Nuclear Momentum Distributions

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March 12, 2015

1 Definitions

1.1 Second quantziation formalism

State vector: how many particles in each single-particle orbital u_α

$$|n_1, n_2, n_3, \ldots\rangle . \tag{1}$$

Creation and annihilation operators: $c_\alpha^\dagger$ adds one particle to one-paricle state u_α and c_α removes one particle from this state.

$$\begin{aligned} c_\alpha^\dagger |n_1, n_2, \ldots, n_i, \ldots\rangle &= \left(\mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes c_\alpha^\dagger \otimes \cdots \otimes \mathbb{1}\right) |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_\alpha\rangle \otimes \cdots && (2)\\ &= (-1)^{s_\alpha} |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes c_\alpha^\dagger |n_\alpha\rangle \otimes \cdots && (3)\\ &= \delta_{0n_\alpha} (-1)^{s_\alpha} |n_1, \ldots, n_{\alpha-1}, n_\alpha + 1, n_{\alpha+1}, \ldots\rangle . && (4)\end{aligned}$$

with

$$s_\alpha = n_1 + n_2 + \ldots + n_{\alpha-1}. \tag{5}$$

The kronecker delta ensures the Pauli principle for fermions is satisfied. The factor $(-1)^{s_\alpha}$ follows from the commutation relations for fermions

$$\begin{aligned} \{c_\alpha, c_\beta^\dagger\} &= \delta_{\alpha\beta} && (6)\\ \{c_\alpha, c_\beta\} &= \{c_\alpha^\dagger, c_\beta^\dagger\} = 0 && (7)\end{aligned}$$

For the same reasons we have

$$c_\alpha |n_1, n_2, \ldots, n_\alpha, \ldots\rangle = \delta_{1n_\alpha} (-1)^{s_\alpha} |n_1, \ldots, n_{\alpha-1}, n_\alpha - 1, n_{\alpha+1}, \ldots\rangle . \tag{8}$$

One can define a number operator

$$\hat{N} = \sum_\alpha c_\alpha^\dagger c_\alpha \tag{9}$$

which has eigenvalues $\mathrm{N} \in \mathbb{N}$ and wave functions with a fixed number of particles as eigenfunctions. A normalized many-body state can now be expressed as

$$|n_1, n_2, n_3, \ldots\rangle = (c_1^\dagger)^{n_1} (c_2^\dagger)^{n_2} \cdots |0\rangle \tag{10}$$

This normalization holds for fermions and the n_is are either 1 or 0. Here we chose a specific order for the single-particle states $\{u_\alpha\}$ and keep it fixed.
One can define a creation operator that creates a particle at position $\vec{r}$

$$\psi^\dagger(\vec{r}) = \sum_\alpha c_\alpha^\dagger u_\alpha^*(\vec{r}) \tag{11}$$

with

$$c_\alpha^\dagger = \int d\vec{r}\psi^\dagger(\vec{r})u_\alpha^*(\vec{r}) \tag{12}$$

To check our definitions one can create a particle in state α

$$
\begin{aligned}
|\alpha\rangle &= \int d\vec{r}u_\alpha(\vec{r})\,|\vec{r}\rangle && (13)\\
&= \int d\vec{r}u_\alpha(\vec{r})\psi^\dagger(\vec{r})\,|0\rangle && (14)\\
&= \int d\vec{r}u_\alpha(\vec{r})\sum_\beta c_\beta^\dagger u_\beta^*(\vec{r})\,|0\rangle && (15)\\
&= c_\alpha^\dagger\,|0\rangle\,. && (16)
\end{aligned}
$$

In the last step we used the orthonormality relation of the single-particle states

$$\int d\vec{r}u_\alpha(\vec{r})u_\beta^*(\vec{r}) = \delta_{\alpha\beta} \tag{17}$$

We also have the following relations

$$
\begin{aligned}
u_\alpha(\vec{r}) &= \langle\vec{r}|\vec{\alpha}\rangle && (18)\\
\langle\vec{r}|\vec{r}\,'\rangle &= \delta(\vec{r}-\vec{r}\,') && (19)\\
\langle\vec{r}|\vec{k}\rangle &= \frac{1}{(2\pi)^{3/2}}e^{i\vec{k}\cdot\vec{r}} && (20)\\
\langle\vec{k}|\vec{r}\rangle &= \frac{1}{(2\pi)^{3/2}}e^{-i\vec{k}\cdot\vec{r}}. && (21)
\end{aligned}
$$

Now we can write down a normailzed A-body Fockstate in first quantization

$$|\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_A\rangle = \frac{1}{\sqrt{N!}}\psi^\dagger(\vec{r}_1)\psi^\dagger(\vec{r}_2)\cdots\psi^\dagger(\vec{r}_A)\,|0\rangle \tag{22}$$

And the wave function in configuration space

$$\langle\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_A|n_1,n_2,n_3,\ldots\rangle = \Psi_A(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_A). \tag{23}$$

1.2 One-particle momentum distribution

The chance of finding a particle with a momentum in the interval $[k,k+dk]$ is $n_1(k)k^2dk$.

