Math 450 Homework 3
Surveying & Differential equations modeling

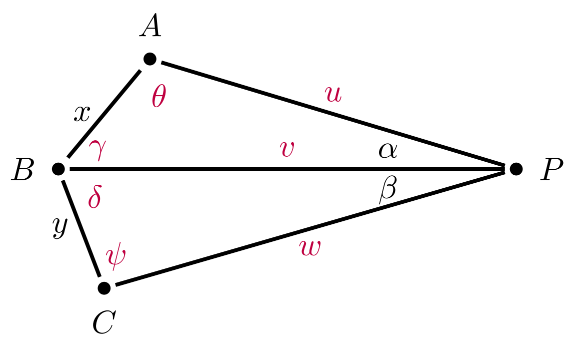
due March 8, 2021

1. Consider the following. An observer knows the coordinates of landmarks A (1,0), B (0,0), and C (0,2). She observes at position P that landmark B appears between A and C, with an angle $\angle APB = 0.1798$ radians and an angle $\angle BPC = 0.3393$ radians. Find the observer's position P, if the observer is farther from B than the other landmarks.

   Solution:

   This is a special case of what is known as Snell's problem because he was the first to solve such. **The answer is that P is at $(3, 4)$.**

   In general: consider the figure below relating the angles and distances between 4 landmarks.



   The observer first measures the distances x and y, as well as the angle ρ between them. Then at point P the observer measures two angles α and β amongst three landmarks. The distances x and y between those landmarks are already known, as is the total angle $\rho = \gamma + \delta$ between the landmarks (but the individual angles γ and δ are not). The goal is to calculate the distances between the observer and each of these cities, thus determining the observer's location. Today, we call this Snel's resection problem. Expressed algebraically, courtesy of the law of sines, we wish to find all unknowns 7 in a system of 7 equations (4 nonlinear, 3 linear)

$$\frac{x}{\sin\alpha} = \frac{v}{\sin\theta} = \frac{u}{\sin\gamma}$$
$$\frac{y}{\sin\beta} = \frac{v}{\sin\phi} = \frac{w}{\sin\delta}$$
$$\rho = \delta + \gamma$$
$$\pi = \gamma + \theta + \alpha,$$
$$\pi = \delta + \psi + \beta.$$

   Dividing the first two pairs of equalities in our initial system, we find

$$Q = \frac{\sin\psi}{\sin\theta} \tag{1}$$

where $Q := (x\sin\beta)/(y\sin\alpha)$ is known. And by adding the last two linear equations and subtracting off the third-to-last, we find

$$K = \theta + \psi,$$

where $K := 2\pi - \rho + (\alpha + \beta)$ is also known. So, we have two equations for θ and ψ. Solving the linear equation and substituting,

$$Q\sin\theta = \sin(K - \theta). \tag{2}$$

Then apply trigonometric expansion,

$$Q\sin\theta = \sin K\cos\theta - \cos K\sin\theta, \tag{3}$$

$$\tan\theta = \frac{\sin K}{Q + \cos K}. \tag{4}$$

So θ can be calculated by tangent inversion, up to multiples of π, with additional geometric information used to select the appropriate solution. Once θ is known, back-substitution for calculation of the rest of the parameters is straight-forward.

2. In surveying a triangular field, you go 5 perches at 53 degrees north of east, 4 perches at 76 degrees south of east, and 4 perches directly west, and end up back where you started. However, when you do the calculations, you discover that this triangle does not close perfectly.

   a. Suppose we have a point in the complex plane given in polar coordinates $z(\delta, \epsilon) = (r - \delta)e^{(\theta-\epsilon)\pi i/180}$, where δ and ϵ are small errors in the observed distance r and bearing θ (in units of degrees). Use Taylor series to find a linear approximation to corrected position $z(\delta, \epsilon)$.

