Stability Analysis Case Study: Simplex Architecture Controlled Inverted Pendulum

Taylor Johnson

University of Illinois at Urbana-Champaign
ECE598SM1
Fall 2008

December 4, 2008

Inverted Pendulum

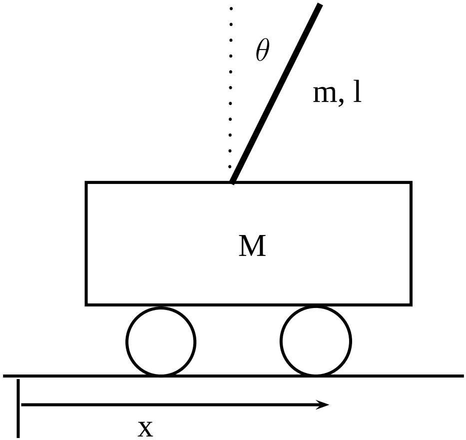


Figure: Inverted Pendulum System

Nonlinear System Model

Nonlinear System Model

- $\dot{X} = f(X, u)$

Nonlinear System Model

- $\dot{X} = f(X, u)$
- Equations of Motion for Plant
  $(m + M)\ddot{x} + \frac{1}{2}ml\cos(\theta)\ddot{\theta} - \frac{1}{2}ml\sin(\theta)\dot{\theta}^2 = F - f_c$
  $\frac{1}{2}ml\theta\ddot{x} + \frac{1}{3}ml^2\ddot{\theta} - \frac{1}{2}mgl\sin(\theta) = -f_p$

Nonlinear System Model

- $\dot{X} = f(X, u)$
- Equations of Motion for Plant
  $(m + M)\ddot{x} + \frac{1}{2}ml\cos(\theta)\ddot{\theta} - \frac{1}{2}ml\sin(\theta)\dot{\theta}^2 = F - f_c$
  $\frac{1}{2}ml\theta\ddot{x} + \frac{1}{3}ml^2\ddot{\theta} - \frac{1}{2}mgl\sin(\theta) = -f_p$
- Full System Model (including Motor Dynamics)
  $\ddot{x} = \frac{1}{D}\left[\frac{1}{3}ml^2(B_lV_a - f_c - C_1) + \frac{1}{2}ml\cos(\theta)(f_p + C_2)\right]$
  $\ddot{\theta} = \frac{1}{D}\left[-\frac{1}{2}ml\cos(\theta)(B_lV_a - f_c - C_1) - \bar{M}(f_p + C_2)\right]$

Nonlinear System Model

- $\dot{X} = f\left(X, u\right)$
- Equations of Motion for Plant
  $\left(m + M\right)\ddot{x} + \frac{1}{2}ml\cos\left(\theta\right)\ddot{\theta} - \frac{1}{2}ml\sin\left(\theta\right)\dot{\theta}^2 = F - f_c$
  $\frac{1}{2}ml\theta\ddot{x} + \frac{1}{3}ml^2\ddot{\theta} - \frac{1}{2}mgl\sin\left(\theta\right) = -f_p$
- Full System Model (including Motor Dynamics)
  $\ddot{x} = \frac{1}{D}\left[\frac{1}{3}ml^2\left(B_l V_a - f_c - C_1\right) + \frac{1}{2}ml\cos\left(\theta\right)\left(f_p + C_2\right)\right]$
  $\ddot{\theta} = \frac{1}{D}\left[-\frac{1}{2}ml\cos\left(\theta\right)\left(B_l V_a - f_c - C_1\right) - \bar{M}\left(f_p + C_2\right)\right]$
  where $D = \frac{1}{3}\bar{M}ml^2 - \frac{1}{4}m^2l^2\cos^2\left(\theta\right)$
  $\bar{M} = \frac{m+M+(K_g * J_m)}{r^2}$
  $C_1 = \bar{B}\dot{x} - \frac{1}{2}ml\sin\left(\theta\right)\dot{\theta}^2$
  $C_2 = -\frac{1}{2}mgl\sin\left(\theta\right)$
  $\bar{B} = \frac{K_g B_m}{r^2} + \frac{K_g^2 K_i K_b}{r^2 R_a}$
  $B_l = \frac{K_g K_i}{r R_a}$

Linearized System Model

Linearized System Model

- $\dot{X} = AX + Bu$

Linearized System Model

- $\dot{X} = AX + Bu$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x - x_s \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix},$$

Linearized System Model

- $\dot{X} = AX + Bu$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x - x_s \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_{22} & -a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & a_{43} & -a_{44} \end{bmatrix},$$

Linearized System Model

- $\dot{X} = AX + Bu$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x - x_s \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_{22} & -a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & a_{43} & -a_{44} \end{bmatrix}, B = \begin{bmatrix} 0 \\ b_2 \\ 0 \\ -b_4 \end{bmatrix}$$

Linearized System Model

- $\dot{X} = AX + Bu$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x - x_s \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_{22} & -a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & a_{43} & -a_{44} \end{bmatrix}, B = \begin{bmatrix} 0 \\ b_2 \\ 0 \\ -b_4 \end{bmatrix}$$

- Notable linearizations are $\sin(\theta) \approx \theta$ for small θ and armature inductance $L_a = 0$ to remove the armature current I_a state variable, reducing the order of the system, thus leaving only armature voltage V_a for control ($u = V_a$)

