This notebook derives some manipulations/special cases of formulae from O'Toole & Woodhouse (2011) "Numerically stable computation of complete synthetic seismograms including the static displacement in plane layered media".

In[1]:= $$\text{\$Assumptions} = v_p > 0 \,\&\&\, v_p \in \text{Reals} \,\&\&\, v_s > 0 \,\&\&\, v_s \in \text{Reals} \,\&\&\, \rho > 0 \,\&\&\, \rho \in \text{Reals} \,\&\&\, k \in \text{Reals} \,\&\&\, k \geq 0 \,\&\&\, h \in \text{Reals} \,\&\&\, h > 0$$

Out[1]= $$v_p > 0 \,\&\&\, v_p \in \mathbb{R} \,\&\&\, v_s > 0 \,\&\&\, v_s \in \mathbb{R} \,\&\&\, \rho > 0 \,\&\&\, \rho \in \mathbb{R} \,\&\&\, k \in \mathbb{R} \,\&\&\, k \geq 0 \,\&\&\, h \in \mathbb{R} \,\&\&\, h > 0$$

In[2]:=

Equation 14 gives:

In[3]:= $$A := \left\{\left\{0,\, k\,\frac{\lambda}{\sigma},\, \frac{1}{\sigma},\, 0\right\},\, \left\{-k,\, 0,\, 0,\, \frac{1}{\mu}\right\},\, \left\{-\rho\,\omega^2,\, 0,\, 0,\, k\right\},\, \left\{0,\, k^2\,(\gamma+\mu) - \rho\,\omega^2,\, -k\,\frac{\lambda}{\sigma},\, 0\right\}\right\}$$

Equation 55 defines Z:

In[4]:= $$Z := \left\{\left\{\frac{1}{\sqrt{\rho}},\, 0,\, 0,\, 0\right\},\, \left\{0,\, 0,\, 0,\, \frac{-1}{\sqrt{\rho}}\right\},\, \left\{0,\, 0,\, \sqrt{\rho},\, -2\,\mu\,\frac{k}{\sqrt{\rho}}\right\},\, \left\{2\,\mu\,\frac{k}{\sqrt{\rho}},\, \sqrt{\rho},\, 0,\, 0\right\}\right\}$$

and we can verify the stated form of the inverse matrix:

In[5]:= $$\text{Inverse}[Z]\ //\ \text{MatrixForm}$$

Out[5]//MatrixForm=

$$\begin{pmatrix} \sqrt{\rho} & 0 & 0 & 0 \\ -\frac{2\,k\,\mu}{\sqrt{\rho}} & 0 & 0 & \frac{1}{\sqrt{\rho}} \\ 0 & -\frac{2\,k\,\mu}{\sqrt{\rho}} & \frac{1}{\sqrt{\rho}} & 0 \\ 0 & -\sqrt{\rho} & 0 & 0 \end{pmatrix}$$

We can then form A' and verify that it is as given in eq.57:

In[6]:= $$\text{Ap} := \text{FullSimplify}\left[\text{Inverse}[Z].A.Z,\, \left\{\sigma == \lambda + 2\,\mu,\, \gamma == \mu\,\frac{3\,\lambda + 2\,\mu}{\lambda + 2\,\mu}\right\}\right]$$

In[7]:= $$\text{Ap}\ //\ \text{MatrixForm}$$

Out[7]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & \frac{\rho}{\sigma} & -k \\ 0 & 0 & -k & \omega^2 \\ -\omega^2 & -k & 0 & 0 \\ -k & -\frac{\rho}{\mu} & 0 & 0 \end{pmatrix}$$

Substituting in for the Lamé parameters we find

In[8]:= $$\text{Ap}\ /.\ \sigma \to \rho\, {v_p}^2\ /.\ \mu \to \rho\, {v_s}^2$$

Out[8]= $$\left\{\left\{0,\, 0,\, \frac{1}{v_p^2},\, -k\right\},\, \left\{0,\, 0,\, -k,\, \omega^2\right\},\, \left\{-\omega^2,\, -k,\, 0,\, 0\right\},\, \left\{-k,\, -\frac{1}{v_s^2},\, 0,\, 0\right\}\right\}$$

Converting to use ζ_p, ζ_s we have

In[9]:= $$\text{Ap} := \left\{\left\{0,\, 0,\, \frac{k^2 - {\zeta_p}^2}{\omega^2},\, -k\right\},\, \left\{0,\, 0,\, -k,\, \omega^2\right\},\, \left\{-\omega^2,\, -k,\, 0,\, 0\right\},\, \left\{-k,\, \frac{{\zeta_s}^2 - k^2}{\omega^2},\, 0,\, 0\right\}\right\}$$

which has the eigendecomposition as given in eqs. 59 & 60 (up to normalisation differences).

