# Spectra of sampled sounds

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This Supporting Information document holds an extension of the exposition in the main document to illustrate the spectra os sampled sounds through the components of traditional waveforms with few samples.

CCS Concepts: •**Applied computing** →**Sound and music computing;** •**Computing methodologies** →*Modeling methodologies;* •**General and reference** →*Surveys and overviews; Reference works;*

Additional Key Words and Phrases: music, acoustics, psychophysics, digital audio, signal processing

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The sinusoidal components in the discretized sound have some particularities. Considering a signal $T$ and its corresponding Fourier decomposition $\mathcal{F}\langle T\rangle = C = \{c_k\}_0^{\Lambda-1} = \left\{\sum_{i=0}^{\Lambda-1} t_i e^{-ji(k\frac{2\pi}{\Lambda})}\right\}_0^{\Lambda-1}$, the recomposition is the sum of the frequency components to yield the temporal samples[1]:

$$
\begin{aligned}
t_i =& \frac{1}{\Lambda}\sum_{k=0}^{\Lambda-1} c_k e^{j\frac{2\pi k}{\Lambda}i} \\
=& \frac{1}{\Lambda}\sum_{k=0}^{\Lambda-1} (a_k + j.b_k)\left[cos(w_k i) + j.sen(w_k i)\right]
\end{aligned}
\tag{SI-B-1}
$$

where $c_k = a_k + j.b_k$ defines the amplitude and phase of each frequency: $w_k = \frac{2\pi}{\Lambda}k$ in radians or $f_k = w_k\frac{f_s}{2\pi} = \frac{f_s}{\Lambda}k$ in Hertz, and are limited by $w_k \leq \pi$ and $f_k \leq \frac{f_s}{2}$ as given by the Nyquist Theorem.

For a sonic signal, samples $t_i$ are real and are given by the real part of Equation SI-B-1:

[1]The factor $\frac{1}{\Lambda}$ can be distributed among the Fourier transform and its reconstruction, as preferred. Note that $j$ here is the imaginary unit $j^2 = -1$.

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$$
\begin{aligned}
t_i &= \frac{1}{\Lambda}\sum_{k=0}^{\Lambda-1}\left[a_k cos(w_k i) - b_k sen(w_k i)\right] \\
&= \frac{1}{\Lambda}\sum_{k=0}^{\Lambda-1}\sqrt{a_k^2+b_k^2}\; cos\left[w_k i - arctan(b_k, a_k)\right]
\end{aligned}
\tag{SI-B-2}
$$

where $arctan(x, y) \in [0, 2\pi]$ is the inverse tangent with the right choice of the quadrant in the imaginary plane.

$\Lambda$ real samples $t_i$ result in $\Lambda$ complex coefficients $c_k = a_k + j.b_k$. The coefficients $c_k$ are equivalent two by two, corresponding to the same frequencies and with the same contribution to its reconstruction. They are complex conjugates: $a_{k1} = a_{k2}$ and $b_{k1} = -b_{k2}$ and, as a consequence, the modules are equal and phases have opposite signs. Recalling that $f_k = k\frac{f_s}{\Lambda},\ k \in \left\{0, ..., \left\lfloor \frac{\Lambda}{2} \right\rfloor\right\}$. When $k > \frac{\Lambda}{2}$, the frequency $f_k$ is mirrored through $\frac{f_s}{2}$ in this way: $f_k = \frac{f_s}{2} - (f_k - \frac{f_s}{2}) = f_s - f_k = f_s - k\frac{f_s}{\Lambda} = (\Lambda - k)\frac{f_s}{\Lambda} \quad \Rightarrow \quad f_k \equiv f_{\Lambda-k}\ ,\ \ \forall\ \ k < \Lambda$.

