FSH512: Falling into a Realistic Black Hole

by

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I hereby declare that all the work presented in this thesis is my own, unless clearly indicated by citation.

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Student Signature

Submission Date: 21/08/20

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Abstract

It has long been known that the universe is geometric in nature. The force of gravity is nothing but the curvature of the underlying fabric of space-time. Einstein's Theory of General Relativity has proved to be the cornerstone of our understanding of the universe. One of the most fascinating predictions of the General Theory of Relativity is the existence of black holes. Black holes are regions in space-time where the curvature is so intense that space-time is divided into two distinct regions, separated by an event horizon. The fact that an infalling observer, once inside the event horizon, can no longer follow a causal geodesic that leads back to the outer region results in an inevitable encounter with the singularity, prompting us to question the nature of this experience. At present, the majority of the scientific community believes that encountering the extreme gravitational strength of a singularity crushes the observer completely. However, the notion of surviving the singularity is interesting to explore, as there have been indications suggesting that the nature of a realistic singularity could indeed be null and weak. This study aims to delve into the question of the nature of a realistic singularity and the experience of an infalling observer in such a black hole. Beginning with a comprehensive mathematical formulation of the gravitational effects of a Schwarzschild black hole on an infalling observer, this study demonstrates the existence of a strong, space-like singularity at the centre that reduces the infalling observer to zero volume with infinite stretching along one spatial dimension and infinite crushing along the other two dimensions. Furthermore, employing rigorous mathematical analysis, this study investigates and establishes the genericity of a class of solutions that indicate the presence of a null and weak spacetime singularity.

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Chapter 1: Introduction

1.1 Motivation For Research

General relativity has been a well established theory for more than a hundred years, it provided a theoretical framework that revolutionised our knowledge and understanding of the Universe to great depths. The basis of science lies in the recognition of flaws in a theory and work towards a more complete theory. Einstein recognized the flaws in Newton's theory of gravity and this motivated him to formulate his own theory of gravity, which has proved to be remarkable. Similarly, It is well known that Einstein's equation breaks down and yields ambiguous results in extreme conditions of gravity, signalling the limitations of this theory. A region of such interest is a singularity that lies within black holes. It has been long known in theory that objects such as black holes can exist but it is now known through observational evidence that black holes do exist. The Event Horizon Telescope was the first to capture a photograph of a black hole in 2017, in this case the one residing at the heart of M87 Galaxy. Further, singularity theorems imply that a singularity is expected to develop inside a black hole. Even after such strong evidence of the existence of space-time singularities, there is very little known about them. Hence, the study of singularities provides us with an opportunity to uncover a new layer of reality that emerges under extreme conditions of gravity.

1.2 Background

General relativity is a fundamental theory of gravitation formulated by Albert Einstein in the early 20th century. It describes how massive objects, like planets, stars, and galaxies, influence the curvature of spacetime around them due to their mass and energy. In simple terms, it explains how objects move in response to this curvature, giving rise to the force of gravity. General relativity has been extensively tested and confirmed through various experiments and observations and has become a cornerstone of modern physics. One of the key features of general relativity is its ability to predict the behaviour of spacetime under different conditions. Depending on the distribution of matter and energy, spacetime can curve in various ways, leading to different solutions or outcomes. These solutions often correspond to specific physical scenarios. One such scenario is the formation of black hole singularities, points of infinite density and curvature within the fabric of spacetime. Singularities are formed when the mass of an object becomes highly concentrated within a small volume. To visualise the density of a black hole, one can imagine a star with a hundred times the mass of our sun compressed into a small city.

1.3 Significance And Relevance Of The Study

The study of singularities holds profound significance in the realm of theoretical physics and our understanding of the fundamental fabric of the universe. Singularities represent points in spacetime where the conventional laws of physics break down, posing interesting questions and challenges that drive scientific exploration. Singularities are expected to develop as the final state of a realistic gravitational collapse, and it is widely believed that these singularities are BKL singularities.

The BKL (Belinsky-Khalatnikov-Lifshitz) singularity is the only known generic singularity and it deals with the behaviour of spacetime near the initial singularity of a generic, anisotropic, and inhomogeneous universe. However, even a single inhomogeneous singular vacuum solution of the BKL type has not yet been proved mathematically [1]. This study provides an in-depth mathematical analysis of the framework used in the formulation of the existence of a generic class of solutions that contains null weak singularities

1.4 The Research Aim And Questions

The study aims to understand the effects of these regions of extreme gravitational strength, i.e. singularities on an extended body following a time-like geodesic that encounters a singularity.

The study answers the following questions:

1. What is the fate of an extended body as it moves along a time-like geodesic that falls into a Schwarzchild blackhole?
2. What is the nature of the singularity that lies at the centre of the Schwarzchild black hole?
3. Can Einstein's equations yield a generic class of solutions that admit a null and weak space-time singularity?
4. What is the fate of an extended body as it moves along a time-like geodesic and falls into a null and weak space-time singularity?

1.5 Research Approach

The study starts with getting to grips with the calculation of tensor's associated with general relativity such as the Riemann curvature tensor and the Ricci tensor. Further, the geodesic equations and the geodesic deviation equations for a radially infalling time-like geodesic are formulated to understand the behaviour of the Jacobi fields as a body falls into a Schwarzchild singularity. The geodesic deviation equations provide us with a set of second order differential equations that describe the behaviour of Jacobi fields. We interpret these Jacobi fields to form a 3-dimensional body and assess the behaviour of the individual Jacobi fields by solving the second order differential equations which shows how the dimensions of the body change as it moves through the geodesic falling into the singularity.

Moving ahead, through rigorous mathematical calculations using Wolfram Mathematica this study proves the mathematical framework that describes the existence of a generic class of solutions to the Einstein field equations that admit the presence of a null space-time singularity [1]. The geodesic and geodesic deviation equations are formulated in a 2-dimensional space with the same constraints that were used to prove the existence of a weak space-time singularity

1.6 Prior Studies

The exploration of space-time singularities and black holes stands at the forefront of theoretical physics and cosmology. These captivating phenomena, arising from the curvature of space and time predicted by Einstein's theory of general relativity, have captured the attention of researchers for decades. In this section, we'll uncover the works that have shaped our knowledge, the innovative methodologies that have led to fascinating concepts and the mathematical gaps in paper

In the context of singularities, which are points or regions of spacetime where certain physical quantities become infinite or undefined, several features are considered: their location, causal character, and strength. In this study we use Tipler’s terminology [2] of gravitational strength. The strength refers to the magnitude of physical quantities near the singularity and determines the severity of the breakdown of our current understanding of physics. As per Tipler, In the presence of weak singularities, the spacetime can be extended smoothly beyond the singularity point. In these cases, the metric tensor, which is a mathematical tool describing spacetime's geometry, remains continuous and well-behaved, without becoming singular or undefined. On the contrary, strong singularities have more serious implications. They represent points or regions where the curvature of spacetime becomes infinite or where physical quantities blow up, indicating the breakdown of known laws of physics. In the presence of a strong singularity, the smooth extension of spacetime is not possible, and the metric tensor becomes singular or degenerate. The severity of a singularity matters because it influences how we can predict and understand physical systems.
The causal character of a singularity describes the type of world lines that can be followed by particles or signals in its vicinity. A spacelike singularity implies that no signal or particle can cross it, a timelike singularity allows particles but not signals to cross, and a null singularity allows both particles and signals to cross. There has been strong belief that the space-like singularity, Balinsky-Khalatnikov-Lifshitz (BKL) singularity, is a realistic one due to its genericity.

However, some recent studies have indicated the emergence of an entirely different kind of spacetime singularity within the context of realistic rotating black holes. Interestingly, this singularity takes on a null and weak form, in contrast to the spacelike properties of the BKL singularity.

