Math 450 Homework 6
Spatial Models

due April 30, 2021

1. Another important form of Zipf's law is the citation patterns found in scientific articles. Rather than doing equations, let's make a simulation of a simplified version of this.

   (a) Start with a list containing the integers 1 and 2.

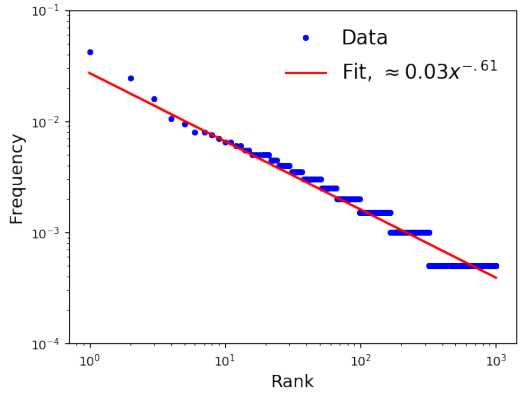
   (b) Choose one number randomly from the list and append it back to the list. You can use the `sample()` function from the "random" module to do this.

   (c) Append the next number (3) to the list.

   (d) Repeat these last two steps 1,000 times, incrementing the new number each time until you get a list of 2,002 integers.

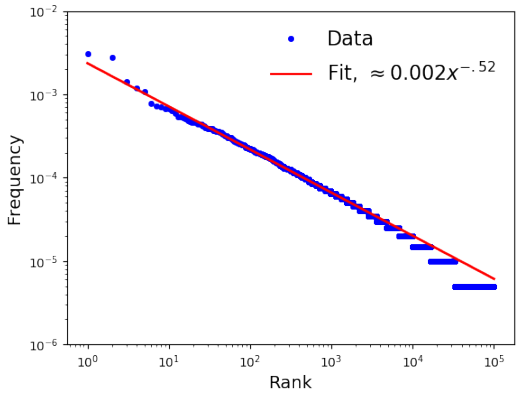
   Now, count up how many times each integer appears in the list and plot a rank-value plot on a log-log scale. Explain how we know that the plot exhibits a power-law scaling, and approximate the scaling exponent.

   After generating our data we create the following rank plot



   We know the plot exhibits a power-law scaling because the data forms a straight line in the log-log plot. We recover the power law $y \sim x^{-0.61}$ (you may get something slightly different).

   If you really want to dive into this problem, can sample 100000 times instead *and* for the fit use only the points that occur frequently (the singletons are an indication of insufficient resolution at those points). Sampling 100000 times and using the 100 most common integers, we recover the power law $\sim x^{-0.52}$, which is near the asymptotic answer of $x^{-}1/2$. The log-log plot looks like:



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2. The "neighborhood" in a regular lattice defines the geometry of the space. In class, we discussed 4 different kinds of neighborhoods for 2-dimensional lattice on a Cartesian grid – 8-neighbor Moore, the 6-neighbor, the 4-neighbor von Neumann, and the 3-neighbor.

   (a) Let $N_k(x, y)$ be the set of neighbors of site (x, y) where x and y are integers. If $k = 3$, then

   $$N_3(x,y) = \begin{cases} \{(x+1,y),(x,y+1),(x,y-1)\} & \text{if } x+y \text{ is even,} \\ \{(x-1,y),(x,y+1),(x,y-1)\} & \text{if } x+y \text{ is odd.} \end{cases}$$

   Find similar formulas for $N_4(x, y)$, $N_6(x, y)$, and $N_8(x, y)$.

   $$N_4(x,y) = \{(x-1,y),(x+1,y),(x,y+1),(x,y-1)\}$$
   $$N_6(x,y) = \{(x-1,y),(x+1,y),(x,y+1),(x,y-1),(x+1,y+1),(x-1,y-1)\}$$
   $$N_8(x,y) = \left\{ \begin{array}{c} (x-1,y),(x+1,y),(x,y+1),(x,y-1), \\ (x+1,y+1),(x-1,y-1),(x+1,y-1),(x-1,y+1) \end{array} \right\}$$

   (b) In a regular lattice, the atomic loop length is the smallest number of neighboring edges in loop from $(0, 0)$ back to itself where the same edge between two neighbors is never traversed more than once. Each such loop with this minimal loop length is called an "atomic loop". For each $k \in \{3, 4, 6, 8\}$, find the atomic loop length and the number of atomic loops containing $(0, 0)$ for lattices with neighborhoods $N_k(x, y)$.

