Exeter Math Club Competition
January 24, 2015

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- **Proctors**
- **Runners**
- **Head Graders**
- **Graders**
- **Judges** Zuming Feng, Greg Spanier

Chapter 1

EMC^2 2015 Problems

1.1 Speed Test

There are 20 problems, worth 3 points each, to be solved in 25 minutes.

1. Matt has a twenty dollar bill and buys two items worth \$7.99 each. How much change does he receive, in dollars?

2. The sum of two distinct numbers is equal to the positive difference of the two numbers. What is the product of the two numbers?

3. Evaluate
$$\frac{1+2+3+4+5+6+7}{8+9+10+11+12+13+14}.$$

4. A sphere with radius r has volume 2π. Find the volume of a sphere with diameter r.

5. Yannick ran 100 meters in 14.22 seconds. Compute his average speed in meters per second, rounded to the nearest integer.

6. The mean of the numbers $2, 0, 1, 5$, and x is an integer. Find the smallest possible positive integer value for x.

7. Let $f(x) = \sqrt{2^2 - x^2}$. Find the value of $f(f(f(f(f(-1)))))$.

8. Find the smallest positive integer n such that 20 divides $15n$ and 15 divides $20n$.

9. A circle is inscribed in equilateral triangle ABC. Let M be the point where the circle touches side AB and let N be the second intersection of segment CM and the circle. Compute the ratio $\frac{MN}{CN}$.

10. Four boys and four girls line up in a random order. What is the probability that both the first and last person in line is a girl?

11. Let k be a positive integer. After making k consecutive shots successfully, Andy's overall shooting accuracy increased from 65% to 70%. Determine the minimum possible value of k.

12. In square $ABCD$, M is the midpoint of side CD. Points N and P are on segments BC and AB respectively such that $\angle AMN = \angle MNP = 90°$. Compute the ratio $\frac{AP}{PB}$.

13. Meena writes the numbers 1, 2, 3, and 4 in some order on a blackboard, such that she cannot swap two numbers and obtain the sequence $1, 2, 3, 4$. How many sequences could she have written?

14. Find the smallest positive integer N such that $2N$ is a perfect square and $3N$ is a perfect cube.

15. A polyhedron has 60 vertices, 150 edges, and 92 faces. If all of the faces are either regular pentagons or equilateral triangles, how many of the 92 faces are pentagons?

16. All positive integers relatively prime to 2015 are written in increasing order. Let the twentieth number be p. The value of
$$\frac{2015}{p} - 1$$
can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $a + b$.

17. Five red lines and three blue lines are drawn on a plane. Given that x pairs of lines of the same color intersect and y pairs of lines of different colors intersect, find the maximum possible value of $y - x$.

18. In triangle ABC, where $AC > AB$, M is the midpoint of BC and D is on segment AC such that DM is perpendicular to BC. Given that the areas of MAD and MBD are 5 and 6, respectively, compute the area of triangle ABC.

19. For how many ordered pairs (x, y) of integers satisfying $0 \leq x, y \leq 10$ is $(x+y)^2 + (xy-1)^2$ a prime number?

20. A solitaire game is played with 8 red, 9 green, and 10 blue cards. Totoro plays each of the cards exactly once in some order, one at a time. When he plays a card of color c, he gains a number of points equal to the number of cards that are *not* of color c in his hand. Find the maximum number of points that he can obtain by the end of the game.

1.2 Accuracy Test

There are 10 problems, worth 9 points each, to be solved in 45 minutes.

1. A number of Exonians took a math test. If all of their scores were positive integers and the mean of their scores was 8.6, find the minimum possible number of students.

2. Find the least composite positive integer that is not divisible by any of 3, 4, and 5.

3. Five checkers are on the squares of an 8×8 checkerboard such that no two checkers are in the same row or the same column. How many squares on the checkerboard share neither a row nor a column with any of the five checkers?

4. Let the operation $x@y$ be $y - x$. Compute $((\cdots((1@2)@3)@\cdots@2013)@2014)@2015$.

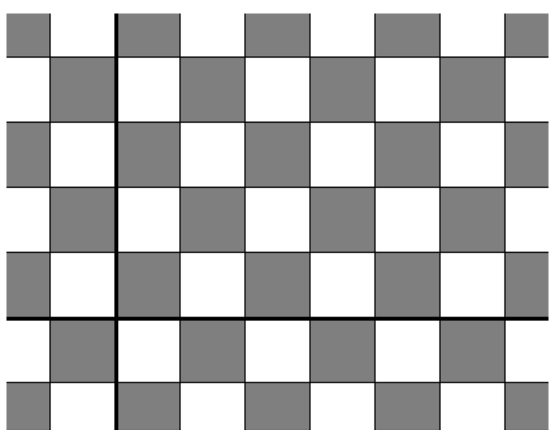
5. In a town, each family has either one or two children. According to a recent survey, 40% of the children in the town have a sibling. What fraction of the families in the town have two children?

6. Equilateral triangles ABE, BCF, CDG and DAH are constructed outside the unit square $ABCD$. Eliza wants to stand inside octagon $AEBFCGDH$ so that she can see every point in the octagon without being blocked by an edge. What is the area of the region in which she can stand?

7. Let S be the string 0101010101010. Determine the number of substrings containing an odd number of 1's. (A substring is defined by a pair of (not necessarily distinct) characters of the string and represents the characters between, inclusively, the two elements of the string.)

8. Let the positive divisors of n be d_1, d_2, $\ldots$ in increasing order. If $d_6 = 35$, determine the minimum possible value of n.

9. The unit squares on the coordinate plane that have four lattice point vertices are colored black or white, as on a chessboard, shown on the diagram below.



For an ordered pair (m, n), let $OXZY$ be the rectangle with vertices $O = (0, 0)$, $X = (m, 0)$, $Z = (m, n)$ and $Y = (0, n)$. How many ordered pairs (m, n) of nonzero integers exist such that rectangle $OXZY$ contains exactly 32 black squares?

10. In triangle ABC, $AB = 2BC$. Given that M is the midpoint of AB and $\angle MCA = 60^\circ$, compute $\frac{CM}{AC}$.

1.3 Team Test

There are 15 problems, worth 20 points each, to be solved in 45 minutes.

1. Nicky is studying biology and has a tank of 17 lizards. In one day, he can either remove 5 lizards or add 2 lizards to his tank. What is the minimum number of days necessary for Nicky to get rid of all of the lizards from his tank?

2. What is the maximum number of spheres with radius 1 that can fit into a sphere with radius 2?

3. A positive integer x is *sunny* if $3x$ has more digits than x. If all sunny numbers are written in increasing order, what is the 50th number written?

4. Quadrilateral $ABCD$ satisfies $AB = 4, BC = 5, DA = 4, \angle DAB = 60°$, and $\angle ABC = 150°$. Find the area of $ABCD$.

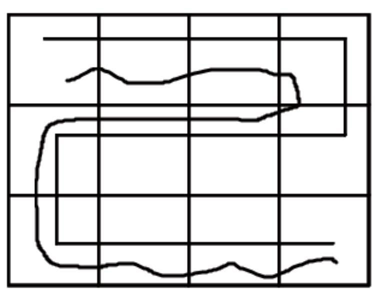
5. Totoro wants to cut a 3 meter long bar of mixed metals into two parts with equal monetary value. The left meter is bronze, worth 10 zoty per meter, the middle meter is silver, worth 25 zoty per meter, and the right meter is gold, worth 40 zoty per meter. How far, in meters, from the left should Totoro make the cut?

6. If the numbers x_1, x_2, x_3, x_4, and x_5 are a permutation of the numbers $1, 2, 3, 4$, and 5, compute the maximum possible value of $|x_1 - x_2| + |x_2 - x_3| + |x_3 - x_4| + |x_4 - x_5|$.

7. In a 3×4 grid of 12 squares, find the number of paths from the top left corner to the bottom right corner that satisfy the following two properties:

   - The path passes through each square exactly once.
   - Consecutive squares share a side.

   Two paths are considered distinct if and only if the order in which the twelve squares are visited is different. For instance, in the diagram below, the two paths drawn are considered the same.



8. Scott, Demi, and Alex are writing a computer program that is 25 lines long. Since they are working together on one computer, only one person may type at a time. To encourage collaboration, no person can type two lines in a row, and everyone must type something. If Scott takes 10 seconds to type one line, Demi takes 15 seconds, and Alex takes 20 seconds, at least how long, in seconds, will it take them to finish the program?

9. A hand of four cards of the form $(c, c, c + 1, c + 1)$ is called a *tractor*. Vinjai has a deck consisting of four of each of the numbers 7, 8, 9 and 10. If Vinjai shuffles and draws four cards from his deck, compute the probability that they form a tractor.

10. The parabola $y = 2x^2$ is the wall of a fortress. Totoro is located at $(0, 4)$ and fires a cannonball in a straight line at the closest point on the wall. Compute the y-coordinate of the point on the wall that the cannonball hits.

11. How many ways are there to color the squares of a 10 by 10 grid with black and white such that in each row and each column there are exactly two black squares and between the two black squares in a given row or column there are exactly 4 white squares? Two configurations that are the same under rotations or reflections are considered different.

