1 General Considerations

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} \neq 0$$

- Wave propagation
- Convective flows with buoancy
- Flows with variable temperature, friction, sources of heat
- High speed flows with Mach numbers $Ma \geq 1$

Compressible flows can still be described through the continuum model and conservation laws. The assumption is also that the thermodynamic state of the fluid is in a local equilibrium.

Assumptions

- Length scale of flows large compared to molecular scales (mean free path λ)
- Length scale of flows small compared to the geometric scales (length L)
- Time scale τ_F of the flow long compared to the molecular process (relaxation) time constants τ_R

Description of the "Continuum" Flow State

- Three components of flow velocity $\underline{u}(\underline{x}, t)$
- The fluid density $\rho(\underline{x}, t)$
- The fluid pressure $p(\underline{x}, t)$
- The energy $e(\underline{x}, t)$

The required equations are the conservation laws for mass, momentum and energy together with suitable thermodynamic equations of state. With corresponding initial and boundary conditions, the evolution can then be computed.

2 Thermodynamic Relations

State Variables

- Density: $\rho = \rho(p, T)$
- Pressure: $p = p(\rho, T)$
- Temperature: $T = T(\rho, p)$
- Internal energy: $e = e(\rho, T)$ $[e] = J/kg$
- Enthalpy: $h = h(p, T)$
- Entropy: $s = s(\rho, T)$

Van der Waals Gas

$$(p + a\rho^2)\left(\frac{1}{\rho} - b\right) = RT$$

Incompressible Fluid

$$\rho = const. \neq \rho(p, T)$$

3 Conservation Laws for Continuum Flows

Mass Conservation

$$\frac{Dm}{Dt} = \frac{D}{Dt}\int_{\tilde{V}} \rho d\tilde{V} = 0 \text{ (material volume)}$$

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho(\mathbf{u}\cdot\mathbf{n}) dS = 0 \text{ (Eulerian Volume)}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \text{ (material volume / index)}$$

$$\frac{D\rho}{Dt} = -\rho \frac{\partial u_i}{\partial x_i} \text{ (Eulerian Volume / index)}$$

Momentum Conservation

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial}{\partial x_j}\sigma_{ij} + \rho f_i$$

$$\rho \frac{Du_i}{Dt} = \frac{\partial}{\partial x_j}\sigma_{ij} + \rho f_i$$

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \rho f_i$$

Energy Conservation

$$\rho \frac{D}{Dt}(e + \frac{1}{2}u_1^2) = \frac{\partial}{\partial x_j}(\sigma_{ij}u_i) + \rho f_i u_i - \frac{\partial q_i}{\partial x_i} + \rho q_v$$

$$\rho \frac{D}{Dt}(e + \frac{1}{2}u_1^2) = -\frac{\partial}{\partial x_i}(pu_i) + \frac{\partial}{\partial x_j}(\tau_{ij}u_i) + \rho f_i u_i - \frac{\partial q_i}{\partial x_i} + \rho q_v$$

$$\rho u_i \frac{Du_i}{Dt} = \rho \frac{D}{Dt}\left(\frac{u_i^2}{2}\right) = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial}{\partial x_j}\tau_{ij} + \rho f_i u_i \quad \rho \frac{De}{Dt} =$$
$$\rho \frac{D}{Dt}\left(e + \tfrac{1}{2}u_i^2\right) - \rho \frac{D}{Dt}\left(\frac{u_i^2}{2}\right) =$$
$$= -p\frac{\partial u_i}{\partial x_i} + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \rho q_v - \frac{\partial q_i}{\partial x_i}$$

Dissipation Function Φ

Insert $h = e + \frac{p}{\rho}$ to obtain Enthalpy equation, introduce $h_t = h + \frac{u_i^2}{2}$ and add kinetic energy (p. 15). For perfect gasses, $h = c_p T$, $q_i = -k\frac{dT}{dx}$, derive the temperature equation.

Entropy Equation

$$\rho T \frac{Ds}{Dt} = \Phi + \rho q_v - \frac{\partial q_i}{\partial x_i}$$

Vorticity Equation

$$\rho \frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho}\right) = (\vec{\omega}\cdot\nabla)\vec{u} + \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nabla \times \left(\frac{1}{\rho}\nabla \cdot \vec{\tau}\right)$$

Crocco Theorem (rewritten momentum equation using Enthalpy and Entropy)

$$\frac{\partial u}{\partial t} + \nabla\left(\frac{1}{2}\vec{u}^2 + h + \psi\right) = \vec{u} \times \vec{\omega} + T\nabla s + \frac{1}{\rho}\nabla \cdot \vec{\tau}$$

Compressible Bernoulli equation (integrate momentum equation law along particle path). Clasical not feasible

$$\rho\left(\frac{Dh_t}{Dt} - f_i u_i\right) = 0$$
$$f_i = -\frac{\partial \psi}{\partial x_i}$$
$$\psi \neq \psi(t)$$
$$\frac{D}{Dt}(h_t + \psi) = 0$$

