Parameters

- Aerodynamic Force $\overline{F_A} = \overline{L} + \overline{D}$
- Lift Force L $[N]$
- Drag Force D $[N]$
- Aerodynamic Moment M_A $[Nm]$
- Dynamic Pressure $q = 1/2\rho V^2$ (Bernoulli) $[Pa]$
- Chord Length c $[m]$
- Surface Area $S = b \cdot c$ $[m^2]$ (Rectangular)
- Wing Span b $[m]$
- Lift Coefficient $C_l = L/(1/2\rho V^2 \cdot S)$
- Drag Coefficient $C_d = D/(1/2\rho V^2 \cdot S)$
- Moment Coefficient $C_m = M_A/(1/2\rho V^2 \cdot c \cdot S)$
- Angle of Attack α $[rad]$ (positive in clockwise direction)
- Lift curve slope $a = C_{l/\alpha} = C_l/\alpha \approx tan(angle\, x - axis\, to\, curve)$
- Pitch angle θ (Rotation w.r.t elastic axis)
- Lunge h (Deflection of elastic axis parallel to lift)

Conventions throughout Course

- If L and D absolute $\rightarrow$ use calculations above
- If L and D per span unit $\rightarrow$ correct via dividing by b
- Sign conventions: Lift positive, Drag positive in x and y direction
- Moments and angles positive in clockwise direction
- Our system coordinate system is defined by the wing. The angle of attack is defined relative to it
- The variables which describe the airfoil motion are the pitch θ and the plunge h which act at the shear centre of the wing

Mathematical Basics

- A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- Inverse of Matrix (2D): $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- Inverse of Matrix (3D):
  1. det (A), then transpose A
  2. Find the ajunct matrix (minors) of
  (cover row and column of element) of A^T
  and multiply with $+-$ matrix
  $A^{-1} = 1/det(A) \cdot Adj(A^T)$
- The solutions of $Ax = 0$ for a matrix A, x cannot be just the trivial solution if A is not invertible
- $[rad] = \frac{\pi}{180}[deg]$

Steady Aerofoil and Wing Section Aerodynamics

- Aerofoil = 2-D wing section with goal to generate lift force perpendicular to the relative airspeed
- Convention: Lift is up, Drag is in direction of windspeed and Aerodynamic moment in clockwise direction acting on the aerodynamic center. Aerodynamic center is normally at the quarter chord position $c_{m,c/4}$ for syymetric airfoils. $x_{ac} = -m_0/2\pi + 0.25$ with m_0 as a shape constant
- Further assumptions: No viscosity, incompressible fluid, $Ma < 0.2, 0.3$, no vortices, potential flow (Navier-Stokes)
- Another centre is the shear center (elastic axis) from mechanics
- $L = 1/2\rho V^2 ca\alpha$, with a from tables (CFD and Wind Tunnel) $[N/m]$
- $M_A = 1/2\rho V^2 c^2 c_{m0}$ with c_{m0} also from tables $[N]$

Lift curve $C_l(\alpha)$ and drag curve $C_d(\alpha)$

- At small ranges of α, both lift and drag increase with: $C_l \propto \alpha$ and $C_d \propto \alpha^2$
- In aeroelasticity and this course, α will be very small, hence drag will be negligble small

The aerodynamic moment M_A

- The aerodynamic moment is much more important than drag C_d
- M_A varies with α in the small ranges of the angle of attack (very small, p. 7)
- **Important to note:** There exist a point at which the aerodynamic moment does not depend on α. This is the the aerodynamic centre
- The aerodynamic centre is not the same as the centre of pressure, which is defined as the point where the aerodynamic moment is zero given a certain angle of attack α
- Symmetric airfoils at $\alpha = 0$ have no aerodynamic moment at all times ($M_A = 0 = const$). At the aerodynamic centre for symmetric foils results into no moment
- Asymmetric airfoils at $\alpha = 0$ have a non-zero aerodynamic moment at all times (all angles α)

Assessment of C_l/α (Correction of value through Mach Number)

