1 Limits and Continuity

Limits

Let f be a real-valued function defined on some interval I and let c be a point in I
Limit existence: $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x)$
Evaluating a limit: $\lim_{x \to c} f(x) = f(c)$ given that f is continuous at $x = c$

- $\lim_{x \to c} (f(x) \pm g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
- $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$
- $\lim_{x \to c} (f(x)g(x)) = (\lim_{x \to c} f(x))(\lim_{x \to c} g(x))$
- $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$
- $\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x))$ if g is continuous at point b and $\lim_{x \to c} f(x) = b$

Continuity

Applicable to endpoints

- $f(c)$ exists
- $\lim_{x \to c} f(x)$ exists
- $\lim_{x \to c} f(x) = f(c)$

Continuity on an interval: f is continuous at $x = c$ for all $c \in I$
If f and g are continuous at $x = c$, then $f + g$, f^n, kf, fg, $\frac{f}{g}$ if $g(c) \neq 0$ are continuous
If f is continuous at $x = g(c)$, then composite function $f \circ g$ is continuous at $x = c$

Limits at infinity

If $\lim_{x \to \infty} f(x) = c \in \mathbb{R}$ or $\lim_{x \to -\infty} f(x) = c \in \mathbb{R}$, then $y = c$ is a horizontal asymptote of $f(x)$
Limit of rational functions:

$$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \lim_{x \to \pm\infty} \frac{\overbrace{Ax^\alpha}^{\text{leading term}} + \cdots}{\underbrace{Bx^\beta}_{\text{leading term}} + \cdots} = \begin{cases} 0, \alpha < \beta \\ \frac{A}{B}, \alpha = \beta \\ \infty / -\infty, \alpha > \beta \end{cases}$$

- $\lim_{x \to \infty} \frac{1}{x^n} = \lim_{x \to -\infty} \frac{1}{x^n} = 0$

Indeterminate forms

$\lim_{x \to c} \frac{f(x)}{g(x)}$ has form $\frac{0}{0}$ or $\frac{\infty}{\infty}$
Replacement rule: If $f(x) = g(x) \forall x \in I$ (except possibly at $x = c$), then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$

Trigonometric limits

If $\lim_{x \to c} g(x) = 0$, then

- $\lim_{x \to c} \frac{\sin g(x)}{g(x)} = \lim_{x \to c} \frac{g(x)}{\sin g(x)} = 1$
- $\lim_{x \to c} \frac{\tan g(x)}{g(x)} = \lim_{x \to c} \frac{g(x)}{\tan g(x)} = 1$

Squeeze theorem

Suppose $g(x) \leq f(x) \leq h(x) \forall x \in I$ where $c \in I$, except possibly at $x = c$. If $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} f(x) = L$
If $\lim_{x \to c} g(x) = 0$, then for any function h, $\lim_{x \to c} g(x) \sin h(x) = 0$ and $\lim_{x \to c} g(x) \cos h(x) = 0$

Intermediate value theorem

If f is continuous on $[a, b]$ and k is a number between $f(a)$ and $f(b)$, then $f(c) = k$ for some $c \in [a, b]$

Precise definition of the limit of a function

Let $f(x)$ be defined on an open interval containing the point c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x \to c} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

Try to represent the LHS as RHS or vice versa to solve for ϵ or δ

Differentiation

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Differentiability implies continuity: If f is differentiable at $x = x_0$, then f is continuous at $x = x_0$ (converse is not always true)
Differentiability on intervals: A function f is differentiable on an interval I if it is differentiable at every point in I

Rules of differentiation

Constant rule	$\frac{d}{dx} c = 0$
Constant multiple rule	$\frac{d}{dx} cu = c \frac{du}{dx}$
Sum rule	$\frac{d}{dx} u + v = \frac{du}{dx} + \frac{dv}{dx}$
Product rule	$\frac{d}{dx} uv = \frac{du}{dx} v + u \frac{dv}{dx}$
Quotient rule	$\frac{d}{dx} \frac{u}{v} = \frac{\frac{du}{dx} v - u \frac{dv}{dx}}{v^2}$
Chain rule	$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Common differentiation identities

