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GLOBAL FUKAYA CATEGORY AND QUANTUM NOVIKOV CONJECTURE I

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ABSTRACT. Using Floer-Fukaya theory for a monotone (M, ω) we construct a natural continuous map

$$BHam(M, \omega) \to (\mathcal{S}, NFuk(M)),$$

with $(\mathcal{S}, NFuk(M))$ denoting the $NFuk(M)$ component of the "space" of ∞-categories, where $NFuk(M)$ is the A_∞-nerve of the Fukaya category $Fuk(M)$. On the level of bundles, this can be understood as functorially translating a Hamiltonian fibre bundle to a certain "derived" vector bundle, over the same space. This construction is very closely related to the theory of the Seidel homomorphism and the quantum Chern classes of the author, and this map is intended to be the deepest expression of their underlying geometric theory. In part II the above map is shown to be non trivial by an explicit calculation. In particular we arrive at a new non-trivial "quantum" invariant of a smooth manifold, and a "quantum" Novikov conjecture.

CONTENTS

1. Introduction

Smooth fibrations over a Lorentz 4-manifold with fiber a Calabi-Yau 6-fold are a model for the physical background in string theory. This suggests that there may be some string theory linked mathematical invariants of such a fibration. Indeed, when the structure group of $M \hookrightarrow P \to X$ can be reduced to the group of Hamiltonian symplectomorphisms of M, in which case P is called a Hamiltonian fibration, there are a couple of basic invariants of such a fibration based on Floer-Gromov-Witten theory. For example the Seidel representation [1], and the related quantum characteristic classes of the author [2], related invariants are also proposed by Hutchings [3]. Even earlier there is work on parametric Gromov-Witten invariants of Hamiltonian fibrations by Le-Ono [4], and Olga Buse [5]. At the same time, Costello's theorem [6] on reconstruction of topological conformal field theories from Calabi-Yau A_∞ categories suggests that the above invariants must have a similar reconstruction principle.

For a given a Hamiltonian fibration P as above, the A_∞ Fukaya categories of the fibers, fit into a "family", although exactly what this "family" should mean is a non-trivial problem by itself. Then our basic idea is that associated to a Hamiltonian fibration there should be a classifying map from X into an appropriate "classifying" space of A_∞ categories, from which the other invariants can be reconstructed, via a version of Toen's derived Morita theory, [7]. We say more on this in Section 1.3.

This paper will be mostly self contained, as we will explain many (especially algebraic) concepts used.

1.1. A functor from the category of smooth simplices of X.

Given P as above, and a choice of analytic perturbation data, to each smooth simplex

$$\Sigma : \Delta^n \to X,$$

we associate an A_∞ category $F(\Sigma)$. The principal step in this is the construction of natural maps, satisfying various axioms, from the universal curves over $\{\overline{\mathcal{R}}_d\}$, into the standard topological simplices Δ^n, where $\overline{\mathcal{R}}_d$ is the moduli spaces of Riemann surfaces which are topologically disks with $d+1$ punctures on the boundary. This topological-combinatorial connection of the universal curves with simplices is new, and is possibly of independent interest. The above data is then extended to functor

$$F : Simp(X) \to A_\infty - Cat^{unit},$$

from the category of smooth non-degenerate simplices of X into the category of small unital $\mathbb{Z}_2$-graded A_∞ categories over $\mathbb{Q}$, with morphisms strict embeddings, which are moreover quasi-equivalences. Let $X_\bullet$ denotes the smooth singular set of X. We shall see that F canonically extends as

$$F : \Delta/X_\bullet \to A_\infty - Cat^{unit},$$

with $\Delta/X_\bullet$ the category of all smooth simplices of X.

We then use the above and the A_∞ nerve functor

$$N : A_\infty - Cat \to sSet$$

to obtain:

Theorem 1.1. *After a choice of auxiliary perturbation data $\mathcal{D}$, there is a natural ∞-category $Fuk_\infty(P, \mathcal{D}) = \operatorname{colim}_{\Delta/X_\bullet} NF$, and a (co)-Cartesian fibration*

$$NFuk(M) \hookrightarrow Fuk_\infty(P, \mathcal{D}) \to X_\bullet,$$

whose equivalence class depends only on the Hamiltonian isomorphism class of P.

