Exeter Math Club Competition
January 25, 2014

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Contest Day Acknowledgments

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- **Graders** Chad Hai Qian, Jeffrey Qiao, Meena Jagadeesan, Ravi Jagadeesan, Ray Li, James Lin, Alexander Wei, Kuo-An (Andy) Wei, Dai Yang, David Yang, Yuan (Yannick) Yao
- **Judges** Zuming Feng, Greg Spanier

Chapter 1

EMC^2 2014 Problems

1.1 Speed Test

There are 25 problems, worth 2 points each, to be solved in 30 minutes.

1. Chad, Ravi, Kevin, and Meena are four of the 551 residents of Chadwick, Illinois. Expressing your answer to the nearest percent, how much of the population do they represent?

2. Points A, B, and C are on a line for which $AB = 625$ and $BC = 256$. What is the sum of all possible values of the length AC?

3. An increasing arithmetic sequence has first term 2014 and common difference 1337. What is the least odd term of this sequence?

4. How many non-congruent scalene triangles with integer side lengths have two sides with lengths 3 and 4?

5. Let a and b be real numbers for which the function $f(x) = ax^2 + bx + 3$ satisfies $f(0) + 2^0 = f(1) + 2^1 = f(2) + 2^2$. What is $f(0)$?

6. A *pentomino* is a set of five planar unit squares that are joined edge to edge. Two pentominoes are considered the same if and only if one can be rotated and translated to be identical to the other. We say that a pentomino is *compact* if it can fit within a 2 by 3 rectangle. How many distinct compact pentominoes exist?

7. Consider a hexagon with interior angle measurements of 91, 101, 107, 116, 152, and 153 degrees. What is the average of the interior angles of this hexagon, in degrees?

8. What is the smallest positive number that is either one larger than a perfect cube and one less than a perfect square, or vice versa?

9. What is the first time after 4:56 (a.m.) when the 24-hour expression for the time has three consecutive digits that form an increasing arithmetic sequence with difference 1? (For example, 23:41 is one of those moments, while 23:12 is not.)

10. Chad has trouble counting. He wants to count from 1 to 100, but cannot pronounce the word "three," so he skips every number containing the digit three. If he tries to count up to 100 anyway, how many numbers will he count?

11. In square $ABCD$, point E lies on side BC and point F lies on side CD so that triangle AEF is equilateral and inside the square. Point M is the midpoint of segment EF, and P is the point other than E on AE for which $PM = FM$. The extension of segment PM meets segment CD at Q. What is the measure of $\angle CQP$, in degrees?

12. One apple is five cents cheaper than two bananas, and one banana is seven cents cheaper than three peaches. How much cheaper is one apple than six peaches, in cents?

13. How many ordered pairs of integers (a, b) exist for which $|a|$ and $|b|$ are at most 3, and $a^3 - a = b^3 - b$?

14. Five distinct boys and four distinct girls are going to have lunch together around a table. They decide to sit down one by one under the following conditions: no boy will sit down when more boys than girls are already seated, and no girl will sit down when more girls than boys are already seated. How many possible sequences of taking seats exist?

15. Jordan is swimming laps in a pool. For each lap after the first, the time it takes her to complete is five seconds more than that of the previous lap. Given that she spends 10 minutes on the first six laps, how long does she spend on the next six laps, in minutes?

16. Chad decides to go to trade school to ascertain his potential in carpentry. Chad is assigned to cut away all the vertices of a wooden regular tetrahedron with sides measuring four inches. Each vertex is cut away by a plane which passes through the three midpoints of the edges adjacent to that vertex. What is the surface area of the resultant solid, in square inches?

    Note: A *tetrahedron* is a solid with four triangular faces. In a *regular* tetrahedron, these faces are all equilateral triangles.

17. Chad and Jordan independently choose two-digit positive integers. The two numbers are then multiplied together. What is the probability that the result has a units digit of zero?

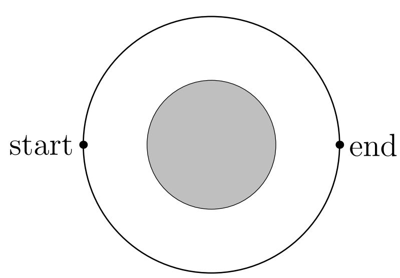
18. For art class, Jordan needs to cut a circle out of the coordinate grid. She would like to find a circle passing through at least 16 lattice points so that her cut is accurate. What is the smallest possible radius of her circle?

    Note: A lattice point is defined as one whose coordinates are both integers. For example, (5, 8) is a lattice point whereas (3.5, 5) is not.

19. Chad's ant Arctica is on one of the eight corners of Chad's toolbox, which measures two decimeters in width, three decimeters in length, and four decimeters in height. One day, Arctica wanted to go to the opposite corner of this box. Assuming she can only crawl on the surface of the toolbox, what is the shortest distance she has to crawl to accomplish this task, in decimeters? (You may assume that the toolbox is floating in the Exeter Space Station, so that Arctica can crawl on all six faces.)

20. Jordan is counting numbers for fun. She starts with the number 1, and then counts onward, skipping any number that is a divisor of the product of all previous numbers she has said. For example, she starts by counting 1, 2, 3, 4, 5, but skips 6, a divisor of $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. What is the 20^{th} number she counts?

21. Chad and Jordan are having a race in the lake shown below. The lake has a diameter of four kilometers and there is a circular island in the middle of the lake with a diameter of two kilometers. They start at one point on the edge of the lake and finish at the diametrically opposite point. Jordan makes the trip only by swimming in the water, while Chad swims to the island, runs across it, and then continues swimming. They both take the fastest possible route and, amazingly, they tie! Chad swims at two kilometers an hour and runs at five kilometers an hour. At what speed does Jordan swim?



22. Cameron has stolen Chad's barrel of oil and is driving it around on a truck on the coordinate grid on his truck. Cameron is a bad truck driver, so he can only move the truck forward one kilometer at a

time along one of the gridlines. In fact, Cameron is so bad at driving the truck that between every two one-kilometer movements, he has to turn exactly 90 degrees. After 50 one-kilometer movements, given that Cameron's first one-kilometer movement was westward, how many points he could be on?

23. Let a, b, and c be distinct nonzero base ten digits. Assume there exist integers x and y for which $\overline{abc} \cdot \overline{cb} = 100x^2 + 1$ and $\overline{acb} \cdot \overline{bc} = 100y^2 + 1$. What is the minimum value of the number $\overline{abbc}$?

    Note: The notation $\overline{pqr}$ designates the number whose hundreds digit is p, tens digit is q, and units digit is r, not the product $p \cdot q \cdot r$.

24. Let r_1, r_2, r_3, r_4 and r_5 be the five roots of the equation $x^5 - 4x^4 + 3x^2 - 2x + 1 = 0$. What is the product of $(r_1 + r_2 + r_3 + r_4), (r_1 + r_2 + r_3 + r_5), (r_1 + r_2 + r_4 + r_5), (r_1 + r_3 + r_4 + r_5)$, and $(r_2 + r_3 + r_4 + r_5)$?

25. Chad needs seven apples to make an apple strudel for Jordan. He is currently at 0 on the metric number line. Every minute, he randomly moves one meter in either the positive or the negative direction with equal probability. Arctica's parents are located at $+4$ and -4 on the number line. They will bite Chad for kidnapping Arctica if he walks onto those numbers. Also, there is one apple located at each integer between -3 and 3, inclusive. Whenever Chad lands on an integer with an unpicked apple, he picks it. What is the probability that Chad picks all the apples without getting bitten by Arctica's parents?

1.2 Accuracy Test

There are 8 problems, worth 5 points each, to be solved in 40 minutes.

1. Chad lives on the third floor of an apartment building with ten floors. He leaves his room and goes up two floors, goes down four floors, goes back up five floors, and finally goes down one floor, where he finds Jordan's room. On which floor does Jordan live?

2. A real number x satisfies the equation $2014x + 1337 = 1337x + 2014$. What is x?

3. Given two points on the plane, how many distinct regular hexagons include both of these points as vertices?

4. Jordan has six different files on her computer and needs to email them to Chad. The sizes of these files are 768, 1024, 2304, 2560, 4096, and 7680 kilobytes. Unfortunately, the email server holds a limit of S kilobytes on the total size of the attachments per email, where S is a positive integer. It is additionally given that all of the files are indivisible. What is the maximum value of S for which it will take Jordan at least three emails to transmit all six files to Chad?

5. If real numbers x and y satisfy $(x + 2y)^2 + 4(x + 2y + 2 - xy) = 0$, what is $x + 2y$?

6. While playing table tennis against Jordan, Chad came up with a new way of scoring. After the first point, the score is regarded as a ratio. Whenever possible, the ratio is reduced to its simplest form. For example, if Chad scores the first two points of the game, the score is reduced from 2:0 to 1:0. If later in the game Chad has 5 points and Jordan has 9, and Chad scores a point, the score is automatically reduced from 6:9 to 2:3. Chad's next point would tie the game at 1:1. Like normal table tennis, a player wins if he or she is the first to obtain 21 points. However, he or she does not win if after his or her receipt of the 21$^{\text{st}}$ point, the score is immediately reduced. Chad and Jordan start at 0:0 and finish the game using this rule, after which Jordan notes a curiosity: the score was never reduced. How many possible games could they have played? Two games are considered the same if and only if they include the exact same sequence of scoring.

7. For a positive integer m, we define m as a *factorial number* if and only if there exists a positive integer k for which $m = k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1$. We define a positive integer n as a *Thai number* if and only if n can be written as both the sum of two factorial numbers and the product of two factorial numbers. What is the sum of the five smallest Thai numbers?

8. Chad and Jordan are in the Exeter Space Station, which is a triangular prism with equilateral bases. Its height has length one decameter and its base has side lengths of three decameters. To protect their station against micrometeorites, they install a force field that contains all points that are within one decameter of any point of the surface of the station. What is the volume of the set of points within the force field and outside the station, in cubic decameters?

1.3 Team Test

There are 10 problems, worth 18 points each, to be solved in 45 minutes.

1. What is the units digit of the product of the first seven primes?

2. In triangle ABC, $\angle BAC$ is a right angle and $\angle ACB$ measures 34 degrees. Let D be a point on segment BC for which $AC = CD$, and let the angle bisector of $\angle CBA$ intersect line AD at E. What is the measure of $\angle BED$?

3. Chad numbers five paper cards on one side with each of the numbers from 1 through 5. The cards are then turned over and placed in a box. Jordan takes the five cards out in random order and again numbers them from 1 through 5 on the other side. When Chad returns to look at the cards, he deduces with great difficulty that the probability that exactly two of the cards have the same number on both sides is p. What is p?

4. Only one real value of x satisfies the equation $kx^2 + (k+5)x + 5 = 0$. What is the product of all possible values of k?

5. On the Exeter Space Station, where there is no effective gravity, Chad has a geometric model consisting of 125 wood cubes measuring 1 centimeter on each edge arranged in a 5 by 5 by 5 cube. An aspiring carpenter, he practices his trade by drawing the projection of the model from three views: front, top, and side. Then, he removes some of the original 125 cubes and redraws the three projections of the model. He observes that his three drawings after removing some cubes are identical to the initial three. What is the maximum number of cubes that he could have removed? (Keep in mind that the cubes could be suspended without support.)

6. Eric, Meena, and Cameron are studying the famous equation $E = mc^2$. To memorize this formula, they decide to play a game. Eric and Meena each randomly think of an integer between 1 and 50, inclusively, and substitute their numbers for E and m in the equation. Then, Cameron solves for the absolute value of c. What is the probability that Cameron's result is a rational number?

7. Let CDE be a triangle with side lengths $EC = 3$, $CD = 4$, and $DE = 5$. Suppose that points A and B are on the perimeter of the triangle such that line AB divides the triangle into two polygons of equal area and perimeter. What are all the possible values of the length of segment AB?

8. Chad and Jordan are raising bacteria as pets. They start out with one bacterium in a Petri dish. Every minute, each existing bacterium turns into 0, 1, 2 or 3 bacteria, with equal probability for each of the four outcomes. What is the probability that the colony of bacteria will eventually die out?

9. Let $a = w + x, b = w + y, c = x + y, d = w + z, e = x + z$, and $f = y + z$. Given that $af = be = cd$ and
$$(x-y)(x-z)(x-w) + (y-x)(y-z)(y-w) + (z-x)(z-y)(z-w) + (w-x)(w-y)(w-z) = 1,$$
what is
$$2(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) - ab - ac - ad - ae - bc - bd - bf - ce - cf - de - df - ef?$$

10. If a and b are integers at least 2 for which $a^b - 1$ strictly divides $b^a - 1$, what is the minimum possible value of ab?

    Note: If x and y are integers, we say that x *strictly* divides y if x divides y and $|x| \neq |y|$.

1.4 Guts Test

There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.

1.4.1 Round 1

1. [4] What is $2 + 22 + 1 + 3 - 31 - 3$?

2. [4] Let $ABCD$ be a rhombus. Given $AB = 5$, $AC = 8$, and $BD = 6$, what is the perimeter of the rhombus?

3. [4] There are 2 hats on a table. The first hat has 3 red marbles and 1 blue marble. The second hat has 2 red marbles and 4 blue marbles. Jordan picks one of the hats randomly, and then randomly chooses a marble from that hat. What is the probability that she chooses a blue marble?

1.4.2 Round 2

4. [5] There are twelve students seated around a circular table. Each of them has a slip of paper that they may choose to pass to either their clockwise or counterclockwise neighbor. After each person has transferred their slip of paper once, the teacher observes that no two students exchanged papers. In how many ways could the students have transferred their slips of paper?