- The linear part of the lift curve is characterised by the slope $a = C_l/\alpha(M) = \frac{C_l/\alpha_{M=0}}{\sqrt{1-M^2}}$
- The Prandtl-Glauert factor is $1/\sqrt{1-M^2}$
- The factor is depending on the Mach number. The slop increases with increasing M (between 0 and 1)
- The dependence on Re is more subtle (p. 8)

Extension to wing aerodynamics (p. 8)

Aerofoil dynamics (2D) refer to the previous topics, however the 3-D case can be also modeled by through a couple examples. A finite wing is less stable and efficient than the airfoil since the tips have vortices on at the wing tips. These "induce" a velocity, which locally reduces the angle of attack. An important parameter is the so called **Aspect Ratio** $AR = b^2/S$. If the wing is assumed to be of surface $S = b \cdot c$, it follows $AR = b/c$.

- The lift curve can become a function of AR if due to the different tips. Approximately, the lift slope a_0 is adjusted via following formula:
- $a = a_0 \frac{AR}{AR+4}$
- The values a and c_{m0} will hence be corrected with a a^* and c^*_{m0}

Strip Theory (p.9)

- If AS is very small (delta wings), the integral of multiple airfoils
- Define multiple airfoils stacked next to each other along the span b
- Example, the wing is an elliptical $f(y) = \sqrt{1 - (\frac{y}{b/2})^2} \cdot \overline{f}_\phi$
- $f(y) = a\alpha c = C_l c$ with $c =$ chord length.

Steady-state (static) Aeroelasticity

Typical Section = 1DOF model

2-D problem with a rigid wing. We can have multiple typical sections stacked onto each other, which would be later adding dimensionality to the variable θ. The idea is later on to model the torsional spring to be a torsional stiffness of a beam (since a real wing is actually a beam with a certain stiffness).

- The torsion acts in a beam section on the shear centre, however in aeroelasticity on the elastic axis
- The goal of engineering is always to move the shear center to the front (comes with risk to thin out the rear longeron and thicken the front longeron)
- In equilibrium, we know that the aerodynamic forces are equal to the spring forces
- $M_t + L \cdot e = \theta k_\theta = (qc^2 c_{m0} + qca\theta e) \cdot b$ (moment equations)
- Pitching moment M_t acting on section with k_θ stiffness
- $k_\theta \theta = qc(C_{l/\alpha} e(\theta + \alpha_0) + cC_{m0})$ (Momentum Equation)
- $k_h h = L = qcC_{l/\alpha}(\theta + \alpha_0)$ (Lift Equation)

Static Instability or Divergence

- If the elastic twist θ would become infinity for a given stiffness if the denominator of equation
- $\theta = qc\frac{C_{l,a} e\alpha_0 + cC_{m,0}}{k_\theta - cqC_{l,a} e}$, $\theta = \infty \Leftrightarrow$ denominator = 0
- If the dynamic pressure $q = \frac{k_\theta}{cC_{l,a} e} = q_{div} \rightarrow$ instable (divergence)
- Divergence = Static Instability
- $M_{tot} = (k_\theta - qSae)\theta - qSa(e\alpha_0 + C_m c)$ (In equilibrium $M_{tot} = 0$) ($M_{tot} > 0$ if in anti-clockwise direction)
- Different interpretation: $\frac{\partial M_{tot}}{\partial \theta} \geq 0 \Leftrightarrow$ increasing total moment in section for increasing $\theta \Leftrightarrow \Delta\theta > 0$
- $\frac{\partial M_{tot}}{\partial \theta} \geq 0 \Leftrightarrow$ overall moment brings blade section back to original position
- The divergent dynamic pressure can be found by differentiating w.r.t. θ

Lagrange Equation (Energy interpretation)