Function	Derivative	Function	Derivative
x^n	nx^{n-1}	$\sin^{-1} x$	$\frac{1}{\sqrt{1 - x^2}}$
$\cos x$	$-\sin x$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1 - x^2}}$
$\sin x$	$\cos x$	$\tan^{-1} x$	$\frac{1}{1 + x^2}$
$\tan x$	$\sec^2 x$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2 - 1}}, \lvert x\rvert > 1$
$\sec x$	$\sec x \tan x$	$\csc^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2 - 1}}, \lvert x\rvert > 1$
$\csc x$	$-\csc x \cot x$		
$\cot x$	$-\csc^2 x$		
e^x	e^x		
$\ln x$	$\frac{1}{x}$		

Implicit differentiation

Differentiate both sides of an equation

$$\frac{d}{dx} g(y) = g'(y) \frac{dy}{d}$$

Derivative of inverse function

Let f be bijective and differentiable on an open interval I

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Higher-order derivatives

$$f^{(n)}(x) = \frac{d^n y}{dx^n} = D^n f(x)$$

Parametric equations

$$x = f(t) \text{ and } y = g(t), \therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

2 Applications of Differentiation

Tangents and normals

Tangent:

$$y - f(x_0) = m(x - x_0)$$

Normal:

$$y - f(x_0) = -\frac{1}{m}(x - x_0), m = f'(x_0)$$

Increasing and decreasing function

- f is increasing on an interval I if $f(x_2) > f(x_1)$ for $x_1, x_2 \in I$ and $x_2 > x_1$
- f is decreasing on an interval I if $f(x_2) < f(x_1)$ for $x_1, x_2 \in I$ and $x_2 > x_1$

If f is differentiable on (a, b) and continuous on $[a, b]$, then

- f is increasing on $[a, b]$ if $f'(x) > 0, \forall x \in (a, b)$
- f is decreasing on $[a, b]$ if $f'(x) < 0, \forall x \in (a, b)$

Concave upward and downward

Let f be differentiable on (a, b) and $c \in (a, b)$

- If $f''(c) > 0$, then f is concave upward at $(c, f(c))$ (tangent function is increasing)
- If $f''(c) < 0$, then f is concave downward at $(c, f(c))$ (tangent function is decreasing)

Point of inflection is a point $(c, f(c))$ where the graph of the function f has a tangent line and where the concavity changes
$f''(c) = 0$ if $(c, f(c))$ is a point of inflection and $f''(c)$ exists

Related rates

Rate of change with relation to time

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

Maximum and minimum values

- **Absolute/global maximum** at $x = c$ if $f(x) \leq f(c), \forall x \in D_f$
- **Absolute/global minimum** at $x = c$ if $f(x) \geq f(c), \forall x \in D_f$
- **Relative/local maximum** on interval I at $x = c$ if $f(x) \leq f(c), \forall x$ in open interval containing $x = c$
- **Relative/local minimum** on interval I at $x = c$ if $f(x) \geq f(c), \forall x$ in open interval containing $x = c$

Extreme value theorem: If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value and an absolute minimum value at some points in $[a, b]$
If f is differentiable on an open interval containing $x = c$ and f has a local extremum at $x = c$, then $f'(c) = 0$
Critical point of f is a) not an endpoint, and b) either $f'(c) = 0$ or $f'(c)$ does not exist

First derivative test

Let f be differentiable on an open interval containing a critical point c except possibly at c and f is continuous at c

- f' goes from positive to negative at $x = c$, $f(c)$ is a local maximum
- f' goes from negative to positive at $x = c$, $f(c)$ is a local minimum
- f' does not change sign $x = c$, $f(c)$ is not a local extremum

Second derivative test

Let f be a twice differentiable function defined in an open interval containing c

- $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a local maximum
- $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a local minimum
- $f''(c) = 0$ is inconclusive

L'Hôpital's Rule

If $\lim_{x \to c} \frac{f(x)}{g(x)}$ is an indeterminate form, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Rolle's Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is at least one number $c \in (a, b)$ such that $f'(c) = 0$

Mean Value Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then, there is at least one number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

