May 6, 2022

SMOOTH SIMPLICIAL SETS AND UNIVERSAL CHERN-WEIL HOMOMORPHISM

YASHA SAVELYEV

ABSTRACT. We start by introducing and developing a basic geometric-categorical notion of a smooth simplicial set. Loosely, this is to diffeological spaces of Chen-Souriau as simplicial sets are to spaces. Given a Frechet Lie group G, we construct abstract classifying spaces $BG^{\mathcal{U}}$ as smooth Kan complexes. Here the index $\mathcal{U}$ is a chosen Grothendieck universe of a certain type. When G in addition has the homotopy type of a CW complex, there is a homotopy equivalence $BG \simeq |BG^{\mathcal{U}}|$, where BG is the usual Milnor classifying space. This leads to our main application that for G a Frechet Lie group, having the homotopy type of a CW complex, there is a universal Chern-Weil homomorphism:

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG, \mathbb{R}),$$

satisfying naturality, and generalizing the classical Chern-Weil homomorphism for compact Lie groups. We give an example in the form of an application to the study of the infinite dimensional symplectomorphism group of a symplectic manifold.

CONTENTS

1. Introduction

We introduce the notion of a smooth simplicial set, which is loosely an analogue in simplicial sets of diffeological spaces of Chen-Souriau, with the latter perhaps a most basic notion of a "smooth space". The language of smooth simplicial sets turn out to be a powerful tool to resolve the problem of the construction of the universal Chern-Weil homomorphism for Frechet Lie groups.

One problem of topology is the construction of a "smooth structure" on the classifying space BG of a Frechet Lie group G. There are specific requirements for what such a notion of a smooth structure should entail. At the very least we hope to be able to carry out Chern-Weil theory universally on BG. That is we want a differential geometric construction of the Chern-Weil homomorphism:

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG, \mathbb{R}),$$

with $\mathbb{R}[\mathfrak{g}]^G$ denoting Ad_G invariant polynomials on the Lie algebra $\mathfrak{g}$ of G. When G is compact BG can be written as a colimit of smooth manifolds and so in that case the existence of the universal Chern-Weil homomorphism is classical.

One candidate for a smooth structure on BG is some kind of diffeology. For example Magnot and Watts [7] construct a natural diffeology on the Milnor classifying space BG. However, this and related diffeologies do not appear to be sufficient to carry out Chern-Weil theory directly. A further specific possible requirement for the above discussed "smooth structures", is that the simplicial set $BG_\bullet$, of smooth maps $\Delta^d \to BG$, should have a geometric realization weakly homotopy equivalent to BG. See for instance [5] for one approach to this particular problem in the context of diffeologies. This kind of requirement is crucial for instance in [13], which may be understood as a kind of "quantum Chern-Weil theory" on $BHam(M, \omega)$ for $Ham(M, \omega)$ the group of Hamiltonian symplectomorphisms of a symplectic manifold. In the language of smooth simplicial sets, the analogue of this latter requirement is always trivially satisfied. The specific content of this is Proposition 3.6.

The structure of a smooth simplicial set is initially more flexible than a space with diffeology, but with further conditions, like the Kan condition, can become forcing. Given a Frechet Lie group G, we construct, for each choice of a particular kind of Grothendieck universe $\mathcal{U}$, a smooth simplicial set $BG^{\mathcal{U}}$ with a specific classifying property, analogous to the classifying property of BG. The simplicial set $BG^{\mathcal{U}}$ is moreover a Kan complex, and so is a basic example of a smooth Kan complex. We then show that if G in addition has the homotopy type of a CW complex then the geometric realization $|BG^{\mathcal{U}}|$ is homotopy equivalent to BG.

All of the dreams of "smoothness" mentioned above then in some sense hold true for $BG^{\mathcal{U}}$ via its smooth Kan complex structure. In particular, as one immediate application we get:

Theorem 1.1. *Let G be a Frechet Lie group having the homotopy type of a CW complex, then there is a universal Chern-Weil algebra homomorphism:*

$$cw : \mathbb{R}[\mathfrak{g}]^G \to H^*(BG, \mathbb{R}).$$

This is natural, so that if $P \to Y$ is a smooth Frechet G-bundle and

$$cw^P : \mathbb{R}[\mathfrak{g}]^G \to H^*(Y, \mathbb{R})$$

is the associated classical Chern-Weil homomorphism, then

$$cw^P = (f_P)^* \circ cw,$$

for $f_P : Y \to BG$ the classifying map of Y.

Here is one concrete example, using Reznikov's polynomials [11] on the Lie algebra of the group of Hamiltonian symplectomorphisms of a compact symplectic manifold. Let $Symp(\mathbb{CP}^k)$ denote the group of symplectomorphisms of $\mathbb{CP}^k$, that is diffeomorphisms $\phi : \mathbb{CP}^k \to \mathbb{CP}^k$ s.t. $\phi^* \omega_0 = \omega_0$ for ω_0 the Fubini-Study symplectic 2-form on $\mathbb{CP}^k$.

Theorem 1.2. *The natural map*

$$i : BPU(n) \to BSymp(\mathbb{CP}^{n-1})$$

is injective on real homology for all $n \geq 2$.

The first result in this direction is due to Reznikov himself as he proves that:

$$i_* : \pi_k(BPU(n)) \otimes \mathbb{Q} \to \pi_k(BHam(\mathbb{CP}^{n-1}, \omega)) \otimes \mathbb{Q}$$

is injective.

Remark 1.3. *Note that Reznikov's theorem does not a priori imply injectivity on rational homology. This is despite both spaces being simply connected (so that rational homotopy theory becomes very powerful). Here is a counterexample. Let X be the 4-manifold with boundary obtained by removing an open 4-ball B^4 from $\mathbb{CP}^2$. In other words $X = \mathbb{CP}^2 - i(D^4)$ for $i : B^4 \to \mathbb{CP}^2$ a smooth embedding of the standard open 4-ball $B^4 \subset \mathbb{R}^4$, with i extending smoothly to the topological boundary ∂B^4. Let $j : S^3 \to X$ be the inclusion map of the boundary. Then both spaces are simply connected, j is injective on rational homotopy groups but vanishes on rational homology. Proving that j is injective on rational homotopy groups does not require more then degree theory and Poincare duality. One such argument was communicated to me by Dennis Sullivan.*

More history and background surrounding these theorems is in Sections 9 and 10. We end this introduction with some natural open questions.

Question 1.4. *Our argument is formalized in ZFC + Grothendieck's axiom of universes, where ZFC is Zermelo-Fraenkel set theory plus axiom of choice. Does theorem 1.1 have a proof in ZFC?*

Probably the answer is yes, on the other hand as communicated to me by Dennis Sullivan there are known set theoretical (ZFC) issues with some questions on universal characteristic and secondary characteristic classes. So that the answer of no may be possible.

Question 1.5. *Is there a full dictionary between smooth simplicial sets and diffeological spaces (or Chen spaces)?*

As will be explained, given a diffeological space (or Chen space) we naturally get a smooth simplicial set. However the other direction seems to be rather complicated, as the naive geometric realization does not remember much of the smooth simplicial set structure. (There might be other possibilities for forming the geometric realization.)

1.1. **Acknowledgements.** I am grateful to Dennis Sullivan and Jean-Pierre Magnot for comments.

2. Preliminaries and notation

We denote by Δ the simplex category, i.e. the category with objects finite sets: $[n] = \{0, 1, \ldots, n\}$, with $\hom_\Delta([n], [m])$ non-decreasing maps. A simplicial set X is a functor

$$X : \Delta \to Set^{op}.$$

We usually write $X(n)$ instead of $X([n])$, and this is called the set of n-simplices of X. Δ^d_{simp} will denote the standard representable d-simplex:

$$\Delta^d_{simp}(n) = hom_\Delta([n], [d]).$$

The element of $\Delta^d_{simp}(0)$ corresponding to the map $i_k : [0] \to [d]$, $i_k(0) = k$ will usually be denoted by just k.

Let Δ^d be the topological d-simplex, i.e.

$$\Delta^d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \,|\, x_1 + \ldots + x_d \leq 1, \text{ and } \forall i : x_i \geq 0\}.$$

The vertices of Δ^d will be assumed ordered in the standard way.

Definition 2.1. *Let X be a smooth manifold with corners. We say that a map $\sigma : \Delta^n \to X$ is smooth if it smooth as a map of manifolds with corners. In particular $\sigma : \Delta^n \to \Delta^d$ is smooth iff it has an extension to a smooth map from a neighborhood in $\mathbb{R}^n$ of Δ^n into a neighborhood of Δ^d in $\mathbb{R}^d$. We say that a smooth $\sigma : \Delta^n \to X$ is* **collared** *if there is a neighborhood $U \supset \partial \Delta^n$ in Δ^n, such that $\Sigma|_U = \Sigma \circ ret$ for $ret : U \to \partial \Delta^n$ some smooth retraction. Here smooth means that ret has an extension to a smooth map $V \subset \mathbb{R}^d \to \mathbb{R}^d$, with $V \supset \partial \Delta^n$ open in $\mathbb{R}^d$.*

We denote by $\Delta^d_\bullet$ the simplicial set of all smooth simplices in Δ^d: $\Delta^d_\bullet(k)$ is the set of smooth maps

$$\sigma : \Delta^k \to \Delta^d.$$

We call an affine map $\Delta^k \to \Delta^d$ taking vertices to vertices in a order preserving way ***simplicial***. And we denote by

$$\Delta^d_{simp} \subset \Delta^d_\bullet$$

the subset consisting of simplicial maps. Note that Δ^d_{simp} is naturally isomorphic to the standard representable d-simplex Δ^d_{simp} as previously defined. Thus we may also understand Δ as the category with objects topological simplices Δ^d, $d \geq 0$ and morphisms simplicial maps.

Notation 2.2. *A morphism $m \in hom_\Delta([n], [k])$ uniquely corresponds to a simplicial map $\Delta^n_{simp} \to \Delta^k_{simp}$, which uniquely corresponds to a simplicial map in the above sense $\Delta^n \to \Delta^k$. The correspondence is by taking the maps $\Delta^n_{simp} \to \Delta^k_{simp}$, $\Delta^n \to \Delta^k$, to be determined by the map of the vertices corresponding to m. We will not notationally distinguish these corresponding morphisms. So that m may will simultaneously refer to all of the above morphisms.*

A morphism or ***map of simplicial sets*** $f : X \to Y$ is a natural transformation f of the corresponding functors. By a d-simplex Σ of a simplicial set X, we may mean, interchangeably, either the element in $X(d)$ or the map of simplicial sets:

$$\Sigma : \Delta^d_{simp} \to X,$$

uniquely corresponding to Σ via the Yoneda lemma. If we write Σ^d for a simplex of X, it is implied that it is a d-simplex.

Definition 2.3. *For X a simplicial set, $\Delta(X)$ will denote the over category of simplices of X, explicitly the category whose set of objects* $\operatorname{obj}\Delta(X)$ *is the set of simplices*

$$\Sigma : \Delta^d_{simp} \to X, \quad d \geq 0$$

and morphisms $f : \Sigma_1 \to \Sigma_2$, commutative diagrams:

$$\begin{tikzcd} \Delta^d_{simp} \arrow[r, "\widetilde{f}"] \arrow[dr, "\Sigma_1"'] & \Delta^n_{simp} \arrow[d, "\Sigma_2"] \\ & X \end{tikzcd} \tag{2.1}$$

with top arrow a simplicial map, which we denote by $\widetilde{f}$. An object $\Sigma : \Delta^d_{simp} \to X$ is likewise called a d-simplex, and such a Σ may be said to have degree d. As noted in the paragraph before the definition, the degree d may also be specified by a superscript.

Definition 2.4. *We say that $\Sigma^n \in \Delta(X)$ is* **non-degenerate** *if there is no morphism $f : \Sigma^n \to \Sigma^m$ in $\Delta(X)$ s.t. $m < n$.*

There is a forgetful functor

$$T : \Delta(X) \to \Delta,$$

$T(\Sigma^d) = \Delta^d_{simp}$, $T(f) = \widetilde{f}$. We denote by $\Delta^{inj}(X) \subset \Delta(X)$ the sub-category with same objects, and morphisms f such that $\widetilde{f}$ are monomorphisms, i.e. are face inclusions.

2.1. **Geometric realization.** Let Top be the category of topological spaces. Let X be a simplicial set, then define as usual the ***geometric realization*** of X by the colimit in Top:

$$|X| := \operatorname{colim}_{\Delta(X)} T,$$

for $T : \Delta(X) \to \Delta \subset Top$ as above, understanding Δ as a subcategory of Top as previously explained.

3. Smooth simplicial sets

If

$$\sigma : \Delta^d \to \Delta^n$$

is a smooth map we then have an induced map of simplicial sets

$$\sigma_\bullet : \Delta^d_\bullet \to \Delta^n_\bullet, \tag{3.1}$$

defined by

$$\sigma_\bullet(\rho) = \sigma \circ \rho.$$

We now give a pair of equivalent definitions of smooth simplicial sets. The first is more hands on, while the second is more concise and conceptual. The equivalence is the content of Proposition 3.12 ahead.

Definition 3.1 (First definition)**.** *A* **smooth simplicial set** *is a data consisting of:*

(1) *A simplicial set X.*

(2) *For each* $\Sigma : \Delta^n_{simp} \to X$ *an* n*-simplex, there is an assigned map of simplicial sets*
$$g(\Sigma) : \Delta^n_\bullet \to X,$$
which satisfies:
$$g(\Sigma)|_{\Delta^n_{simp}} = \Sigma. \tag{3.2}$$
We abbreviate $g(\Sigma)$ *by* Σ_**, when there is no need to disambiguate which structure* g *is meant.*

(3) *The following property will be called* **push-forward functoriality***:*
$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_\bullet$$
where $\sigma : \Delta^k \to \Delta^d$ *is a* k*-simplex of* $\Delta^d_\bullet$*, and where* Σ *as before is a* d*-simplex of* X*.*

Thus, formally a smooth simplicial set is a 2-tuple (X, g), satisfying the axioms above. When there is no need to disambiguate we omit specifying g.

Definition 3.2. *A* **smooth** *map between smooth simplicial sets*
$$(X_1, g_1), (X_2, g_2)$$
is a simplicial map
$$f : X_1 \to X_2,$$
which satisfies the condition:
$$g_2(f(\Sigma)) = f \circ g_1(\Sigma), \tag{3.3}$$
or more compactly:
$$(f(\Sigma))_* = f \circ \Sigma_*.$$
We may also write $f(\Sigma)_*$*, instead of* $(f(\Sigma))_*$ *but the former notation is sometimes unclear.*

A ***diffeomorphism*** between smooth simplicial sets is defined to be a smooth map, with a smooth inverse.