   Solution:

   Taylor series approximation gives

   $$z(\delta,\epsilon) \approx re^{i\pi\theta/180} - \left(\delta + \frac{i\pi r}{180}\epsilon\right)e^{i\pi\theta/180}$$

   b. In polar coordinates, the closure condition for a survey is

   $$\sum_k r_k e^{\theta_k \pi i/180} = 0.$$

   Using the approximation you derived above and the data $(r_1, \theta_1) = (5, 53)$, $(r_2, \theta_2) = (4, -76)$, $(r_3, \theta_3) = (4, 180)$, derive two linear equation for the 6 errors δ_i and ϵ_i. (one from the real part, and one from the imaginary part). Give all numbers to 4 decimal places.

   Solution:

   $$-0.6018\delta_1 - 0.2419\delta_2 + \delta_3 + 0.06969\epsilon_1 - 0.06773\epsilon_2 = 0.02323$$

   %

   $$-0.7986\delta_1 + 0.9702\delta_2 - 0.05251\epsilon_1 - 0.01688\epsilon_2 + 0.06981\epsilon_3 = -0.1119$$

c. Using weighted linear least squares, solve the overdetermined 8 by 6 linear system for the errors δ_i and ϵ_i. Use weights of 10 for each closure equation, 1 for each $\delta_i = 0$ equation, and 1/40 for each $\epsilon_i = 0$ equation. Use `scipy.linalg.lstsq` to solve $WAx = Wb$. Where W is the diagonal weight matrix. Give all answers to 3 decimal places.

Solution:

$$\begin{aligned} \delta_1 &= 0.00661, & \epsilon_1 &= 0.62302, \\ \delta_2 &= -0.00763, & \epsilon_2 &= 0.26027, \\ \delta_3 &= -0.00042, & \epsilon_3 &= -0.88976. \end{aligned}$$

d. Discuss what it means that we solved a linearization of the closure equations rather than the full equations.

Solution:

Because we are using a linear approximation rather than the full equations, our corrections will be imperfect. However, it may be possible to further improve them by re-linearizing around the new corrected values and repeating.

3. One day it started snowing at a heavy and steady rate. A snowplow started out at noon, going 2 miles the first hour and 1 mile the second hour. If the speed of the snowplow is inversely proportional to the depth of the snow, at what time did it start snowing?

Solution:

This is Agnew's Snowplow problem – Agnew was an influential leader of the mathematics department at Cornell. This problem is found in Bender and Orszag. Let's suppose it started snowing at time t_0 at a rate a inches per hour. Then at time t, there were $a(t - t_0)$ inches on the ground. Now, from the description given, we now that the snowplow is slowed down by the snow – the more snow there is, the slower it will go. A natural conjecture, then, is that the speed of the snowplow (say, dx/dt) is inversely proportional to the depth of the snow:

$$\frac{dx}{dt} = \frac{k}{a(t - t_0)}$$

for proportionality constant k. Then, integrating and using $x(12) = 0$ since the snowplow starts out at noon,

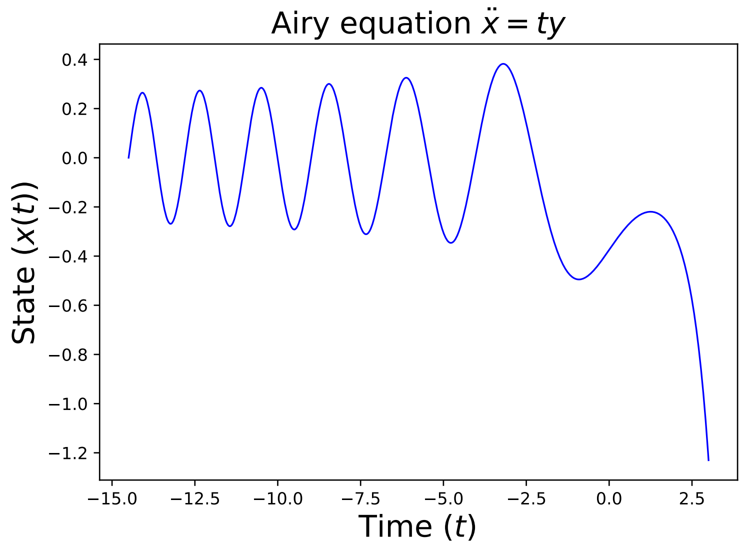
$$x(t) = \frac{k}{a} \log\left(\frac{t - t_0}{12 - t_0}\right)$$