12. In rectangle $ABCD$, points E and F are on sides AB and CD, respectively, such that $AE = CF > AD$ and $\angle CED = 90°$. Lines AF, BF, CE and DE enclose a rectangle whose area is 24% of the area of $ABCD$. Compute $\frac{BF}{CE}$.

13. Link cuts trees in order to complete a quest. He must cut 3 Fenwick trees, 3 Splay trees and 3 KD trees. If he must also cut 3 trees of the same type in a row at some point during his quest, in how many ways can he cut the trees and complete the quest? (Trees of the same type are indistinguishable.)

14. Find all ordered pairs (a, b) of positive integers such that

$$\sqrt{64a + b^2} + 8 = 8\sqrt{a} + b.$$

15. Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle BCD = 108°$, $\angle CDE = 168°$ and $AB = BC = CD = DE$. Find the measure of $\angle AEB$.

1.4 Guts Test

There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.

1.4.1 Round 1

1. [5] Alec rated the movie *Frozen* 1 out of 5 stars. At least how many ratings of 5 out of 5 stars does Eric need to collect to make the average rating for *Frozen* greater than or equal to 4 out of 5 stars?

2. [5] Bessie shuffles a standard 52-card deck and draws five cards without replacement. She notices that all five of the cards she drew are red. If she draws one more card from the remaining cards in the deck, what is the probability that she draws another red card?

3. [5] Find the value of $121 \cdot 1020304030201$.

1.4.2 Round 2

4. [7] Find the smallest positive integer c for which there exist positive integers a and b such that $a \neq b$ and $a^2 + b^2 = c$.

5. [7] A semicircle with diameter AB is constructed on the outside of rectangle $ABCD$ and has an arc length equal to the length of BC. Compute the ratio of the area of the rectangle to the area of the semicircle.

6. [7] There are 10 monsters, each with 6 units of health. On turn n, you can attack one monster, reducing its health by n units. If a monster's health drops to 0 or below, the monster dies. What is the minimum number of turns necessary to kill all of the monsters?

1.4.3 Round 3

7. [9] It is known that 2 students make up 5% of a class, when rounded to the nearest percent. Determine the number of possible class sizes.

8. [9] At 17:10, Totoro hopped onto a train traveling from Tianjin to Urumuqi. At 14:10 that same day, a train departed Urumuqi for Tianjin, traveling at the same speed as the 17:10 train. If the duration of a one-way trip is 13 hours, then how many hours after the two trains pass each other would Totoro reach Urumuqi?

9. [9] Chad has 100 cookies that he wants to distribute among four friends. Two of them, Jeff and Qiao, are rivals; neither wants the other to receive more cookies than they do. The other two, Jim and Townley, don't care about how many cookies they receive. In how many ways can Chad distribute all 100 cookies to his four friends so that everyone is satisfied? (Some of his four friends may receive zero cookies.)

1.4.4 Round 4

10. [11] Compute the smallest positive integer with at least four two-digit positive divisors.

11. [11] Let $ABCD$ be a trapezoid such that AB is parallel to CD, $BC = 10$ and $AD = 18$. Given that the two circles with diameters BC and AD are tangent, find the perimeter of $ABCD$.

12. [11] How many length ten strings consisting of only As and Bs contain neither "BAB" nor "BBB" as a substring?

1.4.5 Round 5

Each of the three problems in this round depends on the answer to two of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.

13. [13] Let B be the answer to problem 14, and let C be the answer to problem 15. A quadratic function $f(x)$ has two real roots that sum to $2^{10} + 4$. After translating the graph of $f(x)$ left by B units and down by C units, the new quadratic function also has two real roots. Find the sum of the two real roots of the new quadratic function.

14. [13] Let A be the answer to problem 13, and let C be the answer to problem 15. In the interior of angle $NOM = 45°$, there is a point P such that $\angle MOP = A°$ and $OP = C$. Let X and Y be the reflections of P over MO and NO, respectively. Find $(XY)^2$.

15. [13] Let A be the answer to problem 13, and let B be the answer to problem 14. Totoro hides a guava at point X in a flat field and a mango at point Y different from X such that the length XY is B. He wants to hide a papaya at point Z such that YZ has length A and the distance ZX is a nonnegative integer. In how many different locations can he hide the papaya?

1.4.6 Round 6

16. [15] Let $ABCD$ be a trapezoid such that AB is parallel to CD, $AB = 4$, $CD = 8$, $BC = 5$, and $AD = 6$. Given that point E is on segment CD and that AE is parallel to BC, find the ratio between the area of trapezoid $ABCD$ and the area of triangle ABE.

17. [15] Find the maximum possible value of the greatest common divisor of $\overline{MOO}$ and $\overline{MOOSE}$, given that S, O, M, and E are some nonzero digits. (The digits S, O, M, and E are not necessarily pairwise distinct.)

18. [15] Suppose that 125 politicians sit around a conference table. Each politician either always tells the truth or always lies. (Statements of a liar are never completely true, but can be partially true.) Each politician now claims that the two people beside them are both liars. Suppose that the greatest possible number of liars is M and that the least possible number of liars is N. Determine the ordered pair (M, N).

1.4.7 Round 7

19. [18] Define a *lucky* number as a number that only contains 4s and 7s in its decimal representation. Find the sum of all three-digit lucky numbers.

20. [18] Let line segment AB have length 25 and let points C and D lie on the same side of line AB such that $AC = 15$, $AD = 24$, $BC = 20$, and $BD = 7$. Given that rays AC and BD intersect at point E, compute $EA + EB$.

21. [18] A 3×3 grid is filled with positive integers and has the property that each integer divides both the integer directly above it and directly to the right of it. Given that the number in the top-right corner is 30, how many distinct grids are possible?

1.4.8 Round 8

22. [22] Define a sequence of positive integers s_1, s_2, $\ldots$, s_{10} to be *terrible* if the following conditions are satisfied for any pair of positive integers i and j satisfying $1 \leq i < j \leq 10$:

   - $s_i > s_j$
   - $j - i + 1$ divides the quantity $s_i + s_{i+1} + \cdots + s_j$

   Determine the minimum possible value of $s_1 + s_2 + \cdots + s_{10}$ over all terrible sequences.

23. [22] The four points (x, y) that satisfy $x = y^2 - 37$ and $y = x^2 - 37$ form a convex quadrilateral in the coordinate plane. Given that the diagonals of this quadrilateral intersect at point P, find the coordinates of P as an ordered pair.

24. [22] Consider a non-empty set of segments of length 1 in the plane which do not intersect except at their endpoints. (In other words, if point P lies on distinct segments a and b, then P is an endpoint of both a and b.) This set is called *3-amazing* if each endpoint of a segment is the endpoint of exactly three segments in the set. Find the smallest possible size of a 3-amazing set of segments.

Chapter 2

EMC^2 2015 Solutions

2.1 Speed Test Solutions

1. Matt has a twenty dollar bill and buys two items worth \$7.99 each. How much change does he receive, in dollars?

   Solution. The answer is $\boxed{\$4.02}$.

   The two items in total cost $2 \times \$7.99 = \$15.98$. The change received is then $\$20.00 - \$15.98 = \$4.02$.

2. The sum of two distinct numbers is equal to the positive difference of the two numbers. What is the product of the two numbers?

   Solution. The answer is $\boxed{0}$.

   Let the two numbers be x and y with x being the greater one; then, we have that $x + y = x - y$, which implies $y = 0$. As a result, we have that $xy = 0$.

3. Evaluate
$$\frac{1+2+3+4+5+6+7}{8+9+10+11+12+13+14}.$$

   Solution. The answer is $\boxed{\frac{4}{11}}$.

   Both the numerator and denominator are arithmetic series with length seven, so each has a value of 7 times its middle term. Thus our answer is the ratio of their middle terms, or $\frac{4}{11}$.

4. A sphere with radius r has volume 2π. Find the volume of a sphere with diameter r.

   Solution. The answer is $\boxed{\frac{\pi}{4}}$.

   A sphere with diameter r has radius $\frac{r}{2}$. Then the volume of this sphere is $(\frac{1}{2})^3 = \frac{1}{8}$ of the volume of the original sphere, or $\frac{2\pi}{8} = \frac{\pi}{4}$.

5. Yannick ran 100 meters in 14.22 seconds. Compute his average speed in meters per second, rounded to the nearest integer.

   Solution. The answer is $\boxed{7}$.

   We have
$$14.22 \times 6.5 < 15 \times 6.5 < 100 < 14 \times 7.5 < 14.22 \times 7.5.$$
   So Yannick's average speed is strictly between 6.5 and 7.5 meters per second, and therefore the answer is 7 meters per second.

6. The mean of the numbers $2, 0, 1, 5,$ and x is an integer. Find the smallest possible positive integer value for x.

   Solution. The answer is $\boxed{2}$.

   We need $2 + 0 + 1 + 5 + x = x + 8$ to a multiple of 5, so we can conclude that $x = 2$.

7. Let $f(x) = \sqrt{2^2 - x^2}$. Find the value of $f(f(f(f(f(-1)))))$.

Solution. The answer is $\boxed{\sqrt{3}}$.