Now let Δ^{sm} denote the category whose objects are the topological simplices Δ^k, $k \geq 0$. And $\hom_{\Delta^{sm}}(\Delta^k, \Delta^n)$ is the set of smooth maps $\Delta^k \to \Delta^n$. Then

Definition 3.3 (Second definition)**.** *A* **smooth simplicial set** X *is a functor* $X : \Delta^{sm} \to Set^{op}$*. A smooth map* $f : X \to Y$ *of smooth simplicial sets is defined to be a natural natural transformation from the functor* X *to* Y*.*

In what follows we use the first definition, to avoid too much abstraction in this first paper. But the second definition would probably be more useful in the future.

Example 3.4 (The tautological smooth simplicial set)**.** $\Delta^n_\bullet$ *has a tautological smooth simplicial set structure, where*
$$g(\Sigma) = \Sigma_\bullet,$$
for $\Sigma : \Delta^k \to \Delta^n$ *a smooth map, hence a* k*-simplex of* $\Delta^n_\bullet$*, and where* $\Sigma_\bullet$ *is as in* (3.1)*.*

Lemma 3.5. *Let* X *be a smooth simplicial set and* $\Sigma : \Delta^n_{simp} \to X$ *an* n*-simplex. Let* $\Sigma_* : \Delta^n_\bullet \to X$ *be the induced simplicial map. Then* Σ_* *is smooth with respect to the tautological smooth simplicial set structure on* $\Delta^n_\bullet$ *as above.*

Proof. Let σ be a k-simplex of $\Delta^n_\bullet$, so $\sigma : \Delta^k \to \Delta^n$ is a smooth map, we need that

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_*.$$

Now $\sigma_* = \sigma_\bullet$, by definition of the tautological smooth structure on $\Delta^n_\bullet$. So we have:

$$\begin{aligned}(\Sigma_*(\sigma))_* &= \Sigma_*(\sigma) \circ \sigma_\bullet \text{ by Axiom 3} \\ &= \Sigma_*(\sigma) \circ \sigma_*.\end{aligned}$$

□

Proposition 3.6. *The set of n-simplices of a smooth simplicial set X is naturally isomorphic to the set of smooth maps $\Delta^n_\bullet \to X$. In fact, define $X_\bullet$ to be the simplicial set whose n-simplices are smooth maps $\Delta^n_\bullet \to X$, with $X_\bullet$ given the obvious simplicial structure. Then $X_\bullet$ is naturally isomorphic to X.*

Proof. Given a simplex $\rho : \Delta^n_{simp} \to X$, we have a uniquely associated to it, by the lemma above, smooth map $\rho_* : \Delta^n_\bullet \to X$. Conversely, suppose we are given a smooth map $m : \Delta^n_\bullet \to X$. Then we get an n-simplex $\rho_m := m|_{\Delta^n_{simp}}$. Let $id^n : \Delta^n \to \Delta^n$ be the identity map. We have that

$$\begin{aligned}m &= m \circ id^n_\bullet = m \circ id^n_* \\ &= (m(id^n))_* \text{ as } m \text{ is smooth} \\ &= (\rho_m(id^n))_* \text{ trivially by definition of } \rho_m \\ &= \rho_{m,*} \circ id^n_* \text{ as } \rho_{m,*} \text{ is smooth by Lemma 3.5} \\ &= \rho_{m,*}.\end{aligned}$$

Thus the map $\rho \mapsto \rho_*$, from the set of n-simplices of X to the set of smooth maps $\Delta^n_\bullet \to X$, is bijective.

The proof of the second part of the proposition is straightforward from the first part and is omitted. □

Lemma 3.7. *Given a smooth $m : \Delta^d_\bullet \to \Delta^n_\bullet$ there is a unique smooth map $f : \Delta^d \to \Delta^n$ such that $m = f_\bullet$.*

Proof. This is of course very trivial. Define f by $m(id)$ for $id : \Delta^d \to \Delta^d$ the identity. Then

$$\begin{aligned}f_\bullet &= (m(id))_\bullet \\ &= (m(id))_* \\ &= m \circ id_* \quad \text{(as } m \text{ is smooth)} \\ &= m.\end{aligned}$$

So f induces m. Now if g induces m then $g_\bullet = m$ hence $g = g_\bullet(id) = m(id)$. □

Definition 3.8. *A smooth simplicial set whose underlying simplicial set is a Kan complex will be called a* **smooth Kan complex**.

Let $Sing^{sm}(Y)$ [1] denote the simplicial set of smooth simplices in X. That is $Sing^{sm}(Y)(k)$ is the set of smooth maps $\Sigma : \Delta^k \to Y$. And where the simplicial structure on $Sing^{sm}(Y)$ is the natural one. $Sing^{sm}(Y)$ will often be abreviated by $Y_\bullet$. Analogously, $Sing^c(Y)$ will be the simplicial set of continuous simplices in X.

---

[1] This is often called the "smooth singular set of Y". However, for us "smooth" is reserved for another purpose, so to avoid confusion we do not use such terminology.

Example 3.9. *Let Y be a smooth d-fold. And set $X = Y_\bullet = Sing^{sm}(Y)$. Then X is naturally a smooth simplicial set, analogously to Example 3.4. This should be a Kan complex but a reference is not known to me. However, if we ask that $\Sigma : \Delta^k \to Y$ are in addition collared (as in Definition 2.1) then the Kan condition is simple to verify. More generally, we may clearly take Y to be a manifold with boundary or with corners, an orbifold or any diffeological space.*

Example 3.10. *One special example is worth attention. Let M be a smooth manifold. Then there is a natural smooth simplicial set LM^Δ whose d-simplices Σ are smooth maps $f_\Sigma : \Delta^d \times S^1 \to M$. The maps Σ_* are defined by*

$$\Sigma_*(\sigma) = f_\Sigma \circ (\sigma \times id),$$

for

$$\sigma \times id : \Delta^d \times S^1 \to \Delta^d \times S^1,$$

the induced map. This LM^Δ is one simplicial model of the free loop space. Naturally the free loop space LM also has the structure of a Frechet manifold, in particular we have the smooth simplicial set $LM_\bullet$, whose n-simplices are Frechet smooth maps $\Sigma : \Delta^n \to LM$. There is a natural simplicial map $LM^\Delta \to LM_\bullet$, which is clearly smooth. (It is indeed a diffeomorphism.)

3.1. **Simplex category of a smooth simplicial set.** Given a smooth simplicial set X, there is an extension of the previously defined category $\Delta(X)$.

Definition 3.11. *For X a smooth simplicial set, $\Delta^{sm}(X)$ will denote the category whose set of objects* $\operatorname{obj} \Delta^{sm}(X)$ *is the set of smooth maps*

$$\Sigma : \Delta^d_\bullet \to X, \quad d \geq 0$$

and morphisms $f : \Sigma_1 \to \Sigma_2$, commutative diagrams:

$$\begin{tikzcd} \Delta^d_\bullet \arrow[r, "\widetilde{f}_\bullet"] \arrow[dr, "\Sigma_1"'] & \Delta^n_\bullet \arrow[d, "\Sigma_2"] \\ & X \end{tikzcd} \tag{3.4}$$

with top arrow any smooth map (for the tautological smooth simplicial set structure on $\Delta^d_\bullet$), which we denote by $\widetilde{f}_\bullet$. By Lemma 3.7, $\widetilde{f}_\bullet$ is induced by a unique smooth map $\widetilde{f} : \Delta^d \to \Delta^n$.

By Proposition 3.6 we have a natural faithful embedding $\Delta(X) \to \Delta^{sm}(X)$ which is an isomorphism on object sets of these categories. We may then likewise call elements of $\Delta^{sm}(X)$ as d-simplices.

Proposition 3.12. *Definitions 3.1, 3.3 are equivalent. In other words there is a natural equivalence of the corresponding categories of smooth simplicial sets.*

Proof. We only sketch the proof, as this result will not be used. Given a smooth simplicial set X, in the sense of Definition 3.3, we have the associated category $\Delta^{sm}(X)$. This then gives a functor $XF : \Delta^{sm} \to Set^{op}$ by setting $XF(\Delta^k) = \Delta^{sm}(X)(k)$ and for $\sigma : \Delta^k \to \Delta^d$ setting

$$XF(\sigma) : \Delta^{sm}(X)(d) \to \Delta^{sm}(X)(k)$$

to be the map $XF(f)(\Sigma) = \Sigma \circ \sigma_\bullet$.

Conversely, given a smooth simplicial set XF, in the sense of Definition 3.1, define $X(k) = XF(\Delta^k)$. And for $\Sigma \in X(k)$ define $\Sigma_* : \Delta^k_\bullet \to X$ to be the map:

$$\Sigma_*(\sigma) = XF(\sigma)(\Sigma).$$

□

3.2. **Products.** Given a pair of smooth simplicial sets $(X_1, g_1), (X_2, g_2)$, the product $X_1 \times X_2$ of the underlying simplicial sets, has the structure of a smooth simplicial set

$$(X_1 \times X_2, g_1 \times g_2),$$

constructed as follows. Denote by $\pi_i : X_1 \times X_2 \to X_i$ the simplicial projection maps. Then for each $\Sigma \in X_1 \times X_2(d)$,

$$g_1 \times g_2(\Sigma) : \Delta^d_\bullet \to X_1 \times X_2$$

is defined by:

$$g_1 \times g_2(\Sigma)(\sigma) := (g_1(\pi_1(\Sigma))(\sigma), g_2(\pi_2(\Sigma))(\sigma)).$$

3.3. **More on smooth maps.** As defined, a smooth map $f : X \to Y$, induces a functor $\Delta^{sm} f : \Delta^{sm}(X) \to \Delta^{sm}(Y)$. This is defined by $\Delta^{sm} f(\Sigma) = f \circ \Sigma$, where $\Sigma : \Delta^d_\bullet \to X$ is in $\Delta^{sm}(X)$. If $m : \Sigma_1 \to \Sigma_2$ is a morphism in $\Delta^{sm}(X)$:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "\Sigma_1"] & \Delta^d_\bullet \arrow[d, "\Sigma_2"] \\ & X, \end{tikzcd}$$

then obviously the diagram below also commutes:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "h_1"] & \Delta^d_\bullet \arrow[d, "h_2"] \\ & X, \end{tikzcd}$$

where $h_i = \Delta^{sm} f(\Sigma_i)$, $i = 1, 2$. And so determines a morphism $\Delta^{sm} f(m) : h_1 \to h_2$.

3.4. **Smooth homotopy.**

Definition 3.13. *Let X, Y be smooth simplicial sets. Set $I := \Delta^1_\bullet$ and let $0_\bullet, 1_\bullet \subset I$ be the images of the pair of inclusions $\Delta^0_\bullet \to I$ corresponding to the pair of endpoints. A pair of smooth maps $f, g : X \to Y$ are called* **smoothly homotopic** *if there exists a smooth map*

$$H : X \times I \to Y$$

such that $H|_{X \times 0_\bullet} = f$ and $H|_{X \times 1_\bullet} = g$.

The following notion will be useful later on.

Definition 3.14. *Let X be a smooth simplicial set. We define $\pi^{sm}_k(X)$ to be the set of smooth homotopy equivalence classes of smooth maps $f : S^k_\bullet \to X$.*

4. Differential forms on smooth simplicial sets and DeRham theorem

Definition 4.1. *Let X be a smooth simplicial set. A **simplicial differential k-form** ω, or just differential form where there is no possibility of confusion, is for each d-simplex Σ of X a smooth differential k-form ω_Σ on Δ^d, such that*

$$i^* \Omega_{\Sigma_2} = \Omega_{\Sigma_1}, \tag{4.1}$$

for every morphism $i : \Sigma_1 \to \Sigma_2$ in $\Delta^{inj}(X)$, (see Section 2). If in addition:

$$\omega_{g(\Sigma)(\sigma)} = \sigma^* \omega_\Sigma, \tag{4.2}$$

for every $\sigma \in \Delta^d_\bullet$, and every d-simplex Σ, then we say that ω is **coherent**.

Example 4.2. *If $X = Y_\bullet$ for Y a smooth d-fold, and if ω is a differential k-form on Y, then $\{\omega_\Sigma = \Sigma^* \omega\}_\Sigma$ is a coherent differential k-form on X called the* **induced simplicial differential form**.

Example 4.3. *Let LM^Δ be the smooth Kan complex of Example 3.10. Then Chen's iterated integrals [1] naturally give coherent differential forms on LM^Δ.*

The above coherence condition is often unnecessary, hence is not part of the basic definition here. This is already one difference with differential forms on diffeological or Chen spaces, where coherence is ostensibly forced.

Let X be a smooth simplicial set. We denote by $\Omega^k(X)$ the $\mathbb{R}$-vector space of differential k-forms on X. Define

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

by

$$d(\{\omega_\Sigma\}) := d\{\omega_\Sigma\} = \{d\omega_\Sigma\}.$$

Clearly we have

$$d^2 = 0.$$

A k-form ω is said to be ***closed*** if $d\omega = 0$, and ***exact*** if for some $(k-1)$-form η, $\omega = d\eta$.

Definition 4.4. *The* **wedge product** *on*

$$\Omega^\bullet(X) = \bigoplus_{k \geq 0} \Omega^k(X)$$

is defined by

$$\omega \wedge \eta = \{\omega_\Sigma \wedge \eta_\Sigma\}_\Sigma.$$

Then $\Omega^\bullet(X)$ has the structure of a differential graded $\mathbb{R}$-algebra with respect to $\wedge$.

We then, as usual, define the ***De Rham cohomology*** of X:

$$H^k_{DR}(X) = \frac{\text{closed k-forms}}{\text{exact k-forms}},$$

which is a graded commutative $\mathbb{R}$-algebra.

The simplicial De Rham complex above is certainly not a new invention, versions of it have been used by Whitney and perhaps most famously by Sullivan [15].

4.1. **Homology and cohomology of a simplicial set.** We go over this mostly to establish notation. For a simplicial set X, we define an abelian group

$$C_k(X,\mathbb{Z}),$$

as the free abelian group generated by the set of k-simplices $X(k)$. Elements of $C_k(X,\mathbb{Z})$ are called k-***chains***. The boundary operator:

$$\partial : C_k(X,\mathbb{Z}) \to C_{k-1}(X,\mathbb{Z}),$$

is defined on a k-simplex σ as classically by

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i d_i \sigma,$$

where d_i are the face maps, this is then extended by linearity to general chains. As classically, $\partial^2 = 0$. The homology of this complex is denoted by $H_k(X,\mathbb{Z})$, called integral homology. The integral cohomology is defined analogously to the classical topology setting, using dual chain groups $C^k(X,\mathbb{Z}) = hom(C_k(X,\mathbb{Z}),\mathbb{Z})$. The corresponding coboundary operator is denoted by d as usual:

$$d : C^k(X,\mathbb{Z}) \to C^{k+1}(X,\mathbb{Z}).$$

Homology and cohomology with other ring coefficients (or modules) are likewise defined analogously. Given a simplicial map $f : X \to Y$ there are natural induced chain maps $f^* : C^k(Y,\mathbb{Z}) \to C^k(X,\mathbb{Z})$, and $f_* : C_k(X,\mathbb{Z}) \to C_k(X,\mathbb{Z})$. We say that $f, g : X \to Y$ are homotopic if there a simplicial map $H : X \times \Delta^1_{simp} \to Y$ so that $f = H \circ i_0$, $g = H \circ i_1$ for $i_0, i_1 : X \to X \times \Delta^1_{simp}$ corresponding to the pair of end point inclusions $\Delta^0_{simp} \to \Delta^1_{simp}$.

