Analysis of price-tracking Kandle

Mangrove Research

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We explore two ideas in this note:
- extracting the deterministic impermanent loss term from the (last price, return) joint distribution of a price-tracking Kandle
- defining a closed-formula (dependent on number of crossings) for the return of a price-tracking Kandle

1 Extracting the deterministic component

Idea: For price-tracking Kandle, as for all variants, there is a strong correlation between the last price and the pay-off (or return). In other words so-called 'impermanent loss' dominates the return.

One can 'neutralise' the price-deterministic part of the pay-off by adding a correction using the regression line with slope b, intercept c.

Write priceChange for the relative price change $\mathrm{lastPrice}/\mathrm{initialPrice} - 1$:

$$(\mathrm{priceChange}, \mathrm{return}) \longrightarrow (\mathrm{lastPrice}, \mathrm{return} - (b \times \mathrm{priceChange} + c))$$

If $b = 1/2$, $c = 0$ we are subtracting the price-deterministic return of the buy-and-hold where half of the initial cash is invested in the underlying. The exact offsets b, c, to be determined by fitting a regression line to a numerical experiment depend on (r, σ).

Concretely that can be done eg by shorting a certain amount b of the underlying on a perp market, or by borrowing some of the underlying on a mending platform. What does c represent: the deterministic cost of financing the strategy (a sort of premium)?

For large ratios $r > 1.1$ (aka grid steps), there are < 1 crossings on average and the constant is very nearly $b = \frac{1}{2}$ (ie a complete buy-and-hold). As r decreases, the mean number of crossings increases and the constant b, c seem to decrease. To investigate empirically.

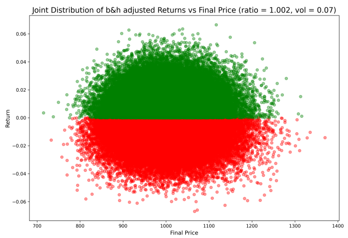

Figure 1: *adjusted return vs exit price (ratio = 1.002, vol = 0.07)*

QS1 - Is there for given σ a value $r^\star(\sigma)$ of r that maximises the mean return? [similar to prior work done for plain Kandle]

Todo: Try same experiments with multiple price points (not just 2)

Can we combine sim with reset with a stop loss? Eg, is it useful to stop resetting after k moves, or a certain minimum residual capital (or on a momentum signal)?

1.1 Examples

The cost of financing a price-tracking Kandle (intercept) with $r = 1.005$, $\sigma = 0.06$ is a negative return of $\sim 4.6\%$. See the comparison with the B&H line which has an intercept of zero (Fig. 2). The same cloud of points is represented with the regression line subtracted Fig. 3.

It would be interesting to understand how the parameters of the regression line depend on r and σ.

Fig. 4 gives the distribution of adjusted returns. From the figure it seems one can decompose the return into a random normal component and a deterministic one :

$$\begin{aligned} ret(r,\sigma) &= \mathcal{N}(0, s(r,\sigma)) + \\ &\quad (c(r,\sigma) + b(r,\sigma)(\Delta p/p_0)) \end{aligned} \tag{1}$$

where $\Delta p/p_0$ is the relative price change at the end of the price sequence.

Todo: explore the coefficients in (Eq 1) above.

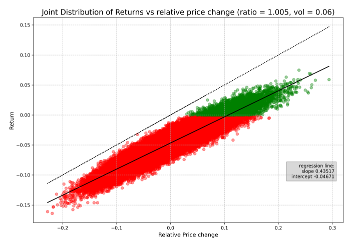

Figure 2: *non-adjusted returns vs relative price change ($r = 1.005$, $\sigma = 0.06$); the dotted line is the buy-and-hold return for half the capital invested in base ($\alpha = 1/2$); the intercept of the regression line is the cost of the strategy per unit of capital provided.*

