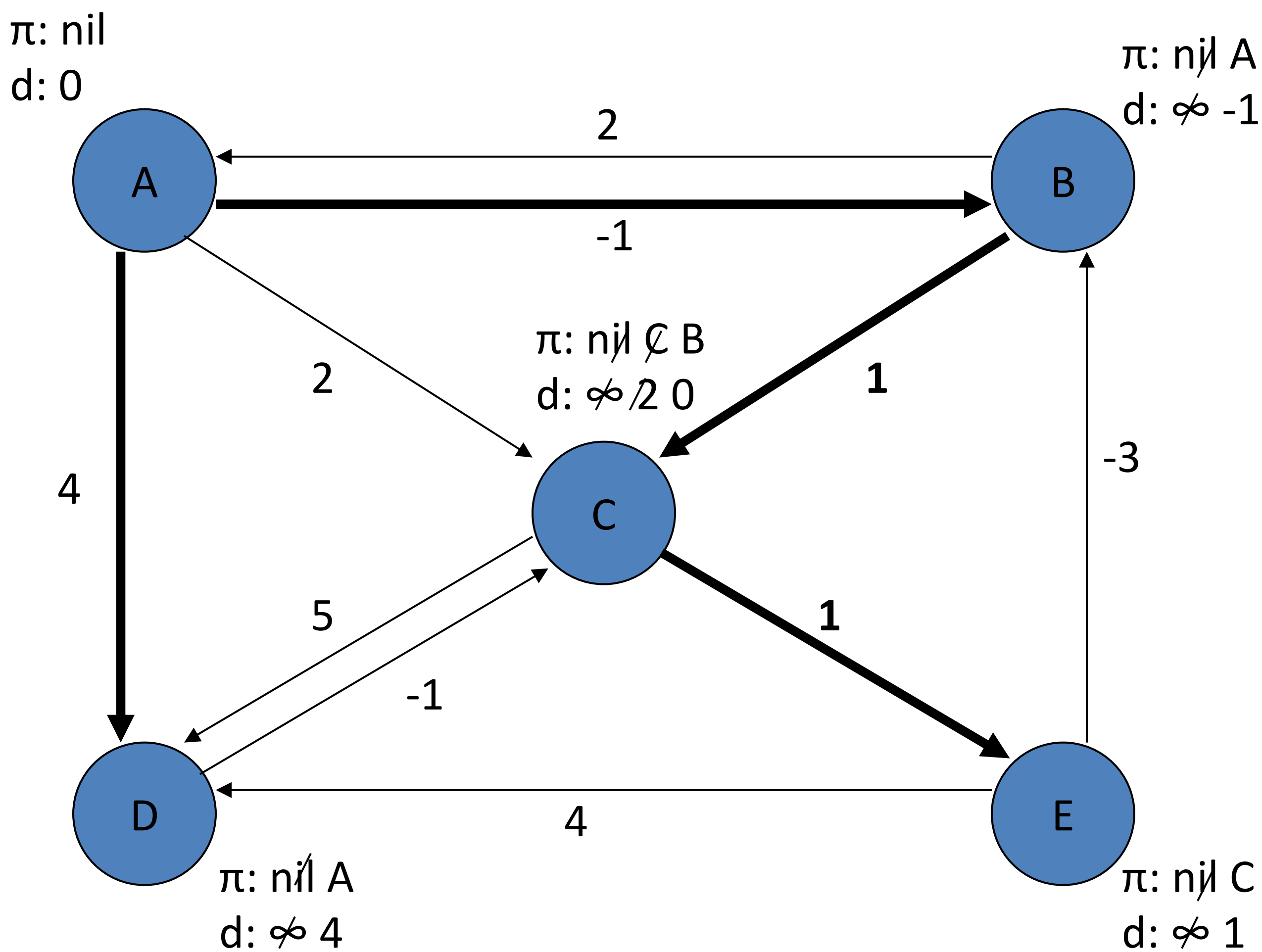
Bellman Ford



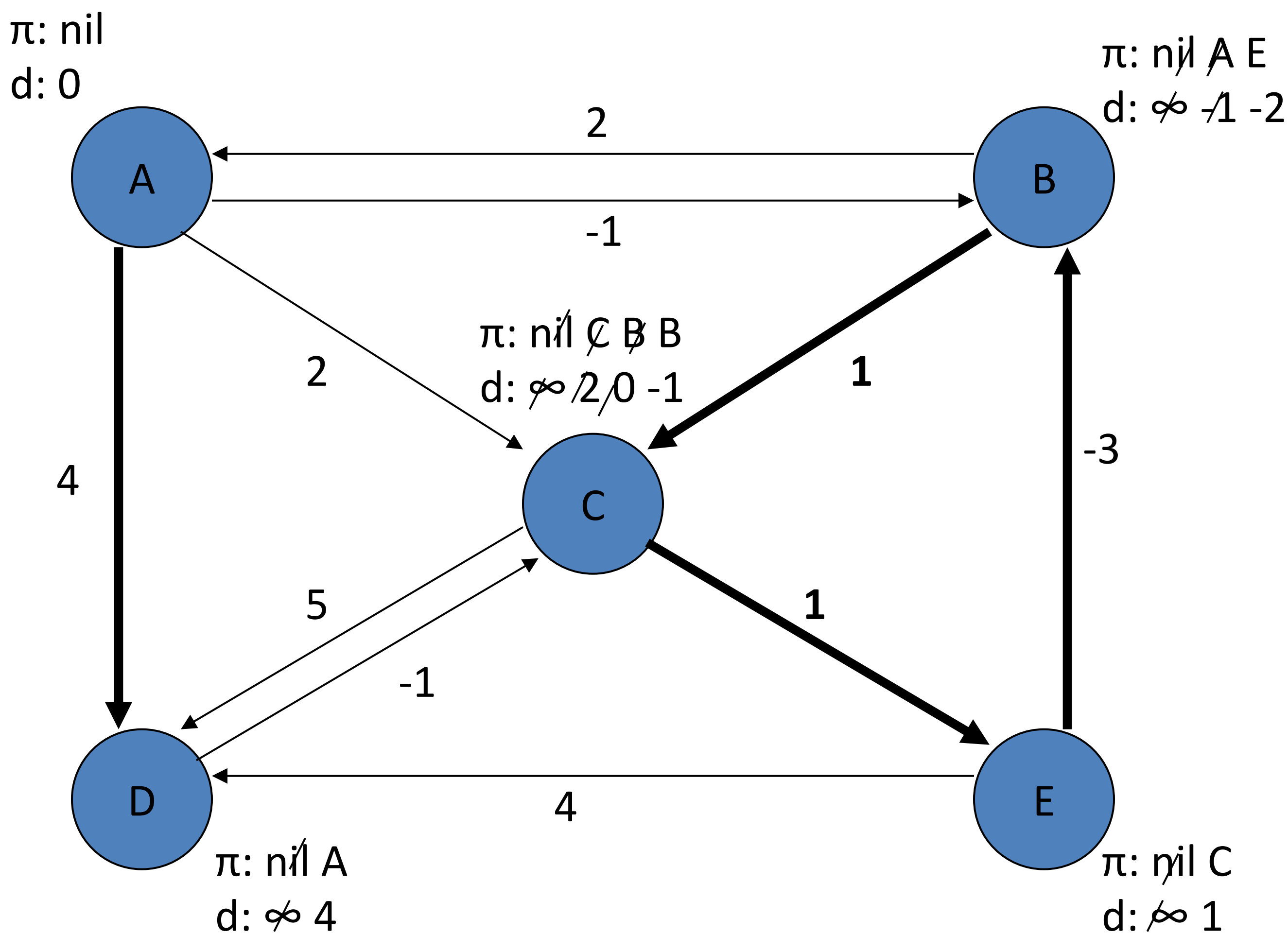
Bellman Ford



Bellman Ford



Bellman Ford



Bellman Ford

```
BellmanFord()
   for each v ∈ V
      d[v] = ∞;
   d[s] = 0;
   for i=1 to |V|-1
      for each edge (u,v) ∈ E
         Relax(u,v, w(u,v));

   for each edge (u,v) ∈ E
      if (d[v] > d[u] + w(u,v))
           return "no solution";
```