Linear State Feedback

Linear State Feedback

- Linear model
  $\dot{X}(t) = AX(t) + Bu(t)$

Linear State Feedback

- Linear model
  $\dot{X}(t) = AX(t) + Bu(t)$
- Variation-of-constants formula (linear solution)
  $X(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$

Linear State Feedback

- Linear model
  $\dot{X}(t) = AX(t) + Bu(t)$
- Variation-of-constants formula (linear solution)
  $X(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$
- Control is of the form
  $u(t) = KX(t)$

Linear State Feedback

- Linear model
  $\dot{X}(t) = AX(t) + Bu(t)$
- Variation-of-constants formula (linear solution)
  $X(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$
- Control is of the form
  $u(t) = KX(t)$
- Linear state feedback
  $\dot{X}(t) = (A + BK)X(t)$

Linear State Feedback

- Linear model
  $\dot{X}(t) = AX(t) + Bu(t)$
- Variation-of-constants formula (linear solution)
  $X(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$
- Control is of the form
  $u(t) = KX(t)$
- Linear state feedback
  $\dot{X}(t) = (A + BK)X(t)$
- Linear state feedback solution
  $X(t) = e^{(A+BK)(t-t_0)}X(t_0)$

Linear State Feedback

- Linear model
  $\dot{X}(t) = AX(t) + Bu(t)$
- Variation-of-constants formula (linear solution)
  $X(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$
- Control is of the form
  $u(t) = KX(t)$
- Linear state feedback
  $\dot{X}(t) = (A + BK)X(t)$
- Linear state feedback solution
  $X(t) = e^{(A+BK)(t-t_0)}X(t_0)$
- First-order approximations for $\dot{x}$ and $\dot{\theta}$
  $\dot{\theta}(t) = \frac{[\theta(t)-\theta(t-mT_s)]}{mT_s}$
  $\dot{x}(t) = \frac{[x(t)-x(t-mT_s)]}{mT_s}$
  where m is an integer greater than one (chosen as 2 by experimentation)

System Constraints

System Constraints

- Plant Constraints

System Constraints

- Plant Constraints
  $x \in [-0.7, 0.7]$ meters

System Constraints

- Plant Constraints
  $x \in [-0.7, 0.7]$ meters
  $\dot{x} \in [-1.0, 1.0]$ meters/second

System Constraints

- Plant Constraints
  $x \in [-0.7, 0.7]$ meters
  $\dot{x} \in [-1.0, 1.0]$ meters/second
  $\theta \in [-30, 30]^{\circ}$

System Constraints

- Plant Constraints
  $x \in [-0.7, 0.7]$ meters
  $\dot{x} \in [-1.0, 1.0]$ meters/second
  $\theta \in [-30, 30]^\circ$
  $\dot{\theta}$ is unconstrained

System Constraints

- Plant Constraints
  $x \in [-0.7, 0.7]$ meters
  $\dot{x} \in [-1.0, 1.0]$ meters/second
  $\theta \in [-30, 30]^\circ$
  $\dot{\theta}$ is unconstrained
- Control Constraints

System Constraints

- Plant Constraints
  $x \in [-0.7, 0.7]$ meters
  $\dot{x} \in [-1.0, 1.0]$ meters/second
  $\theta \in [-30, 30]^\circ$
  $\dot{\theta}$ is unconstrained
- Control Constraints
  $V_a \in [-4.96, 4.96]$ volts

System Constraints

- Plant Constraints
  $x \in [-0.7, 0.7]$ meters
  $\dot{x} \in [-1.0, 1.0]$ meters/second
  $\theta \in [-30, 30]^\circ$
  $\dot{\theta}$ is unconstrained
- Control Constraints
  $V_a \in [-4.96, 4.96]$ volts
- Feasible Region
  Defined by these constraints, that is, only certain physical conditions are feasible for the system

System Constraints

- Plant Constraints
  $x \in [-0.7, 0.7]$ meters
  $\dot{x} \in [-1.0, 1.0]$ meters/second
  $\theta \in [-30, 30]^\circ$
  $\dot{\theta}$ is unconstrained
- Control Constraints
  $V_a \in [-4.96, 4.96]$ volts
- Feasible Region
  Defined by these constraints, that is, only certain physical conditions are feasible for the system
- Stabilizable Region
  This is a subset of the feasible region, determined by solving a linear-matrix inequality (LMI) problem to find controller gains that maximize the region

Simplex Architecture

Simplex Architecture

- Analytical Redundancy

Simplex Architecture

- Analytical Redundancy
- Controllers

Simplex Architecture

- Analytical Redundancy
- Controllers
  - Safety Controller ($u_{sc}(t) = K_{sc}X(t)$)

Simplex Architecture

- Analytical Redundancy
- Controllers
  - Safety Controller ($u_{sc}(t) = K_{sc} X(t)$)
  - Baseline Controller ($u_{bc}(t) = K_{bc} X(t)$)

Simplex Architecture

- Analytical Redundancy
- Controllers
  - Safety Controller ($u_{sc}(t) = K_{sc}X(t)$)
  - Baseline Controller ($u_{bc}(t) = K_{bc}X(t)$)
  - Experimental Controller ($u_{ec}(t) = K_{ec}X(t)$)