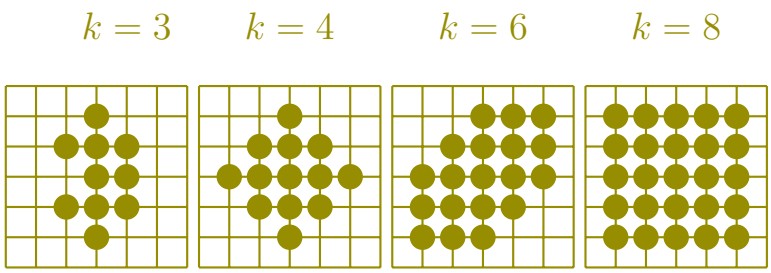
k	atomic loop length	atomic loop count
3	6	3
4	4	4
6	3	6
8	3	12

   When we count loops, we can count them as directed or undirected. If we count them as directed, then each loop can be traversed in two directions, so we would get double the number of atomic loops listed above.

   (c) Given a neighborhood $N_k(x, y)$, we can define a metric $d_k((x, y), (u, v))$ to measure the distance between points (x, y) and (u, v) recursively as follows:

   $$d_k((x,y),(u,v)) = \begin{cases} 0 & \text{if } (x,y) = (u,v), \\ 1 + \min\{d_k((w,z),(u,v)) : (w,z) \in N_k((x,y))\} & \text{otherwise.} \end{cases}$$

   For each $k \in \{3, 4, 6, 8\}$, draw the set of points $\{(u, v) : d_k((0, 0), (u, v)) \leq 2\}$ on the appropriate lattice.



(d) For each $k \in \{3, 4, 6, 8\}$, find $d_k((0,0),(3,3))$.

$d_3((0,0),(3,3)) = 3$, $d_4((0,0),(3,3)) = 6$, $d_6((0,0),(3,3)) \in \{3, 6\}$ depending on picked orientation, $d_8((0,0),(3,3)) = 3$.

3. Apply renormalization theory to estimate the percolation threshold under site percolation with Moore neighborhoods.

The argument is very similar to that given for the von Neumann theory, but the counting of cases a litte different. There are 6 ways to fill in the 2x2 with 2 black marks. With Moore neighborhoods, 4 of them percolate from top to bottom rather than 2. All the rest of the counting is the same, so

$$p_2 = p_1^4 + 4p_1^3(1 - p_1) + 4p_1^2(1 - p_1)^2.$$

Taking $p_2 = p_1 = p$,

$$p = p^4 + 4p^3(1 - p) + 4p^2(1 - p)^2,$$
$$0 = p(p - 1)(p^2 - 3p + 1)$$

Thus, either $p = 0$, $p = 1$, or $p = (3 - \sqrt{5})/2 \approx 0.3819$. This last is the threshold we care about.

4. What is the dual lattice of bond percolation on a hexagonal neighborhood? What does this imply about the percolation thresholds of the two lattices?

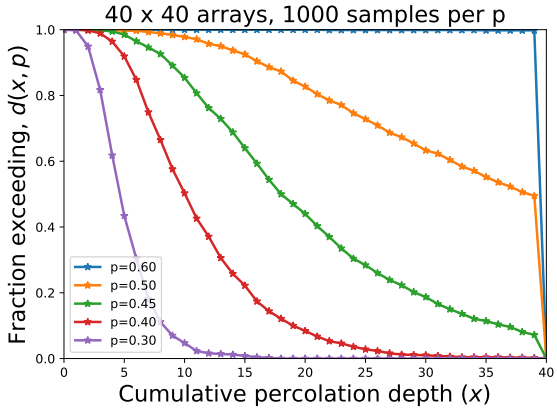
The dual lattice for hexagonal-neighborhood bond percolation is bond percolation on a triangular-neighborhood lattice. Thus, for every six-neighbhor bond lattice, we can find a dual 3-neighbor bond lattice. And just as in our example above, the 6-neighbor lattice will percolate if and only if its dual lattice does not. Thus the sum of the critical void fractions of both must equal 1 ($p_c(3) + p_c(6) = 1$).