As is well known if f, g are homotopic then f^*, g^* and f_*, g_* are chain homotopic.

4.2. **Integration.** Let X be a smooth simplicial set. Given a chain

$$\sigma = \sum_i a_i \Sigma_i \in C_k(X,\mathbb{Z})$$

and a smooth differential form ω, we define:

$$\int_\sigma \omega = \sum_i a_i \int_{\Delta^k} \omega_{\Sigma_i}$$

where the integrals on the right are the classical integrals of a differential form. Thus we obtain a homomorphism:

$$\int : \Omega^k(X) \to C^k(X,\mathbb{R}),$$

$\int(\omega)$ is the k-cochain defined by:

$$\int(\omega)(\sigma) := \int_\sigma \omega,$$

where σ is a k-chain. We will abbreviate $\int(\omega) = \int \omega$.

Lemma 4.5. *For a smooth simplicial set X, the homomorphism $\int$ commutes with d, and so induces a homomorphism:*

$$\int : H^k_{DR}(X) \to H^k(X,\mathbb{R}).$$

Proof. We need that

$$\int d\omega = d \int \omega.$$

Let $\Sigma : \Delta^k_{simp} \to X$ be a k-simplex. Then

$$\begin{aligned}
\int d\omega(\Sigma) &= \int_{\Delta^k} d\omega_\Sigma \text{ by definition} \\
&= \int_{\partial \Delta^k} \omega_\Sigma \text{ by Stokes theorem} \\
&= d(\int \omega)(\Sigma) \text{ by the definition of } d \text{ on co-chains.}
\end{aligned}$$

□

4.3. **Pull-back.** Given a smooth map $f : X_1 \to X_2$ of smooth simplicial sets, we define

$$f^* : \Omega^k(X_2) \to \Omega^k(X_1)$$

naturally by

$$f^*(\omega) = f^*\omega := \{(f^*\omega)_\Sigma\}_\Sigma := \{\omega_{f(\Sigma)}\}_\Sigma.$$

Clearly f^* commutes with d so that we have an induced differential graded $\mathbb{R}$-algebra homomorphism:

$$f^* : \Omega^\bullet(X_2) \to \Omega^\bullet(X_1).$$

And in particular an induced $\mathbb{R}$-algebra homomorphism:

$$f^* : H^\bullet_{DR}(X_2) \to H^\bullet_{DR}(X_1).$$

4.4. **Relation with ordinary homology and cohomology.** Let $s - Set$ denote the category of simplicial sets and Top the category of topological spaces. Let

$$| \cdot | : s - Set \to Top$$

be the geometric realization functor as defined in Section 2.1. Let X be a (smooth) simplicial set. Then for any ring K we have natural chain maps

$$CR : C_k(X, K) \to C_k(|X|, K),$$
$$CR^c : C^k(|X|, K) \to C^k(X, K),$$

as a d-simplex $\Sigma : \Delta^d_{simp} \to X$, by construction of $|X|$ uniquely induces a continuous map $\Delta^d \to |X|$, which is set to be $CR(\Sigma)$.

When X is a Kan complex, the natural map $X \to Sing^c(|X|)$ is a simplicial homotopy equivalence. And so, in this case, the induced maps in homology/cohomology:

$$R : H_k(X, K) \to H_k(|X|, K), \tag{4.3}$$
$$R^c : H^k(|X|, K) \to H^k(X, K), \tag{4.4}$$

are isomorphisms.

Now let Y be a smooth manifold and $X = Y_\bullet = Sing^{sm}(Y)$. As mentioned this is expected to be a Kan complex, but no reference is known to me. However in this case we still have isomorphisms:

$$\begin{aligned}
&R : H_k(Y_\bullet, K) \to H_k(|Y_\bullet|, K), \\
&R^c : H^k(|Y_\bullet|, K) \to H^k(Y_\bullet, K) \text{ if } K \text{ is a field of characteristic } 0.
\end{aligned}$$

First note that we have a natural homotopy equivalence $|Y_\bullet| \simeq Y$. This is just because the natural map $|Y_\bullet| \to Y$ is a weak homotopy equivalence, (by homotopy approximating continuous maps by smooth maps), and so is a homotopy equivalence, by Whitehead theorem. Let us denote by

$$n : Y \to |Y_\bullet|, \tag{4.5}$$

its homotopy inverse. Then factor R and R^c as:

$$H_k(Y_\bullet, K) \xrightarrow{I} H_k(Y, K) \xrightarrow{n_*} H_k(|Y_\bullet|, K), \tag{4.6}$$

$$H^k(|Y_\bullet|, K) \xrightarrow{n^*} H^*(Y, K) \xrightarrow{I^c} H^k(Y_\bullet, K) \tag{4.7}$$

The map I is induced by the chain map CI sending the generator of $C_k(Y_\bullet, K)$ corresponding to a simplex $\Sigma \in Y_\bullet(k)$ to the element of $C_k(Y)$ given by the smooth map $\Sigma : \Delta^d \to Y$ (as Σ by definition corresponds to such a smooth map). I is of course injective, it is surjective because any cycle in $C_k(Y, K)$ is homologous to a smooth cycle (meaning $\sum_{i=1}^n c_i \Sigma_i$ with $\Sigma_i : \Delta^k \to Y$ smooth). Likewise I^c, is induced by the cochain map sending a cochain α to the cochain α' defined by

$$\alpha'(\sigma) := \alpha(CI(\sigma)),$$

where $\sigma \in C_k(Y_\bullet, K)$. I^c is of course surjective. Now suppose that $I^c(\alpha) = 0$, then the functional

$$\langle I^c(\alpha), \cdot \rangle : H_k(Y_\bullet, K) \to K$$

is 0. Since I is an isomorphism, the functional

$$\langle \alpha, \cdot \rangle : H_k(Y, K) \to K$$

is 0. Hence, if K is a field of 0 characteristic we get that $\alpha = 0$, and I^c is injective.

Notation 4.6. *In the case of the paper $K = \mathbb{R}$. Then given a Kan complex X, or $X = Y_\bullet$ for a smooth manifold Y, and given a cohomology class $\alpha \in H^k(X, K)$, we will denote by $|\alpha| \in H^k(|X|, K)$ the class $(R^c)^{-1}(\alpha)$.*

Given a map of simplicial sets $f : X_1 \to X_2$ we let $|f| : |X_1| \to |X_2|$ denote the induced map of geometric realizations. The following is immediate from definitions.

Lemma 4.7. *Let $f : X_1 \to X_2$ be a simplicial map with X_i Kan complexes, or with $X_i = Y_{i,\bullet}$ for Y_i smooth manifolds. Let $f^* : H^k(X_2, K) \to H^k(X_1, K)$ be the induced homomorphism then:*

$$|f^*(\alpha)| = |f|^*(|\alpha|).$$

5. Smooth simplicial G-bundles

Part of our motivation is the construction of the universal Chern-Weil homomorphisms for Frechet Lie groups. A *Frechet Lie group* G is Lie group whose underlying manifold is a possibly infinite dimensional smooth manifold locally modelled on a Frechet space, that is a locally convex, complete Hausdorff vector space. Later on it will also be important that G have the homotopy type of a CW complex. By Milnor [9], a prototypical example of such a Lie group is the group of diffeomorphisms $\mathrm{Diff}(M)$ of a smooth manifold. Another very interesting example for us is the group of Hamiltonian symplectomorphisms $Ham(M, \omega)$ of a symplectic manifold. Particularly because its Lie algebra admits natural bi-invariant polynomials, so that it is possible to define interesting Chern-Weil theory for this group.

In what follows G is always assumed to be a Frechet Lie group. We now introduce the basic building blocks for simplicial G-bundles.

Definition 5.1. *A* **smooth G-bundle P over Δ^n** *is a smooth G-bundle over Δ^n with the latter naturally understood as a smooth manifold with corners, using the natural embedding $\Delta^n \subset \mathbb{R}^n$.*

Remark 5.2. *For concreteness, this can be interpreted as follows. P is a topological principal G-bundle over $\Delta^n \subset \mathbb{R}^n$, together with a choice of an open neighborhood V of Δ^n in $\mathbb{R}^n$ and a choice of a smooth G-bundle $\widetilde{P} \to V$ such that $i^*\widetilde{P} \simeq P$ for $i : \Delta^n \to V$ the inclusion map, and for $\simeq$ an implicitly specified isomorphism of topological G-bundles. We may thus write $(P, \widetilde{P}, V)$ for this structure and in principle all of the subsequent constructions can be made to refer to the above concrete model. So that the generalities of smooth G-bundles over manifolds with corners are not really needed in this paper.*

To warn, at this point our terminology may partially clash with common terminology, in particular a simplicial G-bundle will *not* be a pre-sheaf on Δ with values in the category of smooth G-bundles. Instead, it will be a functor (not a co-functor!) on $\Delta^{sm}(X)$ with additional properties. The latter pre-sheafs will not appear in the paper so that this should not cause confusion.

In the definition of simplicial differential forms we omitted coherence. In the case of simplicial G-bundles, the analogous condition (full functoriality on $\Delta^{sm}(X)$) turns out to be necessary if we want universal simplicial G-bundles with expected behavior.

Notation 5.3. *Given a Frechet Lie group G, let $\mathcal{G}$ denote the category of smooth G-bundles over manifolds with corners, with morphisms smooth G-bundle maps. (See however Remark 5.2 just above.)*

Definition 5.4. *Let G be a Frechet Lie group and X a smooth simplicial set. A* **smooth simplicial G-bundle** *P over X is the following data:*

- *A functor $P : \Delta^{sm}(X) \to \mathcal{G}$, so that for Σ a d-simplex, $P(\Sigma)$ is a smooth G-bundle over Δ^d.*
- *For each morphism f:*

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{f}_\bullet"] \arrow[dr, "\Sigma^k_1"] & \Delta^d_\bullet \arrow[d, "\Sigma^d_2"] \\ & X \end{tikzcd}$$

in $\Delta^{sm}(X)$, we have a commutative diagram:

$$\begin{tikzcd} P(\Sigma^k_1) \arrow[r, "P(f)"] \arrow[d, "p_1"] & P(\Sigma^d_2) \arrow[d, "p_2"] \\ \Delta^k \arrow[r, "\widetilde{f}"] & \Delta^d, \end{tikzcd}$$

where the maps p_1, p_2 are the respective bundle projections, and where $\widetilde{f}$ is the map induced by the map $\widetilde{f}_\bullet : \Delta^k_\bullet \to \Delta^d_\bullet$ as in Lemma 3.7. In other words $P(f)$ is a bundle map over $\widetilde{f}$. We call this condition **compatibility**.

We will only deal with smooth simplicial G-bundles, and so will usually just say ***simplicial*** G***-bundle***, omitting the qualifier 'smooth'.

Notation 5.5. *We often use notation P_Σ for $P(\Sigma)$. If we write a simplicial G-bundle $P \to X$, this just means that P is a simplicial G-bundle over X in the sense above. So that $P \to X$ is just notation not a morphism.*

Example 5.6. *If X is a smooth simplicial set and G is as above, we denote by $X \times G$ the simplicial G-bundle,*

$$\forall n \in \mathbb{N}, \forall \Sigma^n \in \Delta(X) : (X \times G)_{\Sigma^n} = \Delta^n \times G,$$

*with $\Delta^n \times G \to \Delta^n$ the trivial projection. This is called the **trivial simplicial** G**-bundle over** X.*

Example 5.7. *Let $Z \to Y$ be a smooth G-bundle over a smooth manifold Y. Then we have a simplicial G-bundle $Z_\bullet$ over $Y_\bullet$ defined by*

$$Z_\bullet(\Sigma) = \Sigma^* Z.$$

And where for $f : \Sigma_1 \to \Sigma_2$ a morphism, the bundle map

$$N(f) : (Z_\bullet(\Sigma_1) = \Sigma_1^* N) \to (Z_\bullet(\Sigma_2) = \Sigma_2^* Z)$$

is just the natural map $f^* \Sigma_2^* Z \to \Sigma_2^* Z$ in the pull-back square. We say that $Z_\bullet$ is **the simplicial** G**-bundle induced by** Z.

Definition 5.8. *Let $P_1 \to X_1, P_2 \to X_2$ be a pair of simplicial G-bundles. Let $h : X_1 \to X_2$ be a smooth map. A* **smooth simplicial** G**-bundle map over** h *from P_1 to P_2 is a a natural transformation of functors:*

$$\widetilde{h} : P_1 \to P_2 \circ \Delta^{sm} h.$$

This is required to have the following additional property. For each d-simplex $\Sigma \in \Delta^{sm}(X_1)$ the natural transformation $\widetilde{h}$ specifies a morphism in $\mathcal{G}$:

$$\widetilde{h}_\Sigma : P_1(\Sigma) \to P_2(\Sigma),$$

and we ask that this is a bundle map over the identity so that the following diagram commutes:

$$\begin{tikzcd} P_1(\Sigma) \arrow[r, "\widetilde{h}_\Sigma"] \arrow[d, "p_1"] & P_2(\Sigma) \arrow[d, "p_2"] \\ \Delta^d \arrow[r, "id"] & \Delta^d. \end{tikzcd}$$

We will usually just say simplicial G-bundle map instead of smooth simplicial G-bundle map, (as everything is always smooth) when h is not specified it is assumed to be the identity.

Definition 5.9. *Let P_1, P_2 be simplicial G-bundles over X_1, X_2 respectively. A* **simplicial** G**-bundle isomorphism** *is a simplicial G-bundle map*

$$\widetilde{h} : P_1 \to P_2$$

s.t. there is a simplicial G-bundle map

$$\widetilde{h}^{-1} : P_2 \to P_1$$

with

$$\widetilde{h}^{-1} \circ \widetilde{h} = id.$$

This is clearly the same as asking that $\widetilde{h}$ *be a natural isomorphism of the corresponding functors. Usually,* $X_1 = X_2$ *and in this case, unless specified otherwise, it is assumed* $h = id$. *A simplicial* G*-bundle isomorphic to the trivial simplicial* G*-bundle is called* **trivializeable**.

Definition 5.10. *If* $X = Y_\bullet$ *for* Y *a smooth manifold, we say that a simplicial* G*-bundle* P *over* X *is* **induceable by a smooth G-bundle** $N \to Y$ *if there is a simplicial* G*-bundle isomorphism* $N_\bullet \to P$.

The following will be one of the crucial ingredients later on.

