Parameters

- Aerodynamic Force $\overline{F_A} = \overline{L} + \overline{D}$
- Lift Force L $[N]$
- Drag Force D $[N]$
- Aerodynamic Moment M_A $[Nm]$
- Dynamic Pressure $q = 1/2\rho V^2$ (Bernoulli) $[Pa]$
- Chord Length c $[m]$
- Surface Area $S = b \cdot c$ $[m^2]$ (Rectangular)
- Wing Span b $[m]$
- Lift Coefficient $C_l = L/(1/2\rho V^2 \cdot S)$
- Drag Coefficient $C_d = D/(1/2\rho V^2 \cdot S)$
- Moment Coefficient $C_m = M_A/(1/2\rho V^2 \cdot c \cdot S)$
- Angle of Attack α $[rad]$ (positive in clockwise direction)
- Lift curve slope $a = C_{l/\alpha} = C_l/\alpha \approx tan(angle\, x - axis\, to\, curve)$
- Pitch angle θ (Rotation w.r.t elastic axis)
- Lunge h (Deflection of elastic axis parallel to lift)

Conventions throughout Course

- If L and D absolute $\rightarrow$ use calculations above
- If L and D per span unit $\rightarrow$ correct via dividing by b
- Sign conventions: Lift positive, Drag positive in x and y direction
- Moments and angles positive in clockwise direction
- Our system coordinate system is defined by the wing. The angle of attack is defined relative to it
- The variables which describe the airfoil motion are the pitch θ and the plunge h which act at the shear centre of the wing

Mathematical Basics

- A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- Inverse of Matrix (2D): $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- Inverse of Matrix (3D):
  1. det (A), then transpose A
  2. Find the ajunct matrix (minors) of

  (cover row and column of element) of A^T

  and multiply with +- matrix

  $A^{-1} = 1/det(A) \cdot Adj(A^T)$
- The solutions of $Ax = 0$ for a matrix A, x cannot be just the trivial solution if A is not invertible
- $[rad] = \frac{\pi}{180}[deg]$

Steady Aerofoil and Wing Section Aerodynamics

- Aerofoil = 2-D wing section with goal to generate lift force perpendicular to the relative airspeed
- Convention: Lift is up, Drag is in direction of windspeed and Aerodynamic moment in clockwise direction acting on the aerodynamic center. Aerodynamic center is normally at the quarter chord position $c_{m,c/4}$ for syymetric airfoils. $x_{ac} = -m_0/2\pi + 0.25$ with m_0 as a shape constant
- Further assumptions: No viscosity, incompressible fluid, $Ma < 0.2, 0.3$, no vortices, potential flow (Navier-Stokes)
- Another centre is the shear center (elastic axis) from mechanics
- $L = 1/2\rho V^2 ca\alpha$, with a from tables (CFD and Wind Tunnel) $[N/m]$
- $M_A = 1/2\rho V^2 c^2 c_{m0}$ with c_{m0} also from tables $[N]$

Lift curve $C_l(\alpha)$ and drag curve $C_d(\alpha)$

- At small ranges of α, both lift and drag increase with: $C_l \propto \alpha$ and $C_d \propto \alpha^2$
- In aeroelasticity and this course, α will be very small, hence drag will be negligible small

The aerodynamic moment M_A

- The aerodynamic moment is much more important than drag C_d
- M_A varies with α in the small ranges of the angle of attack (very small, p. 7)
- **Important to note:** There exist a point at which the aerodynamic moment does not depend on α. This is the the aerodynamic centre
- The aerodynamic centre is not the same as the centre of pressure, which is defined as the point where the aerodynamic moment is zero given a certain angle of attack α
- Symmetric airfoils at $\alpha = 0$ have no aerodynamic moment at all times ($M_A = 0 = const$). At the aerodynamic centre for symmetric foils results into no moment
- Asymmetric airfoils at $\alpha = 0$ have a non-zero aerodynamic moment at all times (all angles α)

Assessment of C_l/α (Correction of value through Mach Number)

- The linear part of the lift curve is characterised by the slope $a = C_l/\alpha(M) = \frac{C_l/\alpha_{M=0}}{\sqrt{1-M^2}}$
- The Prandtl-Glauert factor is $1/\sqrt{1-M^2}$
- The factor is depending on the Mach number. The slope increases with increasing M (between 0 and 1)
- The dependence on Re is more subtle (p. 8)
- In supersonic regimes, the aerodynamic centre is shifted towards the back (more stability) however $C_{l/\alpha}$ decreases

Extension to wing aerodynamics (p. 8)

Aerofoil dynamics (2D) refer to the previous topics, however the 3-D case can be also modeled by through a couple examples. A finite wing is less stable and efficient than the airfoil since the tips have vortices on at the wing tips. These "induce" a velocity, which locally reduces the angle of attack. An important parameter is the so called **Aspect Ratio** $AR = b^2/S$. If the wing is assumed to be of surface $S = b \cdot c$, it follows $AR = b/c$.

