1 Introduction, Definitions & Overview

Reliability

- ... is a characteristic of an item, expressed by the probability that the item performs its required function under given conditions during a stated time interval, i.e. $(0, t]$
- Item = entity for investigation, i.e. component, assembly, equipment, subsystem, system
- from a **qualitative** point of view, reliability is defined as the ability of an item to **remain functional**
- from a **quantitative** point of view, reliability is defined as the probability that **no operational interruptions** will occur during a stated time interval $R(t)$

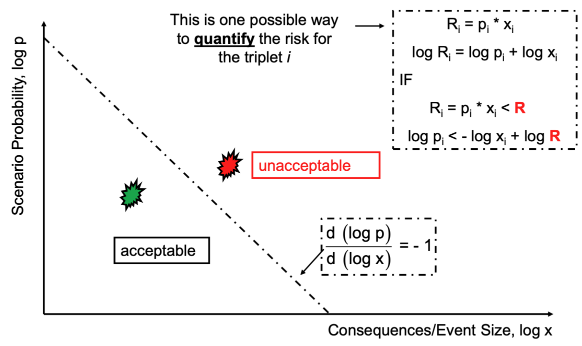
Availability

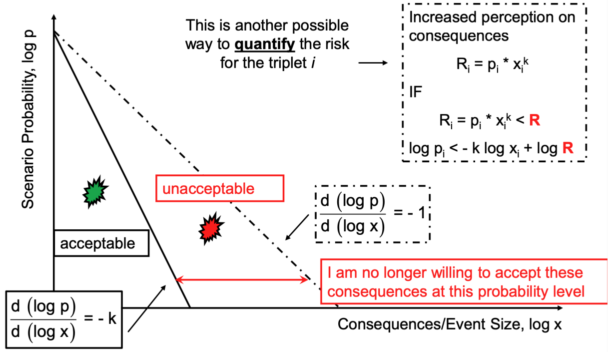
- Point Availability (PA) is a characteristic of an item expressed by the probability that the item performs its function **at an instant of time** t
- Qualitatively, it can be described as the dependebility
- Average Availability (AA), is the expected time at which the item can perform its required function
- Availability is used to express Point Availability in a sloppy way or Average Availability

In comparison to Availability, the Reliability analysis includes the fact that no item is allowed to fail. This means, the function performed cannot be interrupted (redundant system however can be repaired). Availability incurs that failures can happen on the item level.

Risk

- RISK = POTENTIAL DAMAGE x UNCERTAINTY
- Dictionary: RISK = probability of damage to a person or an object
- We define RISK as a function of an...
  - Accident Scenario, S
  - Probability, p
  - Consequence, x
- We can quantify RISK on the so-called "Farmer's Curve"





The total Risk can be calculated as follows:

$$\text{Total Risk} = \sum p_i x_i^k, k \geq 1 \tag{1}$$

It is entirely possible that the risk of different events can be dominated by either it's probability or its consequence.

- A large probability p is prevented of (minimisation based on high probability)
- A large consequence x is mitigated, protected (minimisation given its large impact)

2 Probability Theory and Reliability Analysis

Definitions:

- Experiment ϵ
- Sample space Ω
- Event E

An event E is a subset of the sample space Ω and the experiment ε yields a set of possible outcoms ($= E$) of the experiment

Certain Events follow Boolean Logic, an event E can occur or not occur, meaning an Indicator Variable X_E is 0 when E does not occur and 1 if E occurs

Uncertain Events follow can either be true or false, with each a probability associated to it. Event E in sample space Ω is triggered with a probability that the outcome has happened or not

Classical Probability

- The experiment ϵ has N possible, elementary, mutually exclusive and equally probable outcomes $A_1, A_2, ..., A_N \in \Omega$
- The event $E = A_1 \cup A_2 \cup ... \cup A_M$, $M \leq N$
- The probability of event E is defined as $p(E) = M/N$

