1 General Considerations

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} \neq 0 \quad (1)$$

- Wave propagation
- Convective flows with buoancy
- Flows with variable temperature, friction, sources of heat
- High speed flows with Mach numbers $Ma \geq 1$

Compressible flows can still be described through the continuum model and conservation laws. The assumption is also that the thermodynamic state of the fluid is in a local equilibrium.

Assumptions

- Length scale of flows large compared to molecular scales (mean free path λ)
- Length scale of flows small compared to the geometric scales (length L)
- Time scale τ_F of the flow long compared to the molecular process (relaxation) time constants τ_R

Description of the "Continuum" Flow State

- Three components of flow velocity $\underline{u}(\underline{x}, t)$
- The fluid density $\rho(\underline{x}, t)$
- The fluid pressure $p(\underline{x}, t)$
- The energy $e(\underline{x}, t)$

The required equations are the conservation laws for mass, momentum and energy together with suitable thermodynamic equations of state. With corresponding initial and boundary conditions, the evolution can then be computed.

2 Thermodynamic Relations

State Variables

- Density: $\rho = \rho(p, T)$
- Pressure: $p = p(\rho, T)$
- Temperature: $T = T(\rho, p)$
- Internal energy: $e = e(\rho, T)$ $[e] = J/kg$
- Enthalpy: $h = h(p, T)$
- Entropy: $s = s(\rho, T)$

Van der Waals Gas

$$(p + a\rho^2)\left(\frac{1}{\rho} - b\right) = RT \quad (2)$$

Incompressible Fluid

$$\rho = const. \neq \rho(p, T) \quad (3)$$

3 Conservation Laws for Continuum Flows

$$\frac{Dm}{Dt} = \frac{D}{Dt}\int_{\tilde{V}} \rho d\tilde{V} = 0 \; (material\ volume) \quad (4)$$

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho(\mathbf{u}\cdot\mathbf{n}) dS = 0 \; (Eulerian\ Volume) \quad (5)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \; (material\ volume\ /\ index) \quad (6)$$

$$\frac{D\rho}{Dt} = -\rho \frac{\partial u_i}{\partial x_i} \; (Eulerian\ Volume/\ index) \quad (7)$$

Mass Conservation

Material Volume

$$\frac{Dm}{Dt} = \frac{D}{Dt}\int_{\vec{V}} \rho d\vec{V} = 0 \quad (8)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i}(\rho u_i) = 0 \quad (9)$$

Eulerian Volume

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho(\vec{u}\cdot\vec{n}) dS = 0 \quad (10)$$

$$\frac{D\rho}{Dt} = -\rho \frac{\partial u_i}{\partial x_i} \quad (11)$$

Momentum Conservation

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial}{\partial x_j}\sigma_{ij} + \rho f_i \quad (12)$$

$$\rho \frac{Du_i}{Dt} = \frac{\partial}{\partial x_j}\sigma_{ij} + \rho f_i \quad (13)$$

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \quad (14)$$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\delta_{ij}\frac{\partial u_k}{\partial x_k} \quad (15)$$

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \rho f_i \quad (16)$$

Energy Conservation

$$\rho \frac{D}{Dt}(e + \frac{1}{2}u_1^2) = \frac{\partial}{\partial x_j}(\sigma_{ij}u_i)) + \rho f_i u_i - \frac{\partial q_i}{\partial x_i} + \rho q_v \quad (17)$$

$$\rho \frac{D}{Dt}(e + \frac{1}{2}u_1^2) = -\frac{\partial}{\partial x_i}(pu_i) + \frac{\partial}{\partial x_j}(\tau_{ij}u_i) + \rho f_i u_i - \frac{\partial q_i}{\partial x_i} + \rho q_v \quad (18)$$

$\rho u_i \frac{Du_i}{Dt} = \rho \frac{D}{Dt}\left(\frac{u_i^2}{2}\right) = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial}{\partial x_j}\tau_{ij} + \rho f_i u_i$ $\rho \frac{De}{Dt} =$ $\rho \frac{D}{Dt}\left(e + \frac{1}{2}u_i^2\right) - \rho \frac{D}{Dt}\left(\frac{u_i^2}{2}\right) =$
$= -p\frac{\partial u_i}{\partial x_i} + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \rho q_v - \frac{\partial q_i}{\partial x_i}$

Dissipation Function Φ

Insert $h = e + \frac{p}{\rho}$ to obtain Enthalpy equation, introduce $h_t = h + \frac{u_i^2}{2}$ and add kinetic energy (p. 15). For perfect gasses, $h = c_p T$, $q_i = -k\frac{dT}{dx}$, derive the temperature equation.

