EXTENDED FULLER INDEX, SKY CATASTROPHES AND THE SEIFERT CONJECTURE

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Abstract. We extend the classical Fuller index, to prove what may be considered to be one "corrected" version of the original Seifert conjecture, on the existence of periodic orbits for vector fields on S^{2k+1}. We also give some extensions of this theorem for more general smooth manifolds. As one corollary we get that for a given smooth non-singular vector field X_1 on S^{2n+1} with no periodic orbits, and any homotopy $\{X_t\}$ of smooth non-singular vector fields, with X_0 the Hopf vector field, $\{X_t\}$ has a sky catastrophe, which is a kind of bifurcation originally discovered by Fuller.

1. Introduction

The original Seifert conjecture [?] asked if a non-singular vector field on S^3 must have a periodic orbit. In this formulation the answer was shown to be no for C^1 vector fields by Schweitzer [?], and later for C^∞ vector fields by Kuperberg [?], a C^1 volume preserving counter-example is given by Kuperberg in [?]. For a vector field C^0 close to the Hopf vector field it was shown to hold by Seifert and later by Fuller [?] using his Fuller index. On the other hand Viterbo [?] shows that Reeb vector fields for the standard contact structure on S^{2k+1} always have periodic orbits, in the case of S^3 for an overtwisted contact structure this is shown by Hofer [?], using pseudo-holomorphic curve techniques. In the context of the Seifert conjecture, it is interesting to understand the most basic properties of qualitative-dynamical, or even just topological character, that Reeb vector fields posses, which makes them different from general non-singular vector fields. As stated such an undertaking may be overly ambitious as the Reeb property is too deep geometrically. One initial attempt may be to work relative to a reference Reeb vector field X_0, and in this manner we obtain the following.

Given a homotopy of smooth vector fields $\{X_t\}$ on M we define the space of periodic orbits of $\{X_t\}$:

$$S = S(\{X_t\}) = \{(o, p, t) \in LM \times (0, \infty) \times [0, 1] \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX_t\}. \quad (1.1)$$

Here LM denotes the free loop space. We have an embedding

$$emb : S \hookrightarrow M \times (0, \infty) \times [0, 1]$$

given by $(o, p, t) \mapsto (o(0), p, t)$ and S is given the corresponding subspace topology. Further on the same kind of topology will be used on related spaces. (It is the same as the induced topology from compact-open or Frechet topology on LM.)

Theorem 1.2. *Let $X = X_1$ be a smooth non-singular vector field on S^{2k+1} homotopic to the Hopf vector field $H = X_0$ through homotopy $\{X_t\}$ of smooth non-singular vector fields. Suppose that each point of*

$$S \cap \left(LS^{2k+1} \times (0, \infty) \times \{0\}\right)$$

is contained in a compact, open (in S) subset of S. Then X has periodic orbits.

We point out that a subset of S is compact if and only if it is closed and its projection to the component $(0, \infty)$ is bounded. This of course is immediate by existence of the topological embedding emb above. Let us call a homotopy $\{X_t\}$ satisfying the conditions of the theorem above *partially admissible.*

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Lemma 1.3. *There exists a $\delta > 0$ so that whenever X is δ C^0 close to the Hopf vector field H, $X = X_1$ for a partially admissible homotopy $\{X_t\}$, with $X_0 = H$. And in particular by the above X has periodic orbits.*

Thus Theorem 1.2 may be understood as an extensive generalization of the theorem of Seifert giving existence of periodic orbits for non-singular vector fields C^0 close to the Hopf vector field, on which the Seifert conjecture was based.

We shall say that $\{X_t\}$ has a *sky catastrophe*, if there is an element

$$y \in S \cap \left(LS^{2k+1} \times (0, \infty) \times \{0\}\right)$$

so that there is no open compact subset of S containing y. A sky catastrophe is the last discovered kind of bifurcation originally constructed by Fuller [?]. He constructs a smooth family $\{X_\tau\}, \tau \in [0,1]$ of vector fields on a solid torus, for which there is a continuous (and isolated) family of $\{X_\tau\}$ periodic orbits $\{o_\tau\}$, with the period of o_τ going to infinity as $\tau \mapsto 1$, and so that for $\tau = 1$ the orbit disappears. Clearly this is a special case of the above definition. These bifurcations turned out to be common in many kinds of systems appearing in nature and have been extensively studied, see for instance Shilnikov-Turaev [?]. The following is the contrapositive of the theorem above.

Corollary 1.4. *Given any homotopy $\{X_t\}$ of smooth non-singular vector fields from the Hopf vector field to a vector field with no periodic orbits, $\{X_t\}$ has a sky catastrophe.*

Note that by the main construction in Wilson [?], the Hopf vector field H is homotopic through smooth non-singular vector fields to a vector field with finitely many simple closed orbits. Combining this with the construction in [?] we find that there do exist homotopies of H through smooth non-singular vector fields to a vector field with no closed orbits. Since the property of being non-singular and of having no closed orbits is C^0 open we immediately obtain:

Corollary 1.5. *There exists an open subset $\mathcal{C}$ of the space of homotopies $\{X_t\}$, with X_0 the Hopf vector field, endowed with C^0 topology, such that every $\{X_t\} \in \mathcal{C}$ has a sky catastrophe.*

Thus not only do sky catastrophes exist as Fuller shows but they are a "regular" phenomenon. test new again spank

More general and extended forms of the above theorem are stated in Section 3. To prove them we give a certain natural extension of the classical Fuller index, with the latter giving certain invariant rational counts of periodic orbits of a smooth vector field in "dynamically isolated compact sets". Our extension is $\mathbb{Q} \sqcup \{\pm\infty\}$ valued. This involves some new ideas. First we introduce a notion of perturbation system for a vector field, which will allow us to consider weighted in terms of index and multiplicity "infinite sums" of closed orbits of a vector field. For these sums to have any meaning we impose certain "positivity" or "negativity" conditions. The situation is somewhat like in "positive topological quantum field theories " where infinite sums which are normally meaningless with coefficients in a ring like $\mathbb{C}$ are made meaningful by working with a complete semi-ring in the sense of Samuel Eilenberg.

Can we use Theorem 1.2 and its analogues for more general manifolds in Section 3 to show existence of orbits? Let us assume some minimal regularity on the homotopy $\{X_t\}$ so that connected components of $S(\{X_t\})$ are open. Then ideally we would like to have some a priori upper bounds for the period on connected components coming from some geometry-topology of the manifold and or the vector field. This then means that connected components of $S(\{X_t\})$ would be open and compact, and we can apply our theorems to get existence results. Fuller himself in [?] gives an example of such bounds for vector fields on the 2-torus, in fact his bounds are absolute for the whole S (for a fixed homotopy class of orbits) not just its connected components. He also speculates that if one works with divergence free vector fields one can do more to this effect. Below we examine the case of Reeb vector fields.

1.1. **Reeb vector fields and sky catastrophes.** It is natural to ask in light of Hofer's and Viterbo's results if a homotopy of Reeb vector fields $\{X_t\}$ on S^{2n+1} is free of sky catastrophes. The following theorem puts a very strong restriction on the kinds of sky catastrophes that can happen, for general contact manifolds. (It is fair to say, that if they exist, they must be pathological, and likely very hard to construct). The proof uses only elementary geometry of the Reeb vector fields.