$$n_1(\vec{k}) = \frac{1}{(2\pi)^3}\int d\vec{r}_1\int d\vec{r}_1{}'e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_1{}')}\rho_1(\vec{r}_1,\vec{r}_1{}') \tag{24}$$

with $\rho_1(\vec{r}_1,\vec{r}_1{}')$ the one-body non-diagonal density matrix defined as

$$\rho_1(\vec{r}_1,\vec{r}_1{}') = \int\{d\vec{r}_{2-A}\}\Psi_A^*(\vec{r}_1,\vec{r}_2,\vec{r}_3,...,\vec{r}_A)\Psi_A(\vec{r}_1{}',\vec{r}_2,\vec{r}_3,...,\vec{r}_A). \tag{25}$$

Here, $\Psi_A(\vec{r}_1,\vec{r}_2,\vec{r}_3,...,\vec{r}_A)$ is the ground state wave function of the nucleus A and with the notation

$$\{d\vec{r}_{i-A}\} = d\vec{r}_i d\vec{r}_{i+1}...\vec{r}_A. \tag{26}$$

For $\langle\Psi_A|\Psi_A\rangle = 1$, one has that

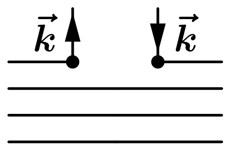

Figure 1: Feynman interpretation for the one-body momentum operator. The operator annihilates and creates a particle with momentum $\vec{k}$, this at the same time because there is no energy transfer

$$\int d\vec{k} n_1(\vec{k}) = 1 \tag{27}$$

In the second quantization formalism one can express the one-particle momentum distribution as

$$n_1(\vec{k}) = \frac{1}{A} \langle \Psi_A | \, \psi^\dagger(\vec{k})\psi(\vec{k}) \, | \Psi_A \rangle \, . \tag{28}$$

Intuitively, the operator $\psi^\dagger(\vec{k})\psi(\vec{k})$ counts the number of particles with momentum $\vec{k}$.
Mathematically, this can be seen as following

$$\begin{aligned}
n_1(\vec{k}) &= \frac{1}{(2\pi)^3} \int d\vec{r}_1 \int d\vec{r}_1{}' \, e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_1{}')} \rho_1(\vec{r}_1, \vec{r}_1{}') && (29)\\
&= \int d\vec{r}_1 \int d\vec{r}_1{}' \int d\{\vec{r}_{2-A}\} \langle \vec{r}_1 | \vec{k} \rangle \langle \vec{k} | \vec{r}_1{}' \rangle \langle \Psi_A | \vec{r}_1, \{\vec{r}_{2-A}\} \rangle \langle \vec{r}_1{}', \{\vec{r}_{2-A}\} | \Psi_A \rangle && (30)\\
&= \frac{1}{A!} \int d\vec{r}_1{}' \int d\{\vec{r}_{2-A}\} \langle \vec{r}_1 | \vec{k} \rangle \langle \vec{k} | \vec{r}_1{}' \rangle \langle \Psi_A | \, \psi^\dagger(\vec{r}_1)\psi^\dagger(\vec{r}_2) \cdots \psi^\dagger(\vec{r}_A) \, |0\rangle \langle 0| \, \psi(\vec{r}_A) \cdots \psi(\vec{r}_2)\psi(\vec{r}_1{}') \, |\Psi_A\rangle && (31)
\end{aligned}$$

The projection on the vacuum state $|0\rangle \langle 0|$ can be subsituted with the identity because the operators $\psi(\vec{r})$ ($\psi^\dagger(\vec{r})$) have already annihilated all the particles contained in the ket (bra) vectors.

$$n_1(\vec{k}) = \frac{1}{A!} \int d\vec{r}_1{}' \int d\vec{r}_{2-A} \langle \vec{r}_1 | \vec{k} \rangle \langle \vec{k} | \vec{r}_1{}' \rangle \langle \Psi_A | \, \psi^\dagger(\vec{r}_1)\psi^\dagger(\vec{r}_2) \cdots \psi^\dagger(\vec{r}_A)\psi(\vec{r}_A) \cdots \psi(\vec{r}_2)\psi(\vec{r}_1{}') \, |\Psi_A\rangle \tag{32}$$

The integration on the coordinates $\vec{r}_2$ to $\vec{r}_A$ can be done by noticing that $\int d\vec{r}\psi^\dagger(\vec{r})\psi(\vec{r})$ is the particle number operator in configuration space. So the integration over $\vec{r}_A$ gives a factor one since it acts one a state vecor $\psi(\vec{r}_{A-1}) \cdots \psi(\vec{r}_2)\psi(\vec{r}_1) \, |\Psi_A\rangle$, so all particles are annihilated except for 1. The integration over $\vec{r}_{A-1}$ gives a factor 2 for the same reasons. So if we integrate over all coordinates $\vec{r}_2$ to $\vec{r}_A$, we get a factor $(A-1)!$. The one-particle momentum distribution can now be written as

$$\begin{aligned}
n_1(\vec{k}) &= \frac{1}{A} \int d\vec{r}_1 \int d\vec{r}_1{}' \langle \vec{r}_1 | \vec{k} \rangle \langle \vec{k} | \vec{r}_1{}' \rangle \langle \Psi_A | \, \psi^\dagger(\vec{r}_1)\psi(\vec{r}_1{}') \, |\Psi_A\rangle && (33)\\
&= \frac{1}{A} \langle \Psi_A | \, \psi^\dagger(\vec{k})\psi(\vec{k}) \, |\Psi_A\rangle && (34)
\end{aligned}$$

