

1 Introduction, Definitions & Overview

Reliability

- ... is a characteristic of an item, expressed by the probability that the item performs its required function under given conditions during a stated time interval, i.e. $(0, t]$
- Item = entity for investigation, i.e. component, assembly, equipment, subsystem, system
- from a **qualitative** point of view, reliability is defined as the ability of an item to **remain functional**
- from a **quantitative** point of view, reliability is defined as the probability that **no operational interruptions** will occur during a stated time interval $R(t)$

Availability

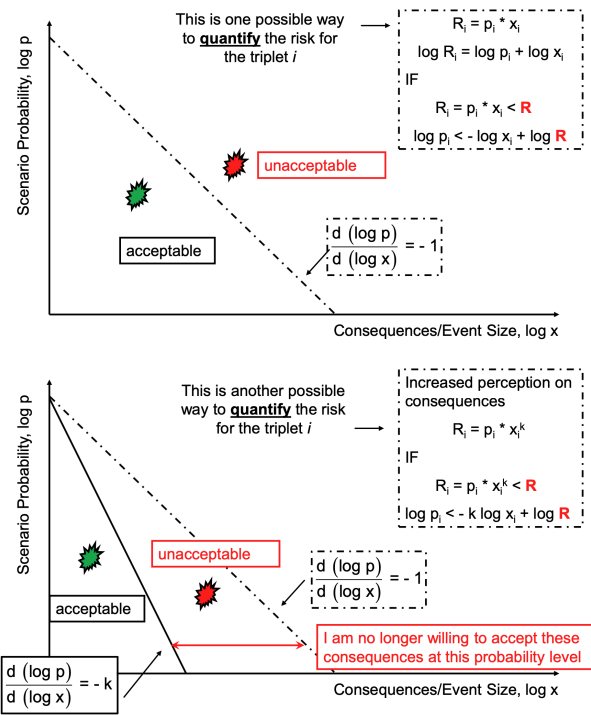
- Point Availability (PA) is a characteristic of an item expressed by the probability that the item performs its function **at an instant of time t**
- Qualitatively, it can be described as the dependability
- Average Availability (AA), is the expected time at which the item can perform its required function
- Availability is used to express Point Availability in a sloppy way or Average Availability

In comparison to Availability, the Reliability analysis includes the fact that no item is allowed to fail. This means, the function performed cannot be interrupted (redundant system however can be repaired). Availability incurs that failures can happen on the item level.

Risk

- RISK = POTENTIAL DAMAGE x UNCERTAINTY
- Dictionary: RISK = probability of damage to a person or an object
- We define RISK as a function of an...
 - Accident Scenario, S
 - Probability, p
 - Consequence, x

- We can quantify RISK on the so-called “Farmer’s Curve”



The total Risk can be calculated as follows:

$$\text{Total Risk} = \sum p_i x_i^k, k \geq 1 \quad (1)$$

It is entirely possible that the risk of different events can be dominated by either it's probability or its consequence.

- A large probability p is prevented of (minimisation based on high probability)
- A large consequence x is mitigated, protected (minimisation given its large impact)

2 Probability Theory and Reliability Analysis

Definitions:

- Experiment ϵ
- Sample space Ω
- Event E

An event E is a subset of the sample space Ω and the experiment ϵ yields a set of possible outcomes ($= E$) of the experiment

Certain Events follow Boolean Logic, an event E can occur or not occur, meaning an Indicator Variable X_E is 0 when E does not occur and 1 if E occurs

Uncertain Events follow can either be true or false, with each a probability associated to it. Event E in sample space Ω is triggered with a probability that the outcome has happened or not

Classical Probability

- The experiment ϵ has N possible, elementary, mutually exclusive and equally probable outcomes $A_1, A_2, \dots, A_N \in \Omega$
- The event $E = A_1 \cup A_2 \cup \dots \cup A_M, M \leq N$
- The probability of event E is defined as $p(E) = M/N$

Kolmogorov Axioms

- $0 \leq P(E) \leq 1$
- $P(\Omega) = 1, P(\emptyset) = 0$
- Mutually exclusive events: $P(\cup_i E_i) = \sum p(E_i)$
- Non-mutually exclusive events:
 $P(A \cup B) = P_A + P_B - P(A \cap B)$
- Conditional probability: $P(A|B) = P(A \cap B)/P(B)$
- Theorem of total probability: Given an event A in Ω where the space is consisting of exclusive and exhaustive events $\cup_j E_j = \Omega$: $P(A) = \sum_i (P(A|E_i)P(E_i))$

Random Variables

- **CDF**: Is a non-decreasing function and returns the probability (state) from random variable X from 0 to a given point A : $F_X(X = A) = P(0 < X \leq A)$
- **pdf**: Probability of per unit x (continuous)
- **pmf**: Histogram, it assigns the probability to discrete values x

Summary

- Distribution Percentile x_α :
 - $F_X(x_\alpha) = \alpha/100 = \int_{-\infty}^{x_\alpha} f_X(x)dx$
- Median:
 - $F_X(x_{50}) = 0.5$
- Mean:
 - $\mu_X = E[X] = \langle X \rangle = \sum_i x_i p_i$ (discrete)
 - $= \int_{-\infty}^{\infty} x f_X(x)dx$ (continuous)
- Variance:
 - $\sigma_X^2 = \sum (x_i - \mu_X)^2 p_i$ (discrete)
 - $= \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)dx$ (continuous)

Hazard Function (Failure Rate)

For risk and reliability analyses, we can use models whereas the time to failure of a component T can be expressed through a CDF $F_T(t)$ and a pdf $f_T(t)$. The complementary, cumulative function is

$$R(t) = 1 - F_T(t) = P(T \geq t) \quad (2)$$

which is described as the **Reliability or Survival Function** of the component T at time t and gives the probability of it surviving up to time t without failures.