Simplex Architecture

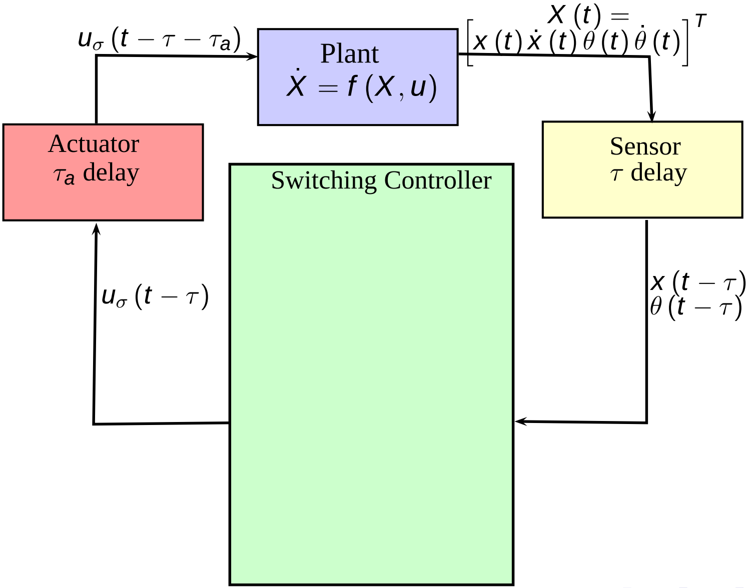
- Analytical Redundancy
- Controllers
  - Safety Controller ($u_{sc}(t) = K_{sc}X(t)$)
  - Baseline Controller ($u_{bc}(t) = K_{bc}X(t)$)
  - Experimental Controller ($u_{ec}(t) = K_{ec}X(t)$)
- Switching Controller
  Choose $u_\sigma \in [u_{sc}, u_{bc}, u_{ec}]$ based on the current (and future) stabilizable region(s) the system is within

Simplex Architecture

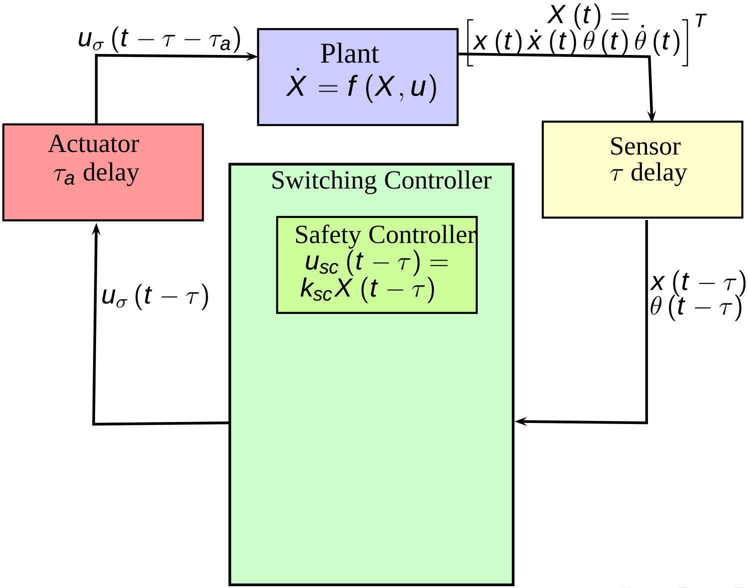
- Analytical Redundancy
- Controllers
  - Safety Controller ($u_{sc}(t) = K_{sc}X(t)$)
  - Baseline Controller ($u_{bc}(t) = K_{bc}X(t)$)
  - Experimental Controller ($u_{ec}(t) = K_{ec}X(t)$)
- Switching Controller
  Choose $u_\sigma \in [u_{sc}, u_{bc}, u_{ec}]$ based on the current (and future) stabilizable region(s) the system is within
- Entire system modeled using HIOA (not shown here due to time-constraints)

