

1 General Considerations

Dρ/Dt = ∂ρ/∂t + u\_i ∂ρ/∂x\_i ≠ 0

- Wave propagation
- Convective flows with buoyancy
- Flows with variable temperature, friction, sources of heat
- High speed flows with Mach numbers Ma ≥ 1

Compressible flows can still be described through the continuum model and conservation laws. The assumption is also that the thermodynamic state of the fluid is in a local equilibrium.

Assumptions

- Length scale of flows large compared to molecular scales (mean free path λ)
- Length scale of flows small compared to the geometric scales (length L)
- Time scale τ\_F of the flow long compared to the molecular process (relaxation) time constants τ\_R

Description of the “Continuum” Flow State

- Three components of flow velocity u(x, t)
- The fluid density ρ(x, t)
- The fluid pressure p(x, t)
- The energy e(x, t)

The required equations are the conservation laws for mass, momentum and energy together with suitable thermodynamic equations of state. With corresponding initial and boundary conditions, the evolution can then be computed.

2 Thermodynamic Relations

State Variables

- Density: ρ = ρ(p, T)
- Pressure: p = p(ρ, T)
- Temperature: T = T(ρ, p)
- Internal energy: e = e(ρ, T) [e] = J/kg
- Enthalpy: h = h(p, T)

- Entropy: s = s(ρ, T)

Van der Waals Gas

(p + aρ²) (1/ρ - b) = RT

Incompressible Fluid

ρ = const. ≠ ρ(p, T)

3 Conservation Laws for Continuum Flows

Mass Conservation

Dm/Dt = D/Dt ∫\_Ṽ ρ dṼ = 0 (material volume)

∫\_V ∂ρ/∂t dV + ∫\_S ρ(u · n) dS = 0 (Eulerian Volume)

∂ρ/∂t + ∂/∂x\_i (ρu\_i) = 0 (material volume / index)

Dρ/Dt = -ρ ∂u\_i/∂x\_i (Eulerian Volume / index)

Momentum Conservation

∂/∂t (ρu\_i) + ∂/∂x\_j (ρu\_i u\_j) = ∂/∂x\_j σ\_ij + ρf\_i

ρ Du\_i/Dt = ∂/∂x\_j σ\_ij + ρf\_i

σ\_ij = -pδ\_ij + τ\_ij

τ\_ij = μ (∂u\_i/∂x\_j + ∂u\_j/∂x\_i) + (μ\_v - 2/3 μ) δ\_ij ∂u\_k/∂x\_k

ρ Du\_i/Dt = -∂p/∂x\_i + ∂/∂x\_j [ μ (∂u\_i/∂x\_j + ∂u\_j/∂x\_i) + (μ\_v - 2/3 μ) δ\_ij ∂u\_k/∂x\_k ] + ρf\_i

Energy Conservation

ρ D/Dt (e + 1/2 u\_1²) = ∂/∂x\_j (σ\_ij u\_i) + ρf\_i u\_i - ∂q\_i/∂x\_i + ρq\_v

ρ D/Dt (e + 1/2 u\_1²) = -∂/∂x\_i (p u\_i) + ∂/∂x\_j (τ\_ij u\_i) + ρf\_i u\_i - ∂q\_i/∂x\_i + ρq\_v

ρ u\_i D u\_i/Dt = ρ D/Dt (u\_i²/2) = -u\_i ∂p/∂x\_i + u\_i ∂/∂x\_j τ\_ij + ρ f\_i u\_i ρ D e/Dt = ρ D/Dt (e + 1/2 u\_i²) - ρ D/Dt (u\_i²/2) = -p ∂u\_i/∂x\_i + τ\_ij ∂u\_i/∂x\_j + ρ q\_v - ∂q\_i/∂x\_i

Dissipation Function Φ

Insert h = e + p/ρ to obtain Enthalpy equation, introduce h\_t = h + u²/2

and add kinetic energy (p. 15). For perfect gasses, h = c\_p T, q\_i = -k dT/dx, derive the temperature equation.

Entropy Equation

ρ T D s/Dt = Φ + ρ q\_v - ∂q\_i/∂x\_i

Vorticity Equation

ρ D/Dt (ω/ρ) = (ω · ∇) u + 1/ρ² ∇ρ × ∇p + ∇ × (1/ρ ∇ · τ)

Crocco Theorem (rewritten momentum equation using Enthalpy and Entropy)

∂u/∂t + ∇ (1/2 u² + h + ψ) = u × ω + T ∇s + 1/ρ ∇ · τ

Compressible Bernoulli equation (integrate momentum equation law along particle path). Classical not feasible

ρ (D h\_t/Dt - f\_i u\_i) = 0

f\_i = -∂ψ/∂x\_i

ψ ≠ ψ(t)

D/Dt (h\_t + ψ) = 0

Between 2 points along stream line

h\_t + ψ = e + p/ρ + u²/2 + ψ = const.