5. Calculating percolation depth algorithmically can be done recursively as described in class. See code posted in 'Spatial Models' section on Percolation.

(a) Let $A(p, N)$ be a random 0/1 matrix with shape $N \times N$ where sites are filled (1) with probability p, and empty (0) with probability $1 - p$. Recall that in class, we generated example matrices like this with the python code

$$A = \text{floor}(\text{rand}(\text{N,N}) + \text{p})$$

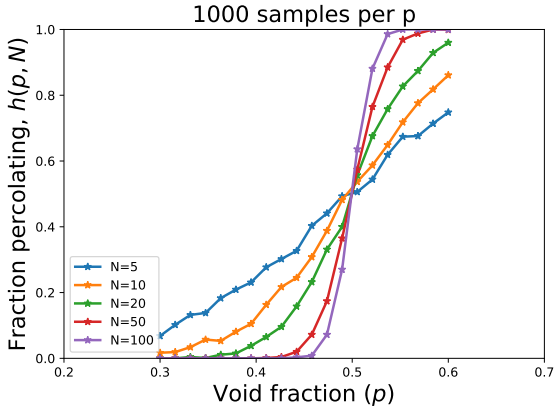
For each value of $p \in \{0.3, 0.4, 0.45, 0.5, 0.6\}$, simulate 1000 40×40 matrices. Let $d(x, p)$ be the fraction of these matrices with percolation depth less than or equal to x. **Assuming a hexagonal neighborhood (6 neighbors)**, plot $d(x, p)$ for each value of p, all in one plot. Remember to label your plot axes and include a legend.



(b) What particular feature of your plot change when p is between 0.45 and 0.5? What does this change mean?

At the right side of the plots, a jump in value appears for all values of $p < 0.5$. This jump corresponds to the maximum percolation depth reaching the full lattice size of 40. This signifies that the maximum percolation depth is exceeding the matrix size for those void-fractions (for small values of p, the percolation depth often reaches its maximum).

(c) Suppose we define $h(p, N)$ as the fraction of $N \times N$ $A(p, N)$ matrices which percolate all the way through. Plot $h(p, N)$ as a function of $p \in [0.3, 0.6]$ (this is an INTERVAL, not a set!) for $N \in \{5, 10, 20, 50, 100\}$. Use atleast 1000 matrices for each. (Warning: This calculation may take you a long time.)



(d) Extrapolating from your plot, what do you think will happen to $\lim_{N\to\infty} h(p, N)$?

The curves are approaching a heaviside-like function with a sharp threshold near void-fraction $p = 0.5$.

(e) Reality, of course, is often three-dimensional, rather than two-dimensional. Describe how you think the percolation threshold will change when we switch from two dimensions to three dimensions and why it will change.

In three dimensions, there are more directions to move relative to two dimensions, so the critical void-fraction won't have to be as it is in two dimensions. That is, for a constant fraction of sites occupied, percolation is easier in 3 dimensions because there are more possible paths between sides, and it's harder to block them all.

6. Search the web for a new cellular automata model that we have not already discussed in class. Your rule should be different from your classmates.

(a) Specify the rules of the cellular automata in enough detail that we can program it.

(b) Describe the emergent phenomena exhibited by your automata. (waves, particles, solitons, oscillators, interfaces, spirals, freezing, clustering, bursting, ...)

