Math 450 Homework 2
Temperature of the earth, Fermi Models; Least squares, Simple laws

due Feb 19, 2021

1. If energy-balance theory of global climate is correct, it should be applicable not just to the earth but to other bodies in our solar system as well. Lets consider the special case of Mars.

   (a) What data would you need to predict the average temperature of Mars?

   In class, we derived the formula

   $$\Theta_{\oplus} = \sqrt[4]{\frac{(1-\alpha)}{4}\frac{R_{\odot}^2 \Theta_{\odot}^4}{D^2}}$$

   for the temperature of the earth $\Theta_{\oplus}$ in terms of the sun's temperature $\Theta_{\odot}$, the sun's radius $R_{\odot}$, the average distance to the sun D, and the albedo of earth α. To apply this to mars, we need the temperature of the sun, the radius of the sun, the albedo of mars and the average distance from the sun to mars. Knowing the radius of mars won't do any harm, since it appears in some of the intermediate calculations.

   (b) Where can you find this data on the internet? (Please choose a source or sources that are trusted and have a verifiable citation.)

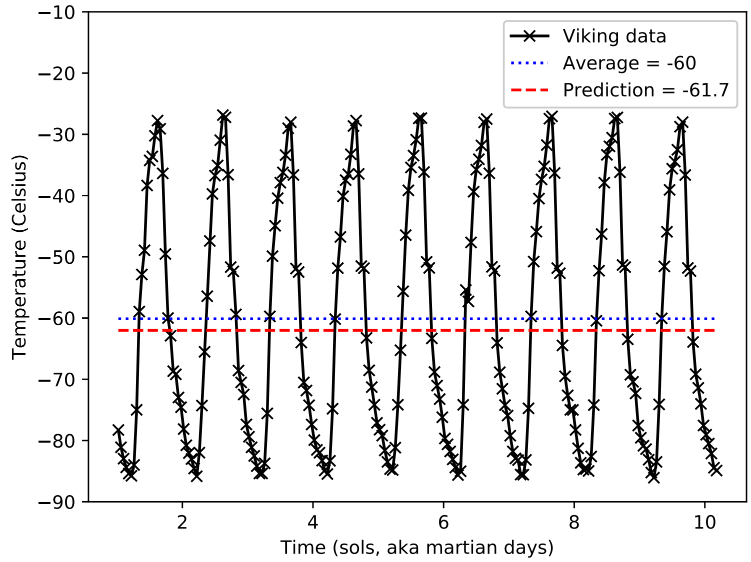
   A *reliable* source is needed – one with a reputation of high standards of scientific integrity, without a clear agenda to bias the data. Someplace with an international reputation will help us assert the wisdom of crowds in trusting their data. For example, `http://nssdc.gsfc.nasa.gov/planetary/factsheet/marsfact.html` Sources like Wikipedia are easily manipulated by bad-actors, and so should not be trusted as a single-source. Double-checking with multiple sources can dramatically improve the confidence of in the data, but only if the sources are truly independent, and we cann't only determine independence when sites clearly document the sources of their information. Printed references, when accessible, are also good for double-checking.

   (c) Using the methods from class, predict the average temperature on Mars.

   First $T_S = 5800$ K, $R_S = 7 \times 10^8$ For mars, the albedo (a.k.a. Bond albedo) $\alpha = 0.250$, and the distance $D = 2.2794 \times 10^{11}$ meters. This leads to a prediction of Mar's mean temperature being $T_{mars} = 211\text{K} = -62\text{C}$.

   (d) In 1976, the Viking 1 lander became the first spacecraft to reach mars. As part of its mission, Viking 1 recorder the temperature on Mars (link). Plot the actual temperature over 10 sols (a Martian day), and discuss the relationship between the predicted and observed values.

   Our prediction (dashed red) falls well within the range of daily variation in martian temperature, so our prediction is confirmed.



2. In our Fermi model for the earth's radius, we only used the crudest estimates.

   (a) List 4 sources of error in our model, ranked (in your opinion) from most to least important.

   This is brain-storming, so you've just got to defend yourself. Ranked from most important to least important, I suggest ...

   i. The flight we picked did not start and end at antipodal points on the globe, so the path wasn't really half a circumference.
   ii. The speed of an airplane varies with altitude and time, as does the speed of sound vary with altitude.
   iii. The airplane actually flies above the surface of the earth, making the radius appear a little larger.
   iv. The earth is only approximately a sphere.

