2013, December

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(Dated: January 22, 2014)

The one-body density operator $\hat{n}^{[1]}(\vec{k})$ and the radial one-body momentum distribution operator $\hat{n}^{[1]}(k)$ are defined as

$$\hat{n}^{[1]}(\vec{k}) = \frac{1}{(2\pi)^3}\sum_i e^{-\imath\vec{k}(\vec{r}_i{}'-\vec{r}_i)}\prod_{j\neq i}\delta(\vec{r}_j{}'-\vec{r}_j) \tag{1}$$

$$\hat{n}^{[1]}(k) = \int \mathrm{d}\Omega_k\ \hat{n}^{[1]}(\vec{k}) \tag{2}$$

The effective (correlated) operator of $\hat{n}^{[1]}(k)$ in the TBC approximation reads

$$\hat{n}^{[1],eff}(k) = \sum_i \hat{n}^{[1]}(i)+\sum_{i<j}[\hat{n}^{[1]}(i)+\hat{n}^{[1]}(j)]\hat{l}(i,j)+\sum_{i<j}\hat{l}^{\dagger}(i,j)[\hat{n}^{[1]}(i)+\hat{n}^{[1]}(j)]+\sum_{i<j}\hat{l}^{\dagger}(i,j)[\hat{n}^{[1]}(i)+\hat{n}^{[1]}(j)]\hat{l}(i,j), \tag{3}$$

where $\hat{n}^{[1]}(i)$ is the part of $\hat{n}^{[1]}(k)$ acting on particle i. The expectation value of the correlated one-body momentum distribution is

$$\begin{aligned}
n^{[1]}(k) &= \langle\Psi\mid\hat{n}^{[1],eff}(k)\mid Psi\rangle \qquad (4)\\
&= \langle\Psi\mid\sum_i\hat{n}^{[1]}(i)\mid\Psi\rangle+\langle\Psi\mid\sum_{i<j}[\hat{n}^{[1]}(i)+\hat{n}^{[1]}(j)]\hat{l}(i,j)\mid\Psi\rangle+\langle\Psi\mid\sum_{i<j}\hat{l}^{\dagger}(i,j)[\hat{n}^{[1]}(i)+\hat{n}^{[1]}(j)]\mid\Psi\rangle\\
&\quad+\langle\Psi\mid\sum_{i<j}\hat{l}^{\dagger}(i,j)[\hat{n}^{[1]}(i)+\hat{n}^{[1]}(j)]\hat{l}(i,j)\mid\Psi\rangle\\
&= \sum_\alpha\langle\alpha\mid\hat{n}^{[1]}(1)\mid\alpha\rangle+\sum_{\alpha<\beta}{}_{nas}\langle\alpha\beta\mid[\hat{n}^{[1]}(1)+\hat{n}^{[1]}(2)]\hat{l}(1,2)\mid\alpha\beta\rangle_{nas}\\
&\quad+\sum_{\alpha<\beta}{}_{nas}\langle\alpha\beta\mid\hat{l}^{\dagger}(1,2)[\hat{n}^{[1]}(1)+\hat{n}^{[1]}(2)]\mid\alpha\beta\rangle_{nas}\\
&\quad+\sum_{\alpha<\beta}{}_{nas}\langle\alpha\beta\mid\hat{l}^{\dagger}(1,2)[\hat{n}^{[1]}(1)+\hat{n}^{[1]}(2)]\hat{l}(1,2)\mid\alpha\beta\rangle_{nas} \qquad (5)
\end{aligned}$$

The calculation of the bare operator $\langle\alpha\mid\hat{n}^{[1]}(1)\mid\alpha\rangle$ is straightforward. The calculations of the correlated two-body operators are more complex.

