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# SMOOTH SIMPLICIAL SETS AND UNIVERSAL CHERN-WEIL FOR INFINITE DIMENSIONAL GROUPS

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ABSTRACT. We give a construction of the universal Chern-Weil differential graded algebra homomorphism (well defined up to homotopy of $dg$ maps):

$$cw : \mathbb{R}[\mathfrak{g}]^G \to \Omega^\bullet(BG, \mathbb{R})$$

for infinite dimensional regular Lie groups $G$, where $\Omega^\bullet(BG, \mathbb{R})$ is a certain De Rham algebra quasi-isomorphic to $C^\bullet(BG, \mathbb{R})$. In particular, this applies to the group of compactly supported diffeomorphisms and the group of compactly supported symplectomorphisms/contactomorphisms, whose Chern-Weil theory has already been much studied. As one consequence, the Cheeger-Simons differential characteristic classes can likewise be defined on $BG$ whenever $G$ satisfies $H^{odd}(BG, \mathbb{R}) = 0$ (in particular for $G$ a finite dimensional Lie group). For the construction of $cw$ we introduce a basic geometric-categorical notion of a smooth simplicial set. Loosely, this is to Chen spaces as simplicial sets are to spaces. We then give a new construction of the classifying space of $G$ as a smooth Kan complex, with geometric realization weakly equivalent to the Milnor $BG$. Other basic ingredients are the theory of Grothendieck universes and the work of Müller-Wockel on smooth structures on $G$-bundles.

## CONTENTS

## 1. Introduction

First, we introduce the notion of a smooth simplicial set, which is most directly an analogue in simplicial sets of Chen spaces [3], and less directly of diffeological spaces of Souriau [38]. The Chen/diffeological spaces are perhaps the most basic notions of a "smooth space". The language of smooth simplicial sets turn out to be a powerful tool to resolve the problem of the construction of the universal Chern-Weil dg algebra homomorphism for infinite dimensional Lie groups. This has been a long standing open problem, even for just the cohomological Chern-Weil level.

1.1. **Generalized Lie groups and classifying space.** More specifically, by an infinite dimensional Lie group $G$, we mean a Lie group whose underlying infinite dimensional smooth manifold is modeled on an locally convex topological vector space, and which is regular, where the latter means there is a suitable exponential map, see Section 5. This encompasses: Banach, Fréchet, $LF$ (locally modeled on limit of Fréchet topological vector spaces) Lie groups. Since this definition also encompasses standard finite dimensional Lie groups, it is better to give this a new working name: ***generalized Lie group***. Essentially, we are looking for just the minimal conditions needed for the definition of Chern-Weil classes for smooth $G$-bundles over ordinary smooth manifolds.

A basic non Fréchet example of a generalized Lie group is the group of compactly supported diffeomorphisms of a finite dimensional smooth manifold. The latter group in addition has the homotopy type of a CW complex, see [12].

One problem of topology is the construction of a "smooth structure" on the Milnor classifying space $BG$ of a (generalized) (real or complex) Lie group $G$. There are specific requirements for what such a notion of a smooth structure should entail. At the very least we hope to be able to carry out Chern-Weil theory universally on $BG$. We now sketch what that that entails.

1.2. **Universal** $dg$ **Chern-Weil homomorphism.** In what follows, we keep to real generalized Lie groups, but there is no essential difficulty to extending our theory to the complex case. Denote by $\mathbb{R}[\mathfrak{g}]^G$ the algebra over $\mathbb{R}$ of $Ad_G$ invariant (required to be continuous when $\mathfrak{g}$ is infinite dimensional) polynomials on the Lie algebra $\mathfrak{g}$ of $G$. Denote by $\Omega^\bullet(BG, \mathbb{R})$ the "$dg$ algebra of differential forms on $BG$", (for the moment left unspecified) quasi-isomorphic to the standard cochain algebra $C^\bullet(BG, \mathbb{R})$. Then we want a "purely" differential geometric construction of the Chern-Weil dg algebra homomorphism (well defined up to homotopy of $dg$ maps):

$$cw : \mathbb{R}[\mathfrak{g}]^G \to \Omega^\bullet(BG, \mathbb{R}),$$

where the differential on the left is trivial (and each generator has degree 2). This of course needs to satisfy suitable naturality. The goal is to set up all structures in such a way that the differential geometry in this construction becomes trivial.

For finite dimensional Lie groups, the cohomological universal Chern-Weil homomorphism:

$$hcw : \mathbb{R}[\mathfrak{g}]^G \to H^\bullet(BG, \mathbb{R}) \tag{1.1}$$

has been studied for instance by Bott [1]. It has also been directly constructed in [5] using simplicial techniques, also important for us here. However, to produce a (natural) $dg$ enhancement of this, some geometric theory is ostensibly required. For example, working on the $dg$ level is important in Freed-Hopkins [8]. However they work on smooth classifying stacks, rather then on the Milnor spaces $BG$, (and with classical Lie groups $G$).

1.2.1. *Smooth structures on $BG$.* One candidate for a smooth structure on $BG$ is some kind of diffeology. For example Magnot and Watts [24] construct a natural diffeology on the Milnor classifying space $BG$. Another approach to this is contained in Christensen-Wu [4], where the authors also state their plan to develop some kind of universal Chern-Weil theory in the future.

A further specific possible requirement for the above discussed "smooth structures", is that the smooth singular simplicial set $BG_\bullet$ should have a geometric realization weakly homotopy equivalent to $BG$. See for instance [19] for one approach to this particular problem in the context of diffeologies. This kind of requirement is crucial for instance in the author's [36], which may be understood as a kind of "quantum Chern-Weil theory" on $B\operatorname{Ham}(M,\omega)$, for $\operatorname{Ham}(M,\omega)$ the group of Hamiltonian symplectomorphisms of a symplectic manifold. The analogue of this in the category of smooth simplicial sets always trivially satisfied. The specific content of this is Proposition 3.7.

The structure of a smooth simplicial set is initially more flexible than a space with diffeology, but we may add further conditions, like the Kan condition, which will be important for us. Given a generalized Lie group $G$, we construct, for each choice of a particular kind of Grothendieck universe $\mathcal{U}$, a smooth simplicial set $BG^{\mathcal{U}}$ with a specific classifying property, analogous to the classifying property of $BG$, but relative to $\mathcal{U}$. We note that this is *not* the Milnor construction. But it will be shown that $|BG^{\mathcal{U}}|$ always has the weak homotopy type of Milnor $BG$, and so homotopy type if $G$ has the homotopy type of a CW complex.

The simplicial set $BG^{\mathcal{U}}$ is moreover a Kan complex, and so is a basic example of a smooth Kan complex. Our constructions, will work naturally on $BG^{\mathcal{U}}$ rather than its geometric realization. And all the desires of "smoothness" mentioned above then is some sense hold true for $BG^{\mathcal{U}}$ via its smooth Kan complex structure.

1.2.2. *Cohomological Chern-Weil homomorphism.* We first note that the cohomological Chern-Weil homomorphism 1.1 has a direct extension for generalized Lie groups.

**Theorem 1.1.** *Let $G$ be a generalized Lie group then there is an algebra homomorphism:*
$$hcw : \mathbb{R}[\mathfrak{g}]^G \to H^\bullet(BG, \mathbb{R}).$$
*This is natural, so that if $P \to Y$ is a smooth $G$-bundle, over a smooth manifold $Y$, and*
$$hcw^P : \mathbb{R}[\mathfrak{g}]^G \to H^\bullet(Y, \mathbb{R})$$
*is the associated Chern-Weil map, then*
$$hcw^P = f_P^* \circ hcw$$
*for*
$$f_P : Y \to BG$$
*the classifying map of $P$ and*
$$f_P^* : H^\bullet(BG, \mathbb{R}) \to H^\bullet(Y, \mathbb{R})$$
*the induced algebra map.*

1.3. **Enhancement to a *dg* homomorphism.** The statement of this is somewhat more involved then the cohomological Chern-Weil. This is because the algebra of differential forms that we need, cannot be naturally constructed on the Milnor model $BG$. It will be constructed on the smooth Kan complexes $BG^{\mathcal{U}}$. In particular, we need to work with a fixed Grothendieck universe $\mathcal{U}$.

Denote by $\Omega^\bullet(BG^{\mathcal{U}}, \mathbb{R})$ the standard Whitney-Sullivan De Rham algebra of the simplicial set $BG^{\mathcal{U}}$, Section 4. Note that since $|BG^{\mathcal{U}}|$ is weakly homotopy equivalent to $BG$, the algebra $\Omega^\bullet(BG^{\mathcal{U}}, \mathbb{R})$ is quasi-isomorphic to the standard co-chain algebra $C^\bullet(BG, \mathbb{R})$.

We need an additional ingredient. If $P \to Y$ is a $\mathcal{U}$-small principal $G$-bundle, then there is a certain classifying simplicial map $f_{P^\Delta} : Y_\bullet \to BG^{\mathcal{U}}$, see Theorem 8.6, where $Y_\bullet$ denotes the smooth singular set of $Y$. Also there is an obvious natural *dg* map:

$$\Theta : \Omega^\bullet(Y, \mathbb{R}) \to \Omega^\bullet(Y_\bullet, \mathbb{R}).$$

**Theorem 1.2.** *Let $G$ be a generalized Lie group and $\mathcal{U}$ a $G$-admissible Grothendieck universe. Then there is an dg algebra homomorphism well defined up to homotopy of dg maps:*

$$cw : \mathbb{R}[\mathfrak{g}]^G \to \Omega^\bullet(BG^{\mathcal{U}}, \mathbb{R}).$$

*This is natural, so that if $P \to Y$ is a $\mathcal{U}$-small smooth $G$-bundle, over a smooth manifold $Y$, and*

$$cw^P : \mathbb{R}[\mathfrak{g}]^G \to \Omega^\bullet(Y, \mathbb{R})$$

*is the associated classical Chern-Weil dg map (well defined up to homotopy), then*

$$\Theta \circ cw^P \simeq f^*_{P^\Delta} \circ cw, \quad (\textit{homotopy of dg maps}).$$

1.4. **Universal Cheeger-Simons characteristic classes.** Cheeger-Simons [2] define a lift of the Chern-Weil homomorphism to differential characters, that may be called differential characteristic classes. We note that this construction can be carried out universally for any generalized Lie group $G$ satisfying $H^{odd}(BG, \mathbb{R}) = 0$. To state this let us first setup some notation.

Denote by $I^G \subset \mathbb{R}[\mathfrak{g}]^G$ the subset of polynomials with integral coefficients. Let $\Lambda \subset \mathbb{R}$ be the sub-ring generated by the periods of all the Chern-Weil forms $cw(\rho)$, $\rho \in I^G$. Denote by $\hat{H}^\bullet(BG, R/\Lambda)$ the ring of Cheeger-Simons differential characters. Note that has similar meaning to $\Omega^\bullet(BG, \mathbb{R})$. That is concretely we are working not with Milnor $BG$ but with a smooth Kan complex $\widetilde{BG}^{\mathcal{U}}$ analogous to $BG^{\mathcal{U}}$, and so that $\widetilde{BG}^{\mathcal{U}}$ carries a $\mathcal{U}$-universal $G$-bundle $\widetilde{EG}^{\mathcal{U}}$ with a $\mathcal{U}$-universal connection $A^{\mathcal{U}}$.

Given this understanding, if $P \to Y$ is a smooth $G$-bundle with a connection $A$, over a smooth manifold $Y$, there is a classifying map $f_{P,A} : Y \to BG$, so that up to isomorphism the pull-back of the universal connection is $A$. Since we have not completely specified $\hat{H}^\bullet(BG, R/\Lambda)$ and $f_{P,A}$ the following will only be called a meta-theorem instead of a theorem.

**Meta-Theorem 1.1.** *Let $G$ be a generalized Lie group with $H^{odd}(BG) = 0$. There is a ring homomorphism $cs : I^G \to \hat{H}^\bullet(BG, R/\Lambda)$, which is natural in the sense*

*that if $P \to Y$ is a smooth $G$-bundle with a connection $A$, over a smooth manifold $Y$, and*
$$cs^{P,A} : I^G \to \hat{H}^\bullet(Y, \mathbb{R}/\Lambda)$$
*is the associated Cheeger-Simons homomorphism, then*
$$cs^{P,A} = f^*_{P,A} \circ cs.$$

The proof and concrete statement of this readily follows from the theory of this paper, however we postpone it for a sequel as we would need to introduce a lot of terminology related to Cheeger-Simons theory and some auxiliary results.

1.5. **An example: the group of Hamiltonian symplectomorphisms.** Here is one concrete example, with more details in Sections 10 and 11. Let $\mathcal{H} = \mathrm{Ham}(M, \omega)$ denote the generalized Lie group of compactly supported Hamiltonian symplectomorphisms of some symplectic manifold. Let $\mathfrak{h}$ denote its lie algebra. To remind the reader, when $M$ is compact $\mathfrak{h}$ is naturally isomorphic to the space of mean 0 smooth functions on $M$, and otherwise it is the space of all smooth compactly supported functions. In [33] Reznikov defined $Ad_{\mathcal{H}}$-invariant polynomials $\{r_k\}_{k \geq 1}$ on the Lie algebra $\mathfrak{h}$. (When $M$ is compact, the class $r_1$ vanishes.) Denote by $\mathcal{R}ez$ the sub-algebra of $\mathbb{R}[\mathfrak{h}]^{\mathcal{H}}$ generated by $\{r_k\}$.

By classical Chern-Weil theory we get cohomology classes $c^{r_k}(P) \in H^{2k}(X, \mathbb{R})$ for any smooth $\mathcal{H}$-bundle $P$ over a smooth manifold $X$. Using Theorem 1.2 we get:

**Corollary 1.3.** *There is a dg algebra homomorphism (well defined up to homotopy)*
$$rez : \mathcal{R}ez \to \Omega^\bullet(B\mathcal{H}^{\mathcal{U}}, \mathbb{R}),$$
*satisfying the naturality as in Theorem 1.2. And in particular, there are universal Reznikov cohomology classes $c^{r_k} \in H^{2k}(B\mathcal{H}, \mathbb{R})$, satisfying the following. Let $Z \to Y$ be a smooth principal $\mathcal{H}$-bundle. Let $c^{r_k}(Z) \in H^{2k}(Y)$ denote the Reznikov class. Then*
$$f^*_Z c^{r_k} = c^{r_k}(Z),$$
*where $f_Z : Y \to B\mathcal{H}$ is the classifying map of the underlying topological $\mathcal{H}$-bundle.*

The second part of the corollary is an explicit form of a statement asserted by Reznikov [33, page 12] on the extension of his classes to the universal level on $B\mathcal{H}$. The above corollary is actually stronger, since we don't require compactness of $M$.

Likewise, we obtain a differential geometric proof that the Guillemin-Sternberg-Lerman coupling class $\mathfrak{c}(P) \in H^2(P)$ [9], [25] of a Hamiltonian fibration (Definition 11.1) has a universal representative. Specifically, let $M^{\mathcal{H}}$ denote the $M$-fibration associated to the universal principal $\mathcal{H}$-fibration $\mathcal{E} \to B\mathcal{H}$. (In other words the universal Hamiltonian $M$-bundle.)

**Theorem 1.4.** *There is a cohomology class $\mathfrak{c} \in H^2(M^{\mathcal{H}})$ so that if $P \to X$ is a smooth Hamiltonian $M$-fibration and $\widetilde{f} : P \to M^{\mathcal{H}}$ the corresponding map then $\widetilde{f}^*_P \mathfrak{c} = \mathfrak{c}(P)$.*

For $M$ closed this is proved by Kedra-McDuff [16, Proposition 3.1] using homotopy theory techniques.

Here is a basic application. Let $Symp(\mathbb{CP}^k)$ denote the group of symplectomorphisms of $\mathbb{CP}^k$, that is diffeomorphisms $\phi : \mathbb{CP}^k \to \mathbb{CP}^k$ s.t. $\phi^* \omega_0 = \omega_0$ for $\omega_0$ the Fubini-Study symplectic 2-form on $\mathbb{CP}^k$. Using the above corollary, we may obtain an elementary proof of the following theorem of Kedra-McDuff:

**Theorem 1.5** (Kedra-McDuff)**.** *Let*

$$i : BPU(n) \to BSymp(\mathbb{CP}^{n-1})$$

*be the natural map. Then*

$$i_* : H_\bullet(BPU(n), \mathbb{R}) \to H_\bullet(BSymp(\mathbb{CP}^{n-1}), \mathbb{R})$$

*is an injection for all* $n \geq 2$.

More history and background surrounding these theorems is in Sections 9 and 10.

1.6. **Other examples.** One other basic set of examples of generalized Lie groups with a wealth of invariant polynomials on the lie algebra, are the loop groups $LG, \Omega G$, for $G$ any Lie group, and $LG$ the free loop space, and $\Omega G$ the based loop space at $id$. See for instance [31] for related computations. Loop groups are prominent in conformal field theory, see for instance [32], for the foundation of the subject. The relevant Chern-Weil theory then has physical connotations. Other examples of infinite dimensional Chern-Weil theory include: [22], [27], [34], [21].

We note that the Chern-Weil homomorphism is known to be an isomorphism when the group $\mathcal{G}$ is compact and connected. In general counterexamples are known. Since we now have the universal Chern-Weil homomorphism for generalized Lie groups, it would be very interesting to characterize when it is an isomorphism. As a special case, I conjecture that it is an isomorphism when $\mathcal{G} = \Omega G$ for $G$ a compact semi-simple Lie group. In this case the geometry and topology of $\mathcal{G}$ is determined by various algebraic/complex analytic data intertwined with Morse theory, [32]. In other words, it is a semi-algebraic object. And so it stands to reason that the real cohomology of $B\mathcal{G}$ could be determined by another semi-algebraic object $\mathbb{R}[\mathfrak{g}]^{\mathcal{G}}$ ('semi' as it is composed of continuous functionals).

There are various precedents in giving a differential geometric definition of the (infinite dimensional group) Chern-Weil homomorphism in some cases, for example Magnot [23].

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## 2. Preliminaries and notation

We denote by $\Delta$ the simplex category:

- The set of objects of $\Delta$ is $\mathbb{N}$.

- $\hom_\Delta(n,m)$ is the set of non-decreasing maps $[n] \to [m]$, where $[n] = \{0, 1, \ldots, n\}$, with its natural order.

A simplicial set $X$ is a functor

$$X : \Delta^{op} \to Set.$$

The set $X(n)$ is called the set of $n$-simplices of $X$. Given a collection of sets $\{X(n)\}_{n\in\mathbb{N}}$, by a ***simplicial structure*** we will mean the extension of this data to a functor: $X : \Delta \to Set^{op}$.

$\Delta^d_{simp}$ will denote a particular simplicial set: the standard representable $d$-simplex, with

$$\Delta^d_{simp}(n) = hom_\Delta(n, d).$$

A morphism or a map of simplicial sets, or a ***simplicial map*** $f : X \to Y$ is a natural transformation $f$ of the corresponding functors. The category of simplicial sets will be denoted by $s - Set$.

By a $d$-simplex $\Sigma$ of a simplicial set $X$, we may mean, interchangeably, either the element in $X(d)$ or the map of simplicial sets:

$$\Sigma : \Delta^d_{simp} \to X,$$

uniquely corresponding to $\Sigma$ via the Yoneda lemma. If we write $\Sigma^d$ for a simplex of $X$, it is implied that it is a $d$-simplex.

With the above identification if $f : X \to Y$ is a map of simplicial sets then

$$f(\Sigma) = f \circ \Sigma. \tag{2.1}$$

## 2.1. Topological simplices and smooth singular simplicial sets.

Let $\Delta^d$ be the topological $d$-simplex, i.e.

$$\Delta^d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \,|\, x_1 + \ldots + x_d \leq 1, \text{ and } \forall i : x_i \geq 0\}.$$

The vertices of $\Delta^d$ will be assumed ordered in the standard way.

**Definition 2.1.** *Let $X$ be a smooth manifold with corners, in the diffeological sense [10]. We say that a map $\sigma : \Delta^n \to X$ is smooth if it smooth as a map of manifolds with corners. In particular, $\sigma : \Delta^n \to \Delta^d$ is smooth iff it has an extension to a smooth map $V \supset \Delta^n \to \mathbb{R}^d$, with $V$ open.*

We denote by $\Delta^d_\bullet$ the smooth singular simplicial set of $\Delta^d$, i.e. $\Delta^d_\bullet(k)$ is the set of smooth maps

$$\sigma : \Delta^k \to \Delta^d.$$

We call an affine map $\Delta^k \to \Delta^d$ taking vertices to vertices, in an order preserving way, ***simplicial***. And we denote by

$$\Delta^d_{simp} \subset \Delta^d_\bullet$$

the sub-simplicial set consisting of these topological simplicial maps. That is $\Delta^d_{simp}(k)$ is the set of simplicial maps $\Delta^k \to \Delta^d$.

Note that $\Delta^d_{simp}$ is naturally isomorphic to the standard representable $d$-simplex $\Delta^d_{simp}$ as previously defined, so that this abuse of notation should not cause issues.