   The easiest way to improve the estimate (within this model) is to make sure we have picked the right flight path, and then nail down flight speeds. The last two points are also complicates, but probably less important.

   (b) Come up with an alternative Fermi model for estimating the earth's radius.

   Anything reasonable will do, but it's got to work, at least approximately. For example, estimating earth's curvature from a mountain top a known height above sea level, Eratothenes method of zenith angle estimation, or measuring the gravitational constant and extrapolating.

3. Do **one** of the following Fermi model problems. That is, for full marks you have to do either (a) or (b). You're welcome to try both of course!

   (a) Thomas Edison started out life as a telegraph operator. How long would it take Edison to transmit the text of an 1865 front page of the New York Times to San Francisco by Morse code? Provide an answer with 80% confidence bounds on your error.

   In 1865, pages were six columns, 40 cm by 55 cm, so around 120 column inches total. Columns are typically around 30 words an inch. An expert telegraph operator like Edison could send around 50 words a minute, but a steady working rate was probably closer to 30 words a minute. So,

$$120 \times 30/30 = 120 \text{ minutes.}$$

The library of congress has a scan of the April 15, 1865 front page posted, including dimensions. The wikipedia entry on "column inches" estimates the number of words per inch. The book "Elements of telegraph operating, telegraphy" from 1901 gives 30 words per minute of Morse as a steady working rate.
Of course, there is also a question of how long it would take the signals to get to San Francisco from NYC. The message would probably have to be relayed many times. That would slow-down its arrival in San Francisco, but Edison's transmission of it would only take around 2 hours.

(b) In *Return of the Jedi*, when Lando Calrissian is piloting the Millenium Falcon through the superstructure of the death star to get to its reactor core, how fast is the ship flying? (see this clip).

There are a bunch of different ways you could try to estimate their speed. Here's one. At the beginning of their escape, the millenium falcon passes though a series of equally spaced frames. It takes about 3 seconds to pass through these. Playing the scene back at quarter-speed, you can count there are about 15 of these frames, and at the midpoint, you can see the gap between the frames is a little less than the length of the falcon. Wikipedia says the Millenium Falcon is 35 meters (114 feet) long, which is consistent with observations of it's size relative to the crew in the movies. Assuming constant velocity, it is travelling at about 120 kilometers an hour (about 80 miles an hour) – hardly the kind of speed you'd need to travel at to escape the explosion, but perhaps the most visually interesting speed. You might get very different estimates based on your method – that's another givaway that it is a movie. For example, the estimate that the death-star is 160 kilometers in diameter, and that it takes the falcon 30 second to escape implies a speed of 10,000 km/hr, approximately, which is very fast but totally inconsistent with shots of the Falcon appearing as anything other than a blur in the escape sequence. And of course, that's the best answer when supported with evidence – you cann't really estimate the speed, because everything is so inconsistent and self-contradictory.

* (Warm-up problem, **NOT FOR CREDIT**) (Gauss) Find the least-squares solution for x, y, and z of the following over-determined linear system.

$$\begin{aligned} x - y + 2z &= 3, \\ 3x + 2y - 5z &= 5, \\ 4x + y + 4z &= 21, \\ -2x + 6y + 6z &= 28. \end{aligned}$$

$x = 2.465$, $y = 3.563$, $z = 1.918$

4. The calculations Schott actually performs are a little more complicated than described above. Based on other information, some measurements were expected to be more accurate than others. Specifically, the measurement of the angle between Burden to Jescelyne is believed to be less accurate than the other three angles. It would be nice to be able to take this inaccuracy into account when calculating our least-squares solution. And there is indeed a way – the method of weighted linear least squares. We can weight the errors produced by each equation such that equations measured to greater accuracy are given larger weights and equations with less accuracy are given smaller weights.

(a) Find the matrix form for the equations of the method of weighted linear least squares by determining the vector x_{closest} that minimizes the total weighted square error

$$J(x) = \sum_i W_{ii}((Ax)_i - b_i)^2,$$

where W is a non-negative diagonal matrix.