4 Simplification Strategies (p.20)

- Unsteady → steady (no wave propagation) (no time dependence)
- 3D → 2D → quasi 1-D
- Viscous, heat conduction → inviscid, adiabatic (isentropic, homentropic)
- Subsonic → transonic → supersonic → hypersonic (Elliptic → hyperbolic)
- Full nonlinear → linearised (solve for small perturbations around predefined flow state unique solvable problem, separation of influencing factors facilitated)

## 5 Conservation Laws for Stream Tubes (p. 22)

Quasi 1D, separate for environment. Outer surface formed by instantaneous streamlines, no flow across boundaries. Inlet + outlet. Shape (t). For small enough  $A$ , flow properties can be treated constant in any cross section.

### Mass Conservation

$$\int_1^2 \frac{\partial}{\partial t} [\rho(s, t) A(s, t)] ds + \rho_2 A_2 u_2 - \rho_1 A_1 u_1 = 0$$

$$\dot{m} = \rho A u = \text{const.}$$

### Momentum Conservation

$$\begin{aligned} \int_1^2 \frac{\partial}{\partial t} [\rho(s, t) A(s, t)] ds + \rho_2 A_2 u_2 \vec{u}_2 - \rho_1 A_1 u_1 \vec{u}_1 = \\ = -p_2 A_2 \vec{n}_2 + p_1 A_1 \vec{n}_1 + F_\tau|_1^2 + F_S \end{aligned}$$

Steady, frictionless

$$\rho_2 u_2^2 + p_2 = \rho_1 u_1^2 + p_1$$

### Energy Conservation (p.20)

Steady, frictionless

$$e_2 + \frac{u_2^2}{2} + \frac{p_2}{\rho_2} = e_1 + \frac{u_1^2}{2} + \frac{p_1}{\rho_1}$$

Enthalpy substitution  $h = e + \frac{p}{\rho} \rightarrow h_{t1} = h_{t2} = \text{const.}$

## 6 Steady one-dimensional Flow without Friction and Heat (p. 25)

Assumptions:

- No friction (inviscid)
- No heat source or transport
- No flow through mantle
- Perfect gas

$$\begin{aligned} Ma &= \frac{u}{a} \\ a^2 &= \gamma R T \end{aligned}$$

Stagnation properties, when  $u = 0$  (Ruhegrösse), subscript 0:

$$\frac{h_0}{h} = \frac{T_0}{T} = \left( \frac{a_0^2}{a^2} \right) = 1 + \frac{\gamma - 1}{2} Ma^2$$

Isentropic flow (p.26):

$$\frac{p_0}{p} = \left( \frac{T_0}{T} \right)^{\frac{\gamma}{\gamma-1}} = \left[ 1 + \frac{\gamma - 1}{2} Ma^2 \right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho} = \left( \frac{T_0}{T} \right)^{\frac{1}{\gamma-1}} = \left[ 1 + \frac{\gamma - 1}{2} Ma^2 \right]^{\frac{1}{\gamma-1}}$$

When  $Ma < 0.3$ , density changes  $< 4.5\%$ : Assumption is: incompressible. The critical state is then ( $Ma = 1$ ), *superscript \**

$$\frac{h^*}{h_0} = \frac{T^*}{T_0} = \left( \frac{a^{*2}}{a_0^2} \right) = \left[ 1 + \frac{\gamma - 1}{2} \right]^{-1} = \frac{2}{\gamma + 1} = 0.8333 (\gamma = 1.4)$$

$$\frac{p^*}{p_0} = \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma-1}} = 0.5283 (\gamma = 1.4)$$

$$\frac{\rho^*}{\rho_0} = \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma-1}} = 0.6339 (\gamma = 1.4)$$

Critical  $Ma^*$  (isentropic flow stays limited when  $Ma \rightarrow \infty$ ). The flow velocity stays finite even if  $Ma$  goes to infinity:

$$\begin{aligned} Ma^* &= \frac{u}{a^*} = \frac{u}{a(Ma=1)} = \frac{u}{a} \frac{a}{a_0} \frac{a_0}{a^*} \\ &= Ma \sqrt{\frac{T}{T_0}} \sqrt{\frac{T_0}{T^*}} = \sqrt{\frac{\frac{\gamma+1}{2} Ma^2}{1 + \frac{\gamma-1}{2} Ma^2}} \end{aligned}$$

$$Ma^* \rightarrow \sqrt{\frac{\gamma+1}{\gamma-1}} (Ma \rightarrow \infty) = 2.4495 (\gamma = 1.4)$$

### Area velocity relation

A velocity increase  $\rightarrow$  density decrease (always). If  $Ma \ll 1$ , then the density changes are small compared to the velocity changes. A small velocity increase at  $Ma \gg 1$  will lead to large density changes.

$$Ma^2 \frac{1}{u} \frac{du}{dx} = -\frac{1}{\rho} \frac{d\rho}{dx} \quad \text{(Mach-density relation)}$$

$$(Ma^2 - 1) \frac{1}{u} \frac{du}{dx} = \frac{1}{A} \frac{dA}{dx} \quad \text{(Mach-Area relation)}$$

If  $Ma < 1$ , then an area increase will result in a velocity reduction. If  $Ma > 1$ , then opposite applies. If  $Ma = 1$ , then a change in Area  $A$  has no effect (choked flow)

