# 1 General Considerations

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + u_i \frac{\partial\rho}{\partial x_i} \neq 0 \tag{1}$$

- Wave propagation
- Convective flows with buoancy
- Flows with variable temperature, friction, sources of heat
- High speed flows with Mach numbers $Ma \geq 1$

Compressible flows can still be described through the continuum model and conservation laws. The assumption is also that the thermodynamic state of the fluid is in a local equilibrium.

### Assumptions

- Length scale of flows large compared to molecular scales (mean free path $\lambda$)
- Length scale of flows small compared to the geometric scales (length $L$)
- Time scale $\tau_F$ of the flow long compared to the molecular process (relaxation) time constants $\tau_R$

### Description of the "Continuum" Flow State

- Three components of flow velocity $\underline{u}(\underline{x}, t)$
- The fluid density $\rho(\underline{x}, t)$
- The fluid pressure $p(\underline{x}, t)$
- The energy $e(\underline{x}, t)$

The required equations are the conservation laws for mass, momentum and energy together with suitable thermodynamic equations of state. With corresponding initial and boundary conditions, the evolution can then be computed.

# 2 Thermodynamic Relations

### State Variables

- Density: $\rho = \rho(p, T)$
- Pressure: $p = p(\rho, T)$
- Temperature: $T = T(\rho, p)$
- Internal energy: $e = e(\rho, T)$ $[e] = J/kg$
- Enthalpy: $h = h(p, T)$
- Entropy: $s = s(\rho, T)$

### Van der Waals Gas

$$(p + a\rho^2)\left(\frac{1}{\rho} - b\right) = RT \tag{2}$$

### Incompressible Fluid

$$\rho = const. \neq \rho(p, T) \tag{3}$$

# 3 Conservation Laws for Continuum Flows

$$\frac{Dm}{Dt} = \frac{D}{Dt}\int_{\tilde{V}} \rho d\tilde{V} = 0 \; (material\ volume) \tag{4}$$

$$\int_V \frac{\partial\rho}{\partial t} dV + \int_S \rho(\mathbf{u}\cdot\mathbf{n}) dS = 0 \; (Eulerian\ Volume) \tag{5}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \; (material\ volume\ /\ index) \tag{6}$$

$$\frac{D\rho}{Dt} = -\rho \frac{\partial u_i}{\partial x_i} \; (Eulerian\ Volume/\ index) \tag{7}$$

### Mass Conservation

Material Volume

$$\frac{Dm}{Dt} = \frac{D}{Dt}\int_{\vec{V}} \rho d\vec{V} = 0 \tag{8}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x_i}(\rho u_i) = 0 \tag{9}$$

Eulerian Volume

$$\int_V \frac{\partial\rho}{\partial t} dV + \int_S \rho(\vec{u}\cdot\vec{n}) dS = 0 \tag{10}$$

$$\frac{D\rho}{Dt} = -\rho\frac{\partial u_i}{\partial x_i} \tag{11}$$

### Momentum Conservation

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial}{\partial x_j}\sigma_{ij} + \rho f_i \tag{12}$$

$$\rho\frac{Du_i}{Dt} = \frac{\partial}{\partial x_j}\sigma_{ij} + \rho f_i \tag{13}$$

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{14}$$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\delta_{ij}\frac{\partial u_k}{\partial x_k} \tag{15}$$

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \rho f_i \tag{16}$$

### Energy Conservation

$$\rho\frac{D}{Dt}(e + \frac{1}{2}u_1^2) = \frac{\partial}{\partial x_j}(\sigma_{ij}u_i)) + \rho f_i u_i - \frac{\partial q_i}{\partial x_i} + \rho q_v \tag{17}$$

$$\rho\frac{D}{Dt}(e + \frac{1}{2}u_1^2) = -\frac{\partial}{\partial x_i}(pu_i) + \frac{\partial}{\partial x_j}(\tau_{ij}u_i) + \rho f_i u_i - \frac{\partial q_i}{\partial x_i} + \rho q_v \tag{18}$$

$\rho u_i \frac{Du_i}{Dt} = \rho\frac{D}{Dt}\left(\frac{u_i^2}{2}\right) = -u_i\frac{\partial p}{\partial x_i} + u_i\frac{\partial}{\partial x_j}\tau_{ij} + \rho f_i u_i \quad \rho\frac{De}{Dt} =$
$\rho\frac{D}{Dt}\left(e + \frac{1}{2}u_i^2\right) - \rho\frac{D}{Dt}\left(\frac{u_i^2}{2}\right) =$
$= -p\frac{\partial u_i}{\partial x_i} + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \rho q_v - \frac{\partial q_i}{\partial x_i}$

### Dissipation Function $\Phi$

Insert $h = e + \frac{p}{\rho}$ to obtain Enthalpy equation, introduce $h_t = h + \frac{u_i^2}{2}$ and add kinetic energy (p. 15). For perfect gasses, $h = c_pT$, $q_i = -k\frac{dT}{dx}$, derive the temperature equation.