A. Correlated one-body density operator

The one-body momentum distribution $n^{[1]}(\vec{k}_1)$ is related to the two-body momentum distribution $n^{[2]}(\vec{k}_1,\vec{k}_2)$,

$$n^{[1]}(\vec{k}_1) = \int \mathrm{d}\vec{k}_2\ n^{[2]}(\vec{k}_1,\vec{k}_2). \tag{6}$$

This can also be seen as the substitution of the delta function $\delta(\vec{r}_2-\vec{r}_2{}')$ by its integral representation in Eq. (1). We take a look at the expectation value of $\hat{n}^{[1]}(i)$ for a pair $\mid\alpha\beta\rangle$,

$$\begin{aligned}
n^{[1]}_{\alpha\beta}(\vec{k}_1) &= {}_{nas}\langle\alpha\beta\mid\hat{n}^{[1]}(1)\mid\alpha\beta\rangle_{nas}\\
&= \frac{1}{(2\pi)^6}\int\mathrm{d}\vec{k}_2\int\mathrm{d}\vec{r}_1\int\mathrm{d}\vec{r}_1{}'\int\mathrm{d}\vec{r}_2\int\mathrm{d}\vec{r}_2{}'\ e^{\imath\vec{k}_1(\vec{r}_1{}'-\vec{r}_1)}e^{\imath\vec{k}_2(\vec{r}_2{}'-\vec{r}_2)}\rho^{[2]}(\vec{r}_1{}',\vec{r}_2{}';\vec{r}_1,\vec{r}_2). \qquad (7)
\end{aligned}$$

where $\rho^{[2]}$ is the two-body non-diagonal density. We introduce the relative $\vec{r}_{12}$ and c.m. coordinates $\vec{R}_{12}$ in its usual way.

$$n^{[1]}_{\alpha\beta}(\vec{k}_1) = \frac{1}{(2\pi)^3}\int\mathrm{d}\vec{r}_{12}\int\mathrm{d}\vec{r}_{12}{}'\int\mathrm{d}\vec{R}_{12}\int\mathrm{d}\vec{R}_{12}{}'\ e^{\imath\vec{k}_1(\vec{r}_1{}'-\vec{r}_1)}\frac{1}{(2\pi)^3}\int\mathrm{d}\vec{k}_2 e^{\imath\vec{k}_2\frac{(\vec{R}_{12}{}'-\vec{r}_{12}{}'-\vec{R}_{12}+\vec{r}_{12})}{\sqrt{2}}}\rho^{[2]}(\vec{r}_{12}{}',\vec{R}_{12}{}';\vec{r}_{12},\vec{R}_{12}) \tag{8}$$

The two-body non-diagonal density in function of the relative and c.m. coordinates is

$$\rho^{[2]}(\vec{r}_{12}', \vec{R}_{12}'; \vec{r}_{12}, \vec{R}_{12}) = \sum_{A,B} C_{\alpha\beta}^{A\,\dagger} C_{\alpha\beta}^{B} \Psi^*_{N_A L_A M_{L_A}}(\vec{R}_{12}') \Psi^*_{n_A l_A S_A j_A m_{j_A}}(\vec{r}_{12}') \Psi^*_{N_B L_B M_{L_B}}(\vec{R}_{12}) \Psi_{n_B l_B S_B j_B m_{j_B}}(\vec{r}_{12}). \tag{9}$$

After performing the integration over $\vec{k}_2$, Eq. (8) becomes

$$\begin{aligned} n_{\alpha\beta}^{[1]}(\vec{k}_1) = & \frac{1}{(2\pi)^3}\sqrt{8}\int \mathrm{d}\vec{r}_{12}\int \mathrm{d}\vec{r}_{12}'\int \mathrm{d}\vec{R}_{12}\int \mathrm{d}\vec{R}_{12}'\; e^{\imath\vec{k}_1(\vec{r}_1{}'-\vec{r}_1)}\delta(\vec{R}_{12}'-\vec{r}_{12}'-\vec{R}_{12}+\vec{r}_{12}) \\ & \times \sum_{A,B} C_{\alpha\beta}^{A\,\dagger} C_{\alpha\beta}^{B} \Psi^*_{N_A L_A M_{L_A}}(\vec{R}_{12}') \Psi^*_{n_A l_A S_A j_A m_{j_A}}(\vec{r}_{12}') \Psi^*_{N_B L_B M_{L_B}}(\vec{R}_{12}) \Psi_{n_B l_B S_B j_B m_{j_B}}(\vec{r}_{12}) \end{aligned} \tag{10}$$