### Stationary normal shock

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = 1 - \frac{2}{\gamma + 1} \left( 1 - \frac{1}{Ma_1^2} \right) = \frac{1}{Ma^{*2}}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1} (Ma_1^2 - 1)$$

$$\frac{T_2}{T_1} = \left[ 1 + \frac{2\gamma}{\gamma + 1} (Ma_1^2 - 1) \right] \left[ 1 - \frac{2}{\gamma + 1} \left( 1 - \frac{1}{Ma_1^2} \right) \right]$$

$$\frac{\Delta s}{R} = \frac{1}{\gamma - 1} \left[ \ln \left( \frac{p_2}{p_1} \right) - \gamma \ln \left( \frac{\rho_2}{\rho_1} \right) \right] =$$

$$\frac{1}{\gamma - 1} \left\{ \left[ 1 + \frac{2\gamma}{\gamma + 1} (Ma_1^2 - 1) \right] \left[ 1 - \frac{2}{\gamma + 1} \left( 1 - \frac{1}{Ma_1^2} \right) \right] \right\}$$

$h_{01} = h_{02}$ ,  $T_{01} = T_{02}$ , and total enthalpy conserved (however stagnation pressure not constant,  $p_{01} \neq p_{02}$ ):

$$\begin{aligned} \frac{p_{02}}{p_{01}} &= \frac{p_{02}}{p_2} \frac{p_2}{p_1} \frac{p_1}{p_{01}} = \frac{p_2}{p_1} \left( \frac{T_{02}}{T_2} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{T_1}{T_{01}} \right)^{\frac{\gamma}{\gamma-1}} = \\ &\left[ 1 + \frac{2\gamma}{\gamma + 1} (Ma_1^2 - 1) \right]^{\frac{1}{\gamma-1}} \left[ 1 - \frac{2}{\gamma + 1} \left( 1 - \frac{1}{Ma_1^2} \right) \right]^{\frac{-\gamma}{\gamma-1}} \end{aligned}$$

As  $s$  increases,  $u$  decreases.  $Ma_2$  is always  $< 1$ , when  $Ma_1 \rightarrow \infty$ :

$$Ma_2 \rightarrow \sqrt{\frac{\gamma - 1}{2\gamma}} = 0.38 (\gamma = 1.4)$$

$$Ma_2^2 = \left( \frac{u_2}{a_2} \right)^2 = \left( \frac{u_2}{u_1} \right)^2 \left( \frac{u_1}{a_1} \right)^2 \left( \frac{a_1}{a_2} \right)^2 = \left( \frac{u_2}{u_1} \right)^2 Ma_1^2 \left( \frac{T_1}{T_2} \right)$$

$$Ma_2 = \sqrt{\frac{1 + \frac{\gamma-1}{\gamma+1} (Ma_1^2 - 1)}{1 + \frac{2\gamma}{\gamma+1} (Ma_1^2 - 1)}}$$

A weak shock occurs at  $Ma_1$  close to one. See page 31 for equation

### Rankine Hugoniot (p.32) - Adiabatic Shock (no $Ma$ dependency)

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma \left( \frac{p_2}{p_1} - 1 \right)}{\gamma + 1 - (\gamma - 1) \frac{p_2}{p_1}}$$

### Moving Shock Wave (p.33)

Switch to reference frame (from frame fixed with moving shock front into a frame moving with shock)

$$u_1 \hat{=} u_s, \quad p_1 \hat{=} p_0, \quad \rho_1 \hat{=} \rho_0$$

Flow behind

$$u_2 \hat{=} u_s - u_d, \quad p_2 \hat{=} p_d, \quad \rho_2 \hat{=} \rho_d$$

Shock  $u_d$

$$u_d = u_s - u_2 = u_1 - u_2 = u_1 \left( 1 - \frac{u_2}{u_1} \right) = u_1 \frac{2}{\gamma + 1} \left( 1 - \frac{1}{Ma_1^2} \right)$$

$$Ma_d = \frac{u_d}{a_d} = \frac{u_1 - u_2}{a_d} = \frac{u_1}{a_1} \frac{a_1}{a_d} \left( 1 - \frac{u_2}{u_1} \right) = Ma_1 \sqrt{\frac{T_1}{T_2}} \left( 1 - \frac{u_2}{u_1} \right)$$

$$u_d = \frac{a_0}{\gamma} \frac{\frac{\Delta p}{p_0}}{\sqrt{1 + \frac{\gamma+1}{2\gamma} \frac{\Delta p}{p_0}}} (a_1 \hat{=} a_0), \quad Ma_s = \frac{u_s}{a_0} = \sqrt{1 + \frac{\gamma+1}{2\gamma} \frac{\Delta p}{p_0}}$$

Pressure increase

$$\frac{\Delta p}{p_0} = \frac{p_d - p_0}{p_0} = \frac{2\gamma}{\gamma + 1} (Ma_S^2 - 1), \quad [Ma_1 = \frac{u_1}{a_1} = \frac{u_s}{a_s} = Ma_s]$$

The ratio (Pressure increase) has an asymptotic limit. For high  $Ma_s$ , the function becomes limited.  $\frac{u_s}{u_d} \rightarrow \frac{\gamma+1}{2}$  (for high pressure differences)

