A CONFORMAL SYMPLECTIC WEINSTEIN CONJECTURE

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Abstract. We introduce a direct generalization of the Weinstein conjecture to certain locally conformally symplectic, or just lcs, manifolds. This conjectures existence of certain 1-d curves in the manifold, we call Reeb curves. We give partial verifications of this CSW conjecture based on extended Gromov-Witten theory of the lcs-fication $C \times S^1$ of a contact manifold (C, λ). In particular we show that the extended Gromov-Witten invariant counting certain charged elliptic curves in $C \times S^1$ is identified with the extended classical Fuller index of the Reeb vector field R^λ. By extended we mean that these invariants can be $\pm\infty$-valued. We also show that in some cases the failure of this conjecture implies existence of sky catastrophes for families of holomorphic curves in a lcs manifold. No examples of the latter phenomenon are known to exist, even in the un-tamed almost complex world. This phenomenon, if it exists, would be analogous to sky catastrophes in dynamical systems discovered by Fuller.

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1. Introduction

The theory of pseudo-holomorphic curves in symplectic manifolds as initiated by Gromov and Floer has revolutionized the study of symplectic and contact manifolds. What the symplectic form gives that is missing for a general almost complex manifold is a priori energy bounds for pseudo-holomorphic curves a fixed class. On the other hand there is a natural structure which directly generalizes both symplectic and contact manifolds, called locally conformally symplectic structure or lcs structure for short.

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Definition 1.1. *A locally conformally symplectic manifold or sometimes just* lcsm *is a smooth* $2n$*-fold* M *with an* lcs *structure: which is a non-degenerate 2-form* ω*, which is locally diffeomorphic to* $f \cdot \omega_{st}$*, for some (non-fixed) positive smooth function* f*, with* ω_{st} *the standard symplectic form on* $\mathbb{R}^{2n}$*. Explicitly, for every* $p \in M$ *there is a smooth chart*

$$\phi : V \subset \mathbb{R}^{2n} \to M,$$

so that $\phi(V) \ni p$*, and* $\phi^* \omega = f \cdot \omega_{st}$*, for some smooth positive* f*.*

It is natural to try to do Gromov-Witten theory for such manifolds. The first problem that occurs is that a priori energy bounds are gone, as since ω is not necessarily closed, the L^2-energy can now be unbounded on the moduli spaces of J-holomorphic curves in such a (M, ω). Strangely a more acute problem is potential presence of holomorphic sky catastrophes - given a smooth family $\{J_t\}$, $t \in [0, 1]$, of $\{\omega_t\}$-compatible almost complex structures, we may have a continuous family $\{u_t\}$ of J_t-holomorphic curves s.t. energy$(u_t) \mapsto \infty$ as $t \mapsto a \in (0, 1)$ and s.t. there are no holomorphic curves for $t \geq a$. These are analogues of sky catastrophes discovered by Fuller [8] for closed orbits of dynamical systems.

Even when it is impossible to tame these problems we show that there can still be an extended Gromov-Witten type theory which is analogous to the theory of extended Fuller index in dynamical systems, [24]. In a very particular situation the relationship with the Fuller index becomes perfect as one of the results of this paper obtains the (extended) Fuller index for Reeb vector fields on a contact manifold C as a certain (extended) genus 1 Gromov-Witten invariant of the Banyaga lcsm $C \times S^1$, see Example 1. The latter also gives a conceptual interpretation for why the Fuller index is rational, as it is reinterpreted as an (virtual) orbifold Euler number.

Inspired by this, we conjecture that certain lcsm's must poses certain 1-d curves that we call Reeb curves. This is a direct generalization of the Weinstein conjecture, which we call the conformal symplectic Weinstein conjecture. We prove this CSW conjecture for certain lcs structures C^3 nearby to Banyaga type lcs structures on $C \times S^1$. This partly uses the above mentioned connection of Gromov-Witten theory of $C \times S^1$ with the classical Fuller index. Since the original Weinstein conjecture is already proved, Taubes [27], for C a closed contact three-fold, the indirect evidence for the CSW conjecture at least on 4-folds of the type $C \times S^1$ is very good. In addition to the C^3 neighborhood version, we also prove a stronger result that relates the CSW conjecture to existence of holomorphic sky catastrophes.

Finally, we should exclaim that the Gromov-Witten theory in this story plays a local (in the space of structures) role, unless addition global geometric control is obtained, as in [23]. This is analogous to what happens with Fuller index in dynamical systems. A global lcs invariant, which takes the form of a homology theory, is under development, but many ingredients for this are already present here. (For example generators, and appropriate almost complex structures.)

1.1. **Some background on locally conformally symplectic manifolds.** These were originally considered by Lee in [11], arising naturally as part of an abstract study of "a kind of even dimensional Riemannian geometry", and then further studied by a number of authors see for instance, [2] and [28]. This is a fascinating object, a lcsm admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and Murphy show that if a closed almost complex $2n$-fold M has $H^1(M, \mathbb{R}) \neq 0$ then it admits a lcs structure, [4]. These lcs structures are Lichnerowitz exact. Another result of Apostolov, Dloussky [1] is that any complex surface with an odd first Betti number admits an lcs structure, which tames the complex structure. In this case the corresponding lcs structures are generally non-exact.

To see the connection with the first cohomology group, let us point out right away the most basic invariant of a lcs structure ω when M has dimension at least 4: the Lee class, $\alpha = \alpha_\omega \in H^1(M, \mathbb{R})$. This has the property that on the associated α-covering space $\widetilde{M}$, the lift $\widetilde{\omega}$ is globally conformally symplectic. The class α may be defined as the following Cech 1-cocycle. Let $\phi_{a,b}$ be the transition map for lcs charts ϕ_a, ϕ_b of (M, ω). Then $\phi_{a,b}^* \omega_{st} = g_{a,b} \cdot \omega_{st}$ for a positive real constant $g_{a,b}$ and $\{\ln g_{a,b}\}$ gives our 1-cocycle. Thus an lcs form is globally conformally symplectic iff its Lee class vanishes.

Again assuming M has dimension at least 4, the Lee class α has a natural differential form representative, called the Lee form and defined as follows. We take a cover of M by open sets U_a in which $\omega = f_a \cdot \omega_a$ for ω_a symplectic, and f_a a positive smooth function. Then we have 1-forms $d(\ln f_a)$ in each U_a which glue to a well defined closed 1-form on M, as shown by Lee. By slight abuse, we denote this 1-form, its cohomology class and the Cech 1-cocycle from before all by α. It is moreover immediate that for an lcs form ω

$$d\omega = \alpha \wedge \omega,$$

for α the Lee form as defined above.

As we mentioned lcsm's can also be understood to generalize contact manifolds. This works as follows. First we have a natural class of explicit examples of lcsm's, obtained by starting with a symplectic cobordism (see [4]) of a closed contact manifold C to itself, arranging for the contact forms at the two ends of the cobordism to be proportional (which can always be done) and then gluing together the boundary components. As a particular case of this we get Banyaga's basic example.

Example 1 (Banyaga). Let (C, λ) be a contact $(2n+1)$-manifold where λ is a contact form, $\forall p \in C$: $\lambda \wedge \lambda^{2n}(p) \neq 0$, and take $M = C \times S^1$ with 2-form

$$\omega_\lambda = d^\alpha \lambda := d\lambda - \alpha \wedge \lambda,$$

for $\alpha := pr_{S^1}^* d\theta$, $pr_{S^1} : C \times S^1 \to S^1$ the projection, and λ likewise the pull-back of λ by the projection $C \times S^1 \to C$. We call (M, ω_λ) as above the ***lcs-fication*** of (C, λ).

The operator $d^\alpha : \Omega^k(M) \to \Omega^{k+1}(M)$ is called the Lichnerowicz differential with respect to a closed 1-form α, and satisfies $d^\alpha \circ d^\alpha = 0$ so that we have an associated Lichnerowicz chain complex.

We assume from now on unless explicitly stated otherwise that our manifolds have dimension at least 4.

1.2. **Conformal symplectic Weinstein conjecture.** An ***exact lcs structure*** on M is a pair (λ, α) with α a closed 1-form, s.t. $\omega = d^\alpha \lambda$ is non-degenerate. This determines a generalized distribution $\mathcal{V}_\lambda$:

$$\mathcal{V}_\lambda(p) = \{v \in T_pM | d\lambda(v, \cdot) = 0\},$$

which we call the ***vanishing distribution***. And we have a generalized distribution ξ_λ, which is defined to be the $d\lambda$-orthogonal complement to $\mathcal{V}_\lambda$, which we call ***co-vanishing distribution***. For each $p \in M$, $\mathcal{V}_\lambda(p)$ has dimension at most 2 since $d\lambda - \alpha \wedge \lambda$ is non-degenerate. If M^{2n} is closed $\mathcal{V}_\lambda$ cannot identically vanish since $(d\lambda)^n$ cannot be non-degenerate by Stokes theorem.

Definition 1.2. *Let (M, λ, α) be an exact* lcs *structure. We have a cone structure $C_\lambda \subset \mathcal{V}_\lambda$, with*

$$C_\lambda(p) := \{v \in \mathcal{V}_\lambda(p) | \lambda(v) > 0\}.$$

We propose that C_λ plays the role of the Reeb distribution in this context. And we say that a smooth map $o : S^1 \to M$ is a **Reeb curve** *for (M, λ, α) if it is tangent to C_λ, in other words*

$$\dot{o}(t) \in C_\lambda(o(t))$$

for each t.

Example 2. Let (C, λ) be a closed contact $(2n+1)$-fold with a contact form λ. The ***Reeb vector field*** R^λ on C is a vector field satisfying

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

A ***Reeb orbit*** of (C, λ) is a smooth map $o : S^1 \to C$ such that

$$\dot{o}(t) = cR_\lambda(o(t)),$$

for some $c > 0$ called period. If $(C \times S^1, d^\alpha \lambda)$ is the lcs-fication of (C, λ), then identifying $S^1 = \mathbb{R}/\mathbb{Z}$, S^1 acts on $C \times S^1$ by $s \cdot (x, \theta) = (x, \theta + s)$. Let

$$\frac{d}{d\theta} \subset \{0\} \oplus TS^1 \subset C \times S^1$$

denote the vector field generating this S^1 action. Then

$$C_\lambda = \{(v, u) \in R \oplus T \,|\, v \neq 0\},$$

where

$$R = \mathbb{R}_{>0} \cdot (R_\lambda \oplus \{0\})$$

$R^\lambda \oplus \{0\}$ is the section of $T(C \times S^1) \simeq TC \oplus \mathbb{R}$, corresponding to R^λ,

$$T = \{0\} \oplus \mathbb{R} \cdot \frac{\partial}{\partial \theta}.$$

We then have the following basic "conformal symplectic Weinstein conjecture", later on we state a stronger form of this conjecture.