Theorem 1.6. *Let $\{X_t\}$, $t \in [0,1]$ be a smooth homotopy through Reeb vector fields on a contact manifold M. Let $S = S(\{X_t\})$ be defined as before, then there is no unbounded locally Lipschitz continuous $p : [0,\infty) \to S$ whose composition with the projection $\pi_3 : LM \times (0,\infty) \times [0,1] \to [0,1]$ has finite length.*

2. Fuller index and its extension

The Fuller index is an analogue for orbits of the fixed point index, but with a couple of new ingredients: we must account for the symmetry groups of the orbits, and since the period is freely varying there is an extra compactness issue to deal with. Let us briefly recall the definition following Fuller's original paper [**?**]. All vector vector fields from now on, everywhere in the paper, will be assumed to be smooth and non-singular and manifolds smooth and oriented (the last for simplicity).

Let X be a vector field on M. Set

$$S(X) = S(X,\beta) = \{(o,p) \in L_\beta M \times (0,\infty) \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\},$$

where $L_\beta M$ denotes the free homotopy class β component of the free loop space. Elements of $S(X)$ will be called orbits. There is a natural S^1 reparametrization action on $S(X)$, and elements of $S(X)/S^1$ will be called *unparametrized orbits*, or just orbits. Slightly abusing notation we write (o,p) for the equivalence class of (o,p). The multiplicity $m(o,p)$ of a periodic orbit is the ratio p/l for $l > 0$ the least period of o. We want a kind of fixed point index which counts orbits (o,p) with certain weights - however in general to get invariance we must have period bounds. This is due to potential existence of sky catastrophes as described in the introduction.

Let $N \subset S(X)$ be a compact open set (the open condition is the meaning in this case of "dynamically isolated" from before). Assume for simplicity that elements $(o,p) \in N$ are isolated. (Otherwise we need to perturb.) Then to such an (N,X,β) Fuller associates an index:

$$i(N,X,\beta) = \sum_{(o,p)\in N/S^1} \frac{1}{m(o,p)} i(o,p),$$

where $i(o,p)$ is the fixed point index of the time p return map of the flow of X with respect to a local surface of section in M transverse to the image of o. Fuller then shows that $i(N,X_t,\beta)$ is invariant for a deformation $\{X_t\}$ of X if N is dynamically isolated for all t, that is if $N \times [0,1]$ is open in $S(\{X_t\})$. (He actually gives a slightly stronger invariance property.)

In the case where X is the R^λ-Reeb vector field on a contact manifold (C^{2n+1},ξ), and if (o,p) is non-degenerate, we have:

$$i(o,p) = \operatorname{sign}\operatorname{Det}(\mathrm{Id}|_{\xi(x)} - F^\lambda_{p,*}|_{\xi(x)}) = (-1)^{CZ(o)-n}, \tag{2.1}$$

where $F^\lambda_{p,*}$ is the differential at x of the time p flow map of R^λ, and where $CZ(o)$ is the Conley-Zehnder index, (which is a special kind of Maslov index) see [**?**].

2.1. Extending Fuller index.

We assume from now on, everywhere in the paper, that M is compact, smooth, and oriented, although we sometimes reiterate for clarity. We will now describe an extension of the Fuller index allowing us to work with the entire Fuller phase space $LM \times \mathbb{R}_+$. We define an index $i(X,\beta)$ that depends only on X,β but which is a priori defined and is invariant only for certain special vector fields and homotopies of vector fields. In fact we have already given the essence of the necessary condition on the homotopy in the special case of Theorem 1.2. Let

$$S(\{X_t\},\beta) = \{(o,p,t) \in L_\beta M \times (0,\infty) \times [0,1] \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX_t\}.$$

Definition 2.2. *For a homotopy $\{X_t\}$ of smooth non-singular vector fields on M, we say that it is* **partially admissible in free homotopy class** *β, if every element of*

$$S(\{X_t\},\beta) \cap (L_\beta M \times (0,\infty) \times \{0\})$$

is contained in a compact open subset of $S(\{X_t\},\beta)$. We say that $\{X_t\}$ is **admissible in free homotopy class** *β if every element of*

$$S(\{X_t\},\beta) \cap (L_\beta M \times (0,\infty) \times \partial[0,1])$$

is contained in a compact open subset of $S(\{X_t\}, \beta)$.

For X a vector field, we set

$$\begin{aligned} S(X) &= \{(o,p) \in LM \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\}. \\ S(X,\beta) &= \{(o,p) \in L_\beta M \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\}. \\ S(X,a,\beta) &= \{(o,p) \in S(X,\beta) \mid p \leq a\}. \end{aligned} \tag{2.3}$$

Definition 2.4. *Suppose that* $S(X,\beta)$ *has open connected components. And suppose that we have a collection of vector fields* $\{X^a\}$, *for each* $a > 0$, *satisfying the following:*

- $S(X^a, a, \beta)$ *consists of isolated orbits for each* a.
- $S(X^a, a, \beta) = S(X^b, a, \beta)$, *(equality of subsets of* $L_\beta M \times \mathbb{R}_+$*) if* $b > a$, *and the index and multiplicity of the orbits corresponding to the identified elements of these sets coincide.*
- *There is a prescribed homotopy* $\{X^a_t\}$ *of each* X^a *to* X, *called* **structure homotopy**, *with the property that for every* $y \in S(\{X^a_t\}) \cap (L_\beta M \times (0,a] \times \partial[0,1])$ *there is an open compact subset* $\mathcal{C}_y$ *of* $S(\{X^a_t\})$ *which is* **non-branching** *which means that* $\mathcal{C}_y \cap (L_\beta M \times (0,\infty) \times \{i\})$, $i = 0,1$ *are connected.*
- $$S(\{X^a_t\}, \beta) \cap (L_\beta M \times (0,a] \times [0,1]) = S(\{X^b_t\}, \beta) \cap (L_\beta M \times (0,a] \times [0,1]),$$
  (equality of subsets of $L_\beta M \times \mathbb{R}_+ \times [0,1]$*) if* $b > a$ *is sufficiently large.*

We will then say that $\{X^a\}$ *is a* **perturbation system** *for* X *in the class* β, *(keeping track of structure homotopies and of* β *implicitly).*

We shall see shortly that a Morse-Bott Reeb vector field always admits a perturbation system.

Definition 2.5. *Suppose that* X *admits a perturbation system* $\{X^a\}$ *so that there exists an* $E = E(\{X^a\})$ *with the property that* $S(X^a, a, \beta) = S(X^E, a, \beta)$ *for all* $a > E$, *where this as before is equality of subsets of* $L_\beta M \times \mathbb{R}_+$, *and the index and multiplicity of identified elements are also identified. Then we say that* X *is* **finite type** *and set:*

$$i(X,\beta) = \sum_{(o,p) \in S(X^E,\beta)/S^1} \frac{1}{m(o,p)} i(o,p).$$

Definition 2.6. *Otherwise, suppose that* X *admits a perturbation system* $\{X^a\}$ *and there is an* $E = E(\{X^a\}) > 0$ *so that the index* $i(o,p)$ *of elements* $(o,p) \in S(X^a,\beta)$ *with* $E \leq p \leq a$ *is positive, respectively negative for every* $a > E$, *and s.t.*

$$\lim_{a \mapsto \infty} \sum_{(o,p) \in S(X,a,\beta)/S^1} \frac{1}{m(o,p)} i(o,p) = \infty, \text{ respectively } \lim_{a \mapsto \infty} \sum_{o \in S(X,a,\beta)/S^1} \frac{1}{m(o,p)} i(o,p) = -\infty.$$

Then we say that X *is* **positive infinite type**, *respectively* **negative infinite type** *and set* $i(X,\beta) = \infty$, *respectively* $i(X,\beta) = -\infty$. *We say it is* **infinite type** *if it is one or the other.*

Definition 2.7. *We say that* X *is* **definite** *type if it is infinite type or finite type.*

With the above definitions

$$i(X,\beta) \in \mathbb{Q} \sqcup \infty \sqcup -\infty,$$

when it is defined.

