A REMARK ON DEFORMATION OF GROMOV NON-SQUEEZING

YASHA SAVELYEV

Abstract. We prove that in dimension 4 the Gromov non-squeezing phenomenon is persistent with respect to C^0 symplectic perturbations of the symplectic form on the range. Given such a perturbation, we can no longer compactify the range and hence the classical Gromov argument breaks down. Our methods consist of a certain trap idea for holomorphic curves, analogous to traps in dynamical systems. The theorem motivates intriguing further questions on further deforming non-squeezing to general nearby forms, or just existence of higher dimensional symplectic deformation.

1. Introduction

One of the most important to this day results in symplectic geometry is the so called Gromov non-squeezing theorem, appearing in the seminal paper of Gromov [3]. Let $\omega_{st} = \sum_{i=1}^n dp_i \wedge dq_i$ denote the standard symplectic form on $\mathbb{R}^{2n}$. Gromov's theorem then says that there does not exist a symplectic embedding

$$(B_R, \omega_{st}) \hookrightarrow (S^2 \times \mathbb{R}^{2n-2}, \omega_{\pi r^2} \oplus \omega_{st}),$$

for $R > r$, with B_R the standard closed radius R ball in $\mathbb{R}^{2n}$ centered at 0, and $\omega_{\pi r^2}$ a symplectic form on S^2 with area πr^2.

We show that in dimension 4 Gromov's non-squeezing is C^0 persistent in the following sense.

Theorem 1.1. *Let $R > r > 0$ be given, and let $\omega = \omega_{\pi r^2} \oplus \omega_{st}$ be the symplectic form on $M = S^2 \times \mathbb{R}^2$ as above. Then there is an $\epsilon > 0$ s.t. for any symplectic form ω' on M, C^0 ϵ-close to ω, there is no symplectic embedding $\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega')$, meaning that $\phi^* \omega' = \omega_{st}$.*

To prove this, we cannot use the classical Gromov-Witten argument since we cannot compactify the range. Another idea is needed to get an appropriate compact moduli space of holomorphic curves. For the moment we are forced to restrict to dimension 4, where we construct a certain holomorphic trap (Definition 2.1) somewhat analogous to traps in dynamical systems.

It is natural to ask if the above theorem continues to hold for general nearby forms. Or formally this translates to:

Question 1. Let $R > r > 0$ be given, and let $\omega = \omega_{\pi r^2} \oplus \omega_{st}$ be the symplectic form on $M = S^2 \times \mathbb{R}^{2n-2}$, as above. For every $\epsilon > 0$ is there a 2-form ω' on M, C^0 ϵ-close to ω, such that there is a symplectic embedding $\phi : B_R \hookrightarrow M$, i.e. s.t. $\phi^* \omega' = \omega_{st}$? Note that if ω' as above exists, it cannot be globally closed (for ϵ sufficiently small) by the theorem above.

We cannot readily reduce this question to just applying Theorem 1.1 (in dimension 4). This is because, while a symplectic form on a subdomain of the form $\phi(B_R) \subset M$ extends to a symplectic form, by a classical theorem of Gromov [2], the extension may not be C^0 close to ω. Indeed, this appears to be rather unlikely to happen, unless there exists a stronger type of h-principle than Gromov's which allows us to control the uniform norm.

The above question seems to be difficult. My opinion is that the answer is 'yes', in part because it is difficult to imagine any obstruction, for example we no longer have Gromov-Witten theory for general ω'. On the other hand, my attempts to construct an example failed, so that 'no' is certainly very possible.

2000 *Mathematics Subject Classification.* 53D45.
Key words and phrases. non-squeezing, Gromov-Witten theory, virtual fundamental class, polyfolds.

A work of Müller [7] explores a different kind of question, by instead relaxing the condition of the map being symplectic. This is a very different idea, and there is no direct connection to our problem, as pull-backs by diffeomorphisms of nearby forms may not be nearby. Hence, there is no way to go from nearby embeddings that we work with to ϵ-symplectic embeddings of Müller.

2. A trap for holomorphic curves

For basic notions of J-holomorphic curves we refer the reader to [5].