Then, we can use our observations of $x(13)$ and $x(14)$ to determine t_0. 11.3819 == 11:23 am == (25-sqrt(5))/2

4. Rewrite Airy's 2nd-order equation $\ddot{x} = tx$ into a system of two first-order equations and use numerical integration to calculate the solution from initial condition $x(-14.5) = 0$ and $\dot{x}(-14.5) = 1$ until $t = 3$. Then plot the curve. Be sure to sample the curve frequently enough that it appears smooth.

Solution:

This is just practice using the python ODE solver. This second-order equation transforms to the nonlinear system $\dot{u} = v$, $\dot{v} = tu$ when we let $u(t) = x(t)$ and $v(t) = \dot{x}(t)$.



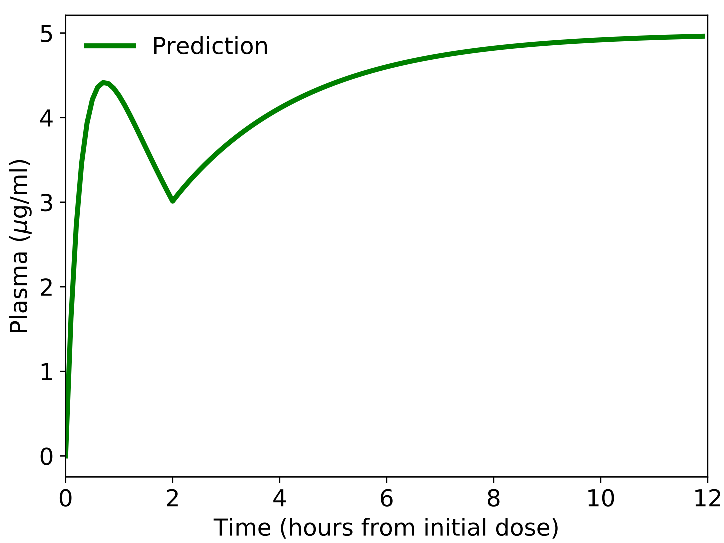
5. Suppose a hospital patient is given a 325 mg dose of acetaminophen orally, and then two hours later is put on an intravenous drip of 10 mg per hour directly into the blood stream. Use our 2-compartment model to predict the acetaminophen plasma concentration over the 12 hours since the oral dose, assume a blood volume of 5 liters.

Solution:

The initial value problem is

$$\begin{aligned}\frac{dx_0}{dt} &= -r_0 x_0 - m_{10} x_0 \\ \frac{dx_1}{dt} &= -r_1 x_1 + m_{10} x_0 + aH(t-12) \\ x_0(0) &= 325\text{mg} \\ x_1(0) &= 0\end{aligned}$$

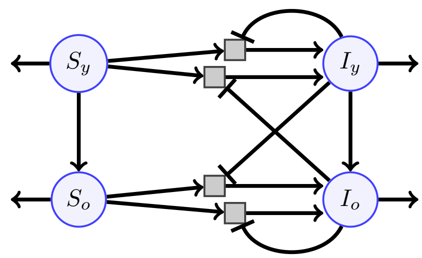
where $H(t-12)$ is the Heaviside function, taking $r_0 = 3$, $m_{10} = 0.3$, $r_1 = 0.4$ as in class, and a=10mg per hour.



6. Little Blue Run is a coal ash pond in Pennsylvania that is leaking selenium into nearby water ways. Make a linear compartmental model that can describe the accumulation of selenium in a nearby lake's bottom sediments, and in fish living in the lake.

   a. Decide on a set of units to be used in your model.
   b. Clearly define a set of state variables for your model and their units.
   c. Identify a set of input, removal, and movement reactions for the selenium in the system, and define rate parameters for each reaction.
   d. Draw and label the reaction network for your model.
   e. Write down a system of ordinary differential equations for your model.
   * **Not for credit** Justify choices for parameter values, solve your model and graph the solution. Don't forget to label axes and include the units in the axis labels.