In the last line we defined the creation and annihilation operators in momentum space

$$\begin{aligned}
\psi(\vec{k}) &= \frac{1}{(2\pi)^{3/2}} \int d\vec{r} e^{-i\vec{k}\cdot\vec{r}} \psi(\vec{r}) && (35)\\
&= \int d\vec{r} \, \langle \vec{k} | \vec{r} \rangle \, \psi(\vec{r}) && (36)
\end{aligned}$$

The density operator in the second quantization formalism can also be written as

$$\rho(\vec{r}_1, \vec{r}_1{}') = \int \{d\vec{r}_{2-A}\} \Psi_A^*(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) \Psi_A(\vec{r}_1{}', \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) \tag{37}$$

$$= \frac{1}{A!} \int \{d\vec{r}_{2-A}\} \langle \Psi_A | \psi^\dagger(\vec{r}_1) \psi^\dagger(\vec{r}_2) ... \psi^\dagger(\vec{r}_A) \psi(\vec{r}_{A-1}) ... \psi(\vec{r}_2) \psi(\vec{r}_1{}') | \Psi_A \rangle \tag{38}$$

$$= \frac{1}{A} \langle \Psi_A | \psi^\dagger(\vec{r}_1) \psi(\vec{r}_1{}') | \Psi_A \rangle \tag{39}$$

$$= \frac{1}{A} \langle \Psi_A | \sum_{\alpha\beta} c_\alpha^\dagger u_\alpha^*(\vec{r}) c_\beta u_\beta(\vec{r}\,')) | \Psi_A \rangle \tag{40}$$

1.3 Two-particle momentum distribution

The chance of finding a particle with momentum in the interval $[k_1, k_1 + dk_1]$ when there is another particle with a momentum in the interval $[k_2, k_2 + dk_2]$ is is $n_2(k_1, k_2) dk_1 dk_2$.

$$n_2(\vec{k}_1, \vec{k}_2) = \frac{1}{(2\pi)^6} \int d\vec{r}_1 \int d\vec{r}_2 \int d\vec{r}_1{}' \int d\vec{r}_2{}' \mathrm{e}^{i\vec{k}_1 \cdot (\vec{r}_1 - \vec{r}_1{}')} \mathrm{e}^{i\vec{k}_2 \cdot (\vec{r}_2 - \vec{r}_2{}')} \rho_2(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') \tag{41}$$

with the two-body non-diagonal density matrix

$$\rho_2(\vec{r}_1, \vec{r}_2, \vec{r}_1{}', \vec{r}_2{}') = \int \{d\vec{r}_{3-A}\} \Psi_A^*(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) \Psi_A(\vec{r}_1{}', \vec{r}_2{}', \vec{r}_3, ..., \vec{r}_A) \tag{42}$$

the two-body non-diagonal density matrix.
One can also define the two-particle momentum distribution in the relative and centre of mass (rcm) coordinates instead of the centre well (cw) coordinates

$$\vec{r} = \frac{1}{\sqrt{2}} (\vec{r}_1 - \vec{r}_2) \tag{43}$$

$$\vec{R} = \frac{1}{\sqrt{2}} (\vec{r}_1 + \vec{r}_2) \tag{44}$$

$$\vec{k} = \frac{1}{\sqrt{2}} \left(\vec{k}_1 - \vec{k}_2 \right) \tag{45}$$

$$\vec{P} = \frac{1}{\sqrt{2}} \left(\vec{k}_1 + \vec{k}_2 \right) \tag{46}$$

$$n(\vec{k}, \vec{P}) = \frac{1}{(2\pi)^6} \int d\vec{r} \int d\vec{R} \int d\vec{r}\,' \int d\vec{R}\,' \mathrm{e}^{i\vec{k} \cdot (\vec{r} - \vec{r}\,')} \mathrm{e}^{i\vec{P} \cdot (\vec{R} - \vec{R}\,')} \rho_2(\vec{r}, \vec{R}; \vec{r}\,', \vec{R}\,') \tag{47}$$

with

$$\rho_2(\vec{r}, \vec{R}; \vec{r}\,', \vec{R}\,') = \rho_2 \left(\vec{r}_1 = \frac{\vec{r} + \vec{R}}{\sqrt{2}}, \vec{r}_2 = \frac{-\vec{r} + \vec{R}}{\sqrt{2}}, \vec{r}_1{}' = \frac{\vec{r}\,' + \vec{R}\,'}{\sqrt{2}}, \vec{r}_2{}' = \frac{-\vec{r}\,' + \vec{R}\,'}{\sqrt{2}} \right) \tag{48}$$

