

Parameters

- Aerodynamic Force $\overline{F_A} = \overline{L} + \overline{D}$
- Lift Force L [N]
- Drag Force D [N]
- Aerodynamic Moment M_A [Nm]
- Dynamic Pressure $q = 1/2\rho V^2$ (Bernoulli) [Pa]
- Chord Length c [m]
- Surface Area $S = b \cdot c$ [m²] (Rectangular)
- Wing Span b [m]
- Lift Coefficient $C_l = L/(1/2\rho V^2 \cdot S)$
- Drag Coefficient $C_d = D/(1/2\rho V^2 \cdot S)$
- Moment Coefficient $C_m = M_A/(1/2\rho V^2 \cdot c \cdot S)$
- Angle of Attack α [rad] (positive in clockwise direction)
- Lift curve slope $a = C_l/\alpha \approx \tan(\text{angle } x - \text{axis to curve})$

Conventions throughout Course

- If L and D absolute → use calculations above
- If L and D per span unit → correct via dividing by b
- Sign conventions: Lift positive, Drag positive in x and y direction
- Moments and angles positive in clockwise direction
- Our system coordinate system is defined by the wing. The angle of attack is defined relative to it
- The variables which describe the airfoil motion are the pitch θ and the plunge h which act at the shear centre of the wing

Mathematical Basics

- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- Inverse of Matrix (2D): $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- Inverse of Matrix (3D):
 1. det (A), then transpose A
 2. Find the adjunct matrix (minors) of (cover row and column of element) of A^T and multiply with +- matrix
 $A^{-1} = 1/\det(A) \cdot \text{Adj}(A^T)$
- The solutions of $Ax = 0$ for a matrix A , x cannot be just the trivial solution if A is not invertible
- [rad] = $\frac{\pi}{180}$ [deg]

Steady Aerofoil and Wing Section Aerodynamics

- Aerofoil = 2-D wing section with goal to generate lift force perpendicular to the relative airspeed
- Convention: Lift is up, Drag is in direction of windspeed and Aerodynamic moment in clockwise direction acting on the aerodynamic center. Aerodynamic center is normally at the quarter chord position $c_{m,c/4}$ for symmetric airfoils. $x_{ac} = -m_0/2\pi + 0.25$ with m_0 as a shape constant
- Further assumptions: No viscosity, incompressible fluid, $Ma < 0.2, 0.3$, no vortices, potential flow (Navier-Stokes)
- Another centre is the shear center (elastic axis) from mechanics
- $L = 1/2\rho V^2 c a \alpha$, with a from tables (CFD and Wind Tunnel) [N/m]
- $M_A = 1/2\rho V^2 c^2 c_{m0}$ with c_{m0} also from tables [N]

Lift curve $C_l(\alpha)$ and drag curve $C_d(\alpha)$

- At small ranges of α , both lift and drag increase with: $C_l \propto \alpha$ and $C_d \propto \alpha^2$
- In aeroelasticity and this course, α will be very small, hence drag will be negligible small

The aerodynamic moment M_A

- The aerodynamic moment is much more important than drag C_d
- M_A varies with α in the small ranges of the angle of attack (very small, p. 7)
- **Important to note:** There exist a point at which the aerodynamic moment does not depend on α . This is the aerodynamic centre
- The aerodynamic centre is not the same as the centre of pressure, which is defined as the point where the aerodynamic moment is zero given a certain angle of attack α
- Symmetric airfoils at $\alpha = 0$ have no aerodynamic moment at all times ($M_A = 0 = \text{const}$). At the aerodynamic centre for symmetric foils results into no moment
- Asymmetric airfoils at $\alpha = 0$ have a non-zero aerodynamic moment at all times (all angles α)

Assessment of C_l/α (Correction of value through Mach Number)

- The linear part of the lift curve is characterised by the slope $a = C_l/\alpha(M) = \frac{C_l/\alpha_{M=0}}{\sqrt{1-M^2}}$
- The Prandtl-Glauert factor is $1/\sqrt{1-M^2}$
- The factor is depending on the Mach number. The slope increases with increasing M (between 0 and 1)
- The dependence on Re is more subtle (p. 8)

Extension to wing aerodynamics (p. 8)

Aerofoil dynamics (2D) refer to the previous topics, however the 3-D case can be also modeled by through a couple examples. A finite wing is less stable and efficient than the airfoil since the tips have vortices on at the wing tips. These “induce” a velocity, which locally reduces the angle of attack. An important parameter is the so called **Aspect Ratio** $AR = b^2/S$. If the wing is assumed to be of surface $S = b \cdot c$, it follows $AR = b/c$.