3 Integrals

$$F'(x) = f(x)$$

$$\int f(x) dx = F(x) + C$$

$$\int \alpha f(x) + \beta g(x) dx = \alpha \int f(x) dx + \beta \int g(x) dx$$

Common integration identities

Function	Antiderivative
$\int (ax + b)^n dx$	$\frac{(ax + b)^{n+1}}{(n + 1)a} + C, (n \neq 1)$
$\int \frac{1}{ax + b} dx$	$\frac{1}{a} \ln \lvert ax + b\rvert + C$
$\int e^{ax+b} dx$	$\frac{1}{a} e^{ax+b} + C$
$\int \sin(ax + b) dx$	$-\frac{1}{a} \cos(ax + b) + C$
$\int \cos(ax + b) dx$	$\frac{1}{a} \sin(ax + b) + C$
$\int \tan(ax + b) dx$	$\frac{1}{a} \ln \lvert \sec(ax + b)\rvert + C$
$\int \sec(ax + b) dx$	$\frac{1}{a} \ln \lvert \sec(ax + b) + \tan(ax + b)\rvert + C$
$\int \csc(ax + b) dx$	$-\frac{1}{a} \ln \lvert \csc(ax + b) + \cot(ax + b)\rvert + C$
$\int \cot(ax + b) dx$	$-\frac{1}{a} \ln \lvert \csc(ax + b)\rvert + C$
$\int \sec^2(ax + b) dx$	$\frac{1}{a} \ln \lvert \tan(ax + b)\rvert + C$
$\int \csc^2(ax + b) dx$	$-\frac{1}{a} \ln \lvert \cot(ax + b)\rvert + C$
$\int \sec(ax + b) \tan(ax + b) dx$	$\frac{1}{a} \sec(ax + b) + C$
$\int \csc(ax + b) \cot(ax + b) dx$	$-\frac{1}{a} \csc(ax + b) + C$
$\int \frac{1}{a^2 + (x + b)^2} dx$	$\frac{1}{a} \tan^{-1}(\frac{x + b}{a}) + C$
$\int \frac{1}{\sqrt{a^2 - (x + b)^2}} dx$	$\frac{1}{a} \sin^{-1}(\frac{x + b}{a}) + C$
$\int \frac{-1}{\sqrt{a^2 - (x + b)^2}} dx$	$\frac{1}{a} \cos^{-1}(\frac{x + b}{a}) + C$
$\int \frac{1}{a^2 - (x + b)^2} dx$	$\frac{1}{2a} \ln \left\lvert \frac{x + b + a}{x + b - a} \right\rvert + C$
$\int \frac{1}{(x + b)^2 - a^2} dx$	$\frac{1}{2a} \ln \left\lvert \frac{x + b - a}{x + b + a} \right\rvert + C$
$\int \frac{1}{\sqrt{(x + b)^2 \pm a^2}} dx$	$\ln \left\lvert (x + b) + \sqrt{(x + b)^2 \pm a^2} \right\rvert + C$
$\int \sqrt{a^2 - x^2} dx$	$\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}(\frac{x}{a}) + C$
$\int \sqrt{x^2 - a^2} dx$	$\frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left\lvert x + \sqrt{x^2 - a^2} \right\rvert + C$

Partial fractions

Decompose a rational expression, $f(x) = \frac{P(x)}{Q(x)}$, into a series of simpler rational expressions.
Only possible if $deg(P(x)) < deg(Q(x))$

1. Factor the denominator
2. Apply the following rules for each factor:

$$\begin{aligned} ax+b &\longrightarrow \frac{A}{ax+b} \\ (ax+b)^k &\longrightarrow \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k} \\ ax^2+bx+c &\longrightarrow \frac{Ax+B}{ax^2+bx+c} \\ (ax^2+bx+c)^k &\longrightarrow \frac{A_1x+B}{ax^2+bx+c} + \frac{A_2x+B}{(ax^2+bx+c)^2} \\ &+ \cdots + \frac{A_kx+B}{(ax^2+bx+c)^k} \end{aligned}$$

3. Break up the rational expression into the sum of terms
4. Solve for every A_k and B_k

If $deg(P(x)) \geq deg(Q(x))$, then perform polynomial division first.

