Nonlinear Hamiltonian analysis of the novel Poincaré gauge theories

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I. INTRODUCTION

Break quadratic action down using irreducible projection operators (IPOs) with respect to SO(1, 3) – six for curvature and three for torsion. We write this as

$$\begin{aligned} L_{\mathrm{T}} = &\sum_{I=1}^{3} m_{\mathrm{p}}{}^{2}\hat{\beta}_{I}\mathcal{T}^{i}_{\ jk}\ {}^{I}\mathcal{P}_{i\ \ l}^{\ jk\ \ nm}\mathcal{T}^{l}_{\ nm} \\ &+\sum_{I=1}^{6}\hat{\alpha}_{I}\mathcal{R}^{ij}_{\ \ kl}\,{}^{I}\mathcal{P}_{ij\ \ nm}^{\ \ kl\ \ pq}\mathcal{R}^{nm}_{\ \ \ pq}+L_{\mathrm{m}} \end{aligned} \tag{1}$$

The *canonical* momenta are

$$\pi_{i}^{\ \mu}\equiv\frac{\partial bL}{\partial\partial_{0}b^{i}_{\ \mu}},\quad \pi_{ij}^{\ \ \mu}\equiv\frac{\partial bL}{\partial\partial_{0}A^{ij}_{\ \ \mu}} \tag{2}$$

from which we define *parallel* momenta $\hat{\pi}_{i}^{\ \overline{k}}\equiv\pi_{i}^{\ \alpha}b^{k}_{\ \alpha}$ and $\hat{\pi}_{ij}^{\ \ \overline{k}}\equiv\pi_{ij}^{\ \ \alpha}b^{k}_{\ \alpha}$, which are

$$J^{-1}\hat{\pi}_{i}^{\ \overline{k}}=\frac{\partial L}{\partial\mathcal{T}^{i}_{\ \perp\overline{k}}}=\sum_{I=1}^{3}4m_{\mathrm{p}}{}^{2}\hat{\beta}_{I}\,{}^{I}\mathcal{P}_{i\ \ n}^{\ \perp\overline{k}\ \ mo}\mathcal{T}^{n}_{\ mo}\,, \tag{3a}$$

$$J^{-1}\hat{\pi}_{ij}^{\ \ \overline{k}}=\frac{\partial L}{\partial\mathcal{R}^{ij}_{\ \ \perp\overline{k}}}=\sum_{I=1}^{6}8\hat{\alpha}_{I}\,{}^{I}\mathcal{P}_{ij\ \ mn}^{\ \ \perp\overline{k}\ \ pq}\mathcal{R}^{mn}_{\ \ \ pq}. \tag{3b}$$

Next we project out O(3) irreps using n_i to break Lorentz symmetry (not really broken because of lapse and shift).

$$\varphi\equiv J^{-1}\hat{\pi}=, \tag{4a}$$

$$\overset{\wedge}{\varphi}_{\overline{kl}}\equiv J^{-1}\overset{\wedge}{\hat{\pi}}_{\overline{kl}}=, \tag{4b}$$

$$\varphi_{\perp\overline{k}}\equiv J^{-1}\hat{\pi}_{\perp\overline{k}}=, \tag{4c}$$

$$\tilde{\varphi}_{\overline{kl}}\equiv J^{-1}\tilde{\hat{\pi}}_{\overline{kl}}=, \tag{4d}$$

$$\varphi_{\perp}\equiv J^{-1}\hat{\pi}_{\perp}=, \tag{4e}$$

$${}^{\mathrm{P}}\varphi\equiv J^{-1}\,{}^{\mathrm{P}}\hat{\pi}=, \tag{4f}$$

$$\overset{\wedge}{\varphi}_{\perp\overline{kl}}\equiv J^{-1}\overset{\wedge}{\hat{\pi}}_{\perp\overline{kl}}=, \tag{4g}$$

$$\vec{\varphi}_{\overline{k}}\equiv J^{-1}\vec{\hat{\pi}}_{\overline{k}}=, \tag{4h}$$

$$\tilde{\varphi}_{\perp\overline{kl}}\equiv J^{-1}\tilde{\hat{\pi}}_{\perp\overline{kl}}=, \tag{4i}$$

$${}^{\mathrm{T}}\varphi_{\overline{klo}}\equiv J^{-1}\,{}^{\mathrm{T}}\hat{\pi}_{\overline{klo}}=. \tag{4j}$$

Next we evaluate Poisson brackets between these. The following very useful identities for this process in ADM are not given in the literature:

$$\begin{aligned} \frac{\partial n_{l}}{\partial b^{k}_{\ \mu}}&=-n_{k}h_{\overline{l}}^{\ \mu},\quad \frac{\partial h_{l}^{\ \nu}}{\partial b^{k}_{\ \mu}}=-h_{k}^{\ \nu}h_{l}^{\ \nu},\\ \frac{\partial b}{\partial b^{k}_{\ \nu}}&=bh_{k}^{\ \nu},\quad \frac{\partial J}{\partial b^{k}_{\ \nu}}=Jh_{\overline{k}}^{\ \nu}. \end{aligned} \tag{5}$$