Simplex Architecture

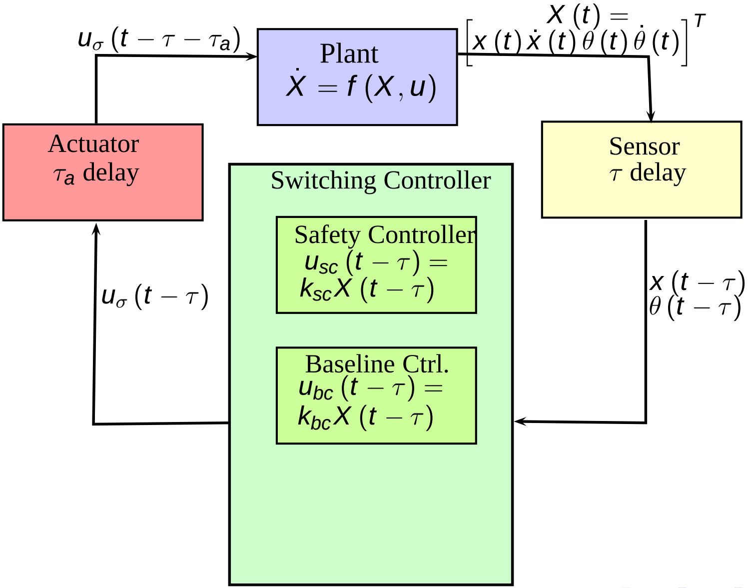
Simplex Architecture



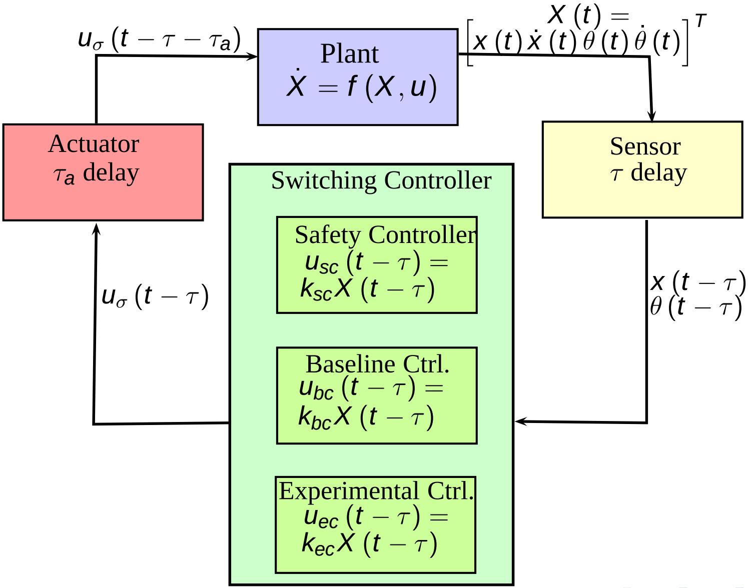
Simplex Architecture



Simplex Architecture



Simplex Architecture



Small-Gain Theorem

Small-Gain Theorem

- If the original system $\dot{X} = f(X, u)$ is input-to-state stable (ISS), and a disturbance input is small enough, then the system with disturbance is ISS.

Small-Gain Theorem

- If the original system $\dot{X} = f(X, u)$ is input-to-state stable (ISS), and a disturbance input is small enough, then the system with disturbance is ISS.
- What is small enough?

Small-Gain Theorem

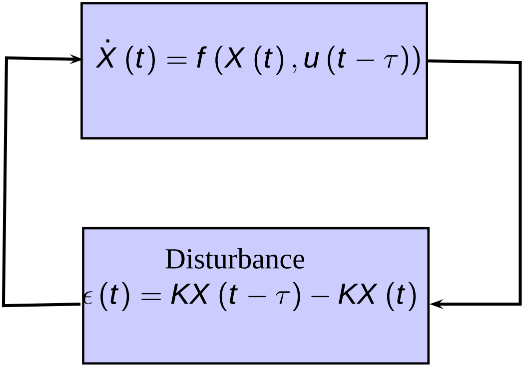
- If the original system $\dot{X} = f(X, u)$ is input-to-state stable (ISS), and a disturbance input is small enough, then the system with disturbance is ISS.
- What is small enough? Effectively less than a system's stability margin, in the LTI case, the real-axis distance of eigenvalues to the right-half plane.

Small-Gain Theorem

- If the original system $\dot{X} = f(X, u)$ is input-to-state stable (ISS), and a disturbance input is small enough, then the system with disturbance is ISS.
- What is small enough? Effectively less than a system's stability margin, in the LTI case, the real-axis distance of eigenvalues to the right-half plane.
- In the case of a disturbance (error) caused by delay, this allows one to bound the delay by bounding the error caused by the delay, allowing one to determine a tolerable (not maximal) delay to ensure ISS.

