GROMOV-WITTEN INVARIANTS OF RIEMANN-FINSLER MANIFOLDS

YASHA SAVELYEV

Abstract. We define a $\mathbb{Q}$-valued deformation invariant of certain complete Riemann-Finsler manifolds. We use this to prove (possibly non-compact but complete) fibration generalizations of Preissman's theorem on non-existence of negative sectional curvature metrics on compact products. In addition we prove that every rational number is the value of this invariant for some compact Riemannian manifold. We also prove that sky catastrophes of smooth dynamical systems are not geodesible by a certain class of forward complete Riemann-Finsler metrics, in particular by Riemannian metrics with non-positive sectional curvature. This partially answers a question of Fuller and gives important examples for our theory here. In a sister paper [9], we study a direct generalization of this metric invariant, by lifting the count of geodesics to a Gromov-Witten count of elliptic curves in an associated locally conformally symplectic manifold.

1. Introduction

We will define certain rational number valued metric invariants for certain complete Riemann-Finsler manifolds, in particular for complete Riemannian metrics with non-positive sectional curvature. These invariants can be directly interpreted as a part of certain elliptic Gromov-Witten invariants in an associated lcs manifold, [9]. However, in the more basic setting here, we can reduce the invariants to counts of closed geodesic strings (equivalence classes of closed unit speed geodesics up to reparametrization S^1 action), via Fuller index [3]. And so this self contained more elementary story is developed separately here.

Terminology 1. All our metrics are Riemann-Finsler metrics unless specified otherwise, and usually denoted by just g. Completeness, always means forward completeness. Curvature always means sectional curvature in the Riemannian case and flag curvature in the Finsler case. Thus we will usually just say complete metric g, for a forward complete Riemann-Finsler metric. A reader may certainly choose to interpret all metrics as Riemannian metrics, completeness as standard completeness, and curvature as sectional curvature.

In what follows $\pi_1(X)$ denotes the set of free homotopy classes of continuous maps $o: S^1 \to X$.

Definition 1.1. *Let X be a smooth manifold. Fix an exhaustion by nested compact sets $\bigcup_{i \in \mathbb{N}} K_i = X$, $K_i \supset K_{i-1}$ for all $i \geq 1$. We say that a class $\beta \in \pi_1(X)$ is boundary compressible if β is in the image of*

$$inc_*: \pi_1(X - K_i) \to \pi_1(X)$$

for all i, where $inc: X - K_i \to X$ is the inclusion map. We say that β is **boundary incompressible** *if it is not boundary compressible.*

Let $\pi_1^{inc}(X)$ denote the set of such boundary incompressible classes. When X is compact, we set $\pi_1^{inc}(X) := \pi_1(X) - const$, where $const$ denotes the set of homotopy classes of constant loops.

It is easily seen that the above is well defined and moreover any homeomorphism $X_1 \to X_2$ of a pair of manifolds induces a set isomorphism $\pi_1^{inc}(X_1) \to \pi_1^{inc}(X_2)$. Denote by $L_\beta X$ the class $\beta \in \pi_1^{inc}(X)$

Key words and phrases. locally conformally symplectic manifolds, Gromov-Witten theory, virtual fundamental class, Fuller index.

component of the free loop space of X, with its compact open topology. Let g be a complete metric on X, and let $S(g,\beta) \subset L_\beta X$ denote the subspace of all unit speed parametrized, smooth, closed g-geodesics in class β.

Definition 1.2. *We say that a metric g on X is* **β-taut** *if g is complete and $S(g,\beta)$ is compact. We will say that g is* **taut** *if it is β-taut for each $\beta \in \pi_1^{inc}(X)$.*

A complete metric all of whose boundary incompressible closed geodesics are minimizing in their homotopy class is taut, see Lemma 5.1. Thus, by the Cartan-Hadamard theorem [1], a basic example of a taut metric is a complete metric with non-positive curvature. It should be emphasized that taut metrics form a much larger class of metrics then just non-positive curvature metrics. One class of examples comes by way of Lemma 1.17 ahead, but this only scratches the surface.

Definition 1.3. *Let $\beta \in \pi_1^{inc}(X)$, and let g_0, g_1 be a pair of β-taut metrics on X. A* **β-taut homotopy or deformation** *between g_0, g_1, is a continuous (in the topology of C^0 convergence on compact sets) family $\{g_t\}$, $t \in [0,1]$ of complete metrics on X, s.t.*

$$S(\{g_t\},\beta) := \{(o,t) \in L_\beta X \times [0,1] \,|\, o \in S(g_t,\beta)\}$$

is compact. We say that $\{g_t\}$ is a **taut homotopy** *if it is β-taut for each $\beta \in \pi_1^{inc}(X)$. The above definitions of tautness are extended naturally to the case of a smooth fibration $X \hookrightarrow P \to [0,1]$, with a smooth fiber-wise family of metrics.*

A useful criterion for β-tautness is the following.

Theorem 1.4. *Let $\{g_t\}_{t\in[0,1]}$ be a continuous family of complete metrics on X. Suppose that:*

$$\sup_t |\max_{o\in S(g_t,\beta)} l_{g_t}(o) - \min_{o\in S(g_t,\beta)} l_{g_t}(o)| < \infty,$$

where l_{g_t} is the length functional with respect to g_t, then $\{g_t\}$ is β-taut. It follow that sky catastrophes of vector fields on closed manifolds are not geodesible by metrics all of whose geodesics are minimal, Appendix A.1.

For example, the hypothesis is trivially satisfied if g_t have the property that all their class β geodesics are minimal. In particular if g_t have non-positive curvature then $\{g_t\}$ is taut, again by the Cartan-Hadamard theorem.

Fuller at the end of [3] has asked for any metric conditions on vector fields to rule out sky catastrophes, see Appendix A.1. By the above, non-positivity of curvature is one such condition.

Remark 1.5. Note that if sky catastrophes were never geodesible then the geodesible Seifert conjecture would follow, by the main result of [8]. Hence, this is a subtle situation and it now seems to be plausible that sky catastrophes for geodesible and Reeb families of vector fields do exist. The qualitative structure of such potential geodesible or Reeb sky catastrophes is somewhat understood by results in [8], and [8, Theorem 1.10] in particular. But this does not greatly aid constructing potential examples, which must be topologically very complex, (there are necessarily infinitely many suitably synchronized bifurcation events). No results prior to the theorem above are known to me aside from those mentioned by Fuller himself in [3].

