Math 450 Homework 1
Numbers, Spherical trig, and Linkages

due Feb 5, 2021

1. What summary statistic about a set of people is most useful in determining each of the following?

   a. Height of a door

      Maximum height, top 10th percentile in height, or other answers along those lines.

   b. Maximum capacity of a ferry

      Maximum weight, top 10th percentile in weight. To be extra safe. But if your answer is reasonable, e.g. mean b/c some people are above it, some people are below it, and there are a lot of people on a boat, that's good too!

   c. Width of a cellphone

      I would say measurements between 1 standard deviation of the mean in hand size – you want it comfortable for most people. But so long as your answer is reasonable you're good!

   d. Width between seat rows on an airplane

      Maximum leg length, top 10th percentile in leg length, answers along those lines. (Though it feels like airlines actually use the shortest leg lengths!)

2. How would the survey design change if Mason and Dixon used 20' secants instead of 10' secants?

   Referring to the diagram in our notes, a secant arc of 20 minutes implies $b = 10' = 2.909 \times 10^{-3}$ radians, and $a = 0.877521$ radians, and $A = 89°51'41.5" = 1.5683794$ radians. So, using $r = 6.29$ megameters, Mason and Dixon would walk 36.59 kilometers west, turn north $16'37.0"$, and repeat. Using $r = 6.37$ megameters, Mason and Dixon would walk 37.06 kilometers west.

3. Suppose you measured the altitudes of the sun as 60 degrees 1 minute in London and 59 degrees 33 minutes in York. Then you measured the distance from York to London as 367,196 feet (about 112 kilometers). What was his estimate for the radius of the earth (in kilometers)?

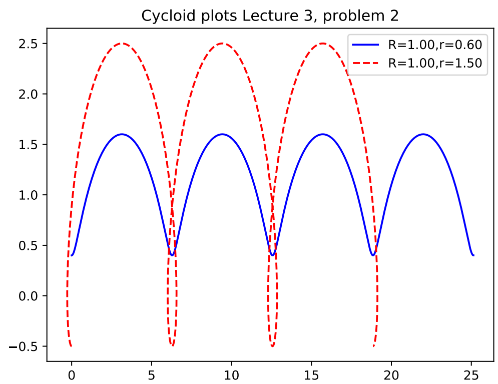
   To solve this, we model the earth as a perfect sphere and determine what fraction of the circumference Norwood observed, rescale to get the full circumference, and then divide by 2π to get the radius, or

   $$360 \times \frac{60}{28} \times 112 \times \frac{1}{2\pi} = 13,510 \text{ kilometers}$$

   No fancy trigonometry required. As an aside, I should mention that these are not actually real measurements. In the 1630s, Richard Norwood measured the altitudes of the sun as 51 degrees 30 minutes in London and 53 degrees 58 minutes in York. He then measured the distance from York to London as 367,200 feet (about 112 kilometers).

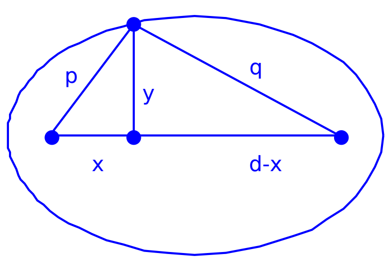
4. Imagine that in our cycloid problem, we could extend the position of the mark beyond the rim of the wheel. **Numerically** plot (i.e., write a little program to plot) the path when $R = 1$ and $r = 1.5$ for 3 revolutions.

   Everybody needs to show their code, so we can see that they did the work for themselves, rather than just stealing the picture from a classmate.



5. Archimedes' trammel is not the only way to construct an ellipse. Imagine two thumb tacks placed on cardboard a distance d apart. Take a string of length L, tie it in a loop, hang it over the thumb tacks, and draw the curve of the farthest the loop can reach. Show using coordinate geometry that this curve is also an ellipse, as long as as $L > 2d$.