Between 2 points along stream line

$$h_t + \psi = e + \frac{p}{\rho} + \frac{u_i^2}{2} + \psi = const.$$

4 Simplification Strategies (p.20)

- Unsteady $\rightarrow$ steady (no wave propagation) (no time dependence)
- 3D $\rightarrow$ 2D $\rightarrow$ quasi 1-D
- Viscous, heat conduction $\rightarrow$ inviscid, adiabatic (isentropic, homentropic)
- Subsonic $\rightarrow$ transonic $\rightarrow$ supersonic $\rightarrow$ hypersonic (Elliptic $\rightarrow$ hyperbolic)
- Full nonlinear $\rightarrow$ linearised (solve for small pertubations around predefined flow state unique solvable problem, separation of influencing factors facilitated)

5 Conservation Laws for Stream Tubes (p. 22)

Quasi 1D, separate for environment. Outer surface formed by instantaneous streamlines, no flow across boundaries. Inlet + outlet. Shape (t). For small enough A, flow properties can be treated constant in any cross section.

Mass Conservation

$$\int_1^2 \frac{\partial}{\partial t}\left[\rho(s,t)A(s,t)\right] ds + \rho_2 A_2 u_2 - \rho_1 A_1 u_1 = 0$$

$$\dot{m} = \rho A u = const.$$

Momentum Conservation

$$\int_1^2 \frac{\partial}{\partial t}\left[\rho(s,t)A(s,t)\right] ds + \rho_2 A_2 u_2 \vec{u}_2 - \rho_1 A_1 u_1 \vec{u}_1 =$$

$$= -p_2 A_2 \vec{n}_2 + p_1 A_1 \vec{n}_1 + F_\tau|_1^2 + F_S$$

Steady, frictionless

$$\rho_2 u_2^2 + p_2 = \rho_1 u_1^2 + p_1$$

Energy Conservation (p.20)
Steady, frictionless

$$e_2 + \frac{u_2^2}{2} + \frac{p_2}{\rho_2} = e_1 + \frac{u_1^2}{2} + \frac{p_1}{\rho_1}$$

Enthalpy subsitution $h = e + \frac{p}{\rho} \rightarrow h_{t1} = h_{t2} = const.$

6 Steady one-dimensional Flow without Friction and Heat (p. 25)

Assumptions:

- No friction (inviscid)
- No heat source or transport
- No flow through mantle
- Perfect gas

$$Ma = \frac{u}{a}$$

$$a^2 = \gamma R T$$

Stagnation properties, when $u = 0$ (Ruhegrösse), subscript 0:

$$\frac{h_0}{h} = \frac{T_0}{T} = \left(\frac{a_0^2}{a^2}\right) = 1 + \frac{\gamma-1}{2}Ma^2$$

Isentropic flow (p.26):

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left[1 + \frac{\gamma-1}{2}Ma^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma-1}} = \left[1 + \frac{\gamma-1}{2}Ma^2\right]^{\frac{1}{\gamma-1}}$$

When $Ma < 0.3$, density changes $< 4.5\%$: Assumption is: incompressible. The critical state is then ($Ma = 1$), *superscript* *

$$\frac{h^*}{h_0} = \frac{T^*}{T_0} = \left(\frac{a^{*2}}{a_0^2}\right) = \left[1 + \frac{\gamma-1}{2}\right]^{-1} = \frac{2}{\gamma+1} = 0.8333(\gamma = 1.4)$$

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = 0.5283(\gamma = 1.4)$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} = 0.6339(\gamma = 1.4)$$

Critical Ma^* (isentropic flow stays limited when $Ma \rightarrow \infty$). The flow velocity stays finite even if Ma goes to infinity:

$$Ma^* = \frac{u}{a^*} = \frac{u}{a(Ma=1)} = \frac{u}{a}\frac{a}{a_0}\frac{a_0}{a^*}$$

$$= Ma\sqrt{\frac{T}{T_0}}\sqrt{\frac{T_0}{T^*}} = \sqrt{\frac{\frac{\gamma+1}{2}Ma^2}{1 + \frac{\gamma-1}{2}Ma^2}}$$

$$Ma^* \rightarrow \sqrt{\frac{\gamma+1}{\gamma-1}}\ (Ma \rightarrow \infty) = 2.4495\ (\gamma = 1.4)$$

Area velocity relation

A velocity increase $\rightarrow$ density decrease (always). If $Ma << 1$, then the density changes are small compared to the velocity changes. A small velocity increase at $Ma >> 1$ will lead to large density changes.

$$Ma^2 \frac{1}{u}\frac{du}{dx} = -\frac{1}{\rho}\frac{d\rho}{dx} \quad \textbf{(Mach-density relation)}$$

$$\left(Ma^2 - 1\right)\frac{1}{u}\frac{du}{dx} = \frac{1}{A}\frac{dA}{dx} \quad \textbf{(Mach-Area relation)}$$