Remark 2.8. It is an elementary exercise that the condition that

$$\lim_{a \mapsto \infty} \sum_{o \in S(X^a,a,\beta)/S^1} \frac{1}{m(o)} i(o) = \infty, \text{ respectively } \lim_{a \mapsto \infty} \sum_{o \in S(X^a,a,\beta)/S^1} \frac{1}{m(o)} i(o) = -\infty,$$

actually follows if the other conditions are satisfied, so is only stated for emphasis.

Definition 2.9. *A vector field* X *is* **admissible** *if it admits a perturbation system, and if it is definite type.*

Remark 2.10. One may be tempted to extend the finite type to include the case when

$$\lim_{a \mapsto \infty} \sum_{o \in S(X^a, a, \beta)/S^1} \frac{1}{m(o)} i(o),$$

exists and the associated series is absolutely convergent. This definitely works for our arguments later, but it is an elementary exercise to show that there are no such X unless it is of the previous finite type.

2.1.1. *Perturbation systems for Morse-Bott Reeb vector fields.*

Definition 2.11. *A contact form λ on M, and its associated flow R^λ are called* Morse-Bott *if the λ action spectrum $\sigma(\lambda)$ - that is the space of critical values of $o \mapsto \int_{S^1} o^* \lambda$, is discreet and if for every $a \in \sigma(\lambda)$, the space $N_a := \{x \in M | \, F_a(x) = x\}$, F_a the time a flow map for R^λ - is a closed smooth manifold such that rank $d\lambda|_{N_a}$ is locally constant and $T_x N_a = \ker(dF_a - I)_x$.*

Proposition 2.12. *Let λ be a contact form of Morse-Bott type, on a closed contact manifold C. Then the corresponding Reeb vector field R^λ admits a perturbation system $\{X^a\}$, for every class β, with each X^a Reeb so that all the structure homotopies are through Reeb vector fields.*

The above very likely extends to more general "Morse-Bott type" and beyond vector fields of non Reeb type, however one must take care to give the right definitions.

Proof. Let

$$O_{\leq E} = O_{\leq E}(R^\lambda) \simeq S(R^\lambda, E)$$

denote the set of points $x \in C$, s.t. $F^\lambda_p(x) = x$, for F^λ_p the time $p \leq E$ flow map for R^λ. Given an a take an $E > a$ s.t. the set $O_{\leq E} = O_{\leq E}(R^\lambda)$ is a union of closed manifolds (of varying dimension), call such an E *appropriate*. Let $\mathcal{O}_{\leq E}$ be the natural S^1-quotient of $O_{\leq E}$. By [**?**, Section 2.2] we may find a smooth function f_E on C with support in a normal neighborhood of $O_{\leq E}$, with $Df_E(R^\lambda) = 0$ on $O_{\leq E}$ descending to a Morse function on the union of closed orbifolds $\mathcal{O}_{\leq E}$.

Let $\lambda_{E,\mu} = (1 + \mu f_E)\lambda$. By [**?**, Section 2.2] we may choose $\mu_0 > 0$ so that elements of $\mathcal{O}_{\leq E}(R^{\lambda_{E,\mu}})$ are non-degenerate and correspond to critical points of f_E, for $0 < \mu \leq \mu(E)$. Let $\{E_n\}$ be an increasing sequence of appropriate levels as above. Since the action spectrum of λ is discreet by the Morse-Bott assumption, we may take $\{f_{E_n}\}$ so that $f_{E_{n'}}|_{E_n}$ coincides with f_{E_n} if $E_{n'} > E_n$. (Note however that the cutoff value $\mu(E_{n'})$ needs to in general be smaller then $\mu(E_n)$.) Given this, we set $X^a = R^{\lambda_{E_n, \mu(E_n)}}$, for any $E_n > a$. For the structure homotopies we take the obvious homotopies induced by the homotopies

$$t \mapsto (1 + (1-t)\mu(E_n) f_{E_n})\lambda, \quad t \in [0,1],$$

of the contact forms. □

Lemma 2.13. *The Hopf vector field H on S^{2k+1} is infinite type.*

Proof. Pick a perfect Morse function f on $\mathbb{CP}^k$. This induces a perfect Morse function f on $\mathcal{O}_{\leq 2\pi n}$, upon identifying $\mathcal{O}_{\leq 2\pi n}$ with the n-fold disjoint union of copies of $\mathbb{CP}^k$ (forgetting the totally non-effective orbifold structure). Use the construction above, to obtain a perturbation system $\{H^{2n\pi}\}$ for H, $n \in \mathbb{Z}_+$, so that the space $\mathcal{O}^{pert}_{\leq 2\pi n} = \mathcal{O}_{\leq 2\pi n}(H^{2n\pi})$, be identified with critical points of f on the space $\mathcal{O}_{\leq 2\pi n}$. Given a critical point p of f on the component $\mathcal{O}_{2\pi i} \simeq \mathbb{CP}^k$, $0 < i \leq 2\pi n$, of $\mathcal{O}_{\leq 2\pi n}$, let o_p denote the corresponding orbit in $\mathcal{O}^{pert}_{2\pi i}$. By [**?**, Lemma 2.4],

$$\mu_{CZ}(o_p) = \mu_{CZ}(\mathcal{O}_{2\pi i}) - \frac{1}{2} \dim_{\mathbb{R}} \mathcal{O}_{2\pi i} + morseindex_f(p),$$

where $\mu_{CZ}(\mathcal{O}_{2\pi i})$ is the generalized Maslov index for an element of $\mathcal{O}_{2\pi i}$, see for instance [**?**, Section 5.2.2]. Let us slightly elaborate as our reader may not be familiar with this. We pick a representative for a class of an orbit o in $\mathcal{O}_{2\pi i}$, pick a bounding disk for the orbit and using this choice trivialize the contact distribution along o. Given this trivialization the R^λ Reeb flow induces a path of symplectic matrices to which we apply the generalized Maslov index. Since $\mu_{CZ}(\mathcal{O}_{2\pi i})$ has even parity for all k, it follows that $\mu_{CZ}(o_p)$ has the same parity for all n, and so by (2.1) H is infinite type. $\square$

3. Extensions of theorem 1.2 and their proofs

Theorem 3.1. *Suppose we have an admissible vector field X_0, with $i(X_0, \beta) \neq 0$ on a closed, oriented manifold M, which is joined to X_1 by a partially admissible homotopy $\{X_t\}$, then X_1 has periodic orbits.*

Theorem 1.2 clearly follows by the above and by Lemma 2.13. What follows is a more precise result.

