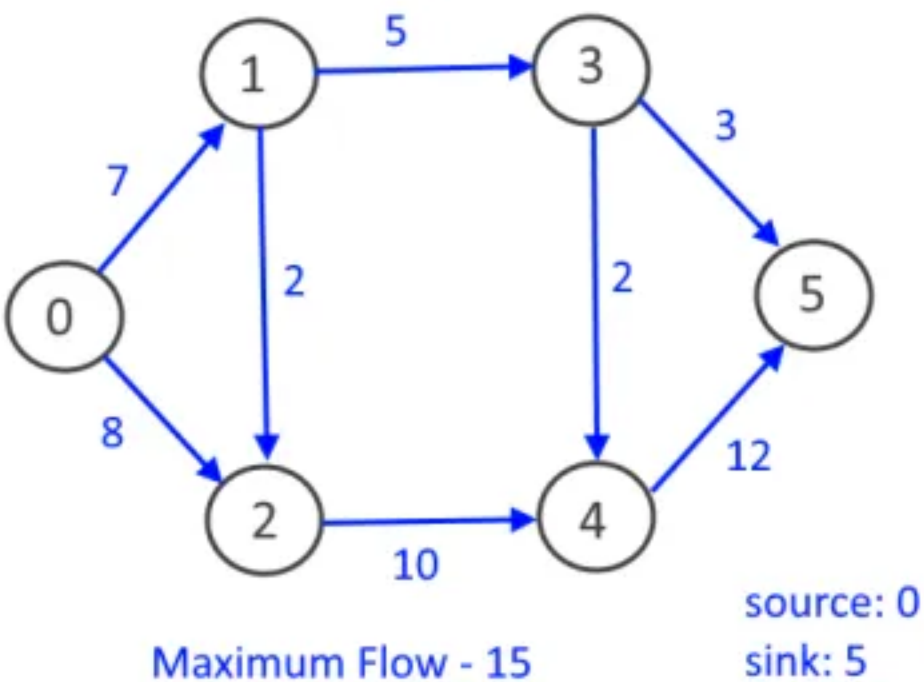
Lecture 9: Graph Algorithms (III)

Dr. Tsung-Wei Huang
Department of Electrical and Computer Engineering
University of Utah, Salt Lake City, UT

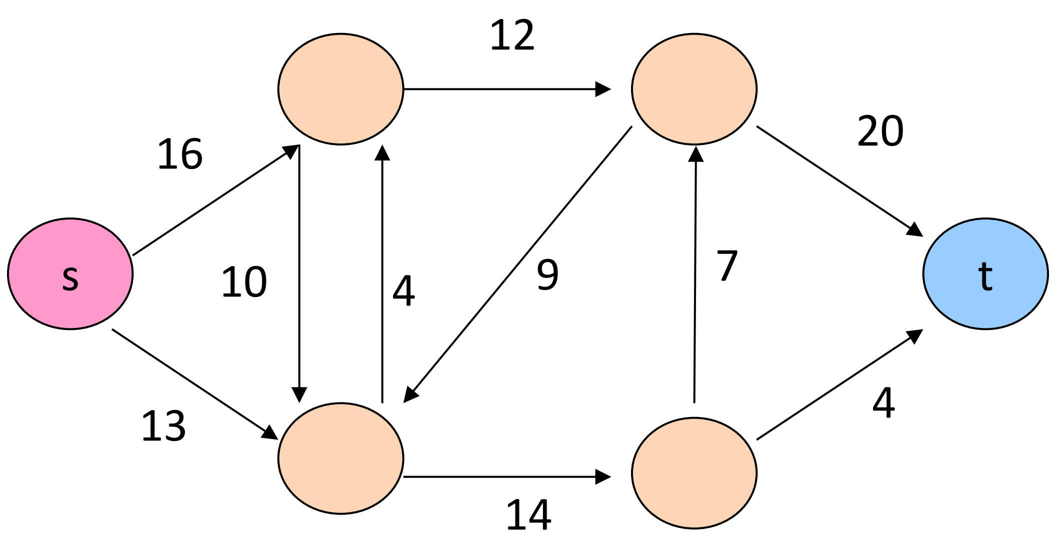
Maximum Flow

- **Find a maximum feasible s-t flow in a graph**
  - s is a source node
  - t is a target (sink) node



Problem Formulation

- **Network flow problem**
  - A flow network G=(V,E): a directed graph, where each edge $(u,v) \in E$ has a nonnegative capacity $c(u,v)>=0$.
  - If $(u,v) \notin E$, we assume that $c(u,v)=0$.
  - two distinct vertices :**a source s and a sink t.**

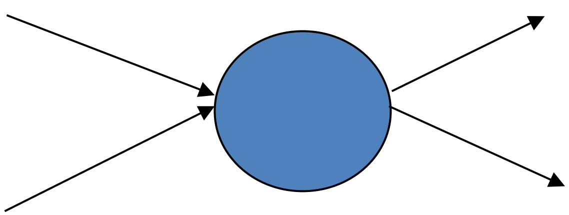


Flow Constraint

- ❑ **G=(V,E): a flow network with capacity function c.**
- ❑ ***s* -- the source and *t* -- the sink.**
- ❑ **A flow f(u, v) in G must satisfy**

1. Capacity constraint
    - For all u,v $\in$V, we require f(u,v) $\leq$ c(u,v).
2. Flow conservation
    - For all u $\in$V-{s,t}, we require

$$\sum_{e.in.v} f(e) = \sum_{e.out.v} f(e)$$



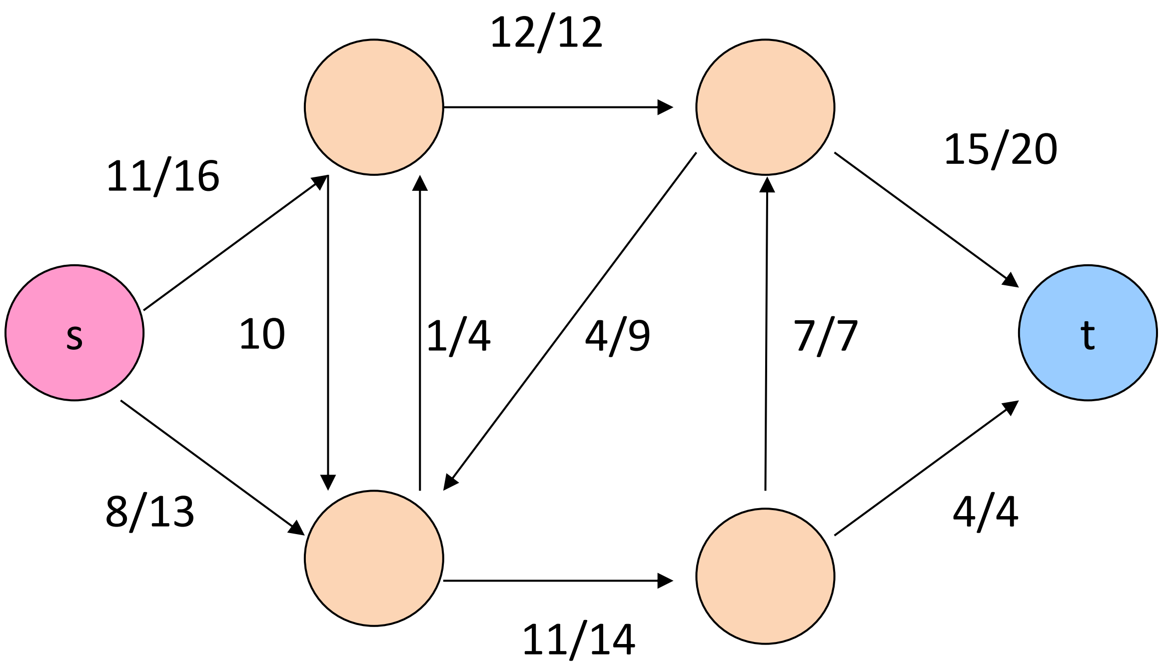
Objective

- ❑ **The quantity f (u,v) is called the net flow from vertex u to vertex v.**
- ❑ **The value of a flow is defined as**

$$\left|f\right| = \sum_{v \in V} f(s,v)$$

  - ❑ The total flow from source to any other vertices.
  - ❑ The same as the total flow from any vertices to **the sink.**

Example



A flow f in G with value $\cdot\ |f| = 19$

Recap

- **Given a flow network G with source s and sink t**
- **Find a flow of maximum flow value from s to t.**
- **How to solve it efficiently ?**
  - Brute force …?

Ford-Fulkerson Framework

FORD-FULKERSON-FRAMEWORK(G, s, t)

initialize flow *f* to *0*

while there exists an *augmenting* path *p*

do *augment* flow *f* along *p*

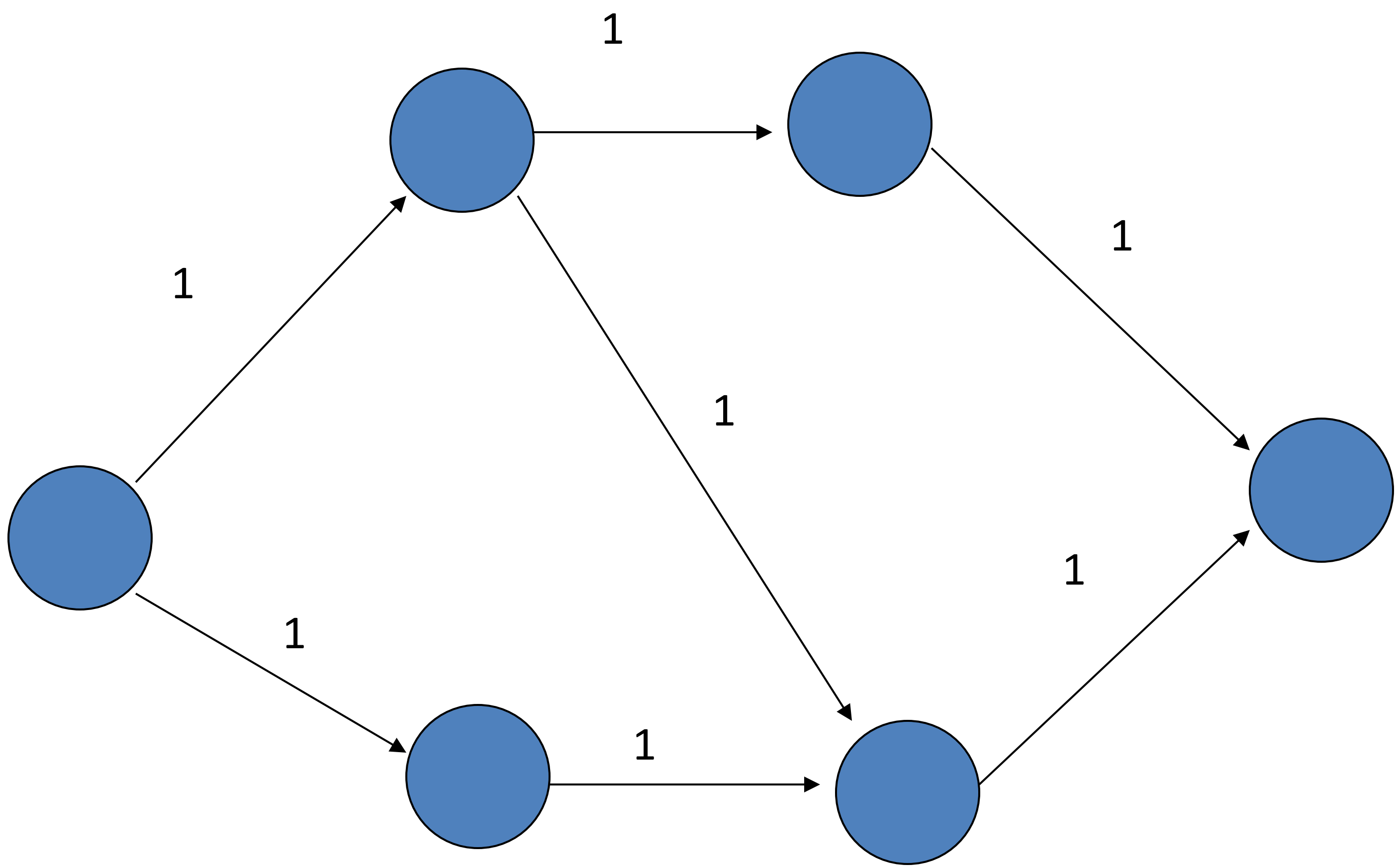
return *f*

Why Framework not Algorithm?