Entropy Equation

$$\rho T \frac{Ds}{Dt} = \Phi + \rho q_v - \frac{\partial q_i}{\partial x_i} \quad (19)$$

Vorticity Equation

$$\rho \frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho}\right) = (\vec{\omega}\cdot\nabla)\vec{u} + \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nabla \times \left(\frac{1}{\rho}\nabla \cdot \vec{\tau}\right) \quad (20)$$

Crocco Theorem (rewritten momentum equation using Enthalpy and Entropy)

$$\frac{\partial u}{\partial t} + \nabla\left(\frac{1}{2}\vec{u}^2 + h + \psi\right) = \vec{u}\times\vec{\omega} + T\nabla s + \frac{1}{\rho}\nabla\cdot\vec{\tau} \quad (21)$$

Compressible Bernoulli

equation (integrate momentum equation law along particle path). Clasical not feasible

$$\rho\left(\frac{Dh_t}{Dt} - f_i u_i\right) = 0 \quad (22)$$

$$f_i = -\frac{\partial\psi}{\partial x_i} \quad (23)$$

$$\psi \neq \psi(t) \quad (24)$$

$$\frac{D}{Dt}(h_t + \psi) = 0 \quad (25)$$

Between 2 points along stream line

$$h_t + \psi = e + \frac{p}{\rho} + \frac{u_i^2}{2} + \psi = const. \quad (26)$$

4 Simplification Strategies (p.20)

- Unsteady $\rightarrow$ steady (no wave propagation) (no time dependence)
- 3D $\rightarrow$ 2D $\rightarrow$ quasi 1-D
- Viscous, heat conduction $\rightarrow$ inviscid, adiabatic (isentropic, homentropic)
- Subsonic $\rightarrow$ transonic $\rightarrow$ supersonic $\rightarrow$ hypersonic (Elliptic $\rightarrow$ hyperbolic)
- Full nonlinear $\rightarrow$ linearised (solve for small pertubations around predefined flow state unique solvable problem, separation of influencing factors facilitated)

5 Conservation Laws for Stream Tubes (p. 22)

Quasi 1D, separate for environment. Outer surface formed by instantaneous streamlines, no flow across boundaries. Inlet + outlet. Shape (t). For small enough A, flow properties can be treated constant in any cross section.

Mass Conservation

$$\int_1^2 \frac{\partial}{\partial t}\left[\rho(s,t)A(s,t)\right] ds + \rho_2 A_2 u_2 - \rho_1 A_1 u_1 = 0 \quad (27)$$

$$\dot{m} = \rho A u = const. \quad (28)$$

Momentum Conservation

$$\int_1^2 \frac{\partial}{\partial t}\left[\rho(s,t)A(s,t)\right] ds + \rho_2 A_2 u_2 \vec{u}_2 - \rho_1 A_1 u_1 \vec{u}_1 = \quad (29)$$

$$= -p_2 A_2 \vec{n}_2 + p_1 A_1 \vec{n}_1 + F_\tau|_1^2 + F_S \quad (30)$$

Steady, frictionless

$$\rho_2 u_2^2 + p_2 = \rho_1 u_1^2 + p_1 \quad (31)$$

Energy Conservation (p.20)

Steady, frictionless

$$e_2 + \frac{u_2^2}{2} + \frac{p_2}{\rho_2} = e_1 + \frac{u_1^2}{2} + \frac{p_1}{\rho_1} \quad (32)$$