7. The following diagram represents a compartmental model of infectious disease transmission in a population with two life-history stages (young and old). Translate the diagram into a set of reactions, labelling each reaction arrow with its corresponding rate constant. Then use the reaction network to construct the corresponding system of differential equations.



8. Compartmental differential equation models have also been applied to the theory of war.

   (a) During World War 1, engineer Frederick Lanchester proposed the following equations for the destruction of rival air-forces during World War I. Let $A(t)$ and $B(t)$ be the sizes of rival air-forces over time.

$$\frac{dA}{dt} = -gB, \quad \frac{dB}{dt} = -rA.$$

   The constants g and r are called "force multipliers", and represent things like technological advantages. The "loser" is the first air-force to be destroyed. Find an inequality involving only the initial sizes of each air force $A(0)$ and $B(0)$, along with the constants g and r that predicts how big air force A has to be to win the battle.

   Solution:
   Solve the phase plane equation $dB/dA = rA/gB$ and note that when A wins, $A > 0$, $B = 0$, so find condition on $A(0)$ and $B(0)$, along with the constants g and r for such a solution to exist.
   The answer is

$$A_0 > \sqrt{\frac{g}{r}} B_0.$$

   (b) Guerilla warfare is often contrasted from traditional warfare because the enemy can be hard to locate when in small numbers. This can be applied in the context of the American revolution or the Vietnam war. One way to represent this is with the law of mass action.

$$\frac{dA}{dt} = -gBA, \quad \frac{dB}{dt} = -rA.$$

Find an inequality involving only the initial sizes of each air force $A(0)$ and $B(0)$, along with the constants g and r that predicts how big air force A has to be to win the battle.

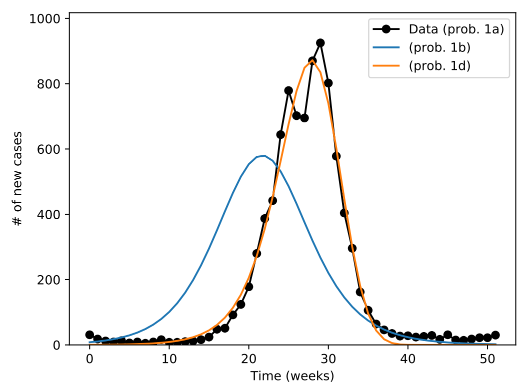
Solution:
Same as (a): solve the phase plane equation $dB/dA = rA/gBA$ and note that when A wins, $A > 0$, $B = 0$, so find condition on $A(0)$ and $B(0)$, along with the constants g and r for such a solution to exist.
The answer is

$$A_0 > \frac{g}{r} B_0^2.$$

9. Data from Kermack and McKendrick's 1927 paper on epidemic modelling includes data on the number of new cases each week during a plague epidemic in Bombay: https://jmconway.org/Math450/Spring2021/Lectures/data/plague_data.txt

(a) Plot the data points (t_i, y_i) where y_i is the number of new cases of plague and t_i is the time, measured in weeks.

Solution:



(b) The simplest modern compartmental model of epidemic dynamics is the SIR model

$$\begin{aligned} S(0) &= S_0, \quad \dot{S} = -\beta S I, \\ I(0) &= I_0, \quad \dot{I} = \beta S I - \gamma I, \end{aligned}$$

where β, γ, S_0, and I_0 are unknown parameters. In this model, the rate of *new* infections is $\beta S I$. Plot $\beta S(t) I(t)$ for the same values of t that are observed in the data when $\beta = 3 \times 10^{-5}$, $\gamma = 0.01$, $S_0 = 9000$, and $I_0 = 30$.

Solution:
See above figure.

(c) Calculate the absolute error

$$E_1 = \sum_i |y_i - \beta S(t_i) I(t_i)|$$

and the square error

$$E_2 = \sum_i (y_i - \beta S(t_i) I(t_i))^2$$

between the data and the model's predicted rate of new cases.