Definition 2.1. *Let (M, J) be an almost complex manifold, and $A \in H_2(M)$ fixed. Let $K \subset M$ be a closed subset. Suppose that for every $x \in \partial K$ (the topological boundary) there is a J-holomorphic, real codimension 2, compact submanifold of M: $H_x \ni x$ satisfying:*

- *$H_x \subset K$.*
- *$A \cdot H_x \leq 0$, where the left-hand side is the homological intersection number.*

We call such a K a **J-holomorphic trap** *(for class A curves).*

Lemma 2.2. *Let M, J and A be as above, and K be a J-holomorphic trap for class A curves. Let $u : \Sigma \to M$ be a J-holomorphic class A curve u with Σ a connected closed Riemann surface. Then*

$$(\operatorname{image} u \cap K) \neq \emptyset \implies \operatorname{image} u \subset K.$$

Proof. Suppose that u intersects ∂K, otherwise we already have $\operatorname{image} u \subset \mathit{interior}(K)$, since $\operatorname{image} u$ is connected (and by elementary topology). Then u intersects H_x as in the definition of a holomorphic trap, for some x. Consequently, as $A \cdot H_x \leq 0$, by positivity of intersections [5, Section 2.6], $\operatorname{image} u \subset H_x \subset K$. □

3. Proof of Theorem 1.1

Definition 3.1. *We say that a pair (ω, J) of a 2-form ω on a smooth manifold M and an almost complex structure J on M are* **compatible** *if $\omega(\cdot, J\cdot)$ defines a J-invariant inner product $g_{\omega,J}$ on M.*

Let us quickly recall the definition of the C^0 distance d_{C^0}, on the set of 2-forms $\Omega^2(M)$ for a fixed metric g on M.

$$d_{C^0}(\omega_0, \omega_1) = \sup_{|v \wedge w|_g = 1} |\omega_0(v, w) - \omega_1(v, w)|,$$

where more specifically, the supremum is over all g-norm 1 simple bivectors $v \wedge w$ in $\Lambda^2(TM)$.

Let ω be the symplectic form on $M = S^2 \times \mathbb{R}^2$ as in the statement. In our case d_{C^0} will be defined with respect to the metric $g_{\omega,J}$ as in Definition 3.1 for J the standard product complex structure.

Suppose by contradiction that for every $\epsilon > 0$ there is an ω_1 s.t. $d_{C^0}(\omega, \omega_1) < \epsilon$ and such that there exists a symplectic embedding

$$\phi : B_R \hookrightarrow (M, \omega_1).$$

Let $\epsilon' > 0$ be s.t. any symplectic form ω_1 on M, C^0 ϵ'-close to ω satisfies:

- $\omega_t = (1 - t)\omega + t\omega_1$ is non-degenerate, for each $t \in [0, 1]$.
- For each $t \in [0, 1]$, ω_t is non-degenerate on all the fibers of the natural projection $p : (M = S^2 \times \mathbb{R}^2) \to \mathbb{R}^2$. In what follows we just call them ***fibers***.

For $\epsilon < \epsilon'$ as above, let ω_1 and $\phi : B_R \to (M, \omega_1)$ be as in our hypothesis. Set $B := \phi(B_R)$ and let $D \supset B$ be an open domain, with compact closure K, s.t. K is the product $S^2 \times D^2$ for $D^2 \subset \mathbb{R}^2$ a (standard) closed disk. In particular, ∂K is smoothly folliated by the fibers. We denote by $T^{vert}\partial K \subset TM$, the sub-bundle of vectors tangent to the leaves of the above-mentioned foliation.

We may extend $\phi_* j$ to an ω_1-compatible almost complex structure J_1 on M, preserving $T^{vert}\partial K$ using:

- $\operatorname{image} \phi$ does not intersect ∂K.
- The non-degeneracy of ω_1 on the fibers, which follows by the defining condition of ϵ.
- The well known existence/flexibility results for compatible almost complex structures on symplectic vector bundles.

We may then extend J_1 to a smooth (in the Frechet manifold of all almost complex structures) family $\{J_t\}$, $t \in [0,1]$, of almost complex structures on M, s.t. J_t is ω_t-compatible for each t, with $J_0 = J$ as above, and such that J_t preserves $T^{vert}\partial K$ for each t. The latter condition can be satisfied by similar reasoning as above, using that ω_t is non-degenerate on the fibers for each t.