Thus we may also understand $\Delta$ as the category with objects topological simplices $\Delta^d$, $d \geq 0$ and morphisms simplicial maps.

**Notation 2.2.** *A morphism $m \in hom_{\Delta}(n,k)$ uniquely corresponds to a simplicial map $\Delta^n_{simp} \to \Delta^k_{simp}$, which uniquely corresponds to a topological simplicial map $\Delta^n \to \Delta^k$ (as defined right above). The correspondence is by taking the maps $\Delta^n_{simp} \to \Delta^k_{simp}$, $\Delta^n \to \Delta^k$, to be determined by the map $m : \{0,\ldots,n\} \to \{0,\ldots,k\}$. We will not notationally distinguish these corresponding morphisms. So that $m$ may simultaneously refer to all of the above morphisms.*

### 2.2. **The simplex category of a simplicial set.**

**Definition 2.3.** *For $X$ a simplicial set, $\Delta(X)$ will denote a certain category called the* **simplex category of** *$X$. This is the category s.t.:*

- *The set of objects* $\operatorname{obj}\Delta(X)$ *is the set of simplices*
$$\Sigma : \Delta^d_{simp} \to X, \quad d \geq 0.$$
- *Morphisms $f : \Sigma_1 \to \Sigma_2$ are commutative diagrams in $s-Set$:*

$$\begin{tikzcd} \Delta^d_{simp} \arrow[r, "\widetilde{f}"] \arrow[dr, "\Sigma_1"'] & \Delta^n_{simp} \arrow[d, "\Sigma_2"] \\ & X \end{tikzcd} \tag{2.2}$$

  *with top arrow a simplicial map, which we denote by $\widetilde{f}$.*

*An object $\Sigma : \Delta^d_{simp} \to X$ is likewise called a $d$-simplex, and such a $\Sigma$ will be said to have degree $d$. We may specify the degree with a superscript, for example $\Sigma^d$ for degree $d$.*

**Definition 2.4.** *We say that $\Sigma^n \in \Delta(X)$ is* **non-degenerate** *if there is no morphism*

$$\begin{tikzcd} \Delta^m_{simp} \arrow[r, "\widetilde{f}"] \arrow[dr, "\Sigma"'] & \Delta^n_{simp} \arrow[d, "\Sigma^n"] \\ & X \end{tikzcd}$$

*s.t. $m > n$ and s.t. the map $\widetilde{f} : \Delta^m \to \Delta^n$ is surjective.*

There is a forgetful functor
$$T : \Delta(X) \to \Delta,$$
$T(\Sigma^d) = \Delta^d_{simp}$, $T(f) = \widetilde{f}$. We denote by $\Delta^{inj}(X) \subset \Delta(X)$ the sub-category with same objects, and morphisms $f$ such that $\widetilde{f}$ are monomorphisms, i.e. are face inclusions.

2.3. **Geometric realization.** Let $Top$ be the category of topological spaces. Let $X$ be a simplicial set, then define as usual the ***geometric realization*** of $X$ by the colimit in $Top$:

$$|X| := \operatorname{colim}_{\Delta(X)} T,$$

for $T : \Delta(X) \to \Delta \subset Top$ as above, understanding $\Delta$ as a subcategory of $Top$ as previously explained.

## 3. Smooth simplicial sets

If

$$\sigma : \Delta^d \to \Delta^n$$

is a smooth map then we have an induced map of simplicial sets

$$\sigma_\bullet : \Delta^d_\bullet \to \Delta^n_\bullet, \tag{3.1}$$

defined by

$$\sigma_\bullet(\rho) = \sigma \circ \rho.$$

We now give a pair of equivalent definitions of smooth simplicial sets. The first is more hands on, and has a close connection to the definition of Chen/diffeological spaces, while the second is more conceptual/categorical.

**Definition 3.1** (First definition)**.** *A* **smooth simplicial set** *consists of the following data:*

(1) *A simplicial set $X$.*

(2) *For each $\Sigma : \Delta^n_{simp} \to X$ an $n$-simplex there is an assigned map of simplicial sets*

$$g(\Sigma) : \Delta^n_\bullet \to X.$$

*This satisfies:*

(a)

$$g(\Sigma)|_{\Delta^n_{simp}} = \Sigma. \tag{3.2}$$

*We abbreviate $g(\Sigma)$ by $\Sigma_*$, when there is no need to disambiguate which structure $g$ is meant.*

(b) *The following property will be called* **push-forward functoriality***:*

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_\bullet \tag{3.3}$$

*where $\sigma : \Delta^k \to \Delta^d$ is a $k$-simplex of $\Delta^d_\bullet$, and where $\Sigma$ as before is a $d$-simplex of $X$.*

Thus, formally a smooth simplicial set is a 2-tuple $(X, g)$, satisfying the axioms above. When there is no need to disambiguate we omit specifying $g$.

**Definition 3.2.** *A* **smooth** *map between smooth simplicial sets*

$$(X_1, g_1), (X_2, g_2)$$

*is a simplicial map*

$$f : X_1 \to X_2,$$

*which satisfies the condition:*

$$\forall n \in \mathbb{N}\, \forall \Sigma \in X_1(n) : g_2(f(\Sigma)) = f \circ g_1(\Sigma), \tag{3.4}$$

*or more succinctly:*

$$\forall n \in \mathbb{N}\, \forall \Sigma \in X_1(n) : (f(\Sigma))_* = f \circ \Sigma_*.$$

A ***diffeomorphism*** between smooth simplicial sets is defined to be a smooth map, with a smooth inverse.

Now let $\Delta^{sm}$ denote the category:

(1) The set of objects of $\Delta^{sm}$ is $\mathbb{N}$.

(2) $\hom_{\Delta^{sm}}(k, n)$ is the set of smooth maps $\Delta^k \to \Delta^n$.

(3) The composition of morphism is the natural composition.

**Definition 3.3** (Second definition)**.** *A* **smooth simplicial set** *$X$ is a functor $X : \Delta^{sm} \to Set^{op}$. A smooth map $f : X \to Y$ of smooth simplicial sets is defined to be a natural transformation from the functor $X$ to $Y$.*

The equivalence of the above definitions is established further ahead, as we need certain preliminaries.

**Remark 3.4.** *There are respective advantages to both definitions. With the second definition we can lean more on category theory. In particular, some of the technical results ahead are incarnations of the Yoneda lemma and other such tools. With the first definition it is simpler to work with the Kan condition, moreover it is simpler to relate it to the existing theory of diffeological spaces, and our primary audience is differential geometers.*

**Example 3.5** (The tautological smooth simplicial set)**.** *$\Delta^n_\bullet$ has a tautological smooth simplicial set structure, where*

$$g(\Sigma) = \Sigma_\bullet,$$

*for $\Sigma : \Delta^k \to \Delta^n$ a smooth map, hence a $k$-simplex of $\Delta^n_\bullet$, and where $\Sigma_\bullet$ is as in (3.1).*

**Lemma 3.6.** *Let $X$ be a smooth simplicial set and $\Sigma : \Delta^n_{simp} \to X$ an $n$-simplex. Let $\Sigma_* : \Delta^n_\bullet \to X$ be the corresponding simplicial map. Then $\Sigma_*$ is smooth with respect to the tautological smooth simplicial set structure on $\Delta^n_\bullet$ as above.*

*Proof.* Let $\sigma$ be a $k$-simplex of $\Delta^n_\bullet$, so $\sigma : \Delta^k \to \Delta^n$ is a smooth map. We need that

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_*.$$

Now $\sigma_* = \sigma_\bullet$, by definition of the tautological smooth structure on $\Delta^n_\bullet$. So we have:

$$\begin{aligned}(\Sigma_*(\sigma))_* &= \Sigma_* \circ \sigma_\bullet \text{ by Axiom 2b} \\ &= \Sigma_* \circ \sigma_*.\end{aligned}$$

□

**Proposition 3.7.** *The set of $n$-simplices of a smooth simplicial set $X$ is naturally isomorphic to the set of smooth maps $\Delta^n_\bullet \to X$. In fact, define $X_\bullet$ to be the simplicial set whose $n$-simplices are smooth maps $\Delta^n_\bullet \to X$, and so that if $i : m \to n$ is a morphism in $\Delta$ then*

$$X_\bullet(i) : X(n) \to X(m)$$

*is the "pull-back" map:*

$$X_\bullet(i)(\Sigma) = \Sigma \circ i_\bullet,$$

*for $i_\bullet : \Delta^m_\bullet \to \Delta^n_\bullet$ the induced map. Then $X_\bullet$ is naturally isomorphic to $X$.*

*Proof.* Let $\rho : \Delta^n_{simp} \to X$ be an $n$-simplex. By the lemma above, we have a uniquely associated to it smooth map $\rho_* : \Delta^n_\bullet \to X$. Conversely, suppose we are given a smooth map $m : \Delta^n_\bullet \to X$. Then we get an $n$-simplex $\rho_m := m|_{\Delta^n_{simp}}$. Let $id^n : \Delta^n \to \Delta^n$ be the identity map. We have that

$$\begin{aligned} m &= m \circ id^n_\bullet = m \circ id^n_* \\ &= (m(id^n))_*, \text{ as } m \text{ is smooth} \\ &= (\rho_m(id^n))_*, \text{ trivially by definition of } \rho_m \\ &= (\rho_m)_* \circ id^n_*, \text{ as } (\rho_m)_* \text{ is smooth by Lemma 3.6} \\ &= (\rho_m)_*. \end{aligned}$$

Thus, the map $I_n(\rho) = \rho_*$, from the set of $n$-simplices of $X$ to the set of smooth maps $\Delta^n_\bullet \to X$, is bijective.

Given an element $m \in hom_\Delta(n, d)$, let $m_{simp} : \Delta^n_{simp} \to \Delta^d_{simp}$ denote the corresponding natural transformation, also identified with an element of $\Delta^d_{simp}(n)$. Then the corresponding map

$$X(m) : X(d) \to X(n)$$

is

$$\rho \mapsto \rho \circ m_{simp},$$

for $\rho : \Delta^n_{simp} \to X$.

With that in mind, the diagram below commutes

$$\begin{tikzcd} X(d) \arrow[r, "X(m)"] \arrow[d, "I_d"] & X(n) \arrow[d, "I_n"] \\ X_\bullet(d) \arrow[r, "X_\bullet(m)"] & X_\bullet(n), \end{tikzcd}$$

as

$$\begin{aligned} X_\bullet(m) \circ I_d(\rho) &= X_\bullet(m)(\rho_*) \\ &= \rho_* \circ m_\bullet \end{aligned}$$

while

$$\begin{aligned} I_n \circ X(m)(\rho) &= (\rho \circ m_{simp})_* \\ &= (\rho_* \circ m_{simp})_*, \text{ by (3.2)} \\ &= (\rho_*(m_{simp}))_*, \text{ by (2.1)} \\ &= \rho_* \circ m_\bullet, \text{ by Axiom 3.3.} \end{aligned}$$

Thus $I$ is a natural transformation and is an isomorphism of simplicial sets $I : X \to X_\bullet$. □

**Lemma 3.8.** *Given a smooth* $m : \Delta^d_\bullet \to \Delta^n_\bullet$ *there is a unique smooth map* $f : \Delta^d \to \Delta^n$ *such that* $m = f_\bullet$.

*Proof.* Define $f$ by $m(id)$ for $id : \Delta^d \to \Delta^d$ the identity. Then

$$\begin{aligned} f_\bullet &= (m(id))_\bullet \\ &= (m(id))_* \\ &= m \circ id_* \quad \text{(as } m \text{ is smooth)} \\ &= m. \end{aligned}$$

So $f$ induces $m$. Now if $g$ induces $m$ then $g_\bullet = m$ hence $g = g_\bullet(id) = m(id)$. □

## 3.1. **Smooth Kan complexes.**

**Definition 3.9.** *A smooth simplicial set whose underlying simplicial set is a Kan complex will be called a* **smooth Kan complex***.*

Let $Y$ be a smooth manifold and let $Sing^{sm}(Y)$ denote the simplicial set of smooth singular simplices in $Y$ [1]. That is $Sing^{sm}(Y)(k)$ is the set of smooth maps $\Sigma : \Delta^k \to Y$, with its natural simplicial structure. $Sing^{sm}(Y)$ will often be abbreviated by $Y_\bullet$.

**Example 3.10.** *Let $Y$ be a smooth d-fold, and set $X = Y_\bullet$. Then $X$ is naturally a smooth simplicial set. This should be a Kan complex but a reference is not known to me.*

**Example 3.11.** *Here is one special example. Let $M$ be a smooth manifold. Then there is a natural smooth simplicial set $LM^\Delta$ whose d-simplices $\Sigma$ are smooth maps $f_\Sigma : \Delta^d \times S^1 \to M$. The maps $\Sigma_*$ are defined by*

$$\Sigma_*(\sigma) = f_\Sigma \circ (\sigma \times id),$$

*for $\sigma \in \Delta^d_\bullet(k)$ and*

$$\sigma \times id : \Delta^k \times S^1 \to \Delta^d \times S^1,$$

*the product map. This $LM^\Delta$ is one simplicial model of the free loop space. Naturally, the free loop space $LM$ also has the structure of a Fréchet manifold, in particular we have the smooth simplicial set $LM_\bullet$, whose n-simplices are Gateaux smooth maps $\Sigma : \Delta^n \to LM$. There is a natural simplicial map $LM^\Delta \to LM_\bullet$, which is readily seen to be smooth. (It is indeed a diffeomorphism.)*

*The above smooth simplicial set structure $LM^\Delta$, in the language of diffeologies, is closely related to the functional diffeology on $C^\infty(Y, Z)$, for which there are diffeomorphisms:*

$$C^\infty(X \times Y, Z) \to C^\infty(X, C^\infty(Y, Z)),$$

*given another diffeological space $X$.*

[1] This is often called the "smooth singular simplicial set of $Y$". However, for us "smooth" is reserved for another purpose, so to avoid confusion we do not use such terminology.

3.2. **Smooth simplex category of a smooth simplicial set.** Given a smooth simplicial set $X$, there is an extension of the previously defined simplex category $\Delta(X)$.

**Definition 3.12.** *For $X$ a smooth simplicial set, $\Delta^{sm}(X)$ will denote the category whose set of objects $\operatorname{obj}\Delta^{sm}(X)$ is the set of smooth maps*

$$\Sigma : \Delta^d_\bullet \to X, \quad d \geq 0$$

*and morphisms $f : \Sigma_1 \to \Sigma_2$, commutative diagrams:*

$$\begin{tikzcd} \Delta^d_\bullet \arrow[r, "\widetilde{f}_\bullet"] \arrow[dr, "\Sigma_1"] & \Delta^n_\bullet \arrow[d, "\Sigma_2"] \\ & X \end{tikzcd} \tag{3.5}$$

*with top arrow any smooth map (for the tautological smooth simplicial set structure on $\Delta^d_\bullet$), which we denote by $\widetilde{f}_\bullet$. By Lemma 3.8, $\widetilde{f}_\bullet$ is induced by a unique smooth map $\widetilde{f} : \Delta^d \to \Delta^n$.*

By Proposition 3.7 we have a natural faithful embedding $\Delta(X) \to \Delta^{sm}(X)$ that is an isomorphism on object sets. We call elements of $\Delta^{sm}(X)$ $d$-simplices.

**Proposition 3.13.** *Definitions 3.1, 3.3 are equivalent.*

*Proof.* Let $\mathcal{C}_1$ denote the category of smooth simplicial sets as given by the Definition 3.1. And let $\mathcal{C}_2$ denote the category of smooth simplicial sets as given by the Definition 3.3.

Given $X \in \mathcal{C}_1$, we define a functor $I(X) : \Delta^{sm} \to Set^{op}$ by setting

$$I(X)(k) = \{\Sigma_\bullet : \Delta^k_\bullet \to X \mid \Sigma_\bullet \text{ is smooth i.e. is a morphism in } \mathcal{C}_1\}.$$

And for $\sigma : \Delta^k \to \Delta^d$ a smooth map setting

$$I(X)(\sigma) : I(X)(d) \to I(X)(k)$$

to be the map

$$I(X)(\sigma)(\Sigma_\bullet) = \Sigma_\bullet \circ \sigma_\bullet. \tag{3.6}$$

This defines

$$I : \mathcal{C}_1 \to \mathcal{C}_2$$

on objects.

Conversely, given $F \in \mathcal{C}_2$, define a simplicial set $I^{-1}(F)$ by the rules:

(1) $I^{-1}(F)(k) := F(k)$.

(2) For $\Sigma \in I^{-1}(F)(k)$, $\Sigma_* : \Delta^k_\bullet \to X$ is the map:

$$\Sigma_*(\sigma) = F(\sigma)(\Sigma).$$

So that we get an element $I^{-1}(F) \in \mathcal{C}_1$. This defines

$$I^{-1} : \mathcal{C}_2 \to \mathcal{C}_1$$

on objects. By Proposition 3.7 $(I^{-1} \circ I(X)) \simeq X$, an isomorphism in $\mathcal{C}_1$.

Suppose now we are given a morphism in $\mathcal{C}_1$: $f : X_0 \to X_1$ i.e. a simplicial map satisfying the condition:

$$\forall n \in \mathbb{N}\, \forall \Sigma \in X(n) : (f(\Sigma))_* = f \circ \Sigma_*. \tag{3.7}$$

Define a natural transformation:

$$I(f) : I(X_0) \to I(X_1),$$

by setting $I(f)_k : I(X_0)(k) \to I(X_1)(k)$ to be the map $I(f)_k(\Sigma_\bullet) = f \circ \Sigma_\bullet$.

This is a natural transformation by the associativity of the composition $f \circ (\Sigma_\bullet \circ \sigma_\bullet) = (f \circ \Sigma_\bullet) \circ \sigma_\bullet$.

It is clear that $I : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor. We show that it is faithful on hom sets. If $f_0, f' : X_0 \to X_1$ are a pair of morphisms in $\mathcal{C}_1$ suppose that $I(f) = I(f')$. Then

$$\forall n \in \mathbb{N}\, \forall \Sigma_\bullet \in I(X)(n) : f \circ \Sigma_\bullet = f' \circ \Sigma_\bullet.$$

In particular,

$$\forall n \in \mathbb{N}\, \forall \Sigma \in X(n) : f \circ \Sigma_* = f' \circ \Sigma_*,$$

as $\Sigma_* \in I(X)(n)$. And so

$$\forall n \in \mathbb{N}\, \forall \Sigma \in (X)(n) : f(\Sigma) = f'(\Sigma).$$

And thus $f = f'$.

We show that $I$ surjective on hom sets. Suppose that $N : I(X_0) \to I(X_1)$ is a morphism in $\mathcal{C}_2$, i.e. a natural transformation of the corresponding functors. So for $\sigma : \Delta^d \to \Delta^k$ smooth, we have a commutative diagram:

$$\begin{tikzcd} I(X_0)(k) \arrow[r, "I(X_0)(\sigma)"] \arrow[d, "N_k"] & I(X_0)(d) \arrow[d, "N_d"] \\ I(X_1)(k) \arrow[r, "I(X_1)(\sigma)"] & I(X_2)(d) \end{tikzcd} \tag{3.8}$$

Define a simplicial map

$$f_N : X_0 \to X_1,$$

by

$$f_N(\Sigma) = N_k(\Sigma_*)|_{\Delta^k_{simp}},$$

for $\Sigma \in X_0(k)$.

We check that $I(f_N) = N$. Let $\Sigma_\bullet : \Delta^d_\bullet \to X_0$ be smooth. For $\sigma : \Delta^k \to \Delta^d$ smooth, we have:

$$\begin{aligned} I(f_N)_d(\Sigma_\bullet)(\sigma) &= (f_N \circ \Sigma_\bullet)(\sigma), \text{ by definition of } I \\ &= f_N(\Sigma_\bullet(\sigma)) \\ &= N_k(\Sigma_\bullet(\sigma)_*)|_{\Delta^k_{simp}}, \text{ by definition of } f_N \\ &= N_k(\Sigma_\bullet \circ \sigma_\bullet)|_{\Delta^k_{simp}}, \text{ as } \Sigma_\bullet \text{ is smooth} \\ &= N_d(\Sigma_\bullet) \circ \sigma_\bullet|_{\Delta^k_{simp}}, \text{ by } N \text{ being a natural transformation, (3.8) and (3.6)} \\ &= N_d(\Sigma_\bullet) \circ \sigma, \text{ notation 2.2} \\ &= N_d(\Sigma_\bullet)(\sigma), \text{ identification (2.1).} \end{aligned}$$

Since $\Sigma_\bullet, \sigma$ were general it follows that $I(f_N) = N$.