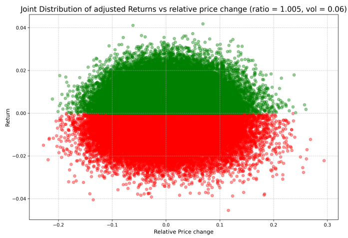

Figure 3: *adjusted returns vs relative price change (r = 1.005, $\sigma = 0.06$).*

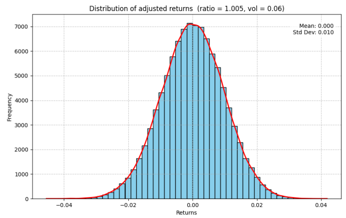

Figure 4: *histograms of adjusted returns ($r = 1.005$, $\sigma = 0.06$).*

2 Analysis of sim reset

Let $\alpha \in [0,1]$ be the fraction of the initial capital kept in quote (cash ratio). Setting $\alpha = 0$ means that one has only asks; $\alpha = 1$ means one has only bids.

Set $\alpha' := 1 - \alpha$ to simplify notations.

Proposition 1 *The total gain under a given price run is:*

$$\Gamma(\alpha) \quad = \quad (\alpha + \alpha' r)^{n_u} (\alpha + \alpha'/r)^{n_d} \qquad (2)$$

where n_u is the number of up-crossings, n_d is the number of down-crossings along the price trajectories.

Proof: We start with a capital v_B^0 of cash (base), of which we keep αv_B^0, and buy $\alpha' v_B^0/p_0$ of base; assuming the current price is p_0 (and no slippage).

Suppose an uc:

$$\begin{pmatrix} \alpha' v_B^0/p_0 \\ \alpha v_B^0 \end{pmatrix} \longrightarrow^c \begin{pmatrix} 0 \\ \alpha v_B^0 + (rp_0)\alpha' v_B^0/p_0 \end{pmatrix}$$

Our new position is worth $v_B^0 \gamma_u$, with the growth rate $\gamma_u = \alpha + \alpha' r > 1$.

Symmetrically for a dc:

$$\begin{pmatrix} \alpha' v_B^0/p_0 \\ \alpha v_B^0 \end{pmatrix} \longrightarrow^d \begin{pmatrix} \alpha' v_B^0/p_0 + \alpha v_B^0/(p_0/r) \\ 0 \end{pmatrix}$$

with a new position worth $v_B^0 \gamma_d$, with $\gamma_d = \alpha + \alpha'/r < 1$. $\square$

NB - Importantly, in the proof above, we assume that on a down-crossing, we mark-to-market the new position at price p_0/r; ie the price at which base was just bought. In reality, on a down-crossing the post-execution re-sell price may be lower: $p_n > p_0/r \geq p_{n+1}$. (In which case we have over-bought so to speak.) Thus there is a loss factor proportional to $p_{n+1}/(p_0/r) < 1$ and we get a more realistic

$$\hat{\gamma}_d = \gamma_d p_{n+1}/(p_0/r) \leq \gamma_d < 1$$

instead.

[Symmetrically, one could have sold higher on an upcrossing with a "manque à gagner" proportional to $p_{n+1}/(rp_0)$. In the simulations we do mark to market at the real price p_{n+1}.]

The corrected compound growth rate is therefore:

$$\begin{array}{rcl} \log \hat{\Gamma} & = & n_u \log(\alpha + \alpha' r) + n_d \log(\alpha + \alpha'/r) + \ell(\mathbf{p}, r) \\ \ell(\mathbf{p}, r) & = & \sum_{i\downarrow} \log(p_{i+1}/(p_0(i)/r)) \end{array}$$

where $p_0(i)$ is the last touched price, ie the mid-point of the strategy right before the ith jump downwards.