- The lift curve can become a function of AR if due to the different tips. Approximately, the lift slope a_0 is adjusted via following formula:
- $a = a_0 \frac{AR}{AR+4}$
- The values a and c_{m0} will hence be corrected with a a^* and c^*_{m0}

Strip Theory (p.9)

- If AS is very small (delta wings), the integral of multiple airfoils
- Define multiple airfoils stacked next to each other along the span b
- Example, the wing is an elliptical $f(y) = \sqrt{1-(\frac{y}{b/2})^2} \cdot \overline{f}_\phi$
- $f(y) = a\alpha c = C_l c$ with $c =$ chord length.

Steady-state (static) Aeroelasticity

Typical Section = 1DOF model

2-D problem with a rigid wing. We can have multiple typical sections stacked onto each other, which would be later adding dimensionality to the variable θ. The idea is later on to model the torsional spring to be a torsional stiffness of a beam (since a real wing is actually a beam with a certain stiffness).

- The torsion acts in a beam section on the shear centre, however in aeroelasticity on the elastic axis
- The goal of engineering is always to move the shear center to the front (comes with risk to thin out the rear longeron and thicken the front longeron)
- In equilibrium, we know that the aerodynamic forces are equal to the spring forces
- $M_t + L \cdot e = \theta k_\theta = (qc^2 c_{m0} + qca\theta e) \cdot b$ (moment equations)
- Pitching moment M_t acting on section with k_θ stiffness
- $k_\theta \theta = qc(C_{l/\alpha} e(\theta+\alpha_0) + cC_{m0})$ (Momentum Equation)
- $k_h h = L = qcC_{l/\alpha}(\theta+\alpha_0)$ (Lift Equation)

Static Instability or Divergence

- If the elastic twist θ would become infinity for a given stiffness if the denominator of equation
- $\theta = qc\frac{C_{l,a} e\alpha_0 + cC_{m,0}}{k_\theta - cqC_{l,a} e}$, $\theta = \infty \Leftrightarrow$ denominator = 0
- If the dynamic pressure $q = \frac{k_\theta}{cC_{l,a} e} = q_{div} \rightarrow$ instable (divergence)
- Divergence = Static Instability
- $M_{tot} = (k_\theta - qSae)\theta - qSa(e\alpha_0 + C_m c)$ (In equilibrium $M_{tot} = 0$) ($M_{tot} > 0$ if in anti-clockwise direction)
- Different interpretation: $\frac{\partial M_{tot}}{\partial \theta} \geq 0 \Leftrightarrow$ increasing total moment in section for increasing $\theta \Leftrightarrow \Delta\theta > 0$
- $\frac{\partial M_{tot}}{\partial \theta} \geq 0 \Leftrightarrow$ overall moment brings blade section back to original position
- The divergent dynamic pressure can be found by differentiating w.r.t. θ

Lagrange Equation (Energy interpretation)

- $L = T - U$ (Kinetic and potential energy)
- $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$
- In statics: $\frac{\partial U}{\partial x_i} = 0$ (Potential energy conservative)
- Here: $\frac{\partial U}{\partial x_i} = \frac{\partial}{\partial x}\frac{\delta W}{\delta x}$ (Virtual work), hence for θ
- $\frac{\partial U}{\partial \theta} = \frac{\partial}{\partial \theta}\frac{\delta W}{\delta \theta} = 0$ (Check for q)
- Resulting q provides the divergence
- $U = 1/2 \cdot k_\theta \theta^2 = \int F(\theta) d\theta$ (For system with only one spring)

Section with more than 1 DOF

1. Number of DOF = Dimensions of Stiffness Matrix and number of equations needed (full rank) K
2. Define a potential energy matrix for mechanical system $K_{i,j} = \frac{\partial U}{\partial x_i \partial x_j} = K_{j,i}$
3. Define aerodynamic matrix K_a based on aerodynamic forces (independent on e for example)
4. If a matrix is not symmetric = Non-conservative forces
5. Similar to before, instead of asking if the system is stable if the denominator is zero, we must know if the determinant of the transfer function is zero
6. Transfer Function: $[K - qK_a] = K_{ael}$ is 'Aeroelastic K'
7. Find a q for which the transfer function determinant becomes zero, which is divergence dynamic pressure. The solution (forces acting) is the so called divergence mode
8. If all eigenvalues are $> 0 \Leftrightarrow$ stable, if one is at least $< 0 \Leftrightarrow$ unstable (for sections)

System (more than 1 section) with multiple DOF

- Define a system with multiple θ_i, whereas the calculations become similar to when when calculating one section with multiple DOF
- Make an Ansatz with the lagrange equations and define stiffness matrix K
- The aerodynamic matrix becomes the identity matrix (if we only speak about θ)
- This implies that the solutions for q are the eigenvalues of K