- **The framework is iterative**
- **Augmenting flow has different implementations**
  - Each implementation is a different algorithm
- **Augmenting flow is equivalent to finding a path**
  - u→v is connected if there remains capacity (non-zero)
  - u→v is disconnected if the capacity is zero

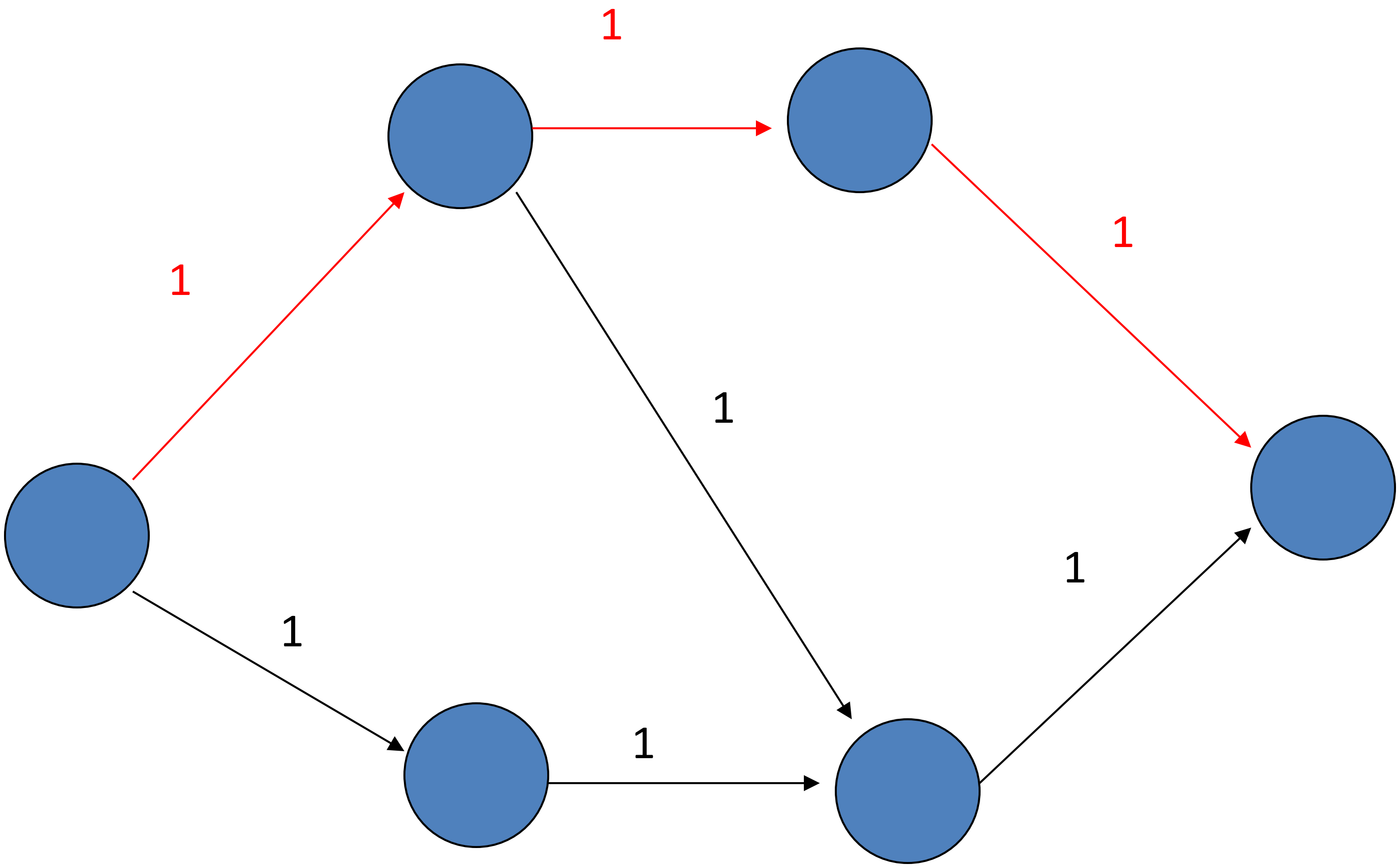
Example

- ❑ **What is the maximum flow?**



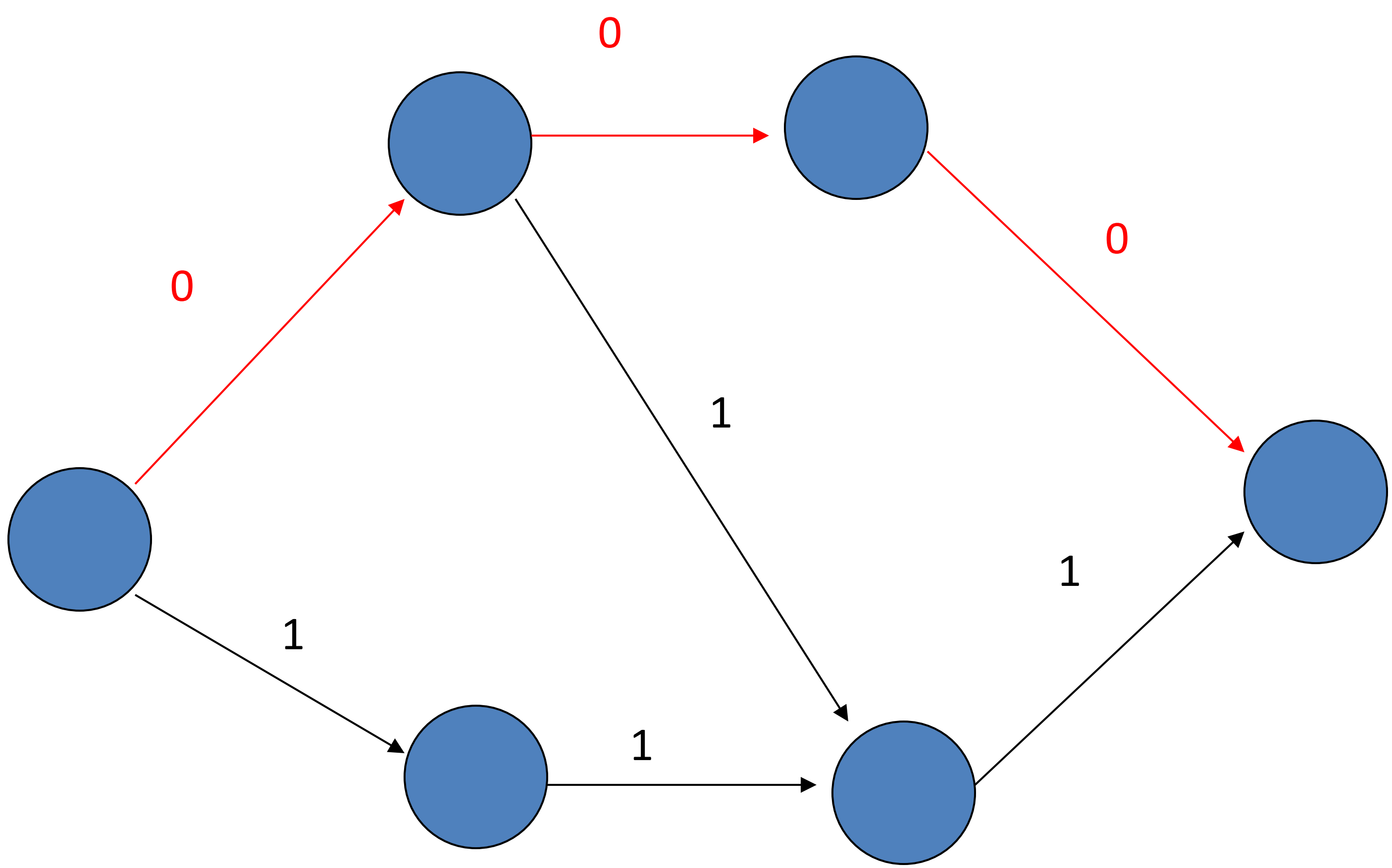
Example

- Let's do DFS to augment flow: iter1 finds flow 1