Conjecture 1. *Let M be closed of dimension at least 4, and (λ, α) an exact* lcs *structure on M with α rational, then there is a Reeb curve for (M, λ, α).*

The dimension 2 case is special but some version (possibly same version) of the conjecture should hold in this case. As one trivial example, whose verification is left to the reader, given an exact lcs 2-manifold (M, λ, α), with $d\lambda = 0$ and with α rational, the conjecture holds automatically, just take the Reeb curve to parametrize a component of a regular fiber of the map $f : \Sigma \to S^1$ classifying α, that is so that $\alpha = q \cdot f^* d\theta$, for $q \in \mathbb{Q}$.

Lemma 1.3. *Conjecture 1 implies the Weinstein conjecture: every closed contact manifold (C, λ) has a Reeb orbit.*

Proof. Let λ be a contact form on a closed manifold C. Let $o : S^1 \to C \times S^1$ be a Reeb curve for lcs-fication $(C \times S^1, d^\alpha \lambda)$. Since $o_*(TS^1) \subset \mathcal{V}_\lambda$, $(pr_C)_* \circ o_*(TS^1) \subset \ker d\lambda \subset TC$. Since in addition $o^* \circ u^* \lambda$ is non-vanishing on TS^1, $pr_C \circ o$ is immersed in C and so is the image of a Reeb orbit. □

In what follows we use the following C^k metric on the space $\mathcal{L}(M)$ of exact lcs structures on M. For $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in \mathcal{L}(M)$ define:

$$d_k((\lambda_1, \alpha_1), (\lambda_2, \alpha_2)) = d_{C^k}(\lambda_1, \lambda_2) + d_{C^k}(\alpha_1, \alpha_2), \tag{1.4}$$

where d_{C^k} on the right side is the usual C^k metric.

For λ_H the standard contact structure on S^{2k+1}, so that its Reeb flow is the Hopf flow, we will call $\omega_H := d^\alpha \lambda_H$ the ***Hopf*** lcs ***structure***.

Theorem 1.5. *Conjecture 1 holds for a d_3 neighborhood of the Hopf* lcs *structure (λ_H, α) on $S^{2k+1} \times S^1$. More generally it holds for a d_3 neighborhood of the* lcs*-fication of a contact manifold (C, λ) whose Reeb flow has non-vanishing extended Fuller index in some homotopy class.*

This is proved in Section 3. Note that Seifert [25] initially found an analogous existence phenomenon of orbits on S^{2k+1} for a non-singular vector field C^0-nearby to the Hopf vector field, [1]. And he asked if the nearby condition can be removed, this became known as the Seifert conjecture. This turned out not to be quite true [10]. Likewise it is natural for us to conjecture that the nearby condition can be removed and this is the CSW conjecture. In our case this has some additional evidence that we discuss in the next section.

Directly extending Theorem 1.5 we have the following.

Theorem 1.6. *Let C be a closed contact manifold with contact form λ, with $i(R^\lambda, \beta) \neq 0$, for some class β, where the latter is the extended Fuller index, as described in Appendix A. Let (λ, α) be the associated exact* lcs*-structure on $M = C \times S^1$, the* lcs*-fication. Then either the conformal symplectic Weinstein conjecture holds for any exact* lcs *structure (λ', α') on M, so that $\omega_1 = d^{\alpha'} \lambda'$ is homotopic through* lcs *forms $\{\omega_t\}$ to $\omega_0 = d^\alpha \lambda$ or holomorphic sky catastrophes exist, (these are further discussed in Section 1.3).*

[1]With more careful analysis we can also likely relax C^3 condition to C^0 condition.

Example 3. Take $C = S^{2k+1}$ and $\lambda = \lambda_H$, then $i(R^\lambda, 0) = \pm\infty$, (sign depends on k), [24]. Or take C to be unit cotangent bundle of a hyperbolic manifold (X, g), λ the associated Louiville form, and (λ, α) the associated Banyaga lcs structure, in this case $i(R^\lambda, \beta) = \pm 1$ for every $\beta \neq 0$.

To motivate the above conjecture we need to introduce pseudo-holomorphic curves in lcs manifolds.

1.2.1. *Pseudo-holomorphic curves in exact* lcs *manifolds.* Banyaga type lcsm's give immediate examples of almost complex manifolds where the L^2 energy functional is unbounded on the moduli spaces of fixed class J-holomorphic curves, as well as where null-homologous J-holomorphic curves can be non-constant. We are going to see this shortly after developing a more general theory.

Definition 1.7. *Let (M, λ, α) be an exact lcs structure, $\omega = d^\alpha \lambda$. We say that an ω-compatible J is* **admissible** *if it preserves the splitting $\mathcal{V}_\lambda \oplus \xi_\lambda$, that is $J(\mathcal{V}_\lambda) \subset \mathcal{V}_\lambda$ and $J(\xi_\lambda) \subset \xi_\lambda$, and if $d\lambda$ tames J on ξ_λ. We call (M, λ, α, J) as above a* **tamed exact** lcs **structure***, this can also be abbreviated as (ω, J) where ω is an exact* lcs *structure.*

Lemma 1.8. *Let (M, λ, α, J) be a tamed exact* lcs *structure. Then given a smooth $u : \Sigma \to M$, where Σ is a closed (nodal) Riemann surface, u is J-holomorphic only if*

$$\operatorname{image} du(z) \subset \mathcal{V}_\lambda(u(z))$$

for all $z \in \Sigma$, in particular $u^* d\lambda = 0$.

Proof. For u J-holomorphic as above, we have

$$\int_\Sigma u^* d\lambda = 0$$

by Stokes theorem. Let $proj_{\xi_\lambda}(p) : T_pM \to \xi_\lambda(p)$ be the projection induced by the splitting $\mathcal{V}_\lambda \oplus \xi_\lambda$. Then if for some $z \in \Sigma$, $proj_{\xi_\lambda} \circ du(z) \neq 0$, since J is tamed by $d\lambda$ on ξ_λ and since J preserves the splitting $\mathcal{V}_\lambda \oplus \xi_\lambda$, we would have $\int_\Sigma u^* d\lambda > 0$. Thus

$$\forall z \in \Sigma : proj_{\xi_\lambda} \circ du(z) = 0,$$

so

$$\forall z \in \Sigma : \operatorname{image} du(z) \subset \mathcal{V}_\lambda(u(z)).$$

□

1.2.2. *Example, lcs-fication of a contact manifold.* Let (C, λ) be a closed contact $(2n+1)$-fold with a contact form λ. We also denote by λ the pull-back of λ by the projection $C \times S^1 \to C$, and by $\xi \subset T(C \times S^1)$ the distribution $\xi(p) = \ker d\lambda(p)$.

We take J to be an almost complex structure on ξ, which is S^1 invariant, and compatible with $d\lambda$. The latter means that

$$g_J(\cdot, \cdot) := d\lambda|_\xi(\cdot, J\cdot)$$

is a J invariant Riemannian metric on the distribution ξ.

There is an induced almost complex structure J^λ on $C \times S^1$, which is S^1-invariant, coincides with J on ξ and which satisfies:

$$J^\lambda(R^\lambda \oplus \{0\}(p)) = \frac{d}{d\theta}(p),$$

where $R^\lambda \oplus \{0\}$ is the section of $T(C \times S^1) \simeq TC \oplus \mathbb{R}$, as previously, and where

$$\frac{d}{d\theta} \subset \{0\} \oplus TS^1 \subset C \times S^1$$

denotes the vector field generating the action of S^1 on $C \times S^1$ as previously.

In previous terms $(C \times S^1, \lambda, \alpha, J^\lambda)$ is a tamed exact lcs structure. We now consider a moduli space of holomorphic tori in $C \times S^1$, which have a certain charge, this charge condition is also studied Oh-Wang [20], and I am grateful to Yong-Geun Oh for related discussions. Partly the reason for introduction of "charge" is that it is now possible for non-constant holomorphic curves to be null-homologous, so we need additional control. Here is a simple example take $S^3 \times S^1$ with $J = J^\lambda$, for the λ the standard contact form, then all the Reeb holomorphic tori (as defined further below) are null-homologous. In many cases we can just work with homology classes $A \neq 0$, but this is inadequate for our setup for conformal symplectic Weinstein conjecture.

Let Σ be a complex torus with a chosen marked point $z \in \Sigma$. These are also known as elliptic curves. An isomorphism $\phi : (\Sigma_1, z_1) \to (\Sigma_2, z_2)$ is a biholomorphism s.t. $\phi(z_1) = z_2$. The set of isomorphism classes forms a smooth orbifold $M_{1,1}$, with a natural compactification, the Deligne-Mumford compactification $\overline{M}_{1,1}$, by adding a point at infinity corresponding to a nodal curve.

Suppose then (M, ω) is an lcs manifold, J ω-compatible almost complex structure, and α the Lee class corresponding to ω. Assuming for simplicity, at the moment, (otherwise take stable maps) that (M, J) does not admit non-constant J-holomorphic maps $(S^2, j) \to (M, J)$, we define:

$$\overline{\mathcal{M}}_{1,1}^{1,0}(J, A)$$

as a set of equivalence classes of tuples (u, S), for $S = (\Sigma, z) \in \overline{M}_{1,1}$, and $u : \Sigma \to M$ a J-holomorphic map satisfying the ***charge (1,0) condition***: there exists a pair of generators ρ, γ for $H_1(\Sigma, \mathbb{Z})$, such that

$$\langle \rho, u_* \alpha \rangle = 1$$
$$\langle \gamma, u_* \alpha \rangle = 0,$$

and with $[u] = A$. The equivalence relation is $(u_1, S_1) \sim (u_2, S_2)$ if there is an isomorphism $\phi : S_1 \to S_2$ s.t. $u_2 \circ \phi = u_1$.

Note that the charge condition directly makes sense for nodal curves. And it is easy to see that the charge condition is preserved under Gromov convergence, and obviously a charge (1,0) J-holomorphic map cannot be constant for any A.

By slight abuse we may just denote such an equivalence class above by u, so we may write $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(J, A)$, with S implicit.

1.2.3. *Reeb holomorphic tori in* $(C \times S^1, J^\lambda)$. For the almost complex structure J^λ as above we have one natural class of charge (1,0) holomorphic tori in $C \times S^1$. Let o be a period c Reeb orbit o of R^λ, that is a map:

$$o : S^1 \to C,$$
$$\dot{o}(s) = c \cdot R^\lambda(o(s)), c > 0.$$

A ***Reeb torus*** u_o for o, is the map

$$u_o(s, t) = (o(s), t),$$

$s, t \in S^1$. A Reeb torus is J^λ-holomorphic for a uniquely determined holomorphic structure j on T^2 defined by:

$$j(\frac{\partial}{\partial s}) = c \frac{\partial}{\partial t}.$$

Let $\widetilde{S}(\lambda)$ denote the space of general period λ-Reeb orbits. There is an S^1 action on this space, with $\theta \cdot o$ the orbit

$$\theta \cdot o(s) = o(s + \theta).$$

Let $S(\lambda) := \widetilde{S}(\lambda)/S^1$ denote the quotient by this action. We have a map:

$$R : S(\lambda) \to \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A), \quad R(o) = u_o.$$

Proposition 1.9. *The map R is a bijection.* [2]

[2]It is in fact an equivalence of the corresponding topological action groupoids, but we do not need this explicitly.

