LOCALLY CONFORMALLY SYMPLECTIC DEFORMATION OF GROMOV NON-SQUEEZING

YASHA SAVELYEV

Abstract. We prove one deformation theoretic extension of the Gromov non-squeezing phenomenon to lcs structures, or locally conformally symplectic structures, which suitably generalize both symplectic and contact structures. We also conjecture an analogue in lcs geometry of contact non-squeezing of Eliashberg-Polterovich and discuss other related questions.

1. Introduction

We study here some analogues of Gromov non-squeezing for locally conformally symplectic manifolds, which generalize both symplectic and contact manifolds. Let us recall the definition.

Definition 1.1. *A* **locally conformally symplectic manifold** *or* **lcs manifold** *is a smooth $2n$-fold M, with a* lcs *structure: a non-degenerate 2-form ω, with the property that for every $p \in M$ there is an open $U \ni p$ such that $\omega|_U = f_U \cdot \omega_U$, for some symplectic form ω_U defined on U and some smooth positive function f_U on U. In the case of our paper we always have $n \geq 2$, as in case $n = 1$ there are other candidates for what should be an* lcs *structure.*

These structures have recently come into focus, for example we have a fascinating recent theorem of Apostolov-Dloussky [1] that every complex surface with an odd first Betti number admits a natural compatible lcs structure. Without compatibility, a more general existence result of this form is in Eliashberg-Murphy [4].

A basic invariant of a lcs structure ω is the Lee class,

$$\alpha = \alpha_\omega \in H^1(M, \mathbb{R}),$$

which we now briefly describe. The class α has the following differential form representative, called the Lee form and also denoted by α for simplicity. If U is an open set so that $\omega|_U = f_U \cdot \omega_U$ for ω_U symplectic, and f_U a positive smooth function, then $\alpha = d(\ln f_U)$ on U. By a simple calculation this can be seen to give well-defined 1-form α, see also Lee [9]. The class α has the property that on the associated α-covering space $\widetilde{M}$, the lift $\widetilde{\omega}$ is globally conformally symplectic, that is $\widetilde{\omega} = f \cdot \omega_0$ with ω_0 symplectic and $f > 0$. By α-covering space we mean the covering space associated to the normal subgroup $\ker\langle\alpha, \cdot\rangle \subset \pi_1(M, x)$, where $\langle\alpha, \cdot\rangle : \pi_1(M, x) \to \mathbb{R}$ is the homomorphism

$$[\gamma] \mapsto \langle\alpha, [\gamma]\rangle = \int_{S^1} \gamma^* \alpha.$$

It is moreover immediate that for a lcs form ω

$$d\omega = \alpha \wedge \omega,$$

for α the Lee form as defined above. For some authors, the pair (ω, α) with α closed s.t. $d\omega = \alpha \wedge \omega$ is the definition of a lcs structure. This has the advantage of being interesting even in dimension 2, but in dimension at least 4 the Lee form is uniquely determined, so that there is no difference of our definition with this second definition.

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Let α be a closed 1-form on a smooth manifold M. The operator

$$d^\alpha : \Omega^k(M) \to \Omega^{k+1}(M),$$
$$d^\alpha(\eta) = d\eta - \alpha \wedge \eta$$

is called the Lichnerowicz differential. It satisfies

$$d^\alpha \circ d^\alpha = 0$$

so that we have an associated chain complex called the ***Lichnerowicz chain complex***. The following is one basic example of an lcs manifold.

Example 1 (Banyaga). Let (C, λ) be a contact $(2n+1)$-manifold where λ is a contact form:

$$\forall p \in C : \lambda \wedge \lambda^{2n}(p) \neq 0.$$

Take $M = C \times S^1$ with the 2-form

$$\omega_\lambda = d^\alpha \lambda$$

for $\alpha := pr_{S^1}^* d\theta$, $pr_{S^1} : C \times S^1 \to S^1$ the projection, and λ likewise the pull-back of λ by the projection $C \times S^1 \to C$. We call (M, ω_λ) as above the ***lcs-fication*** of (C, λ).

1.1. **Symplectic and** lcs **non-squeezing.** One of the most important to this day results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [7]. The most well known formulation of this is that there does not exist a symplectic embedding

$$B_R \hookrightarrow D_r^2 \times \mathbb{R}^{2n-2}$$

for $R > r$, with B_R the standard closed radius R ball in $\mathbb{R}^{2n}$ centered at 0, and D_r^2 the radius r closed disk in $\mathbb{R}^2$. Gromov's non-squeezing is C^0 persistent in the following sense. The proof of this is subsumed by the proof of Theorem 4.1, but is more elementary. It is apparently still a "new theorem".

We say that a symplectic form ω on $M \times N$ is *split* if $\omega = \omega_1 \oplus \omega_2$ for symplectic forms ω_1, ω_2 on M respectively N.

