

RESEARCH STATEMENT

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1. OVERVIEW

Broadly speaking I work in geometry-topology, with much of my work exploring interrelations of algebraic topology with differential geometry, and in particular symplectic geometry via Floer, Gromov-Witten theory, Fukaya categories, Hofer geometry and in part dynamical systems.

My recent research is along four main distinct lines.

- (1) In symplectic geometry I have been working on developing the global Fukaya category. Here the basic object is a symplectic manifold (M, ω) , ω a closed non-degenerate differential 2-form. And this has a fascinating infinite dimensional transformation group of Hamiltonian symplectomorphisms $Ham(M, \omega)$, with a natural bi-invariant Finsler norm called the Hofer metric. One culmination of this program is the proof, [22], of one sense of a conjecture of Teleman on existence of natural “continuous” action of the group of Hamiltonian symplectomorphisms $Ham(M, \omega)$ on the Fukaya category of (M, ω) . In particular this leads to new invariants of smooth manifolds, new Chern-Gauss-Weil type curvature bounds for singular connections, and other applications, [23]. The theory of the global Fukaya category is a deep generalization of my earlier work on quantum characteristic classes, [26], [29], [30], [27], [25], [28]. These quantum characteristic classes, a.k.a. Savelyev-Seidel morphism, have been very recently related by Chi Hong Chow, et al [6], [5], [7] to the theory of Lagrangian correspondences and holomorphic quilts of Wehrheim-Woodward, and he resolved some problems of algebraic geometry. Also very recently, Cheuk Yu Mak, et al [14] develop some connections to quantum K -theory.
- (2) I am also working on the development of the theory of locally conformally symplectic or *lcs* manifolds, which suitably generalize both symplectic and contact manifolds. The idea is to use tools like (pseudo)-holomorphic curves, to find “rigidity phenomena” in *lcs* geometry analogous to, for example: the Gromov non-squeezing theorem, the Weinstein “conjecture” in symplectic/contact geometry. Here my main contribution is toward developing the statement and partial verification of an analogue of the Weinstein conjecture dubbed “conformal symplectic Weinstein conjecture”, [32]. This has an intriguing connection with the theory of elliptic curves of complex algebraic geometry, see also [18]. I have also studied the non-squeezing problem in *lcs* geometry in [33].
- (3) At the intersection of algebraic topology with differential geometry, I have introduced and developed the notion of a “smooth simplicial set”, which in particular leads to a solution of a well known problem of the construction

of the universal Chern Weil homomorphism for Frechet Lie groups (under the condition the group have the homotopy type of CW complex), [34].

- (4) Finally, I have been fascinated for some years by some problems in logic. I have learned the subject in large part by email correspondence with the philosopher-logician Peter Koellner. My contribution here has been the first generalization of the first and second incompleteness theorem of Gödel to stably computably enumerable formal systems, strictly generalizing the classical setting of computably enumerable formal systems, [24].

2. CURRENT RESEARCH IN MORE DETAIL

The following subsections can be read out of order. I will give more details of some of the above mentioned research lines.

2.1. Global Fukaya category. Part of the fascination of symplectic geometry is the interrelations of geometry and dynamics: displacement problems, fixed point problems, theories of closed characteristics, etc., with algebra/geometric analysis: Gromov-Witten invariants, Floer theory and of course the A_∞ Fukaya category. This interrelation lead to a lot of new insights and examples for both geometry and algebra. For example the theory of A_∞ categories likely would not have such explosive growth without the geometric insights obtained from symplectic geometry.

Two highly mysterious and pervasive objects in symplectic geometry are the infinite dimensional group $Ham(M, \omega)$ of Hamiltonian diffeomorphisms, and the Fukaya A_∞ category $Fuk(M, \omega)$. The group $Ham(M, \omega)$ when M is simply connected is just the group of diffeomorphisms preserving ω .

A natural conjecture of Teleman, ICM 2014, is that $Ham(M, \omega)$ should naturally and continuously act on $Fuk(M, \omega)$. Since $Fuk(M, \omega)$ is a category not a continuous object this must be interpreted. One very natural interpretation is to simply take the geometric realization of some kind of nerve of $Fuk(M, \omega)$. I take the so called A_∞ -nerve $NFuk(M, \omega)$, which I will attribute here simply to Lurie [15], but see also Faonte [12]. The absolutely crucial property (for computations and applications) is that $NFuk(M, \omega)$ is always an ∞ -category.