### Detonations ( $Ma_2 > 1$ ) and Deflagrations ( $Ma_2 < 1$ ) (p.36, ZND)

**Assumption: Ignore adiabatic flow, include however heat release**

$$\text{Rayleigh line: } \frac{p_1}{p_0} = 1 + \frac{\rho_0}{p_0} u_0^2 - \frac{\rho_0}{p_0} \frac{p_1}{\rho_0} u_1^2 = 1 + \gamma Ma_0^2 \left( 1 - \frac{\rho_0}{\rho_1} \right),$$

$$\text{Rankine Hugoniot with heat: } \frac{p_2}{p_0} = \frac{(\gamma+1) - (\gamma-1) \frac{\rho_0}{\rho_2} + 2\gamma \hat{q}}{(\gamma+1) \frac{\rho_0}{\rho_2} - (\gamma-1)}, \quad \hat{q} =$$

$\frac{q_{heat}}{c_p T_1}$ , This gives us  $p_1$  and  $p_2$ , the pressure of the shockwave before

the combustion and downstream after the combustion layer

**Chapman-Jouget Point (p.37)**

...is the intersection where  $Ma = 1$ , so  $Ma_2 = 1 = Ma_0 \sqrt{\frac{\rho_0}{\rho_2}} \sqrt{\frac{\rho_0}{\rho_2}}$   
The limiting case for shock cycle (Rayleigh tangent to Hugoniot Line):

$$\frac{\rho_0}{\rho_2}|_c = \frac{u_2}{u_0}|_c = \frac{\gamma Ma_0^2 + 1}{Ma_0^2(\gamma + 1)}$$

Behind the shock, the flow is subsonic  $\leftrightarrow$  strong detonation. There is a weak deflagration if the density ratio  $\frac{\rho_1}{\rho_2} \gg 1$ . The reaction front propagates at subsonic speed. Weak detonation: flow remains supersonic (not explainable through ZND)

**Laval Nozzle (p. 39)**

Varying cross-section:

$$\begin{aligned} \frac{p(x)}{p_0} &= \left[ 1 + \frac{\gamma - 1}{2} Ma^2(x) \right]^{\frac{-\gamma}{\gamma - 1}} \\ \frac{A^*}{A(x)} &= Ma(x) \left[ \frac{2}{\gamma + 1} + \frac{\gamma - 1}{\gamma + 1} Ma^2(x) \right] \\ u(x) &= Ma(x) a_0 \frac{a(x)}{a_0} = Ma(x) a_0 \sqrt{\frac{T(x)}{T_0}} = \frac{a_0 \cdot Ma(x)}{\sqrt{1 + \frac{\gamma - 1}{2} Ma^2(x)}} \\ u^* &= a^*, \text{ if } Ma^* = 1 \end{aligned}$$

In order to increase the  $Ma_{exit}$ , reduce the area ration (tune  $A^*$ ). Different flow regimes are shown on p. 41. A variable exit area is in practice not possible

**7 Unsteady one-dimensional Flows**

**Wave equation for small perturbations** Assuming small perturbations around equilibrium state with first order perturbations will result into following differential equation (enthalpy):

$$\begin{aligned} \frac{\partial p'}{\partial t} - a_0^2 \frac{\partial \rho'}{\partial t} &= 0 \iff p' = a_0^2 \rho' \\ \frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} &= 0 \text{ (mass eq.)} \\ \frac{\partial u'}{\partial t} + \frac{a_0^2}{\rho_0} \frac{\partial \rho'}{\partial x} &= 0 \text{ (momentum eq.)} \end{aligned}$$

Through cross-differentiation (elimination of terms), one arrives at the d'Alembert solution:

$$\begin{aligned} u'(x, t) &= a_0 [F(x - a_0 t) + G(x + a_0 t)] \\ \rho'(x, t) &= \rho_0 [F(x - a_0 t) + G(x + a_0 t)] \end{aligned}$$

Through characteristics one defines left and right propagating waves,  $F(\eta)$  and  $G(\xi)$ . The characteristics are in this case straight lines. Initial conditions are at  $t = 0$ , boundary conditions are at

$x = b.c.$

**Method of characteristics for nonlinear wave propagation** Here, no small perturbations are assumed, while assuming homentropic flow ( $s = const.$ ). The Riemann invariants (characteristics) are not straight anymore, and can be curved. Disturbances are no longer constant, but have a flow dependent value. Given  $a$  and  $u$  are given along a curve  $C$ , find where it intersects with two characteristics, which cross at point  $Q$ . (See p. 48)

**Piston Motion in tube (example for unsteady one-dimensional motion):**

- Boundary Condition: At  $x = x_p(t)$ ,  $u(x = x_p, t) = u_p(t)$
- How to solve: Left propagating wave from rest state, intersects  $P$  at  $u = u_p$ . The characteristic with  $\eta = const$  which then can intersect the other characteristic with  $\xi = const$ . yields point  $Q$
- $x = \left[ a_0 + \frac{\gamma + 1}{2} u_p(\tau) \right] (t - \tau) + x_p(\tau)$

**Simple expansion waves**

In the case for the piston moving to the left, the characteristics are limited by two factors:

- $x = a_0 t$ : Initially, at  $t = 0$ , the characteristic is maximum and can only be as steep as  $a_0$
- $u_p = -U$ : The piston motion can only have a max. velocity at its endpoints ( $x_p = -Ut$  and  $Ut$ )
- This gives an area of solutions, which is called a “centered fan”

$$\begin{aligned} Ma &= \frac{|U|}{a_0} \left[ 1 - \frac{\gamma - 1}{2} \frac{|U|}{a_0} \right]^{-1}, \frac{\rho}{\rho_0} = \left[ 1 - \frac{\gamma - 1}{2} \frac{|U|}{a_0} \right]^{\frac{2}{\gamma - 1}} \\ \frac{p}{p_0} &= \left[ 1 - \frac{\gamma - 1}{2} \frac{|U|}{a_0} \right]^{\frac{2\gamma}{\gamma - 1}} \end{aligned}$$

**Simple Compression Waves**, see p. 54, explained for increasing velocity to the right

**Reflections**

**Reflection from solid wall:**  $G = -F$  if boundary moves with velocity 0

**Reflection from free boundary (contact surface), p.56:** The ratio  $\alpha$  is the impedance, and is the ratio of both  $a$  of two regions

**Reflection from an open end with outflow, p.58:** At an orifice ( $a$  = outer, 0 = stagnation), the characteristics are:

$$G = F - \frac{4}{\gamma - 1} a(p_a)$$

The speed of sound is computed via the isentropic relations:

$$\frac{a_a}{a_0} = \sqrt{\frac{T_a}{T_0}} = \left( \frac{p_a}{p_0} \right)^{\frac{\gamma - 1}{2\gamma}}$$

**8 Two-dimensional steady supersonic Flow**

An oblique shock wave forms around a body with a sharp tip or a long a wall with a sudden profile change (at the turning point). Two parameters describe the problem. The inflow Mach number  $Ma_1$  and turning angle  $\theta$ . The velocity can be described by a component normal to the shock front and tangential to the shock front  $u_n$  and  $u_t$ . The downstream and upstream pressures  $p_1$  and  $p_2$  are assumed constant.

$$\begin{aligned} u_{2t} &= u_{1t} = u_t \\ u_{1n} &= u_1 \sin \beta, \quad u_{2n} = u_2 \sin(\beta - \theta) \\ \frac{u_{2n}}{u_{1n}} &= \frac{(\gamma - 1) Ma_1^2 \sin^2 \beta + 2}{(\gamma + 1) Ma_1^2 \sin^2 \beta} \\ \frac{u_{2t}}{u_{1t}} &= \frac{\tan(\beta - \theta)}{\tan \beta} = \frac{1 - \tan \theta \cot \beta}{1 + \tan \theta \tan \beta} = \frac{1 - (\tan \theta / \tan \beta)}{1 + \tan \theta \tan \beta} \\ \tan \theta &= \frac{1 - \frac{u_{2n}}{u_{1n}}}{\cot \beta + \left( \frac{u_{2n}}{u_{1n}} \right) \tan \beta} = \frac{\left( \sin^2 \beta - \frac{1}{Ma_1^2} \right) \sqrt{1 - \sin^2 \beta}}{\sin \beta \left( \frac{\gamma + 1}{2} - \sin^2 \beta + \frac{1}{Ma_1^2} \right)} \end{aligned}$$

The equation for  $\theta$  is an implicit function for  $\beta$ , whereas there are two solutions for it. Normally,  $\theta$  (geometry) and incident  $Ma_1$  are given. For  $Ma_1 > 1$ ,  $\beta$  is in the range of  $[\arcsin(1/Ma_1), 90^\circ]$ . For a given  $Ma_1$ , there is a maximum turning angle  $\theta_{max}$  beyond which the shock is not anymore at the turning point. The shock is then “detached” (set  $Ma \rightarrow \infty$  in equation for  $\theta$ ).

- Strong and weak shock: The maxima of all shock angle for oblique shocks can be connected. Higher Mach angles  $\beta$  occur with strong shocks
- Separation between  $Ma_2 < 1$  and  $Ma_2 > 1$
- For the changes in pressure, density and other thermodynamic properties, the equations for the normal shock can be used with a new “effective Mach number”  $Ma_{1, new} = Ma_1 \sin \beta$

$$\begin{aligned} u_{2n} &= u_2 \sin(\beta - \theta) \\ Ma_2^2 \sin^2(\beta - \theta) &= \frac{1 + \frac{\gamma - 1}{\gamma + 1} (Ma_1^2 \sin^2 \beta - 1)}{1 + \frac{2\gamma}{\gamma + 1} (Ma_1^2 \sin^2 \beta - 1)} \end{aligned}$$

- Small turning angles  $\theta$ : In the limit of  $\theta \rightarrow 0$ ,  $\beta$  becomes the Mach angle  $\mu = \lim_{\theta \rightarrow 0} \beta = \arcsin \frac{1}{Ma_1}$
- Hypersonic flow  $Ma_1 \gg 1$ : Shock angle and turning angle linearly dependent on gas property  $\gamma$  alone,  $\sin \beta \approx \beta = \frac{\gamma + 1}{2} \theta$