Theorem 5.11. *Let* G *be as above and let* $P \to Y_\bullet$ *be a simplicial* G*-bundle, for* Y *a smooth* d*-manifold. Then* P *is induceable by some smooth* G*-bundle* $N \to Y$.

Proof. We need to introduce an auxiliary notion. Let Z be a smooth d-manifold with corners. And let $\mathcal{D}(Z)$ denote the category whose objects are smooth embeddings $\Sigma : \Delta^d \to Z$, (for the same fixed d) and so that a morphism $f \in hom_{\mathcal{D}(Z)}(\Sigma_1, \Sigma_2)$ is a commutative diagrams:

$$
\begin{tikzcd}
\Delta^d \arrow[r, "\widetilde{f}"] \arrow[dr, "\Sigma_1"] & \Delta^d \arrow[d, "\Sigma_2"] \\
& Z.
\end{tikzcd}
\tag{5.1}
$$

Note that the map $\widetilde{f}$ is unique, when such a diagram exists as Σ_i are embeddings. Thus $hom_{\mathcal{D}(X)}(\Sigma_1, \Sigma_2)$ is either empty or consists of a single element.

Going back to our Y. Let $\{O_i\}_{i \in I}$ be a locally finite open cover of Y, closed under intersections, with each O_i diffeomorphic to an open ball. Such a cover is often called a good cover of a manifold, and the existence of such a cover is folklore theorem, but a proof can be found in [3, Prop A1]. Let $\mathcal{O}$ denote the category with the set of objects $\{O_i\}$ and with morphisms set inclusions. Set $C_i = \mathcal{D}(O_i)$, then we naturally have $C_i \subset \Delta^{sm}(Y_\bullet)$. For each i, we have the functor

$$F_i = P|_{C_i} : C_i \to \mathcal{G}.$$

By assumption that each O_i is diffeomorphic to an open ball, O_i has an exhaustion by embedded d-simplices. Meaning that there is a sequence of smooth embeddings $\Sigma_j : \Delta^d \to O_i$ with $\mathrm{image}(\Sigma_{j+1}) \supset \mathrm{image}(\Sigma_j)$ for each j. And so that $\bigcup_j \mathrm{image}(\Sigma_j) = O_i$. In particular, for each i, the colimit in $\mathcal{G}$:

$$P_i := \mathrm{colim}_{C_i} F_i \tag{5.2}$$

is naturally a smooth G-bundle over O_i, with G-bundle charts defined as follows. Take the collection of maps

$$\{\phi^i_{\Sigma,j}\}_{\Sigma \in C_i, j \in J^\Sigma},$$

satisfying the following.

- Each $\phi^i_{\Sigma,j}$ is the composition map

  $$V^i_{\Sigma,j} \times G \xrightarrow{\xi_{ij}} P_\Sigma \xrightarrow{c_\Sigma} P_i$$

  where $V^i_{\Sigma,j} \subset (\Delta^d)^\circ$ is open, for $(\Delta^d)^\circ$ the topological interior of the subspace $\Delta^d \subset \mathbb{R}^d$. And where $c_\Sigma : (P_\Sigma = F_i(\Sigma)) \to P_i$ is the natural map in the colimit diagram of (5.2).

- The collection
$$\{\xi_{i,j}\}_{j \in J^\Sigma}$$
forms an atlas of smooth G-bundle charts for $P_\Sigma|_{(\Delta^d)^\circ}$.

The collection $\{\phi^i_{\Sigma,j}\}$ then forms a smooth G-bundle atlas for P_i.

So we obtain a functor
$$D : \mathcal{O} \to \mathcal{G},$$
defined by
$$D(O_i) = P_i,$$
and defined naturally on morphisms. Specifically, a morphism $O_{i_1} \to O_{i_2}$ induces a functor $C_{i_1} \to C_{i_2}$ and hence a smooth G-bundle map $P_{i_1} \to P_{i_2}$.

Let $t : \mathcal{O} \to Top$ denote the tautological functor, so that $Y = \operatorname{colim}_{\mathcal{O}} t$, where for simplicity we write equality for natural isomorphisms here and further on in this proof. Now,
$$N := \operatorname{colim}_{\mathcal{O}} D, \tag{5.3}$$
is naturally a topological G-bundle over $\operatorname{colim}_{\mathcal{O}} t = Y$. Let $c_i : P_i \to N$ denote the natural maps in the colimit diagram of (5.3). The collection of charts $\{c_i \circ \phi^i_{\Sigma,j}\}_{i,j,\Sigma}$ forms a smooth atlas on N, giving it a structure of a smooth G-bundle.

We now prove that P is induced by N. Let Σ be a d-simplex of $X := Y_\bullet$, then $\{V_i := \Sigma^{-1}(O_i)\}_{i \in I}$ is a locally finite open cover of Δ^d closed under finite intersections. Let $N_\bullet$ be the simplicial G-bundle induced by N. So
$$N_\bullet(\Sigma) := N_\Sigma := \Sigma^* N.$$

As Δ^d is a convex subset of $\mathbb{R}^d$, the open metric balls in Δ^d, for the induced metric, are convex as subsets of $\mathbb{R}^d$. Consequently, as each $V_i \subset \Delta^d$ is open, it has a basis of convex (as subsets of $\mathbb{R}^d$) metric balls, with respect to the induced metric. By Rudin [12] there is then a locally finite cover of V_i by elements of this basis. In fact, as Rudin shows any open cover of V_i has a locally finite refinement by elements of such a basis.

Let $\{W^i_j\}$ consist of elements of this cover and all intersections of its elements, (which must then be finite intersections). So $W^i_j \subset V_i$ are open convex subsets and $\{W^i_j\}$ is a locally finite open cover of V_i. In particular, W^i_j have an exhaustion by nested images of embedded simplices, that is
$$W^i_j = \bigcup_{k \in \mathbb{N}} \operatorname{image} \sigma^{i,j}_k$$
for $\sigma^{i,j}_k : \Delta^d \to W^i_j$ smooth and embedded, with $\operatorname{image} \sigma^{i,j}_k \subset \operatorname{image} \sigma^{i,j}_{k+1}$ for each k. Alternatively, we can use that each V_i is a manifold with corners, and then take a good cover, however the above is more elementary.

Let C be the small category with objects $I \times J \times \mathbb{N}$, so that there is exactly one morphism from $a = (i,j,k)$ to $b = (i',j',k')$ whenever $\operatorname{image} \sigma^{i,j}_k \subset \operatorname{image} \sigma^{i',j'}_{k'}$, and no morphisms otherwise. Let
$$F : C \to \mathcal{D}(\Delta^d)$$
be the functor $F(a) = \sigma^{i,j}_k$ for $a = (i,j,k)$, (the definition on morphisms is forced). For brevity, we then reset $\sigma_a := F(a)$.

If $\mathcal{O}(Y)$ denotes the category of topological subspaces of Y with morphisms inclusions, then there is a forgetful functor

$$T : \mathcal{D}(Y) \to \mathcal{O}(Y)$$

which takes f to image$(\widetilde{f})$. With all this in place, we obviously have a colimit in Top:

$$\Delta^d = \operatorname{colim}_C T \circ F,$$

Now by construction, for each $a \in C$ we may express:

$$\Sigma \circ \sigma_a = \Sigma_a \circ \sigma_a, \tag{5.4}$$

for some i and some $\Sigma_a : \Delta^d \to U_i \subset Y$ a smooth embedded d-simplex. Then for all $a \in C$ we have a chain of natural isomorphisms, whose composition will be denoted by ϕ_a :

$$P_{\Sigma \circ \sigma_a} = P_{\Sigma_a \circ \sigma_a} \to N_{\Sigma_a \circ \sigma_a} = N_{\Sigma \circ \sigma_a} \tag{5.5}$$

To better explain the second map, note that we have a composition of natural bundle maps:

$$P_{\Sigma_a \circ \sigma_a} \to P_i \to N,$$

with the first map the bundle map in the colimit diagram of (5.2), and the second map the bundle map in the colimit diagram of (5.3). This composition is over $\Sigma_a \circ \sigma_a$. And so, by the defining universal property of the pull-back, there is a uniquely induced universal map

$$P_{\Sigma_a \circ \sigma_a} \to (\Sigma_a \circ \sigma_a)^* N = N_{\Sigma_a \circ \sigma_a},$$

which is a G-bundle isomorphism.

Now we have a natural functor $F_\Sigma : \mathcal{D}(\Delta^d) \to \mathcal{G}$, given by $F_\Sigma(\sigma) = P_{\Sigma \circ \sigma}$, and

$$P_\Sigma = \operatorname{colim}_C F_\Sigma \circ F. \tag{5.6}$$

Similarly,

$$N_\Sigma = \operatorname{colim}_C F'_\Sigma \circ F \tag{5.7}$$

where $F'(\sigma) = N_{\Sigma \circ \sigma}$. Now the maps $\phi_a : P_{\Sigma \circ \sigma_a} \to N_{\Sigma \circ \sigma_a}$ induce a natural transformation of functors

$$\phi : F_\Sigma \circ F \to F'_\Sigma \circ F.$$

So that ϕ naturally induces a map of the colimits:

$$h_\Sigma : P_\Sigma \to N_\Sigma,$$

which is an isomorphism of these smooth G-bundles. It is then clear that $\{h_\Sigma\}$ determines the bundle isomorphism $h : P \to N_\bullet$ we are looking for. $\square$

5.1. **Pull-backs of simplicial bundles.** Let $P \to X$ be a simplicial G-bundle over a smooth simplicial set X. And let $f: Y \to X$ be smooth. We define the pull-back simplicial G-bundle $f^*P \to Y$ by the functor $f^*P = P \circ \Delta^{sm} f$.

Note that the analogue of the following lemma is not true in the category of set fibrations. The pull-back by the composition is not the composition of pull-backs (except up to a natural isomorphism).

Lemma 5.12. *The pull-back is functorial. So that if $f: X \to Y$ and $g: Y \to Z$ are smooth maps of smooth simplicial sets, and $P \to Z$ is a smooth simplicial G-bundle over Z then*

$$(g \circ f)^* P = f^*(g^*(P)) \text{ an actual equality.}$$

Proof. This is of course trivial, as functor composition is associative:

$$(g \circ f)^* P = P \circ \Delta^{sm}(g \circ f) = P \circ (\Delta^{sm} g \circ \Delta^{sm} f) = (P \circ \Delta^{sm} g) \circ \Delta^{sm} f = f^*(g^* P).$$

□

6. Connections on simplicial G-bundles

Definition 6.1. *Let G be a Frechet Lie group. A* **simplicial G-connection** *D on a simplicial G-bundle P over a smooth simplicial set X is for each d-simplex Σ of X, a smooth G-invariant Ehresmann G-connection D_Σ on P_Σ. This data is required to satisfy: if $f: \Sigma_1 \to \Sigma_2$ is a morphism in $\Delta(X)$ then*

$$P(f)^* D_{\Sigma_2} = D_{\Sigma_1}.$$

We say that D is **coherent** *if the same holds for all morphisms $f: \Sigma_1 \to \Sigma_2$ in $\Delta^{sm}(X)$. Will will often just say G-connection instead of simplicial G-connection, where there is no need to disambiguate.*

As with differential forms the coherence condition is very restrictive, and is not part of the basic definition.

Lemma 6.2. *G-connections on simplicial G-bundles exist and any pair of G-connections D_1, D_2 on a simplicial G-bundle P are* **concordant**. *The latter means that there is a G-connection on $\widetilde{D}$ on $P \times I$,*

$$I := [0,1]_\bullet$$

which restricts to D_1, D_2 on $P \times I_0$, respectively on $P \times I_1$, for $I_0, I_1 \subset I$ denoting the images of the two end point inclusions $\Delta^0_\bullet \to I$.

Proof. Suppose that $\Sigma: \Delta^d_{simp} \to X$ is a degeneracy of a 0-simplex $\Sigma_0: \Delta^0_{simp} \to X$, meaning that there is a morphism from Σ to Σ_0 in $\Delta(X)$. Then $P_\Sigma = \Delta^d \times P_{\Sigma_0}$ (as previously equality indicates natural isomorphism) and we fix the corresponding trivial connection D_Σ on P_Σ. We then proceed inductively.

Suppose we have constructed connections D_Σ for all degeneracies of n-simplices, $n \geq 0$. We now extend this to all degeneracies of $(n+1)$-simplices. If Σ is a non-degenerate $(n+1)$-simplex then D_Σ is already determined over the boundary of Δ^{n+1}, as by the hypothesis D_Σ is already defined on all n-simplices, so extend D_Σ over all of Δ^{n+1} arbitrarily. Thus we have extended D_Σ to all $(n+1)$-simplices, as such a simplex is either non-degenerate or is a degeneracy of a n-simplex. If Σ' is a m-simplex that is a degeneracy of a $(n+1)$-simplex Σ^{n+1}, then $P_{\Sigma'} = pr^* P_{\Sigma^{n+1}}$ for a certain determined simplicial projection $pr: \Delta^m \to \Delta^{n+1}$, and we define $D_\Sigma = \widetilde{pr}^* D_{\Sigma^{n+1}}$. For $\widetilde{pr}: P_{\Sigma'} \to P_{\Sigma^{n+1}}$ the natural map in the pull-back square.

The second part of the lemma follows by an analogous argument, since we may just extend D_1, D_2 to a concordance connection $\widetilde{D}$, using the above inductive procedure. $\square$

Example 6.3. *Given a classical smooth G-connection D on a smooth principal G-bundle $Z \to Y$, we obviously get a simplicial G-connection on the induced simplicial G-bundle $N = Z_\bullet$. Concretely, this is defined by setting D_Σ on $N_\Sigma = \Sigma^* Z$ to be $\widetilde{\Sigma}^* D$, for $\widetilde{\Sigma} : \Sigma^* Z \to Z$ the natural map. This is called the* **induced simplicial connection***, and may be denoted by $D_\bullet$. Going in the other direction is always possible if the given simplicial G-connection in addition satisfies coherence, but we will not elaborate.*

7. Chern-Weil homomorphism

7.1. **The classical case.** To establish notation we first discuss classical Chern-Weil homomorphism.

Let G be a Frechet Lie group, and let $\mathfrak{g}$ denote its Lie algebra. Let P be a smooth G-bundle over a smooth manifold Y. Fix a G-connection D on P. Let $\operatorname{Aut} P_y$ denote the group of smooth G-torsor automorphisms of the fiber P_y of P over $y \in Y$. Note that $\operatorname{Aut} P_y \simeq G$ where $\simeq$ means non-canonically isomorphic. Then associated to D we have the classical curvature 2-form R^D on Y, understood as a 2-form valued in the vector bundle $\mathcal{P} \to Y$, whose fiber over $y \in Y$ is $\operatorname{lie} \operatorname{Aut} P_y$ - the Lie algebra of $\operatorname{Aut} P_y$.