5. [5] Chad wants to test David's mathematical ability by having him perform a series of arithmetic operations at lightning-speed. He starts with the number of cubic centimeters of silicon in his 3D printer, which is 109. He has David perform all of the following operations in series each second:

   - Double the number
   - Subtract 4 from the number
   - Divide the number by 4
   - Subtract 5 from the number
   - Double the number
   - Subtract 4 from the number

   Chad instructs David to shout out after three seconds the result of three rounds of calculations. However, David computes too slowly and fails to give an answer in three seconds. What number should David have said to Chad?

6. [5] Points D, E, and F lie on sides BC, CA, and AB of triangle ABC, respectively, such that the following length conditions are true: $CD = AE = BF = 2$ and $BD = CE = AF = 4$. What is the area of triangle ABC?

1.4.3 Round 3

7. [6] In the *2, 3, 5, 7 game*, players count the positive integers, starting with 1 and increasing, which do not contain the digits 2, 3, 5, and 7, and also are not divisible by the numbers 2, 3, 5, and 7. What is the fifth number counted?

8. [6] If A is a real number for which
$$19 \cdot A = \frac{2014!}{1! \cdot 2! \cdot 2013!},$$
what is A?

   Note: The expression $k!$ denotes the product $k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1$.

9. [6] What is the smallest number that can be written as both $x^3 + y^2$ and $z^3 + w^2$ for positive integers x, y, z, and w with $x \neq z$?

1.4.4 Round 4

Each of the three problems in this round depends on the answer to one of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.

In addition, it is given that the answer to each of the following problems is a positive integer less than or equal to the problem number.

10. [7] Let B be the answer to problem 11 and let C be the answer to problem 12. What is the sum of a side length of a square with perimeter B and a side length of a square with area C?

11. [7] Let A be the answer to problem 10 and let C be the answer to problem 12. What is $(C-1)(A+1)-(C+1)(A-1)$?

12. [7] Let A be the answer to problem 10 and let B be the answer to problem 11. Let x denote the positive difference between A and B. What is the sum of the digits of the positive integer $9x$?

1.4.5 Round 5

13. [8] Five different schools are competing in a tournament where each pair of teams plays at most once. Four pairs of teams are randomly selected and play against each other. After these four matches, what is the probability that Chad's and Jordan's respective schools have played against each other, assuming that Chad and Jordan come from different schools?

14. [8] A square of side length 1 and a regular hexagon are both circumscribed by the same circle. What is the side length of the hexagon?

15. [8] From the list of integers $1, 2, 3, \ldots, 30$, Jordan can pick at least one pair of distinct numbers such that none of the 28 other numbers are equal to the sum or the difference of this pair. Of all possible such pairs, Jordan chooses the pair with the least sum. Which two numbers does Jordan pick?

1.4.6 Round 6

16. [9] What is the sum of all two-digit integers with no digit greater than four whose squares also have no digit greater than four?

17. [9] Chad marks off ten points on a circle. Then, Jordan draws five chords under the following constraints:

    - Each of the ten points is on exactly one chord.
    - No two chords intersect.
    - There do not exist (potentially non-consecutive) points A, B, C, D, E, and F, in that order around the circle, for which AB, CD, and EF are all drawn chords.

    In how many ways can Jordan draw these chords?

18. [9] Chad is thirsty. He has 109 cubic centimeters of silicon and a 3D printer with which he can print a cup to drink water in. He wants a silicon cup whose exterior is cubical, with five square faces and an

open top, that can hold exactly 234 cubic centimeters of water when filled to the rim in a rectangular-box-shaped cavity. Using all of his silicon, he prints a such cup whose thickness is the same on the five faces. What is this thickness, in centimeters?

1.4.7 Round 7

19. [10] Jordan wants to create an equiangular octagon whose side lengths are exactly the first 8 positive integers, so that each side has a different length. How many such octagons can Jordan create?

20. [10] There are two positive integers on the blackboard. Chad computes the sum of these two numbers and tells it to Jordan. Jordan then calculates the sum of the greatest common divisor and the least common multiple of the two numbers, and discovers that her result is exactly 3 times as large as the number Chad told her. What is the smallest possible sum that Chad could have said?

21. [10] Chad uses *yater* to measure distances, and knows the conversion factor from yaters to meters precisely. When Jordan asks Chad to convert yaters into meters, Chad only gives Jordan the result rounded to the nearest integer meters. At Jordan's request, Chad converts 5 yaters into 8 meters and 7 yaters into 12 meters. Given this information, how many possible numbers of meters could Jordan receive from Chad when requesting to convert 2014 yaters into meters?

1.4.8 Round 8

22. [11] Jordan places a rectangle inside a triangle with side lengths 13, 14, and 15 so that the vertices of the rectangle all lie on sides of the triangle. What is the maximum possible area of Jordan's rectangle?

23. [11] Hoping to join Chad and Jordan in the Exeter Space Station, there are 2014 prospective astronauts of various nationalities. It is given that 1006 of the astronaut applicants are American and that there are a total of 64 countries represented among the applicants. The applicants are to group into 1007 pairs with no pair consisting of two applicants of the same nationality. Over all possible distributions of nationalities, what is the maximum number of possible ways to make the 1007 pairs of applicants? Express your answer in the form $a \cdot b!$, where a and b are positive integers and a is not divisible by

$b+1$.

Note: The expression $k!$ denotes the product $k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1$.

24. [11] We say a polynomial P in x and y is n-good if $P(x,y)=0$ for all integers x and y, with $x \neq y$, between 1 and n, inclusive. We also define the complexity of a polynomial to be the maximum sum of exponents of x and y across its terms with nonzero coefficients. What is the minimal complexity of a nonzero 4-good polynomial? In addition, give an example of a 4-good polynomial attaining this minimal complexity.

Chapter 2

EMC^2 2014 Solutions

2.1 Speed Test Solutions

1. Chad, Ravi, Kevin, and Meena are four of the 551 residents of Chadwick, Illinois. Expressing your answer to the nearest percent, how much of the population do they represent?

   Solution. The answer is $\boxed{1\%}$.

   We can see that
   $$551 \cdot 1\% > 500 \cdot 0.01 = 5 > 4,$$
   so 1% of the population is more than 4. This means that the four people represent less than 1% of the population. Also,
   $$4 > 3 = 600 \cdot 0.005 > 551 \cdot 0.5\%,$$
   so the percentage is between 0.5% and 1%. Rounding to the nearest percent yields 1%.

2. Points A, B, and C are on a line for which $AB = 625$ and $BC = 256$. What is the sum of all possible values of the length AC?

   Solution. The answer is $\boxed{1250}$.

   If C is the middle point, then $AC = AB - BC$. If B is the middle point, then $AC = AB + BC$. If A is the middle point, then we have that $BC = BA + AC$, which results in a negative value of AC. Therefore there are two possible values of AC, which sum to $(AB - BC) + (AB + BC) = 2 \cdot AB = 2 \cdot 625 = 1250$.

3. An increasing arithmetic sequence has first term 2014 and common difference 1337. What is the least odd term of this sequence?

   Solution. The answer is $\boxed{3351}$.

   The first term 2014 is not odd, but the second term $2014 + 1337 = 3351$ is odd. Because the sequence is increasing, all other terms are greater than 3351, and therefore 3351 is the answer.

4. How many non-congruent scalene triangles with integer side lengths have two sides with lengths 3 and 4?

   Solution. The answer is $\boxed{3}$.

   The length of the third side of the triangle must be between 2 and 6, inclusive; otherwise, the longest side of the triangle would be at least the sum of the other two, and the triangle could not be constructed. However, we must also exclude 3 and 4 as possibilities because the triangle is scalene, which only leaves 2, 5, and 6 as possibilities for the length of the third side. All three of these triangles are valid.

5. Let a and b be real numbers for which the function $f(x) = ax^2 + bx + 3$ satisfies $f(0) + 2^0 = f(1) + 2^1 = f(2) + 2^2$. What is $f(0)$?

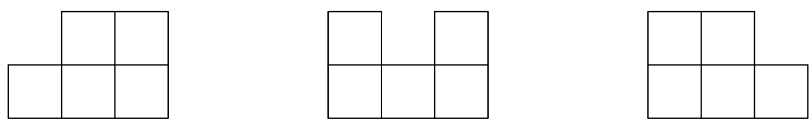
   Solution. The answer is $\boxed{3}$.

   We have $f(0) = a(0)^2 + b(0) + 3 = 3$.

6. A *pentomino* is a set of five planar unit squares that are joined edge to edge. Two pentominoes are considered the same if and only if one can be rotated and translated to be identical to the other. We say that a pentomino is *compact* if it can fit within a 2 by 3 rectangle. How many distinct compact pentominoes exist?

   Solution. The answer is $\boxed{3}$.

   We can get 3 distinct compact pentominoes by removing any square in the top row (with 3 squares) in a 2 by 3 rectangle, as shown below. The 3 configurations with bottom squares removed are the same as the previous 3 under a 180° rotation.



7. Consider a hexagon with interior angle measurements of 91, 101, 107, 116, 152, and 153 degrees. What is the average of the interior angles of this hexagon, in degrees?

   Solution. The answer is $\boxed{120^\circ}$.

   We can sum the six numbers to get a total sum of 720°, so the average is $\frac{720^\circ}{6} = 120^\circ$.

   Alternate solution: The sum of all of the interior angles of any hexagon is $(6-2) \cdot 180^\circ = 720^\circ$. Therefore the average is $\frac{720^\circ}{6} = 120^\circ$.

8. What is the smallest positive number that is either one larger than a perfect cube and one less than a perfect square, or vice versa?

   Solution. The answer is $\boxed{26}$.

   The first few squares are $1, 4, 9, 16, 25, 36, \ldots$, and the first few cubes are $1, 8, 27, 64, \ldots$. We can see that 26 is the first number between a square and a cube.

9. What is the first time after 4:56 (a.m.) when the 24-hour expression for the time has three consecutive digits that form an increasing arithmetic sequence with difference 1? (For example, 23:41 is one of those moments, while 23:12 is not.)

   Solution. The answer is $\boxed{10{:}12}$.

   All times after 4:56 and before 10:00 have three digits. Additionally, their second digit is at most 5, so if they satisfy the desired conditions their first digit must be at most 4. It is clear that none of 4:57, 4:58, and 4:59 satisfies the desired conditions. Therefore, the next time with three consecutive digits that form an increasing arithmetic sequence must be after 10:00. Note that 10:12 is a time with the desired conditions, and it is easy to check that the times after 10:00 and before 10:12 do not contain arithmetic sequences, so 10:12 is the answer.

10. Chad has trouble counting. He wants to count from 1 to 100, but cannot pronounce the word "three," so he skips every number containing the digit three. If he tries to count up to 100 anyway, how many numbers will he count?

Solution. The answer is $\boxed{81}$.

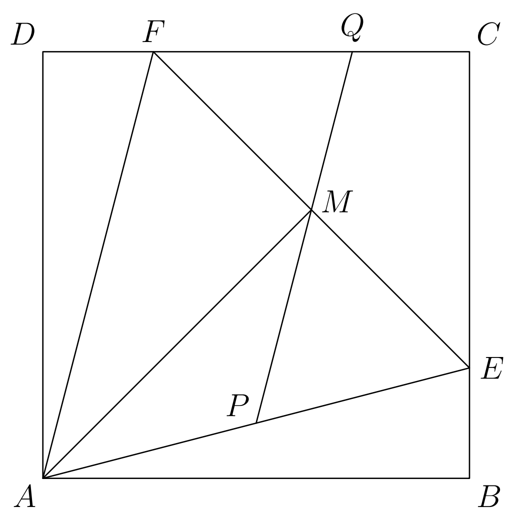
Chad can count 100, so we can replace 100 with the two-digit number 00. We can also add a 0 to the front of all one-digit numbers to make all of the 100 numbers have two digits. There are 9 possibilities for each digit of the numbers Chad can count because he cannot count 3, so there are $9 \cdot 9 = 81$ numbers that Chad can count.

Alternate solution: There are 10 numbers that have 3 as their tens digit and 10 numbers that have 3 as their ones digit. However, these 20 numbers include 33 twice, so there are $10 + 10 - 1 = 19$ numbers that have at least one 3 as a digit. Chad cannot say these numbers, so he says $100 - 19 = 81$ numbers.

11. In square $ABCD$, point E lies on side BC and point F lies on side CD so that triangle AEF is equilateral and inside the square. Point M is the midpoint of segment EF, and P is the point other than E on AE for which $PM = FM$. The extension of segment PM meets segment CD at Q. What is the measure of $\angle CQP$, in degrees?

Solution. The answer is $\boxed{105}$.

The diagram is as follows:



Because M is the midpoint of EF, we have $EM = FM = PM$. Therefore, $\triangle EPM$ is isosceles with $\angle EPM = \angle PEM = 60°$.

Also, $ABCD$ is a square, so $AB = AD$, and $\triangle AEF$ is equilateral, so $AE = AF$. Then, by the Pythagorean Theorem,

$$BE = \sqrt{AE^2 - AB^2} = \sqrt{AF^2 - AD^2} = DF.$$

Combining these, we have $\triangle ABE \cong \triangle ADF$, so $\angle BAE = \angle DAF$. Considering $\angle BAD$, we have $\angle BAE + \angle EAF + \angle FAD = 90°$, or $\angle BAE = \frac{90°-60°}{2} = 15°$ using the fact that $\angle EAF = 60°$. Then, considering $\triangle ABE$, we have $\angle AEB = 90° - \angle BAE = 75°$. Finally, $\angle CEP = 180° - \angle BEA = 105°$.

Now, consider quadrilateral $CQPE$, whose interior angles should sum to 360°. That is,

$$\begin{aligned}\angle CQP + \angle QPE + \angle PEC + \angle ECQ &= 360^\circ\\ \angle CQP + 60^\circ + 105^\circ + 90^\circ &= 360^\circ\\ \angle CQP &= 105^\circ,\end{aligned}$$

because $\angle ECQ = 90^\circ$.

