Two-body momentum distribution for IPM with HO potential

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Two-body momentum distribution in centre of well coordinates is given by

$$n_2(\vec{k}_1, \vec{k}_2) = \frac{1}{2A(A-1)} \sum_{\substack{\alpha\beta \\ \alpha\neq\beta}} \left[\phi^*_\alpha(\vec{x}_1)\phi^*_\beta(\vec{x}_2) - \phi^*_\beta(\vec{x}_1)\phi^*_\alpha(\vec{x}_2)\right] \left[\phi_\alpha(\vec{x}_1)\phi_\beta(\vec{x}_2) - \phi_\beta(\vec{x}_1)\phi_\alpha(\vec{x}_2)\right]. \tag{1}$$

Here, $\vec{x}$ is a shorthand notation for all the coordinates ($\vec{k}$, $\vec{\sigma}$, $\vec{\tau}$). Integration over spin and isospin coordinates is implied. The summation goes over all occupied states (n_α, l_α, m_α, σ_α, τ_α).
The total one-particle wavefunction is given by

$$\phi_\alpha(\vec{x_1}) = \psi_{n_\alpha l_\alpha m_\alpha}(\vec{k})\chi_{\sigma_\alpha}(\vec{\sigma})\xi_{\tau_\alpha}(\vec{\tau}). \tag{2}$$

Consider the product of two momentum-space wave functions

$$\psi_{n_\alpha l_\alpha m_\alpha}(\vec{k}_1)\psi_{n_\beta l_\beta m_\beta}(\vec{k}_2) \tag{3}$$

To change to relative and centre of mass coordinates one needs to couple the angular momenta of the two particles

$$\begin{aligned}
\psi_{n_\alpha l_\alpha m_\alpha}(\vec{k}_1)\psi_{n_\beta l_\beta m_\beta}(\vec{k}_2) &= \sum_{\Lambda M_\Lambda} \left\langle \Lambda M_\Lambda \,\middle|\, l_\alpha m_\alpha l_\beta m_\beta \right\rangle \sum_{m'_\alpha m'_\beta} \left\langle l_\alpha m'_\alpha l_\beta m'_\beta \,\middle|\, \Lambda M_\Lambda \right\rangle \psi_{n_\alpha l_\alpha m'_\alpha}(\vec{k}_1)\psi_{n_\beta l_\beta m'_\beta}(\vec{k}_2) \\
&= \sum_{\Lambda M_\Lambda} \left\langle \Lambda M_\Lambda \,\middle|\, l_\alpha m_\alpha l_\beta m_\beta \right\rangle \left[\psi_{n_\alpha l_\alpha}(\vec{k}_1)\psi_{n_\beta l_\beta}(\vec{k}_2)\right]_{\Lambda M_\Lambda}
\end{aligned}$$

with notation

$$\left[\psi_{n_\alpha l_\alpha}(\vec{k}_1)\psi_{n_\beta l_\beta}(\vec{k}_2)\right]_{\Lambda M_\Lambda} \equiv \sum_{m_\alpha m_\beta} \psi_{n_\alpha l_\alpha m_\alpha}(\vec{k}_1)\psi_{n_\beta l_\beta m_\beta}(\vec{k}_2) \left\langle l_\alpha m_\alpha l_\beta m_\beta \,\middle|\, \Lambda M_\Lambda \right\rangle$$

For harmonic oscillator two-body states there is a simple transformation from cw coordinates to rcm coordinates, namely the Moshinsky transformation

$$\begin{aligned}
\left[\psi_{n_\alpha l_\alpha}(\vec{k}_1)\psi_{n_\beta l_\beta}(\vec{k}_2)\right]_{\Lambda M_\Lambda} &= \sum_{nl}\sum_{NL} \left[\psi_{nl}(\vec{k})\psi_{NL}(\vec{P})\right]_{\Lambda M_\Lambda} \left\langle nlNL;\Lambda \,\middle|\, n_\alpha l_\alpha n_\beta l_\beta;\Lambda \right\rangle_{MB} \\
&= \sum_{nlm_l}\sum_{NLM_L} \left\langle lm_l LM_L \,\middle|\, \Lambda M_\Lambda \right\rangle \left\langle nlNL;\Lambda \,\middle|\, n_\alpha l_\alpha n_\beta l_\beta;\Lambda \right\rangle_{MB} \psi_{nlm_l}(\vec{k})\psi_{NLM_L}(\vec{P})
\end{aligned}$$

where $\left\langle nlNL;\Lambda \,\middle|\, n_\alpha l_\alpha n_\beta l_\beta;\Lambda \right\rangle_{MB}$ is called the Moshinsky Braket.