Comment on eigenvalues

- The eigenvalues of the Aorelastic K_{ael} cannot guarantee that q are always the eigenvalues, since K_a is sometimes non-symmetric (most of the times, only if rotational degrees of freedom present)
- In general, following statement holds true: $det(K - qK_a) = 0$

- $det(K_a^{-1}K - qI) = det(A - \lambda I) = 0$

- If A is not symmetric, we can say: There are less eigenvectors and values than the order (n), can be complex and come in complex conjugate pairs

Active conrol on sections

- With active control, the behaviour of the elastic twist θ can be controlled with for example a trailing edge flap

- As described in the script, a trailing edge flap can influence the lift and the moment as follows:

- $l = qcC_{l/\delta}\delta$, $m = qcC_{m/\delta}\delta$ ($C_{m/\delta} < 0$)

- Both forces contribute to the overall moment, hence will be added to the calculations we did previously

- With a so called 'Gain' G, the controller controls δ proportional to θ, hence a new linear equation system is formed

- Assuming the nose-down motion of controlling the the edge flap, we have to simplify terms, the end result is

- $q_{div,flap} = \frac{k_\theta}{cae - Gca^*}$, with $a^* = -(cC_{m/\delta} + eC_{l/\delta})/(c\delta)$

Ritz Method

... is a energy variational method whereas an equilibrium occurs in correspondence of an extreme of potential energy. A general application is the virtual work. According to the Hamilton's principle and Lagrange equations, we can define a set of equations.

- $\frac{\partial V}{\partial x_i} = 0$ for all i (x_i degrees of freedom)

- From mechanics, we need the bending stiffness (I) and the torsional stiffness (J)

- $I = 1/12 * b * h^3$ (w.r.t x) and $J = \frac{4A^2}{\int ds/t}$ (Integral is perimeter (Umfang) divided by thickness)

- For circular shapes: $I = \frac{\pi}{4}r^4$, $J = \frac{\pi}{2}r^4$

Derivation for a beam section (mechanical part, torsion)

- Torsional Strain Energy: $U = \frac{1}{2}\int_0^l GJ(\frac{\partial\theta}{\partial x})^2 dx$

- Given aerodynamic forces are non-conservative, we use the concept of virtual work

- $\delta U = \delta W$ (Virtual work due to non-conservative forces)

- $\sum_{i=1}^n \frac{\partial U}{\partial x_i}\delta x_i = \sum_{i=1}^n \delta W_i$ (reformulated for small variations of one DOF)

- $\frac{\partial U}{\partial x_i} - \frac{\delta W}{\delta x_i} = 0$ (reformulated)

- The work done by the external forces (aerodynamic) can be rewritten:

- $\delta W = \int_0^l m(x)\delta\theta(x)dx$

- $m(x)$ generated by aerodynamic forces

- Without going into further detail, there are 2 distinct cases from which one has to go on in the calculation, either the functions are given in a generalised form or in matrix form

- $\theta(x) = \sum_{i=1}^N \phi_i(x)a_i = [\Phi]\{a\}$

- $[\Phi]$ is a row vector with elements ϕ_i!

- $\phi_i(x)$ are shape functions and a_i are coefficients and the linear combination of those make up $\theta(x)$

- Finally, the overall equations result in $[K]\{a\} = \{f\}$

- $K_{i,j} = \int_0^l GJ\phi_{i_x}\phi_{j_x}dx = GJ\frac{\partial^2 U}{\partial a_i \partial a_j}$ (Stiffness matrix entries, partial derivatives w.r.t. to x and a)

- $[K] = GJ\int_0^l [\Phi_x]^T[\Phi_x]dx$

- $f_i = \int_0^l m(x)[\Phi]_i(x)dx$ (index i for each element)

- $\{f\} = \int_0^l m(x)[\Phi]^T dx$ (in matrix form)

Derivation for a beam section (aerodynamic part, torsion)

Following assumptions are drawn:

- Elastic axis is perfectly straight

- Aerodynamic center of all sections on a straight line

- External moments as before by aerodynamic forces: $m(x) = qcea(\theta(x) + \alpha_0)$

Replace all $\theta(x)$ with the above solutions and insert insert $m(x)$ into generalised forces vector

- $\{f\} = q\int_0^l cea\alpha_0[\Phi]^T dx + q\int_0^l cea[\Phi]^T[\Phi]dx\{a\}$

- $\{f\} = \{f_0\} + q[K_A]$

- $[K_A] = \int_0^l cea[\Phi]^T[\Phi]dx$

- $\{a\} = ([K] - q[K_A])^{-1}\{f_0\}$ solves for all a

- a gives us the beam gives us the response of the system (pitch θ at all points along x)