We have proved that $I$ is a functor that is essentially surjective on objects, and is fully-faithful on hom sets, it follows by a classical theorem of category theory that $I$ is an equivalence of categories. □

3.3. **Products.** Given a pair of smooth simplicial sets $(X_1, g_1), (X_2, g_2)$, the product $X_1 \times X_2$ of the underlying simplicial sets, has the structure of a smooth simplicial set

$$(X_1 \times X_2, g_1 \times g_2),$$

constructed as follows. Denote by $\pi_i : X_1 \times X_2 \to X_i$ the simplicial projection maps. Then for each $\Sigma \in (X_1 \times X_2)(d)$,

$$(g_1 \times g_2)(\Sigma) : \Delta^d_\bullet \to X_1 \times X_2$$

is defined by:

$$(g_1 \times g_2)(\Sigma)(\sigma) := (g_1(\pi_1(\Sigma))(\sigma), g_2(\pi_2(\Sigma))(\sigma)).$$

3.4. **More on smooth maps.** As defined, a smooth map $f : X \to Y$ of smooth simplicial sets, induces a functor

$$\Delta^{sm} f : \Delta^{sm}(X) \to \Delta^{sm}(Y).$$

This is defined by $\Delta^{sm} f(\Sigma) = f \circ \Sigma$, where $\Sigma : \Delta^d_\bullet \to X$ is in $\Delta^{sm}(X)$. If $m : \Sigma_1 \to \Sigma_2$ is a morphism in $\Delta^{sm}(X)$:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "\Sigma_1"'] & \Delta^d_\bullet \arrow[d, "\Sigma_2"] \\ & X, \end{tikzcd}$$

then obviously the diagram below also commutes:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "h_1"'] & \Delta^d_\bullet \arrow[d, "h_2"] \\ & Y, \end{tikzcd}$$

where $h_i = \Delta^{sm} f(\Sigma_i) = f \circ \Sigma_i$, $i = 1, 2$. And so the latter diagram determines a morphism $\Delta^{sm} f(m) : h_1 \to h_2$ in $\Delta^{sm}(Y)$. Clearly, this determines a functor $\Delta^{sm} f$ as needed.

### 3.5. Smooth homotopy.

**Definition 3.14.** *Let $X, Y$ be smooth simplicial sets. Set $I := \Delta^1_\bullet$ and let $0_\bullet, 1_\bullet \subset I$ be the images of the pair of inclusions $\Delta^0_\bullet \to I$ corresponding to the pair of endpoints. A pair of smooth maps $f, g : X \to Y$ are called* **smoothly homotopic** *if there exists a smooth map*

$$H : X \times I \to Y$$

*such that $H|_{X \times 0_\bullet} = f$ and $H|_{X \times 1_\bullet} = g$.*

The following notion will be useful later on.

**Definition 3.15.** *Let $X$ be a smooth simplicial set. Let $S^k_\bullet = Sing^{sm}(S^k)$. Then $\pi^{sm}_k(X)$ is defined to be the set of (free) smooth homotopy equivalence classes of smooth maps $f : S^k_\bullet \to X$.*

3.6. **Geometric realization.** Geometric realization of a smooth simplicial set $X$ is defined to be the geometric realization of the underlying simplicial set.

## 4. Differential forms on smooth simplicial sets

The theory of differential forms on smooth simplicial sets that we now present, is part of the standard abstract theory of differential forms on simplicial sets. Some of the results of this are folklore, for example the De Rham theorem can be credited to Sullivan [40], but many much more detailed, subsequent expositions have been made, for example DuPont [5]. As such, the theory of differential forms here is a priori *inequivalent* to the theory of differential forms on diffeological spaces in the sense of Souriau [39]. If one wanted to translate our discussion of differential forms into the language of diffeological spaces, then probably it would be similar to the work Katsuhiko [17], see also [18], [14].

First we define smooth differential forms on the topological simplices $\Delta^d$.

**Definition 4.1.** *Set $T\Delta^d := i^* T\mathbb{R}^d$ for $i : \Delta^d \to \mathbb{R}^d$ the natural inclusion. Let $T^*\Delta^d$ denote the dual vector bundle. A* **smooth differential $k$-form** *$\omega$ on $\Delta^d$ is a continuous section of $\Lambda^k(T^*\Delta^d)$, having a smooth extension to a section of $\Lambda^k(T^*N)$ for $N \supset \Delta^d$ an open subset of $\mathbb{R}^d$.*

The above is equivalent to various other possible definitions. For example we may take $\Delta^d$ to be a special case of a smooth manifold with corners, and use a more general theory of differential forms. This can be done, for example, using theory of diffeological spaces [10]. See also Karshon-Watts [15], which establishes one kind of "uniqueness of notions of smooth structures" for the case of simplices, so that our chosen model is canonical up to suitably equivalence.

**Definition 4.2.** *Let $X$ be a simplicial set. A* ***simplicial differential $k$-form*** *$\omega$, or just differential $k$-form where there is no possibility of confusion, is an assignment for each $d$-simplex $\Sigma$ of $X$ a smooth differential $k$-form $\omega(\Sigma) = \omega_\Sigma$ on $\Delta^d$, such that*

$$i^* \omega_{\Sigma_2} = \omega_{\Sigma_1}, \tag{4.1}$$

*for every morphism $i : \Sigma_1 \to \Sigma_2$ in $\Delta(X)$, (see Section 2.2). If in addition $X$ is a smooth simplicial set, and if in addition:*

$$i^* \omega_{\Sigma_2} = \omega_{\Sigma_1}, \tag{4.2}$$

*for every morphism $i : \Sigma_1 \to \Sigma_2$ in $\Delta^{sm}(X)$ then we say that $\omega$ is* **coherent**.

The coherence condition is only meaningful for a smooth simplicial set. As we shall see, this condition is usually unnecessary, but it is acquired naturally in some contexts.

A simplicial differential form $\omega$ may be denoted simply as $\omega = \{\omega_\Sigma\}$. It may also be convenient to use the anonymous function notation $\Sigma \mapsto \omega_\Sigma$.

**Example 4.3.** *If $Y$ is a smooth $d$-fold, and if $\omega$ is a classical differential $k$-form on $Y$, then $\Sigma \mapsto \Sigma^* \omega$ is a coherent simplicial differential $k$-form on $Y_\bullet$ called the* **induced simplicial differential form**. *And this determines a dg map:*

$$\Theta : \Omega^\bullet(Y, \mathbb{R}) \to \Omega^\bullet(Y_\bullet, \mathbb{R}). \tag{4.3}$$

**Example 4.4.** *Let $LM^\Delta$ be the smooth Kan complex of Example 3.11. Then Chen's iterated integrals [3] naturally give coherent differential forms on $LM^\Delta$.*

Let $X$ be a simplicial set. We denote by $\Omega^k(X)$ the $\mathbb{R}$-vector space of differential $k$-forms on $X$. Define

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

by the formula:

$$d\omega(\Sigma) = d(\omega(\Sigma)).$$

In other words $d\omega$ is:

$$\Sigma \mapsto d\omega_\Sigma.$$

Clearly we have

$$d^2 = 0.$$

A $k$-form $\omega$ is said to be ***closed*** if $d\omega = 0$, and ***exact*** if for some $(k-1)$-form $\eta$, $\omega = d\eta$.

**Definition 4.5.** *The* **wedge product** *on*

$$\Omega^\bullet(X) = \bigoplus_{k \geq 0} \Omega^k(X)$$

*is defined by*

$$\omega \wedge \eta = \{\omega_\Sigma \wedge \eta_\Sigma\}.$$

*Then $\Omega^\bullet(X)$ has the structure of a differential graded $\mathbb{R}$-algebra with respect to $\wedge$.*

We then, as usual, define the ***De Rham cohomology*** of $X$:

$$H^k_{DR}(X) = \frac{\text{closed k-forms}}{\text{exact k-forms}},$$

then

$$H^\bullet_{DR}(X) = \bigoplus_{k \geq 0} H^k_{DR}(X)$$

is a graded commutative $\mathbb{R}$-algebra.

Versions of the simplicial De Rham complex have been used by Whitney and perhaps most famously by Sullivan [40]. In particular, the proof of the De Rham theorem (next section) is due to Sullivan.

4.1. **Homology and cohomology of a simplicial set.** We go over this mostly to establish notation. For a simplicial set $X$, we define an abelian group

$$C_k(X, \mathbb{Z}),$$

as the free abelian group generated by the set of $k$-simplices $X(k)$. Elements of $C_k(X, \mathbb{Z})$ are called $k$-***chains***. The boundary operator:

$$\partial : C_k(X, \mathbb{Z}) \to C_{k-1}(X, \mathbb{Z}),$$

is defined on a $k$-simplex $\sigma$ by

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i d_i \sigma,$$

where $d_i : X(k) \to X(k-1)$ are the face maps, this is then extended by linearity to general chains. Then clearly $\partial^2 = 0$.

The homology of this complex is denoted by $H_k(X, \mathbb{Z})$, called integral homology. The integral cohomology is defined analogously to the classical topology setting, using dual chain groups $C^k(X, \mathbb{Z}) = hom(C_k(X, \mathbb{Z}), \mathbb{Z})$. The corresponding coboundary operator is denoted by $d$ as usual:

$$d : C^k(X, \mathbb{Z}) \to C^{k+1}(X, \mathbb{Z}).$$

Homology and cohomology with other ring coefficients (or modules) are likewise defined analogously. Given a simplicial map $f : X \to Y$ there are natural induced chain maps $f^* : C^k(Y, \mathbb{Z}) \to C^k(X, \mathbb{Z})$, and $f_* : C_k(X, \mathbb{Z}) \to C_k(X, \mathbb{Z})$.

We say that a pair of simplicial maps $f, g : X \to Y$ are ***homotopic*** if there a simplicial map $H : X \times \Delta^1_{simp} \to Y$ so that $f = H \circ i_0$, $g = H \circ i_1$ for $i_0, i_1 : X \to X \times \Delta^1_{simp}$ corresponding to the pair of end point inclusions $\Delta^0_{simp} \to \Delta^1_{simp}$. A ***simplicial homotopy equivalence*** is then defined analogously to the topological setting.

As is well known if $f, g$ are homotopic then $f^*, g^*$ and $f_*, g_*$ are chain homotopic.

4.2. **Integration.** Let $X$ be a simplicial set. Given a chain

$$\sigma = \sum_i a_i \Sigma_i \in C_k(X, \mathbb{Z})$$

and a smooth differential form $\omega$, we define:

$$\int_\sigma \omega = \sum_i a_i \int_{\Delta^k} \omega_{\Sigma_i}$$

where the integrals on the right are the classical integrals of a differential form. Thus, we obtain a homomorphism:

$$\int : \Omega^k(X) \to C^k(X, \mathbb{R}),$$

$\int(\omega)$ is the $k$-cochain defined by:

$$\int(\omega)(\sigma) := \int_\sigma \omega,$$

where $\sigma$ is a $k$-chain. We will abbreviate $\int(\omega) = \int \omega$. The following is well known.

**Lemma 4.6.** *For a simplicial set $X$, the homomorphism $\int$ commutes with $d$, and so induces a homomorphism:*

$$DR : H^k_{DR}(X) \to H^k(X, \mathbb{R}).$$

*Proof.* We need that

$$\int d\omega = d \int \omega.$$

Let $\Sigma : \Delta^k_{simp} \to X$ be a $k$-simplex. Then

$$\begin{aligned}
(\int d\omega)(\Sigma) &= \int_{\Delta^k} d\omega_\Sigma, \quad \text{by definition} \\
&= \int_{\partial \Delta^k} \omega_\Sigma, \quad \text{by Stokes theorem} \\
&= d(\int \omega)(\Sigma), \quad \text{by the definition of } d \text{ on co-chains.}
\end{aligned}$$

□

In fact the De Rham theorem tells us that $DR$ is an isomorphism, but we will not need this.

4.3. **Pull-back.** Given a (smooth) map $f : X_1 \to X_2$ of (smooth) simplicial sets, we define

$$f^* : \Omega^k(X_2) \to \Omega^k(X_1)$$

naturally by

$$f^*(\omega)(\Sigma) := \omega(f(\Sigma)). \tag{4.4}$$

Let's check that $f^*$ commutes with $d$. We have:

$$\begin{aligned}
\forall \Sigma : f^*(d\omega)(\Sigma) &= d\omega(f(\Sigma)) \\
&= d(f^*\omega(\Sigma)) \\
&= d(f^*\omega)(\Sigma).
\end{aligned}$$

So we have an induced differential graded $\mathbb{R}$-algebra homomorphism:

$$f^* : \Omega^\bullet(X_2) \to \Omega^\bullet(X_1).$$

And in particular an induced $\mathbb{R}$-algebra homomorphism:

$$f^* : H^\bullet_{DR}(X_2) \to H^\bullet_{DR}(X_1).$$

4.4. **Relation with ordinary homology and cohomology.** Let $s - Set$ denote the category of simplicial sets and $Top$ the category of topological spaces. Let

$$|\cdot| : s - Set \to Top$$

be the geometric realization functor as defined in Section 2.3. Let $X$ be a (smooth) simplicial set. Then for any ring $K$ and any $d \in \mathbb{N}$ we have natural chain maps

$$\begin{aligned} CR &: C_d(X, K) \to C_d(|X|, K), \\ CR^\vee &: C^d(|X|, K) \to C^d(X, K). \end{aligned} \tag{4.5}$$

The chain map $CR$ is defined as follows. A $d$-simplex $\Sigma : \Delta^d_{simp} \to X$, by construction of $|X|$ naturally induces a continuous map $\Sigma_{top} : \Delta^d \to |X|$. Denote by also by $\Sigma_{top}$ the corresponding generator of $C_d(|X|, K)$, then we set $CR(\Sigma) = \Sigma_{top}$ in this notation. Then $CR^\vee$ is the dual chain map.

It is well known that $CR$ and $CR^\vee$ are quasi-isomorphisms, i.e. induce isomorphisms

$$\begin{aligned} R &: H_d(X, K) \to H_d(|X|, K), \\ R^\vee &: H^d(|X|, K) \to H^d(X, K). \end{aligned} \tag{4.6}$$

A proof in an analogous setting of $\Delta$-complexes is in Hatcher [11, Section 2.1], see also [13]. Now let $Y$ be a smooth manifold and $X = Y_\bullet = Sing^{sm}(Y)$. The natural map

$$h : |Y_\bullet| \to Y \tag{4.7}$$

is a weak homotopy equivalence. To see this let $f : S^n \to Y$ represent a class in $\pi_n(Y, y_0)$, then there is a smooth $f' : S^n \to Y$ representing the same class. It readily follows that the class $[f']$ is in the image of $h_* : \pi_k(Y_\bullet, h^{-1}(y_0)) \to \pi_k(Y, y_0)$. Injectivity of $h_*$ is verified similarly.

So $h$ is a homotopy equivalence by the Whitehead's theorem. Let us denote by

$$N : Y \to |Y_\bullet|, \tag{4.8}$$

a homotopy inverse.

Define

$$I : H_d(Y_\bullet, K) \to H_d(Y, K)$$

to be the map induced by the chain map

$$CI : C_d(Y_\bullet, K) \to C_d(Y, K)$$

sending the generator of $C_d(Y_\bullet, K)$, corresponding to a simplex $\Sigma \in Y_\bullet(d)$, to the generator of $C_d(Y)$, corresponding to the smooth map $\Sigma_{top} : \Delta^d \to Y$ (as $\Sigma \in Y_\bullet(d)$ by definition uniquely corresponds to such a smooth map).

Then factor $R$ as:

$$\begin{tikzcd} H_d(Y_\bullet, K) \arrow[r, "I"] \arrow[dr, "R"'] & H_d(Y, K) \arrow[d, "N_*"] \\ & H_d(|Y_\bullet|, K). \end{tikzcd}$$

We may factor $R^\vee$ as:

$$\begin{tikzcd} H^d(|Y_\bullet|, K) \arrow[r, "N^*"] \arrow[dr, "R^\vee"] & H^d(Y, K) \arrow[d, "I^\vee"] \\ & H^d(Y_\bullet, K), \end{tikzcd} \tag{4.9}$$

where $I^\vee$ is induced by the dual $CI^\vee$ of $CI$.

**Notation 4.7.** *Let $\alpha \in H^d(X, K)$.*

(1) *We set*
$$|\alpha| := (R^\vee)^{-1}(\alpha) \in H^d(|X|, K).$$

(2) *If $Y$ is a smooth manifold, and $X = Y_\bullet$. We set*
$$|\alpha|_{sm} := (I^\vee)^{-1}(\alpha) \in H^d(Y, K).$$

Given a map of simplicial sets $f : X_1 \to X_2$ we let $|f| : |X_1| \to |X_2|$ denote the induced map of geometric realizations.

**Lemma 4.8.** *Let $f : X_1 \to X_2$ be a simplicial map of simplicial sets. Let $f^* : H^d(X_2, K) \to H^d(X_1, K)$ be the induced homomorphism then:*
$$|f^*(\alpha)| = |f|^*(|\alpha|).$$

*Proof.* We have a clearly commutative diagram of chain maps (omitting the coefficient ring):

$$\begin{tikzcd} C_d(X_1) \arrow[r, "CR"] \arrow[d, "f_*"] & C_d(|X_1|) \arrow[d, "|f|_*"] \\ C_d(X_2) \arrow[r, "CR"] & C_d(|X_2|), \end{tikzcd}$$

from which the result immediately follows. ∎

The following is immediate from definitions.

**Lemma 4.9.** *If $K = \mathbb{R}$ then*
$$I^\vee \circ DR^{ord} = DR \circ H\Theta,$$

*where:*

- *$DR^{ord} : H^d_{DR}(Y, \mathbb{R}) \to H^d(Y, \mathbb{R})$ is the ordinary De Rham integration isomorphism.*
- *$DR$ is as in Lemma 4.6.*
- *$H\Theta : H^d_{DR}(Y, \mathbb{R}) \to H^d_{DR}(Y_\bullet, \mathbb{R})$ is the cohomology map induced by the map $\Theta$ as in (4.3).*

## 5. Smooth simplicial $G$-bundles

A ***locally convex*** Lie group $G$ is a Lie group whose underlying manifold is modeled on a locally convex topological vector space, for basics on this subject we refer the reader to Neeb [30]. If in addition $G$ is regular, we will call it a ***generalized Lie group***. For the definition of regularity, which is a notion due to Milnor see [30, Definition II.5.2]. This is a condition, which in particular guarantees existence of a suitable exponential map. We need this in order to define the Chern-Weil differential forms.

An interesting example is the group $\mathcal{H} = \mathrm{Ham}(M, \omega)$ of compactly supported Hamiltonian symplectomorphisms of a symplectic manifold. This is not a Fréchet Lie group. But it is modeled on a nuclear LF space and is regular Michor [26]. In particular, it is a generalized Lie group in our sense. As mentioned in the introduction, the Lie algebra of $\mathcal{H}$ admits natural bi-invariant polynomials, which leads to a non-trivial Chern-Weil theory.

In what follows $G$ may be assumed to be either a locally convex Lie group or a diffeological Lie group. In Section 7 we specialize to $G$ being a generalized Lie group. We now introduce the basic building blocks for simplicial $G$-bundles.

**Definition 5.1.** *Let $P$ be a topological principal $G$-bundle over $\Delta^n$, with the embedding $\Delta^n \subset \mathbb{R}^n$ as previously. Suppose we have a choice of a maximal atlas of topological $G$-bundle trivializations $\phi_i : U_i \times G \to P$, $U_i \subset \Delta^n$ open, s.t. the transitions maps*

$$(U_i \cap U_j) \times G \xrightarrow{\phi_{ij} = \phi_j^{-1} \circ \phi_i,} (U_i \cap U_j) \times G$$

*extend to smooth maps $N \times G \to N \times G$, for $N \supset U_i \cap U_j$ some open set in $\mathbb{R}^n$. Then with such a choice of an atlas we call $P$* **a smooth $G$-bundle over $\Delta^n$**. *Smooth bundle maps, and isomorphisms are then defined as with standard smooth bundles.*

The above definition makes sense even just for a diffeological Lie group, but for diffeological Lie groups $G$ we may also directly define a $G$-bundle over $\Delta^n$ as a smooth locally trivial $G$-bundle over $\Delta^n$ in the diffeological sense. These definitions then coincide cf. [10].

At this point our terminology may partially clash with common terminology, in particular a simplicial $G$-bundle will *not* be a pre-sheaf on $\Delta$ with values in the category of smooth $G$-bundles. Instead, it will be a functor (not a co-functor!) on $\Delta^{sm}(X)$ with additional properties. The latter pre-sheafs will not appear in the paper so that this should not cause confusion.

In the definition of simplicial differential forms we omitted coherence. In the case of simplicial $G$-bundles, the analogous condition (full functoriality on $\Delta^{sm}(X)$) turns out to be necessary if we want universal simplicial $G$-bundles with expected behavior.