Kolmogorov Axioms

1. $0 \leq P(E) \leq 1$
2. $P(\Omega) = 1$, $P(\emptyset) = 0$
3. Mutually exclusive events: $P(\cup_i E_i) = \sum p(E_i)$
4. Non-mutually exlusive events: $P(A \cup B) = P_A + P_B - P(A \cap B)$
5. Conditional probability: $P(A|B) = P(A \cap B)/P(B)$
6. Theorem of total probability: Given an event A in Ω where the space is consisting of exclusive and exhaustive events $\cup_j E_j = \Omega$: $P(A) = \Sigma_i(P(A|E_i)P(E_i))$

Random Variables

- **CDF**: Is a non-decreasing function and returns the probabilty (state) from random variable X from 0 to a given point A: $F_X(X = A) = P(0 < X \leq A)$
- **pdf**: Probability of per unit x (continuous)
- **pmf**: Histogram, it assignes the probability to discrete values x

Summary

- Distribution Percentile x_α:
  - $F_X(x_\alpha) = \alpha/100 = \int_{-\infty}^{x_\alpha} f_X(x)dx$
- Median:
  - $F_X(x_50) = 0.5$
- Mean:
  - $\mu_X = E[X] =< X >= \sum_i x_i p_i$ (discrete)
  - $= \int_{-\infty}^{\infty} x f_X(x)dx$ (continuous)
- Variance:
  - $\sigma_X^2 = \sum(x_i - \mu_X)^2 p_i$ (discrete)
  - $= \int_{-\infty}^{\infty}(x - \mu_X)^2 f_X(x)dx$ (continuous)

Hazard Function (Failure Rate)

For risk and reliability analyses, we can use models whereas the time to failure of a component T can be expressed through a CDF $F_T(t)$ and a pdf $f_T(t)$. The complementary, cumulative function is

$$R(t) = 1 - F_T(t) = P(T \geq t) \tag{2}$$

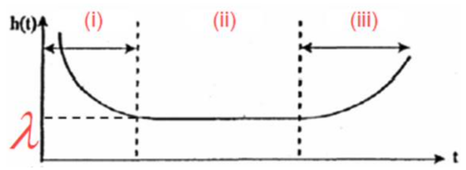
which is decribed as the **Reliability or Survival Function** of the component T at time t and gives the probability of it surviving up to time t without failures.

In order to monitor the failure evolution, given the component has survived up to time t in a time interval dt, one can define a so called **Hazard Function or Failure Rate** $h_T(t)$.

$$h_T(t)dt = P(t < T \leq t + dt | T > t) = \tag{3}$$

$$= \frac{P(t < T \leq t + t + dt)}{P(T > t)} = \frac{f_T(t)dt}{R(t)} \tag{4}$$

The hazard function is depending on time, and is often discribed through the bathtub curve. The failure rate at the beginning is higher (infant mortality, burn in) and decreses after a certain time. The failure rate becomes constant λ and increases at the end through ageing.



Through the definition of $R(t)$ and integrating the hazard function, we receive:

$$F_T(t) = 1 - e^{-\int_0^t h_T(\tilde{t})d\tilde{t}} \tag{5}$$

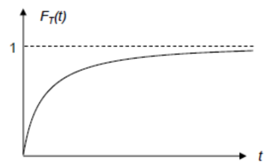
$$R(t) = e^{-\int_0^t h_T(\tilde{t})d\tilde{t}} \tag{6}$$

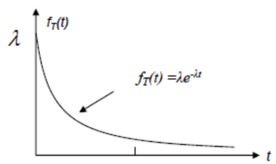
If our hazard function is in its constant phase (constant hazard rate), the failure evolution follows the **Exponential Distribution**:

$$h_T(t)\lambda, t > 0 \tag{7}$$

$$F_T(t) = P(T \leq t) = 1 - e^{-\lambda t} \tag{8}$$

$$R(t) = \begin{cases} f_T(t) = \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{9}$$





The mean time to failure (MTTF) can be found through the expectation value

$$E[T] = \frac{1}{\lambda} = MTTF \tag{10}$$

$$Var[T] = \frac{1}{\lambda^2} \tag{11}$$

The failure process is memoryless. Given a component has survived at least until time t_1, the probability of it failing between time t_1 and t_2 is only depending on the time inbetween and not prior to time t_1

$$P(t_1 < T < t_2) = \frac{P(t_1 < T < t_2)}{P(T > t_1)} = \frac{F_T(t_2) - F_T(t_1)}{1 - F_T(t_1)} = \tag{12}$$

$$\frac{e^{-\lambda t_1} - e^{-\lambda t_2}}{e^{-\lambda t_1}} = 1 - e^{-\lambda(t_2 - t_1)} \tag{13}$$

The influence of an ageing process of the components failure rate shows that is not constant through time and hence can be described through the **Weibull distribution**.