If $Ma < 1$, then an area increase will result in a velocity reduction. If $Ma > 1$, then opposite applies. If $Ma = 1$, then a change in Area A has no effect (chocked flow)

Stationary normal shock

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = 1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right) = \frac{1}{Ma^{*2}}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2 - 1\right)$$

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2 - 1\right)\right]\left[1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)\right]$$

$$\frac{\Delta s}{R} = \frac{1}{\gamma-1}\left[\ln\left(\frac{p_2}{p_1}\right) - \gamma\ln\left(\frac{\rho_2}{\rho_1}\right)\right] =$$

$$\frac{1}{\gamma-1}\left\{\left[1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2 - 1\right)\right]\left[1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)\right]\right\}$$

$h_{01} = h_{02}$, $T_{01} = T_{02}$, and total enthalpy conserved (however stagnation pressure not constant, $p_{01} \neq p_{02}$):

$$\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{01}} = \frac{p_2}{p_1}\left(\frac{T_{02}}{T_2}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{T_1}{T_{01}}\right)^{\frac{\gamma}{\gamma-1}} =$$

$$\left[1 + \frac{2\gamma}{\gamma+1}(Ma_1^2 - 1)\right]^{\frac{1}{\gamma-1}}\left[1 - \frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)\right]^{\frac{-\gamma}{\gamma-1}}$$

As s increases, u decreases. Ma_2 is always < 1, when $Ma_1 \rightarrow \infty$:

$$Ma_2 \rightarrow \sqrt{\frac{\gamma-1}{2\gamma}} = 0.38\ (\gamma = 1.4)$$

$$Ma_2^2 = \left(\frac{u_2}{a_2}\right)^2 = \left(\frac{u_2}{u_1}\right)^2\left(\frac{u_1}{a_1}\right)^2\left(\frac{a_1}{a_2}\right)^2 = \left(\frac{u_2}{u_1}\right)^2 Ma_1^2\left(\frac{T_1}{T_2}\right)$$

$$Ma_2 = \sqrt{\frac{1 + \frac{\gamma-1}{\gamma+1}\left(Ma_1^2 - 1\right)}{1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2 - 1\right)}}$$

A weak shock occurs at Ma_1 close to one. See page 31 for equation

Rankine Hugoniot (p.32) - Adiabatic Shock (no Ma dependency)

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma\left(\frac{\rho_2}{\rho_1} - 1\right)}{\gamma + 1 - (\gamma-1)\frac{\rho_2}{\rho_1}}$$

Moving Shock Wave (p.33)
Switch to reference frame (from frame fixed with moving shock front into a frame moving with shock)

$$u_1 \hat{=} u_s\ ,\ p_1 \hat{=} p_0\ ,\ \rho_1 \hat{=} \rho_0$$

Flow behind

$$u_2 \hat{=} u_s - u_d\ ,\ p_2 \hat{=} p_d\ ,\ \rho_2 \hat{=} \rho_d$$

Shock u_d

$$u_d = u_s - u_2 = u_1 - u_2 = u_1\left(1 - \frac{u_2}{u_1}\right) = u_1\frac{2}{\gamma+1}\left(1 - \frac{1}{Ma_1^2}\right)$$

$$Ma_d = \frac{u_d}{a_d} = \frac{u_1 - u_2}{a_d} = \frac{u_1}{a_1}\frac{a_1}{a_d}\left(1 - \frac{u_2}{u_1}\right) = Ma_1\sqrt{\frac{T_1}{T_2}}\left(1 - \frac{u_2}{u_1}\right)$$

$$u_d = \frac{a_0}{\gamma}\frac{\frac{\Delta p}{p_0}}{\sqrt{1 + \frac{\gamma+1}{2\gamma}\frac{\Delta p}{p_0}}}\ (a_1 \hat{=} a_0),\ \ Ma_s = \frac{u_s}{a_0} = \sqrt{1 + \frac{\gamma+1}{2\gamma}\frac{\Delta p}{p_0}}$$

Pressure increase

$$\frac{\Delta p}{p_0} = \frac{p_d - p_0}{p_0} = \frac{2\gamma}{\gamma+1}\left(Ma_S^2 - 1\right),\ \left[Ma_1 = \frac{u_1}{a_1} = \frac{u_s}{a_s} = Ma_s\right]$$

The ratio (Pressure increase) has an asymptotic limit. For high Ma_s, the function becomes limited. $\frac{u_s}{u_d} \rightarrow \frac{\gamma+1}{2}$ (for high pressure differences)

Detonations ($Ma_2 > 1$) and Deflagrations ($Ma_2 < 1$) (p.36, ZND)
Assumption: Ignore adiabatic flow, include however heat release

Rayleigh line: $\frac{p_1}{p_0} = 1 + \frac{\rho_0}{p_0}u_0^2 - \frac{\rho_0}{p_0}\frac{\rho_1}{\rho_0}u_1^2 = 1 + \gamma Ma_0^2\left(1 - \frac{\rho_0}{\rho_1}\right)$,