- Stability: $det([K] - q[K_A]) = 0$

- Hence the basis of the solution of the eigenvalue problem

- Eigenvalues q: Dynamic pressure where zero stability

- Eigenvectors: Corresponding divergence modes

- Attention: $a \neq \{a\}$! (lift slope vs. coefficients)

One single shape function (1-DOF) and $[\Phi] = \phi(x)$

- $\theta(x) = a \cdot \phi(x)$ is one dimensional, we assume $a = 1$ because we can define it within ϕ

- $U = \frac{1}{2}\int_0^l GJ\phi_x^2 dx$

- $K = \int_0^l GJ\phi_x^2 dx = [K]_{torsion}$

- $K_a = \int_0^l ceC_{l\alpha}\phi^2 dx$ ($C_{l\alpha} = a$ lift curve slope)

- $f_0 = q\int_0^l ceC_{l\alpha}\alpha_0\phi dx$

- $(K - qK_a)a = f_0$

- $a = f_0/(K - qK_a)$ gives us the response by which θ is multiplied

- $q_d = K/K_a$ gives us the divergence

One single shape function, Bending and Twisting

- For bending, the potential energy is: $U = \frac{1}{2}\int_0^l EI\psi_{xx}^2 dx$
- $K = \int_0^l EI\psi_{xx}^2 dx = [K]_{bending}$
- If we assume bending takes place and torsion as well, we assume both to be decoupled
- The stiffness matrix reads
- $K = \begin{bmatrix} [K]_{bending} & 0 \\ 0 & [K]_{torsion} \end{bmatrix}$
- The aerodynamic stiffness matrix is of shape (always for bending and twisting):
- $K_a = \begin{bmatrix} 0 & \int_0^l cC_{l\alpha}\phi\psi dx \\ 0 & \int_0^l ecC_{l\alpha}\psi^2 dx \end{bmatrix}$
- The vector x includes the coefficients for the shape functions ψ and ϕ:
- $x = (K - qK_a)^{-1} f$
- $f = \begin{Bmatrix} q\int_0^l cC_{l\alpha}\phi\psi dx \\ q\int_0^l cC_{l\alpha}\psi^2 dx \end{Bmatrix}$
- $det(K - qK_a) = 0 \Leftrightarrow$ then $q = q_{div}$

Shape functions

Shape functions have to be chosen. In FEA, shape functions are local for each finite element.

- Orthonormal modes: $\int \psi_i\psi_j dx = 0$ for different shape functions in i and j
- Simple polynomials are great shape functions x/l, $(x/l)^n$
- Natural vibration modes or normal modes (eigenvectors of the problem): $K - \lambda M$ with K the stiffness and M the mass matrix (Important for dynamic systems later)

Bending / twisting coupling

In class multiple examples have been shown whereas following are the key takeaways:

- Out of plane bending can exist if for example the shear centre and the principle axes (centre of gravity) are apart from each other significantly
- The conventions for positive and negative e eccentricities: positive if aerodynamic centre in front of shear centre and hence negative if the other way around
- Positive e is detrimental for aeroelastic stability, negative is beneficial
- Helicopter blades have D-spars to shift the elastic axis forward
- Gurney flaps at the end of the wing with length 1% of c make the wing virtually longer

Control effectiveness, typical section

Control effectiveness is described by following term:

- $\frac{L_{elastic}}{L_{rigid}} = \frac{1-\frac{q}{q_r}}{1-\frac{q}{q_{div}}}$ = Control Effectiveness
- $q_r = -q_{div}\frac{e}{c}\frac{C_{l\delta}}{C_{m\delta}}$ ($C_{m\delta}$ is negative!)
- If the control effectiveness is zero, the aileron deflection does not contribute to more lift
- If the control effectiveness is is negative, this means that $0 \leq q_r < q < q_{div}$, the control system pushes the system into the contrary direction of intended use
- Possible goal: As close to q_{div} and below q_r
- Another solution: Outboard ailerons (less stiff due to smaller torsional stiffness) and inboard ailerons (GJ/l)
- The overall equations for future equations will be for equilibria:
- $[K]\{\phi\} = q[K_a]\{\phi\} + \{m_0\} + q\{f_c\}\delta$ (Aileron Equation)
- For mulitple segments: $q[f_c]\delta$

