

# Computational Physics - Project 4

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# Introduction

In project 4 we are dealing with the Ising model in two dimensions without an external magnetic field. We are looking at a lattice of L times L particles, which have spin values  $\pm 1$ . In order to compute different interesting values, we want to use the metropolis algorithm. With our computations, we want to calculate the Energy, the absolute value of the magnetisation, the heat capacity and susceptibility of the system as a function of time. We also want to compare our solutions with the theoretical closed solution. This project may also show the link from statistical physics to macroscopic properties of a given physical system, which is a very interesting relation.

## 1 Theory

### 1.1 General properties of physical systems and their link to statistical physics

#### 1.1.1 physical ensembles

Canonical ensemble is a statistical way to represent the possible states in a system with fixed temperature, whereas the system exchange energy, the energy follows as an expectation value. The probability distribution is given by the Boltzmann distribution.

$$P_i(\beta) = \frac{e^{-\beta E_i}}{Z} \quad (1)$$

$\beta = 1/k_B T$  where T is the temperature,  $k_B$  is the Boltzmann constant,  $E_i$  is the energy of microstate i and Z is the partition function for the canonical ensemble is the sum over all the microstates M.

$$Z = \sum_{i=1}^M e^{-\beta E_i} \quad (2)$$

#### 1.1.2 General properties of canonical ensembles

the canonical ensemble pursues towards an energy minimum and higher entropy expressed by Helmholtz' free energy.

$$F = -k_B T \ln Z = \langle E \rangle - TS \quad (3)$$

where the entropy S is given by

$$S = -k_B \ln Z + k_B T \frac{\partial \ln Z}{\partial T} \quad (4)$$

after running the system for long time the canonical ensemble is uniquely determined and does not depend on the arbitrary choices for a given temperature, having a steady state without being affected by the equilibrium continuous motion. the system uncertainty due to the energy fluctuations in the canonical ensemble give the variance of the energy.

from equation 3, 4 and probability distribution  $P_i$

$$\langle E \rangle = k_B T^2 \frac{\partial \ln Z}{\partial T} = \sum_{i=1}^M E_i P_i(\beta) = \frac{1}{Z} \sum_{i=1}^M E_i e^{-\beta E_i} \quad (5)$$

The heat capacity is how much the energy change due to the change in the temperature. The heat capacity  $C_V$  can be defined as

$$C_V = \frac{\partial E}{\partial T} \quad (6)$$

$$\frac{\partial}{\partial T} \frac{1}{Z} = \frac{\partial}{\partial T} \frac{1}{\sum_{i=1}^M e^{-\frac{1}{k_B T} E_i}} = -\frac{1}{k_B T^2} \frac{\sum_{i=1}^M E_i e^{-\frac{1}{k_B T} E_i}}{\left(\sum_{i=1}^M e^{-\frac{1}{k_B T} E_i}\right)^2} = -\frac{1}{k_B T^2} \frac{\sum_{i=1}^M E_i e^{-\frac{1}{k_B T} E_i}}{(Z)^2} \quad (7)$$

$$\frac{\partial}{\partial T} \sum_{i=1}^M E_i e^{-\frac{1}{k_B T} E_i} = \frac{1}{k_B T^2} \sum_{i=1}^M E_i^2 e^{-\frac{1}{k_B T} E_i} \quad (8)$$

$$C_V = \frac{\partial \langle E \rangle}{\partial T} = \frac{\partial}{\partial T} \left( \frac{1}{Z} \sum_{i=1}^M E_i e^{-\frac{1}{k_B T} E_i} \right) = -\frac{1}{k_B T^2} \frac{\sum_{i=1}^M E_i e^{-\frac{1}{k_B T} E_i}}{Z^2} \sum_{i=1}^M E_i e^{-\frac{1}{k_B T} E_i} + \frac{1}{Z} \frac{1}{k_B T^2} \sum_{i=1}^M E_i^2 e^{-\frac{1}{k_B T} E_i} \quad (9)$$

$$= -\frac{1}{k_B T^2} \left( \frac{\sum_{i=1}^M E_i e^{-\frac{1}{k_B T} E_i}}{Z} \right)^2 + \frac{1}{Z} \frac{1}{k_B T^2} \sum_{i=1}^M E_i^2 e^{-\frac{1}{k_B T} E_i} = \frac{1}{k_B T^2} (\langle E \rangle^2 - \langle E^2 \rangle) \quad (10)$$