Rankine Hugeniot with heat: $\frac{p_2}{p_0} = \frac{(\gamma+1)-(\gamma-1)\frac{\rho_0}{\rho_2}+2\gamma\hat{q}}{(\gamma+1)\frac{\rho_0}{\rho_2}-(\gamma-1)}$, $\hat{q} = \frac{q_{heat}}{c_p T_1}$, This gives us p_1 and p_2, the pressure of the shockwave before the combustion and downstream after the combustion layer

Chapman-Jouget Point (p.37)
...is the intersection where $Ma = 1$, so $Ma_2 = 1 = Ma_0 \sqrt{\frac{\rho_0}{\rho_2}} \sqrt{\frac{\rho_0}{\rho_2}}$
The limiting case for shock cycle (Rayleigh tangent to Hugoniot Line):

$$\frac{\rho_0}{\rho_2}|_c = \frac{u_2}{u_0}|_c = \frac{\gamma Ma_0^2 + 1}{Ma_0^2(\gamma+1)}$$

Behind the shock, the flow is subsonic $\leftrightarrow$ strong detonation. There is a weak deflagration if the density ratio $\frac{\rho_1}{\rho_2} >> 1$. The reaction front propagates at subsonic speed. Weak detonation: flow remains supersonic (not explainable through ZND)

Laval Nozzle (p. 39)
Varying cross-section:

$$\frac{p(x)}{p_0} = \left[1 + \frac{\gamma-1}{2} Ma^2(x)\right]^{\frac{-\gamma}{\gamma-1}}$$
$$\frac{A^*}{A(x)} = Ma(x)\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1} Ma^2(x)\right]$$
$$u(x) = Ma(x) a_0 \frac{a(x)}{a_0} = Ma(x) a_0 \sqrt{\frac{T(x)}{T_0}} = \frac{a_0 \cdot Ma(x)}{\sqrt{1 + \frac{\gamma-1}{2} Ma^2(x)}}$$
$$u^* = a^* \text{ , if } Ma^* = 1$$

In order to increase the Ma_{exit}, reduce the area ration (tune A^*). Different flow regimes are shown on p. 41. A variable exit area is in practice not possible

7 Unsteady one-dimensional Flows

Wave equation for small perturbations Assuming small pertubations around equilibrium state with first order pertubations will result into following differential equation (enthalpy):

$$\frac{\partial p'}{\partial t} - a_0^2 \frac{\partial \rho'}{\partial t} = 0 \Longleftrightarrow p' = a_0^2 \rho'$$
$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0 \text{ (mass eq.)}$$
$$\frac{\partial u'}{\partial t} + \frac{a_0^2}{\rho_0} \frac{\partial \rho'}{\partial x} = 0 \text{ (momentum eq.)}$$

Through cross-differentiation (elimination of terms), one arrives at the d'Alembert solution:

$$u'(x,t) = a_0[F(x - a_0 t) + G(x + a_0 t)]$$
$$\rho'(x,t) = \rho_0[F(x - a_0 t) + G(x + a_0 t)]$$

Through characteristics one defines left and right propagatiting waves, $F(\eta)$ and $G(\xi)$. The characteristics are in this case straight lines. Initial conditions are at $t = 0$, boundary conditions are at $x = b.c.$

Method of characteristics for nonlinear wave propagation Here, no small pertubations are assumed, while assuming homentropic flow ($s = const.$). The Riemann invariants (characteristics) are not straight anymore, and can be curved. Disturbances are no longer constant, but have a flow dependent value. Given a and u are given along a curve C, find where it intersects with two characteristics, which cross at point Q. (See p. 48)

Piston Motion in tube (example for unsteady one-dimensional motion):

- Boundary Condition: At $x = x_p(t)$, $u(x = x_p, t) = u_p(t)$
- How to solve: Left propagating wave from rest state, intersects P at $u = u_p$. The characterisitic with $\eta = const$ which then can intersect the other characteristic with $\xi = const.$ yields point Q
- $x = \left[a_0 + \frac{\gamma+1}{2} u_p(\tau)\right](t - \tau) + x_p(\tau)$

Simple expansion waves
In the case for the piston moving to the left, the characteristics are limited by two factors:

- $x = a_0 t$: Initially, at $t = 0$, the characteristic is maximum and can only be as steep as a_0
- $u_p = -U$: The piston motion can only have a max. velocity at its endpoints ($x_p = -Ut$ and Ut)
- This gives an area of solutions, which is called a "centered fan"

$$Ma = \frac{|U|}{a_0}\left[1 - \frac{\gamma-1}{2}\frac{|U|}{a_0}\right]^{-1}, \frac{\rho}{\rho_0} = \left[1 - \frac{\gamma-1}{2}\frac{|U|}{a_0}\right]^{\frac{2}{\gamma-1}}$$
$$\frac{p}{p_0} = \left[1 - \frac{\gamma-1}{2}\frac{|U|}{a_0}\right]^{\frac{2\gamma}{\gamma-1}}$$