Effects of Sweep Angle on Divergence

- In this course, a 2-DOF model is used (pitching and flapping)
- Spring stiffness for moments on a beam: k_θ and k_ϕ
- $k_\theta = \frac{GJ}{l}$ (l = length of lever/beam b, torsion)
- $k_\phi = \frac{EI}{l}$ (flapping / bending)
- $G/E = \frac{1}{2(1+\nu)}$ (Poisson ratio)
- The angle of attack will be reintroduced:
- $\tan(\alpha) = \frac{V\perp}{V\|} = \frac{-V\sin(\Lambda)\sin(\phi)}{V\cos(\Lambda)} = -\tan(\Lambda)\sin(\phi)$
- $\alpha = -\tan(\Lambda)\phi$ (small angle approx)
- In the script the approach given results are as follows:
- $[K] = \begin{bmatrix} k_\phi & 0 \\ 0 & k_\theta \end{bmatrix}$
- Simplifications: $Q = q_n cbC_{l\alpha}$ and $t = \tan(\Lambda)$ (Q is q redefined)
- $[K_a] = \begin{bmatrix} -tb/2 & b/2 \\ -te & e \end{bmatrix}$
- $\{f\} = \frac{Q\alpha_0}{\cos(\Lambda)}\begin{Bmatrix} b/2 \\ e \end{Bmatrix}$
- $[K]_{ael} = [K] - Q[K_a]$
- Condigtion for divergence:
- $det(K_{ael}) = \Delta = 0 \Leftrightarrow Q_D = \frac{k_\phi k_\theta}{k_\phi e - k\theta bt/2}$
- $\Leftrightarrow q_D = \frac{k_\theta/(SeC_{l/\alpha})}{\cos^2(\Lambda)[1-(b/e)(k_\theta/k_\phi)(\tan(\Lambda)/2)]}$
- This gives us a uniqe solution for two degrees of freedom. If we try to push $q_D \to \infty$, we can do so by setting the denominator of $q_D = 0$
- This allows us to model our wing with geometrical and material parameters such that the system never becomes unstable

- Divergence can already be avoided with small Λ sweep angles
- Because the angle creates a coupling between wing bending and torsion (deformation), and the angle of attack
- $\alpha_{new} = \alpha_0 / \cos(\Lambda) + \theta - \phi \tan(\Lambda)$
- Assuming small angles: $\alpha_{new} \approx \alpha_0 + \theta - \phi\Lambda$ whereas the negative part is larger
- Other approaches: Build wing with unbalanced comosite laminates such that bending/twisting coupling is generated by material
- Adding aerodynamic control surfaces (see flaps) with active control
- Negative sweep angles reduce divergence speed

Sweep Angles and Ritz Method (Class Notes)

- Shape functions are chosen such that one has dependencies along the section
- For twisting: $\theta(y) = f_\theta(y)\Theta$ (Θ constant)
- For bending: $w(y) = f_w(y)B$
- $\phi = \frac{\partial w}{\partial y} = B \cdot \frac{\partial f_w}{\partial y}$
- This will make the angle of attack α dependent on y
- $\alpha(y) = -\tan(\Lambda)\frac{\partial f_w(y)}{\partial y}B + f_\theta(y)\Theta$
- Hence lift and moment become also dependent on y (we also assume $C_{m,0} = 0$ or const):
- $l(y) = qcC_{l,\alpha}(\alpha(y))$, $m(y) = l(y) \cdot e$
- Principle of virtual work:
- $\delta W = \int (l(y)\delta w + m(y)\delta\theta)dy$
- Expanding δw with $f_w \delta B$ and $\delta\theta$ with $\delta\theta$ with $f_\theta \delta\Theta$ will yield an integral whereas
- The aerodynamic stiffness K_a will be of following shape and hence non-symmetric:
- $[K_a] = \begin{bmatrix} \int \ldots \Theta\delta\Theta & \int \ldots B\delta\Theta \\ \int \ldots \Theta\delta B & \int \ldots B\delta B \end{bmatrix}$, $x = \begin{Bmatrix} \Theta \\ B \end{Bmatrix}$

Unsteady Aeroelasticity

Dynamic Systems (Repetition from Bachelor Level)

- $\dot{x} = Ax + Bu$
- $y = Cx + Du$
- x are state variables, u are input variables and y are output
- Aeroelastic System:
- $m\ddot{z} + k_z z =$ Mechanical force in lunge direction
- $I\ddot{\theta} + k\theta =$ Mechanical force in pitch direction
- In compact form:
- $M\ddot{x} + C\dot{x} + Kx =$ Mechanical Forces $= f =$Input
- Canonical Form:
- $A\dot{r} = g$
- $r = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$ and $g = \begin{Bmatrix} 0 \\ \neg\dagger M^{-1} f \end{Bmatrix}$
- r is of dimension $2n$ (twice the No. of DOF) and ordered such that the definition of A is valid
- The roots of the characteristic polynomial of A tell us if the system is stable
- The eigenvectors are the modes of the system and the response of the system is a linear combination of these nodes
- The topic will be looked at again later on, however one can say in general for the mass, damping and stiffness:
- $M = \begin{bmatrix} m & mx_{cg} \\ mx_{cg} & I + mx_{xg}^2 \end{bmatrix}$
- $C = \begin{bmatrix} c_z & 0 \\ 0 & c_\theta \end{bmatrix}$
- $K = \begin{bmatrix} k_z & 0 \\ 0 & k_\theta \neg\dagger \end{bmatrix}$