In order to monitor the failure evolution, given the component has survived up to time t in a time interval dt , one can define a so called **Hazard Function or Failure Rate** $h_T(t)$.

$$h_T(t)dt = P(t < T \leq t + dt | T > t) = \quad (3)$$

$$= \frac{P(t < T \leq t + dt)}{P(T > t)} = \frac{f_T(t)dt}{R(t)} \quad (4)$$

The hazard function is depending on time, and is often described through the bathtub curve. The failure rate at the beginning is higher (infant mortality, burn in) and decreases after a certain time. The failure rate becomes constant λ and increases at the end through ageing.



Through the definition of $R(t)$ and integrating the hazard function, we receive:

$$F_T(t) = 1 - e^{-\int_0^t h_T(\tilde{t})d\tilde{t}} \quad (5)$$

$$R(t) = e^{-\int_0^t h_T(\tilde{t})d\tilde{t}} \quad (6)$$

If our hazard function is in its constant phase (constant hazard rate), the failure evolution follows the **Exponential Distribution**:

$$h_T(t)\lambda, t > 0 \quad (7)$$

$$F_T(t) = P(T \leq t) = 1 - e^{-\lambda t} \quad (8)$$

$$R(t) = \begin{cases} f_T(t) = \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (9)$$



The mean time to failure (MTTF) can be found through the expectation value

$$E[T] = \frac{1}{\lambda} = MTTF \quad (10)$$

$$Var[T] = \frac{1}{\lambda^2} \quad (11)$$

The failure process is memoryless. Given a component has survived at least until time t_1 , the probability of it failing between time t_1 and t_2 is only depending on the time inbetween and not prior to time t_1

$$P(t_1 < T < t_2) = \frac{P(t_1 < T < t_2)}{P(T > t_1)} = \frac{F_T(t_2) - F_T(t_1)}{1 - F_T(t_1)} = \quad (12)$$

$$\frac{e^{-\lambda t_1} - e^{-\lambda t_2}}{e^{-\lambda t_1}} = 1 - e^{-\lambda(t_2 - t_1)} \quad (13)$$

The influence of an ageing process of the components failure rate shows that is not constant through time and hence can be described through the **Weibull distribution**.

Boolean Logic - Fault Tree Analysis

Fault trees = set of Boolean algebraic equations (one for each gate) \rightarrow structure (switching) function Φ

$$X_T = \Phi(X_1, X_2, X_3, \dots, X_N) \quad (14)$$

The top event is connected by an OR-gate, hence if one of each events is true, then the top event will be true. Here some further rules:

- **Negation**: Given event E , given by the indicator variable X_E , its negation is described by $\overline{X_E} = 1 - X_E$
- **Intersection**: The event $A \cap B$ is true, if both A and B are simultaneously true:

$$X_{A \cap B} = X_A X_B \quad (15)$$

$$(X_{A \cap B} = 0 \text{ for mutually exclusive events})$$

- **Union**: The event $A \cup B$ is true, if either A or B are true and false if both are false:

$$X_{A \cup B} = 1 - (1 - X_A)(1 - X_B) \quad (16)$$

$$= 1 - \Pi(1 - X_j) = \Pi X_j \quad (17)$$

$$= X_A + X_B - X_A X_B \quad (18)$$

- **Probability of event E with expected value Operator $E(\cdot)$** : $p(E) = E(X_E)$

$$p(E) = p(X_E = 1) \cdot 1 + p(X_E = 0) \cdot 0 = E(X_E) \quad (19)$$

- **Multiple, non-mutually exclusive events**:

$$X_{\cap} = \Pi X_i \quad (20)$$

- Probability of event E_{\cap} for the intersection of n events:

$$P(E_{\cap}) = E[X_{\cap}] = \Pi P(E_j) \text{ (if events are independant)} \quad (21)$$

- Union of event E_{\cup} for the union of n events:

$$X_{\cup} = 1 - \Pi(1 - X_j) = X_A + X_B + X_C \quad (22)$$

$$-X_A X_B - X_A X_C - X_B X_C - X_A X_B X_C \quad (23)$$

- Probability of even E_{\cup} for the union of n events:

$$P(E_{\cup}) = E[X_{\cup}] = \sum P(E_j) - \quad (24)$$

$$\sum \sum P(E_j \cap E_i) + (-1)^{n+1} P(\cap E_j) \quad (25)$$

Structure Function and Minimal Cut Sets

- Cut Set: Is a logical combination of primary event (**combination of component failures**) which render true the top event (**system failure**)
- Minimal Cut Sets: Cut set that does not have another cut set as a subset. This means repairing one element of the set repairs the entire system. Therefore, removing one element of a MCS makes it no longer a cut set.

Any fault tree $\Phi(X)$ can be equivalently written as an OR-gate in the first level below the top event combining the minimal cut sets, each in return represented by an AND-gate intersecting all elements comprising the given minimal cut set

$$\Phi(X) = 1 - (1 - M_1)(1 - M_2)(1 - M_3) \dots \quad (26)$$

$$\dots(1 - M_{mcs}) = \Pi_j^{mcs} M_j \quad (27)$$

$$P(\Phi(X) = 1) = E[\sum M_j - \sum \sum M_i M_j + \dots \quad (28)$$

$$\dots + (-1)^{mcs+1} \Pi M_j \quad (29)$$

$$= \sum P(M_j) - \sum \sum P(M_i M_j) + \dots \quad (30)$$

$$\dots + (-1)^{mcs+1} P(\Pi M_j) \quad (31)$$

- First event approximation: $\sum P(M_j)$

- Second event approximation:
 $\sum P(M_j) - \sum_i^{mcs-1} \sum_j^{mcs} P(M_i M_j)$