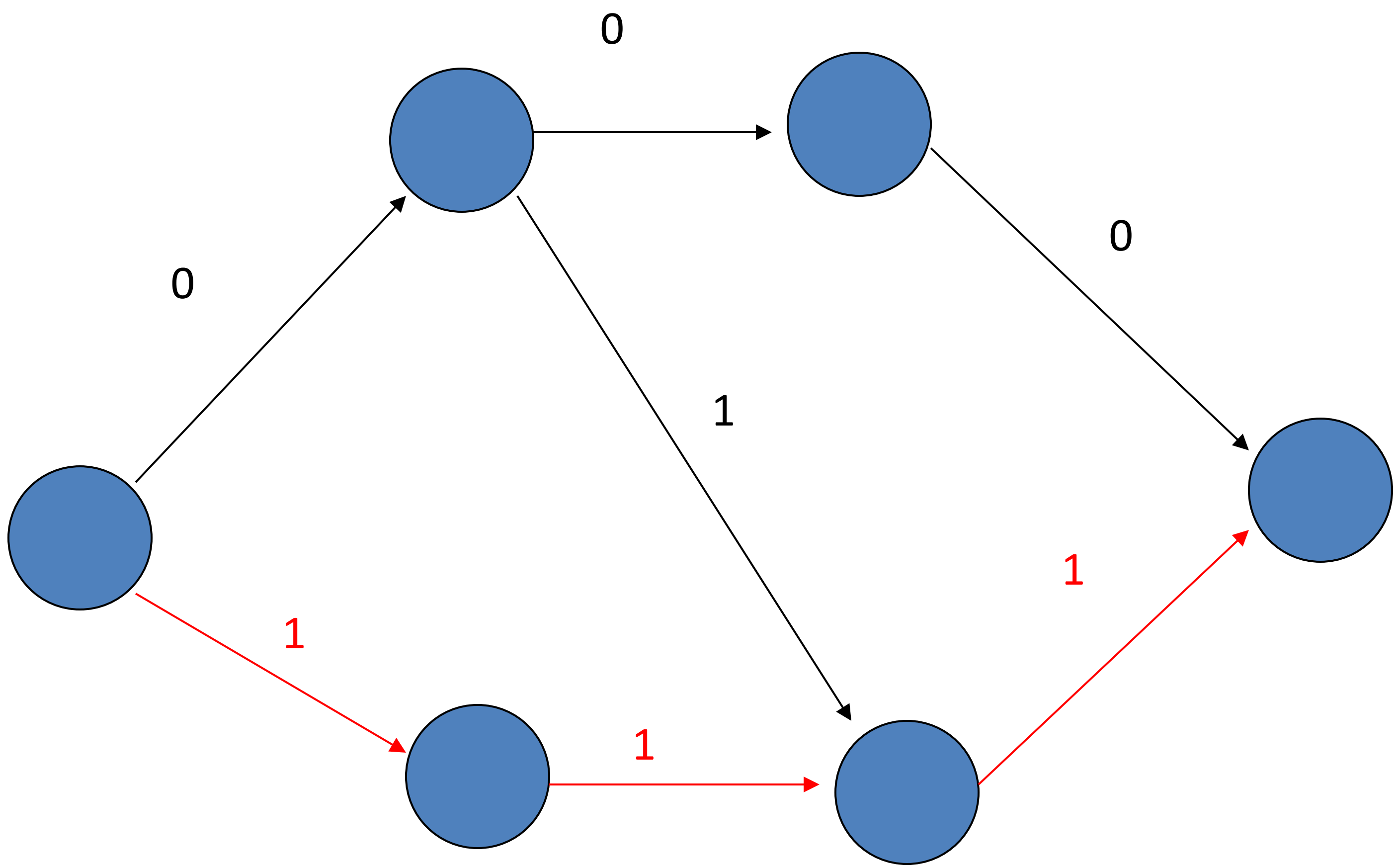
Example

- ❑ **Update remaining capacity**



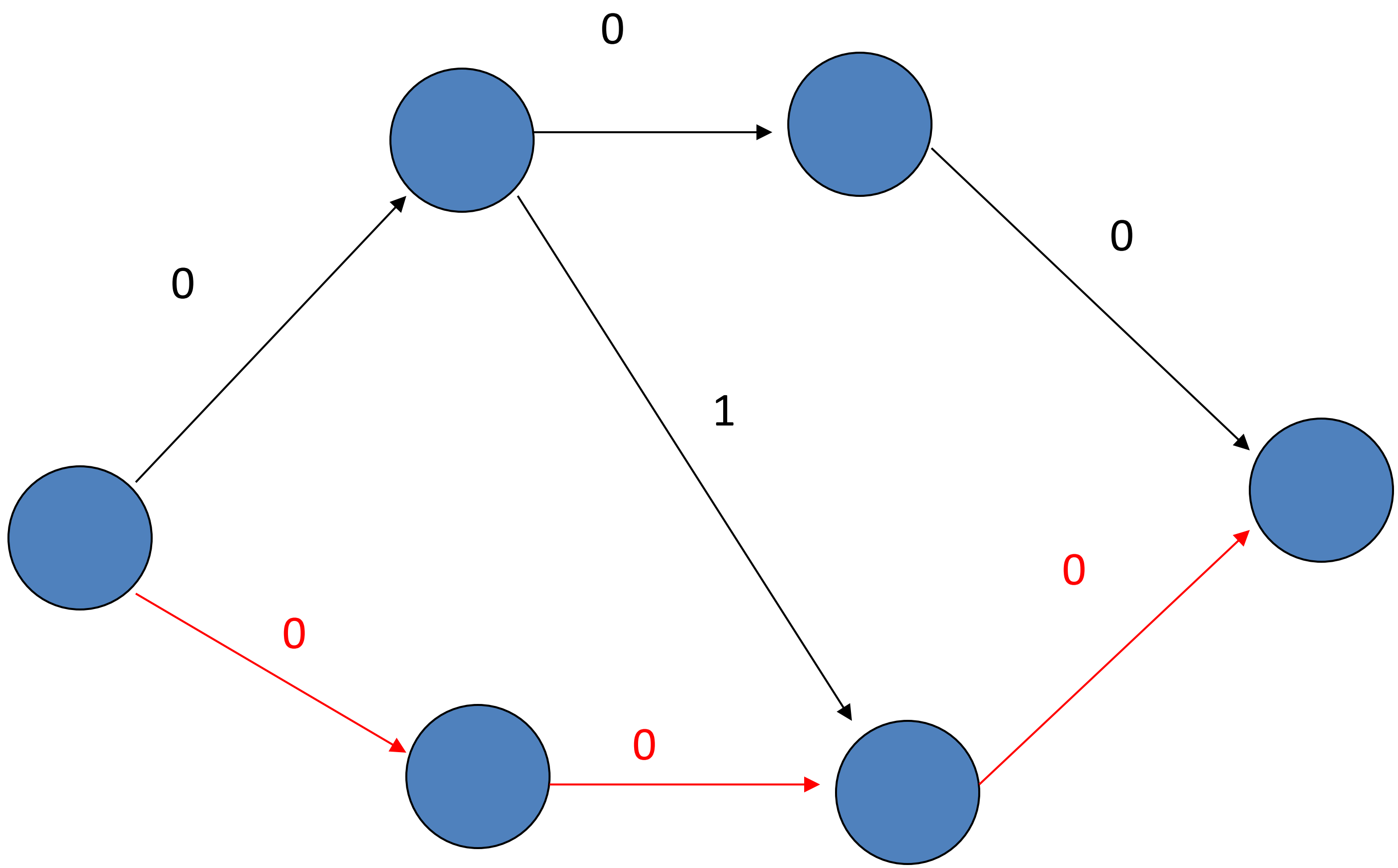
Example

- Let's do DFS to augment flow: iter2 finds flow 1



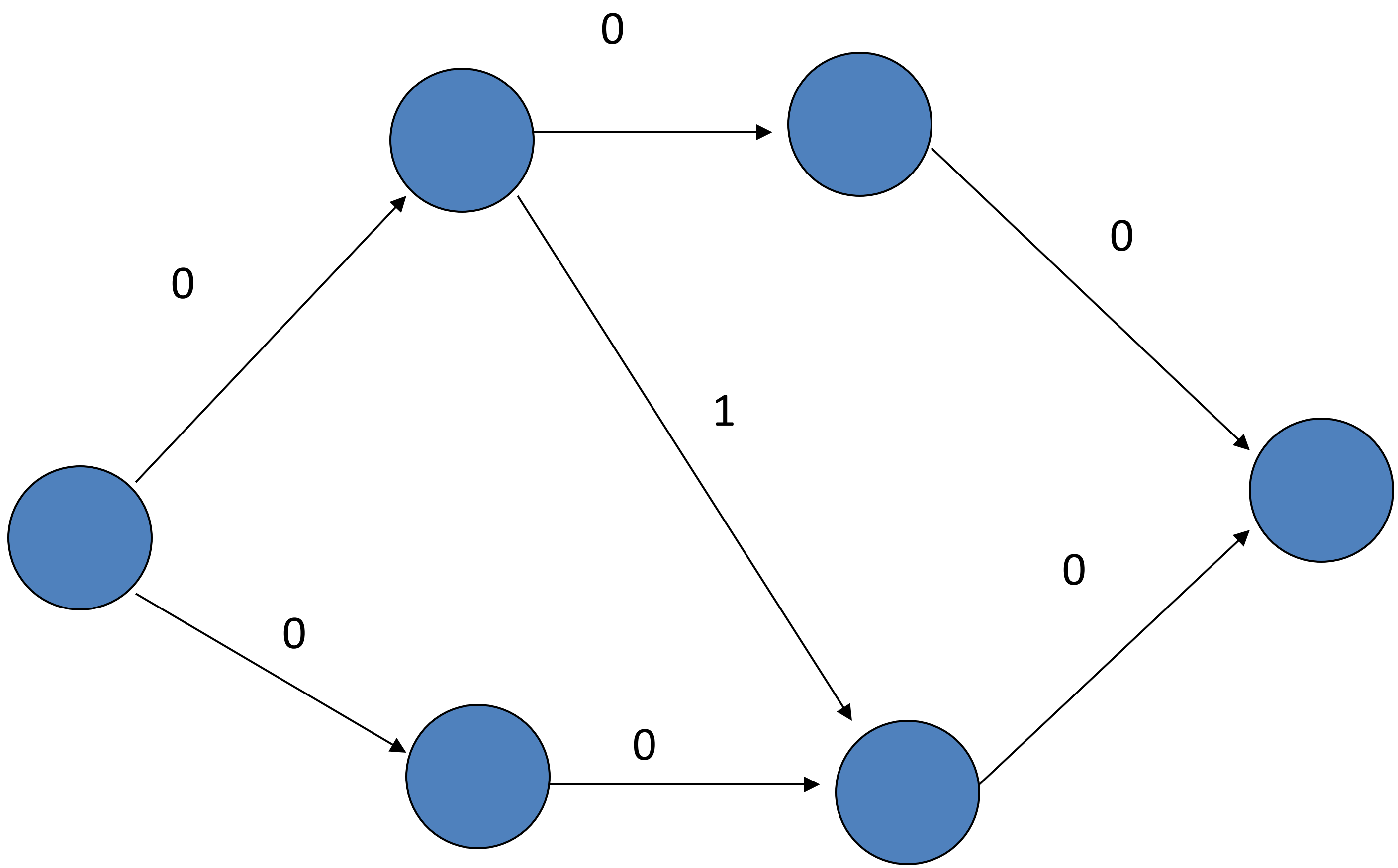
Example

- Update remaining capacity



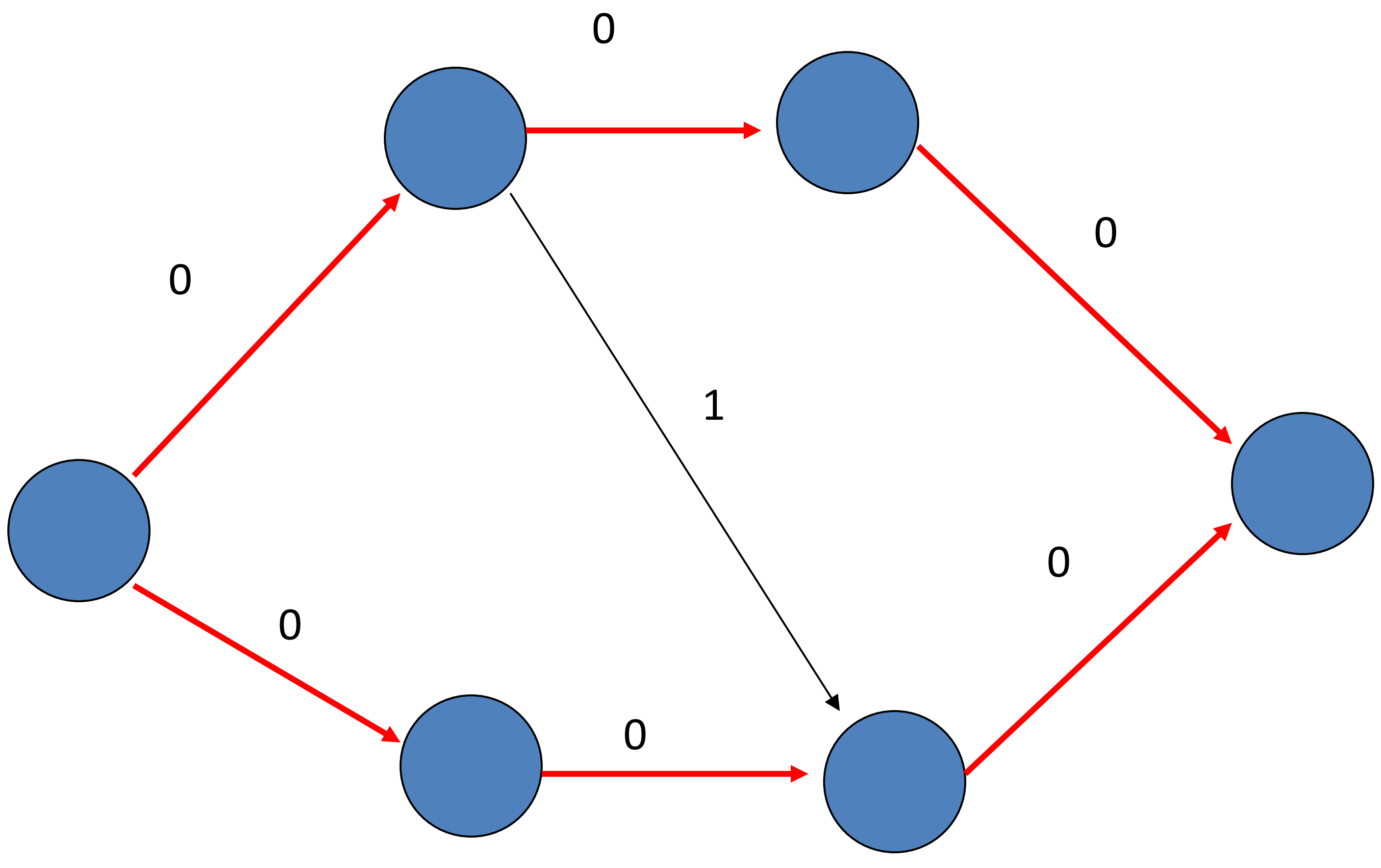
Example

- ❑ **Can we augment any flow through DFS?**



Example

- Maximum flow: 2

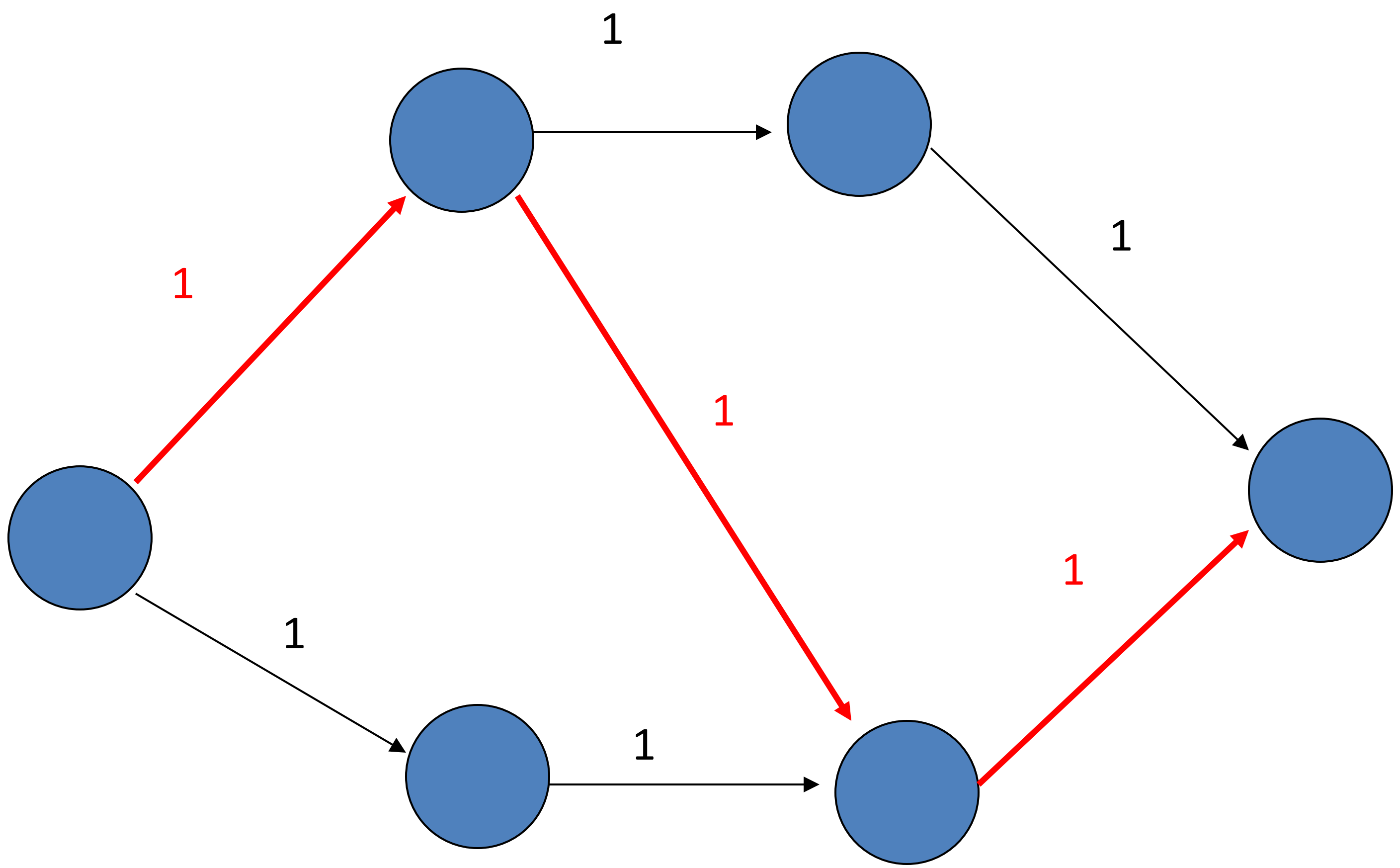


What is the Problem?