**Notation 5.2.** *Let $\mathcal{G}$ denote the category of smooth principal $G$-bundles over the simplices $\Delta^n$, ($n$ not fixed) with morphisms smooth $G$-bundle maps.*

Let $\mathcal{F}_1 : \Delta^{sm}(X) \to \Delta^{sm}$ be the natural forgetful functor. And $\mathcal{F}_2 : \mathcal{G} \to \Delta^{sm}$ the functor taking a $G$-bundle $P \to \Delta^k$ to $k$ and defined on morphisms as follows. If $\widetilde{\phi} : P_1 \to P_2$ is a morphism in $\mathcal{G}$ over a smooth map $\phi : \Delta^k \to \Delta^n$ then $\mathcal{F}_2(\widetilde{\phi})(k, n) = \phi$.

**Definition 5.3.** *Let $G$ be a diffeological Lie group and $X$ a smooth simplicial set. A* **smooth simplicial $G$-bundle** *$P$ over $X$ is a functor $P : \Delta^{sm}(X) \to \mathcal{G}$, so that the diagram:*

$$\begin{tikzcd} \Delta^{sm}(X) \arrow[r, "P"] \arrow[dr, "\mathcal{F}_1"] & \mathcal{G} \arrow[d, "\mathcal{F}_2"] \\ & \Delta^{sm}, \end{tikzcd}$$

*commutes. We will call this condition* **compatibility***.*

We will only deal with smooth simplicial $G$-bundles, and so will usually say ***simplicial $G$-bundle***, omitting the qualifier 'smooth'.

**Notation 5.4.** *We often use notation $P_\Sigma$ for $P(\Sigma)$. If we write a simplicial $G$-bundle $P \to X$, this means that $P$ is a simplicial $G$-bundle over $X$ in the sense above. So that $P \to X$ is just notation not a morphism.*

**Example 5.5.** *If $X$ is a smooth simplicial set and $G$ is as above, we denote by $X \times G$ the simplicial $G$-bundle,*

$$\forall n \in \mathbb{N}, \forall \Sigma^n \in \Delta(X) : (X \times G)_{\Sigma^n} \text{ is the trivial bundle } \Delta^n \times G \to \Delta^n.$$

*And where for $\sigma : \Delta^n \to \Delta^k$, $(X \times G)(\sigma) : \Delta^n \times G \to \Delta^k \times G$ is the map $\sigma \times id$.*

*This is called the* ***trivial simplicial $G$-bundle over $X$****.*

**Example 5.6.** *Let $Z \to Y$ be a smooth $G$-bundle over a smooth manifold $Y$. Then we have a simplicial $G$-bundle $Z^\Delta$ over $Y_\bullet$ defined by the conditions:*

(1) *$Z^\Delta(\Sigma) = \Sigma^* Z$.*
(2) *For $f : \Sigma_1 \to \Sigma_2$ a morphism, the bundle map*
$$Z^\Delta(f) : (Z^\Delta(\Sigma_1) = \Sigma_1^* Z) \to (Z^\Delta(\Sigma_2) = \Sigma_2^* Z)$$
*is the universal map $u : \Sigma_1^* Z \to \Sigma_2^* Z$ corresponding to the universal pull-back property of $\Sigma_2^* Z$.*

*The uniqueness of the universal maps readily implies that $Z^\Delta$ is a functor. We say that $Z^\Delta$ is* **the simplicial $G$-bundle induced by** *$Z$.*

**Definition 5.7.** *Let $P_1 \to X_1, P_2 \to X_2$ be a pair of simplicial $G$-bundles. Let $h : X_1 \to X_2$ be a smooth map. A* **smooth simplicial $G$-bundle map over** *$h$ from $P_1$ to $P_2$ is a natural transformation of functors:*

$$\widetilde{h} : P_1 \to P_2 \circ \Delta^{sm} h.$$

*This is required to have the following additional property. For each $d$-simplex $\Sigma \in \Delta^{sm}(X_1)$ the natural transformation $\widetilde{h}$ specifies a morphism in $\mathcal{G}$:*

$$\widetilde{h}_{\Sigma} : P_1(\Sigma) \to P_2(h \circ \Sigma),$$

*and we ask that this is a bundle map over the identity, so that the following diagram commutes:*

$$\begin{tikzcd} P_1(\Sigma) \arrow[r, "\widetilde{h}_{\Sigma}"] \arrow[d, "p_1"] & P_2(h \circ \Sigma) \arrow[d, "p_2"] \\ \Delta^d \arrow[r, "id"] & \Delta^d . \end{tikzcd}$$

We will usually say simplicial $G$-bundle map instead of smooth simplicial $G$-bundle map, (as everything is always smooth) when $h$ is not specified it is assumed to be the identity.

**Definition 5.8.** *Let $P_1, P_2$ be simplicial $G$-bundles over $X_1, X_2$ respectively. A* **simplicial $G$-bundle isomorphism** *is a simplicial $G$-bundle map*

$$\widetilde{h} : P_1 \to P_2$$

*s.t. there is a simplicial $G$-bundle map*

$$\widetilde{h}^{-1} : P_2 \to P_1$$

*with*

$$\widetilde{h}^{-1} \circ \widetilde{h} = id.$$

Usually $X_1 = X_2$ and in this case, unless specified otherwise, it is assumed $h = id$. A simplicial $G$-bundle isomorphic to the trivial simplicial $G$-bundle is called ***trivializeable***.

**Definition 5.9.** *If $X = Y_\bullet$ for $Y$ a smooth manifold, we say that a simplicial $G$-bundle $P$ over $X$ is* **inducible by a smooth $G$-bundle** *$N \to Y$ if there is a simplicial $G$-bundle isomorphism $N^{\Delta} \to P$.*

The following will be one of the crucial ingredients later on.

**Theorem 5.10.** *Let $G$ be a diffeological Lie group and let $P \to Y_\bullet$ be a simplicial $G$-bundle, for $Y$ a smooth $d$-manifold (or a manifold with corners, understood as previously). Then $P$ is inducible by some smooth $G$-bundle $N \to Y$.*

*Proof.* We need to introduce an auxiliary notion. Let $Z$ be a smooth $d$-manifold with corners, as before understood as a diffeological space. And let $\mathcal{D}(Z)$ denote the category whose objects are smooth (diffeological) embeddings $\Sigma : \Delta^d \to Z$, (for the same fixed $d$). A morphism $f \in hom_{\mathcal{D}(Z)}(\Sigma_1, \Sigma_2)$ is a commutative diagrams:

(5.1)
$$\begin{tikzcd} \Delta^d \arrow[r, "\widetilde{f}"] \arrow[dr, "\Sigma_1"'] & \Delta^d \arrow[d, "\Sigma_2"] \\ & Z. \end{tikzcd}$$

Note that the map $\widetilde{f}$ is unique, when such a diagram exists, as $\Sigma_i$ are embeddings. Thus $hom_{\mathcal{D}(Z)}(\Sigma_1, \Sigma_2)$ is either empty or consists of a single element.

Although, we state the result for manifolds with corners, for simplicity we assume here that $Y$ is a manifold. Let $\{O_i\}_{i \in I}$ be a locally finite open cover of $Y$, closed under intersections, with each $O_i$ diffeomorphic to an open ball in $\mathbb{R}^d$. Such a cover is often called a good cover of a manifold. Existence of such a cover is a folklore theorem, but a proof can be found in [7, Prop A1].

Let $\mathcal{O}$ denote the category with the set of objects $\{O_i\}$ and with morphisms inclusions. And set $C_i = \mathcal{D}(O_i)$, then we naturally have $C_i \subset \Delta^{sm}(Y_\bullet)$. For each $i$, we have the functor

$$F_i = P|_{C_i} : C_i \to \mathcal{G}.$$

By assumption that each $O_i$ is diffeomorphic to an open ball, $O_i$ has an exhaustion by embedded $d$-simplices. This means that there is a sequence of smooth embeddings $\Sigma_j : \Delta^d \to O_i$ satisfying:

- $\operatorname{image}(\Sigma_{j+1}) \supset \operatorname{image}(\Sigma_j)$ for each $j \in \mathbb{N}$.
- $\bigcup_j \operatorname{image}(\Sigma_j) = O_i$.

As each element of $C_i$ is contained in some $\Sigma_j$,

$$\Sigma_0 \to \ldots \to \Sigma_j \to \Sigma_{j+1} \to \ldots$$

forms a final sub-category of $C_i$. Thus, for each $i$, the colimit in $\mathcal{G}$:

$$P_i := \operatorname{colim}_{C_i} F_i \tag{5.2}$$

is the colimit of the sequence

$$P(\Sigma_0) \to \ldots \to P(\Sigma_j) \to P(\Sigma_{j+1}) \ldots$$

And this colimit is naturally a topological $G$-bundle over $O_i = \cup_i \operatorname{image}(\Sigma_i)$.

We may give $P_i$ the structure of a smooth $G$-bundle, with $G$-bundle charts defined as follows. For each $\Sigma \in C_i$, pick a smooth trivialization $\xi_\Sigma : (\Delta^d)^\circ \times G \to P_\Sigma|_{(\Delta^d)^\circ}$.

Then set $\phi_{\Sigma,i}$ to be the composition map

$$(\Delta^d)^\circ \times G \xrightarrow{\xi_\Sigma} P_\Sigma \xrightarrow{c_\Sigma} P_i,$$

where $c_\Sigma : (P_\Sigma = F_i(\Sigma)) \to P_i$ is the natural map in the colimit diagram of (5.2).

The collection $\{\phi_{\Sigma,i}\}$ then forms a smooth $G$-bundle atlas for $P_i$.

So we obtain a functor

$$D : \mathcal{O} \to \mathcal{G},$$

defined by

$$D(O_i) = P_i,$$

and defined naturally on morphisms. Specifically, a morphism $O_{i_1} \to O_{i_2}$ induces a functor $C_{i_1} \to C_{i_2}$ and hence a smooth $G$-bundle map $P_{i_1} \to P_{i_2}$, by the naturality of the colimit.

Let $t : \mathcal{O} \to Top$ denote the tautological functor, sending the subspace $O$ to the corresponding topological space, so that $Y = \operatorname{colim}_{\mathcal{O}} t$, where for simplicity we write equality for natural isomorphisms here and further on in this proof.

Now,

$$N := \mathrm{colim}_{\mathcal{O}} D, \tag{5.3}$$

is naturally a topological $G$-bundle

$$N \xrightarrow{p} \mathrm{colim}_{\mathcal{O}} t = Y.$$

Let $c_i : P_i \to N$ denote the natural maps in the colimit diagram of (5.3). The collection of charts $\{c_i \circ \phi_{\Sigma,i}\}_{i,\Sigma \in C_i}$ forms a smooth atlas on $N$. Let us rename these charts as $\{\rho_k\}$, for $k$ elements of the index set implicit above.

To see that the transition maps are smooth suppose that $z \in \mathrm{image}\, \rho_k \cap \mathrm{image}\, \rho_{k'}$, then we may find an

$$O_{i_0} \ni y = p(z)$$

and a $\Sigma_{k_0} \in \mathcal{D}(O_{i_0})$ s.t. $y \in \mathrm{image}\, \rho_0 = \mathrm{image}\, c_{i_0} \circ \phi_{\Sigma_{k_0}, i_0}$ and s.t.

$$\mathrm{image}\, \rho_0 \subset \mathrm{image}\, \rho_k \cap \mathrm{image}\, \rho_{k'}.$$

Let $m_k : \Sigma_{k_0} \to \Sigma_k$ and $m_{k'} : \Sigma_{k_0} \to \Sigma_{k'}$ denote the pair of natural inclusion morphisms in $\Delta^{sm}(Y_\bullet)$. The transition map $\rho_k \circ \rho_k^{-1}$, in a neighborhood of $z$, is then the map $P(m_{k'}) \circ P(m_k)^{-1}$, which by defining properties of $P$ is smooth.

We now prove that $P$ is induced by $N$. Let $\Sigma : \Delta^n \to Y$ be smooth, then $\{V_i := \Sigma^{-1}(O_i)\}_{i \in I}$ is a locally finite and hence finite open cover of $\Delta^n$ closed under intersections. Let $N^\Delta$ be the simplicial $G$-bundle over $Y_\bullet$ induced by $N$. So

$$N^\Delta_\Sigma := \Sigma^* N.$$

As $\Delta^n$ is a convex subset of $\mathbb{R}^n$, the open metric balls in $\Delta^n$, for the induced metric, are convex as subsets of $\mathbb{R}^n$. Consequently, as each $V_i \subset \Delta^n$ is open, it has a basis of convex (as subsets of $\mathbb{R}^n$) metric balls, with respect to the induced metric. By Rudin [35] there is then a locally finite cover of $V_i$ by elements of this basis. In fact, Rudin shows any open cover of $V_i$ has a locally finite refinement by elements of such a basis.

Let $\{W^i_j\}$ consist of elements of this cover and all intersections of its elements, (which must then be finite intersections). So $W^i_j \subset V_i$ are open convex subsets and $\{W^i_j\}$ is a locally finite open cover of $V_i$, closed under finite intersections.

As each $W^i_j \subset \Delta^n$ is open and convex it has an exhaustion by nested images of embedded simplices. That is

$$W^i_j = \bigcup_{k \in \mathbb{N}} \mathrm{image}\, \sigma^{i,j}_k \tag{5.4}$$

for $\sigma^{i,j}_k : \Delta^d \to W^i_j$ smooth and embedded, with $\mathrm{image}\, \sigma^{i,j}_k \subset \mathrm{image}\, \sigma^{i,j}_{k+1}$ for each $k$.

Let $C$ be the small category with objects $I \times J \times \mathbb{N}$, so that there is exactly one morphism from $a = (i, j, k)$ to $b = (i', j', k')$ whenever $\mathrm{image}\, \sigma^{i,j}_k \subset \mathrm{image}\, \sigma^{i',j'}_{k'}$, and no morphisms otherwise. Let

$$F : C \to \mathcal{D}(\Delta^d)$$

be the functor $F(a) = \sigma^{i,j}_k$ for $a = (i, j, k)$, (the definition on morphisms is forced). For brevity, we denote $\sigma_a := F(a)$.

For a smooth manifold with corners $Y$, if $\mathcal{O}(Y)$ denotes the category of topological subspaces of $Y$ with morphisms inclusions, then there is a forgetful functor

$$T : \mathcal{D}(Y) \to \mathcal{O}(Y)$$

which takes $f$ to image$(\widetilde{f})$. With all this in place, we have:

**Lemma 5.11.**

$$\Delta^d = \text{colim}_C\, T \circ F, \tag{5.5}$$

*as a colimit in $Top$.*

*Proof.* First recall that a general topological space $X$ is the colimit of any open cover $\{O_i\}$ of $X$ closed under intersections. In particular, $\Delta^d$ is the colimit of the cover $\{W^i_j\}_{i,j}$. On the other hand $W^i_j = \text{colim}_{S_{i,j}}\, T \circ F$, for $S_{i,j} \subset C$ a full subcategory corresponding to the exhaustion 5.4. Moreover, $C = \cup_{i,j} S_{i,j}$. The result readily follows. □

It follows that

$$N^{\Delta}_{\Sigma} = \text{colim}_C\, N^{\Delta} \circ \Delta^{sm}\Sigma \circ F.$$

Now, by construction for each $a \in C$, $\Sigma \circ \sigma_a$ is contained in an open set $O_i$ diffeomorphic to the standard open ball in $\mathbb{R}^d$. It follows that we may express:

$$\Sigma \circ \sigma_a = \Sigma_a \circ m_a \circ \sigma_a, \tag{5.6}$$

for some $\Sigma_a : \Delta^d \to O_i \subset Y$ a smooth embedded $d$-simplex. And $m_a : \Delta^n \to \Delta^d$ smooth.

So for all $a \in C$,

$$N^{\Delta} \circ \Delta^{sm}\Sigma \circ F(a) = (m_a \circ \sigma_a)^* P_{\Sigma_a},$$

after naturally identifying $P_{\Sigma_a}$ with $N^{\Delta}_{\Sigma_a}$. More precisely, there is a natural isomorphism $\phi_a : P_{\Sigma_a} \to N^{\Delta}_{\Sigma_a}$ given by the composition:

$$P_{\Sigma_a} \to P_i \to N, \tag{5.7}$$

with the first map the bundle map in the colimit diagram of (5.2), and the second map the bundle map in the colimit diagram of (5.3). The composition (5.7) gives a bundle map over $\Sigma_a$. And so, by the defining universal property of the pull-back, there is a uniquely induced universal map

$$P_{\Sigma_a} \to (\Sigma_a)^* N = N^{\Delta}_{\Sigma_a},$$

which is a $G$-bundle isomorphism.

Also,

$$P_{\Sigma} = \text{colim}_C\, P \circ \Delta^{sm}\Sigma \circ F.$$

Similarly to the above discussion we have that for all $a \in C$:

$$P \circ \Delta^{sm}\Sigma \circ F(a) = P_{\Sigma_a \circ m_a \circ \sigma_a},$$

and by functoriality of $P$ there is a morphism:

$$P(m_a \circ \sigma_a) : P_{\Sigma_a \circ m_a \circ \sigma_a} \to P_{\Sigma_a},$$

over $m_a \circ \sigma_a$ and hence an induced natural morphism:

$$P_{\Sigma_a \circ m_a \circ \sigma_a} \to (m_a \circ \sigma_a)^* P_{\Sigma_a},$$

which is also a $G$-bundle isomorphism.

To summarize, we obtain for all $a \in C$ a natural isomorphism

$$N \circ \Delta^{sm}\Sigma \circ F(a) \xrightarrow{\phi_a} P \circ \Delta^{sm}\Sigma \circ F(a).$$

These fit into a natural transformation of functors:

$$\phi : N \circ \Delta^{sm}\Sigma \circ F \to P \circ \Delta^{sm}\Sigma \circ F.$$

So that $\phi$ induces a map of the colimits:

$$h_\Sigma : P_\Sigma \to N^\Delta_\Sigma,$$

by naturality, and this is an isomorphism of these smooth $G$-bundles. It is then clear that $\{h_\Sigma\}_\Sigma$ determines the bundle isomorphism $h : P \to N^\Delta$ we are looking for. □

5.1. **Pullbacks of simplicial bundles.** Let $P \to X$ be a simplicial $G$-bundle over a smooth simplicial set $X$. And let $f : Y \to X$ be a smooth map of smooth simplicial sets. We define the pull-back simplicial $G$-bundle $f^*P \to Y$ by the functor $f^*P := P \circ \Delta^{sm} f$.

Note that the analogue of the following lemma is not true in the category of topological fibrations. The pull-back by the composition is not the composition of pullbacks (except up to a natural isomorphism).

**Lemma 5.12.** *The pull-back is functorial. So that if $f : X \to Y$ and $g : Y \to Z$ are smooth maps of smooth simplicial sets, and $P \to Z$ is a smooth simplicial $G$-bundle over $Z$ then*

$$(g \circ f)^*P = f^*(g^*(P)) \text{ this is an actual equality.}$$

*Proof.* This is of course elementary, as functor composition is associative:

$$(g \circ f)^*P = P \circ \Delta^{sm}(g \circ f) = P \circ (\Delta^{sm} g \circ \Delta^{sm} f) = (P \circ \Delta^{sm} g) \circ \Delta^{sm} f = f^*(g^*P).$$

□

## 6. Connections on simplicial $G$-bundles

In this section $G$ is a generalized Lie group.

**Definition 6.1.** *A* **simplicial $G$-connection** *$D$ on a simplicial $G$-bundle $P$ over a smooth simplicial set $X$ is for each $d$-simplex $\Sigma$ of $X$, a smooth $G$-invariant Ehresmann $G$-connection $D(\Sigma) = D_\Sigma$ on $P_\Sigma$. This data is required to satisfy: if $f : \Sigma_1 \to \Sigma_2$ is a morphism in $\Delta(X)$ then*

$$P(f)^* D_{\Sigma_2} = D_{\Sigma_1}. \tag{6.1}$$

*We say that $D$ is* **coherent** *if the same holds for all morphisms $f : \Sigma_1 \to \Sigma_2$ in $\Delta^{sm}(X)$. We will often say $G$-connection instead of simplicial $G$-connection, where there is no need to disambiguate.*

As with differential forms the coherence condition is very restrictive, and is not part of the basic definition.

Let $P \to X$ be a simplicial $G$-bundle. Define $P \times I \to X \times I$, for $I := [0,1]_\bullet$, to be the simplicial $G$-bundle $pr^* P$, for $pr : X \times I \to X$ the natural projection.

**Lemma 6.2.** *$G$-connections on simplicial $G$-bundles exist and any pair of $G$-connections $D_1, D_2$ on a simplicial $G$-bundle $P$ are* **concordant**. *The latter means that there is a $G$-connection on $\widetilde{D}$ on $P \times I \to X \times I$, which restricts to $D_1, D_2$ on $P \times I_0$, respectively on $P \times I_1$, for $I_0, I_1 \subset I$ denoting the images of the two end point inclusions $\Delta^0_\bullet \to I$.*

*Proof.* Suppose that $\Sigma : \Delta^d_{simp} \to X$ is a degeneracy of a 0-simplex $\Sigma_0 : \Delta^0_{simp} \to X$, meaning that there is a morphism from $\Sigma$ to $\Sigma_0$ in $\Delta(X)$. Then $P_\Sigma = \Delta^d \times P_{\Sigma_0}$ (as previously equality indicates natural isomorphism) and we fix the corresponding trivial connection $D_\Sigma$ on $P_\Sigma$. This assignment satisfies the condition that for all morphisms $m : \Sigma_1 \to \Sigma_2$ in $\Delta(X)$, for $\Sigma_1, \Sigma_2$ degeneracies of 0-simplices, $D_{\Sigma_1} = P(m)^* D_{\Sigma_2}$. We then proceed inductively.

