GLOBAL FUKAYA CATEGORY II: SINGULAR CONNECTIONS AND OTHER APPLICATIONS

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ABSTRACT. Using the global Fukaya category introduced in Part I, which is a certain Floer and higher category theoretic invariant of a Hamiltonian fibration, we find curvature lower bounds for certain singular $SU(2)$ connections on principal $SU(2)$-bundles over S^4, the latter being of critical interest in physical Yang-Mills theory. This phenomenon is invisible to Chern-Weil theory, and inaccessible to known Yang-Mills theory techniques, which are the sharpest known "classical" techniques in this setting. Loosely speaking our lower bounds are obtained not by integrating some curvature forms, but by counting certain holomorphic curves with Lagrangian boundary conditions. So this can be understood as one application of Floer theory and the theory of ∞-categories in basic differential geometry. On the way we also construct a higher dimensional version of the relative Seidel morphism studied by Hu and Lalonde, compute this in a particular case and discuss an application to Hofer geometry of the space of Lagrangian equators in S^2.

1. INTRODUCTION

A *Hamiltonian fibration* is a smooth fiber bundle

$$M \hookrightarrow P \to X,$$

with structure group $\mathrm{Ham}(M, \omega)$ with its C^∞ Frechet topology. Given a Hamiltonian bundle $M \hookrightarrow P \to X$, for M monotone, in Part I [18]. we have constructed a classifying map

$$f_P : X \to (\mathcal{S}, NFuk(M))$$

where $\mathcal{S}$ denotes the space of ∞-categories, concretely quasi-categories, in the component of the A_∞ nerve $NFuk(M)$ of the Fukaya category of M, having the structure of an ∞-category. This extends to the universal level so that there is a classifying map:

$$B\mathrm{Ham}(M, \omega) \to (\mathcal{S}, NFuk(M).$$

The construction also induces a kind of topological/categorical fibration over X called (co)-Cartesian fibration, with fiber modelled on $NFuk(M)$. We called this the global Fukaya category $Fuk_\infty(P)$ of P. One aim here is to give basic applications of that construction.

As first computational step, we show that for P a non-trivial Hamiltonian S^2 fibration over S^4, the maximal Kan sub-fibration of $Fuk_\infty(P)$, which is just a combinatorial analogue of a Serre fibration, is non-trivial. In particular $Fuk_\infty(P)$ is non-trivial as a (co)-Cartesian fibration and so f_P is homotopically non-trivial. This gives in particular:

Theorem 1.1. *The natural homomorphism as constructed in Part I,*

$$\mathbb{Z} = \pi_4(BHam(S^2), id) \xrightarrow{k} \pi_4(\mathcal{S}, NFuk(S^2)),$$

is injective.

None of the homotopy groups of $\mathcal{S}$ are known, so this is an application of geometry to algebraic topology. Morally, such an application is possible because geometry forces a priori A_∞-associativity of certain structures, which may then have formal consequences in algebraic topology.

The calculation is performed by constructing perturbation data in such a way that we are reduced to calculation of a certain higher product in a certain Fukaya type A_∞ category. Note that this is an actual chain level calculation. To perform it we construct a higher relative Seidel element - a higher dimensional analogue of the relative Seidel element in [8]. The calculation of this higher Seidel element uses a regularization technique based on "virtual Morse theory" for the Hofer length functional [16].

It is likely that k is surjective. Surjectivity is in a sense the statement that up to equivalence there are no exotic (co)-Cartesian fibrations over S^4, with fiber equivalent to $N(Fuk(S^2))$ - they all come from Hamiltonian S^2 fibrations, via the global Fukaya category.

1.1. **Some lower bounds for the curvature of singular connections.** As one less expected application, we can use the computation of Theorem 1.1 to obtain lower bounds for curvature of certain singular connections. First we give a basic definition of a singular connection. Note that in existing literature there may be additional conditions, but the following suffices for the moment.

Definition 1.2. *Let $G \hookrightarrow P \to X$ be a principal G bundle, where G is a Frechet Lie group. A* **singular G-connection** *on P is a closed subset $C \subset X$, and a smooth Ehresmann G-connection $\mathcal{A}$ on $P|_{X-C}$.*

We now introduce a certain related abstraction, which can be understood as a simplicial resolution of a singular connection.

Definition 1.3. *Let $G \hookrightarrow P \to X$ be a principal G bundle, where G is a Frechet Lie group. Denote by $X_\bullet$ the simplicial set whose set of n-simplices, $X_\bullet(n)$, consists smooth maps $\Sigma : \Delta^n \to X$, with Δ^n standard topological n-simplex with vertices ordered $0, \ldots, n$. Let $Y_\bullet \subset X_\bullet$ be a Kan sub-complex so that its geometric realization satisfies $|Y_\bullet| \simeq X$. And denote by $\Delta(Y_\bullet)$ the category with objects $\cup_n Y_\bullet(n)$ and with $hom(\Sigma_0, \Sigma_1)$ commutative diagrams:*

$$\begin{tikzcd} \Delta^n \arrow[r, "mor"] \arrow[dr, "\Sigma_0"'] & \Delta^m \arrow[d, "\Sigma_1"] \\ & X, \end{tikzcd}$$

for mor a simplicial map, that is a linear map preserving order of the simplices. Define a **singular simplicial G-connection** *$\mathcal{A}$ or hereby just simplicial G-connection on P to be the following data:*

- *A choice of $Y_\bullet \subset X_\bullet$ as above.*
- *For each $\Sigma : \Delta^n \to X$ in $Y_\bullet$ a smooth G-connection $\mathcal{A}_\Sigma$ on $\Sigma^* P \to \Delta^n$, (a usual Ehresmann G-connection.)*
- *For a morphism $mor : \Sigma_0 \to \Sigma_1$ in $\Delta(Y_\bullet)$, we ask that $mor^* \mathcal{A}_{\Sigma_1} = \mathcal{A}_{\Sigma_0}$.*

If we try to "push forward" a simplicial connection to get a "classical" connection on X, then we get a kind of multi-valued singular connection. Multi-valued because each $x \in X$ may be in the image of a number of $\Sigma : \Delta^n \to X$ and Σ itself may not be injective, and singular because each Σ is in general singular so that the naive push-forward may have blow up singularities. Versions of singular connections have been studied by a number of authors, see for example [6], [22]. The second reference curiously also deals with $SU(2)$ connections over 4-manifolds as does our main example. The above abstract variant appears to be new, however it should be closely related to previously studied variants. One strength of our variant is its simple push forward functoriality:

Example 1.4. *$P_1 \to Y$, $P_2 \to X$ be principal G-bundles, $\widetilde{f} : P_1 \to P_2$ be a smooth G-bundle map over a smooth homotopy equivalence $f : Y \to X$ and $\mathcal{A}$ a smooth G-connection on P_1. Then there is an induced simplicial G-connection $f_*\mathcal{A}$ on P_2, with respect to $f(K_\bullet)$ for each choice of a Kan subcomplex $K_\bullet \subset Y_\bullet$ with $|K_\bullet| \simeq Y$. More generally we can push-forward any simplicial G-connection by any smooth bundle map over a smooth homotopy equivalence. Specifically for any $\Sigma \in K_\bullet$, by universality property of pull-back diagrams, there is a natural universal map $u : \Sigma^* P_1 \to (f \circ \Sigma)^* P_2$ which is a bundle diffeomorphism, and we define*

$$f_*\mathcal{A}_{(f\circ\Sigma)} = u_*\mathcal{A}_\Sigma.$$

Let $G \hookrightarrow P \to X$ be as above, g a Riemannian metric on X, and $\mathcal{A}$ a simplicial G-connection $\mathcal{A}$ on P. The curvature of each $\mathcal{A}_\Sigma$, for

$$\Sigma : \Delta^n \to X$$

is understood as a 2-form $R^{\mathcal{A}}_\Sigma$ on Δ^n with

$$R^{\mathcal{A}}_\Sigma(v, w) \in \operatorname{lie}\operatorname{Aut} P_x,$$

for $x \in \Delta^n$, $v, w \in T_x\Delta^n$, $P_x = \Sigma^* P|_x$, and $\operatorname{Aut} P_x \simeq G$ the group of G-torsor automorphisms of P_x, where $\simeq$ means non-canonical group isomorphism.

Definition 1.5. *For $M, P, X, g, \mathcal{A}$ as above, and given a Ad-invariant norm $|\cdot|$ on $\operatorname{lie} G$ we define a kind of L^∞ norm:*

$$|\mathcal{A}|_g = \sup_{n\in\mathbb{N},\Sigma\in Y_\bullet(n),x\in\Delta^n,v,w\in T_x\Delta^n} \frac{1}{ar(\Sigma_* v, \Sigma_* w)} |R^{\mathcal{A}}_\Sigma(v, w)| \in \mathbb{R}_{\geq 0} \sqcup \{\infty\}, \tag{1.1}$$

where for each x $ar : T_xX \times T_xX \to \mathbb{R}$ satisfies:

$$ar(v, w) := \sqrt{g(v, v) \cdot g(w, w) - g(v, w)^2}.$$

If for some v, w $R^{\mathcal{A}}_\Sigma(v, w) \neq 0$ but $ar(\Sigma_* v, \Sigma_* w) = 0$ we set $|\mathcal{A}|_g = \infty$. If $R^{\mathcal{A}}_\Sigma(v, w) = 0$ the value of the expression $\frac{1}{ar(\Sigma_* v,\Sigma_* w)}|R^{\mathcal{A}}_\Sigma|$ is set to be 0, irrespectively of the value $ar(\Sigma_* v, \Sigma_* w)$.

Definition 1.6. *Let $f : (N, b) \to (\Omega^2 X, const)$ be a based smooth map, $const \in \Omega^2 X$ the constant map to some $x_0 \in X$, where (X, g) is a Riemannian manifold and N a closed smooth manifold. Define*

$$sys^2(X, N, g, f) = \inf_{f'\in[f]} \sup_{y\in N} \operatorname{area}_g f'(y),$$

where $[f]$ denotes the based homotopy class.

We will set

$$\hbar_g := \frac{1}{sys^2(S^4, S^2, g, gen)} > 0, \tag{1.2}$$

where $gen: S^2 \to \Omega^2 S^4$ is the π_2 generator.

Theorem 1.7. *Let $S^2 \hookrightarrow P \to S^4$ be a non-trivial $SU(2)$ fibration, and g a metric on S^4, then:*

$$\inf_{\mathcal{A}} |\mathcal{A}|_g \geq 1/2 \cdot \hbar_g,$$

where infimum is taken over all simplicial $SU(2)$-connections $\mathcal{A}$ on P, and the norm on $\operatorname{lie} SU(2)$ is taken to be the operator norm normalized so that the Finsler length of the geodesic from id to $-id$ is $\frac{1}{2}$.

We now give a more concrete corollary the above.

Definition 1.8. *Let $(C, \mathcal{A})$ be a singular connection as in Definition 1.2. A* **resolution of** *$\mathcal{A}$ is a smooth G-bundle $P' \to Y$ a smooth G-connection $\mathcal{A}'$ on P', and G-bundle map:*

$$\begin{array}{ccc} P' & \xrightarrow{\widetilde{f}} & P \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

s.t. the set of singular values of f is C, and $\mathcal{A}'(z) = \widetilde{f}^* \mathcal{A}(\widetilde{f}(z))$ whenever $df(p'(z))$ is non-singular, in other words whenever $f(p'(z)) \notin C$.

Remark 1.9. *More generally we can resolve by simplicial connections, we leave this generalization to the reader. In particular the theorem below immediately generalizes to simplicial resolutions.*

Theorem 1.7 has more tangible consequences, for example the following is a partial corollary:

Theorem 1.10. *Let $S^2 \hookrightarrow P \to S^4$ be a non-trivial $SU(2)$ fibration, g a metric on S^4. Let $(C, \mathcal{A})$ be a singular $SU(2)$ connection on P which admits a resolution $(P', \widetilde{f}, \mathcal{A}')$:*

$$\begin{array}{ccc} P' & \xrightarrow{\widetilde{f}} & P \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & S^4 \end{array}$$

s.t. $f: Y \to S^4$ is a homotopy equivalence. Suppose further that $R^{\mathcal{A}'}$ is identically zero on $f^{-1}(C)$. Then

$$|\mathcal{A}|_g \geq 1/2 \cdot \hbar_g,$$

where:

$$|\mathcal{A}|_g = \sup_{x \in S^4 - C, v, w \in T_x S^4} |R^{\mathcal{A}}(v, w)|,$$

where the norm on the right is as in Theorem 1.7, and the supremum is over all orthonormal pairs v, w.

The author is not aware of any analogues of this theorem.

Question 1.11. *Is the lower bound of the two theorems above sharp? If it is not sharp can our technique be made sharper?*

Note that it is not sharp for smooth $SU(2)$ connections, a sharp lower bound is twice the lower bound above, cf. [20]. So that if it is sharp for our singular connections then we can use singularities to manipulate the global curvature conditions. I do not see how to do this at the moment, although it is certainly conceivable. Note also that the Kan condition on $Y_\bullet$ is absolutely essential, it is easy to construct a non Kan $Y_\bullet \subset X_\bullet$ with $|Y_\bullet| \simeq X$, and an $\mathcal{A}$ on P above with $|\mathcal{A}|_g = 0$. For example, we may take $Y_\bullet$ to correspond to a triangulation, with all simplices in $\Sigma \in Y_\bullet$ non-degenerate, and then construct $\mathcal{A}$ so that each $\mathcal{A}_\Sigma$ is flat. Thus, Theorem 1.7 expresses a certain non-trivial marriage of differential geometry and abstract combinatorial topology.

Question 1.12. *Can such a lower bound be obtained for simplicial $SU(2)$ connections without our Floer holomorphic curve machinery? Perhaps with Yang-Mills theory?*

A few notes on this. For usual smooth $SU(2)$ connections Chern-Weil theory by itself will not give such a lower bound, for an explanatory computation of exactly this kind see [13]. However, as already mentioned for smooth $SU(2)$ connections a sharper lower bound can be obtained via Yang-Mills theory [20]. It is not quiet clear if and how Yang-Mills theory of [20] can be extended to simplicial connections. If we attempt to translate the argument of this paper then as a first step one needs to extend Yang-Mills theory to G-fibrations, with G compact, over surfaces with corners, and G-connections with constraints on the boundary holonomy maps. (Of course G is just $SU(2)$ in our specific example, but this shouldn't be needed.) One issue is what to do with the corners. In the setting of this paper, figuratively speaking, our Hamiltonian fibrations over such a surface are resolved over corners using Lagrangian boundary conditions. A direct translation of this to Yang-Mills theory is not known to me, but perhaps it is not really necessary, if one can make analysis work with corners. Notwithstanding, it is not clear what happens past this first step, since in the case of this paper we also need to involve certain abstract algebraic topology to glue all the data, and a translation of this to Yang-Mills theory is not apparent.

1.2. **Hamiltonian rigidity vs flexibility.** By way of the calculation we also obtain an application in Hofer geometry. It can be understood as a relative analogue of a result in [17]. Let $Lag(M, L_0)$ denote the space of oriented Lagrangian submanifolds of M Hamiltonian isotopic to L_0, we may also just write $Lag(M)$. Let $\Omega_{L_0} Lag(M)$ denote the space based smooth loops in $Lag(M)$, constant near end points, and let $\Omega^{taut}_{L_0} Lag(M) \subset \Omega_{L_0} Lag(M)$ be the subspace of loops taut concordant to the constant loop at L_0. The notion of taut concordance is defined in more generality in Definition 6.4. In the case above, two loops p_1, p_2 in $Lag(M)$ are said to be *taut concordant*, if we have a Lagrangian sub-fibration $\mathcal{L}$ of $Cyl \times M$, $Cyl = S^1 \times [0,1]$, so that $\mathcal{L}$ over the boundary circles corresponds to the pair of loops p_1, p_2 and so that there is a Hamiltonian connection $\mathcal{A}$ on $M \times [0,1]$ preserving $\mathcal{L}$, and so that the coupling form $\Omega_\mathcal{A}$ of $\mathcal{A}$ vanishes on $\mathcal{L}$, see Section 6.1 for the definition of coupling forms.

Note that of course $Lag(S^2)$ is homotopy equivalent to $Lag^{eq}(S^2) \simeq S^2$ where $Lag^{eq}(S^2)$ denotes the space of oriented equators in S^2. Moreover, there is an embedding $i : \Omega S^2 \hookrightarrow \Omega^{taut}_{L_0} Lag(S^2)$, this is because two loops $p_1, p_2 \in \Omega Lag^{eq}(S^2) \simeq \Omega S^2$ are taut concordant iff they are homotopic in $Lag^{eq}(S^2)$, see Lemma 10.4.

Theorem 1.13. *Let $L_0 \subset S^2$ be the equator. And let*

$$f : S^2 \to \Omega^{taut}_{L_0} Lag(S^2),$$

represent $i_* g$, for g the generator of

$$\pi_2(\Omega(S^2)) \simeq \pi_3(S^2)) \simeq \mathbb{Z},$$

and i as above. Then we have identity for the systole with respect to L^+:

$$\min_{f', [f']=[f]} \max_{s \in S^2} L^+(f'(s)) = 1/2 \cdot \text{area}(S^2, \omega),$$

where L^+ denotes the positive Hofer length functional, as defined in Section 10.1.1. The minimum is attained on a cycle of equators in S^2.