Continuous turning of supersonic flows

When the surface is moving continuously, the model can be adapted.

$$\frac{dp}{p} = -\frac{\gamma Ma}{1 + \frac{\gamma-1}{2} Ma^2} dMa$$
$$d\theta = -\frac{\sqrt{Ma^2 - 1}}{1 + \frac{\gamma-1}{2} Ma^2} \frac{dMa}{Ma}$$

These equations describe the “Prandtl-Meyer Compression”. In case the flow direction changes positively ( $d\theta > 0$ ), the Mach number decreases ( $dMa < 0$ ), and the pressure as well ( $dp < 0$ ).

$$\nu(Ma) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan \sqrt{\frac{\gamma-1}{\gamma+1} (Ma^2 - 1)} - \arctan \sqrt{Ma^2 - 1}$$
$$\nu_{max} = \nu(Ma \rightarrow \infty) = \frac{\pi}{2} \left( \sqrt{\frac{\gamma+1}{\gamma-1}} - 1 \right) \hat{=} 130.5^\circ \quad (\gamma = 1.4)$$

The above Prandtl-Meyer function outputs degrees, which couple supersonic turns and change in Mach numbers.

$$\Delta\theta = \theta_2 - \theta_1 = -\nu(Ma_2) + \nu(Ma_1)$$

An oblique schock occurs at a distanced point  $P$  away from the wall. A “Prandtl-Meyer Expansion” occurse if the turning angle is negative. The flow accelerates, and Mach lines diverge. If the flow turns around a point, all lines are going along a center.

Reflection and crossing of waves

Use same equations as before. Here, following things must be considered:

- Wedge with bounding wall: Reflections back from wall create additional shocks until  $\theta < \theta_{max}$  (function of  $\beta$ ). The Mach numbers decrease
- Wedge without bounding wall: Point  $P$  is where fluid expands (reflected by the free jet boundary)
- Walls which converge: For two different turning angles  $\theta$ , two oblique shocks meet at point  $P$ :
  - $\theta_1 - \theta_5 = \theta_4 - \theta_2$
  - $p_4(Ma_2, \theta_4, p_2) = p_5(Ma_3, \theta_5, p_3)$
- Prismatic wing: First a oblique shock, then fans, then shock due to turning back flow to inflow direction. Even without friction, a drag is introduced

**Detached Shocks** Normally, the oblique shock occurs for  $\theta < \theta_{max}$ . For larger angles, the shock detaches and shows a near hyperbolic shape. See p.78 for details when it comes to decreasing  $Ma$ .

Supersonic nozzle exit flows

Supersonic nozzle flow into stagnant environment.

- Under-expanded scenario ( $p_{env} < p_{noz}$ ): First PM-expansion due to reduction of pressure. However later on again goes back (rocket plume). Interacting incoming and reflecting expansion fans create the going back.
- Over-expanded scenario ( $p_{env} > p_{noz}$ ): The turning angle  $\beta$  is much smaller, the plume is impinged

9 Method Characteristics for planar homentropic supersonic Flows

For nonlinear wave propagation in  $(x, t)$ -space, one can compute flow fields in steady, two-dimensional homentropic flows. With the Crocco theorem, and the assumption of steady, inviscid, isoenergetic, adiabatic flow, there is no vorticity. Introducing the velocity potential, we receive equations for the speed of sound.

Transformation of equations (p.81)

Mach lines are no longer straight lines over an extended region, but are treated as characteristic curves on which the Riemann invariants remain constant. The characteristics become dependent on the local Mach angle  $\mu = \mu(\nu)$  and the Mach number  $Ma$ , determined by the Prandtl-Meyer Function. The velocity magnitude is then given by the thermodynamic state  $p, \rho, T$

$$F = \nu + \theta = const. \text{ and } G = \nu - \theta = const.$$

Initial and Boundary Value Problems (p.85)

- Initial Value Problem: A curve C is given, as well as the Riemann Invariants. This gives us both Mach numbers and their flow directions in two points 1 and 2. In point 3, both intersect, and this gives the local  $\nu_3$  and  $\theta_3$ , meaning the local Mach number:

$$u_3 = Ma_3 a_3 = Ma_3 a_0 \frac{a_3}{a_0} = Ma_3 a_0 \sqrt{\frac{T_3}{T_0}}$$
$$a_0 = \sqrt{\gamma R T_0}, \quad \frac{T_3}{T_0} = \left( 1 + \frac{\gamma-1}{2} Ma_3^2 \right)^{-1}$$

10 Homentropic Flow around slender Wings

Instead of using methods of characteristics, now a new methodology is used. It uses also linearisations, which make the problems easier to handle and is valid for supersonic and subsonic flows.

Initial Considerations:

A function describes the wing profile with a camber line ( $y_c(x)$ ) and a thickness distribution ( $h(x)$ ), both depending on  $x$ . The boundary condition is that the flows are tangential to the contour, and that in the far field the stream velocity is free stream velocity  $u_\infty$ .