We can rewrite $\Psi^*_{N_A L_A M_{L_A}}(\vec{R}_{12}')$ as

$$\begin{aligned} \Psi^*_{N_A L_A M_{L_A}}(\vec{R}_{12}') &= \int \frac{\mathrm{d}\vec{P}_{12}}{(2\pi)^{3/2}} e^{-\imath\vec{P}_{12}\vec{R}_{12}'} \int \frac{\mathrm{d}\vec{R}_{12}''}{(2\pi)^{3/2}} e^{+\imath\vec{P}_{12}\vec{R}_{12}''} \Psi^*_{N_A L_A M_{L_A}}(\vec{R}_{12}'') \\ &= (\imath)^{L_A}\int \frac{\mathrm{d}\vec{P}_{12}}{(2\pi)^{3/2}} e^{-\imath\vec{P}_{12}\vec{R}_{12}'} \phi_{N_A L_A}(P_{12}) Y^*_{L_A M_{L_A}}(\Omega_P), \end{aligned} \tag{11}$$

where we used the plane wave expansion and defined the radial momentum wave function

$$\phi_{N_A L_A}(P) = \sqrt{\frac{2}{\pi}}\int \mathrm{d}R\; R^2 j_{L_A}(RP) R_{N_A L_A}(R). \tag{12}$$

Performing the integration over $\vec{R}_{12}'$ and as a result thereof substituting $\vec{R}_{12}' = \vec{R}_{12} - \vec{r}_{12} + \vec{r}_{12}'$, gives

$$\begin{aligned} n_{\alpha\beta}^{[1]}(\vec{k}_1) = & \frac{1}{(2\pi)^3}\sqrt{8}\sum_{A,B} C_{\alpha\beta}^{A\,\dagger} C_{\alpha\beta}^{B}\int \mathrm{d}\vec{r}_{12}\int \mathrm{d}\vec{r}_{12}'\; e^{\imath\sqrt{2}\vec{k}_1(\vec{r}_{12}'-\vec{r}_{12})} \\ & \times \Psi^*_{n_A l_A S_A j_A m_{j_A}}(\vec{r}_{12}') \Psi_{n_B l_B S_B j_B m_{j_B}}(\vec{r}_{12}) \\ & \times (\imath)^{L_A}\int \vec{P}_{12} e^{-\imath\vec{P}_{12}(\vec{r}_{12}'-\vec{r}_{12})} \phi_{N_A L_A}(P_{12}) Y^*_{L_A M_{L_A}}(\Omega_P) \\ & \times \int \frac{\mathrm{d}\vec{R}_{12}}{(2\pi)^{3/2}}\; e^{-\imath\vec{P}_{12}\vec{R}_{12}} \Psi_{N_B L_B M_{L_B}}(\vec{R}_{12}). \end{aligned} \tag{13}$$

After applying the plane wave expansions, $n_{\alpha\beta}^{[1]}(k_1) = \int \mathrm{d}\Omega_{k_1}\; n_{\alpha\beta}^{[1]}(\vec{k}_1)$ becomes

$$\begin{aligned} n_{\alpha\beta}^{[1]}(k_1) = & \frac{1}{(2\pi)^3}\sum_{A,B} C_{\alpha\beta}^{A\,\dagger} C_{\alpha\beta}^{B}\int \mathrm{d}\vec{r}_{12}\int \mathrm{d}\vec{r}_{12}'\int \mathrm{d}\vec{P}_{12} \\ & \times (4\pi)^2\sqrt{8}\sum_{l_1 m_{l_1}} j_{l_1}(\sqrt{2}k_1 r_{12}') j_{l_1}(\sqrt{2}k_1 r_{12}) Y^*_{l_1 m_{l_1}}(\Omega_{12}') Y_{l_1 m_{l_1}}(\Omega_{12}) \\ & \times \Psi^*_{n_A l_A S_A j_A m_{j_A}}(\vec{r}_{12}') \Psi_{n_B l_B S_B j_B m_{j_B}}(\vec{r}_{12}) \\ & \times (4\pi)^2\sum_{l m_l l' m_{l'}} (\imath)^{L_A - L_B + l - l'} j_l(P_{12} r_{12}) j_{l'}(P_{12} r_{12}') Y_{l m_l}(\Omega_{12}) Y^*_{l' m_{l'}}(\Omega_{12}') Y^*_{l m_l}(\Omega_P) Y_{l' m_{l'}}(\Omega_P) \\ & \times \phi_{N_A L_A}(P_{12}) Y^*_{L_A M_{L_A}}(\Omega_P) \phi_{N_B L_B}(P_{12}) Y_{L_B M_{L_B}}(\Omega_P) \end{aligned} \tag{14}$$