Even though everything is now smooth, this is not obvious. For by contrast if we measure a related quantity of the "girth" (infimum of the diameter of a representative) of the generator $[g]$ of $\pi_2 Lag(S^2)$, as in [14], then there is an upper bound for this girth which is smaller then the lower bound for girth considered in the subspace of $Lag(S^2)$ consisting of equators. So moving from equators to general oriented S^1 Lagrangians in S^2 we may reduce the girth to less than the classically expected girth. By "classical" we mean for the classical objects: great circles. Indeed, it may be that girth of the generator

$$[g] \in \pi_2 Lag(S^2)$$

is actually 0. (This would rather astonishing however.) On the other hand, our theorem says that this kind of non-classical squeezing does not happen at all for the systole we consider. In other words whereas our systole exhibits Hamiltonian rigidity, the girth in [14] while closely related, exhibits flexibility.

This is proved in section 12. On the way in Section 9 we construct a higher dimensional version of the relative Seidel morphism [8] in the monotone context, and show its non triviality in Section 10. The Sections 12, 9, 10 are mostly logically independent of the ∞-categorical and even the A_∞ setup and may be read independently.

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3. Outline

Notation 3.1. *We use notation Δ^n to denote the standard topological n-simplex. For the standard representable n-simplex as a simplicial set we use the notation $\Delta^n_\bullet$, in general when X is a topological space $X_\bullet$ will mean the singular simplicial set of X. Likewise if $p : X \to Y$ is map of spaces $p_\bullet : X_\bullet \to Y_\bullet$ will mean the induced simplicial map. However the notation $Y_\bullet$ could also mean an abstract simplicial set.*

In what follows when we say Part I we will mean [18]. Let us briefly review what we do in Part I. Let $M \hookrightarrow P \xrightarrow{p} X$ be a Hamiltonian fibration. Denote by $\Delta(X) := \Delta/X_\bullet$ the smooth simplex category of X, with objects smooth maps $\Sigma : \Delta^n \to X$ and morphisms commutative diagrams:

$$\begin{tikzcd} \Delta^n \arrow[r, "mor"] \arrow[dr, "\Sigma_0"'] & \Delta^m \arrow[d, "\Sigma_1"] \\ & X, \end{tikzcd}$$

where $mor : \Delta^n \to \Delta^m$ is a simplicial map, that is a linear map taking vertices to vertices, preserving the order. The distinction with $Simp(X)$ is that in $\Delta(X)$ we allow mor to be degenerate.

Then given certain auxiliary perturbation data $\mathcal{D}$, which in particular involves a choice of a natural system of maps from the universal curves to Δ^n and certain choices of Hamiltonian connections, we construct in Part I a functor

$$F : \Delta(X) \to A_\infty - Cat,$$

where $A_\infty - Cat$ denotes the category of A_∞ categories. The concordance class of this functor suitably defined is the invariant from which everything else is derived. The properties of this functor are such that we may algebraically, naturally construct from this a functor

$$F^{unit}: \Delta(X) \to A_\infty - Cat^{unit},$$

with $A_\infty - Cat^{unit}$ denoting the category of unital A_∞ categories, by taking unital replacements. In what follows we rename F by F^{raw} and F^{unit} by F, as this will visually simplify notation.

We then define

$$Fuk_\infty(P) = \operatorname{colim}_{\Delta(X)} NF,$$

which is shown to be an ∞-category whose equivalence class (under concordance) is independent of all choices. This also has the structure of a co-Cartesian fibration:

$$NFuk(M) \hookrightarrow Fuk_\infty(P) \to X_\bullet,$$

where $NFuk(M)$ is the A_∞ nerve of the Fukaya category of M and $X_\bullet$ denotes the simplicial set of singular simplices in X. The notion of co-Cartesian fibration corresponds to, in categorical language, a relaxation of the notion of Serre/Kan fibrations. Indeed what will do here is extract from the above data a Kan fibration and work with that, since then we can just use standard tools of topology. To this end we will use the following elementary lemma.

Lemma 3.2. *Suppose we have a (co)-Cartesian fibration Y over a Kan complex X. Let $K(Y)$ denote the maximal Kan sub-complex of Y then $K(Y)$ is a Kan fibration over X.*

The proof is given in Appendix A. In particular by the above lemma $K(P) := K(Fuk_\infty(P))$ is a Kan fibration over $X_\bullet$.

Definition 3.3. *We say that a Kan fibration or a (co)-Cartesian fibration P over a Kan complex X is* **non-trivial** *if it is not null-concordant. Here P is* **null-concordant** *means that there is a Kan respectively (co)-Cartesian fibration $Y \to X \times \Delta^1$, whose pull-back $i_0 : \Delta^0 \to \Delta^1$ is trivial and by $i_1 : \Delta^0 \to \Delta^1$ is P. For i_0, i_1 the two vertex inclusions.*

Theorem 3.4. *Suppose that $P \to S^4$ is a non-trivial Hamiltonian S^2 fibration then $p_\bullet : K(P) \to S^4_\bullet$ does not admit a section. In particular $K(P)$ is a non-trivial Kan fibration over $S^4_\bullet$ and so $Fuk_\infty(P)$ is a non-trivial (co)-Cartesian fibration over $S^4_\bullet$.*

This is main technical result of the paper, and its statement may give some intuition on why it may be difficult to reproduce our geometric application to curvature lower bounds with more elementary methods. Although in a sense we just are just deducing existence of a certain holomorphic curve, for this deduction we need a global compatibility condition involving multiple moduli spaces, involved in multiple local datum's of Fukaya categories, so that this computation is not straightforward.

The proof will be aided by constructing suitable perturbation data, and will be split into a number of sections. As previously indicated the arguments are quiet general, however for simplicity we focus on a special case.

4. Qualitative description of perturbation data

A bit of possibly non-standard terminology: we say that A is a *model* for B in some category if there is a morphism $mod : A \to B$ which is an (weak)-equivalence, in an appropriate sense that will be clear from context. The map *mod* will be called a *modelling map*. In our context the modeling map *mod* always turns out to be a monomorphism.

Let $Fuk(S^2)$ denote the $\mathbb{Z}_2$-graded A_∞ category over $\mathbb{Q}$, with objects oriented spin Lagrangian submanifolds Hamiltonian isotopic to the equator. Our particular construction of $Fuk(M)$ is presented in Part I.

Denote by $Fuk^{eq}(S^2) \subset Fuk(S^2)$ the full sub-category obtained by restricting our objects to be equators in S^2. We take our perturbation data $\mathcal{D}_{pt}$ for construction of $Fuk(S^2)$ so that the following is satisfied. All the connections $\mathcal{A}(L, L')$ for $L, L' \in Fuk^{eq}(S^2)$ are $SU(2)$-connections. For L intersecting L' transversally the $SU(2)$ connection $\mathcal{A}(L, L')$ is the trivial flat connection, which is admissible since in this case L, L' intersect transversally. For $L = L'$ the corresponding connection is generated by an autonomous Hamiltonian.

The associated cohomological Donaldson-Fukaya category $DF(S^2)$ is equivalent as a linear category over $\mathbb{Q}$ to $FH(L_0, L_0)$ (considered as a linear category with one object) for $L_0 \in Fuk(S^2)$.

It is easily verified that a morphism (1-edge) f is an isomorphism in $NFuk(S^2)$, see Part I for definitions, if and only if it is the image by N of a morphism in $Fuk(S^2)$ that induces an isomorphism in $DF(S^2)$. Such a morphism will be called a *c-isomorphism*.

Consequently the maximal Kan subcomplex $K(S^2)$ of $NFuk(S^2)$ is characterized as the maximal subcomplex of this nerve with 1-simplices the images by N of c-isomorphisms in $Fuk(S^2)$.

Remark 4.1. *It would be interesting to identify the homotopy type of $K(S^2)$.*

4.1. **Extending $\mathcal{D}_{pt}$ to higher dimensional simplices.** Let us model $D^4_\bullet$ as follows. Take the standard representable 3-simplex $\Delta^3_\bullet$, and the standard representable 0-simplex $\Delta^0_\bullet$. Then collapse all faces of $\Delta^3_\bullet$ to a point, that is take the colimit of the following diagram:

$$\begin{tikzcd}
 & & \Delta^0_\bullet & \\
\Delta^2_\bullet \arrow[urr] \arrow[drr] & \Delta^2_\bullet \arrow[ur] \arrow[dr, "i_0"', "i_1"] & \Delta^2_\bullet \arrow[u] \arrow[d, "i_2"] & \Delta^2_\bullet \arrow[ul] \arrow[dl, "i_3"] \\
 & & \Delta^3_\bullet &
\end{tikzcd} \tag{4.1}$$

Here i_j are the inclusion maps of the non-degenerate 2-faces. This gives a Kan complex $S^{3,mod}_\bullet$ modelling the simplicial set $S^3_\bullet$. Now take the cone on $S^{3,mod}_\bullet$, denoted by $C(S^{3,mod}_\bullet)$, and collapse the one non-degenerate 1-edge. The resulting Kan complex $D^{4,mod}_\bullet$ is our model for $D^4_\bullet$, it may be identified with a subcomplex of $D^4_\bullet$ so that the inclusion map $mod : D^{4,mod}_\bullet \to D^4_\bullet$ induces a homotopy equivalence of pairs

$$(D^{4,mod}_\bullet, S^{3,mod}_\bullet) \to (D^4_\bullet, S^3_\bullet). \tag{4.2}$$

We set $b_0 \in D^4_\bullet$ to be the vertex which is the image by mod of the unique 0-vertex in $D^{4,mod}_\bullet$.

Let $h_\pm : D^4 \to S^4$ be a pair of complementary maps, meaning that

$$h_+(D^4) \cap h_-(D^4)$$

is contained in the image E of a smooth map $S^3 \to S^4$, so that $h_\pm$ takes b_0 to x_0, and so that the minimal Kan subcomplex $Y_\bullet \subset X_\bullet$, s.t. $h_\pm(D^{4,mod}_\bullet) \subset Y_\bullet$, satisfies $|Y_\bullet| \simeq S^4$, with $\simeq$ homotopy equivalence. In practice it will be enough to just let h_- to represent the generator of $\pi_4(S^4_\bullet, x_0)$ and for h_+ to be constant map to x_0.

We set

$$D_\pm := h_\pm(D^{4,mod}_\bullet)$$

and we set $\Sigma_\pm \in S^4_\bullet$ to be the image by $h_\pm$ of the sole non-degenerate 4-simplex of $D^{4,mod}_\bullet$. We also set

$$\partial D_\pm := h_\pm(\partial D^{4,mod}_\bullet).$$

Fix a Hamiltonian frame for the fiber P_{x_0} of P over x_0, in other words a diffeomorphism $S^2 \to P_{x_0}$ that is part of an atlas of trivializations of P as a $Ham(S^2)$ structure group bundle. In particular this allows us to identify $Fuk(S^2)$ with $F^{raw}(x_0)$. Denote by $x_{0,\bullet}$ the image of the map

$$\Delta^0_\bullet \to S^{4,mod}_\bullet,$$

induced by the inclusion of the 0-simplex x_0. Recall from Part I that given perturbation data for a non-degenerate simplex, we assigned extended perturbation data for all degeneracies of this simplex. So our data $\mathcal{D}_{pt}$ for x_0 induces perturbation data for simplices of $x_{0,\bullet}$, it will again by denoted by $\mathcal{D}_{pt}$.

Fix an object $L_0 \in Fuk^{eq}(S^2) \subset F^{raw}(x_0)$. Denote by $\gamma \in \hom_{F^{raw}(x_0)}(L_0, L_0)$ the "fundamental chain" projecting to the identity in $DF(L_0, L_0)$, (this is uniquely determined by our conditions and corresponds to a single geometric section). Denote by L^i_0 the image of L_0 by the embedding

$$F^{raw}(x_0) \to F^{raw}(\Sigma_+)$$

corresponding to the i'th vertex inclusion into Δ^4, $i = 0, \ldots, 4$.

Let m_i be the edge between $i-1, i$ vertices and set

$$\overline{m}_i := \Sigma_+ \circ m_i.$$

Let Σ^0_i denote the 0-simplex obtained by restriction of Σ^4 to the i'th vertex. Note that each $\overline{m}_i$ is degenerate by construction, so we have a induced morphisms

$$F^{raw}(pr) : F^{raw}(\overline{m}_i) \to F^{raw}(x_0)$$

corresponding (F^{raw} is a functor) to the degeneracy morphism in $\Delta(S^4)$,

$$pr : \overline{m}_i \to \Sigma^0_i.$$

Finally, for each L^{i-1}_0, L^i_0 we have a c-isomorphism

$$\gamma_i : L^{i-1}_0 \to L^i_0$$

in $F^{raw}(\overline{m}_i) \subset F^{raw}(\Sigma_+)$ which corresponds to γ, meaning that the fully-faithful projection $F^{raw}(pr)$ takes γ_i to γ. We will denote by $\gamma_{i,j}$ the analogous morphisms $L^i_0 \to L^j_0$.

Notation 4.2. *Let us abbreviate from now the morphism spaces* $hom_{F^{raw}(\Sigma_\pm)}$ *by* $hom_{\Sigma_\pm}$*, and* $\mu^d_{F^{raw}(\Sigma_\pm)}$ *by* $\mu^d_{\Sigma_\pm}$.

Definition 4.3. *We call perturbation data* $\mathcal{D}$ *for* P **small** *if it is extends the data* $\mathcal{D}_{pt}$ *as above, and if with respect to* $\mathcal{D}$

$$\mu^d_{\Sigma_+}(\gamma^1,\ldots,\gamma^d) = 0, \text{ for } 2 < d < 4, \tag{4.3}$$

where $(\gamma^1,\ldots,\gamma^d)$ *is a composable chain, and each* γ^i *is of the form* $\gamma_{i,j}$ *as above.*

We will see further on how to construct such small data, assume for now that it is constructed.

Let $\{f_J\}$, corresponding to an n-simplex, be as in the definition of the A_∞ nerve in Part I, where J is a subset of $[n] = \{0,\ldots,n\}$.

Lemma 4.4. *Let* $\mathcal{D}$ *be small as above, then there is a a 4-simplex* $\sigma \in NF^{raw}(\Sigma_+)$ *with faces determined by the conditions:*

- $f_J = 0$*, for* J *any subset of* $[4]$ *with at least* 3 *elements.*
- $f_{\{i-1,i\}} = \gamma_i$ *for* γ_i *as before.*

Proof. This follows by (4.3) and by the identity $\mu^2_{\Sigma_+}(\gamma,\gamma) = \gamma$. □

If we take our unital replacements so that γ corresponds to the unit then σ induces a section of $K(P_+) \to D_+$, where $K(P_\pm)$ will be shorthand for $K(P)$ restricted over $D_\pm$.

Let

$$i : \left(K(P_+)|_{\partial D_+} := p_\bullet^{-1}(\partial D_+)\right) \to K(P_-),$$

be the natural inclusion map. Set

$$sec = i \circ \sigma \circ h_+|_{\partial D^{4,mod}_\bullet}.$$

Lemma 4.5. *Suppose that* P *is a non-trivial Hamiltonian fibration and* $\mathcal{D}$ *small data for* P *as above, then sec as above does not extend to a section of* $K(P_-)$.

Remark 4.6. *When* P *is obtained by clutching with a generator of* $\pi_3(SU(2))$*, and when* $h_\pm$ *are embeddings, the class* $[sec]$ *in* $\pi_3(K(P_-)) \simeq \pi_3(K(S^2))$ *can be thought of as "quantum" analogue of the class of the classical Hopf map.*

Proof of Theorem 3.4. If we take any small perturbation data $\mathcal{D}$ for P, then the first part follows immediately by Lemma 4.5. So $K(P)$ is non-trivial as a Kan fibration. This then implies that $Fuk_\infty(P)$ is non-trivial as a (co)-Cartesian fibration, which means specifically that its classifying map

$$f_P : S^4 \to (\mathcal{S}, NFuk(S^2))$$

is not null-homotopic.

To see this, suppose otherwise that we have a null-homotopy H of f_P, then this gives a (co)-Cartesian fibration

$$\widetilde{P} \to S^4_\bullet \times I_\bullet,$$

restricting to $Fuk_\infty(P)$ over $S^4_\bullet \times 0_\bullet$ and to $NFuk(S^2) \times S^4_\bullet$ over the other end $S^4_\bullet \times 1_\bullet$. Here $0_\bullet$, $1_\bullet$ are notatation for simplicial set isomorphic Δ^0 corresponding to the boundary of $I_\bullet$. If we take the maximal Kan sub-fibration of $\widetilde{P}$, then by Lemma 3.2 we would obtain a trivialization of $K(P)$ which is a contradiction. □

Proof of Theorem 1.1. Theorem 3.4 implies that the group homomorphism

$$\mathbb{Z} \simeq \pi_4(BHam(S^2, id) \to \pi_4(\mathcal{S}, NFuk(S^2)),$$

has vanishing kernel, so that the result follows. □

5. Towards the proof of Lemma 4.5

We will denote by $L_{0,\bullet}$ the image of the map $\Delta^0_\bullet \to K(P_-)$, induced by the inclusion of L_0 into $K(S^2)$ as a 0-simplex. Suppose that *sec* extends to a section of $K(P_-)$, so we have map $e : D^{4,mod}_\bullet \to K(P_-)$ extending *sec* over $\partial D^{4,mod}_\bullet$. We may assume that e lies over h_-, meaning $p_\bullet \circ e = h_-$. Since it can be homotoped to have this property. To see this, first take a relative homotopy downstairs. Using that we have a homotopy equivalence of pairs (4.2), and then lift it to a relative homotopy upstairs using the defining lifting property of Kan fibrations.