Solution:
$E_1 = 7,620.8, \; E_2 = 2,622,909.9$

(d) Use "scipy.optimize.fmin()" to minimize the square error E_2 over the 4 parameters β, γ, S_0, and I_0. Plot the best fit and the data on the same plot for comparison, and state the best estimates for each of the 4 parameters.

Solution

See above for the plot. The best estimates I found were $\beta = 1.864 \times 10^{-5}, \gamma = -0.1593, S_0 = 8,866, I_0 = 2.1$. However, the parameters are not fully identifiable from the data – you may have different values, with a curve that is about the same as mine. And a negative recovery rate makes no biological sense. Other possible explanations may be that (*) fmin is doing a bad job of minimizing the error, (*) our model is actually a bad model of the data – this is now known to be true. Plague dynamics were largely seasonal in Bombday rather than driven by direct transmission.

(e) One imperfect but commonly used measure of goodness of fit is the "coefficient of determination", which for this problem can be written

$$R^2 = 1 - \frac{\sum_i (y_i - \beta S(t_i) I(t_i))^2}{\sum_i y_i^2}$$

You can think of R^2 as the fraction of the data's information successfully explained by our model; the closer R^2 is to 1, the better the model fits the data.

Calculate the coefficient of determination for your best parameter estimates. (It should be better than 0.95 if you've gotten a good fit.)

Solution:

For my parameters, $R^2 = 0.985$, which is consistent with the graph closely matching the data, but not perfectly matching the data.

Challenge problem, not for credit.

1. Dear Mr. Franklin, As privateering is now so much in Fashion, the printing of the following question may be an amusement, if not to the privateers, yet to some of your correspondents or readers. Suppose a privateer, in the latitude of 10 degrees North, should at 6 in the morning spy a ship due south of her, distant 20 miles; upon which she steers directly for her, and runs at the rate of 8 miles an hour. The ship at the same time sees the privateer, but not being much afraid of her, keeps on her course due west, and sails at the rate of 6 miles an hour; how many hours will it be before the privateer overtakes the ship?
(Letter by Thomas Godfrey to the Ben Franklin's Pennsylvania Gazette, 1743)

Solution:

This problem is a "pursuit" problem, and the solution is called a "curve-of-pursuit". These problems were trendy back in 1740's after Frenchman Pierre Bouguer had revealed their solution. But it would have been extraordinary for Godfrey's problem to have received any solutions from the newspaper's readers.

Suppose the merchant ship starts at position 0, and so, has travelled a distance $6t$ after t hours. Let $(x(t), y(t))$ be the position of the privateer at time t relative to the merchant ship's initial position, so at $t = 0$, $(x(0), y(0)) = (0, 20)$. Since the privateer always steers in the direction of the merchant ship, then by similar triangles,

$$\frac{-\dot{x}}{6t + x} = \frac{-\dot{y}}{y},$$

or, after cross-multiplying,

$$-y\dot{x} + \dot{y}(6t + x) = 0.$$

Now comes the fancy parts. Let's reparameterize our problem in terms of y instead of t. We can divide through by $\dot{y}$ to get

$$-y\frac{dx}{dy} + 6t(y) + x(y) = 0.$$

This looks okay, but we've got this weird $t(y)$ term.

$$-y\frac{d^2x}{dy^2} - \frac{dx}{dy} + 6\frac{dt}{dy} + \frac{dx}{dy} = 0.$$

Since

$$\dot{x}^2 + \dot{y}^2 = 8^2$$

$$\frac{1}{8}\sqrt{\left(\frac{dx}{dy}\right)^2 + 1} = \frac{dt}{dy}$$

Substituting,

$$-y\frac{d^2x}{dy^2} + \frac{6}{8}\sqrt{\left(\frac{dx}{dy}\right)^2 + 1} = 0.$$

This is a second order equation, but can be transformed into a first order equation by taking $f(y) = dx/dy$. The resulting equation is separable and the solution can then be obtained by integration.

$$x(y) = \frac{y}{2}\left[\left(\frac{y}{A}\right)^v - \left(\frac{A}{y}\right)^v\right] + C$$

Numerically, we can calculate the time to intercept and find it is about 5 hours, 43 minutes.