Theorem 3.2. *If M is closed, oriented and X_0, X_1 and $\{X_t\}$ are admissible then $i(X_0, \beta) = i(X_1, \beta)$.*

3.1. **Fuller correspondence.** We need a beautiful construction of Fuller [**?**], which converts contractible orbits of X into non-contractible orbits in an associated space, for an associated vector field, so that the correspondence between the periodic orbits is particularly suitable. Let $\mathbf{M}'$ be the subset of the k-fold product $M \times \ldots \times M$, for k a prime, consisting of points all of whose coordinates $(x_1, \ldots, x_k)$ are distinct. Let $\mathbf{M}$ be the quotient of $\mathbf{M}'$ by the permutation action of $\mathbb{Z}_k$, generated by $P(x_1, x_2, \ldots x_k) = (x_2, \ldots, x_k, x_1)$. As this is a free action the projection map $\mathbf{M}' \to \mathbf{M}$ is a regular k-sheeted covering. And so we have a homomorphism

$$\mu : \pi_1(\mathbf{M}) \to \mathbb{Z}_k,$$

which extends to $H_1(\mathbf{M})$ since $\mathbb{Z}_k$ is abelian.

A vector field X on M determines a vector field $\mathbf{X}$ on $\mathbf{M}$ given by

$$\mathbf{X}(\mathbf{x}) = [(X(x_1), \ldots, X(x_k)], \tag{3.3}$$

$\mathbf{x} = [(x_1, \ldots, x_k)]$. It is easy to see $\mathbf{X}$ is complete when X is complete, which holds in our case by compactness of M. Now for an orbit $(o, p) \in S(X, \beta)$ with multiplicity $m < k$ define a multiplicity m orbit $(\mathbf{o}, \frac{p}{k}) \in S(\mathbf{X})$.

$$\mathbf{o}(t) = [o(t/k), o(t/k + \frac{1}{k}), \ldots, o(t/k + (k-1)/k)] \in L\mathbf{M}.$$

Clearly $\mu([\mathbf{o}]) = \mathbf{1}$, ($\mathbf{1}$ corresponding to the generator P of the permutation group $\mathbb{Z}_k$, with P as above) and $i(\mathbf{o}) = i(o)$ by Fuller [**?**, Lemma 4.5]. If we work over all classes β then it is easy to see that

$$Ful_k : (o, p) \mapsto (\mathbf{o}, \frac{p}{k}), \tag{3.4}$$

is a bijection from the set of all (unparametrized) period p orbits of X with multiplicity less than k to the set of all (unparametrized) period $\frac{p}{k}$ orbits of $\mathbf{X}$ with multiplicity less than k.

Lemma 3.5. *Let $\{X_t\}$ be usual and let $\boldsymbol{m}(\{X_t\}, a)$ denote the least upper bound for the set of multiplicities of elements of*

$$S(\{X_t\}, \beta) \cap (L_\beta M \times (0, a] \times [0, 1])$$

then $\boldsymbol{m}(\{X_t\}, a) < \infty$.

Proof. This is a version of [**?**, Lemma 4.2]. The proof is as follows. Since

$$S(\{X_t\}, \beta) \cap (L_\beta M \times (0, a] \times [0, 1])$$

is compact as it is identified under the embedding $emb : S(\{X_t\}, \beta) \to M \times (0, \infty) \times [0, 1])$, $(o, p, t) \mapsto (o(0), p, t)$ with a closed bounded subset of a finite dimensional manifold, we would otherwise have a convergent sequence $\{(o_k, p_k)\}$ in $S(\{X_t\}, \beta)$ with $\{t_k\}$ also convergent, and so that p_k converges to 0. But this contradicts the assumption that X_t are non-singular. $\square$

We set

$$S(\mathbf{X}, a, \beta) = Ful_k(S(X, a, \beta)), \tag{3.6}$$

for $k > m(X, a)$.

3.2. **Preliminaries on admissible homotopies.**

Definition 3.7. *Let $\{X_t\}$ be a smooth homotopy of non-singular vector fields. For $b > a > 0$ we say that $\{X_t\}$ is* **partially** a, b-**admissible**, *respectively* a, b-**admissible** *(in class β) if for each*

$$y \in (S = S(\{X_t\}, \beta)) \cap (L_\beta M \times (0, a) \times \{0\}),$$

there is a compact open subset $\mathcal{C}_y$ of S containing it and contained in $M \times (0, b) \times [0, 1]$. Respectively, if for each

$$y \in S \cap (L_\beta M \times (0, a) \times \partial [0, 1]),$$

there is a compact open subset $\mathcal{C}_y$ of S containing it and contained in $M \times (0, b) \times [0, 1]$.

Lemma 3.8. *Suppose that $\{X_t\}$ is partially admissible, then for every a there is a $b > a$ so that $\{\widetilde{X}^b_t\} = \{X_t\} \cdot \{X^b_t\}$ is partially a, b-admissible, where $\{X_t\} \cdot \{X^b_t\}$ is the (reparametrized to have t domain $[0, 1]$) concatenation of the homotopies $\{X_t\}, \{X^b_t\}$, and where $\{X^b_t\}$ is the structure homotopy from X^b to X_0.*

Proof. More explicitly

$$\begin{aligned} \{\widetilde{X}^b_t\} &= \{X^b_{2t}\} \text{ for } t \in [0, 1/2] \\ \{\widetilde{X}^b_t\} &= \{X_{2(t-1/2)}\} \text{ for } t \in [1/2, 1]. \end{aligned} \tag{3.9}$$

Let $y \in S(\{X^a_t\}) \cap (L_\beta M \times (0, a) \times \{0\})$. Let $\mathcal{C}'_y$ be a non-branching open compact subset of $S(\{X^a_t\})$ containing y. Let

$$y' \in K_y = \mathcal{C}'_y \cap (L_\beta M \times (0, \infty) \times \{1\}),$$

with K_y by assumptions connected and since it is an open and closed subset of

$$S(\{X^a_t\}) \cap (L_\beta M \times (0, \infty) \times \{1\}),$$

$\pi(K_y)$ coincides with one of the connected components of $S(X_0)$, for

$$\pi : L_\beta M \times (0, \infty) \times [0, 1] \to L_\beta M \times (0, \infty)$$

the projection.

Let $\mathcal{C}_{y'}$ be a compact open subset of $S(\{X_t\})$ containing $\pi(y') \times \{0\} \in L_\beta M \times (0, \infty) \times [0, 1]$, $\mathcal{C}_{y'}$ exists by partial admissibility assumption on $\{X_t\}$. Let

$$M_y = \mathcal{C}_{y'} \cap (L_\beta M \times (0, \infty) \times \{0\})$$

then M_y must contain $\pi(K_y) \times \{0\}$ as these sets are open and closed in $S(\{X_t\}) \cap (L_\beta M \times (0, \infty) \times \{0\})$ and $\pi(K_y) \times \{0\}$ is connected. And $M_y - \pi(K_y) \times \{0\}$ is a finite union (possibly empty) of compact open connected components $\{W^j_y\}$, by the assumption that connected components of $S(X)$ are open, and since M_y is compact.

Let

$$S_a = \bigcup_y \mathcal{C}_{y'}$$

for y, y' as above. Then since there are only finitely many such y S_a is compact and so is contained in $L_\beta M \times (0, b') \times [0, 1]$, for some $b' > a$ and sufficiently large as in the last axiom for a perturbation system.