And so we have a 4-simplex $T = e(\Sigma^4)$ projecting to $\Sigma_- \in D_-$ by $p_\bullet$. Since T is in the image of e, all but possibly one 3-face of T are totally degenerate with image in $L_{0,\bullet}$, and with this 3-face lying in the image of *sec*.

Then if $m_{i,j}, \gamma_{i,j}$ are as in the previous section but corresponding now to Σ_- rather then Σ_+, by the boundary condition on e, the edges of T are all of the form $N(\gamma_{i,j})$ for N the nerve functor as usual, since this is the corresponding condition for the edges of *sec*.

Lemma 5.1. *For $\mathcal{D}$ small as above, and for the unital replacement F of F^{raw} as above, the simplex T exists if and only if $\mu^4_{\Sigma_-}(\gamma_1, \ldots, \gamma_4)$ is exact.*

Proof. The following argument will be over $\mathbb{F}_2$ as opposed to $\mathbb{Q}$ as the signs will not matter. Recall that we take the unital replacement so that $\gamma \in hom_{F^{raw}(P_{x_0})}(L_0, L_0)$ corresponds to the unit in the unital replacement.

Now if $T \in K(P_-)$ as above exists, then it corresponds under unital replacement to a 4-simplex $T' \in NF^{raw}(\Sigma_-)$ satisfying the following condition on its 4-face $f_{[4]} \in hom_{\Sigma_-}(L^0_0, L^4_0)$:

$$\mu^1_{\Sigma_-} f_{[4]} = \sum_{1<i<4} f_{[4]-i} + \sum_s \sum_{(J_1,\ldots J_s) \in decomp_s} \mu^s_{\Sigma_-}(f_{J_1}, \ldots, f_{J_s}). \tag{5.1}$$

By our conditions on the boundary of T and hence T' and by the conditions in Lemma 4.4, we must have $f_J = 0$, for every proper subset $J \subset [4]$, in some length s decomposition of $[4]$, unless $J = \{i, j\}$ in which case $f_{i,j} = \gamma_{i,j}$. Given this (5.1) holds if and only if $\mu^4_{\Sigma_-}(\gamma_1, \ldots, \gamma_4)$ is exact.

□

We are going to show that for certain small $\mathcal{D}_0$, $\mu^4_{\Sigma_-}(\gamma_1, \ldots, \gamma_4)$ does not vanish in homology, which will finish the argument. However the calculation will require significant setup.

6. Hamiltonian fibrations and taut structures, holomorphic sections and area bounds

We collect here some preliminaries on moduli spaces of holomorphic sections with Lagrangian boundary constraints, as well as some facts on area bounds. There is a possibly new discussion involving taut Hamiltonian structures, but otherwise the material here is a straightforward extension of standard discussions. As such it may be just skimmed on a first reading.

6.1. **Coupling forms.** We refer the reader to [10, Chapter 6] for more details on what follows. A Hamiltonian fibration is a smooth fiber bundle

$$M \hookrightarrow P \to X,$$

with structure group $\mathrm{Ham}(M, \omega)$ with its C^∞ Frechet topology. A ***Hamiltonian connection*** is just an Ehresmann connection for a Hamiltonian fibration.

A *coupling form*, originally appearing in [5], for a Hamiltonian fibration $M \hookrightarrow P \xrightarrow{p} X$, is a closed 2-form Ω on P whose restriction to fibers coincides with ω and which has the property:

$$\int_M \Omega^{n+1} = 0 \in \Omega^2(X),$$

with integration being integration over the fiber operation. Such a 2-form determines a Hamiltonian connection $\mathcal{A}_\Omega$, by declaring horizontal spaces to be Ω-orthogonal spaces to the vertical tangent spaces. A coupling form generating a given connection $\mathcal{A}$ is unique. A Hamiltonian connection $\mathcal{A}$ in turn determines a coupling form $\Omega_\mathcal{A}$ as follows. First we ask that $\Omega_\mathcal{A}$ induces the connection $\mathcal{A}$ as above. This determines $\Omega_\mathcal{A}$ up to values on $\mathcal{A}$-horizontal lifts $\widetilde{v}, \widetilde{w} \in T_pP$ of $v, w \in T_xX$. We specify these values by the formula

$$\Omega_\mathcal{A}(\widetilde{v}, \widetilde{w}) = R_\mathcal{A}(v, w)(p), \tag{6.1}$$

where $R_\mathcal{A}$ is the lie algebra valued curvature 2-form of $\mathcal{A}$. Specifically, for each x, $R_\mathcal{A}|_x$ is a 2-form valued in $C^\infty_{norm}(p^{-1}(x))$ - the space of 0-mean normalized smooth functions on $p^{-1}(x)$.

6.2. **Hamiltonian structures on fibrations.** Let S be a Riemann surface with boundary, with punctures in the boundary, and a fixed structure of strip charts at ends, positive or negative, as in Part I. Let $M \hookrightarrow \widetilde{S} \xrightarrow{pr} S$ be a Hamiltonian fiber bundle, with model fiber monotone symplectic manifold (M, ω), with distinguished Hamiltonian bundle trivializations

$$[0, 1] \times (0, \infty) \times M \to \widetilde{S}$$

at the positive ends, and with distinguished Hamiltonian bundle trivializations

$$[0, 1] \times (-\infty, 0) \times M \to \widetilde{S},$$

at the negative ends. These are collectively called called ***strip charts***. Given the structure of such bundle trivializations we say that $\widetilde{S}$ has ***end structure***.

Let

$$\mathcal{L} \subset (\widetilde{S}|_{\partial S} = pr^{-1}(\partial S)) \to \partial S$$

be a Lagrangian sub-bundle, with model fiber an object, in the sense of Part I, (in particular a spin oriented Lagrangian submanifold) so that $\mathcal{L}$ is a constant sub-bundle in the strip chart trivializations above. We say in this case that $\mathcal{L}$ ***respects the end structure***. In the coordinates at the end e_i, let L_i^j be the fibers of $\mathcal{L}$ over

$$\{j\} \times \{t\},\ j = 0, 1, \text{ by assumption } t \text{ independent.}$$

For $\mathcal{L}$ as above, we say that a Hamiltonian connection $\mathcal{A}$ on $\widetilde{S}$ is ***compatible*** with the connections $\{\mathcal{A}_i\}$ on $[0, 1] \times M$, if in the strip coordinate chart at each e_i end, $\mathcal{A}$ is flat and $\mathbb{R}$-translation invariant and so has the form $\overline{\mathcal{A}}_i$ where $\overline{\mathcal{A}}_i$ denotes its $\mathbb{R}$-translation invariant extension $\mathcal{A}_i$ to $(0, \infty) \times \mathbb{R} \times M$. We say that a Hamiltonian connection $\mathcal{A}$ on $\widetilde{S}$ as above, is ***$\mathcal{L}$-exact*** if $\mathcal{A}$ preserves $\mathcal{L}$ (this means that the horizontal spaces of $\mathcal{A}$ are tangent to $\mathcal{L}$).

A family $\{j_z\}$ of fiber wise ω-compatible almost complex structures on $\widetilde{S}$ will be said to ***respect the end structure*** if at each end e_i, in the strip coordinate chart above, the family $\{j_z\}$ is $\mathbb{R}$-translation invariant and is admissible with respect to $\mathcal{A}_i$, in the sense of Part I, Definition 5.3. The data $(\widetilde{S}, S, \mathcal{L}, \mathcal{A}, \{j_z\})$ as above will be called ***Hamiltonian structure***.

We will normally suppress $\{j_z\}$ in the notation and elsewhere for simplicity, as it will be purely in the background in what follows, (we do not need to manipulate it explicitly).

Definition 6.1. *Let $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be a Hamiltonian structure, we say that a smooth section σ of $\widetilde{S} \to S$ is* **asymptotically flat** *if at each end e_i of S, σ C^1-converges to an $\mathcal{A}$-flat section. Specifically, in the strip coordinates for a positive end, this means that there is a flat section $\widetilde{\sigma}$ of $S \to (0,\infty) \times M$, so that for every $\epsilon > 0$ there is a $t > 0$ so that $d_{C^1}(\widetilde{\sigma}|_{[t,\infty)}, \sigma|_{[t,\infty)}) < \epsilon$. Similarly for a negative end. Given a pair of asymptotically flat sections σ_1, σ_2, with boundary in $\mathcal{L}$, we say that they have the same* **relative class** *if they are asymptotic to the same flat sections at each end, and are homologous relative boundary condition and relative the asymptotic constraints at the ends. The set of relative classes will be denoted by $H_2^{sec}(\widetilde{S}, \mathcal{L})$.*

6.2.1. *Families of Hamiltonian structures.*

Definition 6.2. *A* **family Hamiltonian structure** *or henceforth just Hamiltonian structure, consists of the following:*

- *A smooth compact oriented manifold $\mathcal{K}$ with boundary, (or corners).*
- *For each $r \in \mathcal{K}$ a Hamiltonian structure $(\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$, so that $\{(\widetilde{S}_r, S_r, \mathcal{L}_r)\}$ fits into smooth fibrations*
$$\widetilde{S} \hookrightarrow \widetilde{\boldsymbol{S}} \xrightarrow{p_1} \mathcal{K}, M \hookrightarrow \widetilde{\boldsymbol{S}} \xrightarrow{p_2} \boldsymbol{S}, S \hookrightarrow \boldsymbol{S} \xrightarrow{p} \mathcal{K}.$$
The last fibration has fiber a Riemann surface, so that $\{S_r\} = \{p^{-1}r\}$. The second is a Hamiltonian fibration with fiber M. And the first is a fibration whose fibers $p_1^{-1}(r)$ are themselves the total spaces of a smooth Hamiltonian fibration $\widetilde{S}_r \to S_r$, so that the structure group of $\widetilde{\boldsymbol{S}} \xrightarrow{p_1} \mathcal{K}$ can be reduced to smooth Hamiltonian bundle maps of the fiber. To elaborate further, let $M \hookrightarrow \widetilde{S} \to S$ be a Hamiltonian M-fibration over a Riemann surface S. Let Aut denote the group of Hamiltonian M-bundle automorphisms of $\widetilde{S}$. Then $\widetilde{\boldsymbol{S}} \xrightarrow{p_1} \mathcal{K}$ is the associated bundle $P \times_{Aut} \widetilde{S}$ for some principal Aut bundle P over $\mathcal{K}$.
- *The charts*
$$e_{i,r} : [0,1] \times (0,\infty) \times M \to \widetilde{S}_r,$$
for the positive ends, fit into a Hamiltonian M-bundle diffeomorphism onto the image:
$$\widetilde{e}_i : [0,1] \times (0,\infty) \times \mathcal{K} \times M \to \widetilde{\boldsymbol{S}}, \tag{6.2}$$
similarly for the negative ends.
- *We then have an induced smooth r-family of connections $\{e_{i,r}^* \mathcal{A}_r\}$ on $[0,1] \times (0,\infty) \times M$, in case of positive ends, and an induced smooth r-family of Lagrangian subfibrations $\{e_{i,r}^{-1} \mathcal{L}_r\}$ over $\partial[0,1] \times (0,\infty)$, similarly for negative ends. We ask that the families $\{e_{i,r}^* \mathcal{A}_r\}$, and $\{e_{i,r}^{-1} \mathcal{L}_r\}$ are constant in r, and so for each i $\{e_{i,r}^* \mathcal{A}_r\}$, has the form $\overline{\mathcal{A}}_i$ for $\mathcal{A}_i$ as previously.*

- *There is a Hamiltonian connection $\mathcal{A}$ on $\widetilde{\boldsymbol{S}} \to \boldsymbol{S}$ that extends all the connections $\mathcal{A}_r$ (in the natural sense), and preserving $\boldsymbol{L} := \cup_r \mathcal{L}_r$.*

We will write $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ for this data, $\mathcal{K}$ may be omitted from notation when it is implicit.

In the notation above, if there exists a Hamiltonian connection $\mathcal{A}$ on $\widetilde{\mathbf{S}} \to \mathbf{S}$ as in the last point, so that $\Omega_{\mathcal{A}}$ vanishes on $\mathbf{L}$, we will say that $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is a ***hyper taut Hamiltonian structure***.

6.2.2. *Moduli spaces of sections of Hamiltonian structures.* Let $\overline{\mathcal{M}}(\{\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be the Gromov-Floer compactification of the space of $J(\mathcal{A})$-holomorphic sections σ of $\widetilde{\mathcal{S}}$, with finite vertical L^2 energy $e(\sigma)$ (also called Floer energy), and with boundary on $\mathcal{L}$. Note that for any $J_{\mathcal{A}}$-holomorphic σ:

$$e(\sigma) = \int_S \sigma^* \Omega_{\mathcal{A}},$$

and $\Omega_{\mathcal{A}}$ vanishes on $\mathcal{L}$ by the condition that $\mathcal{A}$ preserves $\mathcal{L}$, so that the standard energy controls apply, to deduce Gromov-Floer compactification structure.

Likewise, let $\overline{\mathcal{M}}(\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}})$ be the Gromov-Floer compactification of the space of pairs (σ, r), $r \in \mathcal{K}$ with σ a $J(\mathcal{A}_r)$-holomorphic, finite vertical L^2 energy section of $\widetilde{\mathcal{S}}_r$, with boundary on $\mathcal{L}_r$.

We also denote by

$$\overline{\mathcal{M}}(\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}, A) \subset \overline{\mathcal{M}}(\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\})$$

the subset corresponding to relative class A curves, where the latter is as defined above.

Definition 6.3. *We say that $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is A-***regular*** if the above moduli space is regular, (transversely cut out). We say that it is A-***admissible*** if there are no elements*

$$(\sigma, r) \in \overline{\mathcal{M}}(\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}, A),$$

for r in a neighborhood of the boundary of $\mathcal{K}$.

Definition 6.4. *Given a pair $\{\widetilde{S}^i_r, S^i_r, \mathcal{L}^i_r, \mathcal{A}^i_r\}_{\mathcal{K}}$, $i = 1, 2$, we say that they are* **concordant** *if there is a Hamiltonian structure*

$$\{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times [0,1]},$$

with an oriented diffeomorphism (in the natural sense, preserving all structure)

$$\{\widetilde{S}^0_r, S^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}_{\mathcal{K}^{op}} \sqcup \{\widetilde{S}^1_r, S^1_r, \mathcal{L}^1_r, \mathcal{A}^1_r\}_{\mathcal{K}} \to \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times \partial I},$$

where op denotes the opposite orientation for $\mathcal{K}$.

Definition 6.5. *We say that a Hamiltonian structure $\{\Theta_r\} = \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is* **taut** *if for any pair $r_1, r_2 \in \mathcal{K}$, Θ_{r_1} is concordant to Θ_{r_2} by a concordance $\{\widetilde{T}_\tau, T_\tau, \mathcal{L}'_\tau, \mathcal{A}'_\tau\}_{[0,1]}$ which is a hyper taut Hamiltonian structure.*

Definition 6.6. *Given an A-admissible pair $\{\widetilde{S}^i_r, S^i_r, \mathcal{L}^i_r, \mathcal{A}^i_r\}_{\mathcal{K}}$, $i = 1, 2$, we say that they are A-***admissibly concordant*** if there is an A-admissible Hamiltonian data*

$$\{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times [0,1]},$$

*which furnishes a concordance. If this concordance is in addition a taut Hamiltonian structure, then we say that these pairs are A-***admissibly taut concordant***.*

Lemma 6.7. *Let $\Theta_r = \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}$ be regular and A-admissible, with S_r having one distinguished negative end e_0. Define*

$$\mathcal{M}(\{\Theta_r\}, A, \gamma_0)$$

to be like the space $\mathcal{M}(\{\Theta_r\}, A)$ but we add the constraint that at the e_0 end our sections are asymptotic to γ_0. For L_0^j, $j = 0, 1$ as above and $CF(L_0^0, L_0^1)$ defined with respect to the connection $\mathcal{A}_0$ define

$$ev = ev(\{\Theta_r\}, A) \in CF(L_0^0, L_0^1)$$

by:

$$\langle ev(\{\Theta_r\}, A), \gamma_0 \rangle = \#\mathcal{M}(\{\Theta_r\}, A, \gamma_0),$$

where $\#\mathcal{M}(\{\Theta_r\}, A, \gamma_0)$ means signed count of elements when dimension is zero, and is otherwise set to be zero. Then $ev(\{\Theta_r\}, A)$ is a cycle, and its homology class depends only on the A-admissible concordance class of $\{\Theta_r\}$.

Proof. This is very standard and follows by the same kind of arguments as given in the construction of the relative Seidel morphism in [8], keeping in mind that we are now dealing with families of surfaces and the parameter family has boundary. For this reason we only indicate the argument.

To show that ev is closed suppose that $\langle \mu^1(ev), \gamma' \rangle \neq 0$ for a geometric generator γ', then $\mathcal{M}(\{\Theta_r\}, A, \gamma')$ is a compact 1-dimensional manifold with boundary. By our various assumptions, the elements of its boundary can only correspond to Floer degenerations. And by gluing any Floer degeneration that could happen does happen. So the signed count of boundary points of this moduli space which must be 0, is in correspondence with $\langle \mu^1(ev), \gamma' \rangle$, and this is a contradiction.