Let $\mathcal{G}(X)$ be the set of equivalence classes of taut metrics g, where g_0 is equivalent to g_1 whenever there is a taut homotopy between them. We may denote an equivalence class by its representative g by a slight abuse of notation.

Theorem 1.6. *For each manifold X, there is a natural, functional*

$$\mathrm{F} : \mathcal{G}(X) \times \pi_1^{inc}(X) \to \mathbb{Q}.$$

The value $\mathrm{F}(g,\beta)$ can be interpreted as a count of the set of closed g-geodesic strings in class β. But one must take care of exactly how to count, as in general this set should be understood as an orbifold or rather a Kuranishi space of a certain kind, hence this is why F is $\mathbb{Q}$ valued.

Definition 1.7. *Let $\beta \in \pi_1(X)$. For any based point $x_0 \in \operatorname{image}\beta \subset X$ (for $\operatorname{image}\beta$ the image of some representative of β) there is a naturally determined element $\beta_{x_0} \in \pi_1(X, x_0)$ well defined up to an inner automorphism, (concatenate a representative of β with a path from x_0 to a point in $\operatorname{image}\beta$). We say that a class $\beta \in \pi_1(X, x_0)$ is at* **most a k-power**, *if whenever $\beta = \alpha^n$, with $n > 0$ then $n \leq k$. Similarly, $\beta \in \pi_1(X)$ is at most a k-power if for any x_0 as above, β_{x_0} is at most a k-power. We say that $\beta \in \pi_1(X)$ is* **not a power** *if it is at most a 1-power.*

Example 1. Let g be a Riemannian metric with negative sectional curvature on a closed manifold X and $\beta \in \pi_1(X)$ a class represented by a multiplicty n closed geodesic, then

$$F(g,\beta) = \frac{1}{n}. \tag{1.8}$$

In particular, if β is not a power then $F(g,\beta) = 1$. More generally, (1.8) holds whenever g has a unique and non-degenerate closed geodesic string in class β. Here and throughout the paper, a closed geodesic string is ***non-degenerate*** if the corresponding S^1 family of closed geodesics is Morse-Bott non-degenerate.

Theorem 1.9. *Every rational number has the form $F(g,\beta)$ for some β-taut Riemannian g on some compact manifold X and for some $\beta \in \pi_1^{inc}(X)$.*

If $\beta \in \pi_1^{inc}(X)$ is not a power, then it is easy to see that that the reparametrization S^1 action on $L_\beta X$ is free, so that $H_*^{S^1}(L_\beta X, \mathbb{Z}) \simeq H_*(L_\beta X/S^1, \mathbb{Z})$, where $H_*^{S^1}(L_\beta X, \mathbb{Z})$ denotes the S^1-equivariant homology. Moreover, we have:

Theorem 1.10. *Suppose that $\beta \in \pi_1^{inc}(X)$ is not a power, and X admits a β-taut metric, then $H_*^{S^1}(L_\beta X, \mathbb{Z})$ is finite dimensional. Denote by $\chi^{S^1}(L_\beta X)$ the Euler characteristic of this homology. Then for any β-taut metric g on X:*

$$F(g,\beta) = \chi^{S^1}(L_\beta X).$$

Aside from the main example of manifolds admitting a metric of non-positive curvature, other non-trivial examples for the theorem above can be found by the proof of Theorem 1.9. For these types of examples any negative integer may appear as the value of $F(g,\beta)$. We leave out the details.

Remark 1.11. If β is a power, the idea behind Theorem 1.10 breaks down, as the S^1-equivariant homology of $L_\beta X$ may then be infinite dimensional even if X admits a β-taut g. As a trivial example this homology is already infinite dimensional when g is negatively curved, and the class β geodesic is k-covered, as then this homology is the group homology of $\mathbb{Z}_k$. In particular the connection with the Euler characteristic a priori breaks down. It is thus an interesting open problem if the functional F remains topological. This looks unlikely. A related question follows the proof of Corollary 1.15.

A celebrated theorem of Preissman [6] says that there are no negative sectional curvature metrics on compact products. Fibration counterexamples to Preissman's product theorem certainly exist. In fact, every closed 3-manifold X^3, for which there is no injection $\mathbb{Z}^2 \to \pi_1(X, x_0)$, and which fibers over a circle has a hyperbolic structure g_h, Thurston [10]. We are going to give a certain generalization of Preissman's theorem to fibrations, with possibly non-compact fibers, also replacing the negative sectional curvature condition by much weaker tautness conditions.

Definition 1.12. *Let $Z \hookrightarrow X \xrightarrow{p} Y$ be a smooth fiber bundle with X having a β-taut Riemannian metric g, for $\beta \in \pi_1^{inc}(X)$, and let g_Y be a metric on Y. Suppose that*

(1) *The fibers $Z_y = p^{-1}(y)$ are totally g-geodesic, for closed geodesics in class β. We denote by g_y the metric g restricted to Z_y.*

(2) *The fibers are parallel (the distribution $T^{vert}X = \ker p_*$ is parallel along any smooth curve in X with respect to the Levi-Civita connection of g).*

(3) *For any pair of fibers (Z_{y_0}, g_{y_0}), (Z_{y_1}, g_{y_1}), and a path $\gamma : [0,1]$ from y_0 to y_1 the fiber family $\{(Z_{\gamma(t)}, g_{\gamma t})\}$ furnishes a taut homotopy.*

(4) *p projects g-geodesics to geodesics of Y, g_Y.*

*We then call $p : X \to Y$ a β-**taut submersion**, with the metrics g, g_Y and g_Z all possibly implicit.*

Definition 1.13. *For $Z \hookrightarrow X \to Y$ as above, we say that $\beta \in \pi_1(X)$ is a* **fiber class** *if it is in the image of the inclusion $i_Z : \pi_1(Z) \to \pi_1(X)$.*

In the above definition of a taut submersion and the following theorem we need the auxiliary metric g on X to be Riemannian, and there is no obvious extension of the theorem to the Riemann-Finsler case. However, the conclusions of the theorem are for Riemann-Finsler metrics.