Suppose we have constructed connections $D_\Sigma$ for all $k$-simplices, $0 \leq k \leq n$, and all their degeneracies, satisfying the condition $S(n)$, which is as follows. For all morphisms $m : \Sigma_1 \to \Sigma_2$ in $\Delta(X)$, for $\Sigma_1, \Sigma_2$ $k$-simplices or their degeneracies with $0 \leq k \leq n$, $D_{\Sigma_1} = P(m)^* D_{\Sigma_2}$. We construct an extension $D_\Sigma$ for all $(n+1)$-simplices and their degeneracies, so that this extension satisfies $S(n+1)$.

If $\Sigma$ is a non-degenerate $(n+1)$-simplex then $D_\Sigma$ is already determined over the boundary of $\Delta^{n+1}$ by the defining condition (6.1). For by the hypothesis, $D_\Sigma$ is already defined on all $n$-simplices. Then extend $D_\Sigma$ over all of $\Delta^{n+1}$ arbitrarily. An explicit construction of such an extension is not totally trivial. We may however use the same argument as in the extension construction for differential forms defined on the boundary of a simplex, see proof of [40, Theorem 7.1].

Thus, we have extended $D_\Sigma$ to all $(n+1)$-simplices, as such a simplex is either non-degenerate or is a degeneracy of an $n$-simplex, and in the latter case $D_\Sigma$ is defined by the hypothesis.

Now, suppose we have a degeneracy $mor : \Sigma^m \to \Sigma^k$, $k < m$, $k \leq n+1$ ($\Sigma^k$ can itself be degenerate). Then we have bundle map:

$$P(mor) : P(\Sigma^m) \to P(\Sigma^k).$$

And we define $D_{\Sigma^m} = P(mor)^* D_{\Sigma^k}$. The property $S(n)$ insures that this is well defined. And so we have constructed an assignment $D_\Sigma$ for all degeneracies of $(n+1)$-simplices. By construction this satisfies $S(n+1)$. And so we complete the inductive step.

The second part of the lemma follows by an analogous argument, since we may extend $D_1, D_2$ to a concordance connection $\widetilde{D}$, using the above inductive procedure.

□

**Example 6.3.** *Given a classical smooth $G$-connection $D$ on a smooth principal $G$-bundle $Z \to Y$, we naturally get a simplicial $G$-connection on the induced simplicial*

*$G$-bundle $Z^\Delta$. Concretely, this is defined by setting $D_\Sigma$ on $Z^\Delta_\Sigma = \Sigma^* Z$ to be $\widetilde{\Sigma}^* D$, for $\widetilde{\Sigma} : \Sigma^* Z \to Z$ the natural map (in the pull-back diagram). The pull-back $\widetilde{\Sigma}^* D$, is the pre-image by $\widetilde{\Sigma}$ of the corresponding distribution. This is called the* **induced simplicial connection***, and it will be denoted by $D^\Delta$. Going in the other direction is always possible if the given simplicial $G$-connection in addition satisfies coherence, but we will not elaborate.*

## 7. Chern-Weil homomorphism

7.1. **The classical case.** To establish notation we first discuss classical Chern-Weil homomorphism. In this section again $G$ will be a generalized Lie group. Let $\mathfrak{g}$ denote its Lie algebra. Let $P$ be a smooth principal $G$-bundle over a smooth manifold $Y$. Fix a $G$-connection $D$ on $P$. Let $\operatorname{Aut} P_y$ denote the group of smooth $G$-equivariant diffeomorphisms of the fiber $P_y$ of $P$ over $y \in Y$. Note that $\operatorname{Aut} P_y \simeq G$ where $\simeq$ means non-canonically isomorphic. Then associated to $D$ we have the classical curvature 2-form $R^D$ on $Y$, understood as a 2-form valued in the vector bundle $\mathcal{P} \to Y$, whose fiber over $y \in Y$ is $\mathfrak{lie}(AutP_y)$ - the Lie algebra of $\operatorname{Aut} P_y$.

Thus,

$$\forall v, w \in T_y Y : R^D(v, w) \in \mathcal{P}_y = \mathfrak{lie}(AutP_y).$$

Now, let $\rho$ be a continuous symmetric multilinear functional:

$$\rho : \prod_{i=1}^{i=k} \mathfrak{g} \to \mathbb{R},$$

satisfying

$$\rho(Ad_g(v)) = \rho(v), \quad \forall g \in G, \forall v \in \prod_{i=1}^{i=k} \mathfrak{g}$$

Here if $v = (\xi_1, \ldots, \xi_n)$, $Ad_g(v) = (Ad_g(\xi_1), \ldots, Ad_g(\xi_n))$ is the adjoint action by the element $g \in G$. As $\rho$ is $Ad$ invariant, it uniquely determines multilinear maps with the same name:

$$\rho : \prod_{i=1}^{i=k} \mathfrak{lie}(AutP_y) \to \mathbb{R},$$

by taking a Lie-group isomorphism $\operatorname{Aut} P_y \to G$, defined by fixing a point in the fiber $P_y$. We may now define a closed $\mathbb{R}$-valued $2k$-form $\omega^{\rho,D}$ on $Y$:

(7.1)

$$\omega^{\rho,D}(v_1, \ldots, v_{2k}) = \frac{1}{2k!} \sum_{\eta \in S_{2k}} \operatorname{sign} \eta \cdot \rho(R^D(v_{\eta(1)}, v_{\eta(2)}), \ldots, R^D(v_{\eta(2k-1)}, v_{\eta_{2k}})),$$

for $S_{2k}$ the permutation group of a set with $2k$ elements, and where $v_1, \ldots, v_{2k} \in T_y Y$.

Set $cw^{P,D}(\rho) = \omega^{\rho,D}$. Let $\mathbb{R}[\mathfrak{g}]$ denote the algebra of continuous polynomial functions on $\mathfrak{g}$, which we define to be the symmetric algebra on the continuous dual $\mathfrak{g}^*$. And let $\mathbb{R}[\mathfrak{g}]^G$ denote the sub-algebra of fixed points by the adjoint $G$ action.

Then we get a $dg$ algebra homomorphism:

$$cw^{P,D} : \mathbb{R}[\mathfrak{g}]^G \to \Omega^\bullet(Y, \mathbb{R}).$$

Set
$$\alpha^{\rho,D} := \int \omega^{\rho,D} \in C^{2k}(Y,\mathbb{R}).$$
Then we define the classical Chern-Weil characteristic class:
$$c^\rho(P) := [\alpha^{\rho,D}] \in H^{2k}(Y,\mathbb{R}). \tag{7.2}$$

7.2. **Chern-Weil homomorphism for smooth simplicial bundles.** Now let $P$ be a simplicial $G$-bundle over a smooth simplicial set $X$. Fix a simplicial $G$-connection $D$ on $P$.

For each simplex $\Sigma^d$, we have the curvature 2-form $R^D_\Sigma$ of the connection $D_\Sigma$ on $P_\Sigma$, defined as in the section just above. For concreteness:
$$\forall v, w \in T_z\Delta^d : R^D_\Sigma(v,w) \in \mathfrak{lie}(Aut P_y),$$
for $P_z$ the fiber of $P_\Sigma$ over $z \in \Delta^d$.

As above, let $\rho$ be an $Ad$ invariant continuous symmetric multilinear functional:
$$\rho : \prod_{i=1}^{i=k} \mathfrak{g} \to \mathbb{R}.$$
We may now define a closed, $\mathbb{R}$-valued, simplicial differential $2k$-form $\omega^{\rho,D}$ on $X$:
$$\omega^{\rho,D}_\Sigma(v_1,\ldots,v_{2k}) = \frac{1}{2k!}\sum_{\eta\in S_{2k}} \operatorname{sign}\eta \cdot \rho(R^D_\Sigma(v_{\eta(1)},v_{\eta(2)}),\ldots,R^D_\Sigma(v_{\eta(2k-1)},v_{\eta_{2k}})).$$
Set $cw^P(\rho) = \omega^{\rho,D}$ for any $D$ as above. This defines a $dg$ algebra homomorphism:
$$cw^{P,D} : \mathbb{R}[\mathfrak{g}]^G \to \Omega^\bullet(X,\mathbb{R}).$$

**Lemma 7.1.** *For $P \to X$ as above*
$$cw^{P,D} \simeq cw^{P,D'},$$
*for any pair of simplicial $G$-connections $D, D'$ on $P$, where $\simeq$ is homotopy equivalence of dg maps.*

*Proof.* For $D, D'$ as in the statement, fix a concordance simplicial $G$-connection $\widetilde{D}$, between $D, D'$, on the $G$-bundle $P \times I \to X \times I$, as in Lemma 6.2.

Let $\Omega^\bullet_{pol}(I) \subset \Omega^\bullet(I,\mathbb{R})$ be the sub-algebra of polynomial forms on $I$. We have a commutative diagram of $dg$ maps:
$$\begin{tikzcd}
 & \Omega^\bullet(X,\mathbb{R}) \otimes \Omega^\bullet_{pol}(I) \arrow[r, "\widetilde{r}_i"] \arrow[d] & \Omega^\bullet(X,\mathbb{R}) \arrow[d, "id"] \\
\mathbb{R}[\mathfrak{g}]^G \arrow[r, "cw^{P,\widetilde{D}}"] & \Omega^\bullet(X\times I,\mathbb{R}) \arrow[r, "r_i"] & \Omega^\bullet(X,\mathbb{R}),
\end{tikzcd}$$
where

- $\Omega^\bullet(X,\mathbb{R}) \otimes \Omega^\bullet_{pol}(I) \to \Omega^\bullet(X\times I,\mathbb{R})$ is the natural map $\omega\otimes\kappa \mapsto pr^*_X\omega \wedge pr^*_I\kappa$, and is a quasi-isomorphism.

- $\widetilde{r}_i$, $i = 0, 1$ correspond to the pair of augmentations $\epsilon_i : \Omega^{\bullet}_{pol}(I) \to \mathbb{R}$: $\epsilon_0(t) = 0$, $\epsilon_0(dt) = 0$, $\epsilon_1(t) = 1$, $\epsilon_1(dt) = 0$.
- $r_i$ are the restriction maps corresponding to the pair of natural inclusions $X \to X \times I$.

As $\mathbb{R}[\mathfrak{g}]^G$ is Sullivan, by basic algebra [6, Proof of Proposition 12.6] there is a "lifting":
$$\mathbb{R}[\mathfrak{g}]^G \to \Omega^{\bullet}(X, \mathbb{R}) \otimes \Omega^{\bullet}_{pol}(I).$$
And it is immediate that this forms a *dg* homotopy (see [6, Section 12]) between $cw^{P,D}$ and $cw^{P,D'}$.

□

Set
$$\alpha^{\rho,D} := \int \omega^{\rho,D} \in C^{2k}(X, \mathbb{R}).$$
Then in particular, the cohomology class:
$$c^{\rho}(P) := [\alpha^{\rho,D}] \in H^{2k}(X, \mathbb{R}),$$
is well defined. And this is called the Chern-Weil characteristic class.

**Notation 7.2.** *Let us denote $cw^P$ any representative of the homotopy class $[cw^{P,D}]$. (For smooth or simplicial $G$-bundles $P$.)*

We have the expected naturality:

**Lemma 7.3.** *Let $P$ be a simplicial $G$-bundle over $Y$, $\rho$ as above and $f : X \to Y$ a smooth simplicial map. Then*
$$f^* \circ cw^P \simeq cw^{f^*P},$$
*where $\simeq$ as before means homotopic.*

*Proof.* Let $D$ be a simplicial $G$-connection on $P$. Define the pull-back connection $f^*D$ on $f^*P$ by $f^*D(\Sigma) = D_{f(\Sigma)}$. Then $f^*D$ is a simplicial $G$-connection on $f^*P$. Now,
$$\begin{aligned} \forall \Sigma : \omega^{\rho, f^*D}(\Sigma) &= \omega^{\rho,D}(f(\Sigma)), \text{ by definition of } f^*D \\ &= f^*\omega^{\rho,D}(\Sigma), \text{ definition (4.4)}. \end{aligned}$$
And consequently, $\omega^{\rho,f^*D} = f^*\omega^{\rho,D}$. It follows that:
$$f^* \circ cw^{P,D} = cw^{f^*P, f^*D}.$$
The result the readily follows by Lemma 7.1. □

**Proposition 7.4.** *Let $G \hookrightarrow Z \to Y$ be an ordinary smooth principal $G$-bundle, and $\rho$ as above. Let $Z^{\Delta}$ be the induced simplicial $G$-bundle over $Y_{\bullet}$ as in Example 5.6. Then:*

(1) *The form $\omega^{\rho,D^{\Delta}}$ is the simplicial differential form induced by $\omega^{\rho,D}$, where induced is as in Example 4.3. In particular, $cw^{Z^{\Delta}} \simeq \Theta \circ cw^Z$, where $\Theta$ is as in (4.3).*

(2) *If $c^\rho(Z) \in H^{2k}(Y, \mathbb{R})$ is the classical Chern-Weil characteristic class as in* (7.2)*, then*

$$|c^\rho(Z^\Delta)|_{sm} = c^\rho(Z), \tag{7.3}$$

*where $|c^\rho(Z^\Delta)|_{sm}$ is as in part 2 of Notation 4.7.*

*Proof.* Fix a smooth $G$-connection $D$ on $Z$. This induces a simplicial $G$-connection $D^\Delta$ on $Z^\Delta$, as in Example 6.3. Let $\omega^{\rho,D}$ denote the classical smooth Chern-Weil differential $2k$-form on $Y$, as in (7.1).

Now,

$$\begin{aligned} \forall \Sigma : \omega^{\rho,D^\Delta}(\Sigma) &= \omega^{\rho,\widetilde{\Sigma}^* D} \text{ by definitions} \\ &= \Sigma^* \omega^{\rho,D} \text{ by classical naturality of Chern-Weil forms.} \end{aligned}$$

So we obtain the first part of the Proposition.

Let $\alpha^{\rho,D} = \int \omega^{\rho,D} \in C^{2k}(Y, \mathbb{R})$. It readily follows by Lemma 4.9 that:

$$|c^\rho(Z^\Delta)|_{sm} = (I^\vee)^{-1}([\alpha^{\rho,D^\Delta}]) = [\alpha^{\rho,D}] = c^\rho(Z),$$

where $I^\vee$ is as in (4.9). □

## 8. The universal simplicial $G$-bundle

Briefly, a Grothendieck universe is a set $\mathcal{U}$ forming a model for set theory. That is if we interpret all terms of set theory as elements of $\mathcal{U}$, then all the set theoretic constructions keep us within $\mathcal{U}$. We will assume Grothendieck's axiom of universes which says that for any (pure) set $X$ there is a Grothendieck universe $\mathcal{U} \ni X$. Intuitively, such a universe $\mathcal{U}$ is formed by all possible set theoretic constructions starting with $X$. For example if $\mathcal{P}(X)$ denotes the power set of $X$, then $\mathcal{P}(X) \in \mathcal{U}$ and if $\{Y_i \in \mathcal{P}(X)\}_{i \in I}$ for $I \in \mathcal{U}$ is a collection then $\bigcup_i Y_i \in \mathcal{U}$. This may appear very natural, but we should note that this axiom is beyond $ZFC$. Although it is now a common axiom of modern set theory, especially in the context of category theory, c.f. [20]. In some contexts one works with universes implicitly. This is impossible here, as we need to establish certain universe independence.

Let $G$ be a locally convex Lie group. Let $\mathcal{U}$ be a Grothendieck universe satisfying:

$$G \in \mathcal{U}, \quad \forall n \in \mathbb{N} : \Delta^n \in \mathcal{U},$$

where $\Delta^n$ are the usual topological $n$-simplices. Such a $\mathcal{U}$ will be called ***$G$-admissible***.

We will construct smooth Kan complexes $BG^{\mathcal{U}}$ for each $G$-admissible $\mathcal{U}$. Moreover, we will construct a weak equivalence $|BG^{\mathcal{U}}| \to BG$ for each $\mathcal{U}$, where $BG$ the classical Milnor classifying space.

If $G$ has the homotopy type of a CW complex, then $BG$ has the homotopy type of a CW complex. In particular, by Whitehead's theorem homotopy type of $|BG^{\mathcal{U}}|$ is the independent of $\mathcal{U}$, and in fact $|BG^{\mathcal{U}}|$ is $BG$ up to homotopy.

**Definition 8.1.** *A* **$\mathcal{U}$-small set** *is an element of $\mathcal{U}$. For $X$ a smooth simplicial set, a smooth simplicial $G$-bundle $P \to X$ will be called* **$\mathcal{U}$-small** *if for each simplex $\Sigma$ of $X$ the bundle $P_\Sigma$ is $\mathcal{U}$-small.*

8.1. **The classifying spaces $BG^{\mathcal{U}}$.** Let $\mathcal{U}$ be $G$-admissible. We define a simplicial set $BG^{\mathcal{U}}$, whose set of $k$-simplices $BG^{\mathcal{U}}(k)$ is the set of $\mathcal{U}$-small smooth simplicial $G$-bundles over $\Delta^k_\bullet$. The simplicial maps are defined by pull-back so that given a map $i \in hom_\Delta(m, n)$ the map

$$BG^{\mathcal{U}}(i) : BG^{\mathcal{U}}(n) \to BG^{\mathcal{U}}(m)$$

is the natural pull-back:

$$BG^{\mathcal{U}}(i)(P) = i^*_\bullet P,$$

for $i_\bullet$, the induced map $i_\bullet : \Delta^m_\bullet \to \Delta^n_\bullet$, $P \in BG^{\mathcal{U}}(n)$ a simplicial $G$-bundle over $\Delta^n_\bullet$, and where the pull-back map $i^*_\bullet$ is as in Section 5.1. Then Lemma 5.12 insures that $BG^{\mathcal{U}} : \Delta^{op} \to s - Set$ is a functor, so that we get a simplicial set $BG^{\mathcal{U}}$.

We define a smooth simplicial set structure $g$ on $BG^{\mathcal{U}}$ as follows. Given a $d$-simplex $P \in BG^{\mathcal{U}}(d)$ the induced map

$$(g(P) = P_*) : \Delta^d_\bullet \to BG^{\mathcal{U}},$$

is defined naturally by

$$P_*(\sigma) := \sigma^*_\bullet P. \tag{8.1}$$

where $P$ on the right is the initial simplicial $G$-bundle $P \to \Delta^d_\bullet$. More explicitly, $\sigma \in \Delta^d_\bullet(k)$ is a smooth map $\sigma : \Delta^k \to \Delta^d$, $\sigma_\bullet : \Delta^k_\bullet \to \Delta^d_\bullet$ denotes the induced map and the pull-back is as previously defined. We need to check the push-forward functoriality Axiom 2b.

Let $\sigma \in \Delta^d_\bullet(k)$, then for all $j \in \mathbb{N}, \rho \in \Delta^k_\bullet(j)$:

$$\begin{aligned}(P_*(\sigma))_*(\rho) &= (\sigma^*_\bullet P)_*(\rho)\\ &= \rho^*_\bullet(\sigma^*_\bullet P), \text{ by definition of } g.\end{aligned}$$

And

$$\begin{aligned}P_* \circ \sigma_\bullet(\rho) &= (\sigma_\bullet(\rho))^*_\bullet P\\ &= (\sigma_\bullet \circ \rho_\bullet)^* P, \text{ as } \sigma_\bullet \text{ is smooth}\\ &= \rho^*_\bullet(\sigma^*_\bullet P).\end{aligned}$$

And so

$$(P_*(\sigma))_* = P_* \circ \sigma_\bullet,$$

so that $BG^{\mathcal{U}}$ is indeed a smooth simplicial set.

8.2. **The universal smooth simplicial $G$-bundle $EG^{\mathcal{U}} \to BG^{\mathcal{U}}$.** To abbreviate already complex notation, in what follows $V$ denotes $BG^{\mathcal{U}}$ for a general, $G$-admissible $\mathcal{U}$. There is a tautological functor

$$E : \Delta^{sm}(V) \to \mathcal{G} \tag{8.2}$$

that we now describe.

A smooth map $P : \Delta^d_\bullet \to V$, uniquely corresponds to a $d$-simplex $P^b$ of $V$ via Proposition 3.7, i.e. a simplicial $G$-bundle $P^b \to \Delta^d_\bullet$. In other words $P^b$ is the bundle:

$$P^b = P(id^d), \tag{8.3}$$

for $id^d : \Delta^d \to \Delta^d$ the identity, and where the equality is an equality of simplicial $G$-bundles, in other words functors.