For each W^j_y as above let $\mathcal{C}^j_y$ be a non-branching open compact subset of $S(\{X^{b'}_t\})$ intersecting $\pi(W^j_y) \times \{1\}$, and hence so that

$$\mathcal{C}^j_y \cap (L_\beta M \times (0, \infty) \times \{1\}) = \pi(W^j_y) \times \{1\}.$$

This equality again follows by $\mathcal{C}^j_y \cap (L_\beta M \times (0, \infty) \times \{1\})$ and $\pi(W^j_y) \times \{1\}$ being open closed and connected subsets of $S(\{X^{b'}_t\}) \cap (L_\beta M \times (0, \infty) \times \{1\})$.

Let

$$T_y = (\bigcup_j \mathcal{C}^j_y) \cup \mathcal{C}'_y \cup S_a,$$

where this union is taken in

$$\mathcal{U} = \mathcal{U}_- \sqcup \mathcal{U}_+ / \sim, \tag{3.10}$$

where $\mathcal{U}_\pm$ are two names for $L_\beta M \times (0,\infty) \times [0,1]$ and the equivalence relation the identification map of $M \times (0,\infty) \times \{1\}$ in the first component with $M \times (0,\infty) \times \{0\}$ in the second component. And here $(\bigcup_j \mathcal{C}_y^j) \cup \mathcal{C}'_y$ is understood as being a subset of $\mathcal{U}_-$ and S_a of $\mathcal{U}_+$.

Let ϕ be the "linear" (linear, if one naturally identifies $[0,1]_- \sqcup [0,1]_+ / \sim$ with $[0,2]$) homeomorphism of $[0,1]_- \sqcup [0,1]_+ / \sim$, with $[0,1]$, with $[0,1]_\pm$ two names for $[0,1]$ for $\sim$ identifying $1 \in [0,1]_-$ with $0 \in [0,1]_+$, with $\phi(0) = 0$, $0 \in [0,1]_-$, $\phi(1) = 1/2$, $1 \in [0,1]_-$, $\phi(0) = 1/2$, $0 \in [0,1]_+$, and $\phi(1) = 1$, $1 \in [0,1]_+$. Then by the above discussion T_y is a compact subset of (3.10) and can clearly be identified with a certain compact and open $C_y \subset S(\{\widetilde{X}_t^b\})$, containing y via the homeomorphism of $\mathcal{U}$ with $L_\beta M \times (0,\infty) \times [0,1]$, induced by ϕ. Again since there are only finitely many such y

$$\bigsqcup_y C_y$$

is contained in $L_\beta M \times (0,b) \times [0,1]$, for some b sufficiently large. So our assertion follows. $\square$

The analogue of Lemma 3.8 in the admissible case is the following:

Lemma 3.11. *Suppose that X_0, X_1 and $\{X_t\}$ are admissible, then for every a there is a $b > a$ so that $\{\widetilde{X}_t^b\} = \{X_{1,t}^b\}^{-1} \cdot \{X_t\} \cdot \{X_{0,t}^b\}$ is a,b-admissible, where $\{X_{i,t}^b\}$ are the structure homotopies from X_i^b to X_i.*

The proof of this is completely analogous to the proof of Lemma 3.8.

3.3. **Proof of Theorem 3.1.** Suppose that X_0 is admissible with $i(X_0,\beta) \neq 0$, $\{X_t\}$ is partially admissible and X_1 has no periodic orbits. Let a be given and b determined so that $\{\widetilde{X}_t^b\}$ is a partially (a,b)-admissible homotopy. Set $\mathbf{m} = \mathbf{m}(\{\widetilde{X}_t^b\}, b)$. Take a prime $k = k(a) > \mathbf{m}$ and define $\mathbf{M}$ as above with respect to k. As M is compact the flow $\widetilde{X}_t^b$ is complete for every t, and consequently as previously observed the flow of $\mathbf{X}_t$ is complete for every t.

Let $\mathbf{F}_{t,p}$ denote the time p flow map of $\mathbf{X}_t$. Define

$$\mathbf{F} : \mathbf{M} \times (0,b] \times [0,1] \to \mathbf{M} \times \mathbf{M},$$

by

$$\mathbf{F}(\mathbf{x}, p, t) = (\mathbf{F}_{t,p}(\mathbf{x}), \mathbf{x}).$$

Set

$$\widetilde{\mathbf{S}} = \mathbf{F}^{-1}(\Delta), \tag{3.12}$$

for Δ the diagonal. And set

$$\mathbf{S} = emb \circ Ful_k(S(\{\widetilde{X}_t^b\}, \beta)) \subset \mathbf{M} \times (0,\infty) \times [0,1],$$

for

$$emb : Ful_k(S(\{\widetilde{X}_t^b\}, \beta)) \to \mathbf{M} \times (0,\infty) \times [0,1], \tag{3.13}$$

the map $(\mathbf{o}, p, t) \mapsto (\mathbf{o}(0), p, t)$. Let

$$\mathbf{S}^i = (\mathbf{S} \cap \mathbf{M} \times \mathbb{R}_+ \times \{i\}).$$

By assumptions $\mathbf{S}^1$ is empty.

Let $\{X_t\}$ be partially a,b-admissible and let

$$S_{a,\beta,0} = \bigcup_{y \in S(\{X_t\},\beta) \cap (M \times (0,a) \times \{0\})} \mathcal{C}_y. \tag{3.14}$$

Here $\mathcal{C}_y$ is as in the Definition 3.7 with respect to a,b. Then $S_{a,\beta,0}$ is an open compact subset of $S(\{X_t\},\beta) \cap (L_\beta M \times (0,b) \times [0,1])$. Define

$$\mathbf{S}_{a,0} = \mathbf{S}_{a,\beta,0} = emb \circ Ful_k(S_{a,\beta,0}).$$

A given orientation on M orients $\mathbf{M}$, $\mathbf{M} \times \mathbf{M}$, and the diagonal $\Delta \subset \mathbf{M} \times \mathbf{M}$. Let $\{\nabla_r\}$ be a sequence of forms C^∞ dual to the diagonal Δ, with support of ∇_r converging to the diagonal as $r \mapsto \infty$, uniformly on compact sets. $\{\nabla_r\}$ are characterized by the condition that $C \cdot \Delta = \int_C f^* \nabla_r$, where $f : C \to \mathbf{M} \times \mathbf{M}$ is a chain whose boundary is disjoint from Δ, and from the support of ∇_r and where $C \cdot \Delta$ is the intersection number.

Lemma 3.15. *We may choose an r so that $\boldsymbol{F}^* \nabla_r$ breaks up as the sum:*

$$\gamma^r + \sum_y \alpha_y^r,$$

for y as in (3.14), where each α_y^r has compact support in open sets U_y,

$$\boldsymbol{M} \times (0, \frac{b}{k}) \times [0,1] \supset U_y \supset emb \circ Ful_k(\mathcal{C}_y),$$

and where γ^r has support which does not intersect any of the U_y.

Proof. Note that $\mathbf{S}_{a,0}$ is an open and compact subset of $\widetilde{\mathbf{S}} \cap \big(\mathbf{M} \times (0, \frac{b}{k}) \times [0,1]\big)$ by construction, and by the fact that maps (3.13) are open. (The latter, by the definition of the topology on these spaces as discussed following (1.1)). Likewise

$$\widetilde{\mathbf{S}} - \mathbf{S}_{a,0}$$

is compact as $\widetilde{\mathbf{S}}$ is compact, and $\mathbf{S}_{a,0}$ is open.