So in the particular case of J^λ, the domains of elliptic curves in $C \times S^1$ are "rectangular", that is are quotients of the complex plane by a rectangular lattice, however for a more general almost complex structure on $C \times S^1$, tamed by more general lcs forms as we soon consider, the domain almost complex structure on our curves can in principle be arbitrary, in particular we might have nodal degenerations. Also note that the expected dimension of $\overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$ is 0. It is given by the Fredholm index of the operator (3.2) which is 2, minus the dimension of the reparametrization group (for non-nodal curves) which is 2. That is given an elliptic curve $S = (\Sigma, z)$, let $\mathcal{G}(\Sigma)$ be the 2-dimensional group of biholomorphisms ϕ of Σ. And given a J-holomorphic map $u : \Sigma \to M$, (Σ, z, u) is equivalent to $(\Sigma, \phi(z), u \circ \phi)$ in $\overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$, for $\phi \in \mathcal{G}(\Sigma)$.

In Theorem 3.5 we relate the (extended) count (Gromov-Witten invariant) of these curves to the (extended) Fuller index, which is reviewed in the Appendix A. This will be one ingredient for the following.

Theorem 1.10. *Let $M = S^{2k+1} \times S^1$, $\omega_H := d^\alpha \lambda_H$ the Hopf lcs structure. Then there exists a $\delta > 0$ s.t. for any exact* lcs *structure (λ', α') on M d_0 δ-close to (λ_H, α), and J compatible with $\omega' = d^{\alpha'} \lambda'$ and C^2 δ-close to J^{λ_H}, there exists an elliptic, charge (1,0), J-holomorphic curve in $S^{2k+1} \times S^1$. Moreover, if $k = 1$ and J is admissible then this curve may be assumed to be non-nodal and embedded.*

The following is to be proved in Section 3.

Theorem 1.11. *Let (M, λ, α, J) be a tamed exact* lcs *structure, if α is rational then every non-constant J-holomorphic curve $u : \Sigma \to M$ contains a Reeb curve, meaning that there is a $S_0 \simeq S^1 \subset \Sigma$ s.t. $u|_{S_0}$ is a Reeb curve. If moreover Σ is smooth, connected and immersed then $\Sigma \simeq T^2$.*

In a sense the above discussion tells us that J-holomorphic curves strictify Reeb curves, in the sense that Reeb curves satisfy a partial differential relation while J-holomorphic curves satisfy a partial differential equation, but given a solution of the former we also the latter. Strictifying could be helpful because the "strict" objects may possibly be counted in some way.

It makes sense to try to partially strictify Reeb curves more directly.

Definition 1.12. *Let (M, λ, α) be an exact* lcs *structure, Σ a closed possibly nodal Riemann surface. A smooth map $u : \Sigma \to M$ is called a* **Reeb 2-curve** *if $u_*(T\Sigma) \subset \mathcal{V}_\lambda$ and if there is a smooth map $o : S^1 \to \Sigma$ s.t. $\forall s \in S^1 : o^* u^* \lambda(s) \neq 0$.*

By Theorem 1.11 J-holomorphic curves give examples of Reeb 2-curves. More generally, for u satisfying the first condition, the second condition is satisfied for example if α is rational and $u^*\alpha \wedge u^*\lambda$ is symplectic except at finitely many points. The proofs of theorems 1.5, 1.6 actually produce Reeb 2-curves, through which we then deduce existence of Reeb curves. So it makes sense to further conjecture the following.

Conjecture 2. *Let M be closed, of dimension at least 4, and ω an exact* lcs *form on M whose Lee form α is rational, then there is a Reeb 2-curve in M.*

The above conjectures are not just a curiosity. In contact geometry, rigidity is based on existence phenomena of Reeb orbits, and lcs manifolds may be understood as generalized contact manifolds. To attack rigidity questions in lcs geometry, it is then likely that we need an analogue of Reeb orbits, we propose that this analogue is Reeb curves.

1.2.4. *Connection with the extended Fuller index.* One of the main ingredients for the above is a connection of extended Fuller index with certain extended Gromov-Witten invariants. If β is a free homotopy class of a loop in C set

$$A_\beta = [\beta] \times [S^1] \in H_2(C \times S^1).$$

Then we have:

Theorem 1.13. *Suppose that λ is a contact form on a closed manifold C, so that its Reeb flow is definite type, see Appendix A, then*

$$GW_{1,1}(A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(R^\lambda, \beta),$$

where both sides are certain extended rational numbers $\mathbb{Q} \sqcup \{\pm\infty\}$ valued invariants, so that in particular if either side does not vanish then there are λ Reeb orbits in class β.

What about higher genus invariants of $C \times S^1$? Following the proof of Proposition 1.9, it is not hard to see that all J^λ-holomorphic curves must be branched covers of Reeb tori. If one can show that these branched covers are regular when the underlying tori are regular, the calculation of invariants would be fairly automatic from this data, see [32], [30] where these kinds of regularity calculation are made.

1.3. **Sky catastrophes.** This final introductory section will be of a more technical nature. The following is well known.

Theorem 1.14. *[[17, Proposition 4.1.4], [29]] Let (M, J) be a compact almost complex manifold, and $u : (S^2, j) \to M$ a J-holomorphic map. Given a Riemannian metric g on M, there is an $\hbar = \hbar(g, J) > 0$ s.t. if $e_g(u) < \hbar$ then u is constant, where e_g is the L^2-energy functional,*

$$e_g(u) = \text{energy}_g(u) = \int_{S^2} |du|^2 dvol.$$

Using this we get the following (trivial) extension of Gromov compactness. Let

$$\mathcal{M}_{g,n}(J, A) = \mathcal{M}_{g,n}(M, J, A)$$

denote the moduli space of isomorphism classes of class A, J-holomorphic curves in M, with domain a genus g closed Riemann surface, with n marked labeled points. Here an isomorphism between $u_1 : \Sigma_1 \to M$, and $u_2 : \Sigma_2 \to M$ is a biholomorphism of marked Riemann surfaces $\phi : \Sigma_1 \to \Sigma_2$ s.t. $u_2 \circ \phi = u_1$.

The following follows by the same argument as [17, Theorem 5.6.6]. We claim no originality.

Theorem 1.15. *Let (M, J) be an almost complex manifold. Then $\mathcal{M}_{g,n}(J, A)$ has a pre-compactification*

$$\overline{\mathcal{M}}_{g,n}(J, A),$$

by Kontsevich stable maps, with respect to the natural metrizable Gromov topology see for instance [17, Chapter 5.6], for genus 0 case, [21] for general case. Moreover given $E > 0$, the subspace $\overline{\mathcal{M}}_{g,n}(J, A)_E \subset \overline{\mathcal{M}}_{g,n}(J, A)$ consisting of elements u with $e(u) \leq E$ is compact, where e is the L^2 energy with respect to an auxillary metric. In other words e is a proper function.

Thus the most basic situation where we can talk about Gromov-Witten "invariants" of (M, J) is when the energy function is bounded on $\overline{\mathcal{M}}_{g,n}(J, A)$, and we shall say that J is ***bounded*** (in class A), later on we generalize this in terms of what we call ***finite type***. In this case $\overline{\mathcal{M}}_{g,n}(J, A)$ is compact, and has a virtual moduli cycle as in the original approach of Fukaya-Ono [7], or the more algebraic approach [21]. So we may define functionals:

$$GW_{g,n}(A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}, \tag{1.16}$$

where $\overline{M}_{g,n}$ denotes the compactified moduli space of Riemann surfaces. Of course symplectic manifolds with any tame almost complex structure is one class of examples, another class of examples comes from some locally conformally symplectic manifolds. (We can take for instance the lcs-fication of (C, λ) with the latter the unit cotangent bundle of a hyperbolic manifold, with λ the canonical Louisville form, and J as in Section 1.2.2).

Given a continuous in the C^∞ topology family $\{J_t\}$, $t \in [0, 1]$ we denote by $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ the space of pairs (u, t), $u \in \overline{\mathcal{M}}_{g,n}(J_t, A)$.

Definition 1.17. *We say that a continuous family $\{J_t\}$, $t \in [0,1]$ on a compact manifold M has a* **holomorphic sky catastrophe** *in class A if there is an element $u \in \overline{\mathcal{M}}_{g,n}(J_i, A)$, $i = 0, 1$ which does not belong to any open compact (equivalently energy bounded) subset of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$.*

Let us slightly expand this definition. If $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ is locally connected, so that the connected components are open, then we have a sky catastrophe in the sense above if and only if there is a $u \in \overline{\mathcal{M}}_{g,n}(J_i, A)$ which has a non-compact connected component in $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$. At this point in time there are no known examples of families $\{J_t\}$ with sky catastrophes.

Question 1. Do holomorphic sky catastrophes exist?

Really what we are interested in is whether they exist generically. The author's opinion is that they may appear even generically. However, if we further constrain the geometry to exact lcs structures as in Section 1.2, then the question becomes much more subtle, see also [24] for a related discussion on possible obstructions to sky catastrophes.

If holomorphic sky catastrophes are discovered, this would be a very interesting discovery. The original discovery by Fuller [8] of sky catastrophes in dynamical systems is one of the most important in dynamical systems, see also [26] for an overview.

2. Elements of Gromov-Witten theory of an lcs manifold

Suppose (M, J) is a compact almost complex manifold, where the almost complex structures J are assumed throughout the paper to be C^∞, and let $N \subset \overline{\mathcal{M}}_{g,k}(J, A)$ be an open compact subset with energy positive on N. The latter condition is only relevant when $A = 0$. We shall primarily refer in what follows to work of Pardon in [21], only because this is what is more familiar to the author, due to greater comfort with algebraic topology. But we should mention that the latter is a follow up to a profound theory that is originally created by Fukaya-Ono [7], and later expanded with Oh-Ohta [6].

The construction in [21] of implicit atlas, on the moduli space $\mathcal{M}$ of curves in a symplectic manifold, only needs a neighborhood of $\mathcal{M}$ in the space of all curves. So more generally if we have an almost complex manifold and an *open* compact component N as above, this will likewise have a natural implicit atlas, or a Kuranishi structure in the setup of [7]. And so such an N will have a virtual fundamental class in the sense of Pardon [21], (or in any other approach to virtual fundamental cycle, particularly the original approach of Fukaya-Oh-Ohta-Ono). This understanding will be used in other parts of the paper, following Pardon for the explicit setup. We may thus define functionals:

$$GW_{g,n}(N, A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}. \tag{2.1}$$

How do these functionals depend on N, J?