Observing that

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = m_{11}p_1^2 + m_{22}p_2^2,$$

it follows that in matrix-vector form, $E(x) = (Ax - b)^T W(Ax - b)$. So, using matrix algebra rules,

$$\begin{aligned} E(x) &= (Ax - b)^T W(Ax - b) \\ &= (x^T A^T - b^T)W(Ax - b) \\ &= x^T A^T W Ax - x^T A^T W b - b^T W Ax + b^T W b \end{aligned}$$

And since each term in this sum is a scalar (not a vector or matrix) $x^T A^T W b = (x^T A^T W b)^T = b^T W Ax$, so

$$E(x) = x^T A^T W Ax - 2x^T A^T W b + b^T W b.$$

Now, if we calculate the gradient with respect to x, we find

$$\frac{dE(x)}{dx} = 2A^T W Ax - 2A^T W b.$$

Setting equal, and rearranging, we find the weighted normal equations

$$A^T W Ax = A^T W b.$$

(b) Schott's weights the first 3 measurements as 3 times more accurate than the 4th, and the observation that angles must sum to 360 degrees with (say) 1,000 times more accurate. Use the method of weighted linear least equations obtained above to find the angles between stations to the nearest thousands of an arc-second.

We want to resolve the linear least squares problem with weights $3, 3, 3, 1, 1000$. The weighted normal equations $A^T W Ax = A^T W b$ are then

$$A^T W A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 10^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} t \\ u \\ v \\ w \end{bmatrix}$$

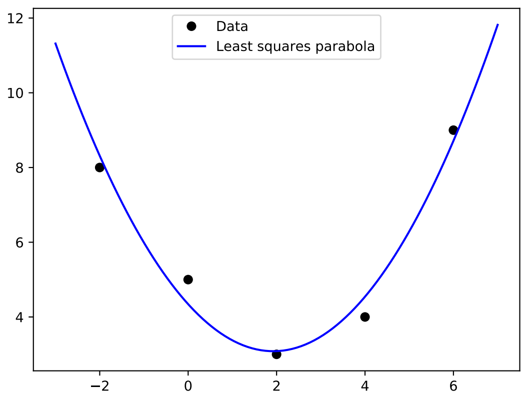
$$A^T W b = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 10^3 \end{bmatrix} \begin{bmatrix} 65.197917 \\ 66.404320 \\ 87.040195 \\ 141.356044 \\ 360 \end{bmatrix}.$$

The corrected weighted least squares results are

$$\begin{bmatrix} t \\ u \\ v \\ w \end{bmatrix} \approx (A^T W A)^{-1}(A^T W b) = \begin{bmatrix} 65.198170 \\ 66.404574 \\ 87.040449 \\ 141.356805 \end{bmatrix} = \begin{bmatrix} 65 & 11' & 53.4145" \\ 66 & 24' & 16.4675" \\ 87 & 2' & 25.6175" \\ 141 & 21' & 24.5005" \end{bmatrix}$$

5. The method of linear least squares can be applied to fitting certain polynomial curves to data. The parabola $y = p_0 + p_1 x + p_2 x^2$ can be fit to data by letting the columns of the matrix A be the monomial powers x^0, x^1, and x^2, with the coefficients p_i being our unknowns. Use the method of linear least squares to find the parabola that best fits the points $(-2, 8)$, $(0, 5)$, $(2, 3)$, $(4, 4)$, and $(6, 9)$.

   We find $c_0 = 4.34285714$, $c_1 = -1.30714286$, and $c_2 = 0.33928571$. Here's a plot, for reference, comparing the parabola to the data.



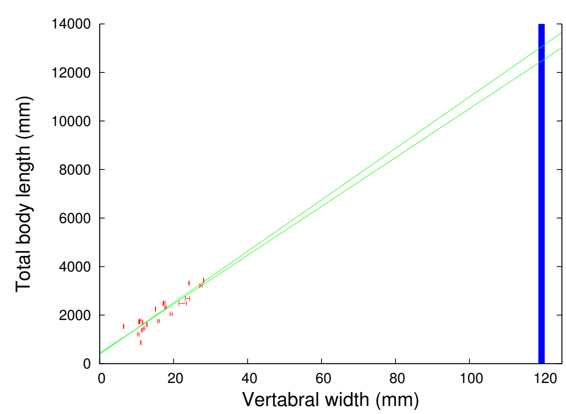
6. In a 2009 article, Head and other authors use a fossil vertebrae to deduce that about 60 million years ago, there existed a neotropical snake that grew to 13 meters in length and weighted more than 1,000 kilograms (42 feet and more than 2,000 pounds). They called this snake *Titanoboa cerrejonensis*. This discovery and others like it inspired the PBS Secrets of the Dead episode Graveyard of the giant beasts in 2016. Let's see if we agree with the hype by looking through this scientific paper and checking their numbers.