- **DFS has no order guarantee!**
  - Order you visit vertices is up to the graph data structure
  - Different orders may update capacity differently
    - In turn affect the solution

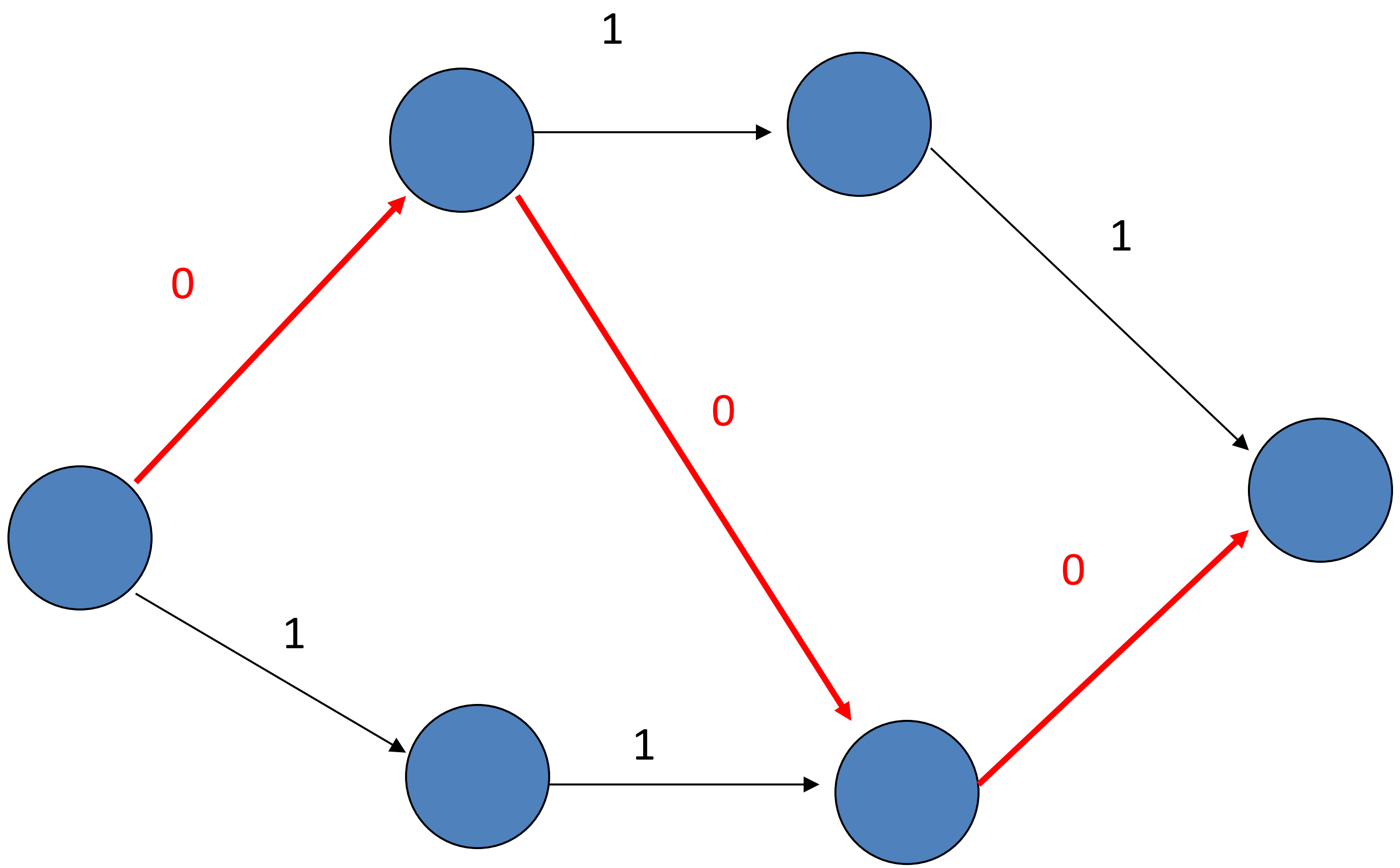
Example

- DFS finds another route in the first iteration



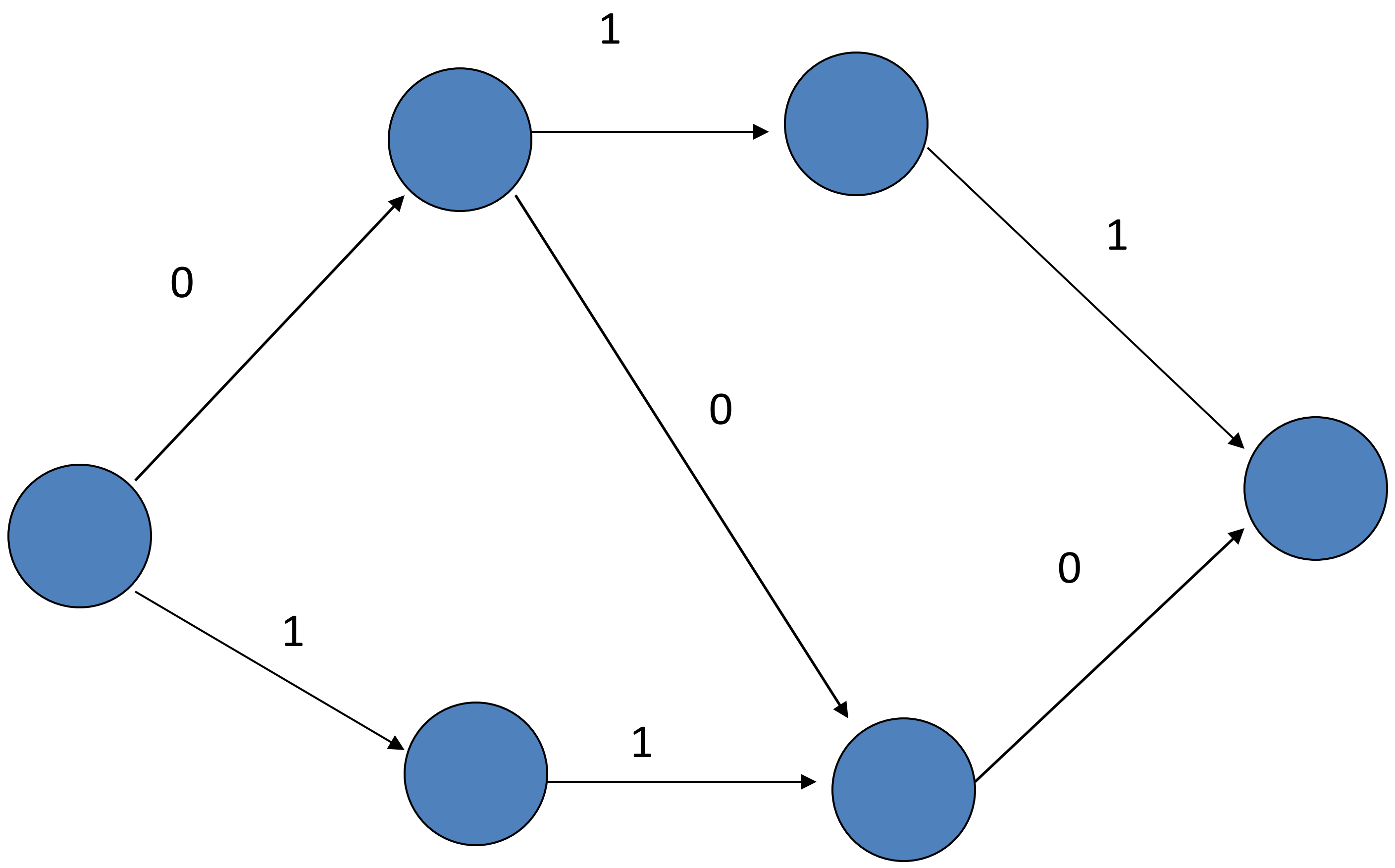
Example

- ❑ **Update remaining capacity**



Example

- DFS to augment the flow? Maximum flow = 1?

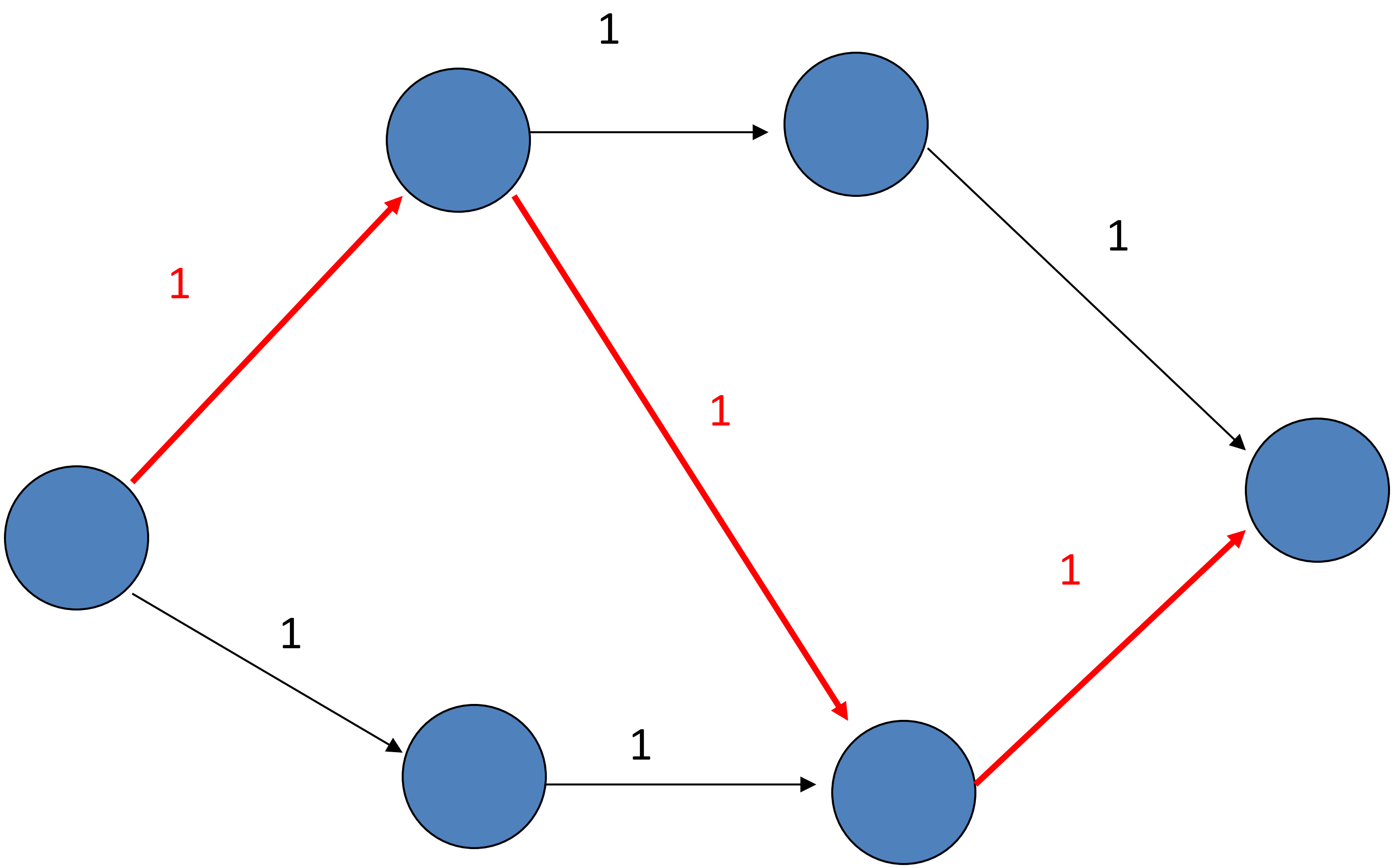


Residual Network

- **Residual network defines edges to admit net flow**
  - The amount of additional net flow from u to v before exceeding the capacity c(u,v) is the residual capacity of (u,v), given by:
    - In the regular direction: $c_f(u,v)=c(u,v)-f(u,v)$
    - In the opposite direction: $c_f(v, u)=c(v, u)+f(u, v)$.