Boolean Logic - Fault Tree Analysis

Fault trees = set of Boolean algebraic equations (one for each gate) → structure (switching) function Φ

$$X_T = \Phi(X_1, X_2, X_3, ..., X_N) \tag{14}$$

The top event is connected by an OR-gate, hence if one of each events is true, then the top event will be true. Here some further rules:

- Negation: Given event E, given by the indicator variable X_E, its negation is described by $\overline{X_E} = 1 - X_E$
- Intersection: The event $A \cap B$ is true, if both A and B are simultaneously true:

  $$X_{A\cap B} = X_A X_B \tag{15}$$

  ($X_{A\cap B} = 0$ for mutually exclusive events)
- Union: The event $A \cup B$ is true, if either A or B are true and false if both are false:

  $$X_{A\cup B} = 1 - (1 - X_A)(1 - X_B) \tag{16}$$

  $$= 1 - \Pi(1 - X_j) = \amalg X_j \tag{17}$$

  $$= X_A + X_B - X_A X_B \tag{18}$$
- Probability of event E with expected value Operator $E(\cdot)$: $p(E) = E(X_E)$

  $$p(E) = p(X_E = 1)\cdot 1 + p(X_E = 0)\cdot 0 = E(X_E) \tag{19}$$
- Multiple, non-mutually exclusive events:

  $$X_\cap = \Pi X_i \tag{20}$$

- Probability of event $E_\cap$ for the intersection of n events:

$$P(E_\cap) = E[X_\cap] = \Pi P(E_j)(\text{if events are independant}) \tag{21}$$

- Union of event $E_\cup$ for the union of n events:

$$X_\cup = 1 - \Pi(1 - X_j) = X_A + X_B + X_C \tag{22}$$
$$-X_A X_B - X_A X_C - X_B X_C - X_A X_B X_C \tag{23}$$

- Probability of even $E_\cup$ for the union of n events:

$$P(E_\cup) = E[X_\cup] = \sum P(E_j) - \tag{24}$$
$$\sum\sum P(E_j \cap E_i) + (-1)^{n+1} P(\cap E_j) \tag{25}$$

Structure Function and Minimal Cut Sets

- Cut Set: Is a logical combination of primary event (**combination of component failures**) which render true the top event (**system failure**)

- Minimal Cut Sets: Cut set that does not have another cut set as a subset. This means repairing one element of the set repairs the entire system. Therefore, removing one element of a MCS makes it no longer a cut set.

Any fault tree $\Phi(X)$ can be equivalently written as an OR-gate in the first level below the top event combining the minimal cut sets, each in return represented by an AND-gate intersecting all elements comprising the given minimal cut set

$$\Phi(X) = 1 - (1 - M_1)(1 - M_2)(1 - M_3)... \tag{26}$$
$$...(1 - M_{mcs}) = \Pi_j^{mcs} M_j \tag{27}$$
$$P(\Phi(X) = 1) = E[\sum M_j - \sum\sum M_i M_j + ... \tag{28}$$
$$... + (-1)^{mcs+1} \Pi M_j \tag{29}$$
$$= \sum P(M_j) - \sum\sum P(M_i M_j) + ... \tag{30}$$
$$... + (-1)^{mcs+1} P(\Pi M_j) \tag{31}$$

- First event approximation: $\sum P(M_j)$

- Second event approximation: $\sum P(M_j) - \sum_i^{mcs-1} \sum_j^{mcs} P(M_i M_j)$