The magnetic susceptibility is a measurable quantity which indicate if the material is attracted or repelled out of a magnetic field. magnetic materials can be classified as paramagnetic, diamagnetic or ferromagnetic based on their susceptibility.

$$\chi = \frac{\partial \langle M \rangle}{\partial H} \quad (11)$$

we can evaluate the mean magnetization through

$$\langle M \rangle = \sum_i^M M_i P_i(\beta) = \frac{1}{Z} \sum_i^M M_i e^{-\frac{E_i}{k_B T}} \quad (12)$$

The total energy of the system in addition of external magnetic field H

$$E = -\sum_{i,j} J_{s_i s_j} - H \sum_i s_i \quad (13)$$

the magnetization is the sum of all spin for a given configuration.

$$\frac{\partial E_i}{\partial H} = -\sum_i s_i = -M_i \quad (14)$$

$$\frac{\partial}{\partial H} \frac{1}{Z} = \frac{\partial}{\partial H} \frac{1}{\sum_{i=1}^M e^{-\frac{1}{k_B T} E_i}} = -\frac{1}{k_B T} \frac{\sum_{i=1}^M M_i e^{-\frac{1}{k_B T} E_i}}{\left( \sum_{i=1}^M e^{-\frac{1}{k_B T} E_i} \right)^2} = -\frac{1}{k_B T} \frac{\sum_{i=1}^M M_i e^{-\frac{1}{k_B T} E_i}}{(Z)^2} \quad (15)$$

$$\frac{\partial}{\partial H} \sum_{i=1}^M M_i e^{-\frac{1}{k_B T} E_i} = \frac{1}{k_B T} \sum_{i=1}^M M_i^2 e^{-\frac{1}{k_B T} E_i} \quad (16)$$

$$\chi = \frac{\partial \langle M \rangle}{\partial H} = \frac{\partial}{\partial H} \left( \frac{1}{Z} \sum_{i=1}^M M_i e^{-\frac{1}{k_B T} E_i} \right) = -\frac{1}{k_B T} \frac{\sum_{i=1}^M M_i e^{-\frac{1}{k_B T} E_i}}{Z^2} \sum_{i=1}^M M_i e^{-\frac{1}{k_B T} E_i} + \frac{1}{Z} \frac{1}{k_B T} \sum_{i=1}^M M_i^2 e^{-\frac{1}{k_B T} E_i} \quad (17)$$

$$= -\frac{1}{k_B T} \left( \frac{\sum_{i=1}^M M_i e^{-\frac{1}{k_B T} E_i}}{Z} \right)^2 + \frac{1}{Z} \frac{1}{k_B T} \sum_{i=1}^M M_i^2 e^{-\frac{1}{k_B T} E_i} = \frac{1}{k_B T} (\langle M \rangle^2 - \langle M^2 \rangle) \quad (18)$$

### 1.1.3 Ferromagnetic order

A ferromagnet have a spontaneous magnetic moment even with the absence of an external magnetic field. Due the existence of a spontaneous moment the electron spin and magnetic moments must be arranged in a regular manner. ferromagnet all spin aligned, antiferromagnet all spin align with neighboring pointing in opposite directions, ferrimagnet the opposing moments are unequal, etc.

### 1.1.4 link from the Macroscopic values to statistical physics

## 1.2 theoretical numerical solutions

### 1.2.1 Ising model

Ising model is a mathematical model for ferromagnetism studies of phase transitions for magnetic system at given a temperature. The model consists the interaction between two neighbouring spins is related by the interaction energy

$$-J s_k s_l \quad (19)$$

where the  $s$  can be in two states +1 or -1, where  $s_k$  and  $s_l$  are the nearest neighbors. Which give a low energy (-J) if the two spin aligned and high energy (J) for spin pointing in opposite direction. The total energy to a system with N number of spins and with the absence of magnetic field can be expressed as

$$E = -J \sum_{\langle kl \rangle}^N s_k s_l \quad (20)$$

...probability distribution with expectation value  $\langle E \rangle$  ...