- Mass-inflation model: the Kerr black hole background is approximated by the spherically symmetric Reissner-Nordstrom solution, and the gravitational perturbations are described using two cross-flowing null fluids [3,4].
- Replacing the null fluids in Mass inflation model with a spherically symmetric scalar field[5].
- nonlinear perturbation analysis of the internal structure of rotating black holes [6].

These collective findings suggest a departure from the conventional idea that the BKL singularity solely dictates the nature of realistic singularities.

1.7 Literature Gap

Despite extensive research in the field, the discussion about the structure and characteristics of general and realistic singularities continues. Some arguments propose that the non-linear nature of Einstein's equations might transform a null and weak curvature singularity into a strong and space-like one [7]. However, a thorough mathematical investigation is still necessary to prove the consistency of Einstein's equations with a null and weak curvature singularity. Ori and Flanagan have addressed this issue and introduced a mathematical approach in [1], suggesting that a weak singularity can be consistent with Einstein field equations. This can be determined by using the evolution equations on some initial data. Nevertheless, the detailed mathematical analysis supporting these claims has not been published [8]. This study provides an in-depth analysis of the mathematical framework proposed by Ori and Flanagan in [1]

1.8 Fundamentals

This section is vital for our study's exploration of spacetime singularities and their complexities. This section outlines the tools and concepts we'll use to understand these phenomena. We'll rely on differential geometry to describe curved spacetime, tensor calculus to mathematically express the relationships between curvature, matter, and gravity and differential equations will help us analyse how different quantities change in space and time. Furthermore, the study dives into the domain of black holes—a class of astrophysical objects characterised by their immense gravitational pull, where the curvature of spacetime becomes extremely high. The intricacies of black hole physics require the integration of differential geometry, tensor calculus, and differential equations to understand the behaviour of matter and energy within these extreme regions.

In curved spacetime, the traditional methods of classical calculus break down due to the fact that the spacetime geometry is no longer flat and Euclidean. Classical calculus is built on the assumption of working in flat Euclidean space, where the concept of straight lines, distances, and angles is well-defined. However, in curved spacetime, these concepts are not as straightforward. Hence, a different framework of mathematics is required.

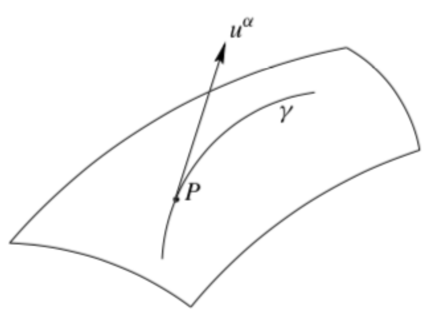

Figure 1 Tensors on Manifolds. [10]

Tensors are mathematical tools used in curved geometry. They provide a consistent framework for expressing and manipulating quantities such as vectors and matrices. There are several reasons tensors are essential tools in General Relativity:

- In curved spacetime, distances, angles, and other geometric properties change as you move from point to point. Tensors help us define these properties in a way that respects the local curvature. By using tangent spaces, we can treat the curved manifold as if it were locally flat, allowing us to perform calculations using familiar tools from linear algebra.

- The concept of differentiation is more complex in curved spacetimes than in flat space. Tensors allow us to define covariant derivatives that take into account the curvature of the manifold. These derivatives ensure that tensor equations maintain their form under coordinate transformations and accurately describe the physics of curved spacetime.
- Tensors have well-defined transformation laws under coordinate changes. When working in curved spaces, we often need to switch between different coordinate systems to account for the curvature. The components of a tensor in one coordinate system can be related to the components in another coordinate system using specific transformation rules. These rules ensure that the physical laws remain consistent regardless of the chosen coordinates.

Tensors play a vital role in describing the geometry of the spacetime and the behaviour of physical quantities within it. The tensors that are essential to the study of General relativity are:

- **Metric Tensor**

The metric tensor in general relativity is denoted by $\boldsymbol{g_{\mu\nu}}$ and it encapsulates the geometry of spacetime. It defines the relationship between the spacetime interval and the coordinate differentials. The metric tensor is used to calculate distances, angles, and other geometric properties within curved spacetime. The equation that relates the metric tensor to the spacetime interval ds and coordinate differentials dx^μ is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- **Christoffel Symbols**

Christoffel symbols are denoted by $\boldsymbol{\Gamma^\alpha_{\beta\gamma}}$ and are used to differentiate vectors in curved spacetimes. They are also called connection coefficients. While they are not tensors themselves, they are derived from the metric tensor. They help us understand how vectors change as they are moved around in curved spaces. These symbols are necessary because the usual rules of calculus and differentiation that we use in flat spaces don't work the same way in curved spaces. The formula for calculating Christoffel symbols using the metric and inverse metric is as follows:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right)$$

- **Riemann Curvature Tensor**

The Riemann curvature tensor is a four-index mathematical object. It is denoted by $\boldsymbol{R^\alpha_{\beta\mu\nu}}$ and characterises the curvature of a manifold, providing insights into the geometry of the space and how it deviates from flat Euclidean space. It encodes information about how the geometry changes as you move around on the surface and tells us whether a space is curved, how much it's curved, and how that curvature changes from point to point. The formula for calculating the curvature using Riemann tensor is as follows:

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\C} \Gamma^\C_{\nu\beta} - \Gamma^\alpha_{\nu\C} \Gamma^\C_{\mu\beta}$$

The Riemann tensor satisfies a set of symmetries which impact its properties. Although it has 256 components, these symmetries reduce the number of independent components to 20. These 20 independent components encapsulate the essential information about the curvature of spacetime and the gravitational interactions, making the manipulation and analysis of the Riemann tensor more manageable within the context of general relativity.

- $R_{\alpha\beta\mu\nu} = - R_{\beta\alpha\mu\nu}$
- $R_{\alpha\beta\mu\nu} = R_{\nu\mu\alpha\beta}$
- $R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0$

- **Ricci Tensor**

The Riemann curvature tensor contains a lot of information about how vectors change direction in a curved space. However, it has four indices and can be quite complicated to work with. The Ricci tensor is a "contracted" version of the Riemann tensor that compresses some of this information into a more manageable form. This contraction "squeezes" the information in the Riemann tensor to provide a summary of how space is curved.

- **Geodesic Equation**

A geodesic is the generalisation of a straight line to curved spaces. In flat Euclidean space, a straight line is the shortest path between two points. However, in curved spaces, this concept changes due to the curvature of the space itself. A geodesic is the path that locally minimises distance between neighbouring points on a curved surface or manifold. The Geodesic equation can be expressed as:

$$u^\beta \nabla_\beta u^\alpha = (u^\cdot)^\alpha + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu$$

The geodesic equation is a mathematical expression that describes how a particle or an object moves along a geodesic in a curved space. It's derived from the principle of least action, which states that objects follow paths that minimise a certain quantity (action). The Euler-Lagrange equation represents this fundamental principle, where the derivative of the Lagrangian with respect to the generalised velocity $\dot{x}^\alpha$ is equated to the partial derivative of the Lagrangian with respect to the generalised coordinate x^α. The equation ensures that the motion of a particle follows the path that extremizes the action, which is described by the Lagrangian L.

$$\frac{d}{ds} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0$$

- **Geodesic Deviation Equation**

In the previous section, we explored the geodesic equation, which describes the paths that particles follow in curved spacetimes. However, the universe is not just a collection of isolated particles moving along geodesics; it also contains neighbouring geodesics that interact due to the curvature of spacetime. The geodesic deviation equation provides insight into how nearby geodesics separate or converge as they move through curved spacetimes.