### Entropy Equation

$$\rho T\frac{Ds}{Dt} = \Phi + \rho q_v - \frac{\partial q_i}{\partial x_i} \tag{19}$$

### Vorticity Equation

$$\rho\frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho}\right) = (\vec{\omega}\cdot\nabla)\vec{u} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\left(\frac{1}{\rho}\nabla\cdot\vec{\tau}\right) \tag{20}$$

#### Crocco Theorem (rewritten momentum equation using Enthalpy and Entropy)

$$\frac{\partial u}{\partial t} + \nabla\left(\frac{1}{2}\vec{u}^2 + h + \psi\right) = \vec{u} \times \vec{\omega} + T\nabla s + \frac{1}{\rho}\nabla \cdot \vec{\tau} \qquad (21)$$

### Compressible Bernoulli

equation (integrate momentum equation law along particle path). Clasical not feasible

$$\rho\left(\frac{Dh_t}{Dt} - f_i u_i\right) = 0 \qquad (22)$$

$$f_i = -\frac{\partial \psi}{\partial x_i} \qquad (23)$$

$$\psi \neq \psi(t) \qquad (24)$$

$$\frac{D}{Dt}(h_t + \psi) = 0 \qquad (25)$$

Between 2 points along stream line

$$h_t + \psi = e + \frac{p}{\rho} + \frac{u_i^2}{2} + \psi = const. \qquad (26)$$

## 4 Simplification Strategies (p.20)

- Unsteady $\rightarrow$ steady (no wave propagation) (no time dependence)
- 3D $\rightarrow$ 2D $\rightarrow$ quasi 1-D
- Viscous, heat conduction $\rightarrow$ inviscid, adiabatic (isentropic, homentropic)
- Subsonic $\rightarrow$ transonic $\rightarrow$ supersonic $\rightarrow$ hypersonic (Elliptic $\rightarrow$ hyperbolic)
- Full nonlinear $\rightarrow$ linearised (solve for small pertubations around predefined flow state unique solvable problem, separation of influencing factors facilitated)

## 5 Conservation Laws for Stream Tubes (p. 22)

Quasi 1D, separate for environment. Outer surface formed by instantaneous streamlines, no flow across boundaries. Inlet + outlet. Shape (t). For small enough $A$, flow properties can be treated constant in any cross section.

#### Mass Conservation

$$\int_1^2 \frac{\partial}{\partial t}\left[\rho(s,t)A(s,t)\right] ds + \rho_2 A_2 u_2 - \rho_1 A_1 u_1 = 0 \qquad (27)$$

$$\dot{m} = \rho A u = const. \qquad (28)$$

#### Momentum Conservation

$$\int_1^2 \frac{\partial}{\partial t}\left[\rho(s,t)A(s,t)\right] ds + \rho_2 A_2 u_2 \vec{u}_2 - \rho_1 A_1 u_1 \vec{u}_1 = \qquad (29)$$

$$= -p_2 A_2 \vec{n}_2 + p_1 A_1 \vec{n}_1 + F_\tau|_1^2 + F_S \qquad (30)$$

Steady, frictionless

$$\rho_2 u_2^2 + p_2 = \rho_1 u_1^2 + p_1 \qquad (31)$$

#### Energy Conservation (p.20)

Steady, frictionless

$$e_2 + \frac{u_2^2}{2} + \frac{p_2}{\rho_2} = e_1 + \frac{u_1^2}{2} + \frac{p_1}{\rho_1} \qquad (32)$$

Enthalpy subsitution $h = e + \frac{p}{\rho} \rightarrow h_{t1} = h_{t2} = const.$

## 6 Steady one-dimensional Flow without Friction and Heat

Assumptions:

- No friction (inviscid)
- No heat source or transport
- No flow through mantle
- Perfect gas

$$Ma = \frac{u}{a} \qquad (33)$$

$$a^2 = \gamma R T \qquad (34)$$

Stagnation properties, when $u = 0$:

$$\frac{h_0}{h} = \frac{T_0}{T} = \left(\frac{a_0^2}{a^2}\right) = 1 + \frac{\gamma-1}{2}Ma^2 \qquad (35)$$

