Laplacian in Spherical Coordinates

We want to write the Laplacian functional

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1}$$

in spherical coordinates

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \tag{2}$$

To do so we need to invert the previous transformation rules and repeatedly use the chain rule

《高数下》9.4：多元复合函数的推导
详见 jpg

$$\begin{aligned} \frac{\partial}{\partial x(r,\theta,\phi)} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial y(r,\theta,\phi)} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z(r,\theta,\phi)} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} \end{aligned} \tag{3}$$

and

《高数下》p-81.例4使用的方法

$$\begin{aligned} \frac{\partial^2}{\partial x^2(r,\theta,\phi)} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \\ &= \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}\right) = \\ &= \left(\frac{\partial^2 r}{\partial x^2}\right)\frac{\partial}{\partial r} + \left(\frac{\partial r}{\partial x}\frac{\partial r}{\partial x}\right)\frac{\partial^2}{\partial r^2} + \left(\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x}\right)\frac{\partial^2}{\partial r\partial \theta} + \left(\frac{\partial r}{\partial x}\frac{\partial \phi}{\partial x}\right)\frac{\partial^2}{\partial r\partial \phi} + \\ &\quad + \left(\frac{\partial^2 \theta}{\partial x^2}\right)\frac{\partial}{\partial \theta} + \left(\frac{\partial \theta}{\partial x}\frac{\partial \theta}{\partial x}\right)\frac{\partial^2}{\partial \theta^2} + \left(\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x}\right)\frac{\partial^2}{\partial \theta\partial r} + \left(\frac{\partial \theta}{\partial x}\frac{\partial \phi}{\partial x}\right)\frac{\partial^2}{\partial \theta\partial \phi} + \\ &\quad + \left(\frac{\partial^2 \phi}{\partial x^2}\right)\frac{\partial}{\partial \phi} + \left(\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial x}\right)\frac{\partial^2}{\partial \phi^2} + \left(\frac{\partial \phi}{\partial x}\frac{\partial \theta}{\partial x}\right)\frac{\partial^2}{\partial \phi\partial \theta} + \left(\frac{\partial r}{\partial x}\frac{\partial \phi}{\partial x}\right)\frac{\partial^2}{\partial r\partial \phi} \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{\partial^2}{\partial y^2(r,\theta,\phi)} &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \\ &= \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}\right) = \\ &= \left(\frac{\partial^2 r}{\partial y^2}\right)\frac{\partial}{\partial r} + \left(\frac{\partial r}{\partial y}\frac{\partial r}{\partial y}\right)\frac{\partial^2}{\partial r^2} + \left(\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y}\right)\frac{\partial^2}{\partial r\partial \theta} + \left(\frac{\partial r}{\partial y}\frac{\partial \phi}{\partial y}\right)\frac{\partial^2}{\partial r\partial \phi} + \\ &\quad + \left(\frac{\partial^2 \theta}{\partial y^2}\right)\frac{\partial}{\partial \theta} + \left(\frac{\partial \theta}{\partial y}\frac{\partial \theta}{\partial y}\right)\frac{\partial^2}{\partial \theta^2} + \left(\frac{\partial \theta}{\partial y}\frac{\partial r}{\partial y}\right)\frac{\partial^2}{\partial \theta\partial r} + \left(\frac{\partial \theta}{\partial y}\frac{\partial \phi}{\partial y}\right)\frac{\partial^2}{\partial \theta\partial \phi} + \\ &\quad + \left(\frac{\partial^2 \phi}{\partial y^2}\right)\frac{\partial}{\partial \phi} + \left(\frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial y}\right)\frac{\partial^2}{\partial \phi^2} + \left(\frac{\partial \phi}{\partial y}\frac{\partial \theta}{\partial y}\right)\frac{\partial^2}{\partial \phi\partial \theta} + \left(\frac{\partial r}{\partial y}\frac{\partial \phi}{\partial y}\right)\frac{\partial^2}{\partial r\partial \phi} \end{aligned} \tag{5}$$