Alternate solution: As in the previous solution, we can find that $\angle AEB = \angle AFD = 75^\circ$ and $\angle EPM = 60^\circ$. Because $\angle EAF = 60^\circ = \angle EPM$, lines AF and PQ are parallel. Therefore, $\angle CQP = \angle CFA = 180^\circ - \angle DFA = 180^\circ - 75^\circ = 105^\circ$.

12. One apple is five cents cheaper than two bananas, and one banana is seven cents cheaper than three peaches. How much cheaper is one apple than six peaches, in cents?

    Solution. The answer is $\boxed{19}$.

    Suppose the prices of an apple, a banana, and a peach are A, B, and P cents, respectively. We have

    $$\begin{aligned}2B &= A + 5\\ 3P &= B + 7,\end{aligned}$$

    so the answer is

    $$\begin{aligned}6P - A &= 2(3P) - A\\ &= 2(B+7) - A\\ &= 2B + 14 - A\\ &= (A+5) + 14 - A\\ &= 19.\end{aligned}$$

13. How many ordered pairs of integers (a, b) exist for which $|a|$ and $|b|$ are at most 3, and $a^3 - a = b^3 - b$?

    Solution. The answer is $\boxed{13}$.

    We can create a table of values of $n^3 - n$ for when n is between -3 and 3:

n	$n^3 - n$
-3	-24
-2	-6
-1	0
0	0
1	0
2	6
3	24

    So, the solutions for (a, b) are $(-3, -3)$, $(-2, -2)$, $(2, 2)$, $(3, 3)$ and all (a, b) in which a and b are both any of -1, 0, or 1. Thus, in total there are $4 + 3 \cdot 3 = 13$ solutions.

14. Five distinct boys and four distinct girls are going to have lunch together around a table. They decide to sit down one by one under the following conditions: no boy will sit down when more boys than girls are already seated, and no girl will sit down when more girls than boys are already seated. How many possible sequences of taking seats exist?

Solution. The answer is $\boxed{46080}$.

Pair the first person to sit with the second, the third with the fourth, and so on, excluding the last person. In order to satisfy the conditions, there must be one boy and one girl in each pair. So, in the four pairs, there are exactly 4 boys and 4 girls, hence the last person to be seated must be a boy. There are 2 possibilities for the order of genders in each pair, which yields a total of $2^4 = 16$ gender arrangements. Then, considering each gender separately, there are five choices for the first boy to sit down, four choices for the second, and so on. Similarly, there are four choices for the first girl to sit down, three choices for the second, and so on. So the total number of sequences is $16 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1) = 16 \cdot 120 \cdot 24 = 46080$.

15. Jordan is swimming laps in a pool. For each lap after the first, the time it takes her to complete is five seconds more than that of the previous lap. Given that she spends 10 minutes on the first six laps, how long does she spend on the next six laps, in minutes?

Solution. The answer is $\boxed{13}$.

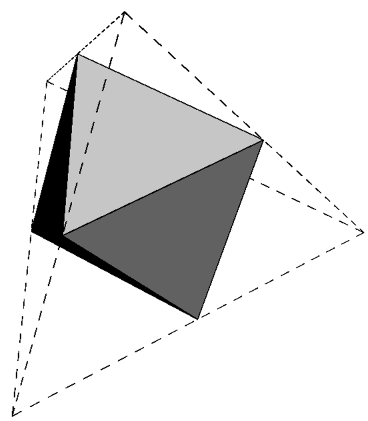
Match the n-th lap with the $(n + 6)$-th lap for each positive integer $n \leq 6$. In each pair, the latter lap takes $6 \cdot 5 = 30$ seconds longer to finish than the former lap, so it would take $6 \cdot 30 = 180$ seconds more to finish the next 6 laps. Therefore, the answer is $10 + \frac{180}{60} = 13$ minutes.

16. Chad decides to go to trade school to ascertain his potential in carpentry. Chad is assigned to cut away all the vertices of a wooden regular tetrahedron with sides measuring four inches. Each vertex is cut away by a plane which passes through the three midpoints of the edges adjacent to that vertex. What is the surface area of the resultant solid, in square inches?

Note: A *tetrahedron* is a solid with four triangular faces. In a *regular* tetrahedron, these faces are all equilateral triangles.

Solution. The answer is $\boxed{8\sqrt{3}}$.

After each cut, a new equilateral triangular face with side length 2 inches is created. After all four cuts, each original face is also an equilateral triangle with side length 2 inches.



Each of these triangles has area $\frac{2^2 \cdot \sqrt{3}}{4} = \sqrt{3}$ square inches. The resultant solid consists of eight of these triangles, so its total surface area is $8\sqrt{3}$ square inches.

Alternate solution: Each of the four cuts adds an equilateral triangle with side length 2 inches to the surface area, but also takes away three such equilateral triangles. The new surface area can be computed by adding to and subtracting from the surface area of the original solid. The uncut tetrahedron has four equilateral triangles each with area $4\sqrt{3}$, so the final surface area is $4 \cdot 4\sqrt{3} + 4 \cdot \sqrt{3} - 4 \cdot 3 \cdot \sqrt{3} = 8\sqrt{3}$.

17. Chad and Jordan independently choose two-digit positive integers. The two numbers are then multiplied together. What is the probability that the result has a units digit of zero?

Solution. The answer is $\boxed{\frac{27}{100}}$.

For each given digit between 0 and 9, inclusive, there are exactly 9 two-digit numbers ending in that digit. Because this is the same for all digits, the problem can be regarded as randomly choosing two one-digit integers from 0 to 9. We can see that there are a total of $10 \cdot 10 = 100$ possible choices for the two units digits.

Now suppose the product of the two numbers ends in zero. If Chad chooses 2, 4, 6, or 8, Jordan must choose 0 or 5. If Chad chooses 5, Jordan's number is one of 0, 2, 4, 6 or 8. If Chad's number is 0, Jordan can choose any of the ten digits. Finally, if Chad chooses 1, 3, 7, or 9, Jordan can only choose 0. Therefore, there are $4 \cdot 2 + 1 \cdot 5 + 1 \cdot 10 + 4 \cdot 1 = 27$ ways for Chad and Jordan to have numbers with product ending in 0. Thus, the desired probability is $\frac{27}{100}$.

Alternate solution: We assign to each two-digit integer an ordered pair (a, b) as follows: the remainder upon division of the number by 2 is a, and upon division by 5 is b. The product of the two numbers has units digit 0 if and only if at least one of the a values and at least one of the b values are 0, as the number must be divisible by both 2 and 5 to have units digit 0. We treat each value independently. For the a-values, the probability that one value is 0 is $\frac{1}{2}$, and that it is not 0 is the same. Thus, the probability that at least one of the two numbers has a-value 0 is $1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$. Similarly, the probability that one b-value is 0 is $\frac{1}{5}$, so the probability that at least one of the two numbers has b-value 0 is $1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}$. Multiplying the two together gives an answer of $\frac{27}{100}$.

Note that this solution requires that the a-value and b-value are indeed probabilistically independent of each other. However, this is also easy to verify because there exist an equal number of integers assigned to each of the ten possible pairs (a, b).

18. For art class, Jordan needs to cut a circle out of the coordinate grid. She would like to find a circle passing through at least 16 lattice points so that her cut is accurate. What is the smallest possible radius of her circle?

    Note: A lattice point is defined as one whose coordinates are both integers. For example, (5, 8) is a lattice point whereas (3.5, 5) is not.

    Solution. The answer is $\boxed{\frac{\sqrt{130}}{2}}$.

    Let the smallest possible radius be r and the center of one circle containing 16 lattice points with radius r have center (p, q).

    First, suppose $r < \frac{15}{2}$. Then the diameter is less than 15, so any two points on the circle are less than 15 units apart. Thus, it is impossible for all 16 lattice points on the circle to have different x-coordinates, as the leftmost and rightmost points would then be at least 15 units apart. So, the circle contains two points (a, b) and (a, c), where a, b, and c are integers, with the same x-coordinate. Then, (p, q) must lie on the perpendicular bisector of the line segment between these points, so $q = \frac{b+c}{2}$. By a similar argument, we find that p is an integer divided by two. Without loss of generality, we may assume $(p, q) = \left(\frac{1}{2}, \frac{1}{2}\right)$ because a circle with different center could be translated to satisfy this.

    Now, suppose a lattice point (m, n) is on the circle. Then, $\left(m - \frac{1}{2}\right)^2 + \left(n - \frac{1}{2}\right)^2 = r^2$. Also, we have $\left(m - \frac{1}{2}\right)^2 = \left((1 - m) - \frac{1}{2}\right)^2$ and $\left(n - \frac{1}{2}\right)^2 = \left((1 - n) - \frac{1}{2}\right)^2$. So, (m, n), $(m, (1 - n))$, $((1 - m), n)$, and $((1 - m), (1 - n))$ are all on the circle. In addition, the four points produced by switching the x- and y-coordinates of those points are also on the circle. Note that at least one of the points (s, t) in such a family has the property that $s \geq t \geq 1$. Thus, we can count the number of lattice points on the circle by only considering those with this property. If $s = t$, there are only 4 distinct points to be counted; otherwise, there are 8. However, note that the circle may only contain one lattice point (u, u) with $u \geq 1$, in which case there still must be at least two other points (s_1, t_1) and (s_2, t_2) on the circle with $s_i > t_i \geq 1$ for $i = 1, 2$ if the circle is to have more than 12 lattice points.

    For the circle to contain 16 lattice points, we must have $\left(s_1 - \frac{1}{2}\right)^2 + \left(t_1 - \frac{1}{2}\right) = r^2$ and $\left(s_2 - \frac{1}{2}\right)^2 + \left(t_2 - \frac{1}{2}\right) = r^2$. If we substitute $x_i = 2s_i - 1$ and $y_i = 2t_i - 1$ for $i = 1, 2$, so that x_i and y_i are positive integers, we obtain

    $$x_1^2 + y_1^2 = x_2^2 + y_2^2 = (2r)^2.$$

    Thus, we must find the smallest number expressible as the sum of two perfect squares in two ways. We can check that $65 = 1^2 + 8^2 = 4^2 + 7^2$ satisfies this and no smaller integer does. This means $(2r)^2 = 65$, or $r = \frac{\sqrt{130}}{2}$. Indeed, the circle given by

    $$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{65}{2}$$

    contains 16 lattice points.

    Note that we do not need to check the cases in which $r \geq \frac{15}{2}$ because such an r cannot be the smallest possible radius.

19. Chad's ant Arctica is on one of the eight corners of Chad's toolbox, which measures two decimeters in width, three decimeters in length, and four decimeters in height. One day, Arctica wanted to go to the opposite corner of this box. Assuming she can only crawl on the surface of the toolbox, what is the shortest distance she has to crawl to accomplish this task, in decimeters? (You may assume that the toolbox is floating in the Exeter Space Station, so that Arctica can crawl on all six faces.)

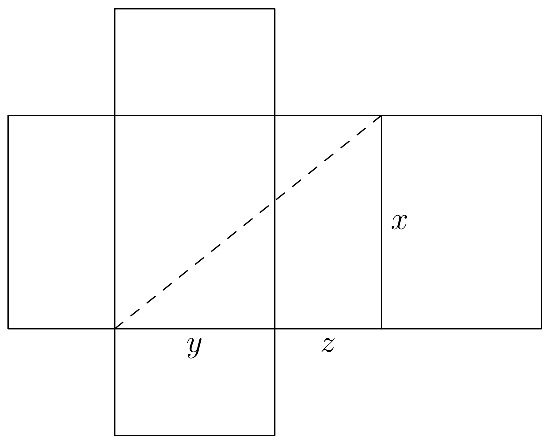
Solution. The answer is $\boxed{\sqrt{41}}$.

Given a path from one corner to the opposite corner, observe that the length of the path does not change if we unfold the box.

After unfolding, the shortest path is a straight line between A and B, where A and B are opposite corners. Depending on how we unfold, AB can have different lengths, so we want the shortest possible value of AB. From the Pythagorean Theorem, we have

$$AB = \sqrt{x^2 + (y+z)^2},$$

where x, y and z are the side lengths of the box, in some order.



This is minimized when x is the longest side. Therefore, our answer is

$$\sqrt{4^2 + (3+2)^2} = \boxed{\sqrt{41}}.$$

20. Jordan is counting numbers for fun. She starts with the number 1, and then counts onward, skipping any number that is a divisor of the product of all previous numbers she has said. For example, she starts by counting 1, 2, 3, 4, 5, but skips 6, a divisor of $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. What is the 20^{th} number she counts?

    Solution. The answer is $\boxed{47}$.

    We can show that the only integers that are counted are either 1, a prime, or a prime raised to the 2^n-th power, where n is a positive integer.

    It's easy to show that 1 and any prime will be counted, as a prime p wouldn't be a divisor of $(p-1) \cdot (p-2) \cdots 1$. By playing with smaller numbers we can arrive at the result regarding 2^n-th powers of primes, but a rigorous proof is included below. Using this claim, we can list the numbers that Jordan counts: 1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47, and so on. Therefore, the 20^{th} number would be 47.

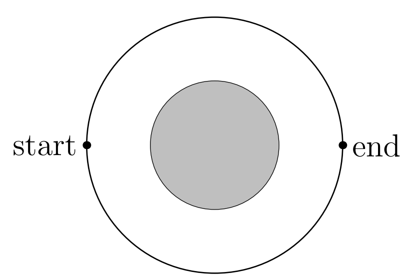
    We now prove that the only numbers other than 1 and primes that are counted are primes raised to the 2^n-th power. No numbers with at least 2 distinct prime factors are counted, because if there exists such a number $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ (where $p_1, p_2, \cdots, p_k$ are all distinct primes and $e_1, e_2, \cdots, e_k$ are all positive integers) then for each integer i from 1 to k, either $p_i^{e_i}$ is counted or the product of all the numbers counted from 1 to $p_i^{e_i} - 1$ is divisible by $p_i^{e_i}$. Therefore, m divides the product of all counted numbers between 1 and $(m-1)$, inclusive, and would not be counted.