- $L = T - U$ (Kinetic and potential energy)
- $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$
- In statics: $\frac{\partial U}{\partial x_i} = 0$ (Potential energy conservative)
- Here: $\frac{\partial U}{\partial x_i} = \frac{\partial}{\partial x}\frac{\delta W}{\delta x}$ (Virtual work), hence for θ
- $\frac{\partial U}{\partial \theta} = \frac{\partial}{\partial \theta}\frac{\delta W}{\delta \theta} = 0$ (Check for q)
- Resulting q provides the divergence
- $U = 1/2 \cdot k_\theta \theta^2 = \int F(\theta) d\theta$ (For system with only one spring)

Section with more than 1 DOF

1. Number of DOF = Dimensions of Stiffness Matrix and number of equations needed (full rank) K
2. Define a potential energy matrix for mechanical system $K_{i,j} = \frac{\partial U}{\partial x_i \partial x_j} = K_{j,i}$
3. Define aerodynamic matrix K_a based on aerodynamic forces (independent on e for example)
4. If a matrix is not symmetric = Non-conservative forces
5. Similar to before, instead of asking if the system is stable if the denominator is zero, we must know if the determinant of the transfer function is zero
6. Transfer Function: $[K - qK_a] = K_{ael}$ is 'Aeroelastic K'
7. Find a q for which the transfer function determinant becomes zero, which is divergence dynamic pressure. The solution (forces acting) is the so called divergence mode
8. If all eigenvalues are $> 0 \Leftrightarrow$ stable, if one is at least $< 0 \Leftrightarrow$ unstable (for sections)

System (more than 1 section) with multiple DOF

- Define a system with multiple θ_i, whereas the calculations become similar to when when calculating one section with multiple DOF
- Make an Ansatz with the lagrange equations and define stiffness matrix K
- The aerodynamic matrix becomes the identity matrix (if we only speak about θ)
- This implies that the solutions for q are the eigenvalues of K

Comment on eigenvalues

- The eigenvalues of the Aeorelastic K K_{ael} cannot guarantee that q are always the eigenvalues, since K_a is sometimes non-symmetric (most of the times, only if rotational degrees of freedom present)
- In general, following statement holds true: $det(K - qK_a) = 0$

- $det(K_a^{-1}K - qI) = det(A - \lambda I) = 0$

- If A is not symmetric, we can say: There are less eigenvectors and values than the order (n), can be complex and come in complex conjugate pairs

Active conrol on sections

- With active control, the behaviour of the elastic twist θ can be controlled with for example a trailing edge flap

- As described in the script, a trailing edge flap can influence the lift and the moment as follows:

- $l = qcC_{l/\delta}\delta$, $m = qcC_{m/\delta}\delta$ ($C_{m/\delta} < 0$)

- Both forces contribute to the overall moment, hence will be added to the calculations we did previously

- With a so called 'Gain' G, the controller controls δ proportional to θ, hence a new linear equation system is formed

- Assuming the nose-down motion of controlling the the edge flap, we have to simplify terms, the end result is

- $q_{div,flap} = \frac{k_\theta}{cae - Gca^*}$, with $a^* = -(cC_{m/\delta} + eC_{l/\delta})/(c\delta)$

Ritz Method

... is a energy variational method whereas an equilibrium occurs in correspondence of an extreme of potential energy. A general application is the virtual work. According to the Hamilton's principle and Lagrange equations, we can define a set of equations.

- $\frac{\partial V}{\partial x_i} = 0$ for all i (x_i degrees of freedom)

- From mechanics, we need the bending stiffness (I) and the torsional stiffness (J)

- $I = 1/12 * b * h^3$ (w.r.t x) and $J = \frac{4A^2}{\int ds/t}$ (Integral is perimeter (Umfang) divided by thickness)

- For circular shapes: $I = \frac{\pi}{4}r^4$, $J = \frac{\pi}{2}r^4$

Derivation for a beam section (mechanical part, torsion)

- Torsional Strain Energy: $U = \frac{1}{2}\int_0^l GJ(\frac{\partial\theta}{\partial x})^2 dx$

- Given aerodynamic forces are non-conservative, we use the concept of virtual work

- $\delta U = \delta W$ (Virtual work due to non-conservative forces)