In[10]:=
```
Eigenvalues [Ap] // FullSimplify
```

Out[10]= $\{-\zeta_p, \zeta_p, -\zeta_s, \zeta_s\}$

In[11]:=
```
Eigenvectors[Ap] // FullSimplify
```

Out[11]= $\left\{\left\{\frac{\zeta_p}{k}, 0, \frac{\omega^2}{k}, 1\right\}, \left\{-\frac{\zeta_p}{k}, 0, \frac{\omega^2}{k}, 1\right\}, \left\{\frac{k}{\zeta_s}, -\frac{\omega^2}{\zeta_s}, 0, 1\right\}, \left\{-\frac{k}{\zeta_s}, \frac{\omega^2}{\zeta_s}, 0, 1\right\}\right\}$

We can then take the matrix exponential and obtain eq. 62

In[12]:=
```
MatrixExp[h Ap] // FullSimplify // MatrixForm
```

Out[12]//MatrixForm=

$$\begin{pmatrix} \text{Cosh}[h\,\zeta_p] & \frac{k\,(\text{Cosh}[h\,\zeta_p]-\text{Cosh}[h\,\zeta_s])}{\omega^2} & \frac{-\text{Sinh}[h\,\zeta_p]\,\zeta_p+\frac{k^2\,\text{Sinh}[h\,\zeta_s]}{\zeta_s}}{\omega^2} & -\frac{k\,\text{Sinh}[h\,\zeta_s]}{\zeta_s} \\ 0 & \text{Cosh}[h\,\zeta_s] & -\frac{k\,\text{Sinh}[h\,\zeta_s]}{\zeta_s} & \frac{\omega^2\,\text{Sinh}[h\,\zeta_s]}{\zeta_s} \\ -\frac{\omega^2\,\text{Sinh}[h\,\zeta_p]}{\zeta_p} & -\frac{k\,\text{Sinh}[h\,\zeta_p]}{\zeta_p} & \text{Cosh}[h\,\zeta_p] & 0 \\ -\frac{k\,\text{Sinh}[h\,\zeta_p]}{\zeta_p} & \frac{-\frac{k^2\,\text{Sinh}[h\,\zeta_p]}{\zeta_p}+\text{Sinh}[h\,\zeta_s]\,\zeta_s}{\omega^2} & \frac{k\,(\text{Cosh}[h\,\zeta_p]-\text{Cosh}[h\,\zeta_s])}{\omega^2} & \text{Cosh}[h\,\zeta_s] \end{pmatrix}$$