The Ω_P dependent part of (14) gives

$$\begin{aligned} & \int \mathrm{d}\Omega_P\; Y^*_{l m_l}(\Omega_P) Y^*_{L_A M_{L_A}}(\Omega_P) Y_{l' m_l'}(\Omega_P) Y_{L_B M_{L_B}}(\Omega_P) = \\ & \qquad \sum_{q m_q} \frac{\hat{L}_A \hat{l}\hat{q}}{\sqrt{4\pi}} \begin{pmatrix} L_A & l & q \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_A & l & q \\ M_{L_A} & m_l & m_q \end{pmatrix} \frac{\hat{L}_B \hat{l}'\hat{q}}{\sqrt{4\pi}} \begin{pmatrix} L_B & l' & q \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_B & l' & q \\ M_{L_B} & m_{l'} & m_q \end{pmatrix}. \end{aligned} \tag{15}$$

The integration over Ω_{12} and Ω'_{12} gives respectively

$$\int \mathrm{d}\Omega_{12}\; Y_{l_1 m_{l_1}}(\Omega_{12}) Y_{l_B m_{l_B}}(\Omega_{12}) Y_{l m_l}(\Omega_{12}) = \frac{\sqrt{\hat{l}_1 \hat{l}_B \hat{l}}}{\sqrt{4\pi}} \begin{pmatrix} l_1 & l_B & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_B & l \\ m_{l_1} & m_{l_B} & m_l \end{pmatrix}, \tag{16}$$

$$\int \mathrm{d}\Omega'_{12}\; Y_{l_1 m_{l_1}}(\Omega'_{12}) Y_{l_A m_{l_A}}(\Omega'_{12}) Y_{l' m_{l'}}(\Omega'_{12}) = \frac{\sqrt{\hat{l}_1 \hat{l}_B \hat{l}'}}{\sqrt{4\pi}} \begin{pmatrix} l_1 & l_B & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_B & l' \\ m_{l_1} & m_{l_B} & m_{l'} \end{pmatrix}. \tag{17}$$

We generalize Eq. (14) by inserting the general correlation operator $\hat{f}_B(\vec{r}_{12}) = f_B(r_{12})\hat{\mathcal{O}}$ and $\hat{f}_A^\dagger(\vec{r}\,'_{12})$. We can return to the non-correlated expression by taking $\hat{f}_A^\dagger(\vec{r}\,'_{12}) = \hat{f}_B(\vec{r}_{12}) = 1$. For now we suppose $\hat{\mathcal{O}} = 1$, extension to $\hat{\mathcal{O}}$ equal to tensor or spin-isospin operator is straightforward, but complicates the expression slightly (extra summations possible). The final expression for the expectation value of $n^{[1]}(k)$ for a nucleon pair $\mid \alpha\beta\rangle$ is then

$$\begin{aligned}
n^{[1]}_{\alpha\beta}(k_1) = {} & \frac{1}{(2\pi)^3}(4\pi)^2(4\pi)^2\sqrt{8}\sum_{A,B} C^{A\;\dagger}_{\alpha\beta} C^{B}_{\alpha\beta} \sum_{l_1 m_{l_1}} \sum_{l m_l l' m_{l'}} \\
& \times (\imath)^{L_A - L_B + l - l'} \frac{\sqrt{\hat{l}_1 \hat{l}_B \hat{l}}}{\sqrt{4\pi}} \begin{pmatrix} l_1 & l_B & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_B & l \\ m_{l_1} & m_{l_B} & m_l \end{pmatrix} \frac{\sqrt{\hat{l}_1 \hat{l}_B \hat{l}'}}{\sqrt{4\pi}} \begin{pmatrix} l_1 & l_B & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_B & l' \\ m_{l_1} & m_{l_B} & m_{l'} \end{pmatrix}. \\
& \times \sum_{q m_q} \frac{\hat{L}_A \hat{l} \hat{q}}{\sqrt{4\pi}} \begin{pmatrix} L_A & l & q \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_A & l & q \\ M_{L_A} & m_l & m_q \end{pmatrix} \frac{\hat{L}_B \hat{l}' \hat{q}}{\sqrt{4\pi}} \begin{pmatrix} L_B & l' & q \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_B & l' & q \\ M_{L_B} & m_{l'} & m_q \end{pmatrix} \\
& \times \int \mathrm{d}r_{12}\, r_{12}^2 \int \mathrm{d}r'_{12}\, {r'_{12}}^2 \int \mathrm{d}P_{12}\, P_{12}^2\, j_{l_1}(\sqrt{2}k_1 r'_{12}) j_{l_1}(\sqrt{2}k_1 r_{12}) j_l(P_{12} r_{12}) j_{l'}(P_{12} r'_{12}) \\
& \times \phi_{N_A L_A}(P_{12}) \phi_{N_B L_B}(P_{12})\; f_A(r'_{12}) R_{n_A l_A}(r'_{12}) f_B(r_{12}) R_{n_B l_B}(r_{12}).
\end{aligned} \tag{18}$$