- $\sum_{i=1}^n \frac{\partial U}{\partial x_i}\delta x_i = \sum_{i=1}^n \delta W_i$ (reformulated for small variations of one DOF)

- $\frac{\partial U}{\partial x_i} - \frac{\delta W}{\delta x_i} = 0$ (reformulated)

- The work done by the external forces (aerodynamic) can be rewritten:

- $\delta W = \int_0^l m(x)\delta\theta(x)dx$

- $m(x)$ generated by aerodynamic forces

- Without going into further detail, there are 2 distinct cases from which one has to go on in the calculation, either the functions are given in a generalised form or in matrix form

- $\theta(x) = \sum_{i=1}^N \phi_i(x)a_i = [\Phi]\{a\}$

- $[\Phi]$ is a row vector with elements ϕ_i!

- $\phi_i(x)$ are shape functions and a_i are coefficients and the linear combination of those make up $\theta(x)$

- Finally, the overall equations result in $[K]\{a\} = \{f\}$

- $K_{i,j} = \int_0^l GJ\phi_{i_x}\phi_{j_x}dx = GJ\frac{\partial^2 U}{\partial a_i \partial a_j}$ (Stiffness matrix entries, partial derivatives w.r.t. to x and a)

- $[K] = GJ\int_0^l [\Phi_x]^T[\Phi_x]dx$

- $f_i = \int_0^l m(x)[\Phi]_i(x)dx$ (index i for each element)

- $\{f\} = \int_0^l m(x)[\Phi]^T dx$ (in matrix form)

Derivation for a beam section (aerodynamic part, torsion)

Following assumptions are drawn:

- Elastic axis is perfectly straight

- Aerodynamic center of all sections on a straight line

- External moments as before by aerodynamic forces: $m(x) = qcea(\theta(x) + \alpha_0)$

Replace all $\theta(x)$ with the above solutions and insert insert $m(x)$ into generalised forces vector

- $\{f\} = q\int_0^l cea\alpha_0[\Phi]^T dx + q\int_0^l cea[\Phi]^T[\Phi]dx\{a\}$

- $\{f\} = \{f_0\} + q[K_A]$

- $[K_A] = \int_0^l cea[\Phi]^T[\Phi]dx$

- $\{a\} = ([K] - q[K_A])^{-1}\{f_0\}$ solves for all a

- a gives us the beam gives us the response of the system (pitch θ at all points along x)

- Stability: $det([K] - q[K_A]) = 0$

- Hence the basis of the solution of the eigenvalue problem

- Eigenvalues q: Dynamic pressure where zero stability

- Eigenvectors: Corresponding divergence modes

- Attention: $a \neq \{a\}$! (lift slope vs. coefficients)

One single shape function (1-DOF) and $[\Phi] = \phi(x)$

- $\theta(x) = a \cdot \phi(x)$ is one dimensional, we assume $a = 1$ because we can define it within ϕ

- $U = \frac{1}{2}\int_0^l GJ\phi_x^2 dx$

- $K = \int_0^l GJ\phi_x^2 dx = [K]_{torsion}$

- $K_a = \int_0^l ceC_{l\alpha}\phi^2 dx$ ($C_{l\alpha} = a$ lift curve slope)