The functor N is an analogue for A_∞ categories of the classical nerve construction, and is originally due to Lurie [8]. It was then clarified in the A_∞ context by Tanaka [9] and Faonte [10]. Using this we get using Lurie's straightening theorem for such fibrations:

Theorem 1.2. *For (M, ω) a monotone symplectic manifold, a Hamiltonian fibration $M \hookrightarrow P \to X$ together with some choice of auxiliary perturbation data $\mathcal{D}$ induces a classifying map*

$$cl(P) : X \to (\mathcal{S}, NFuk(M)),$$

where $(\mathcal{S}, NFuk(M))$ denotes the space of ∞-categories, in the component of $NFuk(M)$. (See Section A.1) The homotopy class of $cl(P)$ is independent of the choice of $\mathcal{D}$.

The above can also be formulated on the universal level.

Theorem 1.3. *For (M, ω) a monotone symplectic manifold, there is a homotopy natural map*

$$cl : BHam(M, \omega) \to (\mathcal{S}, NFuk(M)).$$

This can be understood as the maxim that $Ham(M, \omega)$ "acts" on the Fukaya category. A discreet version of such action can be found in Seidel [11]. But above the action is in a suitable sense continuous. The proof is in Section 6.3.

1.2. **Towards new invariants and quantum Novikov conjecture.** By the above discussion we automatically obtain a new invariant of a Hamiltonian fibration $M \hookrightarrow P \to X$ as the homotopy class of its classifying map to $\mathcal{S}$.

It may difficult to get intrinsic motivation for Hamiltonian fibrations for a reader outside of symplectic geometry, as a start one may read [12]. However as one particular case we can fiberwise projectivize the complexified tangent bundle:

$$P(X) = P(TX \otimes \mathbb{C})$$

of a smooth manifold X. This $P(X)$ in particular has the structure of a smooth Hamiltonian fibration. In this way we also get a new invariant of a smooth r manifold X, given by the homotopy class of the classifying map

$$cl(P(X)) : X \to (\mathcal{S}, NFuk(\mathbb{CP}^{r-1})).$$

It should be remarked that a priori the homotopy class of $cl(P(X))$ depends on the smooth structure of the tangent bundle. However it immediately follows from the universal construction in Section 6.3, (Theorem 6.3) that only the topological type of the tangent bundle is detected by the homotopy class $[cl(P(X))]$.

Recall that Pontryagin classes of a smooth manifold are defined as Chern classes of its complexified tangent bundle. Novikov has shown that rational Pontryagin classes are topologically invariant. It is then very natural to ask the following, "quantum" variant of the Novikov conjecture:

Question 1.4. *Suppose that $f : X \to Y$ is a homeomorphism of smooth manifolds. Is $cl(P(X))$ homotopic to $cl(P(Y)) \circ f$?*

Why may one expect the answer of yes? One substantial reason is that like the rational Pontryagin classes of X the invariant $[cl(P(X))]$ is based on a rational algebraic theory, (we passed through rational Fukaya categories). But of course the answer of "no" is also very interesting, since it means that our construction gives new smooth invariants via holomorphic curves. Indeed the question is so interesting that we feel it is one of the main reasons for existence of this paper.

1.3. **Hochschild and geometric Hochschild cohomology and homotopy groups of $Ham(M, \omega)$.** This section is an excursion, meant to relate our geometry theory with the algebraic derived Morita theory of Toen. For an A_∞ category C we define

$$HH^{2-i}_{geom}(C) = \pi_i(\mathcal{S}, NC).$$

The left hand side is named geometric Hochschild cohomology, the name and notation will be justified shortly. By Theorem 1.3 above we then get:

Theorem 1.5. *For (M, ω) monotone, there is a natural group homomorphism*

$$\pi_{i-1}(Ham(M, \omega), id) \to HH^{2-i}_{geom}(Fuk(M)). \tag{1.1}$$

$HH^*(Fuk(M))$ is known to be isomorphic to $QH^*(M)$ for a wide class of cases, and so the above, when $i > 2$ has the same formal form as the author's quantum characteristic classes [2], provided there is a connection between $HH^*(Fuk(M))$ and $HH^*_{geom}(Fuk(M))$. This would be the most basic form of the "reconstruction" that was mentioned before. Such a connection is described further below.