**Notation 8.2.** *Although we disambiguate in the discussion just below, later on we may conflate the notation $P, P^b$ with just $P$.*

Recalling that $P^b$ is a certain functor $\Delta^{sm}(\Delta^d_\bullet) \to \mathcal{G}$ we then set:

$$E(P) = P^b(id^d_\bullet).$$

We now define the action of $E$ on morphisms. Suppose we have a morphism $m \in \Delta^{sm}(V)$:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "P_1"] & \Delta^d_\bullet \arrow[d, "P_2"] \\ & V, \end{tikzcd}$$

then we have an equality:

$$\begin{aligned} P_1^b &= P_1(id^k) \ (8.3) \\ &= (P_2 \circ \widetilde{m}_\bullet)(id^k) \\ &= P_2(\widetilde{m}) \\ &= (P_2^b)_*(\widetilde{m}), \text{ thinking of } P_2^b \text{ as a simplex of } V \\ &= P_2^b \circ \Delta^{sm}\widetilde{m}_\bullet, \text{ by } (8.1). \end{aligned} \tag{8.4}$$

So that

$$P_1^b(id^k_\bullet) = P_2^b(\widetilde{m}_\bullet \circ id^k_\bullet) = P_2^b(\widetilde{m}_\bullet).$$

We have a tautological morphism $e_m \in \Delta^{sm}(\Delta^d_\bullet)$ corresponding to the diagram:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "\widetilde{m}_\bullet"] & \Delta^d_\bullet \arrow[d, "id^d_\bullet"] \\ & \Delta^d_\bullet. \end{tikzcd}$$

So we get a smooth $G$-bundle map:

$$P_2^b(e_m) : (E(P_1) = P_2^b(\widetilde{m}_\bullet)) \to (E(P_2) = P_2^b(id^d_\bullet)),$$

which is over the smooth map $\widetilde{m} : \Delta^k \to \Delta^d$ induced by $\widetilde{m}_\bullet$. And we set $E(m) = P_2^b(e_m)$.

We need to check that with these assignments $E$ is a functor. Suppose we have a diagram:

$$\begin{tikzcd} \Delta^l_\bullet \arrow[r, "\widetilde{m}^0_\bullet"] \arrow[drr, bend right, "P_0"] & \Delta^k_\bullet \arrow[r, "\widetilde{m}^1_\bullet"] \arrow[dr, "P_1"] & \Delta^d_\bullet \arrow[d, "P_2"] \\ & & V. \end{tikzcd}$$

In other words, we have a diagram for the composition $m = m^1 \circ m^0$ in $\Delta^{sm}(V)$. Then $e_m = e_{m^1} \circ e'_{m^0}$ where $e'_{m^0}$ is the diagram:

$$\begin{tikzcd} \Delta^l_\bullet \arrow[r, "\widetilde{m}^0_\bullet"] \arrow[dr, "\widetilde{m}_\bullet"] & \Delta^k_\bullet \arrow[d, "\widetilde{m}^1_\bullet"] \\ & \Delta^d_\bullet, \end{tikzcd}$$

and $e_{m^1}$ is the diagram:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}^1_\bullet"] \arrow[dr, "\widetilde{m}^1_\bullet"] & \Delta^d_\bullet \arrow[d, "id^d_\bullet"] \\ & \Delta^d_\bullet. \end{tikzcd}$$

So

$$E(m) = P^b_2(e_m) = P^b_2(e_{m^1}) \circ P^b_2(e'_{m^0}) = E(m^1) \circ P^b_2(e'_{m^0}).$$

Now,

$$\begin{aligned} E(m_0) &= P^b_1(e_{m^0}) \\ &= (P^b_2 \circ \Delta^{sm}\widetilde{m}^1_\bullet)(e_{m^0}), \text{ analogue of (8.4)} \\ &= P^b_2(e'_{m_0}). \end{aligned}$$

And so we get: $E(m) = E(m_1) \circ E(m_0)$. Thus, $E$ is a functor.

By construction the functor $E$ satisfies the compatibility condition, and hence determines a simplicial $G$-bundle.

**Definition 8.3.** *Given $G, \mathcal{U}$ as previously, the universal simplicial $G$-bundle $EG^{\mathcal{U}}$ is defined to be the functor $E$ as constructed above.*

**Proposition 8.4.** *$BG^{\mathcal{U}}$ is a Kan complex.*

*Proof.* In what follows we again abbreviate $BG^{\mathcal{U}}$ as $V$. Recall that $\Lambda^n_k \subset \Delta^n_{simp}$, denotes the sub-simplicial set corresponding to the "boundary" of $\Delta^n$ with the $k$'th face removed, where by $k$'th face we mean the face opposite to the $k$'th vertex. Let $h : \Lambda^n_k \to V$, $0 \leq k \leq n$, be a simplicial map, this is also called a horn. We need to construct an extension of $h$ to $\Delta^n_{simp}$.

For simplicity we assume $n = 2$, and $k = 1$ as the general case is identical. There are three natural inclusions

$$i_j : \Delta^0_{simp} \to \Delta^2_{simp},$$

$j = 0, 1, 2$, with $i_1$ corresponding to the inclusion of the horn vertex. The corresponding 0-simplices will be denoted by $0, 1, 2$. Let

$$\sigma_{i,j} : \Delta^1_{simp} \to \Delta^2$$

be the edge between vertexes $i, j$, that is $\sigma_{i,j}(0) = i$, $\sigma_{i,j}(1) = j$.

Let us denote by $L$ the smooth sub-simplicial set of $\Delta^2_\bullet$ corresponding to the simplices whose images lie in image $\sigma_{0,1}$ or image $\sigma_{1,2}$. There are then smooth maps $\sigma_{i,j} : \Delta^1_\bullet \to L$ extending $\sigma_{i,j}$ above.

The map $h$ above, induces a smooth map $h_\bullet : L \to V$, and we denote $P := h_\bullet^* E$. So $P \to L$ is a simplicial $G$-bundle. The extension of $h$ to $\Delta^2_{simp}$ will be accomplished once we extend the $P$ over $\Delta^2_\bullet$.

**Lemma 8.5.** *The bundle $P \to L$ is trivializable.*

*Proof.* Set $P_{i,j} := \sigma^*_{i,j} P$, then by Theorem 5.10 $P_{i,j}$ is induced by a smooth $G$-bundle over $\Delta^1$. The latter is trivializable, and hence $P_{i,j}$ is trivializable as a simplicial G-bundle. Denote by $\phi_{i,j} : \Delta^1_\bullet \times G \to P_{i,j}$ the corresponding trivialization over the $id : \Delta^1_\bullet \to \Delta^1_\bullet$.

Denoting $0, 1 \in \Delta^1_\bullet(0)$ the end-point vertices as previously, $\phi_{0,1}$ induces a $G$-equivariant smooth map:

$$G = (\Delta^1_\bullet \times G)(1) \to P_{0,1}(1),$$

and we denote this map by $\phi_1$. Likewise, $\phi_{1,2}$ induces a smooth map:

$$G = (\Delta^1_\bullet \times G)(0) \to P_{1,2}(0),$$

and we denote this map by $\phi_2$.

Now the map $\phi_1^{-1} \circ \phi_2$ may not be the $id : G \to G$, but clearly we may adjust $\phi_{1,2}$ so that it is. And so we may assume this holds.

Then $\phi_{0,1}$ and $\phi_{1,2}$ clearly induce a trivialization $tr : T \to P$, for $T \to L$ the trivial simplicial $G$-bundle.

□

We have the trivial extension of $T$ to the trivial simplicial $G$-bundle over $\Delta^2_\bullet$. And so by the lemma above it should be clear that $P$ likewise has an extension $\widetilde{P}$ over $\Delta^2_\bullet$, but we need this extension to be $\mathcal{U}$-small so that we must be explicit.

We proceed inductively. Let $D^0$ denote the full sub-category of $\Delta^{sm}(\Delta^2_\bullet)$ with the set objects $\operatorname{obj} L \cup \Delta^2_\bullet(0)$ (non-disjoint union).

We extend $P$ to a functor $\widetilde{P}^0 : D^0 \to \mathcal{G}$. For $\sigma \in \Delta^2_\bullet(d)$, if $\sigma$ has image in the horn $\Lambda^2_1 \subset \Delta^2$, then set $\widetilde{P}^0(\sigma) = P(\sigma)$. The extension of $\widetilde{P}^0$ to morphisms in $D^0$ is then taken to be the trivial extension.

Let $T^0 : D^0 \to \mathcal{G}$ be the trivial functor (as in the definition of a trivial bundle in Example 5.5). Then in addition, there is clearly a natural transformation $tr^0 : T^0 \to \widetilde{P}^0$ extending the natural transformation $tr$ of the lemma above.

Let $S(n)$ be the statement:

(1) There is an extension $\widetilde{P}^n$ of $P$ over the full-subcategory $D^n \subset \Delta^{sm}(\Delta^2_\bullet)$, defined analogously to $D^0$, with objects

$$\operatorname{obj} L \cup \bigcup_{0 \le k \le n} \Delta^2_\bullet(k).$$

(2) $\widetilde{P}^n$ satisfies compatibility.

(3) There is a natural transformation $tr^n : T^n \to \widetilde{P}^n$ extending the natural transformation $tr$ of the lemma above. Where $T^n : D^n \to \mathcal{G}$ is the trivial functor defined analogously to $T^0$ above.

We prove
$$S(n) \implies S(n+1),$$
and that moreover the corresponding functor $\widetilde{P}^{n+1}$ can be chosen to extend $\widetilde{P}^n$. Then natural induction implies the existence of the needed extension $\widetilde{P}$ over $\Delta^2_\bullet$.

Let $\sigma \in \Delta^2_\bullet(n+1)$. By the hypothesis $S(n)$ the functor:
$$\sigma^*_\bullet \widetilde{P}^n : C_{n+1} \to \mathcal{G},$$
is defined, where $C_{n+1}$ denotes the sub-category of $\Delta(\partial \Delta^{n+1}_{simp})$ with objects all non-degenerate objects and with morphisms injections. (The pull-back is as in Section 5.1).

We then have a topological bundle
$$p' : N'_\sigma \to \partial \Delta^{n+1}$$
defined as the colimit of $\sigma^*_\bullet \widetilde{P}^n$ over $C_{n+1}$.

Let
$$inc : \partial \Delta^{n+1} \to \Delta^{n+1}$$
denote the inclusion. We first construct a principal $G$-bundle with discreet topology
$$N_\sigma \xrightarrow{p} \Delta^{n+1},$$
by the following conditions:

$$N_\sigma|_{\partial \Delta^{n+1}} := p^{-1}(\partial \Delta^{n+1}) = N'_\sigma, \tag{8.5}$$

$$N_\sigma|_{(\Delta^{n+1})^\circ} = (\Delta^{n+1})^\circ \times G, \tag{8.6}$$

where the projection map $p$ is determined by the maps $p' : N'_\sigma \to \partial \Delta^{n+1}$, and the projection map $(\Delta^{n+1})^\circ \times G \to (\Delta^{n+1})^\circ$.

By the inductive hypothesis $S(n)$ part 3, there is a distinguished trivialization $h_n : \partial \Delta^{n+1} \times G \to N'_\sigma$, corresponding to $tr^n$. The map $h_n$ and the identity map $(\Delta^{n+1})^\circ \times G \to (\Delta^{n+1})^\circ$ induce a discreet $G$-bundle isomorphism $\Delta^{n+1} \times G \to N_\sigma$. Then push-forward the smooth $G$-bundle structure and the topology along this map. The resulting smooth $G$-bundle is then set to be $\widetilde{P}^{n+1}(\sigma)$.

We have thus defined the extension $\widetilde{P}^{n+1}$ on objects. We now need to treat morphisms in $D^{n+1} \subset \Delta(\Delta^2_\bullet)$. For any $d$-simplex $\rho$ of $\Delta^2$ let $\rho_i$ denote the $i$'th face of $\rho$, $0 \le i \le d$. For clarity, this means that we have an inclusion morphism $inc_i : \rho_i \to \rho$ corresponding to the diagram:

$$\begin{tikzcd}
\Delta^n \arrow[r, "\widetilde{inc_i}"] \arrow[dr, "\rho_i"'] & \Delta^d \arrow[d, "\rho"] \\
& \Delta^2,
\end{tikzcd}$$

where $\widetilde{inc_i}$ is the topological face inclusion map corresponding to the face opposite the vertex $i$, also called the $i$-face of $\Delta^d$.

By construction we have natural maps:

$$\widetilde{P}^{n+1}(inc_i) : \widetilde{P}^n(\sigma_i) = \widetilde{P}^{n+1}(\sigma_i) \to \widetilde{P}^{n+1}(\sigma),$$

for each $i \in \{0, \ldots, n+1\}$.

If $m : \sigma^{n+1} \to \rho^{n+1}$ is an identity morphism, then set $\widetilde{P}^{n+1}(m)$ to be the *id*.

Suppose we given a morphism $m : \sigma \to \rho$, $\sigma \in \Delta^2_\bullet(n+1)$, $\rho \in \Delta^2_\bullet(n)$. We then define $\widetilde{P}^{n+1}(m)$ as follows. First we define a map:

$$\partial : inc^* \widetilde{P}^{n+1}(\sigma) \to \widetilde{P}^{n+1}(\rho),$$

for $inc : \partial\Delta^{n+1} \to \Delta^{n+1}$ the inclusion.

Over a $j$-face of $\Delta^{n+1}$, $\widetilde{P}^{n+1}(\sigma)$ is naturally identified with $\widetilde{P}^n(\sigma_j)$. The composition $m_j = m \circ inc_j$,

$$\sigma_j \xrightarrow{inc_j} \sigma \xrightarrow{m} \rho,$$

is a morphism in $D^n$. So we have the map

$$\widetilde{P}^n(m_j) : \widetilde{P}^n(\sigma_j) \to \widetilde{P}^{n+1}(\rho).$$

The collection of these maps for $0 \leq j \leq n+1$ then naturally induces the map $\partial$.

We then define $\widetilde{P}^{n+1}(m)$ using the identifications (8.5), (8.6) as follows. Set $\widetilde{P}^{n+1}(m)$ to be the map $\partial$ on

$$\widetilde{P}^{n+1}(\sigma)|_{\partial\Delta^{n+1}} = p^{-1}(\partial\Delta^{n+1}).$$

The hypothesis $S(n)$ part 3 insures that $\widetilde{P}^{n+1}(m)$ has an extension to a map $\widetilde{P}^{n+1}(\sigma) \to \widetilde{P}^{n+1}(\rho)$.

As $\widetilde{P}^{n+1}$ must extend $\widetilde{P}^n$, combined with the construction above, we have thus specified the functor $\widetilde{P}^{n+1}$ on a generating set of morphisms in $D^{n+1}$, which defines $\widetilde{P}^{n+1}$ completely. For example, given a degeneracy $m : \sigma^n \to \rho^{n-2}$, we may factorize it as $\sigma^n \xrightarrow{m'} \rho^{n-1} \xrightarrow{pr} \rho^{n-2}$, and then set:

$$\widetilde{P}^{n+1}(m) = (\widetilde{P}^{n+1}(pr) = \widetilde{P}^n(pr)) \circ \widetilde{P}^{n+1}(m').$$

Functoriality of $\widetilde{P}^n$, implies that this is well defined. And by construction $\widetilde{P}^{n+1}$ will be a functor satisfying compatibility. So we are done.

□

**Theorem 8.6.** *Let $X$ be a smooth simplicial set. $\mathcal{U}$-small simplicial $G$-bundles $P \to X$ are "classified by" smooth maps*

$$f_P : X \to BG^{\mathcal{U}}.$$

*Specifically:*

(1) *For every $\mathcal{U}$-small $P$ there is a natural smooth map $f_P : X \to BG^{\mathcal{U}}$ so that*

$$f_P^* EG^{\mathcal{U}} = P$$

*as simplicial $G$-bundles. We say in this case that $f_P$* **classifies** *$P$.*

(2) *If $P_1, P_2$ are isomorphic $\mathcal{U}$-small smooth simplicial $G$-bundles over $X$ then the classifying maps $f_{P_1}, f_{P_2}$ are smoothly homotopic, as in Definition 3.14.*

(3) *If $X = Y_\bullet$ for $Y$ a smooth manifold and $f, g : X \to BG^{\mathcal{U}}$ are smoothly homotopic then $P_f = f^* EG^{\mathcal{U}}, P_g = g^* EG^{\mathcal{U}}$ are isomorphic simplicial $G$-bundles.*

Note that the above is a partly stronger and partly weaker than just saying that isomorphism classes of $\mathcal{U}$-small bundles over $X$ are in correspondence with smooth homotopy classes of maps $X \to BG^{\mathcal{U}}$. It is strictly stronger when $X = Y_\bullet$ for $Y$ a smooth manifold.

*Proof.* Set $V = BG^{\mathcal{U}}$, $E = EG^{\mathcal{U}}$. Let $P \to X$ be a $\mathcal{U}$-small simplicial $G$-bundle. Define $f_P : X \to V$ by:

$$f_P(\Sigma) = \Sigma_*^* P, \tag{8.7}$$

where $\Sigma \in \Delta^d(X)$, $\Sigma_* : \Delta^d_\bullet \to X$, the induced map, and the pull-back $\Sigma_*^* P$ our usual simplicial $G$-bundle pull-back. We check that the map $f_P$ is simplicial.

Let $m : k \to d$ be a morphism in $\Delta$. We need to check that the following diagram commutes:

$$\begin{array}{ccc} X(d) & \xrightarrow{X(m)} & X(k) \\ \downarrow f_P & & \downarrow f_P \\ V(d) & \xrightarrow{V(m)} & V(k). \end{array}$$

Let $\Sigma \in X(d)$, then by push-forward functoriality Axiom 2b $(X(m)(\Sigma))_* = \Sigma_* \circ m_\bullet$ where $m_\bullet : \Delta^k_\bullet \to \Delta^d_\bullet$ is the simplicial map induced by $m : \Delta^k \to \Delta^m$. And so

$$f_P(X(m)(\Sigma)) = (\Sigma_* \circ m_\bullet)^* P = m_\bullet^*(\Sigma_*^* P) = V(m)(f_P(\Sigma)),$$

where the second equality uses Lemma 5.12. Hence the diagram commutes.

We now check that $f_P$ is smooth. Let $\Sigma \in X(d)$, then we have:

$$\begin{aligned} (f_P(\Sigma))_*(\sigma) &= \sigma_\bullet^*(\Sigma_*^* P) \\ &= (\Sigma_* \circ \sigma_\bullet)^* P, \quad \text{Lemma 5.12} \\ &= (\Sigma_*(\sigma))_*^* P, \quad \text{as } \Sigma_* \text{ is smooth, Lemma 3.6} \\ &= f_P(\Sigma_*(\sigma)), \quad \text{by (8.7)} \\ &= (f_P \circ \Sigma_*)(\sigma), \end{aligned}$$

and so $f_P$ is smooth.

We check that $f_P^* E = P$. Let $\Sigma : \Delta^d_\bullet \to X$ be smooth, and $\sigma \in \Delta^d_\bullet$. First, we need the identity:

$$\begin{aligned} \Delta^{sm} f_P(\Sigma)(\sigma) = (f_P \circ \Sigma)(\sigma) = f_P(\Sigma(\sigma)) &= (\Sigma(\sigma))_*^* P \text{ by definition of } f_P \\ &= (\Sigma^* \circ \sigma_\bullet)^* P \text{ as } \Sigma \text{ is smooth} \\ &= \sigma_\bullet^*(\Sigma^* P) \text{ Lemma 5.12} \\ &= g(\Sigma^* P)(\sigma). \end{aligned} \tag{8.8}$$

So

$$\Delta^{sm} f_P(\Sigma) = g(\Sigma^* P). \tag{8.9}$$

Then

$$\begin{aligned} f_P^* E(\Sigma) = (E \circ \Delta^{sm} f_P)(\Sigma) &= E(g(\Sigma^* P)), \text{ by (8.8)} \\ &= (\Sigma^* P)(id^d_\bullet), \text{ definition of } E \\ &= P(\Sigma). \end{aligned}$$

So $f_P^* E = P$ on objects.

Now let $m$ be a morphism:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "\Sigma_1"] & \Delta^d_\bullet \arrow[d, "\Sigma_2"] \\ & X, \end{tikzcd}$$

in $\Delta^{sm}(X)$. We then have, for $e_m$ is as in the definition of $E$:

$$\begin{aligned} f_P^* E(m) &= E(\Delta^{sm} f_P(m)) \\ &= (\Delta^{sm} f_P(\Sigma_2))^b(e_m) \text{ by definition of } E \\ &= \Sigma_2^* P(e_m) \text{ by (8.9)} \\ &= (P \circ \Delta^{sm} \Sigma_2)(e_m). \end{aligned}$$

But $\Delta^{sm}\Sigma_2(e_m)$ is the diagram:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, bend right, "\Sigma_2 \circ \widetilde{m}_\bullet"'] & \Delta^d_\bullet \arrow[d, "\Sigma_2 \circ id^d_\bullet"] \\ & X, \end{tikzcd}$$

i.e. it is the diagram $m$. So $(P \circ \Delta^{sm}\Sigma_2)(e_m) = P(m)$. Thus, $f_P^* E = P$ on morphisms.