Simple Compression Waves, see p. 54, explained for increasing velocity to the right

Reflections
Reflection from solid wall: $G = -F$ if boundary moves with velocity 0

Reflection from free boundary (contact surface), p.56: The ratio α is the impendance, and is the ratio of both a of two regions

Reflection from an open end with outflow, p.58: At an orifice (a = outer, 0 = stagnation), the characteristics are:

$$G = F - \frac{4}{\gamma - 1} a(p_a)$$

The speed of sound is computed via the isentropic relations:

$$\frac{a_a}{a_0} = \sqrt{\frac{T_a}{T_0}} = \left(\frac{p_a}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}$$

8 Two-dimensional steady supersonic Flow

An oblique shock wave forms around a body with a sharp tip or a long a wall with a sudden profile change (at the turning point). Two parameters describe the problem. The inflow Mach number Ma_1 and turning angle θ. The velocity can be described by a component normal to the shock front and tangential to the shock front u_n and u_t. The downstream and upstream pressures p_1 and p_2 are assumed constant.

$$u_{2t} = u_{1t} = u_t$$
$$u_{1n} = u_1 \sin\beta \text{ , } u_{2n} = u_2 \sin(\beta - \theta)$$
$$\frac{u_{2n}}{u_{1n}} = \frac{(\gamma-1)Ma_1^2 \sin^2\beta + 2}{(\gamma+1)Ma_1^2 \sin^2\beta}$$
$$\frac{u_{2t}}{u_{1t}} = \frac{\tan(\beta-\theta)}{\tan\beta} = \frac{1 - \tan\theta \cot\beta}{1 + \tan\theta\tan\beta} = \frac{1 - (\tan\theta / \tan\beta)}{1 + \tan\theta\tan\beta}$$
$$\tan\theta = \frac{1 - \frac{u_{2n}}{u_{1n}}}{\cot\beta + (\frac{u_{2n}}{u_{1n}})\tan\beta} = \frac{\left(\sin^2\beta - \frac{1}{Ma_1^2}\right)\sqrt{1 - \sin^2\beta}}{\sin\beta\left(\frac{\gamma+1}{2} - \sin^2\beta + \frac{1}{Ma_1^2}\right)}$$

The equation for θ is an implicit function for β, whereas there are two solutions for it. Normally, θ (geometry) and incident Ma_1 are given. For $Ma_1 > 1$, β is in the range of $[\arcsin(1/Ma_1), 90°]$. For a given Ma_1, there is a maximum turning angle θ_{max} beyond which the shock is not anymore at the turning point. The shock is then "detached" (set $Ma \to \infty$ in equation for θ).

- Strong and weak shock: The maxima of all shock angle for oblique shocks can be connected. Higher Mach angles β occur with strong shocks
- Separation between $Ma_2 < 1$ and $Ma_2 > 1$
- For the changes in pressure, density and other thermodynamic properties, the equations for the normal shock can be used with a new "effective Mach number" $Ma_{1,new} = Ma_1 \sin\beta$

$$u_{2n} = u_2 \sin(\beta - \theta)$$
$$Ma_2^2 \sin^2(\beta - \theta) = \frac{1 + \frac{\gamma-1}{\gamma+1}(Ma_1^2 \sin^2\beta - 1)}{1 + \frac{2\gamma}{\gamma+1}(Ma_1^2 \sin^2\beta - 1)}$$

- Small turning angles θ: In the limit of $\theta \to 0$, β becomes the Mach angle $\mu = \lim_{\theta\to 0}\beta = \arcsin\frac{1}{Ma_1}$
- Hypersonic flow $Ma_1 >> 1$: Shock angle and turning angle linearly dependent on gas property γ alone, $\sin\beta \approx \beta = \frac{\gamma+1}{2}\theta$

Continuous turning of supersonic flows
When the surface is moving continuously, the model can be adapted.

$$\frac{dp}{p} = -\frac{\gamma Ma}{1+\frac{\gamma-1}{2}Ma^2}dMa$$
$$d\theta = -\frac{\sqrt{Ma^2-1}}{1+\frac{\gamma-1}{2}Ma^2}\frac{dMa}{Ma}$$

These equations describe the "Prandtl-Meyer Compression". In case the flow direction changes positively ($d\theta > 0$), the Mach number decreases ($dMa < 0$), and the pressure as well ($dp < 0$).

$$\nu(Ma) = \sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\sqrt{\frac{\gamma-1}{\gamma+1}(Ma^2-1)} - \arctan\sqrt{\mathrm{Ma}^2-1}$$
$$\nu_{max} = \nu(Ma\to\infty) = \frac{\pi}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}}-1\right) \hat{=} 130.5^\circ \ (\gamma=1.4)$$

The above Prandtl-Meyer function outputs degrees, which couple supersonic turns and change in Mach numbers.

$$\Delta\theta = \theta_2 - \theta_1 = -\nu(Ma_2) + \nu(Ma_1)$$

An oblique schock occurs at a distanced point P away from the wall. A "Prandtl-Meyer Expansion" occurse if the turning angle is negative. The flow accelerates, and Mach lines diverge. If the flow turns around a point, all lines are going along a center.