We have that

$$f(f(x)) = \sqrt{2^2 - (\sqrt{2^2 - x^2})^2} = \sqrt{x^2} = |x|.$$

Therefore $f(f(f(f(f(-1))))) = f(1) = \sqrt{4-1} = \sqrt{3}$.

8. Find the smallest positive integer n such that 20 divides $15n$ and 15 divides $20n$.

   Solution. The answer is $\boxed{12}$.

   20 dividing $15n$ is equivalent to 4 dividing $3n$, and 15 dividing $20n$ is equivalent to 3 dividng $4n$. Thus n must be a multiple of 12, so the smallest possible value of n is 12.

9. A circle is inscribed in equilateral triangle ABC. Let M be the point where the circle touches side AB and let N be the second intersection of segment CM and the circle. Compute the ratio $\frac{MN}{CN}$.

   Solution. The answer is $\boxed{2}$.

   Let O be the center of the circle. Since it is an equilateral triangle, M is the midpoint of AB and CM goes through O. Notice that $\angle OAM = \angle MCA = 30°$, so $\frac{AM}{MO} = \frac{CM}{AM} = \sqrt{3}$ and $\frac{CM}{MO} = 3$. Therefore we have $CN = NO = OM$ and $\frac{NM}{CN} = 2$.

10. Four boys and four girls line up in a random order. What is the probability that both the first and last person in line is a girl?

    Solution. The answer is $\boxed{\frac{3}{14}}$.

    When lining up the 8 people, we can always determine the first and last person before ordering the other 6 people. There are $8 \times 7 = 56$ ways to choose two people, and $4 \times 3 = 12$ ways chooses 2 girls. So the probablility is $\frac{12}{56} = \frac{3}{14}$.

11. Let k be a positive integer. After making k consecutive shots successfully, Andy's overall shooting accuracy increased from 65% to 70%. Determine the minimum possible value of k.

    Solution. The answer is $\boxed{10}$.

    Assume that Andy had made a shots before the k shots and $0.65a$ of them were successful. So we have $\frac{0.65a+k}{a+k} = 0.7$. Solving the equation for a, we get $a = 6k$, and therefore $0.65a = 3.9k$. It suffices to make $3.9k$ an integer, so the minimum value of $k = 10$.

12. In square $ABCD$, M is the midpoint of side CD. Points N and P are on segments BC and AB respectively such that $\angle AMN = \angle MNP = 90°$. Compute the ratio $\frac{AP}{PB}$.

    Solution. The answer is $\boxed{\frac{5}{3}}$.

    Assume the side length of the square is 1. Since $\angle AMN = \angle MNP = \angle D = \angle C = \angle B = 90°$, we have $\angle DAM = \angle CMN = \angle BNP$ by simple angle-chasing, and therefore triangle DAM, CMN, BNP are all similar to each other. It is clear that $AD = 1, DM = \frac{1}{2}$. From this we have $\frac{AD}{DM} = \frac{MC}{CN} = \frac{NB}{BP} = 2$. Then $MC = \frac{1}{2}, CN = \frac{1}{4}, NB = \frac{3}{4}, BP = \frac{3}{8}, PA = \frac{5}{8}$, and finally $\frac{AP}{PB} = \frac{5}{3}$.

13. Meena writes the numbers 1, 2, 3, and 4 in some order on a blackboard, such that she cannot swap two numbers and obtain the sequence $1, 2, 3, 4$. How many sequences could she have written?

Solution. The answer is $\boxed{18}$.

Observe that the condition makes the problem equivalent to counting the number of permutations that can be obtained by swapping two distinct elements of $(1, 2, 3, 4)$. This is because we can always undo/redo the swap to move between the permutation we're counting and $(1, 2, 3, 4)$. Since we all we need to do is choose two of the four elements to swap, the number of ways is $\binom{4}{2} = 6$. Since the total number of permutations is $4! = 24$, our answer is $24 - 6 = 18$.

14. Find the smallest positive integer N such that $2N$ is a perfect square and $3N$ is a perfect cube.

Solution. The answer is $\boxed{72}$.

Assume $N = 2^a 3^b P$, where a, b are nonnegative integers and P is an integer that is a multiple of neither 2 or 3. From the problem statement we see that P must be an integer to the sixth power, so we can safely assume that it is at least 1. Also since $2^{a+1}3^b$ is a perfect square, and $2^a 3^{b+1}$ is a perfect cube, we see that $a + 1, b$ are even and $a, b + 1$ are multiples of 3.It is easy to see that a is at least 3 and b is at least 2, so N is at least $2^3 \cdot 3^2 = 72$. Indeed, $2 \times 72 = 144 = 12^2$ and $3 \times 72 = 216 = 6^3$, so $N = 72$ satisfies the requirements.

15. A polyhedron has 60 vertices, 150 edges, and 92 faces. If all of the faces are either regular pentagons or equilateral triangles, how many of the 92 faces are pentagons?

Solution. The answer is $\boxed{12}$.

Since each edge of the polyhedron is an edge of exactly two faces, the total number of edges of all the faces must be $2 \times 150 = 300$. Letting t be the number of triangular faces and p the number of pentagonal faces, we have a system $\{t+p = 92, 3t+5p = 300\}$ from which we find that $t = 80$ and $p = 12$.

Note: The polyhedron described in the problem is known as a snub dodecahedron.

16. All positive integers relatively prime to 2015 are written in increasing order. Let the twentieth number be p. The value of
$$\frac{2015}{p} - 1$$
can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $a + b$.

Solution. The answer is $\boxed{2015}$.

The fraction we want to compute is equivalent to $\frac{2015-p}{p}$. Since p is coprime with 2015, by Euler's algorithm it is clear that $\gcd(2015 - p, p) = \gcd(2015, p) = 1$. Therefore $a = 2015 - p$ and $b = p$, and $a + b = 2015$, as desired.

17. Five red lines and three blue lines are drawn on a plane. Given that x pairs of lines of the same color intersect and y pairs of lines of different colors intersect, find the maximum possible value of $y - x$.

Solution. The answer is $\boxed{15}$.

The maximum possible value of y is $5 \cdot 3 = 15$, since each blue line can intersect at most five red lines. The minimum possible value of x is 0, which occurs when all lines of the same color are parallel. This implies that $y - x \leq 15$. Note we can achieve this bound by making all red lines parallel to the x-axis and all blue lines parallel to the y-axis on the coordinate plane. Thus our answer is 15.

18. In triangle ABC, where $AC > AB$, M is the midpoint of BC and D is on segment AC such that DM is perpendicular to BC. Given that the areas of MAD and MBD are 5 and 6, respectively, compute the area of triangle ABC.

    Solution. The answer is $\boxed{22}$.

    Since M is the midpoint of BC, we have $[MCD] = [MBD] = 6$, and $[ABM] = [ACM] = [MAD] + [MCD] = 5 + 6 = 11$, and finally $[ABC] = [ABM] + [ACM] = 11 + 11 = 22$.

19. For how many ordered pairs (x, y) of integers satisfying $0 \leq x, y \leq 10$ is $(x+y)^2 + (xy-1)^2$ a prime number?

    Solution. The answer is $\boxed{10}$.

    We can expand and factor the given expression to obtain

    $$(x+y)^2 + (xy-1)^2 = x^2y^2 + x^2 + y^2 + 1 = (x^2+1)(y^2+1).$$

    For this value to be prime, exactly one of x^2+1 and y^2+1 must be 1, so one of x and y is 0. Without loss of generality, let $x = 0$, Then we know that $y \leq 10$ and $y^2 + 1$ is a prime. If $y^2 + 1$ is even, then it must be 2, giving us the solution $(0, 1)$. Otherwise, $y^2 + 1$ is odd, and y is even.

    We can now check all the even numbers less than or equal to 10: $2^2 + 1 = 5$ is prime. $4^2 + 1 = 17$ is prime. $6^2 + 1 = 37$ is prime. $8^2 + 1 = 65 = 5 \cdot 13$ is not prime. $10^2 + 1 = 101$ is prime. This gives us a total of 5 solutions of the form $(0, y)$. If we swap x and y, we get another 5 solutions, yielding a final answer of 10.

20. A solitaire game is played with 8 red, 9 green, and 10 blue cards. Totoro plays each of the cards exactly once in some order, one at a time. When he plays a card of color c, he gains a number of points equal to the number of cards that are *not* of color c in his hand. Find the maximum number of points that he can obtain by the end of the game.

    Solution. The answer is $\boxed{242}$.

    I claim that regardless of how the cards are played, the total number of points rewarded starting with r red cards, g green cards, and b blue cards is always $rg + gb + br$. I prove this by induction with base case $(r, b, g) = (0, 0, 0)$. Assume that for all ordered triples (r, g, b) with $r + g + b \leq s$ that the number of points achieved is always $rg + gb + br$.