Isentropic flow:

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left[1 + \frac{\gamma-1}{2}Ma^2\right]^{\frac{\gamma}{\gamma-1}} \qquad (36)$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma-1}} = \left[1 + \frac{\gamma-1}{2}Ma^2\right]^{\frac{1}{\gamma-1}} \qquad (37)$$

When $Ma < 0.3$, density changes $< 4.5\%$: Assumption is: incompressible. The critical state is then $(Ma = 1)$, $superscript\ ^*$

$$\frac{h^*}{h_0} = \frac{T^*}{T_0} = \left(\frac{a^{*2}}{a_0^2}\right) = \left[1 + \frac{\gamma-1}{2}\right]^{-1} = \frac{2}{\gamma+1} = 0.8333(\gamma = 1.4)$$

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = 0.5283(\gamma = 1.4) \qquad (38)$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} = 0.6339(\gamma = 1.4) \qquad (39)$$

Critical $Ma^*$:

$$Ma^* = \frac{u}{a^*} = \frac{u}{a(Ma=1)} = \frac{u}{a}\frac{a}{a_0}\frac{a_0}{a^*} \qquad (40)$$

$$= Ma\sqrt{\frac{T}{T_0}}\sqrt{\frac{T_0}{T^*}} = \sqrt{\frac{\frac{\gamma+1}{2}Ma^2}{1 + \frac{\gamma-1}{2}Ma^2}} \qquad (41)$$

$$Ma^* \rightarrow \sqrt{\frac{\gamma+1}{\gamma-1}}\,(Ma \rightarrow \infty) = 2.4495\ (\gamma = 1.4) \qquad (42)$$

#### Area velocity relation

A velocity increase $\rightarrow$ density decrease. If $Ma << 1$, then the density changes are small compared to the velocity changes.

$$Ma^2 \frac{1}{u}\frac{du}{dx} = -\frac{1}{\rho}\frac{d\rho}{dx} \qquad (43)$$

$$\left(Ma^2 - 1\right)\frac{1}{u}\frac{du}{dx} = \frac{1}{A}\frac{dA}{dx} \qquad (44)$$

If $Ma < 1$, then an area increase will result in a velocity reduction. If $Ma > 1$, then opposite applies. If $Ma = 1$, then a change has no effect (chocked flow)

**Stationary normal shock**

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = 1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right) = \frac{1}{Ma^{*2}} \tag{45}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2 - 1\right) \tag{46}$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2-1\right)\right]\left[1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)\right] \tag{47}$$

$$\frac{\Delta s}{R} = \frac{1}{\gamma-1}\left[\ln\left(\frac{p_2}{p_1}\right) - \gamma\ln\left(\frac{\rho_2}{\rho_1}\right)\right] =$$
$$\frac{1}{\gamma-1}\left\{\left[1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2-1\right)\right]\left[1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)\right]\right\}$$

$h_{01} = h_{02}$, $T_{01} = T_{02}$, and total enthalpy conserved (however stagnation pressure not constant, $p_{01} \neq p_{02}$):

$$\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{01}} = \frac{p_2}{P-1}\left(\frac{T_{02}}{T_2}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{T_1}{T_{01}}\right)^{\frac{\gamma}{\gamma-1}} =$$
$$\left[1 + \frac{2\gamma}{\gamma+1}(Ma_1^2-1)\right]^{\frac{1}{\gamma-1}}\left[1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)\right]^{\frac{-\gamma}{\gamma-1}}$$

As $s$ increases, $u$ decreases. $Ma_2$ is always $< 1$, when $Ma_1 \to \infty$:

$$Ma_2 \to \sqrt{\frac{\gamma-1}{2\gamma}} = 0.38\ (\gamma = 1.4) \tag{48}$$

$$Ma_2^2 = \left(\frac{u_2}{a_2}\right)^2 = \left(\frac{u_2}{u_1}\right)^2\left(\frac{u_1}{a_1}\right)^2\left(\frac{a_1}{a_2}\right)^2 = \left(\frac{u_2}{u_1}\right)^2 Ma_1^2\left(\frac{T_1}{T_2}\right)$$

$$Ma_2 = \sqrt{\frac{1 + \frac{\gamma-1}{\gamma+1}\left(Ma_1^2-1\right)}{1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2-1\right)}} \tag{49}$$