- $f_0 = q\int_0^l ceC_{l\alpha}\alpha_0\phi dx$

- $(K - qK_a)a = f_0$

- $a = f_0/(K - qK_a)$ gives us the response by which θ is multiplied

- $q_d = K/K_a$ gives us the divergence

One single shape function, Bending and Twisting

- For bending, the potential energy is: $U = \frac{1}{2}\int_0^l EI\psi_{xx}^2 dx$
- $K = \int_0^l EI\psi_{xx}^2 dx = [K]_{bending}$
- If we assume bending takes place and torsion as well, we assume both to be decoupled
- The stiffness matrix reads
- $K = \begin{bmatrix} [K]_{bending} & 0 \\ 0 & [K]_{torsion} \end{bmatrix}$
- The aerodynamic stiffness matrix is of shape (always for bending and twisting):
- $K_a = \begin{bmatrix} 0 & \int_0^l cC_{l\alpha}\phi\psi dx \\ 0 & \int_0^l ecC_{l\alpha}\psi^2 dx \end{bmatrix}$
- The vector x includes the coefficients for the shape functions ψ and ϕ:
- $x = (K - qK_a)^{-1} f$
- $f = \begin{Bmatrix} q\int_0^l cC_{l\alpha}\phi\psi dx \\ q\int_0^l cC_{l\alpha}\psi^2 dx \end{Bmatrix}$
- $det(K - qK_a) = 0 \Leftrightarrow$ then $q = q_{div}$

Shape functions

Shape functions have to be chosen. In FEA, shape functions are local for each finite element.

- Orthonormal modes: $\int \psi_i\psi_j dx = 0$ for different shape functions in i and j
- Simple polynomials are great shape functions x/l, $(x/l)^n$
- Natural vibration modes or normal modes (eigenvectors of the problem): $K - \lambda M$ with K the stiffness and M the mass matrix (Important for dynamic systems later)

Bending / twisting coupling

In class multiple examples have been shown whereas following are the key takeaways:

- Out of plane bending can exist if for example the shear centre and the principle axes (centre of gravity) are apart from each other significantly
- The conventions for positive and negative e eccentricities: positive if aerodynamic centre in front of shear centre and hence negative if the other way around
- Positive e is detrimental for aeroelastic stability, negative is beneficial
- Helicopter blades have D-spars to shift the elastic axis forward
- Gurney flaps at the end of the wing with length 1% of c make the wing virtually longer

Control effectiveness, typical section

Control effectiveness is described by following term:

- $\frac{L_{elastic}}{L_{rigid}} = \frac{1-\frac{q}{q_r}}{1-\frac{q}{q_{div}}}$ = Control Effectiveness
- $q_r = -q_{div}\frac{e}{c}\frac{C_{l\delta}}{C_{m\delta}}$ ($C_{m\delta}$ is negative!)
- If the control effectiveness is zero, the aileron deflection does not contribute to more lift
- If the control effectiveness is is negative, this means that $0 \leq q_r < q < q_{div}$
- Possible goal: As close to q_{div} and below q_r
- Another solution: Outboard ailerons (less stiff due to smaller torsional stiffness) and inboard ailerons (GJ/l)
- The overall equations for future equations will be for equilibria:
- $[K]\{\phi\} = q[K_a]\{\phi\} + \{m_0\} + q\{f_c\}\delta$ (Aileron Equation)
- For mulitple segments: $q[f_c]\delta$

Effects of Sweep Angle on Divergence

- In this course, a 2-DOF model is used (pitching and flapping)
- Spring stiffness for moments on a beam: k_θ and k_ϕ
- $k_\theta = \frac{GJ}{l}$ (l = length of lever/beam b, torsion)
- $k_\phi = \frac{EI}{l}$ (flapping / bending)
- $G/E = \frac{1}{2(1+\nu)}$ (Poisson ratio)
- The angle of attack will be reintroduced:
- $\tan(\alpha) = \frac{V\perp}{V\|} = \frac{-V\sin(\Lambda)\sin(\phi)}{V\cos(\Lambda)} = -\tan(\Lambda)\sin(\phi)$
- $\alpha = -\tan(\Lambda)\phi$ (small angle approx)
- In the script the approach given results are as follows:
- $[K] = \begin{bmatrix} k_\phi & 0 \\ 0 & k_\theta \end{bmatrix}$
- Simplifications: $Q = q_n cbC_{l\alpha}$ and $t = \tan(\Lambda)$ (Q is q redefined)
- $[K_a] = \begin{bmatrix} -tb/2 & b/2 \\ -te & e \end{bmatrix}$
- $\{f\} = \frac{Q\alpha_0}{\cos(\Lambda)}\begin{Bmatrix} b/2 \\ e \end{Bmatrix}$
- $[K]_{ael} = [K] - Q[K_a]$
- Condigtion for divergence:
- $det(K_{ael}) = \Delta = 0 \Leftrightarrow Q_D = \frac{k_\phi k_\theta}{k_\phi e - k\theta bt/2}$
- $\Leftrightarrow q_D = \frac{k_\theta/(SeC_{l/\alpha})}{\cos^2(\Lambda)[1-(b/e)(k_\theta/k_\phi)(\tan(\Lambda)/2)]}$
- This gives us a uniqe solution for two degrees of freedom. If we try to push $q_D \to \infty$, we can do so by setting the denominator of $q_D = 0$
- This allows us to model our wing with geometrical and material parameters such that the system never becomes unstable