Small-Gain Theorem - Coupling with Disturbance from Delay

Small-Gain Theorem - Coupling with Disturbance from Delay



Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t - \tau)$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t - \tau) = KX(t) + \epsilon(t)$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t - \tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t - \tau) - KX(t)$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t - \tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t - \tau) - KX(t)$
- System model with disturbance from delay
  $\dot{X}(t) = (A + BK)X(t) + B\epsilon(t)$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t - \tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t - \tau) - KX(t)$
- System model with disturbance from delay
  $\dot{X}(t) = (A + BK)X(t) + B\epsilon(t)$
- Small Gain Argument
  $\epsilon = -\int_{t-\tau}^{t} K(AX(s) + BKX(s - \tau))\,ds$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t-\tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t-\tau) - KX(t)$
- System model with disturbance from delay
  $\dot{X}(t) = (A+BK)X(t) + B\epsilon(t)$
- Small Gain Argument
  $\epsilon = -\int_{t-\tau}^{t} K(AX(s) + BKX(s-\tau))\,ds$
  $|\epsilon| \leq \tau(||KA|| + ||KBK||) \cdot ||X||_{[t-2\tau,t]}$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t-\tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t-\tau) - KX(t)$
- System model with disturbance from delay
  $\dot{X}(t) = (A + BK)X(t) + B\epsilon(t)$
- Small Gain Argument
  $\epsilon = -\int_{t-\tau}^{t} K(AX(s) + BKX(s-\tau))\,ds$
  $|\epsilon| \leq \tau(||KA|| + ||BK||) \cdot ||X||_{[t-2\tau,t]}$
  $d = (||KA|| + ||KBK||)$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t - \tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t - \tau) - KX(t)$
- System model with disturbance from delay
  $\dot{X}(t) = (A + BK)X(t) + B\epsilon(t)$
- Small Gain Argument
  $\epsilon = -\int_{t-\tau}^{t} K(AX(s) + BKX(s - \tau))\,ds$
  $|\epsilon| \leq \tau(||KA|| + ||KBK||) \cdot ||X||_{[t-2\tau,t]}$
  $d = (||KA|| + ||KBK||)$
- Lyapunov function satisfied by $\dot{V} = -X^T QX + X^T PB\epsilon$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t - \tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t - \tau) - KX(t)$
- System model with disturbance from delay
  $\dot{X}(t) = (A + BK)X(t) + B\epsilon(t)$
- Small Gain Argument
  $\epsilon = -\int_{t-\tau}^{t} K(AX(s) + BKX(s - \tau))\,ds$
  $|\epsilon| \leq \tau(||KA|| + ||KBK||) \cdot ||X||_{[t-2\tau,t]}$
  $d = (||KA|| + ||KBK||)$
- Lyapunov function satisfied by $\dot{V} = -X^T QX + X^T PB\epsilon$
  Conservative bound
  $\dot{V} \leq -X^T QX + |X| \cdot |\epsilon| \cdot ||PB||$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t-\tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t-\tau) - KX(t)$
- System model with disturbance from delay
  $\dot{X}(t) = (A + BK)X(t) + B\epsilon(t)$
- Small Gain Argument
  $\epsilon = -\int_{t-\tau}^{t} K(AX(s) + BKX(s-\tau))\,ds$
  $|\epsilon| \leq \tau(||KA|| + ||KBK||) \cdot ||X||_{[t-2\tau,t]}$
  $d = (||KA|| + ||KBK||)$
- Lyapunov function satisfied by $\dot{V} = -X^T QX + X^T PB\epsilon$
  Conservative bound
  $\dot{V} \leq -X^T QX + |X| \cdot |\epsilon| \cdot ||PB||$
  $\dot{V} \leq -\lambda_{min}(Q)|X|^2 + |X| \cdot |\epsilon| \cdot ||PB||$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t-\tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t-\tau) - KX(t)$
- System model with disturbance from delay
  $\dot{X}(t) = (A + BK)X(t) + B\epsilon(t)$
- Small Gain Argument
  $\epsilon = -\int_{t-\tau}^{t} K(AX(s) + BKX(s-\tau))\,ds$
  $|\epsilon| \leq \tau(||KA|| + ||KBK||) \cdot ||X||_{[t-2\tau,t]}$
  $d = (||KA|| + ||KBK||)$
- Lyapunov function satisfied by $\dot{V} = -X^T QX + X^T PB\epsilon$
  Conservative bound
  $\dot{V} \leq -X^T QX + |X| \cdot |\epsilon| \cdot ||PB||$
  $\dot{V} \leq -\lambda_{min}(Q)|X|^2 + |X| \cdot |\epsilon| \cdot ||PB||$
  $\dot{V} = -|X|(\lambda_{min}(Q)|X| - ||PB|| \cdot |\epsilon|)$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t-\tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t-\tau) - KX(t)$
- System model with disturbance from delay
  $\dot{X}(t) = (A+BK)X(t) + B\epsilon(t)$
- Small Gain Argument
  $\epsilon = -\int_{t-\tau}^{t} K(AX(s) + BKX(s-\tau))\,ds$
  $|\epsilon| \leq \tau(||KA|| + ||KBK||) \cdot ||X||_{[t-2\tau,t]}$
  $d = (||KA|| + ||KBK||)$
- Lyapunov function satisfied by $\dot{V} = -X^T QX + X^T PB\epsilon$
  Conservative bound
  $\dot{V} \leq -X^T QX + |X| \cdot |\epsilon| \cdot ||PB||$
  $\dot{V} \leq -\lambda_{min}(Q)|X|^2 + |X| \cdot |\epsilon| \cdot ||PB||$
  $\dot{V} = -|X|(\lambda_{min}(Q)|X| - ||PB|| \cdot |\epsilon|)$
  $|x| > \frac{||PB||}{\lambda_{min}(Q)} \cdot |\epsilon|$

Bounding Our System for ISS and Small-Gains

- Delayed control $u(t) = KX(t-\tau) = KX(t) + \epsilon(t)$
- Error from delay
  $\epsilon(t) = KX(t-\tau) - KX(t)$
- System model with disturbance from delay
  $\dot{X}(t) = (A + BK)X(t) + B\epsilon(t)$
- Small Gain Argument
  $\epsilon = -\int_{t-\tau}^{t} K(AX(s) + BKX(s-\tau))\,ds$
  $|\epsilon| \leq \tau(||KA|| + ||KBK||) \cdot ||X||_{[t-2\tau,t]}$
  $d = (||KA|| + ||KBK||)$
- Lyapunov function satisfied by $\dot{V} = -X^T QX + X^T PB\epsilon$
  Conservative bound
  $\dot{V} \leq -X^T QX + |X| \cdot |\epsilon| \cdot ||PB||$
  $\dot{V} \leq -\lambda_{min}(Q)|X|^2 + |X| \cdot |\epsilon| \cdot ||PB||$
  $\dot{V} = -|X|(\lambda_{min}(Q)|X| - ||PB|| \cdot |\epsilon|)$
  $|x| > \frac{||PB||}{\lambda_{min}(Q)} \cdot |\epsilon|$
  Define $\rho(r) = cr$ where $c = \frac{||PB||}{\lambda_{min}(Q)}$

Small-Gain Theorem for Our Case

- Assume the undelayed (undisturbed) system is ISS, then if there exists some τ such that $\tau cd < 1$,

Small-Gain Theorem for Our Case

- Assume the undelayed (undisturbed) system is ISS, then if there exists some τ such that $\tau cd < 1$, the disturbed system is ISS with respect to the disturbance caused by the delay error (ϵ), and a tolerable delay is $\tau < \frac{1}{cd}$.