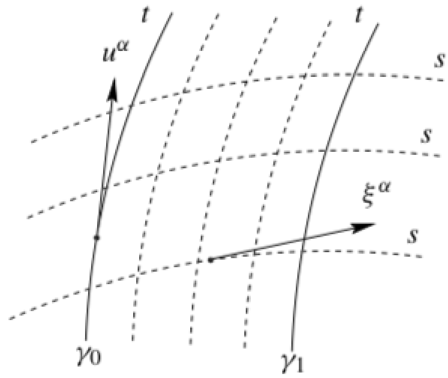

Figure 2 Deviation vector between neighbouring geodesics [11].

The geodesic deviation equation essentially states that the second derivative of the Jacobi field with respect to the affine parameter, combined with the effect of the Riemann curvature tensor, results in the Jacobi field's evolution along the reference geodesic. It is a fundamental concept in understanding the tidal forces experienced by extended objects in a gravitational field.

The Geodesic equation can be written as:

\[u^\beta \nabla_\beta u^\alpha = (u^\cdot)^\alpha + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu \]

- **Functional Genericity**

Functional genericity deals with proving that a certain property or behaviour holds for a generic set of functions within a functional space. This means that while there might be some functions that do not exhibit the property, they are exceptional cases that can be ignored for most purposes. The concept of functional genericity is often used to make statements about the behaviour of functions within a space without needing to consider every individual function. It allows us to focus on typical behaviours and properties while acknowledging that exceptional cases exist but have little impact on the overall results.

The concept of functional genericity is essential in the context of realistic black holes and space-time singularities. It is necessary that a realistic black hole solution should be generic. In this study the idea of functional genericity parallels that used by BKL [9]. Assume that a given solution exhibits a specific characteristic, such as the nature of the singularity, then for a group of solutions to remain stable under small yet generic perturbations, this group should rely on a number of arbitrary functions/degrees of freedom (denoted by k) to maintain its functional genericity. Functional genericity is a prerequisite for stability and is further essential to ensure the existence of an open set in the solution space with the desired characteristic.

1.9 Black Hole and Singularities

In the vastness of the universe, there are regions where gravity becomes incredibly powerful – so potent that even light gets trapped and cannot escape. These gravitational traps are black holes. They're like voids in space and time, regions in space where everything gets pulled in and nothing gets out. This makes it impossible to know the interior of these mysterious objects. Although they were mathematical predictions of general theory of relativity by Einstein, they were long back conceptualised by a british clergyman and underrated scientist John Michell, who imagined a star with the escape velocity equal to the speed of light. He coined those astrophysical objects as Dark stars.

Anyhow, the existence of these astrophysical objects, now known as black holes, is supported by observational evidence. Such a space-time, containing a black hole, possesses two distinct regions of space-time.

(I) Space-time between a distant point and the event horizon.

(II) Space-time between the event horizon and the singularity.

The two space-time regions can be seen as causally disconnected because once an observer has crossed the event horizon, it can no longer follow a causal geodesic that can lead him back to the region (I). This is the same reason that the surface of the event horizon is a null hypersurface, it marks the boundary where light rays can barely escape. This is due to the fact that the gravitational strength in the region (II) is so intense that the curvature diverges and limits any geodesic to it, resulting in the geodesics converging to form a caustic or singularity.

Apart from the Event horizon, Black holes possess yet another horizon. The Cauchy horizon, located within a black hole and behind the event horizon. The Cauchy horizon signifies a boundary where the curvature of spacetime starts to escalate significantly. It separates the interior region of the black hole into two distinct domains: one where predictability holds (within the Cauchy horizon) and another where predictability dissolves (beyond the Cauchy horizon and near to singularity). Within the Cauchy horizon, spacetime is still influenced by a set of initial data—the snapshot of the black hole's state at a particular moment. In this region, the evolution of spacetime can be predicted based on this initial data using the Cauchy-Kowalevski Theorem. However, just beyond the Cauchy horizon, the Cauchy-Kowalevski theorem encounters limitations. These limitations arise due to the uncertainties in prediction caused by the diverging curvature, Thus breaking down the region of predictability.

A series of singularity theorems were put forth by Roger Penrose and later refined by Stephen Hawking and Penrose himself. They provide insight into the inevitability of the development of singularities inside black holes. These theorems are crucial in understanding the nature of black holes and the behaviour of spacetime near their cores. Roger Penrose formulated his singularity theorems in 1965, which stated that under specific conditions, singularities are a natural consequence of the gravitational collapse of massive objects. These conditions include the existence of trapped surfaces, which are surfaces from which no light rays can escape to infinity. This indicates that once a trapped surface forms, the collapse will eventually lead to the formation of a singularity. Afterwards, In collaboration with Roger Penrose, Stephen Hawking refined the singularity theorems in 1970, extending their scope and applicability. The Hawking-Penrose theorems incorporated stronger energy conditions and introduced additional technical assumptions. These theorems broadened the scenarios under which singularities were expected to arise, emphasising their generality.

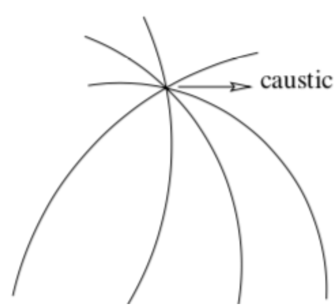

Figure 3 Singularity/Caustic formation in a Geodesic congruence. [12]

Initially, the Schwarzchild black hole was thought to be an archetype for a space-time singularity.

However, it’s understood now that the Schwarzchild black hole is not an archetype for a realistic black hole as:

- In reality most astrophysical objects possess some level of non-spherical structure. The assumption that the Schwarzchild black hole is spherically symmetric is an oversimplification considering the dynamic processes involved in the evolution and collapse of massive stars.

- Most astrophysical objects, including stars and galaxies, possess angular momentum or rotation. The majority of observed black holes in the universe are likely to have some level of rotation, which significantly affects their properties, such as the shape of the event horizon and the formation of an ergosphere (a region of spacetime associated with the dragging of spacetime by the rotating black hole). The Schwarzschild solution's lack of rotation doesn't adequately represent the diversity of black holes found in nature.
- Since it is unstable to deviations from spherical symmetry. This means that even the slightest perturbations or disturbances to its perfectly spherical geometry would lead to its collapse or transformation into more complex structures. This instability contradicts the nature of observed black holes.

The Schwarzchild black hole, not being showcased as an archetype of a realistic black hole essentially lies in the fact that it is unstable to deviations. Realistic black holes typically exhibit long-term stability and complex behaviours such as accretion of matter, jet formation, and interactions with their surroundings, they essentially live in violent environments, and perturbations or disturbances to them are inevitable. Hence, A realistic black hole solution should be functionally generic, in the sense that it can adjust and reconfigure itself to accommodate minor perturbations.

1.10 Summary

Throughout this chapter, we presented the motivation behind our research, demonstrated knowledge of the state-of-the-art in relation to the topic, how the project was approached from a larger picture and the tools and concepts used to explore spacetime singularities, focusing on black holes and their complexities.

Chapter 2: Results and Discussion

2.1 Introduction

In this section, we present and discuss the results formulated throughout the study. This section involves in depth mathematical analysis of Schwarzchild singularity and a generic class of solutions which have a null and weak curvature singularity which are consistent with Einstein Field equations.

2.2 Results and Discussion

- ***Schwarzschild Black Hole***

Karl Schwarzschild provided the first exact solution to Einstein's field equations. The Schwarzschild solution describes the presence of a black hole, named the Schwarzschild black hole. It represents the simplest form of a spherically symmetric, non-rotating black hole, devoid of angular momentum and charge.

In this section of the study, we delve into understanding the gravitational properties of the Schwarzschild black hole and the nature of the singularity at its centre. The main focus is on solving the geodesic equation and geodesic deviation equation to obtain exact solutions for Jacobi fields. These solutions allow us to comprehend the behaviour of Jacobi fields along time-like geodesics. Additionally, we can discern the singularity's nature at the centre by calculating the volume of a parallel-propagated tetrad formed by Jacobi fields. The concept involves transporting the Jacobi fields along radial geodesics while preserving their relative directions, revealing intrinsic manifold properties.