Important fact: Idempotent law follows for Boolean values - $X * X = X$

$$P(M_1 M_2) = P(M_1 \cap M_2) = E[M_1 \cap M_2] = \quad (32)$$

$$E[X_1 X_2 \cap X_2 X_3 X_4] = E[X_1 X_2 X_2 X_3 X_4] \quad (33)$$

$$= E[X_1 X_2 X_3 X_4] \quad (34)$$

if both events M are independent, then

$$P(M_1 M_2) = P(X_1)P(X_2)P(X_3)P(X_4) = p_k(T_m)^4 \quad (35)$$

The occurrence probability of X_2 will only be appearing once in the calculation of the intersection of M_1 and M_2 and hence the Unreliability (Failure Rate) calculations become more exact through subtraction of the intersection (second event approximation) and will be bounded by both approximations:

$$\sum P(M_j) - \sum \sum P M_i M_j < U_{mcs}(T_m) < \sum P(M_j) \quad (36)$$

Reliability Analysis

Series System

- All components must function in order of a functioning system

$$R(t) = \Pi R_i(t) \quad (37)$$

- For N exponential components:

$$R(t) = e^{-\lambda t} \quad (38)$$

$$\lambda = \sum \lambda_i \text{ (System Failure Rate)} \quad (39)$$

$$E[T] = 1/\lambda(\text{MTTF}) \quad (40)$$

Parallel System

- All components must fail in order the system to fail

$$R(t) = 1 - \Pi [1 - R_i(t)] \quad (41)$$

- For N exponential components:

$$R(t) = 1 - \Pi [1 - e^{-\lambda_i t}] \quad (42)$$

$$MTTF = \sum 1/\lambda_i \sum \sum 1/(\lambda_i + \lambda_j) + \dots \quad (43)$$

$$\dots + \sum \sum \sum 1/(\lambda_i + \lambda_j + \lambda_k) + \dots \quad (44)$$

$$\dots + (-1)^{N-1} 1/\sum \lambda_i \quad (45)$$

- Example with two exponential units of failure rates λ_1 and λ_2 :

$$MTTF = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{[\lambda_1 + \lambda_2]} \quad (46)$$

- For N identical elements, compare series and parallel:

$$\text{parallel : } MTTF = \sum_n^N \frac{1}{n\lambda} \quad (47)$$

$$\text{series : } MTTF = \frac{1}{N\lambda} \quad (48)$$

$$MTTF_{\text{parallel}} < MTTF_{\text{series}} \quad (49)$$

Markov Processes: Basic Elements

System

- The system can occupy a finite/countable number of states N
- The states are mutually exclusive, i.e. the system is in one state at each time
- The states are exhaustive, i.e. the system must be in one state at all times

Transitions between States occur **stochastically**, i.e. randomly in time

Mathematical Representation

- The random process of the state transition can be described by an integer random variable, e.g. $X(t) = 5$, the system occupies state 5 at time t
- The stochastic process can be observed at discrete times, and we assume that time intervals between states is such small that only one transition occurs between two states

The conceptual Model: Discrete States

- The state transition will be described by an integer random variable $X(n)$ is the system state at time t_n , $X(3) = 5$ means that the system occupies state 5 at time t_3
- The objective is: Compute the probability that the system is in a given state at given time t , for all possible states and times
- **In general stochastic processes:** The probability of a future state depends on its entire life history

- **In Markov processes:** The probability of a future state only depends on the present state. The process has no memory

- The transition probability that the system in state i at time t_m moves to state j at time t_n

$$p_{ij}(m, n) = P[X(n) = j | X(m) = i], n > m \geq 0 \quad (50)$$

$$i = 0, 1, 2, \dots, N, j = 0, 1, 2, \dots, N \quad (51)$$

Properties of Transition Probabilities

- Transition probabilities are greater or equal to 0
- Transitions must sum up to 1

$$p_{ij}(m, n) = p[X(n) = j, X(m) = i] \\ = \sum_k p_{ik}(m, r) p_{kj}(r, n), \\ i = 0, 1, 2, \dots, N, j = 0, 1, 2, \dots, N$$

Stationary Transition Probabilities

- If the transition probability $p_{ij}(m, n)$ depends on the time interval $(t_n - t_m)$ and not on the individual times t_m and t_n , then

- The transition probabilities are stationary
- The Markov Process is homogeneous in time

$$p_{ij}(m, n) = p_{ij}(k), k \geq 0, i, j = 0, 1, 2, \dots, N \quad (52)$$

- We only need to know the **stationary one-step transition probabilities** $p_{ij}(1) = p_{ij}$

The Transition Probability Matrix:

$$\underline{A}_{ij} = \begin{pmatrix} p_{00} & p_{01} & \dots & p_{0N} \\ p_{10} & p_{11} & \dots & p_{1N} \\ \dots & \dots & \dots & \dots \\ p_{N0} & p_{N1} & \dots & p_{NN} \end{pmatrix} \quad (53)$$

- $\dim(A) = (N + 1) \times (N + 1)$
- $0 \leq p_{ij} \leq 1, \forall i, j \in \{0, 1, \dots, N\}$
- $\sum_j p_{ij} = 1$, hence only $(N + 1) \times N$ elements must be known
- \underline{A} is a stochastic Matrix

Unconditional State Probabilities:

$$\underline{P}(n) = [P_0(n), P_1(n), \dots, P_j(n), \dots, P_N(n)] = \quad (54)$$

$$\text{probabilities of being in state } 0, 1, \dots, N \text{ at time step } n \quad (55)$$

$$\underline{P}(0) = \underline{C} = [C_0, C_1, \dots, C_j, \dots, C_N] = \quad (56)$$

$$\text{initial vector at time step } n=0 \quad (57)$$

At the first time step $n = 1$:

$$P_j(1) = P[X(1) = j] \quad (58)$$

$$= \sum_{i=0}^N P[X(1) = j | X(0) = i] \cdot P[X(0) = i] \quad (59)$$

$$= \sum_{i=0}^N p_{ij} C_i, \text{ with } j = 0, 1, 2, \dots, N \quad (60)$$

The fundamental equation follows:

$$\underline{P}(n) = \underline{P}(0) \cdot \underline{A}^n = \underline{C} \cdot \underline{A}^n \quad (61)$$

$$\underline{A}^n = \begin{pmatrix} p_{00}(n) & p_{01}(n) & \dots & p_{0N}(n) \\ p_{10}(n) & p_{11}(n) & \dots & p_{1N}(n) \\ \dots & \dots & \dots & \dots \\ p_{N0}(n) & p_{N1}(n) & \dots & p_{NN}(n) \end{pmatrix} \quad (62)$$

$$p_{ij}(n) = P[X(n) = j | X(0) = i] \quad (63)$$

The probability of arriving in state j after n steps given the initial state was i