### 1.2.2 Periodic boundary conditions

Periodic boundary conditions is used for approximating a large or infinite system by using smaller repeating system, we will impose PBCs on our spin lattice in x and y directions. satisfied  $s(L+1, y) = s(1, y)$

$$s(x, L+1) = s(x, 1)$$

### 1.2.3 Metropolis algorithm in the two dimensional Ising model

Ising model with Metropolis algorithm generate a sequence of states with Monte Carlo path, where the transition between states depends on the transition probability between the next and current state. The probability distribution is given by Boltzmann distribution which is the probability for finding the system in a state  $s$ .

$$P_s = \frac{e^{-\beta E_i}}{\sum_{i=1}^M e^{-\beta E_i}} \quad (21)$$

it's difficult to compute since we need the sum of all states, if we have a 10 x 10 spin lattice interacting in our Ising model, there are  $2^{100}$  possible states. This is not that efficient, but luckily the Metropolis algorithm need only the ratios between the states probability and we do not need to compute the sum of all the states after all. The Metropolis algorithm in this case can be implement by establishing two dimensional Ising model with random lattice configuration, then we flip a randomly chosen spin and compute the energy difference  $\Delta E$ . If  $\Delta E \leq 0$  we accept the flip, otherwise we compute the transition probability  $w = e^{-\beta \Delta E}$  and compare with a random number  $r$ .

If  $r \leq w$  we accept the flip otherwise we keep the old configuration. We can keep choosing new random spins until we are satisfied with a good representation of the states."lecture notes Computational physics 2015 at University of Oslo by Morten Hjorth-Jensen page 435)"

### 1.2.4 critical temperature (Lars Onsager)

in 1944 the Norwegian chemist Lars Onsager made very important discovery in theoretical physics, namely the exact solution of the Ising spin model in two dimensions, his work stands to this day, Onsager's solutions achieved the thermodynamic properties of interaction systems and phase transitions at  $T_c$ . however in 1942 Lars Onsager solved the two dimensional model for zero field energy which been published two years after, in 1948 he wrote the solution for the zero field magnetization in a conference at Cornell. however Onsager showed that the partition function for the canonical ensemble with zero external magnetic field  $Z(B = 0, T)$  with N spins.

$$Z_N = (2 \cosh(\beta J) e^I)^N \quad (22)$$

where  $I = \frac{1}{2\pi} \int_0^\pi d\phi \ln \left[ \frac{1}{2} \left( 1 + \sqrt{1 - \kappa^2 \sin^2 \phi} \right) \right]$  where  $\kappa = \frac{2 \sinh(2\beta J)}{\cosh^2(2\beta J)}$  and the energy is given by

$$\langle E \rangle = -J \coth(2\beta J) \left[ 1 + \frac{2}{\pi} (2 \tanh^2(2\beta J) - 1) K_1(q) \right] \quad (23)$$

where  $q = \sinh(2\beta J) / \cosh^2(2\beta J)$  and the complete elliptic integral of the first kind

$$K_1(q) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} \quad (24)$$

and differentiating the energy with the respect to temperature we obtain the specific heat

$$C_v = \frac{\partial \langle E \rangle}{\partial T} = \frac{4K_B}{\pi} (\beta J \coth(2\beta J))^2 \left\{ K_1(q) - K_2(q) - (1 - \tanh^2(2\beta J)) \left[ \frac{\pi}{2} + (2 \tanh^2(2\beta J) - 1) K_1(q) \right] \right\} \quad (25)$$

where

$$K_2(q) = \int_0^{\frac{\pi}{2}} d\phi \sqrt{1 - q^2 \sin^2 \phi} \quad (26)$$

near the critical temperature  $T_c$  the specific heat behaves as

$$C_v \approx -\frac{2}{\pi} \left( \frac{2J}{K_B T_c} \right)^2 \ln \left[ 1 - \frac{T}{T_c} \right] + \text{const.} \quad (27)$$