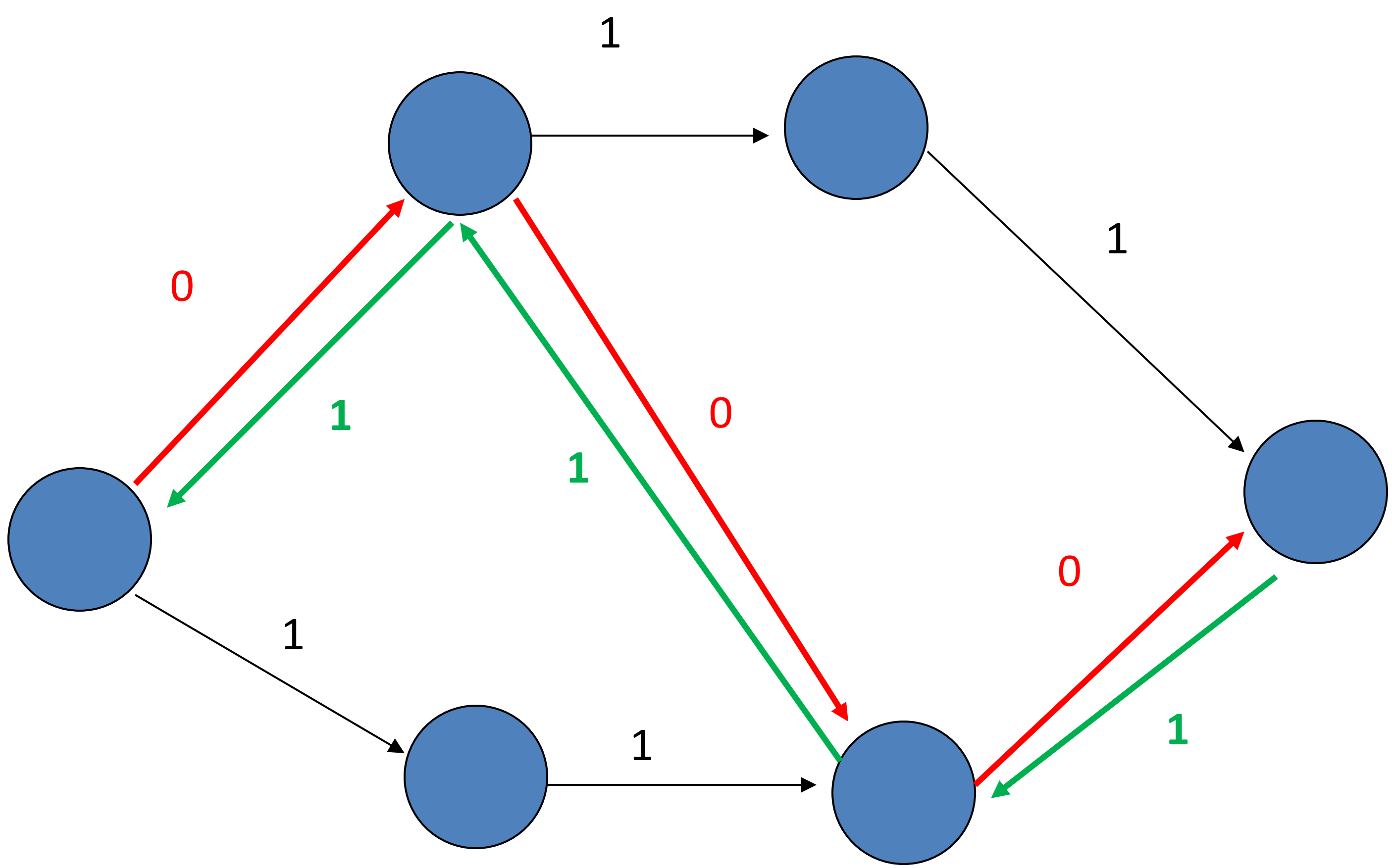
Example

- DFS augments a unit flow in the first iteration



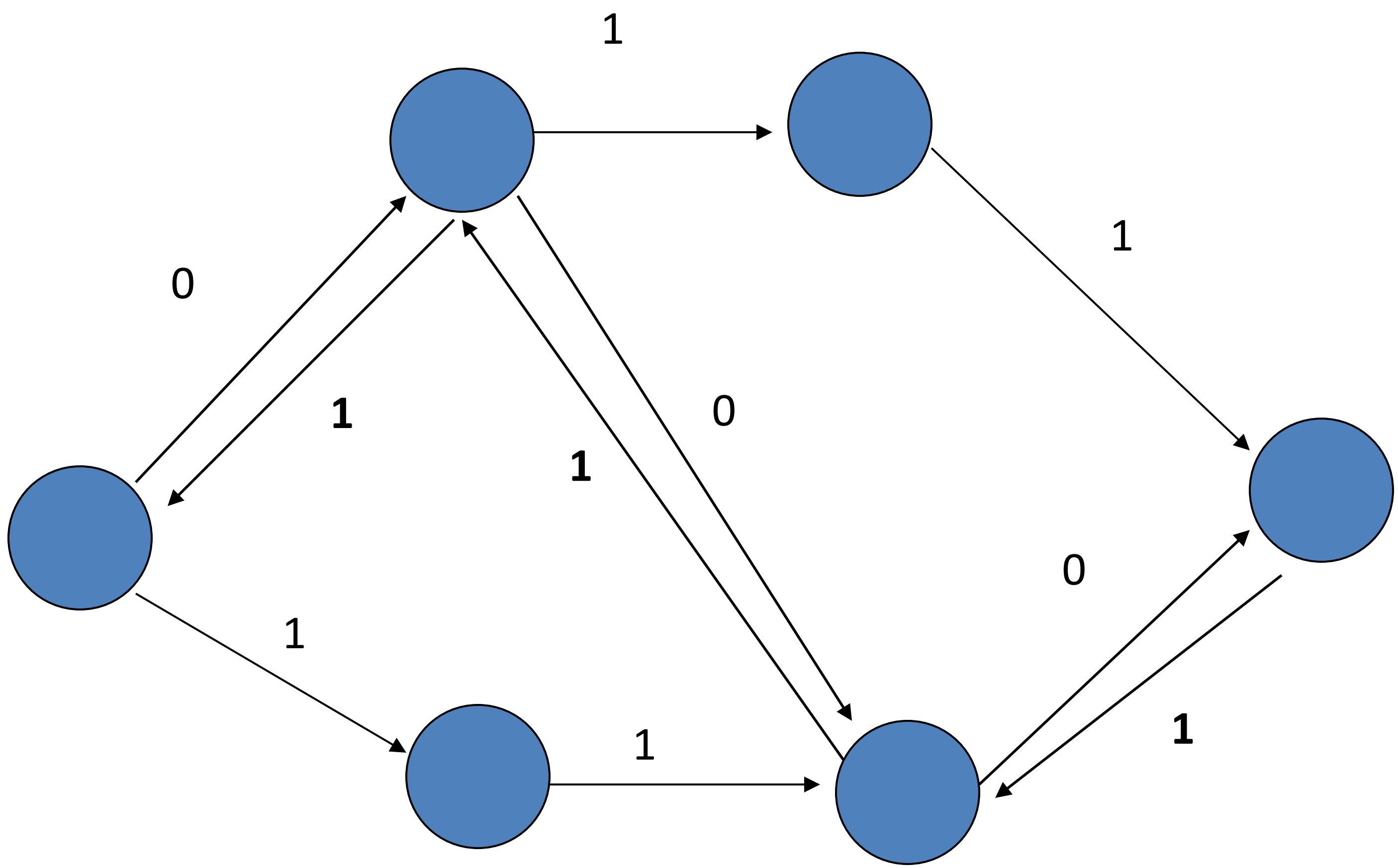
Example

- ❑ **Update the residual network**



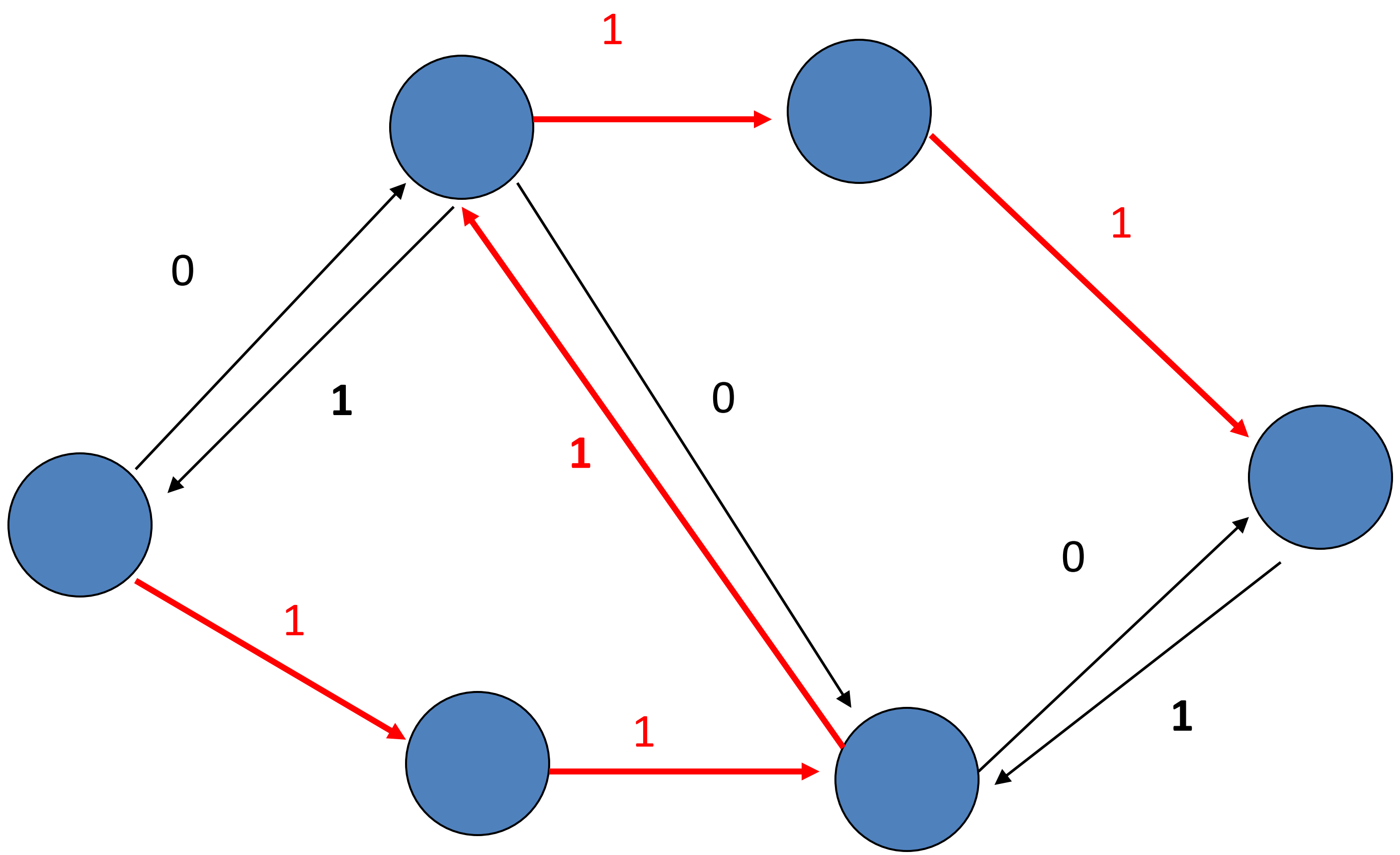
Example

- ❑ **Residual network gives a chance to "circle back"**



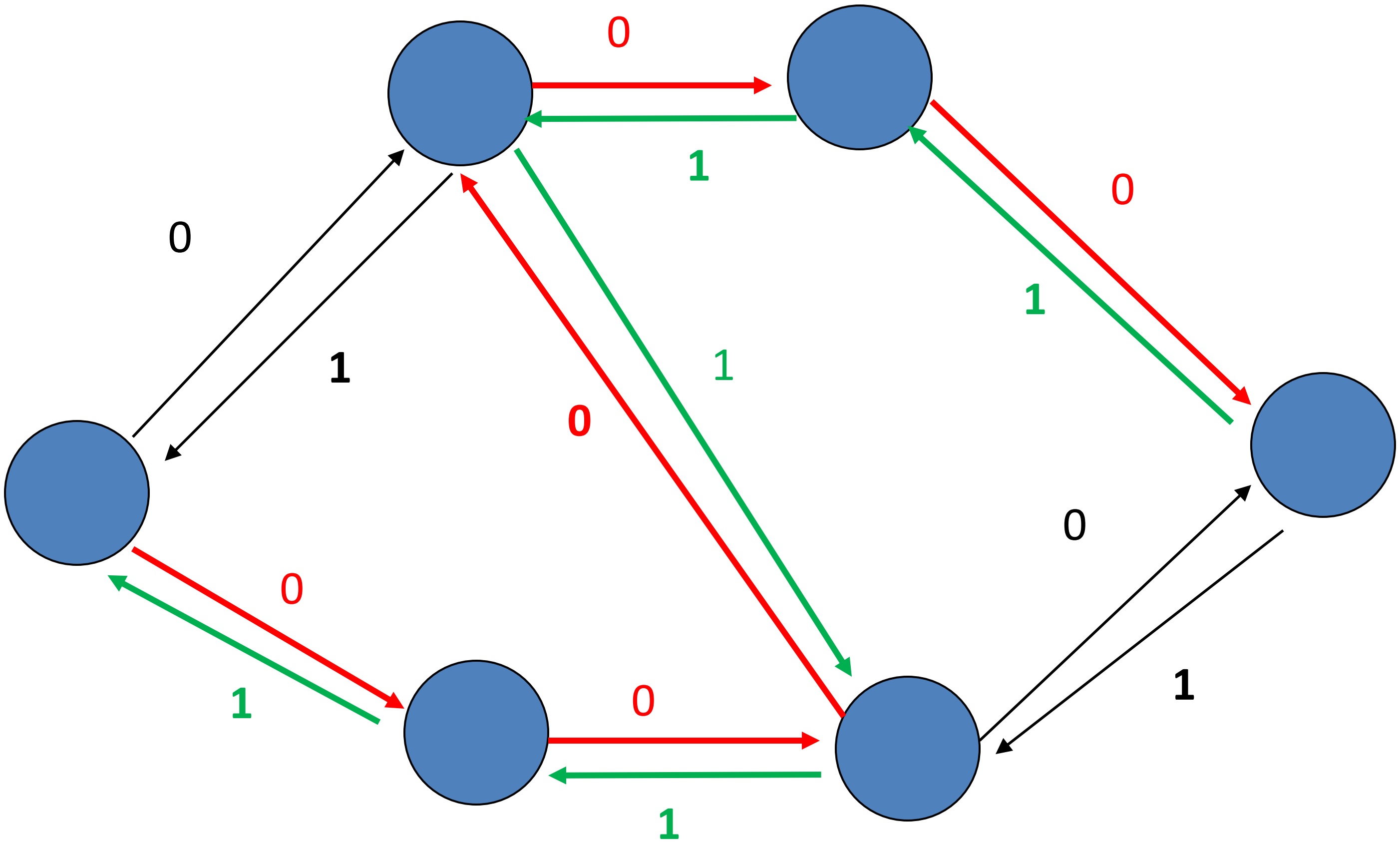
Example

- DFS augments another unit flow in the second iter



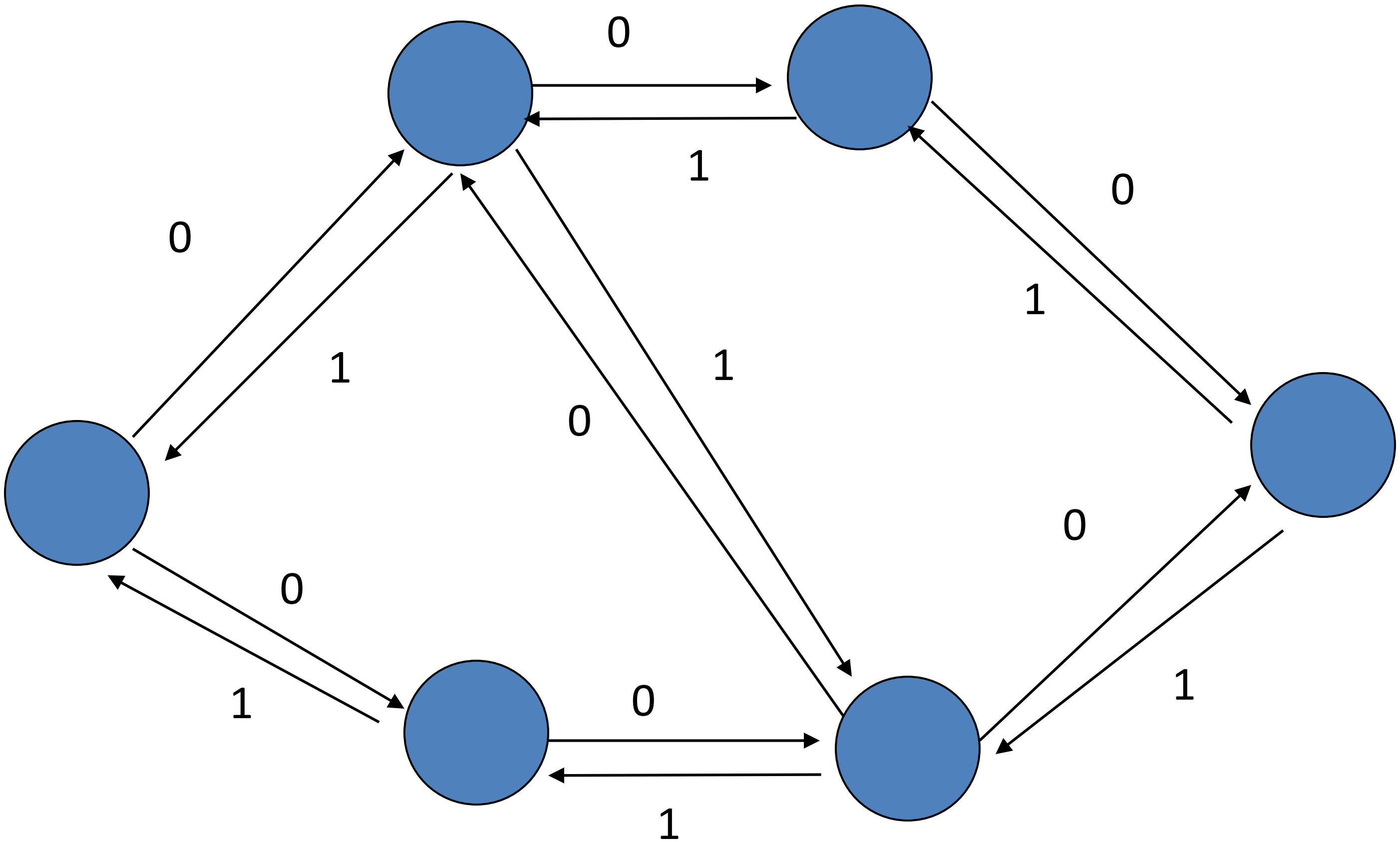
Example

- ❑ **Update residual network**

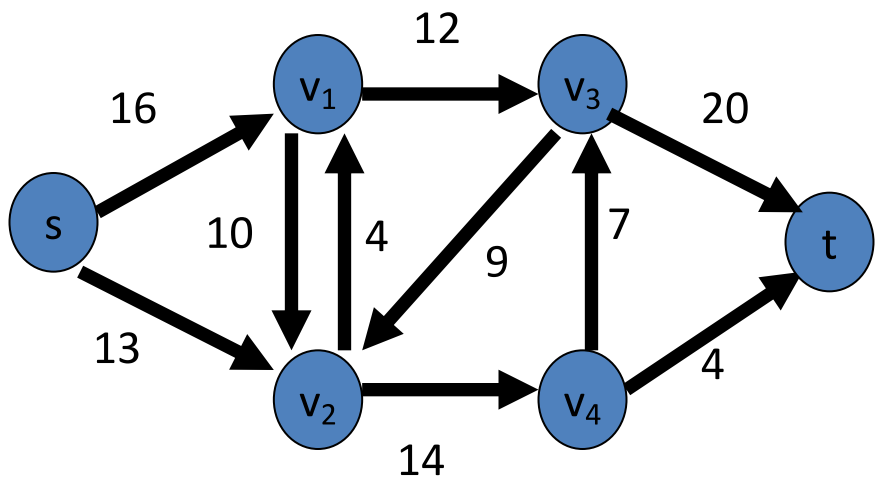


Example

- Maximum flow: 2

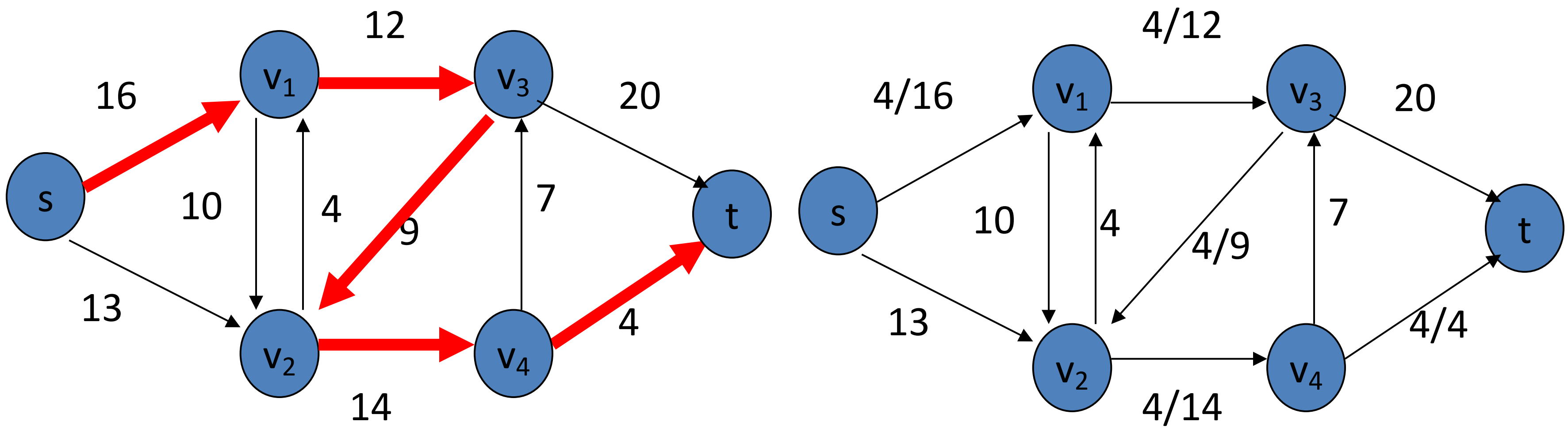


Slightly Complicated Example



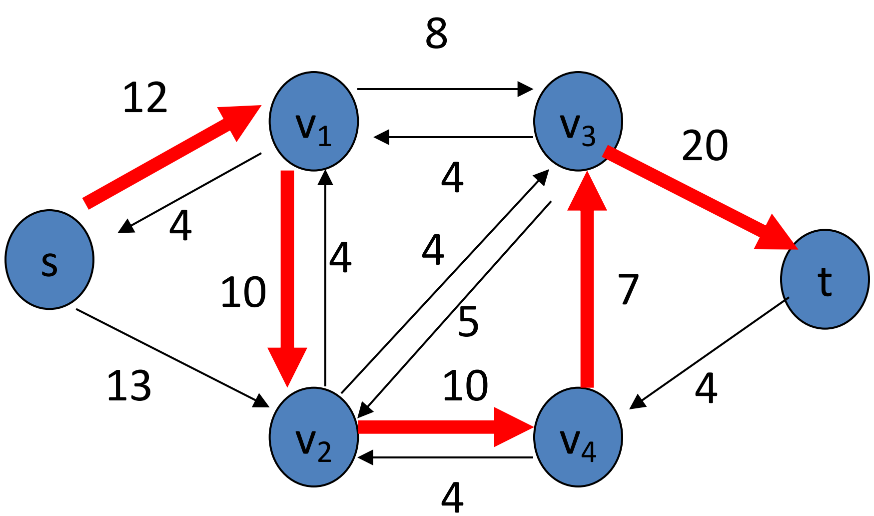
Initial

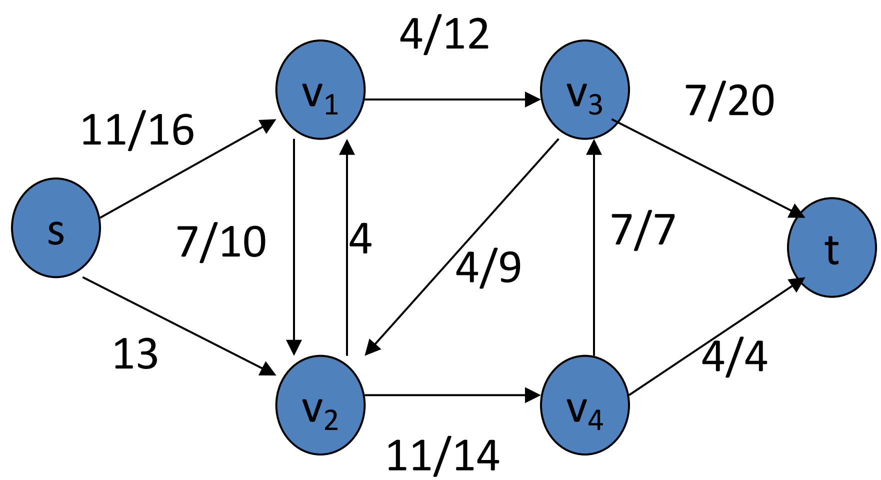
Slightly Complicated Example



(a) Maximum flow: 4

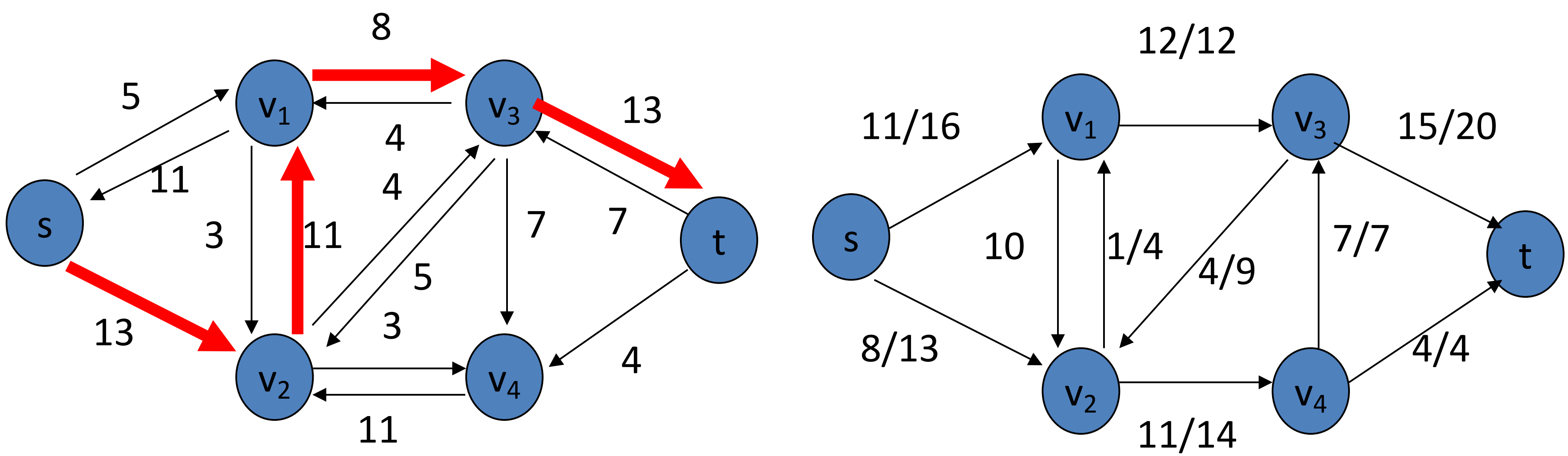
Slightly Complicated Example





(b) Maximum flow: 4 + 7

Slightly Complicated Example



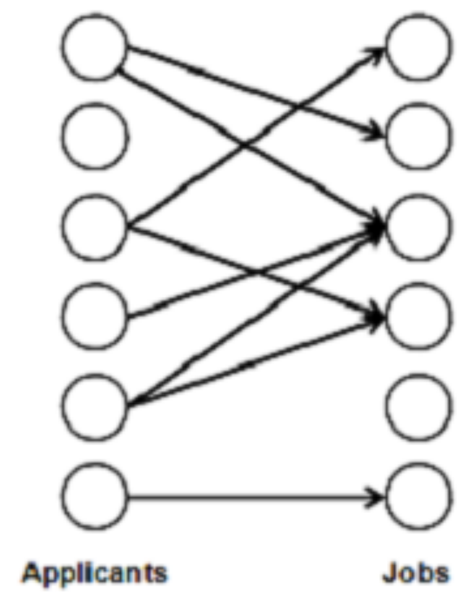
(c) Maximum flow: 4 + 7 + 8

Code Snippet

```
while(1) {
   // … initialize BFS storage

   while(!BFS.empty()) {
      now = BFS.front();
      for(next=0; next<n; next++) {
         if(visited[next])continue;