Small-Gain Theorem for Our Case

- Assume the undelayed (undisturbed) system is ISS, then if there exists some τ such that $\tau cd < 1$, the disturbed system is ISS with respect to the disturbance caused by the delay error (ϵ), and a tolerable delay is $\tau < \frac{1}{cd}$.
- For this system, the following tolerable delays were found for each controller $\tau_{sc} = 9$ms, $\tau_{bc} = 12.3$ms, $\tau_{ec} = 18.6$ms.

Conclusions and Future Work

Conclusions and Future Work

- Control period is $T_s = 20$ms, so this method cannot guarantee stability with respect to the actual delay experienced, so model-based state projection is necessary.

Conclusions and Future Work

- Control period is $T_s = 20$ms, so this method cannot guarantee stability with respect to the actual delay experienced, so model-based state projection is necessary.
- Note that small-gain is only a sufficient condition, so it could be the case that it is stable without state projection. A future line of work could be to define and solve an optimization problem to find the maximum tolerable τ.

Conclusions and Future Work

- Control period is $T_s = 20$ms, so this method cannot guarantee stability with respect to the actual delay experienced, so model-based state projection is necessary.
- Note that small-gain is only a sufficient condition, so it could be the case that it is stable without state projection. A future line of work could be to define and solve an optimization problem to find the maximum tolerable τ.
- When perfect state projection is considered, stability is then guaranteed for a delay $T_s + \tau$. State projection is not perfect here however, due primarily to the first-order approximation used to reconstruct $\dot{x}$ and $\dot{\theta}$ since they are unobservable, in addition to other errors like quantization not explicitly shown here. Further work could be done here as well.

Questions

? ? ? ? ?

Questions: Lyapunov Stability

Questions: Lyapunov Stability

- Stable in the Sense of Lyapunov (isL) if $\forall \epsilon > 0 . \exists \delta = \delta(\epsilon) > 0$ such that, if $||x(0)|| < \delta$, then $||x(t)|| < \epsilon, \forall t \geq 0$

Questions: Lyapunov Stability

- Stable in the Sense of Lyapunov (isL) if $\forall \epsilon > 0. \exists \delta = \delta(\epsilon) > 0$ such that, if $||x(0)|| < \delta$, then $||x(t)|| < \epsilon, \forall t \geq 0$
- Asymptotically Stable
  If isL and if $\exists \delta > 0$ such that if $||x(0)|| < \delta$, then $x(t) \rightarrow 0$ as $t \rightarrow \inf$

Questions: Lyapunov Stability

- Stable in the Sense of Lyapunov (isL) if $\forall \epsilon > 0. \exists \delta = \delta(\epsilon) > 0$ such that, if $||x(0)|| < \delta$, then $||x(t)|| < \epsilon, \forall t \geq 0$
- Asymptotically Stable
If isL and if $\exists \delta > 0$ such that if $||x(0)|| < \delta$, then $x(t) \rightarrow 0$ as $t \rightarrow \inf$
- LTI Case
If $\exists P = P^T > 0$ for some $Q = Q^T > 0$ that satisfies Lyapunov equation $A^T P + PA + Q = 0$, then the system is asymptotically stable, and the Lyapunov function is $V = X^T PX$

Questions: Input-to-State Stability (ISS)

Questions: Input-to-State Stability (ISS)

- Comparison Functions of Hahn
  Class $\mathcal{K}$: continuous, strictly increasing
  Class $\mathcal{K}_\infty$: same as class $\mathcal{K}$ and unbounded
  Class $\mathcal{KL}$: continuous, strictly decreasing

Questions: Input-to-State Stability (ISS)

- Comparison Functions of Hahn
  Class $\mathcal{K}$: continuous, strictly increasing
  Class $\mathcal{K}_\infty$: same as class $\mathcal{K}$ and unbounded
  Class $\mathcal{KL}$: continuous, strictly decreasing
- Consider $\dot{x}(t) = f(x(t), w(t))$ with solutions $\varphi(t, x, w)$

Questions: Input-to-State Stability (ISS)

- Comparison Functions of Hahn
  Class $\mathcal{K}$: continuous, strictly increasing
  Class $\mathcal{K}_\infty$: same as class $\mathcal{K}$ and unbounded
  Class $\mathcal{KL}$: continuous, strictly decreasing
- Consider $\dot{x}(t) = f(x(t), w(t))$ with solutions $\varphi(t, x, w)$
- System is ISS if $\exists \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that $\forall x(t_0)$, $\forall w$, and $\forall t \geq 0$, then
  $||\varphi(t, x, w)|| \leq \max\{\beta(||x||, t), \gamma(||w||_\infty)\}$

Questions: Nonlinear System Model Derived from Energy

- Coordinates of Small Portion of Bar with Mass *dm*
  $x_{dm} = x + q \sin(\theta) \Rightarrow \dot{x}_{dm} = \dot{x} + q \cos(\theta)\,\dot{\theta}$
  $y_{dm} = q \cos(\theta) \Rightarrow \dot{y}_{dm} = -q \sin(\theta)\,\dot{\theta}$

