

Compressed Sensing

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1 Preliminaries

We define useful mappings used in convex optimization, namely the proximal mapping

$$\text{Prox}_f^\lambda(y) = \arg \min_x \left(f(x) + \frac{1}{2\lambda} \|x - y\| \right)$$

2 Overview

We are given an incomplete matrix M with known entries $(i, j) \in \Omega$. In the general form matrix completion reads

$$\min \|X\|_* \quad s.t. \quad \mathcal{A}(X) = B$$

where \mathcal{A} is a linear map $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{|\Omega|}$

and B encodes our knowledge of M . In our case we choose $\mathcal{A} = \mathcal{P}_\Omega$ with

$$[\mathcal{P}_\Omega(X)]_{i,j} = \begin{cases} X_{j+mi} & \text{if } (i, j) \in \Omega \\ 0, & \text{otherwise} \end{cases}$$

We denote $A \in \mathbb{R}^{|\Omega| \times mn}$ the matrix corresponding to the mapping \mathcal{P}_Ω . With this we can rewrite the objective above:

$$\min \|X\|_* \quad s.t. \quad \mathcal{P}_\Omega(X) = \text{Avec}(X) = B = \mathcal{P}_\Omega(M)$$

3 Douglas- Rachford Splitting

In the noiseless case we can use DRS to solve this numerically via a fixed-point iteration:

$$z^{(k+1)} = z^{(k)} + \text{Prox}_g^\gamma \left(2 \text{Prox}_f^\gamma(z^{(k)}) - z^{(k)} \right) - \text{Prox}_g^\gamma(z^{(k)})$$

To do this, we express our objective as a sum of two functions $f + g$ where $f(X) = \delta_C(X)$ with $C = \{X | AX = B\}$ and $g(X) = \|X\|_*$. We then obtain

$$\begin{aligned}\text{Prox}_f^\gamma(x) &= \Pi_C(x) = x + A^+(b - Ax) \\ &= x + A^T(AA^T)^{-1}(b - Ax) \\ &= x + A^T(b - Ax)\end{aligned}$$

To express the proximal mapping of the nuclear norm, we define the singular value thresholding operator \mathcal{D}_γ . For a matrix Y with the SVD $Y = U\Sigma V^T = U \text{diag}(\sigma(Y))V^T$ it is expressed by $\mathcal{D}_\gamma(Y) = U \text{diag}(\sigma(Y) - \gamma)_+ V^T$. Thus we get

$$\begin{aligned}\text{Prox}_g^\gamma(x) &= U(\sigma(x) - \gamma)_+ V^T \\ &= \mathcal{D}_\gamma(x)\end{aligned}$$

4 FISTA

In the noisy case we are trying to minimize the following objective (in the Lagrangian formulation):

$$\min_X \lambda \|X\|_* + \frac{1}{2} \|AX - B\|_2^2 \quad (4.1)$$

For some $\lambda > 0$. The FISTA algorithm is formulated with respect to the following objective:

$$\min_X f(x) + g(x)$$

where we assume f and g to be sufficiently smooth, i.e. $f \in C^{1,1}(\mathbb{R}^n)$, which means

$$\exists L(f) : \|\nabla f(x) - \nabla f(y)\| \leq L(f) \|x - y\| \forall x, y \in \mathbb{R}^n$$

In our case we have $f(x) = \frac{1}{2} \|AX - B\|_2^2 = \frac{1}{2} \langle AX - B, AX - B \rangle$ and $g(x) = \lambda \|X\|_*$. By simple calculation we get

$$\nabla f(X) = A^T AX - A^T B = A^T (AX - B) \quad (4.2)$$

$$L(f) = 2\lambda_{\max}(A^T A) \quad (4.3)$$

Algorithm 1 FISTA with constant step size

Input Lipschitz- constant $L(f)$ of ∇f , $y_1 = x_0 \in \mathbb{R}^n$, $t_1 = 1$

- 1: **for** $k = 1, \dots$ **do**
 - 2: $x_k = p_L(y_k)$
 - 3: $t_{k+1} = \frac{1 + \sqrt{1 + 4 * t_k^2}}{2}$
 - 4: $y_{k+1} = x_k + \frac{t_k - 1}{t_{k+1}} (x_k - x_{k-1})$
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In this algorithm we def.

$$\begin{aligned}
Q_L(x, y) &= f(y) + \langle x - y, \nabla f(y) \rangle + \frac{L(f)}{2} \|x - y\|^2 + g(x) \\
p_L(y) &= \arg \min_x Q_L(x, y) \\
&= \arg \min_x \left(g(x) + \frac{L(f)}{2} \left\| x - \left(y - \frac{1}{L} \nabla f(y) \right) \right\|^2 \right) \\
&= \arg \min_x \left(\frac{g(x)}{\lambda} + \frac{L(f)}{2\lambda} \left\| x - \left(y - \frac{1}{L} \nabla f(y) \right) \right\|^2 \right) \\
&= \text{Prox}_{g/\lambda}^{\lambda/L} \left(y - \frac{1}{L} \nabla f(y) \right)
\end{aligned}$$

Since we are trying to find the proximal mapping for the nuclear norm, we can apply the thresholding operator just like in our iteration step for DRS. We get

$$\begin{aligned}
\text{Prox}_{g/\lambda}^{\lambda/L} \left(y - \frac{1}{L} \nabla f(y) \right) &= U \text{diag} \left(\sigma \left(y - \frac{1}{L} \nabla f(y) \right) - \frac{\lambda}{L} \right)_+ V^T \\
&= \mathcal{D}_{\lambda/L} \left(y - \frac{1}{L} \nabla f(y) \right) \\
&= \mathcal{D}_{\lambda/L} \left(y - \frac{1}{L} A^T (Ay - B) \right)
\end{aligned}$$

I'm not sure if this is correct