    Now consider an ordered triple (r', g', b') with $r' + g' + b' = s + 1$. If a blue card is played first, generating $r' + g'$ points, then by the induction hypothesis the total number of points is

    $$r'g' + g'(b'-1) + (b'-1) + r' + g' = r'g' + g'b' + b'r'.$$

    One can verify that playing a red or green card generates the same total, thus completing the induction argument. Therefore the total number of points he can get is $8 \cdot 9 + 9 \cdot 10 + 10 \cdot 8 = 242$.

2.2 Accuracy Test Solutions

1. A number of Exonians took a math test. If all of their scores were positive integers and the mean of their scores was 8.6, find the minimum possible number of students.

   Solution. The answer is $\boxed{5}$.

   Observe that the sum of all scores is $8.6n = \frac{43n}{5}$. Since this is an integer, n must be divisible by 5 and therefore at least 5. One example where $n = 5$ works is when the students' scores are $8, 8, 9, 9$ and 9, respectively.

2. Find the least composite positive integer that is not divisible by any of 3, 4, and 5.

   Solution. The answer is $\boxed{14}$.

   For an integer n to be composite, it must be expressible as the product of 2 integers greater than 1. One of these integers can be 2, but not both, because otherwise n would be a multiple of 4. Thus the second integer is at least 7, giving us $n = 14$. It is easy to check that none of the positive integers below 14 satisfy the requirements.

3. Five checkers are on the squares of an 8×8 checkerboard such that no two checkers are in the same row or the same column. How many squares on the checkerboard share neither a row nor a column with any of the five checkers?

   Solution. The answer is $\boxed{9}$.

   It is clear that the checkers occupy 5 distinct rows and 5 distinct columns. Thus 3 rows and 3 columns are unoccupied. Since each pair of an unoccupied row and an unoccupied column correspond to exactly one desired square, we have $3 \cdot 3 = 9$ such squares in total.

4. Let the operation $x@y$ be $y - x$. Compute $((\cdots((1@2)@3)@\cdots@2013)@2014)@2015$.

   Solution. The answer is $\boxed{1008}$.

   Since $x@y = y - x$, we can rewrite the original expression as

   $$2015 - (2014 - (\cdots - (3 - (2 - 1))\cdots)) = 2015 - 2014 + 2013 - \cdots + 3 - 2 + 1.$$

   Therefore, the original expression is equal to

   $$(2015 - 2014) + (2013 - 2012) + \cdots + (3 - 2) + 1 = \frac{2014}{2} + 1 = 1008.$$

5. In a town, each family has either one or two children. According to a recent survey, 40% of the children in the town have a sibling. What fraction of the families in the town have two children?

   Solution. The answer is $\boxed{\frac{1}{4}}$.

   Suppose that there are n children in the town. Then $0.4n$ children are from families with two children and $0.6n$ children are from families with one child. Therefore, there are $\frac{0.4n}{2} = 0.2n$ families with two children and $0.6n$ families with one child, for a total $0.2n + 0.6n = 0.8n$ families. Thus the desired fraction is $\frac{0.2n}{0.8n} = \frac{1}{4}$.

6. Equilateral triangles ABE, BCF, CDG and DAH are constructed outside the unit square $ABCD$. Eliza wants to stand inside octagon $AEBFCGDH$ so that she can see every point in the octagon without being blocked by an edge. What is the area of the region in which she can stand?

   Solution. The answer is $\boxed{\frac{3+\sqrt{3}}{3}}$.

   For the whole octagon to be visible to her, observe that Eliza only has to be able to see each point on the perimeter of the octagon. In order to do so, she must be inside the extensions of each acute angle of the octagon. (The acute angles we are referring to are $\angle AEB$, $\angle BFC$, $\angle CGD$ and $\angle DHA$.) Note that this is sufficient, since each point on the perimeter belongs to at least one such acute angle. Thus our answer is the area of the intersection of all four of these acute angles. We can cut this region into square $ABCD$ and four smaller triangles. It is not difficult to see that the rays intersect at the center of each equilateral triangle, so each smaller triangle has area $\frac{1}{3}$ that of an equilateral triangle. Each equilateral triangle has area $\frac{\sqrt{3}}{4}$, so we can sum these parts and obtain

   $$1 + 4 \cdot \frac{1}{3} \cdot \frac{\sqrt{3}}{4} = \frac{3+\sqrt{3}}{3}$$

   as our answer.

7. Let S be the string 0101010101010. Determine the number of substrings containing an odd number of 1's. (A substring is defined by a pair of (not necessarily distinct) characters of the string and represents the characters between, inclusively, the two elements of the string.)

   Solution. The answer is $\boxed{48}$.

   Call a substring *minimal* if it starts and ends with 1. We see that adding a 0 to either the front, back, or both of the minimal substring produces another substring with the same number of ones. Since there are 4 substrings corresponding to a single minimal substring, we only need to find the number of minimal substrings with an odd number of ones.

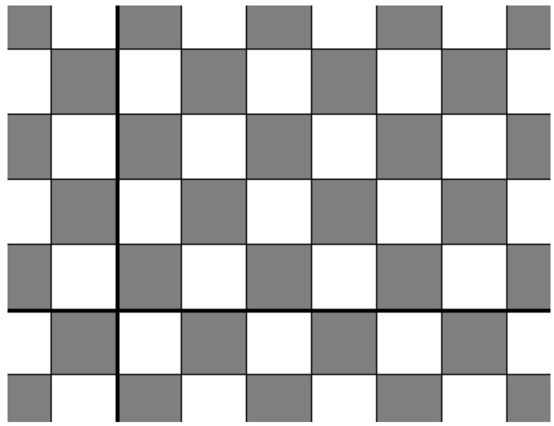
   It is not difficult to see that there are 6 ways to choose a minimal substring with 1 one, 4 ways to choose one with 3 ones and 2 ways to choose one with 5 ones. Therefore there are $6 + 4 + 2 = 12$ minimal substrings with desired property and thus $4 \cdot 12 = 48$ such substrings in total.

8. Let the positive divisors of n be d_1, d_2, $\ldots$ in increasing order. If $d_6 = 35$, determine the minimum possible value of n.

   Solution. The answer is $\boxed{1925}$.

   Since 35 is a divisor of n, $1, 5, 7$ are also divisors of n. We need to find exactly 2 more divisors that are less than 35. If n is divisible by another prime factor p that is 2 or 3, then $p, 5p, 7p$ are all divisors of n and 35 can't be the sixth smallest divisor of n. If n is divisible by 5^2, then 25 is a divisor of n. If n is divisible by 7^2, then no new divisors less than 35 will be produced. If n is divisible by another prime factor p that is greater than 7 and less than 35, then p will be the only new factor less than 35. We see then that making n divisible by $5^2, 7, 11$ is the optimal strategy and thus the smallest n is $5^2 \cdot 7 \cdot 11 = 1925$.

9. The unit squares on the coordinate plane that have four lattice point vertices are colored black or white, as on a chessboard, shown on the diagram below.



For an ordered pair (m,n), let $OXZY$ be the rectangle with vertices $O=(0,0)$, $X=(m,0)$, $Z=(m,n)$ and $Y=(0,n)$. How many ordered pairs (m,n) of nonzero integers exist such that rectangle $OXZY$ contains exactly 32 black squares?

Solution. The answer is $\boxed{48}$.

We have the following cases:

Case 1: The area is even. In this case, the number of black squares is always equal to the number of white squares. Thus if there are 32 black squares, there must also be 32 white squares, which implies the area of $OXZY$ is 64. Therefore $|mn|=64$, which means the possible ordered pairs (m,n) are $(\pm1,\pm64)$, $(\pm2,\pm32)$, $(\pm4,\pm16)$, $(\pm8,\pm8)$, $(\pm16,\pm4)$, $(\pm32,\pm2)$ and $(\pm64,\pm1)$, giving us $7\cdot4=28$ possibilities.

Case 2: Both sides have odd length and the four corner squares are all black. Observe that the number of black squares is one more than the number of white squares. Thus when there are 32 black squares, there must be 31 white squares, giving us an area of 63. In this case, point Z must be in the first or third quadrant, so m and n must have the same sign. Now, we can compute the first-quadrant pairs (m,n) to be $(1,63)$, $(3,21)$, $(7,9)$, $(9,7)$, $(21,3)$ and $(63,1)$. The reflections of these points over the origin into the third quadrant also yield solutions, thus there are a total $6\cdot2=12$ pairs for this case.

Case 3: Both sides have odd length and the four corner squares are all white. This case is essentially the opposite of Case 2. The number of white squares is one more than the number of black squares. Thus when there are 32 black squares, there must be 33 white squares, giving us an area of 65 for $OXZY$. In this case, point Z must be in the second or fourth quadrant, so m and n must have opposite signs. Observe that the second quadrant pairs (m,n) are $(-1,65)$, $(-5,13)$, $(-13,5)$ and $(-65,1)$. Since these points and their reflections over the origin into the fourth quadrant are all solutions, we have $4\cdot2=8$ pairs here.

Summing the results for all the cases, we obtain a total of $28+12+8=48$ ordered pairs.

10. In triangle ABC, $AB=2BC$. Given that M is the midpoint of AB and $\angle MCA=60^\circ$, compute $\frac{CM}{AC}$.

Solution. The answer is $\boxed{\frac{1}{3}}$.