$$\begin{aligned}
&\psi_{n_\alpha l_\alpha m_\alpha}(\vec{k}_1)\psi_{n_\beta l_\beta m_\beta}(\vec{k}_2) = \\
&\qquad \sum_{\substack{nlm_l \\ NLM_L}} \sum_{\Lambda M_\Lambda} \left\langle \Lambda M_\Lambda \,\middle|\, l_\alpha m_\alpha l_\beta m_\beta \right\rangle \left\langle nlNL;\Lambda \,\middle|\, n_\alpha l_\alpha n_\beta l_\beta;\Lambda \right\rangle_{MB} \left\langle lm_l LM_L \,\middle|\, \Lambda M_\Lambda \right\rangle \psi_{nlm_l}(\vec{k})\psi_{NLM_L}(\vec{P})
\end{aligned} \tag{4}$$

One wants to write down the anti-symmetric state

$$\psi_{n_\alpha l_\alpha m_\alpha}(\vec{k}_1)\psi_{n_\beta l_\beta m_\beta}(\vec{k}_2) - \psi_{n_\alpha l_\alpha m_\alpha}(\vec{k}_2)\psi_{n_\beta l_\beta m_\beta}(\vec{k}_1) \tag{5}$$

To find an expression for the second term one can choose between interchanging the momentum coordinates $\vec{k}_1$ and $\vec{k}_2$ or interchanging the quantum numbers $n_\alpha l_\alpha m_\alpha$ and $n_\beta l_\beta m_\beta$. Interchanging $\vec{k}_1$ and $\vec{k}_2$ result in the tranfformation $\vec{k} \to -\vec{k}$. one can use the parity realtion of the spherical harmonics $Y_{lm_l}(\theta,\varphi) \to Y_{lm_l}(\pi-\theta,\pi+\varphi) = (-1)^l Y_{lm_l}(\theta,\varphi)$

$$\psi_{nlm_l}(\vec{k}) = K_{nl}(k)Y_{lm_l}(\theta,\varphi) \to \psi_{nlm_l}(-\vec{k}) = (-1)^l K_{nl}(k)Y_{lm_l}(\theta,\varphi) \tag{6}$$

If one interchanges the quantum numbers $n_\alpha l_\alpha m_\alpha$ and $n_\beta l_\beta m_\beta$ one needs to use the symmetry realtions of the Clebsch-Gordan brackets and the Moshinsky brackets. one has, respectively,

$$\left\langle \Lambda M_\Lambda \,\middle|\, l_\alpha m_\alpha l_\beta m_\beta \right\rangle \to \left\langle \Lambda M_\Lambda \,\middle|\, l_\beta m_\beta l_\alpha m_\alpha \right\rangle = (-1)^{l_\alpha+l_\beta-\Lambda} \left\langle \Lambda M_\Lambda \,\middle|\, l_\alpha m_\alpha l_\beta m_\beta \right\rangle \tag{7}$$

and

$$\left\langle nlNL;\Lambda \,\middle|\, n_\alpha l_\alpha n_\beta l_\beta;\Lambda \right\rangle_{MB} \to \left\langle nlNL;\Lambda \,\middle|\, n_\beta l_\beta n_\alpha l_\alpha;\Lambda \right\rangle_{MB} = (-1)^{L-\Lambda} \left\langle nlNL;\Lambda \,\middle|\, n_\alpha l_\alpha n_\beta l_\beta;\Lambda \right\rangle_{MB} \tag{8}$$

So the factor for interchanging the quantum number becomes $(-1)^{l_\alpha+l_\beta+L-2\Lambda} = (-1)^{l_\alpha+l_\beta+L}$. Energy should be conserved in the transformation from cw to rcm coordinates, so one has $2n_\alpha + l_\alpha + 2n_\beta + l_\beta = 2n + l + 2N + L$. If one uses this relation, one gets a factor $(-1)^l$. The same factor we got when interchaning the momentum coordinates. So this should be correct. So the two-body antisymmetric state (only momentum wave functions) can be written as

$$\begin{aligned}&\psi_{n_\alpha l_\alpha m_\alpha}(\vec{k}_1)\psi_{n_\beta l_\beta m_\beta}(\vec{k}_2) - \psi_{n_\alpha l_\alpha m_\alpha}(\vec{k}_2)\psi_{n_\beta l_\beta m_\beta}(\vec{k}_1) = \\ &\sum_{\substack{nlm_l\\NLM_L}} \sum_{\Lambda M_\Lambda} \left[1-(-1)^l\right] \left\langle \Lambda M_\Lambda \,\middle|\, l_\alpha m_\alpha l_\beta m_\beta \right\rangle \left\langle nlNL;\Lambda \,\middle|\, n_\alpha l_\alpha n_\beta l_\beta;\Lambda \right\rangle_{MB} \langle lm_l LM_L \,|\, \Lambda M_\Lambda \rangle\, \psi_{nlm_l}(\vec{k})\psi_{NLM_L}(\vec{P})\end{aligned} \tag{9}$$

For the spin and isospin part of the two-body anti-symmetric wave function one has, for example

$$\chi_{\sigma_\alpha}(\vec{\sigma}_1)\chi_{\sigma_\beta}(\vec{\sigma}_2) = \sum_{SM_S} \left\langle SM_S \,\middle|\, \frac{1}{2}\sigma_\alpha \frac{1}{2}\sigma_\beta \right\rangle \sum_{\sigma'_\alpha\sigma'_\beta} \left\langle \frac{1}{2}\sigma'_\alpha \frac{1}{2}\sigma'_\beta \,\middle|\, SM_S \right\rangle \chi_{\sigma'_\alpha}(\vec{\sigma}_1)\chi_{\sigma'_\beta}(\vec{\sigma}_2).$$