Important fact: Idemponent law follows for Boolean values - $X * X = X$

$$P(M_1 M_2) = P(M_1 \cap M_2) = E[M_1 \cap M_2] = \tag{32}$$
$$E[X_1 X_2 \cap X_2 X_3 X_4] = E[X_1 X_2 X_2 X_3 X_4] \tag{33}$$
$$= E[X_1 X_2 X_3 X_4] \tag{34}$$

if both events M are independent, then

$$P(M_1 M_2) = P(X_1)P(X_2)P(X_3)P(X_4) = p_k(T_m)^4 \tag{35}$$

The occurence probability of X_2 will only be appearing once in the calculation of the intersection of M_1 and M_2 and hence the Unreliability (Failure Rate) calculations become more exact through subtraction of the intersection (second event approximation) and will be bounded by both approximations:

$$\sum P(M_j) - \sum\sum P M_i M_j < U_{mcs}(T_m) < \sum P(M_j) \tag{36}$$

Reliability Analysis

Series System

- All components must function in order of a functioning system

$$R(t) = \Pi R_i(t) \tag{37}$$

- For N exponential components:

$$R(t) = e^{-\lambda t} \tag{38}$$
$$\lambda = \sum \lambda_i (\text{System Failure Rate}) \tag{39}$$
$$E[T] = 1/\lambda (\text{MTTF}) \tag{40}$$

Parallel System

- All components must fail in order the system to fail

$$R(t) = 1 - \Pi\left[1 - R_i(t)\right] \tag{41}$$

- For N exponential components:

$$R(t) = 1 - \Pi\left[1 - e^{\lambda_i t}\right] \tag{42}$$

$$MTTF = \sum 1/\lambda_i \sum\sum 1/(\lambda_i + \lambda_j) + ... \tag{43}$$

$$... + \sum\sum\sum 1/(\lambda_i + \lambda_j + \lambda_k) + ... \tag{44}$$

$$... + (-1)^{N-1} 1/\sum \lambda_i \tag{45}$$

- Example with two exponential units of failure rates λ_1 and λ_2:

$$MTTF = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{[\lambda_1 + \lambda_2]} \tag{46}$$

- For N identical elements, compare series and parallel:

$$\text{parallel}: \; MTTF = \sum_n^N \frac{1}{n\lambda} \tag{47}$$
$$\text{series}: \; MTTF = \frac{1}{N\lambda} \tag{48}$$
$$MTTF_{parallel} < MTTF_{series} \tag{49}$$

Markov Processes: Basic Elements

System

- The system can occupy a finite/countable number of states N

- The states are mutually exclusive, i.e. the system is in one state at each time

- The states are exhaustive, i.e. the system must be in one state at all times

Transitions between States occur **stochastically**, i.e. randomly in time

Mathematical Representation

- The random process of the state transition can be described by an integer random variable, e.g. $X(t) = 5$, the system occupies state 5 at time t

- The stochastic process can be observed at discrete times, and we assume that time intervals between states is such small that only one transition occurs between two states

The conceptual Model: Discrete States

- The state transition will be described by an integer random variable $X(n)$ is the system state at time t_n, $X(3) = 5$ means that the system occupies state 5 at time t_3

- The objective is: Compute the probability that the system is in a given state at given time t, for all possible states and times

- **In general stochastic processes:** The probability of a future state depends on its entire life history

- **In Markov processes:** The probability of a future state only depends on the present state. The process has no memory
- The transition probability that the system in state i at time t_m moves to state j at time t_n

$$p_{ij}(m,n) = P\left[X(n)=j|X(m)=i\right], n > m \geq 0 \tag{50}$$

$$i = 0,1,2,...,N, j = 0,1,2,...,N \tag{51}$$

Properties of Transition Probabilities

- Transition probabilities are greater or equal to 0
- Transitions must sum up to 1
- $p_{ij}(m,n) = p[X(n)=j, X(m)=i]$ $= \sum_k p_{ik}(m,r)p_{kj}(r,n),$ $i = 0,1,2,...,N, j = 0,1,2,...,N$

Stationary Transition Probabilities

- If the transition probability $p_{ij}(m,n)$ depends on the time interval $(t_n - t_m)$ and not on the individual times t_m and t_n, then
  - The transition probabilities are stationary
  - The Markov Process is homogeneous in time

$$p_{ij}(m,n) = p_{ij}(k), k \geq 0, i,j = 0,1,2,...,N \tag{52}$$

- We only need to know the **stationary one-step transition probabilities** $p_{ij}(1) = p_{ij}$