We establish the last part by induction, considering the k-th instance of a power of a prime p being counted and inducting on k. The base case for $k = 1$ of $p^{2^0} = p$ was already observed above. Now assume that the only powers of p that were counted are $p^1, p^2, p^4, \ldots, p^{2^{k-1}}$. Then, the next power of p that will be the smallest one that does not divide

$$\begin{aligned} p^1 \cdot p^2 \cdot p^4 \cdots p^{2^{k-1}} &= p^{1+2+4+\cdots+2^{k-1}} \\ &= p^{2^k - 1}. \end{aligned}$$

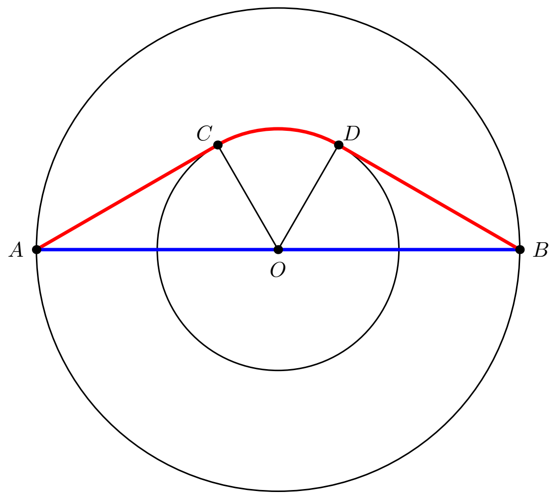
Therefore, the next power of p to be counted is p^{2^k}, proving the claim.

21. Chad and Jordan are having a race in the lake shown below. The lake has a diameter of four kilometers and there is a circular island in the middle of the lake with a diameter of two kilometers. They start at one point on the edge of the lake and finish at the diametrically opposite point. Jordan makes the trip only by swimming in the water, while Chad swims to the island, runs across it, and then continues swimming. They both take the fastest possible route and, amazingly, they tie! Chad swims at two kilometers an hour and runs at five kilometers an hour. At what speed does Jordan swim?



Solution. The answer is $\boxed{\frac{30\sqrt{3}+5\pi}{21}}$.

In the diagram below, Chad's path is shown in blue and Jordan's in red:



Chad swims a total of 2 kilometers and runs 2 kilometers, so he takes $\frac{2}{2} + \frac{2}{5} = \frac{7}{5}$ hours to finish the race. Because Chad and Jordan finish at the same time, this is also the amount of time Jordan requires to swim across the lake.

The distance Jordan swims is $AC + \widehat{CD} + BD$. Because $\triangle ACO$ and $\triangle BDO$ are 30-60-90 triangles, $AC = BD = \sqrt{3}$. Also, $\angle AOC = \angle BOD = 60^\circ$, so $\angle COD = 180^\circ - 60^\circ - 60^\circ = 60^\circ$ as well. Therefore, $\widehat{CD} = \frac{60^\circ}{360^\circ} \cdot 2\pi = \frac{\pi}{3}$. Summing these lengths and dividing by the time, we obtain the speed at which Jordan swims:

$$\frac{\sqrt{3} + \frac{\pi}{3} + \sqrt{3}}{\frac{7}{5}} = \boxed{\frac{30\sqrt{3} + 5\pi}{21}}.$$

22. Cameron has stolen Chad's barrel of oil and is driving it around on a truck on the coordinate grid on his truck. Cameron is a bad truck driver, so he can only move the truck forward one kilometer at a time along one of the gridlines. In fact, Cameron is so bad at driving the truck that between every two one-kilometer movements, he has to turn exactly 90 degrees. After 50 one-kilometer movements, given that Cameron's first one-kilometer movement was westward, how many points he could be on?

    Solution. The answer is $\boxed{650}$.

    Number the movements from 1 to 50, and let Cameron start at $(0, 0)$ on the coordinate plane. Note that odd-numbered movements only affect the x-coordinate, while even-numbered movements only affect the y-coordinate. This means the x- and y-coordinates are independent of each other, so our answer will be the product of the number of possible x values and the number of possible y values. Each movement either increases or decreases the coordinate it affects by one. We have no choice about Cameron's first movement, since he starts facing west and moves to $(-1, 0)$. Thus we have 24 x-coordinate movements and 25 y-coordinate movements to consider. Observe that after 24 x-coordinate movements, Cameron's x-coordinate must be odd, and must be between -25 and 23. This yields $\frac{23-(-25)}{2} + 1 = 25$ possible x-coordinates. After 25 y-coordinate movements, Cameron's y-coordinate must be odd as well, and must be between -25 and 25. Thus Cameron can end up at $\frac{25-(-25)}{2} + 1 = 26$ different y-coordinates. Multiplying these two values gives us $25 \cdot 26 = 650$ as our answer.

23. Let a, b, and c be distinct nonzero base ten digits. Assume there exist integers x and y for which $\overline{abc} \cdot \overline{cb} = 100x^2 + 1$ and $\overline{acb} \cdot \overline{bc} = 100y^2 + 1$. What is the minimum value of the number $\overline{abbc}$?

    Note: The notation $\overline{pqr}$ designates the number whose hundreds digit is p, tens digit is q, and units digit is r, not the product $p \cdot q \cdot r$.

    Solution. The answer is $\boxed{1337}$.

    Note that both $\overline{abc} \cdot \overline{cb}$ and $\overline{acb} \cdot \overline{bc}$ have 01 as their last two digits, which means bc ends in 1. This can only happen in four ways, namely $(b, c) = (1, 1), (3, 7), (7, 3)$, and $(9, 9)$.

    If $(b, c) = (1, 1)$, then the last two digits of both products would be the same as those of $\overline{bc} \times \overline{cb} = 11 \times 11 = 121$, which means both products end with 21, not 01.

    For $(b, c) = (3, 7)$ and $(b, c) = (7, 3)$, a valid value of a for one case is also valid for the other case. Clearly $\overline{a337} < \overline{a773}$, so we only consider the case where $b = 3$ and $c = 7$. We have $37 \cdot 73 = 2701$, so $x^2 = 73a + 27$ and $y^2 = 37a + 27$. Trying $a = 1$ yields $x = 10$ and $y = 8$, and no smaller value of a is allowed, so the smallest value of $\overline{abbc}$ in this case is 1337.

    If $(b, c) = (9, 9)$, then $\overline{abbc} \geq 1999 > 1337$ because $a \geq 1$. So we do not need to consider this case, as it cannot achieve the minimum.

    Therefore, the answer is 1337.

24. Let r_1, r_2, r_3, r_4 and r_5 be the five roots of the equation $x^5 - 4x^4 + 3x^2 - 2x + 1 = 0$. What is the product of $(r_1 + r_2 + r_3 + r_4), (r_1 + r_2 + r_3 + r_5), (r_1 + r_2 + r_4 + r_5), (r_1 + r_3 + r_4 + r_5)$, and $(r_2 + r_3 + r_4 + r_5)$?

    Solution. The answer is $\boxed{41}$.

    Let $s = r_1 + r_2 + r_3 + r_4 + r_5$ and $f(x) = x^5 - 4x^4 + 3x^2 - 2x + 1$. Note that $f(x)$ can also be written as $(x - r_1)(x - r_2)(x - r_3)(x - r_4)(x - r_5)$, whose expansion has an x^4 term with coefficient $-(r_1 + r_2 + r_3 + r_4 + r_5) = -s$. Therefore, we must have $-s = -4$, or $s = 4$.

    Now, we can rewrite the desired product of the quantities as

$$\begin{aligned}(s - r_1)(s - r_2)(s - r_3)(s - r_4)(s - r_5) &= f(s) \\ &= f(4) \\ &= 4^5 - 4 \cdot 4^4 + 3 \cdot 4^2 - 2 \cdot 4 + 1 \\ &= 41.\end{aligned}$$

25. Chad needs seven apples to make an apple strudel for Jordan. He is currently at 0 on the metric number line. Every minute, he randomly moves one meter in either the positive or the negative direction with equal probability. Arctica's parents are located at $+4$ and -4 on the number line. They will bite Chad for kidnapping Arctica if he walks onto those numbers. Also, there is one apple located at each integer between -3 and 3, inclusive. Whenever Chad lands on an integer with an unpicked apple, he picks it. What is the probability that Chad picks all the apples without getting bitten by Arctica's parents?

    Solution. The answer is $\boxed{\frac{1}{7}}$.

    At some point in time, Chad will reach the apple at -3 or 3 (as he can only get bitten after passing one of these points). By symmetry, we may assume that he reaches the apple at -3 first. Then, we may rephrase the probability as that of Chad reaching either 3 before reaching -4. Shifting the numbers by 4 to the right, this could be rephrased as Chad starting at the number 1 and needing to reach 7 before 0.

We use the process of mathematical induction to show that the probability of Chad reaching n before 0 when starting at 1 is precisely $\frac{1}{n}$. This statement is clearly true for $n = 2$. Assume the probability that Chad reaches $n-1$ before 0 when starting at 1 is $\frac{1}{n-1}$. Let the probability that he reaches n before 0 be p. Since he must first reach $n-1$ to ever reach n, we have that $p = \frac{1}{n-1} \cdot (1-p)$, where $1-p$ is the probability that Chad reaches n before 0 when starting at $n-1$, as the situation is exactly reversed. Solving, we arrive at the conclusion that $p = \frac{1}{n}$, completing the induction.

Thus, the answer is $\frac{1}{7}$.

Alternate solution: Again, we can reduce the problem to finding the probability that, starting at 1, Chad reaches 7 before 0. Note that because moving in either direction is equally likely, the expected value of Chad's position is 1 at all times. In particular, when Chad first steps on one of 0 and 7, his expected position is again 1. If his actual position is 7 with probability p, then his expected position can also be written as $7 \cdot +0 \cdot (1-p) = 1$, so $p = \frac{1}{7}$.

2.2 Accuracy Test Solutions

1. Chad lives on the third floor of an apartment building with ten floors. He leaves his room and goes up two floors, goes down four floors, goes back up five floors, and finally goes down one floor, where he finds Jordan's room. On which floor does Jordan live?

   Solution. The answer is $\boxed{5}$.

   Chad finds Jordan's room at floor number $3 + 2 - 4 + 5 - 1 = 5$.

2. A real number x satisfies the equation $2014x + 1337 = 1337x + 2014$. What is x?

   Solution. The answer is $\boxed{1}$.

   Rearranging the terms yields

$$\begin{aligned} 2014x - 1337x &= 2014 - 1337 \\ 677x &= 677 \\ x &= 1. \end{aligned}$$

   Alternate solution: Note that the given equation is linear, so it must have at most one solution. Rewriting the equation as

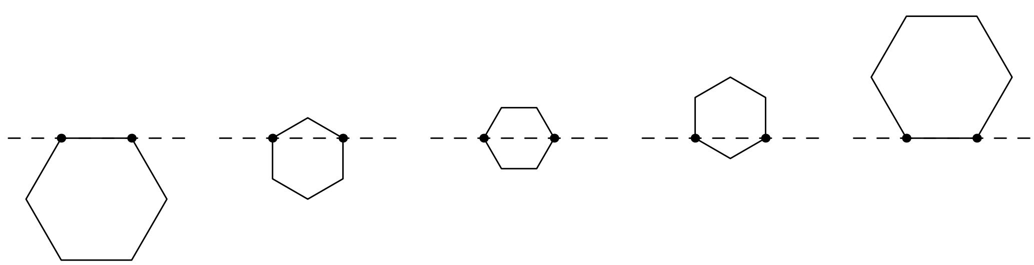
$$2014(x-1) = 1337(x-1),$$

   it is clear that $x = 1$ is a solution, and it must be the only solution.

3. Given two points on the plane, how many distinct regular hexagons include both of these points as vertices?

   Solution. The answer is $\boxed{5}$.

   Consider the line going through the two points. This line could split the other four points of the hexagon so that there are 0, 1, 2, 3, or 4 points above the line. Each way yields one distinct hexagon, as shown below.



   There are five ways in which the line can split the other four points so there are five possible hexagons.

4. Jordan has six different files on her computer and needs to email them to Chad. The sizes of these files are 768, 1024, 2304, 2560, 4096, and 7680 kilobytes. Unfortunately, the email server holds a limit of S kilobytes on the total size of the attachments per email, where S is a positive integer. It is additionally given that all of the files are indivisible. What is the maximum value of S for which it will take Jordan at least three emails to transmit all six files to Chad?

Solution. The answer is $\boxed{9471}$.

Noticing that all the file sizes are multiples of 256 kilobytes, we define one chadbyte to be 256 kilobytes so that the files have sizes 3, 4, 9, 10, 16, and 30 chadbytes. Thus, the total size of all files is $3 + 4 + 9 + 10 + 16 + 30 = 72$ chadbytes.

Note that S must be smaller than the minimum email size limit for which two emails can send all files. This minimum limit must therefore be at least 36 chadbytes, because otherwise, two completely full emails would not be able send all 72 chadbytes. If two emails of size 36 chadbytes could send all of the files, the sizes of the other files on the same email as the 30 chadbyte file would sum to 6 chadbytes, which is clearly impossible. All file sizes have an integral number of chadbytes, so we next check if two emails with sizes at most 37 chadbytes can send all files. Indeed, sending the files of sizes 3, 4, and 30 chadbytes in one email and the rest in the other, achieves this.

Therefore, the email size limit must be smaller than 37 chadbytes to require three emails. Converting back to kilobytes, this means $S < 37 \cdot 256 = 9472$ kilobytes. Because S is an integer, the maximum possible value of S is 9471 kilobytes.

5. If real numbers x and y satisfy $(x + 2y)^2 + 4(x + 2y + 2 - xy) = 0$, what is $x + 2y$?

Solution. The answer is $\boxed{-4}$.