Thus,

$$\forall v, w \in T_y Y : R^D(v, w) \in \mathcal{P}_y = \operatorname{lie} \operatorname{Aut} P_y.$$

Now, let ρ be a symmetric multi-linear functional:

$$\rho : (V = \prod_{i=1}^{i=k} \mathfrak{g}) \to \mathbb{R},$$

satisfying

$$\forall g \in G, \forall v \in V : \rho(Ad_g(v)) = \rho(v).$$

Here if $v = (\xi_1, \ldots, \xi_n)$, $Ad_g(v) = (Ad_g(\xi_1), \ldots, Ad_g(\xi_n))$ is the adjoint action by the element $g \in G$. As ρ is Ad invariant, it uniquely determines multi-linear maps with the same name:

$$\rho : (V = \prod_{i=1}^{i=k} \operatorname{lie} \operatorname{Aut} P_y) \to \mathbb{R},$$

by fixing any Lie-group isomorphism $\operatorname{Aut} P_y \to G$. We may now define a closed $\mathbb{R}$-valued $2k$-form $\omega^{\rho,D}$ on Y:

(7.1)
$$\omega^{\rho,D}(v_1, \ldots, v_{2k}) = \frac{1}{2k!} \sum_{\eta \in P_{2k}} \operatorname{sign} \eta \cdot \rho(R^D(v_{\eta(1)}, v_{\eta(2)}), \ldots, R^D(v_{\eta(2k-1)}, v_{\eta_{2k}})),$$

for P_{2k} the permutation group of a set with $2k$ elements, and where $v_1, \ldots, v_{2k} \in T_y Y$. Set

$$\alpha^{\rho,D} := \int \omega^{\rho,D}.$$

Then we define the classical Chern-Weil characteristic class:

$$c^\rho(P) = c^\rho_{2k}(P) := [\alpha^{\rho,D}] \in H^{2k}(X, \mathbb{R}). \tag{7.2}$$

7.2. **Chern-Weil homomorphism for smooth simplicial bundles.** Now let P be a simplicial G-bundle over a smooth simplicial set X. Fix a simplicial G-connection D on P.

For each simplex Σ^d, we have the the curvature 2-form R^D_Σ of the connection D_Σ on P_Σ, defined as in the section just above. For concreteness:

$$\forall v, w \in T_z \Delta^d : R^D_\Sigma(v, w) \in \operatorname{lie}\operatorname{Aut} P_z,$$

for P_z the fiber of P_Σ over $z \in \Delta^d$.

As above, let ρ be a Ad invariant symmetric multi-linear functional:

$$\rho : (V = \prod_{i=1}^{i=k} \mathfrak{g}) \to \mathbb{R}.$$

Thus ρ uniquely determines for each $z \in \Delta^d$ a symmetric multi-linear map with the same name:

$$\rho : (V = \prod_{i=1}^{i=k} \operatorname{lie}\operatorname{Aut} P_z) \to \mathbb{R},$$

by fixing any Lie-group isomorphism $\operatorname{Aut} P_z \to G$. We may now define a closed (simplicial) $\mathbb{R}$-valued $2k$-form $\omega^{\rho,D}$ on X:

$$\omega^{\rho,D}_\Sigma(v_1, \ldots, v_{2k}) = \frac{1}{2k!} \sum_{\eta \in P_{2k}} \operatorname{sign}\eta \cdot \rho(R^D_\Sigma(v_{\eta(1)}, v_{\eta(2)}), \ldots, R^D_\Sigma(v_{\eta(2k-1)}, v_{\eta_{2k}})),$$

for P_{2k} as above the permutation group of a set with $2k$ elements. Set

$$\alpha^{\rho,D} := \int \omega^{\rho,D}.$$

Lemma 7.1. *For $P \to X$ as above*

$$[\alpha^{\rho,D}] = [\alpha^{\rho,D'}] \in H^{2k}(X, \mathbb{R}),$$

for any pair of G-connections D, D' on P.

Proof. For D, D' as in the statement, fix a concordance G-connection $\widetilde{D}$, between D, D', on the G-bundle $P \times I \to X \times I$, as in Lemma 6.2. Then $\alpha^{\rho,\widetilde{D}}$ is a $2k$ cocycle on $X \times I$ restricting to $\alpha^{\rho,D}, \alpha^{\rho,D'}$ on $X \times I_0, X \times I_1$.

Now the pair of inclusions

$$i_j : X \to X \times I \quad j = 0, 1$$

corresponding to the end points of I are homotopic and so $\alpha^{\rho,D}$, $\alpha^{\rho,D'}$ are cohomologous cocycles, cf. Section 4.1. □

Then we define the associated Chern-Weil characteristic class:

$$c^\rho(P) = c^\rho_{2k}(P) := [\alpha^{\rho,D}] \in H^{2k}(X, \mathbb{R}),$$

(the subscript $2k$ is implicitly given by ρ.) We have the expected naturality:

Lemma 7.2. *Let P be a simplicial G-bundle over Y, ρ as above and $f : X \to Y$ a smooth simplicial map. Then*

$$f^* c^\rho(P) = c^\rho(f^* P).$$

Proof. Let D be a simplicial G-connection on P then f^*D is a simplicial G-connection on f^*P and clearly $\omega^{\rho,f^*D} = f^*\omega^{\rho,D}$, so that passing to cohomology we obtain our result. □

Proposition 7.3. *Let $G \hookrightarrow Z \to Y$ be an ordinary smooth principal G-bundle, and ρ as above. Let $Z_\bullet$ be the induced simplicial G-bundle over $Y_\bullet$ as in Example 5.7. Then the classes $c^\rho(Z_\bullet) \in H^{2k}(Y_\bullet, \mathbb{R})$ coincide with the classical Chern-Weil classes of Z. More explicitly, if $c^\rho(Z) \in H^{2k}(Y, \mathbb{R})$ is the classical Chern-Weil characteristic class as in (7.2), then*

$$n^*(|c^\rho(Z_\bullet)|) = c^\rho(Z), \tag{7.3}$$

where $|c^\rho(Z_\bullet)|$ is as in Notation 4.6, and n is as in (4.5).

Proof. Fix a smooth G-connection D on Z. This induces a simplicial G-connection $D_\bullet$ on $Z_\bullet$, as in Example 6.3. Let $\omega^{\rho,D}$ denote the classical smooth Chern-Weil differential $2k$-form on Y, as in (7.1). Let $\alpha^{\rho,D} = \int \omega^{\rho,D} \in H^{2k}(Y, \mathbb{R})$. By its construction $\omega^{\rho,D_\bullet}$ is the simplicial differential form induced by $\omega^{\rho,D}$, where induced is as in Example 4.2. Consequently,

$$(I^c)^{-1}([\alpha^{\rho,D_\bullet}]) = [\alpha^{\rho,D}] = c^\rho(Z),$$

where I^c is as in (4.7). But

$$(I^c)^{-1}([\alpha^{\rho,D_\bullet}]) = n^*(|[\alpha^{\rho,D_\bullet}]|),$$

and so

$$n^*(|c^\rho(Z_\bullet)|) = c^\rho(Z).$$

□

8. The universal simplicial G-bundle

Briefly, a Grothendieck universe is a set $\mathcal{U}$ forming a model for set theory. That is if we interpret all terms of set theory as elements of $\mathcal{U}$, then all the set theoretic constructions keep us within $\mathcal{U}$. We will assume Grothendieck's axiom of universes which says that for any set X there is a Grothendieck universe $\mathcal{U} \ni X$. Intuitively, such a universe $\mathcal{U}$ is formed by from all possible set theoretic constructions starting with X. For example if $\mathcal{P}(X)$ denotes the power set of X, then $\mathcal{P}(X) \in \mathcal{U}$ and if $\{Y_i \in \mathcal{P}(X)\}_{i \in I}$ for $I \in \mathcal{U}$ is a collection then $\bigcup_i Y_i \in \mathcal{U}$. This may appear very natural, but we should note that this axiom is beyond ZFC. Although it is now a common axiom of modern set theory, especially in the context of category theory, c.f. [6]. In some contexts one works with universes implicitly. This is impossible here, as we need to establish certain universe independence.

Let G be a Frechet Lie group. Let $\mathcal{U}$ be a Grothendieck universe satisfying:

$$\mathcal{U} \ni \{G\}, \quad \forall n \in \mathbb{N} : \mathcal{U} \ni \{\Delta^n\},$$

where Δ^n are the usual topological n-simplices. These conditions are of course partly redundant by intent. Such a $\mathcal{U}$ will be called ***G-admissible***. We construct smooth Kan complexes $BG^{\mathcal{U}}$ for each G-admissible $\mathcal{U}$. The homotopy type of $BG^{\mathcal{U}}$ will then be shown to be independent of $\mathcal{U}$, provided G has the homotopy type of a CW complex. Moreover, in this case we will show that $BG^{\mathcal{U}} \simeq BG$, for BG the classical classifying space.

Definition 8.1. *A* **$\mathcal{U}$-small set** *is an element of $\mathcal{U}$. For X a smooth simplicial set, a smooth simplicial G-bundle $P \to X$ will be called* **$\mathcal{U}$-small** *if for each simplex Σ of X the bundles P_Σ are $\mathcal{U}$-small.*

8.1. **The classifying spaces $BG^{\mathcal{U}}$.** Let $\mathcal{U}$ be G-admissible. We define a simplicial set $BG^{\mathcal{U}}$, whose set of k-simplices $BG^{\mathcal{U}}(k)$ is the set of $\mathcal{U}$-small smooth simplicial G-bundles over $\Delta^k_\bullet$. The simplicial maps are just defined by pull-back so that given a map $i \in hom_\Delta([m],[n])$ the map

$$BG^{\mathcal{U}}(i) : BG^{\mathcal{U}}(n) \to BG^{\mathcal{U}}(m)$$

is just the natural pull-back:

$$BG^{\mathcal{U}}(i)(P) = i^*_\bullet P,$$

for $i_\bullet$, the induced map $i_\bullet : \Delta^m_\bullet \to \Delta^n_\bullet$, $P \in BG^{\mathcal{U}}(n)$ a simplicial G-bundle over $\Delta^n_\bullet$, and where the pull-back map $i^*_\bullet$ is as in Section 5.1. Then Lemma 5.12 insures that $BG^{\mathcal{U}}$ is a functor, so that we get a simplicial set $BG^{\mathcal{U}}$.

We define a smooth simplicial set structure on $BG^{\mathcal{U}}$ as follows. Given a d-simplex $P \in BG^{\mathcal{U}}(d)$ the induced map

$$\Sigma_* : \Delta^d_\bullet \to BG^{\mathcal{U}},$$

is defined naturally by

$$P_*(\sigma) := \sigma^*_\bullet P,$$

where P on the right is corresponding simplicial G-bundle $P \to \Delta^d_\bullet$. More explicitly, $\sigma \in \Delta^d_\bullet(k)$ is a smooth map $\sigma : \Delta^k \to \Delta^d$, $\sigma_\bullet : \Delta^k_\bullet \to \Delta^d_\bullet$ denotes the induced map and the pull-back is as previously. We need to check the push-forward functoriality Axiom 3.

For all $\sigma \in \Delta^d_\bullet(k)$:

$$(P_*(\sigma))_* = (P \circ \sigma_\bullet)_*,$$

and for all $\rho \in \Delta^k_\bullet(j)$

$$\begin{aligned}(P \circ \sigma_\bullet)_*(\rho) &= P \circ \sigma_\bullet \circ \rho_\bullet \\ &= P \circ (\sigma_\bullet(\rho))_\bullet \\ &= P_*(\sigma_\bullet(\rho)) \\ &= (P_* \circ \sigma_\bullet)(\rho).\end{aligned}$$

And so

$$(P_*(\sigma))_* = P_* \circ \sigma_\bullet,$$

so that $BG^{\mathcal{U}}$ is indeed a smooth simplicial set.

8.2. **The universal smooth simplicial G-bundle $EG^{\mathcal{U}}$.** In what follows V denotes $BG^{\mathcal{U}}$ for a general, G-admissible $\mathcal{U}$. There is a natural functor

$$E : \Delta^{sm}(V) \to \mathcal{G},$$

which we now describe.

A smooth map $P : \Delta^d_\bullet \to V$, corresponds to a simplex P of V via Lemma 3.5 [2], which by construction corresponds to a simplicial G-bundle $P \to \Delta^d_\bullet$. Recalling

[2]This is a slight abuse of notation, but helps to lower visual complexity.

that P is a functor on $\Delta^{sm}(\Delta^d_\bullet)$ we then set: $E(P) = P(id^d_\bullet)$, for $id^d_\bullet \in \Delta^{sm}(\Delta^d_\bullet)$, the identity map $id^d_\bullet : \Delta^d_\bullet \to \Delta^d_\bullet$. Now if we have a morphism $m \in \Delta^{sm}(V)$,

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "P_1"] & \Delta^d_\bullet \arrow[d, "P_2"] \\ & V, \end{tikzcd}$$

then by construction of V we have an equality of functors on $\Delta^{sm}(\Delta^k_\bullet)$:

$$P_1 = \widetilde{m}^*_\bullet P_2 = P_2 \circ \Delta^{sm}\widetilde{m}_\bullet.$$

So that

$$P_1(id^k_\bullet) = P_2(\widetilde{m}_\bullet \circ id^k_\bullet)) = P_2(\widetilde{m}_\bullet).$$

We have a tautological morphism $e_m \in \Delta^{sm}(\Delta^d_\bullet)$ corresponding to the diagram:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "\widetilde{m}_\bullet"] & \Delta^d_\bullet \arrow[d, "id^d_\bullet"] \\ & \Delta^d_\bullet, \end{tikzcd}$$

So we get a smooth G-bundle map:

$$P_2(e_m) : P_2(\widetilde{m}_\bullet) \to P_2(id^d_\bullet),$$

which is over the smooth map $\widetilde{m} : \Delta^k \to \Delta^d$ induced by $\widetilde{m}_\bullet$. And we set $E(m) = P_2(e_m)$.

We need to check that with these assignments, E is a functor. Suppose we have a diagram:

$$\begin{tikzcd} \Delta^l_\bullet \arrow[r, "\widetilde{m}^0_\bullet"] \arrow[drr, bend right, "P_0"] & \Delta^k_\bullet \arrow[r, "\widetilde{m}^1_\bullet"] \arrow[dr, "P_1"] & \Delta^d_\bullet \arrow[d, "P_2"] \\ & & V, \end{tikzcd}$$

Then $e_m = e_{m_1} \circ e'_{m_0}$ where e'_{m_0} is the diagram:

$$\begin{tikzcd} \Delta^l_\bullet \arrow[r, "\widetilde{m}^0_\bullet"] \arrow[dr, "\widetilde{m}_\bullet"] & \Delta^k_\bullet \arrow[d, "\widetilde{m}^1_\bullet"] \\ & \Delta^d_\bullet, \end{tikzcd}$$

and e_{m_1} is the diagram:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}^1_\bullet"] \arrow[dr, "\widetilde{m}^1_\bullet"] & \Delta^d_\bullet \arrow[d, "id^d_\bullet"] \\ & \Delta^d_\bullet. \end{tikzcd}$$

So

$$E(m) = P_2(m) = P_2(e_{m_1}) \circ P_2(e'_{m_0}).$$

Now $P_2(e'_{m_0}) = P_1(e_{m_0})$, since we have an equality of functors on $\Delta^{sm}(\Delta^k_\bullet)$:

$$P_1 = (\widetilde{m}^1_\bullet)^* P_2 = P_2 \circ \Delta^{sm}\widetilde{m}^1_\bullet.$$

And so we get: $E(m) = E(m_1) \circ E(m_0)$. Thus E is a functor.