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7. In class, we used separation of variables to come up with two approximate formulas for the age of the earth in terms of the initial uniform temperature θ_0, the thermal diffusivity κ, the temperature gradient near the earth's surface ($\theta'(t)$) and a size parameter. Under the linear-bar model, we arrived at

$$t_L = \frac{L^2}{\kappa\pi^2} \ln\left(\frac{4\theta_0}{L\frac{\partial\theta(x=0,t)}{\partial x}}\right)$$

where the length L may be approximated by the diameter of the earth. On a sphere, we arrived at

$$t_r = \frac{r_{\text{earth}}^2}{\kappa\pi^2} \ln\left(\frac{\theta_0}{-\theta_r(r_{\text{earth}}, t)\, r_{\text{earth}}}\right)$$

where r_{earth} is the earth's radius.

(a) Using the estimates of $\theta_0 = 1,200$ degrees K, $\kappa = 2.5 \times 10^{-3}$ meters square per year, $L = 2r_{\text{earth}} = 1.28 \times 10^7$ meters, and $\theta'(t) = 3.6 \times 10^{-2}$ degrees per meter increase as we descend, calculate both t_L and t_r.

Using the given parameters, we get $t_L = -3.03 \times 10^{16}$ and $t_r = -8.72 \times 10^{15}$, two very large negative numbers, which don't seem to make physical sense.

(b) Do your answers make sense? Explain.

First, the numbers should be positive, not negative. Second, they are way to big – the universe is only 10^{10} years old, but these are several magnitudes bigger than even that.

(c) Can you fix the problem by choosing better parameter estimates?

Not completely. There is a typo above – the thermal conductivity should be closer to 2.5×10^3, about 1 million times bigger than listed. But even then, we still get negative numbers for our answers: $t_L = -3.03 \times 10^{10}$ and $t_r = -8.72 \times 10^9$

(d) Give an intuitive explanation of why our calculations failed to make sense.

We arrived at our formulas by using only the first term in the Eigenfunction series solutions of the heat equations we were considering. Unfortunately, ignoring those other terms is the problem – the time it would take those higher-order terms to decay is already longer than the age of the earth. As converge to that

8. As I mentioned in class, there are other ways to solve the heat equation and apply the solution to the question of the earth's age. Imagine an infinite plane, where one side of the plane is empty space, and the other side is solid rock – we call this an "infinite slab", like a foundation slab. If the slab starts at uniform temperature θ_0 and space always has temperature 0, then the solution of the heat equation in the slab at a depth y is given by

$$\theta(y,t) = \theta_0 \operatorname{erf}\left(\frac{y}{2\sqrt{\kappa t}}\right)$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

(a) Show by direct calculation that this solves the heat equation

$$\frac{d\theta}{dt} = \kappa \frac{d^2\theta}{dy^2}$$

and that $\theta(0,t) = 0$.

(b) Use Maclaurin series to find the coefficients of the series approximation

$$\text{erf}(x) \approx a_0 + a_1 x + a_2 x^2 + O(x^3)$$

$$\text{erf}(x) \approx 0 + \frac{2}{\sqrt{\pi}}x - \frac{2}{3\sqrt{\pi}}x^3 + \frac{1}{5\sqrt{\pi}}x^5 \dots$$

(c) Using only the first non-vanishing term in this series approximation, show that

$$\frac{d\theta(0,t)}{dy} \approx \frac{\theta_0}{\sqrt{\pi\kappa t}}$$

Based on the McLaurin series,

$$\theta(x,t) \approx \frac{2\theta_0}{\sqrt{\pi}}\left(\frac{x}{2\sqrt{\kappa t}} - \frac{1}{3}\left(\frac{x}{2\sqrt{\kappa t}}\right)^3 + O(x^5)\right)$$

$$\frac{d\theta(y,t)}{dy} \approx \frac{\theta_0}{\sqrt{\pi\kappa t}}$$

(d) Approximate the age of the earth using this formula and the parameter estimates given above.

$$t = \frac{1}{\kappa\pi}\left(\frac{\theta_0}{\frac{d\theta(0,t)}{dy}}\right)^2 \approx 1.4 \times 10^{11}$$

Using the corrected thermal diffusivity of 2.5×10^3, we get about $141,471$ years old, a number similar to Kelvin's result.