The super-Hamiltonian is

$$\mathcal{H}_{\perp}\equiv\hat{\pi}_{i}^{\ \overline{k}}\mathcal{T}^{i}_{\ \perp\overline{k}}+\frac{1}{2}\hat{\pi}_{ij}^{\ \ \overline{k}}\mathcal{R}^{ij}_{\ \ \perp\overline{k}}-JL-n^{k}D_{\alpha}\pi_{k}^{\ \alpha} \tag{6}$$

from which we eventually get

$$\begin{aligned} \mathcal{H}_{\perp}=&\sum_{I=1}^{3}m_{\mathrm{p}}{}^{2}J\hat{\beta}_{I}\Big[4\mathcal{T}^{i}_{\ \perp\overline{k}}\ {}^{I}\mathcal{P}_{i\ \ j}^{\ \perp\overline{k}\ \ \perp\overline{l}}\mathcal{T}^{j}_{\ \perp\overline{l}}\\ &-\mathcal{T}^{i}_{\ \overline{mk}}\ {}^{I}\mathcal{P}_{i\ \ j}^{\ \overline{mk}\ \ \overline{nl}}\mathcal{T}^{j}_{\ \overline{nl}}\Big]\\ &+\sum_{I=1}^{6}J\hat{\alpha}_{I}\Big[4\mathcal{R}^{ip}_{\ \ \perp\overline{k}}\,{}^{I}\mathcal{P}_{ip\ \ jq}^{\ \ \perp\overline{k}\ \ \perp\overline{l}}\mathcal{R}^{jq}_{\ \ \perp\overline{l}}\\ &-\mathcal{R}^{ip}_{\ \ \overline{mk}}\,{}^{I}\mathcal{P}_{ip\ \ jq}^{\ \ \overline{mk}\ \ \overline{nl}}\mathcal{R}^{jq}_{\ \ \overline{nl}}\Big]-n^{k}D_{\alpha}\pi_{k}^{\ \alpha} \end{aligned} \tag{7}$$

– thus easy to see that super-Hamiltonian decomposes into quadratic canonical terms and terms which can be made canonical by substituting with any nonvanishing constraint functions.

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II. MASSIVE THEORIES

The simplest theories (constraints are only functions of momenta) turn out to all be massive-only in the weak regime. Obviously this is quite useless, but we can use it as a staging area for the supposedly viable theories. We construct the Poisson matrices of primary if-constraints (PICs) to be strictly on-shell. The entries are purely schematic, and represent some usually long expressions which currently exist only on paper. As expected, all nonzero Poisson brackets for the selected cases are linear in the momenta. The labels around the edge give the O(3) irreps of the constraint fields, with multiplicities. For ease, the matrices are divided into quadrants (should more than one quadrant exist) for translational, rotational and mixed brackets. The mixed brackets are useful for chain termination.

Case 20:

$$\begin{array}{c|ccccc|c}
 & \varphi_{\perp} & \widehat{\varphi}_{\perp\overline{kl}} & \vec{\varphi}_{\overline{k}} & \widetilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klo}} & \\
\varphi_{\perp} & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\widehat{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\vec{\varphi}_{\overline{k}} & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\widetilde{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\
{}^{\mathrm{T}}\varphi_{\overline{klo}} & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\
 & 1 & 3 & 3 & 5 & 5 &
\end{array} \tag{8}$$

Case 24:

$$\begin{array}{c|cccc|c}
 & \widetilde{\varphi}_{\overline{kl}} & \varphi_{\perp} & \widetilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klo}} & \\
\widetilde{\varphi}_{\overline{kl}} & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi} & 5 \\
\varphi_{\perp} & \cdot & \cdot & \cdot & \cdot & 1 \\
\widetilde{\varphi}_{\perp\overline{kl}} & \hat{\pi} & \cdot & \cdot & \cdot & 5 \\
{}^{\mathrm{T}}\varphi_{\overline{klo}} & \hat{\pi} & \cdot & \cdot & \cdot & 5 \\
 & 5 & 1 & 5 & 5 &
\end{array} \tag{9}$$

Case 25:

$$\begin{array}{c|cccccc|c}
 & \widetilde{\varphi}_{\overline{kl}} & \varphi_{\perp} & \widehat{\varphi}_{\perp\overline{kl}} & \vec{\varphi}_{\overline{k}} & \widetilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klo}} & \\
\widetilde{\varphi}_{\overline{kl}} & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & \hat{\pi} & 5 \\
\varphi_{\perp} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\widehat{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\vec{\varphi}_{\overline{k}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\widetilde{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\
{}^{\mathrm{T}}\varphi_{\overline{klo}} & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\
 & 5 & 1 & 3 & 3 & 5 & 5 &
\end{array} \tag{10}$$