By construction the functor E satisfies the compatibility condition, and hence determines a simplicial G-bundle. The universal simplicial G-bundle $EG^{\mathcal{U}}$ is then another name for E above, for some $G, \mathcal{U}$.

Proposition 8.2. *V is a Kan complex.*

Proof. Let

$$E : \Delta(V) \to \mathcal{G}$$

be the restriction of E, as above, to $\Delta(V) \subset \Delta^{sm}(V)$. Recall that $\Lambda^n_k \subset \Delta^n_{simp}$, denotes the sub-simplicial set corresponding to the "boundary" of Δ^n with the k'th face removed, where by k'th face we mean the face opposite to the k'th vertex. Let $h : \Lambda^n_k \to V$, $0 \leq k \leq n$, be a simplicial map, this is also called a horn. We need to construct an extension of h to Δ^n_{simp}. For simplicity we assume $n = 2$, the general case is identical. Let

$$\Delta(h) : \Delta(\Lambda^n_k) \to \Delta(V)$$

be the induced functor. Set $P = E \circ \Delta(h)$. Clearly, to construct our extension we just need an appropriate extension of P over $\Delta(\Delta^n_{simp})$. (Appropriate, means that we need the compatibility condition of Definition 5.4 be satisfied.)

Lemma 8.3. *There is a natural transformation of $\mathcal{G}$ valued functors $tr : T \to P$, where T is the trivial functor $T : \Delta(\Lambda^n_k) \to \mathcal{G}$, $T(\sigma^d) = \Delta^d \times G$.*

Proof. Set $L := \Lambda^2_k$, with $k = 1$, again without loss of generality. There are three natural inclusions

$$i_j : \Delta^0_{simp} \to L,$$

$j = 0, 1, 2$, with i_1 corresponding to the inclusion of the horn vertex. The corresponding 0-simplices will just be denoted by $0, 1, 2$. Fix a G-bundle map (in this case just smooth G-torsor map):

$$\phi_1 : \Delta^0 \times G \to P(i_1).$$

Let

$$\sigma_{1,2} : \Delta^1_{simp} \to L$$

be the edge between vertexes $1, 2$, that is $\sigma_{1,2}(0) = 1$, $\sigma_{1,2}(1) = 2$. Then $P(\sigma_{1,2})$ is a smooth bundle over the contractible space Δ^1 and so we may find a G-bundle map

$$\phi_{1,2} : \Delta^1 \times G \to P(\sigma_{1,2}),$$

whose restriction to $\{0\} \times G$ is ϕ_1. Meaning:

$$\phi_{1,2} \circ (i_0 \times id) = \phi_1,$$

where

$$i_0 : \Delta^0 \to \Delta^1,$$

is the map $i_0(0) = 0$.

We may likewise construct a G-bundle map

$$\phi_{0,1} : \Delta^1 \times G \to P(\sigma_{0,1}),$$

(where $\sigma_{0,1}$ is defined analogously to $\sigma_{1,2}$), whose restriction to $\{1\} \times G$ is ϕ_1.

Then $\phi_{0,1}$, $\phi_{1,2}$ obviously glue to a natural transformation:

$$tr : T \to P.$$

□

To continue, we have the trivial extension of T,

$$\widetilde{T} : \Delta(\Delta^2_{simp}) \to \mathcal{G},$$

defined by

$$\widetilde{T}(\sigma^d) = \Delta^d \times G.$$

And so by the lemma above it is clear that P likewise has an extension $\widetilde{P}$ to $\Delta(\Delta^2_{simp})$, but we need this extension to be $\mathcal{U}$-small so that we must be explicit. Let σ^2 denote the non-degenerate 2-simplex of Δ^2. It suffices to construct $\widetilde{P}_{\sigma^2} := \widetilde{P}(\sigma^2)$. Let

$$\sigma_{0,1}, \sigma_{1,2} : \Delta^1 \to \Delta^2$$

be the edge inclusions of the edges between the vertices $0,1$, respectively $1,2$. And let $e_{0,1}, e_{1,2}$ denote their images.

We then define a set theoretic (for the moment no topology) G-bundle

$$\widetilde{P}_{\sigma^2} \xrightarrow{p} \Delta^2$$

by the following conditions:

$$\begin{aligned}
\sigma^*_{0,1}\widetilde{P}_{\sigma^2} &= P(\sigma_{0,1}),\\
\sigma^*_{1,2}\widetilde{P}_{\sigma^2} &= P(\sigma_{1,2}),\\
P_{\sigma^2}|_{(\Delta^2)^\circ} &= (\Delta^2)^\circ \times G,
\end{aligned}$$

where $(\Delta^2)^\circ$ denotes the topological interior of $\Delta^2 \subset \mathbb{R}^2$, and where the projection map p is natural.

We now discuss the topology. We have the smooth G-bundle maps

$$\phi^{-1}_{0,1} : P(\sigma_{0,1}) \to \Delta^2 \times G,$$

$$\phi^{-1}_{1,2} : P(\sigma_{1,2}) \to \Delta^2 \times G,$$

over $\sigma_{0,1}, \sigma_{1,2}$, as in the proof of the lemma above. Let d_0 be any metric on $\Delta^2 \times G$ inducing the natural product topology. The topology on $\widetilde{P}_{\sigma^2}$ will be given by the d-metric topology, for d extending d_0 on $(\Delta^2)^\circ \times G \subset \widetilde{P}_{\sigma^2}$, and defined as follows. For $y_1 \in \widetilde{P}_{\sigma^2}$ with $p(y_1) \in e_{0,1}$, y_2 arbitrary, $d(y_1, y_2) = d_0(\phi^{-1}_{0,1}(y_1), y_2)$. Likewise, for $y_1 \in \widetilde{P}_{\sigma^2}$ with $p(y_1) \in e_{1,2}$, y_2 arbitrary, $d(y_1, y_2) = d_0(\phi^{-1}_{1,2}(y_1), y_2)$. This defines $\widetilde{P}_{\sigma^2}$ as a topological G-bundle over Δ^2.

There is a natural topological G-bundle trivialization

$$\xi : \widetilde{P}_{\sigma^2} \to \Delta^2 \times G$$

defined as follows. $\xi(y) = y$ when $p(y) \in (\Delta^2)^\circ$ and $\xi(y) = \phi^{-1}_{0,1}(y)$ when $p(y) \in e_{0,1}$, $\xi(y) = \phi^{-1}_{1,2}(y)$ when $p(y) \in e_{0,2}$. We then take the smooth structure on $\widetilde{P}_{\sigma^2}$ to be the smooth structure pulled back by ξ. By construction $\widetilde{P}_{\sigma^2}$ is $\mathcal{U}$-small, as all of the constructions take place in $\mathcal{U}$. Moreover, by construction $\sigma^*_{0,1}\widetilde{P}_{\sigma^2} = P_{\sigma_{0,1}}$ as a smooth G-bundle and $\sigma^*_{1,2}\widetilde{P}_{\sigma^2} = P_{\sigma_{1,2}}$ as a smooth G-bundle, which readily follows by the fact that the maps $\phi_{0,1}, \phi_{1,2}$ are smooth G-bundle maps. Thus, we have constructed the needed extension. □

Theorem 8.4. *Let X be a smooth simplicial set. $\mathcal{U}$-small simplicial G-bundles $P \to X$ are classified by smooth maps*

$$f_P : X \to BG^{\mathcal{U}}.$$

Specifically:

(1) *For every $\mathcal{U}$-small P there is a natural smooth map $f_P : X \to BG^{\mathcal{U}}$ so that*
$$f_P^* EG^{\mathcal{U}} \simeq P$$
as simplicial G-bundles. We say in this case that f_P **classifies** *P.*

(2) *If P_1, P_2 are isomorphic $\mathcal{U}$-small smooth simplicial G-bundles over X then any classifying maps f_{P_1}, f_{P_2} for P_1, respectively P_2 are smoothly homotopic, as in Definition 3.13.*

(3) *If $X = Y_\bullet$ for Y a smooth manifold and $f, g : X \to BG^{\mathcal{U}}$ are smoothly homotopic then $P_f = f^* EG^{\mathcal{U}}, P_g = g^* EG^{\mathcal{U}}$ are isomorphic simplicial G-bundles.*

Proof. Set $V = BG^{\mathcal{U}}$, $E = EG^{\mathcal{U}}$. Let $P \to X$ be a $\mathcal{U}$-small simplicial G-bundle. Define $f_P : X \to V$ naturally by:

$$f_P(\Sigma) = \Sigma_*^* P, \tag{8.1}$$

where $\Sigma \in \Delta^d(X)$, $\Sigma_* : \Delta^d_\bullet \to X$, the induced map, and the pull-back $\Sigma_*^* P$ our usual simplicial G-bundle pull-back. Let us check that this map is simplicial. Let $m : [k] \to [d]$ be a morphism in Δ. We need to check that the following diagram commutes:

$$\begin{tikzcd} X(d) \arrow[r, "X(m)"] \arrow[d, "f_P"] & X(k) \arrow[d, "f_P"] \\ V(d) \arrow[r, "V(m)"] & V(k). \end{tikzcd}$$

Let $\Sigma \in X(d)$, then $(X(m)(\Sigma))_* = \Sigma_* \circ m_\bullet$ where $m_\bullet : \Delta^k_\bullet \to \Delta^d_\bullet$ is the simplicial map induced by $m : \Delta^k \to \Delta^m$. And so $f_P(X(m)(\Sigma)) = m_\bullet^*(\Sigma_*^* P) = V(m)(f_P(\Sigma))$, using Lemma 5.12. And so the diagram commutes.

Let us check that f_P is smooth. Let $\Sigma \in X(d)$, then we have:

$$\begin{aligned} (f_P(\Sigma))_*(\sigma) &= \sigma_\bullet^*(\Sigma_*^* P) \\ &= (\Sigma_* \circ \sigma_\bullet)^* P \text{ Lemma 5.12} \\ &= (\Sigma_*(\sigma))_*^* P \text{ as } \Sigma_* \text{ is smooth, Lemma 3.5} \\ &= (f_P \circ \Sigma_*)(\sigma), \end{aligned}$$

and so f_P is smooth.

Lemma 8.5. $f_P^* E = P$.

Proof. Let $\Sigma : \Delta^d_\bullet \to X$ be smooth. Then we have:

$$\begin{aligned} \Delta^{sm} f_P(\Sigma)(\sigma) = (f_P \circ \Sigma)(\sigma) = f_P(\Sigma(\sigma)) &= (\Sigma(\sigma)_*^*) P \text{ by definition of } f_P \\ &= (\Sigma^* \circ \sigma_\bullet)^* P \text{ as } \Sigma \text{ is smooth} \\ &= \sigma_\bullet^*(\Sigma^* P) \text{ Lemma 5.12} \\ &= (\Sigma^* P)_*(\sigma). \end{aligned} \tag{8.2}$$

So $\Delta^{sm} f_P(\Sigma) = (\Sigma^* P)_*$. Then

$$\begin{aligned} f_P^* E(\Sigma) = (E \circ \Delta^{sm} f_P)(\Sigma) &= E((\Sigma^* P)_*) \text{ by the above} \\ &= (\Sigma^* P)(id^d_\bullet) \text{ definition of } E \\ &= P(\Sigma). \end{aligned}$$

So $f_P^* E = P$ on objects. Now let m be the morphism:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "\Sigma_1"] & \Delta^d_\bullet \arrow[d, "\Sigma_2"] \\ & X, \end{tikzcd}$$

in $\Delta^{sm}(X)$. We then have:

$$\begin{aligned} f_P^* E(m) &= E(\Delta^{sm} f_P(m)) \\ &= \Sigma_2^* P(e_m) \text{ by definition of } E \text{ and } (8.2) \\ &= (P \circ \Delta^{sm} \Sigma_2)(e_m), \end{aligned}$$

where e_m is as in the definition of E. But $\Delta^{sm}\Sigma_2(e_m)$ is the diagram:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, bend right, "\Sigma_2 \circ \widetilde{m}_\bullet"] & \Delta^d_\bullet \arrow[d, "\Sigma_2 \circ id^d_\bullet"] \\ & X, \end{tikzcd}$$

i.e. it is the diagram m. So $(P \circ \Delta^{sm}\Sigma_2)(e_m) = P(m)$. Thus $f_P^* E = P$ on morphisms. $\square$

So we have proved the first part of the theorem. We now prove the second part of the theorem. Suppose that P_1', P_2' are isomorphic $\mathcal{U}$-small simplicial G-bundles over X. Let $f_{P_1'}, f_{P_2'}$ be some classifying maps for P_1', P_2'. In particular, there is an isomorphism of $\mathcal{U}$-small simplicial G-bundles

$$\phi : (P_1 := f_{P_1'}^* E) \to (P_2 := f_{P_2'}^* E).$$

We construct a $\mathcal{U}$-small simplicial G-bundle $\widetilde{P}$ over $X \times I$ as follows, where $I = \Delta^1_\bullet$ as before. Let σ be a k-simplex of X. Then ϕ specifies a G-bundle diffeomorphism $\phi_\sigma : P_1(\sigma) \to P_2(\sigma)$ over the identity map $\Delta^k \to \Delta^k$. Let M_σ be the mapping cylinder of ϕ_σ. So that

$$M_\sigma = (P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma))/\sim, \tag{8.3}$$

for $\sim$ the equivalence relation generated by the condition $(x, 1) \in P_1(\sigma) \times \Delta^1 \sim \phi(x) \in P_2(\sigma)$. Then M_σ, is a smooth G-bundle over $\Delta^k \times \Delta^1$. Let pr_X, pr_I be the natural projections of $X \times I$, to X respectively I. Let Σ be a d-simplex of $X \times I$. Let $\sigma_1 = pr_X \Sigma$, and $\sigma_2 = pr_I(\Sigma)$. Let $id^d : \Delta^d \to \Delta^d$ be the identity, so

$$(id^d, \sigma_2) : \Delta^d \to \Delta^d \times \Delta^1,$$

is a smooth map, where σ_2 is the corresponding smooth map $\sigma_2 : \Delta^d \to \Delta^1 = [0, 1]$. We then define

$$\widetilde{P}_\Sigma := (id^d, \sigma_2)^* M_{\sigma_1},$$

which is a smooth G-bundle over Δ^d.