Integration by substitution

$$\int f(g(x))g'(x)dx = \int f(u)du$$

$$\int_a^b f(g(x))g'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

Integration by parts

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

Order of differentiation

1. **Logarithmic function:** $\ln(ax+b)$ or its higher powers
2. **Inverse trigonometric function:** $\sin^{-1}(ax+b)$, $\cos^{-1}(ax+b)$, $\tan^{-1}(ax+b)$
3. **Algebraic function:** Power functions x^a, polynomials
4. **Trigonometric function:** $\sin(ax+b)$, $\cos(ax+b)$, $\tan(ax+b)$, $\csc(ax+b)$, $\sec(ax+b)$, $\cot(ax+b)$ or any combinations of these

Order of integration

1. **Exponential function:** e^{ax+b}
2. **Trigonometric function:** $\sin(ax+b)$, $\cos(ax+b)$, $\tan(ax+b)$, $\csc(ax+b)$, $\sec(ax+b)$, $\cot(ax+b)$ or any combinations of these

Riemann sums

Approximating the area under function f from a to b by dividing each section into rectangles of width $\frac{1}{n}$

$$\int_a^b f(x)dx = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\left(\frac{b-a}{n}\right) f\left(a + k\left(\frac{b-a}{n}\right)\right)\right\}$$

Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))g'(x)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$

Definite integrals

Let $c \in [a,b]$ and $\alpha, \beta \in \mathbb{R}$

1. $\int_a^b \alpha dx = \alpha(b-a)$
2. $\int_c^c f(x)dx = 0$
3. $\int_a^b (\alpha f(x) + \beta g(x))dx = \int_a^b \alpha f(x)dx + \int_a^b \beta g(x)dx$
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
6. $\int_a^b f(x)dx \geq 0$ if $f(x) \geq 0$ for $a \leq x \leq b$
7. $\int_a^b f(x)dx \leq 0$ if $f(x) \leq 0$ for $a \leq x \leq b$
8. $\int_a^b f(x)dx \geq \int_a^b g(x)dx$ if $f(x) \geq g(x)$ for $a \leq x \leq b$
9. $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ if $f(x) \leq g(x)$ for $a \leq x \leq b$
10. $m(b-a) \leq \int_a^b g(x)dx \leq M(b-a)$ if $m \leq f(x) \leq M$ for $a \leq x \leq b$
11. $\int_{-a}^a f(x)dx = 0$ if f is an odd function defined on $[-a,a]$
12. $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if f is an even function defined on $[-a,a]$

Improper integrals

If limit does not exist, improper integral diverges, else it converges

Type 1

If $f(x)$ is continuous on given range,

1. $\displaystyle\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$
2. $\displaystyle\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$
3. $\displaystyle\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$

Type 2

1. $\displaystyle\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$ if $f(x)$ is continuous on $(a,b]$ and discontinuous at a
2. $\displaystyle\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$ if $f(x)$ is continuous on $[a,b)$ and discontinuous at b
3. $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ if $f(x)$ is discontinuous at c with $a < c < b$

4 Applications of Integration

Area between curves

Regardless of whether $f(x) \geq g(x)$ or vice versa

$$A = \int_a^b |f(x) - g(x)|\,dx$$

If lower bound by x-axis, then $g(x)$ is a constant function of 0

$$A = \int_c^d |f(y) - g(y)|\,dy$$

for curves bound by the y-axis
Note: split into smaller integrals if necessary

Volume of solid of revolution

Assuming revolution about x-axis or y-axis only

Disk method

$$V = \pi\int_a^b [f(x)]^2dx - \pi\int_a^b [g(x)]^2dx$$

where $f(x)$ and $g(x)$ is the radius of the outer and inner disk and $f(x) \geq g(x)$

$$V = \pi\int_c^d [f(y)]^2dy - \pi\int_c^d [g(y)]^2dy$$

where $f(y)$ and $g(y)$ is the radius of the outer and inner disk and $f(y) \geq g(y)$