So the fibers above are J_t-holomorphic hypersurfaces for each t, and smoothly foliate ∂K. Moreover, if $A = [S^2] \otimes [pt]$ is as in the statement, then the intersection number of A with a fiber is 0. That is $A \cdot p^{-1}(z) = 0$, for $\forall z \in \mathbb{R}^2$. And so K is a compact J_t-holomorphic trap for class A curves, for each t.

Set $x_0 := \phi(0)$. Denote by $\mathcal{M}_t$ the space of equivalence classes of maps $u : \mathbb{CP}^1 \to M$, where u is a J_t-holomorphic, class A curve passing through x_0. The equivalence relation is by the usual biholomorphism reparametrization group action, so that $u \sim u'$ if there exists a biholomorphism $\phi : \mathbb{CP}^1 \to \mathbb{CP}^1$ s.t. $u' = u \circ \phi$. Then $\mathcal{M} = \cup_t \mathcal{M}_t$ is compact by energy minimality of A (which rules out bubbling), by Lemma 2.2, and by compactness of K.

We need to regularize. We may use the "standard" Banach approach. This has the advantage of being readily understood by experts but a possible disadvantage of being opaque to new-comers to the field, and being rather ad-hock and elaborate. For this reason we will also give an independent argument using polyfold theory.

3.1. **Banach approach.** This is based on [5] and the picture is as follows. There is an evaluation map $ev : \mathcal{B} \to M$, $u \mapsto u(0)$, where $\mathcal{B}$ is the universal Banach moduli space of class A curves.

$$\mathcal{B} = \mathcal{M}^*(A, \mathcal{J}^l) := \{(u, J) \,|\, J \in \mathcal{J}^l, u : \mathbb{CP}^1 \to M \text{ is a simple class } A \text{ } J\text{-holomorphic curve}\},$$

where $\mathcal{J}^l$ is the space of class C^l almost complex structures, taking l to be sufficiently large. There is a natural projection $\pi : \mathcal{B} \to \mathcal{J}^l$. The product map

$$\mathcal{B} \xrightarrow{ev \times \pi} M \times \mathcal{J}^l$$

is a Fredholm map. There is one immediate problem: given $(x_0, J) \in M \times \mathcal{J}^l$ we may not be able to perturb it to a regular value of the form (x_0, J') (that is we may need to perturb x). This would break the last step of the proof of the theorem, which needs specifically a holomorphic curve through x_0. But we can fix this with an approximation argument, taking a sequence of regular (x^k, J^k) converging to (x_0, J).

Similarly, let $\{J_t\}$, $t \in [0,1]$, be the family as constructed above. We may approximate, (with respect to the compact open topology on $M \times \mathcal{J}^l$) the path $t \mapsto (x_0, J_t)$, in $M \times \mathcal{J}^l$ by a sequence of paths $p^k(t) = (x^k(t), J^k(t))$ so that:

- ev is transverse to each p^k.
- $J^k(t)$ is $\{\omega_t\}$-compatible for each t.
- $J^k(t)$ preserves $T^{vert}\partial K$ for each t.
- $J^k(0) = J$ for each k, $x^k(t) = x_0$ for each k, t.

Then $\mathcal{M}^k = ev^{-1}(p^k)/\sim$ is a compact one dimensional manifold for each k, where $\sim$ is the equivalence relation, corresponding to the reparametrization action of the group of biholomorphisms of $\mathbb{CP}^1$ fixing 0. The boundary component $ev^{-1}(x_0, J)/\sim$ is a point corresponding to the single J-holomorphic, class A curve passing through x_0. It follows, that the boundary component $ev^{-1}(x^k(1), J^k(1))/\sim$ is likewise non-empty, and so by standard compactness that $\mathcal{M}_1$ has some element u_0.

3.2. **Polyfold approach.** Alternatively, we may use Hofer-Wysocki-Zehnder polyfold regularization in Gromov-Witten theory, especially as recently worked out in this present context by the team of Franziska Beckschulte, Ipsita Datta, Irene Seifert, Anna-Maria Vocke, and Katrin Wehrheim. We can also of course use other virtual approaches, but this is not instantaneous, for example if we were to invoke [6] then we would have needed to adapt construction of the implicit atlases to the constrained case (this can be done of course).