Suppose that $\rho : \sigma \to \sigma'$ is a morphism in $\Delta^{sm}(X)$, for σ a k-simplex and σ' a d-simplex. As ϕ is a simplicial G-bundle map, we have a commutative diagram:

$$\begin{tikzcd} P_1(\sigma) \arrow[r, "P_1(\rho)"] \arrow[d, "\phi_\sigma"] & P_1(\sigma') \arrow[d, "\phi_{\sigma'}"] \\ P_2(\sigma) \arrow[r, "P_2(\rho)"] & P_2(\sigma'). \end{tikzcd} \tag{8.4}$$

And so we get a naturally induced (by the pair of maps $P_1(\rho), P_2(\rho)$) bundle map:

$$\begin{tikzcd} M_\sigma \arrow[r, "g_\rho"] \arrow[d] & M_{\sigma'} \arrow[d] \\ \Delta^k \times \Delta^1 \arrow[r, "\widetilde{\rho} \times id"] & \Delta^d \times \Delta^1. \end{tikzcd} \tag{8.5}$$

More explicitly, let $q_\sigma : P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma) \to M_\sigma$ denote the quotient map. Then

$$g_\rho : P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma) \to M_{\sigma'}$$

is defined by:

$$g_\rho(x, t) = q_{\sigma'}((P_1(\rho)(x), t)) \in M_{\sigma'},$$

for

$$(x, t) \in P_1(\sigma) \times \Delta^1,$$

while $g_\rho(y) = q_{\sigma'}(P_2(\rho)(y))$ for $y \in P_2(\sigma)$. By commutativity of (8.4) g_ρ induces the map $g_\rho : M_\sigma \to M_{\sigma'}$, appearing in (8.5).

Now suppose we have a morphism $m : \Sigma \to \Sigma'$ in $\Delta^{sm}(X \times I)$, where Σ is a k-simplex and Σ' is a d-simplex. Then we have a commutative diagram:

$$\begin{tikzcd} M_\sigma \arrow[r, "g_{pr_X(m)}"] \arrow[d] & M_{\sigma'} \arrow[d] \\ \Delta^k \times \Delta^1 \arrow[r, "\widetilde{m} \times id"] & \Delta^d \times \Delta^1 \\ \Delta^k \arrow[u, "h_1"] \arrow[r, "\widetilde{m}"] & \Delta^d \arrow[u, "h_2"'] \\ \widetilde{P}_\Sigma \arrow[u] \arrow[uuu, bend left=60] & \widetilde{P}_{\Sigma'} \arrow[u] \arrow[uuu, bend right=60] \end{tikzcd} \tag{8.6}$$

where $h_1 = (id^k, pr_I(\Sigma))$ and $h_2 = (id^d, pr_I(\Sigma'))$. We then readily get an induced natural bundle map:

$$\widetilde{P}(m) : \widetilde{P}_\Sigma \to \widetilde{P}_{\Sigma'},$$

as left most and right most arrows in the above commutative diagram are the natural maps in pull-back squares, and so by universality of the pull-back such a map exists and is uniquely determined. Of course $\widetilde{P}(m)$ is the unique map making the whole diagram (8.6) commute.

With the above assignments, it is immediate that $\widetilde{P}$ is indeed a functor, by the uniqueness of the assignment $\widetilde{P}(m)$. And this determines our $\mathcal{U}$-small smooth simplicial G-bundle $\widetilde{P} \to X \times I$. By the first part of the theorem, we have an induced smooth classifying map $f_{\widetilde{P}} : X \times I \to V$. By construction it is a homotopy between $f_{P'_1}, f_{P'_2}$. [3] So we have verified the second part of the theorem.

We now prove the third part of the theorem. Suppose that $f, g : X \to V$ are smoothly homotopic, and let $H : X \times I \to V$ be the corresponding smooth homotopy. By Lemma 5.11, the bundles P_f, P_g are induced by smooth G-bundles P'_f, P'_g over Y. Now $P_H = H^* E$ is a simplicial G-bundle over $X \times I = (Y \times [0,1])_\bullet$ and hence by Lemma 5.11 P_H is also induced by a smooth G-bundle P'_H over $Y \times [0,1]$. We may clearly in addition arrange that P'_H restricts to $P'_f \sqcup P'_g$ over $Y \times \partial[0,1]$. It follows that P'_f, P'_g are smoothly concordant and hence isomorphic smooth G-bundles, and so P_f, P_g are isomorphic simplicial G-bundles.

□

We now study the dependence on a Grothendieck universe $\mathcal{U}$.

Theorem 8.6. *Let G be a Frechet Lie group having the homotopy type of a CW complex. Let $\mathcal{U}$ be a G-admissible universe, let $|BG^{\mathcal{U}}|$ denote the geometric realization of $BG^{\mathcal{U}}$ and let BG^{top} denote the classical classifying space of G as defined by the Milnor construction [8]. Then there is a homotopy equivalence*

$$e^{\mathcal{U}} : |BG^{\mathcal{U}}| \to BG^{top},$$

which is natural in the sense that if $\mathcal{U} \ni \mathcal{U}'$ then

$$[e^{\mathcal{U}'} \circ |i^{\mathcal{U},\mathcal{U}'}|] = [e^{\mathcal{U}}], \tag{8.7}$$

where $|i^{\mathcal{U},\mathcal{U}'}| : |BG^{\mathcal{U}}| \to |BG^{\mathcal{U}'}|$ is the map of geometric realizations, induced by the natural inclusion $i^{\mathcal{U},\mathcal{U}'} : BG^{\mathcal{U}} \to BG^{\mathcal{U}'}$ and where $[\cdot]$ denotes the homotopy class. In particular, for G as above the homotopy type of $BG^{\mathcal{U}}$ is independent of the choice of G-admissible $\mathcal{U}$.

Proof. For $\mathcal{U}$ G-admissible let $\mathcal{U}'$ be a universe enlargement of $\mathcal{U}$, that is $\mathcal{U}'$ is a universe with $\mathcal{U}' \ni \mathcal{U}$. Set $V := BG^{\mathcal{U}}$, $V' := BG^{\mathcal{U}'}$, $E := EG^{\mathcal{U}}$, $E' := EG^{\mathcal{U}'}$. There is a natural inclusion map $i = i^{\mathcal{U},\mathcal{U}'} : V \to V'$, and $i^* E' = E$ [4].

Lemma 8.7. *Let G be any Frechet Lie group and V as above.*

$$i_* : \pi_k^{sm}(V) \to \pi_k^{sm}(V')$$

is a set isomorphism for all $k \in \mathbb{N}$, where π_k^{sm} are as in Definition 3.14.

Proof. We show that i_* is injective. Let $f, g : S^k_\bullet \to V$ be a pair of smooth maps. Let P_f, P_g denote the smooth bundles over S^k induced via Lemma 5.11 by f^*E, g^*E. Set $f' = i \circ f$, $g' = i \circ g$ and suppose that $F : S^k_\bullet \times I \to V'$ is a smooth homotopy between f', g'. By Lemma 5.11 the simplicial bundle F^*E' is induced by a smooth bundle $P_F \to S^k \times I$. In particular P_f, P_g are classically isomorphic smooth $\mathcal{U}$-small G-bundles. Taking the mapping cylinder for the corresponding G-bundle isomorphism gives us a smooth G-bundle $P' \to S^k \times I$ that is $\mathcal{U}$-small by construction. Finally, P' induces a smooth simplicial G-bundle H over $S^k_\bullet \times I$

---

[3] To be perfectly formal, this is not exactly right. For the same reason that fixing the standard construction of the pull-back, a bundle $P \to B$ is not set theoretically equal to the bundle $id^* P \to B$, for $id : B \to B$ the identity, (but they are of course naturally isomorphic.) However this slight ambiguity can be fixed following the same simple idea as in the proof of Proposition 8.2.

[4] This is indeed an equality, not just a natural isomorphism.

that by construction is $\mathcal{U}$-small. The classifying map $f_H : S^k_\bullet \times I \to V$ then gives a smooth homotopy between f, g.

We now show surjectivity of i_*. Let $f : S^k_\bullet \to V'$ be smooth. By Lemma 5.11 the simplicial G-bundle $f^* E'$ is induced by a smooth G-bundle $P' \to S^k$. Any such bundle is obtained by the clutching construction, that is P' is isomorphic as a smooth G-bundle to the bundle:

$$C = D^k_- \times G \sqcup D^k_+ \times G / \sim,$$

where D^k_+, D^k_- are two copies of the standard closed k-ball in $\mathbb{R}^k$, and $\sim$ is the following equivalence relation: for

$$(d, g) \in D^k_- \times G$$

$$(d, g) \sim \widetilde{f}(d, g) \in D^k_+ \times G,$$

where

$$\widetilde{f} : \partial D^k_- \times G \to \partial D^k_+ \times G, \quad \widetilde{f}(d, x) = (d, f(d)^{-1} \cdot x),$$

for some smooth $f : S^{k-1} \to G$. Then C is $\mathcal{U}$-small, since this gluing construction is carried out in $\mathcal{U}$.

Let

$$C_\bullet \to S^k_\bullet$$

denote the induced $\mathcal{U}$-small smooth simplicial G-bundle. Now $C_\bullet$ and $f^* E'$ are induced by isomorphic $\mathcal{U}'$-small smooth G-bundles, hence are isomorphic $\mathcal{U}'$-small simplicial G-bundles. And so by Theorem 8.4, the classifying map $f_{C_\bullet} \to V'$ is smoothly homotopic to f.

Since $C_\bullet$ is $\mathcal{U}$-small it is also classified by a smooth map $f' : S^k_\bullet \to V$. It is immediate that $[i \circ f'] = [f_{C_\bullet}]$, since $i^* E' = E$, and so $i_*([f']) = [f]$. $\square$

Corollary 8.8. *Let G be any Frechet Lie group, and V as above. Simplicial G-bundles over $S^k_\bullet$, up to smooth isomorphism, are classified by smooth homotopy classes of maps $f : S^k_\bullet \to V$. That is the mapping c_V:*

$$[f] \mapsto [P_f := f^* E]$$

is a set bijection from the set of smooth homotopy classes of maps $f : S^k_\bullet \to V$ to the set of isomorphism classes of simplicial G-bundles over $S^k_\bullet$.

Proof. c_V is well defined by the third part of Theorem 8.4. It is injective by the second part Theorem 8.4. Let P be a simplicial G-bundle over $S^k_\bullet$, then P is $\mathcal{U}'$ small for some G-admissible universe $\mathcal{U}' \supset \mathcal{U}$. So by the first part of Theorem 8.4, P is classified by some smooth map:

$$f' : S^k_\bullet \to BG^{\mathcal{U}'}.$$

By the preceding lemma there is a smooth map $f_P : S^k_\bullet \to V$ so that $[i \circ f_P] = [f']$, where $i : V \to BG^{\mathcal{U}'}$ is the inclusion. In particular by the second part of Theorem 8.4 $f_P^* E$ is isomorphic to P as a simplicial G-bundle. Thus c_V is surjective. $\square$

We now show the second part of the theorem. Set as before $V := BG^{\mathcal{U}}$, $E := EG^{\mathcal{U}}$ and set

$$|E| := \operatorname{colim}_{\Delta(V)} E$$

where $E : \Delta(V) \to \mathcal{G}$ is as previously discussed, and where the colimit is understood to be in the category of topological G-bundles. Let $|V|$ be the geometric realization as previously defined. Then we have a topological G-fibration

$$|E| \to |V|,$$

which is classified by some

$$e = e^{\mathcal{U}} : |V| \to BG^{top},$$

uniquely determined up to homotopy. In particular,

$$|E| \simeq e^* EG^{top}, \tag{8.8}$$

where EG^{top} is the universal G-bundle over BG^{top} and where $\simeq$ in this argument will always mean G-bundle isomorphism. We will show that e induces an isomorphism of all homotopy groups. At this point we will use the assumption that G has the homotopy type of a CW complex, so that BG^{top} has the homotopy type of a CW complex, and so e must then be a homotopy equivalence by Whitehead theorem, which will finish the proof.

Let $f : S^k \to BG^{top}$ be continuous. By Müller-Wockel [10], main result, the bundle $P_f := f^* EG^{top}$ is topologically isomorphic to a smooth G-bundle $P' \to S^k$. By the axiom of universes P' is $\mathcal{U}_0$-small for some G-admissible $\mathcal{U}_0$. So we obtain a $\mathcal{U}_0$-small simplicial G-bundle $P'_\bullet \to S^k_\bullet$.

By Lemma 8.7 $P'_\bullet \simeq g^* E$ for some

$$g : S^k_\bullet \to V,$$

where $\simeq$ is an isomorphism of simplicial G-bundles. Let $|P'_\bullet|$ denote the colimit

$$\operatorname{colim}_{\Delta(S^k_\bullet)} P'_\bullet,$$

where $P'_\bullet : \Delta(S^k_\bullet) \to \mathcal{G}$ is functor determined by $P'_\bullet$. And where as before the colimit is understood to be in the category of topological G-bundles.

Then $|P'_\bullet| \to |S^k_\bullet|$ is a topological G-bundle classified by $e \circ |g|$, for

$$|g| : |S^k_\bullet| \to |V|,$$

the naturally induced topological map.

By construction, there is a topological G-bundle map $|P'_\bullet| \to P'$, over the natural map $|S^k_\bullet| \to S^k$ as P' is a co-cone for the corresponding colimit diagram in $\mathcal{G}$. And so P' and hence P_f, as a topological G-bundle is isomorphic to $h^*|P'_\bullet|$, where

$$h : S^k \to |S^k_\bullet|$$

represents the generator of $\pi_k(|S^k_\bullet|)$. Thus $e \circ |g| \circ h$ represents the homotopy class $[f]$ and so $e_* : \pi_k(V) \to \pi_k(BG^{top})$ is surjective. Here, the notation $\pi_k(Y)$ means the set of free homotopy classes of maps $S^k \to Y$.

We prove injectivity. Let $f : S^k \to |V|$ be continuous. Let $P \to S^k$ be a smooth G-bundle topologically isomorphic to $f^*|E|$. Again P exists by [10]. By Corollary 8.8, $P_\bullet$ is classified by a smooth map:

$$g : S^k_\bullet \to V.$$

As before we then represent the class $[f]$, by $|g| \circ h$ for $h : S^k \to |S^k_\bullet|$ as above. Now suppose that $e \circ f$ is null-homotopic. Then by [10] P is smoothly isomorphic to the trivial G-bundle. Thus by Corollary 8.8 g is smoothly null-homotopic, so that $|g|$ is null-homotopic and so $[f] = [|g| \circ h]$ is the trivial class. So $e_* : \pi_k(V) \to \pi_k(BG^{top})$ is an isomorphism. It follows that e_* is an isomorphism of all homotopy groups.