Cylindrical shell method

$$V = 2\pi\int_a^b x\,|f(x) - g(x)|\,dx$$

where x is the radius of the shell, $f(x) - g(x)$ is the height of the shell, and $f(x) \geq g(x)$

$$V = 2\pi\int_c^d y\,|f(y) - g(y)|\,dy$$

where y is the radius of the shell, $f(y) - g(y)$ is the width of the shell, and $f(y) \geq g(y)$

Arc length of curve

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}dx$$

applicable to $x = f(y)$ as well

5 Basics

Interval notation

$x \in [a,b] = a \leq x \leq b$
$x \in (a,b) = a < x < b$
$x \in (a,b] = a < x \leq b$ OR $x \in [a,b) = a \leq x < b$

Properties of absolute values

- $(|-x| = |x|)\forall x \in \mathbb{R}$
- $(|xy| = |x||y|)\forall x,y \in \mathbb{R}$
- $(-|x| \leq x \leq |x|)\forall x \in \mathbb{R}$
- For a fixed $r > 0, |x| < r$ iff $x \in (-r,r)$
- $(\sqrt{x^2} = |x|)\forall x \in \mathbb{R}$
- Triangle inequality: $(|x+y| \leq |x| + |y|)\forall x,y \in \mathbb{R}$

Properties of inequality

- $a \leq b \wedge c \leq d \to a + c \leq b + d$
- $0 < a \wedge b \leq c \to ab \leq ac$
- $0 > a \wedge b \leq c \to ab \geq ac$

Solving quadratic inequalities

Find the roots, select x between roots, apply x to quadratic inequality to check if applicable. If applicable, include root in solution

Solving absolute inequalities

Applicable to $< and >$
$|f(x)| \leq g(x) \Leftrightarrow -g(x) \leq f(x) \leq g(x)$
$|f(x)| \geq g(x) \Leftrightarrow f(x) \leq -g(x) \vee f(x) \geq g(x)$

Solving rational inequalities

Bring all terms with x to one side and solve accordingly

Functions

$f : A \to B = a \mapsto f(a)$ assigns to each $a \in A$ one specific member $f(a) \in B$
A is the *domain* of f and B is the *codomain* of f
Range (R) of $f = \{f(x) \in B | x \in A\}$
Note $R \subseteq B$

Restricting domains

$\mathbb{R} \setminus \{\dots\}$

Composite function

$(f : A \to B \wedge g : B \to C) \to g \circ f : A \to C = g(f(x))$

Inverse function

$f(g(x)) = x \to f = g^{-1}(x)$
To solve, swap x and y in the equation and isolate for x in terms of y

Function type

- **Injective**: $f(x) = f(y) \Rightarrow x = y$ (one-to-one)
- **Surjective**: for any $z \in B$, there is an $x \in A$ such that $f(x) = z$ (every input has an output)
- **Bijective**: f is both injective and surjective (strictly one-to-one across all input to output)

Rational Functions

$$\frac{p(x)}{q(x)}$$

$p(x)$ and $q(x)$ are polynomials
Domain: $\mathbb{R} \setminus$ {roots of q(x)}

Exponential and Logarithmic Functions

$$f(x) = a^x, a > 0$$

Inverse: $\log_a x$
Inverse identities: $e^{\ln x} = x, x > 0$ and $\ln e^x = x, \forall x$
Domain: $\mathbb{R}$
Range: $\mathbb{R}^+$

Change of base formulae

$$\log_a x = \frac{\ln x}{\ln a}, a > 0 \text{ and } a \neq 1$$

Range of a Function

Determine the range of a function using basic algebraic techniques like finding the inverse and solving the domain of the inverse (equals to the range of the original)

6 Appendix

Trigonometric identities

- $\sec^2 x - 1 = \tan^2 x$
- $\csc^2 x - 1 = \cot^2 x$
- $\sin A\cos A = \frac{1}{2}\sin 2A$
- $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$
- $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$
- $\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$
- $\cos A\sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$
- $\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$
- $\sin A\sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$
- $\sqrt{a^2 - (x+b)^2} \to x + b = a\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{a^2 + (x+b)^2} \to x + b = a\tan\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{(x+b)^2 - a^2} \to x + b = a\sec\theta, 0 \leq \theta \leq \frac{\pi}{2}$ or $\pi \leq < \frac{3\pi}{2}$