$$C_v \sim \left[ 1 - \frac{T}{T_c} \right]^\alpha \quad (28)$$

the limiting form of the function

$$\lim_{\alpha \rightarrow 0} \frac{1}{\alpha} (Y^{-\alpha} - 1) = -\ln Y \quad (29)$$

can be used to infer that closed-form result in low singularity with  $\alpha = 0$ . list of Onsager's result

$$\left\langle \frac{M(T)}{N} \right\rangle = \left[ 1 - \frac{(1 - \tanh^2(\beta J))^4}{16 \tanh^4(\beta J)} \right]^{\frac{1}{8}} \quad (30)$$

"lecture notes Computational physics 2015 at University of Oslo by Morten Hjorth-Jensen page 435)"

### 1.3 Closed solution for a 2 dimensional 2 x 2 lattice

We want now to look at a 2 x 2 lattice and we want to calculate the partition function, the energy, magnetisation, heat capacity and susceptibility of the system dependent of T. The partition function for a canonical ensemble with periodic boundary conditions can be computed by:

$$Z = \sum_{i=1}^M e^{-\beta E_i} \quad (31)$$

Here,  $\beta$  is  $\frac{1}{k_b \cdot T}$ , where  $k_b$  is the Boltzmann constant. In this expression we sum over all microstates m. The Energy of the system in configuration i is then:

$$E_i = -J \sum_{\langle kl \rangle}^N s_k s_l \quad (32)$$

The sum over  $\langle kl \rangle$  means that we only sum over nearest neighbours. In our 2 x 2 case, we have for each "particle" two possible values  $\pm 1$ . This means that we have all in all  $2^{2 \cdot 2} = 2^4 = 16$  micro states. We have to compute the Energy of the micro states in order to compute the partition function. We also want to introduce the magnetisation, which is simply the sum over all the spins of the system:

$$M_i = \sum_{j=1}^N s_j \quad (33)$$

We want also to introduce the so called degeneracy, which counts the number of micro states for a given micro energy. We get the following table: We can now write the expression of the partition function as in equation 35. We used the

Figure 1: Energy of the different micro states

Number of spins up (+1)	Degeneracy	Energy	Magnetization
4	1	$-8J$	4
3	4	0	2
2	4	0	0
2	2	$8J$	0
1	4	0	-2
0	1	$-8J$	-4

Table 1 to calculate the sum over the micro states.

$$Z = \sum_{i=1}^M e^{-\beta E_i} = 12 \cdot e^{-\beta \cdot 0} + 2 \cdot e^{-8J\beta} + 1 \cdot e^{8J\beta} + 1 \cdot e^{8J\beta} \quad (34)$$

$$= 12 + 2 \cdot e^{-8J\beta} + 2 \cdot e^{8J\beta} \quad (35)$$

$$= 12 + 4 \cdot \cosh(8J\beta) \quad (36)$$

We can now calculate the expectation value of the energy. There are two possible ways of calculating it. the first way of calculating the expectation value of the energy can be seen in equation 37.

$$\langle E \rangle = -\frac{\partial \ln(Z)}{\partial \beta} = -\frac{1}{Z} \cdot 32J \cdot \sinh(8J\beta) \quad (37)$$

$$= -\frac{32J \cdot \sinh(8J\beta)}{Z} \quad (38)$$

$$= -\frac{8 \cdot J \cdot \sinh(8J\beta)}{3 + \cosh(8J\beta)} \quad (39)$$

Alternatively, we can calculate the expectation value of the Energy by looking at the micro states:

$$\langle E \rangle = -\frac{1}{Z} \sum_{i=1}^M E_i e^{-\beta E_i} = -\frac{8 \cdot J \cdot \sinh(8J\beta)}{3 + \cosh(8J\beta)} \quad (40)$$

Both expressions are equal. Next, we want to determine the expectation value of the magnetisation. We use the formula 41. We can see that we get 0 for the expectation value of the magnetisation.