   This construction, first found in the work of Hagia Sophia architect Anthemius of Tralles in 6th century Byzantium, was independently rediscovered by Kepler and contributed to his theory of the planets. % Suppose we start with the following ellipse. Place one focus at the origin, and the second a distance d down the x-axis. A point (x, y) on the curve forms two triangles with hypotenuse lengths p and q.



   Now, we have 3 equations relating our 2 variables x and y and 4 parameters p, q, d, and L.

$$p^2 = x^2 + y^2, \tag{1}$$
$$q^2 = (d - x)^2 + y^2 \tag{2}$$
$$p + q + d = L \tag{3}$$

   Since we have 3 equations, we should be able to eliminate the two parameters p and q, and get a single equation of the form $F(x, y, d, L) = 0$, where d and L are our fundamental parameters. The algebra here can be tricky – you have to be a little creative, it seems, to get everything to work. For example, if we subtract the first and second equations from each other, and then use the last equation to eliminate q, we can get $L^2 - 2Ld - 2(L - d)p + 2dx = 0$. Now, if we solve for p, square both sides, and substitute $p^2 = x^2 + y^2$, we get an equation without q or

p any more. With a little further factoring,

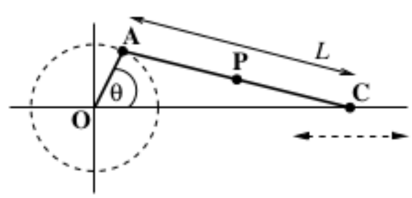
$$L\,(L-2d)\left(x-\frac{d}{2}\right)^2+(L-d)^2\,y^2=\frac{L}{4}\,(L-2d)\,(L-d)^2$$
$$\frac{4}{(L-d)^2}\left(x-\frac{d}{2}\right)^2+\frac{4}{L(L-2d)}y^2=1$$

This quadratic equation form falls into the general class conic sections you should have studied previously. A conic section is an ellipse if the variable coefficients are strictly positive. So, this equation is an ellipse as long as $L > 2d$. If $L < 2d$, equation would become a hyperbola, but the physical motivation wouldn't make sense any more.

6. A linkage consists of a handle turning in a circle around a pivot point, and a slider connected to the handle by a connecting rod of fixed length. The handle pivot point is the origin and the slider is initially to the right of the origin, and moves allong the x-axis. A pen is passed through a point on the connecting rod between the slider and the handle. Find an equation for the curve drawn by the pen, depending only on the physical constants of the mechanism.

   a. How many physical constants are needed to define this mechanism?
   b. How many variables are specify the position of the linkage?
   c. Find a set of equations sufficient to specify the curve drawn by the pen.
   d. Treating slider position as a free parameter, find a pair of parametric equations for the pen curve.
   e. Plot the pen curve when the radius of the circle is 2, the length of the connecting rod is 1 percent longer, and the pen is located at the three middle quarter points of the connecting rod (so your picture has 3 curves).

   This linkage is known as Gray's oograph.



   Let the circle be centered at the origin, and (u, v) be the position of the handle A, with $u^2 + v^2 = r^2$, where r is the distance of the handle from the circle center. Let the connecting rod have length L, and the slider C have position $(s, 0)$ on the x-axis, so $v^2 + (s-u)^2 = L^2$. Let (x, y) be the position of the pen P, which is always a fixed distance k from the slider, with $0 < k < L$. By similar triangles, we also have $(s-x)/(s-u) = y/v = k/L$. We have 5 variables $\{u, v, s, x, y\}$, and 3 constants $\{r, L, k\}$. We've listed 4 equations, so if we've picked them all independent, that will leave us with one degree of freedom for the x by y curve, which is what we would like. Using these last ratios to eliminate u and v, we can derive the parametric form

$$x(s)=\frac{k\left(r^2-L^2\right)+s^2\,(2L-k)}{2Ls},$$
$$y(s)=\pm\sqrt{k^2-(s-x(s))^2}.$$