Initialize d[], which will converge to shortest-path value δ

Relaxation: Make |V|-1 passes, relax each edge

Test for solution: have we converged yet? ie, ∃ negative cycle?

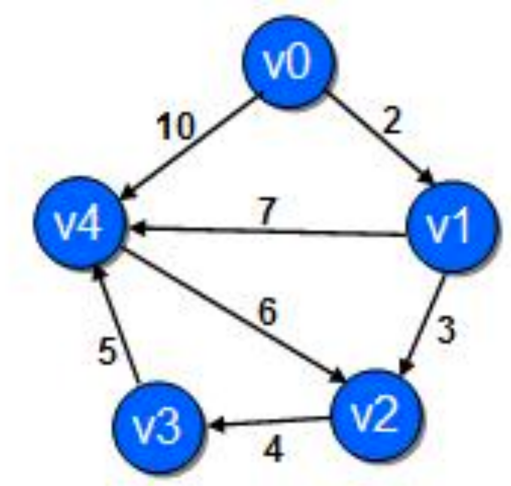
```
Relax(u,v,w): if (d[v] > d[u]+w) then d[v]=d[u]+w
```

Bellman-Ford Variant: SPFA

- **Shortest path faster algorithm (SPFA)**
  - Modified Bellman-Ford using either BFS pattern
    - Maintain the “frontier” vertices to relax
  - Practical performance is excellent

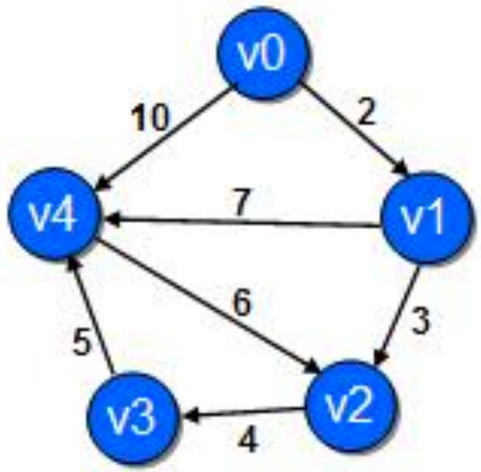
```
input G,v
for each u ∈ V(G)
      let dist[u] = ∞
let dist[v] = 0
let Q be an initially empty queue
push(Q,v)
while not empty(Q)
      let u = pop(Q)
      for each (u,w) ∈ E(G)
            if dist[w] > dist[u]+wt(u,w)
                  dist[w] = dist[u]+wt(u,w)
                  if w is not in Q
                        push(Q,w)
```