Since M is the midpoint of AB and $AB = 2BC$, we know that $AM = MB = BC$. Draw BX perpendicular to MC at X and extend CM to a point Y such that $MX = MY$. Now, observe that $CX = MX$ and $\triangle AYM \cong \triangle BXM$. Therefore, $\angle AYM = 90^\circ$ and $CA = 2CY$. Since X and M trisect CY, $CY = \frac{3}{2}CM$, which implies $CA = 3CM$ and $\frac{CM}{AC} = \frac{1}{3}$ as desired.

2.3 Team Test Solutions

1. Nicky is studying biology and has a tank of 17 lizards. In one day, he can either remove 5 lizards or add 2 lizards to his tank. What is the minimum number of days necessary for Nicky to get rid of all of the lizards from his tank?

   Solution. The answer is $\boxed{9}$.

   Assume for x days he removed 5 lizards and for y days he added 2 lizards. So we have $5x - 2y = 17$ or $x = \frac{17+2y}{5}$. The minimum positive integer value of x and y is achieved when $y = 4$, and thus $x = 5, x + y = 9$, so Nicky needs 9 days to get rid of all 17 lizards.

2. What is the maximum number of spheres with radius 1 that can fit into a sphere with radius 2?

   Solution. The answer is $\boxed{2}$.

   It's easy to see that 2 unit spheres can fit inside the big sphere by putting the two centers along a diameter. Now we show that we cannot fit 3 or more inside. Clearly if we want to fit three inside the optimal situation is when they are mutually tangent and therefore the centers are vertices of an equilateral triangle with side length 2. By cutting through the plane with the three centers, we get three mutually tangent unit circles and we see that they cannot fit inside a circle with radius 2 (in fact, the smallest circle that encloses the three unit circles has a radius of $\frac{2}{\sqrt{3}} + 1$ and we leave the proof to the readers), which is the largest cross-section of the large sphere. Therefore we can't fit three spheres in a sphere with radius 2.

3. A positive integer x is *sunny* if $3x$ has more digits than x. If all sunny numbers are written in increasing order, what is the 50th number written?

   Solution. The answer is $\boxed{77}$.

   The one-digit sunny numbers can be computed to be $\{4, 5, 6, 7, 8, 9\}$, and the two-digit sunny numbers are $\{34, 35, 36, ..., 99\}$. The problem is equivalent to finding the $50 - 6 = 44$-th two digit number, which is $34 + 44 - 1 = 77$.

4. Quadrilateral $ABCD$ satisfies $AB = 4, BC = 5, DA = 4, \angle DAB = 60°$, and $\angle ABC = 150°$. Find the area of $ABCD$.

   Solution. The answer is $\boxed{10 + 4\sqrt{3}}$.

   Connect DB. Notice that triangle ABD is an equilateral triangle with area $\frac{\sqrt{3}}{4} \cdot 4^2 = 4\sqrt{3}$. Then $\angle DBC = 150° - 60° = 90°$. Then triangle BCD is an equilateral triangle with area $\frac{4 \cdot 5}{2} = 10$. Combining the two areas, we get $10 + 4\sqrt{3}$.

5. Totoro wants to cut a 3 meter long bar of mixed metals into two parts with equal monetary value. The left meter is bronze, worth 10 zoty per meter, the middle meter is silver, worth 25 zoty per meter, and the right meter is gold, worth 40 zoty per meter. How far, in meters, from the left should Totoro make the cut?

   Solution. The answer is $\boxed{\frac{33}{16}}$.

Note that the entire bar is worth $10 + 25 + 40 = 75$ zoty, so each part should be worth $\frac{75}{2} = 37.5$ zoty. Also since the gold part of the bar is already worth 40 zoty, we should make the cut in the gold part, and therefore the right part is entirely gold. From this we can find that the right part should be $\frac{37.5}{40} = \frac{15}{16}$ meters long, leaving the left part to be $3 - \frac{15}{16} = \frac{33}{16}$ meters long.

6. If the numbers x_1, x_2, x_3, x_4, and x_5 are a permutation of the numbers $1, 2, 3, 4$, and 5, compute the maximum possible value of $|x_1 - x_2| + |x_2 - x_3| + |x_3 - x_4| + |x_4 - x_5|$.

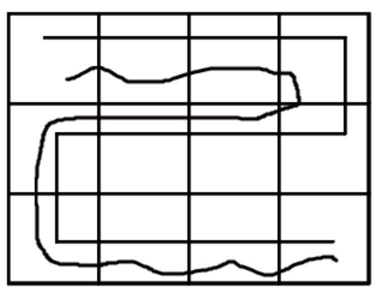
   Solution. The answer is $\boxed{11}$.

   If x_i is between x_{i+1} and x_{i-1}, we can remove x_i from the sequence and add it to the end. Since $|x_{i-1} - x_{i+1}| = |x_{i-1} - x_i| + |x_i - x_{i+1}|$, we did not decrease the sum of difference in the middle and added a positive difference at the end, and therefore increased the desired sum. Thus we can assume the sequence alternates between increasing and decreasing. Now, we can also assume $x_1 > x_2 < x_3 > x_4 < x_5$, so the sum is equal to $x_1 - 2x_2 + 2x_3 - 2x_4 + x_5$, which is maximized when $x_3 = 5$, $x_1, x_5 = 3, 4$, and $x_2, x_4 = 1, 2$, giving a sum of 11.

7. In a 3×4 grid of 12 squares, find the number of paths from the top left corner to the bottom right corner that satisfy the following two properties:

   - The path passes through each square exactly once.
   - Consecutive squares share a side.

   Two paths are considered distinct if and only if the order in which the twelve squares are visited is different. For instance, in the diagram below, the two paths drawn are considered the same.



   Solution. The answer is $\boxed{4}$.

   Label the squares on each row ABCD, EFGH, IJKL from left to right. We want to find the number of paths that start from A and ends with L. We start by examining the possible moves starting from the first step.
   Case 1: First move is to B. Then there are still 2 possibilities: If next move is to C, then the path must go to D, H, and then GFEIJKL; if the next move is to F, then the path must go to EIJK, then GCDHL. There are 2 ways in all.
   Case 2: First move is to E, then we are forced to go to I, and then J. If next move is to F, then we have to go to B, C, then DHGKL; if next move is to K, then we have to go to G, F, B, C, then DHL. There are 2 ways in all in this case.
   Summing up the two cases, we have $2 + 2 = 4$ paths in total.

8. Scott, Demi, and Alex are writing a computer program that is 25 lines long. Since they are working together on one computer, only one person may type at a time. To encourage collaboration, no person can type two lines in a row, and everyone must type something. If Scott takes 10 seconds to type one line, Demi takes 15 seconds, and Alex takes 20 seconds, at least how long, in seconds, will it take them to finish the program?

   Solution. The answer is $\boxed{315}$.

   To minimize the total time we need to maximize the number of lines typed by Scott and minimize the number of lines typed by Alex. Since Scott cannot type twice in a row, he can type at most 13 lines of code. Thus Demi and Alex together must type at least 12 lines. Additionally, at least one of these lines must be Alex's. Thus it must take at least $13 \cdot 10 + 11 \cdot 15 + 1 \cdot 20 = 315$ seconds to finish the program. One way this can happen is SASDS...SDSDS, with Scott typing every even line, and Demi typing the rest of the lines except for one line which Alex types.

9. A hand of four cards of the form $(c, c, c+1, c+1)$ is called a *tractor*. Vinjai has a deck consisting of four of each of the numbers 7, 8, 9 and 10. If Vinjai shuffles and draws four cards from his deck, compute the probability that they form a tractor.

   Solution. The answer is $\boxed{\frac{27}{455}}$.

   Assume that the hand of four cards is ordered and all cards are distinct (even if they have the same number), so all tractors should be $(7,7,8,8)$, $(8,8,9,9)$, $(9,9,10,10)$, or their permutations. If Matt draws $7,7,8,8$, there are $\binom{4}{2} = 6$ ways to choose two 7s and 6 ways to choose two 8s, and there are $4! = 24$ ways to order them, so in total there are $3 \cdot 6 \cdot 6 \cdot 24$ ways to draw a tractor. On the other hand, there are $16 \cdot 15 \cdot 14 \cdot 13$ ways to draw four cards. So the probability is $\frac{3 \cdot 12 \cdot 12 \cdot 24}{16 \cdot 15 \cdot 14 \cdot 13} = \frac{27}{455}$.

10. The parabola $y = 2x^2$ is the wall of a fortress. Totoro is located at $(0,4)$ and fires a cannonball in a straight line at the closest point on the wall. Compute the y-coordinate of the point on the wall that the cannonball hits.

    Solution. The answer is $\boxed{\frac{15}{4}}$.

    The cannonball hits the fortress at a point $(x, 2x^2)$ for some real number x, and we wish to minimize the distance from $(0,4)$ to $(x, 2x^2)$. By the distance formula, this is

    $$\sqrt{(x-0)^2 + (2x^2-4)^2} = \sqrt{4x^4 - 15x^2 + 16} = \sqrt{\left(2x^2 - \frac{15}{4}\right)^2 + \frac{31}{16}}.$$

    The expression is minimized when $y = 2x^2 = \frac{15}{4}$.