With the use of the Clebsch-Gordan symmetry relation (7), now for half-integer spin values, one can write

$$\chi_{\sigma_\alpha}(\vec{\sigma}_2)\chi_{\sigma_\beta}(\vec{\sigma}_1) = \sum_{SM_S} (-1)^{1+S} \left\langle SM_S \,\middle|\, \frac{1}{2}\sigma_\alpha \frac{1}{2}\sigma_\beta \right\rangle \sum_{\sigma'_\alpha\sigma'_\beta} \left\langle \frac{1}{2}\sigma'_\alpha \frac{1}{2}\sigma'_\beta \,\middle|\, SM_S \right\rangle \chi_{\sigma'_\alpha}(\vec{\sigma}_1)\chi_{\sigma'_\beta}(\vec{\sigma}_2). \tag{10}$$

An analogue expression is found for the isospin part. If one puts results $(9,10)$ together one gets

$$\begin{aligned}\phi_\alpha(\vec{x}_1)\phi_\beta(\vec{x}_2) - \phi_\beta(\vec{x}_1)\phi_\alpha(\vec{x}_2) = &\sum_{\substack{nlm_l\\NLM_L}} \sum_{\Lambda M_\Lambda} \sum_{SM_S} \sum_{\sigma'_\alpha\sigma'_\beta} \sum_{TM_T} \sum_{\tau'_\alpha\tau'_\beta} \left[1-(-1)^{l+S+T}\right] \left\langle \Lambda M_\Lambda \,\middle|\, l_\alpha m_\alpha l_\beta m_\beta \right\rangle \\ &\left\langle SM_S \,\middle|\, \frac{1}{2}\sigma_\alpha \frac{1}{2}\sigma_\beta \right\rangle \left\langle \frac{1}{2}\sigma'_\alpha \frac{1}{2}\sigma'_\beta \,\middle|\, SM_S \right\rangle \left\langle TM_T \,\middle|\, \frac{1}{2}\tau_\alpha \frac{1}{2}\tau_\beta \right\rangle \left\langle \frac{1}{2}\tau'_\alpha \frac{1}{2}\tau'_\beta \,\middle|\, TM_T \right\rangle \\ &\left\langle nlNL;\Lambda \,\middle|\, n_\alpha l_\alpha n_\beta l_\beta;\Lambda \right\rangle_{MB} \langle lm_l LM_L \,|\, \Lambda M_\Lambda \rangle\, \phi_{nlm_l}(\vec{k})\phi_{NLM_L}(\vec{P})\chi_{\sigma'_\alpha}(\vec{\sigma}_1)\chi_{\sigma'_\beta}(\vec{\sigma}_2)\xi_{\tau'_\alpha}(\vec{\tau}_1)\xi_{\tau^prime_\beta}(\vec{\tau}_2)\end{aligned} \tag{11}$$

Thus, one has (integrated over spin and isospin variables)

$$n_2(\vec{k},\vec{P}) = \frac{1}{2A(A-1)} \sum_{\substack{\alpha\beta \\ \alpha\neq\beta}} \sum_{\substack{nlm_l \\ NLM_L}} \sum_{\Lambda M_\Lambda} \sum_{SM_S} \sum_{\sigma'_\alpha \sigma'_\beta} \sum_{TM_T} \sum_{\tau'_\alpha \tau'_\beta} \left[1-(-1)^{l+S+T}\right]^2 \left\langle \Lambda M_\Lambda \,\middle|\, l_\alpha m_\alpha l_\beta m_\beta \right\rangle$$
$$\left\langle SM_S \,\middle|\, \frac{1}{2}\sigma_\alpha \frac{1}{2}\sigma_\beta \right\rangle \left\langle \frac{1}{2}\sigma'_\alpha \frac{1}{2}\sigma'_\beta \,\middle|\, SM_S \right\rangle \left\langle TM_T \,\middle|\, \frac{1}{2}\tau_\alpha \frac{1}{2}\tau_\beta \right\rangle \left\langle \frac{1}{2}\tau'_\alpha \frac{1}{2}\tau'_\beta \,\middle|\, TM_T \right\rangle$$
$$\left\langle nlNL;\Lambda \,\middle|\, n_\alpha l_\alpha n_\beta l_\beta;\Lambda \right\rangle_{MB} \left\langle lm_l LM_L \,\middle|\, \Lambda M_\Lambda \right\rangle \phi_{nlm_l}(\vec{k}) \phi_{NLM_L}(\vec{P}) \chi_{\sigma'_\alpha}(\vec{\sigma}_1) \chi_{\sigma'_\beta}(\vec{\sigma}_2) \xi_{\tau'_\alpha}(\vec{\tau}_1) \xi_{\tau^prime_\beta}(\vec{\tau}_2) \quad (12)$$