Remark 1.6. *Note that the case of $i = 2$, which "corresponds to" the Seidel homomorphism [1], is a bit special since the correspondence in Theorem 1.8 works differently when $i = 2$.*

In Part II we calculate with Hamiltonian S^2 fibrations over S^4 to get:

Theorem 1.7. *The map*

$$\mathbb{Z} \to HH^{-2}_{geom}(Fuk(S^2)) = \pi_4(\mathcal{S}, NFuk(S^2),$$

determined by (1.1) is an injection.

1.3.1. *Geometric Hochschild cohomology and Toen's derived Morita theory.* A small disclaimer. $HH^*_{geom}(C)$ is just a name for an object whose construction is immediate from work of Joyal and Lurie, and quiet possibly appears elsewhere. The author is far from an expert in the subject, and we claim no originality for its construction. What may however be interesting is the connection to symplectic geometry that we discover in these papers, and perhaps $HH^*_{geom}(C)$ deserves a more careful study on its own.

Let us then very briefly indicate the connection of $HH^*_{geom}(C)$ with Hochschild cohomology via Toen's derived Morita theory. First we recall:

Theorem 1.8 (Corollary 8.4)**.** *[7] For a small dg-category C, (with cohomological grading conventions) there are natural isomorphisms*

$$\pi_i|(dg - Cat, C)| \simeq HH^{2-i}(C), \text{ for } i > 2, \tag{1.2}$$

$$\pi_2|(dg - Cat, C)| \simeq HH^0(C)^*, \tag{1.3}$$

with $HH^0(C)^*$ denoting the multiplicative group of invertible elements, $(dg-Cat, C)$ denoting the C component of the subcategory of $dg - Cat$ with morphisms quasi-equivalences, and with $|\cdot|$ the geometric realization or the classical nerve functor.

On the other hand the nerve functor N naturally induces a homomorphism,

$$N_* : \pi_i|(dg - Cat, C)| \to \pi_i|(\infty - \mathcal{C}at, NC)| \simeq \pi_i(\mathcal{S}, NC).$$

When C is $\mathbb{Z}$-graded, rational and (pre)-triangulated there are folklore theorems of Lurie (personal communication) to the effect that this is an isomorphism. Thus, in this case for $i > 2$

$$HH^{2-i}(C) = \pi_i(\mathcal{S}, NC) = HH^{2-i}_{geom}(C),$$

by our definition. This should extend without issues to $\mathbb{Z}$-graded rational (pre)-triangulated A_∞ categories. Beyond that it seems nothing is known.

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2. Notations and conventions and large categories

We will use terms ∞-category and quasi-category interchangeably, usually the term quasi-category will be used when we want to do something concrete.

We use diagrammatic order for composition of morphisms in the Fukaya category, and quasi-categories so $f \circ g$ means

$$\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot,$$

as reversing order for composition in quasi-categories is geometrically very confusing, since morphisms are identified with edges of simplices. Elsewhere we use the more common Leibnitz convention. Although this is somewhat contradictory in practice things should be clear from context. By simplex and notation Δ^n we will interchangeably mean the topological n-simplex and the standard representable n-simplex as a simplicial set, for the latter we may also write $\Delta^n_\bullet$.

Given a category C the over-category of an object $c \in C$ is denoted by C/c. We say that a morphism in C is *over* c exactly if it is a morphism in the over-category of c.

Given an A_∞ category by the nerve we always mean the A_∞ nerve.

Some of our quasi-categories are "large" with proper classes of simplices instead of sets. The standard formal treatment of this is to work with Grothendieck universes. This is a mostly accepted extension of set theory, and to some people, like this author, set theory without universes makes no sense, on a platonic level. We shall not however need to make this explicit. For reference one paper that does make this kind of thing explicit is [7] also previously cited.

3. Preliminaries

3.1. **The category** $Simp(X)$**.** A simplicial set $S_\bullet$ is a functor $S_\bullet : \Delta^{op} \to Set$, where Δ^{op} denotes the opposite category to Δ: the category of combinatorial simplices, whose objects are non negative integers and morphisms non-strictly increasing maps

$$\{0 < 1 < \ldots < n\} \to \{0 < 1 < \ldots < m\}.$$

We will denote the objects of Δ^{op} by $[n]$. Later on it will convenient to think of objects $[n]$ as totally ordered finite sets and so talk about membership.