In the second quantization formalism one can write the two-patricle momemtum distribution as

$$n_2(\vec{k}_1, \vec{k}_2) = \frac{1}{A(A-1)} \langle \Psi_A | \, \psi^\dagger(\vec{k}_1) \psi^\dagger(\vec{k}_2) \psi(\vec{k}_1) \psi(\vec{k}_2) \, | \Psi_A \rangle \tag{49}$$

and the two-body non-diagonal density matrix as

$$\rho(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') = \int \{d\vec{r}_{3-A}\} \Psi_A^*(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) \Psi_A(\vec{r}_1{}', \vec{r}_2{}', \vec{r}_3, ..., \vec{r}_A) \tag{50}$$

$$= \frac{1}{A!} \int \{d\vec{r}_{3-A}\} \langle \Psi_A | \psi^\dagger(\vec{r}_1) \psi^\dagger(\vec{r}_2) \psi^\dagger(\vec{r}_3) ... \psi^\dagger(\vec{r}_A) \psi(\vec{r}_A) ... \psi(\vec{r}_3) \psi(\vec{r}_2{}') \psi(\vec{r}_1{}') | \Psi_A \rangle \tag{51}$$

$$= \frac{1}{A(A-1)} \langle \Psi_A | \psi^\dagger(\vec{r}_1) \psi^\dagger(\vec{r}_2) \psi(\vec{r}_2{}') \psi(\vec{r}_1{}') | \Psi_A \rangle \tag{52}$$

$$= \frac{1}{A(A-1)} \langle \Psi_A | \sum_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger u_\alpha^*(\vec{r}_1) u_\beta^*(\vec{r}_2) u_\gamma(\vec{r}_1{}') u_\delta(\vec{r}_2{}') c_\gamma c_\delta | \Psi_A \rangle \,. \tag{53}$$

1.3.1 Two-particle momentum distribution in rcm coordinates

Two particles can be described either by centre of well (cw) coordinates $\vec{r}_1$ en $\vec{r}_2$ or by relative and centre of mass coordinates $\vec{r}$ and $\vec{R}$, respectively. So the two-particle wave function can either be written as $\langle \vec{r}_1 | n_1 l_1 m_1 \rangle \langle \vec{r}_2 | n_2 l_2 m_2 \rangle$ or as $\langle \vec{r} | nlm \rangle \langle \vec{R} | NLM \rangle$. In rcm coordinates the form of the harmonic oscillator wave functions is still te same as in centre of well coordinates but with other quantum numbers. Now we have $n,\ l$ which characterize the relative motion and $N,\ L$ which characterize the centre-of-mass motion. The orbital momenta of the two subsystems can be coupled to a total angular momentum L

$$|l_1 - l_2| \leq \Lambda \leq l_1 + l_2 \tag{54}$$

$$|l - L| \leq \Lambda \leq l + L \tag{55}$$

Two particles are coupled to a well-defined total orbital momentum Λ with projection M_Λ

$$|n_1 l_1 n_2 l_2; \Lambda M_\Lambda\rangle = \sum_{m_1, m_2} |n_1 l_1 m_1 n_2 l_2 m_2\rangle \langle l_1 m_1, l_2 m_2 \,|\, \Lambda M_\Lambda \rangle \tag{56}$$

$$|nlNL; \Lambda M_\Lambda\rangle = \sum_{m, M} |nlmNLM\rangle \langle lmLM \,|\, \Lambda M_\Lambda \rangle \tag{57}$$

For the two body wave function which has the total angular momentum Λ and projection M_Λ there is a orthogonal transformation between the center-of-well and the relative and center-of-mass coordinates

$$|n_1 l_1 n_2 l_2; \Lambda M_\Lambda\rangle = \sum_{nl, N\Lambda} |nlNL; \Lambda M_\Lambda\rangle \langle nlNL; \Lambda \,|\, n_1 l_1 n_2 l_2; \Lambda \rangle \,. \tag{58}$$

the transformation coefficients $\langle nlNL; \Lambda \,|\, n_1 l_1 n_2 l_2; \Lambda \rangle$ are known as the Moshinsky brackets and are independent of M_Λ. In a harmonic oscillator potential the total energy of a particle is given by $E = \hbar\omega(2n_1 + l_1 + \frac{3}{2})$. The total energy of two particles must be the same in both coordinates systems, so we have

$$2n_1 + l_1 + 2n_2 + l_2 = 2n + l + 2N + L \tag{59}$$

The Moshinsky brackets yield zero if this equality is not satisfied.

2 Momentum distributions for IPM

2.1 General properties

In an independent particle model (IPM)the total wave function of the nucleus is a Slater determinant of the one-particle wave functions. A nucleon moves independent in a mean field potential created by all the other nucleons.

$$\Psi_A(\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_A) = \frac{1}{\sqrt{A!}} \sum_{i_1 i_2 ... i_A} \varepsilon_{i_1 i_2 ... i_A} \phi_{i_1}(\vec{r}_1)\phi_{i_2}(\vec{r}_2)...\phi_{i_A}(\vec{r}_A). \tag{60}$$