Expanding and rearranging the equation yields

$$\begin{aligned}\left(x^2 + 4xy + 4y^2\right) + (4x + 8y + 8 - 4xy) &= 0\\ \left(x^2 + 4x + 4\right) + 4\left(y^2 + 2y + 1\right) &= 0\\ (x + 2)^2 + 4\,(y + 1)^2 &= 0\end{aligned}$$

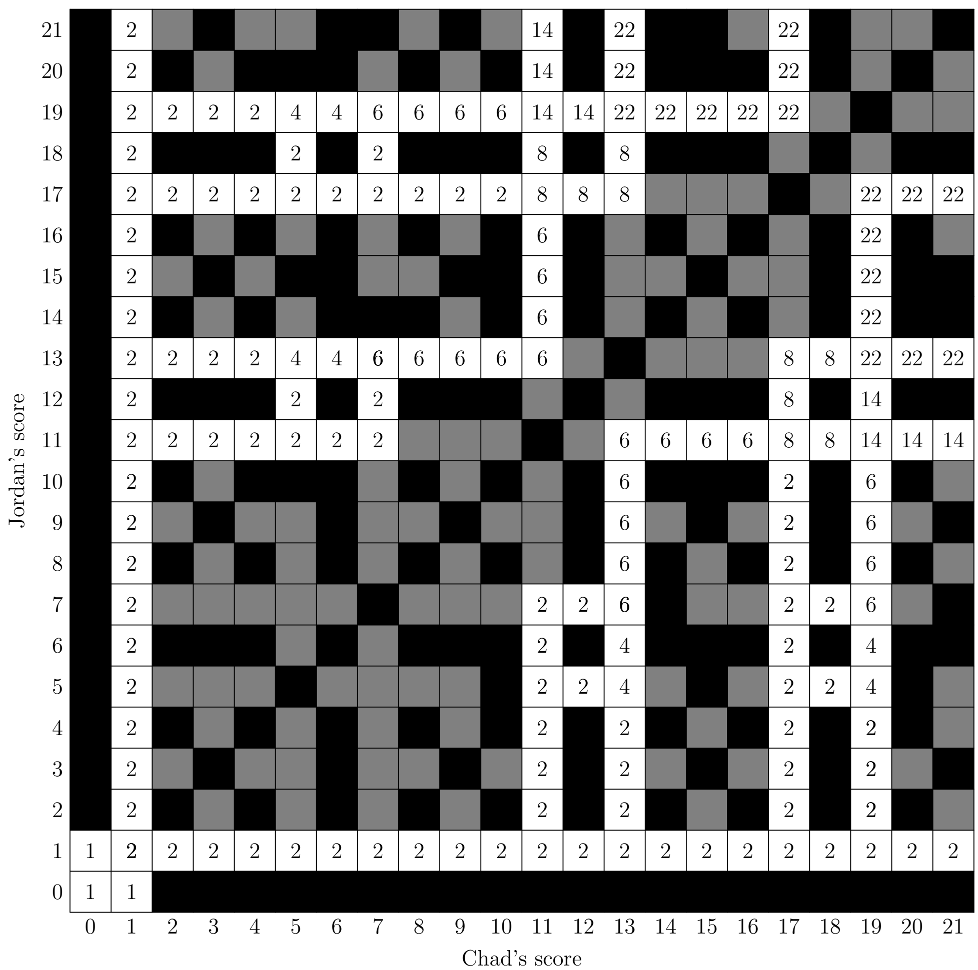
Squares of real numbers are nonnegative, so the only way the equation can be satisfied is if $(x+2)^2 = 0$ and $(y + 1)^2 = 0$. Therefore, $x = -2$ and $y = -1$, so $x + 2y = -4$.

6. While playing table tennis against Jordan, Chad came up with a new way of scoring. After the first point, the score is regarded as a ratio. Whenever possible, the ratio is reduced to its simplest form. For example, if Chad scores the first two points of the game, the score is reduced from 2:0 to 1:0. If later in the game Chad has 5 points and Jordan has 9, and Chad scores a point, the score is automatically reduced from 6:9 to 2:3. Chad's next point would tie the game at 1:1. Like normal table tennis, a player wins if he or she is the first to obtain 21 points. However, he or she does not win if after his or her receipt of the 21^{st} point, the score is immediately reduced. Chad and Jordan start at 0:0 and finish the game using this rule, after which Jordan notes a curiosity: the score was never reduced. How many possible games could they have played? Two games are considered the same if and only if they include the exact same sequence of scoring.

Solution. The answer is $\boxed{120}$.

We create a 22 by 22 grid in which the rows and columns are each numbered from 0 to 21, and each cell represents the score of the game. So, for example, the cell in column 11 and row 5 represents Chad having 11 points and Jordan having 5. Thus, a game without any score reductions is represented by a path from the cell in row 0 and column 0 to any cell in row 21 or column 21 that does not pass through a cell in which the score is reduced. Note that one point is added to one person's score each rally, so moves along the grid consist of either one step in the positive row direction or one step in the positive column direction.

We color black the cells at which the score would be reduced, so that a path may not pass through a black square. We also color grey those cells which are not reachable from the starting score of 0:0, or do not have a valid path to either row 21 or column 21. The remaining squares are colored white. The diagram below shows the resulting grid:



In addition to the colors, we write in each white cell the number of ways in which that score could be reached. We compute these numbers by starting with 1 in the cell in row 0 and column 0. Then, for each empty white cell we write in it the sum of the numbers below it and to its left, if those cells contain numbers. This gives the desired number because a score can only be reached by either increasing Chad's score by one or increasing Jordan's score by one. So, a 1 would be written for the scores 0:1 and 1:0, and then a 2 for the score 1:1, and so on.

Finally, we count the total number of valid games by adding the number of ways one of the players' scores could reach 21. That is, we sum the numbers in row 21 and column 21, which yields $2 \cdot (2 + 14 + 22 + 22) = 120$.

7. For a positive integer m, we define m as a *factorial number* if and only if there exists a positive integer k for which $m = k \cdot (k-1) \cdot ... \cdot 2 \cdot 1$. We define a positive integer n as a *Thai number* if and only if n can be written as both the sum of two factorial numbers and the product of two factorial numbers. What is the sum of the five smallest Thai numbers?

   Solution. The answer is $\boxed{210}$.

   Notice that 2 is a factorial number, so all numbers of the form $2m$, where m is a factorial number, are Thai numbers: $2m$ is the sum of m and m as well as the product of 2 and m. The first few factorial numbers are 1, 2, 6, 24, and 120, from which we can find five Thai numbers: 2, 4, 12, 48, and 240.

   However, we must check if these are the smallest five. We can do this by examining the possible products of two factorial numbers less than 240, and determine if two factorial numbers sum to each product. The products in which the smaller factorial number is 2 were already examined above. Note that no two factorial numbers sum to another factorial number (other than $1 + 1 = 2$, which was accounted for above), so smaller factorial number in the product cannot be 1. The smaller number also cannot be larger than 6, because then the product would be at least $24 \cdot 24 > 240$. If the smaller number is 6, we must check $6 \cdot 6 = 36$, which is not a sum of two factorial numbers, and $6 \cdot 24 = 144$, which is $120 + 24$. Hence, 144 is also a Thai number.

   The smallest five Thai numbers are 2, 4, 12, 48, and 144, which sum to 210.

8. Chad and Jordan are in the Exeter Space Station, which is a triangular prism with equilateral bases. Its height has length one decameter and its base has side lengths of three decameters. To protect their station against micrometeorites, they install a force field that contains all points that are within one decameter of any point of the surface of the station. What is the volume of the set of points within the force field and outside the station, in cubic decameters?

   Solution. The answer is $\boxed{\frac{9\sqrt{3}}{2} + 9 + \frac{41\pi}{6}}$.

   We can split the shape between the Exeter Space Station and the force field into five types of regions:

   - Three rectangular prisms along the three rectangular faces of the Exeter Space Station, with a height of 1, length of 3, and width of 1, each have volume $1 \cdot 3 \cdot 1 = 3$, for a total volume of 9 for this type.
   - Along each of the two triangular faces of the station is a region of the force field in the shape of a triangular prism with height 1 and base side length 3. The total volume for the two regions of this type is $2 \cdot \left(\frac{3^2\sqrt{3}}{4} \cdot 1\right) = \frac{9\sqrt{3}}{2}$.
   - The third type of region consists of six shapes that are quarter sectors of a cylinder with height 3 and radius 1, formed along the edges of the Exeter Space Station that have length 3. This type of region has a total volume of $6 \cdot \left(3 \cdot \frac{\pi}{4}\right) = \frac{9\pi}{2}$.
   - The fourth set of regions consists of three shapes that are each $\frac{360^\circ - 90^\circ - 90^\circ - 60^\circ}{360^\circ} = \frac{120^\circ}{360^\circ} = \frac{1}{3}$ of a cylinder with height 1 and radius 1, found along the edges of the Exeter Space Station with length 1. This type of region has a total volume of $3 \cdot \frac{\pi}{3} = \pi$.
   - The fifth category consists of six regions that have the shape of a 120° slice of a sphere cut in half perpendicular to the the exposed diameter, and are located near the six corners of the station. Together, these regions have volume $6 \cdot \left(\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{4\pi}{3}\right) = \frac{4}{3}\pi$.

Summing these values, the total desired volume is $9 + \frac{9\sqrt{3}}{2} + \frac{9\pi}{2} + \pi + \frac{4\pi}{3} = \frac{9\sqrt{3}}{2} + 9 + \frac{41\pi}{6}$.

2.3 Team Test Solutions

1. What is the units digit of the product of the first seven primes?

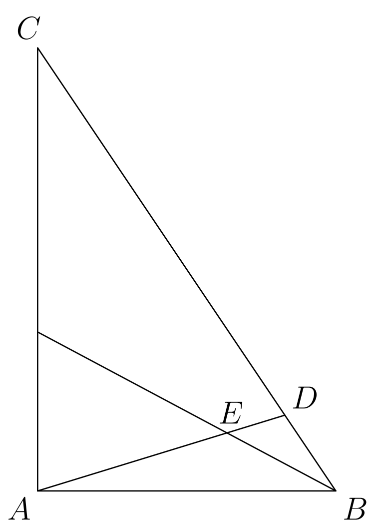
   Solution. The answer is $\boxed{0}$.

   We know that the first three primes are 2, 3, 5, so the product of the first seven primes is divisible by 2 and 5, and therefore is a multiple of 10. Hence, the units digit must be 0.

2. In triangle ABC, $\angle BAC$ is a right angle and $\angle ACB$ measures 34 degrees. Let D be a point on segment BC for which $AC = CD$, and let the angle bisector of $\angle CBA$ intersect line AD at E. What is the measure of $\angle BED$?

   Solution. The answer is $\boxed{45^\circ}$.

   The diagram is as follows:



   Note that $AC = CD$ means $\triangle ACD$ is isosceles, so $\angle ADC = \frac{180^\circ - \angle ACD}{2} = \frac{180^\circ - 34^\circ}{2} = 73^\circ$. Therefore, $\angle BDE = 180^\circ - \angle ADC = 107^\circ$. Also, because BE bisects $\angle ABC$, we have $\angle DBE = \frac{90^\circ - \angle ACB}{2} = 28^\circ$. Finally, considering $\triangle BED$, we have $\angle BED = 180^\circ - \angle BDE - \angle DBE = 180^\circ - 107^\circ - 28^\circ = 45^\circ$.

3. Chad numbers five paper cards on one side with each of the numbers from 1 through 5. The cards are then turned over and placed in a box. Jordan takes the five cards out in random order and again numbers them from 1 through 5 on the other side. When Chad returns to look at the cards, he deduces with great difficulty that the probability that exactly two of the cards have the same number on both sides is p. What is p?

   Solution. The answer is $\boxed{\frac{1}{6}}$.

   There are $\binom{5}{2} = 10$ possibilities for the set two numbers that are written on the pair of cards that have the same number on both sides. For each of those possibilities, we have 3 mismatched numbers a, b, and c written on the front of three cards with 2 ways to write numbers on the backs, in order: b,

c, and a; or c, a, and b. In total, there are $5! = 120$ ways to write the five numbers on the back of the cards, so the probability is $p = \frac{10\cdot 2}{120} = \frac{1}{6}$.

4. Only one real value of x satisfies the equation $kx^2 + (k+5)x + 5 = 0$. What is the product of all possible values of k?

   Solution. The answer is $\boxed{0}$.

   If $k = 0$, the equation becomes $5x + 5 = 0$, which clearly has exactly one solution, $x = -1$. Because 0 is among the possible values of k, the desired product is 0.

   While it doesn't affect the answer, we can find all other possible values for $k \neq 0$ by noting that for the given equation to have one solution, the discriminant of the quadratic, $(k+5)^2 - 4(k)(5) = (k-5)^2$, must be 0. Therefore, $k = 5$ is the only other possible value of k.

5. On the Exeter Space Station, where there is no effective gravity, Chad has a geometric model consisting of 125 wood cubes measuring 1 centimeter on each edge arranged in a 5 by 5 by 5 cube. An aspiring carpenter, he practices his trade by drawing the projection of the model from three views: front, top, and side. Then, he removes some of the original 125 cubes and redraws the three projections of the model. He observes that his three drawings after removing some cubes are identical to the initial three. What is the maximum number of cubes that he could have removed? (Keep in mind that the cubes could be suspended without support.)

   Solution. The answer is $\boxed{100}$.

   Note that in each view, 25 cubes are observed, so there must be at least 25 cubes in the model with some cubes removed. We can show that 25 is indeed the minimum number of cubes required, which would make the number of cubes removed and the answer 100, by constructing a valid model with 25 cubes, as follows.