So we have proved the first part. We now prove the second part. Suppose that $\phi : P_1 \to P_2$ is an isomorphism of $\mathcal{U}$-small simplicial $G$-bundles over $X$. We construct a $\mathcal{U}$-small simplicial $G$-bundle $\widetilde{P}$ over $X \times I$ as follows, where $I = \Delta^1_\bullet$ as before.

Let $\sigma$ be a $k$-simplex of $X$. Then $\phi$ specifies a $G$-bundle diffeomorphism $\phi_\sigma : P_1(\sigma) \to P_2(\sigma)$ over the identity map $\Delta^k \to \Delta^k$. Let $M_\sigma$ be the mapping cylinder of $\phi_\sigma$. So that

$$M_\sigma = (P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma))/\sim, \tag{8.10}$$

for $\sim$ the equivalence relation generated by the condition

$$(x, 1) \in P_1(\sigma) \times \Delta^1 \sim \phi(x) \in P_2(\sigma).$$

Then $M_\sigma$ is a smooth $G$-bundle over $\Delta^k \times \Delta^1$.

Let $pr_X, pr_I$ be the natural projections of $X \times I$, to $X$ respectively $I$. Let $\Sigma$ be a $d$-simplex of $X \times I$, for any $d$. Set $\sigma_1 = pr_X \Sigma$, and $\sigma_2 = pr_I(\Sigma)$. Let $id^d : \Delta^d \to \Delta^d$ be the identity, so

$$(id^d, \sigma_2) : \Delta^d \to \Delta^d \times \Delta^1,$$

is a smooth map, where $\sigma_2$ is the corresponding smooth map $\sigma_2 : \Delta^d \to \Delta^1 = [0,1]$. We then define

$$\widetilde{P}_\Sigma := (id^d, \sigma_2)^* M_{\sigma_1},$$

which is a smooth $G$-bundle over $\Delta^d$.

Notice that if $\Sigma$ is in $X \times 0_\bullet \subset X \times I$, then we do *not* have $\widetilde{P}(\Sigma) = P_1(\Sigma)$, instead there is a natural isomorphism. This is for the same reason that fixing the standard construction of the set theoretic pull-back, a bundle $P \to B$ is not set theoretically equal to the bundle $id^* P \to B$, for $id : B \to B$ the identity, (but they are of course naturally isomorphic.) However, we can adjust the construction of $\widetilde{P}_\Sigma$ so that $\widetilde{P}(\Sigma) = P_1(\Sigma)$ does hold, similarly to the inductive procedure in the Proposition 8.4. In what follows, we ignore this minor ambiguity.

Suppose that $\rho : \sigma \to \sigma'$ is a morphism in $\Delta^{sm}(X)$, for $\sigma$ a $k$-simplex and $\sigma'$ a $d$-simplex. As $\phi$ is a simplicial $G$-bundle map, we have a commutative diagram:

$$\begin{array}{ccc} P_1(\sigma) & \xrightarrow{P_1(\rho)} & P_1(\sigma') \\ \big\downarrow{\scriptstyle \phi_\sigma} & & \big\downarrow{\scriptstyle \phi_{\sigma'}} \\ P_2(\sigma) & \xrightarrow{P_2(\rho)} & P_2(\sigma'). \end{array} \tag{8.11}$$

And so we get a naturally induced (by the pair of maps $P_1(\rho), P_2(\rho)$) bundle map:

$$\begin{array}{ccc} M_\sigma & \xrightarrow{\quad g_\rho \quad} & M_{\sigma'} \\ \big\downarrow & & \big\downarrow \\ \Delta^k \times \Delta^1 & \xrightarrow{\widetilde{\rho} \times id} & \Delta^d \times \Delta^1 . \end{array} \tag{8.12}$$

More explicitly, let $q_\sigma : P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma) \to M_\sigma$ denote the quotient map. Define

$$\widetilde{g}_\rho : P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma) \to M_{\sigma'}$$

by:

$$\widetilde{g}(x,t) = q_{\sigma'}((P_1(\rho)(x), t)) \in M_{\sigma'},$$

for

$$(x,t) \in P_1(\sigma) \times \Delta^1,$$

while $\widetilde{g}_\rho(y) = q_{\sigma'}(P_2(\rho)(y))$ for $y \in P_2(\sigma)$. By commutativity of (8.11) $\widetilde{g}_\rho$ induces the map $g_\rho : M_\sigma \to M_{\sigma'}$, appearing in (8.12).

Now suppose we have a morphism $m : \Sigma \to \Sigma'$ in $\Delta^{sm}(X \times I)$, where $\Sigma$ is a $k$-simplex and $\Sigma'$ is a $d$-simplex. Then we have a commutative diagram:

$$\begin{tikzcd}
M_\sigma \arrow[r, "g_{pr_X}(m)"] \arrow[d] & M_{\sigma'} \arrow[d] \\
\Delta^k \times \Delta^1 \arrow[r, "\widetilde{m} \times id"] & \Delta^d \times \Delta^1 \\
\Delta^k \arrow[r, "\widetilde{m}"] \arrow[u, "h_1"] & \Delta^d \arrow[u, "h_2"'] \\
\widetilde{P}_\Sigma \arrow[u] \arrow[uuu, bend left=60] & \widetilde{P}_{\Sigma'} \arrow[u] \arrow[uuu, bend right=60]
\end{tikzcd} \tag{8.13}$$

where $h_1 = (id^k, pr_I(\Sigma))$ and $h_2 = (id^d, pr_I(\Sigma'))$. We then readily get an induced natural bundle map:

$$\widetilde{P}(m) : \widetilde{P}_\Sigma \to \widetilde{P}_{\Sigma'},$$

as left most and right most arrows in the above commutative diagram are the natural maps in pull-back squares, and so by universality of the pull-back such a map exists and is uniquely determined. Of course $\widetilde{P}(m)$ is the unique map making the whole diagram (8.13) commute.

With the above assignments, it is immediate that $\widetilde{P}$ is indeed a functor, by the uniqueness of the assignment $\widetilde{P}(m)$. And this determines our $\mathcal{U}$-small smooth simplicial $G$-bundle $\widetilde{P} \to X \times I$. By the first part of the theorem, we have an induced smooth classifying map $f_{\widetilde{P}} : X \times I \to V$. By construction, it is a homotopy between $f_{P_1}, f_{P_2}$. So we have verified the second part of the theorem.

We now prove the third part of the theorem. Let $X = Y_\bullet$. Suppose that $f, g : X \to V$ are smoothly homotopic, and let $H : X \times I \to V$ be the corresponding smooth homotopy. Now $P_H = H^* E$ is a simplicial $G$-bundle over $X \times I = (Y \times [0,1])_\bullet$ and hence by Theorem 5.10 $P_H$ is induced by a smooth $G$-bundle $P'_H$ over $Y \times [0,1]$.

Now by construction

$$P_f \simeq (P'_H|_{Y \times \{0\}})^\Delta$$

and

$$P_g \simeq (P'_H|_{Y \times \{1\}})^\Delta.$$

And

$$P'_H|_{Y \times \{0\}} \simeq P'_H|_{Y \times \{1\}}$$

by standard smooth bundle theory and hence

$$(P'_H|_{Y \times \{0\}})^\Delta \simeq (P'_H|_{Y \times \{1\}})^\Delta.$$

And so $P_f \simeq P_g$.

$\square$

We now study the dependence on a Grothendieck universe $\mathcal{U}$.

**Theorem 8.7.** *Let $G$ be a locally convex Lie group. Let $\mathcal{U}$ be a $G$-admissible universe, let $|BG^{\mathcal{U}}|$ denote the geometric realization of $BG^{\mathcal{U}}$ and let $BG^{top}$ denote the classifying space of $G$ as defined by the Milnor construction* [28]*. Then there is a weak homotopy equivalence*

$$e^{\mathcal{U}} : |BG^{\mathcal{U}}| \to BG^{top},$$

*which is natural in the sense that if $\mathcal{U} \in \mathcal{U}'$ then*

$$[e^{\mathcal{U}'} \circ |i^{\mathcal{U},\mathcal{U}'}|] = [e^{\mathcal{U}}], \tag{8.14}$$

*where $|i^{\mathcal{U},\mathcal{U}'}| : |BG^{\mathcal{U}}| \to |BG^{\mathcal{U}'}|$ is the map of geometric realizations, induced by the natural inclusion $i^{\mathcal{U},\mathcal{U}'} : BG^{\mathcal{U}} \to BG^{\mathcal{U}'}$ and where $[\cdot]$ denotes the homotopy class. In particular, if $G$ has the homotopy type of a CW complex, then for all $\mathcal{U}$ $BG^{\mathcal{U}}$ has the homotopy type of $BG^{top}$.*

*Proof.* Set $V := BG^{\mathcal{U}}$, and $E := EG^{\mathcal{U}}$. For latter use, we also denote by $v_0 \in V(0)$ the 0-simplex corresponding to the classifying map of the trivial $G$-bundle $G \times \Delta^0_\bullet \to \Delta^0_\bullet$. In other words, $E(v_0)$ is the bundle $G \to \Delta^0$.

**Lemma 8.8.** *Let $P \to X$ be a simplicial $G$-bundle. Set $|P| = \operatorname{colim}_{\Delta(X)} P$, where the colimit is understood to be in the category of topological $G$-bundles, and recalling that $P$ is a functor $\Delta^{sm}(X) \to \mathcal{G}$, and so restricts to a functor $\Delta(X) \to \mathcal{G}$. Then there is a natural topological $G$-bundle $|P| \to |X|$. This is called the* **geometric realization of** *$P$.*

*Proof.* Set $P' = \mathcal{F}_1 \circ P$, where $\mathcal{F}_1$ is as in Definition 5.3. So there is a natural transformation $N : P \to P'$ of $Top$ valued functors. By naturality of the colimit there is an induced map:

$$\operatorname{colim}_{\Delta(X)} P \to \operatorname{colim}_{\Delta(X)} P',$$

i.e. a continuous map $|P| \to |X|$.

We check that the map $|p|$ is a locally trivial fibration. By a topological $d$-simplex $\Sigma : \Delta^d \to |X|$ we shall mean the natural map $\Delta^d \to |X|$ corresponding to a $d$-simplex of $X$, (in the colimit diagram). If $v \in |X|$ is a vertex, i.e. the image of a topological 0-simplex $\Delta^0 \to |X|$, we construct a contractible open neighborhood $U_v \ni v$ as follows.

Let $S_d$ be the set of the topological $d$-simplices containing $v$. For $\Sigma \in S_d$, let $\overline{\Sigma} = \operatorname{image} \Sigma - \operatorname{image} \Sigma_v$, where $\Sigma_v : \Delta^{d-1} \to |X|$ is the face of $\Sigma$ not containing $v$. Then define:

$$U_v = \bigcup_d \bigcup_{\Sigma \in S_d} \overline{\Sigma}.$$

Over each $\overline{\Sigma}$ the bundle $|p|$ is obviously a trivializeable topological $G$-bundle. Moreover, any trivialization over the star of $v$ (the collection of the remaining faces of $\overline{\Sigma}$) may be extended to a trivialization over $\overline{\Sigma}$.

It is then a simple inductive argument to construct a topological $G$-bundle trivialization:

$$U_v \times G \to X.$$

Now, it clear that $\{U_v\}$ form an open cover of $|X|$. So we are done.

□

By the lemma we have a topological $G$-fibration
$$|E| \to |V|.$$
Then
$$|E| \simeq e^* EG^{top}, \tag{8.15}$$
where

- $EG^{top}$ is the universal $G$-bundle over $BG^{top}$.
- $$e = e^{\mathcal{U}} : |V| \to BG^{top}$$ is uniquely determined up to homotopy.
- $\simeq$ here and the rest of this argument will mean $G$-bundle isomorphism or simplicial $G$-bundle isomorphism, depending on context.

We say that $e$ ***classifies*** $|E| \to |V|$. We will show that $e$ induces an isomorphism of all homotopy groups.

We first prove an auxiliary lemma. Let $\mathcal{U}'$ be a universe enlargement of $\mathcal{U}$, that is $\mathcal{U}'$ is a universe with $\mathcal{U} \in \mathcal{U}'$. There is a natural inclusion map
$$i = i^{\mathcal{U},\mathcal{U}'} : BG^{\mathcal{U}} \to BG^{\mathcal{U}'},$$
and
$$i^* EG^{\mathcal{U}'} = E\,[2].$$

**Lemma 8.9.** *Let $G$ be a locally convex Lie group, then*
$$i_* : \pi_k^{sm}(BG^{\mathcal{U}}) \to \pi_k^{sm}(BG^{\mathcal{U}'})$$
*is a set isomorphism for all $k \in \mathbb{N}$, where $\pi_k^{sm}$ are the free homotopy groups as in Definition 3.15.*

*Proof.* We show that $i_*$ is injective. Let's abbreviate $V = BG^{\mathcal{U}}$, $V' = BG^{\mathcal{U}'}$, $E = EG^{\mathcal{U}}$, $E' = EG^{\mathcal{U}'}$. Let $f, g : S^k_\bullet \to V$ be a pair of smooth maps. Let $P_f, P_g$ denote the smooth bundles over $S^k$ induced via Lemma 5.10 by $f^*E, g^*E$. Set $f' = i \circ f$, $g' = i \circ g$ and suppose that $f', g'$ are smoothly homotopic. By Part 3 of Theorem 8.6, $P_{f'}, P_{g'}$ are isomorphic and so $P_f$ and $P_g$ are isomorphic simplicial $G$-bundles.

Then by Part 2 of Theorem 8.6, $f, g$ are smoothly homotopic, which proves injectivity.

We now show surjectivity of $i_*$. Let $f : S^k_\bullet \to V'$ be smooth. By Lemma 5.10 the simplicial $G$-bundle $f^*E'$ is induced by a smooth $G$-bundle $P' \to S^k$. Recall that any smooth bundle over $S^k$ is obtained by the clutching construction corresponding

[2] This is indeed an equality, not just a natural isomorphism.

to some smooth map $\phi : S^{k-1} \to G$. Specifically, $P'$ is isomorphic as a smooth $G$-bundle to the bundle:

$$C = D^k_- \times G \sqcup D^k_+ \times G / \sim,$$

where:

(1) $D^k_+, D^k_-$ are two copies of the standard closed $k$-ball in $\mathbb{R}^k$.

(2) $\sim$ is the equivalence relation generated by the relation: for $(d, g) \in D^k_- \times G$,

$$(d, g) \sim \widetilde{\phi}(d, g) \in D^k_+ \times G,$$

where

$$\widetilde{\phi} : \partial D^k_- \times G \to \partial D^k_+ \times G, \text{ is the map } \widetilde{\phi}(d, x) = (d, \phi(d)^{-1} \cdot x).$$

This gluing construction can be carried out in $\mathcal{L}$, for any $G$-admissible Grothendieck universe $\mathcal{L}$. In particular, $C$ is $\mathcal{U}$-small.

Let

$$C^\Delta \to S^k_\bullet$$

denote the induced $\mathcal{U}$-small smooth simplicial $G$-bundle. Now $C^\Delta$ and $f^*E'$ are induced by isomorphic $\mathcal{U}'$-small smooth $G$-bundles, hence are isomorphic $\mathcal{U}'$-small simplicial $G$-bundles.

By Part 2 of Theorem 8.6, the classifying map $f_{C^\Delta} : S^k_\bullet \to V'$ is smoothly homotopic to $f$. Since $C^\Delta$ is $\mathcal{U}$-small, it is also classified by a smooth map $f' : S^k_\bullet \to V$. It is immediate that $[i \circ f'] = [f_{C^\Delta}]$, since $i^*E' = E$. And so $i_*([f']) = [f]$. □

**Corollary 8.10.** *Let $G$ be a locally convex Lie group, and let $\mathcal{U}, \mathcal{U}'$ be as in the previous lemma. Let $\mathcal{P}^{\mathcal{U}}$, $\mathcal{P}^{\mathcal{U}'}$ denote the set of isomorphism classes of $\mathcal{U}$-small, respectively $\mathcal{U}'$-small simplicial $G$-bundles $P$ over $S^k_\bullet$. Then the natural map*

$$j : \mathcal{P}^{\mathcal{U}} \to \mathcal{P}^{\mathcal{U}'},$$

*is a set bijection.*

*Proof.* It is immediate from Theorem 8.6 that for any admissible Grothendieck universe $G$ the map $c : \pi^{sm}_k(BG^{\mathcal{U}}) \to \mathcal{P}^U$, $c([f]) = [f^*EG^{\mathcal{U}}]$ is a bijection. The corollary then readily follows by the lemma.

□

We now return to the proof of the theorem, and specifically to the proof of surjectivity of $e$.

Let $f : S^k \to BG^{top}$ be a continuous map. By Müller-Wockel [29], main result, the bundle $P_f := f^*EG^{top}$ is topologically isomorphic to a smooth $G$-bundle $P' \to S^k$. By the axiom of universes $P'$ is $\mathcal{U}_0$-small for some $G$-admissible $\mathcal{U}_0 \ni \mathcal{U}$. So we obtain a $\mathcal{U}_0$-small simplicial $G$-bundle $(P')^\Delta \to S^k_\bullet$.

By Corollary 8.10 $[(P')^\Delta] = j([g^*E])$ for some smooth

$$g : S^k_\bullet \to V.$$

Then $|(P')^{\Delta}| \to |S^k_\bullet|$ is a topological $G$-bundle classified by $e \circ |g|$, for

$$|g| : |S^k_\bullet| \to |V|,$$

the naturally induced topological map.

By construction, there is a topological $G$-bundle map $u : |(P')^{\Delta}| \to P'$, over the natural map $|S^k_\bullet| \to S^k$ as $P'$ is a co-cone for the corresponding colimit diagram in $\mathcal{G}$. Let

$$h : S^k \to |S^k_\bullet|$$

represents the generator of $\pi_k(|S^k_\bullet|)$, where the notation $\pi_k(Y)$ means the set of free homotopy classes of maps $S^k \to Y$. Then we get a diagram of $G$-bundle maps:

$$\begin{array}{ccccc} h^*|(P')^{\Delta}| & \longrightarrow & |(P')^{\Delta}| & \longrightarrow & P' \\ \downarrow & & \downarrow & & \downarrow \\ S^k & \xrightarrow{\quad h \quad} & |S^k_\bullet| & \xrightarrow{\quad u \quad} & S^k. \end{array}$$

Now, $u \circ h \simeq id$, and so $P'$ and hence $P_f$, as a topological $G$-bundle is isomorphic to $h^*|(P')^{\Delta}|$.

Also, we have a diagram of $G$-bundle maps:

$$\begin{array}{ccccc} h^*|(P')^{\Delta}| & \longrightarrow & |(P')^{\Delta}| & \longrightarrow & EG^{top} \\ \downarrow & & \downarrow & & \downarrow \\ S^k & \xrightarrow{\quad h \quad} & |S^k_\bullet| & \xrightarrow{e \circ |g|} & BG^{top}. \end{array}$$

Thus, $e \circ |g| \circ h$ represents the free homotopy class $[f]$ and so $e_* : \pi_k(V) \to \pi_k(BG^{top})$ is surjective.

We prove injectivity. Let $f_0, f_1 : S^k \to |V|$ be continuous. Let $P_j \to S^k$ be smooth $G$-bundles topologically isomorphic to $f_j^*|E|$, $j = 0, 1$. Again $P_j$ exists by the main result of [29].

By Corollary 8.10, $P_j^{\Delta}$ are classified by smooth maps:

$$g_j : S^k_\bullet \to V.$$

Similarly, to the proof of surjectivity, we may represent the classes $[f_j]$, by $|g_j| \circ h$ for $h : S^k \to |S^k_\bullet|$ as above.

Now suppose that $[e \circ f_0] = [e \circ f_1]$. Then by [29] $P_j$ are smoothly isomorphic $G$-bundles. Thus, $P_j^{\Delta}$ are isomorphic and so by Part 2 of Theorem 8.6 $g_j$ are smoothly homotopic. Consequently, $|g_j|$ are homotopic and so $[f_0] = [f_1]$.

It follows that:

$$\forall k \in \mathbb{N} : e_* : \pi_k(|V|) \to \pi_k(BG^{top}) \tag{8.16}$$

is a set isomorphism. If $G$ is connected, then $BG^{top}$ is simply connected. An argument similar to the proof of Proposition 8.4 shows that $V$ is a simply connected Kan complex, and so $|V|$ is simply connected. In this case, it is elementary topology that

$$\forall k \in \mathbb{N} : e_* : \pi_k(|V|, v_0) \to \pi_k(BG^{top}, e(v_0)) \tag{8.17}$$

is a group isomorphism. And so we may conclude that $e$ is a homotopy equivalence.