11. How many ways are there to color the squares of a 10 by 10 grid with black and white such that in each row and each column there are exactly two black squares and between the two black squares in a given row or column there are exactly 4 white squares? Two configurations that are the same under rotations or reflections are considered different.

    Solution. The answer is $\boxed{120}$.

    Notice that in each row there should be exactly 1 black square in the first 5 squares and 1 black square in the last 5 squares, and when one cut this row into two equal halves, the two black squares are on the same relative position of each half.

Now cut the grid into four 5 by 5 sub-grids and consider the sub-grid on the top-left corner. There should be 1 black square on each row and column in this sub-grid, so there are $5! = 120$ ways to color this sub-grid. It's then obvious that the other three sub-grids are simply the translations of this sub-grid. So there are 120 ways to color the entire grid.

12. In rectangle $ABCD$, points E and F are on sides AB and CD, respectively, such that $AE = CF > AD$ and $\angle CED = 90°$. Lines AF, BF, CE and DE enclose a rectangle whose area is 24% of the area of $ABCD$. Compute $\frac{BF}{CE}$.

Solution. The answer is $\boxed{\frac{\sqrt{6}}{2}}$.

Since $AE = CF, AB = CD$, it is clear that $BE = DF$. By SAS congruence we have ADF being congruent to CBE, and ADE is congruent to CBF. Then we have $AF||CE, DE||BF$. Because $\angle CED = 90°$, AF, CE are perpendicular to DE, BF respectively. By angle-chasing we have triangle ADE is similar to DFA. Since $AE > AD$, we get that $AD > DF$, so $CF > DF$ as well.

Assume the intersection of AF and DE, BF and CE are X and Y respectively. It is clear that

$$\frac{XE}{DE} = \frac{CF}{CD}, \frac{YE}{CE} = \frac{DF}{CD},$$

$$\frac{XE}{DE} + \frac{YE}{CE} = \frac{CF}{CD} + \frac{DF}{CD} = 1.$$

Also because $CF > DF$, $\frac{XE}{DE} > \frac{YE}{CE}$. Let $\frac{XE}{DE} = k$, so $\frac{YE}{CE} = 1 - k$, and $k > 0.5$. From problem statement we have $\frac{[XEYF]}{[ABCD]} = 0.24$, so

$$\frac{[XEYF]}{[CDE]} = \frac{[XEYF]}{1/2[ABCD]} = 0.48,$$

$$\frac{[XEYF]}{[CDE]} = \frac{k(1-k)}{1/2}.$$

Setting the two equations equal to each other and solving for k, we have $k = 0.6$ (the other solution $k = 0.4$ is disregarded because $k > 0.5$.)

Noticing that $\frac{k}{1-k} = \frac{CF}{DF} = \frac{FC}{BE}$, and that

$$\frac{BF}{CE} = \frac{FC}{BC} = \frac{BC}{BE} = \sqrt{\frac{FC}{BE}},$$

which follows from the fact that triangle FCB and CBE are similar, we can get that $\frac{BF}{CE} = \sqrt{\frac{k}{1-k}} = \frac{\sqrt{6}}{2}$, as desired.

13. Link cuts trees in order to complete a quest. He must cut 3 Fenwick trees, 3 Splay trees and 3 KD trees. If he must also cut 3 trees of the same type in a row at some point during his quest, in how many ways can he cut the trees and complete the quest? (Trees of the same type are indistinguishable.)

Solution. The answer is $\boxed{366}$.

If a solution cuts a type of tree three times in a row, we say that this solution *clears* that type of tree. We can now use the principle of inclusion-exclusion to count number of ways for Link to cut the trees. We start off by estimating our answer to be $3 \cdot 7 \cdot \binom{6}{3} = 420$: 3 ways to choose which type of tree is cleared, 7 places to put this group of three cuts, and 6 choose 3 ways to cut the remaining trees. However, we count twice each solution that clears two types of tree, and thrice each solution that clears three types of tree. Now, we subtract out the solutions that clear two types of trees, or $\binom{3}{2} \cdot 4 \cdot 5 = 60$: 3 choose 2 ways to choose the trees that are cleared, 4 ways to place the first group of cleared trees and 5 ways to place the second. However, we are not done yet, since we subtracted the solutions that clear all three types of trees three times. We have to add the $3! = 6$ ways to do so back, getting an answer of $420 - 60 + 6 = 366$.

14. Find all ordered pairs (a, b) of positive integers such that

$$\sqrt{64a + b^2} + 8 = 8\sqrt{a} + b.$$

Solution. The answer is $\boxed{(4, 12), (9, 10), (25, 9)}$.

Rewrite the equation as

$$\sqrt{64a + b^2} = 8\sqrt{a} + b - 8.$$

Squaring and cancelling terms yields

$$0 = 64 - 128\sqrt{a} - 16b + 16b\sqrt{a},$$

or $b = 8 + \frac{4}{\sqrt{a}-1}$. Notice that if $a > 25$ then $8 < b < 9$, giving no solutions. Also, if a is not a perfect square, then b is irrational. It remains to check $a = 1, 4, 9, 16, 25$, which give the solutions $(4, 12), (9, 10), (25, 9)$. It is not difficult to check that all three solutions work.

15. Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle BCD = 108^\circ$, $\angle CDE = 168^\circ$ and $AB = BC = CD = DE$. Find the measure of $\angle AEB$.

Solution. The answer is $\boxed{24^\circ}$.

Construct a point F such that $ABCDF$ is a regular pentagon. Note that $\triangle FDE$ is equilateral. In addition, observe that $\angle AEB = \angle FEB - \angle FEA$. By symmetry, we know that $\angle FEB$ is half of $\angle FED$ or 30°. Because $\triangle AFE$ is isosceles, $\angle FEA = \frac{180^\circ - \angle AFE}{2} = \frac{180^\circ - 168^\circ}{2} = 6^\circ$. This allows us to compute $\angle AEB$ as $30^\circ - 6^\circ = 24^\circ$.

2.4 Guts Test Solutions

2.4.1 Round 1

1. [5] Alec rated the movie *Frozen* 1 out of 5 stars. At least how many ratings of 5 out of 5 stars does Eric need to collect to make the average rating for *Frozen* greater than or equal to 4 out of 5 stars?

   Solution. The answer is $\boxed{3}$.

   If Eric collects one 5 star vote, then the average is 3, and if he collects two, then the average is $\frac{11}{3}$. Thus he needs to collect at least 3 votes, which will bring the average rating to exactly 4.

2. [5] Bessie shuffles a standard 52-card deck and draws five cards without replacement. She notices that all five of the cards she drew are red. If she draws one more card from the remaining cards in the deck, what is the probability that she draws another red card?

   Solution. The answer is $\boxed{\frac{21}{47}}$.

   After Bessie draws five red cards, there are $26 - 5 = 21$ red cards among the remaining $52 - 5 = 47$ cards. Thus the probability that her next card is red is $\dfrac{21}{47}$.

3. [5] Find the value of $121 \cdot 1020304030201$.

   Solution. The answer is $\boxed{123456787654321}$.

   Observe that $121 = 11^2$ and $1020304030201 = 1010101^2$. Thus the product is equal to $(11 \cdot 1010101)^2 = (11111111)^2 = 123456787654321$.

2.4.2 Round 2

4. [7] Find the smallest positive integer c for which there exist positive integers a and b such that $a \neq b$ and $a^2 + b^2 = c$.

   Solution. The answer is $\boxed{5}$.

   Assume $a > b$. Then $a \geq 2$ and $b \geq 1$, so $a^2 + b^2 \geq 5$, so the smallest such integer c is $c = 5$.

5. [7] A semicircle with diameter AB is constructed on the outside of rectangle $ABCD$ and has an arc length equal to the length of BC. Compute the ratio of the area of the rectangle to the area of the semicircle.

   Solution. The answer is $\boxed{4}$.

   Suppose $AB = 2r$, and the length of arc AB and line segment BC are both l. Then $[ABCD] = 2rl$, and the area of semicircle AB is $\frac{rl}{2}$, so the ratio is 4.

6. [7] There are 10 monsters, each with 6 units of health. On turn n, you can attack one monster, reducing its health by n units. If a monster's health drops to 0 or below, the monster dies. What is the minimum number of turns necessary to kill all of the monsters?

Solution. The answer is $\boxed{13}$.

For every turn after turn 6, the attack will kill one monster no matter what. Therefore, to achieve the minimum number of turns, we should spend the first 5 turns completely killing as many monsters as we can. Since we do 15 damage on the first 5 turns, we can kill 2 monsters, but not 3. It takes 8 more turns to kill the remaining 8 monsters, for a total of 13 turns.

2.4.3 Round 3

7. [9] It is known that 2 students make up 5% of a class, when rounded to the nearest percent. Determine the number of possible class sizes.

   Solution. The answer is $\boxed{8}$.