The only integral is 'only' three dimensional, but is depends on 11 parameters for every correlation operator combination. Advantage of this expression is that each correlated term in (5) has a similar shape.

B. $\hat{n}^{[1]}(2)$ term

Eq. (18) gives expression for ${}_{nas}\langle\alpha\beta \mid \hat{l}^\dagger(1,2)\hat{n}^{[1]}(1)\hat{l}(1,2) \mid \alpha\beta\rangle_{nas}$. In this section we derive the expression for the full term ${}_{nas}\langle\alpha\beta \mid \hat{l}^\dagger(1,2)[\hat{n}^{[1]}(1) + \hat{n}^{[2]}]\hat{l}(1,2) \mid \alpha\beta\rangle_{nas}$. The expectation value of Eq.(8) for the $n^{[1]}(2)$ term has

$$\int \mathrm{d}\vec{k}_2\; e^{\vec{k}_2(\vec{r}_1{}' - \vec{r}_1)} e^{\vec{k}_1(\vec{r}_2{}' - \vec{r}_2)} \tag{19}$$

instead of

$$\int \mathrm{d}\vec{k}_2\; e^{\vec{k}_1(\vec{r}_1{}' - \vec{r}_1)} e^{\vec{k}_2(\vec{r}_2{}' - \vec{r}_2)}. \tag{20}$$

The integration over $\vec{k}_2$ will give $\delta(\vec{r}_1{}' - \vec{r}_1)$ instead of $\delta(\vec{r}_2{}' - \vec{r}_2)$. The exponent $e^{\vec{k}_1(\vec{r}_1{}' - \vec{r}_1)}$ becomes $e^{-\sqrt{2}\vec{k}_1(\vec{r}\,'_{12} - \vec{r}_{12})}$ in Eq. (13). After integrating over Ω_{k_1}, the sign difference will make no difference in the expression for $n^{[1]}(k_1)$ in Eq. (14).

The exponent $e^{-\vec{P}\vec{R}'_{12}}$ becomes $e^{-\vec{P}(\vec{R}_{12} - \vec{r}_{12} + \vec{r}\,'_{12})}$ in Eq. (13) instead of $e^{-\vec{P}(\vec{R}_{12} + \vec{r}_{12} - \vec{r}\,'_{12})}$. This will result in Eq. (18) in a factor $(\imath)^{l'-l}$ instead of $(\imath)^{l-l'}$.

C. A second method

A second method to calculate the correlated momentum distribution uses the expansion for the correlation function. In case of the central correlation, we have

$$\begin{aligned}
g_c(r_{12}) &= 2\sqrt{\pi} Y_{00}(\Omega_{12}) g_c(r_{12}) \\
&= 8 \sum_{l_1 m_{l_1}} \sum_{l_2 m_{l_2}} \hat{l}_1 \hat{l}_2 \begin{pmatrix} l_1 & l_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & 0 \\ m_{l_1} & m_{l_2} & 0 \end{pmatrix} Y^*_{l_1 m_{l_1}}(\Omega_1) Y_{l_2 m_{l_2}}(\Omega_2) \\
&\quad \times \imath^{l_1 - l_2} \int \mathrm{d}q\; q^2 j_{l_1}(\frac{q r_1}{\sqrt{2}}) j_{l_2}(\frac{q r_2}{\sqrt{2}}) g_c(q)
\end{aligned} \tag{21}$$

Or,

$$g_c(r_{12}) \propto \int \mathrm{d}x\ P_l(x) f(\sqrt{r_1^2 + r_2^2 - 2r_1 r_2}) \tag{22}$$

This works well for central correlation, left or right, but it becomes complicated for tensor correlation and computational expensive for left-right correlations.