Theorem 1.2. *Given $R > r$, there is an $\epsilon > 0$ s.t. for any symplectic form ω' on $M = S^2 \times T^{2n-2}$, C^0-close to a split symplectic form ω and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt] \in H_2(M),$$

(for $\langle , \rangle$ the usual pairing of homology and cohomology classes) there is no symplectic embedding $\phi : B_R \hookrightarrow (M, \omega')$.

On the other hand it is natural to ask if the above theorem continues to hold for general nearby forms. Or formally this translates to:

Question 1. Given $R > r$ and every $\epsilon > 0$ is there a (necessarily non-closed by above) 2-form ω' on $S^2 \times T^{2n-2}$ C^0 or even C^∞ ϵ-close to a split symplectic form ω, satisfying $\langle \omega, A \rangle = \pi r^2$, and such that there is an embedding $\phi : B_R \hookrightarrow S^2 \times T^{2n-2}$, with $\phi^* \omega' = \omega_{st}$? We likewise call such a map ϕ ***symplectic embedding***.

We cannot reduce this question to just applying Theorem 1.2. This is because:

(1) A symplectic form on a subdomain of the form $\phi(B_R) \subset M$ may not extend to a symplectic form on M (even if M has a symplectic form!).
(2) When an extension to a symplectic form on M does exist, it may not be C^0-close to a split form ω of the form above.

This appears to be a very difficult question, my opinion is that at least in the C^0 case the answer is yes, in part because it is difficult to imagine any obstruction, for example we no longer have Gromov-Witten theory for such a general ω'.

In the following theorem we show that if ω' is lcs then the answer to Question 1 is no. One may think that recent work of Müller [13] may be related to the present discussion. But there seems to be no obvious such relation as pull-backs by diffeomorphisms of nearby forms may not be nearby. Hence, there is no way to go from nearby embeddings that we work with to ϵ-symplectic embeddings of Müller.

The following theorem is a more elementary precursor to Theorem 4.1.

Theorem 1.3. *Let ω be a split symplectic form on $M = S^2 \times T^{2n-2}$. Let A be as above with $\langle \omega, A \rangle = \pi r^2$. Set $R > r$, then there is an $\epsilon > 0$ (depending only on R, r, ω) s.t. if $\{\omega_t\}$, $t \in [0,1]$, $\omega_0 = \omega$ is a C^1-continuous family of* lcs *forms on M, with $d_0(\omega_t, \omega_0) < \epsilon$ for all t, then there is no symplectic embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1)$$

meaning an embedding ϕ such that $\phi^* \omega_1 = \omega_{st}$. Here $\omega_{st} = \sum_{i=1}^n dp_i \wedge dq_i$ is the standard symplectic form on $\mathbb{R}^{2n}$, with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$.

Let us elaborate a bit. Assuming there is no volume obstruction, (and this can be arranged by modifying ω) then for any ϵ there is a volume preserving counterexample ϕ to the theorem.

The C^1 continuity is used to establish energy controls for certain pseudo-holomorphic curves, as Gromov-Witten theory behaves very differently in lcs setting. This is relaxed in Theorem 4.1 to certain $\mathcal{T}^0$-continuity, close to C^0-continuity. Relaxing this further to C^0 continuity would probably require substantially new ideas.

Note that Frechet smooth lcs deformations $\{\omega_t\}$ of our symplectic form ω, with Lee forms α_t likewise smoothly varying in t, are obstructed unless α_t are DeRham exact, as pointed out to me by Kevin Sackel. This can be verified by an elementary calculation by taking the t derivative at 0 of the equation:

$$d^\alpha \omega_t = \alpha_t \wedge \omega_t.$$

But our families are not required to be smooth so that non-trivial lcs deformations of a symplectic form may exist. This motivates the question:

Question 2. Do there exist (continuous) lcs deformations $\{(\omega_t, \alpha_t)\}$ of the standard product symplectic form on $S^2 \times T^{2n-2}$, α_t the Lee form of ω_t, so that α_t are not DeRham exact?

Either the answer of yes or no would be very intriguing, as neither is approachable by any known techniques.

1.1.1. *Toward direct generalization of contact non-squeezing.* The Eliashberg-Kim-Polterovich contact non-squeezing theorem as stated by Fraser [5] has the following form. Let $C = R^{2(n-1)} \times S^1$, $S^1 = \mathbb{R}/\mathbb{Z}$, be the prequantization space of R^{2n-2}, or in other words the contact manifold with the contact form $d\theta - \lambda$, for $\lambda = \frac{1}{2}(ydx - xdy)$. Let B_R denote the open radius R ball in $\mathbb{R}^{2n-2}$, and $\overline{B}_R$ its topological closure.