Example



Let’s start from v0

Example

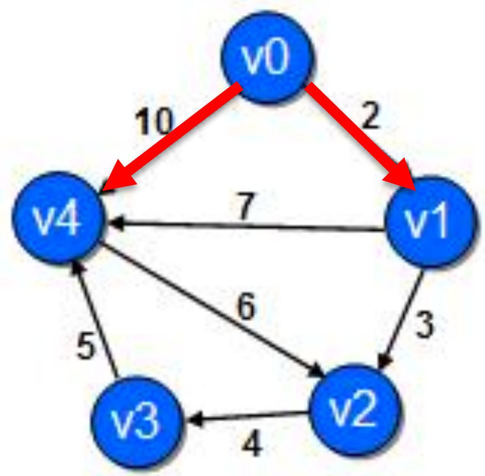


q	v0				

	v0	v1	v2	v3	v4
dis	0	∞	∞	∞	∞

Insert v0 into the queue

Example

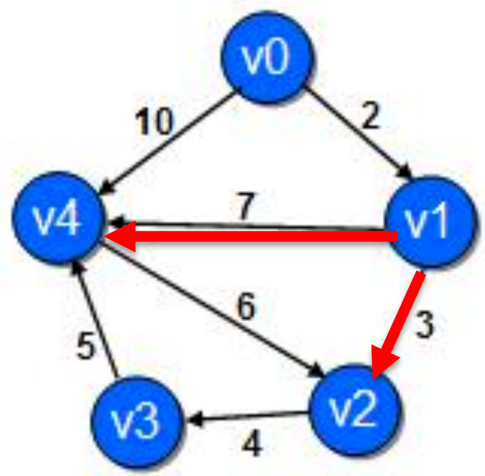


q	v1	v4			

	v0	v1	v2	v3	v4
dis	0	2	∞	∞	10

Relax v1 and v4 and insert them to the queue

Example

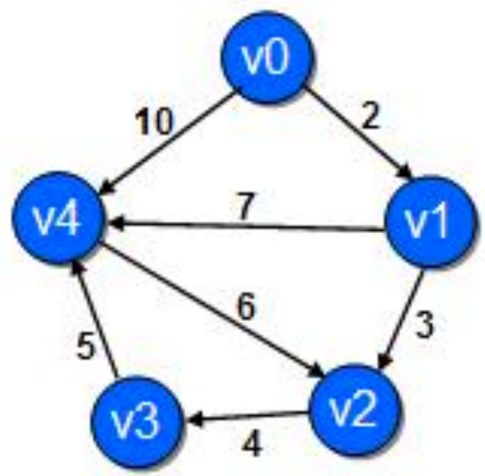


q	v4	v2			

	v0	v1	v2	v3	v4
dis	0	2	5	∞	9

Relax v2 and v4 and insert them to the queue

Example

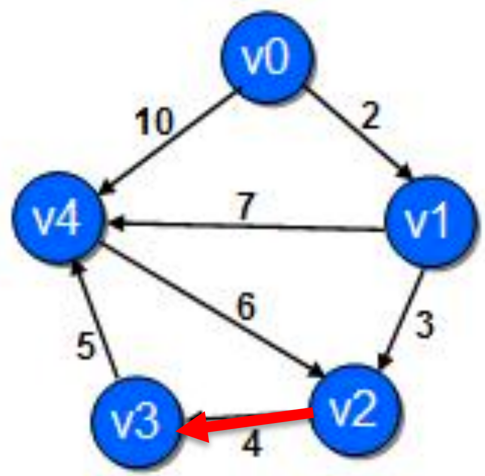


q	v2				

	v0	v1	v2	v3	v4
dis	0	2	5	∞	9

Nothing to relax from v4

Example

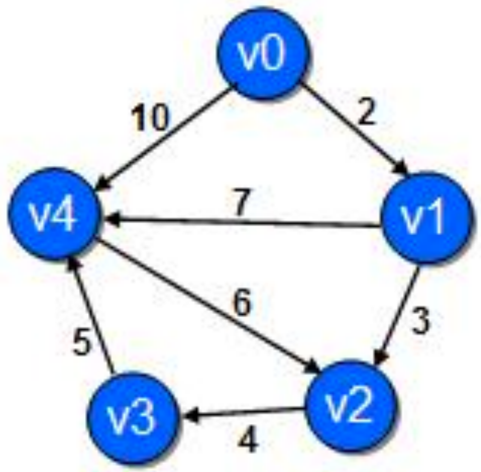


q	v3				

	v0	v1	v2	v3	v4
dis	0	2	5	9	9

Relax v3 and insert it to the queue

Example



q					

	v0	v1	v2	v3	v4
dis	0	2	5	9	9

Nothing to relax from v3, done

SPFA Properties

- **If a vertex is inserted into the queue N times**
  - Negative cycle found!
- **What is the time complexity of SPFA?**