The same applies to $w_k = f_k\frac{2\pi}{f_s}$ and the periodicity $2\pi$: it follows that $w_k = -w_{\Lambda-k}\ ,\ \ \forall\ \ k < \Lambda$. Given the cosine (an even function) and the inverse tangent (an odd function), the components in $w_k$ and $w_{\Lambda-k}$ contribute with coefficients $c_k = c^*_{\Lambda-k}$ in the reconstruction of the real samples. In summary, in a decomposition of $\Lambda$ samples, the $\Lambda$ frequency components $\{c_i\}_0^{\Lambda-1}$ are equivalent in pairs, except for $f_0$, and, when $\Lambda$ is even, for $f_{\Lambda/2} = f_{\max} = \frac{f_s}{2}$. Both these components are isolated, i.e. there is one and only one component at frequency $f_0$ or $f_{\Lambda/2}$ (if $\Lambda$ is even). In fact, when $k = 0$ or $k = \Lambda/2$ the mirror of the frequencies are themselves: $f_{\Lambda/2} = f_{(\Lambda-\Lambda/2)=\Lambda/2}$ and $f_0 = f_{(\Lambda-0)=\Lambda} = f_0$. Furthermore, these two frequencies (zero and Nyquist frequency) do not have a phase offset: their coefficients are strictly real. Therefore, the number $\tau_\Lambda$ of equivalent coefficient pairs in a decomposition of $\Lambda$ samples is:

$$
\tau_\Lambda = \frac{\Lambda - \Lambda\%2}{2} - 2 + \Lambda\%2 = \left\lceil \frac{\Lambda}{2} \right\rceil - 2
\tag{SI-B-3}
$$

This discussion can be summarized in the following equivalences:

$$
f_k \equiv f_{\Lambda-k}\quad ,\quad w_k \equiv -w_{\Lambda-k} \tag{SI-B-4}
$$

$$
a_k = a_{\Lambda-k}\quad ,\quad b_k = -b_{\Lambda-k} \tag{SI-B-5}
$$

$$
\sqrt{a_k^2+b_k^2} = \sqrt{a_{\Lambda-k}^2+b_{\Lambda-k}^2} \tag{SI-B-6}
$$

$$
arctan(b_k, a_k) = -arctan(b_{\Lambda-k}, a_{\Lambda-k}) \tag{SI-B-7}
$$

with $\forall\ 1 \le k \le \tau_\Lambda, k \in \mathbb{N}$.

To express the general case for components combination in each sample $t_i$, one can gather the relations for the reconstruction of a real signal (Equation SI-B-2), for the number of paired coefficients (Equation SI-B-3), and for the equivalences of modules (Equation SI-B-6) and phases (Equation SI-B-7):

$$
t_i = \frac{a_0}{\Lambda} + \frac{a_{\Lambda/2}}{\Lambda}(1 - \Lambda\%2) + \frac{2}{\Lambda}\sum_{k=1}^{\tau_\Lambda}\sqrt{a_k^2+b_k^2}\; cos\left[w_k i - arctan(b_k, a_k)\right]
\tag{SI-B-8}
$$

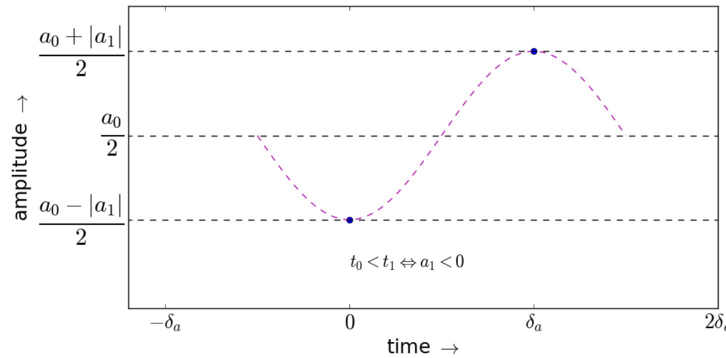


Fig. SI-B-1. Oscillation of 2 samples (maximum frequency for any $f_s$). The first coefficient determines a constant detachment (called *offset*, *bias* or *DC component*) and the second coefficient specifies the oscillation amplitude.