General Solution of the Fundamental Equation

1. Solve the eigenvalue (ω) problem: $\underline{V} \cdot \underline{A} = \omega \cdot \underline{V}$
2. With the eigenvectors: $\underline{V}_j \cdot \underline{A} = \omega_j \cdot \underline{V}_j$
3. $\underline{P}(n) = \sum_{j=0}^N \alpha_j \cdot \underline{V}_j$ and $\underline{C} = \sum_{j=0}^N c_j \cdot \underline{V}_j$ are linear combinations of the eigenvectors
4. The coefficients can be found through the adjoint eigenvalue problem

$$5. \text{ The general solution yields: } \underline{P}(n) = \sum_{j=0}^N c_j \cdot \omega_j^n \cdot \underline{V}_j$$

$$6. \text{ The availability at time } k: A(k) = \sum_{j \in \text{success states}} P_j(k)$$

Steady State Probabilities:

1. Solve $\underline{\Pi} = \underline{\Pi} \cdot \underline{A}$ with $\sum_{j=0}^N \Pi_j = 1$ (2 equations)

First-passage Probabilities

- Probability that the system arrives **for the first time** in state j **after n steps**, given that it was in state i at the initial time 0

$$f_{ij}(n) = P[X(n) = j \text{ for the first time} | X(0) = i] \\ f_{ij}(n) = P[X(n) = j, X(m) \neq j, 0 < m < n | X(0) = i]$$

- Notice that $f_{ij}(n) \neq p_{ij}(n)$ = probability that the system reaches state j **after n steps** starting from state i , but **not necessarily for the first time**

General Formula for the first-passage probability

$$f_{ij}(k) = \underbrace{p_{ij}(k)}_{(I)} - \underbrace{\sum_{l=1}^{k-1} f_{ij}(k-l) p_{jj}(l)}_{(II)} \quad (64)$$

$$R(k) = 1 - \sum_{l=1}^k \sum_{j \in \text{failed states}} f_{ij}(l) \quad (65)$$

The probability of passing the state j from i is the transition probability at time step k , given it was in i at 0 (I) whereas (II) is the probability of staying in j as it reached the state prior to time step k

The reliability is defined as the probability at time k where the system has not yet failed, hence we subtract all first passage probabilities of all failed states up to time k

Recurrent, Transient and Absorbing States

- **Recurrent:** The system at state i will surely return to i **sooner or later:**

$$\Pi_i \neq 0 \quad (66)$$

- **Transient:** The system at state i has a **finite probability** of **never** returning to it:

$$\Pi_i = 0 \quad (67)$$

- **Absorbing:** The state i is absorbing if the system cannot leave it once it enters:

$$p_{ii} = 1 \quad (68)$$

Average Occupation Time of a State

- The average occupation time of a state i for l_i steps = Average number of steps before the system exits the state i :

$$l_i[\text{steps}] = \frac{1}{P(\text{System exits state } i)} \quad (69)$$

$$= \frac{1}{1 - P(\text{System remains in } i)} = \frac{1}{1 - p_{ii}} \quad (70)$$

Continuous-time Discrete-state Markov Processes

What are they useful for?

- Model the evolution of one component or system as a structure of components
- To find the probability that the system will be functioning at time t^*

The transition probability matrix

$$p_{ij}(dt) = \alpha_{ij} \cdot dt + \underbrace{\theta(dt)}_{O(dt)}, \lim_{dt \rightarrow 0} \frac{\theta(dt)}{dt} = 0 \quad (71)$$

In analogy with the discrete-time formulation:

$$\underline{A} = \begin{pmatrix} 1 - dt \cdot \sum_{j=1}^N \alpha_{0j} & \alpha_{01} \cdot dt & \dots & \alpha_{0N} \cdot dt \\ \alpha_{10} \cdot dt & 1 - dt \cdot \sum_{j=0, \neq 1}^N \alpha_{1j} & \dots & \alpha_{1N} \cdot dt \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad (72)$$

$$\underline{P}(t + dt) = \underline{P}(t) \cdot \underline{A} \quad (73)$$

$$\frac{d\underline{P}}{dt} = \underline{P}(t) \cdot \underline{A}^* \quad (74)$$

$$\underline{P}(0) = \underline{C} \quad (75)$$

$$\underline{A}^* = \begin{pmatrix} -\sum_{j=1}^N \alpha_{0j} = \alpha_{00} & \alpha_{01} & \dots & \alpha_{0N} \\ \alpha_{10} & -\sum_{j=0, \neq 1} \alpha_{1j} = \alpha_{11} & \dots & \alpha_{1N} \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad (76)$$

The sum of all rows in \underline{A}^* is 0. From now on, the matrix \underline{A} will be called the **Transition Rate Matrix**. The system of linear, first-order differential equations in the unknown state probabilities will be solved through the Laplace Transform

- Definitions of Laplace Transform $L(\cdot)$:

$$\tilde{P}_j(s) = L[P_j(t)] = \int_0^\infty e^{-st} P_j(t) dt, \quad j = 0, 1, \dots, N \quad (77)$$

$$L\left[\frac{dP_j(t)}{dt}\right] = s \cdot \tilde{P}_j(s) - P_j(0), \quad j = 1, 1, \dots, N \quad (78)$$

- Applying $L(\cdot)$ on $\frac{d\underline{P}}{dt} = \underline{P} \cdot \underline{A}$ yields:

$$\underline{\tilde{P}}(s) = \underline{C} \cdot [s\underline{I} - \underline{A}]^{-1} \quad (79)$$

- $\underline{P}(t)$ = inverse transform of $\underline{\tilde{P}}(s)$
- Compute for example (one component, one repairman, 2x2):

$$(s\underline{I} - \underline{A})^{-1} = \frac{1}{s^2 + s\lambda + s\mu} \begin{pmatrix} s + \mu & \lambda \\ \mu & s + \lambda \end{pmatrix} \quad (80)$$