Theorem 1.4 (Eliashberg-Kim-Polterovich [3], Fraser [5], Chiu [2])**.** *For $R \geq 1$ there is no contactomorphism $\phi : C \to C$, isotopic to the identity, so that $\phi(\overline{B}_R \times S^1) \subset B_R \times S^1$.*

A Hamiltonian conformal symplectomorphism of an lcs manifold (M, ω), which we just abbreviate by the short name: ***Hamiltonian lcs map***, is a lcs diffeomorphism ϕ_H generated analogously to the symplectic case by a smooth function $H : M \times [0,1] \to \mathbb{R}$. Specifically, we define the time dependent vector field X_t by:

$$\omega(X_t, \cdot) = d^\alpha H_t,$$

for α the Lee form, and then taking ϕ_H to be the time 1 flow map of $\{X_t\}$. For example, let $(C \times S^1, \omega_\lambda)$ be the lcs-fication of a contact manifold (C, λ) as above.

If $\forall t : H_t = -1$ then $d^\alpha(H_t) = \alpha$ and clearly

$$X_t = (R^\lambda \oplus 0),$$

as a section of $TC \oplus TS^1$ with R^λ the λ-Reeb vector field. The latter is the vector field defined by:

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

Thus in this case the associated flow is naturally induced by the Reeb flow. More generally, given a smooth contact isotopy $\{\phi_t\}$, $\phi_t : C \to C$ contactomorphism of a closed contact manifold C, s.t. $\phi_0 = id$, there is a similarly induced Hamiltonian isotopy $\{\widetilde{\phi}_t\}$ on the lcs-fication $C \times S^1$, s.t. $\{pr_C \circ$

$\widetilde{\phi}_t\} = \{\phi_t\}$, for $pr_C : C \times S^1 \to C$ the projection. This is left as an exercise for the reader. Thus, the following conjecture is a direct generalization of the contact non-squeezing Theorem 1.4.

Conjecture 1 (see also Oh-Savelyev [11]). *If $R \geq 1$ there is no compactly supported, Hamiltonian lcs map*

$$\phi : \mathbb{R}^{2n} \times S^1 \times S^1 \to \mathbb{R}^{2n} \times S^1 \times S^1,$$

so that $\phi(\overline{U}) \subset U$, for $U := B_R \times S^1 \times S^1$ and $\overline{U}$ the topological closure.

2. Topology on the space of lcs forms and J-holomorphic curves

Theorem 1.3 is stated for the standard C^1 topology on the space of differential forms. However, this can be relaxed to use a certain natural C^0 style topology $\mathcal{T}_0$, specific to lcs forms. We will now discuss this. Let M be a closed smooth manifold of dimension at least 4. The metric topology $\mathcal{T}^0$ on the set $LCS(M)$ of smooth lcs 2-forms on M will be defined with respect to the following metric.

Definition 2.1. *Fix a Riemannian metric g on M. For $\omega_1, \omega_2 \in LCS(M)$ define*

$$d_0(\omega_1, \omega_2) = d_{C^0}(\omega_1, \omega_2) + d_{C^0}(\alpha_1, \alpha_2),$$

for α_i the Lee forms of ω_i and d_{C^0} the usual C^0 metric induced by g. In general d_{C^k} will denote the usual C^k metric. Recall that for $M = \mathbb{R}^n$ the C^k norm on the space $\Omega^m_c(\mathbb{R}^n)$ of compactly supported forms is defined as follows. Expand $\omega \in \Omega^m_c(\mathbb{R}^n)$ as $\omega = \Sigma_\alpha f_\alpha dx^\alpha$, so that $\{dx^\alpha(p)\}$ is the standard basis of $\Lambda^m(T^*_p\mathbb{R}^n)$ for each p. Then take the sum of the C^k norms of the coefficient functions f_α. For a general compact manifold M, a (class of equivalent) C^k metric on the space of forms, is analogously defined by fixing a Riemmanian metric g on M.

Proposition 2.2. *The metric d_0 on $LCS(M)$ is continuous with respect to the usual C^1 metric.*

Proof. The following argument was suggested to me by Vestislav Apostolov. Let $\Lambda(TM)$ be the vector bundle over M with fiber $\Lambda(TM)_p$ over p, the alternating tensor algebra $\Lambda(T_pM)$. Let $\Lambda^2(TM)$ denote the sub-bundle of degree 2 elements. Let $\Phi^2(M) = \Omega(\Lambda^2(TM))$ denote the space of C^∞ sections of $\Lambda^2(TM)$ with C^0 topology. Likewise, $\Lambda(T^*M)$ will denote the bundle whose fiber over p is the alternating tensor algebra $\Lambda(T^*_pM)$.

Let $\Theta^2(M)$ denote the space of non-degenerate C^∞ differential 2-forms on M with C^0 topology. We first construct a continuous map:

$$\phi : \Theta^2(M) \to \Phi^2(M).$$

Let ω be a non-degenerate 2-form, so that for each $p \in M$ we get an isomorphism $i_\omega : T_pM \to T^*_pM$, $i_\omega = \omega(v, \cdot)$. Let i^{-1}_ω denote the inverse of this map. Then for each $p \in M$ we have a bi-linear form ω^{-1}_p on $T^*_p(M)$ defined by $\omega^{-1}_p(\eta, \mu) = \eta(i^{-1}(\mu))$. This is readily seen to be skew-symmetric. Hence determines a section $\omega^{-1} \in \Phi^2(M)$. We then set $\phi(\omega) = \omega^{-1}$, so that ϕ is continuous by construction.