Enthalpy subsitution $h = e + \frac{p}{\rho} \rightarrow h_{t1} = h_{t2} = const.$

6 Steady one-dimensional Flow without Friction and Heat (p. 25)

Assumptions:

- No friction (inviscid)
- No heat source or transport
- No flow through mantle
- Perfect gas

$$Ma = \frac{u}{a} \quad (33)$$

$$a^2 = \gamma R T \quad (34)$$

Stagnation properties, when $u = 0$ (Ruhegrösse), subscript 0:

$$\frac{h_0}{h} = \frac{T_0}{T} = \left(\frac{a_0^2}{a^2}\right) = 1 + \frac{\gamma-1}{2}Ma^2 \quad (35)$$

Isentropic flow (p.26):

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left[1 + \frac{\gamma-1}{2}Ma^2\right]^{\frac{\gamma}{\gamma-1}} \quad (36)$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma-1}} = \left[1 + \frac{\gamma-1}{2}Ma^2\right]^{\frac{1}{\gamma-1}} \quad (37)$$

When $Ma < 0.3$, density changes $< 4.5\%$: Assumption is: incompressible. The critical state is then $(Ma = 1)$, $superscript$ *

$$\frac{h^*}{h_0} = \frac{T^*}{T_0} = \left(\frac{a^{*2}}{a_0^2}\right) = \left[1 + \frac{\gamma-1}{2}\right]^{-1} = \frac{2}{\gamma+1} = 0.8333(\gamma = 1.4)$$

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = 0.5283(\gamma = 1.4) \quad (38)$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} = 0.6339(\gamma = 1.4) \quad (39)$$

Critical Ma^* (isentropic flow stays limited when $Ma \rightarrow \infty$). The flow velocity stays finite even if Ma goes to infinity:

$$Ma^* = \frac{u}{a^*} = \frac{u}{a(Ma=1)} = \frac{u}{a}\frac{a}{a_0}\frac{a_0}{a^*} \quad (40)$$

$$= Ma\sqrt{\frac{T}{T_0}}\sqrt{\frac{T_0}{T^*}} = \sqrt{\frac{\frac{\gamma+1}{2}Ma^2}{1 + \frac{\gamma-1}{2}Ma^2}} \quad (41)$$

$$Ma^* \rightarrow \sqrt{\frac{\gamma+1}{\gamma-1}}\,(Ma \rightarrow \infty) = 2.4495\,(\gamma = 1.4) \quad (42)$$

Area velocity relation

A velocity increase $\rightarrow$ density decrease (always). If $Ma << 1$, then the density changes are small compared to the velocity changes. A small velocity increase at $Ma >> 1$ will lead to large density changes.

$$Ma^2\frac{1}{u}\frac{du}{dx} = -\frac{1}{\rho}\frac{d\rho}{dx} \quad \textbf{(Mach-density relation)} \quad (43)$$

$$\left(Ma^2 - 1\right)\frac{1}{u}\frac{du}{dx} = \frac{1}{A}\frac{dA}{dx} \quad \textbf{(Mach-Area relation)} \quad (44)$$

If $Ma < 1$, then an area increase will result in a velocity reduction. If $Ma > 1$, then opposite applies. If $Ma = 1$, then a change in Area A has no effect (chocked flow)

Stationary normal shock

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = 1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right) = \frac{1}{Ma^{*2}} \tag{45}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2 - 1\right) \tag{46}$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2 - 1\right)\right]\left[1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)\right] \tag{47}$$

$$\frac{\Delta s}{R} = \frac{1}{\gamma-1}\left[\ln\left(\frac{p_2}{p_1}\right) - \gamma\ln\left(\frac{\rho_2}{\rho_1}\right)\right] =$$
$$\frac{1}{\gamma-1}\left\{\left[1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2 - 1\right)\right]\left[1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)\right]\right\}$$