If $\Delta^n_\bullet$ denotes the standard n-simplex: $\Delta^n_\bullet = hom_\Delta(\cdot, [n])$, we have the category of simplices over $S_\bullet$, $\Delta/S_\bullet$ whose set of objects is the set of natural transformations $Nat(\Delta^n_\bullet, S_\bullet)$ and morphisms commutative diagrams,

$$\begin{tikzcd} \Delta^n_\bullet \arrow[r] \arrow[dr] & \Delta^m_\bullet \arrow[d] \\ & S_\bullet, \end{tikzcd}$$

with natural transformation $\Delta^n_\bullet \to \Delta^m_\bullet$ induced by maps $[n] \to [m]$.

Let X be a smooth manifold, and let Δ^n denote the standard topological n-simplex. Denote by $X_\bullet$ the total smooth singular set of X. In other words this is the simplicial set defined by: $X_\bullet([n]) = C^\infty(\Delta^n, X)$: the set of all smooth maps $\Delta^n \to X$. Consider the category of simplices over $X_\bullet$, $\Delta/X_\bullet$. This is clearly the same as the category with objects smooth maps $\Delta^n \to X$, and morphisms commutative diagrams,

$$\begin{tikzcd} \Delta^n \arrow[r] \arrow[dr] & \Delta^m \arrow[d] \\ & X, \end{tikzcd}$$

with Δ^n denoting the standard topological n-simplex and top horizontal arrow a simplicial map, that is a linear map taking vertices to vertices. We say that $\Sigma : \Delta^n \to X$ is *non-degenerate* if it does not fit into a commutative diagram

$$\begin{tikzcd} \Delta^n \arrow[r] \arrow[dr, "\Sigma"] & \Delta^m \arrow[d] \\ & X, \end{tikzcd}$$

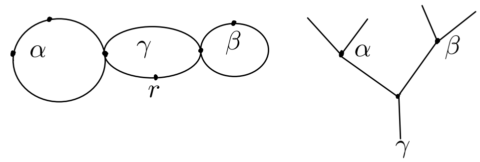

FIGURE 1.

with $m < n$. To simplify notation we will denote by $Simp(X)$ the subcategory of $\Delta/X_\bullet$, obtained from $\Delta/X_\bullet$ by restricting to non-degenerate objects and to monomorphisms in $\Delta/X_\bullet$ for morphisms.

3.2. **Preliminaries on Riemann surfaces.** Let S be a nodal Riemann surface, with $d+1$ cyclically ordered punctures on the boundary. Here a ***puncture*** is a specific kind of end structure $\{e_i\}_{i \in I}$ corresponding to the removed marked points on the boundary of S, and is to be explained. The nodal points are denoted by $\{n_j\}_{j \in J}$. For each j we have a pair $S_{j,\pm}$ of smooth components which are topologically disks with punctures

$$\{e_i\}_{i \in I_{j,\pm}} \subset \{e_i\}_{i \in I},$$

we explain the signs $\pm$ shortly, for now they just distinguish the pair of components. If we remove the nodes from S then each

$$S^\circ_{j,\pm} := S_{j,\pm} - n_j$$

has an additional puncture $n_{j,\pm}$ called the node end. We distinguish one puncture of S as the root, and use the cyclic ordering to order the punctures, $0, \ldots, d$ going clockwise, starting from the root.

It is sometimes convenient to depict such Riemann surfaces as stable, rooted semi-infinite trees, embedded in the plane, where stable means that the valency of each vertex is at least 3. We do this by assigning a vertex to each smooth component as above, a half infinite edge to each marked point, and an edge to each nodal point, as depicted in Figure 1.

To make some arguments and notation cleaner we also introduce a linear ordering on the smooth components of S, or vertices by order "of composition", defined as follows. The component with the root semi-infinite edge e_0 will be called the root vertex denoted by ω. In terms of the associated tree for the surface, we have a pre-order on vertices given by the distance to the root vertex, (by giving each edge length 1). To get an actual order, first isometrically embed the tree in the plane while preserving the cyclic ordering. Then clockwise order vertices equidistant to the root, as in Figure 1. We shall denote by α the furthermost component from ω, by β the next furthermost component, etc. (Pretending that we can't run out of letters.) We will mostly make use of this in Section 4.1.

For $d \geq 2$ let $\overline{\mathcal{S}}_d \to \overline{\mathcal{R}}_d$ denote the universal family of the Riemann surfaces S, as above. (Note that Seidel [11] calls our $\overline{\mathcal{R}}_d$ by $\overline{\mathcal{R}}_{d+1}$.) We will also denote by

$$\rho : \overline{\mathcal{S}}^\circ_d \to \overline{\mathcal{R}}_d,$$

this universal family where the nodal points of the surfaces have been removed.