There is still unlikely to be an interesting true continuous group action of $Ham(M, \omega)$ on $|NFuk(M, \omega)|$, partly because standard constructions of $Fuk(M, \omega)$ involve choices which are difficult to make “gauge invariant”. But there is a kind of “homotopy coherent action”. More specifically, there is a natural up to homotopy map

$$cl : BHam(M, \omega) \rightarrow \mathbb{S},$$

where \mathbb{S} can be understood as the $NFuk(M, \omega)$ component of a certain natural space of ∞ -categories, (this space being suitably “classifying”.) This gives rise to a certain ∞ -categorical fibration $\mathcal{E}_M \rightarrow BHam(M, \omega)$, or in terminology of Joyal-Lurie it is just a categorical fibration. And this is what I call the universal global Fukaya category. The word “global” refers to the fact that the restriction of \mathcal{E}_M over $x \in BHam(M, \omega)$, is $NFuk(M, \omega)$. This is then one sense of Teleman conjecture, and this is roughly what he proposes himself. ¹

Despite the apparent abstraction in the above story, there are direct applications to problems of “classical” differential geometry. For example in [23] I obtain curvature bounds for singular connections for certain $PU(2)$ bundles over S^4 . For

¹We greatly simplify notation and discussion compared to [22].

smooth $PU(2)$ connections similar bounds can be obtained via Yang-Mills theory as in [31]. The theory of the global Fukaya category also naturally leads to invariants of smooth manifolds. Similarly to the theory of Pontryagin classes, one starts by complexifying the tangent bundle of a smooth manifold X . But we then fiberwise projective to obtain a Hamiltonian fibration

$$\mathbb{CP}^{r-1} \hookrightarrow P(X) \rightarrow X,$$

with fiber \mathbb{CP}^{r-1} for r the real dimension of X . The global Fukaya category of this fibration $P(X)$, which itself is a combinatorial object, is tautologically a smooth invariant of X , but related, simpler invariants can also be constructed from this data. In particular there are integer valued invariants in connection with this theory. At the moment it is not known if any of these invariants detect the smooth structure. Either possibility is interesting.

2.1.1. Future directions. In [6], Chi Hong Chow relates quantum characteristic classes to the theory of Lagrangian correspondences and holomorphic quilts of Wehrheim-Woodward. The culmination of his program is a very efficient calculation of these classes. As mentioned the theory of the global Fukaya category is a generalization of this theory of quantum classes. The idea of Chow should generalize to the setting of the global Fukaya category. This would be very exciting and lead to much more efficient computations, and new immediate applications, particularly to the study of the topology of $BHam(M, \omega)$. Even more specifically, there are a number of questions about $BHam(\mathbb{CP}^n, \omega)$ in [35], which can potentially be studied using these kinds of techniques.

A more long term, and possibly the most intriguing, research program is to discover whether the smooth invariants of manifolds outlined above, coming by way of the Global Fukaya category, can detect the smooth structure. Either answer could be interesting as we would gain new insights into what geometric information can be gleaned from the theory of holomorphic curves/Fukaya categories in the theory of smooth manifolds, and possibly get new connection to classical theory. For example in [29] I relate some of this theory with the theory of Chern classes, which also gave interesting new rigidity phenomena in Hofer geometry.

Indeed it would also be very interesting to compare these invariants to other invariants like the Donaldson/Seiberg-Witten invariants. To proceed with this program we first need more computational techniques, possibly of the kind outlined in the paragraph above, using holomorphic quilts. Particularly for this program I would welcome any collaboration.

Another immediate line of research is to better understand the relationship with the theory of singular connections. This is particularly interesting because there are not many ways in which symplectic geometry has direct mathematical implications in outside fields.

Finally, it would be good to develop connections of the global Fukaya category with central topics of symplectic geometry like Mirror symmetry. Early first steps in this direction are in the work of Oh-Tanaka [17], where for example they draw connections with string topology.