- Then:

$$\underline{\tilde{P}}(s) = \underline{C} \cdot (s\underline{I} - \underline{A})^{-1} = \begin{bmatrix} \frac{s + \mu}{s(s + \lambda + \mu)} & \frac{\lambda}{s(s + \lambda + \mu)} \end{bmatrix} \quad (81)$$

- Anti-transform $\tilde{P}(s)$, whereas following definitions are used:

$$L^{-1}\left[\frac{1}{s + a}\right] = e^{-at} \quad (82)$$

$$L^{-1}\left[\frac{1}{s(s + a)}\right] = \frac{1}{a} (1 - e^{-at}) \quad (83)$$

- The State Probability Vector is found:

$$\underline{P}(t) = \begin{pmatrix} \underbrace{\frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \cdot e^{-(\lambda + \mu) \cdot t}}_{P_0(t)}, \underbrace{\frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} \cdot e^{-(\lambda + \mu) \cdot t}}_{P_1(t)} \end{pmatrix} \quad (84)$$

whereas

- $P_0(t)$ is the system point **availability** (prob. of being in operational state 0 at time t)
- $P_1(t)$ is the system point **unavailability** (prob. of being in failed state 1 at time t)

- Set $\frac{d\underline{P}(t)}{dt} = 0$ in the fundamental equation and solve the system of equations:

$$\underline{P} \cdot \underline{A} = 0 \quad (85)$$

$$\sum_{j=0}^N \Pi_j = 1 \quad (86)$$

- the Steady State Probabilities become:

$$\begin{aligned} \Pi_0 &= \lim_{t \rightarrow \infty} P_0(t) = \frac{\mu}{\mu + \lambda} = \frac{1/\lambda}{1/\mu + 1/\lambda} = \frac{MTBF}{MTTR + MTBF} \\ &= \text{average fraction of time the system is functioning} \end{aligned} \quad (87)$$

$$\begin{aligned} \Pi_1 &= \lim_{t \rightarrow \infty} P_1(t) = \frac{\lambda}{\mu + \lambda} = \frac{1/\mu}{1/\mu + 1/\lambda} = \frac{MTTR}{MTTR + MTBF} \\ &= \text{average fraction of time the system is down (i.e. under repair)} \end{aligned} \quad (88)$$

- Frequency of **departure** from a state i to state j

$$\nu_{ij}^{dep}(t) = \lim_{\Delta t \rightarrow 0} \frac{p_{ij}(dt) \cdot P_i(t)}{dt} = \alpha_{ij} \cdot P_i(t) \underbrace{=}_{\text{steady state}} \alpha_{ij} \cdot \Pi_i \quad (89)$$

- Total Frequency of **departure** from state i to any other state j

$$\nu_i(t) = \sum_{j=0, \neq i} \alpha_{ij} P_i(t) = \alpha_{ii} \cdot P_i(t) \underbrace{=}_{\text{steady state}} \nu_i = \alpha_{ii} \cdot \Pi_i \quad (90)$$

- Frequency of **arrival** to state i from k

$$\nu_i^{arr}(t) = \sum_{k=0, \neq i} \alpha_{ki} \cdot P_k(t) \underbrace{=}_{\text{steady state}} \alpha_{ki} \cdot \Pi_k \quad (91)$$

- From some algebra, it can be followed that at steady state, that the number of arrivals to state i equals the departures from i

- System failure intensity** W_f

- Rate at which system failures occur = expected number of failures per unit time = rate of exiting a success state to go into one of fault:

$$W_f(t) = \sum_{i \in S} P_i(t) \cdot \lambda_{i \rightarrow F} \quad (94)$$

whereas S is the set of success states in the system, F of the failure states. $\lambda_{i \rightarrow F}$ describes the conditional (transition) probability of leaving success state i towards a failure state

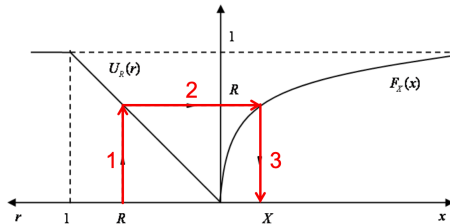
- **System repair intensity** W_r (same principal as for failure intensity)

$$W_r(t) = \sum_{j \in F} P_j(t) \cdot \mu_{j \rightarrow S} \quad (95)$$

Introduction to Monte Carlo Simulation - The experimental view

Inverse Transform Method (analytical way)

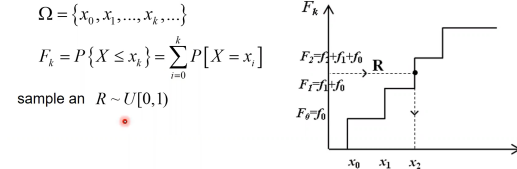
- By taking a sample R from $U_R(r)$ and compute X
- $X = F_X^{-1}(R)$
- For the exponential distribution, this would result in
- $X = F_X^{-1}(R) = -\frac{1}{\lambda} \ln(1 - R)$
- This means, that from a random sample R , one can will receive a X which is transformed through the new shape of the curve from the exponential distribution



For example, we used minimal cut sets in the past to do rare event approximations. With Monte Carlo, we would have to identify states and then do inverse transforms to find the operating states. The arrival state for all arcs in the network are then calculated. The simulation is done for N trials. The estimation of the failure probability is done by dividing the count of failed network states by the number of samples.

Sampling from discrete distributions (non-analytical)

- This is the same concept as for the analytical way
- However, now we only have discrete distributions
- This means, that samples R can only achieve a discrete number of outcomes (X)



Simulation of System Transport

- Plant = system of N_C connected components
- Component = a subsystem of the plant
- State of the plant at time t = the set/vector of the states in which the N_C components are at t . The plant may be labeled by a scalar which enumerates
- Plant transition = When any of the components performs a state transition we say that the plant has performed a transition. The time t_n and state k_n are defined at the n -th transition.

Failure Probability Estimation: Sampling from discrete distribution

In class, an example with a network is shown. The rare event approximation is made, as well as the Monte Carlo simulations. With the number of simulations of Monte Carlo, the estimate over a certain number of time will achieve a closer result with a smaller confidence interval. In order to achieve a certain confidence interval, a number of simulations have to be done with a certain number of samples.