Now for $\omega \in LCS(M)$ define the one-form η on M as follows. Let $v \in T_pM$ then

$$\eta_p(v) = (d\omega)_p(v \wedge \phi(\omega)_p),$$

so that $v \wedge \phi(\omega)_p \in \Lambda^3(T_pM)$ and $(d\omega)_p \in \Lambda^3(T^*_pM)$ identified with a functional in $(\Lambda^3(T_pM))^*$. Taking a basis for T_pM so that ω_p in this basis is the the standard symplectic form, it is easily verified that

$$\forall p \in M : \eta_p = (n-1)\alpha_p,$$

for α the Lee form satisfying $d\omega = \alpha \wedge \omega$, and where $2n$ is the dimension of M. We have thus obtained a map $LCS(M) \to \Omega(T^*M)$, which takes an lcs form and produces its Lee form, and which is continuous with respect to the C^1 topology on $LCS(M)$ and the C^0 topology on the space of 1-forms. Clearly the result follows. $\square$

The following characterization of convergence will be helpful.

Lemma 2.3. *Let M be as above and let $\{\omega_k\} \subset LCS(M)$ be a sequence $\mathcal{T}^0$ converging to a symplectic form ω. Denote by $\{\widetilde{\omega}_k\}$ the lift sequence on the universal cover $\widetilde{M}$. Then there is a sequence $\{\widetilde{\omega}_k^{symp}\}$ of symplectic forms on $\widetilde{M}$, and a sequence $\{f_k\}$ of positive functions pointwise converging to 1, such that $\widetilde{\omega}_k = f_k \widetilde{\omega}_k^{symp}$.*

Proof. We may assume that M is connected. Let α_k be the Lee form of ω_k, and g_k functions on $\widetilde{M}$ defined by $g_k([p]) = \int_{[0,1]} p^* \alpha_k$, where the universal cover $\widetilde{M}$ is understood as the set of equivalence classes of paths p starting at a fixed $x_0 \in M$, with a pair p_1, p_2 equivalent if $p_1(1) = p_2(1)$ and $p_2^{-1} \cdot p_1$ is null-homotopic, where $\cdot$ is the path concatenation.

Then we get:

$$d\widetilde{\omega}_k = dg_k \wedge \widetilde{\omega}_k,$$

so that if we set $f_k := e^{g_k}$ then

$$d(f_k^{-1} \widetilde{\omega}_k) = 0.$$

Since by assumption $|\alpha_k|_{C^0} \to 0$, then pointwise $g_k \to 0$ and pointwise $f_k \to 1$, so that if we set

$$\widetilde{\omega}_k^{symp} := f_k^{-1} \widetilde{\omega}_k$$

then we are done. □

Definition 2.4. *We say that a pair (ω, J) of an* lcs *form ω on M and an almost complex structure J on M are* **compatible** *if $\omega(\cdot, J\cdot)$ defines a J-invariant inner product on M. For other basic notions of J-holomorphic curves we refer the reader to* [10].

Definition 2.5. *Let G be a discreet group properly acting on a smooth manifold M. An open $D \subset M$ is called a* fundamental domain *if:*

- *The interior of the closure of D equals D.*
- *G-orbit of $cl(D)$ equals M.*
- *$g(D) \cap D = \emptyset$ for all $g \in G - id$.*
- *The collection of subsets $\{g(cl(D)) \mid g \in G\}$ is locally finite.*

Suppose $N \subset M$ is a closed connected subset satisfying:

$$\forall x \in N \, \forall g \in G : gx \neq x. \tag{2.6}$$

Then there is a fundamental domain $D \supset N$. A more general theorem of the kind can be found in Kapovich [8].

In the rest of the paper we may conflate the notation for D and $D/G \subset M/G$. Also we may refer to $D \in M/G$ as the fundamental domain.

Theorem 2.7. *Let M be as above, $A \in H_2(M)$ fixed, and $\{\omega_t\}$, $t \in [0,1]$, a $\mathcal{T}^0$-continuous family of* lcs *forms on M. Let $\{J_t\}$ be a Frechet smooth family of almost complex structures, with J_t compatible with ω_t for each t. Let $\pi : \widetilde{M} \to M$ be the universal cover of M, and let $D \subset \widetilde{M}$ be a fundamental domain (i.e. a fundamental domain for the action of $\pi_1(M)$ on $\widetilde{M}$). Set $K := \overline{D}$ to be the topological closure. Suppose that for each t, and for every $x \in \partial K$ (the topological boundary) there is a $\widetilde{J}_t$-holomorphic hyperplane H_x through x, with $H_x \subset K$, such that $\pi(H_x) \subset M$ is a closed submanifold and such that $A \cdot \pi_*([H_x]) \leq 0$. Define:*

$$e_t(u) := \int_{\mathbb{CP}^1} u^* \omega_t.$$

Then

$$\sup_{u,t} e_t(u) < \infty,$$

where the supremum is over all pairs (u,t), $u : \mathbb{CP}^1 \to M$ is J_t-holomorphic and in class A.

