Nuclear Momentum Distributions

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1 Definitions

1.1 One-particle momentum distribution

The One-particle momentum distribution gives the chance of finding a particle with a momentum in the interval $[\vec{k}, \vec{k} + d\vec{k}]$. It is given by the following expression

$$n_1(\vec{k}) = \frac{1}{(2\pi)^3} \int d\vec{r}_1 \int d\vec{r}'_1 e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}'_1)} \rho_1(\vec{r}_1, \vec{r}'_1) \tag{1}$$

where $\rho_1(\vec{r}_1, \vec{r}'_1)$ is the one-body non-diagonal density matrix

$$\rho_1(\vec{r}_1, \vec{r}'_1) = \int \{d\vec{r}_{2-A}\} \Psi_A^*(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) \Psi_A(\vec{r}'_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A). \tag{2}$$

Here, $\Psi_A(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A)$ is the ground state wave function of the nucleus A and with the notation

$$\{d\vec{r}_{i-A}\} = d\vec{r}_i d\vec{r}_{i+1} ... \vec{r}_A. \tag{3}$$

For $\langle \Psi_A \,|\, \Psi_A \rangle = 1$, one has that

$$\int d\vec{k} n_1(\vec{k}) = 1 \tag{4}$$

In the second quantization formalism one can express the one-particle momentum distribution as

$$n_1(\vec{k}) = \langle \Psi_A | \, c_k^\dagger c_k \, | \Psi_A \rangle \,. \tag{5}$$

For the one ony-body non-diagonal density matrix one can write

$$\langle \vec{r}'_1, \vec{r}'_2, \ldots, \vec{r}'_A \,|\, \hat{\rho} \,|\, \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_A \rangle = \left(\prod_{\substack{j=1 \\ j \neq i}}^{A} \delta(\vec{r}_j - \vec{r}'_j) \right) \sum_{i=1}^{A} \rho(\vec{r}_i) \tag{6}$$

$$\langle \Psi_A \,|\, \hat{\rho} \,|\, \Psi_A \rangle = \sum_{i=1}^{A} \int d\vec{r}_1 \cdots d\vec{r}_i \cdots d\vec{r}_A \int d\vec{r}'_1 \cdots d\vec{r}'_i \cdots d\vec{r}'_A \, \langle \Psi_A \,|\, \vec{r}_1, ..., \vec{r}'_i, ..., \vec{r}_A \rangle \, \rho(\vec{r}_i, \vec{r}'_i) \, \langle \vec{r}_1, ..., \vec{r}_i, ..., \vec{r}_A \,|\, \Psi_A \rangle \tag{7}$$

$$= \frac{A}{A!} \int d\vec{r}_1 \cdots d\vec{r}_A \int d\vec{r}'_1 \left\langle \Psi_A \,\middle|\, \psi^\dagger(\vec{r}'_1) \psi^\dagger(\vec{r}_2) ... \psi^\dagger(\vec{r}_A) \rho(\vec{r}_1, \vec{r}'_1) \psi(\vec{r}_1) ... \psi(\vec{r}_A) \,\middle|\, \Psi_A \right\rangle \tag{8}$$

$$= \int d\vec{r} \int d\vec{r}' \left\langle \Psi_A \,\middle|\, \psi^\dagger(\vec{r'}) \rho(\vec{r}, \vec{r}') \psi(\vec{r}) \,\middle|\, \Psi_A \right\rangle \tag{9}$$

So the one-body non-diagonal operator in second quantization becomes

$$\hat{\rho}_{off-diag} = \int d\vec{r} d\vec{r}' \psi^\dagger(\vec{r}) \rho(\vec{r}, \vec{r}') \psi(\vec{r}') \tag{10}$$