Lemma 2.2. *Let $\{J_t\}$, $t \in [0,1]$ be a Frechet smooth family. Suppose that $\widetilde{N}$ is an open compact subset of the cobordism moduli space $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ and that the energy function is positive on $\widetilde{N}$, (the latter only relevant when $A = 0$). Let*

$$N_i = \widetilde{N} \cap \left(\overline{\mathcal{M}}_{g,n}(J_i, A)\right),$$

then

$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1).$$

In particular if $GW_{g,n}(N_0, A, J_0) \neq 0$, there is a class A J_1-holomorphic stable map in M.

Proof of Lemma 2.2. We may construct exactly as in [21] a natural implicit atlas on $\widetilde{N}$, with boundary $N_0^{op} \sqcup N_1$, (op denoting opposite orientation). And so

$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1),$$

as functionals. □

The most basic lemma in this setting is the following, and we shall use it in the following section.

Definition 2.3. *An* **almost symplectic pair** *on M is a tuple (M, ω, J), where ω is a non-degenerate 2-form on M, and J is ω-compatible, meaning that $\omega(\cdot, J\cdot)$ defines J-invariant Riemannian metric. When ω is lcs we call such a pair an* **lcs pair***.*

Definition 2.4. *We say that a pair of almost symplectic pairs (ω_i, J_i) are* ***δ-close****, if $\{\omega_i\}$ are C^0 δ-close, and $\{J_i\}$ are C^2 δ-close, $i = 0, 1$. Define this similarly for a pair (g_i, J_i) for g a Riemannian metric and J an almost complex structure.*

Definition 2.5. *For an almost symplectic pair (ω, J) on M, and a smooth map $u : \Sigma \to M$ define:*

$$e_\omega(u) = \int_\Sigma u^*\omega.$$

By an elementary calculation this coincides with the L^2 g_J-energy of u, for $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$. That is $e_\omega(u) = e_{g_J}(u)$. In what follows by $f^{-1}(a, b)$, with f a function, we mean the preimage by f of the open set (a, b).

Lemma 2.6. *Given a compact M and an almost symplectic pair (ω, J) on M, suppose that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = e_\omega^{-1}(E^0, E^1)\right) \subset \left(V = e_\omega^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

with $\epsilon > 0$, $E^0 > 0$ and with $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$. Suppose also that $GW_{g,n}(N, J, A) \neq 0$. Then there is a $\delta > 0$ s.t. whenever (ω', J') is a compatible almost symplectic pair δ-close to (ω, J), there exists $u \in \overline{\mathcal{M}}_{g,n}(J', A) \neq \emptyset$, with

$$E^0 < e_{\omega'}(u) < E^1.$$

Proof of Lemma 2.6.

Lemma 2.7. *Given a Riemannian manifold (M, g), and J an almost complex structure, suppose that $N \subset \overline{\mathcal{M}}_{d,n}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = e_g^{-1}(E^0, E^1)\right) \subset \left(V = e_g^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

with $\epsilon > 0$, $E_0 > 0$, and with $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$. Then there is a $\delta > 0$ s.t. whenever (g', J') is δ-close to (g, J) if $u \in \overline{\mathcal{M}}_{g,n}(J', A)$ and

$$E^0 - \epsilon < e_{g'}(u) < E^1 + \epsilon$$

then

$$E^0 < e_{g'}(u) < E^1.$$

Proof of Lemma 2.7. Suppose otherwise then there is a sequence $\{(g_k, J_k)\}$ converging to (g, J), and a sequence $\{u_k\}$ of J_k-holomorphic stable maps satisfying

$$E^0 - \epsilon < e_{g_k}(u_k) \leq E^0$$

or

$$E^1 \leq e_{g_k}(u_k) < E^1 + \epsilon.$$

By Gromov compactness, specifically theorems [17, B.41, B.42], we may find a Gromov convergent subsequence $\{u_{k_j}\}$ to a J-holomorphic stable map u, with

$$E^0 - \epsilon \leq e_g(u) \leq E^0$$

or

$$E^1 \leq e_g(u) \leq E^1 + \epsilon.$$

But by our assumptions such a u does not exist. ∎

Lemma 2.8. *Let M be compact, and let (M, ω, J) be an almost symplectic triple, so that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is exactly as in the lemma above with respect to some $\epsilon > 0$. Then, there is a $\delta' > 0$ s.t. the following is satisfied. Let (ω', J') be δ'-close to (ω, J), then there is a continuous in the C^∞ topology family of almost symplectic pairs $\{(\omega_t, J_t)\}$, $(\omega_0, J_0) = (g, J)$, $(\omega_1, J_1) = (g', J')$ s.t. there is open compact subset*

$$\widetilde{N} \subset \overline{\mathcal{M}}_{g,n}(\{J_t\}, A),$$

and with

$$\widetilde{N} \cap \overline{\mathcal{M}}(J, A) = N.$$

Moreover if $(u, t) \in \widetilde{N}$ then

$$E^0 < e_{g_t}(u) < E^1.$$

Proof. For ϵ as in the hypothesis, let δ be as in Lemma 2.7.

Lemma 2.9. *Given a $\delta > 0$ there is a $\delta' > 0$ s.t. if (ω', J') is δ'-near (ω, J) there is an interpolating, continuous in C^∞ topology family $\{(\omega_t, J_t)\}$ with (ω_t, J_t) δ-close to (ω, J) for each t.*

Proof. Let $\{g_t\}$ be the family of metrics on M given by the convex linear combination of $g = g_{\omega_J}, g' = g_{\omega', J'}$. Clearly g_t is δ'-close to g_0 for each t. Likewise the family of 2 forms $\{\omega_t\}$ given by the convex linear combination of ω, ω' is non-degenerate for each t if δ' was chosen to be sufficiently small and is δ'-close to $\omega_0 = \omega_{g,J}$ for each moment.

Let

$$ret : Met(M) \times \Omega(M) \to \mathcal{J}(M)$$

be the "retraction map" (it can be understood as a retraction followed by projection) as defined in [16, Prop 2.50], where $Met(M)$ is space of metrics on M, $\Omega(M)$ the space of 2-forms on M, and $\mathcal{J}(M)$ the space of almost complex structures. This map has the property that the almost complex structure $ret(g, \omega)$ is compatible with ω, and that $ret(g_J, \omega) = J$ for $g_J = \omega(\cdot, J\cdot)$. Then $\{(\omega_t, ret(g_t, \omega_t)\}$ is a compatible family. As ret is continuous in C^2-topology, δ' can be chosen so that $\{ret_t(g_t, \omega_t\}$ are δ-nearby. ☐

Let δ' be chosen with respect to δ as in the above lemma and $\{(\omega_t, J_t)\}$ be the corresponding family. Let $\widetilde{N}$ consist of all elements $(u, t) \in \overline{\mathcal{M}}(\{J_t\}, A)$ s.t.

$$E^0 - \epsilon < e_{\omega_t}(u) < E^1 + \epsilon.$$

Then by Lemma 2.7 for each $(u, t) \in \widetilde{N}$, we have:

$$E^0 < e_{\omega_t}(u) < E^1.$$

In particular $\widetilde{N}$ must be closed, it is also clearly open, and is compact as the energy e is a proper function, as discussed. ☐

To finish the proof of the main lemma, let N be as in the hypothesis, δ' as in Lemma 2.8, and $\widetilde{N}$ as in the conclusion to Lemma 2.8, then by Lemma 2.2

$$GW_{g,n}(N_1, J', A) = GW_{g,n}(N, J, A) \neq 0,$$

where $N_1 = \widetilde{N} \cap \overline{\mathcal{M}}_{g,n}(J_1, A)$. So the conclusion follows. ☐

While not having sky catastrophes gives us a certain compactness control, the above is not immediate because we can still in principle have total cancellation of the infinitely many components of the moduli space $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$. In other words a virtual 0-dimension Kuranishi space $\overline{\mathcal{M}}^{1,0}(J^\lambda, A)$, with an infinite number of compact connected components, can certainly be null-cobordant, by a cobordism all of whose components are compact. So we need a certain additional algebraic and geometric control to preclude such a total cancellation.

Proof of Theorem 1.15. (Outline, as the argument is standard.) Suppose that we have a sequence u^k of J-holomorphic maps with L^2-energy $\leq E$. By [17, 4.1.1], a sequence u^k of J-holomorphic curves has a convergent subsequence if $sup_k ||du^k||_{L^\infty} < \infty$. On the other hand when this condition does not hold rescaling argument tells us that a holomorphic sphere bubbles off. The quantization Theorem 1.14, then tells us that these bubbles have some minimal energy, so if the total energy is capped by E, only finitely many bubbles may appear, so that a subsequence of u^k must converge in the Gromov topology to a Kontsevich stable map. □

3. Genus 1 curves in the lcsm $C \times S^1$ and the Fuller index

Proof of Proposition 1.9. Suppose we a have a curve without spherical nodal components $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$, represented by $u : \Sigma \to M = C \times S^1$. Since by Lemma 1.8, $u_*(T\Sigma) \subset \mathcal{V}_\lambda$, we get that

$$(pr_C \circ u)_*(T\Sigma) \subset \ker d\lambda \subset TC,$$

where $pr_C : C \times S^1 \to C$ is the projection. Note that this implies in particular that Σ is non-nodal.

By the charge $(1, 0)$ condition $pr_{S^1} \circ u$ is surjective and so by the Sard theorem we have a regular value $\theta_0 \in S^1$, so that $u^{-1} \circ pr^{-1}_{S^1}(\theta_0)$ contains an embedded circle $S_0 \subset \Sigma$, where $pr_{S^1} : C \times S^1 \to S^1$ is the projection. Now $d(pr_{S^1} \circ u)$ is surjective along $T(\Sigma)|_{S_0}$, which means, since u is J^λ-holomorphic, that $pr_C \circ u|_{S_0}$ has non-vanishing differential. From this and the discussion above it follows that image of $pr_C \circ u$ is the image of some Reeb orbit. Consequently, by assumption that u has charge $(1, 0)$, u is equivalent to a Reeb torus for a uniquely determined Reeb orbit o_u.

The statement of the lemma follows when u has no spherical nodal components. On the other hand non-constant J^λ-holomorphic spheres are impossible, which can be seen as follows. Any such a J^λ-holomorphic sphere u lifts to the covering space $\widetilde{M} = C \times \mathbb{R}$ of M, as a $\widetilde{J}$-holomorphic map $\widetilde{u}$, where $\widetilde{J}$ is the lift of J^λ, and is compatible with the lift $\widetilde{\omega}$ of $\omega = d^\alpha \lambda$. On the other had $\widetilde{\omega} = d\lambda - dt \wedge \lambda$ is conformally symplectomorphic to the exact symplectic form $d(e^t \lambda)$, for $t : C \times \mathbb{R} \to \mathbb{R}$ the projection. So that $\widetilde{u}$ is constant by Stokes theorem.