   Group the 125 original cubes into five layers of 5 by 5 squares, and number the layers from 1 to 5, so that only one layers is visible when the original cube is viewed from the top. Then, remove all cubes except those specified in the diagram below:

2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4
1	2	3	4	5

   In the grid, each cell represents one cube. The number is the layers number of the cell and the position in the grid is the same as the position of the cube as viewed from the top.

   This construction is clearly valid for the top view, as 25 cubes are visible. For the other views, we must check that there is one of each number (which represents row number in the 5 by 5 by 5 arrangement) in each row and column of the grid. This is indeed true, so our construction is valid.

Alternate solution: By the same reasoning as the previous solution, we must have at least 25 cubes, but we provide a different description of a valid construction for removing 100 cubes.

Align the original 5 by 5 by 5 cube on a three-dimensional coordinate grid so that there is a cube at each point (x, y, z) for all x, y, and z between 0 and 4, inclusive. Then, remove all cubes except those for which the sum of their coordinates is divisible by 5. For each of the three views to have 25 cubes, there must be a cube located at one of the five points with two fixed coordinates. For example, there must be a cube at a point $(1, 3, z)$, for some z, if a cube is observed in the corresponding spot when viewed parallel to the z-axis. Our construction satisfies this because for any given x and y, exactly one of $(x, y, -x-y)$, $(x, y, 5-x-y)$, and $(x, y, 10-x-y)$ is kept. The same is true for the other two views, so this construction satisfies the conditions. There will be 25 cubes remaining because for each choice of x and y, there is a unique z for which $5|x+y+z$.

6. Eric, Meena, and Cameron are studying the famous equation $E = mc^2$. To memorize this formula, they decide to play a game. Eric and Meena each randomly think of an integer between 1 and 50, inclusively, and substitute their numbers for E and m in the equation. Then, Cameron solves for the absolute value of c. What is the probability that Cameron's result is a rational number?

Solution. The answer is $\boxed{\frac{69}{1250}}$.

There exist integers p, q, r, and s such that $E = p \cdot q^2$ and $m = r \cdot s^2$ and p and r have no perfect square divisors other than 1. Then, we have

$$c = \sqrt{\frac{E}{m}} = \frac{\sqrt{p \cdot q^2}}{\sqrt{r \cdot s^2}} = \frac{q}{s} \cdot \sqrt{\frac{p}{r}},$$

so c is rational if and only if $\sqrt{\frac{p}{r}}$ is rational. Suppose $\sqrt{\frac{p}{r}} = \frac{a}{b}$, where a and b are relatively prime integers. Therefore,

$$pb^2 = ra^2,$$

so $a^2 \mid p$ and $b^2 \mid r$. Thus, if c is rational we must have $a = b = 1$, which means $p = r$.

Now, for each possible value p, any choice for E and m from the numbers between 1 and 50 that are exactly p times a perfect square produces a rational c. We proceed with casework on the value of p:

- If $p = 1$, we choose E and m from 1, 4, 9, 16, 25, 36, and 49, which we can do in $7^2 = 49$ ways.
- If $p = 2$, we can choose E and m from 2, 8, 18, 32, and 50, in $5^2 = 25$ ways.
- If $p = 3$, we can choose E and m from 3, 12, 27, and 48 in $4^2 = 16$ ways.
- If $p = 5$, we can choose E and m from 5, 20, and 45 in $3^2 = 9$ ways.
- For $p = 6$, 7, 10, and 11, we can choose E and m from p and $4p$ in $2^2 = 4$ ways for each p.
- For each of the 23 values of p between 12 and 50 that do not have perfect square divisors, there is one choice for each of E and m, namely p.

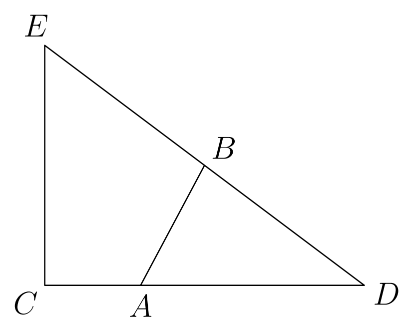
In total, there are $49+25+16+9+4\cdot4+23\cdot1 = 138$ possibilities for E and m so that c is rational. Also, $50^2 = 2500$ ways exist to assign values to E and m without considering c, so the desired probability is $\frac{138}{2500} = \frac{69}{1250}$.

7. Let CDE be a triangle with side lengths $EC = 3$, $CD = 4$, and $DE = 5$. Suppose that points A and B are on the perimeter of the triangle such that line AB divides the triangle into two polygons of equal area and perimeter. What are all the possible values of the length of segment AB?

Solution. The answer is $\boxed{2\sqrt{3}}$.

Note that there are three possible configurations for AB, depending on which side contains neither A nor B. We consider each separately.

Suppose CE does not contain either endpoint of AB. Then the diagram looks like the following:



Let $DA = p$ and $DB = q$. We have

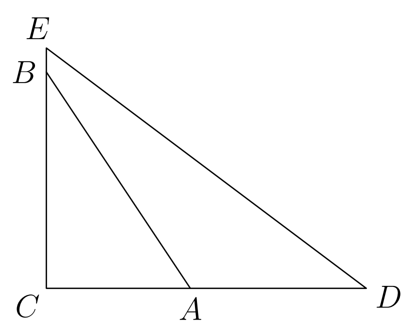
$$[DAB] = \frac{p}{4} \cdot [CBD] = \frac{q}{5} \cdot \frac{p}{4} \cdot [CDE] = \frac{pq}{20} \cdot 2[DAB],$$

or $pq = 10$. Also, due to the perimeter condition, we have

$$p + q + AB = AB + BE + EC + CA = AB + (12 - p - q),$$

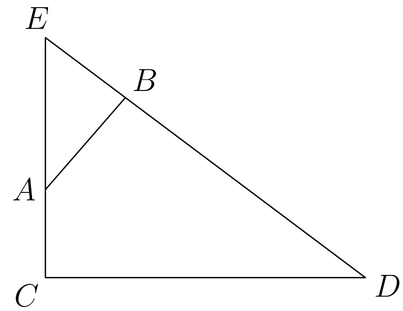
so $p+q = 6$. Substituting for q, this yields $p(6-p) = 10$, or $(p-3)^2+1 = 0$, which has no real solutions.

Now consider the configuration as depicted in the diagram below:



Let $CA = r$ and $CB = s$. Using a similar argument as above, we can derive the equations $r + s = 6$ and $rs = \frac{3 \cdot 4}{2} = 6$. Solving for r and s yields $(r, s) = (3 + \sqrt{3}, 3 - \sqrt{3}), (3 - \sqrt{3}, 3 + \sqrt{3})$. However, $3 + \sqrt{3} > 4 > 3$, so in either case either A or B lies outside the perimeter of $\triangle CDE$, so this configuration does not produce a valid AB.

Finally, consider the third possible arrangement:



Let $BE = x$ and $CE = y$. Again, we can find that $x + y = 6$ and $xy = \frac{15}{2}$. Solving yields $\{x, y\} = \left\{3 + \frac{\sqrt{6}}{2}, 3 - \frac{\sqrt{6}}{2}\right\}$ (either order for x and y is valid). By the Law of Cosines, we find that

$$\begin{aligned} AB &= \sqrt{\left(3 + \frac{\sqrt{6}}{2}\right)^2 + \left(3 - \frac{\sqrt{6}}{2}\right)^2 - 2\left(3 + \frac{\sqrt{6}}{2}\right)\left(3 - \frac{\sqrt{6}}{2}\right)\cos\angle CED} \\ &= \sqrt{2 \cdot 3^2 + 2 \cdot \left(\frac{\sqrt{6}}{2}\right)^2 - 2\left(9 - \frac{6}{4}\right)\frac{3}{5}} \\ &= \sqrt{21 - 9} \\ &= 2\sqrt{3}, \end{aligned}$$

which is the only possible value of AB.

8. Chad and Jordan are raising bacteria as pets. They start out with one bacterium in a Petri dish. Every minute, each existing bacterium turns into 0, 1, 2 or 3 bacteria, with equal probability for each of the four outcomes. What is the probability that the colony of bacteria will eventually die out?

Solution. The answer is $\boxed{\sqrt{2} - 1}$.

Suppose that the probability of a colony eventually dying out is x. Then for a colony with n bacteria, the probability of eventually dying out is x^n because we can consider each single bacterium as a separate colony. Comparing bacteria before and after the first minute, we have

$$\begin{aligned} x &= \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot x + \frac{1}{4} \cdot x^2 + \frac{1}{4} \cdot x^3 \\ x^3 + x^2 - 3x + 1 &= 0 \\ (x - 1)(x^2 + 2x - 1) &= 0 \\ x &= 1, -1 - \sqrt{2}, \text{ or } -1 + \sqrt{2} \end{aligned}$$

We claim that x cannot be 1, which would mean that all colonies eventually die out. The number of bacteria in the colony is, on average, multiplied by $\frac{0+1+2+3}{4} = 1.5$ every minute, which means in general the bacteria do not die out. (A more rigorous line of reasoning is included below.) Because x is not negative, the only valid solution is $x = \sqrt{2} - 1$.

To show that x cannot be 1, we show that it is at most $\sqrt{2}-1$. Let x_n be the probability that a colony of one bacteria will die out after at most n minutes. Then, we have the relation

$$x_{n+1} = \frac{1}{4}\left(1 + x_n + x_n^2 + x_n^3\right).$$

We claim that $x_n \le \sqrt{2}-1$ for all n, which we will prove using induction.

It is clear that $x_1 = \frac{1}{4} \le \sqrt{2}-1$. Now, assume $x_k \le \sqrt{2}-1$ for some k. We have

$$\begin{aligned} x_{k+1} &\le \frac{1}{4}\left(1 + x_k + x_k^2 + x_k^3\right) \\ &\le \frac{1}{4}\left(1 + \left(\sqrt{2}-1\right) + \left(\sqrt{2}-1\right)^2 + \left(\sqrt{2}-1\right)^3\right) \\ &= \sqrt{2}-1, \end{aligned}$$

which completes the proof that $x_n \le \sqrt{2}-1$ for all n.

Now, we note that as n becomes large, x_n approaches x. Using formal notation, this is $x = \lim_{n\to\infty} x_n \le \sqrt{2}-1$, so x cannot be 1.

9. Let $a = w+x, b = w+y, c = x+y, d = w+z, e = x+z$, and $f = y+z$. Given that $af = be = cd$ and

$$(x-y)(x-z)(x-w) + (y-x)(y-z)(y-w) + (z-x)(z-y)(z-w) + (w-x)(w-y)(w-z) = 1,$$

what is

$$2(a^2+b^2+c^2+d^2+e^2+f^2) - ab - ac - ad - ae - bc - bd - bf - ce - cf - de - df - ef?$$

Solution. The answer is $\boxed{3}$.

Let $af = be = cd = p$ and $a+f = b+e = c+d = q$. Then a and f are distinct roots of the polynomial $t^2 - qt + p = 0$. Similarly, so are b and e, as well as c and d. Therefore, $\{a, f\} = \{b, e\} = \{c, d\}$.

As x, y, z, and w are symmetric, we may assume without loss of generality that $x \le y \le z \le w$. By the second equation, x, y, z, and w are not all equal, so $x \ne w$. Then $c = x+y$ cannot equal $b = w+y$, so $c = e$, which means $x+y = x+z$, or $y = z$. Now, if $c = a$, we have $y = z = w$; otherwise $c = f$ and $x = y = z$. In the first case, the given equation yields $(x-y)^3 = 1$, implying that $x - y = 1$, contradicting $x \le y$. In the second case, $(w-x)^3 = 1$, implying that $w - x = 1$. Thus, we must have $x = y = z = w - 1$.

Now, the value we need to compute simplifies to

$$\begin{aligned} &\frac{1}{2}\cdot\left[(a-b)^2 + (a-c)^2 + (a-d)^2 + (a-e)^2 + (b-c)^2 + (b-d)^2\right. \\ &\quad \left. + (b-f)^2 + (c-e)^2 + (c-f)^2 + (d-e)^2 + (d-f)^2 + (e-f)^2\right] \\ &\quad = (x-y)^2 + (x-z)^2 + (x-w)^2 + (y-z)^2 + (y-w)^2 + (z-w)^2 \\ &\quad = 3. \end{aligned}$$

10. If a and b are integers at least 2 for which $a^b - 1$ strictly divides $b^a - 1$, what is the minimum possible value of ab?

Note: If x and y are integers, we say that x *strictly* divides y if x divides y and $|x| \ne |y|$.

Solution. The answer is $\boxed{32}$.

We claim that the pair $(a, b) = (16, 2)$, with product 32, satisfies the above conditions, and that no pair with smaller product can satisfy those conditions.

If $(a, b) = (16, 2)$, we have $a^b - 1 = 16^2 - 1 = 2^8 - 1$, while $b^a - 1 = 2^{16} - 1 = (2^8 - 1)(2^8 + 1)$, so it is clear that $16^2 - 1$ strictly divides $2^{16} - 1$.

We prove that no pair with smaller product can satisfy the conditions.

If $3 \le a \le b$, we have $a^b - 1 \ge b^a - 1$, contrary to the strict divisibility condition. Therefore, either $a = 2$ or $a > b$. If $a = 2$, we again have $a^b - 1 \ge b^a - 1$ for $b \ge 4$, and we can check that $b = 2$ and $b = 3$ do not satisfy the conditions. Thus, we may limit our considerations to when $a > b$. In addition, $b^2 < ab < 32$, so $b < 6$.

When $b = 2$, we have that $a^2 - 1$ strictly divides $2^a - 1$. Note that a must be even, because otherwise an even number would divide an odd number. It is easy to check that none of $a = 6$, 8, 10, 12, and 14 satisfies the conditions.

When $b = 3$, we can similarly check the values for a: The values $a = 4$, 5, 6, 7, 8, 9, and 10 do not satisfy the conditions.

When $b = 4$, we have that a is even, but $a = 4$ does not satisfy the conditions.