For a more general $G$, we must directly show the existence of group isomorphisms $\forall k \in \mathbb{N} : e_* : \pi_k(V, v_0) \to \pi_k(BG^{top}, e(v_0))$. However, this can be done by an argument analogous to proof of (8.16), by working with simplicial $G$-bundles $P$ over $S^k_\bullet$, s.t. $P(s_0)$ is equal to the bundle $G \to \Delta^0$, for some fixed $s_0 \in S^k_\bullet(0)$. In common terminology, we need a fixed trivialization of our bundles over $s_0$. And we need to extend Theorem 8.6 to the setting of bundles with a fixed trivialization over a point. To keep the length of the paper manageable we will not elaborate on this elementary extension.

Finally, we show naturality. Let

$$|i^{\mathcal{U},\mathcal{U}'}| : |V| \to |V'|$$

denote the map induced by the inclusion $i^{\mathcal{U},\mathcal{U}'}$. Since $E = (i^{\mathcal{U},\mathcal{U}'})^* E'$, we have that

$$|E| \simeq |i^{\mathcal{U},\mathcal{U}'}|^* |E'|$$

and so

$$|E| \simeq |i^{\mathcal{U},\mathcal{U}'}|^* \circ (e^{\mathcal{U}'})^* EG^{top},$$

by (8.15), from which the conclusion immediately follows. And we are done with the proof of the theorem. □

**Corollary 8.11.** *Let $G$ be a Lie group modeled on an LF space.*

## 9. The universal Chern-Weil homomorphism

In this section $G$ is a generalized Lie group and $\mathfrak{g}$ its lie algebra. Pick any simplicial $G$-connection $\mathcal{A}$ on $EG^{\mathcal{U}} \to BG^{\mathcal{U}}$. Then by the discussion of Section 7 we obtain a $dg$ homomorphism well defined up to homotopy:

$$cw^{EG^{\mathcal{U}}} : \mathbb{R}[\mathfrak{g}]^G \to \Omega^\bullet(BG^{\mathcal{U}}, \mathbb{R}). \tag{9.1}$$

This satisfies the following naturality:

**Proposition 9.1.** *Let $\mathcal{U}$ be a $G$-admissible Grothendieck universe. Let $P \to X$ be a $\mathcal{U}$-small simplicial $G$-bundle and let*

$$cw^P : \mathbb{R}[\mathfrak{g}]^G \to \Omega^\bullet(X, \mathbb{R}),$$

*be a representative of the homotopy class $[cw^{P,D}]$. Then*

$$f_P^* \circ cw^{EG^{\mathcal{U}}} \simeq cw^P,$$

*where $f_P : X \to BG^{\mathcal{U}}$ is the classifying map.*

*Proof.* This follows immediately from Lemma 7.3. □

*Proof of Theorem 1.1.* □

Let $e^{\mathcal{U}}$ be as in Theorem 8.7, then this is a weak equivalence, by Corollary 8.11. And so induces an isomorphism

$$e^{\mathcal{U}}_* : H^\bullet(|BG^{\mathcal{U}}|, \mathbb{R}) \to H^\bullet(BG^{top}, \mathbb{R}),$$

Hatcher [11, Proposition 4.21].

9.1. **Universal Chern-Weil classes.** Then we define the cohomology class

$$c^\rho := e_*^{\mathcal{U}}(|c^{\rho,\mathcal{U}}|) \in H^{2k}(BG^{top}, \mathbb{R}),$$

where the $G$-admissible universe $\mathcal{U}$ is chosen arbitrarily and where

$$|c^{\rho,\mathcal{U}}| \in H^{2k}(|BG^{\mathcal{U}}|, \mathbb{R})$$

is as in Notation 4.7.

**Lemma 9.2.** *The cohomology class $c^\rho$ is well-defined.*

*Proof.* Given another choice of a $G$-admissible universe $\mathcal{U}'$, let $\mathcal{U}'' \supset \{\mathcal{U}, \mathcal{U}'\}$ be a common universe enlargement. By Lemma 7.3 and Lemma 4.8

$$|i^{\mathcal{U},\mathcal{U}''}|^*(|c^{\rho,\mathcal{U}''}|) = |c^{\rho,\mathcal{U}}|.$$

Now $|i^{\mathcal{U},\mathcal{U}''}|$ is a weak equivalence by Theorem 8.7. Let

$$|i^{\mathcal{U},\mathcal{U}''}|^* : H^\bullet(|BG^{\mathcal{U}'}|, \mathbb{R}) \to H^\bullet(|BG^{\mathcal{U}}|, \mathbb{R})$$

be the corresponding algebra isomorphism, and let $|i^{\mathcal{U},\mathcal{U}''}|_*$ denote its inverse.

Then we have:

$$|i^{\mathcal{U},\mathcal{U}''}|_*(|c^{\rho,\mathcal{U}}|) = |c^{\rho,\mathcal{U}''}|. \tag{9.2}$$

Consequently,

$$\begin{aligned} e_*^{\mathcal{U}}(|c^{\rho,\mathcal{U}}|) &= e_*^{\mathcal{U}''} \circ |i^{\mathcal{U},\mathcal{U}''}|_*(|c^{\rho,\mathcal{U}})|, \text{ by the naturality part of Theorem 8.7} \\ &= e_*^{\mathcal{U}''}(|c^{\rho,\mathcal{U}''}|), \text{ by (9.2)}. \end{aligned}$$

In the same way we have:

$$e_*^{\mathcal{U}'}(|c^{\rho,\mathcal{U}'}|) = e_*^{\mathcal{U}''}(|c^{\rho,\mathcal{U}''}|).$$

So

$$e_*^{\mathcal{U}}(|c^{\rho,\mathcal{U}}|) = e_*^{\mathcal{U}'}(|c^{\rho,\mathcal{U}'}|),$$

and so we are done.

□

We call $c^\rho \in H^{2k}(BG^{top}, \mathbb{R})$ ***the universal Chern-Weil characteristic class associated to*** $\rho$.

9.2. **Universal cohomological Chern-Weil homomorphism.** Let

$$hcw : \mathbb{R}[\mathfrak{g}]^G \to H^*(BG^{top}, \mathbb{R}),$$

be the algebra map sending $\rho$ to $c^\rho$ as above. Then to summarize, we have the following theorem purely about the classical classifying space $BG^{top}$, reformulating Theorem 1.1 of the introduction:

**Theorem 9.3.** *Let $G$ be a generalized Lie group. The homomorphism cw satisfies the following naturality property. Let $G \hookrightarrow Z \to Y$ be a smooth principal $G$-bundle. Let $c^\rho(Z) \in H^{2k}(Y)$ denote the standard Chern-Weil class associated to $\rho$. Then*

$$f_Z^* hcw(\rho) = c^\rho(Z),$$

*where $f_Z : Y \to BG^{top}$ is the classifying map of the underlying topological $G$-bundle.*

*Proof.* Let $\mathcal{U}_0 \ni Z$ be a $G$-admissible Grothendieck universe. By Proposition 9.1

$$c^\rho(Z^\Delta) = f^*_{Z^\Delta}(c^{\rho,\mathcal{U}_0}).$$

And by Proposition 7.4, $|c^\rho(Z^\Delta)|_{sm} = c^\rho(Z)$. So we have

$$\begin{aligned} c^\rho(Z) &= |c^\rho(Z^\Delta)|_{sm} \\ &= |f^*_{Z^\Delta}(c^{\rho,\mathcal{U}_0})|_{sm} \\ &= N^*(|f^*_{Z^\Delta} c^{\rho,\mathcal{U}_0}|), \text{ Part 2 of Notation 4.7} \\ &= N^* \circ |f_{Z^\Delta}|^*(|c^{\rho,\mathcal{U}_0}|), \text{ by Lemma 4.8} \\ &= N^* \circ |f_{Z^\Delta}|^* \circ (e^{\mathcal{U}_0})^* c^\rho, \text{ by definition of } c^\rho. \end{aligned}$$

Now, we have a diagram of topological $G$-bundle maps:

$$\begin{tikzcd} {|Z^\Delta|} \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & EG \arrow[d] \\ {|Y_\bullet|} \arrow[r, "h"] & Y \arrow[r, "f_Z"] & BG^{top}, \end{tikzcd}$$

for $h$ as in (4.7). And so $e^{\mathcal{U}} \circ f_{|Z^\Delta|}$ being the classifying map for $|Z^\Delta| \to |Y_\bullet|$, is homotopic to $f_Z \circ h$.

Thus, $e^{\mathcal{U}_0} \circ |f_{Z^\Delta}| \circ N$ is homotopic to $f_Z$. So that

$$c^\rho(Z) = f_Z^* c^\rho = f_Z^* cw(\rho),$$

and we are done. □

## 9.3. **Universal $dg$ Chern-Weil homomorphism.**

*Proof of Theorem 1.2.* We set $cw = cw^{EG^{\mathcal{U}}}$, where the right hand side is as in equation 9.1. Let $P \to Y$ be a smooth, $\mathcal{U}$-small $G$-bundle over a smooth manifold, and set $X = Y_\bullet$. Let $P^\Delta \to X$ denote the induced $\mathcal{U}$-small simplicial $G$-bundle, and $f_{P^\Delta}$ its classifying map. We need to show that $cw$ satisfies the naturality condition:

$$\Theta \circ cw^P \simeq f^*_{P^\Delta} \circ cw.$$

Now, by Proposition 9.1 we have:

$$f^*_{P^\Delta} \circ cw \simeq cw^{P^\Delta}.$$

And by Part 1 of Proposition 7.4 $cw^{P^\Delta} \simeq \Theta \circ cw^P$. And so we are done. □

# 10. Universal Chern-Weil theory for the group of Hamiltonian symplectomorphisms

Let $(M, \omega)$ be a possibly non-compact symplectic manifold of dimension $2n$, so that $\omega$ is a closed non-degenerate 2-form on $M$. Let $\mathcal{H} = \mathrm{Ham}(M, \omega)$ denote the group of its compactly supported Hamiltonian symplectomorphisms, and $\mathfrak{h}$ its Lie algebra. When $M$ is simply connected this is the group $Symp(M, \omega)$ of (compactly supported) diffeomorphisms $\phi : M \to M$ s.t. $\phi^*\omega = \omega$.

For example, take $M = \mathbb{CP}^{n-1}$ with its Fubini-Study symplectic 2-form $\omega_{st}$. Then the natural action of $PU(n)$ on $\mathbb{CP}^{n-1}$ is by Hamiltonian symplectomorphisms.

In [33] Reznikov constructs polynomials

$$\{r_k\}_{k \geq 1} \subset \mathbb{R}[\mathfrak{h}]^{\mathcal{G}},$$

each $r_k$ homogeneous of degree $k$. These polynomials come from the $k$-multilinear functionals: $\mathfrak{h}^{\oplus k} \to \mathbb{R}$,

$$(H_1, \ldots, H_k) \mapsto \int_M H_1 \cdot \ldots \cdot H_k \, \omega^n,$$

upon identifying:

$$\mathfrak{h} = \begin{cases} C_0^\infty(M), & \text{if } M \text{ is compact} \\ C_c^\infty(M), & \text{if } M \text{ is non-compact,} \end{cases}$$

where $C_0^\infty(M)$ denotes the set of smooth functions $H$ satisfying $\int_M H \, \omega^n = 0$. And where $C_c^\infty(M)$ denotes the set of smooth, compactly supported functions. In the case $k = 1$, the associated class vanishes whenever $M$ is compact.

When $M$ is compact the group $\mathcal{H}$ is a Fréchet Lie group having the homotopy type of a countable CW complex by the discussion in the preamble of Section 5. Otherwise, it is a group locally modeled on a nuclear LF space, [26], in particular a generalized Lie group.

Thus, Theorem 9.3 implies the Corollary 1.3 of the introduction, and in particular we get induced Reznikov cohomology classes

$$c^{r_k} \in H^{2k}(B\mathcal{H}, \mathbb{R}). \tag{10.1}$$

As mentioned, the group $PU(n)$ naturally acts on $\mathbb{CP}^{n-1}$ by Hamiltonian symplectomorphisms. So we have an induced map

$$i : BPU(n) \to B \operatorname{Ham}(\mathbb{CP}^{n-1}, \omega_0).$$

Then as one application we prove Theorem 1.5 of the introduction, reformulated as follows:

**Theorem 10.1.** *[Originally Kedra-McDuff [16]]*

$$i^* : H^k(B \operatorname{Ham}(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}) \to H^k(BPU(n), \mathbb{R})$$

*is surjective for all $n \geq 2$, $k \geq 0$ and so*

$$i_* : H_k(BPU(n), \mathbb{R}) \to H_k(B \operatorname{Ham}(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}),$$

*is injective for all $n \geq 2$, $k \geq 0$.*

*Proof.* Let $\mathfrak{g}$ denote the Lie algebra of $PU(n)$, and $\mathfrak{h}$ the Lie algebra of $\operatorname{Ham}(\mathbb{CP}^{n-1}, \omega_0)$. Let $j : \mathfrak{g} \to \mathfrak{h}$ denote the natural Lie algebra map induced by the homomorphism $PU(n) \to \operatorname{Ham}(\mathbb{CP}^{n-1}, \omega_0)$. Reznikov [33] shows that $\{j^* r_k\}_{k>1}$ are the Chern polynomials. In other words, the classes

$$c^{j^* r_k} \in H^{2k}(BPU(n), \mathbb{R}),$$

are the Chern classes $\{c_k\}_{k>1}$, which generate real cohomology of $BPU(n)$, as is well known. But $c^{j^* r_k} = i^* c^{r_k}$, for $c^{r_k}$ as in (10.1), and so the result immediately follows. $\square$

In Kedra-McDuff [16] a proof of the above is given via homotopical techniques. Theirs is a difficult argument, but their technique, as they show, is also partially applicable to study certain generalized, homotopical analogues of the group $\mathcal{H}$. Our argument is elementary, but does not obviously have homotopical ramifications as in [16].

In Savelyev-Shelukhin [37] there are a number of results about induced maps in (twisted) $K$-theory. These further suggest that the map $i$ above should be a monomorphism in the homotopy category. For a start we may ask:

**Question 10.2.** *Is the map $i$ above an injection on integral homology?*

For this one may need more advanced techniques like [36].

10.1. **Beyond $\mathbb{CP}^n$.** Theorem 10.1 extends to completely general compact semi-simple Lie groups $G$, with $\mathbb{CP}^n$ replaced by co-adjoint orbits $M$ of $G$. We just need to compute the pullbacks to $\mathfrak{g}$ of the associated Reznikov polynomials in $\mathbb{R}[\mathfrak{h}]^{\mathcal{G}}$. We can no longer expect injection in general. But the failure to be injective should be solely due to effects of classical representation theory. In other words it is an algebraic problem.

## 11. Universal coupling class for Hamiltonian fibrations

Although we use here some language of symplectic geometry no special expertise should be necessary. As the construction here is a partial reformulation of our general constructions, for the special case of $G = \mathcal{H} = \mathrm{Ham}(M, \omega)$, we will not give exhaustive details.

Let $(M, \omega)$ and $\mathcal{H}$ be as in the previous section, (keeping in mind our $M$ is not assumed to be compact) and let $2n$ be the dimension of $M$.

**Definition 11.1.** *A* **Hamiltonian $M$-fibration** *is a smooth fiber bundle $M \hookrightarrow P \to X$, with structure group $\mathcal{H}$.*

Each $\mathcal{H}$-connection $\mathcal{A}$ on such $P$ uniquely induces a *coupling 2-form* on $P$, as originally appearing in [9]. Specifically, this is a closed 2-form $C_{\mathcal{A}}$ on $P$ whose restriction to fibers coincides with $\omega$ and which has the following property. Let $\omega_{\mathcal{A}} \in \Omega^2(X)$ denote the 2-form defined by:

$$\omega_{\mathcal{A}}(v, w) = n \int_{P_x} R_{\mathcal{A}}(v, w)\, \omega_x^n,$$

for $v, w \in T_x X$. Here $R_{\mathcal{A}}$ as before is the curvature 2-form of $\mathcal{A}$, so that

$$R_{\mathcal{A}}(v, w) \in \mathrm{lie}\, \mathrm{Ham}(M_x, \omega_x) = \begin{cases} C_0^\infty(P_x), & \text{if } M \text{ is compact} \\ C_c^\infty(P_x) & \text{if } M \text{ is non-compact.} \end{cases}$$

Note of course that $\omega_{\mathcal{A}} = 0$ when $M$ is compact. The characterizing property of $C_{\mathcal{A}}$ is then:

$$\int_M C_{\mathcal{A}}^{n+1} = \omega_{\mathcal{A}},$$

where the left-hand side is integration along the fiber. [3]

It can then be shown that the cohomology class $\mathfrak{c}(P)$ of $C_{\mathcal{A}}$ is uniquely determined by $P$ up to $\mathcal{H}$-bundle isomorphism. This is called the ***coupling class of*** $P$, and it has important applications in symplectic geometry. See for instance [25] for more details and some applications.

By replacing the category $\mathcal{G}$ with other fiber bundle categories we may define other kinds of simplicial fibrations over a smooth simplicial set. For example, we may replace $\mathcal{G}$ by the category of smooth Hamiltonian $M$-fibrations, keeping the other axioms in the Definition 5.3 intact. This then gives us the notion of a Hamiltonian simplicial $M$-bundle over a smooth simplicial set.

Let $\mathcal{U}$ be a $\mathcal{H}$-admissible Grothendieck universe. Let $M^{\mathcal{U},\mathcal{H}}$ denote the Hamiltonian simplicial $M$-fibration, naturally associated to $E\mathcal{H}^{\mathcal{U}} \to B\mathcal{H}^{\mathcal{U}}$. So that for each $k$-simplex $\Sigma \in B\mathcal{H}^{\mathcal{U}}$ we have a Hamiltonian $M$-fibration $M^{\mathcal{U},\mathcal{H}}_{\Sigma} \to \Delta^k$, which is the associated $M$-bundle to the principal $\mathcal{H}$-bundle $E\mathcal{H}^{\mathcal{U}}_{\Sigma}$.

Fix a (simplicial) $\mathcal{H}$-connection $\mathcal{A}$ on the universal $\mathcal{H}$-bundle $E\mathcal{H}^{\mathcal{U}} \to B\mathcal{H}^{\mathcal{U}}$. This induces a (simplicial) connection with the same name $\mathcal{A}$ on $M^{\mathcal{U},\mathcal{H}}$.

By the discussion above, for each $k$-simplex $\Sigma \in B\mathcal{H}^{\mathcal{U}}$ we have the associated coupling 2-form $C_{\mathcal{A},\Sigma}$ on the Hamiltonian $M$-bundle $M^{\mathcal{U},\mathcal{H}}_{\Sigma} \to \Delta^k$. The collection of these 2-forms then readily induces a cohomology class $\mathfrak{c}^{\mathcal{U}}$ on the geometric realization:

$$|M^{\mathcal{U},\mathcal{H}}| = \operatorname{colim}_{\Sigma \in \Delta(B\mathcal{H}^{\mathcal{U}})} M^{\mathcal{U},\mathcal{H}}_{\Sigma}.$$

This is similar to the discussion in Section 4.4.

Now by the proof of Theorem 8.7 we have an $\mathcal{H}$-structure preserving, $M$-bundle map over the homotopy equivalence $e^{\mathcal{U}}$:

$$g^{\mathcal{U}} : |M^{\mathcal{U},\mathcal{H}}| \to M^{\mathcal{H}},$$

where $M^{\mathcal{H}}$ denotes the universal Hamiltonian $M$-fibration over $B\mathcal{H}$. And these $g^{\mathcal{U}}$ are natural, so that if $\mathcal{U} \ni \mathcal{U}'$ then

$$[g^{\mathcal{U}'} \circ |\widetilde{i}^{\mathcal{U},\mathcal{U}'}|] = [g^{\mathcal{U}}], \tag{11.1}$$

where $|\widetilde{i}^{\mathcal{U},\mathcal{U}'}| : |M^{\mathcal{U},\mathcal{H}}| \to |M^{\mathcal{U},\mathcal{H}'}|$ is the natural $M$-bundle map over $i^{\mathcal{U},\mathcal{U}'}$ (as in Theorem 8.7 ), and where $[\cdot]$ denotes the homotopy class.

Each $g^{\mathcal{U}}$ is a homotopy equivalence, so we may set

$$\mathfrak{c} := g^{\mathcal{U}}_*(\mathfrak{c}^{\mathcal{U}}) \in H^2(M^{\mathcal{H}}).$$

**Lemma 11.2.** *The class $\mathfrak{c}$ is well-defined, (independent of the choice $\mathcal{U}$).*

The proof is analogous to the proof of Lemma 9.2. Given this definition of the universal coupling class $\mathfrak{c}$, the proof of Theorem 1.4 is analogous to the proof of Theorem 9.3.

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[3] $C_{\mathcal{A}}$ is not generally compactly supported but $C^{n+1}_{\mathcal{A}}$ is, which is a consequence of taking $\mathcal{H}$ to be compactly supported Hamiltonian symplectomorphisms.

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