To simplify the problems, the coordinates are simplified as follows:

$$\tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{L}, \quad \sigma = \frac{c_{max}}{L}, \quad \tau = \frac{h_{max}}{L}, \quad \tilde{h}(\tilde{x}) = \frac{1}{h_{max}} h\left(\frac{x}{L}\right)$$
$$\frac{y_p}{L} = \sigma \tilde{y}_C(\tilde{x}) \pm \frac{\tau}{2} \tilde{h}(\tilde{x}), \quad \tilde{y}_C(\tilde{x}) = \frac{1}{c_{max}} y_C\left(\frac{x}{L}\right)$$

Solving the equations for  $c_p$  is not trivial, as they are non-linear. The parameters, on which the solution depends on are:

- Thickness ratio  $\tau$

- Camber ratio  $\sigma$
- Angle of attack  $\alpha$
- Free stream Mach number  $Ma_\infty$
- Ratio of specific heats  $\gamma$

Linearized Theory

The equations are simplified, assuming slender profiles ( $\tau, \sigma, \alpha \ll 1$ ). This implies  $h$  and  $y_c$  remain small. Also, we assume small angles:  $\alpha \approx \sin \alpha$

$$c_p = \frac{2}{\gamma Ma_\infty^2} \left\{ \left[ 1 - (\gamma - 1) Ma_\infty^2 \tilde{u}_x \right]^{\frac{\gamma-1}{\gamma}} - 1 \right\}$$
$$= -2 \tilde{u}_x(\tilde{x}, \tilde{y}) = -2 \frac{\partial \phi}{\partial \tilde{x}}$$

The pressure distribution is evaluated along the body surface, which in the linearised framework is  $\tilde{y} = 0$

Linearised subsonic flows ( $Ma_\infty < 1$ )

With the so-called **Prandtl Factor**, one can find a simple Laplace equation in the scaled coordinates  $(\tilde{x}, m\tilde{y})$ :

$$m = \sqrt{1 - Ma_\infty^2}$$
$$\frac{\partial^2 \phi}{\partial \tilde{x}^2} + \frac{\partial^2 \phi}{\partial (m\tilde{y})^2} = 0$$

Solutions can be found by superposition of fundamental solutions:

- Source/Sink of magnitude  $Q$  in  $(\xi, 0)$ :  
 $\phi(\tilde{x}, \tilde{y}) = Q \ln \sqrt{(\tilde{x} - \xi)^2 + m^2 \tilde{y}^2}$
- Vortex with circulation  $\Gamma$  in  $(\xi, 0)$ :  
 $\phi(\tilde{x}, \tilde{y}) = \Gamma \arctan \left( \frac{m\tilde{y}}{\tilde{x} - \xi} \right)$

Symmetric profile w/o angle of attack ( $\alpha = 0, \sigma = 0$ )

The source  $q$  chosen here is distribution per unit length, which can be defined as

$$\lim_{\tilde{y} \rightarrow \pm 0} \frac{\partial \phi}{\partial \tilde{y}} = \pm q(\tilde{x}) = \pm \frac{\tau}{2} \frac{d\tilde{h}(\tilde{x})}{d\tilde{x}} = \pm \frac{\tau}{2} \tilde{h}'(\tilde{x})$$
$$\tilde{u}_x(\tilde{x}, \tilde{y}) = \frac{\tau}{2m\pi} \int_0^1 \frac{\tilde{h}'(\xi)(\tilde{x} - \xi)}{(\tilde{x} - \xi)^2 + m^2 \tilde{y}^2} d\xi$$
$$\tilde{u}_y(\tilde{x}, \tilde{y}) = \frac{\tau}{2\pi} \int_0^1 \frac{\tilde{h}'(\xi)m\tilde{y}}{(\tilde{x} - \xi)^2 + m^2 \tilde{y}^2} d\xi$$
$$c_p(\tilde{x}) = -2 \tilde{u}_x(\tilde{x}, \pm 0) =$$
$$= -\frac{\tau}{m\pi} \lim_{\varepsilon \rightarrow 0} \left[ \int_0^{\tilde{x}-\varepsilon} \frac{\tilde{h}'(\xi)}{\tilde{x} - \xi} d\xi + \int_{\tilde{x}+\varepsilon}^1 \frac{\tilde{h}'(\xi)}{\tilde{x} - \xi} d\xi \right]$$

In the far field, the specific shape of the profile has no influence on the velocity components in  $x$  and  $y$ . See eq. 10.37 and 10.40 for far-field behaviour.

**Cambered and flat plate in pitched flow (p.102)** ( $\tau = 0, \sigma \neq 0, \alpha \neq 0$ )  
 A thin plate which is cambered and a circulation term  $\gamma$  with a circulation per unit length is chosen here:

$$\gamma(\tilde{x}) = \frac{1}{\sqrt{\tilde{x}(1-\tilde{x})}} \left\{ C_1 - \frac{2}{m\pi} \int_0^1 [\theta_P(\xi) - \alpha] \frac{\sqrt{\xi(1-\xi)}}{\tilde{x} - \xi} d\xi \right\}$$

The so-called **Betz-Integral** describes this distribution. For flat plates with  $\Theta_P = 0$ , the equations simplify. The **Kutta Condition** postulates, that  $C_1$  is chosen such that  $u_x$  remains finite (at the end of the wing or profile).