We work within a 4-dimensional spacetime, utilising the spherical coordinate system:

$$x^\alpha = \{ r, t, \theta, \phi \}$$

The geodesic congruence is parameterized by the parameter s. For convenience, we assume the geodesic starts from infinity at $s = s_0$ and converges to a singularity at $s = 0$. The tangent vector to the geodesic congruence is $u^\alpha = (\dot{x})^\alpha$, while the Jacobi field

J^α is orthogonal to the tangent vector. This Jacobi field can be decomposed into three fields:

$$
J^\alpha_1 = (a, b, 0, 0);
J^\alpha_2 = (0, 0, c, 0);
J^\alpha_3 = (0, 0, 0, d)
$$

The metric tensor and spacetime interval describing the geometry outside a Schwarzschild black hole take the forms:

$$
g_{\mu\nu} = \begin{pmatrix}
\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\
0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
$$

$$
ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2
$$

- ***Geodesic Equation***

We can formulate the Lagrangian using our metric:

$$
\mathcal{L} = - \left(1 - \frac{2M}{r}\right) \dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2
$$

The Euler-Lagrange equation is expressed as:

$$
\frac{d}{ds} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0
$$

Note that we have used `$s$` as the parameter along our geodesic congruence. Furthermore, we use the following facts which help us simplify our Euler-Lagrange Equations and the Lagrangian:

The spherical symmetry allows us to work in the equatorial plane and fix $\theta = \pi/2$, resulting in $\dot{\theta} = 0$. For radial geodesics, the angular momentum vanishes and so does $\dot{\phi}$. For a time-like geodesic, the Lagrangian can be set to -1: $f = 1 - \frac{2M}{r}$.

Using the Euler-Lagrange equation for $x^\alpha = \{ r, t, \theta, \phi \}$ and using the simplifications above, we obtain the following equations for $\dot{x}^\alpha$:

$$
\dot{t} = \frac{E}{f},
\dot{\phi} = 0,
(\dot{r})^2 = E^2 - f,
\dot{\theta} = 0
$$

Considering that the observer starts its fall from infinity, where it is at rest, we can find an explicit form for $\dot{r}$. As the observer is at rest, the expression $\lim_{{r \to \infty}} (\dot{r})^2 = 0$. Putting in the expression $(\dot{r})^2 = E^2 - f$, we can conclude that $E = 1$. Hence, $\dot{t} = \frac{1}{f}$.

Noting that r decreases along the geodesic congruence, we get the expression $\dot{r} =

-\sqrt{\frac{2M}{r}}$. Further solving the expression for $\dot{r}$, we obtain an explicit expression for $r(s)$:

\[
\dot{r} = -\sqrt{\frac{2M}{r}} \]
\[\sqrt{r} \dot{r} = -\sqrt{2M} \]
\[\int_{s_0}^{s} \sqrt{r} \dot{r} \, ds = \int_{s_0}^{s} -\sqrt{2M} \, ds \]
\[\int_{r_0}^{r} \sqrt{r'} \, dr' = -\sqrt{2M} \, (s - s_0) \]
\(when, (r_0)^{3/2} + \frac{3}{2} \sqrt{2M} \, s_0 = \frac{3}{2} \sqrt{2M} \, s
\)
\[r^{3/2} = (r_0)^{3/2} - \frac{3}{2} \sqrt{2M} \, (s - s_0) = 0 \]

We shift the origin of s such that we get the following expression:
\[
r^{3/2} = -\frac{3}{2} \sqrt{2M} \, s, \quad s < 0 \]

\[\\r = \left(-\frac{3}{2} \sqrt{2M} \, s \right)^{\frac{2}{3}}, \quad s < 0 \]

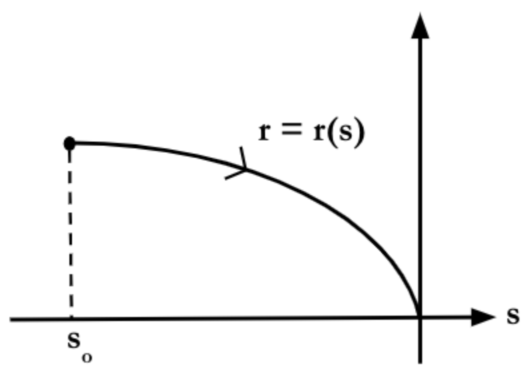

Figure 4 Radial geodesic as a function of affine parameter.

Since the Jacobi fields are orthogonal to the tangent vector, we can imply the following condition:
\[g_{\alpha \beta} u^\alpha J^\beta = 0 \]
where \(u^\alpha = (\dot{x})^\alpha = \{ \dot{t}, \dot{r}, 0, 0 \}\),
and the Jacobi field \(J^\alpha = \{ a ,b ,c ,d \}\),
\[-f a \dot{t} + f^{-1} b \dot{r} = 0 \]
\[a(s) = \frac{b(s) \dot{r}}{f^2 \dot{t}} \]

Simplifying further by putting $$\dot{t}=\frac{1}{f}$$
we obtain, $$a=\frac{b\dot{r}}{f}$$

- ***Geodesic Deviation Equation***

Since we have three Jacobi fields, namely J_{i}^α, where $i=1,2,3$, we will have a geodesic deviation equation for each of them. We will work out the solutions for each of them separately.

(i) $J^\alpha_1 = (a, b, 0, 0)$}

$$J^\alpha_1=a\delta^\alpha_0+b\delta^\alpha_1$$

LHS:

We start by calculating the first covariant derivative along the tangent u^β.

$$DJ^\alpha_1 = \nabla_{u^\beta}J^\alpha_1$$

$$DJ^\alpha_1 = u^\beta \partial_\beta J^\alpha_1 + \Gamma^\alpha_{\beta\mu} u^\mu J^\beta_1$$

Since J^β_1 is non-zero only when $\beta = 0,1$, we get

$$DJ^\alpha_1 =\frac{d(J^\alpha_1)}{ds} + a \Gamma^\alpha_{00} \dot{t}+b\Gamma^\alpha_{10} \dot{t}+a \Gamma^\alpha_{01} \dot{r}+b \Gamma^\alpha_{11} \dot{r}$$

$$DJ^\alpha_1 =\dot{a}\delta^\alpha_0+\dot{b}\delta^\alpha_1 + a \Gamma^\alpha_{00} \dot{t}+b\Gamma^\alpha_{10} \dot{t}+a \Gamma^\alpha_{01} \dot{r}+b \Gamma^\alpha_{11} \dot{r}$$

Substituting in the corresponding values of Christoffel symbols and Simplifying, we get

$$DJ^\alpha_1 =\delta^\alpha_0(\dot{a}+\frac{Mb\dot{t}}{fr^2}+\frac{Ma\dot{r}}{fr^2})+\delta^\alpha_1(\dot{b}+\frac{Mfa\dot{t}}{r^2}-\frac{Mb\dot{r}}{fr^2})$$

Using, $$ a=\frac{b\dot{r}}{f}, \quad r^{\dot{}} = -\left(\frac{2M}{r}\right)^{\frac{1}{2}}, \dot{t}=\frac{1}{f}$$

and Simplifying, we obtain

$$DJ^\alpha_1=\delta^\alpha_0(\dot{a}+\frac{Ma(1+\frac{2M}{r})}{fr^2\dot{r}})+\delta^\alpha_1(\dot{b})$$

We can define, $$ \widetilde{a}=(\dot{a}+\frac{Ma(1+\frac{2M}{r})}{fr^2\dot{r}}), \widetilde{b}=\dot{b}$$