The density operator in the second quantization formalism can also be written as

$$\begin{aligned}
\rho(\vec{r}_1, \vec{r}_1') &= \int \{d\vec{r}_{2-A}\} \Psi_A^*(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) \Psi_A(\vec{r}_1', \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) && (11)\\
&= \int \{d\vec{r}_{2-A}\} \langle \Psi_A \,|\, \vec{r}_1', \vec{r}_2, ..., \vec{r}_A \rangle \langle \vec{r}_1, \vec{r}_2, ..., \vec{r}_A \,|\, \Psi_A \rangle && (12)\\
&= \frac{1}{A!} \int \{d\vec{r}_{2-A}\} \left\langle \Psi_A \,\middle|\, \psi^\dagger(\vec{r}_1') \psi^\dagger(\vec{r}_2) ... \psi^\dagger(\vec{r}_A) \psi(\vec{r}_1) ... \psi(\vec{r}_A) \,\middle|\, \Psi_A \right\rangle && (13)\\
&= \left\langle \Psi_A \,\middle|\, \psi^\dagger(\vec{r}_1') \psi(\vec{r}_1) \,\middle|\, \Psi_A \right\rangle && (14)\\
&= \left\langle \Psi_A \,\middle|\, \sum_{\alpha\beta} c_\alpha^\dagger u_\alpha(\vec{r}') c_\beta u_\beta(\vec{r})) \,\middle|\, \Psi_A \right\rangle && (15)
\end{aligned}$$

1.2 Two-particle momentum distribution

The two-particle momentum distribution gives the chance of finding a particle with momentum in the interval $[\vec{k}_1, \vec{k}_1 + d\vec{k}]$ when there is another particle with a momentum in the interval $[\vec{k}_2, \vec{k}_2 + d\vec{k}]$. It is given by the following expression

$$n(\vec{k}_1, \vec{k}_2) = \frac{1}{(2\pi)^6} \int d\vec{r}_1 \int d\vec{r}_2 \int d\vec{r}_1' \int d\vec{r}_2' \mathrm{e}^{i\vec{k}_1 \cdot (\vec{r}_1 - \vec{r}_1')} \mathrm{e}^{i\vec{k}_2 \cdot (\vec{r}_2 - \vec{r}_2')} \rho_2(\vec{r}_1, \vec{r}_2; \vec{r}_1', \vec{r}_2') \tag{16}$$

where $\rho_2(\vec{r}_1, \vec{r}_2, \vec{r}_1', \vec{r}_2')$ is the two-body non-diagonal density matrix

$$\rho_2(\vec{r}_1, \vec{r}_2, \vec{r}_1', \vec{r}_2') = \int \{d\vec{r}_{3-A}\} \Psi_A^*(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) \Psi_A(\vec{r}_1', \vec{r}_2', \vec{r}_3, ..., \vec{r}_A). \tag{17}$$

One can also define the two-particle momentum distribution in the relative and centre of mass (rcm) coordinates instead of the centre well (cw) coordinates

$$\vec{r}_{12} = \frac{1}{\sqrt{2}} (\vec{r}_1 - \vec{r}_2) \tag{18}$$

$$\vec{R}_{12} = \frac{1}{\sqrt{2}} (\vec{r}_1 + \vec{r}_2) \tag{19}$$

$$\vec{p} = \frac{1}{\sqrt{2}} \left(\vec{k}_1 - \vec{k}_2\right) \tag{20}$$

$$\vec{P} = \frac{1}{\sqrt{2}} \left(\vec{k}_1 + \vec{k}_2\right) \tag{21}$$

$$n(\vec{p}, \vec{P}) = \frac{1}{(2\pi)^6} \int d\vec{r}_{12} \int d\vec{R}_{12} \int d\vec{r}_{12}' \int d\vec{R}_{12}' \mathrm{e}^{i\vec{p} \cdot (\vec{r}_{12} - \vec{r}_{12}')} \mathrm{e}^{i\vec{P} \cdot (\vec{R}_{12} - \vec{R}_{12}')} \rho_2(\vec{r}_{12}, \vec{R}_{12}; \vec{r}_{12}', \vec{R}_{12}') \tag{22}$$

where

$$\rho_2(\vec{r}_{12}, \vec{R}_{12}; \vec{r}_{12}', \vec{R}_{12}') = \rho_2 \left(\vec{r}_1 = \frac{\vec{r}_{12} + \vec{R}_{12}}{\sqrt{2}}, \vec{r}_2 = \frac{-\vec{r}_{12} + \vec{R}_{12}}{\sqrt{2}}, \vec{r}_1' = \frac{\vec{r}_{12}' + \vec{R}_{12}'}{\sqrt{2}}, \vec{r}_2' = \frac{-\vec{r}_{12}' + \vec{R}_{12}'}{\sqrt{2}} \right) \tag{23}$$

In the second quantization formalism one can write the two-patricle momemtum distribution as

$$n_2(\vec{k_1}, \vec{k_2}) = \langle \Psi_A | \, c^\dagger_{k_1} c^\dagger_{k_2} c_{k_1} c_{k_2} \, | \Psi_A \rangle \tag{24}$$