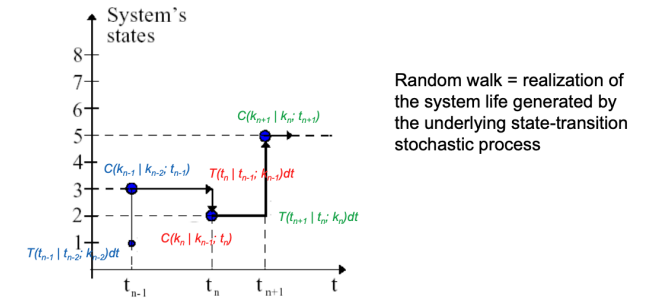
Stochastic Transitions: Governing probabilities

- $T(t|t'; k')dt$ = conditional probability of a transition at $t \in dt$, given that the previous transition as occurred in time step t' and that the state previously entered was k'

- $C(k|k'; t)$ = conditional probability of the plant entering state k from previous time step t . Both probabilities form the "Transport Kernel" K

$$K(t; k|t'; k')dt = T(t|t'; k')dtC(k|k'; t) \quad (96)$$

The ingoing transition density or PDF of a system transition at t , resulting in the entrance in state k . The goal is to get this PDF (global information) $\psi(t; k)$



Availability and Reliability Estimation

The availability is estimated through the metric $U(t_j)$, which is the unavailability of the system, whereas the function $C^A(t_j)$ is a counter which only increases as long as the system is in a failed state which however can be again repaired ($X(t_j) = 2$), compared to a failed state where the system cannot be recovered and hence

$$U(t_j) = \frac{C^A(t_j)}{M} = P(X(t_j) = 2) \quad (97)$$

The unreliability counter is added after the first time, the failed state is reached (the system cannot be recovered anymore). The unavailability counter is added until the system is available again.

Sample from exponential distribution

Probability Density Function:

$$f_T(t) = \lambda e^{-\lambda t}, t \geq 0 \quad (98)$$

$$= 0, t < 0 \quad (99)$$

Cumulative Distribution Function:

$$F_T(t) = 1 - e^{-\lambda t}, t \geq 0 \quad (100)$$

$$= 0, t < 0 \quad (101)$$

Expected value and variance:

$$E[T] = \frac{1}{\lambda} \quad (102)$$

$$Var[T] = \frac{1}{\lambda^2} \quad (103)$$

Simulation for reliability/availability analysis of a sub-system

Indirect Monte Carlo: Example

Components' times of transition between states are exponentially distributed, this means that $\lambda_{j_i \rightarrow m_1}^i$ is the rate of transition of component i going from its state j_i to the state m_i . On

a component level, this means that the transition rate matrix (same as in continuous time Markov) is defined and that the time of transition can be sampled through the summation out of all transitions λ from the nominal state for each component, giving a proxy

$$t_1 = t_0 - \frac{1}{\lambda^{(1,1,1)}} \ln(1 - R_t), R_t \in [0, 1) \quad (104)$$

$$\lambda^{(1,1,1)} = \lambda_1^A + \lambda_1^B + \lambda_1^C \quad (105)$$

Assuming that the transition time t_1 is smaller than the mission time T_M (otherwise successive trial necessary), it needs to be determined which transition probabilities of each component is:

$$\frac{\lambda_1^A}{\lambda^{(1,1,1)}}, \frac{\lambda_1^B}{\lambda^{(1,1,1)}}, \frac{\lambda_1^C}{\lambda^{(1,1,1)}} \quad (106)$$

The inverse transform method yields a component which undergoes a transition, for example B . Hence, given that at t_1 the component has undergone a transition, the discrete probabilities give mutually exclusive and exhaustive arrival states (since for example only 1 and 3 can be reached) from the inverse transform method.

The result of the first transition can be found, and hence the update transition rate could look as follows:

$$\lambda^{(1,3,1)} = \lambda_1^A + \lambda_3^B + \lambda_1^C \quad (107)$$

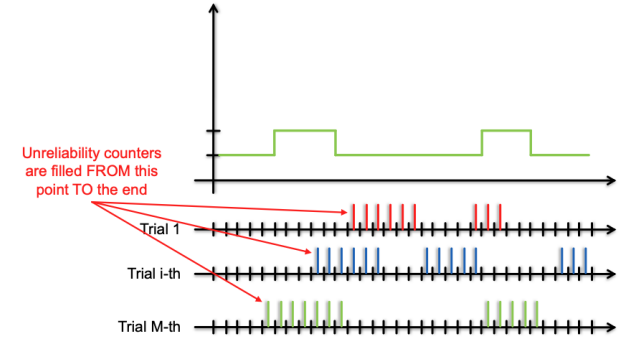
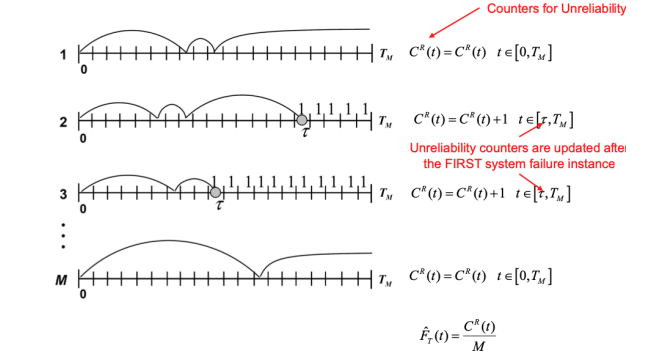
Finally, the next transition will be calculated as follows:

$$t_2 = t_1 - \frac{1}{\lambda^{(1,3,1)}} \ln(1 - R_t) \quad (108)$$

The trial simulation proceeds through various transitions from one system configuration to another up to the mission time T_M . When the system enters a failed configuration (when C fails) or the minimal cut set, then tallies are appropriately collected for the unreliability (until the end) and instantaneous unavailability estimates (discrete times $t_j \in [0, T_M]$)

In the end, after performing a number of trials M , the estimates for unavailability can and unreliability can be computed by simply dividing by M the accumulated contents of $C^R(t_j)$ and $C^A(t_j)$