Finally, we show naturality. Let

$$|i^{\mathcal{U},\mathcal{U}'}| : |V| \to |V'|$$

denote the map induced by the inclusion $i^{\mathcal{U},\mathcal{U}'}$. Since $E = (i^{\mathcal{U},\mathcal{U}'})^* E'$, (an actual equality), we have that

$$|E| \simeq |i^{\mathcal{U},\mathcal{U}'}|^* |E'|$$

and so

$$|E| \simeq |i^{\mathcal{U},\mathcal{U}'}|^* \circ (e^{\mathcal{U}'})^* EG^{top},$$

by (8.8), from which the conclusion immediately follows. □

9. The universal Chern-Weil homomorphism

Let G be a Frechet Lie group and $\mathfrak{g}$ its lie algebra. Pick any simplicial G-connection D on $EG^{\mathcal{U}} \to BG^{\mathcal{U}}$. Then given any Ad invariant symmetric multilinear functional:

$$\rho : (V = \prod_{i=1}^{i=k} \mathfrak{g}) \to \mathbb{R},$$

applying the theory of Section 7 we obtain the simplicial Chern-Weil differential $2k$-form $\omega^{\rho,D}$ on $BG^{\mathcal{U}}$. And we obtain an associated cohomology class $c^{\rho,\mathcal{U}} \in H^{2k}(BG^{\mathcal{U}}, \mathbb{R})$. We thus first arrive at an abstract form of the universal Chern-Weil homomorphism.

Proposition 9.1. *Let G be a Frechet Lie group and $\mathcal{U}$ a G-admissible Grothendieck universe. There is an algebra homomorphism:*

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG^{\mathcal{U}}, \mathbb{R}),$$

sending ρ as above to $c^{\rho,\mathcal{U}}$ and satisfying the following. Let $G \hookrightarrow Z \to Y$ be a $\mathcal{U}$-small smooth principal G-bundle. Let $c^\rho(Z_\bullet) \in H^{2k}(Y_\bullet)$ denote the Chern-Weil class associated to ρ. Then

$$f^*_{Z_\bullet} c^{\rho,\mathcal{U}} = c^\rho(Z_\bullet),$$

where $f_{Z_\bullet} : Y \to BG^{\mathcal{U}}$ is the classifying map of $Z_\bullet$.

Proof. This follows immediately by Lemma 7.2. □

Suppose now that G has the homotopy type of a CW complex. Let $e^{\mathcal{U}}$ be as in Theorem 8.6. We define the associated cohomology class

$$c^\rho := e^{\mathcal{U}}_*(|c^{\rho,\mathcal{U}}|) \in H^{2k}(BG^{top}, \mathbb{R}),$$

where the G-admissible universe $\mathcal{U}$ is chosen arbitrarily, where the pushforward means pull-back by the homotopy inverse, and where $|c^{\rho,\mathcal{U}}| \in H^{2k}(|BG^{\mathcal{U}}|, \mathbb{R})$ is as in Notation 4.6.

Lemma 9.2. *The cohomology class c^ρ is well defined.*

Proof. Given another choice of a G-admissible universe $\mathcal{U}'$, let $\mathcal{U}'' \ni \{\mathcal{U}, \mathcal{U}'\}$ be a common universe enlargement. By Lemma 7.2 and Lemma 4.7

$$|i^{\mathcal{U},\mathcal{U}''}|^*(|c^{\rho,\mathcal{U}''}|) = |c^{\rho,\mathcal{U}}|.$$

Since $|i^{\mathcal{U},\mathcal{U}''}|$ is a homotopy equivalence we conclude that

$$|i^{\mathcal{U},\mathcal{U}''}|_*(|c^{\rho,\mathcal{U}}|) = |c^{\rho,\mathcal{U}''}|,$$

where $|i^{\mathcal{U},\mathcal{U}''}|_*$ denotes the pull-back by the homotopy inverse. Consequently, by the naturality part of Theorem 8.6 and the equation above, we have

$$e^{\mathcal{U}}_*(|c^{\rho,\mathcal{U}}|) = e^{\mathcal{U}''}_* \circ |i^{\mathcal{U},\mathcal{U}''}|_*(|c^{\rho,\mathcal{U}})| = e^{\mathcal{U}''}_*(|c^{\rho,\mathcal{U}''}|).$$

In the same way we have:

$$e^{\mathcal{U}'}_*(|c^{\rho,\mathcal{U}'}|) = e^{\mathcal{U}''}_*(|c^{\rho,\mathcal{U}''}|).$$

So

$$e^{\mathcal{U}}_*(|c^{\rho,\mathcal{U}}|) = e^{\mathcal{U}'}_*(|c^{\rho,\mathcal{U}'}|),$$

and so we are done.

□

We call $c^\rho \in H^{2k}(BG^{top}, \mathbb{R})$ ***the universal Chern-Weil characteristic class associated to*** ρ.

Let $\mathbb{R}[\mathfrak{g}]$ denote the algebra of polynomial functions on $\mathfrak{g}$. And let $\mathbb{R}[\mathfrak{g}]^G$ denote the sub-algebra of fixed points by the adjoint G action. By classical algebra, degree k homogeneous elements of $\mathbb{R}[\mathfrak{g}]^G$ are in correspondence with symmetric G-invariant multi-linear functionals $\Pi_{i=1}^k \mathfrak{g} \to \mathbb{R}$. Then to summarize we have the following theorem purely about the classical classifying space BG^{top} and reformulating Theorem 1.1 of the introduction:

Theorem 9.3. *Let G be a Frechet Lie group having the homotopy type of a CW complex. There is an algebra homomorphism:*

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG^{top}, \mathbb{R}),$$

sending ρ as above to c^ρ as above and satisfying the following. Let $G \hookrightarrow Z \to Y$ be a smooth principal G-bundle. Let $c^\rho(Z) \in H^{2k}(Y)$ denote the classical Chern-Weil class associated to ρ. Then

$$f_Z^* c^\rho = c^\rho(Z),$$

where $f_Z : Y \to BG^{top}$ is the classifying map of the underlying topological G-bundle.

Proof. Let $\mathcal{U}_0 \ni Z$ be a G-admissible Grothendieck universe. By Lemma 7.2

$$c^\rho(Z_\bullet) = f^*_{Z_\bullet}(c^{\rho,\mathcal{U}_0}).$$

And by Proposition 7.3, $n^*(|c^\rho(Z_\bullet)|) = c^\rho(Z)$. So we have

$$\begin{aligned} c^\rho(Z) &= n^*(|c^\rho(Z_\bullet)|) \\ &= n^*(|f^*_{Z_\bullet}(c^{\rho,\mathcal{U}_0})|) \\ &= n^*(|f_{Z_\bullet}|^*(|c^{\rho,\mathcal{U}_0}|)) \text{ by Lemma 4.7} \\ &= n^* \circ |f_{Z_\bullet}|^* \circ (e^{\mathcal{U}_0})^* c^\rho, \text{ by definition of } c^\rho. \end{aligned}$$

Now $e^{\mathcal{U}_0} \circ |f_{Z_\bullet}| \circ n \simeq f_Z$ as by construction $e^{\mathcal{U}} \circ |f_{Z_\bullet}| \circ n$ classifies the topological G-bundle Z. So that

$$c^\rho(Z) = f_Z^* c^\rho,$$

and we are done. □

In other words we have constructed the universal Chern-Weil homomorphism for Frechet Lie groups with homotopy type of CW complexes. Another, related approach to the universal Chern-Weil homomorphism is contained in the book of Dupont [2]. Dupont only states the theorem above for compact Lie groups. Like us Dupont makes heavy use of simplicial techniques, for example the simplicial DeRham complex. However, the main thrust of his argument appears to be rather different, essentially arguing that all of the necessary differential geometry can be indirectly carried out on the Milnor classifying bundle $EG \to BG$, without endowing it with extra structure, beyond the tautological structures inherent in the Milnor construction. On the other hand we need the extra structure of a smooth simplicial set, and so work with the smooth Kan complexes $BG^{\mathcal{U}}$ to do our differential geometry, and then transfer the cohomological data to BG using technical ideas like [10]. So we have a more conceptually involved space, with a certain "smooth structure", but our differential geometry is rendered trivial, and in Dupont's case the space is the ordinary BG but the differential geometry is more involved.

10. Universal Chern-Weil theory for the group of symplectomorphisms

Let (M, ω) be a symplectic manifold, so that ω is a closed non-degenerate 2-form on M. Let $\mathcal{G} = Ham(M, \omega)$ denote the group of its Hamiltonian symplectomorphisms, and $\mathfrak{h}$ its Lie algebra. When M is simply connected this is just the group $Symp(M, \omega)$ of diffeomorphisms $\phi : M \to M$ s.t. $\phi^*\omega = \omega$.

For example $M = \mathbb{CP}^{n-1}$ with its Fubini-Study symplectic 2-form ω_{st}. Then the natural action of $PU(n)$ on $\mathbb{CP}^{n-1}$ is by Hamiltonian symplectomorphisms.

In [11] Reznikov constructs polynomials

$$\{\rho_k\}_{k>1} \subset \mathbb{R}[\mathfrak{h}]^{\mathcal{G}},$$

each ρ_k homogeneous of degree k. In particular given a principal bundle $\mathcal{G} \hookrightarrow P \to X$ for X a smooth manifold we obtain characteristic classes

$$c^{\rho_k}(P) \in H^{2k}(X, \mathbb{R}), \quad k > 1,$$

which were already used by Reznikov in [11] to great effect.

The group $Ham(M, \omega)$ is a Frechet Lie group having the homotopy type of a CW complex by Milnor [9]. In particular, Theorem 9.3 immediately tells us that there are induced cohomology classes

$$c^{\rho_k} \in H^{2k}(BHam(M, \omega), \mathbb{R}). \tag{10.1}$$

As mentioned, the group $PU(n)$ naturally acts on $\mathbb{CP}^{n-1}$ by Hamiltonian symplectomorphisms. So we have an induced map

$$i : BPU(n) \to BHam(\mathbb{CP}^{n-1}, \omega_0).$$

Then as one application we have the following reformulation of Theorem 1.2 of the introduction:

Theorem 10.1.

$$i^* : H^k(BHam(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}) \to H^k(BPU(n), \mathbb{R})$$

is surjective for all $n \geq 2$, $k \geq 0$ *and so*

$$i_* : H_k(BPU(n), \mathbb{R}) \to H_k(BHam(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}),$$

is injective for all $n \geq 2$, $k \geq 0$.

Proof. Let $\mathfrak{g}$ denote the Lie algebra of $PU(n)$. Let $j : \mathfrak{g} \to \mathfrak{h}$ denote the natural Lie algebra map induced by the homomorphism $PU(n) \to Ham(\mathbb{CP}^{n-1}, \omega_0)$. Reznikov [11] shows that $\{j^* \rho_k\}_{k>1}$ are the Chern polynomials. Specifically, the classes

$$c^{j^* \rho_k} \in H^{2k}(BPU(n), \mathbb{R}),$$

are the Chern classes $\{c_k\}_{k>1}$, which generate real cohomology of $BPU(n)$, as is well known. But $c^{j^* \rho_k} = i^* c^{\rho_k}$, for c^{ρ_k} as in (10.1), and so the result immediately follows. $\square$

As mentioned in the introduction, Reznikov [11] proves that

$$i_* : \pi_k(BPU(n)) \otimes \mathbb{R} \to \pi_k(BHam(\mathbb{CP}^{n-1}, \omega_0) \otimes \mathbb{R}, \tag{10.2}$$

is an injection for all k. See also Kedra-McDuff [4] where this is extended to somewhat more general groups of automorphisms. [5]

In Savelyev-Shelukhin [14] there are a number of results about induced maps in (twisted) K-theory. These are significant extensions as one goes from $\mathbb{R}$ coefficients homology/cohomology to integral K-theory. Indeed the following may be very interesting as it should require new ideas.

Question 10.2. *Is the map i above an injection integral homology?*

Theorem 10.1 surely extends to completely general compact semi-simple Lie groups G, with $\mathbb{CP}^n$ replaced by co-adjoint orbits M of G. We just need to compute the associated Reznikov polynomials in $\mathbb{R}[\mathfrak{h}]^{\mathcal{G}}$ and their pull-backs to $\mathfrak{g}$ as above. We can no longer expect injection in general. But the failure to be injective should be solely due to effects of classical representation theory, rather than transcendental effects of extending the structure group to $Ham(M, \omega)$, from a compact Lie group.

References

[1] K.-T. Chen, *Iterated path integrals*, Bull. Am. Math. Soc., 83 (1977), pp. 831–879.

[2] J. L. Dupont, *Curvature and characteristic classes*, vol. 640, Springer, Cham, 1978.

[3] D. Fiorenza, U. Schreiber, and J. Stasheff, *Čech cocycles for differential characteristic classes: an ∞-Lie theoretic construction*, Adv. Theor. Math. Phys., 16 (2012), pp. 149–250.

[4] J. Kedra and D. McDuff, *Homotopy properties of Hamiltonian group actions*, Geom. Topol., 9 (2005), pp. 121–162.

[5] H. Lee Tanaka and Y.-G. Oh, *Smooth constructions of homotopy-coherent actions*, To appear in Algebraic and Geometric Topology, arXiv:2003.06033.

[6] J. Lurie, *Higher topos theory.*, Annals of Mathematics Studies 170. Princeton, NJ: Princeton University Press. , 2009.

[7] J.-P. Magnot and J. Watts, *The diffeology of Milnor's classifying space*, Topology Appl., 232 (2017), pp. 189–213.

[8] J. W. Milnor, *Construction of universal bundles. II.*, Ann. Math. (2), 63 (1956), pp. 430–436.

[9] ———, *On spaces having the homotopy of a CW-complex*, Trans. Am. Math. Soc., 90 (1959), pp. 272–280.

[10] C. Müller and C. Wockel, *Equivalences of smooth and continuous principal bundles with infinite-dimensional structure group*, Adv. Geom., 9 (2009), pp. 605–626.

[11] A. G. Reznikov, *Characteristic classes in symplectic topology*, Selecta Math. (N.S.), 3 (1997), pp. 601–642. Appendix D by L Katzarkov.

[12] M. E. Rudin, *A new proof that metric spaces are paracompact*, Proc. Am. Math. Soc., 20 (1969), p. 603.

[5] It should be noted that in [4] it is stated that Reznikov's theorem above implies Theorem 10.1, this is an error as explained in Remark 1.3

[13] Y. Savelyev, *Global Fukaya category I: quantum Novikov conjecture*, To appear in IMRN, arXiv:1307.3991v5.

[14] Y. Savelyev and E. Shelukhin, *K-theoretic invariants of Hamiltonian fibrations*, J. Symplectic Geom., 18 (2020), pp. 251–289.

[15] D. Sullivan, *Infinitesimal computations in topology.*, Publ. Math., Inst. Hautes Étud. Sci, (1977).

University of Colima, CUICBAS
Email address: yasha.savelyev@gmail.com