As explained in [1, Section 3.5], in a essentially identical situation, we may embed $\mathcal{M}$ into a natural polyfold setup of Hofer-Wysocki-Zehnder [4]. More to the point, we express $\mathcal{M}$ as the zero set of an sc-Fredholm section of a suitable (tame, strong) M-polyfold bundle. The only difference with the setup of [1, Section 3.5] is that they compactify M to $S^2 \times T^2$, to get a compact moduli space. We of course cannot compactify, but remember that we used the holomorphic trap idea to force compactness of $\mathcal{M}$. And so we are in an equivalent situation.

Again as in [1], we take the M-polyfold regularization of $\mathcal{M}$. This gives a one dimensional compact cobordism $\mathcal{M}^{reg}$ between $\mathcal{M}_0^{reg}$ and $\mathcal{M}_1^{reg}$.

Now $\mathcal{M}_0^{reg}$ is a point: corresponding to the unique $(J = J_0)$-holomorphic class A, curve $u : \mathbb{CP}^1 \to M$ passing through x_0. Consequently, $\mathcal{M}_1^{reg}$ is non-empty, that is there is a J_1-holomorphic class A curve $u_0 : \mathbb{CP}^1 \to M$ passing through x_0.

3.3. **Finishing the proof.** Now we have:

$$|\langle \omega_1, A\rangle - \pi \cdot r^2| = |\langle \omega_1, A\rangle - \langle \omega, A\rangle| \leq \epsilon\pi \cdot r^2,$$

as $\langle \omega, A\rangle = \pi r^2$, and as $d_{C^0}(\omega, \omega_1) < \epsilon$, (also using that we can find a representative for A whose g-area is πr^2). So choosing ϵ appropriately we get

$$|\int_{\mathbb{CP}^1} u_0^* \omega_1 - \pi r^2| < \pi R^2 - \pi r^2,$$

And consequently,

$$\int_{\mathbb{CP}^1} u_0^* \omega_1 < \pi R^2.$$

We may then proceed exactly as in the now classical proof of Gromov [3] of the non-squeezing theorem to get a contradiction and finish the proof. A bit more specifically, $\phi^{-1}(\mathrm{image}\,\phi \cap \mathrm{image}\,u_0)$ is a minimal surface in B_R, with boundary on the boundary of B_R, and passing through $0 \in B_R$. By construction it has area strictly less than πR^2, which is impossible by the classical monotonicity theorem of differential geometry. See also [1] where the monotonicity theorem is suitably generalized, to better fit the present context. $\square$

4. Acknowledgements

I am grateful to Helmut Hofer, Boulen Touson, and Simon for an interesting discussion as well as Felix Schlenk, Misha Gromov and Dusa McDuff for some feedback.

References

[1] F. Beckschulte, I. Datta, I. Seifert, A.-M. Vocke, and K. Wehrheim, *A polyfold proof of Gromov's non-squeezing theorem*, in Research directions in symplectic and contact geometry and topology. Selected papers based on the presentations at the 2019 women in symplectic and contact geometry and topology workshop, WiSCon, Providence, RI, USA, July 22–26, 2019, Cham: Springer, 2021, pp. 1–45.

[2] M. Gromov, *Stable mappings of foliations into manifolds*, Math. USSR, Izv., 3 (1971), pp. 671–694.

[3] ———, *Pseudo holomorphic curves in symplectic manifolds.*, Invent. Math., 82 (1985), pp. 307–347.

[4] H. Hofer, K. Wysocki, and E. Zehnder, *Applications of polyfold theory. I: The polyfolds of Gromov-Witten theory*, vol. 1179 of Mem. Am. Math. Soc., Providence, RI: American Mathematical Society (AMS), 2017.

[5] D. McDuff and D. Salamon, *J–holomorphic curves and symplectic topology*, no. 52 in American Math. Society Colloquium Publ., Amer. Math. Soc., 2004.

[6] J. Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves.*, Geom. Topol., 20 (2016), pp. 779–1034.

[7] Stefan Müller, *Epsilon-non-squeezing and C_0-rigidity of epsilon-symplectic embeddings*, arXiv:1805.01390, (2018).

Email address: yasha.savelyev@gmail.com

University of Colima, CUICBAS