2.2. Conformal symplectic geometry. A locally conformally symplectic manifold or just *lcs* manifold, is a smooth $2n$ -fold M with an *lcs* structure: which is a non-degenerate 2-form ω , which is locally diffeomorphic to $f \cdot \omega_{st}$, for some

(non-fixed) positive smooth function f , with ω_{st} the standard symplectic form on \mathbb{R}^{2n} .

This is very unexplored kind of geometry, which is a bit surprising considering that *lcs* manifolds naturally generalize both contact and symplectic manifolds in appropriate sense. But very recently there has been a number of fascinating developments. For example Eliashberg and Murphy show that if a closed almost complex $2n$ -fold M has $H^1(M, \mathbb{R}) \neq 0$ then it admits an *lcs* structure, [11]. Another result of Apostolov, Dloussky [1] is that any complex surface with an odd first Betti number admits an *lcs* structure, which tames the complex structure. This is particularly interesting because it is known that some complex surfaces satisfying the Betti number condition do not admit a Kahler structure.

The main current problem of the field is to find examples of rigidity for *lcs* manifolds, analogous to rigidity results in symplectic and contact geometry like: the Arnold conjecture on the fixed points of contactomorphisms, the non-squeezing theorem of Gromov, the Weinstein conjecture on existence of closed Reeb orbits for closed contact manifolds, which in case of contact three manifolds is now a theorem of Hutchings-Taubes [37]. This problem is essentially open but in [32] I formulate an *lcs* analogue of the Weinstein conjecture and give some partial verifications. In particular the conjecture is verified for suitable neighborhoods in the space of *lcs* structures of any non-degenerate contact form on a closed three manifold.

What is particularly interesting about this “conformal symplectic Weinstein conjecture” is that there is a close connection with the theory of elliptic curves of complex algebraic geometry. I believe this is a first such connection coming from symplectic geometry (in the general sense).

2.2.1. Future directions. The principal current goal is the construction of a suitable *lcs* homology theory, analogous to the contact homology of Eliashberg-Givental-Hofer [9]. See also [19] for details in a modern language. In addition to the relevance to the above-mentioned conjecture, this will give new isotopy invariants of contactomorphisms, and an approach to the non-squeezing conjecture in *lcs* geometry as given in [33]. The latter conjecture is a direct generalization of the non-squeezing phenomenon discovered in Eliashberg-Kim-Polterovich [10], extended by Fraser [13], and Chiu [4].

The connection with contactomorphisms works as follows. Let (X, λ) be a closed $2n + 1$ -manifold with λ a contact form $\lambda \wedge d\lambda^n \neq 0$. A contactomorphism ϕ of X naturally induces an *lcs* structure ω_ϕ on the mapping torus M_ϕ of ϕ . This is sometimes called the Banyaga *lcs* structure, [2]. This *lcs* structure is very special, it is Lichnerowicz exact, first kind and rational (this means that the Lee form of ω_ϕ is rational). What this entails is that, for this kind of *lcs* structure ω and for appropriate Symplectic Field Theory [9] style almost complex structure J compatible with ω , the theory of J -holomorphic curves in M_ϕ is exceptionally well-behaved. This is already in part demonstrated in [32]. In particular, for these Banyaga *lcs* structures the above-mentioned *lcs* homology should be possible to construct in close analogy to the contact case.

We may briefly outline what this homology theory must look like. Recall, that in contact homology $CH(X)$ as in Morse homology we have two main ingredients: the generators and “instantons”. The generators are closed Reeb orbits for a particularly chosen contact form λ on X . That is the generators are smooth maps $o : S^1 \rightarrow X$

solving the equation, $\dot{o}(t) = c \cdot R_\lambda(o(t))$ for $c > 0$. Meanwhile, the instantons are J -holomorphic maps of a punctured Riemann surface into the symplectization $X \times \mathbb{R}$ of X , asymptotic at the ends to the generators in a suitable sense. Here, J is a suitable \mathbb{R} -invariant almost complex structure on the symplectization $X \times \mathbb{R}$.