□

Proposition 3.1. *Let (C, ξ) be a general contact manifold. If λ is a non-degenerate contact 1-form for ξ then all the elements of $\overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$ are regular curves. Moreover, if λ is degenerate then for a period c Reeb orbit o the kernel of the associated real linear Cauchy-Riemann operator for the Reeb torus u_o is naturally identified with the 1-eigenspace of $\phi^\lambda_{c,*}$ - the time c linearized return map $\xi(o(0)) \to \xi(o(0))$ induced by the R^λ Reeb flow.*

Proof. We already know by Proposition 1.9 that all $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$ are equivalent to Reeb tori. In particular have representation by a J^λ-holomorphic map

$$u : (T^2, j) \to (Y = C \times S^1, J^\lambda).$$

Since each u is immersed we may naturally get a splitting $u^*T(Y) \simeq N \times T(T^2)$, using the g_J metric, where $N \to T^2$ denotes the pull-back, of the g_J-normal bundle to image u, and which is identified with the pullback of the distribution ξ_λ on Y, (which we also call the co-vanishing distribution).

The full associated real linear Cauchy-Riemann operator takes the form:

$$D^J_u : \Omega^0(N \oplus T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2), N \oplus T(T^2)). \tag{3.2}$$

This is an index 2 Fredholm operator (after standard Sobolev completions), whose restriction to $\Omega^0(N \oplus T(T^2))$ preserves the splitting, that is the restricted operator splits as

$$D \oplus D' : \Omega^0(N) \oplus \Omega^0(T(T^2)) \to \Omega^{0,1}(T(T^2), N) \oplus \Omega^{0,1}(T(T^2), T(T^2)).$$

On the other hand the restricted Fredholm index 2 operator

$$\Omega^0(T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2)),$$

is surjective by classical Teichmuller theory, see also [31, Lemma 3.3] for a precise argument in this setting. It follows that D^J_u will be surjective if the restricted Fredholm index 0 operator

$$D : \Omega^0(N) \to \Omega^{0,1}(N),$$

has no kernel.

The bundle N is symplectic with symplectic form on the fibers given by restriction of $u^* d\lambda$, and together with J^λ this gives a Hermitian structure (g_λ, j_λ) on N. We have a linear symplectic connection $\mathcal{A}$ on N, which over the slices $S^1 \times \{t\} \subset T^2$ is induced by the pullback by u of the linearized R^λ Reeb flow. Specifically the $\mathcal{A}$-transport map from the fiber $N_{(s_0,t)}$ to the fiber $N_{(s_1,t)}$ over the path $[s_0, s_1] \times \{t\} \subset T^2$, is given by

$$(u_*|_{N_{(s_1,t)}})^{-1} \circ (\phi^\lambda_{c(s_1-s_0)})_* \circ u_*|_{N_{(s_0,t)}},$$

where $\phi^\lambda_{c(s_1-s_0)}$ is the time $c \cdot (s_1 - s_0)$ map for the R^λ Reeb flow, where c is the period of the Reeb orbit o_u, and where $u_* : N \to TY$ denotes the natural map, (it is the universal map in the pull-back diagram.)

The connection $\mathcal{A}$ is defined to be trivial in the θ_2 direction, where trivial means that the parallel transport maps are the id maps over θ_2 rays. In particular the curvature $R_\mathcal{A}$, understood as a lie algebra valued 2-form, of this connection vanishes. The connection $\mathcal{A}$ determines a real linear CR operator $D_\mathcal{A}$ on N in the standard way, take the complex anti-linear part of the vertical differential of a section. Explicitly,

$$D_\mathcal{A} : \Omega^0(N) \to \Omega^{0,1}(N),$$

is defined by

$$D_\mathcal{A}(\mu)(p) = j_\lambda \circ \pi^{vert}(\mu(p)) \circ d\mu(p) - \pi^{vert}(\mu(p)) \circ d\mu(p) \circ j,$$

where

$$\pi^{vert}(\mu(p)) : T_{\mu(p)}N \to T^{vert}_{\mu(p)}N \simeq N$$

is the $\mathcal{A}$-projection, and where $T^{vert}_{\mu(p)}N$ is the kernel of the projection $T_{\mu(p)}N \to T_p\Sigma$. It is elementary to verify from the definitions that this operator is exactly D. See also [19, Section 10.1] for a computation of this kind in much greater generality.

We have a differential 2-form Ω on the total space of N defined as follows. On the fibers $T^{vert}N$, $\Omega = u_*\omega$, for $\omega = d^\alpha\lambda$, and for $T^{vert}N \subset TN$ denoting the vertical tangent space, or subspace of vectors v with $\pi_* v = 0$, for $\pi : N \to T^2$ the projection. While on the $\mathcal{A}$-horizontal distribution Ω is defined to vanish. The 2-form Ω is closed, which we may check explicitly by using that $R_\mathcal{A}$ vanishes to obtain local symplectic trivializations of N in which $\mathcal{A}$ is trivial. Clearly Ω must vanish on the 0-section since it is a $\mathcal{A}$-flat section. But any section is homotopic to the 0-section and so in particular if $\mu \in \ker D$ then Ω vanishes on μ. But then since $\mu \in \ker D$, and so its vertical differential is complex linear, it must follow that the vertical differential vanishes, since $\Omega(v, J^\lambda v) > 0$, for $0 \neq v \in T^{vert}N$ and so otherwise we would have $\int_\mu \Omega > 0$. So μ is $\mathcal{A}$-flat, in particular the restriction of μ over all slices $S^1 \times \{t\}$ is identified with a period c orbit of the linearized at o R^λ Reeb flow, and which does not depend on t as $\mathcal{A}$ is trivial in the t variable. So the kernel of D is identified with the vector space of period c orbits of the linearized at o R^λ Reeb flow, as needed. ☐

Proposition 3.3. *Let λ be a contact form on a $(2n+1)$-fold C, and o a non-degenerate, period c, R^λ-Reeb orbit, then the orientation of $[u_o]$ induced by the determinant line bundle orientation of $\overline{\mathcal{M}}^{1,0}_{1,1}(J^\lambda, A)$, is $(-1)^{CZ(o)-n}$, which is*

$$\operatorname{sign} \operatorname{Det}(\operatorname{Id}|_{\xi(o(0))} - \phi^\lambda_{c,*}|_{\xi(o(0))}).$$

Proof of Proposition 3.3. Abbreviate u_o by u. Let $N \to T^2$ be associated to u as in the proof of Proposition 3.1. Fix a trivialization ϕ of N induced by any trivialization of the contact distribution ξ along o in the obvious sense: N is the pullback of ξ along the composition

$$T^2 \to S^1 \xrightarrow{o} C.$$

Let the symplectic connection $\mathcal{A}$ on N be defined as before. Then the pullback connection $\mathcal{A}' := \phi^* \mathcal{A}$ on $T^2 \times \mathbb{R}^{2n}$ is a connection whose parallel transport paths $p_t : [0,1] \to \mathrm{Symp}(\mathbb{R}^{2n})$, along the closed loops $S^1 \times \{t\}$, are paths starting at 1, and are t independent. And so the parallel transport path of $\mathcal{A}'$ along $\{s\} \times S^1$ is constant, that is $\mathcal{A}'$ is trivial in the t variable. We shall call such a connection $\mathcal{A}'$ on $T^2 \times \mathbb{R}^{2n}$ *induced by p*.

By non-degeneracy assumption on o, the map $p(1)$ has no 1-eigenvalues. Let $p'' : [0,1] \to \mathrm{Symp}(\mathbb{R}^{2n})$ be a path from $p(1)$ to a unitary map $p''(1)$, with $p''(1)$ having no 1-eigenvalues, and s.t. p'' has only simple crossings with the Maslov cycle. Let p' be the concatenation of p and p''. We then get

$$CZ(p') - \frac{1}{2} \operatorname{sign} \Gamma(p', 0) \equiv CZ(p') - n \equiv 0 \mod 2,$$

since p' is homotopic relative end points to a unitary geodesic path h starting at id, having regular crossings, and since the number of negative, positive eigenvalues is even at each regular crossing of h by unitarity. Here $\operatorname{sign} \Gamma(p', 0)$ is the index of the crossing form of the path p' at time 0, in the notation of [22]. Consequently

$$CZ(p'') \equiv CZ(p) - n \mod 2, \tag{3.4}$$

by additivity of the Conley-Zehnder index.

Let us then define a free homotopy $\{p_t\}$ of p to p', p_t is the concatenation of p with $p''|_{[0,t]}$, reparametrized to have domain $[0,1]$ at each moment t. This determines a homotopy $\{\mathcal{A}'_t\}$ of connections induced by $\{p_t\}$. By the proof of Proposition 3.1, the CR operator D_t determined by each $\mathcal{A}'_t$ is surjective except at some finite collection of times $t_i \in (0,1)$, $i \in N$ determined by the crossing times of p'' with the Maslov cycle, and the dimension of the kernel of D_{t_i} is the 1-eigenspace of $p''(t_i)$, which is 1 by the assumption that the crossings of p'' are simple.

The operator D_1 is not complex linear. To fix this we concatenate the homotopy $\{D_t\}$ with the homotopy $\{\widetilde{D}_t\}$ defined as follows. Let $\{\widetilde{\mathcal{A}}_t\}$ be a homotopy of $\mathcal{A}'_1$ to a unitary connection $\widetilde{\mathcal{A}}_1$, where the homotopy $\{\widetilde{\mathcal{A}}_t\}$ is through connections induced by paths $\{\widetilde{p}_t\}$, giving a path homotopy of $p' = \widetilde{p}_0$ to h. Then $\{\widetilde{D}_t\}$ is defined to be induced by $\{\widetilde{\mathcal{A}}_t\}$.

Let us denote by $\{D'_t\}$ the concatenation of $\{D_t\}$ with $\{\widetilde{D}_t\}$. By construction in the second half of the homotopy $\{D'_t\}$, D'_t is surjective. And D'_1 is induced by a unitary connection, since it is induced by unitary path $\widetilde{p}_1$. Consequently D'_1 is complex linear. By the above construction, for the homotopy $\{D'_t\}$, D'_t is surjective except for N times in $(0,1)$, where the kernel has dimension one. In particular the sign of $[u]$ by the definition via the determinant line bundle is exactly

$$-1^N = -1^{CZ(p)-n},$$

by (3.4), which was what to be proved. □

Theorem 3.5.
$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(\widetilde{N}, R^\lambda, \beta),$$
where $N \subset \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A_\beta)$ is an open compact set, $\widetilde{N}$ the corresponding subset of periodic orbits of R^λ, $i(\widetilde{N}, R^\lambda, \beta)$ is the Fuller index as described in the appendix below, and where the left hand side of the equation is a certain Gromov-Witten invariant, that we discuss in Section 2.