   (a) Fit a line through the data given in the Least Squares Lecture Notes, Exercise #4.

   (b) What were the two regression lines the team obtained for the relationship between between vertebra width and total body length using the extreme values of 60% and 65% for vertebral position?

   The paper obtains 4 regression lines, but only two are for total body length: $y = 100.7x + 436.2$ and $y = 106.0x + 390$.

   (c) Create a plot of width vs total body length using known 60% and 65% position data. Add to this plot the regression lines from parts (a) and (b). Make sure your axes scales are large enough to show the predicted body length of *Titanoboa*, given that the discovered vertebra was 12 cm wide.



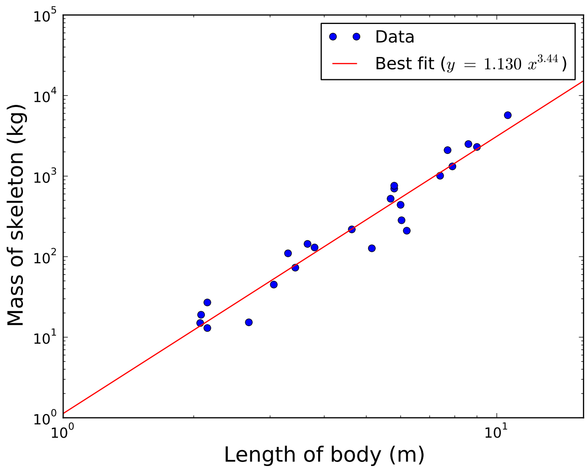
(d) Is your line consistent with the lines found by the team?

Well, this all depends on your choice for part (a). It's okay if it wasn't – you weren't given much guidance.

(e) Do you think Head et al's conclusion about body length are supported by the evidence, are contradicted by the evidence, or that the evidence is inconclusive? Defend your opinion.

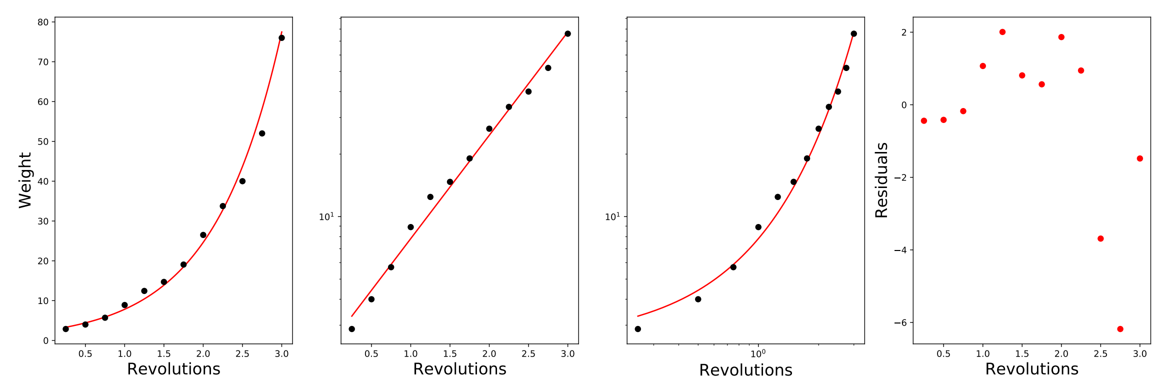
To me, this feels inconclusive. That is, the calculations of paper do support their conclusion, but the new bone is far out-of-sample, and it seems unwise to express much confidence without more evidence.

7. Theropods were an order of dinosaurs that included Tyranosaurous Rex and were also the ancestors of birds. Simple Laws Lecture Notes Exercise #9 provides a data set containing records of length and mass of a variety of species of theropods. Fit a power-law model to these data. Estimate the parameters to 3 decimal places.