In *lcs* homology theory for Banyaga type *lcs* manifold (M, ω) , our generators are to be Reeb curves as defined in [32], lying in a chosen fixed regular fiber M_θ of the natural map $M \rightarrow S^1$ classifying the Lee form α of ω . Usually, such a fiber is naturally contact, with contact form λ_θ the primitive of ω for the α -Lichnerowicz differential. And in this case Reeb curves are identified with closed Reeb orbits for λ_θ . What is still being researched is how to define the corresponding instantons, as there potentially new analytical difficulties. One naive approach is to simply take J -holomorphic maps of punctured Riemann surfaces into M as before. There is indeed, a very natural type of almost complex structures J in this context, studied in [32] and called ω -admissible. However, with this approach instead of \mathbb{R} -symmetry as in contact case we only have \mathbb{Z} -symmetry on our elliptic pde solution spaces, which is a one potential new difficulty for the theory of such instantons.

2.3. Smooth simplicial sets and Chern-Weil theory. Let G be a Lie group, the universal Chern-Weil homomorphism is an algebra homomorphism:

$$\mathbb{R}[\mathfrak{g}]^G \rightarrow H^*(BG, \mathbb{R}),$$

where $\mathbb{R}[\mathfrak{g}]^G$ denotes Ad_G invariant polynomials on the lie algebra \mathfrak{g} .

As observed by Reznikov [21] the infinite dimensional Frechet Lie group $\mathcal{G} = Ham(M, \omega)$ admits very interesting Ad_G invariant polynomials. Other typical examples of Frechet Lie groups G with non-trivial Ad_G invariant polynomials on the lie algebra \mathfrak{g} , are the quantomorphism groups of pre-quantization spaces, as well as the loop groups $G = LH$, that is the free loop space of a finite dimensional Lie group H . Loop groups are very important objects in quantum field theory, [20].

If $P \rightarrow X$ is a (Frechet) smooth G -bundle over a smooth manifold then by basic differential geometry we again have an algebra homomorphism

$$(2.1) \quad cw_P : \mathbb{R}[\mathfrak{g}]^G \rightarrow H^*(X, \mathbb{R}),$$

when G is Frechet. But it has been an open problem to construct the universal Chern-Weil homomorphism

$$(2.2) \quad cw : \mathbb{R}[\mathfrak{g}]^G \rightarrow H^*(BG, \mathbb{R}),$$

the issue being that BG in this case cannot be modeled by a smooth infinite dimensional manifold. The homomorphism (2.2) needs to be natural, in other words we should have

$$cw_P = f_P^* \circ cw,$$

for $f_P : X \rightarrow BG$ the classifying map of P , and f_P^* the pull-back map in cohomology.

This problem is solved in my [34]. First I introduce the notion of a smooth simplicial set, which is to Chen/diffeological spaces of Chen/Souriau [3], [36] as simplicial sets are to spaces. I then give a construction of a class of homotopy models of BG as (geometric realization of) smooth simplicial sets $BG^{\mathcal{U}}$. (The indexes \mathcal{U} are certain Grothendieck universes.) One then readily obtains homomorphisms

$$\mathbb{R}[\mathfrak{g}]^G \rightarrow H^*(BG^{\mathcal{U}}, \mathbb{R}),$$

which are suitably natural so that there is an induced homomorphism

$$\mathbb{R}[\mathfrak{g}]^G \rightarrow H^*(BG, \mathbb{R}).$$

It should be noted that Magnot-Watts [16] construct a diffeology on BG , another diffeology is constructed by Christensen-Wu [8]. At the moment the relationship of these diffeologies to the smooth simplicial set structure mentioned above, is completely mysterious.

2.3.1. Future directions. Probably, the most interesting current problem is to relate the theory of smooth simplicial sets to the theory of diffeological spaces. In particular, there should be a natural adjunction of the corresponding categories, similar to the adjunction of the categories of simplicial sets and topological spaces.

Related problems are: construction of a model category structure and development of abstract homotopy theory for smooth simplicial sets. Study of generalized smooth spaces like “smooth simplicial sets” in the long run may give insights into very long-standing problems of differential topology like the Novikov conjecture.

Finally, the construction of the universal Chern-Weil homomorphism in [34] is formalized in ZFC together with the axiom of universes of Grothendieck. It is very curious to know if it can be formalized in ZFC alone.

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