   Suppose that there are N students in the class. From the problem statement we have $4.5\% \leq \frac{2}{N} < 5.5\%$. So by taking reciprocals, we get

$$\frac{100}{4.5} \geq \frac{N}{2} > \frac{100}{5.5};$$

$$\frac{400}{9} > 44 \geq N \geq 37 > \frac{400}{11}.$$

   Therefore N can be any integer between 37 and 44 inclusive, so there are $44 - 37 + 1 = 8$ possibilities in all.

8. [9] At 17:10, Totoro hopped onto a train traveling from Tianjin to Urumuqi. At 14:10 that same day, a train departed Urumuqi for Tianjin, traveling at the same speed as the 17:10 train. If the duration of a one-way trip is 13 hours, then how many hours after the two trains pass each other would Totoro reach Urumuqi?

   Solution. The answer is $\boxed{8}$.

   Let t be the number of hours that Totoro has traveled when the two trains pass each other. Since the train from Urumuqi departs 3 hours earlier, it has been traveling for $t + 3$ hours. We are also given that the one way trip time is 13 hours, so $t + (t + 3) = 13 \to t = 5$. Thus Totoro has $13 - 5 = 8$ hours left on his train.

9. [9] Chad has 100 cookies that he wants to distribute among four friends. Two of them, Jeff and Qiao, are rivals; neither wants the other to receive more cookies than they do. The other two, Jim and Townley, don't care about how many cookies they receive. In how many ways can Chad distribute all 100 cookies to his four friends so that everyone is satisfied? (Some of his four friends may receive zero cookies.)

   Solution. The answer is $\boxed{2601}$.

   We see that Jeff and Qiao must receive the same number of cookies. Thus if each receives k cookies, then there are $100 - 2k$ cookies to split between the two other friends, which we can do in $101 - 2k$ ways. By summing $101 - 2k$ over all k, $0 \leq k \leq 50$, we see that our answer is the sum of the first 51 odd numbers, equal to $51^2 = 2601$.

2.4.4 Round 4

10. [11] Compute the smallest positive integer with at least four two-digit positive divisors.

Solution. The answer is $\boxed{48}$.

Let the number be N. If $N < 50$, then its 4 two-digit divisors must be $N, \frac{N}{2}, \frac{N}{3}, \frac{N}{4}$. This requires that $N \geq 40$ and that N is a multiple of 2, 3 and 4. It is easy to see that $N = 48$ is the smallest such inteeger, and indeed 48, 24, 16, 12 are all its two-digit divisors.

11. [11] Let $ABCD$ be a trapezoid such that AB is parallel to CD, $BC = 10$ and $AD = 18$. Given that the two circles with diameters BC and AD are tangent, find the perimeter of $ABCD$.

Solution. The answer is $\boxed{56}$.

Let M be the midpoint of BC and N be the midpoint of AD. Because the circles are tangent, we have $MN = \frac{10}{2} + \frac{18}{2} = 14$. In addition, observe that $MN = \frac{AB+CD}{2}$. Thus the perimeter of $ABCD$ is $AB + BC + CD + AD = 10 + 18 + 2 \cdot MN = 56$.

12. [11] How many length ten strings consisting of only As and Bs contain neither "BAB" nor "BBB" as a substring?

Solution. The answer is $\boxed{169}$.

The condition is equivalent to the statement that no two Bs have exactly one letter between them. Consider the two length five strings derived from the even-indexed and odd-indexed letters of the original string. Observe that neither of these two strings can have two consecutive Bs.

Now, suppose the total number of length n strings with no two consecutive Bs is F_n. If the first letter is A, then there are F_{n-1} possibilities; if the first letter is B, then the second letter must be A, leaving us with F_{n-2} more possibilities. Therefore, $F_n = F_{n-1} + F_{n-2}$, which yields the Fibonacci recursion with $F_1 = 2$ and $F_2 = 3$. Thus, it is clear that $F_5 = 13$ and that the answer to this problem is $(F_5)^2 = 169$.

2.4.5 Round 5

Each of the three problems in this round depends on the answer to two of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.

13. [13] Let B be the answer to problem 14, and let C be the answer to problem 15. A quadratic function $f(x)$ has two real roots that sum to $2^{10} + 4$. After translating the graph of $f(x)$ left by B units and down by C units, the new quadratic function also has two real roots. Find the sum of the two real roots of the new quadratic function.

Solution. The answer is $\boxed{4}$.

Note that translating the quadratic down by C does not affect the sum of the roots. However, translating the quadratic to the left by B decreases the value of both roots by B, making the new sum $1028 - 2B$.

14. [13] Let A be the answer to problem 13, and let C be the answer to problem 15. In the interior of angle $NOM = 45°$, there is a point P such that $\angle MOP = A°$ and $OP = C$. Let X and Y be the reflections of P over MO and NO, respectively. Find $(XY)^2$.

   Solution. The answer is $\boxed{512}$.

   Because of reflections, $\angle XOM = A°$ and $\angle NOY = (45 - A)°$, which means that $\angle XOY = \angle XOM + \angle MON + \angle NOY = 90°$. Also, $XO = YO = PO = C$, so we conclude that $(XY)^2 = (\sqrt{2}C)^2 = 2C^2$.

15. [13] Let A be the answer to problem 13, and let B be the answer to problem 14. Totoro hides a guava at point X in a flat field and a mango at point Y different from X such that the length XY is B. He wants to hide a papaya at point Z such that YZ has length A and the distance ZX is a nonnegative integer. In how many different locations can he hide the papaya?

   Solution. The answer is $\boxed{16}$.

   First we note that the answer to this problem, C must be a nonnegative integer. We also have $B = 2C^2$ and $A = 1028 - 2B$, so A and B are also nonnegative integers. We also notice that X different from Y means that B is positive. By the triangle inequality,

$$|A - B| \leq ZX \leq A + B.$$

   If $ZX = |A - B|$, there is only one location Z he can hide the papaya, namely, the point on XY such that $YZ = A$ and $ZX = |A - B|$. Similarly, if $ZX = A + B$, there is only one location. For all

$$|A - B| < ZX < A + B,$$

   there are 2 locations, reflections of each other over XY. In total, there are

$$2 + 2(A + B - |A - B| - 1) = 4\min(A, B)$$

   locations to hide the papaya.

   Note: It remains to solve a system of three equations generated by the answers to the problems:

$$\{A = 1028 - 2B, B = 2C^2, C = 4\min(A, B)\}.$$

   If $B \geq A$, then

$$C = 4\min(A, B) = 4B \to B = 2C^2 = 2(4B)^2 = 32B^2,$$

   implying that $B = 0$ or $B = \frac{1}{32}$. None of these values work, so thus $A < B$, yielding

$$C = 4A \to B = 2C^2 = 32A^2 \to A = 1028 - 2B = 1028 - 64A^2.$$

   Solving the quadratic yields $A = 4$ or $A = -\frac{257}{64}$. We conclude that $A = 4, B = 512, C = 16$.

2.4.6 Round 6

16. [15] Let $ABCD$ be a trapezoid such that AB is parallel to CD, $AB = 4$, $CD = 8$, $BC = 5$, and $AD = 6$. Given that point E is on segment CD and that AE is parallel to BC, find the ratio between the area of trapezoid $ABCD$ and the area of triangle ABE.

Solution. The answer is $\boxed{3}$.

Suppose the height of the trapezoid is h. Then the altitude of triangle ABE from E to AB also has length h. Thus the ratio between the area of trapezoid $ABCD$ and the area of triangle ABE equals

$$\frac{(AB + CD) \cdot h/2}{AB \cdot h/2} = \frac{4+8}{4} = 3.$$

17. [15] Find the maximum possible value of the greatest common divisor of $\overline{MOO}$ and $\overline{MOOSE}$, given that S, O, M, and E are some nonzero digits. (The digits S, O, M, and E are not necessarily pairwise distinct.)

Solution. The answer is $\boxed{98}$.

Note that $\overline{MOOSE} = 100 \cdot \overline{MOO} + \overline{SE}$. Thus $\gcd(\overline{MOO}, \overline{MOOSE}) = \gcd(\overline{MOO}, \overline{SE})$. Since M is a non-zero digit, $\overline{SE}$ is less than $\overline{MOO}$. This means that the gcd is at most $\overline{SE}$, which occurs when $\overline{MOO}$ is a multiple of $\overline{SE}$. Therefore, the largest possible gcd is 99. However, it is not difficult to check that 99 has no three-digit multiple that ends with two of the same digit. Looking at the next largest number, we see that we do have a solution, with $\gcd(588, 98) = 98$.

Note: We can check 98 and 99 by writing them as $100-k$, in which case we want $n\cdot(100-k) = \overline{n00}-nk$ to end with two of the same digit. When checking 99, $k = 1$, so n has to be 1 for the last two digits to be the same, which clearly doesn't work. When checking 98, we have $k = 2$. This works if we want the last two digits to be 88: $nk = 2n = 100 - 88 \Rightarrow n = 6$.)

18. [15] Suppose that 125 politicians sit around a conference table. Each politician either always tells the truth or always lies. (Statements of a liar are never completely true, but can be partially true.) Each politician now claims that the two people beside them are both liars. Suppose that the greatest possible number of liars is M and that the least possible number of liars is N. Determine the ordered pair (M, N).