Theorem 1.14. *Let $p : (X, g) \to (Y, g_Y)$ be a β-taut submersion, where $\beta \in \pi_1^{inc}(X)$ is a fiber class. Suppose further that Y is connected and closed, and is such that all smooth closed contractible g_Y geodesics in Y are constant, then the following holds.*

(1) *If β is at most a k-power, and if $|\chi(Y)| > \operatorname{lcm}(1, \ldots, k)$, where $\operatorname{lcm}(1, \ldots, k)$ denotes the least common multiple of $1, \ldots, k$, then g cannot be β-taut deformed to a Riemann-Finsler metric g' on X with a unique non-degenerate geodesic string in class β, in particular to a complete metric with negative curvature.*

(2) *If $\chi(Y) = 0$ then for any $\beta \in \pi_1^{inc}(X)$, and any β-taut g: $F(g, \beta) = 0$. If in addition $\emptyset \neq \pi_1^{inc}(X)$, then X does not admit a complete Riemann-Finsler metric with negative curvature. More generally, X does not admit a β-taut Riemann-Finsler metric with a unique and non-degenerate geodesic string in any not a power class $\beta \in \pi_1^{inc}(X)$.*

As one corollary we may generalize a special case of Preissman's theorem, a stronger form of this corollary is further ahead.

Corollary 1.15. *Let $X = Z \times Y$ where Z, Y admit a complete metric of non-positive curvature, and Y is closed with vanishing Euler characteristic. Then X does not admit a complete (Riemann-Finsler) metric with a unique and non-degenerate geodesic string in any, not a power, fiber class $\beta \in \pi_1^{inc}(X)$. In particular, if Z satisfies $\emptyset \neq \pi_1^{inc}(Z)$, then X does not admit a metric of negative curvature.*

Proof. Let g_Z, g_Y be complete metrics on Z, Y with non-positive curvature. Take the product metric $g = g_Z \times g_Y$ on $X = Z \times Y$. By Lemma 1.17 and the Cartan-Hadamard theorem, g is taut. Moreover, for any fiber class $\beta \in \pi_1(Z)$, the natural projection $X \to S^1$ is automatically a β-taut submersion satisfying conditions of the theorem above. Hence, the first part follows by Part 2 of the theorem above.

The second part follows by the first part. If $\emptyset \neq \pi_1^{inc}(Z)$ then in particular we may find a not a power, fiber class $\beta \in \pi_1^{inc}(X)$. As previously observed, a complete metric with negative curvature is taut, and has a unique and non-degenerate class β geodesic string. So by the first part such a metric cannot exist. ∎

Question 1. Do there exist a pair of taut metrics g_1, g_2 on a manifold X which are not taut homotopic?

Probably both possibilities are interesting. If the answer is yes then in Part 1 of the above theorem we may replace the conclusion on non existence of β-taut homotopy between g, g' to just non-existence of g', with the same conditions. On the other hand if the answer is no then the previous Theorem 1.6 becomes far more intriguing.

A basic set of examples for the theorem is obtained by starting with any homomorphism

$$\phi : \pi_1(Y, y_0) \to \mathrm{Isom}(Z, g_Z), \quad \text{(the group of all isometries)}. \tag{1.16}$$

where g_Z is a taut metric, and there is a class $\beta_Z \in \pi_1^{inc}(Z)$. Suppose further that (Y, g_Y) satisfies:

- Y is closed and connected.
- All contractible g_Y geodesics in Y are constant.

We have the obvious induced diagonal action

$$\pi_1(Y, y_0) \to \mathrm{Diff}(Z \times \widetilde{Y}), \text{ (the group of all diffeomorphisms)},$$
$$\gamma \mapsto ((z, y) \mapsto (\phi(\gamma)(z), \gamma \cdot y)), \text{ (anonymous function notation)}$$

for $\widetilde{Y}$ the universal cover of Y. Taking the quotient of $Z \times \widetilde{Y}$ by this action, we get an associated "flat" bundle $Z \hookrightarrow X_\phi \xrightarrow{p} Y$, with a metric g_ϕ induced from the product metric $\widetilde{g} = g_Z \oplus g_Y$, on the covering space $q : Z \times \widetilde{Y} \to Z \times Y$.

Lemma 1.17. *Let $p : (X_\phi, g_\phi) \to (Y, g_Y)$ be as above, then this is a β-taut submersion, where $\beta = i_*(\beta_Z)$, for $i_* : \pi_1^{inc}(Z) \to \pi_1^{inc}(X_\phi)$ induced by inclusion.*

By the lemma above, $p : (X_\phi, g_\phi) \to (Y, g_Y)$ satisfies the hypothesis of the theorem above. Yet more concretely:

Example 2. Suppose we have $\beta_Z \in \pi_1^{inc}(Z)$, and let $\phi : Z \to Z$ be an isometry of a taut metric g_Z. Then by the construction above, the mapping torus $(Z, g_Z) \hookrightarrow (X_\phi, g_\phi) \xrightarrow{\pi} S^1$ has the structure of a β-taut submersion, satisfying the hypothesis of the theorem, for $\beta = i_*(\beta_Z)$ as above.

The next corollary of Theorem 1.14 is proved analogously to Corollary 1.15.

Corollary 1.18. *Let $(X_\phi, g_\phi) \to (Y, g_Y)$ be as in the construction above for Z, g_Z having non-positive curvature, and with a class $\beta_Z \in \pi_1^{inc}(Z)$. Then X_ϕ does not admit a β-taut Riemann-Finsler metric with a single non-degenerate class β geodesic string, for $\beta = i_*(\beta_Z)$ as above. In particular, X_ϕ does not admit a complete Riemann-Finsler metric with negative curvature. As a special case, this applies to the mapping tori X_ϕ, for $\phi : Z \to Z$ an isometry of a complete non-positively curved metric on Z. (The non-positive curvature hypothesis is for concreteness we may of course replace this condition by tautness.)*

Remark 1.19. In the Riemannian case, the conclusions on non-existence of negative sectional curvature metrics can likely be deduced with some work from Preissman's theorem, previously mentioned. (We need to do some extensions to non-compact setting.) In special cases we can obtain similar conclusions using S^1-equivariant Morse theory. Specifically, we need at least that the g-energy function on the free loop space LX,

$$e_g(o) = \int_{S^1} \langle \dot{o}(t), \dot{o}(t) \rangle_g dt \tag{1.20}$$

is Morse-Bott. Take for instance $X = T^n$ with the product metric. It would of course be very interesting to understand the relationship of the functional F with S^1-equivariant Morse homology in general. This relationship is very complex, since as previously mentioned, the latter equivariant Morse homology may be infinite dimensional even when X admits a taut metric g.