Notation 3.1. *We denote by $\mathcal{S}_{d,r}$ and sometimes just by $\mathcal{S}_r$ the fiber $\rho^{-1}(r)$, for $r \in \overline{\mathcal{R}}_d$.*

As part of the data, at the i'th puncture we ask for a holomorphic diffeomorphism (having the name of the removed point or the end)

$$e_i : [0,1] \times [0,\infty) \to S,$$

$i \neq 0$. And at the 0'th puncture we ask for a holomorphic diffeomorphism

$$e_0 : [0,1] \times (-\infty, 0] \to S.$$

Let e_i^t denote the restriction of the maps above to $[0,1] \times [t,\infty)$, and let $S^{\circ}_{j,\pm}, S_{j,\pm}$ be as above. We further specify the $\pm$ distinction so that $S_{j,-} > S_{j,+}$ with respect to the linear order above. And we ask for a similar pair of holomorphic diffeomorphisms

$$e_{j,-} : [0,1] \times (-\infty, 0] \to S^{\circ}_{j,-}, \tag{3.1}$$

$$e_{j,+} : [0,1] \times [0,\infty) \to S^{\circ}_{j,+} \tag{3.2}$$

at the $n_{j,\pm}$ ends. Likewise $e^t_{j,\pm}$ will denote the restrictions of the above diffeomorphisms to $[0,1] \times [t,\infty)$, respectively to $[0,1] \times (-\infty, t]$. The data of such diffeomorphisms for a given possibly nodal surface, will be called a *a strip end structure*, and the particular diffeomorphisms *strip diffeomorphisms.*

Choose r-smooth families $\{e_{i,r}\}$, $\{e_{j,\pm,r}\}$ of strip diffeomorphisms, for the entire universal family $\overline{\mathcal{S}}_d \to \overline{\mathcal{R}}_d$, (note that further on r is suppressed). These choices have to be consistent with gluing in the natural sense as explained in [11, Section 9g]. We will keep track of these systems of choices of strip end structures only implicitly.

Although we won't use the following in any essential way, for instructional purposes it will be helpful to recall the following metric characterization of the moduli space $\overline{\mathcal{R}}_d$. The family $\{\mathcal{S}_{d,r}\}$ is in a correspondence with a suitably universal compactified family $\{\mathcal{M}et_{d,r}\}$ of constant curvature -1 metrics on the disk with $d+1$ punctures on the boundary. Under this correspondence, the complex structure on $\mathcal{S}_r$ is just the conformal structure induced by $\mathcal{M}et_r$. This is of course classical, to see all this use Schwartz reflection to "double" each $\mathcal{S}_r$ to a possibly nodal Riemann surface without boundary $\mathcal{D}_r$ with $d+1$ punctures. This determines an embedding of $\overline{\mathcal{R}}_d$ into the Grothendieck-Knudsen moduli space $\overline{M}_{0,d+1}$ of Riemann surfaces which are topologically S^2 wtih $d+1$ points removed. As $d > 2$, for r in the interior of $\overline{\mathcal{R}}_d$, by uniformization theorem $\mathcal{D}_r$ is a quotient of the disk by a subgroup of $PSL(2,\mathbb{R})$, which must also preserve the hyperbolic metric. Therefore $\mathcal{S}_r$ inherits a hyperbolic metric.

The metric point of view gives an illuminating description of the compactification $\overline{\mathcal{R}}_d$, for $d \geq 2$: starting with some $\mathcal{S}_r$ and taking r to a boundary stratum, corresponds to having some fixed collection of embedded, disjoint geodesics on $\mathcal{S}_r$, with boundary in boundary of $\mathcal{S}_r$ will have their length shrunk to zero. Each boundary stratum is completely determined by such a collection of geodesics.

The reverse of this degeneration process is the so called gluing construction, (see for example [11]), which takes a surface in $\overline{\mathcal{R}}_d$ and produces a surface with one less node. This gluing is determined by gluing parameters which we parametrize by $[0,1)$, assigned to each node. For us 0 means don't glue, and 1 is meant to correspond to some small value of the gluing parameter used in actual gluing, so this is a reparametrizations of parameters used in actual gluing. We will write $d_{\alpha,\beta}$ for the parameter used in the gluing of components α, β, and likewise with other components.