Quasi-steady approach

Quasi-steady approaches take into account that the angle of attack changes due to the velocity component normal to the free stream direction

- The aerodynamic equations for 2-DOF systems is in general:
- $l(\theta, \dot{\theta}, \dot{z}) = qcC_{l/\alpha}\left(\theta - \frac{\dot{z}}{V} + (c/2 - e)\frac{\dot{\theta}}{V}\right)$
- $m(\theta, \dot{\theta}, \dot{z}) = l(\theta, \dot{\theta}, \dot{z}) \cdot e$
- This aerodynamic forces hence modify the damping and stiffness matrices of the overall equations of motion:
- $[C_a] = \frac{1}{V}cbC_{l/\alpha}\begin{bmatrix} -1 & c/2 - e \\ -e & e(c/2 - e) \end{bmatrix}$
- $[K_a] = cC_{l/\alpha}\begin{bmatrix} 0 & 1 \\ 0 & e \end{bmatrix}$
- For state vector $x = \begin{Bmatrix} z \\ \neg\dagger\theta \end{Bmatrix}$
- The mechanical forces do not change, hence:
- $[M_{ael}] = [M]$, $[C_{ael}] = -q[C_a]$, and $[K_{ael}] = [K] - q[K_a]$
- To check stability: Check eigenvalues of A from created from the matrices (dynamical)
- $det(A - I\lambda) = det(-M^{-1}K - I\lambda) \stackrel{!}{=} 0$ From linear algebra

---

Comment on Stability

- If a system is <u>statically unstable</u>, it is also <u>dynamically unstable</u>
- If a system is <u>dynamically unstable</u>, it must not be necessarily <u>statically unstable</u>

Check first if following conditions hold true:

1. Static stability via determinant of $[K_{ael}]$ (if unstable, also dynamically unstable)
2. If statically stable, check if dynamically unstable (eigenvalues of A)
3. If both are stable, then overall stable at all times

Approach for different models than airfoil in quasi-steady approach

- For a flapping wing with 1-DOF $\rightarrow$ model a high stiffness $k_z \rightarrow \infty$ and check response
- For a helicopter, we have contributions from a 2-DOF model (pitching θ and flapping β)
- Additionally, the velocity V is a function of the blade's radius ($V(r) = \Omega r$, sometimes ω)
- The angle of attack is hence $\alpha = \theta - \dot{\beta}/\Omega$
- Derivation: $\frac{\dot{\beta}r}{V} = \frac{\dot{\beta}r}{\Omega r}$ and $q(r) = \frac{1}{2}\rho V^2 = \frac{1}{2}\rho\Omega^2 r^2$
- The aerodynamic forces can be calculated first as a function of r
- $l(r) = q(r)cC_{l/\alpha}\left(\theta - \dot{\beta}/\Omega\right)$ with $l = [N/m]$
- $m(r) = l(r) \cdot e$
- The overall moments have to be integrated over the whole radius (span) for the momentum contributions from flapping and pitching
- Eqution of motion:
- $I_\beta\ddot{\beta} + I_\beta\Omega^2\beta = M_\beta^{aero}$ with centrifugal force (par. axis)
- $I_\theta\ddot{\theta} + k_\theta\theta = M_\theta^{aero}$ as usual
- $M_\beta^{aero} = \int_0^R l(r) \cdot \underline{r}dr = \frac{1}{8}cC_{l/\alpha}\Omega^2 R^4\left(\theta - \frac{\dot{\beta}}{\Omega}\right)$
- $M_\theta^{aero} = \int_0^R m(r) \cdot dr = \frac{1}{6}ecC_{l/\alpha}\Omega^2 R^3\left(\theta - \frac{\dot{\beta}}{\Omega}\right)$
- Natural frequency is $\omega = \sqrt{K/M} = \sqrt{I_\beta\Omega^2/I_\beta} = \Omega$
- Reduced frequency is $k = \frac{\omega c}{2V} = \frac{c}{2r}$ in this case
- The reduced frequency checks corrects the natural frequency by the ratio between the half-chord and the velocity, making it dimensionless

Unsteady Aerodynamics

Important Parameters for Unsteady Aerodynamics

- The natural frequency ω is the the highest eigenfrequency of the system which can be excited in a given situation
- The reduced frequency k is a dimenionless frequency and used in the frequency domain
- If $k < 0.05 \rightarrow$ quasi-steady approach
- If $k > 0.05 \rightarrow$ unsteady approach
- The reduced time $\tau = t\frac{2V}{c}$ is a dimensionless time used to describe transient phenomena in the time domain
- Critical damping $\overline{c} = 2\sqrt{MK}$
- Damping ratio $\zeta = C/\overline{c}$
- Unsteady aerodynamics: The reduced frequency measures the unsteadiness of aerodynamics: Air is not able to adjust instantly to a change in α