Proof.

Lemma 2.8. *Let M, A be as above, let $D \subset \widetilde{M}$, with $\pi : \widetilde{M} \to M$ the universal cover of M, be a fundamental domain, and $K := \overline{D}$ its topological closure. Let (ω, J) be a compatible* lcs *pair on M such that for every $x \in \partial K$ there is a $\widetilde{J}$-holomorphic (real codimension 2) hyperplane $H_x \subset K \subset \widetilde{M}$ through x, such that $\pi(H_x) \subset M$ is a closed submanifold and such that $A \cdot [\pi(H_x)] \leq 0$. Then any genus 0, J-holomorphic class A curve u in M has a lift $\widetilde{u}$, to the universal cover, with image in K.*

Proof. For u as in the statement, let $\widetilde{u}$ be a lift intersecting the fundamental domain D, (as in the statement of main theorem). Suppose that $\widetilde{u}$ intersects ∂K, otherwise we already have image $\widetilde{u} \subset D = int(K)$, since image $\widetilde{u}$ is connected (and by elementary topology). Then $\widetilde{u}$ intersects u_x as in the statement, for some x. So u is a J-holomorphic map intersecting the closed hyperplane $\pi(H_x)$ with $A \cdot [\pi(H_x)] \leq 0$. By positivity of intersections [10, Section 2.6], which in this case is just a simple exercise, image $u \subset \pi(H_x)$, and so image $\widetilde{u} \subset H_x$. And so image $\widetilde{u} \subset \partial K$. □

Now, let $u : \mathbb{CP}^1 \to M$ be a J_t-holomorphic class A curve. By the lemma above u has a lift $\widetilde{u}$ contained in the compact $K \subset \widetilde{M}$. Then we have:

$$e_t(u) = \int_{\mathbb{CP}^1} \widetilde{u}^* \widetilde{\omega}_t \leq C_t \langle \widetilde{\omega}_t^{symp}, A \rangle,$$

where $\widetilde{\omega}_t = f_t \widetilde{\omega}_t^{symp}$, for $\widetilde{\omega}_t^{symp}$ symplectic on $\widetilde{M}$, and $f_t : \widetilde{M} \to \mathbb{R}$ positive function constructed as in the proof of Lemma 2.3, and where $C_t = \max_K f_t$. Since $\{\omega_t\}$ is continuous in $\mathcal{T}_0$, we have that $\{f_t\}$, $\{\widetilde{\omega}_t^{symp}\}$ are C_0 continuous families in t. In particular

$$C = \sup_t \max_K f_t$$

and

$$E = \sup_t \langle \widetilde{\omega}_{t'}^{symp}, A \rangle$$

are finite. And so

$$\sup_{(u,t)} e_t(u) \leq C \cdot E,$$

where the supremum is over all pairs (u, t), u is J_t-holomorphic, class A, curve in M as above. □

3. Quick review of genus 0 Gromov-Witten theory

Let M be a compact smooth manifold with a pair (ω, J) for ω a non-degenerate smooth 2-form and J an almost complex structure. We assume that $\omega(\cdot, J\cdot)$ is a J-invariant inner product on M. We will call the above data (M, ω, J) an ***almost symplectic triple***.

Let

$$\mathcal{M}_{0,n}(J, A) = \mathcal{M}_{0,n}(M, J, A)$$

denote the moduli space of isomorphism classes of class A, J-holomorphic curves in M, with domain the Riemann sphere, with n marked labeled points $\{x_1, \ldots x_n\}$. In other words, $\mathcal{M}_{0,n}(J, A)$ is the set of isomorphism classes of tuples $(u, \{x_1, \ldots, x_n\})$, where $u : \mathbb{CP}^1 \to M$ is a J-holomorphic map. Here an isomorphism between $(u_1, \{x_1, \ldots, x_n\})$ and $(u_2, \{x'_1, \ldots, x'_n\})$ is a biholomorphism $\phi : \mathbb{CP}^1 \to \mathbb{CP}^1$, s.t. $\phi(x_i) = x'_i$ and s.t. $u_2 \circ \phi = u_1$. Let

$$e_\omega : \mathcal{M}_{0,n}(J, A) \to \mathbb{R},$$

be the energy:

$$e_\omega([u]) := e_\omega(u) := \int_{\mathbb{CP}^1} u^* \omega,$$

where we take any representative u of the class $[u]$. (Note that this (up to a factor) is the L^2 energy of the map u with respect to appropriate inner products, see [10, Section 2.2]).