2. Proof of Theorem 1.4

The first part of the theorem clearly follows by the second part. So let $\{g_t\}$, $t \in [0,1]$ be as in the hypothesis, with

$$\sup_t |\max_{o \in S(g_t, \beta)} l_{g_t}(o) - \min_{o \in S(g_t,\beta)} l_{g_t}(o)| < c,$$

and suppose that

$$\sup_{(o,t) \in \mathcal{O}(\{g_t\},\beta)} l_{g_t}(o) = \infty.$$

Then we have a sequence $\{o_k\}$, $k \in \mathbb{N}$, of closed class β g_{t_k}-geodesics in X, satisfying:

(1) $\lim_{k \to \infty} t_k = t_\infty \in [0,1]$.

(2) $\lim_{k\to\infty} l_{g_{t_k}}(o_k) = \infty$, where $l_{g_{t_k}}(o_{t_k})$ is the length with respect to g_{t_k}.

Let o_∞ be a minimal, class β, $g_\infty = g_{t_\infty}$ geodesic in X, with g_∞ length L. Let g_{aux} be a fixed auxiliary metric on X, and let L_{aux} be the g_{aux} length of o_∞.

Define a pseudo-metric on the space of metrics on X as follows. Let $K \subset X$ be a fixed compact set containing image o_∞, and set

$$V \subset TX = \{v \in TX \,|\, \pi(v) \in K \text{ for } \pi: TX \to X \text{ the canonical projection, and } |v|_{aux} = 1\},$$

where $|v|_{aux}$ is the norm taken with respect to g_{aux}.

Then define:

$$d_{C^0}(g_1, g_2) = \sup_{v \in V} ||v|_{g_1} - |v|_{g_2}|.$$

By properties 1 and 2 we may find a $k > 0$ such that

$$d_{C^0}(g_{t_k}, g_{t_\infty}) < \epsilon \tag{2.1}$$

and

$$l_{g_{t_k}}(o_k) > c + L + L_{aux} \cdot \epsilon.$$

As $L + L_{aux} \cdot \epsilon \geq l_{g_{t_k}}(o_\infty)$ by (2.1), we have:

$$l_{g_{t_k}}(o_k) > l_{g_{t_k}}(o_\infty) + c.$$

Since we may find a closed g_{t_k}-geodesic o' satisfying $l_{g_{t_k}}(o') \leq l_{g_{t_k}}(o_\infty)$, we get that

$$|\max_{o \in S(g_{t_k},\beta)} l_{g_{t_k}}(o) - \min_{o \in S(g_{t_k},\beta)} l_{g_{t_k}}(o)| > c,$$

and so we are in contradiction.

Thus,

$$\sup_{(o,t) \in \mathcal{O}(\{g_t\},\beta)} l_{g_t}(o) < \infty.$$

It follows, by an analogue of Lemma 5.2, that the images of all elements $o \in S(\{g_t\}, \beta)$ are contained in a fixed compact $K \subset X$. Compactness of $S(\{g_t\}, \beta)$ then follows by the Arzella-Ascolli theorem. $\square$

3. Proof of Lemma 1.17

This is elementary so we limit ourselves to a sketch. Let $\phi_* : \pi_1(Y, y_0) \to \mathrm{Aut}(\pi_1^{inc}(Z))$ be the natural induced action, where $\mathrm{Aut}(\pi_1^{inc}(Z))$ denotes the group of set isomorphisms of $\pi_1^{inc}(Z))$. The orbit

$$O := \bigcup_{\gamma \in \pi_1(Y,y_0)} \phi_*(\gamma)(\beta_Z)$$

is readily seen to be finite by the assumption that the action ϕ is by isometries, and by the fact that $\beta_Z \in \pi_1^{inc}(Z)$. In fact, only finitely many distinct classes in $\pi_1^{inc}(Z)$ have representatives by g_Z-geodesics of fixed length, as otherwise we may find a sequence of boundary incompressible g_Z-geodesics of fixed length, with no convergent subsequence. The latter yields a contradiction by an analogue of Lemma 5.1.

As g_Z is taut, $S(g_Z, \phi_*(\gamma)(\beta_Z))$ is compact for each γ, where $S(g_Z, \phi_*(\gamma)(\beta_Z))$ is the space of geodesics as in Definition 1.2. By the condition on contractible geodesics of g_Y, we get:

$$\begin{aligned} S(g_\phi, \beta) &= q_*(S(g_Z \oplus g_Y, \beta))) \\ &= \bigcup_{\beta \in O} q_*(S(g_Z, \beta) \times \widetilde{Y}), \end{aligned}$$

for $q_* : L(Z \times \widetilde{Y}) \to L(Z \times Y)$ induced by the quotient map $q : Z \times \widetilde{Y} \to Z \times Y$, (as in the preamble to the statement of the lemma) and where $S(g_Z, \gamma) \times \widetilde{Y}$ is identified as a subset $S(g_Z, \gamma) \times \widetilde{Y} \subset L(Z) \times \widetilde{Y} \subset L(Z \times \widetilde{Y})$. Given that O is finite, this then readily implies our claim. □

4. Preliminaries on Reeb flow

Let (C^{2n+1}, λ) be a contact manifold with λ a contact form, that is a one form s.t. $\lambda \wedge (d\lambda)^n \neq 0$. Denote by R^λ the Reeb vector field satisfying:

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

Recall that a ***closed λ-Reeb orbit*** (or just Reeb orbit when λ is implicit) is a smooth map

$$o : (S^1 = \mathbb{R}/\mathbb{Z}) \to C$$

such that

$$\dot{o}(t) = cR^\lambda(o(t)),$$

with $\dot{o}(t)$ denoting the time derivative, for some $c > 0$ called period. Let $S(R^\lambda, \beta)$ denote the space of all closed λ-Reeb orbits in free homotopy class β, with its compact open topology. And set

$$\mathcal{O}(R^\lambda, \beta) = S(R^\lambda, \beta)/S^1,$$

where $S^1 = \mathbb{R}/\mathbb{Z}$ acts by reparametrization $t \cdot o(\tau) = o(t + \tau)$.