- The lift curve can become a function of AR if due to the different tips. Approximately, the lift slope a_0 is adjusted via following formula:
- $a = a_0 \frac{AR}{AR+4}$
- The values a and c_{m0} will hence be corrected with a a^* and c_{m0}^*

Strip Theory (p.9)

- If AS is very small (delta wings), the integral of multiple airfoils
- Define multiple airfoils stacked next to each other along the span b
- Example, the wing is an elliptical $f(y) = \sqrt{1 - (\frac{y}{b/2})^2} \cdot \bar{f}_\phi$
- $f(y) = a\alpha c = C_l c$ with c = chord length.

Steady-state (static) Aeroelasticity

Typical Section = 1DOF model

2-D problem with a rigid wing. We can have multiple typical sections stacked onto each other, which would be later adding dimensionality to the variable θ . The idea is later on to model the torsional spring to be a torsional stiffness of a beam (since a real wing is actually a beam with a certain stiffness).

- The torsion acts in a beam section on the shear centre, however in aeroelasticity on the elastic axis
- The goal of engineering is always to move the shear center to the front (comes with risk to thin out the rear longeron and thicken the front longeron)

- In equilibrium, we know that the aerodynamic forces are equal to the spring forces
- $M_t + L \cdot e = \theta k_\theta = (qc^2 c_{m0} + qca\theta e) \cdot b$ (moment equations)
- Pitching moment M_t acting on section with k_θ stiffness
- $k_\theta = qc(C_{l/a} e(\theta + \alpha_0) + cC_{m0})$

Static Instability or Divergence

- If the elastic twist θ would become infinity for a given stiffness if the denominator of equation
- $\theta = qc \frac{C_{l/a} e \alpha_0 + cC_{m0}}{k_\theta - qcC_{l/a} e}$, $\theta = \infty \Leftrightarrow \text{denominator} = 0$
- If the dynamic pressure $q = \frac{k_\theta}{cC_{l/a} e} = q_{div} \rightarrow \text{instable (divergence)}$
- Divergence = Static Instability
- $M_{tot} = (k_\theta - qSae)\theta - qSa(e\alpha_0 + C_{m0}c)$ (In equilibrium $M_{tot} = 0$) ($M_{tot} > 0$ if in anti-clockwise direction)
- Different interpretation: $\frac{\partial M_{tot}}{\partial \theta} \geq 0 \Leftrightarrow \text{increasing total moment in section for increasing } \theta \Leftrightarrow \Delta\theta > 0$
- $\frac{\partial M_{tot}}{\partial \theta} \geq 0 \Leftrightarrow \text{overall moment brings blade section back to original position}$
- The divergent dynamic pressure can be found by differentiating w.r.t. θ

Lagrange Equation (Energy interpretation)

- $L = T - U$ (Kinetic and potential energy)
- $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$
- In statics: $\frac{\partial U}{\partial x_i} = 0$ (Potential energy conservative)
- Here: $\frac{\partial U}{\partial x_i} = \frac{\partial}{\partial x} \frac{\delta W}{\delta x}$ (Virtual work), hence for θ
- $\frac{\partial U}{\partial \theta} = \frac{\partial}{\partial \theta} \frac{\delta W}{\delta \theta} = 0$ (Check for q)
- Resulting q provides the divergence
- $U = 1/2 \cdot k_\theta \theta^2 = \int F(\theta) d\theta$ (For system with only one spring)

Section with more than 1 DOF

1. Number of DOF = Dimensions of Stiffness Matrix and number of equations needed (full rank) K
2. Define a potential energy matrix for mechanical system $K_{i,j} = \frac{\partial^2 U}{\partial x_i \partial x_j} = K_{j,i}$
3. Define aerodynamic matrix K_a based on aerodynamic forces (independent on e for example)
4. If a matrix is not symmetric = Non-conservative forces
5. Similar to before, instead of asking if the system is stable if the denominator is zero, we must know if the determinant of the transfer function is zero
6. Transfer Function: $[K - qK_a] = K_{ael}$ is ‘Aeroelastic K ’
7. Find a q for which the transfer function determinant becomes zero, which is divergence dynamic pressure. The solution (forces acting) is the so called divergence mode
8. If all eigenvalues are $> 0 \Leftrightarrow \text{stable}$, if one is at least $< 0 \Leftrightarrow \text{unstable}$ (for sections)