Notation 1. In what follows we usually neglect to distinguish classes and representatives. As this should be clear from context. So from now on we just write u.

Let $\{(M,\omega_t,J_t)\}$, $t\in[0,1]$, be a family of almost symplectic triples with $\{(\omega_t,J_t)\}$ varying smoothly in t. We will say that $\{(M,\omega_t,J_t)\}$ is a ***smooth family of almost symplectic triples***. Given a smooth family of almost symplectic triples $\{(M,\omega_t,J_t)\}$, $t\in[0,1]$, we denote by

$$\mathcal{M}_{0,n}(\{J_t\},A)$$

the space of pairs (u,t), $u\in\mathcal{M}_{0,n}(J_t,A)$. (Dropping the marked points from the notation.)

The following is well known and follows by the same argument as [10, Theorem 5.6.6].

Theorem 3.1. *Let (M,ω,J) be as above. Then $\mathcal{M}_{0,n}(M,J,A)$ has a pre-compactification*

$$\overline{\mathcal{M}}_{0,n}(M,J,A),$$

by Kontsevich stable maps, with respect to the natural metrizable Gromov topology [10, Chapter 5.6]. *Moreover given $E>0$, the subspace $\overline{\mathcal{M}}_{g,0}(J,A)_E\subset\overline{\mathcal{M}}_{g,0}(J,A)$ consisting of elements u with $e_\omega(u)\leq E$ is compact. In other words $e=e_\omega$ is a proper function on $\overline{\mathcal{M}}_{g,0}(J,A)$. Similarly, if $\{(M,\omega_t,J_t)\}$ is a smooth family of almost symplectic triples, and we define*

$$e:\overline{\mathcal{M}}_{0,n}(\{J_t\},A)\to\mathbb{R}$$

by

$$e(u,t)=e_{\omega_t}(u),$$

then e is a proper function.

Thus the most basic situation where we can talk about Gromov-Witten "invariants" of (M,J) is when the energy function is bounded on $\overline{\mathcal{M}}_{g,0}(J,A)$. In this case $\overline{\mathcal{M}}_{g,n}(J,A)$ is compact, and has a virtual moduli cycle as in the original approach of Fukaya-Ono [6], or the more algebraic approach of Pardon [12]. So we may define, as usual, functionals called the Gromov-Witten invariants:

$$GW_{g,n}(A,J):H_*(\overline{M}_{g,n})\otimes H_*(M^n)\to\mathbb{Q}, \tag{3.2}$$

where $\overline{M}_{g,n}$ denotes the compactified moduli space of Riemann surfaces. Of course closed symplectic manifolds with any tame almost complex structure is one class of examples, where these functionals are defined, as in that case we have a priori bounds on the energy of holomorphic curves in a fixed class.

Even when defined, these functionals will not in general be J-invariant, but it is immediate, again by Pardon [12], that they are invariant for a smooth family $\{J_t\}$, $t\in[0,1]$ such that the corresponding "cobordism moduli space": $\overline{\mathcal{M}}_{g,0}(\{J_t\},A)$, is compact.

4. Main argument

We will first state and prove a more general result, from which Theorem 1.3 will be deduced.

Theorem 4.1. *Let ω be a split symplectic form on $M=S^2\times T^{2n-2}$, let A be as above with $\langle\omega,A\rangle=\pi r^2$. Let $\{\omega_t\}$, $t\in[0,1]$, $\omega_0=\omega$ be a $\mathcal{T}^0$-continuous family of* lcs *forms on M. Set $R>r$, then there is an $\epsilon>0$ (depending only on R,r,ω) s.t. if $d_0(\omega_t,\omega_0)<\epsilon$ for all t, then there is no symplectic embedding*

$$\phi:(B_R,\omega_{st})\hookrightarrow(M,\omega_1)$$

meaning an embedding ϕ such that $\phi^*\omega_1=\omega_{st}$.

Proof of Theorem 4.1. We will first setup certain kinds of fundamental domains, in the sense of Theorem 2.7, that will be useful. We have real codimension 1 hypersurfaces

$$\Sigma_i=S^2\times(S^1\times\ldots\times S^1\times\{pt\}\times S^1\times\ldots\times S^1)\subset M,$$

where the singleton $\{pt\}\subset S^1$ replaces the i'th factor of $T^{2n-2}=S^1\times\ldots\times S^1$. The hypersurfaces Σ_i are naturally folliated by the ω-symplectic submanifolds

$$M^i_\theta=S^2\times(S^1\times\ldots\times S^1\times\{pt\}\times\{\theta\}\times S^1\times\ldots\times S^1)\simeq S^2\times T^{2n-2},$$

$\theta \in S^1$. We denote by $T^{fol}\Sigma_i \subset TM$, the distribution of vectors tangent to the leaves of the above mentioned folliation. In other words

$$T^{fol}\Sigma_i = \cup_\theta i_* TM^i_\theta,$$

where $i : M^i_\theta \to M$ are the inclusion maps. Set $\Sigma = \bigcup_i \Sigma_i$, and $U = M - \Sigma$. Then U is the interior of a fundamental domain we call D.