         if(mat[now][next]-flow[now][next]>0)  {    // Positive direction
            p[next] = now, visited[next] = true;
            BFS.push(next);
         }
         else if(flow[next][now]>0)  {          // Opposite direction
            p[next] = -now, visited[next] = true;
            BFS.push(next);
         }
      }
      BFS.pop();
   }

  if(!visited[sink]) break;     //If not find the augmenting path.

   for(minf=INF, i=sink; i!=source; i=abs(p[i])) {
      if(p[i]>=0) minf = min(minf, mat[p[i]][i]-flow[p[i]][i]);
      else minf = min(minf, flow[i][-p[i]]);

      for(i=sink; i!=source; i=abs(p[i])) {
         if(p[i]>=0)  flow[p[i]][i]  += minf;
         else flow[i][-p[i]] -= minf;
      }
   }
   for(i=0; i<n; i++)   MAX_FLOW += flow[source][i];
   return MAX_FLOW;
}
```

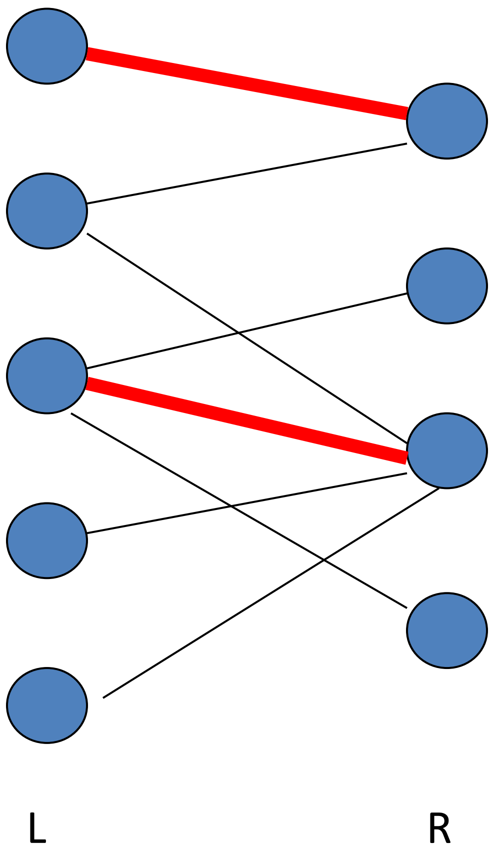
Bipartite Matching

- ❑ A matching in a *Bipartite Graph* is a set of the edges chosen in such a way that no two edges share an endpoint.
- ❑ A maximum matching is a matching of maximum size (maximum number of edges).
- ❑ In a maximum matching, if any edge is added to it, it is no longer a matching. There can be more than one maximum matchings for a given Bipartite Graph.