8. A string was wrapped around a metal bar. A 2 ounce weight was hung at one end of the string, and weights W at the other, so as to counterbalance the weight of 2 oz. The more revolutions of string used, the greater an imbalance the friction between string and bar can sustain. The data is in Simple Laws Lecture Notes Exercise #10. Find a law relating the number of revolutions to the maximum sustained weight.

The data look most like a line when plotting the abscissa against the log of the ordinate. The best-fit exponential curve is given by $\ln y = 0.91237 + 1.14588x$ or $y = 2.49021e^{1.14588x}$.



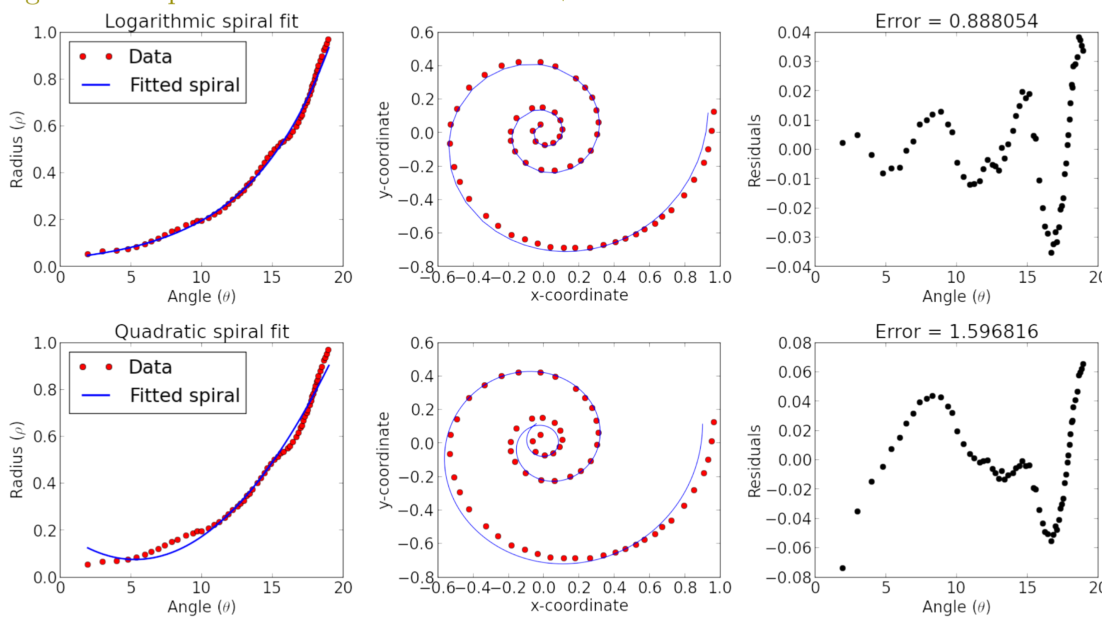
9. The chambered nautilus shell is clearly a spiral, but what kind of spiral? The data for this problem is in Simple Laws Lecture Notes Exercise #15.

(a) Fit a logarithmic spiral $r = \exp(a + b\theta)$ to the data. Find the best-fit values of a and b. Plot the $r\theta$-curve, the predicted spiral over the observed data, and the radial residuals of the model as a function of the angle θ.

(b) Fit a quadratic polynomial spiral $r = p_0 + p_1\theta + p_2\theta^2$ to the data. Find the best-fit values of p_1, p_2, p_3. Plot the $r\theta$-curve, the predicted spiral over the observed data, and the radial residuals of the model as a function of the angle θ.

(c) Which model to you think is the best explanation of the nautilus's shell shape? Defend.

The closest quadratic spiral fit is $r = 0.19896 - 0.046449\theta + 0.004407\theta^2$. The closest logarithmic spiral fit is $\ln r = -3.34146615 + 0.17255612\theta$.



The residuals of the logarithmic spiral are about half that of the quadratic spiral. In addition, the quadratic spiral has problems near the center of the shell outline, where the predicted curve crosses itself – a biologically unrealistic effect. So the log spiral seems to be the best fit model.