Questions: Nonlinear System Model Derived from Energy

- Coordinates of Small Portion of Bar with Mass *dm*
  $x_{dm} = x + q\sin(\theta) \Rightarrow \dot{x}_{dm} = \dot{x} + q\cos(\theta)\,\dot{\theta}$
  $y_{dm} = q\cos(\theta) \Rightarrow \dot{y}_{dm} = -q\sin(\theta)\,\dot{\theta}$
- Kinetic Energy of Small Portion of Bar with Mass *dm*
  $K_{dm} = \frac{1}{2} dm \left(\dot{x}_{dm}^2 + \dot{y}_{dm}^2\right)$

Questions: Nonlinear System Model Derived from Energy

- Coordinates of Small Portion of Bar with Mass *dm*
  $x_{dm} = x + q\sin(\theta) \Rightarrow \dot{x}_{dm} = \dot{x} + q\cos(\theta)\dot{\theta}$
  $y_{dm} = q\cos(\theta) \Rightarrow \dot{y}_{dm} = -q\sin(\theta)\dot{\theta}$
- Kinetic Energy of Small Portion of Bar with Mass *dm*
  $K_{dm} = \frac{1}{2}dm\left(\dot{x}_{dm}^2 + \dot{y}_{dm}^2\right)$
  $= \frac{1}{2}(dm)\left(\dot{x}^2 + 2q\cos(\theta)\dot{x}\dot{\theta} + q^2\dot{\theta}^2\right)$

Questions: Nonlinear System Model Derived from Energy

- Coordinates of Small Portion of Bar with Mass dm
  $x_{dm} = x + q\sin(\theta) \Rightarrow \dot{x}_{dm} = \dot{x} + q\cos(\theta)\dot{\theta}$
  $y_{dm} = q\cos(\theta) \Rightarrow \dot{y}_{dm} = -q\sin(\theta)\dot{\theta}$
- Kinetic Energy of Small Portion of Bar with Mass dm
  $K_{dm} = \frac{1}{2}dm\left(\dot{x}_{dm}^2 + \dot{y}_{dm}^2\right)$
  $= \frac{1}{2}(dm)\left(\dot{x}^2 + 2q\cos(\theta)\dot{x}\dot{\theta} + q^2\dot{\theta}^2\right)$
- Potential Energy of Small Portion of Bar with Mass dm
  $P_{dm} = (dm)g\cos(\theta)$

Questions: Nonlinear System Model Derived from Energy

- Coordinates of Small Portion of Bar with Mass *dm*
  $x_{dm} = x + q\sin(\theta) \Rightarrow \dot{x}_{dm} = \dot{x} + q\cos(\theta)\dot{\theta}$
  $y_{dm} = q\cos(\theta) \Rightarrow \dot{y}_{dm} = -q\sin(\theta)\dot{\theta}$
- Kinetic Energy of Small Portion of Bar with Mass *dm*
  $K_{dm} = \frac{1}{2}dm\left(\dot{x}_{dm}^2 + \dot{y}_{dm}^2\right)$
  $= \frac{1}{2}(dm)\left(\dot{x}^2 + 2q\cos(\theta)\dot{x}\dot{\theta} + q^2\dot{\theta}^2\right)$
- Potential Energy of Small Portion of Bar with Mass *dm*
  $P_{dm} = (dm)g\cos(\theta)$
- Integrating over Bar Length 0 to l
  $K = K_c + K_p = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml\cos(\theta)\dot{x}\dot{\theta} + \frac{1}{6}ml^2\dot{\theta}^2$
  $P = \frac{1}{2}mgl\cos(\theta)$

Questions: Nonlinear System Model Derived from Energy

- Coordinates of Small Portion of Bar with Mass *dm*
  $x_{dm} = x + q\sin(\theta) \Rightarrow \dot{x}_{dm} = \dot{x} + q\cos(\theta)\,\dot{\theta}$
  $y_{dm} = q\cos(\theta) \Rightarrow \dot{y}_{dm} = -q\sin(\theta)\,\dot{\theta}$
- Kinetic Energy of Small Portion of Bar with Mass *dm*
  $K_{dm} = \frac{1}{2}dm\left(\dot{x}_{dm}^2 + \dot{y}_{dm}^2\right)$
  $= \frac{1}{2}(dm)\left(\dot{x}^2 + 2q\cos(\theta)\,\dot{x}\dot{\theta} + q^2\dot{\theta}^2\right)$
- Potential Energy of Small Portion of Bar with Mass *dm*
  $P_{dm} = (dm)\,g\cos(\theta)$
- Integrating over Bar Length 0 to l
  $K = K_c + K_p = \frac{1}{2}(M+m)\,\dot{x}^2 + \frac{1}{2}ml\cos(\theta)\,\dot{x}\dot{\theta} + \frac{1}{6}ml^2\dot{\theta}^2$
  $P = \frac{1}{2}mgl\cos(\theta)$
- Lagrangian $L = K - P$