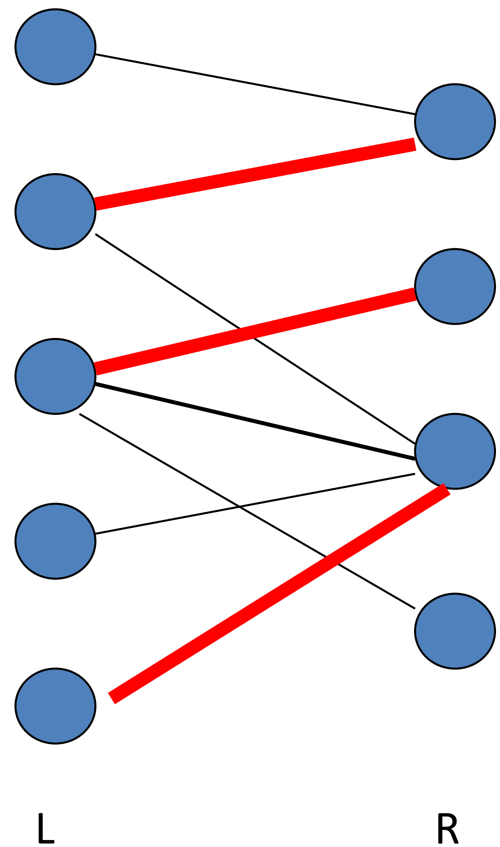


Tremendous real applications …

Maximum Bipartite Matching

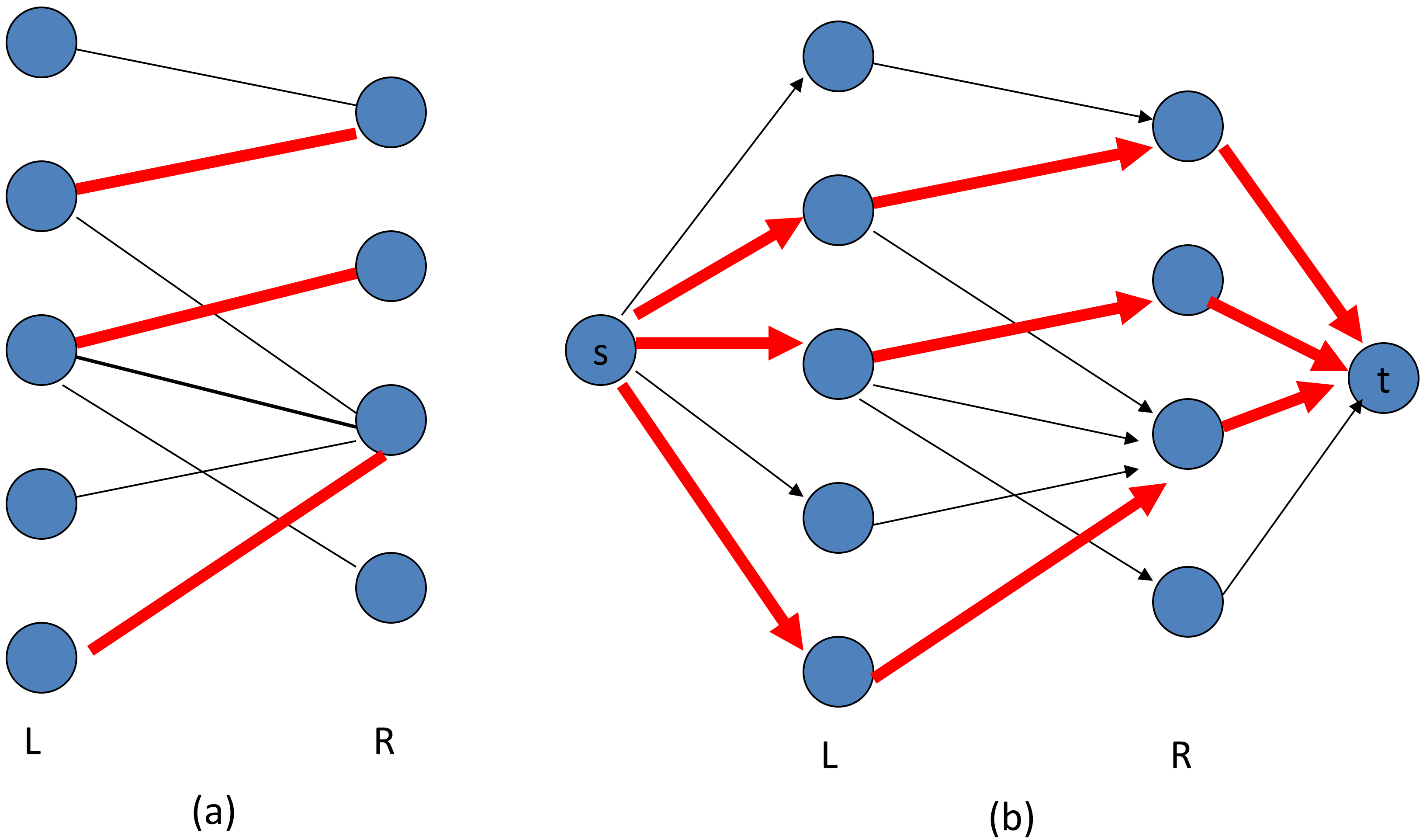


(a)



(b)

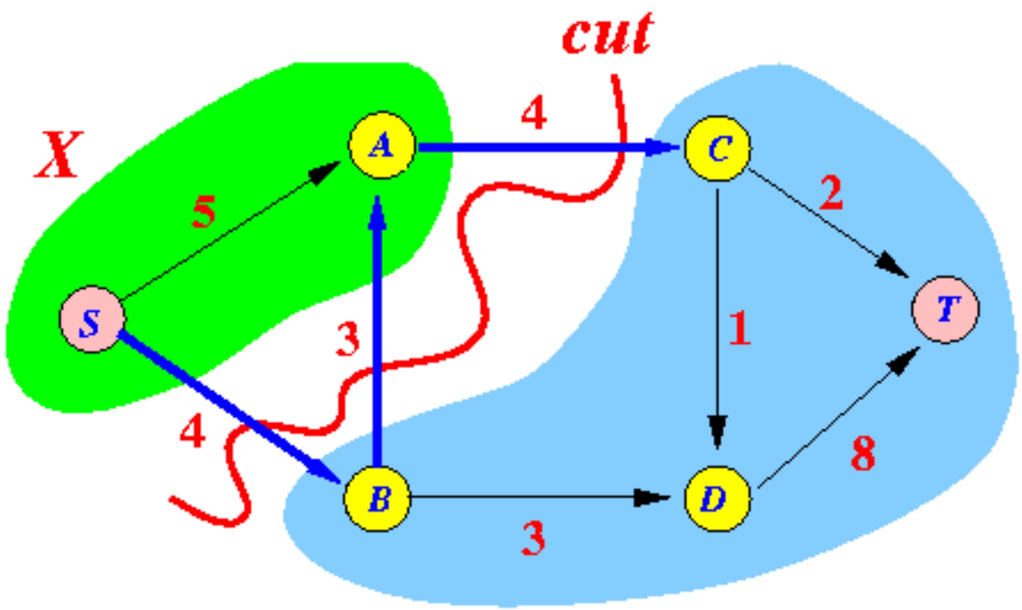
Maximum Bipartite Matching



We can solve this by running maximum flow!

s-t Cut

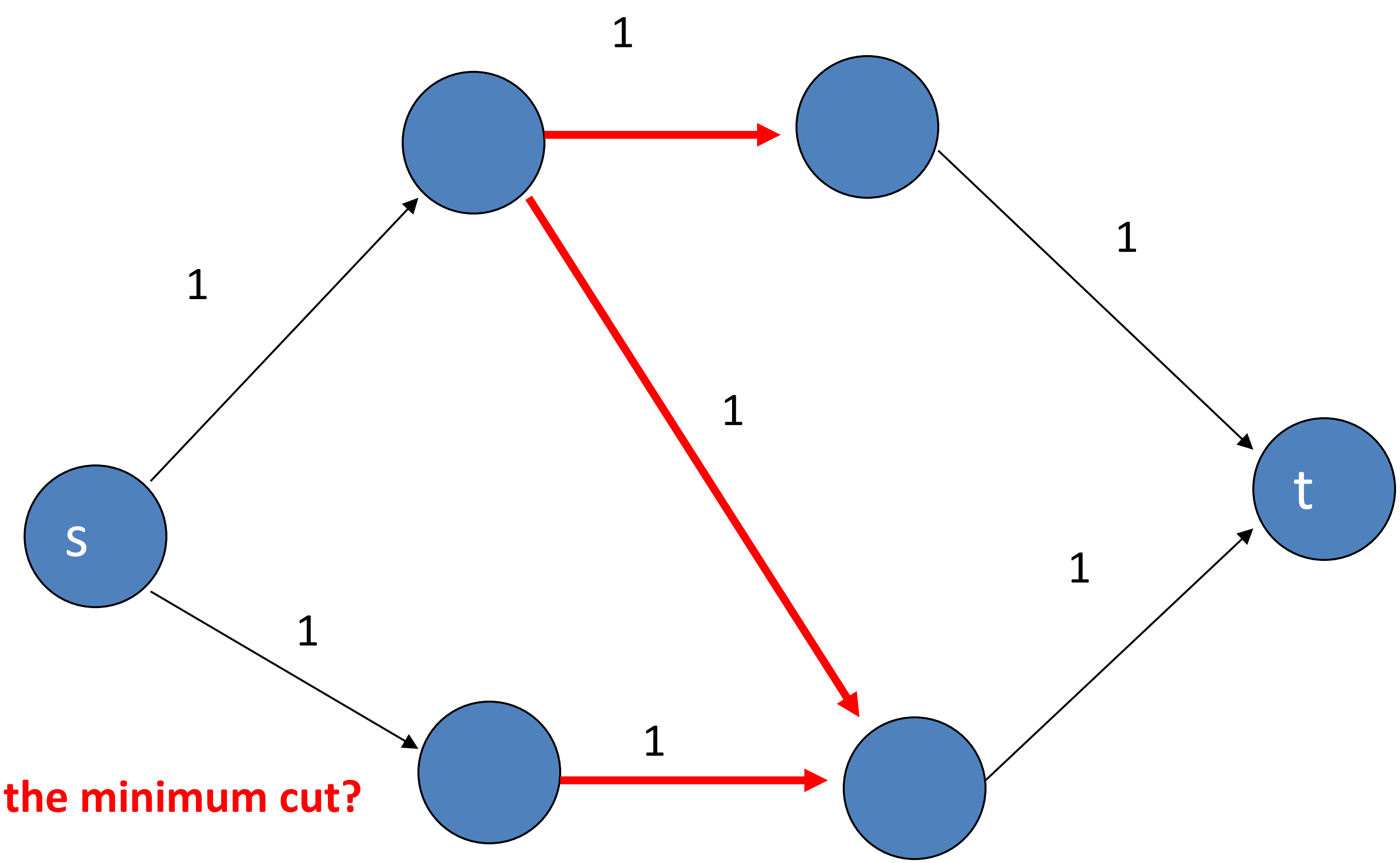
- A s-t cut of a graph G consists of an edge set E such that G - E separate s and t in two components