5. Definition of the functional F

Let X be a manifold with a complete taut metric g. Let C be the unit cotangent bundle of X, with its Louiville contact 1-form λ_g. If $o : S^1 = \mathbb{R}/\mathbb{Z} \to X$ is a unit speed closed geodesic, it has a canonical lift $\widetilde{o} : S^1 \to C$. If $\beta \in \pi_1^{inc}(X)$, let $\widetilde{\beta} \in \pi_1(C)$ denote class $[\widetilde{o}] \in \pi_1(C)$, where o is a unit speed closed geodesic representing β.

Let $S(R^{\lambda_g}, \widetilde{\beta})$ be the orbit space as in Section 4, for the Reeb flow of the contact form λ_g. And set

$$\mathcal{O}_{g,\beta} = \mathcal{O}(R^{\lambda_g}, \widetilde{\beta}) := S(R^{\lambda_g}, \widetilde{\beta})/S^1,$$

i.e. this can be identified with the space of class β g-geodesic strings. By the tautness assumptions $\mathcal{O}_{g,\beta}$ is compact.

We then define

$$F(g, \beta) = i(\mathcal{O}_{g,\beta}, R^{\lambda_g}, \widetilde{\beta}) \in \mathbb{Q}$$

where the right hand side is the Fuller index of R^{λ_g} in class $\widetilde{\beta}$. As a basic example we have:

Lemma 5.1. *Suppose that g is a complete metric on X, all of whose boundary incompressible geodesics are minimal, then g is taut.*

Proof. First we state a more basic lemma.

Lemma 5.2. *Suppose that g is a complete metric on X, $\beta \in \pi_1^{inc}(X)$ and let $S \subset L_\beta X$ be a subset on which the g-length functional is bounded. Then the images in X of elements of S are contained in a fixed compact subset of X.*

Proof. Suppose otherwise. Fix an exhaustion by nested compact sets

$$\bigcup_{i \in \mathbb{N}} K_i = X, \quad K_i \supset K_{i-1}.$$

Then either there is sequence $\{o_i\}_{i \in \mathbb{N}}$, $o_i \in S$ s.t. $o_i \in K_i^c$, for K_i^c the complement of K_i, which contradicts the fact that β is incompressible. Or there is a sequence $\{o_k\}_{k \in \mathbb{N}}$, $o_k \in S$ s.t.:

(1) Each o_k intersects K_{i_0} for some i_0 fixed.

(2) For each $i \in \mathbb{N}$ there is a $k_i > i$ s.t. o_{k_i} is not contained in K_i.

Now if $\operatorname{diam}(o_k)$ is bounded in k, then condition 1 implies that o_k are contained in a set of bounded diameter. (Here $\operatorname{diam}(o_k)$ denotes the diameter of image o_k.) Consequently, by Hopf-Rinow theorem [1], o_k are contained in a compact set. But this contradicts condition 2, and the fact that K_i form an exhaustion of X.

Thus, we conclude that $\operatorname{diam}(o_k)$ is unbounded, but this contradicts the hypothesis. □

Returning to the proof of the main lemma. By assumption, closed, class $\beta \in \pi_1^{inc}(X)$ geodesics are g-minimizing in their homotopy class and in particular have fixed length. By the lemma above there is a fixed $K \subset X$ s.t. every class β closed geodesic has image contained in K. Then compactness of $S(g, \beta)$ follows by Arzella-Ascolli theorem.

□

Proof Theorem 1.6. Let $\beta \in \pi_1^{inc}(X)$, be given and let g be β-taut. We just need to prove that $F(g, \beta)$ is invariant under a β-taut deformation of g. So let $\{g_t\}$, $t \in [0,1]$ be a β-taut deformation of metrics on a compact manifold X. Let $R^{\lambda_{g_t}}$ be the geodesic flow on the g_t unit cotangent bundle C_t. Trivializing the family $\{C_t\}$ we get a family $\{R_t\}$ of flows on $C \simeq C_t$, with R_t conjugate to $R^{\lambda_{g_t}}$.

Let $\mathcal{O}(\{R_t\}, \widetilde{\beta})$ be the cobordism as in (A.2), where $\widetilde{\beta} \in \pi_1(C)$ is as above. Then $\mathcal{O}(\{R_t\}, \widetilde{\beta})$ is compact as $S(\{g_t\}, \beta)$ is compact by assumption.

Basic invariance of the Fuller index, that is (A.3), immediately yields: $F(g_0, \beta) = F(g_1, \beta)$. □

Proof of Theorem 1.10. This is an application of Morse theory. As g is β-taut, and so $S(g, \beta)$ is compact, we may find a C^∞-nearby metric g', s.t. g' has finitely many class β closed geodesic strings, all of which are non-degenerate. The notation $L_\beta X$ now denotes the Hilbert manifold of H^1 loops, as used for example in the classical work of Gromoll-Meyer [4]. This Hilbert manifold is well known to be homotopy equivalent to the standard free loop space with its compact open topology.

The energy function $e_{g'} : L_\beta X \to S^1$, as in (1.20), is smooth, S^1 invariant and satisfies the Palais-Smale condition. The flow for its negative gradient vector field V is complete, and we can do Morse theory mostly as usual. This is understood starting with the work of Klingenberg [5], and even before that by Palais and Smale. In our case, $e_{g'}$ is moreover a Morse-Bott function with critical manifolds C_o corresponding to S^1 families of geodesics, for each closed geodesic string o.