We can see that we have obtained a form

$$ DJ^\alpha_1 = \widetilde{a}\delta^\alpha_0+\widetilde{b}\delta^\alpha_1$$

Similarly, we can deduce that the second covariant derivative should look something like:

$$D^2J^\alpha_1=\widetilde{\widetilde{a}}\delta^\alpha_0+\widetilde{\widetilde{b}}\delta^\alpha_1$$

Where, $$\widetilde{\widetilde{b}}=\ddot{b}$$

We are more interested in the component 'b' , as if we can get a solution for 'b', we can obtain the expression for 'a' by using use the formula $$ a=\frac{b\dot{r}}{f}$$

textbf{RHS:} $-R^\alpha_{\beta\mu\delta} u^\beta J_{1}^\mu u^\delta$

Expanding appropriately, We get the expression for RHS as:

$$-[R^0_{101}a(\dot{r})^2\delta^\alpha_0+R^0_{110}\dot{t}\dot{r}b\delta^\alpha_0+R^1_{001}\dot{t}\dot{r}a\delta^\alpha_1+R^1_{010}(\dot{t})^2b\delta^\alpha_1]$$

$$\text{Putting in the appropriate values of the Riemann Tensor, } \dot{t} \text{, } \dot{r} \text{ and } b = \frac{af}{\dot{r}} $$

$$ \text{RHS} = \delta^\alpha_0 \left(\frac{2Ma}{r^3}\right) + \delta^\alpha_1 \left(\frac{2Mb}{r^3}\right) $$

Comparing

$$\text{LHS:}D^2J^\alpha_1=\widetilde{\widetilde{a}}\delta^\alpha_0+\widetilde{\widetilde{b}}\delta^\alpha_1$$

$$\text{RHS: } \delta^\alpha_0 \left(\frac{2Ma}{r^3}\right) + \delta^\alpha_1 \left(\frac{2Mb}{r^3}\right)$$

We can see that,

$$\ddot{b}=\frac{2Mb}{r^3}$$

Note that solving the ODE for 'b' automatically gives us the solution for 'a'.

Using,

$$r = \left(-\frac{3}{2}\sqrt{2M}s\right)^{\frac{2}{3}}$$

$$\ddot{b}=\frac{4b}{9s^2}$$

$$\ddot{b}-\frac{4b}{9s^2}=0$$

This is a Second Order Linear ODE of Euler Type, we can solve this by the transformation:

$$s=e^t,lns=t,\frac{dt}{ds}=\frac{1}{s}$$

We find that our Differential Equation transforms to:

$$9 \frac{d^2b}{dt^2} - 9 \frac{db}{dt} - 4b = 0$$

and has the solution of form:

$$b(t) = b_1 e^{\frac{4t}{3}} + b_2 e^{-\frac{t}{3}}$$

Replacing our original coordinates, we get,

$$b(s)=b_1 e^{\frac{4lns}{3}}+b_2 e^{\frac{-lns}{3}}$$

Simplifying,

$$b(s)=b_1s^{\frac{4}{3}}+b_2s^{\frac{-1}{3}}$$

(ii) $J^\alpha_2 = (0, 0, c, 0)$}

LHS:

We start by calculating the first covariant derivative along the tangent u^β.

$$DJ^\alpha_2 = \nabla_{u^\beta}J^\alpha_2$$

$$DJ^\alpha_2 = u^\beta \partial_\beta J^\alpha_2 + \Gamma^\alpha_{\beta\mu} u^\mu J^\beta_2$$

Since J^β_2 is non-zero only when $\beta = 2$, we get

$$\frac{d(J^\alpha_2)}{ds} + c \Gamma^\alpha_{\mu 2} u^\mu$$

$$\dot{c} \delta^\alpha_2 + c \dot{t} \Gamma^\alpha_{02} + c \dot{r} \Gamma^\alpha_{12}$$

Putting in the values for the Christoffel symbols, we obtain

$$DJ^\alpha_2 = (c^{\dot{}} + \frac{c\dot{r}}{r}) \delta^\alpha_2$$

For simplification purposes, we write

$$(c^{\dot{}} + \frac{c\dot{r}}{r}) \delta^\alpha_2 = c^{\widetilde{}} \delta^\alpha_2$$

where $c^{\widetilde{}} = (c^{\dot{}} + \frac{c\dot{r}}{r})$

We can see that we have obtained a form

$$DJ^\alpha_2 = c^{\widetilde{}} \delta^\alpha_2$$

Similarly, we can deduce that the second covariant derivative should look something like:

$$D^2 J^\alpha_2 = (c^{\widetilde{}} + \frac{c^{\widetilde{}}\dot{r}}{r}) \delta^\alpha_2$$

Replacing $c^{\widetilde{}} = (c^{\dot{}} + \frac{c\dot{r}}{r})$ and simplifying the above equation, we obtain:

$$\text{LHS} = D^2 J^\alpha_2 = (rc)^{\ddot{}} \delta^\alpha_2 \left(\frac{1}{r}\right)$$

RHS: $-R^\alpha_{\beta\mu\delta} u^\beta J_{2}^\mu u^\delta$

Since the only non-zero component of J_{2}^μ is when $\mu=2$, and the only non-zero values of u^δ and u^β occur when δ and β are 0 and 1 respectively, we can expand our RHS as follows:

$$ c[R^2_{021} \dot{t} \dot{r} + R^2_{120} \dot{t} \dot{r} + R^2_{020} (\dot{t})^2 + R^2_{121} (\dot{r})^2] $$

Using,

$$ R^2_{021} = R^2_{120} = 0 $$

$$ R^2_{121} = -\frac{M}{fr^3} $$

$$ R^2_{020} = \frac{fM}{r^3} $$

On simplification,

$$ \mathrm{RHS} = \left[\frac{M}{fr^3}(\dot{r})^2 - \frac{fM}{r^3}(\dot{t})^2 \right]c $$

Final Geodesic Deviation Equation:

$$ (rc)^{\ddot{}}\delta^\alpha_2\left(\frac{1}{r}\right) = -R^\alpha_{\beta\mu\delta}u^\beta J_{2}^\mu u^\delta $$

LHS and RHS contain non-zero components when $\alpha = 2$ and $\mu = 2$,

Hence our equation becomes:

$$ (rc)^{\ddot{}}\delta^\alpha_2\left(\frac{1}{r}\right) = \left[\frac{M}{fr^3}(\dot{r})^2 - \frac{fM}{r^3}(\dot{t})^2 \right]c $$

Using the expressions $ \dot{r} = -\sqrt{\frac{2M}{r}} $ and $ \dot{t} = \frac1{f} $, and simplifying, we obtain:

$$ (rc)^{\ddot{}} = -\frac{M}{r^2}c $$

We can break LHS down to:

$$ (rc)^{\ddot{}} = (\dot{rc}+\dot{r}c)^{\dot{}} = \ddot{rc}+2\dot{r}\dot{c} + \ddot{r}\dot{c} $$