Similarly, given an A-admissible concordance (which we may assume to be regular)

$$\{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times [0,1]}$$

between $\{\widetilde{S}_r^0, S_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}$ and $\{\widetilde{S}_r^1, S_r^1, \mathcal{L}_r^1, \mathcal{A}_r^1\}$ will get a chain c,

$$\langle c, \gamma_0 \rangle = \#\mathcal{M}(\{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}, A, \gamma_0),$$

so that

$$\mu^1(c) = ev(\{\widetilde{S}_r^1, S_r^1, \mathcal{L}_r^1, \mathcal{A}_r^1\}) - ev(\{\widetilde{S}_r^0, S_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}).$$

□

6.3. **Area of fibrations.**

Definition 6.8. *For a Hamiltonian connection $\mathcal{A}$ on a bundle $M \hookrightarrow \widetilde{S} \to S$, with S a Riemann surface, define*

$$\text{area}(\mathcal{A}) = \inf_{\alpha} \{ \int_S \alpha \,|\, \Omega_{\mathcal{A}} + \pi^*(\alpha) \textit{ is nearly symplectic}\}, \tag{6.3}$$

where $\Omega_{\mathcal{A}}$ is the coupling form of $\mathcal{A}$, α is a 2-form on S so that $\alpha(\cdot, j\cdot)$ is a Riemannian metric, and orientation on S is given by j. Here $\Omega_{\mathcal{A}} + \pi^*(\alpha)$ nearly symplectic means that

$$(\Omega_{\mathcal{A}} + \pi^*(\alpha))(\widetilde{v}, \widetilde{jv}) \geq 0,$$

for $\widetilde{v}, \widetilde{jv}$ horizontal lifts with respect to $\Omega_{\mathcal{A}}$, of $v, jv \in T_z S$, for all $z \in S$.

Note that area($\mathcal{A}$) could be infinite if there are no constraints on $\mathcal{A}$ at the ends. However, when the infimum above is finite it is attained on the uniquely defined 2-form

$$\alpha_{\mathcal{A}}(v,w) := |R_{\mathcal{A}}(v,w)|_+, \tag{6.4}$$

where $v, w \in T_z S$, $R_{\mathcal{A}}(v,w)$ as before identified with zero mean smooth function on the fiber $\widetilde{S}_z$ over z and $|\cdot|_+$ is operator: $|H|_+ = \max_{\widetilde{S}_z} H$.

Lemma 6.9. *Let $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be Hamiltonian data. For $\sigma \in \overline{\mathcal{M}}(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ we have a lower bound*

$$-\int_S \sigma^* \Omega_{\mathcal{A}} \leq \operatorname{area}(\mathcal{A}).$$

Proof. We have

$$\int_S \sigma^*(\Omega_{\mathcal{A}} + \pi^*\alpha) \geq 0,$$

whenever $\Omega_{\mathcal{A}} + \pi^*(\alpha)$ is nearly symplectic, by the defining properties of $J_{\mathcal{A}}$ and by σ being $J_{\mathcal{A}}$-holomorphic. From which our conclusion follows. $\square$

Lemma 6.10. *Let $\{(\widetilde{S}_t, S_t, \mathcal{L}_t, \mathcal{A}_t)\}_{[0,1]}$ be a taut concordance. Let σ_j, $j = 0, 1$ be asymptotically flat sections of $\widetilde{S}_j$ in relative class A. Then*

$$-\int_{S_1} \sigma_1^* \Omega_{\mathcal{A}_1} = -\int_{S_0} \sigma_0^* \Omega_{\mathcal{A}_0},$$

whenever both integrals are finite. In particular, for a Hamiltonian structure $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ $\int_S \sigma^* \Omega_{\mathcal{A}}$ depends only on the relative class of A, whenever the integral is finite.

Proof. By the hypothesis, there is a connection $\mathcal{A}$ on $\widetilde{\mathbf{S}}$, extending each $\mathcal{A}_t$ and so that $\Omega_{\mathcal{A}}$ vanishes on $\mathbf{L} \subset \widetilde{\mathbf{S}}$. The first part then follows by Stokes theorem. Here are the details. For σ_j as above and for each end e_i, cut off the part of the section σ_j lying over $[0,1] \times (t_{\delta_1,\delta_2}, \infty)$ in the corresponding strip chart at the end. Here t_{δ_1,δ_2} is such that $\sigma_0|_{[0,1]\times\{t\}}$ is C^1 δ_1-close to $\sigma_1|_{[0,1]\times\{t\}}$ for all $t > t_{\delta_1,\delta_2}$ and for each end, and is such that

$$\int_{[0,1]\times(t_{\delta_1,\delta_2},\infty)} \sigma_j^*|_{[0,1]\times(t_{\delta_1,\delta_2},\infty)} \Omega_{\mathcal{A}_j} < \delta_2,\ j = 1, 2,$$

for each end e_i. Call the sections with the ends cut off as above by $\sigma_j^{\delta_1,\delta_2}$, they are sections over the compact surfaces S_j^{cut}, with ends correspondingly cut off. Then by Stokes theorem, using that $\Omega_{\mathcal{A}}$ is closed and using the vanishing of $\Omega_{\mathcal{A}}$ on $\mathbf{L}$: for each ϵ there exists δ_1, δ_2 such that

$$\int_{S_1^{cut}} (\sigma_1^{\delta_1,\delta_2})^* \Omega_{\mathcal{A}} - \int_{S_0^{cut}} (\sigma_0^{\delta_1,\delta_2})^* \Omega_{\mathcal{A}} < \epsilon,$$

and

$$\int_{S_j^{cut}} (\sigma_j^{\delta_1,\delta_2})^* \Omega_{\mathcal{A}_j} - \int_{S_j} \sigma_j^* \Omega_{\mathcal{A}_j} < \epsilon,\ j = 1, 2.$$

The last part of the lemma follows, as $\mathcal{A}$ preserving $\mathcal{L}$ immediately implies that $\Omega_{\mathcal{A}}$ vanishes on $\mathcal{L}$, so that a constant concordance

$$\{(\widetilde{S}_t, S_t, \mathcal{L}_t, \mathcal{A}_t)\}_{[0,1]}$$

is taut. $\square$

Definition 6.11. *For σ a relative class A section of $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ let us call:*

$$-\int_S \sigma^* \Omega_{\mathcal{A}},$$

*the $\mathcal{A}$-**coupling area** of σ, denoted by $carea(\Theta, \sigma)$, we may also write $carea(\Theta, A)$ for the same quantity.*

Definition 6.12. *Given Hamiltonian structure $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ set*

$$\hbar(\Theta) := \inf_A carea(\Theta, A),$$

where the infimum is taken over all relative classes A such that $carea(\Theta, A) > 0$.

Lemma 6.13. *$\hbar(\Theta)$ is an invariant of the taut concordance class of Θ.*

Proof. This follows immediately by Lemma 6.10. $\square$

Definition 6.14. *Given Hamiltonian structure $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ suppose that $\hbar(\Theta) > 0$. We will say that Θ is **small** if*

$$\text{area}(\Theta) < \hbar(\Theta).$$

*Similarly, given taut Hamiltonian structure $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ we say that it is is **small near boundary** if $(\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$ is small for r in a neighborhood of the $\partial\mathcal{K}$.*

Lemma 6.15. *Suppose that $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ is small then $\overline{\mathcal{M}}(\Theta, A)$ is empty for every class A such that $carea(\Theta, A) > 0$.*

Proof. For A with $carea(\Theta, A) > 0$ if $\sigma \in \overline{\mathcal{M}}(\Theta, A)$, then by Lemma 6.9, and Lemma 6.7 we get that

$$\hbar(\Theta) \leq carea(\Theta, A) \leq area(\Theta),$$

which would be a contradiction to Θ being small. $\square$

Lemma 6.16. *Let $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ be a taut Hamiltonian structure with $\mathcal{K}$ connected, so that in particular, for each r, $\Theta_r = (\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$ is taut concordant to a fixed Θ. Suppose that $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is small near boundary then $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is A-admissible for all A such that $carea(\Theta, A) > 0$.*

Proof. Follows immediately by the lemma above. $\square$

6.4. **Gluing Hamiltonian structures.** A Hamiltonian connection $\mathcal{A}$ on $[0,1] \times M$ is determined by a choice of a function $H : [0,1] \times M \to \mathbb{R}$, normalized to have mean zero at each moment. The holonomy path of $\mathcal{A}$ is a path $\phi_{\mathcal{A}} : [0,1] \to Ham(M, \omega)$, generated by the Hamiltonian H. Given $L_0 \in Lag(M)$ we get a path $\widetilde{\phi}_{\mathcal{A}} : [0,1] \to Lag(M)$ starting at L_0, defined by $\widetilde{\phi}_{\mathcal{A}}(t) = \phi_{\mathcal{A}}(t)(L_0)$. We will say that these paths are ***generated by $\mathcal{A}$ or by H***, with the latter as above.

Let $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be a Hamiltonian structure, where for simplicity at each end e_i the corresponding Lagrangians L_i^0, L_i^1 coincide. And where the corresponding connection $\mathcal{A}_i$ is determined by a Hamiltonian with Hofer length κ_i, where the Hofer length functional is as in Section 10.1.1.

Let $\mathcal{D}$ denote the surface which is topologically $D^2 - z_0$, $z_0 \in \partial D^2$, endowed with a choice of a strip chart at the end (positive or negative depending on context). We may then cap off some of the open ends $\{e_i\}_{i=0}^n$ of S by gluing at the ends copies of $\mathcal{D}$. More explicitly, in the strip coordinate charts at some, say positive, end e_i of S, excise $[0,1] \times (t, \infty)$ for some $t > 0$, call the resulting surface $S - e_i$. Likewise excise the end of $\mathcal{D}$, call this surface $\mathcal{D} - end$. Then glue $S - e_i$ with $\mathcal{D} - end$, along their new smooth boundary components.

Let us denote the capped off surface by $S^/$. Since $\widetilde{S}$ is naturally trivialized at the ends, we may similarly cap off $\widetilde{\mathcal{S}}_r$ over the e_i end by gluing with a bundle $\mathcal{D} \times M$ at the end obtaining a Hamiltonian M bundle $\widetilde{S}^/$ over $S^/$. We may then trivially extend $\mathcal{L}$ to a Lagrangian subbundle over $\partial \mathcal{S}^/$.

In other words we have a certain gluing operation of Hamiltonian structures. In the case of "capping off" as above we glue $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ with the Hamiltonian structure $\Theta_0 = (\mathcal{D} \times M, \mathcal{D}, L_0 \times \partial \mathcal{D}, \mathcal{A}')$ at the e_i end, provided $\mathcal{A}'$ is compatible with the connection $\mathcal{A}_i$, in the sense of Section 6.2. Let us name the result of this gluing $\Theta \#_i \Theta_0$. The following is immediate:

Lemma 6.17. *Suppose that $\{\Theta_r\}_{\mathcal{K}}$ is a taut Hamiltonian structure and let Θ_0 be as above. Then:*

$$\{\Theta_r \#_i \Theta_0\}$$

is taut, whenever the gluing operation is well defined, that is whenever we have compatibility of connections.

Lemma 6.18. *Let $\mathcal{L} \subset \partial \mathcal{D} \times M$ be the trivial Lagrangian sub-bundle of $\mathcal{D} \times M$, with fiber L_0, over the boundary of $\mathcal{D}$. Let $\mathcal{A}_0$ be a Hamiltonian connection on $[0,1] \times M$, generated by a Hamiltonian with L^+ length κ, constant near the end points. Then there is a Hamiltonian connection $\widetilde{\mathcal{A}}_0$ on $\mathcal{D} \times M$, preserving $\mathcal{L}$, and compatible with respect to $\mathcal{A}_0$, with $area(\widetilde{\mathcal{A}}_0) = \kappa$. The construction is natural in the sense that $\mathcal{A}_0 \mapsto \widetilde{\mathcal{A}}_0$ is a smooth map.*

Proof. Let $p : [0,1] \to Ham(M, \omega)$ be the holonomy path of $\mathcal{A}_0$, $p(0) = id$, generated by H. Define a coupling form Ω' on $D^2 \times M$:

$$\Omega' = \omega - d(\eta(rad) \cdot H d\theta),$$

for (rad, θ) the modified angular coordinates on D^2, $\theta \in [0,1]$, $0 \leq rad \leq 1$, and $\eta : [0,1] \to [0,1]$ is a smooth function satisfying

$$0 \leq \eta'(rad),$$

and

$$\eta(rad) = \begin{cases} 1 & \text{if } 1 - \delta \leq rad \leq 1, \\ rad^2 & \text{if } rad \leq 1 - 2\delta, \end{cases} \tag{6.5}$$

for a small $\delta > 0$. By elementary calculation

$$\text{area}(\mathcal{A}') = L^+(p),$$

where $\mathcal{A}'$ is the connection induced by Ω'.

By assumptions on p the connection $\mathcal{A}'$ is trivial over the arc $arc = \{(1, \theta)\}$, $-\epsilon \leq \theta \leq \epsilon$ for some $\epsilon > 0$. Let arc^c denote the complement of arc in ∂D^2. Fix a smooth embedding $i : D^2 \hookrightarrow \mathcal{D}$ so that the image of the embedding contains $\partial \mathcal{D} - end$ where end is the image of the distinguished strip chart

$$[0,1] \times (0, \infty) \to \mathcal{D},$$

so that $i(arc) \subset end^c$, and so that $i(arc^c) \subset end$ as illustrated in the Figure 1.

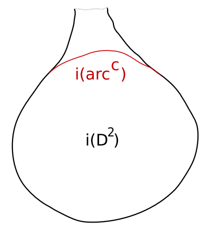

FIGURE 1.

Next fix a deformation retraction ret of $\mathcal{D}$ onto $i(D^2)$, so that in the strip chart above ret, for $r \geq 1$, is the composition $i \circ param \circ pr$, for

$$pr : [0,1] \times (0,\infty) \to [0,1]$$

the projection and for

$$param : [0,1] \to arc^c \subset D^2$$

a diffeomorphism.

Finally set $\Omega = ret^* \Omega'$ on $\mathcal{D} \times S^2$, and set $\widetilde{\mathcal{A}}_0$ to be the induced Hamiltonian connection. $\square$

Using the above lemma, we may then put a $\mathcal{L}^/$-exact Hamiltonian connection, $\mathcal{A}^/$ on $\widetilde{S}^/$ (see Definition 6.2), with

$$\text{area}(\mathcal{A}^/) = \text{area}(\mathcal{A}) + \kappa_i. \tag{6.6}$$

This connection is obtained by gluing area κ_i connection on $\mathcal{D}$ as in the lemma above with $\mathcal{A}$.

Lemma 6.19. *Let $L_0 \subset M$ be a monotone Lagrangian submanifold with monotonicity constant $const > 0$: $\omega(A) = const \cdot \mu(A)$, μ the Maslov number. Let $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be a Hamiltonian structure with $\mathcal{L}$ the constant Lagrangian subbundle with fiber L_0. Let $\mathcal{A}$ be trivial over the boundary of $\mathcal{S}$. Suppose in addition that the Hamiltonians H_i generating $\mathcal{A}_i$, corresponding to each end e_i are sufficiently C^2-small and quasi-autonomous. Let σ be an asymptotically flat section in class $A \in H_2^{sec}(\widetilde{S}, \mathcal{L})$, then*

$$carea(\Theta, A) \geq -const \cdot Maslov(A) - \sum_i \kappa_i,$$

where $Maslov$ is as in Appendix B.

Proof. $H_i : M \times [0,1] \to \mathbb{R}$ being quasi-autonomous means here that all periodic orbits of the time dependent flow of H_i are constant, which means in particular that they correspond to critical points $\{x_{i,j}\}$ of H_i. The sufficiently C^2 small condition is needed to force the Conley-Zehnder-Maslov index of the corresponding orbits to be given by the Morse index. In this case the asymptotic limit γ_i of σ at each end e_i corresponds to some critical point x_i of H_i.

Let $\widetilde{\mathcal{A}}_i$ be the extension of $\mathcal{A}_i$ as in Lemma 6.18. And let $\Theta^/ = (\widetilde{S}^/, S^/, \mathcal{L}^/, \mathcal{A}^/)$ be obtained from Θ by capping off each end e_i with $(\mathcal{D} \times M, \mathcal{D}, L_0 \times \partial \mathcal{D}, \widetilde{\mathcal{A}}_i)$. Define, for each end e_i, the constant flat section section σ_i' of $(\mathcal{D} \times M, \widetilde{\mathcal{A}}_i)$, $\forall z : z \mapsto x_i$. Then we obtain a smooth section $\sigma^/$ of $\Theta^/$ by smooth gluing σ with σ_i' for each i.

By properties of the Maslov number and by the conditions on x_i, $Maslov(\sigma^/) = Maslov(\sigma)$.

Now $\Theta^/$ is taut concordant to

$$\Theta_0 := (D^2 \times M, D^2, \mathcal{L}, \mathcal{A}^{tr}),$$

with $\mathcal{L}$ trivial with fiber L_0, and for $\mathcal{A}^{tr}$ the trivial connection. And

$$carea(\Theta_0, \cdot) = -const \cdot Maslov(\cdot)$$

as functionals on $H_2^{sec}(D^2 \times M, \mathcal{L})$ with $const > 0$. It follows by Lemma 6.10 that

$$carea(\Theta^/, \sigma^/) = carea(\Theta_0, \sigma^/) = -const \cdot Maslov(\sigma^/) = -const \cdot Maslov(\sigma).$$

While:

$$carea(\Theta^/, \sigma^/) \leq carea(\Theta, \sigma) + \sum_i \kappa_i.$$

And so the conclusion follows. □

Lemma 6.20. *Let*

$$\Theta := \{\Theta_r\} := \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$$

be taut Hamiltonian data with $\mathcal{K}$ connected, and such that $\mathcal{L}_r$ is trivial for each r. Suppose that the Floer chain complex $CF(L_i^0, L_i^1)$ is perfect for each i, (with respect to $\mathcal{A}_i$), and that $\mathcal{A}_i$ is generated by a C^2 ϵ-small autonomous Hamiltonian.

