Fuglede Conjecture in $\mathbb{Z}_P^2$

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Definition 0.1. A set $E \subset \mathbb{R}^d$ is *spectral* if there exists a set $A \in \mathbb{R}^d$ such that $\{\chi(a \cdot x)\}_{a \in A}$ is a basis of $L^2(E)$.

Definition 0.2. A set $E \in \mathbb{R}^d$ tiles $\mathbb{R}^d$ if there exists a set $T \in \mathbb{R}^d$ such that

$$\sum_{\tau \in T} 1_E(x - \tau) \equiv 1.$$

Lemma 0.3 (Magic)**.** *Let $E \subset \mathbb{Z}_p^2$ and $m \neq (0,0)$. Then if $\hat{1_E}(m) = 0$,*

(i) $\hat{1_E}(rm) = 0$ for all $r \in \mathbb{Z}_p$, and

(ii) E is equi-distributed along lines perpendicular to m.

Proof. Let $\zeta = e^{-2\pi i/p}$. We have

$$\begin{aligned}
\hat{1_E}(m) &= \sum_{x \in \mathbb{Z}_p^2} \chi(-x \cdot m) 1_E(x) \\
&= \sum_{t \in \mathbb{Z}_p} \sum_{x \cdot m = t} \chi(-x \cdot m) 1_E(x) \\
&= \sum_{t \in \mathbb{Z}_p} \chi(-t) n(t) \\
&= \sum_{t \in \mathbb{Z}_p} \zeta^t n(t) = 0
\end{aligned}$$

where $n(t) = |E \cap \{x \in \mathbb{Z}_p^2 : x \cdot m = t\}|$. Since ζ is a pth root of unity, its minimal characteristic polynomial is $\sum_{n=0}^{p-1} x^n$, the the coefficients $n(t)$ above are independent of t, giving (ii).

Further, let $r \in \mathbb{Z}_p^*$.

$$\hat{1_E}(rm) = \sum_{x \in \mathbb{Z}_p^2} \chi(-rx \cdot m) 1_E(x)$$
$$= \sum_{t \in \mathbb{Z}_p} \sum_{x \cdot m = t/r} \chi(-x \cdot (rm) 1_E(x)$$
$$= \sum_{t \in \mathbb{Z}_p} \chi(-t) n(t/r)$$
$$= \sum_{t \in \mathbb{Z}_p} \zeta^t n(t/r) = 0$$

because $n(t/r)$ is constant. □

Theorem 0.4. *Suppose p is prime and $p \equiv 3 \mod 4$. Then a set $E \in \mathbb{Z}_p^2$ is spectral if and only if it tiles $\mathbb{Z}_p^2$.*

Proof. Suppose E has spectrum A. If $|A| = 1$, E is a single point and the result is trivial. Else, take $a, a' \in A, a - a' \neq \mathbf{0}$. Then,

$$\hat{1_E}(a - a') = p^{-2} \sum_{x \in \mathbb{Z}_p^2} \chi((a' - a) \cdot x) 1_E(x)$$
$$= \qquad = p^{-2} \sum_{x \in E} \chi(a' \cdot x) \chi(-a \cdot x) = 0$$

by orthogonality of the basis.

Let $m = a - a'$ and so $\hat{1_E}(m) = 0$ and Lemma 0.3 applies. Thus, $|E| = |A| = kp$.

If $k > 1$, then $|A| > p$ and thus contains every direction. By the above (and Lemma 0.3 again), thus implies $\hat{1_E}(m) = 0$ for all nonzero m, but then

$$1_E(x) = \sum_m \chi(x \cdot m) \hat{1_E}(m) = \hat{1_E}(\mathbf{0})$$

which is a constant and thus can only be 1, so $E = \mathbb{Z}_p^2$ and thus trivially tiles the whole space.

Thus, the only interesting case is $k = 1$, which means $n(t) = 1$ from the proof of Lemma 0.3. That is, E has exactly one point on each line orthogonal to m, or the map

$$\ell_m : x \mapsto m \cdot x$$

is a bijection $E \to \mathbb{Z}_p$. We can then index the elements of E by

$$\ell_m(e_t) = t.$$

Choose nonzero m' such that $m \cdot m' = 0$. Let $T = \{tm' : t \in \mathbb{Z}_p\}$. For all $x \in \mathbb{Z}_p^2$, $x - e_j \in T$ if and only if $j = x \cdot m$. Thus T is a tiling set for E.

Now, suppose E has tiling set T.

$$1 \equiv \sum_{\tau \in T} 1_E(x-\tau) = \sum_{x \in \mathbb{Z}_p^2} 1_E(x-\tau) 1_T(\tau) = 1_E * 1_T$$

Taking the Fourier transform of both sides, we see

$$\hat{1}(m) = \hat{1_E}(m)\hat{1_T}(m)$$

which must be 0 for all $m \neq \mathbf{0}$. The only interesting cases are $1 < |T| < p^2$ so there must be some $m \neq \mathbf{0}$ such that $\hat{1_T}(m) \neq 0$, and thus $\hat{1_E}(m) = 0$.

Lemma 0.3 once again applies, and so as above $|E| = p$, and E has a unique element on each line orthogonal to m. Let $A = \{tm : t \in \mathbb{Z}_p\}$. To see this is a spectrum for E, we need only show it's an orthogonal set. Indeed, $t, s \in Z_p, t \neq s$.

$$\sum_{x \in E} \chi(tm \cdot x)\chi(-s \cdot m) = \hat{1_E}((t-s)m)$$

which is 0 by the lemma.

$\square$