Reflection and crossing of waves
Use same equations as before. Here, following things must be considered:

- Wedge with bounding wall: Reflections back from wall create additional shocks until $\theta < \theta_{max}$ (function of β). The Mach numbers decrease
- Wedge without bounding wall: Point P is where fluid expands (reflected by the free jet boundary)
- Walls which converge: For two different turning angles θ, two oblique shocks meet at point P:
  - $\theta_1 - \theta_5 = \theta_4 - \theta_2$
  - $p_4(Ma_2, \theta_4, p_2) = p_5(Ma_3, \theta_5, p_3)$
- Prismatic wing: First a oblique shock, then fans, then shock due to turning back flow to inflow direction. Even without friction, a drag is introduced

Detached Shocks Normally, the oblique shock occurs for $\theta < \theta_{max}$. For larger angles, the shock detaches and shows a near hyperbolic shape. See p.78 for details when it comes to decreasing Ma.

Supersonic nozzle exit flows
Supersonic nozzle flow into stagnant environment.

- Under-expanded scenario ($p_{env} < p_{noz}$): First PM-expansion due to reduction of pressure. However later on again goes back (rocket plume). Interacting incoming and reflecting expansion fans create the going back.
- Over-expanded scenario ($p_{env} > p_{noz}$): The turning angle β is much smaller, the plume is impinged

9 Method Characteristics for planar homentropic supersonic Flows

For nonlinear wave propagation in (x,t)-space, one can compute flow fields in steady, two-dimensional homentropic flows. With the Crocco theorem, and the assumption of steady, inviscid, isoenergetic, adiabit flow, there is no vorticity. Introducing the velocity potential, we receive equations for the speed of sound.

Transformation of equations (p.81)
Mach lines are no longer straight lines over an extended region, but are treated as characterisitic curves on which the Riemann invariants remain constant. The characteristics become dependent on the local Mach angle $\mu = \mu(\nu)$ and the Mach number Ma, determined by the Prandtl-Meyer Function. The velocity magnitute is then given by the thermodynamic state p, ρ, T

$$F = \nu + \theta = const. \text{ and } G = \nu - \theta = const.$$

Initial and Boundary Value Problems (p.85)

- Initial Value Problem: A curve C is given, as well as the Riemann Invariants. This gives us both Mach numbers and their flow directions in two points 1 and 2. In point 3, both intersect, and this gives the local ν_3 and θ_3, meaning the local Mach number:

$$u_3 = Ma_3 a_3 = Ma_3 a_0 \frac{a_3}{a_0} = Ma_3 a_0 \sqrt{\frac{T_3}{T_0}}$$
$$a_0 = \sqrt{\gamma R T_0}, \quad \frac{T_3}{T_0} = \left(1 + \frac{\gamma-1}{2}Ma_3^2\right)^{-1}$$

10 Homentropic Flow around slender Wings

Instead of using methods of characteristics, now a new methodoligy is used. It uses also linearisations, which make the problems easier to handle and is valid for supersonic and subsonic flows.

Initial Considerations:
A function describes the wing profile with a camber line ($y_c(x)$) and a thickness distribution ($h(x)$), both depending on x. The boundary condition is that the flows are tangential to the contour, and that in the far field the stream velocity is free stream velocity u_∞.

To simplify the problems, the coordinates are simplified as follows:

$$\tilde{x} = \frac{x}{L},\ \tilde{y} = \frac{y}{L},\ \sigma = \frac{c_{max}}{L},\ \tau = \frac{h_{max}}{L},\ \tilde{h}(\tilde{x}) = \frac{1}{h_{max}}h\left(\frac{x}{L}\right)$$
$$\frac{y_p}{L} = \sigma\tilde{y}_C(\tilde{x}) \pm \frac{\tau}{2}\tilde{h}(\tilde{x}),\ \tilde{y}_C(\tilde{x}) = \frac{1}{c_{max}}y_C\left(\frac{x}{L}\right)$$

Solving the equations for c_p is not trivial, as they are non-linear. The parameters, on which the solution depends on are:

- Thickness ratio τ
- Camber ratio σ
- Angle of attack α
- Free stream Mach number Ma_∞
- Ratio of specific heats γ

Linearized Theory
The equations are simplified, assuming slender profiles (τ, σ, α all $\ll 1$). This implies h and y_c remain small. Also, we assume small angles: $\alpha \approx \sin\alpha$

$$c_p = \frac{2}{\gamma Ma_\infty^2}\left\{\left[1-(\gamma-1)Ma_\infty^2\tilde{u}_x\right]^{\frac{\gamma-1}{\gamma}} - 1\right\}$$
$$= -2\tilde{u}_x(\tilde{x},\tilde{y}) = -2\frac{\partial\phi}{\partial\tilde{x}}$$