$h_{01} = h_{02}$, $T_{01} = T_{02}$, and total enthalpy conserved (however stagnation pressure not constant, $p_{01} \neq p_{02}$):

$$\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{01}} = \frac{p_2}{p_1}\left(\frac{T_{02}}{T_2}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{T_1}{T_{01}}\right)^{\frac{\gamma}{\gamma-1}} =$$
$$\left[1 + \frac{2\gamma}{\gamma+1}(Ma_1^2 - 1)\right]^{\frac{1}{\gamma-1}}\left[1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)\right]^{\frac{-\gamma}{\gamma-1}}$$

As s increases, u decreases. Ma_2 is always < 1, when $Ma_1 \to \infty$:

$$Ma_2 \to \sqrt{\frac{\gamma-1}{2\gamma}} = 0.38\ (\gamma = 1.4) \tag{48}$$

$$Ma_2^2 = \left(\frac{u_2}{a_2}\right)^2 = \left(\frac{u_2}{u_1}\right)^2\left(\frac{u_1}{a_1}\right)^2\left(\frac{a_1}{a_2}\right)^2 = \left(\frac{u_2}{u_1}\right)^2 Ma_1^2\left(\frac{T_1}{T_2}\right)$$

$$Ma_2 = \sqrt{\frac{1 + \frac{\gamma-1}{\gamma+1}\left(Ma_1^2 - 1\right)}{1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2 - 1\right)}} \tag{49}$$

A weak shock occurs at Ma_1 close to one. See page 31 for equation

Rankine Hugoniot (p.32) - Adiabatic Shock (no Ma dependency)

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma\left(\frac{\rho_2}{\rho_1} - 1\right)}{\gamma + 1 - (\gamma-1)\frac{\rho_2}{\rho_1}}$$

Moving Shock Wave (p.33)

Switch to reference frame (from frame fixed with moving shock front into a frame moving with shock)

$$u_1 \widehat{=} u_s\ ,\ p_1 \widehat{=} p_0\ ,\ \rho_1 \widehat{=} \rho_0$$

Flow behind

$$u_2 \widehat{=} u_s - u_d\ ,\ p_2 \widehat{=} p_d\ ,\ \rho_2 \widehat{=} \rho_d$$

Shock u_d

$$u_d = u_s - u_2 = u_1 - u_2 = u_1\left(1 - \frac{u_2}{u_1}\right) = u_1\frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)$$

$$Ma_d = \frac{u_d}{a_d} = \frac{u_1 - u_2}{a_d} = \frac{u_1}{a_1}\frac{a_1}{a_d}\left(1 - \frac{u_2}{u_1}\right) = Ma_1\sqrt{\frac{T_1}{T_2}}\left(1 - \frac{u_2}{u_1}\right)$$

$$u_d = \frac{a_0}{\gamma}\frac{\frac{\Delta p}{p_0}}{\sqrt{1 + \frac{\gamma+1}{2\gamma}\frac{\Delta p}{p_0}}}\ (a_1 \widehat{=} a_0),\ \ Ma_s = \frac{u_s}{a_0} = \sqrt{1 + \frac{\gamma+1}{2\gamma}\frac{\Delta p}{p_0}}$$

Pressure increase

$$\frac{\Delta p}{p_0} = \frac{p_d - p_0}{p_0} = \frac{2\gamma}{\gamma+1}\left(Ma_S^2 - 1\right),\ \left[Ma_1 = \frac{u_1}{a_1} = \frac{u_s}{a_s} = Ma_s\right]$$

The ratio (Pressure increase) has an asymptotic limit. For high Ma_s, the function becomes limited. $\frac{u_s}{u_d} \to \frac{\gamma+1}{2}$ (for high pressure differences)

Detonations ($Ma_2 > 1$) and Deflagrations ($Ma_2 < 1$) (p.36, ZND)

Assumption: Ignore adiabatic flow, include however heat release
Rayleigh line: $\frac{p_1}{p_0} = 1 + \frac{\rho_0}{p_0}u_0^2 - \frac{\rho_0}{p_0}\frac{\rho_1}{\rho_0}u_1^2 = 1 + \gamma Ma_0^2\left(1 - \frac{\rho_0}{\rho_1}\right)$,
Rankine Hugeniot with heat: $\frac{p_2}{p_0} = \frac{(\gamma+1)-(\gamma-1)\frac{\rho_0}{\rho_2}+2\gamma\hat{q}}{(\gamma+1)\frac{\rho_0}{\rho_2}-(\gamma-1)}$, $\hat{q} = \frac{q_{heat}}{c_pT_1}$, This gives us p_1 and p_2, the pressure of the shockwave before the combustion and downstream after the combustion layer