System Reliability Estimation (I)



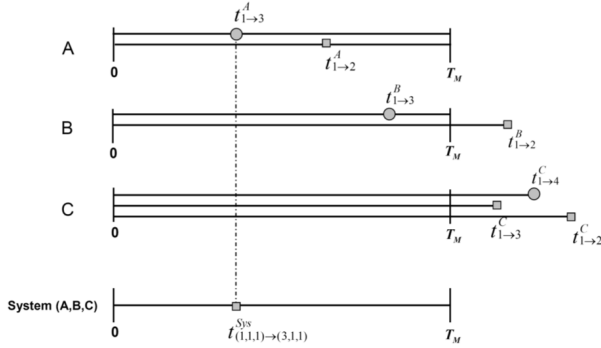
Direct Monte Carlo: Example

For direct Monte Carlo methods, from starting time $t = 0$ the system in nominal configuration will be used and all transition times are sampled (not step by step):

$$t_{1 \rightarrow m_i}^i = t_0 - \frac{1}{\lambda_{1 \rightarrow m_i}^i} \ln(1 - R_{t,1 \rightarrow m_i}^i), R_{t,1 \rightarrow m_i}^i \sim U[0, 1) \quad (109)$$

Sampling the Time and the Type of Transition

For each and every component, the mission times including time to transition can be estimated. The shortest transition time will then be defined as the new transition time.



The simulation proceeds to the successive times in the list, in correspondence of which a system transition occurs. After each transition the timeline is updated with the times of the transition that the component has undergone the last transition can do from its new state. The unreliability and unavailability estimates are computed each time the system enters a failed configuration.

Offshore Installation: Example of indirect Simulation for a multi-state system

Structure of Monte Carlo Code

- Set the number of MC repetitions, the mission time
- Set the input for each component state and transition rate across states
- Set the counter structure (availability and reliability)
- For loop for MC trials (repetitions, histories)
- Initialization of system state vector, reset time)
- While $t < \text{Mission time}$
 - Mapping the state of the components to the system state
 - Take the step to the random walk \rightarrow Compute the transition rate out of initial state
 - Sample from the T and C the transition time and the arrival state
 - Update the counters

The analytical approach of a Markov model becomes impractical, as soon as the system becomes highly complex. This means, that if for example the number of components, their states

(hence many plant states) and for example repair teams (which can modify plant states) can yield a inhomogeneous Markov chain. Therefore, Monte Carlo is used whereas system faults will be grouped. Different minimum cut sets are defined. The transport kernel is defined as always, through sampling from the exponential distribution.

Monte Carlo - The theoretical view

Sampling random variables

Sampling in uniform

- cdf: $U_R(r) = P(R \leq r) = r$
- pdf: $u_R = 1 = \frac{dU_R}{dr}$

Pseudo random numbers in uniform

- $x_i = (ax_{i-1} + c) \bmod m$
- where $a, c \in [0, m-1)$ and $m \gg 1$ are free parameters
- $r_i = \frac{x_i}{m}$
- As a result, the pdf of the analytical function can be well approximated

Buffon's Needle is a method which lets us approximate π . The idea is stemming from the fact that a needle of length L with a size smaller than D spaced lines has a certain probability of intersecting with one line. The needle is randomly oriented by its angle ϕ and its center height y which are independent from each other. For a fixed ϕ , the probability is just a function of y :

$$P(Y \leq L \sin \phi) = \frac{L \sin \phi}{D}$$

For a random variable ϕ , the probability is:

$$P = \frac{2L}{\pi D}$$

The counter is initialised via $N_S = 0$ at the beginning. Since the pdf is $f_Y(y) = \frac{1}{D}$ the reverse transform method will yield: $R = \frac{Y}{D}$ and $Y = R_1 D$, whereas the angle is similarly sampled: $\Phi = R_2 \pi$. If the needle intercepts the line, the $N_S = N_S + 1$. In the long term, $P \approx \frac{N_S}{N}$ and hence π can be approximated through the given formula.

Constant rate exponential distribution

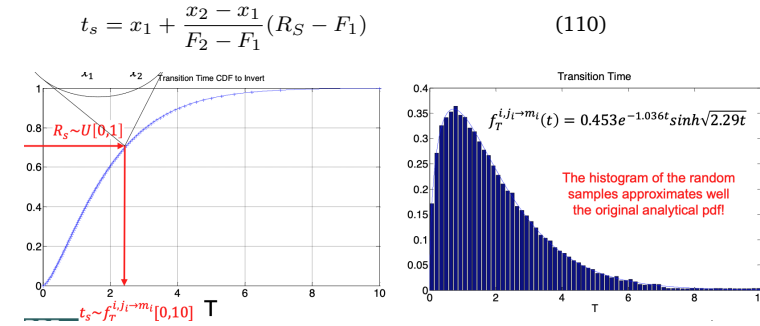
Assuming we have a markovian state (2 states, good and failed), and a constant hazard rate $\lambda = \text{const}$. The following equations hold true:

- $R = 1 - e^{-\lambda t}$
- inverted: $T = F^{-1} = -\frac{1}{\lambda} \ln(1 - R)$

Time-dependent rate Weibull distribution - if the hazard rate is not constant

- $R = 1 - e^{-\beta t^\alpha}$
- inverted: $T = F^{-1} = \left(-\frac{1}{\beta} \ln(1 - R)\right)^{\frac{1}{\alpha}}$

Sampling from non-invertible transition time pdf - if for example the pdf is not analytically invertible (mathematically for example $f_T^{i,j_i \rightarrow m_i}(t) = 0.453e^{-1.036t} \sinh \sqrt{2.29t}$, or if goint to infinite somewhere when x is zero or so). Then, the t_S is sampled via interpolation between two points in the distribution.



The histogram of random samples approximates the function pretty well.

Von Neumann Algorithm I (Sampling by Rejection)

If a pdf is complicated, then normalize the maximal point such that it becomes one and a point $f_X(x)$ is between 0 and 1:

$$h(x) = \frac{f_X(x)}{f_M} \quad (111)$$

Then sample a x' between the beginning and end of the function and calculate its function value $h(x')$, which is called a tentative value. Sample a R between 0 and 1 and accept it iff it is smaller than $h(x')$.