For the non-diagonal two-body density operator

$$\hat{\rho} = \sum_{i<j} \hat{\rho}(\vec{r}_i, \vec{r}_j; \vec{r}'_i, \vec{r}'_j) \tag{25}$$

with the corresponding matrix element between two states in position space

$$\langle \vec{r}'_1, \vec{r}'_2, \ldots, \vec{r}'_A \,|\, \hat{\rho} \,|\, \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_A \rangle = \left(\prod_{\substack{k \neq i \\ k \neq j}}^{A} \delta(\vec{r}_k - \vec{r}'_k) \right) \sum_{i<j} \hat{\rho}(\vec{r}_i, \vec{r}_j; \vec{r}'_i, \vec{r}'_j) \tag{26}$$

$$\begin{aligned}
\rho(\vec{r}_1, \vec{r}_2; \vec{r}'_1, \vec{r}'_2) &= \int \{d\vec{r}_{3-A}\} \Psi^*_A(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) \Psi_A(\vec{r}'_1, \vec{r}'_2, \vec{r}_3, ..., \vec{r}_A) & (27) \\
&= \frac{1}{A!} \int \{d\vec{r}_{3-A}\} \left\langle \Psi_A \,\middle|\, \psi^\dagger(\vec{r}'_1) \psi^\dagger(\vec{r}'_2) \psi^\dagger(\vec{r}_3) ... \psi^\dagger(\vec{r}_A) \psi(\vec{r}_A) ... \psi(\vec{r}_3) \psi(\vec{r}_2) \psi(\vec{r}_1) \,\middle|\, \Psi_A \right\rangle & (28) \\
&= \frac{1}{A(A-1)} \left\langle \Psi_A \,\middle|\, \psi^\dagger(\vec{r}'_1) \psi^\dagger(\vec{r}'_2) \psi(\vec{r}_2) \psi(\vec{r}_1) \,\middle|\, \Psi_A \right\rangle & (29) \\
&= \frac{1}{A(A-1)} \left\langle \Psi_A \,\middle|\, \sum_{\alpha\beta\gamma\delta} c^\dagger_\alpha c^\dagger_\beta u *_\alpha (\vec{r}'_1) u *_\beta (\vec{r}'_2) u_\gamma(\vec{r}_1) u_\delta(\vec{r}_2) c_\gamma c_\delta \,\middle|\, \Psi_A \right\rangle & (30)
\end{aligned}$$

2 Momentum distributions for IPM

2.1 General properties

In an independent particle model the total wave function of the nucleus is a slater determinant of the one-particle wave functions. A nucleon moves independent in a sort of mean field potential created by all the other nucleons.

$$\Psi_A(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) = \frac{1}{\sqrt{A!}} \sum_P (-1)^P \phi_{P_1}(\vec{r}_1) \phi_{P_2}(\vec{r}_2) ... \phi_{P_A}(\vec{r}_A) \tag{31}$$

where the sum is over all permutations of the indices of the one-particle wave functions. We also have

$$\int d\vec{r}_i \phi^*_l(\vec{r}_i) \phi_m(\vec{r}_i) = \delta_{lm} \tag{32}$$

The one-particle non-diagonal density matrix becomes

$$\begin{aligned}
\rho_1(\vec{r}_1, \vec{r}'_1) &= \frac{1}{A!} \sum_P \sum_L (-1)^{P+L} \int d\vec{r}_2 d\vec{r}_3 ... d\vec{r}_A \phi^*_{P_1}(\vec{r}_1) \phi^*_{P_2}(\vec{r}_2) ... \phi^*_{P_A}(\vec{r}_A) \phi_{L_1}(\vec{r}'_1) \phi_{L_2}(\vec{r}_2) ... \phi_{L_A}(\vec{r}_A) & (33) \\
&= \frac{1}{A!} \sum_P \sum_L (-1)^{P+L} \phi^*_{P_1}(\vec{r}_1) \phi_{L_1}(\vec{r}'_1) \delta_{P_2,L_2} \delta_{P_3,L_3} ... \delta_{P_A,L_A} & (34) \\
&= \sum_i \phi^*_i(\vec{r}_1) \phi_i(\vec{r}'_1). & (35)
\end{aligned}$$