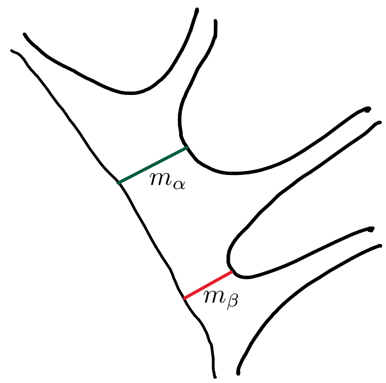

FIGURE 2. This diagram is only schematic. The embedding into the plane is not mean to be holomorphic or isometric for the natural hyperbolic structure on the surface.

The gluing construction for parameters in $[0,1)$ determines an open neighborhood called ***gluing neighborhood*** of the boundary. We will call by N the ***gluing normal neighborhood*** of the boundary: an open neighborhood of the boundary, deformation retracting to the boundary, contained in the gluing neighborhood.

The gluing construction also induces a kind of thick-thin decomposition of the surface, with thin parts conformally identified with $[0,1] \times [0,l]$ for l determined by the corresponding gluing parameter. This decomposition is not intrinsic, as it depends in particular on the choice of the family of strip like coordinate charts. However, instructively these gluing parameters can be thought of as lengths of geodesic segments, for example m_α, m_β in figure 2, and the thin parts are in principle closely related to thin parts of thick-thin decomposition in hyperbolic geometry.

4. A system of natural maps from the universal curve to Δ^n

We explain here a remarkable connection between the universal curve over $\overline{\mathcal{R}}_d$ and the standard topological simplex Δ^n, which will be used in our construction, but may be of independent interest.

Let $\Pi(\Delta^n)$ be the groupoid whose objects are vertices of Δ^n and morphisms simplicial maps $m : [0,1] \to \Delta^n$ (possibly constant). As there is a unique edge between a pair of vertices, the composition maps in $\Pi(\Delta^n)$ are obvious. We say that $(m_1, \ldots, m_d)$ is a *composable chain* of morphisms $m_i \in hom(\Pi(\Delta^n))$ if $t(m_{i-1}) = s(m_i)$, for $s(m)$ the source object and $t(m)$ the target object of the morphism m, For future use let $m_{i-1,i}$ denote the unique morphism from the $i-1$ vertex to the i vertex in Δ^n.

The goal is to construct a "natural" system of maps $u(m_1, \ldots, m_d, n) : \overline{\mathcal{S}}^\circ_d \to \Delta^n$, $d \geq 2$ for each composable chain $(m_1, \ldots, m_d)$ which in particular will satisfy the following properties.

- Let $u(m_1, \ldots, m_d, n, r)$ denote the restriction of $u(m_1, \ldots, m_d, n)$ to $\mathcal{S}_r$, and let m_0 denote the composition $m_1 \cdot \ldots \cdot m_d$ in $\Pi(\Delta^n)$. Then in the strip coordinates $e^1_k : [0,1] \times [1, \infty) \to \mathcal{S}_r$ at the k'th end, $u(m_1, \ldots, m_d, n, r)$ has the form of the projection to $[0,1]$ composed with m_k.

- For $1 \leq k \leq d$, the component of the boundary of $\mathcal{S}_r$ between e_{k-1}, e_k ends is mapped to $s(m_k)$, and the component of the boundary between e_d and e_0 is mapped by $u(m_1, \ldots, m_d, n, r)$ to $t(m_d)$, for each $r \in \overline{\mathcal{R}}_d$.

Let us explain naturality. First denote by $\mathcal{T}(m_1, \ldots, m_d, n)$ the space of maps satisfying the pair of properties above. We have the natural gluing map

$$St_i : \overline{\mathcal{R}}_{s_1} \times \overline{\mathcal{R}}_{s_2} \times [0,1) \to \overline{\mathcal{R}}_{s_1+s_2-1}, \tag{4.1}$$

whose value on (r, r', τ) is given by gluing the surface $\mathcal{S}_r$ at the root and $\mathcal{S}_{r'}$ at its i'th marked point, with gluing parameter $\tau \in [0,1)$, and then following with the classifying map to $\overline{\mathcal{R}}_{s_1+s_2-1}$. (When the value of the gluing parameter is 0, this is the composition map in the Stasheff topological A_∞ operad).

