Computational Physics: N-Body Gravitational Interactions

Ujwal Kumar

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1 Introduction

In this report, we study simulations of objects subject to to gravitational forces and its application to planetary orbits. Firstly, we recall Newton's laws and the equation for the motion of a planet where m refers to the mass of the object, **v** refers to the velocity vector of the object and in equation 3, G refers to the gravitational constant and $M_\odot$ is the mass of the Sun:

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m} \tag{1}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \tag{2}$$

$$\ddot{\mathbf{r}} = \frac{GM_\odot}{r^2}(-\hat{\mathbf{r}}) \tag{3}$$

Using this background, we aim to simulate the motion of the Earth around the Sun by evaluating equations 1 and 2 using the Runge Kutta 2nd order numerical method to evaluate differential equations. The second-order Runge Kutta formula is given by:

$$\frac{d\phi}{dt} = f(\phi, t)$$

$$k_1 = \Delta t \cdot f(\phi_n, t_n)$$

$$k_2 = \Delta t \cdot f\left(\phi_n + \frac{k_1}{2}, t_n + \frac{\Delta t}{2}\right)$$

$$\phi_{n+1} = \phi_n + k_2$$

Where ϕ is a 4-element vector containing $\langle x, y, v_x, v_y \rangle$.

2 Problems

Question 1

When working with gravitational distances and timescales, it is helpful to use units appropriate for such scales. We use Astronomical Units (1 AU = Earth-Sun distance) and years. In Equation 3, we see that $GM_\odot$ has units of $\frac{Nm^2}{kg}$. In units of AU and years, we recompute $GM_\odot$ to be:

$$GM_\odot = 6.67 \times 10^{-11} \left(\frac{3.15 \times 10^7 s}{1\text{yr}}\right)^2 \left(\frac{1\text{AU}}{1.49 \times 10^{11} m}\right)^3 = 39.794487 \ \frac{AU^3}{yr^2}$$

The tangential velocity of the Earth around the Sun can then be given by simply $2\pi r$ where r = 1 AU. Therefore, the tangential velocity is 2π AU/year.

Question 2

Now, we verify our program by looking at the total energy per unit mass or specific orbital energy defined as:

$$\epsilon = \epsilon_k + \epsilon_p = \frac{1}{2}v^2 - \frac{GM_\odot}{r} \tag{4}$$

First, we look at the plot of the orbit which we can verify is qualitatively correct due to its approximate circular shape and trajectory:

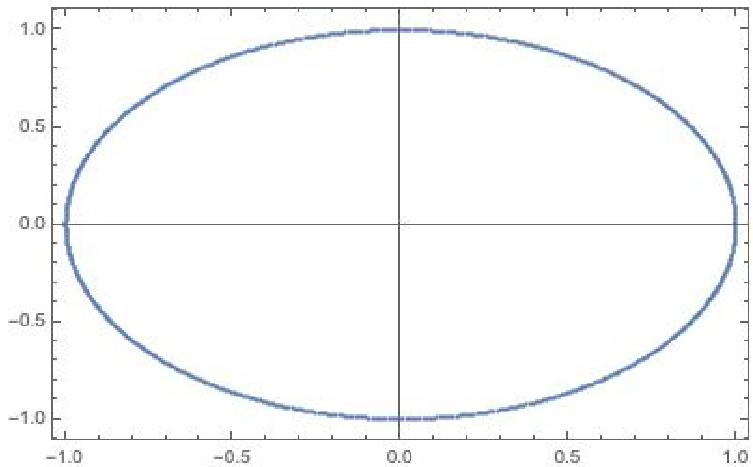

Figure 1: Plot of orbit following circular path as expected

Note at end of conclusion regarding rest of questions.

Question 3

Now we consider a case where the initial radius is slightly farther out at 1.3 AU. We see a new plot of the orbit:

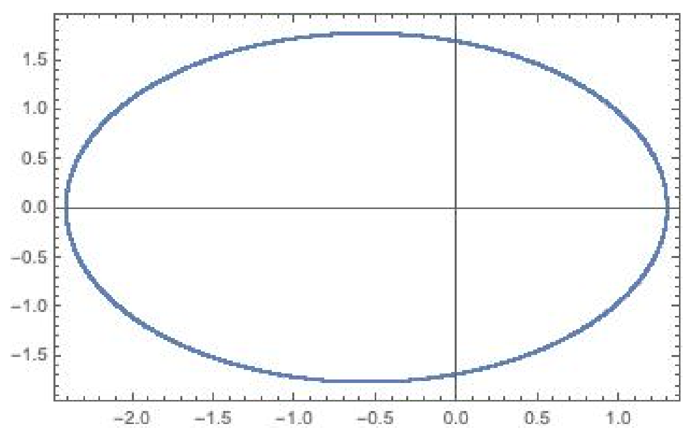

Figure 2: Plot of orbit at 1.3 AU with greater eccentricity

The perihelion is at 1.3 AU and the aphelion is at 2.414 AU. Thus, the aphelion to perihelion ratio is $\frac{r_a}{r_p} = 1.86$.

The energies can also be plotted as seen below. The kinetic energy is seen in yellow, the potential energy in blue and the total energy in green.