The Transition Probability Matrix:

$$\underline{A}_{ij} = \begin{pmatrix} p_{00} & p_{01} & ... & p_{0N} \\ p_{10} & p_{11} & ... & p_{1N} \\ \dots & \dots & \dots & \dots \\ p_{N0} & p_{N1} & ... & p_{NN} \end{pmatrix} \tag{53}$$

- $dim(A) = (N+1)x(N+1)$
- $0 \leq p_{ij} \leq 1, \forall i,j \in \{0,1,...,N\}$
- $\sum_j p_{ij} = 1$, hence only $(N+1)xN$ elements must be known
- $\underline{A}$ is a stochastic Matrix

Unconditional State Probabilities:

$$\underline{P}(n) = [P_0(n), P_1(n), ..., P_j(n), ..., P_N(n)] = \tag{54}$$

$$\text{probabilities of being in state 0,1,...,N at time step n} \tag{55}$$

$$\underline{P}(0) = \underline{C} = [C_0, C_1, ..., C_j, ..., C_N] = \tag{56}$$

$$\text{initial vector at time step n=0} \tag{57}$$

At the first time step $n = 1$:

$$P_j(1) = P[X(1)=j] \tag{58}$$

$$= \sum_{i=0}^{N} P\left[X(1)=j|X(0)=i\right] \cdot P\left[X(0)=i\right] \tag{59}$$

$$= \sum_{i=0}^{N} p_{ij}C_i, \text{with } j = 0,1,2,...,N \tag{60}$$

The fundamental equation follows:

$$\underline{P}(n) = \underline{P}(0) \cdot \underline{A}^n = \underline{C} \cdot \underline{A}^n \tag{61}$$

$$\underline{A}^n = \begin{pmatrix} p_{00}(n) & p_{01}(n) & ... & p_{0N}(n) \\ p_{10}(n) & p_{11}(n) & ... & p_{1N}(n) \\ \dots & \dots & \dots & \dots \\ p_{N0}(n) & p_{N1}(n) & ... & p_{NN}(n) \end{pmatrix} \tag{62}$$

$$p_{ij}(n) = P[X(n)=j|X(0)=i] \tag{63}$$

The probability of arriving in state j after n steps given the initial state was i

General Solution of the Fundamental Equation

1. Solve the eigenvalue (ω) problem: $\underline{V} \cdot \underline{A} = \omega \cdot \underline{V}$
2. With the eigenvectors: $\underline{V_j} \cdot \underline{A} = \omega_j \cdot \underline{V_j}$
3. $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V_j}$ and $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V_j}$ are linear combinations of the eigenvectors
4. The coefficients can be found through the adjoint eigenvalue problem
5. The general solution yields: $\underline{P}(n) = \sum_{j=0}^{N} c_j \cdot \omega_j^n \cdot \underline{V_j}$
6. The availability at time k: $A(k) = \sum_{j \in \text{success states}} P_j(k)$

Steady State Probabilities:

1. Solve $\underline{\Pi} = \underline{\Pi} \cdot \underline{A}$ with $\sum_{j=0}^{N} \Pi_j = 1$ (2 equations)

First-passage Probabilities

- Probability that the system arrives **for the first time** in state j **after n steps**, given that it was in state i at the initial time 0

$$f_{ij}(n) = P\left[X(n)=j \text{ for the first time}|X(0)=i\right]$$
$$f_{ij}(n) = P\left[X(n)=j, X(m) \neq j, 0 < m < n|X(0)=i\right]$$

- Notice that $f_{ij}(n) \neq p_{ij}(n)$ = probability that the system reaches state j **after** n **steps** starting from state i, but **not necessarily for the first time**

General Formula for the first-passage probability

$$f_{ij}(k) = \underbrace{p_{ij}(k)}_{(I)} - \underbrace{\sum_{l=1}^{k-1} f_{ij}(k-l)p_{jj}(l)}_{(II)} \tag{64}$$

$$R(k) = 1 - \sum_{l=1}^{k} \sum_{j \in \text{ failed states}} f_{ij}(l) \tag{65}$$