Dynamic Systems in the Frequency Domain

- In Fourier space $F(\omega)$:
- $x = Xe^{i\omega t}$, $y = Ye^{i\omega t}$ and $u = Ue^{i\omega t}$
- $X = (i\omega I - A)^{-1}BU$
- $Y = C((i\omega I - A)^{-1}B + D)U = H(\omega)U$
- We can define Transfer Functions = Operators for the structure and for the aerodynamics:
- $H_S(\omega) = -\omega^2 m + k_z = -\omega^2[M] + [K]$
- $H_A(\omega) = i\omega q c_a + q k_a = i\omega q[C_a] + q[K_a]$
- $Z = (H_S - H_A)^{-1}F_z(\omega)$
- mapping the input F_z (aero forces, disturbances) to the output Z (state variable) from before.

Example for frequency domain "Flying Door"

- Assuming only flapping β DOF:
- The angle of attack $\alpha_G(t) = v_G(t)/V$
- The angle of attack directly depends on the gust velocity normal to the free stream velocity
- Same calculation as before in the example in quasi-steady approach where β becomes the only variable

Modal Decomposition

- Free vibrations of the system are solutions of the simple system:
- $M\ddot{x} + Kx = 0$
- whereas the eigenvalues ω and eigenvectors Φ can be obtained
- According to linear algebra, we can define new coordinates:
- $r = \Phi x$ hence a coordinate transform gives us following equation:
- $\Phi^T M\Phi\ddot{r} + \Phi^T M\Phi r = \Phi^T f =$ GAF
- GAF are the Generalized Aerodynamic Forces (if only aero forces present in f)
- The behavious of the system is given by a linear combination of its modes. For wings, we will consider mainly two modes (torsion and bending), which we deem most important and hence the entire calculations are simplified
- However, the main criteria of deciding which to keep and which not to keep are based on:
- Frequency: Will certain oscillations be reached? Consider also multiple higher frequencies (for example during buffeting) This is when shockwaves hit our system or we experience airflow separation
- Modeshape: Certain modes must be within the system to represent something physical (Fuselage must be able to move)
- For problems of this type, we have to calculate the new matrices such that it has the same shape as in the beginning:
- $\overline{M} = \Phi^T M\Phi$, $\overline{K} = \Phi^T K\Phi$, $\overline{f} = \Phi^T f$

Ritz Method for dynamic Systems

Since we have non-conservative forces, we have to write:

- $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{\delta W}{\delta x}$
- $\delta W = \int_0^b (l(y)\delta w + m(y)\delta\theta)dy$
- The trial/shape functions are linear combinations with time depending weights
- $w(y,t) = \sum \psi_i(y)a_i(t)$, $\theta(y,t) = \sum \phi_i(y)b_i(t)$
- We can use static solutions, polynomials, normal modes or a combination

Theodorsen (Unsteady Aerodynamics in frequency domain)

Unsteady aerodynamics stems from the fact that a delay is found for $C_{l/\alpha}$ if we change the angle of attack α very quick. The reason behind this is that the distribution of the airfield has to adjust since the angle changes quickly in time (instationary airflow)

We want to quantify aerodynamic lag and reduction of magnitude by introducing some functions depending on reduced frequency k:

- $C(k) = \frac{H_1^{(2)}}{H_1^{(2)}(k) + iH_0^{(2)}(k)}$
- Hankel functions of second kind are denoted as H (on basis of Bessel functions)
- We can use Theodorsen only for harmonic motions
- $H_n^{(2)} = J_n(k) - iY_n(k)$
- We will use following simplification/approximation (Theodorsen Function):
- $C(k) = 1 - \sum_{j=1}^{2} \frac{A_j}{1 - B_j/(ki)}$
- Where we use them for the reduced frequency (i imaginary)
- Theodorsen follows from thin airflow theory and exploits ideal non-rotational aerodynamics (no viscosity and vortices)
- The matrices are corrected via (in the frequency domain):
- $[C_a] \to C(k)[C_a]$, $[K_a] \to C(k)[K_a]$