The pressure distribution is evaluated along the body surface, which in the linearised framework is $\tilde{y} = 0$

Linearised subsonic flows ($Ma\infty < 1$)
With the so-called **Prandtl Factor**, one can find a simple Laplace equation in the scaled coordinates $(\tilde{x}, m\tilde{y})$:

$$m = \sqrt{1 - Ma_\infty^2}$$
$$\frac{\partial^2\phi}{\partial\tilde{x}^2} + \frac{\partial^2\phi}{\partial(m\tilde{y})} = 0$$

Solutions can be found by superposition of fundamental solutions:

- Source/Sink of magnitude Q in $(\xi, 0)$:
  $\phi(\tilde{x},\tilde{y}) = Q\ln\sqrt{(\tilde{x}-\xi)^2 + m^2\tilde{y}^2}$
- Vortex with circulation Γ in $(\xi, 0)$:
  $\phi(\tilde{x},\tilde{y}) = \Gamma\arctan\left(\frac{m\tilde{y}}{\tilde{x}-\xi}\right)$

Symmetric profile w/o angle of attack ($\alpha = 0$, $\sigma = 0$)
The source q chosen here is distribution per unit length, which can be defined as

$$\lim_{\tilde{y}\to\pm 0}\frac{\partial\phi}{\partial\tilde{y}} = \pm q(\tilde{x}) = \pm\frac{\tau}{2}\frac{d\tilde{h}(\tilde{x})}{d\tilde{x}} = \pm\frac{\tau}{2}\tilde{h}'(\tilde{x})$$
$$\tilde{u}_x(\tilde{x},\tilde{y}) = \frac{\tau}{2m\pi}\int_0^1\frac{\tilde{h}'(\xi)(\tilde{x}-\xi)}{(\tilde{x}-\xi)^2+m^2\tilde{y}^2}d\xi$$
$$\tilde{u}_y(\tilde{x},\tilde{y}) = \frac{\tau}{2\pi}\int_0^1\frac{\tilde{h}'(\xi)m\tilde{y}}{(\tilde{x}-\xi)^2+m^2\tilde{y}^2}d\xi$$
$$c_p(\tilde{x}) = -2\tilde{u}_x(\tilde{x},\pm 0) =$$
$$= -\frac{\tau}{m\pi}\lim_{\varepsilon\to 0}\left[\int_0^{\tilde{x}-\varepsilon}\frac{\tilde{h}'(\xi)}{\tilde{x}-\xi}d\xi + \int_{\tilde{x}+\varepsilon}^1\frac{\tilde{h}'(\xi)}{\tilde{x}-\xi}d\xi\right]$$

In the far field, the specific shape of the profile has no influence on the velocity components in x and y. See eq. 10.37 and 10.40 for far-field behaviour.

Cambered and flat plate in pitched flow (p.102) ($\tau = 0, \sigma \neq 0, \alpha \neq 0$)
A think plate which is cambered and a circulation term γ with a circulation per unit length is chosen here:

$$\gamma(\tilde{x}) = \frac{1}{\sqrt{\tilde{x}(1-\tilde{x})}} \left\{ C_1 - \frac{2}{m\pi} \int_0^1 [\theta_P(\xi) - \alpha] \frac{\sqrt{\xi(1-\xi)}}{\tilde{x}-\xi} d\xi \right\}$$

The so-called **Betz-Integral** describes this distribution. For flat plates with $\Theta_P = 0$, the equations simplify. The **Kutta Condition** postulates, that C_1 is chosen such that u_x remains finite (at the end of the wing or profile).

$$\gamma(\tilde{x}) = -\frac{2\alpha}{m}\sqrt{\frac{1-\tilde{x}}{\tilde{x}}} \ , \ C_1 = -\alpha/m$$
$$\tilde{u}_x(\tilde{x}, \pm 0) = \pm \frac{\alpha}{m}\sqrt{\frac{1-\tilde{x}}{\tilde{x}}} \ , c_p(\tilde{x}, \pm 0) = -2\tilde{u}_x(\tilde{x}, \pm 0) = \pm\gamma(\tilde{x})$$

In the far field, the velocity components are however also depending on α. See equations (10.53)

Thin plate with camber (p.106): Pure camber problem, c_p only depending on σ

Linearised superonic flows (p.108)
A new scaling parameter is introduced. Instead of the Prandtl Factor, one uses the **Scaling Parameter** λ. The equation for the wave equation is now hyperbolic. Similar to the linear, one-dimensional wave equation, a general solution can be defined in the form of two invariants.

$$\lambda = \sqrt{Ma_\infty^2 - 1} = \frac{1}{\tan(\mu_\infty)}, \ \frac{\partial^2\phi}{\partial\tilde{x}^2} - \frac{\partial^2\phi}{\partial(\lambda\tilde{y})^2} = 0$$
$$\phi(\tilde{x}, \tilde{y}) = F(\tilde{x} - \lambda y) + G(\tilde{x} + \lambda\tilde{y}) = F(\xi) + G(\eta)$$