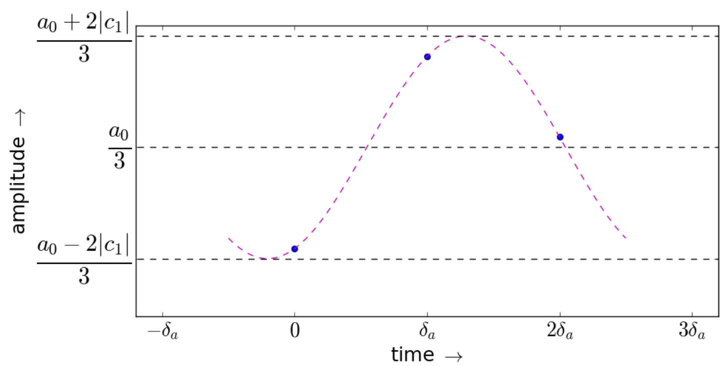


Fig. SI-B-2. Three fixed samples present only one non-null frequency. $c_1 = c_2^*$ and $w_1 \equiv w_2$.

Figure SI-B-1 shows two samples and their spectral component. When there is only two samples, the Fourier decomposition has only one pair of coefficients $\{c_k = a_k - j.b_k\}_0^{\Lambda-1=1}$ relative to frequencies $\{f_k\}_0^1 = \left\{w_k \frac{f_s}{2\pi}\right\}_0^1 = \left\{k \frac{f_s}{\Lambda=2}\right\}_0^1 = \left\{0, \frac{f_s}{2} = f_{\max}\right\}$ with energies $e_k = \frac{(c_k)^2}{\Lambda=2}$. The role of amplitudes $a_k$ is clearly observed with $\frac{a_0}{2}$, the fixed offset (also called *bias* or *DC component*), and $\frac{a_1}{2}$ for the oscillation with frequency $f_1 = \frac{f_s}{\Lambda=2}$. This case has special relevance: at least 2 samples are necessary to represent an oscillation and it yields the Nyquist frequency $f_{\max} = \frac{f_s}{2}$, which is the maximum frequency in a sound sampled with $f_s$ samples per second.

All fixed sequences $T$ of only 3 samples also have just 1 frequency, since the first harmonic would have 1.5 samples and exceeds the bottom limit of 2 samples, i.e. the frequency of the harmonic would exceed the Nyquist frequency: $\frac{2.f_s}{3} > \frac{f_s}{2}$. The coefficients $\{c_k\}_0^{\Lambda-1=2}$ are present in 3 frequency components. One is relative to frequency zero ($c_0$), and the other two ($c_1$ and $c_2$) have the same role for reconstructing a sinusoid with $f = f_s/3$. This case is illustrated in Figure SI-B-2.

With 4 samples it is possible to represent 1 or 2 frequencies with independence of magnitude and phase. Figure SI-B-3 depicts the contribution of each of the two (possible) components. The

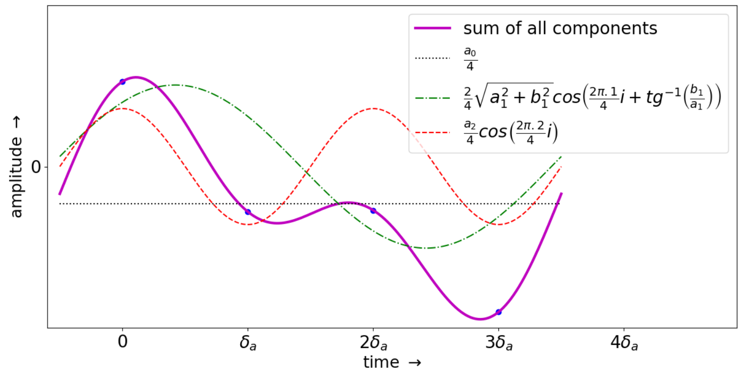


Fig. SI-B-3. Frequency components for 4 samples.