\]

$$ \ddot{rc}+2\dot{r}\dot{c} + \ddot{r}\dot{c} = -\frac{M}{r^2}c $$

We know,

$$ (\dot{r})^2 = \frac{2M}{r} $$

$$ 2\dot{r} \ddot{r} = -\frac{2M \dot{r}}{r^2} $$

$$ \ddot{r} = -\frac{M}{r^2} $$

Using this in

$$ (\dot{rc}+\dot{r}c)^{\dot{}} = \ddot{rc}+2\dot{r}\dot{c}+\ddot{r}\dot{c} = -\frac{M}{r^2}c $$

we get,

$$ rc^{\ddot{}} + 2{r}^{\dot{}}{c}^{\dot{}} = 0 $$

$$ \frac{c^{\ddot{}}}{c} + \frac{2\dot{r}}{r} = 0 $$

$$ \frac{d}{ds} \left(\ln(c^{\dot{}}) + 2\ln(r) \right) = 0 $$

$$ \ln(c^{\dot{}}) + 2\ln(r) = \ln(C_1) $$

$$ c^{\dot{}} = \frac{C_1}{r^2} $$

Using $r = \left(\left(-\frac{3}{2}\sqrt{2M}s \right)^{\frac{2}{3}} \right)$ for $s < 0$,

$$ c^{\dot{}} = C_2 s^{-\frac{4}{3}} $$

where $C_2 = C_1 \left(\frac{2}{9M} \right)^{\frac{2}{3}}$

$$ \int c^{\dot{}} ds = \int C_2 s^{-\frac{4}{3}} ds $$

Finally,

$$ c = -3C_2 s^{-\frac{1}{3}} + C_3 $$

$$ c = C_4 s^{-\frac{1}{3}} + C_3 $$

where $C_4 = -3C_2$

(iii) $J^\alpha_3 = (0, 0, 0, d)$

As we have calculated for **$J^\alpha_2 = (0, 0, c, 0)$},** We can obtain the solution for 'd' through exactly the same procedure. The final solution for 'd' can be expected to be of the same form as that of 'c' due to the similar form of these Jacobi Fields **J^α_1 and J^α_2** and it indeed comes out to be of the same form.

$$ d = D_1 s^{-\frac{1}{3}} + D_2 $$

- ***Volume Encompassed by Jacobi Fields***

$$ g_{\alpha\beta} J^\alpha_1 J^\beta_1 = -f a^2 + f^{-1} b^2 $$

$$ = -\frac{b^2 (\dot{r})^2}{f} + \frac{b^2}{f} = \frac{b^2 (1 - \frac{2M}{r})}{f} = b^2 $$

Magnitude of Individual Jacobi Field's:

$$ \| \mathbf{J^\alpha_1} \|^2=g_{\alpha\beta} J^\alpha_1 J^\beta_1 = -f a^2 + f^{-1} b^2 $$

$$ = -\frac{b^2 (\dot{r})^2}{f} + \frac{b^2}{f} = \frac{b^2 (1 - \frac{2M}{r})}{f} = b^2 $$

$$\| \mathbf{J^\alpha_1} \| = b$$

Similarly,

$$\| \mathbf{J^\alpha_2} \| = rc$$

Similarly,

$$\| \mathbf{J^\alpha_3} \| = rd$$

Volume encompassed by Jacobi fields,

$$ \text{V(s)=}\mathbf{J_1 \cdot (J_2 \times J_3)} $$

\[V(s) = \begin{vmatrix}

b & 0 & 0 \\

0 & rc & 0 \\

0 & 0 & rd \\

\end{vmatrix}\]

\[V(s)=r^2(bcd)\]

Using,

\[r = \left(-\frac{3}{2}\sqrt{2M}s\right)^{\frac{2}{3}}\]

\[V(s)=[\left(-\frac{3}{2}\sqrt{2M}s\right)^{\frac{4}{3}}]*[b_1s^{\frac{4}{3}}+b_2s^{\frac{-1}{3}}]*[C_4 s^{-\frac{1}{3}} + C_3]*[D_1 s^{-\frac{1}{3}} + D_2]\]

We can rewrite the above expression as:

\[V(s) = \left(\frac{9M}{2}\right)^{\frac{2}{3}} (b_1 + b_2s^{\frac{5}{3}})(C_4 + C_3s^{\frac{1}{3}})(D_1 + D_2s^{\frac{1}{3}})\]

The definition of our parameter s is such that, at s=0, the geodesic congruence encounters a singularity. We can see that as s approaches zero, V(s) approaches zero. Hence, as we move along the congruence, the volume encompassed by the Jacobi fields tends to zero. For an observer falling into a black hole, we can conclude that an observer would be crushed completely. Hence, calculating the volume of these Jacobi fields at s=0 provides us a way to examine the nature of the singularity. In the case of a Schwarzchild black hole, The volume goes to zero at the point of singularity; the nature of the singularity is deemed to be strong and space-like.

Further, we can look at the individual behaviour of the Jacobi fields at s=0 as well. Since,

\[\| \mathbf{J^\alpha_1} \| = b=b_1s^{\frac{4}{3}}+b_2s^{\frac{-1}{3}}\]

\[\| \mathbf{J^\alpha_2} \| = rc=\left(-\frac{3}{2}\sqrt{2M}s\right)^{\frac{2}{3}}(C_4 s^{-\frac{1}{3}}+C_3)=\left(-\frac{3}{2}\sqrt{2M}\right)^{\frac{2}{3}}(C_4 s^{\frac{1}{3}} + C_3s^{\frac{2}{3}})\]

\[\|\mathbf{J^\alpha_3}\|=rd=\left(-\frac{3}{2}\sqrt{2M}s\right)(D_1 s^{-\frac{1}{3}} + D_2)=\left(-\frac{3}{2}\sqrt{2M}\right)^{\frac{2}{3}}(D_1s^{\frac{1}{3}}+D_2s^{\frac{2}{3}})\]

Putting s=0 in the above equations, we can see that

J^\alpha_1 diverges to infinity

J^\alpha_2 and J^\alpha_3 go to zero.

We can infer that an observer falling into a Schwarzschild black hole will be infinitely stretched along the radial direction and will be completely crushed along the tangential and azimuthal directions. This phenomenon is called Spaghettification, where the tidal forces create a gradient of gravitational attraction across your body. The side of the body closer to the black hole experiences a stronger gravitational pull than the other side. This difference in gravitational pull can stretch and compress the body.

- ***Functionally Generic Null Singularity***

In this section of study, the framework used by Ori and Flangan in [1] is discussed and the mathematical gaps that remain unpublished up-to-date are filled in through rigorous mathematical analysis using the software Wolfram Mathematica. The Mathematica notebook which works out the mathematical analysis is attached in the Appendix.

The main objective of this section is to showcase that Einstein field equations admit a functionally generic class of solutions with a null and weak curvature singularity.

Einstein field equations in vacuum are represented by: R_{\alpha,\beta}=0

Since, our Ricci tensor is a symmetric tensor, it has 10 independent components. Each component corresponds to a field equation. These Einstein field equations are divided into a set of evolution equations and constraint equations.

The evolution equations are derived from the equations of motion for the metric tensor. They describe how the spacetime curvature evolves over time. These equations involve double time derivatives of the metric tensor.

The constraint equations, on the other hand, are derived from the geometry of spacetime. These equations are used to ensure consistency between the metric tensor and the curvature of spacetime. They help determine the initial values of the metric tensor and its derivatives, ensuring that the solutions to the evolution equations are physically meaningful and consistent with the underlying geometry. These equations do not involve double time derivatives, but first order space, first order time, combination of space-time and second order space derivatives.

We start by working in the coordinate system (x, y, u, v) and adopt the gauge

$$g_{ux} = g_{uy} = g_{uu} = g_{uv}=0$$

This implies that $g^{vv}=0$ and ensures that v=constant is a null hypersurface.

Our metric tensor takes the form:

$$g_{\mu\nu} = \begin{pmatrix}
0 & g_{6} & 0 & 0 \\
g_{6} & 0 & g_{4} & g_{5} \\
0 & g_{4} & g_{1} & g_{2} \\
0 & g_{5} & g_{2} & g_{3}
\end{pmatrix}$$

where,

$g_{xx}=g_1$, $g_{xy}=g_2$, $g_{yy}=g_3$, $g_{vx}=g_4$, $g_{vy}=g_5$ and $g_{uv}=g_6$

To determine the degrees of freedom 'K' required to uniquely determine $D^{+}(S)$, we can define variables

$T=v+u$ and $Z=v-u$

Using Cauchy-Kowaleski theorem at the hypersurface T=constant, we can say that there are 12 functions namely g_i(x,y,z) and g_{i,t}(x,y,x), where i goes from 1 to 6, that need to be specified to determine the solution of evolution equations. These 12 functions also need to satisfy 4 constraint equations. Hence, the number of arbitrary functions that need to be specified are 8 and hence, K=8.