- Divergence can already be avoided with small Λ sweep angles
- Because the angle creates a coupling between wing bending and torsion (deformation), and the angle of attack
- $\alpha_{new} = \alpha_0 / \cos(\Lambda) + \theta - \phi \tan(\Lambda)$
- Assuming small angles: $\alpha_{new} \approx \alpha_0 + \theta - \phi\Lambda$ whereas the negative part is larger
- Other approaches: Build wing with unbalanced comosite laminates such that bending/twisting coupling is generated by material
- Adding aerodynamic control surfaces (see flaps) with active control
- Negative sweep angles reduce divergence speed

Sweep Angles and Ritz Method (Class Notes)

- Shape functions are chosen such that one has dependencies along the section
- For twisting: $\theta(y) = f_\theta(y)\Theta$ (Θ constant)
- For bending: $w(y) = f_w(y)B$
- $\phi = \frac{\partial w}{\partial y} = B \cdot \frac{\partial f_w}{\partial y}$
- This will make the angle of attack α dependent on y
- $\alpha(y) = -\tan(\Lambda)\frac{\partial f_w(y)}{\partial y} B + f_\theta(y)\Theta$
- Hence lift and moment become also dependent on y (we also assume $C_{m,0} = 0$ or const):
- $l(y) = qcC_{l,\alpha}(\alpha(y))$, $m(y) = l(y) \cdot e$
- Principle of virtual work:
- $\delta W = \int (l(y)\delta w + m(y)\delta\theta) dy$
- Expanding δw with $f_w \delta B$ and $\delta\theta$ with $\delta\theta$ with $f_\theta\delta\Theta$ will yield an integral whereas
- The aerodynamic stiffness K_a will be of following shape and hence non-symmetric:
- $[K_a] = \begin{bmatrix} \int \ldots \Theta\delta\Theta & \int \ldots B\delta\Theta \\ \int \ldots \Theta\delta B & \int \ldots B\delta B \end{bmatrix}, x = \begin{Bmatrix} \Theta \\ B \end{Bmatrix}$

Unsteady Aeroelasticity

Dynamic Systems (Repetition from Bachelor Level)

- $\dot{x} = Ax + Bu$
- $y = Cx + Du$
- x are state variables, u are input variables and y are output
- Aeroelastic System:
- $m\ddot{z} + k_z z =$ some aerodynamic forces
- $I\ddot{\theta} + k\theta =$ some aerodynamic forces
- In compact form:
- $M\ddot{x} + C\dot{x} + Kx =$ Forces/Moments
- Canonical Form:
- $\dot{r} = Ar + Bu$ with $A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$
- $y = Cr + Du$
- $r = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$
- r is of dimension $2n$ (twice the No. of DOF) and ordered such that the definition of A is valid
- The roots of the characteristic polynomial of A tell us if the system is stable
- The eigenvectors are the modes of the system and the response of the system is a linear combination of these nodes
- The topic will be looked at again later on, however one can say in general for the mass, damping and stiffness:
- $M = \begin{bmatrix} m & mx_{cg} \\ mx_{cg} & I + mx_{xg}^2 \end{bmatrix}$
- $C = \begin{bmatrix} c_z & 0 \\ 0 & c_\theta \end{bmatrix}$
- $K = \begin{bmatrix} k_z & 0 \\ 0 & k_\theta\neg\dagger \end{bmatrix}$

Quasi steady approach