Symmetric profile w/o angle of attack (p.109) The solutions are deteremined by the profile slope $\tau\tilde{h}'(\tilde{x}) \approx \theta(\tilde{x})$. Along the surface

$$\frac{|u|}{u_\infty} \approx 1 + \tilde{u}_x(\tilde{x}, 0) = 1 - \frac{\tau\tilde{h}'(\tilde{x})}{2\sqrt{Ma_\infty^2 - 1}} = 1 - \frac{\theta(\tilde{x})}{\sqrt{Ma_\infty^2 - 1}}$$
$$\frac{d}{d\tilde{x}}\left(\frac{|u|}{u_\infty}\right) = -\frac{\tau}{2\lambda}\tilde{h}''(\tilde{x}) \geq 0$$

The flow is always accelerated after the shock ($|u| < u_\infty$) and reaches u_∞ at the highest point of the profile ($\tilde{h}' = 0$) and is subsequently again accelerated until the trailing mach line at the end tip.

$$\tilde{u}_x = \mp \frac{\tilde{u}_y}{\sqrt{Ma_\infty^2 - 1}} \ \textbf{(Ackeret)}, \ c_p(\tilde{x}) = \frac{\tau}{\sqrt{Ma_\infty^2 - 1}}\tilde{h}'(\tilde{x})$$

As the c_p increases (overpressure) and then decreases (underpressure) due to the change in profile height, appearance of **wave drag** can be explained.

Cambered plate with angle of attack (p.111)
With a vanishing thickness $\tau = 0$, following equations can be computed:

$$c_p(\tilde{x}) = -2\tilde{u}_x(\tilde{x}, \pm 0) = \mp \frac{2}{\sqrt{Ma_\infty^2 - 1}} [\alpha - \theta_P(\tilde{x})]$$

Images show, that only for an inclination angle α, lift can be achieved, independent of a camber with angle θ

Lift and drag (p.114)

$$C_L = C_{py}\cos\alpha - C_{px}\sin\alpha \approx C_{py} - \alpha C_{px} \approx C_{py}$$
$$\approx \int_0^1 [c_p(\tilde{x}, -0) - c_p(\tilde{x}, +0)] \, d\tilde{x} = \frac{F_y}{\frac{\rho_\infty}{2} u_\infty^2 Lb}$$
$$C_D = C_{py}\sin\alpha + C_{px}\cos\alpha \approx \alpha C_{py} + C_{px} \approx C_{px}$$
$$\approx \int_0^1 \left[c_p(\tilde{x}, +0)\left(\sigma\tilde{y}_C' + \frac{\tau}{2}\tilde{h}'\right) - c_p(\tilde{x}, -0)\left(\sigma\tilde{y}' - \frac{\tau}{2}\tilde{h}'\right)\right] d\tilde{x}$$

For the problems, following simplifications can be made:

- Symmetric profile w/o angle of attack: $C_L = 0$, $C_D = \tau\int_0^1 c_p(\tilde{x})\tilde{h}'(\tilde{x})d\tilde{x}$
- Flat plate with angle of attack: $C_L = 2\int c_p(\tilde{x}, -0)d\tilde{x}$. For subsonic, there is no drag (D'Alembert). For supersonic, there is a wave drag.
- Camber problem without pitch and thickness: C_L same as for flat plate with angle of attack, but no Lift total.

11 Homentropic Flow around axisymmetric slender Bodies

Axisymmetric bodies with radial components are studied here with axial, radial, and azimuthal (in axisymmetric problems, the azimuthal component is not effecting the flow) components. To simplify the problems, the coordinates are defines as follows:

$$\Phi(x, r) = u_\infty L\,[\tilde{x} + \phi(\tilde{x}, \tilde{r})] \ \ , \ \tilde{x} = \frac{x}{L} \ , \ \tilde{r} = \frac{r}{L}$$
$$u_x = \frac{\partial\Phi}{\partial x} \ , \ u_r = \frac{\partial\Phi}{\partial r} \ , \ \tilde{R}(\tilde{x}) = \frac{R(x)}{L} \ , \ A(x) = \pi R(x)^2$$
$$c_p = \frac{p - p_\infty}{\frac{\rho_\infty}{2} u_\infty^2} = -2\tilde{u}_x - \tilde{u}_r^2$$

Linearised axisymmetric subsonic flow (p.123)

$$m^2 = 1 - Ma_\infty^2$$
$$\phi(\tilde{x}, \tilde{r}) = -\frac{1}{4\pi}\int_0^1 \frac{\tilde{A}'(\xi)}{\sqrt{(\tilde{x} - \xi)^2 + m^2\tilde{r}^2}} d\xi$$

12 Similarity Relations

13 Steady flows with friction and heat transport