System (more than 1 section) with multiple DOF

- Define a system with multiple θ_i , whereas the calculations become similar to when calculating one section with multiple DOF
- Make an Ansatz with the lagrange equations and define stiffness matrix K
- The aerodynamic matrix becomes the identity matrix (if we only speak about θ)
- This implies that the solutions for q are the eigenvalues of K

Comment on eigenvalues

- The eigenvalues of the Aeroelastic K_{ael} cannot guarantee that q are always the eigenvalues, since K_a is sometimes non-symmetric (most of the times, only if rotational degrees of freedom present)
- In general, following statement holds true: $\det(K - qK_a) = 0$

- $\det(K_a^{-1}K - qI) = \det(A - \lambda I) = 0$
- If A is not symmetric, we can say: There are less eigenvectors and values than the order (n), can be complex and come in complex conjugate pairs

Active control on sections

- With active control, the behaviour of the elastic twist θ can be controlled with for example a trailing edge flap
- As described in the script, a trailing edge flap can influence the lift and the moment as follows:
- $l = qcC_{l/\delta}\delta$, $m = qcC_{m/\delta}\delta$ ($C_{m/\delta} < 0$)
- Both forces contribute to the overall moment, hence will be added to the calculations we did previously
- With a so called 'Gain' G , the controller controls δ proportional to θ , hence a new linear equation system is formed
- Assuming the nose-down motion of controlling the the edge flap, we have to simplify terms, the end result is
- $q_{div, flap} = \frac{k_\theta}{cae - Gca^*}$, with $a^* = -(cC_{m/\delta} + eC_{l/\delta})/(c\delta)$

Ritz Method

... is a energy variational method whereas an equilibrium occurs in correspondence of an extreme of potential energy. A general application is the virtual work. According to the Hamilton's principle and Lagrange equations, we can define a set of equations.

- $\frac{\partial V}{\partial x_i} = 0$ for all i (x_i degrees of freedom)
- From mechanics, we need the bending stiffness (I) and the torsional stiffness (J)
- $I = 1/12 * b * h^3$ (w.r.t x) and $J = \frac{4A^2}{\int ds/t}$ (Integral is perimeter (Umfang) divided by thickness)

Derivation for a beam section

- Torsional Strain Energy: $U = \frac{1}{2} \int_0^l GJ \left(\frac{\partial \theta}{\partial x} \right)^2 dx$
- Given aerodynamic forces are non-conservative, we use the concept of virtual work

- $\delta U = \delta W$ (Virtual work due to non-conservative forces)
- $\sum_{i=1}^n \frac{\partial U}{\partial x_i} \delta x_i = \sum_{i=1}^n \delta W_i$ (reformulated for small variations of one DOF)
- $\frac{\partial U}{\partial x_i} - \frac{\delta W}{\delta x_i} = 0$ (reformulated)
- The work done by the external forces (aerodynamic) can be rewritten:
- $\delta W = \int_0^l m(x) \delta \theta(x) dx$
- $m(x)$ generated by aerodynamic forces
- Without going into further detail, there are 2 distinct cases from which one has to go on in the calculation, either the functions are given in a generalised form or in matrix form
- $\theta(x) = \sum_{i=1}^N \phi_i(x) a_i = [\Phi] \{a\}$
- $[\Phi]$ is a row vector with elements ϕ_i !
- $\phi_i(x)$ are shape functions and a_i are coefficients and the linear combination of those make up $\theta(x)$
- Finally, the overall equations result in $[K] \{a\} = \{f\}$
- $K_{i,j} = \int_0^l GJ \phi_{i,x} \phi_{j,x} dx = GJ \frac{\partial^2 U}{\partial a_i \partial a_j}$ (Stiffness matrix entries, partial derivatives w.r.t. to x and a)
- $[K] = GJ \int_0^l [\Phi_x]^T [\Phi_x] dx$
- $f_i = \int_0^l m(x) [\Phi]_i(x) dx$ (index i for each element)
- $\{f\} = \int_0^l m(x) [\Phi]^T dx$ (in matrix form)

Following assumptions are drawn:

- Elastic axis is perfectly straight
- Aerodynamic center of all sections on a straight line
- External moments as before by aerodynamic forces:
 $m(x) = qcea(\theta(x) + \alpha_0)$

Replace all $\theta(x)$ with the above solutions

One single shape function (1-DOF)

One single shape function, Bending and Twisting

- For bending, the potential energy is: $U = \frac{1}{2} \int_0^l EI \psi_{xx}^2 dx$
- $\psi =$