- A system with following characteristic equation (2-DOF):
- $\begin{Bmatrix} L \\ \neg\dagger M \end{Bmatrix} = q[K_a]\begin{Bmatrix} h \\ \alpha \end{Bmatrix} + q[C_a]\begin{Bmatrix} \dot{h} \\ \dot{\alpha} \end{Bmatrix} + q[M_a]\begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix}$
- $K_a = \overline{C_{l/\alpha}}S \begin{bmatrix} C(k) & 0 \\ C(k)c(1/4-\gamma) & 0 \end{bmatrix}$
- $C_a = \frac{\overline{C_{l/\alpha}}S}{V} \begin{bmatrix} c(\frac{1}{4} + C(k)(\frac{1}{4}+\gamma)) & C(k) \\ -\frac{c}{4}(\frac{1}{4}+\gamma) + C(k)(\frac{1}{16} - \gamma^2) & C(k)c(\frac{1}{4}-\gamma) \end{bmatrix}$
- $M_a = \frac{\overline{C_{l/\alpha}}Sc}{V^2} \begin{bmatrix} c\gamma/4 & 1/4 \\ -c(1/32+\gamma^2) & -c\gamma/4 \end{bmatrix}$
- $\gamma = 1/4 - e/c$
- γ is the distance of the shear centre from the mid-chord point
- $\overline{C_{l/\alpha}}$ = steady-state lift curve slope
- The aerodynamic damping and stiffness matrices are closely related to the steady-state case
- The aerodynamic mass matrix is the so-called "added mass" which stem from the fluid's inertia
- Theodorsen function is mainly used for the frequency domain, not time domain (hence the interpretation of them being a correction of time domain system is misleading)
- The matrices are now depending on k, hence the response can be different for different k's

Analysis of unsteady aeroelastic systems

A system is unsteady: We excite the system by unsteady forces (forced oscillations) or we model the system as unsteady in order to assess the dynamic stability

- In time domain: We check the response to a given excitation, through dynamic stability, ABCD Matrices, Numerical (Direct time integration) with response in unsteady excitation or canocial inputs and lastly the the flutter stabiltiy is given where the first value of q where the system diverges
- In frequency domain: Same as time domain, however later through transfer function from ABCD, poles form the transfer function and excitation response in frequency domain

Unsteady aerodynamics: We correct the quasi steady formulation with Theodorsen, in time domain there is another formulation, the so called Wagner function

Wagner function (Unsteady Aerodynamics in time Domain)

In practice, the response depends on the Aspect Ratio AR, the Ma Mach number. We assume the behaviour as follows:

- $C_l \approx C_{l/\alpha}\Delta\alpha(1 - A_1 e^{-B_1\tau} - A_2 e^{-B_2\tau})$
- $A_1 = 0.165$ and $A_2 = 0.335$
- $B_1 = 0.0455$ and $B_2 = 0.3$
- The solution of a differential equation:
- $\dot{y} = -By$ with initial condition $y(0) = A$
- The corresponding system:
- $\frac{dy_1}{dt} = -2V/cB_1 y_1 + \alpha(t)$ with b.c. $y_1(0) = A_1$
- $\frac{dy_2}{dt} = -2V/cB_2 y_2 + \alpha(t)$ with b.c. $y_2(0) = A_2$
- The states are artificial "lag states" which we introduce (like degrees of freedom)
- We use as many as needed to approximate the response in the time domain (training data)
- $C_l(t) = \overline{C_{l/\alpha}}(2V/c)(A_1B_1y_1 + A_2B_2y_2)$ with steady-state lift curve slope $\overline{C_{l/\alpha}}$
- Transfer Function in Laplace Domain:
- $H(s) = \frac{(A_1B_1 + A_2B_2)s + B_1B_2}{s^2 + (B_1+B_2)s + B_1B_2}$
- To derive A_i and B_i, one does wind tunnel or numerical experiments und derive "best values" (= ROM or Reduced Order Model)
- Wagner function and Theodorsen are 100% consistent with each other (different domains)

Flutter Tracking

We search for the "flutter dynamic pressure" q_f, which is where the stability of the system is not given anymore. At first one has to find the eigenvalues and vectors as a function of q and the frequencies in the system's modes

- To begin with let's define the system matrix A:
- $A = \begin{bmatrix} 0 & I \\ -M^{-1}(K - qK_a) & -M^{-1}(-qC_a) \end{bmatrix} = A(q)$
- The eigenvalues of A and the modes (eigenvectors) are hence also depending on q (or one could also vary V)
- Two plots are created, where the eigenvalues (frequencies) are plotted against V or q
- For the real and imaginary part of the eigenvalues $Re(\lambda)$ and $Im(\lambda)$
- If the real part of the eigenvalues become positive, the system reaches its flutter speed for a given q_f
- On the imaginary axis, one can see that the eigenvalue associated to the diverging one is converging to another one (coalescence)
- The other mode's real part becomes more negative
- Both modes (associated to the eigenvalues) are coupled (or start to interacti with each other). One becomes heavily damped, the other one becomes unstable
- Stable modes dissipate energy into the flow
- Unstable modes absorbs energy from the flow
- The overall energy of the system increases with q due to the non-conservative aerodynamic forces

Introduction of the Centre of Gravity

By introducing the centre of gravity, one can see that the kinetic energy of the system changes:

-

Windtunnel Testing

Mathematical Appendix

- $e^{[A]t} = T^{-1}e^{[D]t}T$ for the Matrix Exponential
- T is the basis of eigenvectors in the diagonal matrix D
- $y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$ (time-domain)
- $Y(s) = \sum(s)U(s) = (C(sI - A)^{-1}B + D)U(s)$ (frequency domain)