Proof. If $N \subset \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A_\beta)$ is open-compact and consists of isolated regular Reeb tori $\{u_i\}$, corresponding to orbits $\{o_i\}$ we have:

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = \sum_i \frac{(-1)^{CZ(o_i)-n}}{mult(o_i)},$$

where the denominator $mult(o_i)$ is there because our moduli space is understood as a non-effective orbifold, see Appendix B.

The expression on the right is exactly the Fuller index $i(\widetilde{N}, R^\lambda, \beta)$. Thus the theorem follows for N as above. However in general if N is open and compact then perturbing slightly we obtain a smooth family $\{R^{\lambda_t}\}$, $\lambda_0 = \lambda$, s.t. λ_1 is non-degenerate, that is has non-degenerate orbits. And such that there is an open-compact subset $\widetilde{N}$ of $\overline{\mathcal{M}}_{1,1}^{1,0}(\{J^{\lambda_t}\}, A_\beta)$ with $(\widetilde{N} \cap \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A_\beta)) = N$, cf. Lemma 2.8. Then by Lemma 2.2 if

$$N_1 = (\widetilde{N} \cap \overline{\mathcal{M}}_{1,1}^{1,0}(J^{\lambda_1}, A_\beta))$$

we get

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}_{1,1}] \otimes [C \times S^1]).$$

By the previous discussion

$$GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(N_1, R^{\lambda_1}, \beta),$$

but by the invariance of Fuller index (see Appendix A),

$$i(N_1, R^{\lambda_1}, \beta) = i(N, R^\lambda, \beta).$$

□

Proof of Theorem 1.11. Let $u : \Sigma \to M$ be a non-constant J-curve. We first show that $[u^*\alpha] \neq 0$. Suppose otherwise. Let $\widetilde{M}$ denote the α-covering space of M, that is the space of equivalence classes of paths p starting at $x_0 \in M$, with a pair p_1, p_2 equivalent if $p_1(1) = p_2(1)$ and

$$\int_{[0,1]} p_1^*\alpha = \int_{[0,1]} p_2^*\alpha.$$

Then the lift of ω to $\widetilde{M}$ is

$$\widetilde{\omega} = \frac{1}{f} d(f\lambda),$$

where $f = e^g$ and where g is a primitive for the lift $\widetilde{\alpha}$ of α to $\widetilde{M}$, that is $\widetilde{\alpha} = dg$. In particular $\widetilde{\omega}$ is conformally symplectomorphic to an exact symplectic form on $\widetilde{M}$. So if $\widetilde{J}$ denotes the lift of J, any closed $\widetilde{J}$-curve is constant by Stokes theorem. Now $[u^*\alpha] = 0$, so u has a lift to a $\widetilde{J}$-holomorphic map $\widetilde{u} : \Sigma \to \widetilde{M}$. Since Σ is closed, it follows by the above that $\widetilde{u}$ is constant, which is a contradiction.

Since α is rational we may construct a smooth $p : M \to S^1$, so that $\alpha = c \cdot p^* d\theta$ for $c \in \mathbb{Q}$. Let $u : \Sigma \to M$ be a non-constant J-curve. Let $s_0 \in S^1$ be a regular value of $p \circ u$, and let $S_0 \subset \Sigma$, $S_0 \simeq S^1$ be a component of $(p \circ u)^{-1}(s_0)$. Since the critical points of u are isolated we may suppose that u is non-critical along S_0. In particular $u^*\omega$ is non-vanishing everywhere on $T\Sigma|_{S_0}$, which together with Lemma 1.8 implies that $u^*\lambda \wedge u^*\alpha$ is non-vanishing everywhere on $T\Sigma|_{S_0}$. So if $o : S^1 \to S_0$ is any parametrization, $u \circ o$ is a Reeb curve.

Now if u is an immersion then $u^*\omega$ is symplectic and by Lemma 1.8 $u^* d\lambda = 0$, so that $\omega_0 = u^*\alpha \wedge u^*\lambda$ is non-degenerate on Σ. Let $\widetilde{\Sigma}$ be the $u^*\alpha$-covering space of Σ so that $\omega_0 = dH \wedge u^*\lambda$ for some proper $H : \widetilde{\Sigma} \to \mathbb{R}$. Since ω_0 is non-degenerate, H has no critical points so that $\widetilde{\Sigma} \simeq S^1 \times \mathbb{R}$ by basic Morse theory. It follows that $\Sigma \simeq T^2$. □

Lemma 3.6. *Let (M, λ, α, J) be a tamed exact* lcs *structure. Suppose that α is rational, then every non-constant J-curve $u : \Sigma \to M$, with Σ a closed possibly nodal Riemann surface, is smooth, that is Σ is a smooth Riemann surface.*

Proof. Since α is rational we may construct a smooth $p : M \to S^1$, so that $\alpha = c \cdot p^* d\theta$ for $c \in \mathbb{Q}$. Let $u : \Sigma \to M$ be a non-constant J-curve. Let $s_0 \in S^1$ be a regular value of $p \circ u$, and let $S_0 \subset \Sigma$, $S_0 \simeq S^1$ be a component of $(p \circ u)^{-1}(s_0)$. Since the critical points of u are isolated we may suppose that u is non-critical along S_0. Suppose by contradiction that Σ is nodal. We may then find an embedded disk $i : D^2 \to \Sigma$ with $\partial i(D^2) = S$.

Since $u^* d\lambda = 0$ by Lemma 1.8, $\int_{S^1} i^* u^* \lambda = 0$ by Stokes theorem, and so $u^*\lambda(v) = 0$ for some $v \in TS_0(z) \subset T_z\Sigma$, $z \in S_0$. And let $w \in T_z\Sigma$ be such that v, w form a basis for $T_z\Sigma$. Now $u^*\omega$ is symplectic along S_0 so that $u^*\omega(v, w) \neq 0$ which implies that $u^*\alpha \wedge u^*\lambda(v, w) \neq 0$ since $u^* d\lambda(v, w) = 0$, but $u^*\alpha(v) = 0$ and $u^*\lambda(v) = 0$, so that we have a contradiction. □

Proof of Theorem 1.10. Let $N \subset \overline{\mathcal{M}}_{1,1}^{1,0}(A, J^\lambda)$, be the subspace corresponding, (under the correspondence of Proposition 1.9) to the subspace $\widetilde{N}$ of all period 2π R^λ-orbits. It is easy to compute, see for instance [9],

$$i(\widetilde{N}, R^\lambda) = \pm\chi(\mathbb{CP}^k) \neq 0.$$

By Theorem 3.5 $GW_{1,1}(N, J^\lambda, A) \neq 0$. The first part of the theorem then follows by Lemma 2.6.

We now verify the second part. Let U be a δ-neighborhood of $(d^\alpha \lambda_H, J^{\lambda_H})$ guaranteed by the first part of the theorem. Let $(\lambda', \alpha', J) \in U$ and $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(A, J)$ guaranteed by the first part of the theorem, with J admissible. Let $\underline{u}$ be a simple J-holomorphic curve covered by u, which is non-nodal by Lemma 3.6. Let us recall for convenience the adjunction inequality.

Theorem 3.7 (McDuff-Micallef-White [18], [13])**.** *Let (M, J) be an almost complex 4-manifold and $A \in H_2(M)$ be a homology class that is represented by a simple J-holomorphic curve u. Then*

$$2\delta(u) - \chi(\Sigma) \leq A \cdot A - c_1(A),$$

with equality if and only if u is an immersion with only transverse self-intersections.

In our case $A = 0$, $\chi(\Sigma) = 0$, so that $\delta(\underline{u}) = 0$, and so $\underline{u}$ is an embedding. □

Proof of Theorem 1.5. Define a metric ρ_0 measuring distance between subspaces W_1, W_2, of same dimension, of an inner product space (T, g) as follows.

$$\rho_0(W_1, W_2) := |P_{W_1} - P_{W_2}|,$$

for $|\cdot|$ the g-operator norm, and P_{W_i} g-projection operators onto W_i. We may of course generalize this to a C^2 metric ρ_2 again in terms of these projection operators.

Let U be a neighborhood of

$$(\omega_H, J_H := J^{\lambda_H})$$

consisting of pairs (ω, J), $\omega \in \mathcal{L}(M)$, an exact lcs structure d_0 ϵ-close to ω_H, J ω-compatible, and C^2 ϵ-close to J^{λ_H}, where ϵ is chosen as in the first part of Theorem 1.10. To prove the theorem we need to construct a tamed exact lcs structure (λ, α, J), with $(d^\alpha \lambda, J) \in U$ as Theorem 1.10 then tells us that there is a class A, J-holomorphic elliptic curve u in M, and since J is admissible, by Theorem 1.11 there is a Reeb curve for (λ, α).

Suppose that $\omega = d^{\alpha'}\lambda'$ is δ-close to ω_H for the C^3 metric d_3 as in the statement of the theorem. Then if δ was chosen to be sufficiently small $\mathcal{V}_{\lambda'}, \xi_{\lambda'}$ are smooth distributions (not just generalized distributions). This is because the contact condition $\lambda \wedge \lambda^{2k} \neq 0$ is open, so that $d\lambda'(p)$ has kernel for each $p \in M$. Moreover, for each $p \in M$,

$$\rho_2(\mathcal{V}_{\lambda'}(p), \mathcal{V}_{\lambda_H}(p)) < \epsilon_\delta$$

and

$$\rho_2(\xi_{\lambda'}(p), \xi_{\lambda_H}(p)) < \epsilon_\delta$$

where $\epsilon_\delta \to 0$ as $\delta \to 0$, and where ρ_2 is the metric as defined above for subspaces of the inner product space $(T_p M, g)$.

Then choosing δ to be suitably be small, for each $p \in M$ we have an isomorphism

$$\phi(p) : T_p M \to T_p M,$$

$\phi_p := P_1 \oplus P_2$, for $P_1 : \mathcal{V}_{\lambda_H}(p) \to \mathcal{V}_{\lambda'}(p)$, $P_2 : \xi_{\lambda_H}(p) \to \xi_{\lambda'}(p)$ the g-projection operators. Define $J(p) := \phi(p)_* J_H$. In addition, if δ was chosen to be sufficiently small (ω, J) is a compatible pair, and $d\lambda'$ tames J on $\xi_{\lambda'}$. □

Proof of Theorem 1.6. Let $\{\omega_t\}$, $t \in [0, 1]$, be a continuous in usual C^∞ topology homotopy of lcs forms on $M = C \times S^1$, as in the hypothesis. Fix an almost complex structure J_1 on M admissible with respect to (α', λ'). Extend to a Frechet smooth family $\{J_t\}$ of almost complex structures on M, so that J_t is ω_t-compatible for each t. Then in the absence of holomorphic sky catastrophes, by Theorem 4.11, there is a non-constant elliptic J_1-holomorphic curve in M, so that the result follows by Theorem 1.11.