There is an induced Morse-Bott cell decomposition on $L_\beta X$, meaning a stratification formed by V unstable manifolds of the above mentioned critical manifolds C_o. This is Bott's extension of the fundamental Morse decomposition theorem. Now the S^1 action on $L_\beta X$ is free by the condition that β is not a power. This action is not smooth, but is continuous and so taking the topological S^1 quotient we get a standard CW cell decomposition of $L_\beta X/S^1$ with one k-cell for each closed g'-geodesic string o of index $\mathrm{morse}(o) = k$. (The "index" means the Morse-Bott index of the critical manifold C_o.) All of the above is understood, see for instance [4].

From the above cell decomposition, we readily get that the homology $H_*(L_\beta X/S^1, \mathbb{Z}) = H_*(L_\beta^{S^1} X, \mathbb{Z})$ is finite dimensional. We also get that:

$$\begin{aligned}\chi(L_\beta X/S^1) &= \sum_{o \in \mathcal{O}(R^{\lambda_{g'}}, \beta)} -1^{\mathrm{morse}(o)} \quad \text{(immediate from the cell decomposition)}\\ &= F(g', \beta) \quad \text{(basic properties of the fixed point index, see for instance [8, Section 2])}\\ &= F(g, \beta) \quad \text{(by the local invariance (A.4) of the Fuller index).}\end{aligned}$$

□

6. Proof of Theorem 1.14

We first prove:

Theorem 6.1. *Let $p : X \to Y$ be a β-taut submersion as in the statement of Theorem 1.14 and $\beta \in \pi_1^{inc}(X)$ a fiber class. Then*

$$F(g, \beta) = card \cdot \chi(Y) \cdot F(g_Z, \beta_Z),$$

where $card \in \mathbb{N}$ is the cardinality of a certain orbit of the holonomy group (as explained in the proof), and where β_Z is as in Lemma 1.17.

Proof. We have a natural subset of $\mathcal{O}' \subset \mathcal{O}g, \beta$, consisting of all vertical geodesics, that is g-geodesics contained in fibers $p^{-1}(y) = Z_y$. In fact, $\mathcal{O}' = \mathcal{O}g, \beta$, for if o is any class β geodesic, the projection $p(o)$ is a contractible g_Y geodesic, and by assumptions is constant.

In particular there a natural continuous projection $\widetilde{p} : \mathcal{O}g, \beta \to Y$, $\widetilde{p}(o) = y$ where y is determined by the condition that $Z_y \supset \mathrm{image}\, o$. We will use this to construct a suitable (in a sense abstract i.e. not Reeb) perturbation of the vector field R^{λ_g}, using which we can calculate the invariant $F(g, \beta)$.

Fix a Morse function on f on Y, let $C = S^*X$ denote the g-unit cotangent bundle of X. For $v \in T_xX$ let $\langle v|$ denote the functional $T_xX \to \mathbb{R}$, $w \mapsto \langle v, w\rangle_g$. Define $\widetilde{f} : C \to \mathbb{R}$ by $\widetilde{f}(\langle v|) := f(p(v))$, also define $P : C \to \mathbb{R}$ by $P(\langle v|) := |P^{vert}(v)|_g^2$, where $P^{vert}(v)$ denotes the g-orthogonal projection of v onto the $T_x^{vert}X \subset T_xX$, for $T^{vert}X$ the vertical tangent bundle of X, i.e. the kernel of the map $p_* : TX \to TY$. And define $F : C \to \mathbb{R}$ by:

$$F(\langle v|) := P(\langle v|) + \widetilde{f}(\langle v|).$$

Set

$$V_t = R^{\lambda_g} - t\,\mathrm{grad}_{g_S} F,$$

where the gradient is taken with respect to the Sasaki metric g_S on C [7] induced by g. The latter Sasaki metric is the natural metric for which we have an orthogonal splitting $TC = T^{vert}C \oplus T^{hor}C$, where $T^{vert}C$ is the kernel of $pr_* : TC \to TX$, induced by the natural projection $pr : C \to X$, and where $T^{hor}C$ is the g Levi-Civita horizontal sub-bundle.

Set $\mathcal{O}_t = \mathcal{O}(V_t, \widetilde{\beta})$, where $\widetilde{\beta}$ is as in Section 5.

Lemma 6.2. (1) *For all $t \in [0,1]$, $N_t := \mathcal{O}_t \cap \mathcal{O}_{g,\beta}$ is open and closed in $\mathcal{O}_t$.*

(2) *For all $t \in (0,1]$, $N_t = \cup_{y \in \operatorname{crit}(f)} \widetilde{p}^{-1}(y)$, where $\operatorname{crit}(f)$ is the set of critical points of f.*

Proof. It is easy to see that V_t is complete and without zeros. Suppose that $t > 0$. Let $\langle v_\tau|$, $\tau \in \mathbb{R}$ be the flow line of V_t, through $\langle v_0|$, i.e. $\langle v_\tau| = \phi_\tau(\langle v_0|)$, for ϕ_τ the time τ flow map of V_t. By the fact that the fibers of p are assumed to be parallel, we have that

$$R^{\lambda_g}(P) = 0, \quad \text{using the derivation notation.}$$

Also,

$$\operatorname{grad}_{g_S} \widetilde{f}(P) = 0,$$

which readily follows by the conjunction of g_S being Sasaki and the fibers of p being parallel. Consequently, the function

$$\tau \mapsto P(\langle v_\tau|) = |P^{vert}(v_\tau)|^2_g$$

is monotonically decreasing unless either:

(1) v_0 is tangent to $T^{vert}X$, in which case for all τ, v_τ are tangent to $T^{vert}X$ and $|P^{vert}(v_\tau)|^2_g = 1$.

(2) For all τ, $|P^{vert}(v_\tau)|^2_g = 0$.

In particular, the closed orbits of V_t split into two types.

(1) Closed orbits $o(\tau) = \langle v_\tau|$ with v_τ always tangent to $T^{vert}X$. In this case we may immediately, conclude that o is a lift to C of a closed g-geodesic in contained in the fiber over a critical point of f.

(2) Closed orbits $o(\tau) = \langle v_\tau|$ for which v_τ is always g-orthogonal to $T^{vert}X$.

Clearly, the conclusion follows. $\square$

Remark 6.3. It would be very fruitful to remove the condition on the fibers of p being parallel. But our argument would need to substantially change.