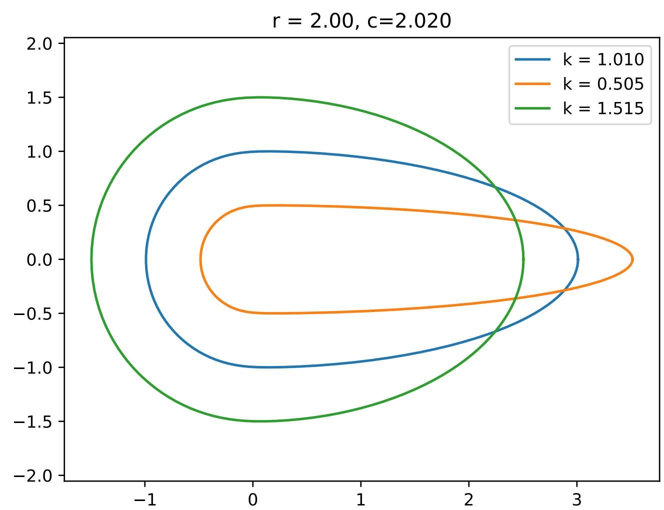
   An implicit formula for screening can be derived, but is too lengthy to present.

   Alternatively, we can use the angle θ as our parameter. Draw vertical lines from A and P intersecting the x-axis at right angles, call the points A_0 and P_0 respectively. Since

AA_0C and PP_0C are similar triangles, then $|PP_0|/k = |AA_0|/L$, and since $|PP_0| = y$ and $|AA_0| = r\sin\theta$, we find that $y = (kr/c)\sin\theta$. Obtaining x is a bit trickier. First note that the x-coordinate of P can be written as $x = |OA_0| + |A_0P_0|$. Trig immediately gives us that $|OA_0| = r\cos\theta$. To get $|A_0P_0|$, note that $|A_0P_0| = |A_0C| - |P_0C|$ and again, since AA_0C and PP_0C are similar triangles, $|PP_0|/|P_0C| = |AA_0|/|A_0C| \Rightarrow y(\theta)/|P_0C| = r\sin\theta/(|A_0P_0| + |P_0C|)$. But $|P_0C|$ can be obtained using the Pyth. theorem, $|P_0C| = \sqrt{k^2 - y^2}$. Using these two last equations we can solve for $|A_0P_0|$, finding $|A_0P_0| = (c-k)\sqrt{1 - y^2/k^2}$. Thus our coordinate (x, y) is given by

$$x(\theta) = r\cos\theta + (c-k)\sqrt{1 - \frac{y(\theta)^2}{k^2}} = r\cos\theta + (c-k)\sqrt{1 - \frac{r^2}{c^2}\sin\theta}$$
$$y(\theta) = \frac{kr}{c}\sin\theta$$

When we draw the graphs, it is a very good idea to force the axes to use equal units to avoid distorting the physical picture. Note how the curves are egg-like in shape – sharp at one end, and blunt at the other.



7. **(Challenge – NOT FOR CREDIT)** A pair of sophomore students are moving a large mural painting on plaster board into a new apartment, but to get to the room, you have to turn the rectangular mural around a tight corner between two hallways. The mural is about as tall as the hallway, so tilting it only makes the fit harder.

   a. What 3 variables control whether or not you can get the couch

   b. Find an equation for the possible width of the first and second hallways for which you will be able to get the couch to your apartment.

   Solving this requires thinking through some pieces. Let L be the length of mural, a be the width of the first hallway, and b be the width of the second hallway. As the mural moves around the corner, it must be shorter enough at each angle to fit. Since this must be true for all angles, then using trigonometry, we can show we must have

$$L \leq \operatorname{argmin}_{\theta\in(0,\pi/2)} \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$$

   Finding the minimum, and solving, the critical angle $\theta^* = \arctan\sqrt[3]{\frac{b}{a}}$ Substituting and simplifying using trigonometry identities, we find we must have

$$L \leq \left(b^{2/3} + a^{2/3}\right)^{3/2}$$