Von Neumann Algorithm II

We assume that now we can split up the function $f_X(x)$ into a product of $g'_X(x)$ and $H(x)$, which is possible based on theoretical assumptions. The maximum will be defined as previously, hence $B_H = \max(H)$ and therefore $h(x) = \frac{H(x)}{B_H}$, the same

procedure is done as shown previously, where if the sample is inbetween the tentative value and 0, we accept the sample. The efficiency of the rejection method is:

$$\epsilon = P[\text{accepted}] = \frac{1}{B_H} \quad (112)$$

In the following example, a trick is used to decouple two functions. One trick would be to define $X = R^2$, meaning that the random variable is a square of another one. This eliminates the terms with squareroots in the denominator of a pdf due to the derivative a square root.

MC Evaluation of Definite Integrals (1D)

The function which yields an award, such as G can be defined as follows:

$$G = \int_a^b g(x)f(x)dx \quad (113)$$

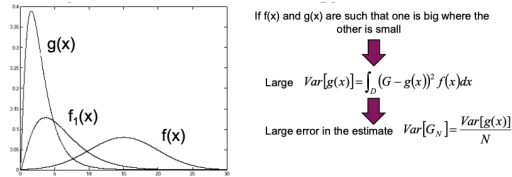
assuming $f(x) > 0$ and a pdf $\int f dx = 1$, G is equal the **Expectation Value** $E(g(x))$, meaning the expected outcome of a function g which awards a sampled x_i . Considering N trials, the expected win is the average of sampled g

$$G_N = \frac{1}{N} \sum_{i=1}^N g(x_i) = \bar{g} \quad (114)$$

In order to prove why G_N is a good estimator, the **Expectation and Variance** of G_N can be computed. The expectation value is exactly G and the variance is zero, when using the above definition of G_N . With $N \rightarrow \infty$, the variance vanishes to zero. Even though one could always compute the values numerically, those are not always known in practical cases due to the complexity of the problem.

Biased case (Importance Sampling)

If however f and g are such that one is big and the other is small, then for example the variance becomes huge nad hence error of the estimate as well.



The expression for G may be written

$$G = \int_D \left[\frac{f(x)}{f_1(x)} g(x) \right] f_1(x) dx = \int_D g_1(x) f_1(x) dx$$

In order to mitigate the error, the following equation is used:

$$G = \int_D \left[\frac{f(x)}{f_1(x)} g(x) \right] f_1(x) dx = \int_D g_1(x) f_1(x) dx \quad (115)$$

The biasing happens by shifting the center towards the other center through scaling and translation. The award is now a function of g_1 instead of g and the award is G_{1N} . By using the taylor expansion of $g(x)$ (a polynomial) and then through a normalisation procedure, g is ultimately approximated.

Simulation of System Transport

The von Neumann approach of the random walk is motivated by having the system transport kernel, a function of the conditional probability transition and the conditional probability that a system enters a state - whereas a new function ψ is defined as a series of transitions and independent of previous states and time steps. **The transition density $\psi(t; k)$ is expanded in a series of the partial transition densities $\psi^n(t; k)$ (n-th transition at t , entering k):**

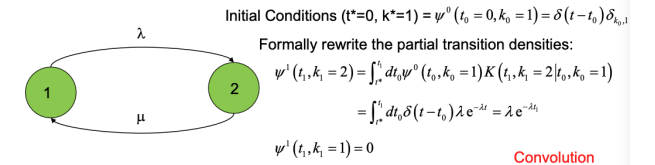
$$\psi(t, k) = \psi^0(t, k) + \sum_{k'} \int_{t_0}^t \psi(t', k') K(t, k|t', k') dt \quad (116)$$

The term in the integral is a convolution term, whereas the transport kernel is convoluted with ψ from the previous time and state. The transition density of the previous time step is however again a transport kernel (see figure) and so this results into a multidimensional convolution for higher order system transport kernel.

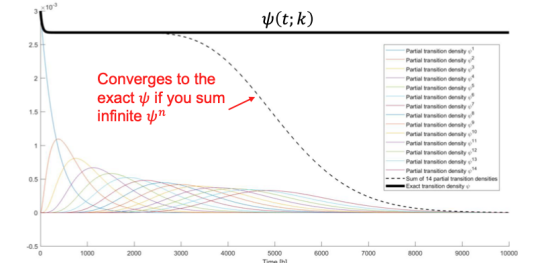
Initial Conditions: (t^*, k^*)

Formally rewrite the partial transition densities:

$$\begin{aligned} \psi^1(t_1, k_1) &= \sum_{k_0} \int_{t^*}^{t_1} dt_0 \psi^0(t_0, k_0) K(t_1, k_1|t_0, k_0) = K(t_1, k_1|t^*, k^*) \\ \psi^2(t_2, k_2) &= \sum_{k_1} \int_{t^*}^{t_2} dt_1 \psi^1(t_1, k_1) K(t_2, k_2|t_1, k_1) = \\ &= \sum_{k_1} \int_{t^*}^{t_2} dt_1 K(t_1, k_1|t^*, k^*) K(t_2, k_2|t_1, k_1) \\ &\dots\dots\dots \\ \psi^n(t, k) &= \sum_{k_1, k_2, \dots, k_{n-1}} \int_{t^*}^{t_n} dt_{n-1} \int_{t^*}^{t_{n-1}} dt_{n-2} \dots \\ &\dots \int_{t^*}^{t_2} dt_2 K(t_1, k_1|t^*, k^*) K(t_2, k_2|t_1, k_1) \dots K(t, k|t_{n-1}, k_{n-1}) \end{aligned}$$



Therefore, we can compute the partial transition densities $\psi^n(t; k)$ by applying sequentially the convolution operator between the previous $\psi^{n-1}(t; k)$ and the transition kernel $K(t_n, k_n|t_{n-1}, k_{n-1})$ pertaining to the current transition n-th.



Simulation for Reliability and Availability Analysis