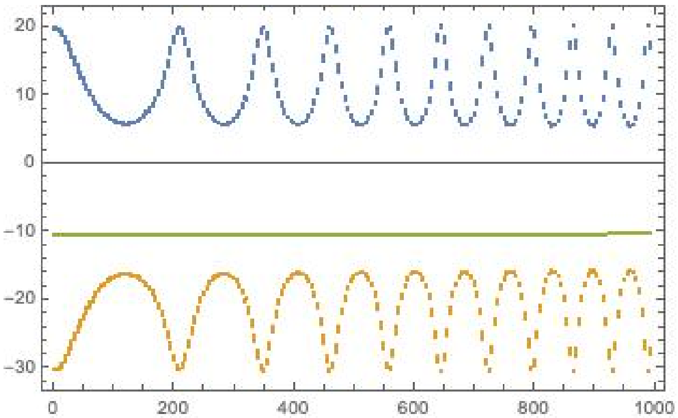

Figure 3: Kinetic Energy, Potential Energy and Total Energy for orbit at 1.3 AU

Question 4

Now, we consider the case for highly eccentric orbits such as that of Haley's comet. We simulate such orbits with a radius of 1.995 AU and a stepsize of 10^{-3} yr. We see the orbit for several periods:

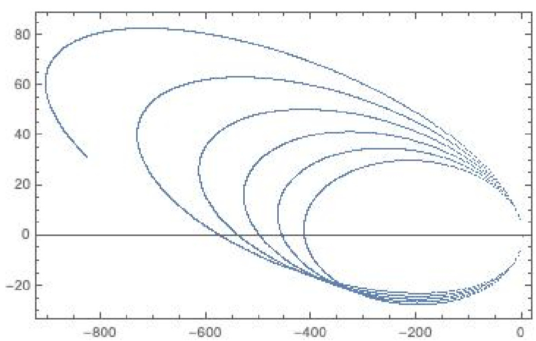

Figure 4: Plot of eccentric orbit for several periods such as that of Haley's Comet

Just as before, the energies can also be plotted as seen below:

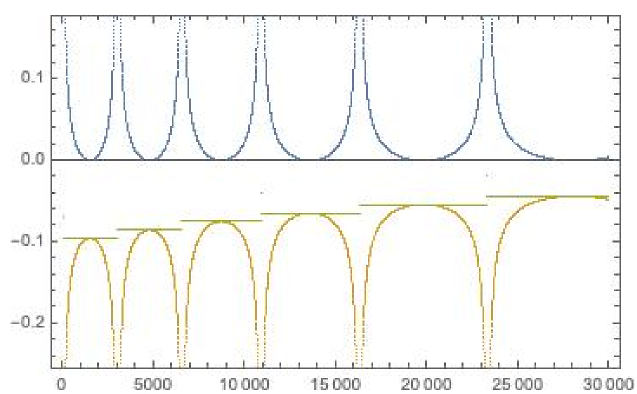

Figure 5: Energies of highly eccentric orbits

Based on this diagram, the approximate period is ≈2700 years.

Question 5

We rerun the situation from Question 4 with the highly eccentric orbit with a step size of 0.05 for a period of 30,000 years to yield the plot of its orbit:

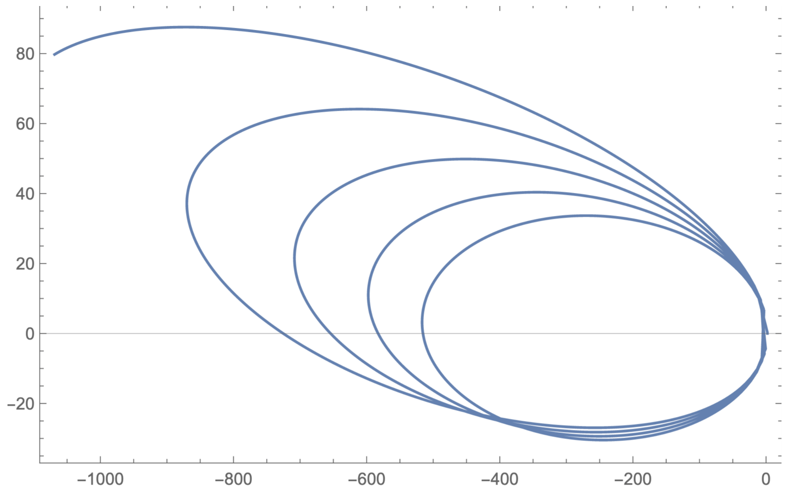

Figure 6: Eccentric orbit using dt = 0.05, T = 30,000

Note at end of conclusion regarding rest of questions.

3 Conclusions

This report gave me the ability to see how simulations are created and how numerical integrators can provide a fast and easy method to solving complex differential equations.

Firstly, on the note of modelling simulations; I was always interested to see how simulations were created. I began to understand how we can use vectors to model positions and velocities, using a numerical integrator to predict future behaviour of each vector element and then simulate this over time.

Secondly, on the note of solving differential equations, I noticed how accurate and effective numerical integrators have been to solving differential equations and hope to apply this to future work and problems.

Note: Was unable to get Mathematica to work when plotting specific energy so I could not recreate those plots, I did create a code that worked to simulate it, however.