Let *sup* denote the support of $\mathbf{F}^* \nabla_r$. Then for r sufficiently large *sup* is contained in an ϵ-neighborhood of $\widetilde{\mathbf{S}}$, for ϵ arbitrarily small (just by compactness). Since $\mathbf{S}_{a,0}$ and $\widetilde{\mathbf{S}} - \mathbf{S}_{a,0}$ are both compact and disjoint they have disjoint metric ϵ-neighborhoods for ϵ sufficiently small. The lemma then clearly follows. □

As $\gamma^r + \sum_y \alpha_y^r$ is closed, $\omega^r = \sum_y \alpha_y^r$ is closed and has compact support. Let ω_0^r, ω_1^r be the restrictions of ω^r to $\mathbf{M} \times \mathbb{R}_+ \times \{0\}$, respectively $\mathbf{M} \times \mathbb{R}_+ \times \{1\}$, with ω_1^r by assumption identically vanishing, for r sufficiently large.

Let $S(\mathbf{X}^b, b, \beta)/S^1$, be as in (3.6), whose elements by slight abuse of notation we denote by $\mathbf{o} = (\mathbf{o}, p)$. Let $[(\omega_0^r)^*]$ denote the Poincare dual class of ω_0^r and given $\mathbf{o}$ as above we denote by $[\mathbf{o}]$ the class of the 1-cycle in $\mathbf{M} \times \mathbb{R}_+$ represented by the (strictly speaking S^1-equivalence class of) map $t \mapsto (\mathbf{o}(t), p)$, $t \in [0,p]/0 \sim p$. If r is taken to be large, then by construction and since all elements of $S(\mathbf{X}^b, b, \beta)/S^1$, are isolated, the support of ω_0^r breaks up as the disjoint union of sets contained in ϵ_r-neighborhoods of the images of the orbits $\mathbf{o} \in S(\mathbf{X}^b, b, \beta)/S^1$, for $\epsilon_r \mapsto 0$ as $r \mapsto \infty$. We shall for short that the support is *localized* at these images. So we may write $\omega_0^r = \sum_l \omega_{0,l}$, with $\omega_{0,l}$ having support localized at the image of $\mathbf{o}_l^0$, where $\{\mathbf{o}_l^0\}$ is the enumeration of $S(\mathbf{X}^b, b, \beta)/S^1$.

By (proof of) [?, Theorem 1] (for r sufficiently large) the Poincare dual class $[\omega_{0,l}^*]$ is given by the class

$$i(\mathbf{o}_l^0)\frac{1}{\mathrm{mult}(\mathbf{o}_l^0)}[\mathbf{o}_l^0],$$

and by Fuller's correspondence $i(\mathbf{o}_l^0) = i(o_l^0)$, and $\mathrm{mult}(\mathbf{o}_l^0) = \mathrm{mult}(o_l^0)$, where o_l^0 is the corresponding element of $S(X_0^b, \beta, b)$. So

$$[(\omega_0^r)^*] = c_a + \sum_{\mathbf{o}^0 \in S(\mathbf{X}^a, a, \beta)/S^1} i(o^0)\frac{1}{\mathrm{mult}(o^0)}[\mathbf{o}^0],$$

where

$$c_a = \sum_{\mathbf{o} \in Q_a} i(o)\frac{1}{\mathrm{mult}(o)}[\mathbf{o}],$$

for

$$Q_a \subset S(\mathbf{X}^b, b, \beta)/S^1 - S(\mathbf{X}^a, a, \beta)/S^1$$

(possibly empty). Thus

$$c_a + \sum_{\mathbf{o}^0 \in S(\mathbf{X}^a, a, \beta)/S^1} i(o^0) \frac{1}{\mathrm{mult}(o^0)} [\mathbf{o}^0] = 0,$$

as for r sufficiently large ω_0^r, ω_1^r are cohomologous with compact support in $\mathbf{M} \times (0, \frac{b}{k}) \times [0,1]$ and as ω_1^r is identically vanishing. Applying μ we get

$$l(\sum_{\mathbf{o} \in Q_a} i(o) \frac{1}{\mathrm{mult}(o)} + \sum_{\mathbf{o}^0 \in S(\mathbf{X}^a, a, \beta)/S^1} i(o^0) \frac{1}{\mathrm{mult}(o^0)}) = 0 \mod k, \tag{3.16}$$

where l is the least common denominator for all the fractions, and this holds for all $a, k = k(a)$ (going higher in the perturbation system and adjusting the least common denominator).

3.4. **Case I, X_0 is finite type.** Let $E = E(\{X^a\})$ be the corresponding cutoff value in the definition of finite type, and take any $a > E$. Then $Q_a = \emptyset$ and

$$\sum_{\mathbf{o} \in Q_a} i(o) \frac{1}{\mathrm{mult}(o)} + \sum_{\mathbf{o}^0 \in S(\mathbf{X}^a, a, \beta)/S^1} i(o^0) \frac{1}{\mathrm{mult}(o^0)} = i(X_0, \beta) \neq 0.$$

Clearly this gives a contradiction to (3.16).

3.5. **Case II, X_0 is infinite type.** We may assume that $i(X_0, \beta) = \infty$, and take $a > E$, where $E = E(\{X^a\})$ is the corresponding cutoff value in the definition of infinite type. Then

$$\sum_{\mathbf{o} \in Q_a} i(o) \frac{1}{\mathrm{mult}(o)} \geq 0,$$

as $a > E(\{X_0^e\})$. While

$$\lim_{a \mapsto \infty} \sum_{\mathbf{o}^0 \in S(\mathbf{X}^a, a, \beta)/S^1} i(o^0) \frac{1}{\mathrm{mult}(o^0)} = \infty,$$

by $i(X_0, \beta) = \infty$. This also contradicts (3.16).

□

3.6. **Proof of Theorem 3.2.** The proof is very similar to the proof of Theorem 3.1, and all the same notation is used. Suppose that X_0, X_1 and $\{X_t\}$ are admissible. Let a be given and b determined so that $\{\widetilde{X}_t^b\}$ is a (a,b)-admissible homotopy. Let $\mathbf{F}$, $\widetilde{\mathbf{S}}$, $\mathbf{S}$ and $\mathbf{S}^i$, be as before.

Let

$$S_{a,\beta} = S_{a,\beta}(F) = \bigcup_{y \in S_\beta(F) \cap (M \times (0,a) \times \partial[0,1])} \mathcal{C}_y,$$

where $\mathcal{C}_y$ are as in the Definition 3.7. Then $S_{a,\beta,0}$ is an open compact subset of $S(\{X_t\}, \beta) \cap (L_\beta M \times (0,b) \times [0,1])$. Define

$$\mathbf{S}_a = \mathbf{S}_{a,\beta} = emb \circ Ful_k(S_{a,\beta}).$$

Let $\{\nabla_r\}$ be a sequence of forms C^∞ dual to the diagonal $\Delta \subset \mathbf{M} \times \mathbf{M}$, as before.

Lemma 3.17. *We may choose an r so that $\boldsymbol{F}^* \nabla_r$ breaks up as the sum:*

$$\gamma^r + \sum_y \alpha_y^r,$$

for y as in (3.14), where each α_y^r has compact support in open sets U_y,

$$\boldsymbol{M} \times (0, \frac{b}{k}) \times [0,1] \supset U_y \supset emb \circ Ful_k(\mathcal{C}_y),$$

and where γ^r has support which does not intersect any of the U_y.