Let D' be a fundamental domain of the following type.

- Similarly to Σ, the boundary $\Sigma' = \partial D'$, is of the form $\Sigma' = \cup_i \Sigma'_i$ and there is a distinguished folliation of each Σ'_i as in the case of Σ.
- For each i, the leafs of the above folliation of Σ'_i are ω-symplectic, dimension $2n-2$ submanifolds L identified with

$$S^2 \times S^1 \times \ldots \times S^1 \times \{pt_1\} \times \{pt_2\}, \tag{4.2}$$

  for some points $pt_1, pt_2 \in S^1$, up to an action of the natural diffeomorphism of M, given by permuting the factors S^1, in the expression $M = S^2 \times (S^1 \times \ldots \times S^1)$.

We call D' a ***good*** fundamental domain.

We are ready for the main argument. Fix an $\epsilon' > 0$ s.t. any 2-form ω_1 on M, C^0 ϵ'-close to ω satisfies:

- ω_1 is non-degenerate.
- ω_1 is non-degenerate on submanifolds of type L above. In other words these are the submanifolds identified with the submanifold (4.2) up to an action of the natural diffeomorphism of M, permuting the factors S^1, in the expression $M = S^2 \times (S^1 \times \ldots \times S^1)$.

Suppose by contradiction that for every $\epsilon > 0$ there is a $\mathcal{T}^0$-continuous homotopy $\{\omega_t\}$ of lcs forms, with $\omega_0 = \omega$, such that $\forall t : d_0(\omega_t, \omega) < \epsilon$ and such that there exists a symplectic embedding

$$\phi : B_R \hookrightarrow (M, \omega_1).$$

Take $\epsilon < \epsilon'$, and let $\{\omega_t\}$ be as in the hypothesis above, and let ϕ be as in the hypothesis. Then $B = \operatorname{image} \phi$ is contractible in M, hence its lift $\widetilde{B}$ to $\widetilde{M}$ satisfies the condition 2.6. (With respect to $G = \pi_1(M)$ action.) And hence, B is contained in some good fundamental domain D'. The latter claim follows by basic topology (given the previously mentioned existence result for fundamental domains), the reader may first consider the simpler case $M = S^2 \times T^2$, where the claim can be checked by hand.

As the only thing relevant in the following argument is the symplectic folliation of boundary Σ' of D', as above, we may assume for simplicity that $D' = D$, and so $\Sigma' = \Sigma$.

We may extend $\phi_* j$ to an ω_1-compatible almost complex structure J_1 on M, preserving $T^{fol}\Sigma_i$ for each i using:

- $\operatorname{image} \phi$ does not intersect Σ.
- The non-degeneracy of ω_t on the leaves of Σ_i, for each i, t, which follows by the condition $\epsilon < \epsilon'$, and the defining condition of ϵ'.
- The well known existence/flexibility results for almost complex structures on symplectic vector bundles.

We may then extend this to a family $\{J_t\}$ of almost complex structures on M, s.t. J_t is ω_t-compatible for each t, with J_0 the standard split complex structure on M and such that J_t preserves $T^{fol}\Sigma_i$ for each t, i. The latter condition can be satisfied by similar reasoning as above.

Then the family $\{(\omega_t, J_t)\}$ satisfies the hypothesis of Theorem 2.7 for the class $A = [S^2] \otimes [pt]$ as in the statement of the theorem we are proving. Then by Theorem 2.7, L^2 energy e is bounded on

$$C = \overline{\mathcal{M}}_{0,1}(\{J_t\}, A)$$

and hence C is compact by Theorem 3.1.

Now we have the classical Gromov-Witten invariant counting class A, J_0-holomorphic, genus 0 curves passing through a fixed point:

$$GW_{0,1}(A, J_0)([pt]) = 1,$$

whose calculation already appears in [7]. Then by compactness of C, and the discussion preceding the proof:

$$GW_{0,1}(A, J_1)([pt]) = 1.$$

In particular there is a class A J_1-holomorphic curve $u : \mathbb{CP}^1 \to M$ passing through $\phi(0)$.

By Lemma 2.8 we may choose a lift $\widetilde{u}$ of u to $\widetilde{M}$, with homology class $[\widetilde{u}]$ also denoted by A so that the image of $\widetilde{u}$ is contained in a compact set $K \subset \widetilde{M}$, (independent of the choice of $\{\omega_t\}, \{J_t\}$ satisfying above conditions). Let $\widetilde{\omega}_t^{symp}$ and f_t be as in Lemma 2.3, then by this lemma for every $\delta > 0$ we may find an $\epsilon > 0$ so that if $d_0(\omega_1, \omega) < \epsilon$ then $d_{C^0}(\widetilde{\omega}^{symp}, \widetilde{\omega}_1^{symp}) < \delta$ on K, and $\sup_K |f_1 - 1| < \delta$.