For a given $A \in H_2^{sec}(\widetilde{S}, \mathcal{L})$, if

$$\forall r : area(\mathcal{A}_r) < -const \cdot Maslov(A) - \sum_i \kappa_i,$$

and if ϵ is sufficiently small then $\overline{\mathcal{M}}(\{\Theta_r\}, A)$ is empty.

Proof. This follows immediately by Lemma 6.15 and by Lemma 6.19. □

7. Construction of small data

To forewarn, we use here notation and notions from Part I, especially from Sections 4, 5 in Part I. Let m_i denote the edges of Δ^4 as before.

Let $\{m^k\}_{k=1}^{k=d}$ be a composable sequence in $\Pi(\Delta^n)$, which we recall means that the target of m_{i-1} is the source of m_i for each i. Recall from Part I that the perturbation data $\mathcal{D}$ in particular specifies for each n and for each such composable sequence certain maps

$$u(\{m^k\}, n) : \mathcal{E}_d^\circ \to \Delta^n,$$

where $\mathcal{E}_d$ denotes the universal curve over $\overline{\mathcal{R}}_d$, and $\mathcal{E}_d^\circ$ denotes $\mathcal{E}^d$ with nodal points of the fibers removed. The restriction of $u(\{m^k\}, n)$ over the fiber $\mathcal{S}_r$ of $\mathcal{E}_r^\circ$ over r, is denoted by $u(\{m^k\}, n, r)$, which may also be abbreviated by u_r. These maps u satisfy certain naturality axioms which would be too lengthy to reproduce here.

$\mathcal{D}$ also specifies for each r a Hamiltonian connection on (in particular)

$$\widetilde{\mathcal{S}}_r := (\Sigma_+ \circ u(\{m^k\}, 4, r))^* P \to \mathcal{S}_r, \tag{7.1}$$

for every composable d-chain $\{m^k\}$ in $\Pi(\Delta^4)$. We name these connections here by $\mathcal{A}_r^+(\{m^k\})$, further abbreviated by $\mathcal{A}_r^+$ as $\{m^k\}$ will be implicit in what follows.

Suppose that $\mathcal{D}$ extends $\mathcal{D}_{pt}$ from before. If $\mathcal{L}_r \subset \widetilde{\mathcal{S}}_r|_{\partial \mathcal{S}_r}$ denotes the trivial Lagrangian sub-bundle with fiber L_0, then we obtain Hamiltonian structure (for each composable d-chain $\{m^k\}$) $\Theta^+ = \{\Theta_r^+\} = \{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r^+\}_{\overline{\mathcal{R}}_d}$ as defined in the previous section. At each end e_i of $\mathcal{S}_r$, $\mathcal{A}_r^+$ extends the connection $\overline{\mathcal{A}}(L_0, L_0)$

defined in the strip chart at the end. Here $\mathcal{A}(L_0, L_0)$ is the connection on $[0,1] \times S^2$ part of our Floer data $\mathcal{D}$, and $\overline{\mathcal{A}}(L_0, L_0)$ is as in Section 6.2. Then Θ^+ is trivially taut since $\mathcal{L}_r$ is naturally trivial and $\mathcal{A}_r$ is likewise trivial over $\partial \mathcal{S}_r$, for each r, by assumed properties of these connections.

Set

$$\hbar := \frac{1}{2} \operatorname{area}(S^2, \omega).$$

Let κ denote the L^+ length of the holonomy path in $Ham(S^2)$ of $\mathcal{A}_0 = \mathcal{A}(L_0, L_0)$. By assumptions $\mathcal{A}_0$ is generated by an autonomous Hamiltonian, let ϵ be its C^2 norm. We suppose that ϵ is taken small enough so that:

$$\forall r : \operatorname{area}(\mathcal{A}_r^+) < \hbar - 5\kappa, \tag{7.2}$$

so that $\mathcal{A}_r^+$ is small for each r, with small as defined in the previous section. We can assure this inequality, since if κ is small enough (adjusting initial data $\mathcal{D}_{pt}$) the curvature and hence $\operatorname{area}(\mathcal{A}_r^+)$ can be made as small as we like.

Fix a complex structure j_0 on M, and let $\{J_r\}$ be the family of complex structures on $\{\widetilde{\mathcal{S}}_r\}$ induced by $(\{\mathcal{A}_r^+\}, j_0)$.

Lemma 7.1. *As in Part I, let*

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}(\gamma^1, \ldots, \gamma^d; \gamma^0, \Sigma_+, \{J_r\}, A),$$

denote the set of elements of $\overline{\mathcal{M}}(\Theta^+, A)$ with asymptotic constraints γ^i at the e_i end. Here each γ^k, $k \neq 0$, is of the form $\gamma_{i,j}$ where this is as before. If ϵ is taken to be sufficiently small then whenever the class A is such that $\overline{\mathcal{M}}$ has virtual dimension 0, and d satisfies $2 < d \leq 4$, $\overline{\mathcal{M}}$ is empty.

Proof. For a fixed r, by Riemann-Roch (Appendix B) we get that the expected dimension of $\mathcal{M}(\Theta^+, A)$ is

$$1 + Maslov^{vert}(A).$$

Consequently, when $\gamma^0 = \gamma$, the expected dimension of $\mathcal{M}$ is:

$$1 + Maslov^{vert}(A) - 1 + (\dim \mathcal{R}_d = d - 2). \tag{7.3}$$

We need the expected dimension to be 0, and $d \geq 3$, so $Maslov^{vert}(A) \leq -1$. But $Maslov^{vert}(A) = -1$ is impossible as the minimal positive Maslov number is 2. Consequently, the result follows by Lemma 6.20 and the property (7.2).

When γ^0 is the Poincare dual to γ, we would get $Maslov^{vert}(A) \leq -2$ so for the same reason the conclusion follows. □

So if we choose our data $\mathcal{D}$ so that the hypothesis of the lemma above are satisfied, then with respect to $\mathcal{D}$:

$$\mu^2_{\Sigma^4_\pm}(\gamma_{i,j}, \gamma_{j,k}) = \gamma_{i,k} \tag{7.4}$$

$$\mu^3_{\Sigma^4_\pm}(\gamma^1, \ldots, \gamma^3) = 0, \text{for } \gamma^i \text{ as above} \tag{7.5}$$

$$\mu^4_{\Sigma^4_+}(\gamma_1, \ldots, \gamma_4) = 0. \tag{7.6}$$

In particular this $\mathcal{D}$ is small.

8. The product $\mu^4_{\Sigma^4_-}(\gamma_1, \ldots, \gamma_4)$ and the higher Seidel morphism

We now take h_+ to be the constant map to x_0 and

$$h_- : (D^4, \partial D^4) \to (S^4, x_0)$$

the complementary map, that is representing the generator of $\pi_4(S^4, x_0) \simeq \mathbb{Z}$. Let Σ_- be the corresponding 4-simplex of $S^4_\bullet$ as before. We need to analyze the moduli spaces

$$\overline{\mathcal{M}}(\gamma_1, \ldots, \gamma_4; \gamma^0, \Sigma_-, \{\mathcal{A}_r\}, A), \tag{8.1}$$

where $\mathcal{A}_r$ now denotes the connections on

$$\widetilde{\mathcal{S}}_r := (\Sigma_- \circ u(m_1, \ldots, m_4, 4, r))^* P \to \mathcal{S}_r, \tag{8.2}$$

part of some small data $\mathcal{D}_0$ as above. We abbreviate $u(m_1, \ldots, m_4, 4, r)$ by u_r in what follows.

By the dimension formula (7.3), since we need the expected dimension of (8.1) to be zero, the class A must have vertical Maslov number -2 and $\gamma^0 = \gamma_{0,4}$, in other words the latter morphism corresponds to the fundamental chain.

Notation 8.1. *From now on A_0 refers to this Maslov number -2 class, but it may be a section class in different fibrations, with identification clear from context.*

8.1. Constructing suitable $\{\mathcal{A}_r\}$.

To get a handle on (8.1) we want to construct very special small data $\mathcal{D}_0$.

A Hamiltonian S^2 fibration over S^4 is classified by an element

$$[g] \in \pi_3(\mathrm{Ham}(S^2), id) \simeq \pi_3(SU(2), id) \simeq \mathbb{Z}.$$

Such an element determines a fibration P_g over S^4 via the clutching construction:

$$P_g = D^4_- \times S^2 \sqcup D^4_+ \times S^2 \sim,$$

with D^4_-, D^4_+ being 2 different names for the standard closed 4-ball D^4, and the equivalence relation $\sim$ is $(d, x) \sim \widetilde{g}(d, x)$,

$$\widetilde{g} : \partial D^4_- \times S^2 \to \partial D^4_+ \times S^2, \quad \widetilde{g}(d, x) = (d, g(d)^{-1}(x)).$$

From now on P_g will denote such a fibration for a non-trivial class $[g]$. Note that the fiber of P_g over the base point $x_0 \in S^3 \subset D^4_\pm$ (chosen for definition of the homotopy group $\pi_3(Ham(S^2), id)$) has a distinguished, by the construction, identification with S^2.

Take $\mathcal{A}$ to be a connection on $P \simeq P_g$ which is trivial in the distinguished trivialization over D^4_+. This gives connections $\mathcal{A}'_r := (\widetilde{u}_r)^* \mathcal{A}$ on $\widetilde{\mathcal{S}}_r$, $\widetilde{u}_r = \Sigma_- \circ u_r$.

By Lemma 6.7, so long as our data $\mathcal{D}_0$ is small the product $\mu^4_{\Sigma^4_-}(\gamma_1, \ldots, \gamma_4)$ is independent of all choices. In particular we may take h_- so that it is an embedding in the interior of Δ^4, and we may take our maps u_r so that each $\widetilde{u}_r$ has the following form. Denote by E the subset $S^3 \subset S^4$ bounding $D^4_\pm$. The preimage by $\widetilde{u}_r$ of E contains a smoothly embedded curve c_r as in Figure 2, and $\widetilde{u}_r$ takes c_r into E. We will also denote by $c_r : \mathbb{R} \to \mathcal{S}_r$ a chosen trivialization of this curve, with the property that c_r maps $(-\infty, 0)$ diffeomorphically onto the image by e_0 of $\{0\} \times (-\infty, 0)$. Likewise c_r maps $(1, \infty)$ diffeomorphically onto the image by e_0 of $\{1\} \times (-\infty, 0)$. We set:

$$\widetilde{c}_r := \widetilde{u}_r \circ c_r.$$

In Figure 2, the regions $R_\pm$ are the preimages by $\widetilde{u}_r$ of $D^4_\pm \subset S^4$, and c_r bounds R_-. The curves $\{c_r\}$ are assumed to vary continuously in r. Moreover, by the

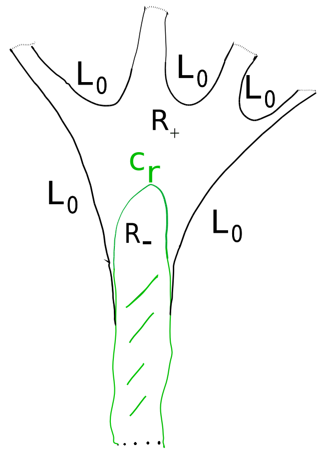

FIGURE 2. The labels L_0 indicate that the Lagrangian subbundle is constant with corresponding fiber L_0. The curve c_r bounds R_-.

last naturality axiom for the maps $\{u_r\}$, we may assume that the family $\{\widetilde{u}_r(\mathcal{S}_r)\}$ induces a singular foliation of S^4, (a smooth foliation outside a single point) then we may likewise assume that $\{\widetilde{c}_r\}$ likewise induces a singular foliation of the equator $E \simeq S^3$.

So each $\mathcal{A}'_r$ is flat in the region R_+, in fact is trivial in the distinguished trivialization of $\widetilde{\mathcal{S}}_r$ over R_+, corresponding to the distinguished trivialization of P over D^4_+. Likewise we have a distinguished trivialization of $\widetilde{\mathcal{S}}_r$ over R_-, corresponding to the distinguished trivialization of P over D^4_-. And in this trivialization the holonomy path of $\mathcal{A}'_r$ over c_r generates an element $f(r) \in \Omega_{L_0} Lag(M, L_0)$. Here by generates we mean that $f(r)(t)$ is given by parallel transport of L_0 by $\mathcal{A}'_r$ over $c_r|_{[0,t]}$. Explicitly, we compute by construction that

$$f(r)(t) := g(\widetilde{c}_r(t))(L_0),$$

$t \in [0, 1]$, where the right hand side means apply an element of $Ham(S^2)$ to L_0 to get a new Lagrangian, and where $\widetilde{c}_r(t)$ is identified with an element of S^3.

Let $D^2_0 \subset \overline{\mathcal{R}}_4$ be an embedded closed disk D^2, not intersecting the boundary $\partial\overline{\mathcal{R}}_4$, so that ∂D^2_0 is in the normal gluing neighborhood N of $\partial\overline{\mathcal{R}}_4$, where N is as in Part I. This gives a map

$$lag' : D^2_0 \to \Omega_{L_0} Lag(S^2)$$

so that $lag'(\partial D^2_0) = L_0$, with the right hand side denoting the constant loop at L_0. Then $lag' \simeq lag$, where $\simeq$ is a homotopy equivalence, and where

$$lag = lag_g : S^2 \to \Omega_{L_0} Lag(S^2) \tag{8.3}$$

is the composition

$$S^2 \xrightarrow{g'} \Omega_{id} SU(2) \to \Omega_{L_0} Lag^{eq}(S^2),$$

for g' naturally induced by g, and for the second map naturally induced by the map

$$SU(2) \to Lag^{eq}(S^2), \quad \phi \mapsto \phi(L_0).$$

We then deform each $\mathcal{A}'_r$ to a connection $\mathcal{A}_r$, which is as follows. In the region R_+ $\mathcal{A}_r$ is still flat, but at each end e_i, $i \neq 0$, in the strip coordinate charts, $\mathcal{A}_r$ has the form $\overline{\mathcal{A}}(L_0^{i-1}, L_0^i)$, where as before the latter means the flat $\mathbb{R}$-translation invariant extension of $\mathcal{A}(L_0^{i-1}, L_0^i)$. Note that $\mathcal{A}(L_0^{i-1}, L_0^i)$ are actually identified for all i, by the axioms for perturbation data, but this is not really material here.

Likewise, at the end e_0, in the corresponding strip chart, $\mathcal{A}_r$ has the form $\overline{\mathcal{A}}(L_0^0, L_0^d)$. Each $\mathcal{A}_r$ preserves the constant Lagrangian sub-bundle $\mathcal{L}_r \subset \partial \mathcal{S}_r \times S^2$ with fiber L_0. Since $\widetilde{\mathcal{S}}_r$ and $\mathcal{A}'_r$ are trivial for $r \in \overline{\mathcal{R}}_d - D_0^2$, with trivialization induced by the trivialization of P_+, and since the condition (7.2) holds, we may insure that

$$\operatorname{area}(\mathcal{A}_r) < \hbar - 5\kappa, \tag{8.4}$$

for r in the complement of D_0^2. In other words $\{\mathcal{A}_r\}$ extend to a system of connections corresponding to small data $\mathcal{D}_0$ for P, as intended.

8.2. **Restructuring the data $\{\mathcal{A}_r\}$.** Applying Lemma 6.20 we see that the resulting Hamiltonian structure $\mathcal{H} := \{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r\}$ is A_0-admissible. We now further modify this for the purposes of computation.

First cap off the ends e_i, $i \neq 0$, of $\mathcal{S}_r$ as in Section 6.4. This gives a Hamiltonian structure

$$\mathcal{H}^\wedge := \{\widetilde{S}_r^\wedge, S_r^\wedge, \mathcal{L}_r^\wedge, \mathcal{A}_r^\wedge\}_{\mathcal{K}=D_0^2},$$

satisfying

$$\operatorname{area}(\mathcal{A}_r^\wedge) + \kappa < \hbar,$$

for each r. And so by Lemma 6.20 it is A_0-admissible.

By the now classical gluing of holomorphic curves it follows that

$$[\mu^4_{\Sigma^4}(\gamma_1, \ldots, \gamma_4)] = [ev(\{\widetilde{S}_r^\wedge, S_r^\wedge, \mathcal{L}_r^\wedge, \mathcal{A}_r^\wedge, D_0^2\}, A_0)].$$

It remains to compute $[ev(\{\widetilde{S}_r^\wedge, S_r^\wedge, \mathcal{L}_r^\wedge, \mathcal{A}_r^\wedge\}, A_0)]$.