Here, $\varepsilon_{i_1 i_2 ... i_A}$ is the Levi-Civita symbol and the summations is over the indices i_1 to i_A go from one to A. We also have

$$\int d\vec{r}_i \phi_l^*(\vec{r}_i)\phi_m(\vec{r}_i) = \delta_{lm} \tag{61}$$

The one-particle non-diagonal density matrix becomes

$$\rho_1(\vec{r}_1, \vec{r}_1{}') = \frac{1}{A!} \sum_{i_1 i_2 ... i_A} \sum_{j_1 j_2 ... j_A} \varepsilon_{i_1 i_2 ... i_A} \varepsilon_{j_1 j_2 ... j_A} \int \{d\vec{r}_{2-A}\} \phi_{i_1}^*(\vec{r}_1)\phi_{i_2}^*(\vec{r}_2)...\phi_{i_A}^*(\vec{r}_A)\phi_{j_1}(\vec{r}_1{}')\phi_{j_2}(\vec{r}_2)...\phi_{j_A}(\vec{r}_A) \tag{62}$$

$$= \frac{1}{A!} \sum_{i_1 i_2 ... i_A} \sum_{j_1 j_2 ... j_A} \varepsilon_{i_1 i_2 ... i_A} \varepsilon_{j_1 j_2 ... j_A} \phi_{i_1}^*(\vec{r}_1)\phi_{j_1}(\vec{r}_1{}')\delta_{i_2,j_2}\delta_{i_3,j_3}...\delta_{i_A,j_A} \tag{63}$$

$$= \frac{1}{A} \sum_i \phi_i^*(\vec{r}_1)\phi_i(\vec{r}_1{}'). \tag{64}$$

We can plug this into (24)

$$n_1(\vec{k}) = \frac{1}{A(2\pi)^3} \sum_i \int d\vec{r}_1 \int d\vec{r}_1{}' e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_1{}')} \phi_i^*(\vec{r}_1)\phi_i(\vec{r}_1{}') \tag{65}$$

$$= \frac{1}{A} \sum_i \tilde{\phi}_i^*(\vec{k})\tilde{\phi}_i(\vec{k}). \tag{66}$$

To find an expression for the two-body non-diagonal density one can plug the slater deteminant (60) into equation (48). Taking into account the orthogonality relation (61) one has

$$\rho_2(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') = \frac{1}{A!} \sum_{i_1 i_2 ... i_A} \sum_{j_1 j_2 ... j_A} \varepsilon_{i_1 i_2 ... i_A} \varepsilon_{j_1 j_2 ... j_A} \phi_{i_1}^*(\vec{r}_1)\phi_{i_2}^*(\vec{r}_2)\phi_{j_1}(\vec{r}_1{}')\phi_{j_2}(\vec{r}_2{}')\delta_{i_3,j_3}...\delta_{i_A,j_A} \tag{67}$$

$$= \frac{1}{A(A-1)} \sum_{i_1 i_2} \sum_{j_1 j_2} \left(\delta_{i_1 j_1}\delta_{i_2 j_2} - \delta_{i_1 j_2}\delta_{i_2 j_1}\right) \phi_{i_1}^*(\vec{r}_1)\phi_{i_2}^*(\vec{r}_2)\phi_{j_1}(\vec{r}_1{}')\phi_{j_2}(\vec{r}_2{}') \tag{68}$$

$$= \frac{1}{A(A-1)} \sum_{ij} \left[\phi_i^*(\vec{r}_1)\phi_j^*(\vec{r}_2)\phi_i(\vec{r}_1{}')\phi_j(\vec{r}_2{}') - \phi_i^*(\vec{r}_1)\phi_j^*(\vec{r}_2)\phi_j(\vec{r}_1{}')\phi_i(\vec{r}_2{}')\right]. \tag{69}$$

The two-particle momentum distribution can be written as

$$n_2(\vec{k}_1, \vec{k}_2) = \frac{1}{A(A-1)} \sum_{ij} \left[\phi_i^*(\vec{k}_1)\phi_j^*(\vec{k}_2)\right] \left[\phi_i(\vec{k}_1)\phi_j(\vec{k}_2) - \phi_j(\vec{k}_1)\phi_i(\vec{k}_2)\right] \tag{70}$$

$$= \frac{1}{2A(A-1)} \sum_{ij} \left[\phi_i^*(\vec{k}_1)\phi_j^*(\vec{k}_2) - \phi_j^*(\vec{k}_1)\phi_i^*(\vec{k}_2)\right] \left[\phi_i(\vec{k}_1)\phi_j(\vec{k}_2) - \phi_j(\vec{k}_1)\phi_i(\vec{k}_2)\right] \tag{71}$$

2.2 IPM for harmonic oscillator potential

2.2.1 General properties of the spherical Harmonic Oscillator

We consider the nucleons moving independently in a spherical symmetric harmonic oscillator potential. From the above we know that we only need to calculate the one-particle wave functions and their fourier transforms. The 3D time independent Schrodinger equation for one patricle is

$$\left(-\frac{\hbar^2}{2M_N}\nabla^2 + \frac{1}{2}M_N\omega^2 r^2\right)\phi_{nlm}(\vec{r}) = E\phi_{nlm}(\vec{r}). \tag{72}$$