I. EXTENDED TBC APPROXIMATION FOR RELATIVE TWO-BODY MOMENTUM DISTRIBUTION

In the extended TBC approximation, we add the terms

$$\begin{aligned}\sum_{i<j<k} & [\hat{\Omega}(i,j) + \hat{\Omega}(i,k)]\hat{l}(j,k) + [\hat{\Omega}(i,j) + \hat{\Omega}(j,k)]\hat{l}(i,k) + [\hat{\Omega}(i,k) + \hat{\Omega}(j,k)]\hat{l}(i,j) \\ & + \hat{l}^\dagger(j,k)[\hat{\Omega}(i,j) + \hat{\Omega}(i,k)]\hat{l}(j,k) + \hat{l}^\dagger(i,k)[\hat{\Omega}(i,j) + \hat{\Omega}(j,k)]\hat{l}(i,k) + \hat{l}^\dagger(i,j)[\hat{\Omega}(i,k) + \hat{\Omega}(j,k)]\hat{l}(i,j) \\ & + \hat{l}^\dagger(j,k)[\hat{\Omega}(i,j) + \hat{\Omega}(i,k)] + \hat{l}^\dagger(i,k)[\hat{\Omega}(i,j) + \hat{\Omega}(j,k)] + \hat{l}^\dagger(i,j)[\hat{\Omega}(i,k) + \hat{\Omega}(j,k)]\end{aligned} \tag{23}$$

The expectation value of the term $\sum_{i<j<k} \hat{\Omega}(i,j)\hat{l}(j,k)$ for the MF Slater determinant $\mid \Psi_A\rangle$ is

$$\sum_{\alpha<\beta<\gamma} {}_{nas}\langle \alpha\beta\gamma \mid \hat{\Omega}(1,2)\hat{l}(2,3) \mid \alpha\beta\gamma\rangle_{nas} \tag{24}$$

For further calculation, we write $\mid \alpha\beta\gamma\rangle$ for this term as

$$\mid \alpha\beta\gamma\rangle = (1 - P_{23})(\mid \alpha\beta\gamma\rangle + \mid \beta\gamma\alpha\rangle + \mid \gamma\alpha\beta\rangle). \tag{25}$$

The correlation operator $l(\hat{2,3})$ depends on the relative coordinates $\vec{r}_{23}$ between the two-particles it acts on. Therefore, a transformation of the antisymmetric 3N states from the particle coordinates $(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ to the internal Jacobi coordinates $(\vec{r}_{23}, \vec{r}_{1(23)}, \vec{R}_{123})$,

$$\vec{r}_{1(23)} = \frac{\vec{R}_{23} - \sqrt{2}\vec{r}_1}{\sqrt{3}}, \tag{26}$$

$$\vec{R}_{123} = \frac{\sqrt{2}\vec{R}_{23} + \vec{r}_1}{\sqrt{3}}. \tag{27}$$

One readily finds that for the uncoupled three-nucleon state in a HO basis

$$\begin{aligned}(1 - P_{23}) \mid \alpha(\vec{r}_1)\beta(\vec{r}_2)\gamma(\vec{r}_3)\rangle = & \sum_{A_{23}=n_{23}l_{23}S_{23}j_{23}m_{j_{23}}T_{23}M_{T_{23}}} \sum_{B_{123}=N_{123}L_{123}M_{L_{123}}} \sum_{\Gamma_{1(23)}=n_{1(23)}l_{1(23)}m_{l_{1(23)}}} \sum_{m_{s_\alpha}} \\ & \times \langle A_{23}B_{123}\Gamma_{1(23)}m_{s_\alpha}t_\alpha \mid (1 - P_{23}) \mid \alpha\beta\gamma\rangle \mid A_{23}B_{123}\Gamma_{1(23)}m_{s_\alpha}t_\alpha\rangle.\end{aligned} \tag{28}$$

First we take a look at the two-body norm operator $\hat{n}^{[2]} = \frac{2}{A(A-1)}$. The expectation value of the term $[\hat{\Omega}(1,2) + \hat{\Omega}(1,3)]\hat{l}(2,3)$ is

$$\frac{4}{A(A-1)} \sum_{\alpha<\beta<\gamma} {}_{nas}\langle \alpha\beta\gamma \mid \hat{l}(2,3) \mid \alpha\beta\gamma\rangle_{nas}. \tag{29}$$