Proof. Analogous to the proof of Lemma 3.15. □

As $\gamma^r + \sum_y \alpha_y^r$ is closed, $\omega^r = \sum_y \alpha_y^r$ is closed and has compact support. Let ω_0^r, ω_1^r be the restrictions of ω^r to $\mathbf{M} \times \mathbb{R}_+ \times \{0\}$, respectively $\mathbf{M} \times \mathbb{R}_+ \times \{1\}$. Let ω_0^r, ω_1^r be the restrictions of ω^r to $\mathbf{M} \times \mathbb{R}_+ \times \{0\}$, respectively $\mathbf{M} \times \mathbb{R}_+ \times \{1\}$. We may write $\omega_i^r = \sum_l \omega_{i,l}$, with $\omega_{i,l}$ having support localized at the image of $\mathbf{o}_l^i$, where $\{\mathbf{o}_l^i\}$ is the enumeration of $S(\mathbf{X}_i^b, b, \beta)/S^1$.

By (proof of) [**?**, Theorem 1] (for r sufficiently large) the Poincare dual class $[\omega_{i,l}^*]$ is given by the class

$$i(\mathbf{o}_l^i)\frac{1}{\mathrm{mult}(\mathbf{o}_l^i)}[\mathbf{o}_l^i],$$

and by Fuller's correspondence $i(\mathbf{o}_l^i) = i(o_l^i)$, and $\mathrm{mult}(\mathbf{o}_l^i) = \mathrm{mult}(o_l^i)$, where o_l^i is the corresponding element of $S(X_i^b, \beta, b)$.

So

$$[(\omega_0^r)^*] = c_a + \sum_{\mathbf{o}^0 \in S(\mathbf{X}_0^a, a, \beta)/S^1} i(o^0)\frac{1}{\mathrm{mult}(o^0)}[\mathbf{o}^0],$$

where

$$c_a = \sum_{\mathbf{o} \in Q_a} i(o)\frac{1}{\mathrm{mult}(o)}[\mathbf{o}],$$

for

$$Q_a \subset S(\mathbf{X}_0^b, b, \beta)/S^1 - S(\mathbf{X}_0^a, a, \beta)/S^1$$

(possibly empty).

Likewise

$$[(\omega_1^r)^*] = c_a' + \sum_{\mathbf{o}^1 \in S(\mathbf{X}_1^a, a, \beta)/S^1} i(o^1)\frac{1}{\mathrm{mult}(o^1)}[\mathbf{o}^1],$$

where

$$c_a' = \sum_{\mathbf{o} \in Q_a'} i(o)\frac{1}{\mathrm{mult}(o)}[\mathbf{o}],$$

for

$$Q_a' \subset S(\mathbf{X}_1^b, b, \beta)/S^1 - S(\mathbf{X}_1^a, a, \beta)/S^1$$

(possibly empty). We have:

$$c_a + \sum_{\mathbf{o}^0 \in S(\mathbf{X}_0^a, a, \beta)/S^1} i(o^0)\frac{1}{\mathrm{mult}(o^0)}[\mathbf{o}^0] = c_a' + \sum_{\mathbf{o}^1 \in S(\mathbf{X}_1^a, a, \beta)/S^1} i(o^1)\frac{1}{\mathrm{mult}(o^1)}[\mathbf{o}^1],$$

as ω_0^r, ω_1^r are cohomologous with compact support in $\mathbf{M} \times (0, \frac{b}{k}) \times [0,1]$. Applying μ we get

$$\begin{aligned} l\Big(\sum_{\mathbf{o} \in Q_a} i(o)\frac{1}{\mathrm{mult}(o)} + \sum_{\mathbf{o}^0 \in S(\mathbf{X}_0^a, a, \beta)/S^1} i(o^0)\frac{1}{\mathrm{mult}(o^0)}\Big) \\ = l\Big(\sum_{\mathbf{o} \in Q_a'} i(o)\frac{1}{\mathrm{mult}(o)} + \sum_{\mathbf{o}^1 \in S(\mathbf{X}_1^a, a, \beta)/S^1} i(o^1)\frac{1}{\mathrm{mult}(o^1)}\Big) \mod k \end{aligned} \tag{3.18}$$

where l is the least common denominator for all the fractions, and this holds for all a, k (changing l appropriately and going higher in the perturbation system).

Suppose by contradiction that $i(X_0, \beta) \neq i(X_1, \beta)$.

Case I, X_i are finite type. Let $E_i = E(\{X_i^a\})$ be the corresponding cutoff values, and take $a > \max(E_0, E_1)$. Then $Q_a = Q_a' = \emptyset$ and

$$\sum_{\mathbf{o} \in Q_a} i(o)\frac{1}{\mathrm{mult}(o)} + \sum_{\mathbf{o}^0 \in S(\mathbf{X}_0^a, a, \beta)/S^1} i(o^0)\frac{1}{\mathrm{mult}(o^0)} = i(X_0, \beta),$$

and

$$\sum_{\mathbf{o} \in Q_a'} i(o)\frac{1}{\mathrm{mult}(o)} + \sum_{\mathbf{o}^1 \in S(\mathbf{X}_1^a, a, \beta)/S^1} i(o^1)\frac{1}{\mathrm{mult}(o^1)} = i(X_1, \beta).$$

This gives a contradiction to (3.18).

Case II, X_i are infinite type. Let $E_i = E(\{X_i^a\})$ be the corresponding cutoff values, and take $a > \max(E_0, E_1)$. Suppose in addition (WLOG) that $i(X_0) = \infty$, $i(X_1) = -\infty$. Then

$$\sum_{\mathbf{o} \in Q_a} i(o) \frac{1}{\mathrm{mult}(o)} \geq 0,$$

and

$$\sum_{\mathbf{o} \in Q'_a} i(o) \frac{1}{\mathrm{mult}(o)} \leq 0,$$

while

$$\lim_{a \mapsto \infty} \sum_{\mathbf{o}^0 \in S(\mathbf{X}_0^a, a, \beta)/S^1} i(o^0) \frac{1}{\mathrm{mult}(o^0)}) = \infty,$$

and

$$\lim_{a \mapsto \infty} \sum_{\mathbf{o}^1 \in S(\mathbf{X}_1^a, a, \beta)/S^1} i(o^1) \frac{1}{\mathrm{mult}(o^1)} = -\infty.$$

Clearly this also gives contradiction.

Case III, X_0 is infinite type and X_1 is finite type. Let $E_i = E(\{X_i^a\})$ be the corresponding cutoff values, and take $a > \max(E_0, E_1)$. Suppose in addition that $i(X_0) = \infty$. Then

$$\sum_{\mathbf{o} \in Q_a} i(o) \frac{1}{\mathrm{mult}(o)} \geq 0,$$

while

$$\lim_{a \mapsto \infty} \sum_{\mathbf{o}^0 \in S(\mathbf{X}_0^a, a, \beta)/S^1} i(o^0) \frac{1}{\mathrm{mult}(o^0)} = \infty,$$

and $Q'_a = \emptyset$ so

$$\sum_{\mathbf{o} \in Q'_a} i(o) \frac{1}{\mathrm{mult}(o)} + \sum_{\mathbf{o}^1 \in S(\mathbf{X}_1^a, a, \beta)/S^1} i(o^1) \frac{1}{\mathrm{mult}(o^1)} = i(X_1, \beta).$$

Again this is a contradiction.