The parameter $\hbar\omega$ can be parameterized as

$$\hbar\omega(MeV) = 45A^{-1/3} - 25A^{-2/3}, \tag{73}$$

with A the mass number of the nucleus. The general solution of (72) is given by

$$\phi_{nlm}(\vec{r}) \equiv \langle\vec{r}|nlm\rangle = R_{nl}(r)Y_{lm}(\Omega) \tag{74}$$

where $Y_{lm}(\Omega)$ are the spherical harmonics and the radial wave functions are given in function of the generalized Laguerre polynomals $L_n^\alpha(r)$ by

$$R_{nl}(r) = \left[\frac{2n!}{\Gamma(n+l+\frac{3}{2})}\nu^{l+\frac{3}{2}}\right]^{\frac{1}{2}} r^l e^{-\frac{\nu r^2}{2}} L_n^{l+\frac{1}{2}}(\nu r^2) \tag{75}$$

where

$$\nu \equiv \frac{M_N\omega}{\hbar} \tag{76}$$

One can calculate the fourier transform of these wave functions explicitly or one can transform equation 72, which is written in configuration space, into momentum space

$$\left(-\frac{M_N\omega^2}{2}\nabla^2 + \frac{\hbar^2}{2M_N}k^2\right)\tilde{\phi}_{nlm}(\vec{k}) = E\tilde{\phi}_{nlm}(\vec{k}). \tag{77}$$

One can now see that this equation has the same form as equation (72). So the solutions have the same form

$$\phi_{nlm}(\vec{k}) \equiv \langle\vec{k}|nlm\rangle = K_{nl}(k)Y_{lm}(\Omega) \tag{78}$$

where $Y_{lm}(\Omega)$ are the spherical harmonics and the radial wave functions are given in function of the generalized Laguerre polynomals $L_n^\alpha(k)$ by

$$K_{nl}(k) = \left[\frac{2n!}{\Gamma(n+l+\frac{3}{2})}\nu'^{l+\frac{3}{2}}\right]^{\frac{1}{2}} k^l e^{-\frac{\nu' k^2}{2}} L_n^{l+\frac{1}{2}}(\nu' k^2) \tag{79}$$

where

$$\nu\,' \equiv \frac{\hbar}{M_N\omega}. \tag{80}$$

2.2.2 One-body momentum distribution for HO potential

Because the harmonic oscillator potential is spherically symmetric we can split the solution in radial and angular parts. For graphical illustration one can plot the radial part of the momentum distribution

$$n_1(k) = \frac{1}{A}\int d\Omega \sum_{nlm,spin} \phi^*_{nlm}(\vec{k})\phi_{nlm}(\vec{k}) \tag{81}$$

$$= \frac{2}{A}\sum_{nlm} K^2_{nl}(k)\int d\Omega Y^*_{lm}(\Omega)Y_{lm}(\Omega) \tag{82}$$

$$= \frac{2}{A}\sum_{nl}(2l+1)K^2_{nl}(k) \tag{83}$$

The sum over the spin variables gives a factor 2 because there is no spin dependence in the wave functions. The other sums go over all occupied states.

2.2.3 Two-body momentum distribution for HO potential

For two-body states one has (insertion of identity operator $\mathbb{1}$)

$$\phi_{n_1l_1m_1}(\vec{k}_1)\phi_{n_2l_2m_2}(\vec{k}_2) = \sum_{\Lambda M_\Lambda} \langle \Lambda M_\Lambda \,|\, l_1m_1l_2m_2\rangle \sum_{m_1'm_2'} \left\langle l_1m_1'l_2m_2' \,\middle|\, \Lambda M_\Lambda\right\rangle \phi_{n_1l_1m_1'}(\vec{k}_1)\phi_{n_2l_2m_2'}(\vec{k}_2) \tag{84}$$

$$= \sum_{\Lambda M_\Lambda} \langle \Lambda M_\Lambda \,|\, l_1m_1l_2m_2\rangle \left[\phi_{n_1l_1}(\vec{k}_1)\phi_{n_2l_2}(\vec{k}_2)\right]_{\Lambda M_\Lambda} \tag{85}$$

with notation

$$\left[\phi_{n_1l_1}(\vec{k}_1)\phi_{n_2l_2}(\vec{k}_2)\right]_{\Lambda M_\Lambda} \equiv \sum_{m_1m_2} \phi_{n_1l_1m_1}(\vec{k}_1)\phi_{n_2l_2m_2}(\vec{k}_2)\,\langle l_1m_1l_2m_2 \,|\, \Lambda M_\Lambda\rangle$$

For harmonic oscillator two-body states there is a simple transformation from cw coordinates to rcm coordinates, namely the Moshinsky transformation

$$\left[\phi_{n_1l_1}(\vec{k}_1)\phi_{n_2l_2}(\vec{k}_2)\right]_{\Lambda M_\Lambda} = \sum_{nl}\sum_{NL}\left[\phi_{nl}(\vec{k})\phi_{NL}(\vec{P})\right]_{\Lambda M_\Lambda} \langle nlNL;\Lambda \,|\, n_1l_1n_2l_2;\Lambda\rangle_{MB}$$

where $\langle nlNL;\Lambda \,|\, n_1l_1n_2l_2;\Lambda\rangle_{MB}$ is called the Moshinsky Braket.
The product of four wavefunctions in centre of well coordinates can thus be written as