The probability of passing the state j from i is the transition probability at time step k, given it was in i at 0 (I) whereas (II) is the probability of staying in j as it reached the state prior to time step k

The reliability is defined as the probability at time k where the system has not yet failed, hence we subtract all first passage probabilities of all failed states up to time k

Recurrent, Transient and Absorbing States

- Recurrent: The system at state i will surely return to i **sooner or later**:

$$\Pi_i \neq 0 \tag{66}$$

- Transient: The system at state i has a **finite probability** of **never** returning to it:

$$\Pi_i = 0 \tag{67}$$

- Absorbing: The state i is absorbing if the system cannot leave it once it enters:

$$p_i i = 1 \tag{68}$$

Average Occupation Time of a State

- The average occupation time of a stat i for l_i steps = Average number of steps before the systems exits the state i:

$$l_i[\text{steps}] = \frac{1}{P(\text{System exits state i})} \tag{69}$$

$$= \frac{1}{1 - P(\text{System remains in i})} = \frac{1}{1 - p_{ii}} \tag{70}$$

Continuous-time Discrete-state Markov Processes

What are they useful for?

- Model the evolution of one component or system as a structure of components
- To find the probability that the system will be functioning at time t^*

The transition probability matrix

$$p_{ij}(dt) = \alpha_{ij} \cdot dt + \underbrace{\theta(dt)}_{O(dt)}, \lim_{dt \to 0} \frac{\theta(dt)}{dt} = 0 \tag{71}$$

In analogy with the discrete-time formulation:

$$\underline{A} = \begin{pmatrix} 1 - dt \cdot \sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} \cdot dt & \dots & \alpha_{0N} \cdot dt \\ \alpha_{10} \cdot dt & 1 - dt \cdot \sum_{j=0,\neq 1}^{N} \alpha_{1j} & \dots & \alpha_{1N} \cdot dt \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{72}$$

$$\underline{P}(t + dt) = \underline{P}(t) \cdot \underline{A} \tag{73}$$

$$\frac{d\underline{P}}{dt} = \underline{P}(t) \cdot \underline{A}^* \tag{74}$$

$$\underline{P}(0) = \underline{C} \tag{75}$$

$$\underline{A}^* = \begin{pmatrix} -\sum_{j=1}^{N} \alpha_{0j} = \alpha_{00} & \alpha_{01} & \dots & \alpha_{0N} \\ \alpha_{10} & -\sum_{j=0,\neq 1} \alpha_{1j} = \alpha_{11} & \dots & \alpha_{1N} \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{76}$$

The sum of all rows in $\underline{A}^*$ is 0. From now on, the matrix $\underline{A}$ will be called the **Transition Rate Matrix**. The system of linear, first-order differential equations in the unknown state probabilities will be solved through the Laplace Transform

- Definitions of Laplace Transform $L(\cdot)$:

$$\tilde{P}_j(s) = L[P_j(t)] = \int_0^{\infty} e^{-st} P_j(t) dt,\ j = 0, 1, , ..., N \tag{77}$$

$$L\left[\frac{dP_j(t)}{dt}\right] = s \cdot \tilde{P}_j(s) - P_j(0),\ j = 1, 1, ..., N \tag{78}$$

- Applying $L(\cdot)$ on $\frac{dP}{dt} = P \cdot A$ yields:

$$\underline{\tilde{P}}(s) = \underline{C} \cdot [s\underline{I} - \underline{A}]^{-1} \tag{79}$$

- $\underline{P}(t)$ = inverse transform of $\underline{\tilde{P}}(s)$
- Compute for example (one component, one repairman, 2x2):

$$(s\underline{I} - \underline{A})^{-1} = \frac{1}{s^2 + s\lambda + s\mu} \begin{pmatrix} s + \mu & \lambda \\ \mu & s + \lambda \end{pmatrix} \tag{80}$$

- Then:

$$\underline{\tilde{P}}(s) = \underline{C} \cdot (s\underline{I} - \underline{A})^{-1} = \left[\frac{s + \mu}{s(s + \lambda + \mu)} \quad \frac{\lambda}{s(s + \lambda + \mu)}\right] \tag{81}$$

