Simplifying Mathematical Notations in Recurrence Equations

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1 Introduction

In the book "Introduction to Algorithms" written by CLRS, section 4.4, the following recurrence equation is given:

$$T(n) = \begin{cases} \Theta(1), & \text{if } n = 1 \\ 3T(\lfloor n/4 \rfloor) + \Theta(n^2), & \text{otherwise} \end{cases} \tag{1}$$

which later on was simplified as

$$T(n) = \begin{cases} \Theta(1), & \text{if } n = 1 \\ 3T(\lfloor n/4 \rfloor) + cn^2, & \text{otherwise} \end{cases} \tag{2}$$

we will argue that even if $\Theta(n^2)$ is a set, and $c * n^2$ is a scalar and they generally cannot be swapped one with the other, in this case it can be done and will help the equation to be more tractable.

2 Proof

Let's consider the general case only, so we have

$$3T(\lfloor n/4 \rfloor) + \Theta(n^2)$$

which expanded is

$$3T(\lfloor n/4 \rfloor) + f(n) \text{ where } f(n) \in \Theta(n^2)$$

$f(n) \in \Theta(n^2)$ means that
$\exists\ c_1 > 0, c_2 > 0, n_0 > 0$ such that $c_1 * n^2 \leq f(n) \leq c_2 * n^2\ \forall n \geq n_0$

So we have

$$3T(\lfloor n/4 \rfloor) + c_1 * n^2 \leq 3T(\lfloor n/4 \rfloor) + f(n) \leq 3T(\lfloor n/4 \rfloor) + c_2 * n^2 \ \forall n \geq n_0$$
$$=$$
$$\underbrace{3T(\lfloor n/4 \rfloor) + c_1 * n^2}_{\alpha} \leq T(n) \leq \underbrace{3T(\lfloor n/4 \rfloor) + c_2 * n^2}_{\beta} \ \forall n \geq n_0$$

So if we resolve α only, we'll have a lower bound for $T(n)$. If we resolve β only, we'll have an upper bound for $T(n)$.

But α and β are asymptotically equivalent, that means that $T(n)$ is asymptotycally equivalent to α and β too.

So we just need to resolve either α or β to obtain the asymptotic value of $T(n)$, so we can just resolve for a generic c :

$$T(n) = \begin{cases} \Theta(1), & \text{if } n = 1 \\ 3T(\lfloor n/4 \rfloor) + cn^2, & \text{otherwise} \end{cases} \tag{3}$$

since using a generic constant won't change the asymptotic value.

3 Conclusions

The concept described in this document and the relative proof can be easily extended to any similar recurrence, and generalized.