and taking the second-order minors yields eq. 66

In[13]:=
```
Minors[MatrixExp[h Ap], 2] // FullSimplify // MatrixForm
```

Out[13]//MatrixForm=

$$\begin{pmatrix} \text{Cosh}[h\,\zeta_p]\,\text{Cosh}[h\,\zeta_s] & -\frac{k\,\text{Cosh}[h\,\zeta_p]\,\text{Sinh}[h\,\zeta_s]}{\zeta_s} & \frac{\omega^2\,\text{Cosh}[h\,\zeta_p \ldots}{\zeta \ldots} \\ -\frac{k\,\text{Cosh}[h\,\zeta_s]\,\text{Sinh}[h\,\zeta_p]}{\zeta_p} & 1+\frac{k^2\,\text{Sinh}[h\,\zeta_p]\,\text{Sinh}[h\,\zeta_s]}{\zeta_p\,\zeta_s} & -\frac{k\,\omega^2\,\text{Sinh}[h \ldots}{\zeta \ldots} \\ \frac{-\frac{k^2\,\text{Cosh}[h\,\zeta_s]\,\text{Sinh}[h\,\zeta_p]}{\zeta_p}+\text{Cosh}[h\,\zeta_p]\,\text{Sinh}[h\,\zeta_s]\,\zeta_s}{\omega^2} & \frac{k\left(1-\text{Cosh}[h\,\zeta_p]\,\text{Cosh}[h\,\zeta_s]+\frac{k^2\,\text{Sinh}[h\,\zeta_p]\,\text{Sinh}[h\,\zeta_s]}{\zeta_p\,\zeta_s}\right)}{\omega^2} & \text{Cosh}[h\,\zeta_p]\,\text{Cosh}[h\,\zeta_s \ldots \\ \frac{\omega^2\,\text{Cosh}[h\,\zeta_s]\,\text{Sinh}[h\,\zeta_p]}{\zeta_p} & -\frac{k\,\omega^2\,\text{Sinh}[h\,\zeta_p]\,\text{Sinh}[h\,\zeta_s]}{\zeta_p\,\zeta_s} & \frac{\omega^4\,\text{Sinh}[h\,\zeta_p \ldots}{\zeta_p \ldots} \\ \frac{k\,\text{Cosh}[h\,\zeta_s]\,\text{Sinh}[h\,\zeta_p]}{\zeta_p} & -\frac{k^2\,\text{Sinh}[h\,\zeta_p]\,\text{Sinh}[h\,\zeta_s]}{\zeta_p\,\zeta_s} & \frac{k\,\omega^2\,\text{Sinh}[h\,\zeta \ldots}{\zeta_p \ldots} \\ -\frac{\text{Sinh}[h\,\zeta_p]\,\text{Sinh}[h\,\zeta_s]\,\zeta_s}{\zeta_p} & \frac{k\,\text{Cosh}[h\,\zeta_s]\,\text{Sinh}[h\,\zeta_p]}{\zeta_p} & -\frac{\omega^2\,\text{Cosh}[h\,\zeta \ldots}{\ldots} \end{pmatrix}$$

A naïve implementation of the above expression is not numerically stable as $\omega \to 0$. To obtain a version useful in this case we begin by finding the zero-th order term in the Taylor expansion:

In[14]:=

$$\texttt{ZeroFreq} = \texttt{Series}\left[\texttt{Minors[MatrixExp[h Ap], 2] /. } \zeta_p \to \sqrt{k^2 - \frac{\omega^2}{{v_p}^2}} \texttt{ /. } \zeta_s \to \sqrt{k^2 - \frac{\omega^2}{{v_s}^2}}, \{\omega, 0, 0\}\right] \texttt{ // Normal}$$