- Anti-transform $\tilde{P}(s)$, whereas following definitions are used:

$$L^{-1}\left[\frac{1}{s + a}\right] = e^{-at} \tag{82}$$

$$L^{-1}\left[\frac{1}{s(s + a)}\right] = \frac{1}{a}\left(1 - e^{-at}\right) \tag{83}$$

- The State Probability Vector is found:

$$\underline{P}(t) = \left(\underbrace{\frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \cdot e^{-(\lambda+\mu)\cdot t}}_{P_0(t)}, \underbrace{\frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} \cdot e^{-(\lambda+\mu)\cdot t}}_{P_1(t)}\right) \tag{84}$$

whereas

- $P_0(t)$ is the system point **availability** (prob. of being in operational state 0 at time t)
- $P_1(t)$ is the system point **unavailability** (prob. of being in failed state 1 at time t)

- Set $\frac{dP(t)}{dt} = 0$ in the fundamental equation and solve the system of equations:

$$\underline{P} \cdot \underline{A} = 0 \tag{85}$$

$$\sum_{j=0}^{N} \Pi_j = 1 \tag{86}$$

- the Steady State Probabilities become:

$$\Pi_0 = \lim_{t \to \infty} P_0(t) = \frac{\mu}{\mu + \lambda} = \frac{1/\lambda}{1/\mu + 1/\lambda} = \frac{MTBF}{MTTR + MTBF} \tag{87}$$

$$= \textbf{average fraction of time the system is functioning} \tag{88}$$

$$\Pi_1 = \lim_{t \to \infty} P_0(t) = \frac{\lambda}{\mu + \lambda} = \frac{1/\mu}{1/\mu + 1/\lambda} = \frac{MTTR}{MTTR + MTBF} \tag{89}$$

$$= \textbf{average fraction of time the system is down (i.e. under repair)} \tag{90}$$

- Frequency of **departure** from a state i to state j

$$\nu_{ij}^{dep}(t) = \lim_{\Delta t \to 0} \frac{p_i j(dt) \cdot P_i(t)}{dt} = \alpha_{ij} \cdot P_i(t) \underbrace{=}_{\text{steady state}} \alpha_{ij} \cdot \Pi_i \tag{91}$$

- Total Frequency of **departure** from state i to any other state j

$$\nu_i(t) = \sum_{j=0,\neq i} \alpha_{ij} P_i(t) = \alpha_{ii} \cdot P_i(t) \underbrace{=}_{\text{steady state}} \nu_i = \alpha_{ii} \cdot \Pi_i \tag{92}$$

- Frequency of **arrival** to state i from k

$$\nu_i^{arr}(t) = \sum_{k=0,\neq i} \alpha_{ki} \cdot P_k(t) \underbrace{=}_{\text{steady state}} \alpha_{ki} \cdot \Pi_k \tag{93}$$

- From some algebra, it can be followed that at steady state, that the number of arrivals to state i equals the departures from i
- **System failure intensity** W_f

- Rate at which system failures occur = expected number of failures per unit time = rate of exiting a success state to go into one of fault:

$$W_f(t) = \sum_{i \in S} P_i(t) \cdot \lambda_{i \to F} \tag{94}$$

whereas S is the set of success states in the system, F of the failure states. $\lambda_{i \to F}$ describes the conditional (transition) probability of leaving success state i towards a failure state

- **System repair intensity** W_r (same principal as for failure intensity)

$$W_r(t) = \sum_{j \in F} P_i(t) \cdot \mu_{j \to S} \tag{95}$$

Introduction to Monte Carlo Simulation - The experimental view

Inverse Transform Method (analytical way)

- By taking a sample R from $U_R(r)$ and compute X
- $X = F_X^{-1}(R)$
- For the exponential distribution, this would result in
- $X = F_X^{-1}(R) = -\frac{1}{\lambda} ln(1-R)$
- This means, that from a random sample R, one can will receive a X which is transformed through the new shape of the curve from the exponential distribution

Sampling from discrete distributions (non-analytical)

- This is the same concept as for the analytical way
- However, now we only have dicrete distributions
- This means, that samples R can only achieve a discrete number of outcomes (X)

Failure Probabilty Estimation: Sampling from discrete distribution