Let δ as above be chosen, and let ϵ correspond to this δ. Now we have:

$$|\langle \widetilde{\omega}_1^{symp}, A\rangle - \pi \cdot r^2| = |\langle \widetilde{\omega}_1^{symp}, A\rangle - \langle \widetilde{\omega}^{symp}, A\rangle| \leq \delta\pi \cdot r^2,$$

as $\langle \widetilde{\omega}^{symp}, A\rangle = \pi r^2$, and as $d_{C^0}(\widetilde{\omega}^{symp}, \widetilde{\omega}_1^{symp}) < \delta$. And we have

$$\max_K f_1 \leq 1 + \delta.$$

So choosing ϵ, δ appropriately we get

$$|\int_{\mathbb{CP}^1} u^*\omega_1 - \pi r^2| \leq |\max_K f_1 \langle \widetilde{\omega}_1^{symp}, A\rangle - \pi \cdot r^2| < \pi R^2 - \pi r^2.$$

Consequently,

$$\int_{\mathbb{CP}^1} u^*\omega_1 < \pi R^2.$$

We may then proceed exactly as in the now classical proof of Gromov [7] of the non-squeezing theorem to get a contradiction and finish the proof. A bit more specifically, $\phi^{-1}(\operatorname{image}\phi \cap \operatorname{image} u)$ is a minimal surface in B_R, with boundary on the boundary of B_R, and passing through $0 \in B_R$. By construction it has area strictly less then πR^2 which is impossible by the classical monotonicity theorem of differential geometry. □

Proof of Theorem 1.3. Let ϵ be as given by the Theorem 4.1. By Proposition 2.2 there is a ϵ' s.t. whenever $\omega_0, \omega_1 \in LCS(M)$ are C^1 ϵ'-close, they are $\mathcal{T}_0$ ϵ-close.

Let $\{\omega_t\}$ be given as in the hypothesis, and such that $d_{C^1}(\omega_0, \omega_t) < \epsilon'$ for all t. By Proposition 2.2. $\{\omega_t\}$ is $\mathcal{T}^0$-continuous, and by the discussion above

$$\forall t : d_0(\omega_0, \omega_t) < \epsilon.$$

So applying Theorem 4.1 we are done. □

Proof of Theorem 1.2. We only sketch the proof, as it is basically just a special case of Theorem 4.1. For ϵ taken to be sufficiently small, the family $\omega_t = t\omega_0 + (1-t)\omega'$ is a family of symplectic forms on M. Then proceed as in the proof Theorem 4.1, upon noting that we do not need additional assumptions on the embedding ϕ or the family ω_t, to have compactness of the relevant moduli spaces. So compactness is automatic, and the proof goes through as before. □

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References

[1] V. Apostolov and G. Dloussky, *Locally conformally symplectic structures on compact non-Kähler complex surfaces*, Int. Math. Res. Not., 2016 (2016), pp. 2717–2747.

[2] S.-F. Chiu, *Nonsqueezing property of contact balls*, Duke Math. J., 166 (2017), pp. 605–655.

[3] Y. Eliashberg, S. S. Kim, and L. Polterovich, *Geometry of contact transformations and domains: orderability versus squeezing.*, Geom. Topol., 10 (2006), pp. 1635–1748.

[4] Y. Eliashberg and E. Murphy, *Making cobordisms symplectic*, J. Am. Math. Soc., 36 (2023), pp. 1–29.

[5] M. Fraser, *Contact non-squeezing at large scale in $\mathbb{R}^{2n} \times S^1$.*, Int. J. Math., 27 (2016), p. 25.

[6] K. Fukaya and K. Ono, *Arnold Conjecture and Gromov–Witten invariant*, Topology, 38 (1999), pp. 933 – 1048.

[7] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds.*, Invent. Math., 82 (1985), pp. 307–347.

[8] M. Kapovich, *A note on properly discontinuous actions*, url = https://arxiv.org/abs/2301.05325, (2023).

[9] H.-C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus.*, Am. J. Math., 65 (1943), pp. 433–438.

[10] D. McDuff and D. Salamon, *J–holomorphic curves and symplectic topology*, no. 52 in American Math. Society Colloquium Publ., Amer. Math. Soc., 2004.

[11] Y.-G. Oh and Y. Savelyev, *Pseudoholomorphic curves on the lcs-fication of contact manifolds*, https://arxiv.org/pdf/2107.03551.pdf, Accepted by Advances in geometry, (2022).

[12] J. Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of J-holomorphic curves*, Geometry and Topology.

[13] Stefan Müller, *Epsilon-non-squeezing and C_0-rigidity of epsilon-symplectic embeddings*, arXiv:1805.01390, (2018).

Email address: yasha.savelyev@gmail.com

University of Colima, CUICBAS