Questions: Nonlinear System Model Derived from Energy

- Coordinates of Small Portion of Bar with Mass *dm*
  $x_{dm} = x + q\sin(\theta) \Rightarrow \dot{x}_{dm} = \dot{x} + q\cos(\theta)\,\dot{\theta}$
  $y_{dm} = q\cos(\theta) \Rightarrow \dot{y}_{dm} = -q\sin(\theta)\,\dot{\theta}$
- Kinetic Energy of Small Portion of Bar with Mass *dm*
  $K_{dm} = \frac{1}{2}dm\left(\dot{x}_{dm}^2 + \dot{y}_{dm}^2\right)$
  $= \frac{1}{2}(dm)\left(\dot{x}^2 + 2q\cos(\theta)\,\dot{x}\dot{\theta} + q^2\dot{\theta}^2\right)$
- Potential Energy of Small Portion of Bar with Mass *dm*
  $P_{dm} = (dm)\,g\cos(\theta)$
- Integrating over Bar Length 0 to *l*
  $K = K_c + K_p = \frac{1}{2}(M+m)\,\dot{x}^2 + \frac{1}{2}ml\cos(\theta)\,\dot{x}\dot{\theta} + \frac{1}{6}ml^2\dot{\theta}^2$
  $P = \frac{1}{2}mgl\cos(\theta)$
- Lagrangian $L = K - P$
- Euler-Lagrange Equations for Forces on System
  $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F - f_c$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = -f_p$

Questions: Plant HIOA

automaton Plant(A : `Real`$^{4\times4}$, B : `Real`$^{4\times1}$)
variables
  input u : `Real`;
  output θ : `Real`; x : `Real`;
  internal θ_h : `Real` $:= 0$; $\dot{\theta}_h$: `Real` $:= 0$;
    x_h : `Real` $:= 0$; $\dot{x}_h$: `Real` $:= 0$;

trajectories
  trajdef plantDynamics
    evolve $d(x_h) = \dot{x}_h$;
      $d(\dot{x}_h) = -a_{22}\dot{x}_h - a_{23}\theta_h + a_{24}\dot{\theta}_h + b_2 u$;
      $d(\theta_h) = \dot{\theta}_h$;
      $d(\dot{\theta}_h) = a_{42}\dot{x}_h + a_{43}\theta_h - a_{34}\dot{\theta}_h - b_4 u$;
      $\theta = \theta_h$; $x = x_h$;

Figure: Linearized Plant Model

Questions: Sensor HIOA

automaton Sensor(T_s : `Real`) **where** $T_s > 0$
signature
  output sample(θ', x' : `Real`)

variables
  input θ : `Real`; x : `Real`;
  internal θ_s : `Real`; x_s : `Real`;
    now_s : `Real` $:= 0$;
    $next\_sample$: `AugmentedReal` $:= T_s$;
    let $time\_left := next\_sample - now_s$

transitions
  output sample(θ', x')
    pre $now_s = next\_sample \wedge \theta' = \theta_s \wedge x' = x_s$;
    eff $next\_sample := next\_sample + T_s$;

trajectories
  trajdef periodicSample
    stop when $now_s = next\_sample$
    evolve $d(now_s) = 1$; $\theta_s = \theta$; $x_s = x$;

Figure: Sensor

Questions: Actuator HIOA

automaton Actuator(T_a : $\texttt{Real}$) **where** $T_a > 0$
  signature
    input controllerOutput(u' : $\texttt{Real}$)

  variables
    output u : $\texttt{Real}$;
    internal u_a : $\texttt{Discrete Real} := 0$;
      $ready_a$: $\texttt{Bool} := false$;
      now_a : $\texttt{Real} := 0$;

  transitions
    input switchingOutput(u')
      eff $u_a = u'$;
        $ready_a := true$;

  trajectories
    trajdef hold
      evolve $d(now_a) = 1$; $u = u_a$;

Figure: Actuator

Questions: Safety, Baseline, Experimental Controller HIOAs

automaton SafetyController(K_{sc} : $\texttt{Real}^{4\times 1}$, T_s, T_{safety} : $\texttt{Real}$, m : $\texttt{Int}$)
signature
input sample(θ', x' : $\texttt{Real}$)
output safetyOutput(u'_{sc} : $\texttt{Real}$)

variables
internal θ_{sc} : $\texttt{Real} := 0$; $\dot{\theta}_{sc}$: $\texttt{Real} := 0$;
x_{sc} : $\texttt{Real} := 0$; $\dot{x}_{sc}$: $\texttt{Real} := 0$;
u_{sc} : $\texttt{Real} := 0$; rt : $\texttt{Real} := 0$;
next_cycle : $\texttt{AugmentedReal} := T_{safety}$;
buffer: $\texttt{Seq}[prevTheta : \texttt{Real}, prevX : \texttt{Real}] := \{\}$;
let *time_left* $:=$ *next_cycle* $-$ *rt*;
let *length* $:=$ *length*(*buffer*)

Figure: Safety Controller

Questions: Safety, Baseline, Experimental Controller HIOAs (cont)

transitions
  input sample(θ', x')
  eff $buffer := buffer \vdash [\theta_{sc}, x_{sc}]$
    $\theta_{sc} := \theta'$; $x_{sc} := x'$;

  output safetyOutput(u'_{sc})
    pre $rt = next\_cycle \wedge u'_{sc} = u_{sc}$
    eff $next\_cycle := next\_cycle + T_{safety}$;
      $\dot{\theta}_{sc} := [\theta_{sc} - head(buffer).prevTheta]/(mT_s)$;
      $\dot{x}_{sc} := [x_{sc} - head(buffer).prevX]/(mT_s)$;
      $u'_{sc} := K_{sc1} * x_{sc} + K_{sc2} * \dot{x}_{sc} + K_{sc3} * \theta_{sc} + K_{sc4} * \dot{\theta}_{sc}$;
      $buffer := tail(buffer)$;

trajectories
  trajdef periodicControl
    stop when $rt = next\_cycle$
    evolve $d(rt) = 1$;

Figure: Safety Controller