□

4. Extended Gromov-Witten invariants and the extended Fuller index

In what follows M is a closed oriented $2n$-fold, $n \geq 2$, and J an almost complex structure on M. Much of the following discussion extends to general moduli spaces $\mathcal{M}_{g,n}(J, A, a_1, \ldots, a_n)$ with $a_1, \ldots, a_n$ homological constraints in M. We shall however restrict for simplicity to the case (ω, J) is a compatible lcs pair on M, $g = 1, n = 1$, the homological constraint is $[M]$, as this is the main interest in this paper. Moreover, we restrict our moduli space to consist of non-zero charge pair (for example $(1, 0)$) curves, with charge defined with respect to the Lee form α of ω as in Section 1.2.1, and this will be implicit, so that we no longer specify this in notation.

In what follows $e(u)$ denotes the energy of a map $u : \Sigma \to M$, with respect to the metric induced by an lcs pair (ω, J).

Definition 4.1. *Let $h = \{(\omega_t, J_t)\}$ be a homotopy of* lcs *pairs on M, so that $\{J_t\}$ is Frechet smooth, and $\{\omega_t\}$ C^0 continuous. We say that it is* **partially admissible for** A *if every element of*

$$\overline{\mathcal{M}}_{1,1}(M, J_0, A)$$

is contained in a compact open subset of $\overline{\mathcal{M}}_{1,1}(M, \{J_t\}, A)$. We say that h is **admissible for** A *if every element of*

$$\overline{\mathcal{M}}_{1,1}(M, J_i, A),$$

$i = 0, 1$ is contained in a compact open subset of $\overline{\mathcal{M}}_{1,1}(M, \{J_t\}, A)$.

Thus in the above definition, a homotopy is partially admissible if there are sky catastrophes going one way, and admissible if there are no sky catastrophes going either way.

Partly to simplify notation, we denote by a capital X a compatible general lcs triple (M, ω, J), then we introduce the following simplified notation.

$$\begin{aligned} S(X, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(X, A)\} \\ S(X, a, A) &= \{u \in S(X, A) \mid e(u) \leq a\} \\ S(h, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(h, A)\}, \text{ for } h = \{(\omega_t, J_t)\} \text{ a homotopy as above} \\ S(h, a, A) &= \{u \in S(h, A) \mid e(u) \leq a\} \end{aligned} \tag{4.2}$$

Definition 4.3. *For an isolated element u of $S(X, A)$, which means that $\{u\}$ is open as a subset, we set $gw(u) \in \mathbb{Q}$ to be the local Gromov-Witten invariant of u. This is defined as:*

$$gw(u) = GW_{1,1}(\{u\}, A, J)([\overline{M}_{1,1}] \otimes [M]),$$

with the right hand side as in (2.1).

Denote by $\mathcal{S}(M, A)$ the set of equivalence classes of all smooth stable maps $\Sigma \to M$, in class A, for Σ an (non-fixed) elliptic curve, and where equivalence has the same meaning as in Section 1.2.1.

Definition 4.4. *Suppose that $S(X, A)$ has open connected components. And suppose that we have a collection of* lcs *pairs*

$$\{X^a = (M, \omega^a, J^a)\},\ a \in \mathbb{R}_+$$

satisfying the following:

- *$S(X^a, a, A)$ consists of isolated curves for each a.*
- $$S(X^a, a, A) = S(X^b, a, A),$$
  (equality of subsets of $\mathcal{S}(M, A)$) if $b > a$,
- *For $b > a$, and for each $u \in S(X^a, a, A) = S(X^b, a, A)$:*
  $$GW_{1,1}(\{u\}, A, J^a) = GW_{1,1}(\{u\}, A, J^b),$$
  thus we may just write $gw(u)$ for the common number.

- *There is a prescribed homotopy* $h^a = \{X^a_t\}$ *of each* X^a *to* X, *called* **structure homotopy**, *with the property that for every*
$$y \in S(X^a_0, A)$$
there is an open compact subset $\mathcal{C}_y \subset S(h^a, A)$, $y \in C_y$, *which is* **non-branching** *which means that*
$$\mathcal{C}_y \cap S(X^a_i, A),$$
$i = 0, 1$ *are connected.*
-
$$S(h^a, a, A) = S(h^b, a, A),$$
(similarly equality of subsets) if $b > a$ *is sufficiently large.*

We will then say that
$$\mathcal{P}(A) = \{(X^a, h^a)\}$$
is a **perturbation system** *for* X *in the class* A.

We shall see shortly that, given a contact (C, λ), the associated Banyaga lcs structure on $C \times S^1$ always admits a perturbation system for the moduli spaces of charge (1,0) curves in any class, if λ is Morse-Bott.

Definition 4.5. *Suppose that* X *admits a perturbation system* $\mathcal{P}(A)$ *so that there exists an* $E = E(\mathcal{P}(A))$ *with the property that*
$$S(X^a, a, A) = S(X^E, a, A)$$
for all $a > E$, *where this as before is equality of subsets, and the local Gromov-Witten invariants of the identified elements are also identified. Then we say that* X *is* **finite type** *and set:*
$$GW(X, A) = \sum_{u \in S(X^E, A)} gw(u).$$

Definition 4.6. *Suppose that* X *admits a perturbation system* $\mathcal{P}(A)$ *and there is an* $E = E(\mathcal{P}(A)) > 0$ *so that* $gw(u) > 0$ *for all*
$$\{u \in S(X^a, A) \,|\, E \leq e(u) \leq a\}$$
respectively $gw(u) < 0$ *for all*
$$\{u \in S(X^a, A) \,|\, E \leq e(u) \leq a\},$$
and every $a > E$. *Suppose in addition that*
$$\lim_{a \mapsto \infty} \sum_{u \in S(X, a, A)} gw(u) = \infty, \text{ respectively } \lim_{a \mapsto \infty} \sum_{u \in S(X, a, \beta)} gw(u) = -\infty.$$
Then we say that X *is* **positive infinite type**, *respectively* **negative infinite type** *and set*
$$GW(X, A) = \infty,$$
respectively $GW(X, A) = -\infty$. *These are meant to be interpreted as extended Gromov-Witten invariants, counting elliptic curves in class* A. *We say that* X *is* **infinite type** *if it is one or the other.*

Definition 4.7. *We say that* X *is* **definite** *type if it admits a perturbation system and is infinite type or finite type.*

With the above definitions
$$GW(X, A) \in \mathbb{Q} \sqcup \infty \sqcup -\infty,$$
when it is defined.

Proof of Theorem 1.13. Given the definitions above, and the definition of the extended Fuller index in [24], this follows by the same argument as the proof of Theorem 3.5. □

4.0.1. *Perturbation systems for Morse-Bott Reeb vector fields.*

Definition 4.8. *A contact form λ on M, and its associated flow R^λ are called* Morse-Bott *if the λ action spectrum $\sigma(\lambda)$ - that is the space of critical values of $o \mapsto \int_{S^1} o^* \lambda$, is discreet and if for every $a \in \sigma(\lambda)$, the space*

$$N_a := \{x \in M | \, F_a(x) = x\},$$

F_a the time a flow map for R^λ - is a closed smooth manifold such that rank $d\lambda|_{N_a}$ is locally constant and $T_x N_a = \ker(dF_a - I)_x$.

Proposition 4.9. *Let λ be a contact form of Morse-Bott type, on a closed contact manifold C. Then the corresponding* lcs *pair $X_\lambda = (C \times S^1, d^\alpha \lambda, J^\lambda)$ admits a perturbation system $\mathcal{P}(A)$, for moduli spaces of charge (1,0) curves for every class A.*

Proof. This follows immediately by [24, Proposition 2.12], and by Proposition 1.9. □

Lemma 4.10. *The Hopf* lcs *pair $(S^{2k+1} \times S^1, d^\alpha \lambda_H, J^{\lambda_H})$, for λ_H the standard contact structure on S^{2k+1} is infinite type.*

Proof. This follows immediately by [24, Lemma 2.13], and by Proposition 1.9. □

Theorem 4.11. *Let (C, λ) be a closed contact manifold so that R^λ has definite type, and suppose that $i(R^\lambda, \beta) \neq 0$. Let $\omega_0 = d^\alpha \lambda$ be the Banyaga structure, and suppose we have a partially admissible homotopy $h = \{(\omega_{t,}, J_t)\}$, for class A_β, then there in an element $u \in \overline{\mathcal{M}}^{1,0}_{1,1}(J_1, A_\beta)$.*

The proof of this will follow.

4.1. Preliminaries on admissible homotopies.

Definition 4.12. *Let $h = \{X_t\}$ be a smooth homotopy of* lcs *pairs. For $b > a > 0$ we say that h is* **partially** a, b**-admissible**, *respectively* a, b**-admissible** *(in class A) if for each*

$$y \in S(X_0, a, A)$$

there is a compact open subset $\mathcal{C}_y \subset S(h, A)$, $y \in \mathcal{C}_y$ with $e(u) < b$, for all $u \in \mathcal{C}_y$. Respectively, if for each

$$y \in S(X_i, a, A),$$

$i = 0, 1$ there is a compact open subset $\mathcal{C}_y \ni y$ of $S(h, A)$ with $e(u) < b$, for all $u \in \mathcal{C}_y$.

Lemma 4.13. *Suppose that X_0 has a perturbation system $\mathcal{P}(A)$, and $\{X_t\}$ is partially admissible, then for every a there is a $b > a$ so that $\{\widetilde{X}^b_t\} = \{X_t\} \cdot \{X^b_t\}$ is partially a, b-admissible, where $\{X_t\} \cdot \{X^b_t\}$ is the (reparametrized to have t domain $[0, 1]$) concatenation of the homotopies $\{X_t\}, \{X^b_t\}$, and where $\{X^b_t\}$ is the structure homotopy from X^b to X_0.*

Proof. This is a matter of pure topology, and the proof is completely analogous to the proof of [24, Lemma 3.8]. □

The analogue of Lemma 4.13 in the admissible case is the following:

Lemma 4.14. *Suppose that X_0, X_1 and $\{X_t\}$ are admissible, then for every a there is a $b > a$ so that*

$$\{\widetilde{X}^b_t\} = \{X^b_{1,t}\}^{-1} \cdot \{X_t\} \cdot \{X^b_{0,t}\} \tag{4.15}$$

is a, b-admissible, where $\{X^b_{i,t}\}$ are the structure homotopies from X^b_i to X_i.