Let $p_1 : [0,1] \to Lag(S^2, L_0)$ be the path generated by $\mathcal{A}(L_0^0, L_0^1)$, with p_1 starting at L_0. Suppose we have defined p_{i-1}, set $L_{i-1} := p_{i-1}(1)$ and define p_i to be the path in $Lag(S^2, L_0)$ starting at L_{i-1}, generated by $\mathcal{A}(L_0^{i-1}, L_0^i)$. Set $p_0 := p_1 \cdot \ldots \cdot p_d$, where $\cdot$ is path concatenation in diagrammatic order. We may assume that L_0 is transverse to $L_4 = p_0(1)$ by adjusting the connections $\mathcal{A}(L_0^{i-1}, L_0^i)$ if necessary. Then deform $\mathcal{L}_r^\wedge$ in the Hamiltonian data $\{\widetilde{S}_r^\wedge, S_r^\wedge, \mathcal{L}_r^\wedge, \mathcal{A}_r^\wedge\}$ as in Figure 3. The resulting Lagrangian subbundle over $\partial \mathcal{S}_r$ will be denoted $\mathcal{L}_r^{\mathfrak{n}}$, $\mathfrak{n}$ here stands for 'new'.

We simultaneously deform $\mathcal{A}_r^\wedge$ to an $\mathcal{L}_r^{\mathfrak{n}}$ exact Hamiltonian connection $\mathcal{A}_r^{\mathfrak{n}}$ which satisfies the following conditions. $\mathcal{A}_r^{\mathfrak{n}}$ is flat in the entire region R_+ (which includes the red shaded finger regions). Along the dotted line (which is contained in the strip chart at the e_0 end) $\mathcal{A}_r^{\mathfrak{n}}$ is the trivial connection in the distinguished trivialization at the end, and such that at the e_0 end, which is down in the Figure 3, the connection is unchanged over $[0,1] \times (t, \infty)$, for t large. In order to get such a deformation, we introduce curvature in the blue stripped region of Figure 3. We name this new Hamiltonian data by

$$\mathcal{H}^{\mathfrak{n}} := \{\widetilde{S}_r^\wedge, S_r^\wedge, \mathcal{L}_r^{\mathfrak{n}}, \mathcal{A}_r^{\mathfrak{n}}\}.$$

As each $L^+(p_i)$ can be arranged to be arbitrarily small, it is clear that we may choose the deformation from $\mathcal{H}^\wedge$ to $\mathcal{H}^{\mathfrak{n}}$ to be small near boundary (Definition 6.14)

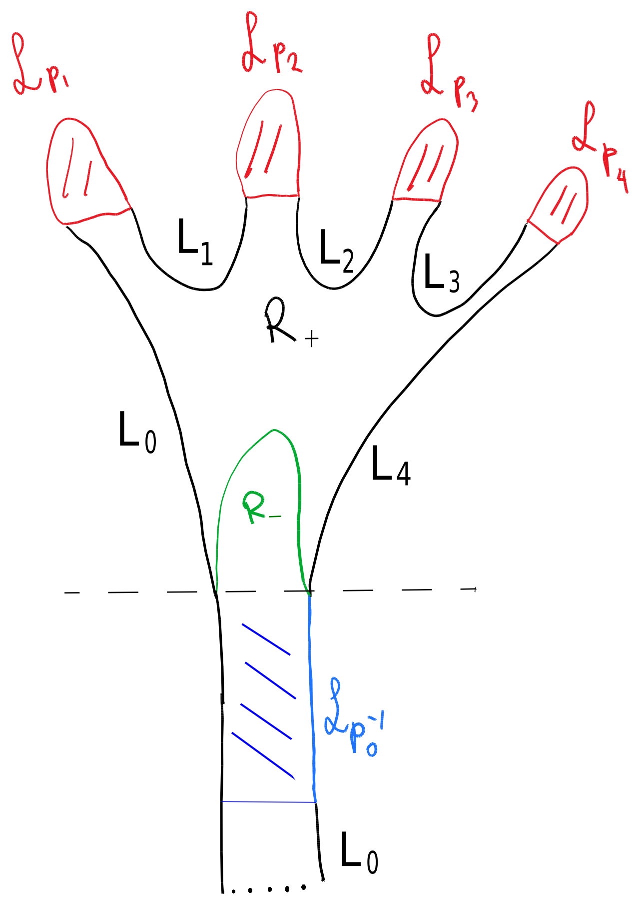

FIGURE 3. Over the boundary components with black labels L_i the Lagrangian subbundle $\mathcal{L}_r^{\mathfrak{n}}$ is constant with corresponding fiber L_i. Over the i'th red boundary component the Lagrangian subbundle corresponds to the path of Lagrangians p_i, analogously to Definition 8.2 further below. Likewise over the blue boundary component the Lagrangian subbundle corresponds to the path of Lagrangians p_0^{-1}. In the red striped regions we have removed the curvature of the connection, the blue striped regions we have added it.

and hence be an A_0-admissible concordance. More specifically, we may choose a concordance from $\mathcal{H}^\wedge$ to $\mathcal{H}^{\mathfrak{n}}$ so that for the associated family of connections $\{\mathcal{A}_{r,t}\}$,

$$\mathcal{A}_{r,0} = \mathcal{A}_r^\wedge, \mathcal{A}_{r,1} = \mathcal{A}_r^{\mathfrak{n}},$$

the L^+ norm of the curvature is everywhere pointwise decreasing in t, except in the region which is blue striped in Figure 3. However, the area increase in this region is bounded from above by $L^+(p_0^{-1})$, so that

$$\forall t : |\operatorname{area}(\mathcal{A}_{r,t}) - \operatorname{area}(\mathcal{A}_r^\wedge)| \leq L^+(p_0^{-1}).$$

In fact we can arrange that

$$\forall t_0 \in [0,1] : \frac{d}{dt}|_{t_0} \operatorname{area}(\mathcal{A}_{r,t}) = 0,$$

since the gain of area in the blue striped region is exactly equal to the loss of area in the red striped regions, but this extra precision is not necessary.

Of course:

$$[ev(\{\widetilde{S}_r^\wedge, S_r^\wedge, \mathcal{L}_r^\wedge, \mathcal{A}_r^\wedge\}, A_0)] = [ev(\{\widetilde{S}_r^\wedge, S_r^\wedge, \mathcal{L}_r^{\mathfrak{n}}, \mathcal{A}_r^{\mathfrak{n}}\}, A_0)]$$

since the corresponding Hamiltonian data are A_0-admissibly concordant. If we stretch the neck along the dashed line in Figure 3, the upper half of the resulting building gives us new Hamiltonian data

$$\{\widetilde{S}_r^0, S_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}.$$

By the classical theory of continuation maps in Floer homology we clearly have that

$$[ev_0] := [ev(\{\widetilde{S}_r^\wedge, S_r^\wedge, \mathcal{L}_r^n, \mathcal{A}_r^n\}, A_0)] \in FH(L_0, L_0)$$

is non-zero iff

$$[ev_{p_0}] := [ev(\{\widetilde{S}_r^0, S_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}, A_0)] \in FH(L_0, L_4)$$

is non-zero.

Let $\mathcal{P}_{L_0,L_4} Lag(M)$ denote the space of smooth paths in $Lag(M)$ from L_0 to L_4. Let

$$f_{p_0} : D_0^2 \to \mathcal{P}_{L_0,L_4} Lag(S^2),$$

be like f but defined with respect to $\{\mathcal{A}_r^0\}$. In this case f_{p_0} takes ∂D_0^2 to $p' \in \mathcal{P}_{L_0,L_4} Lag(M)$, where p' is p_0 up to parametrization. So that f_{p_0} represents a class $a \in \pi_2(\mathcal{P}_{L_0,L_4} Lag(M), p')$. It is clear from construction that $a = [f']$ with f' defined by

$$f'(r)(t) := g(\widetilde{c}_r(t))(p_0(t)),$$

where the right hand side means as before: apply an element of $Ham(S^2)$ to a Lagrangian to get a new Lagrangian.

In what follows we omit specifying the parameter space D_0^2 for r, since it will be the same everywhere.

Definition 8.2. *Let $\pi : \mathbb{R} \to [0,1]$ denote the retraction map, sending $(-\infty, 0]$ to 0, and sending $[1,\infty)$ to 1. Fix a parametrization $\zeta : \mathbb{R} \to \partial\mathcal{D}$, which satisfies $\zeta(t) \in End$ for $t \in (-\infty, 0] \sqcup [1,\infty)$, where End denotes the image of the strip chart $e_0 : [0,1] \times (0,\infty) \to \mathcal{D}$. Given a smooth path*

$$p : [0,1] \to Lag(M, L)$$

constant near $0, 1$, let $\mathcal{L}_p \subset \partial\mathcal{D} \times M$ denote the Lagrangian subfibration over $\partial\mathcal{D}$, with fiber over $r \in \partial\mathcal{D}$ given by $p \circ \pi(r)$. We say that a Lagrangian subfibration $\mathcal{L}$ as above is **determined by** *p if $\mathcal{L} = \mathcal{L}_p$, after a fixed choice of parametrization of boundary of $\mathcal{D}$ by $\mathbb{R}$.*

Thus for $p : [0,1] \to Lag(M, L)$ as above, there is an associated Hamiltonian data $(\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_p)$.

Lemma 8.3. *The A_0-admissible Hamiltonian structure $\{\widetilde{\mathcal{S}}_r^0, \mathcal{S}_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}$ is A_0-admissibly concordant to $\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f_{p_0}(r)}, \mathcal{B}_r\}$, for certain Hamiltonian connections $\{\mathcal{B}_r\}$ (which are not implicitly relevant yet).*

Proof. Let $R_\pm \subset \mathcal{S}_r^\wedge$ be as before. Fix a deformation retraction

$$ret_r : \mathcal{S}_r^0 \times I \to \mathcal{S}_r^0,$$

of $\mathcal{S}_r^0$ onto R^-, smoothly in r. Since $\mathcal{A}_r^0$ is flat over R^+, the pull-back by ret_r of the data

$$\{\widetilde{\mathcal{S}}_r^0, \mathcal{S}_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}$$

then induces an A_0-admissible concordance between the Hamiltonian structure $\{\widetilde{\mathcal{S}}_r^0, \mathcal{S}_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}$ and $\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f_{p_0}(r)}, \mathcal{B}_r = ret_r^* \mathcal{A}_r^0\}$, once we use smooth Riemann mapping theorem to identify each $R^- \subset \mathcal{S}_r^0$ with its induced complex structure j_r with $(\mathcal{D}, j_{st})$, smoothly in r. □

9. Higher relative Seidel morphism

The class of $ev(\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f_{p_0}(r)}, \mathcal{B}_r\}, A_0)$ is related to the notion of higher relative Seidel morphism, and it is worth making this more explicit, even if formally we will not need these generalities.

The relative Seidel morphism appears in Seidel's [21] in the exact case and further developed in [8] in the monotone case. Let $Lag(M)$ denote the space whose components are objects of $Fuk(M)$ in the previous sense, so in particular oriented, spin, Hamiltonian isotopic Lagrangian submanifolds of M. We may also denote the component of L by $Lag(M, L)$. Then the relative Seidel morphism is a functor

$$S : \Pi Lag(M) \to DF(M),$$

where $\Pi Lag(M)$ is the fundamental groupoid of $Lag(M)$ and $DF(M)$ is the Donaldson-Fukaya category of M, see also [4], [3] which can be understood as an extension.

We sketch how this works. To a path p in $Lag(M)$ from L_0 to L_1 we have have an associated Lagrangian subbundle $\mathcal{L}_p$ of $\mathcal{D} \times M$ over the boundary, as in Definition 8.2. Extend this to a Hamiltonian data $(\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_p, \mathcal{A})$ as in Definition 6.2. We define $S([p]) \in DF(L_0, L_1)$ by

$$S([p]) = \sum_A [ev((\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_p, \mathcal{A}), A)],$$

where by monotonicity only finitely many A can have non-zero contribution.

9.1. Definition of the higher (relative) Seidel morphism.

Let M be as before, and let $\mathcal{P}(L_0, L_1)$ denote the space of smooth paths in $Lag(M)$ from L_0 to L_1, constant in $[0, \epsilon] \cup [1 - \epsilon, 1]$ for some $0 < \epsilon < 1$. There is then an additive group homomorphism:

$$\Psi : H_*(\mathcal{P}(L_0, L_1), \mathbb{Q}) \to FH(L_0, L_1) \tag{9.1}$$

defined analogously to above and to [15] in non-relative context. Although formally we will only need the restriction of Ψ to spherical classes.

This works as follows. To a smooth cycle

$$f : B \to \mathcal{P}(L_0, L_1)$$

for B a smooth closed oriented manifold, we may associate a Hamiltonian structure

$$\{\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_b\}_B,$$

$\mathcal{L}_b := \mathcal{L}_{f(b)}$ a Lagrangian subbundle of $M \times \mathcal{D}$ over $\partial \mathcal{D}$ determined by $f(b)$ as before. Now let $\mathcal{A}_0$ be a Hamiltonian connection on $[0, 1] \times M$, so that $\mathcal{A}_0(L_0)$ is transverse to L_1 where $\mathcal{A}_0(L_0) \subset \{1\} \times M$ denotes the $\mathcal{A}_0$-transport over $[0,1]$ of $L_0 \subset \{0\} \times M$.

For each b the space of Hamiltonian connections $\mathcal{L}_b$-exact with respect to $\mathcal{A}_0$, (as in Section 6.2) is contractible, c.f. [1]. So we get induced Hamiltonian structure $\{\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}$ well defined up to concordance, automatically admissible since $\partial B = \emptyset$.

We may then define $\Psi([f])$ by:

$$\Psi([f]) = \sum_A [ev(\{\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}, A)],$$

where by monotonicity only finitely many A can give non-zero contribution. It is immediate that Ψ is an additive group homomorphism. In addition, the morphism Ψ extends to a certain functor to $DF(M)$, but this functoriality will not be needed so we do not elaborate, see however [3] for a similar discussion.

Given the definition above,

$$[ev(\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f_p(r)}, \mathcal{B}_r\}, A_0)] = \Psi(a)$$

clearly holds, as A_0 is the only class that can contribute to $\Psi(a)$, since by the dimension formula (B.1) only a class A with $Maslov^{vert}(A) = -2$ can contribute.

10. Computation of the higher Seidel element $\Psi(a)$

10.1. **Morse theory for the Hofer length functional.** Under certain conditions the spaces of perturbation data for certain problems in Gromov-Witten theory admit a Hofer like functional. Although these spaces of perturbations are usually contractible, there may be a gauge group in the background that we have to respect, so that working equivariantly there is topology. The reader may think of the analogous situation in Yang-Mills theory [2].

Without elaborating too much, the basic idea of the computation that we will perform consists of cooling the perturbation data as much as possible (in the sense of the functional) to obtain a mini-max (for the functional) data, using which we may write down our moduli spaces explicitly. This idea was first used in [16].

10.1.1. *Hofer length.* For $p : [0,1] \to Ham(M, \omega)$ a smooth path, define

$$L^+(p) := \int_0^1 \max_M H_t^p dt,$$

where $H^p : M \times [0,1] \to \mathbb{R}$ generates p normalized by the condition that for each t, $H_t^p := H^p|_{M \times \{t\}}$ has mean 0, that is $\int_M H_t^p dvol_\omega = 0$. Also define

$$L^+_{lag} : \mathcal{P}Lag(M) \to \mathbb{R},$$

$$L^+_{lag}(p) := \int_0^1 \max_{p(t)} H_t^p dt,$$

$p(0) = L$ and where $H^p : M \times [0,1] \to \mathbb{R}$ is normalized as above and generates a lift $\widetilde{p}$ of p to $Ham(M)$ starting at id. By lift we mean that $p(t) = \widetilde{p}(t)(p(0))$. (That is H^p generates a path in $Ham(M)$, which moves L_0 along p.) Some theory of this latter functional is developed in [9].

We may however omit the subscript lag from notation, as usually there can be no confusion which functional we mean.

Note that $Lag^{eq}(S^2)$ is naturally diffeomorphic to S^2 and moreover it is easy to see that the functional $L^+|_{Lag^{eq}(S^2)}$ is proportional to the Riemannian length functional L_{met} on the path space of S^2, with its standard round metric *met*.

Let now $L_0, L_1 \in Lag^{eq}(S^2)$ be any transverse pair, and

$$f' : S^2 \to \mathcal{P}(L_0, L_1) := \mathcal{P}_{L_0,L_1} Lag^{eq}(S^2),$$

be the generator of the group $H_2(\mathcal{P}(L_0, L_1), \mathbb{Z})$. The idea of the computation is then this: perturb f' to be transverse to the (infinite dimensional) stable manifolds for the Riemannian length functional on

$$\mathcal{P}(L_0, L_1) := \mathcal{P}_{L_0,L_1} Lag^{eq}(S^2),$$

push the cycle down by the "infinite time" negative gradient flow for this functional, and use the resulting representative to compute $\Psi(a = [f'])$. Although, we will not actually need infinite dimensional topology.

10.1.2. *The "energy" minimizing perturbation data.* Classical Morse theory [12] tells us that the energy functional

$$E(p) = \int_{[0,1]} \langle \dot{p}(t), \dot{p}(t) \rangle_{met} \, dt$$

on $\mathcal{P}(L_0, L_1)$ is Morse non-degenerate with a single critical point in each degree. Consequently a (as a homology class) has a representative in the 2-skeleton of $\mathcal{P}(L_0, L_1)$, for the Morse cell decomposition induced by E. This follows by Whitehead's compression lemma which is as follows.

Lemma 10.1 (Whitehead, see [7]). *Let (X, A) be a CW pair and let (Y, B) be any pair with $B \neq \emptyset$. For each n such that $X - A$ has cells of dimension n, assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f : (X, A) \to (Y, B)$ is homotopic relative to A to a map $X \to B$.*

Suppose that a has a representative $f' : S^2 \to \mathcal{P}_{L_0,L_1}(S^2)$ mapping into the n-skeleton B^n for the Morse cell decomposition for E, $n > 2$. Apply the lemma above with $(X, A) = (S^2, pt)$, $Y = B^n$ and $B = B^{n-1}$ as above. Then the quotient B^n/B^{n-1} is a wedge of n-spheres and since $\pi_2(S^n) = 0$ for $n > 2$, f can be homotoped into B^{n-1} by the Whitehead lemma. Descend this way until we get a representative mapping into B^2.