$$\gamma(\tilde{x}) = -\frac{2\alpha}{m} \sqrt{\frac{1-\tilde{x}}{\tilde{x}}}, \quad C_1 = -\alpha/m$$

$$\tilde{u}_x(\tilde{x}, \pm 0) = \pm \frac{\alpha}{m} \sqrt{\frac{1-\tilde{x}}{\tilde{x}}}, \quad c_p(\tilde{x}, \pm 0) = -2\tilde{u}_x(\tilde{x}, \pm 0) = \pm \gamma(\tilde{x})$$

In the far field, the velocity components are however also depending on  $\alpha$ . See equations (10.53)

**Thin plate with camber (p.106):** Pure camber problem,  $c_p$  only depending on  $\sigma$

**Linearised superonic flows (p.108)**

A new scaling parameter is introduced. Instead of the Prandtl Factor, one uses the **Scaling Parameter**  $\lambda$ . The equation for the wave equation is now hyperbolic. Similar to the linear, one-dimensional wave equation, a general solution can be defined in the form of two invariants.

$$\lambda = \sqrt{Ma_\infty^2 - 1} = \frac{1}{\tan(\mu_\infty)}, \quad \frac{\partial^2 \phi}{\partial \tilde{x}^2} - \frac{\partial^2 \phi}{\partial (\lambda \tilde{y})^2} = 0$$

$$\phi(\tilde{x}, \tilde{y}) = F(\tilde{x} - \lambda \tilde{y}) + G(\tilde{x} + \lambda \tilde{y}) = F(\xi) + G(\eta)$$

**Symmetric profile w/o angle of attack (p.109)** The solutions are determined by the profile slope  $\tau \tilde{h}'(\tilde{x}) \approx \theta(\tilde{x})$ . Along the surface

$$\frac{|u|}{u_\infty} \approx 1 + \tilde{u}_x(\tilde{x}, 0) = 1 - \frac{\tau \tilde{h}'(\tilde{x})}{2\sqrt{Ma_\infty^2 - 1}} = 1 - \frac{\theta(\tilde{x})}{\sqrt{Ma_\infty^2 - 1}}$$

$$\frac{d}{d\tilde{x}} \left( \frac{|u|}{u_\infty} \right) = -\frac{\tau}{2\lambda} \tilde{h}''(\tilde{x}) \geq 0$$

The flow is always accelerated after the shock ( $|u| < u_\infty$ ) and reaches  $u_\infty$  at the highest point of the profile ( $\tilde{h}' = 0$ ) and is subsequently again accelerated until the trailing mach line at the end tip.

$$\tilde{u}_x = \mp \frac{\tilde{u}_y}{\sqrt{Ma_\infty^2 - 1}} \text{ (Ackeret), } c_p(\tilde{x}) = \frac{\tau}{\sqrt{Ma_\infty^2 - 1}} \tilde{h}'(\tilde{x})$$

As the  $c_p$  increases (overpressure) and then decreases (underpressure) due to the change in profile height, appearance of **wave drag** can be explained.

**Cambered plate with angle of attack (p.111)**

With a vanishing thickness  $\tau = 0$ , following equations can be computed:

$$c_p(\tilde{x}) = -2\tilde{u}_x(\tilde{x}, \pm 0) = \mp \frac{2}{\sqrt{Ma_\infty^2 - 1}} [\alpha - \theta_P(\tilde{x})]$$

Images show, that only for an inclination angle  $\alpha$ , lift can be achieved, independent of a camber with angle  $\theta$

**Lift and drag**

$$C_L = C_{py} \cos \alpha - C_{px} \sin \alpha \approx C_{py} - \alpha C_{px} \approx C_{py}$$

$$\approx \int_0^1 [c_p(\tilde{x}, -0) - c_p(\tilde{x}, +0)] d\tilde{x} = \frac{F_y}{\frac{\rho_\infty}{2} u_\infty^2 L b}$$

$$C_D = C_{py} \sin \alpha + C_{px} \cos \alpha \approx \alpha C_{py} + C_{px} \approx C_{px}$$

$$\approx \int_0^1 \left[ c_p(\tilde{x}, +0) \left( \sigma \tilde{y}'_C + \frac{\tau}{2} \tilde{h}' \right) - c_p(\tilde{x}, -0) \left( \sigma \tilde{y}' - \frac{\tau}{2} \tilde{h}' \right) \right] d\tilde{x}$$

For the problems, following simplifications can be made:

- Symmetric profile w/o angle of attack:  $C_L = 0, C_D = \tau \int_0^1 c_p(\tilde{x}) \tilde{h}'(\tilde{x}) d\tilde{x}$
- Flat plate with angle of attack:  $C_L = 2 \int c_p(\tilde{x}, -0) d\tilde{x}$ . For subsonic, there is no drag (D'Alembert). For supersonic, there is a wave drag.
- Camber problem without pitch and thickness:  $C_L$  same as for flat plate with angle of attack, but no Lift total.

## 11 Homentropic Flow around axisymmetric slender Bodies

Axisymmetric bodies with radial components are studied here with axial, radial, and azimuthal components.

## 12 Similarity Relations

## 13 Steady flows with friction and heat transport