$$\begin{aligned}
&\phi^*_{n_1l_1m_1}(\vec{k}_1)\phi^*_{n_2l_2m_2}(\vec{k}_2)\phi_{n_1l_1m_1}(\vec{k}_1)\phi_{n_2l_2m_2}(\vec{k}_2)\\
&\qquad = \sum_{\Lambda M_\Lambda}\sum_{\Lambda' M_\Lambda'} \langle \Lambda M_\Lambda \,|\, l_1m_1l_2m_2\rangle \left\langle l_1m_1l_2m_2 \,\middle|\, \Lambda' M_\Lambda'\right\rangle\\
&\times \sum_{nlm_l}\sum_{NLM_L}\sum_{n'l'm_l'}\sum_{N'L'M_L'} \langle nlNL;\Lambda \,|\, n_1l_1n_2l_2;\Lambda\rangle_{MB} \langle lm_lLM_L \,|\, \Lambda M_\Lambda\rangle \left\langle n_1l_1n_2l_2;\Lambda' \,\middle|\, n'l'N'L';\Lambda'\right\rangle_{MB}\\
&\qquad \times \left\langle \Lambda' M_\Lambda' \,\middle|\, l'm_l'L'M_L'\right\rangle \phi^*_{n'l'm_l'}(\vec{k})\phi^*_{N'L'M_L'}(\vec{P})\phi_{nlm}(\vec{k})\phi_{NLM_L}(\vec{P})
\end{aligned} \tag{86}$$

Integrated over $\vec{P}$ and Ω_k gives

$$\begin{aligned}
&\int d^3\vec{P}\int d\Omega_k \phi^*_{n_1l_1m_1}(\vec{k}_1)\phi^*_{n_2l_2m_2}(\vec{k}_2)\phi_{n_1l_1m_1}(\vec{k}_1)\phi_{n_2l_2m_2}(\vec{k}_2)\\
&\quad = \sum_{\Lambda M_\Lambda}\sum_{\Lambda' M_\Lambda'}\sum_{nlm_l}\sum_{NLM_L}\sum_{n'} \langle \Lambda M_\Lambda \,|\, l_1m_1l_2m_2\rangle \left\langle l_1m_1l_2m_2 \,\middle|\, \Lambda' M_\Lambda'\right\rangle \langle nlNL;\Lambda \,|\, n_1l_1n_2l_2;\Lambda\rangle_{MB}\\
&\qquad \times \langle lm_lLM_L \,|\, \Lambda M_\Lambda\rangle \left\langle n_1l_1n_2l_2;\Lambda' \,\middle|\, n'lNL;\Lambda'\right\rangle_{MB} \left\langle \Lambda' M_\Lambda' \,\middle|\, lm_lLM_L\right\rangle K_{n'l}(k)K_{nl}(k)
\end{aligned} \tag{87}$$

n'	l'	N'	L'	$\langle nlNL;\Lambda \mid -r_1^2 \mid n'l'N'L';\Lambda\rangle$
$n-1$	l	N	L	$\frac{1}{2}\left[n\left(n+l+\frac{1}{2}\right)\right]^{1/2}$
n	l	$N-1$	L	$\frac{1}{2}\left[N\left(N+L+\frac{1}{2}\right)\right]^{1/2}$
$n-1$	$l+1$	$N-1$	$L+1$	$\left[nN\left(l+1\right)\left(L+1\right)\right]^{1/2}(-1)^{\Lambda+L+l}W(ll+1LL+1;1\Lambda)$
$n-1$	$l+1$	N	$L-1$	$\left[n(N+L+1/2)\left(l+1\right)L\right]^{1/2}(-1)^{\Lambda+L+l}W(ll+1LL-1;1\Lambda)$
n	$l-1$	$N-1$	$L+1$	$\left[(n+l+1/2)Nl\left(L+1\right)\right]^{1/2}(-1)^{\Lambda+L+l}W(ll-1LL+1;1\Lambda)$
n	$l-1$	N	$L-1$	$\left[(n+l+1/2)(N+L+1/2)lL\right]^{1/2}(-1)^{\Lambda+L+l}W(ll-1LL-1;1\Lambda)$

Table 1: The matrix elements of $-r_1^2$

If one now considers the total one-particle wavefunction

$$\psi_\alpha(\vec{x_1}) = \phi_{n_\alpha l_\alpha m_\alpha}(\vec{k})\chi_{\sigma_\alpha}(\vec{\sigma})\xi_{\tau_\alpha}(\vec{\tau}), \tag{88}$$

one can write the two-body relative momentum distribution as

$$\begin{aligned} n_2(k) = \frac{1}{2A(A-1)} &\sum_\alpha\sum_\beta\sum_{\Lambda M_\Lambda}\sum_{\Lambda' M'_\Lambda}\sum_{nlm_l}\sum_{NLM_L}\sum_{n'}\sum_{SM_S}\sum_{TM_T}\left[1-(-1)^{l+S+T}\right]^2 \\ &\left\langle \frac{1}{2}\sigma_\alpha\frac{1}{2}\sigma_\beta \,\middle|\, SM_S\right\rangle \left\langle \frac{1}{2}\tau_\alpha\frac{1}{2}\tau_\beta \,\middle|\, TM_T\right\rangle \left\langle SM_S \,\middle|\, \frac{1}{2}\sigma_\alpha\frac{1}{2}\sigma_\beta\right\rangle \left\langle TM_T \,\middle|\, \frac{1}{2}\tau_\alpha\frac{1}{2}\tau_\beta\right\rangle \\ &\left\langle \Lambda' M'_\Lambda \,\middle|\, l_\alpha m_\alpha l_\beta m_\beta\right\rangle \left\langle LM_L l m_l \,\middle|\, \Lambda' M'_\Lambda\right\rangle \left\langle l_\alpha m_\alpha l_\beta m_\beta \,\middle|\, \Lambda M_\Lambda\right\rangle \left\langle \Lambda M_\Lambda \,\middle|\, LM_L l m_l\right\rangle \\ &\left\langle nlNL;\Lambda \,\middle|\, n_\alpha l_\alpha n_\beta l_\beta;\Lambda\right\rangle_{MB} \left\langle n_\alpha l_\alpha n_\beta l_\beta;\Lambda' \,\middle|\, n'lNL;\Lambda'\right\rangle_{MB} K_{n'l}(k)K_{nl}(k) \end{aligned} \tag{89}$$