Using the expression of Eq. (25) and retaining only the first term,

$$\begin{aligned}\langle \alpha\beta\gamma \mid (1 - P_{23})^\dagger \hat{l}(2,3)(1 - P_{23}) \mid \alpha\beta\gamma\rangle = & \\ \frac{4}{A(A-1)} \sum_{\alpha<\beta<\gamma} \sum_{A_{23},A'_{23}} \sum_{B_{123}} \sum_{\Gamma_{1(23)}} \sum_{m_{s_\alpha}} & \langle \alpha\beta\gamma \mid (1 - P_{23})^\dagger \mid A'_{23}B_{123}\Gamma_{1(23)}m_{s_\alpha}t_\alpha\rangle \\ & \times \langle A_{23}B_{123}\Gamma_{1(23)}m_{s_\alpha}t_\alpha \mid (1 - P_{23}) \mid \alpha\beta\gamma\rangle\langle A'_{23} \mid \hat{l}(2,3) \mid A_{23}\rangle\end{aligned} \tag{30}$$

where we applied the transformation of Eq. (28). The second and third term in Eq. (25) will give a similar contribution to the total expectation value.

Similar to the one-body momentum distribution, we start for the relative two-body momentum distribution $n^{[2]}(\vec{k}_{12})$ from the operator

$$\hat{n}^{[2]}(\vec{k}_{12}) = \int \mathrm{d}\vec{P}_{12} \int \mathrm{d}\vec{k}_3 e^{\imath\vec{k}_1(\vec{r}_1{}' - \vec{r}_1)} e^{\imath\vec{k}_2(\vec{r}_2{}' - \vec{r}_2)} e^{\imath\vec{k}_3(\vec{r}_3{}' - \vec{r}_3)} \tag{31}$$

The expectation value of the correlated operator $\hat{n}^{[2]}(\vec{k}_{12})\hat{l}(2,3)$ for the first term of Eq. (25) then reads

$$\begin{aligned}
&\sum_{\alpha\beta\gamma} \langle \alpha\beta\gamma \mid (1-P_{23})^\dagger \hat{n}^{[2]}(\vec{k}_{12}) \hat{l}(2,3)(1-P_{23}) \mid \alpha\beta\gamma \rangle \\
&= \sum_{\alpha\beta\gamma} \int \mathrm{d}\vec{P}_{12} \int \mathrm{d}\vec{k}_3 \int \mathrm{d}\vec{r}_{1\ldots3}' \int \mathrm{d}\vec{r}_{1\ldots3} \langle \alpha\beta\gamma \mid (1-P_{23})^\dagger \mid \vec{r}_{1\ldots3}' \rangle \\
&\quad \times e^{\imath\vec{k}_1(\vec{r}_1{}' - \vec{r}_1)} e^{\imath\vec{k}_2(\vec{r}_2{}' - \vec{r}_2)} e^{\imath\vec{k}_3(\vec{r}_3{}' - \vec{r}_3)} \langle \vec{r}_{1\ldots3} \mid \hat{l}(2,3)(1-P_{23}) \mid \alpha\beta\gamma \rangle .
\end{aligned} \tag{32}$$

The transformation of Eq. (28) gives

$$\begin{aligned}
&\sum_{\alpha\beta\gamma} \int \mathrm{d}\vec{P}_{12} \int \mathrm{d}\vec{k}_3 \int \mathrm{d}\vec{r}_{1\ldots3}' \int \mathrm{d}\vec{r}_{1\ldots3} \langle \alpha\beta\gamma \mid (1-P_{23})^\dagger \mid \vec{r}_{1\ldots3}' \rangle \\
&\qquad \times e^{\imath\vec{k}_1(\vec{r}_1{}' - \vec{r}_1)} e^{\imath\vec{k}_2(\vec{r}_2{}' - \vec{r}_2)} e^{\imath\vec{k}_3(\vec{r}_3{}' - \vec{r}_3)} \langle \vec{r}_{1\ldots3} \mid \hat{l}(2,3)(1-P_{23}) \mid \alpha\beta\gamma \rangle .
\end{aligned} \tag{33}$$