□

4. Proof of Lemma 1.3

Lemma 4.1. *Let X be a smooth a non-singular vector field on a manifold M, and F_p denote the time p flow map of X. Set*

$$S(X) = \{(x, p) \in M \times (0, \infty) \,|\, F_p(x, p) = x\}.$$

Suppose that S consists of compact isolated components $\{S_i\}$, meaning that for every $S_{i_0} \in \{S_i\}$ there exists an $\kappa > 0$ so that there is a neighborhood U_i of S_{i_0} in $M \times (0, \infty)$, whose closure $\overline{U}_i$ is compact and so that $\overline{U}_i \cap S(X) = S_i$. Then for every $\epsilon > 0$ there exists an δ so that whenever X_1 is a smooth vector field C^0 δ close to X, and $p \in S(X_1)$ is contained in $\overline{U}_i$, then p is in the ϵ-neighborhood of S_i.

Proof. Let

$$F(X) : M \times (0, \infty) \to M \times M,$$

be the map

$$F(X)(x, p) = (F_p(x), x).$$

Then $S(X)$ is the preimage of the diagonal by $F(X)$. Let ϵ be given, so that the ϵ-neighborhood V_i of S_i is contained in U_i. Suppose otherwise by contradiction, then there exists a sequence $\{X_i\}$ of vector fields C^0 converging to X, and a sequence $\{p_i\} \subset U_i - V_i$, $p_i \in S(X_i)$. We may then find a convergent subsequence $p_{i_k} \mapsto p \in \overline{U}_i - V_i$. But $\{F(X_{i_k})\}$ is uniformly on $\overline{U}_i$ convergent to $F(X)$, so that p must be in $S(X)$. But this is a contradiction to the hypothesis that $\overline{U}_i \cap S(X) = S_i$. □

Lemma 1.3 then readily follows. Let us leave out the details. □

5. Proof of Theorem 1.6

Suppose that $\lambda_t = f_t\lambda$, $f_t > 0$, and let X_t be the Reeb vector field for λ_t. Let $p : [0, \infty) \to S$ be a locally Lipschitz path, so that $p_3 = \pi_3 \circ p$ has finite length L. This means that the metric derivative function $|\frac{d}{d\tau} p_3|$:

$$|\frac{d}{d\tau} p_3(\tau_0)| = \limsup_{s \mapsto \tau_0} |\rho_3(\tau_0) - \rho_3(s)|/|\tau_0 - s|,$$

satisfies

$$\int_0^\infty |\frac{d}{d\tau} p_3|\, d\tau = L < \infty.$$

Note that $|\frac{d}{d\tau} p_3|$ is always locally Riemann integrable by the Lipschitz condition. From now on if we write expression of the form $|\frac{d}{dx} f(x_0)|$ we shall mean the metric derivative evaluated at x_0.

Consequently, for any $D > 0$, we may reparametrize (with same domain) $\rho = p|_{[0,D]}$ so that the reparametrized path (notationally unchanged) satisfies:

$$|\frac{d}{d\tau} \rho_3(\tau_0)| \leq \frac{L}{D}, \text{ for almost all } \tau_0, \quad \rho_3 = \pi_3 \circ \rho.$$

Set

$$K = \max_{t \in [0,1], M} |\frac{df_t}{dt}| \cdot \max_{t,M} f_t.$$

Let $\rho_i = \pi_i \circ \rho$, for π_i the projections of $LM \times (0, \infty) \times [0, 1]$ onto the i'th factor. Clearly our theorem follows by the following lemma.

Lemma 5.1. *The metric length of the path $\rho_2 = \pi_2 \circ \rho$ is bounded from above by $\exp(L \cdot K)$ for any D.*

Proof. After reparametrization of ρ to have domain in $[0, 1]$ and keeping the same notation for the path, we have

$$|\frac{d}{d\tau} \rho_3(\tau_0)| \leq L, \text{ for almost all } \tau_0\ . \tag{5.2}$$

For every τ we have a loop $\gamma_\tau : [0, 1] \to M$ given by

$$\gamma_\tau(t) = F^{\rho_3(\tau)}_{\rho_2(\tau) \cdot t}(\rho_1(\tau)),$$

where the flow maps F are defined as before. In other words γ_τ is the closed orbit of $\rho_2 \cdot R^{\lambda_{\rho_3(\tau)}}$ - a $R^{\lambda_{\rho_3(\tau)}}$-Reeb orbit. So we get a locally Lipschitz path $\widetilde{\rho}$ in $LM \times [0, 1]$, $\widetilde{\rho}(\tau) = (\gamma_\tau, \rho_3(\tau))$, where LM is the free loop space of M with uniform metric.

We have the smooth functional:

$$\Lambda : LM \times [0, 1] \to \mathbb{R}, \quad \Lambda(\gamma, t) = \langle \lambda_t, \gamma \rangle,$$

where $\langle , \rangle$ denotes the integration pairing, and clearly $\Lambda(\widetilde{\rho}(\tau)) = \rho_2(\tau)$.

We also have the restricted functionals $\lambda_t : LM \to \mathbb{R}$, $\quad \lambda_t(\gamma) = \langle \lambda_t, \gamma \rangle$, called the λ_t functionals. Let $\xi(\tau) = \pi_{1,*} \frac{d}{d\tau} \widetilde{\rho}(\tau)$, for π_i the projections of $LM \times [0, 1]$ onto the i'th factor. The differential of Λ at $(o, t_0) \in LM \times [0, 1]$ is

$$D\Lambda(o, t_0)(\xi, \frac{\partial}{\partial \tau}) = D\lambda_t(o)(\xi) + \left.\frac{d}{dt}\right|_{t_0} \lambda_t(o),$$

where $\xi \in T_o LM$, and if o is a $R^{\lambda_{t_0}}$- Reeb orbit the first term vanishes. Consequently,

$$|\frac{d}{d\tau} \Lambda \circ \widetilde{\rho}(\tau_0)| = |\frac{d}{d\tau} \lambda_{\pi_2(\tau)}(\pi_1 \circ \widetilde{\rho}(\tau_0))|.$$

On the other hand

$$|\frac{d}{d\tau}\lambda_{\pi_2(\tau)}(\pi_1 \circ \widetilde{\rho}(\tau_0))| \leq |\frac{d}{dt}\lambda_t(\widetilde{\rho}(\tau_0))(\rho_3(\tau_0))| \cdot |\frac{d}{d\tau}\rho_3(\tau_0)|.$$

We have by direct calculation for any $\tau_0 \in [0,1]$:

$$|\frac{d}{dt}\lambda_t(\widetilde{\rho}(\tau_0))(\rho_3(\tau_0))| \leq \max_{t\in[0,1],M} |\frac{df_t}{dt}| \cdot \int_{\widetilde{\rho}(\tau_0)} \lambda.$$

On the other hand

$$\int_{\widetilde{\rho}(\tau_0)} \lambda \leq \max_{M,t} f_t \cdot \Lambda(\widetilde{\rho}(\tau_0)).$$

Consequently:

$$\frac{d}{d\tau}\rho_2(\tau_0) = \frac{d}{d\tau}\left(\Lambda \circ \widetilde{\rho}(\tau)\right)(\tau_0) \leq \max_{t\in[0,1],M} |\frac{df_t}{dt}| \cdot \max_{t,M} f_t \cdot \Lambda(\widetilde{\rho}(\tau_0)) \cdot L = \max_{t\in[0,1],M} |\frac{df_t}{dt}| \cdot \max_{t,M} f_t \cdot L \cdot \rho_2(\tau_0),$$

for almost all $\tau_0 \in [0,1]$.

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