A Calculation Moshinsky Brackets

Transformation from centre of well coordinates to relative and centre of mass coordinates of a two-body state coupled to total orbital angular momentum Λ and projection M_Λ can be done by the Moshinsky transformation brackets [1]

$$|n_1l_1n_2l_2;\Lambda M_\Lambda\rangle = \sum_{nl,N\Lambda} |nlNL;\Lambda M_\Lambda\rangle \langle nlNL;\Lambda \mid n_1l_1n_2l_2;\Lambda\rangle_{MB}\,. \tag{90}$$

One can calculate the Moshinsky brackets with the following recursive relation [2]

$$\begin{aligned} \langle nlNL;\Lambda \mid n_1+1l_1n_2l_2;\Lambda\rangle_{MB} &= \left[(n_1+1)(n_1+l_1+3/2)\right]^{-1/2} \\ &\times \sum_{n'l'N'L'} \left\langle nlNL;\Lambda \,\middle|\, -r_1^2 \,\middle|\, n'l'N'L';\Lambda\right\rangle \left\langle n'l'N'L';\Lambda \,\middle|\, n_1l_1n_2l_2;\Lambda\right\rangle_{MB} \end{aligned} \tag{91}$$

where $2n_1+l_1+2n_2+l_2 = 2n+l+2N+L$. The matrix elements in relation 92 differ from zero for just six two-particle states $|n'l'N'L';\Lambda\rangle$. These are tabulated in 1.
Moshinsky brackets have the following symmetry relation:

$$\langle nlNL;\Lambda \mid n_1l_1n_2l_2;\Lambda\rangle_{MB} = (-1)^{L-\Lambda}\langle nlNL;\Lambda \mid n_2l_2n_1l_1;\Lambda\rangle_{MB}\,. \tag{92}$$

Thus one can calculate $\langle nlNL;\Lambda \mid n_1l_1n_2l_2;\Lambda\rangle_{MB}$ starting from the Moshinsky bracket with $n_1 = 0$ and

$n_2 = 0$. An expression for this bracket is found in [2]

$$\begin{aligned}
&\langle nlNL;\Lambda \,|\, 0l_1 0l_2;\Lambda\rangle_{MB} \\
&\qquad = \left[\frac{l_1! l_2!}{(2l_1)!(2l_2)!}\frac{(2l+1)(2L+1)}{2^{l+L}}\frac{(n+l)!}{n!(2n+2l+1)!}\frac{(N+L)!}{n!(2N+2L+1)!}\right]^{1/2} \\
&\qquad\quad \times (-1)^{n+l+L-\Lambda}\sum_x (2x+1)A(l_1 l, l_2 L, x)W(lLl_1l_2;\Lambda x)
\end{aligned} \tag{93}$$

where $W(lLL_1l_2;\Lambda x)$ is the Racah W coefficient that is related to the Wigner 6-j symbols by

$$W(lLl_1l_2;\Lambda x) = (-1)^{l+L+l_1+l_2}\begin{Bmatrix} l & L & \Lambda \\ l_2 & l_1 & x \end{Bmatrix}. \tag{94}$$

The coefficient A is given by

$$\begin{aligned}
A(l_1 l, l_2 L, x) = &\left[\frac{(l_1+l+x+1)!(l_1+l-x)!(l_1+x-l)!}{(l+x-l_1)!}\frac{(l_2+L+x+1)!(l_2+L-x)!(l_2+x-L)!}{(L+x-l_2)!}\right]^{1/2} \\
&\sum_q (-1)^{\frac{l+q-l_1}{2}}\frac{(l+q-1_1)!}{\left(\frac{(l+q-l_1)}{2}\right)!\left(\frac{(l+l_1-q)}{2}\right)!}\frac{1}{(q-x)!(q+x+)!}\frac{(L+q-l_2)!}{\left(\frac{(L+q-l_2)}{2}\right)!\left(\frac{(L+l_2-q)}{2}\right)!}.
\end{aligned} \tag{95}$$

The summation over q is restricted to those values of q that are non-negative and for which the arguments of the factorials are non-negative integers. The restrictions on x are given by

$$|l-l_1| \leq x \leq l+l_1 \quad |L-l_2| \leq x \leq L+l_2. \tag{96}$$

These last relations follow from the properties of the Wigner 6-j symbol.

References

[1] Marcos Moshinsky. Transformation brackets for harmonic oscillator functions. *Nuclear Physics*, 13(1):104–116, 1959.

[2] D Ursescu, M Tomaselli, T Kuehl, and S Fritzsche. Symbolic algorithms for the computation of moshinsky brackets and nuclear matrix elements. *Computer physics communications*, 173(3):140–161, 2005.