Chapman-Jouget Point (p.37)
...is the intersection where $Ma = 1$, so $Ma_2 = 1 = Ma_0\sqrt{\frac{\rho_0}{\rho_2}}\sqrt{\frac{\rho_0}{\rho_2}}$
The limiting case for shock cycle (Rayleigh tangent to Hugoniot Line):

$$\frac{\rho_0}{\rho_2}|_c = \frac{u_2}{u_0}|_c = \frac{\gamma Ma_0^2 + 1}{Ma_0^2(\gamma+1)}$$

Behind the shock, the flow is subsonic $\leftrightarrow$ strong detonation. There is a weak deflagration if the density ratio $\frac{\rho_1}{\rho_2} >> 1$. The reaction front propagates at subsonic speed. Weak detonation: flow remains supersonic (not explainable through ZND)

Laval Nozzle (p. 39)
Varying cross-section:

$$\frac{p(x)}{p_0} = \left[1 + \frac{\gamma-1}{2}Ma^2(x)\right]^{\frac{-\gamma}{\gamma-1}}$$
$$\frac{A^*}{A(x)} = Ma(x)\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}Ma^2(x)\right]$$
$$u(x) = Ma(x)a_0\frac{a(x)}{a_0} = Ma(x)a_0\sqrt{\frac{T(x)}{T_0}} = \frac{a_0 \cdot Ma(x)}{\sqrt{1 + \frac{\gamma-1}{2}Ma^2(x)}}$$
$$u^* = a^*\ ,\ \text{if } Ma^* = 1$$

In order to increase the Ma_{exit}, reduce the area ration (tune A^*). Different flow regimes are shown on p. 41. A variable exit area is in practice not possible

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7 Unsteady one-dimensional Flows

Wave equation for small perturbations Assuming small pertubations around equilibrium state with first order pertubations will result into following differential equation (enthalpy):

$$\frac{\partial p'}{\partial t} - a_0^2\frac{\partial \rho'}{\partial t} = 0 \Longleftrightarrow p' = a_0^2\rho'$$

$$\frac{\partial \rho'}{\partial t} + \rho_0\frac{\partial u'}{\partial x} = 0\ \ \text{(mass eq.)}$$
$$\frac{\partial u'}{\partial t} + \frac{a_0^2}{\rho_0}\frac{\partial \rho'}{\partial x} = 0\ \ \text{(momentum eq.)}$$

Through cross-differentiation (elimination of terms), one arrives at the d'Alembert solution:

$$u'(x,t) = a_0[F(x - a_0t) + G(x + a_0t)]$$
$$\rho'(x,t) = \rho_0[F(x - a_0t) + G(x + a_0t)]$$

Through characteristics one defines left and right propagatiting waves, $F(\eta)$ and $G(\xi)$. The characteristics are in this case straight lines. Initial conditions are at $t = 0$, boundary conditions are at $x = b.c.$

Method of characteristics for nonlinear wave propagation Here, no small pertubations are assumed, while assuming homentropic flow ($s = const.$). The Riemann invariants (characteristics) are not straight anymore, and can be curved. Disturbances are no longer constant, but have a flow dependent value. Given a and u are given along a curve C, find where it intersects with two characteristics, which cross at point Q. (See p. 48)

Piston Motion in tube (example for unsteady one-dimensional motion):

- Boundary Condition: At $x = x_p(t)$, $u(x = x_p, t) = u_p(t)$
-

8 Two-dimensional steady supersonic Flow

9 Method Characteristics for planar homentropic supersonic Flows

10 Homentropic Flow around slender Wings

11 Homentropic Flow around axisymmetric slender Bodies

12 Similarity Relations