Cut = { (S,B), (B,A), (A,C) }

Minimum s-t Cut

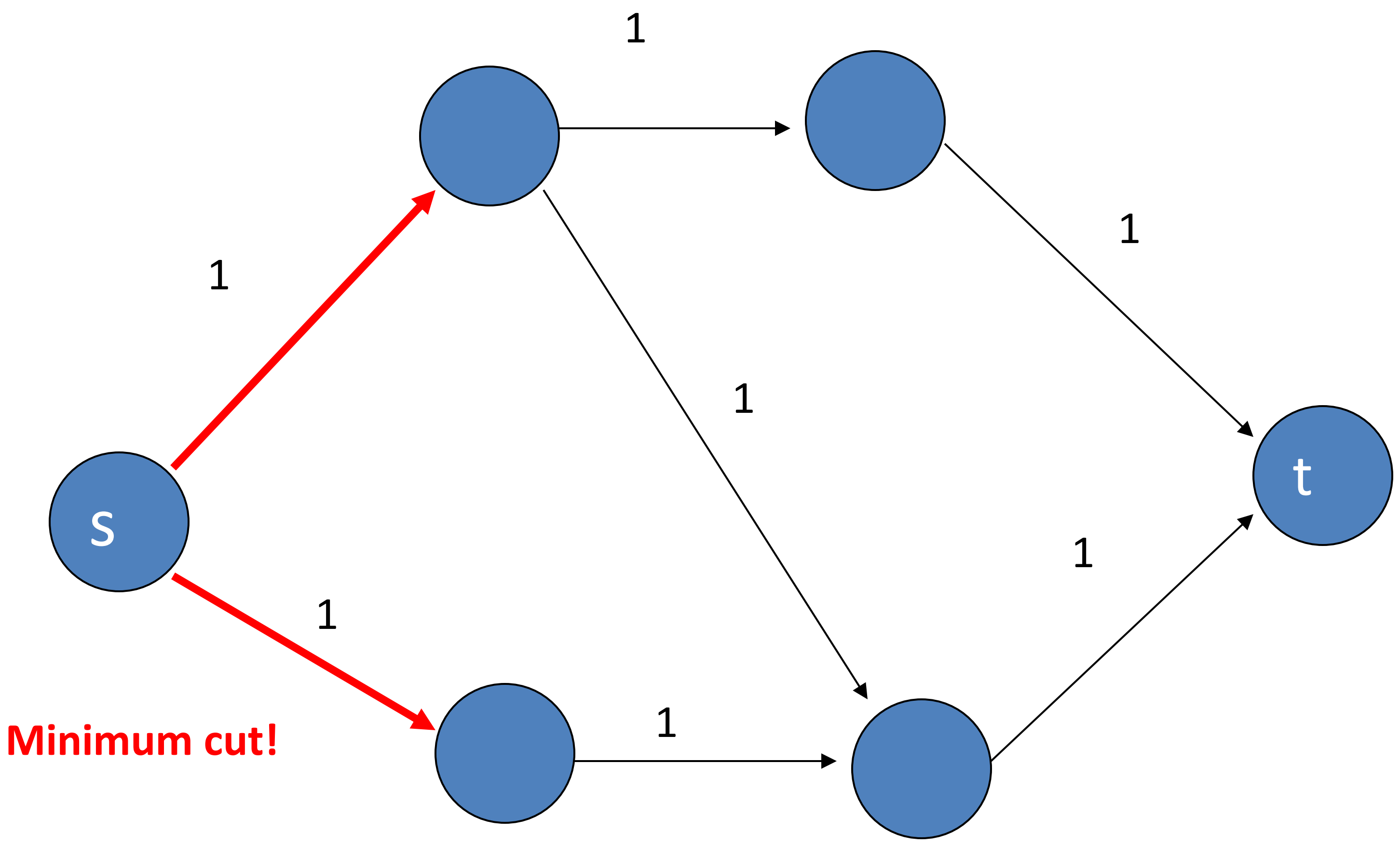
- The s-t cut set with the minimum weight



Is this the minimum cut?

Minimum s-t Cut

- The s-t cut set with the minimum weight



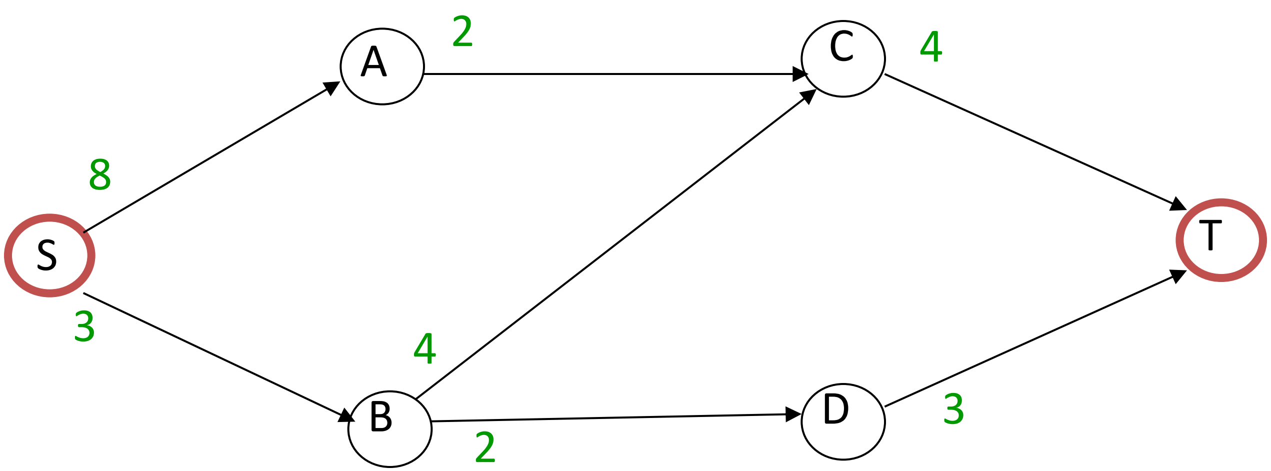
Observation

- Can we give upper bounds on the maximum flow value before finding any augmenting paths?
- One possible upper bound is the total capacity of the arcs leaving the source:
- Another upper bound is the total capacity of the arcs entering the sink:

Note that this upper bound is equal to the maximum flow value.

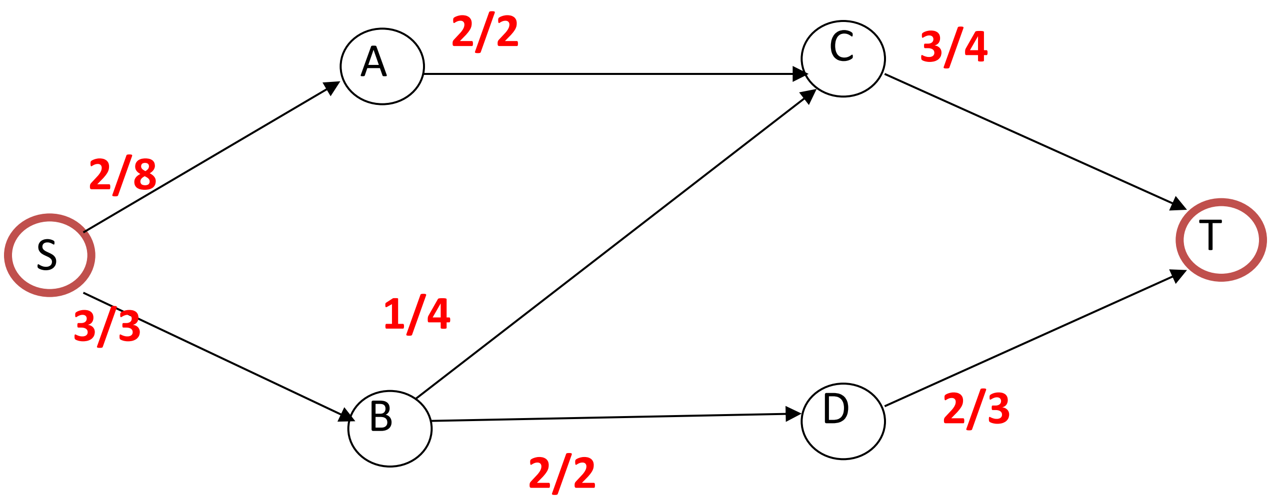
Thus, we could recognize that the algorithm output is optimal simply by **comparing the flow value with the upper bound.**

Example



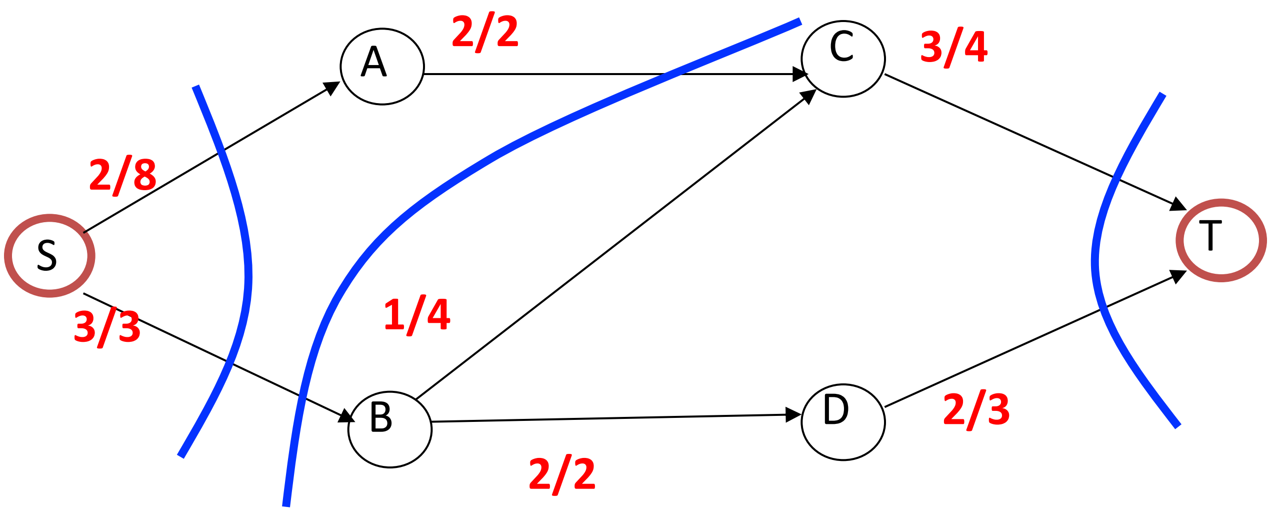
Example

- Maximum flow: 5



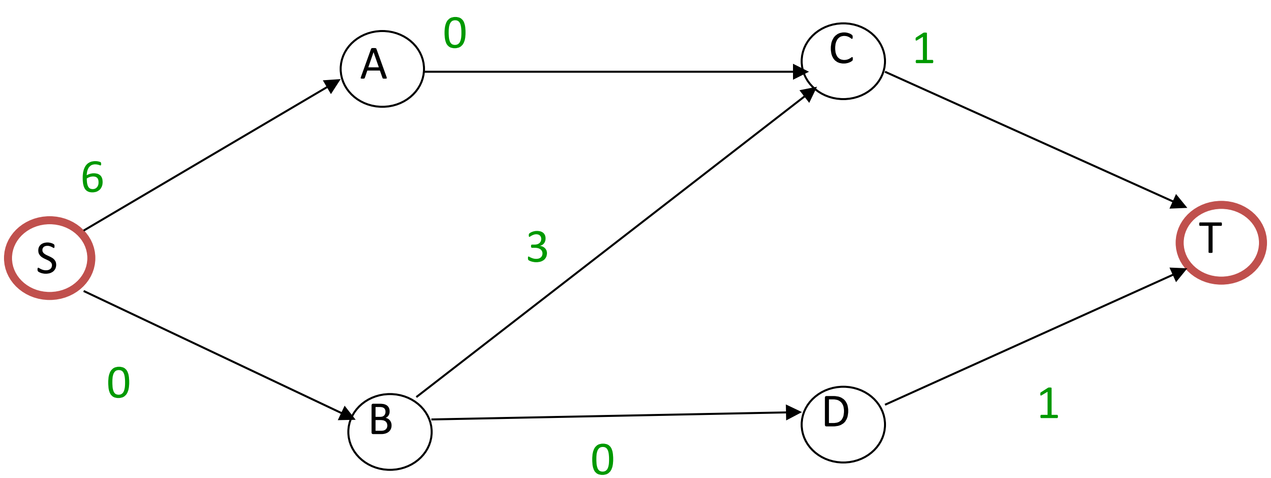
Example

- Find the upper bound of the flow value



Example

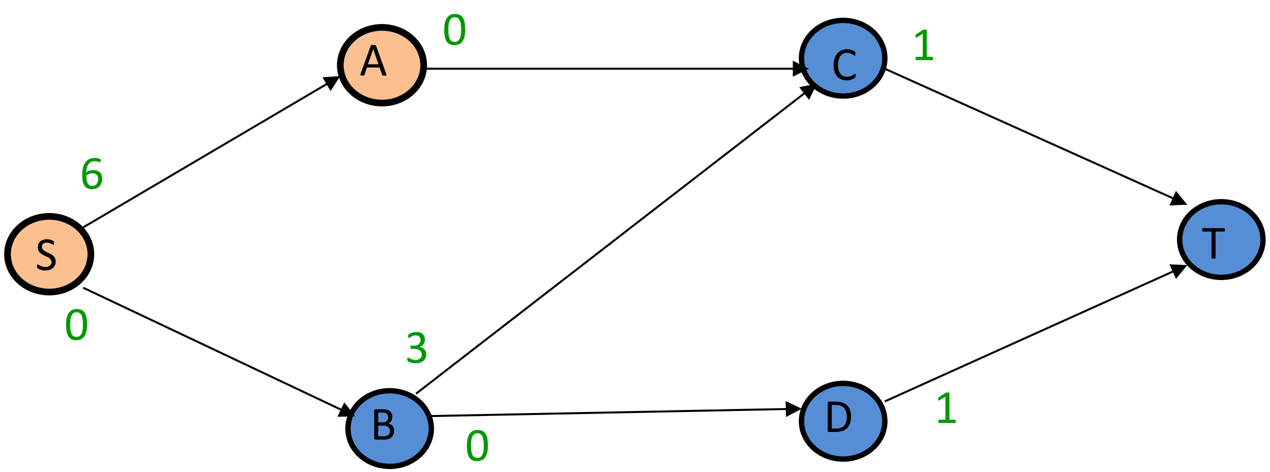
- **Residual network tells us the separation!**
  - For brevity, we only show normal directions



  - All nodes reachable from s belong to the same side!

Example

- **We can do another traversal to find the partition**
  - Cut on s side: {S, A}
  - Cut on t side: {B, C, D, T}



Summary

- Maximum flow
- Bipartite matching
- Minimum s-t cut