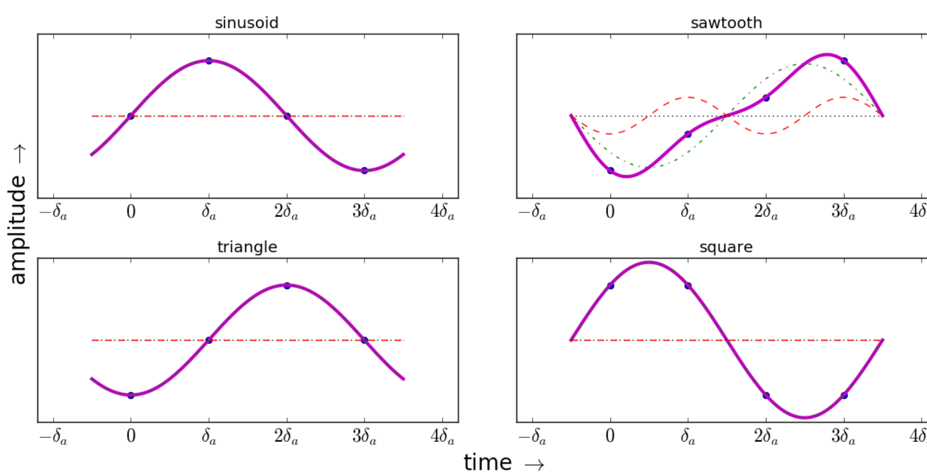


Fig. SI-B-4. Basic waveforms with 4 samples.

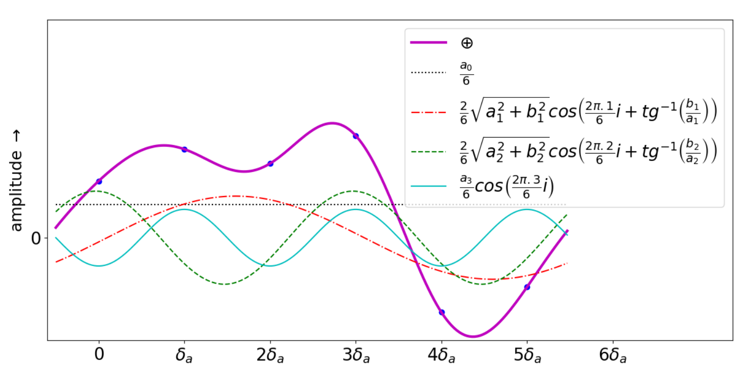


Fig. SI-B-5. Frequency components for 6 samples: 4 sinusoids, one of them is the *bias* with zero frequency.

individual components sum to the original waveform and a brief inspection reveals the major curvatures resulting from the higher frequency, while the fixed offset is captured in the component with frequency $f_0 = 0$. Figure SI-B-4 shows the harmonics for the basic waveforms of Equations 10, 11, 12 and 13 in the case of 4 samples. There is only 1 sinusoid for each waveform, with the exception of the sawtooth, which has the even harmonics.

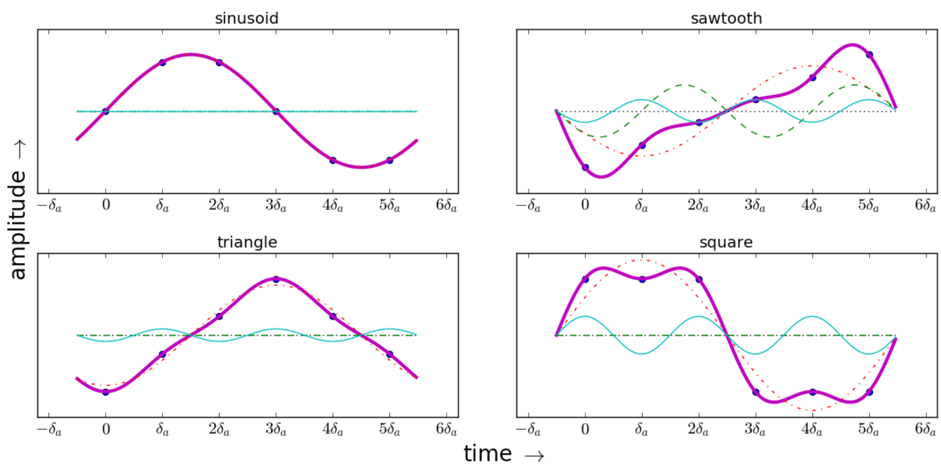


Fig. SI-B-6. Basic waveforms with 6 samples: triangular and square waveforms have odd harmonics, with different proportions and phases; the sawtooth has even harmonics.

Figure SI-B-5 exposes the sinusoidal components within 6 samples, while Figure SI-B-6 presents the decomposition of the basic waveforms: square and triangular have the same components but in different proportions, while the sawtooth has an extra component.

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