Given an element $u \in \mathcal{T}(m_1, \ldots, m_{s_1}, n)$ and an element

$$u' \in \mathcal{T}(m'_1, \ldots, m'_{i-1}, m_1 \cdot \ldots \cdot m_{s_1} = m'_i, m'_{i+1}, \ldots, m'_{s_2}),$$

we have a naturally induced map

$$u \star_i u'_0 : \overline{\mathcal{S}}^{\circ}_{s_1,s_2,0} \to \Delta^n,$$

where

$$\overline{\mathcal{S}}^{\circ}_{s_1,s_2,0} \to \overline{\mathcal{R}}_{s_1} \times \overline{\mathcal{R}}_{s_2} \times \{0\}$$

is the pullback of the family $\overline{\mathcal{S}}^{\circ}_{d=s_1+s_2-1}$ by $St_i|_{\overline{\mathcal{R}}_{s_1} \times \overline{\mathcal{R}}_{s_2} \times \{0\}}$.

We can extend $u \star_i u'_0$ to a map

$$u \star_i u'_1 : \overline{\mathcal{S}}^{\circ}_{s_1,s_2,1} \to \Delta^n,$$

where

$$\overline{\mathcal{S}}^{\circ}_{s_1,s_2,1} \to \overline{\mathcal{R}}_{s_1} \times \overline{\mathcal{R}}_{s_2} \times [0,1)$$

is the pullback of the universal family by $St_i|_{\overline{\mathcal{R}}_{s_1} \times \overline{\mathcal{R}}_{s_2} \times [0,1)}$. To get this extension we specify $u \star_i u'_1|_{\mathcal{S}_{r,r',\tau}}$, for $\mathcal{S}_{r,r',\tau}$ the fiber of $\overline{\mathcal{S}}^{\circ}_{s_1,s_2,1}$ over $(r, r', \tau) \in \overline{\mathcal{R}}_{s_1} \times \overline{\mathcal{R}}_{s_2} \times [0,1)$, $\tau \neq 0$. Recall that the surface $\mathcal{S}_{r,r',\tau}$ glued from $\mathcal{S}_r, \mathcal{S}_{r'}$ has a sub-domain which we denote by $thin = thin_{\tau,i}$, which has a determined conformal identification with a strip of the form $[0,1] \times [-\phi(\tau), \phi(\tau)]$, for some function ϕ. $\mathcal{S}_{r,r'} - thin$ has 2 components, which can be holomorphically identified with regions

$$Reg_r \subset \mathcal{S}_r, Reg_{r'} \subset \mathcal{S}_{r'},$$

so that Reg_r is identified with the complement of $[0,1] \times (-\infty, 0]$ in $\mathcal{S}_r$, for the strip coordinates on the root end. And likewise, so that $Reg_{r'}$ is identified with the complement of $[0,1] \times [0, \infty)$ in $\mathcal{S}_{r'}$, for the strip coordinates on the i'th end. We then define $u \star_i u'_1$ to coincide with u, u' on Reg_r respectively $Reg_{r'}$, while on $thin$ in the coordinates $[0,1] \times [-\phi(\tau), \phi(\tau)]$ $u \star u'_1$ is the map given by the projection $[0,1] \times [-\phi(\tau), \phi(\tau)] \to [0,1]$ followed by the map m'_i.

In order state the naturality axioms we need more geometry. Let N be a normal gluing neighborhood of the boundary $\partial \overline{\mathcal{R}}_d$, as previously explained, where $d \geq 2$. Let $S^{d-3}_0 \subset N$ be an embedded sphere of dimension $d-3$, not intersecting $\partial \overline{\mathcal{R}}_d$. Let R^{d-2}_0 be the dimension $d-2$ ball sub-domain of $\overline{\mathcal{R}}_d$ bounded by S^{d-3}_0. Finally, let $\mathcal{S}_r - ends$ denote the compact Riemann surface with boundary obtained from $\mathcal{S}_r$ by removing the ends, that is images of the charts $e_i|_{[0,1] \times (0,\infty)}$. For $r \notin \partial \overline{\mathcal{R}}_d$ set

$$S^2_r := (\mathcal{S}_r - ends)/\partial(\mathcal{S}_r - ends)$$

so that S^2_r is homeomorphic to S^2. So we get a fiber bundle

$$S^2 \hookrightarrow \widetilde{F} \to R_0^{d-2},$$

with fiber over r: S^2_r. Trivialize this and let

$$S^2 \hookrightarrow F \to (S^{d-2} := R_0^{d-2}/S_0^{d-3}),$$

denote the associated trivial fiber bundle. Let $\infty_r \in S^2_r$ denote the image of $\partial(\mathcal{S}_r - ends)$ in the quotient, and

$$s_\infty(r) = \infty_r,$$

be the corresponding section.