We return to the proof of the theorem. Set

$$\widetilde{N} = \{(o,t) \in L_{\widetilde{\beta}}C \times [0,\epsilon] \,|\, o \in N_t\},$$

where $L_{\widetilde{\beta}}C$ denotes the $\widetilde{\beta}$ component of the free loop space as previously. By part I of Lemma 6.2, this is an open compact subset of $\mathcal{O}(\{V_t\}, \widetilde{\beta})$ s.t.

$$\widetilde{N} \cap (L_{\widetilde{\beta}}C \times \{0\}) = \mathcal{O}(R^{\lambda_g}, \widetilde{\beta}),$$

(equalities throughout are up to natural set theoretic identifications.)

By definitions:

$$N_t = \widetilde{N} \cap (L_{\widetilde{\beta}}C \times \{t\}).$$

Now the invariance of the Fuller index gives:

$$i(N_0, R^{\lambda_g}, \widetilde{\beta}) = i(N_1, V_1, \widetilde{\beta}).$$

We proceed to compute the right hand side. Fix any smooth Ehresmann connection $\mathcal{A}$ on the fiber bundle $p : X \to Y$. This induces a holonomy homomorphism:

$$hol_y : \pi_1(Y, y) \to \operatorname{Aut} \pi_1(Z_y) \text{ (the right-hand side is the group of set automorphisms),}$$

with image denoted $\mathcal{H}_y \subset \operatorname{Aut} \pi_1(Z_y)$.

Let β_Z denote a class in $\pi_1(Z_y)$ s.t. $(i_{Z_y})_*(\beta_Z) = \beta$, for $i_{Z_y}: Z_y \to X$ the inclusion map. Set

$$S_y := \bigcup_{g \in \mathcal{H}_y} g(\beta_Z) \subset \pi_1(Z_y).$$

Then for another $y' \in Y$,

$$h_*: S_{y'} \to S_y, \tag{6.4}$$

is an isomorphism, where $h: Z_y \to Z_{y'}$ is a smooth map given by the $\mathcal{A}$-holonomy map determined by some path from y to y', and where h_* is the naturally induced map.

Denoting by g_y the restriction of g to the fiber Z_y, let R^y denote the λ_{g_y} Reeb vector field on the g_{Z_y}-unit cotangent bundle C_y of Z_y. The cardinality *card* of S_y is finite, as otherwise we get a contradiction to the compactness of $S(g,\beta)$. Now

$$\widetilde{p}^{-1}(y) = \bigcup_{\alpha \in S_y} \mathcal{O}(R^{\lambda_y}, \alpha).$$

From part 2 of Lemma 6.2 and by straightforward index computations we get:

$$i(N_1, V_1, \widetilde{\beta}) = \sum_{y \in \mathrm{crit}(f)} (-1)^{\mathrm{morse}(y)} \cdot i(\widetilde{p}^{-1}(y), R^{\lambda_y}, \widetilde{\beta}),$$

where $\mathrm{morse}(y)$ denote the Morse index of y. Now

$$\begin{aligned} i(\widetilde{p}^{-1}(y), R^{\lambda_y}, \widetilde{\beta}) &= \sum_{\alpha \in S_y} i(\mathcal{O}(R^{\lambda_y}), R^{\lambda_y}, \widetilde{\alpha}) \\ &= \sum_{\alpha \in S_y} F(g_y, \alpha). \\ &= card \cdot F(g_Z, \beta_Z), \end{aligned}$$

where the last equality follows by (6.4), and by the condition 3 in the Definition 1.2. And so the result follows. □

Now returning to the proof of the main theorem. If β is at most a k-power then elements of $\mathcal{O}(R^{\lambda_{g_Z}}, \beta_Z)$ have multiplicity at most k. It follows that either

$$|\operatorname{lcm}(1, \ldots, k) \cdot F(g_Z, \beta_Z)| \geq 1,$$

or $F(g_Z, \beta_Z) = 0$. So if $|\chi(X)| > \operatorname{lcm}(1, \ldots, k)$, by the theorem above $|F(g,\beta)| \geq 1$, or $F(g,\beta) = 0$. In either case the first conclusion follows by Theorem 1.6 and by Example 1.

If $\chi(X) = 0$, then by the theorem above for all β as in the statement: $F(g,\beta) = 0$. So if we take β to be not a power, then by Theorem 1.10 we have

$$0 = F(g,\beta) = \chi^{S_1}(L_\beta X) = F(g', \beta) = 1,$$

if g' is β-taut and has a unique and non-degenerate class β geodesic string. But this is impossible, so that the second part of the theorem readily follows. □

Proof of Theorem 1.9. By Theorem 6.1 0 is certainly a value of the invariant F. We first prove that every negative rational number is the value of the invariant. Let p, q be positive integers. Let Y be a closed surface of genus $(p+1) > 1$ with a hyperbolic metric g_Y, let Z be the genus 2 closed surface with a hyperbolic metric g_Z and let $\beta_Z \in \pi_1^{inc}(Z)$ be the class represented by a $2 \cdot q$-fold covering of a simple loop representing a generator of the fundamental group of Z.

Let $X = Y \times Z$ with the product metric $g = g_Y \times g_Z$ and $p: X \to Y$ the canonical projection. By Theorem 6.1

$$F(g,\beta) = \chi(Y) \cdot F(g_Z, \beta_Z) = (-2p) \cdot \frac{1}{2q} = -\frac{p}{q},$$

where β is as in Lemma 1.17. So we proved our first claim.

Let again p, q be positive integers. Let Y be closed surface of genus 2, with a hyperbolic metric g_Y. And let Z be a manifold satisfying $F(g_Z, \beta_Z) = -\frac{p}{2q}$ for some β_Z-taut metric g_Z on Z and for some class $\beta_Z \in \pi_1^{inc}(Z)$. This exist by the discussion above. Let $g = g_Y \times g_Z$ be the product metric on $Y \times Z$, and β as above. Analogously to the discussion above we get:

$$F(g, \beta) = \chi(Y) \cdot F(g_Z, \beta_Z) = (-2) \cdot \frac{-p}{2q} = \frac{p}{q}.$$

□

A. Fuller index and sky catastrophes

Let X be a complete vector field without zeros on a manifold M. Set

$$S(X, \beta) = \{o \in L_\beta M \mid \exists p \in (0, \infty), \ o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\}. \tag{A.1}$$

The above p is uniquely determined and we denote it by $p(o)$ called the period of o.