Furthermore since $\pi_2(S^1) = 0$ such a representative cannot entirely lie in the 1-skeleton. It follows, since we have a single Morse 2-cell that there is a representative $f : S^2 \to \mathcal{P}_{L_0,L_1}(S^2)$, for a, s.t. the function f^*E is Morse with a maximizer max, of index 2, and s.t. $\gamma_0 = f(max)$ is the index 2 geodesic.

Remark 10.2. *In principle there maybe more than one such maximizer* max*, but recall that we assumed that a is the generator, so by further deformation we may insure that there is only one maximizer. The relevant representative f, with a single maximizer* max *as above, can also be constructed by hand.*

It follows that max is likewise the unique index 2 maximizer of the function f^*L_{met} by the classical relation between the energy functional and length functional. And so max is the index 2 maximizer of f^*L^+.

10.1.3. *The corresponding minimizing data.* In what follows f is minimizing as above. Let $\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b\}$, $\mathcal{L}_b := \mathcal{L}_{f(b)}$, be the associated Hamiltonian structure as in Section 9.1, compatible with the trivial connection $\mathcal{A}_0$ on $[0,1] \times M$ at the e_0 end.

Lemma 10.3. *There is a taut Hamiltonian structure* $\{\Theta_b\} = \{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}$, *satisfying:*

$$\text{area}(\mathcal{A}_b) = L^+(f(b)). \tag{10.1}$$

Proof. Note that a geodesic segment $p : [0,1] \to S^2$ for the round metric met on S^2 has a unique lift $\widetilde{p} : [0,1] \to SU(2)$, $\widetilde{p}(0) = id$ with $\widetilde{p}$ a segment of a one parameter subgroup, and in this case $L^+_{lag}(p) = L^+(\widetilde{p})$. It then follows that for a piecewise geodesic path p in S^2, there is likewise a unique lift $\widetilde{p}$ satisfying $L^+_{lag}(p) = L^+(\widetilde{p})$. We may assume that each $f(b)$ is piecewise geodesic, this follows by the piecewise geodesic approximation theorem Milnor [12, Theorem 16.2] of the loop space.

Let H^b generate the lift $\widetilde{f}(b)$ of $f(b)$ as above, so that $L^+_{lag}(f(b)) = L^+(\widetilde{f}(b))$. The construction of $\{\mathcal{A}_b\}$ is then similar to the one in Lemma 6.18. For each $b \in S^2$ we define the coupling form Ω'_b on $D^2 \times S^2$:

$$\Omega'_b = \omega - d(\eta(rad) \cdot H^b d\theta),$$

for (rad, θ) the modified angular coordinates on D^2, $rad, \theta \in [0,1]$ and $\eta : [0,1] \to [0,1]$ is a smooth function satisfying

$$0 \leq \eta'(rad),$$

and

$$\eta(rad) = \begin{cases} 1 & \text{if } 1 - \delta \leq rad \leq 1, \\ rad^2 & \text{if } rad \leq 1 - 2\delta, \end{cases} \tag{10.2}$$

for a small $\delta > 0$. By assumptions on our paths the associated connection $\mathcal{A}'_b$ is trivial over the arc $arc = \{(1, \theta)\}$, $-\epsilon \leq \theta \leq \epsilon$. And by elementary calculation using (6.4)

$$\text{area}(\mathcal{A}'_b) = L^+(f(b)).$$

Now, fix a smooth embedding $i : D^2 \hookrightarrow \mathcal{D}$ so that the image of the embedding contains $\partial\mathcal{D} - end$ where end is the image of the distinguished strip chart

$$[0,1] \times (0, \infty) \to \mathcal{D}.$$

And so that $i(arc) = [0,1] \times \{1\}$ in this strip chart, as illustrated in the Figure 1.

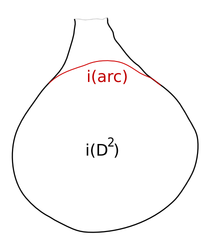

FIGURE 4.

Next fix a deformation retraction ret of $\mathcal{D}$ onto $i(D^2)$, so that in the strip chart above, ret, for $r \geq 1$, is the composition $i \circ param \circ pr$, for $pr : [0,1] \times (0, \infty) \to [0,1]$

the projection and $param : [0, 1] \to arc \subset D^2$ diffeomorphism. And set $\Omega_b = ret^* \Omega'_b$ on $\mathcal{D} \times S^2$. For each b the connection induced by Ω_b is defined to be our $\mathcal{A}_b$.

It remains to verify that $\{\Theta_b\} = \{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}$ is taut, (it is in fact hyper taut but we do not need this). This follows by the following more general lemma.

Lemma 10.4. *Let $Lag^{eq}(\mathbb{CP}^n)$ denote the space of oriented Lagrangian submanifolds Hamiltonian isotopic to $\mathbb{RP}^n$, then two loops $p_1, p_2 : S^1 \to Lag^{eq}(\mathbb{CP}^n)$ are taut concordant as defined in Section 1.2 iff they are homotopic.*

Proof. Let $\mathcal{L}$ be a sub-fibration of $Cyl \times M$ as in the definition of taut concordance of loops. Let $\mathcal{A}$ be any unitary connection on $Cyl \times \mathbb{CP}^n$ which preserves $\mathcal{L}$ (there are no obstructions to constructing this). Then R_A is a lie $U(n)$ valued 2-form, such that for all $v, w \in T_z Cyl$ the vector field $X = R_{\mathcal{A}}(z)(v, w)$ is tangent to $\mathcal{L}_z$. In particular if H_X is the Hamiltonian generating X, then since X is an infinitesimal unitary isometry preserving $\mathcal{L}_z$, H_X vanishes on $\mathcal{L}_z$. It follows by the definition of $\Omega_{\mathcal{A}}$, that it vanishes on $\mathcal{L}$ and so we are done. $\square$

$\square$

By construction each $\mathcal{A}_b$ satisfies the property $\mathrm{area}(\mathcal{A}_b) = L^+(f(b))$, so we immediately deduce:

Lemma 10.5. *The function $area : b \mapsto \mathrm{area}(\mathcal{A}_b)$ has a unique maximizer, coinciding with the maximizer $\max$ of $f^* L_{met}$ and area is Morse at $\max$ with index 2.*

10.1.4. *Finding class A_0 holomorphic sections for the data.* As $f(\max)$ is a geodesic for met, its lift $\widetilde{f}(\max)$ to $SO(3)$ is a rotation around an axis intersecting $L_0 = f(\max)(0)$ in a pair of points, in particular there there is a unique point

$$x_{\max} \in \bigcap_t (L_t = f(\max)(t))$$

maximizing $H_t^{\max}$ for each t. In our case this follows by elementary geometry but there is a more general phenomenon of this form c.f. [9].

Define

$$\sigma_{\max} : \mathcal{D} \to \mathcal{D} \times S^2$$

to be the constant section $z \mapsto x_{\max}$. Then $\sigma_{\max}$ is a $\mathcal{A}_{\max}$-flat section with boundary on $\mathcal{L}_{\max}$, and is consequently $J(\mathcal{A}_{\max})$-holomorphic.

Let $\gamma_0 \in CF(L_0, L_1)$ be the generator corresponding to the intersection point $x_{\max}$ of L_0, L_1.

Lemma 10.6. *$\sigma_{\max}$ has vertical Maslov number -2 so that $\sigma_{\max} \in \overline{\mathcal{M}}(\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}, A_0, \gamma_0)$.*

Proof. Set

$$T_z^{vert} \mathcal{L}_{\max} := \{v \in T\mathcal{L} \subset T_z(\mathcal{D} \times S^2) \,|\, pr_* v = 0\}$$

where $pr : \mathcal{D} \times S^2 \to \mathcal{D}$ is the projection. Denote by

$$Lag(T_{x_{\max}} S^2 \simeq Lag(\mathbb{R}^2) \simeq S^1$$

the space of oriented linear Lagrangian subspaces of $T_{x_{\max}} S^2$. Let ρ be the path in $Lag(T_{x_{\max}} S^2)$ defined by

$$\rho(t) = T^{vert}_{(\zeta(t), x_{\max})} \mathcal{L}_{\max}, \quad t \in [0, 1]$$

where $\zeta : \mathbb{R} \to \partial\mathcal{D}$ is a fixed parametrization as in Definition 8.2.

By our conventions for the Hamiltonian vector field:

$$\omega(X_H, \cdot) = -dH(\cdot),$$

ρ is a clockwise path from $T_{x_{\max}} L_0 = T^{vert}_{(\zeta(0), x_{\max})} \mathcal{L}_{\max}$ to $T_{x_{\max}} L_1 = T^{vert}_{(\zeta(1), x_{\max})} \mathcal{L}_{\max}$ for the orientation induced by the complex orientation on $T_{x_{\max}} S^2$.

By the Morse index theorem in Riemannian geometry [12] and by the condition that $f(\max)$ has Morse index 2, ρ visits initial point $\rho(0)$ exactly twice if we count the start, as this corresponds to the geodesic $f(\max)$ passing through two conjugate points in S^2. So the concatenation of ρ with the minimal counter-clockwise path from $T_{x_{\max}} L_1$ back to $T_{x_{\max}} L_0$ is a degree -1 loop, if $S^1 \simeq Lag(\mathbb{R}^2)$ is given the counter-clockwise orientation. Consequently $\sigma_{\max}$ has Maslov number -2, cf. Appendix B. So $[\sigma_{\max}] = A_0$. □

Proposition 10.7. *$(\sigma_{\max}, \max)$ is the sole element of $\overline{\mathcal{M}}(\{\Theta_b\}, A_0, \gamma_0)$.*

Proof. By Stokes theorem, since ω vanishes on $\sigma_{\max}$, it is immediate:

$$carea(\Theta_{\max}, A_0) = -\int_{\mathcal{D}} \sigma^*_{\max} \widetilde{\Omega}_{\max} = L^+(f(\max)). \tag{10.3}$$

Moreover, since $\{\Theta_b\}$ is taut $carea(\Theta_b, A_0) = L^+(f(\max))$. So by (10.1) and by Lemmas 6.9, 6.10 we have:

$$L^+(f(\max)) \leq \text{area}(\mathcal{A}_b) = L^+(f(b)),$$

whenever there is an element

$$(\sigma, b) \in \overline{\mathcal{M}}(\{\Theta_b\}, A_0, \gamma_0).$$

But clearly this is impossible unless $b = \max$, since $L^+(f(b)) < L^+(f(\max))$ for $b \neq \max$. So to finish the proof of the proposition we just need:

Lemma 10.8. *There are no elements σ other than $\sigma_{\max}$ of the moduli space*

$$\overline{\mathcal{M}}(\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{\max}, \mathcal{A}_{\max}, A_0, \gamma_0).$$

Proof. We have by (10.3), and by (10.1)

$$0 = \langle [\widetilde{\Omega}_{\max} + \alpha_{\widetilde{\Omega}_{\max}}], [\sigma_{\max}] \rangle,$$

and so given another element σ we have:

$$0 = \langle [\widetilde{\Omega}_{\max} + \alpha_{\widetilde{\Omega}_{\max}}], [\sigma] \rangle.$$

It follows that σ is necessarily $\widetilde{\Omega}_{\max}$-horizontal, since

$$(\widetilde{\Omega}_{\max} + \alpha_{\widetilde{\Omega}_{\max}})(v, J_{\widetilde{\Omega}_{\max}} v) \geq 0.$$

Since $J_{\widetilde{\Omega}_{\max}}$ by assumptions preserves the vertical and $\mathcal{A}_{\max}$-horizontal subspaces of $T(\mathcal{D} \times S^2)$, and since the inequality is strict for v in the vertical tangent bundle of

$$S^2 \hookrightarrow \mathcal{D} \times S^2 \to \mathcal{D},$$

the above inequality is strict whenever v is not horizontal. So σ must be $\mathcal{A}_{\max}$-horizontal. But then $\sigma = \sigma_{\max}$ since $\sigma_{\max}$ is the only flat section asymptotic to γ_0. □

□

10.1.5. *Regularity.* It will follow that

$$\Psi(a) = \pm[\gamma_0]$$

if we knew that $(\sigma_{\max}, \max)$ was a regular element of

$$\overline{\mathcal{M}}(\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}, A_0, \gamma_0).$$

We won't answer directly if $(\sigma_{\max}, \max)$ is regular, although it likely is. But it is regular after a suitably small Hamiltonian perturbation of the family $\{\mathcal{A}_r\}$ vanishing at $\mathcal{A}_{\max}$. We call this essentially automatic regularity.

Lemma 10.9. *There is a family $\{\mathcal{A}_b^{reg}\}$ arbitrarily C^∞-close to $\{\mathcal{A}_b\}$ with $\mathcal{A}_{\max}^{reg} = \mathcal{A}_{\max}$ and such that*

$$\overline{\mathcal{M}}(\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b^{reg}\}, A_0, \gamma_0), \tag{10.4}$$

is regular, with $(\sigma_{\max}, \max)$ its sole element. In particular $\Psi(a) = \pm[\gamma_0]$.

Proof. The associated real linear Cauchy-Riemann operator

$$D_{\sigma_{\max}} : \Omega^0(\sigma_{\max}^* T^{vert}\mathcal{D} \times S^2_{\max}) \to \Omega^{0,1}(\sigma_{\max}^* T^{vert}\mathcal{D} \times S^2_{\max}),$$

has no kernel, by Riemann-Roch [11, Appendix C], as the vertical Maslov number of $[\sigma_{\max}]$ is -2. And the Fredholm index of $(\sigma_{\max}, \max)$ which is -2, is -1 times the Morse index of the function area at max, by Lemma 10.5. Given this, our lemma follows by a direct translation of [19, Theorem 1.20], itself elaborating on the argument in [16]. □

To summarize:

Theorem 10.10. *For $0 \neq a \in H_2(\mathcal{P}_{L_0,L_1} Lag(S^2), \mathbb{Z})$,*

$$0 \neq \Psi(a) \in HF(L_0, L_1).$$

Proof. We have shown that $0 \neq \Psi(a) \in HF(L_0, L_1)$, for a the generator of the group $H_2(\mathcal{P}_{L_0,L_1} Lag(S^2), \mathbb{Z})$. Since Ψ is an additive group homomorphism the conclusion follows. □

11. Proof of Lemma 4.5

We first identified $[\mu^4_{\Sigma^4}(\gamma_1, \ldots, \gamma_4)]$ with a certain Floer class $[ev_{p_0}] \in HF(L_0, L_1)$. We then use Lemma 8.3 to identify $[ev_{p_0}]$ with $\Psi(a)$, for a certain spherical 2-class a. Finally, in Section 10 we compute $\Psi(a)$ and show that it is non-zero. This together with Lemma 5.1 imply Lemma 4.5. □

12. Proof of Theorem 1.13

Suppose otherwise, so that

$$\min_{f \in a} \max_{s \in S^2} L^+(f(s)) = U < \hbar,$$

for $a = i_* g$ as in the statement of the theorem. Fix $L_1 \in Lag(S^2)$ so that L_0 intersects L_1 transversally, and so that there is a a path p_0 from L_0 to L_1 with

$$\kappa := L^+(p_0) < \epsilon = (\hbar - U)/2.$$

Then concatenating f with p_0 we obtain a smooth family of paths

$$f' : S^2 \to \mathcal{P}(L_0, L_1)$$

so that the corresponding Hamiltonian structure $\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}$, where $\mathcal{A}_b$ is as in Lemma 10.3 and $\mathcal{L}_b := \mathcal{L}_{f'(b)}$, is taut and satisfies:

$$\forall b \in S^2 : \text{area}(\mathcal{A}_b) = L^+(f'(b)) < \hbar - \kappa. \tag{12.1}$$

Since $\Psi(a') \neq 0$ the moduli space

$$\overline{\mathcal{M}}(\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}, A_0),$$

is non-empty. But by (12.1)

$$\text{area}(\mathcal{A}_b) < \hbar((\mathcal{D} \times S^2)^/, \mathcal{D}^/, \mathcal{L}_b^/) - \kappa, \text{ for each } b,$$

since $\hbar((\mathcal{D} \times S^2)^/, \mathcal{D}^/, \mathcal{L}_b^/) = \hbar$ for each b, by Lemma 6.13. But this contradicts Lemma 6.20. $\square$

13. Proof of Theorem 1.7

Let ω be the Fubini-Study 2-form on S^2 with area 1. Then the pull-back by the natural map

$$\text{lie}\, SU(2) \to \text{lie}\, Ham(S^2, \omega) \simeq C_0^\infty(S^2)$$

of the norm: $|H|_+ = \max_{S^2} H$ is the operator norm on $SU(2)$, with normalization as in statement of Theorem 1.7. This is a simple exercise.