4.2. Invariance.

Theorem 4.16. *Suppose we have a definite type* lcs *pair X_0, with $GW(X_0, A) \neq 0$, which is joined to X_1 by a partially admissible homotopy $\{X_t\}$, then X_1 has non-constant elliptic class A curves.*

Proof of Theorem 4.11. This follows by Theorem 4.16 and by Theorem 1.13. □

We also have a more a more precise result.

Theorem 4.17. *If X_0, X_1 are definite type* lcs *pairs and $\{X_t\}$ is admissible then $GW(X_0, A) = GW(X_1, A)$.*

Proof of Theorem 4.16. Suppose that X_0 is definite type with $GW(X_0, A) \neq 0$, $\{X_t\}$ is partially admissible and $\overline{\mathcal{M}}_{1,1}(X_1, A) = \emptyset$. Let a be given and b determined so that $\widetilde{h}^b = \{\widetilde{X}_t^b\}$ is a partially (a, b)-admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\widetilde{h}^b, A),$$

for $y \in S(X_0^b, a, A)$. Here we use a natural identification of $S(X^b, a, A) = S(\widetilde{X}_0^b, a, A)$ as a subset of $S(\widetilde{h}^b, A)$ by its construction. Then S_a is an open-compact subset of $S(h, A)$ and so admits an implicit atlas (Kuranishi structure) with boundary, (with virtual dimension 1) s.t:

$$\partial S_a = S(X^b, a, A) + Q_a,$$

where Q_a as a set is some subset (possibly empty), of elements $u \in S(X^b, b, A)$ with $e(u) \geq a$. So we have for all a:

$$\sum_{u \in Q_a} gw(u) + \sum_{u \in S(X^b, a, A)} gw(u) = 0. \tag{4.18}$$

4.3. Case I, X_0 is finite type.

Let $E = E(\mathcal{P})$ be the corresponding cutoff value in the definition of finite type, and take any $a > E$. Then $Q_a = \emptyset$ and by definition of E we have that the left side is

$$\sum_{u \in S(X^b, E, A)} gw(u) \neq 0.$$

Clearly this gives a contradiction to (4.18).

4.4. Case II, X_0 is infinite type.

We may assume that $GW(X_0, A) = \infty$, and take $a > E$, where $E = E(\mathcal{P}(A))$ is the corresponding cutoff value in the definition of infinite type. Then

$$\sum_{u \in Q_a} gw(u) \geq 0,$$

as $a > E(\mathcal{P}(A))$. While

$$\lim_{a \mapsto \infty} \sum_{u \in S(X^b, a, A)} gw(u) = \infty,$$

as $GW(X_0, A) = \infty$. This also contradicts (4.18). □

Proof of Theorem 4.17. This is somewhat analogous to the proof of Theorem 4.16. Suppose that $X_i, \{X_t\}$ are definite type as in the hypothesis. Let a be given and b determined so that $\widetilde{h}^b = \{\widetilde{X}_t^b\}$, see (4.15) is an (a, b)-admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\widetilde{h}^b, A)$$

for $y \in S(X_i^b, a, A)$. Then S_a is an open-compact subset of $S(h, A)$ and so has admits an implicit atlas (Kuranishi structure) with boundary, (with virtual dimension 1) s.t:

$$\partial S_a = (S(X_0^b, a, A) + Q_{a,0})^{op} + S(X_1^b, a, A) + Q_{a,1},$$

with *op* denoting opositite orientation and where $Q_{a,i}$ as sets are some subsets (possibly empty), of elements $u \in S(X_i^b, b, A)$ with $e(u) \geq a$. So we have for all a:

$$\sum_{u \in Q_{a,0}} gw(u) + \sum_{u \in S(X_0^b, a, A)} gw(u) = \sum_{u \in Q_{a,1}} gw(u) + \sum_{u \in S(X_1^b, a, A)} gw(u) \tag{4.19}$$

4.5. **Case I, X_0 is finite type and X_1 is infinite type.** Suppose in addition $GW(X_1, A) = \infty$ and let $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$, for $\mathcal{P}_i$, the perturbation systems of X_i. Take any $a > E$. Then $Q_{a,0} = \emptyset$ and the left hand side of (4.19) is

$$\sum_{u \in S(X_0^b, E, A)} gw(u).$$

While the right hand side tends to ∞ as a tends to infinity since,

$$\sum_{u \in Q_{a,1}} gw(u) \geq 0,$$

as $a > E(\mathcal{P}_1(A))$, and

$$\lim_{a \mapsto \infty} \sum_{u \in S(X_1^b, a, A)} gw(u) = \infty,$$

Clearly this gives a contradiction to (4.19).

4.6. **Case II, X_i are infinite type.** Suppose in addition $GW(X_0, A) = -\infty$, $GW(X_1, A) = \infty$ and let $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$, for $\mathcal{P}_i$, the perturbation systems of X_i. Take any $a > E$. Then $\sum_{u \in Q_{a,0}} gw(u) \leq 0$, and $\sum_{u \in Q_{a,1}} gw(u) \geq 0$. So by definition of $GW(X_i, A)$ the left hand side of (4.18) tends to $-\infty$ as a tends to ∞, and the right hand side tends to ∞. Clearly this gives a contradiction to (4.19).

4.7. **Case III, X_i are finite type.** The argument is analogous. □

A. Fuller index

Let X be a vector field on M. Set

$$S(X) = S(X, \beta) = \{(o, p) \in L_\beta M \times (0, \infty) \mid \ o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\},$$

where $L_\beta M$ denotes the free homotopy class β component of the free loop space. Elements of $S(X)$ will be called orbits. There is a natural S^1 reparametrization action on $S(X)$, and elements of $S(X)/S^1$ will be called *unparametrized orbits*, or just orbits. Slightly abusing notation we write (o, p) for the equivalence class of (o, p). The multiplicity $m(o, p)$ of a periodic orbit is the ratio p/l for $l > 0$ the least period of o. We want a kind of fixed point index which counts orbits (o, p) with certain weights - however in general to get invariance we must have period bounds. This is due to potential existence of sky catastrophes as described in the introduction.

Let $N \subset S(X)$ be a compact open set. Assume for simplicity that elements $(o, p) \in N$ are isolated. (Otherwise we need to perturb.) Then to such an (N, X, β) Fuller associates an index:

$$i(N, X, \beta) = \sum_{(o,p) \in N/S^1} \frac{1}{m(o, p)} i(o, p),$$

where $i(o, p)$ is the fixed point index of the time p return map of the flow of X with respect to a local surface of section in M transverse to the image of o. Fuller then shows that $i(N, X, \beta)$ has the following invariance property. Given a continuous homotopy $\{X_t\}$, $t \in [0, 1]$ let

$$S(\{X_t\}, \beta) = \{(o, p, t) \in L_\beta M \times (0, \infty) \times [0, 1] \mid \ o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX_t\}.$$

Given a continuous homotopy $\{X_t\}$, $X_0 = X$, $t \in [0, 1]$, suppose that $\widetilde{N}$ is an open compact subset of $S(\{X_t\})$, such that

$$\widetilde{N} \cap (LM \times \mathbb{R}_+ \times \{0\}) = N.$$

Then if

$$N_1 = \widetilde{N} \cap (LM \times \mathbb{R}_+ \times \{1\})$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta).$$

In the case where X is the R^λ-Reeb vector field on a contact manifold (C^{2n+1}, ξ), and if (o, p) is non-degenerate, we have:

$$i(o, p) = \operatorname{sign} \operatorname{Det}(\operatorname{Id}|_{\xi(x)} - F^\lambda_{p,*}|_{\xi(x)}) = (-1)^{CZ(o)-n}, \tag{A.1}$$

where $F^\lambda_{p,*}$ is the differential at x of the time p flow map of R^λ, and where $CZ(o)$ is the Conley-Zehnder index, (which is a special kind of Maslov index) see [22].

There is also an extended Fuller index $i(X, \beta) \in \mathbb{Q} \sqcup \{\pm\infty\}$, for certain X having definite type. This is constructed in [24], and is conceptually completely analogous to the extended Gromov-Witten invariant constructed in this paper.

B. Remark on multiplicity

This is a small note on how one deals with curves having non-trivial isotropy groups, in the virtual fundamental class technology. We primarily need this for the proof of Theorem 3.5. Given a closed oriented orbifold X, with an orbibundle E over X Fukaya-Ono [7] show how to construct using multi-sections its rational homology Euler class, which when X represents the moduli space of some stable curves, is the virtual moduli cycle $[X]^{vir}$. When this is in degree 0, the corresponding Gromov-Witten invariant is $\int_{[X]^{vir}} 1$. However they assume that their orbifolds are effective. This assumption is not really necessary for the purpose of construction of the Euler class but is convenient for other technical reasons. A different approach to the virtual fundamental class which emphasizes branched manifolds is used by McDuff-Wehrheim, see for example McDuff [12], [15] which does not have the effectivity assumption, a similar use of branched manifolds appears in [3]. In the case of a non-effective orbibundle $E \to X$ McDuff [14], constructs a homological Euler class $e(E)$ using multi-sections, which extends the construction [7]. McDuff shows that this class $e(E)$ is Poincare dual to the completely formally natural cohomological Euler class of E, constructed by other authors. In other words there is a natural notion of a homological Euler class of a possibly non-effective orbibundle. We shall assume the following black box property of the virtual fundamental class technology.

Axiom B.1. *Suppose that the moduli space of stable maps is cleanly cut out, which means that it is represented by a (non-effective) orbifold X with an orbifold obstruction bundle E, that is the bundle over X of cokernel spaces of the linearized CR operators. Then the virtual fundamental class $[X]^{vir}$ coincides with $e(E)$.*

Given this axiom it does not matter to us which virtual moduli cycle technique we use. It is satisfied automatically by the construction of McDuff-Wehrheim, (at the moment in genus 0, but surely extending). It can be shown to be satisfied in the approach of John Pardon [21]. And it is satisfied by the construction of Fukaya-Oh-Ono-Ohta [5], the latter is communicated to me by Kaoru Ono. When X is 0-dimensional this does follow immediately by the construction in [7], taking any effective Kuranishi neighborhood at the isolated points of X, (this actually suffices for our paper.)

As a special case most relevant to us here, suppose we have a moduli space of elliptic curves in X, which is regular with expected dimension 0. Then its underlying space is a collection of oriented points. However as some curves are multiply covered, and so have isotropy groups, we must treat this is a non-effective 0 dimensional oriented orbifold. The contribution of each curve $[u]$ to the Gromov-Witten invariant $\int_{[X]^{vir}} 1$ is $\frac{\pm 1}{[\Gamma([u])]}$, where $[\Gamma([u])]$ is the order of the isotropy group $\Gamma([u])$ of $[u]$, in the McDuff-Wehrheim setup this is explained in [12, Section 5]. In the setup of Fukaya-Ono [7] we may readily calculate to get the same thing taking any effective Kuranishi neighborhood at the isolated points of X.

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