This brings a series of remarks:

(1) The smaller the integration step δt, the smaller the loss factor.[1] We should therefore incorporate δt as a bona fide parameter $\ell(\mathbf{p}, r, \delta t)$. In the limit where $\delta t \to 0^+$, $\log(p_{i+1}/(p_0(i)/r)) \to 0^-$, which is paradoxical! In that case δt has to be understood not just as an integration step, but as a market reactivity time or a lag vs a driving price. Price can lag for economical reasons, eg a fee, or physical reasons such as the interblock time -eg $\delta t = 0.2s$ on arbitrum-one. [eg a fee,]maybe check the LvR paper to get a model of the lag [1]]

(2) one could always renege on bids (for a cost q), to mitigate the loss, and just re-sell the remaining inventory [a behaviour which corresponds to $\alpha = 0$?]

(3a) we can also forward the bids (if we do not want to renege) but we are going to pay the same $\log(p_{i+1}/(p_0(i)/r))$ or about where p_{i+1} is now the price on the source dex, or

(3b) re-sell instantly on an other dex (the semi-book is closed for execution) [we may want to generate the volume]

Note that the crossing numbers are random and only depend on vol-and-grid, σ and r.

As $\alpha + \alpha' r > 1$ (barycenter of 1 and $r > 1$), and $\alpha + \alpha'/r < 1$ (barycenter of 1 and $1/r < 1$), it follows that $\Gamma > 1$ ie the strategy is profitable along a given price trajectory as soon as $n_u > n_d$.

In particular if $\alpha = \alpha' = \frac{1}{2}$, we get:

$$\begin{array}{rcl} \log(\Gamma(1/2)) & = & n_u \log(\frac{1+r}{2}) + n_d \log(\frac{1+1/r}{2}) \\ & = & N(f_u \log(\frac{1+r}{2}) + f_d \log(\frac{1+1/r}{2})) \end{array}$$

where in the last line we assume $N = n_u + n_d > 0$ (not always true! especially for large ranges).

The above is > 0 iff:

$$f_d/f_u \leq \frac{\log(1+r) - \log 2}{\log 2 - \log(1 + r^{-1})} =: k(r)$$

where $k(r) \geq 1$ is increasing and converges to $\log(1 + r)/\log(2) - 1$ for large r. This suggests that larger rs are

[1] that is to say the there may be more small crossings is cheaper than fewer large one by concavity of log

more likely to generate profits. To make sure we need to study how the f_d, f_u distribution depends on r. It seems unlikey but it could be that larger r are biased downwards, ie f_d/f_u is increasing function of r. [in sims up and down crossings look equally likely]

Todo: study the distribution of f_u as a function of r and σ. [Also look at std]

2.1 changing α

We can see the effect of changing α.

If $\alpha = 0$ (base only), $\Gamma(0) = r^{n_u - n_d}$. If we go full base, we win iff the process crosses more up than down.

If $\alpha' = 0$ (quote only), $\Gamma(1) = 1$. Indeed if we never buy the underlying, all that can happen is the execution of a bid, the output of which that strategy re-sells immmediately at same price (assuming no fees, no slippage), and therefore the initial wealth never changes. Ie (modulo fees and slippage), setting $\alpha = 0$ amounts to not playing.

3 Random remarks

A small remark: concretely, when running on chain, price-tracking Kandle does not have access to the next price (as the simulation does) only that $p_{n+1} > rp_n$; to correct this one can use multiple offers to improve the upper bound, and/or use an oracle at update time (eg consult another liquidity source).

Euler-Maruyama for large $\sqrt{dt}$, large σ can lead to negative prices in a GBM; so better use the integral form.

Why is Γ an easier notation for growth of mtm:

$$\begin{aligned}
\Gamma &= \textstyle\prod_i \gamma_i \\
R &= \textstyle\prod_i (1 + r_i) - 1 \\
r \star s &= r + s + rs
\end{aligned}$$

if r, s are small then rs is negligible which explains why people talk with returns.

References

[1] Jason Milionis, Ciamac C Moallemi, Tim Roughgarden, and Anthony Lee Zhang. Automated market making and loss-versus-rebalancing. arXiv preprint arXiv:2208.06046, 2022.