$$\langle M \rangle = \frac{1}{Z} \sum_i^M M_i \cdot e^{-\beta E_i} \quad (41)$$

$$= \frac{1}{Z} \cdot \left( 4 \cdot 1 \cdot e^{-8J\beta} + 2 \cdot 4 + (-2) \cdot 4 + (-4) \cdot 1 \cdot e^{-8J\beta} \right) \quad (42)$$

$$= 0 \quad (43)$$

However, we are interested in the expectation value of the absolute value of magnetisation, which is  $\langle |M| \rangle$ . This expression can be determined as follows:

$$\langle |M| \rangle = \frac{1}{Z} \sum_i^M |M_i| \cdot e^{-\beta E_i} \quad (44)$$

$$= \frac{1}{Z} \cdot \left( |4| \cdot 1 \cdot e^{-8J\beta} + |2| \cdot 4 + |(-2)| \cdot 4 + |(-4)| \cdot 1 \cdot e^{-8J\beta} \right) \quad (45)$$

$$= \frac{1}{Z} \cdot \left( 8 \cdot e^{-8J\beta} + 8 \right) \quad (46)$$

$$= \frac{2 \cdot e^{-8J\beta} + 2}{3 + \cosh(8J\beta)} \quad (47)$$

In order to describe how the temperature will change when thermal energy is added to the system, we want to look at a quantity called heat capacity. ( $C_v$ ) The bigger this quantity is the less heats the system up by a given amount of thermal energy, which is added to the system.

$$C_v = \frac{1}{k_b T^2} \left( \frac{1}{Z} \sum_{i=1}^M E_i^2 e^{-\beta E_i} - \left( \frac{1}{Z} \sum_{i=1}^M E_i e^{-\beta E_i} \right)^2 \right) \quad (48)$$

$$= \frac{1}{k_b T^2} \left( \frac{1}{Z} \left( 2 \cdot (8J)^2 \cdot e^{8J\beta} + 2 \cdot (-8J)^2 \cdot e^{-8J\beta} \right) - \left( \frac{8 \cdot J \cdot \sinh(8J\beta)}{3 + \cosh(8J\beta)} \right)^2 \right) \quad (49)$$

$$= \frac{1}{k_b T^2} \left( \frac{64 \cdot J \cdot \cosh(8J\beta)}{3 + \cosh(8J\beta)} - \left( \frac{8 \cdot J \cdot \sinh(8J\beta)}{3 + \cosh(8J\beta)} \right)^2 \right) \quad (50)$$

$$= \frac{1}{k_b T^2} \left( \frac{64 \cdot J + 3 \cdot J \cdot 64 \cosh(8J\beta)}{(3 + \cosh(8J\beta))^2} \right) \quad (51)$$

$$= \frac{64}{k_b T^2} \left( \frac{J + 3J \cdot \cosh(8J\beta)}{(3 + \cosh(8J\beta))^2} \right) \quad (52)$$

$$(53)$$

At last, we want to have a look at the magnetic susceptibility. This quantity is a magnetic property of the material. The

magnetic susceptibility describes the response of the material to an applied magnetic field.

$$\chi = \frac{1}{k_b T} \cdot \left( \frac{1}{Z} \sum_{i=1}^M M_i^2 e^{-\beta E_i} - \left( \frac{1}{Z} \sum_{i=1}^M M_i e^{-\beta E_i} \right)^2 \right) \quad (54)$$

$$= \frac{1}{k_b T} \cdot \left( \frac{1}{Z} \cdot \left( 4^2 \cdot 1 \cdot e^{-8J\beta} + 2^2 \cdot 4 + (-2)^2 \cdot 4 + (-4)^2 \cdot 1 \cdot e^{-8J\beta} \right) - (0)^2 \right) \quad (55)$$

$$= \frac{1}{k_b T} \cdot \frac{32e^{-8J\beta} + 32}{12 + 4 \cdot \cosh(8J\beta)} \quad (56)$$

$$= \frac{1}{k_b T} \cdot \frac{8e^{-8J\beta} + 8}{3 + \cosh(8J\beta)} \quad (57)$$

$$(58)$$

## 1.4 infinite size effect

## 1.5 everything related to parts c and later

This part is not structured yet

## 2 Execution

## 3 Comparison and discussion of results

## 4 source code