There is a natural S^1 reparametrization action on $S(X, \beta)$: $t \cdot o$ is the loop $t \cdot o(\tau) = o(t + \tau)$. The elements of $\mathcal{O}(X, \beta) := S(X, \beta)/S^1$ will be called ***orbit strings*** or just closed orbits. Slightly abusing notation we just write o for the equivalence class of o.

The multiplicity $m(o)$ of an orbit string is the ratio $p(o)/l$ for $l > 0$ the period of a simple orbit string covered by o.

We want a kind of fixed point index which counts orbit strings o with certain weights. Assume for simplicity that $N \subset \mathcal{O}(X, \beta)$ is finite. (Otherwise, for a general open compact $N \subset \mathcal{O}(X, \beta)$, we need to perturb.) Then to such an (N, X, β) Fuller associates an index:

$$i(N, X, \beta) = \sum_{o \in N} \frac{1}{m(o)} i(o),$$

where $i(o)$ is the fixed point index of the time $p(o)$ return map of the flow of X with respect to a local surface of section in M transverse to the image of o.

Fuller then shows that $i(N, X, \beta)$ has the following invariance property. For a continuous homotopy $\{X_t\}$, $t \in [0, 1]$ set

$$S(\{X_t\}, \beta) = \{(o, t) \in L_\beta M \times [0, 1] \mid o \in S(X_t)\}.$$

And given a continuous homotopy $\{X_t\}$, $X_0 = X$, $t \in [0, 1]$, suppose that $\widetilde{N}$ is an open compact subset of

$$\mathcal{O}(\{X_t\}, \beta) := S(\{X_t\}, \beta)/S^1, \tag{A.2}$$

such that

$$\widetilde{N} \cap (L_\beta M \times \{0\}) / S^1 = N.$$

Then if

$$N_1 = \widetilde{N} \cap (L_\beta M \times \{1\}) / S^1$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta). \tag{A.3}$$

We call this ***basic invariance***. In the case $\mathcal{O}(X_0, \beta)$ is compact, $\mathcal{O}(X_1, \beta)$ is compact for any sufficiently C^0 nearby X_1, and in this case basic invariance implies (see for instance [8, Proof of Lemma 1.6]):

$$i(\mathcal{O}(X_0, \beta), X, \beta) = i(\mathcal{O}(X_1, \beta), X_1, \beta). \tag{A.4}$$

A.1. **Blue sky catastrophes.**

Definition A.5 (Preliminary). *A* **sky catastrophe** *for a smooth family* $\{X_t\}$, $t \in [0,1]$, *of non-vanishing vector fields on a closed manifold* M *is a continuous family of closed orbit strings* $\tau \mapsto o_{t_\tau}$, o_{t_τ} *is an orbit string of* X_{t_τ}, $\tau \in [0,\infty)$, *such that the period of* o_{t_τ} *is unbounded from above.*

A sky catastrophe as above was initially constructed by Fuller [2]. Or rather his construction essentially contained this phenomenon. A more general definition appears in [8], we slightly extend it here to the case of non-compact manifolds. All these definitions become equivalent given certain regularity conditions on the family $\{X_t\}$ and assuming M is compact.

Definition A.6. *Let* $\{X_t\}$, $t \in [0,1]$ *be a continuous family of non-zero, complete smooth vector fields on a manifold* M *and* $\beta \in \pi_1^{inc}(X)$.

We say that $\{X_t\}$ *has a* **catastrophe in class** β, *if there is an element*

$$y \in \mathcal{O}(X_0,\beta) \sqcup \mathcal{O}(X_1,\beta) \subset \mathcal{O}(\{X_t\},\beta)$$

such that there is no open compact subset of $\mathcal{O}(\{X_t\},\beta)$ *containing* y.

A vector field X on M is ***geodesible*** if there exists a metric g on M s.t. every flow line of X is a unit speed g-geodesic. A family $\{X_t\}$ is ***geodesible*** if there is a continuous family $\{g_t\}$ of metrics, with X_t geodesible with respect to g_t for each t. A family $\{X_t\}$ is ***geodesible*** if there is a continuous family $\{g_t\}$ of metrics with X_t geodesible with respect to g_t for each t. A ***geodesible sky catastrophe*** is a geodesible family $\{X_t\}$ with a sky catastrophe. A ***Reeb sky catastrophe*** is a family of Reeb vector fields $\{X_t\}$ with a sky catastrophe.

7. Acknowledgements

Larry Guth for related discussions. Dennis Sullivan and Misha Gromov for much welcome supportive comments.

References

[1] D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, vol. 200 of Grad. Texts Math., New York, NY: Springer, 2000.
[2] F. Fuller, *Note on trajectories in a solid torus.*, Ann. Math. (2), 56 (1952), pp. 438–439.
[3] ———, *An index of fixed point type for periodic orbits.*, Am. J. Math., 89 (1967), pp. 133–145.
[4] D. Gromoll and W. Meyer, *Periodic geodesics on compact Riemannian manifolds*, J. Differ. Geom., 3 (1969), pp. 493–510.
[5] W. Klingenberg, *Closed geodesics*, Ann. Math. (2), 89 (1969), pp. 68–91.
[6] A. Preissmann, *Quelques propriétés globales des espaces de Riemann*, Commentarii Mathematici Helvetici, (1942).
[7] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J. (2), 10 (1958), pp. 338–354.
[8] Y. Savelyev, *Extended Fuller index, sky catastrophes and the Seifert conjecture*, International Journal of mathematics, 29 (2018).
[9] ———, *Elliptic curves in lcs manifolds and metric invariants*, http://yashamon.github.io/web2/papers/conformalsymplectic.pdf, (2023).
[10] W. P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Am. Math. Soc., New Ser., 19 (1988), pp. 417–431.

Email address: yasha.savelyev@gmail.com

Faculty of Science, University of Colima, Mexico