Let $\mathcal{A}$ be an abstract simplicial connection on P, which in particular specifies a Kan $Y_\bullet \subset S^4_\bullet$, $|Y_\bullet| \simeq S^4$ and we suppose (wlog) that $x_0 \in Y_\bullet$. Let $\Sigma \in Y_\bullet$, $\Sigma : \Delta^4 \to S^4$, generate $\pi_4(Y_\bullet, x_0) \simeq \pi_4(S^4_\bullet, x_0) \simeq \mathbb{Z}$. Set

$$\widetilde{u}_r := \Sigma \circ u(m_1, \ldots, m_4, r, 4) : \mathcal{S}_r \to S^4,$$

where $u_r = u(m_1, \ldots, m_4, r, 4)$ are part of the chosen perturbation data $\mathcal{D}$, which are assumed to be embeddings for $r \notin \partial\overline{\mathcal{R}}_4$. We also set $\widetilde{S}_r = \widetilde{u}_r^* P$. Let $\Sigma_0^n : \Delta^n \to X$ denote the composition

$$\Delta^n \to \Delta^0 \xrightarrow{\Sigma_0} X,$$

where Σ_0 is induced by inclusion of x_0 into X.

By inductive procedure as in Part I, we may find perturbation data $\mathcal{D}$ for P so that the associated connections satisfy:

$$\mathcal{A}_r' := pr_1 \mathcal{F}(m_1, \ldots, m_4, \Sigma, r) \simeq_\delta \mathcal{A}_r := u_r^* \mathcal{A}_{\Sigma_-} \tag{13.1}$$

$$\forall n \in \mathbb{N} : pr_1 \mathcal{F}(m_1, \ldots, m_n, \Sigma_n^0, r) \simeq_\delta u(m_1, \ldots, m_n, r, n)^* \mathcal{A}_{\Sigma_0^n} \tag{13.2}$$

with $\simeq_\delta$ meaning δ-close in the metrized C^∞ topology.

We may assume that

$$\forall n : R_{\mathcal{A}_{\Sigma_0^n}} = 0$$

as otherwise if for some n and $v, w \in T_x \Delta^n$ $R_{\mathcal{A}_{\Sigma_0^n}}(v, w) \neq 0$, then since

$$(\Sigma_0^n)_* v = (\Sigma_0^n)_* w = 0,$$

by our definition we would have $|R_{\mathcal{A}}| = \infty$. It follows that if δ is chosen to be small enough then the resulting data $\mathcal{D}$ is small in the sense of Definition 4.3.

Take the unital replacement as in Lemma 5.1. Since we know that $K(P)$ does not admit a section by Theorem 3.4, the simplex T of the Lemma 5.1 does not exist. Hence again by this lemma

$$[\mu^4_\Sigma(\gamma_1, \ldots, \gamma_4)] \neq 0,$$

so that $\overline{\mathcal{M}}(\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}'_r\}, A_0) \neq \emptyset$. And this holds for every $\{\mathcal{A}'_r\}$ as above. So by Lemma 6.20 for every $\{\mathcal{A}'_r\}$ as above there exists an r_0 so that

$$\text{area}(\mathcal{A}'_{r_0}) \geq \hbar - 5\kappa, \tag{13.3}$$

where κ as before denotes the L^+ length of the holonomy path in $Ham(S^2)$ of $\mathcal{A}_0 = \mathcal{A}(L_0, L_0)$, and can be made as small as we like by taking δ above to be suitably small.

Then by the above for every $\epsilon > 0$ there exists an r_0 so that

$$\text{area}(\mathcal{A}_{r_0}) \geq \hbar - \epsilon, \tag{13.4}$$

and so for some r_0:

$$\text{area}(\mathcal{A}_{r_0}) \geq \hbar. \tag{13.5}$$

Let $ar : TX \oplus TX \to \mathbb{R}_{\geq 0}$ be as in the statement of the theorem. We then have an upper bound:

$$\sup_r \left(\text{area}(\widetilde{u}_r) := \int_{\mathcal{S}_r} dA \right) < C,$$

where dA is the 2-form

$$dA(z) := ar(d\widetilde{u}_r(v), d\widetilde{u}_r(w)) dvol_{g_r}(z),$$

for any g_r-orthonormal pair $v, w \in T_z\mathcal{S}_r$, for g_r an auxillary Riemannian metric on $\mathcal{S}_r$, and for $dvol_{g_r}$ the associated area 2-form.

Now:

$$\text{area}(\mathcal{A}_r) = \int_{\mathcal{S}_r} \alpha_r \leq \int_{\mathcal{S}_r} *\alpha_r dA, \tag{13.6}$$

where

$$\alpha_r(v, w) := \max_{S^2} R_{\mathcal{A}_r}(v, w),$$

and where $*\alpha_r$ is the $\mathbb{R} \sqcup \{\infty\}$ valued function on $\mathcal{S}_r$:

$$*\alpha_r(z) := \frac{1}{dA(v, w)} \alpha_r(v, w),$$

for any pair v, w with $dvol_{g_r}(v, w) > 0$, whenever right hand side is defined. Further specified by

$$*\alpha_r(z) := \begin{cases} \infty, \text{ if } dA(v, w) = 0 \text{ while } \alpha_r(v, w) \neq 0 \\ 0, \text{ if } \alpha_r(v, w) = 0. \end{cases}$$

Moreover, it is immediate from definitions that the inequality in (13.6) is an equality unless $|R_{\mathcal{A}}| = \infty$ in which case we would already be done. So if

$$\text{area}(\mathcal{A}_{r_0}) = \int_{\mathcal{S}_{r_0}} *\alpha_{r_0} dA$$

then for some z_0

$$*\alpha_{r_0}(z_0) \geq \hbar / C > 0,$$

which implies what we want since $\hbar = \frac{1}{2} area(S^2) = \frac{1}{2}$. $\square$

14. Proof of Theorem 1.10

Let $Y_\bullet$ be as in the statement of the theorem. Let $\Delta^4_\bullet \to Y_\bullet$ represent the generator of $\pi_4(Y_\bullet)$, let $K_\bullet \subset Y_\bullet$ denote the image of this map, and $\Sigma : \Delta^4 \to K_\bullet$ the sole non-degenerate simplex. Given the resolution as in the statement, we have the pushforward simplicial connection $f_* \mathcal{A}'$ on P, as described in Example 1.4, with respect to $f(K_\bullet)$.

Clearly (from definitions),

$$|f_* \mathcal{A}'|_g = \sup_{x \in \Delta^4, v, w \in T_x \Delta^4} \frac{1}{ar(\Sigma_* v, \Sigma_* w)} |R^{\mathcal{A}'}_\Sigma(v,w)| \in \mathbb{R}_{\geq 0} \sqcup \{\infty\}. \tag{14.1}$$

By the hypothesis $R^{\mathcal{A}'}_\Sigma(v,w) = 0$ whenever $ar(\Sigma_* v, \Sigma_* w) = 0$. It follows by the meaning of the symbol $\frac{1}{ar(\Sigma_* v, \Sigma_* w)}|R^{\mathcal{A}'}_\Sigma(v,w)|$ as in the statement of the theorem, and by the defining properties of $\mathcal{A}'$, that

$$\sup_{x \in \Delta^4, v, w \in T_x \Delta^4} \frac{1}{ar(\Sigma_* v, \Sigma_* w)} |R^{\mathcal{A}'}_\Sigma(v,w)| = \sup_{x \in S^4, v, w \in T_x (S^4 - C)} \frac{1}{ar(v,w)} |R^{\mathcal{A}}(v,w)|$$
$$= L := \sup\{R^{\mathcal{A}}(v,w) : x \in S^4 - C, v, w \in T_x, v, w \text{ are orthonormal}\}.$$

Finally, using Theorem 1.7 we get that

$$L = |f_* \mathcal{A}'|_g \geq 1/2 \cdot \hbar_g,$$

and so we are done. □

Appendix A. Homotopy groups of Kan complexes

For convenience let us quickly review Kan complexes just to set notation. Let

$$\Delta^n_\bullet(k) := hom_\Delta([k],[n]),$$

be the standard representable n-simplex, where Δ is the simplicial category. Let $\Lambda^n_k \subset \Delta^n_\bullet$ denote the sub-simplicial set corresponding to the "boundary" of $\Delta^n_\bullet$ with the k'th face removed, $0 \leq k \leq n$. By *k'th* face we mean the face opposite to the k'th vertex.

A simplicial map

$$h : \Lambda^n_k \subset \Delta^n_\bullet \to X_\bullet$$

will be called a ***horn***. A simplicial set $S_\bullet$ is said to be a ***Kan complex*** if for all $n, k \in \mathbb{N}$ given a diagram with solid arrows

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "h"] & S_\bullet \\ \Delta^n \arrow[ur, dotted, "\widetilde{h}"] & \end{tikzcd},$$

there is a dotted arrow making the diagram commute. The map $\widetilde{h}$ will be called ***the Kan filling*** of the horn h. The k'th face of $\widetilde{h}$ will be called ***Kan filled face along*** h.

Given a pointed Kan complex $(X_\bullet, x)$ and $n \geq 1$ the *n'th simplicial homotopy group* of $(X_\bullet, x)$: $\pi_n(X_\bullet, x)$ is defined to be the set of equivalence classes of morphisms

$$\Sigma : \Delta^n_\bullet \to X_\bullet,$$

such that Σ takes $\partial\Delta^n_\bullet$ to $x_\bullet$, with the latter denoting the image of $\Delta^0_\bullet \to X$, induced by the vertex inclusion $x \to X_\bullet$.

More precisely, we have a commutative diagram:

$$\begin{tikzcd} \Delta^n_\bullet \arrow[r] \arrow[dr, "\Sigma"] & \Delta^0_\bullet \arrow[d, "x"] \\ & X_\bullet, \end{tikzcd}$$

Since for us $X_\bullet$ is often the singular set associated to a topological space X, we note that such morphisms are in complete correspondence with maps:

$$\Sigma : \Delta^n \to X,$$

taking the topological boundary of $\Delta^n_\bullet$ to x.

Two such maps are equivalent if there is a diagram (simplicial homotopy):

$$\begin{tikzcd} \Delta^n_\bullet \arrow[d, "i_0"] \arrow[dr, "\Sigma_1"] & \\ \Delta^n_\bullet \times I_\bullet \arrow[r, "H"] & X_\bullet \\ \Delta^n_\bullet. \arrow[u, "i_1"] \arrow[ur, "\Sigma_2"] & \end{tikzcd}$$

such that $\partial\Delta^n_\bullet \times I_\bullet$ is taken by H to $x_\bullet$. The simplicial homotopy groups of a Kan complex $(X_\bullet, x)$ coincide with the classical homotopy groups of the geometric realization $(|X_\bullet, x|)$. But the power of the above definition is that if we know our Kan complex well, (like in the example of the present paper) the simplicial homotopy groups are very computable since they are completely combinatorial in nature.

Proof of Lemma 3.2. We only outline the argument, as this kind of thing is well known to experts. Since $Y_\bullet$ is a (co)-Cartesian fibration,

$$p : K(Y) \to X_\bullet$$

has a lifting property for edges, essentially by definition of a (co)-Cartesian fibration, and so

$$p : K(Y) \subset Y_\bullet \to X_\bullet$$

is itself a (co)-Cartesian fibration, in particular an inner fibration.

Now given a general horn $h : \Lambda^n_k \to X_\bullet$, $0 \leq k \leq n$ and a filling:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "h"] \arrow[d] & X_\bullet \\ \Delta^n_\bullet \arrow[ur, dotted, "\Sigma"] & \end{tikzcd} ,$$

let

$$\widetilde{h} : \Lambda^n_k \to K(Y)$$

be a chosen lift of h. Since $K(Y)$ is a Kan complex we may find a Kan filling $\widetilde{\Sigma}'$ of $\widetilde{h}$. Now $\widetilde{\Sigma}'$ may not lift Σ, specifically the k'th face (the face missing the k'th vertex) $\widetilde{\Sigma}'_k$ of $\widetilde{\Sigma}'$ may not lift the k'th face Σ_k of Σ. But

$$\Sigma'_k := p(\widetilde{\Sigma}'_k)$$

is clearly "homotopic" relative boundary to Σ_k, meaning that they are both Kan filled faces along h. Then the above more specifically means that there is an *inner horn*

$$h' : \Lambda^n_{k'} \to X_\bullet,$$

with one $(n-1)$-face identified with Σ'_k the rest of which $(n-1)$-faces are degenerate sub-faces of Σ'_k and so that Σ_k is a Kan filled face along h'. (This is what means one is homotopic to the other.)

Let

$$\Sigma_{h'} : \Delta^n_\bullet \to X_\bullet$$

denote the corresponding Kan filling of h'. Since $K(Y)$ is an inner fibration, and since we have a lift $\widetilde{h}'$ of h' determined by the lift $\widetilde{\Sigma}'_k$, we may lift $\Sigma_{h'}$ to $\widetilde{\Sigma}_{h'}$ in $K(Y)$, extending the lift $\widetilde{h}'$. Then take a Kan filling in $K(Y)$ of the horn determined by $\widetilde{\Sigma}'$ and $\widetilde{\Sigma}_{h'}$. The corresponding Kan filled face is the corrected lift $\widetilde{\Sigma}$ of Σ, extending $\widetilde{h}$ that we are looking for. □

Appendix B. On the Maslov number

Let S be obtained from a compact connected Riemann surface S' with boundary, by removing a finite number of points $\{e_i\}$ removed from the boundary of S'.

Let $\mathcal{V} \to S$ be a rank r complex vector bundle, trivialized at the open ends $\{e_i\}$, that is so we have distinguished bundle charts $\mathbb{C}^r \times [0,1] \times [0,\infty) \to \mathcal{V}$ at the ends.

Let

$$\Xi \to \partial S \subset S$$

be a totally real rank r subbundle of $\mathcal{V}$, which is constant in the coordinates

$$\mathbb{C}^r \times [0,1] \times [0,\infty),$$

at the ends. For each end e_i and its distinguished chart $e_i : [0,1] \times [0,\infty) \to S$ let $b^j_i : [0,\infty) \to \partial S$, $j = 0,1$ be the restrictions of e_i to $\{i\} \times [0,\infty)$.

We then have a pair of real vector spaces

$$\Xi^j_i = \lim_{\tau \mapsto \infty} \Xi|_{b^j_i(\tau)}.$$

There is a Maslov number $Maslov(\mathcal{V}, \Xi, \{\Xi^j_i\})$ associated to this data coinciding with the boundary Maslov index in the sense of [11, Appendix C3], in the case $\Xi^0_i = \Xi^1_i$, for the modified pair $(\mathcal{V}^/, \Xi^/)$ obtained from $(\mathcal{V}, \Xi, \{\Xi^j_i\})$ by naturally closing off each e_i end of $\mathcal{V} \to S$.

When Ξ^0_i is transverse to Ξ^1_i for each i, $Maslov(\mathcal{V}, \Xi, \{\Xi^j_i\})$ is obtained as the Maslov index for the modified pair $(\mathcal{V}^/, \Xi^/)$ by again closing off the ends e_i via gluing (at each end e_i) with

$$(\mathbb{C}^r \times \mathcal{D}, \widetilde{\Xi}, \{\widetilde{\Xi}^j_0\}),$$

where $\mathcal{D}$ as before is diffeomorphic to D^2 with a point e_0 on the boundary removed. Here $\widetilde{\Xi}^0_i = \Xi^1_0$ and $\widetilde{\Xi}^1_i = \Xi^0_0$, while $\widetilde{\Xi}$ over the boundary of $\mathcal{D}$ is determined by the "shortest path" from $\widetilde{\Xi}^0_0$ to $\widetilde{\Xi}^1_0$, meaning that since these are a pair of transverse totally real subspaces up to a complex isomorphism of $\mathbb{C}^r$ (whose choice will not matter) we may identify them with the subspaces $\mathbb{R}^r$, and $i\mathbb{R}^r$ after this identification our shortest path is just $e^{i\theta}\mathbb{R}^r$, $\theta \in [0, \pi_2]$.

For a real linear Cauchy-Riemann operator on $\mathcal{V}$, with suitable asymptotics, the Fredholm index is given by:

$$r \cdot \chi(S) + Maslov(\mathcal{V}, \Xi, \{\Xi_i\}).$$

The proof of this is analogous to [11, Appendix C], we can also reduce it to that statement via a gluing argument. (This kind of argument appears for instance in [21])

B.1. **Dimension formula for moduli space of sections.** Suppose we are given a Hamiltonian fiber bundle $M^{2r} \hookrightarrow \widetilde{S} \to S$, with end structure and S as above. Let $\mathcal{L}$ be a Lagrangian sub-bundle of $\widetilde{S}$ over the boundary of S, compatible with the end structure, and such that the Lagrangian submanifolds

$$L_i^j = \lim_{\tau \mapsto \infty} \mathcal{L}|_{b_i^j(\tau)},$$

intersect transversally (identifying the corresponding fibers) or coincide.

Given an $\mathcal{L}$-exact Hamiltonian connection $\mathcal{A}$, on $\widetilde{S}$, (see Definition 6.2) which is additionally assumed to be trivial in the strip coordinate charts at the ends, and a choice of a family $\{j_z\}$ of compatible almost complex structures on the fibers of $\widetilde{S}$, set $\mathcal{M}(A)$ to be the moduli space of (relative) class A finite vertical L^2 energy holomorphic sections of $\widetilde{S} \to S$ with boundary in $\mathcal{L}$. Define

$$Maslov^{vert}(A)$$

to be the Maslov number of the triple $(\mathcal{V}, \Xi, \{\Xi_i\})$ determined by the pullback by $\sigma \in \mathcal{M}(A)$ of the vertical tangent bundle of $\widetilde{S}$, $\mathcal{L}$. Then the expected dimension of $\mathcal{M}(A)$ is:

$$r \cdot \chi(S) + Maslov^{vert}(A). \tag{B.1}$$

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