When $b = 5$, we have that $a = 6$ does not satisfy the conditions.

We have accounted for all cases where $ab \le 31$, so we are done.

2.4 Guts Test Solutions

2.4.1 Round 1

1. What is $2 + 22 + 1 + 3 - 31 - 3$?

   Solution. The answer is $\boxed{-6}$.

   Canceling out two terms, we have $(2 + 22 + 1 - 31) + (3 - 3) = -6$.

2. Let $ABCD$ be a rhombus. Given $AB = 5$, $AC = 8$, and $BD = 6$, what is the perimeter of the rhombus?

   Solution. The answer is $\boxed{20}$.

   Because $ABCD$ is a rhombus, all four of its sides have the same length. We are given that one side, AB, has length 5, so the perimeter is $4 \cdot 5 = 20$.

3. There are 2 hats on a table. The first hat has 3 red marbles and 1 blue marble. The second hat has 2 red marbles and 4 blue marbles. Jordan picks one of the hats randomly, and then randomly chooses a marble from that hat. What is the probability that she chooses a blue marble?

   Solution. The answer is $\boxed{\frac{11}{24}}$.

   There is a $\frac{1}{2}$ probability that Jordan chooses the first hat. If she chooses the first hat, there is a $\frac{1}{4}$ probability that she chooses a blue marble. Otherwise, she can choose the second hat with probability $\frac{1}{2}$, in which case there is a $\frac{4}{6}$ probability that she chooses a blue marble. Therefore, the total probability that she chooses a blue marble is $\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{4}{6} = \frac{3}{24} + \frac{8}{24} = \frac{11}{24}$.

2.4.2 Round 2

4. There are twelve students seated around a circular table. Each of them has a slip of paper that they may choose to pass to either their clockwise or counterclockwise neighbor. After each person has transferred their slip of paper once, the teacher observes that no two students exchanged papers. In how many ways could the students have transferred their slips of paper?

   Solution. The answer is $\boxed{2}$.

   Suppose one of the students passes to the left. Then the student to that student's left must have also passed to the left, and then the student to this student's left must have also passed the left, and so on. Therefore, all students passed their slip to their left. The same applies if the original student passed the slip to the right, so in total there are 2 ways to pass the slips.

5. Chad wants to test David's mathematical ability by having him perform a series of arithmetic operations at lightning-speed. He starts with the number of cubic centimeters of silicon in his 3D printer, which is 109. He has David perform all of the following operations in series each second:

   - Double the number
   - Subtract 4 from the number

- Divide the number by 4
- Subtract 5 from the number
- Double the number
- Subtract 4 from the number

Chad instructs David to shout out after three seconds the result of three rounds of calculations. However, David computes too slowly and fails to give an answer in three seconds. What number should David have said to Chad?

Solution. The answer is $\boxed{61}$.

Suppose David starts the series of operations with a number n. After the six operations, the number is

$$2\left(\frac{2n-4}{4}-5\right)-4=n-16,$$

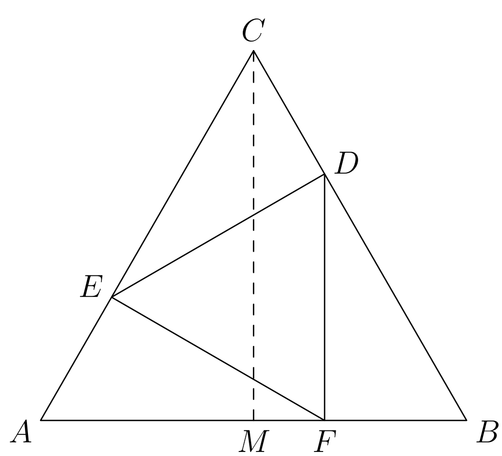
so the series of operations is equivalent to subtracting 16 from the number.

In each of the three rounds of calculations, 16 is subtracted from David's number. Thus, Davids number, originally 109, becomes $109-16\cdot 3=61$.

6. Points D, E, and F lie on sides BC, CA, and AB of triangle ABC, respectively, such that the following length conditions are true: $CD = AE = BF = 2$ and $BD = CE = AF = 4$. What is the area of triangle ABC?

   Solution. The answer is $\boxed{9\sqrt{3}}$.

   The diagram is as follows:



Adding $CD = AE = BF = 2$ and $BD = CE = AF = 4$, we have $BC = AC = AB = 2 + 4 = 6$. This means that $\triangle ABC$ is an equilateral triangle with side length 6.

To find the area of $\triangle ABC$, we find the lengths of an altitude and its corresponding base. Let M be the point on AB such that CM and AB are perpendicular. Then, we have $\angle CMA = \angle CMB = 90^\circ$ and $\angle CAM = \angle CBM = 60^\circ$. Together with $CM = CM$, this shows that $\triangle CAM \cong \triangle CBM$, so $AM = BM = \frac{AB}{2} = 3$. By the Pythagorean Theorem, we then have $CM = \sqrt{6^2-3^2} = 3\sqrt{3}$. Thus, the area of $\triangle ABC$ is $\frac{AB\cdot CM}{2} = \frac{6\cdot 3\sqrt{3}}{2} = 9\sqrt{3}$.

2.4.3 Round 3

7. In the *2, 3, 5, 7 game*, players count the positive integers, starting with 1 and increasing, which do not contain the digits 2, 3, 5, and 7, and also are not divisible by the numbers 2, 3, 5, and 7. What is the fifth number counted?

   Solution. The answer is $\boxed{61}$.

   Following the problem's conditions, we find that the first five numbers counted are 1, 11, 19, 41, and 61.

8. If A is a real number for which
$$19 \cdot A = \frac{2014!}{1! \cdot 2! \cdot 2013!},$$
   what is A?

   Note: The expression $k!$ denotes the product $k \cdot (k-1) \cdot ... \cdot 2 \cdot 1$.

   Solution. The answer is $\boxed{53}$.

   We have
$$A = \frac{2014!}{2 \cdot 19 \cdot 2013!} = \frac{2014 \cdot 2013!}{2 \cdot 19 \cdot 2013!} = \frac{2014}{2 \cdot 19} = 53.$$

9. What is the smallest number that can be written as both $x^3 + y^2$ and $z^3 + w^2$ for positive integers x, y, z, and w with $x \neq z$?

   Solution. The answer is $\boxed{17}$.

   We can see that $17 = 3^2 + 2^3 = 4^2 + 1^3$ satisfies the conditions. After checking that all numbers less than 17 cannot be expressed in this form in two ways, we conclude that 17 is the answer.

2.4.4 Round 4

Each of the three problems in this round depends on the answer to one of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.

In addition, it is given that the answer to each of the following problems is a positive integer less than or equal to the problem number.

10. Let B be the answer to problem 11 and let C be the answer to problem 12. What is the sum of a side length of a square with perimeter B and a side length of a square with area C?

    Solution. The answer is $\boxed{5}$.

    The perimeter of a square is four times its side length, so the side length of the first square is $\frac{B}{4}$. In the second square, C must be the square of the side length, so the side length is $\sqrt{C}$. Adding, the answer to this problem is $A = \frac{B}{4} + \sqrt{C}$.

    After determining C, the equations from problems 10 and 11 form a system of equations, which when solved yields $A = 5$.

11. Let A be the answer to problem 10 and let C be the answer to problem 12. What is $(C-1)(A+1)-(C+1)(A-1)$?

    Solution. The answer is $\boxed{8}$.

    If the answer to this problem is B, we have

    $$\begin{aligned} B &= (C-1)(A+1)-(C+1)(A-1) \\ &= (CA-A+C-1)-(CA+A-C-1) \\ &= 2(C-A). \end{aligned}$$

    After determining C, the equations from problems 10 and 11 form a system of equations, which when solved yields $B = 8$.

12. Let A be the answer to problem 10 and let B be the answer to problem 11. Let x denote the positive difference between A and B. What is the sum of the digits of the positive integer $9x$?

    Solution. The answer is $\boxed{9}$.

    Using the divisibility rule for 9, we can conclude that because $9x$ is divisible by 9, the sum of the digits of $9x$ must also be divisible by 9. The only positive integer less than or equal to 12 that is divisible by 9 is 9, so the answer to this question is 9.

    The divisibility rule for 9 states that a number is divisible by 9 if and only if the sum of its digits is also divisible by 9. This is true because

    $$\overline{a_n a_{n-1} \cdots a_1 a_0} = 10^n \cdot a_n + \cdots + 10 \cdot a_1 + a_0$$

    and $10^n, \cdots, 1$ all have remainder 1 when divided by 9, so we can rewrite the equation as

    $$\overline{a_n a_{n-1} \cdots a_1 a_0} = 9\left(\frac{10^n - 1}{9} a_n + \frac{10^{n-1} - 1}{9} a_{n-1} + \cdots\right) + a_n + \cdots + a_0.$$

    From this it is clear that $\overline{a_n a_{n-1} \cdots a_1 a_0}$ and $a_n + \cdots + a_0$ have the same remainder when divided by 9.

2.4.5 Round 5

13. Five different schools are competing in a tournament where each pair of teams plays at most once. Four pairs of teams are randomly selected and play against each other. After these four matches, what is the probability that Chad's and Jordan's respective schools have played against each other, assuming that Chad and Jordan come from different schools?

    Solution. The answer is $\boxed{\frac{2}{5}}$.

    In this round robin, there are $\binom{5}{2} = 10$ possible matches. The probability that the first match is not between Jordan's and Chad's teams is $\frac{9}{10}$. Similarly, the probability that Chad doesn't play Jordan in the second round is $\frac{8}{9}$. Considering all four matches, the probability that the two schools that

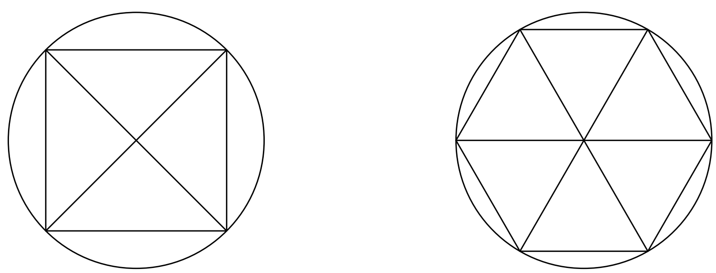
Chad and Jordan come from have not played against each other is $\frac{9}{10} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{6}{7} = \frac{6}{10} = \frac{3}{5}$. Therefore, the probability that they have played against each other is $1 - \frac{3}{5} = \frac{2}{5}$.

Alternate solution: There are a total of $\binom{5}{2} = 10$ pairs of teams, so there are a total of $\binom{10}{4} = 210$ possible sets of four pairs of teams. If Chad's and Jordan's respective schools play against each other, there will be a total of $\binom{9}{3} = 84$ choices for the other three pairs of teams, since Chad's and Jordan's schools can no longer play against each other. Therefore, the probability that they play against each other is $\frac{84}{210} = \frac{2}{5}$.

14. A square of side length 1 and a regular hexagon are both circumscribed by the same circle. What is the side length of the hexagon?

Solution. The answer is $\boxed{\frac{\sqrt{2}}{2}}$.

If we connect the center of the circle with all vertices on the square and the hexagon, we get 4 right isosceles triangles in the square and 6 equilateral triangles in the hexagon.



Because the side length of the square is 1, the radius of the circle would be $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Because the side length of the hexagon is equal to the radius, it would also be $\frac{\sqrt{2}}{2}$.

15. From the list of integers $1, 2, 3, \ldots, 30$, Jordan can pick at least one pair of distinct numbers such that none of the 28 other numbers are equal to the sum or the difference of this pair. Of all possible such pairs, Jordan chooses the pair with the least sum. Which two numbers does Jordan pick?

Solution. The answer is $\boxed{\text{11 and 22}}$.

Call the numbers Jordan picks x and y with $x > y$. Because $x - y$ is an integer between 1 and x, and no integers other than y could be equal to this value, we must have $x - y = y$, or $x = 2y$. Also, $x + y > x > y$, so $x + y$ must be greater than 30; otherwise an integer in this set would be equal to this sum. So we also have $2y + y > 30$, or $y \geq 11$. Thus, the sum is $x + y = 2y + y \geq 33$, so the smallest possible sum is 33, which is produced by the pair $(x, y) = (22, 11)$.

2.4.6 Round 6

16. What is the sum of all two-digit integers with no digit greater than four whose squares also have no digit greater than four?

Solution. The answer is $\boxed{106}$.

The first digit of such a number cannot be greater than four, so it must be 1, 2, 3, or 4. Similarly, the last digit is limited to 0, 1, 2, 3, and 4. However, if the last digit is 3 or 4, then the last digit of the square is greater than four. Thus, we only have to check the 12 numbers with first digits 1, 2, 3, and 4, and last digits 0, 1, and 2. We find that six of these numbers have squares with no digits greater than four: $10^2 = 100$, $11^2 = 121$, $12^2 = 144$, $20^2 = 400$, $21^2 = 441$, and $32^2 = 1024$. The desired sum is $10 + 11 + 12 + 20 + 21 + 32 = 106$.

17. Chad marks off ten points on a circle. Then, Jordan draws five chords under the following constraints:

- Each of the ten points is on exactly one chord.
- No two chords intersect.
- There do not exist (potentially non-consecutive) points A, B, C, D, E, and F, in that order around the circle, for which AB, CD, and EF are all drawn chords.

In how many ways can Jordan draw these chords?

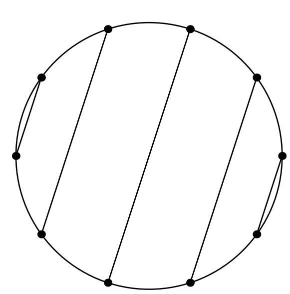
Solution. The answer is $\boxed{5}$.