Out[14]=

$$\begin{aligned}
&\Bigg\{\Bigg\{\frac{1}{4} e^{-2hk}\left(1+e^{2hk}\right)^2,\ -\frac{1}{4} e^{-2hk}\left(-1+e^{2hk}\right)\left(1+e^{2hk}\right),\\
&\quad 0,\ -\frac{e^{-2hk}\left(-v_p^2+e^{4hk}v_p^2-4e^{2hk}h\,k\,v_p^2-v_s^2+e^{4hk}v_s^2+4e^{2hk}h\,k\,v_s^2\right)}{8\,k\,v_p^2\,v_s^2},\\
&\quad \frac{1}{4} e^{-2hk}\left(-1+e^{4hk}\right),\ -\frac{1}{4} e^{-2hk}\left(-1+e^{2hk}\right)^2\Bigg\},\\
&\Bigg\{-\frac{1}{4} e^{-2hk}\left(-1+e^{4hk}\right),\ \frac{1}{4} e^{-2hk}\left(1+e^{2hk}\right)^2,\ 0,\ \frac{e^{-2hk}\left(-1+e^{2hk}\right)^2\left(v_p^2+v_s^2\right)}{8\,k\,v_p^2\,v_s^2},\\
&\quad -\frac{1}{4} e^{-2hk}\left(-1+e^{2hk}\right)^2,\ \frac{1}{4} e^{-2hk}\left(-1+e^{2hk}\right)\left(1+e^{2hk}\right)\Bigg\},\\
&\Bigg\{-\frac{e^{-2hk}\left(-v_p^2+e^{4hk}v_p^2+4e^{2hk}h\,k\,v_p^2-v_s^2+e^{4hk}v_s^2-4e^{2hk}h\,k\,v_s^2\right)}{8\,k\,v_p^2\,v_s^2},\\
&\quad \frac{e^{-2hk}\left(-1+e^{2hk}\right)^2\left(v_p^2+v_s^2\right)}{8\,k\,v_p^2\,v_s^2},\ 1,\ \frac{1}{16\,k^2\,v_p^4\,v_s^4}\\
&\quad e^{-2hk}\Big(v_p^4-2e^{2hk}v_p^4+e^{4hk}v_p^4-4e^{2hk}h^2k^2v_p^4+2v_p^2v_s^2-4e^{2hk}v_p^2v_s^2+2e^{4hk}v_p^2v_s^2+8e^{2hk}\\
&\qquad h^2k^2v_p^2v_s^2+v_s^4-2e^{2hk}v_s^4+e^{4hk}v_s^4-4e^{2hk}h^2k^2v_s^4\Big),\ -\frac{e^{-2hk}\left(-1+e^{2hk}\right)^2\left(v_p^2+v_s^2\right)}{8\,k\,v_p^2\,v_s^2},\\
&\quad \frac{e^{-2hk}\left(-v_p^2+e^{4hk}v_p^2-4e^{2hk}h\,k\,v_p^2-v_s^2+e^{4hk}v_s^2+4e^{2hk}h\,k\,v_s^2\right)}{8\,k\,v_p^2\,v_s^2}\Bigg\},\\
&\{0,0,0,1,0,0\},\ \Bigg\{\frac{1}{4} e^{-2hk}\left(-1+e^{2hk}\right)\left(1+e^{2hk}\right),\ -\frac{1}{4} e^{-2hk}\left(-1+e^{2hk}\right)^2,\ 0,\\
&\quad -\frac{e^{-2hk}\left(-1+e^{2hk}\right)^2\left(v_p^2+v_s^2\right)}{8\,k\,v_p^2\,v_s^2},\ \frac{1}{4} e^{-2hk}\left(1+e^{2hk}\right)^2,\ -\frac{1}{4} e^{-2hk}\left(-1+e^{4hk}\right)\Bigg\},\\
&\Bigg\{-\frac{1}{4} e^{-2hk}\left(-1+e^{2hk}\right)^2,\ \frac{1}{4} e^{-2hk}\left(-1+e^{4hk}\right),\ 0,\\
&\quad \frac{e^{-2hk}\left(-v_p^2+e^{4hk}v_p^2+4e^{2hk}h\,k\,v_p^2-v_s^2+e^{4hk}v_s^2-4e^{2hk}h\,k\,v_s^2\right)}{8\,k\,v_p^2\,v_s^2},\\
&\quad -\frac{1}{4} e^{-2hk}\left(-1+e^{2hk}\right)\left(1+e^{2hk}\right),\ \frac{1}{4} e^{-2hk}\left(1+e^{2hk}\right)^2\Bigg\}\Bigg\}
\end{aligned}$$

We can rewrite this substituting c,s for the hyperbolic functions as in the paper - some effort seems to be required to force Mathematica to do this completely.

In[15]:=
```
ZeroFreqCS = Simplify[
  Simplify[FullSimplify[(ZeroFreq /. vp → Sqrt[σ/ρ] /. vs → Sqrt[μ/ρ]), {Cosh[h k]^2 == cc,
      Cosh[h k] == c, Sinh[h k]^2 == ss, Sinh[h k] == s, Cosh[h k] Sinh[h k] == cs}] //
    TrigExpand, {Cosh[h k]^2 == cc, Cosh[h k] == c, Sinh[h k]^2 == ss,
    Sinh[h k] == s, Cosh[h k] Sinh[h k] == cs}, Trig → False] /.
   cc → c^2 /. ss → s^2 /. cs → c s, c^2 + s^2 == 1 + 2 s^2]
```