We can plug this into (1)

$$n_1(\vec{k}) = \frac{1}{(2\pi)^3}\sum_i \int d\vec{r}_1 \int d\vec{r}_1' e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_1')}\phi_i^*(\vec{r}_1)\phi_i(\vec{r}_1') \tag{36}$$

$$= \sum_i \tilde{\phi}_i^*(\vec{k})\tilde{\phi}_i(\vec{k}). \tag{37}$$

To find an expression for the two-body non-diagonal density one can plug the slater deteminant (31) into equation (23). Taking into account the orthogonality relation (32) one has

$$\rho_2(\vec{r}_1,\vec{r}_2,\vec{r}_1',\vec{r}_2') = \frac{1}{A(A-1)}\sum_{i\neq j}\phi_i^*(\vec{r}_1)\phi_i^*(\vec{r}_2)\phi_j(\vec{r}_1')\phi_j(\vec{r}_2'). \tag{38}$$

2.2 IPM for harmonic oscillator potential

We consider the nucleons moving independently in a spherical symmetric harmonic oscillator potential. From the above we know that we only need to calculate the one-particle wave functions and their fourier transforms. The 3D time independent Schrodinger equation for one patricle is

$$\left(-\frac{\hbar^2}{2M_N}\nabla^2 + \frac{1}{2}M_N\omega^2 r^2\right)\phi_{nlm}(\vec{r}) = E\phi_{nlm}(\vec{r}) \tag{39}$$

where the parameter $\hbar\omega$ can be parameterized as

$$\hbar\omega(MeV) = 45A^{-1/3} - 25A^{-2/3}, \tag{40}$$

where A is the mass number of the nucleus. The general solution of (39) is given by

$$\phi_{nlm}(\vec{r}) \equiv \langle \vec{r}\,|\,nlm\rangle = R_{nl}(r)Y_{lm}(\Omega) \tag{41}$$

where $Y_{lm}(\Omega)$ are the spherical harmonics and the radial wave functions are given in function of the generalized Laguerre polynomals $L_n^\alpha(r)$ by

$$R_{nl}(r) = \left[\frac{2n!}{\Gamma(n+l+\frac{3}{2})}\nu^{l+\frac{3}{2}}\right]^{\frac{1}{2}} r^l e^{-\frac{\nu r^2}{2}} L_n^{l+\frac{1}{2}}(\nu r^2) \tag{42}$$

where

$$\nu \equiv \frac{M_N\omega}{\hbar} \tag{43}$$

One can calculate the fourier transform of these wave functions explicitly or one can transform equation 39, which is written in configuration space, into momentum space

$$\left(-\frac{M_N\omega^2\hbar^2}{2}\nabla^2 + \frac{1}{2M_N}k^2\right)\tilde{\phi}_{nlm}(\vec{k}) = E\tilde{\phi}_{nlm}(\vec{k}). \tag{44}$$

One can now see that this equation has the same form as equation (39). So the solutions have the same form

$$\phi_{nlm}(\vec{k}) \equiv \left\langle \vec{k}\,\middle|\,nlm\right\rangle = K_{nl}(k)Y_{lm}(\Omega) \tag{45}$$

where $Y_{lm}(\Omega)$ are the spherical harmonics and the radial wave functions are given in function of the generalized Laguerre polynomals $L_n^\alpha(k)$ by

$$K_{nl}(k) = \left[\frac{2n!}{\Gamma(n+l+\frac{3}{2})}\nu'^{l+\frac{3}{2}}\right]^{\frac{1}{2}} k^l e^{-\frac{\nu' k^2}{2}} L_n^{l+\frac{1}{2}}(\nu' r^2) \tag{46}$$

where

$$\nu' \equiv \frac{\hbar}{M_N \omega} \tag{47}$$

Now one can calculate the radial one-body momentum distribution

$$n_1(k) = \int d\Omega n_1(\vec{k}) \tag{48}$$

$$= 2 \sum_{nlm} K_{nl}^2(k) \int d\Omega Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) \tag{49}$$

$$= 2 \sum_{nl} (2l+1) K_{nl}^2(k) \tag{50}$$

$$\tag{51}$$

where the factor 2 is for spin degeneracy and each state (n, l) is $(2l + 1)$ times degenerate. Fist we need to calculate the energy of each state. The sum goes over the lowest energy states (n, l) until al nucleons are in a state. We consider the proton and the neutron to be identical in mass.