Note that each chord must divide the other eight points of the circle into two groups each with an even number of points. Otherwise, there would be an unpaired point on one side of that chord that must be the endpoint of a different drawn chord that intersects that chord. Thus, each chord cuts off 0, 2, or 4 other points on the side with fewer points.

Consider the number of chords between pairs of adjacent points. If three or more such chords exist, they clearly contradict the third constraint. Thus, at most two of the chords connect adjacent points.

Now consider the chords that cut off 2 points. For each of these chords, the two points cut off are adjacent and must be connected by a chord. Therefore, we cannot have more than two chords cutting off 2 points, as then we will also have more than two chords cutting off 0 points, which we previously stated was contradictory to the constraints. As a result, there can only be at most 2 chords cutting off 2 points.

Because there are 5 chords in total, there must be at least one chord splitting the other eight points into two groups of four. This chord can be chosen in 5 ways. Once it is chosen, the four points on either side can either be connected by two chords each cutting off 0 points, or one chord cutting off 2 and one chord cutting off 0 points. However, connecting either group of four with two chords that join adjacent points produces three chords that connect adjacent points, as there must be at least one such chord on the other side of the longest chord. Thus, the two chords on either side have to cut off 2 and 0 points, respectively, and there is only one way to draw the five chords in each of the 5 ways to choose the first and longest chord.



18. Chad is thirsty. He has 109 cubic centimeters of silicon and a 3D printer with which he can print a cup to drink water in. He wants a silicon cup whose exterior is cubical, with five square faces and an open top, that can hold exactly 234 cubic centimeters of water when filled to the rim in a rectangular-box-shaped cavity. Using all of his silicon, he prints a such cup whose thickness is the same on the five faces. What is this thickness, in centimeters?

    Solution. The answer is $\boxed{\frac{1}{2}}$.

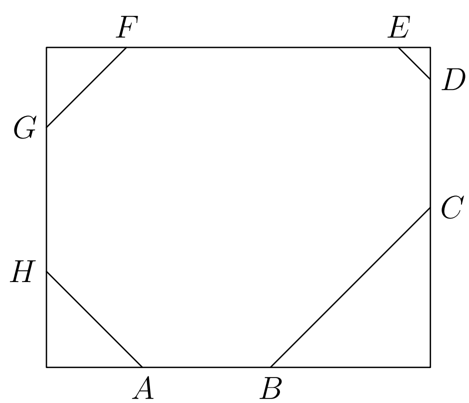
    The total volume of the cup (including the cavity) is $109 + 234 = 343 = 7^3$, so the cup has an outer side length of 7 centimeters. Suppose the thickness of the cup is x centimeters. Then, we have $(7-x)(7-2x)(7-2x) = 234 = 6.5 \cdot 6 \cdot 6$. Hence, $x = \frac{1}{2}$ is a solution, and our answer. Furthermore, we can see that the expression $(7-x)(7-2x)(7-2x)$ is strictly decreasing when x is within the limits of the problem; that is, $0 \le x \le \frac{7}{2}$, so no other sensible solutions exist.

2.4.7 Round 7

19. Jordan wants to create an equiangular octagon whose side lengths are exactly the first 8 positive integers, so that each side has a different length. How many such octagons can Jordan create?

    Solution. The answer is $\boxed{0}$.

    We know that all interior angles of such an octagon would measure 135°. Call the octagon $ABCDEFGH$ and extend sides AB, CD, EF, and GH into lines, making a rectangle which consists of the octagon and 4 isosceles right triangles.



Suppose the lengths of AB, BC, CD, DE, EF, FG, GH, HA are a, b, c, d, e, f, g, and h, respectively. These 8 sides all have integer lengths and none of them are equal. Consider the two sides of the rectangle that contain AB and EF. These two sides have equal length, so

$$a + \frac{\sqrt{2}}{2}(h+b) = e + \frac{\sqrt{2}}{2}(d+f).$$

Since a, h, b, e, d, f are all integers and $\frac{\sqrt{2}}{2}$ is irrational, we see that the two sides must have equal rational parts and equal irrational parts. Therefore we get $a = e$, which contradicts the fact that no two of the 8 sides are equal. Thus Jordan cannot create any such octagons.

20. There are two positive integers on the blackboard. Chad computes the sum of these two numbers and tells it to Jordan. Jordan then calculates the sum of the greatest common divisor and the least common multiple of the two numbers, and discovers that her result is exactly 3 times as large as the number Chad told her. What is the smallest possible sum that Chad could have said?

    Solution. The answer is $\boxed{12}$.

    Suppose the two integers on the board are a and b, with $a \geq b$. Let their greatest common divisor be d. The two integers can then be expressed as $a = pd, b = qd$, where p, q are relatively prime integers and $p \geq q$. So the least common multiple of a and b is pqd. Therefore, we have

$$\begin{aligned} pqd + d &= 3(pd + qd) \\ pq - 3p - 3q + 1 &= 0 \\ (p-3)(q-3) &= 8. \end{aligned}$$

    Because $p \geq q$ we have $p - 3 \geq q - 3$, so the possible solutions are $(p-3, q-3) = (8,1), (4,2), (-2,-4)$, and $(-1,-8)$. We see that the last two make q negative, which is invalid. The other two yield $(p, q) = (11, 4)$, and $(7, 5)$. Since the sum of the two integers is $(p+q)d$, the minimum is achieved when $d = 1$ and $p + q$ is the smallest. We have $11 + 4 > 7 + 5 = 12$, so the smallest sum is 12.

21. Chad uses *yater* to measure distances, and knows the conversion factor from yaters to meters precisely. When Jordan asks Chad to convert yaters into meters, Chad only gives Jordan the result rounded to the nearest integer meters. At Jordan's request, Chad converts 5 yaters into 8 meters and 7 yaters into 12 meters. Given this information, how many possible numbers of meters could Jordan receive from Chad when requesting to convert 2014 yaters into meters?

    Solution. The answer is $\boxed{116}$.

    Suppose 1 yater is exactly y meters long. From the information we know, we have:

$$7.5 \leq 5y < 8.5$$

$$11.5 \leq 7y < 12.5$$

    Simplifying, we have $\frac{3}{2} \leq y < \frac{17}{10}$ and $\frac{23}{14} \leq y < \frac{25}{14}$. Because $\frac{17}{10} < \frac{25}{14}$ and $\frac{3}{2} < \frac{23}{14}$, we can combine the inequalities:

$$\frac{23}{14} \leq y < \frac{17}{10}$$

    Therefore,

$$\frac{23}{14} \cdot 2014 \leq 2014y < \frac{17}{10} \cdot 2014.$$

    Now, $\frac{23}{14} \cdot 2014 = \frac{23 \cdot 1007}{7} = \frac{23161}{7} = 3308\frac{5}{7}$, and $\frac{17}{10} \cdot 2014 = \frac{34238}{10} = 3423.8 = 3423\frac{4}{5}$, so

$$3308.5 < \frac{23}{14} \cdot 2014 \leq 2014y < \frac{17}{10} \cdot 2014 < 3424.5.$$

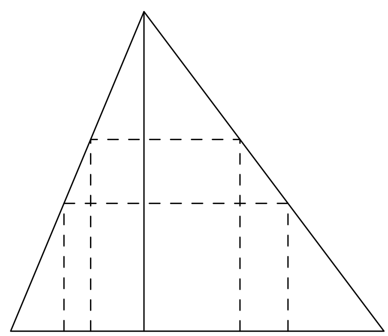
    We see that, when rounded to the nearest integer, $2014y$ can be any integer between 3309 and 3424, inclusive. Thus, the number of possible responses Chad could give is $3424 - 3309 + 1 = 116$.

2.4.8 Round 8

22. Jordan places a rectangle inside a triangle with side lengths 13, 14, and 15 so that the vertices of the rectangle all lie on sides of the triangle. What is the maximum possible area of Jordan's rectangle?

Solution. The answer is $\boxed{42}$.

The answer is 42. We claim that no matter which side of the triangle is shared with the rectangle, the maximum area of it is half the area of the triangle as long as the triangle is acute.



Suppose the base of the triangle is b and the height is h. Let the height of the rectangle be kh, where $0 < k < 1$, so the small triangle above the rectangle has height $(1-k)h$, and, by similar triangles, the base of the small triangle, which coincides with the top side of the rectangle, is $(1-k)b$. it follows that the area of the rectangle is $k(1-k)bh$. We see that

$$k(1-k) = -k^2 + k = -(k - \frac{1}{2})^2 + \frac{1}{4} \leq \frac{1}{4}.$$

So the maximum area of the rectangle is $\frac{1}{4}bh$, which is half of $\frac{1}{2}bh$, the area of the triangle. In addition, this optimal area can be achieved by using $k = \frac{1}{2}$.

Now it suffices to find the area of the triangle. Draw an altitude to the side with length 14, which divides it into two parts with lengths x and $(14-x)$, with x adjacent to the side with length 13. Letting the altitude be y, we have

$$y^2 = 13^2 - x^2 = 15^2 - (14-x)^2.$$

Solving the equation between the second and third expressions, we have $x = 5$, and therefore $h = 12$. So, the area of the triangle is $\frac{12 \cdot 14}{2} = 84$, and the maximum area of the rectangle is $\frac{84}{2} = 42$.

23. Hoping to join Chad and Jordan in the Exeter Space Station, there are 2014 prospective astronauts of various nationalities. It is given that 1006 of the astronaut applicants are American and that there are a total of 64 countries represented among the applicants. The applicants are to group into 1007 pairs with no pair consisting of two applicants of the same nationality. Over all possible distributions of nationalities, what is the maximum number of possible ways to make the 1007 pairs of applicants? Express your answer in the form $a \cdot b!$, where a and b are positive integers and a is not divisible by $b+1$.

Note: The expression $k!$ denotes the product $k \cdot (k-1) \cdot ... \cdot 2 \cdot 1$.

Solution. The answer is $\boxed{499968 \cdot 1006!}$.

Suppose the number of astronauts from countries other than America are $n_1, n_2, \ldots, n_{63}$. Clearly, the 1006 Americans must all be in different pairs, which leaves us 1 pair with two non-Americans, and 1006 pairs with one non-American. Note that once astronauts are assigned to the pair with two non-Americans, there are 1006! ways to assign astronauts to the remaining 1006 pairs, as there are 1006 ways to pair the first non-American with an American, 1005 ways to pair the second, and so on. Thus, we aim to maximize the number of ways to assign astronauts to the pair with two non-Americans.

We know that if the one of them is from the i-th nation, there would be $n_i \cdot (1008 - n_i) = 1008n_i + {n_i}^2$ ways to assign the pair. Adding this up for all 63 values of i and dividing it by 2 (because each way is counted twice), we have

$$\frac{1008(n_1 + n_2 + \cdots + n_{63}) - ({n_1}^2 + {n_2}^2 + \cdots + {n_{63}}^2)}{2} = \frac{1008 \cdot 1008 - ({n_1}^2 + {n_2}^2 + \cdots + {n_{63}}^2)}{2}$$

ways to do so. It remains to minimize the sum of the squares of n_i. In order to minimize this, we evenly distribute 1008 to all nationalities (a difference of 1 is allowed in case even distribution is impossible). This is optimal because if we have two integers a and b among n_i with $a - b > 1$, we can replace them with $a - 1$ and $b + 1$, keeping the same sum but smaller sum of squares, because

$$(a-1)^2 + (b+1)^2 = (a^2 + b^2) + 2(b - a) + 2 > a^2 + b^2 - 2 + 2 = a^2 + b^2.$$

Because $\frac{1008}{63} = 16$, the maximum number of ways to assign astronauts to the pair with two non-Americans is

$$\frac{1008 \cdot 1008 - 63 \cdot 16^2}{2} = 1008 \cdot \frac{1008 - 16}{2} = 499968.$$

Multiplying by 1006!, as noted above, and noticing that 499968 is not a multiple of 1007, we get an answer of $499968 \cdot 1006!$.

24. We say a polynomial P in x and y is n-good if $P(x, y) = 0$ for all integers x and y, with $x \neq y$, between 1 and n, inclusive. We also define the complexity of a polynomial to be the maximum sum of exponents of x and y across its terms with nonzero coefficients. What is the minimal complexity of a nonzero 4-good polynomial? In addition, give an example of a 4-good polynomial attaining this minimal complexity.

Solution. The answer is $\boxed{\text{3 (and example must be provided)}}$.

We first show that there exists a polynomial of complexity 3 satisfying the above conditions, and then that no polynomial of lower complexity can satisfy those conditions.

Consider the polynomial $f(x, y) = (x + y - 5)((x - 2.5)^2 + (y - 2.5)^2 - 2.5)$. At each of the points (1, 4), (2, 3), (3, 2), and (4, 1), the first term $x + y - 5$ is equal to 0, and at the remaining eight points the second term $(x - 2.5)^2 + (y - 2.5)^2 - 2.5$ is equal to 0. Thus, at each of the twelve points, $f(x, y)$ is equal to 0.

Next, assume that a polynomial $g(x, y)$ with complexity at most 2 exists satisfying the conditions. Consider the one-variable polynomial $g(x, 1)$. Since $g(x, y)$ has complexity at most 2, $g(x, 1)$ has degree at most 2. In addition to this, $g(x, 1)$ has roots at $x = 2, 3, 4$. Since the number of roots is greater than the degree, $g(x, 1)$ must be the zero polynomial. Now, let $g(x, y) = (y - 1)(h(x, y)) + i(x)$ for polynomials h and i; these can be found by long division by $y - 1$. Since $g(x, 1) = i(x)$ is the zero polynomial, we have that $y-1$ divides $g(x,y)$. By a similar argument, $y-2, y-3$, and $y-4$ each divide $g(x,y)$. Hence, $g(x,y) = (y-1)(y-2)(y-3)(y-4)k(x,y)$. This has a complexity of at least 4, which contradicts our original assumption. Thus, we have ascertained that no such polynomial $g(x,y)$ exists.