Out[15]=

$$\left\{\left\{c^2, -c\,s, 0, -\frac{\rho\,(h\,k\,(\mu-\sigma)+c\,s\,(\mu+\sigma))}{2\,k\,\mu\,\sigma}, c\,s, -s^2\right\}, \left\{-c\,s, c^2, 0, \frac{s^2\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, -s^2, c\,s\right\}, \left\{-\frac{\rho\,(h\,k\,(-\mu+\sigma)+c\,s\,(\mu+\sigma))}{2\,k\,\mu\,\sigma}, \frac{s^2\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, 1, -\frac{\rho^2\left(h^2\,k^2\,(\mu-\sigma)^2-s^2\,(\mu+\sigma)^2\right)}{4\,k^2\,\mu^2\,\sigma^2}, -\frac{s^2\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, \frac{\rho\,(h\,k\,(\mu-\sigma)+c\,s\,(\mu+\sigma))}{2\,k\,\mu\,\sigma}\right\}, \{0,0,0,1,0,0\}, \left\{c\,s, -s^2, 0, -\frac{s^2\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, c^2, -c\,s\right\}, \left\{-s^2, c\,s, 0, \frac{\rho\,(h\,k\,(-\mu+\sigma)+c\,s\,(\mu+\sigma))}{2\,k\,\mu\,\sigma}, -c\,s, c^2\right\}\right\}$$

Most terms in this expression feature a product of two hyperbolic functions, but not all do. We want to separate the two:

In[16]:=
```
ZFCS0 = ZeroFreqCS /. c → 0 /. s → 0 // FullSimplify
```

Out[16]=

$$\left\{\left\{0,0,0,\frac{h\,\rho\,(-\mu+\sigma)}{2\,\mu\,\sigma},0,0\right\}, \{0,0,0,0,0,0\}, \left\{\frac{h\,\rho\,(\mu-\sigma)}{2\,\mu\,\sigma}, 0, 1, -\frac{h^2\,\rho^2\,(\mu-\sigma)^2}{4\,\mu^2\,\sigma^2}, 0, \frac{h\,\rho\,(\mu-\sigma)}{2\,\mu\,\sigma}\right\}, \{0,0,0,1,0,0\}, \{0,0,0,0,0,0\}, \left\{0,0,0,\frac{h\,\rho\,(-\mu+\sigma)}{2\,\mu\,\sigma},0,0\right\}\right\}$$

In[17]:=
```
ZFCS1 = (ZeroFreqCS - ZFCS0) // FullSimplify
```

Out[17]=

$$\left\{\left\{c^2, -c\,s, 0, -\frac{c\,s\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, c\,s, -s^2\right\}, \left\{-c\,s, c^2, 0, \frac{s^2\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, -s^2, c\,s\right\}, \left\{-\frac{c\,s\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, \frac{s^2\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, 0, \frac{s^2\,\rho^2\,(\mu+\sigma)^2}{4\,k^2\,\mu^2\,\sigma^2}, -\frac{s^2\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, \frac{c\,s\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}\right\}, \{0,0,0,0,0,0\}, \left\{c\,s, -s^2, 0, -\frac{s^2\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, c^2, -c\,s\right\}, \left\{-s^2, c\,s, 0, \frac{c\,s\,\rho\,(\mu+\sigma)}{2\,k\,\mu\,\sigma}, -c\,s, c^2\right\}\right\}$$

In[18]:=

At this point we have ZeroFreqCS = ZFCS0+ZFCS1. We do some rescaling of rows and columns to get things into a nice format:

In[19]:=

```
SL = DiagonalMatrix[{1, 1, 2 k μ σ, 1, 1, 1}]
```

Out[19]= $\{\{1,0,0,0,0,0\},\{0,1,0,0,0,0\},\{0,0,2k\mu\sigma,0,0,0\},\{0,0,0,1,0,0\},\{0,0,0,0,1,0\},\{0,0,0,0,0,1\}\}$

In[20]:=
```
SR = DiagonalMatrix[{1, 1, 1, 2 k μ σ, 1, 1}]
```

Out[20]= $\{\{1,0,0,0,0,0\},\{0,1,0,0,0,0\},\{0,0,1,0,0,0\},\{0,0,0,2k\mu\sigma,0,0\},\{0,0,0,0,1,0\},\{0,0,0,0,0,1\}\}$