Solution. The answer is $\boxed{(83, 63)}$.

If a politician always tells truth, then his/her two neighbors are liars. On the other hand, if a politician is a liar, then at least one of his/her neighbors always tells truth. From this we deduce that there cannot be two truth-tellers in a row, nor can there be three liars in a row. So between two truth-tellers there is either one or two liars. So if there are L liars, we require that $1 \le \frac{L}{125-L} \le 2$. Solving this inequality, we get that $\frac{250}{3} \ge L \ge \frac{125}{2}$. So $M = 83, N = 63$.

Now we need to construct the case where we achieve maximum and minimum respectively. For ease of reference label each person 1, 2, $\cdots$, 125 in clockwise order. The minimum can be achieved when 1, 3, 5, $\cdots$, 123 are all truth-tellers, for 63 liars in all. The maximum can be achieved when 1, 4, 7, $\cdots$, 124 are all truth-tellers, for 83 liars in all.

2.4.7 Round 7

19. [18] Define a *lucky* number as a number that only contains 4s and 7s in its decimal representation. Find the sum of all three-digit lucky numbers.

Solution. The answer is $\boxed{4884}$.

Observe that the average of all of the numbers is $\frac{4+7}{2} \cdot 111$. Since there are $2^3 = 8$ lucky numbers, their sum must be $8 \cdot \frac{4+7}{2} \cdot 111 = 4884$.

20. [18] Let line segment AB have length 25 and let points C and D lie on the same side of line AB such that $AC = 15$, $AD = 24$, $BC = 20$, and $BD = 7$. Given that rays AC and BD intersect at point E, compute $EA + EB$.

    Solution. The answer is $\boxed{55}$.

    It is trivial to verify that $\angle BDA = \angle ACB = 90°$. So we see that triangle EDA is similar to triangle ECB because they share angle E and both have a right angle. Now let $EC = x, ED = y$, and we have the following set of equations (by using the pair of similar triangles):

$$\frac{x}{20} = \frac{y}{24},$$

$$\frac{x+15}{24} = \frac{y+7}{20}.$$

    Solving the set of equations gives $x = 15, y = 18$, so $EA + EB = (x + 15) + (y + 7) = 55$.

21. [18] A 3×3 grid is filled with positive integers and has the property that each integer divides both the integer directly above it and directly to the right of it. Given that the number in the top-right corner is 30, how many distinct grids are possible?

    Solution. The answer is $\boxed{6859}$.

    It is clear that all integers in the grid should have no prime factors other than 2, 3, or 5, and these prime factors should only divide each of them at most once. Define a *0-1 grid* as a 3 by 3 grid filled with integers 0 or 1, with the top-right corner filled with 1, such that each integer is no larger than the one above or to the right of it. It is clear that for each prime factor, we can use a 0-1 grid to show the divisibility of each number in the grid by this factor (for example, if a number is even, it can be replaced by 1 in the 0-1 grid for the prime 2, and otherwise it can be replaced by 0 in the 0-1 grid). Also we can see that we can set up a one-to-one correspondence from the original 3 by 3 grid to an ordered triple of 0-1 grids (each 0-1 grid is for one prime factor). Therefore, it suffices to find the number of 0-1 grids and cube that number to get our desired answer.

    If we observe each row of 0-1 grid, it should be one of 000, 001, 011, or 111. And a row should contain at least as many 1s as the row below it. Therefore, the number of 0-1 grids is equal to the number of ordered triples (a, b, c) of integers between 0 and 3 inclusive, such that $a \geq 1$ and $a \geq b \geq c$.

    It is not difficult to see that when $a = 3$, there are 10 ways to set a value for b and c, and there are 6 and 3 ways when $a = 2$, $a = 1$ respective, so the number of such triples is $10+6+3 = 19$, and therefore there are 19 0-1 grids in all. By cubing the number, we get the answer $19^3 = 6859$.

2.4.8 Round 8

22. [22] Define a sequence of positive integers $s_1, s_2, \ldots, s_{10}$ to be *terrible* if the following conditions are satisfied for any pair of positive integers i and j satisfying $1 \leq i < j \leq 10$:

- $s_i > s_j$
- $j - i + 1$ divides the quantity $s_i + s_{i+1} + \cdots + s_j$

Determine the minimum possible value of $s_1 + s_2 + \cdots + s_{10}$ over all terrible sequences.

Solution. The answer is $\boxed{100}$.

The second condition is equivalent to the mean of any set of consecutive numbers in the sequence is an integer. Observe that for all integer $9 \geq i \geq 1$, we have $s_i - s_{i+1} \geq 2$. Otherwise if $s_i - s_{i+1} = 1$, 2 won't divide $s_i + s_{i+1}$, which is odd. Since $s_{10} \geq 1$, from the preceeding observation we see that $s_9, s_8, s_7, \cdots, s_1$ are at least $3, 5, 7, \cdots, 19$ respectively.

Now we show that this minimization is optimal, that is, the sequence $19, 17, \cdots, 1$ is truly terrible. Observe that $s_i = (10-i)^2 - (9-i)^2$ for all i between 1 and 10 inclusive. Thus $s_i + s_{i+1} + \cdots + s_j = (10-i)^2 - (9-j)^2 = (19-i-j)(1+j-i)$, and the sum is clearly divisible by $j - i + 1$. Noting that the sum of the entire sequence is $10^2 = 100$, the desired answer is therefore 100.

23. [22] The four points (x, y) that satisfy $x = y^2 - 37$ and $y = x^2 - 37$ form a convex quadrilateral in the coordinate plane. Given that the diagonals of this quadrilateral intersect at point P, find the coordinates of P as an ordered pair.

    Solution. The answer is $\boxed{(-\frac{1}{2}, -\frac{1}{2})}$.

    Subtracting the two equations gives us $x - y = y^2 - x^2 \Leftrightarrow (x-y)(x+y-1) = 0$. Thus each of the solutions to this system satisfy at least one of $x - y = 0$ and $x + y - 1 = 0$. These, then, must be two of the lines that the vertices of the quadrilateral lie on. By symmetry, if (x, y) is a solution, then (y, x) is also a solution, therefore the solutions on the line $x + y = 1$ must be symmetric about $y = x$, and thus lie on different sides of the line. Because we are given that the quadrilateral formed by the solutions is convex, these lines must also be the diagonals of the quadrilateral. Hence we can solve this system of linear equations to find that the diagonals intersect at $(-\frac{1}{2}, -\frac{1}{2})$.

24. [22] Consider a non-empty set of segments of length 1 in the plane which do not intersect except at their endpoints. (In other words, if point P lies on distinct segments a and b, then P is an endpoint of both a and b.) This set is called *3-amazing* if each endpoint of a segment is the endpoint of exactly three segments in the set. Find the smallest possible size of a 3-amazing set of segments.

    Solution. The answer is $\boxed{12}$.

    Since each endpoint is an endpoint of exactly 3 segments, and each segment has 2 endpoints, if we assume that there are $2k$ endpoints there would be $\frac{2k \cdot 3}{2} = 3k$ segments. Now we claim that there doesn't exist a 3-amazing construction for $k < 4$.

    First when $k = 1$, we have 2 points and 3 segments, but there can be at most 1 segment between 2 endpoints, so there's no such 3-amazing set. When $k = 2$, we have 4 points and 6 segments, implying that each two points should be 1 apart from each other, yielding the only possible configuration of a regular tetrahedron, which is not planar.

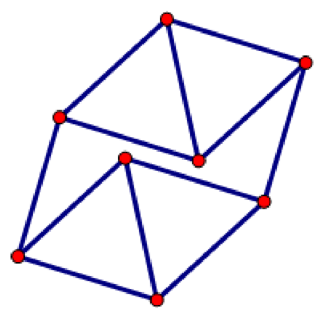
    When $k = 3$, we have 6 points and 9 segments. Call these points A, B, C, D, E, F, and WLOG assume A is connected to B, C, D. From the previous case we see that at least one of BC, CD, DB is not 1, so WLOG assume CD is not connected. We now split into cases.

Case 1: BC and BD are both connected. So C and D both needs to connect to one more point while E and F needs to connect to three more points. This implies that both E and F are connected to both C and D, yielding contradiction.

Case 2: One of BC, BD is connected. WLOG assume BC is connected. So D needs to be connected to both E and F. Moreover, B and C needs to be connected to 1 more point, and E and F both needs 2. So WLOG assume BE, CF, EF are connected. Therefore ABC and DEF are both equilateral triangles, with AD, BE, CF connected. It is easy to show that this configuration must involve intersections, so it is not possible to achieve this case.

Case 3: Neither BC not BD is connected. So all of B, C, D needs to be connected to E and F. This means that $BE = BF = CE = CF = DE = DF = 1$. But given two points E, F in the plane, there can be at most two distinct points P such that $PE = PF = 1$, which means that at least two of B, C, D are the same point, yielding contradiction.

However, there exists a construction of 3-amazing sets with 12 segments:



We showed that there are no 3-amazing sets with less than 12 segments and there is such a set with exactly 12 segments, so the answer is 12.