$$
\begin{aligned}
\frac{\partial^2}{\partial z^2(r,\theta,\phi)} &= \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\right) = \\
&= \frac{\partial}{\partial z}\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}\right) = \\
&= \left(\frac{\partial^2 r}{\partial z^2}\right)\frac{\partial}{\partial r} + \left(\frac{\partial r}{\partial z}\frac{\partial r}{\partial z}\right)\frac{\partial^2}{\partial r^2} + \left(\frac{\partial r}{\partial z}\frac{\partial \theta}{\partial z}\right)\frac{\partial^2}{\partial r\partial \theta} + \left(\frac{\partial r}{\partial z}\frac{\partial \phi}{\partial z}\right)\frac{\partial^2}{\partial r\partial \phi} + \\
&\quad + \left(\frac{\partial^2 \theta}{\partial z^2}\right)\frac{\partial}{\partial \theta} + \left(\frac{\partial \theta}{\partial z}\frac{\partial \theta}{\partial z}\right)\frac{\partial^2}{\partial \theta^2} + \left(\frac{\partial \theta}{\partial z}\frac{\partial r}{\partial z}\right)\frac{\partial^2}{\partial \theta\partial r} + \left(\frac{\partial \theta}{\partial z}\frac{\partial \phi}{\partial z}\right)\frac{\partial^2}{\partial \theta\partial \phi} + \\
&\quad + \left(\frac{\partial^2 \phi}{\partial z^2}\right)\frac{\partial}{\partial \phi} + \left(\frac{\partial \phi}{\partial z}\frac{\partial \phi}{\partial z}\right)\frac{\partial^2}{\partial \phi^2} + \left(\frac{\partial \phi}{\partial z}\frac{\partial \theta}{\partial z}\right)\frac{\partial^2}{\partial \phi\partial \theta} + \left(\frac{\partial r}{\partial z}\frac{\partial \phi}{\partial z}\right)\frac{\partial^2}{\partial r\partial \phi}
\end{aligned} \tag{6}
$$

where we have

$$
\begin{cases}
r = \sqrt{x^2+y^2+z^2} \\
\theta = \arctan\left(\sqrt{\frac{x^2+y^2}{z^2}}\right) \\
\phi = \arctan\left(\frac{y}{x}\right)
\end{cases} \tag{7}
$$

Then we have

$$
\frac{\partial}{\partial r} \to \left(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}\right)\frac{\partial}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r} \tag{8}
$$

此处有错：1/r^2; 应改为：2/r
但最后结果作者又改对了

$$
\frac{\partial^2}{\partial r^2} \to \left(\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2\right)\frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r^2} \tag{9}
$$

$$
\frac{\partial}{\partial \theta} \to \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}\right)\frac{\partial}{\partial \theta} = \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial \theta} \tag{10}
$$

$$
\frac{\partial^2}{\partial \theta^2} \to \left(\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2\right)\frac{\partial^2}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{11}
$$

$$
\frac{\partial}{\partial \phi} \to \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \partial}{\partial z^2}\right)\frac{\partial}{\partial \phi} = 0 \tag{12}
$$

$$
\frac{\partial^2}{\partial \phi^2} \to \left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right)\frac{\partial^2}{\partial \phi^2} = \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \tag{13}
$$

The mixed derivative terms, $\frac{\partial^2}{\partial r\partial\theta}$, $\frac{\partial^2}{\partial r\partial\phi}$ and $\frac{\partial^2}{\partial\theta\partial\phi}$, cancel out.

Putting all the terms together we get

$$
\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} = \\
&= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}r^2\sin\theta\frac{\partial}{\partial r} + \frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)
\end{aligned} \tag{14}
$$

$$
\nabla^2 = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}r^2\sin\theta\frac{\partial}{\partial r} + \frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right) \tag{15}
$$

仅为 球极坐标系 表达的 拉普拉斯公式，内容上没啥特别的