In[21]:=
```
SL.ZFCS0.SR // MatrixForm
```

Out[21]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & h k \rho (-\mu+\sigma) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ h k \rho (\mu-\sigma) & 0 & 2 k \mu \sigma & -h^2 k^2 \rho^2 (\mu-\sigma)^2 & 0 & h k \rho (\mu-\sigma) \\ 0 & 0 & 0 & 2 k \mu \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h k \rho (-\mu+\sigma) & 0 & 0 \end{pmatrix}$$

In[22]:=
```
SL.ZFCS1.SR // MatrixForm
```

Out[22]//MatrixForm=

$$\begin{pmatrix} c^2 & -c s & 0 & -c s \rho (\mu+\sigma) & c s & -s^2 \\ -c s & c^2 & 0 & s^2 \rho (\mu+\sigma) & -s^2 & c s \\ -c s \rho (\mu+\sigma) & s^2 \rho (\mu+\sigma) & 0 & s^2 \rho^2 (\mu+\sigma)^2 & -s^2 \rho (\mu+\sigma) & c s \rho (\mu+\sigma) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ c s & -s^2 & 0 & -s^2 \rho (\mu+\sigma) & c^2 & -c s \\ -s^2 & c s & 0 & c s \rho (\mu+\sigma) & -c s & c^2 \end{pmatrix}$$

In[23]:=

The above two matrices represent S0L(exp(h A'))S0R. We want to form Z(exp(hA'))Z^{-1}. We therefore need to roll the inverse scalings into our definitions of Z,Z^{-1}.

In[24]:=
```
Minors[Z, 2].Inverse[SL] // FullSimplify
```

Out[24]= $\left\{\left\{0,0,-\frac{1}{2k\mu\rho\sigma},0,0,0\right\},\left\{0,1,-\frac{1}{\rho\sigma},0,0,0\right\},\{1,0,0,0,0,0\},\{0,0,0,0,0,1\},\left\{0,0,\frac{1}{\rho\sigma},0,1,0\right\},\left\{0,-2k\mu,\frac{2k\mu}{\rho\sigma},-\rho,2k\mu,0\right\}\right\}$

In[25]:=

```
Inverse[SR].Minors[Inverse[Z], 2] // FullSimplify
```

Out[25]= $\left\{\{0,0,1,0,0,0\},\{-2k\mu,1,0,0,0,0\},\{-\rho,0,0,0,0,0\},\left\{\frac{2k\mu}{\rho\sigma},-\frac{1}{\rho\sigma},0,0,\frac{1}{\rho\sigma},-\frac{1}{2k\mu\rho\sigma}\right\},\{2k\mu,0,0,0,1,0\},\{0,0,0,1,0,0\}\right\}$

For the 4D system at zero frequency things are simpler:

In[26]:=
```
FullSimplify[
 ((Series[MatrixExp[h Ap] /. ζp → Sqrt[k^2 - ω^2/vp^2] /. ζs → Sqrt[k^2 - ω^2/vs^2], {ω, 0, 0}] //
      Normal) /. vp → Sqrt[σ/ρ] /. vs → Sqrt[μ/ρ])] // MatrixForm
```

Out[26]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cosh}[h\,k] & \frac{h\,\rho\,(-\mu+\sigma)\,\mathrm{Sinh}[h\,k]}{2\,\mu\,\sigma} & \frac{h\,k\,\rho\,(\mu-\sigma)\,\mathrm{Cosh}[h\,k]+\rho\,(\mu+\sigma)\,\mathrm{Sinh}[h\,k]}{2\,k\,\mu\,\sigma} & -\mathrm{Sinh}[h\,k] \\ 0 & \mathrm{Cosh}[h\,k] & -\mathrm{Sinh}[h\,k] & 0 \\ 0 & -\mathrm{Sinh}[h\,k] & \mathrm{Cosh}[h\,k] & 0 \\ -\mathrm{Sinh}[h\,k] & \frac{h\,k\,\rho\,(\mu-\sigma)\,\mathrm{Cosh}[h\,k]-\rho\,(\mu+\sigma)\,\mathrm{Sinh}[h\,k]}{2\,k\,\mu\,\sigma} & \frac{h\,\rho\,(-\mu+\sigma)\,\mathrm{Sinh}[h\,k]}{2\,\mu\,\sigma} & \mathrm{Cosh}[h\,k] \end{pmatrix}$$