Now, we focus on the Einstein field Evolution equations to determine the evolution of our initial data. There are a total of 6 evolution equations which correspond to R_i = 0. We start by defining the following variables:

w=v^(1/n)

t=w+u ; z= w- u

Now, we transform our independent variables from (u, v, x, y) to (t, z, x, y).

Our metric takes up the new form,

\[

\left(\begin{array}{cccc}\frac{1}{2}\text{g6}(t,z,x,y)&0&\frac{1}{2}\text{g4}(t,z,x,y)&\frac{1}{2}\text{g5}(t,z,x,y)\\0&-\frac{1}{2}\text{g6}(t,z,x,y)&\frac{1}{2}\text{g4}(t,z,x,y)&\frac{1}{2}\text{g5}(t,z,x,y)\\\frac{1}{2}\text{g4}(t,z,x,y)&\frac{1}{2}\text{g4}(t,z,x,y)&\text{g1}(t,z,x,y)&\text{g2}(t,z,x,y)\\\frac{1}{2}\text{g5}(t,z,x,y)&\frac{1}{2}\text{g5}(t,z,x,y)&\text{g2}(t,z,x,y)&\text{g3}(t,z,x,y)\\\end{array}\right) \]

We find that the field equation Ricci[2,2] =0 can be written as:

\[R_{22} = A \cdot g_{6,tt} + B \] =0

Where, B contains single order derivatives of time and space, double derivatives of space and derivatives of space-time combinations. Note that B is devoid of any double time derivatives.

Further, we find the expression for A as follows:

\[

\frac{2\text{g2}(t,z,x,y)\text{g4}(t,z,x,y)\text{g5}(t,z,x,y)}{\text{g6}(t,z,x,y)^2\left(\text{g2}(t,z

,x,y)^2-\text{g1}(t,z,x,y)\text{g3}(t,z,x,y)\right)}-\frac{\text{g3}(t,z,x,y)\text{g4}(t,z,x,y)^2}{\text{g6}(t,z,x,y)^2\left(\text{g2}(t,z,x,y)^2-\text{g1}(t,z,x,y)\text{g3}(t,z,x,y)\right)}-\frac{\text{g1}(t,z,x,y)\text{g5}(t,z,x,y)^2}{\text{g6}(t,z,x,y)^2\left(\text{g2}(t,z,x,y)^2-\text{g1}(t,z,x,y)\text{g3}(t,z,x,y)\right)}+\frac{2\text{g2}(t,z,x,y)^2}{\text{g6}(t,z,x,y)\left(\text{g2}(t,z,x,y)^2-\text{g1}(t,z,x,y)\text{g3}(t,z,x,y)\right)}-\frac{2\text{g1}(t,z,x,y)\text{g3}(t,z,x,y)}{\text{g6}(t,z,x,y)\left(\text{g2}(t,z,x,y)^2-\text{g1}(t,z,x,y)\text{g3}(t,z,x,y)\right)} \]

We can see that the function A is an analytic function of the metric functions g_{i}.

Simplifying further, We can rewrite the field equation as,

$g_{i,tt} = F_i$; where F_i will contain all combinations of space, time and space-time derivatives except the second order time derivatives.

Using the above equation we can rewrite to generalise the evolution of any metric component in the following form:

$g_{i,tt}= f_{i} [g_{j},g_{j,t} ,g_{j,A} , g_{j,AB}, g_{j,At},z,t]$

This report encapsulates the formulation of this equation for a particular function g_{6}, but it is easy to find out that we can generalise this into the above equation using similar methods.

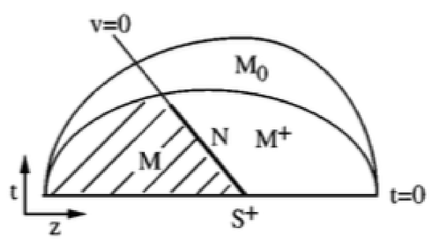

Figure 5 Evolution of Initial data [1].

We will now turn our attention to the evolution of our initial data under the influence of the above field equations. Consider an initial hypersurface t=0 and its intersection with the neighbourhood of origin represented by S^{+}. The Cauchy Kowaleski theorem states that if we know the analytic functions $h_{i}=g_{i}(t=0, x, y, z)$ and $p_{i}=g_{i,t}(t=0, x, y, z)$ then we can uniquely determine the solution to the field equations in the region around the hypersurface S^{+}, Let’s say M^{+}.

It is easy to see the reasoning behind this, since we have the metric at all points at the hypersurface, we know the space derivatives for the metric. Further, using the function p_{i}, we know how the metric changes with time. Using both these statements, we can conclude that we can uniquely determine the solution to the field equations.

Therefore, we can determine $g_i(t, x, y, z)$ at every point in M^{+} and hence, $g_i(t, x, y, z)$ is continuous in M^{+}. Further, we know that the field equations are analytic in nature, so we can comment that the solution to them, $g_i(t, x, y, z)$ will also be analytic in the region around the hypersurface. We are interested in the region M, denoted by $v < 0$ and $t \geq 0$.

Returning to our original variable (u, v, x, y). Since $M \subseteq M^{+}$ involves the neighbourhood of origin and we have seen that $g_i(x,y,z,t)$ is analytic and continuous throughout M^{+}, we can say that $g_i(u, v, x, y)$ will remain continuous and analytic throughout M^{+}, particularly at $v=0$. However, since $w=v^{(1/n)}$, we can say

$$g_{i,v}=g_{i,w}\cdot\left(\frac{dw}{dv}\right)=\left(\frac{1}{n}\right)v^{\left(\frac{1-n}{n}\right)} g_{i,w}$$

we can see that $g_{i,v}$ will diverge at $v=0$ (we impose the inequality that $g_{i,w}$ does not go to zero at $v=0$).

Since, our metric functions $g_i(x,y,z,t)$ were continuous throughout M^{+}, on returning to our initial coordinates (u, v, x, y), we find that $g_i(x, y, u, v)$ remains continuous because the metric properties are fundamental to spacetime geometry and not on the coordinate used. We note that our metric is, therefore, continuous at $v = 0$ as well.

To visualise the nature of the singularity, we can work in a 2-Dimensional space (u, v).

The metric in this 2 dimensional space corresponds to,

$$\left(\begin{array}{cc}0&\text{g6}(u,v)\\\text{g6}(u,v)&0\\\end{array}\right)$$

Since, $g_{6,v}$ diverges at $v=0$. We can obtain an approximate expression for g_{6} using it's analytic nature

\[\g_{6} = h(u) + j(u) v^p\] , where 0 < p < 1

The Riemann tensor has 4-non zero components and only one independent component which is,

\[
2\left(h(u)+j(u)v^p\right)\left(\frac{pj(u)v^{p-1}\left(h'(u)+v^pj'(u)\right)}{\left(h(u)+j(u)v^p\right)^2}-\frac{pv^{p-1}j'(u)}{h(u)+j(u)v^p}\right)2\left(h(u)+j(u)v^p\right)\left(\frac{pj(u)v^{p-1}\left(h'(u)+v^pj'(u)\right)}{\left(h(u)+j(u)v^p\right)^2}-\frac{pv^{p-1}j'(u)}{h(u)+j(u)v^p}\right)]
\

Looking at the expression we see that, the curvature diverges at v= as v^(-1+p)

Further, The Kretschmann Scalar also diverges at v=0, but at the rate of v^(-2+2p) and has the form:
\[\frac{16p^2v^{2p-2}\left(j(u)h'(u)-h(u)j'(u)\right)^2}{\left(h(u)+j(u)v^p\right)^6}]\

Consider a time-like geodesic approaching the null hypersurface v=0,The geodesic is parameterized by proper time s