**Rankine Hugoniot (p.32) - Adiabatic Shock**

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma\left(\frac{\rho_2}{\rho_1} - 1\right)}{\gamma + 1 - (\gamma-1)\frac{\rho_2}{\rho_1}}$$

**Moving Shock Wave**

Switch to reference frame

$$u_1 \widehat{=} u_s\ ,\ p_1 \widehat{=} p_e\ ,\ \rho_1 \widehat{=} \rho_0$$

Flow behind

$$u_2 \widehat{=} u_s - u_d\ ,\ p_2 \widehat{=} p_d\ ,\ \rho_2 \widehat{=} \rho_d$$

Shock $u_d$

$$u_d = u_s - u_2 = u_1 - u_2 = u_1\left(1 - \frac{u_2}{u_1}\right) = u_1\frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)$$

$$Ma_d = \frac{u_d}{a_d} = \frac{u_1 - u_2}{a_d} = \frac{u_1}{a_1}\frac{a_1}{a_d}\left(1 - \frac{u_2}{u_1}\right) = Ma_1\sqrt{\frac{T_1}{T_2}}\left(1 - \frac{u_2}{u_1}\right)$$

$$u_d = \frac{a_0}{\gamma}\frac{\frac{\Delta p}{p_0}}{\sqrt{1 + \frac{\gamma+1}{2\gamma}\frac{\Delta p}{p_0}}}$$

Pressure increase

$$\frac{\Delta p}{p_0} = \frac{p_d - p_0}{0_0} = \frac{2\gamma}{\gamma+1}\left(Ma_S^2 - 1\right)$$

The ratio (Pressure increase) has an asymptotic limit. For high $Ma_s$, the function becomes limited.

**Detonations ($Ma_2 > 1$) and Deflagrations ($Ma_2 < 1$)**

Assumption: Ignore adiabatic flow, include however heat release
Rayleigh line: $\frac{p_1}{p_0} = 1 + \frac{\rho_0}{p_0}u_0^2 - \frac{\rho_0}{p_0}\frac{\rho_1}{\rho_0}u_1^2 = 1 + \gamma Ma_0^2\left(1 - \frac{\rho_0}{\rho_1}\right)$,
Rankine Hugeniot with heat: $\frac{p_2}{p_0} = \frac{(\gamma+1)-(\gamma-1)\frac{\rho_0}{\rho_2}+2\gamma\hat{q}}{(\gamma+1)\frac{\rho_0}{\rho_2}-(\gamma-1)}$ , $\hat{q} = \frac{q_{heat}}{c_p T_1}$

**Chapman Jouget Point**

...is the intersection where $Ma = 1$, so $Ma_2 = 1 = Ma_0\sqrt{\frac{\rho_0}{\rho_2}}\sqrt{\frac{\rho_0}{\rho_2}}$
The limiting case for shock cycle:

$$\frac{\rho_0}{\rho_2}|_c = \frac{u_2}{u_0}|_c = \frac{\gamma Ma_0^2 + 1}{Ma_0^2(\gamma+1)}$$

Behind the shock, the flow is subsonic $\leftrightarrow$ strong detonation. There is a weak deflagration if the density ratio $\frac{\rho_1}{\rho_2} > 1$ is large

**Laval Nozzle (p. 39)**

Varying cross-section:

$$\frac{p(x)}{p_0} = \left[1 + \frac{\gamma-1}{2}Ma^2(x)\right]^{\frac{-\gamma}{\gamma-1}}$$
$$\frac{A^*}{A(x)} = Ma(x)\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}Ma^2(x)\right]$$
$$u(x) = Ma(x)a_0\frac{a(x)}{a_0} = Ma(x)a_0\sqrt{\frac{T(x)}{T_0}} = \frac{a_0 \cdot Ma(x)}{\sqrt{1 + \frac{\gamma-1}{2}Ma^2(x)}}$$
$$u^* = a^*\ ,\ \text{if } Ma^* = 1$$

In order to increase the $Ma_{exit}$, reduce the area ration (tune $A^*$). Different flow regimes are shown on p. 41. A variable exit area is in practice not possible

# 7 Unsteady one-dimensional Flows

# 8 Two-dimensional steady supersonic Flow

# 9 Method Characteristics for planar homentropic supersonic Flows

# 10 Homentropic Flow around slender Wings

# 11 Homentropic Flow around axisymmetric slender Bodies

# 12 Similarity Relations