Case 26:

$$\begin{array}{c|cccccccc|c}
 & \varphi & \varphi_{\perp\overline{k}} & \widetilde{\varphi}_{\overline{kl}} & \varphi_{\perp} & \widehat{\varphi}_{\perp\overline{kl}} & \vec{\varphi}_{\overline{k}} & \widetilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klo}} & \\
\varphi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\varphi_{\perp\overline{k}} & \cdot & \hat{\pi} & \cdot & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & 3 \\
\widetilde{\varphi}_{\overline{kl}} & \cdot & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & \hat{\pi} & 5 \\
\varphi_{\perp} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\widehat{\varphi}_{\perp\overline{kl}} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\vec{\varphi}_{\overline{k}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\widetilde{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\
{}^{\mathrm{T}}\varphi_{\overline{klo}} & \cdot & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\
 & 1 & 3 & 5 & 1 & 3 & 3 & 5 & 5 &
\end{array} \tag{11}$$

Case 28:

$$\begin{array}{c|cccccc|c}
 & \varphi & \varphi_{\perp\overline{k}} & \widetilde{\varphi}_{\overline{kl}} & \varphi_{\perp} & \widetilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klo}} & \\
\varphi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\varphi_{\perp\overline{k}} & \cdot & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi} & \hat{\pi} & 3 \\
\widetilde{\varphi}_{\overline{kl}} & \cdot & \cdot & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi} & 5 \\
\varphi_{\perp} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\
\widetilde{\varphi}_{\perp\overline{kl}} & \cdot & \hat{\pi} & \hat{\pi} & \cdot & \cdot & \cdot & 5 \\
{}^{\mathrm{T}}\varphi_{\overline{klo}} & \cdot & \hat{\pi} & \hat{\pi} & \cdot & \cdot & \cdot & 5 \\
 & 1 & 3 & 5 & 1 & 5 & 5 &
\end{array} \tag{12}$$

Case 32:

$$\begin{array}{c|cccccc|c}
 & \varphi & \varphi_{\perp} & \widehat{\varphi}_{\perp\overline{kl}} & \vec{\varphi}_{\overline{k}} & \widetilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klo}} & \\
\varphi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\varphi_{\perp} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\widehat{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\vec{\varphi}_{\overline{k}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\widetilde{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\
{}^{\mathrm{T}}\varphi_{\overline{klo}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\
 & 1 & 1 & 3 & 3 & 5 & 5 &
\end{array} \tag{13}$$

III. MASSLESS THEORIES

From general considerations, there are very serious problems with all the novel theories which have massless modes.

Case 3:

$$\begin{array}{c|ccccc|c}
 & \varphi & \widehat{\varphi}_{\overline{kl}} & \varphi_{\perp} & \widetilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klo}} & \\
\varphi & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & 1 \\
\widehat{\varphi}_{\overline{kl}} & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi} & \hat{\pi} & 3 \\
\varphi_{\perp} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & 1 \\
\widetilde{\varphi}_{\perp\overline{kl}} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & 5 \\
{}^{\mathrm{T}}\varphi_{\overline{klo}} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & 5 \\
 & 1 & 3 & 1 & 5 & 5 &
\end{array} \tag{14}$$

Case 17:

$$\begin{array}{c|cccccc|c}
 & \varphi & \widehat{\varphi}_{\overline{kl}} & \varphi_{\perp} & {}^{\mathrm{P}}\varphi & \widetilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klo}} & \\
\varphi & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\
\widehat{\varphi}_{\overline{kl}} & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi} & \hat{\pi} & \hat{\pi} & 3 \\
\varphi_{\perp} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\
{}^{\mathrm{P}}\varphi & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\
\widetilde{\varphi}_{\perp\overline{kl}} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 5 \\
{}^{\mathrm{T}}\varphi_{\overline{klo}} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 5 \\
 & 1 & 3 & 1 & 1 & 5 & 5 &
\end{array} \tag{15}$$

Currently evaluating

$$
\begin{array}{r|cccccc|l}
 & \varphi & \hat{\varphi}_{\overline{kl}} & \varphi_{\perp} & {}^{\mathrm{P}}\varphi & \widetilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klo}} & \\
\varphi & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\
\hat{\varphi}_{\overline{kl}} & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi} & \hat{\pi} & \hat{\pi} & 3 \\
\varphi_{\perp} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\
{}^{\mathrm{P}}\varphi & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\
\widetilde{\varphi}_{\perp\overline{kl}} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 5 \\
{}^{\mathrm{T}}\varphi_{\overline{klo}} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 5 \\
\multicolumn{1}{r}{} & 1 & 3 & 1 & 1 & 5 & 5 & \multicolumn{1}{l}{}
\end{array}
\tag{16}
$$

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