The tangent vector is given by

\[u^\alpha = (u^\cdot, v^\cdot) \]

The Jacobi field is given by

\[J^\alpha = (J^1, J^2) \]

Since, s is a proper time on the geodesic, and s goes to zero as v goes to zero, we can write,

\[g_{\alpha\beta} u^\alpha u^\beta = 2F u^\cdot v^\cdot = -1\]

\[\dot{u}^{\alpha} + \Gamma^\alpha_{11} \dot{u} \dot{u} + \Gamma^\alpha_{12} \dot{u} \dot{v} + \Gamma^\alpha_{22} \dot{v} \dot{v} = 0\]

\[\alpha = 1 &: \ddot{u} + \frac{d_u F}{F}\dot{u}\dot{u} = 0 \\\]

\[\alpha = 2 &: \ddot{v} + \frac{d_v F}{F}\dot{v}\dot{v} = 0 \\\]

\text{Using } F = h(u) + k(u) v^p, \text{ where } 0 < p < 1.

\[\frac{d_u F}{F}=\frac{h'(u)+k'(u)v^p}{h(u)+k(u)v^p}\]

as v tends to zero, This is approximately equal to

\[\frac{h'(u)}{h(u)}\]

Similarly,

\[\frac{d_v F}{F}=\frac{pk(u)v^{(p-1)}}{h(u)+k(u)v^{p}}\]

as v tends to zero, This is approximately equal to

\[\frac{pk(u)v^{p-1}}{h(u)}\]

Putting in the geodesic equations for u and v

\[0=\frac{\ddot{u}}{\dot{u}}+\frac{h'(u_0)\dot{u}}{h(u_0)}\]

\[0=\frac{d}{ds}(ln(\dot{u})+\frac{h'(u_0)}{h(u0)}\]

\[ln(\dot{u})+\frac{h'(u_0)}{h(u_0)}=C_0\]

Hence, as v tends to 0, we can approximately say

\[\dot{u} = exp(C_0- \frac{h'(u_0)}{h(u_0)})\]

Similarly, using the geodesic equation corresponding to v, we obtain the following

$$0=\frac{\ddot{v}}{\dot{v}}+\frac{pk(u_0)v^{p-1}\dot{v}}{h(u_0)}$$

Solving, we get

$$\dot{v}=exp(C_1 - \frac{k(u_0v^{p})}{h(u_0)}$$

We can also establish a connection between the constants C_0 and C_1 by considering the fact that when $v = 0$:

$$2F \dot{u} \dot{v} = -1$$

Moving on to the Geodesic Deviation Equation's for Jacobi field components correspond to,

$$\frac{d^2 (J^1)}{ds^2} + \frac{dJ^1}{ds} \left(2\Gamma^1_{11} \dot{u} \right) + J^1 \left[\frac{d(\Gamma^1_{11})}{ds} \dot{u} + \Gamma^1_{11} \ddot{u} + \Gamma^1_{11} \dot{u}^2 \right] = -FR_{2112} \dot{u} (\dot{v} J^1 - \dot{u} J^2)$$

$$\frac{d^2 (J^2)}{ds^2} + \frac{dJ^2}{ds} \left(2\Gamma^2_{22} \dot{v} \right) + J^2 \left[\frac{d(\Gamma^2_{22})}{ds} \dot{v} + \Gamma^2_{22} \ddot{v} + \Gamma^2_{22} {\dot{v}}^2 \right]=-FR_{1212} \dot{v}(\dot{v} J^1 - \dot{u} J^2)$$

Further, using the fact that the Jacobi field is orthogonal to the tangent vector, we can imply the condition

$$g_{\alpha\beta} J^\alpha J^\beta = 0$$

This give us,

$$J^2=\frac{-\dot{v}J^1}{\dot{u}}$$

This helps us in getting rid of a system of two coupled 2nd order differential equations and obtain a single 2nd order differential equation:

$$\frac{d^2 (J^1)}{ds^2} + \frac{dJ^1}{ds} \left(2\Gamma^1_{11} \dot{u} \right) + J^1 \left[\frac{d(\Gamma^1_{11})}{ds} \dot{u} + \Gamma^1_{11} \ddot{u} + \Gamma^1_{11} \dot{u}^2 \right] = -FR_{2112} \dot{u} (\dot{v} J^1 - \dot{u} \frac{-\dot{v}J^1}{\dot{u}})$$

Simplifying, we get

$$\frac{d^2 (J^1)}{ds^2} + \frac{dJ^1}{ds} \left(2\Gamma^1_{11} \dot{u} \right) + J^1 \left[\frac{d(\Gamma^1_{11})}{ds} \dot{u} + \Gamma^1_{11} \ddot{u} + \Gamma^1_{11} \dot{u}^2 \right] =0$$

2.3 Summary

In this section, we've effectively derived the nature of Schwarzschild's singularity as strong and space-like. Additionally, we've established the presence of a generic set of solutions to Einstein's Field Equations. These solutions encompass a particular class that displays a weak and null singularity at the v=0 hypersurface. Furthermore, we've developed the geodesic equations and geodesic deviation equations within a two-dimensional spatial context. These equations pertain to the motion of a timelike geodesic as it descends towards the v=0 hypersurface.

Chapter 3: Conclusion

3.1 Introduction

This chapter encapsulates the accumulation of comprehensive inquiry that has been undertaken throughout the study, merging together different threads of knowledge to understand essential insights in the realm of spacetime singularities and black hole phenomena.

3.2 Summary of the Research

A cornerstone of this study lies in the insights drawn from prior research in the realm of black hole physics and spacetime singularities. The investigation starts with comprehending the nature of the singularity residing at the heart of a Schwarzschild black hole, serving as a fundamental example. This exploration then extends its reach into the focal point of this research, a more intricate and rigorous realm of functionally generic null and weak singularities, in this part we proved the presence of the corresponding singularity and we also derived the geodesic and geodesic deviation equations which showcases how the geodesic congruences are affected in regions of singularities. The heart of this study involves meticulous mathematical analysis, drawing insights from diverse mathematical disciplines such as tensor calculus, differential geometry, and partial differential equations. These varied tools of mathematics are then practically employed through the utilisation of Mathematica, a user-friendly software by Wolfram that provides an accessible means of executing computational mathematics. The objective of this research was to fill in the mathematical propositions presented in [1], a work that has remained unpublished. This study achieved its goal by crafting a clear and concise formulation for these mathematical assertions. Through these concerted efforts, this study has bridged the gap in knowledge, enriching the discourse in this domain.

3.3 Limitations of the Research

As with any study, the path undertaken in this research is not devoid of limitations. The limitations encompassed in this work—whether stemming from the scope of mathematical formalism, the intricacies of theoretical assumptions or time—serve as signposts for future explorations. These limitations, while guiding the trajectory of this study, also pave the way for future scholars to tread further upon this trajectory, to unveil even greater layers of insight and understanding. An essential aspect to acknowledge in our research lies in the constraint of our construction to analytic initial functions. Although it is recognized that this is a technical limitation arising from the mathematical theorems. However, I firmly believe that the physical scenario we've explored—a null weak singularity—will indeed manifest even if the initial functions on S are not analytic. This belief holds, provided that these initial functions remain adequately smooth for $v < 0$ [1]. It's noteworthy that the progression of time has also exerted an influence on the trajectory of our research, influencing the depth to which we could delve. As we conclude this phase of investigation, we recognize the potential for future explorations to transcend these limitations, unveiling even greater depths of understanding in the captivating realm of theoretical physics.

3.4 Summary

In summary, this chapter offers a glimpse into the comprehensive journey undertaken within the domain of spacetime singularities and black hole phenomena. The endeavours and insights accumulated throughout this exploration. Additionally, the research's inherent limitations are acknowledged, and the potential for future advancements that transcend these boundaries is highlighted.

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