Let $C_s(\Delta^n)$ denote the set of composable chains $(m_1, \ldots, m_s)$ of length s in $\Pi(\Delta^n)$. A system of maps: $\mathcal{U}$ is an element of:

$$\prod_{n \in \mathbb{N}} \prod_{s \in \mathbb{N}_{>2}} \prod_{(m_1,\ldots,m_s) \in C_s(\Delta^n)} \mathcal{T}(m_1, \ldots m_s, n). \tag{4.2}$$

Given a system $\mathcal{U}$ its projection onto $(n, s, (m_1, \ldots, m_s))$ component will be denoted by $u(m_1, \ldots, m_s, n)$.

Definition 4.1. *We say that $\mathcal{U}$ is* **natural** *if it satisfies the following axioms:*

(1) *For all s_1, s_2 and for all i if $m'_i = m_1 \cdot \ldots \cdot m_{s_1}$ then the map*

$$u(m_1, \ldots, m_{s_1}, n) \star_i u(m'_1, \ldots, m'_{s_2}, n)_1, \tag{4.3}$$

coincides with the composition

$$\overline{\mathcal{S}}_{s_1,s_2,1} \xrightarrow{St_i} \overline{\mathcal{S}}_{s_1+s_2-1} \xrightarrow{u(m'_1,\ldots,m'_{s_2},n)} \Delta^n. \tag{4.4}$$

(2) *Given a face map $f : \Delta^{n-1} \to \Delta^n$,*

$$f \circ u(m_1, \ldots, m_s, n-1) = u(f(m_1), \ldots, f(m_s), n).$$

(3) *Let*

$$pr : \Delta^{n+k} \to \Delta^n,$$

$k > 0$ be a simplicial projection, then there is an induced functor

$$pr : \Pi(\Delta^{n+k}) \to \Pi(\Delta^n),$$

and

$$pr \circ u(m_1, \ldots, m_s, n+k) = u(pr(m_1), \ldots, pr(m_s), n).$$

(4) *For $(m_1, \cdots, m_s) \in C_s(\Delta^n)$ let $D(m_1, \ldots, m_s)$ denote the minimal dimension of a subsimplex of Δ^n which contains the edges corresponding to the morphisms m_i. Suppose that $D(m_1, \ldots, m_d) = d$ then $u(m_1, \ldots, m_d, d)$ induces a map of pairs:*

$$\widetilde{u} : (R_0^{d-2} \times S^2, S_0^{d-3} \times S^2) \to (\Delta^d, \partial\Delta^d),$$

and so induces a map with same name:

$$\widetilde{u} : S^{d-2} \times S^2 \to \Delta^d/\partial\Delta^d \simeq S^d, \tag{4.5}$$

and we ask that $\widetilde{u}$ is a homological degree 1 map.

Note that $\widetilde{u}$ factors through $S^{d-2} \wedge S^2 \simeq S^d$. To see this, note that by construction $\widetilde{u}(s_\infty) \subset \partial\Delta^d$, for s_∞ the section of $S^{d-2} \times S^2$ as above. Also if $\infty \in S^{d-2}$ denotes the image of S_0^{d-3} in the quotient R_0^{d-2}/S_0^{d-3}, then the fiber over ∞ is likewise mapped to $\partial\Delta^d$ by the Axioms 1 and 2.

Remark 4.2. *Using an inductive procedure as in the proof of the following proposition, it should be possible to show that the final axiom actually follows by the previous axioms.*

Theorem 4.3. *A natural system $\mathcal{U}$ exists, and is unique up to homotopy (through natural systems).*

We first give a not explicitly constructive proof in all generality, and afterwards describe a partial explicit construction.

Proof of 4.3. To construct our maps $u(m_1, \ldots, m_s, n)$ we will proceed by induction. When $n = 0$, there is nothing to do, as we have unique maps for all $s \geq 2$.

Given a composable sequence $(m_1, \ldots, m_s)$ of morphisms in $\Pi(\Delta^n)$ for some n, let $D(m_1, \ldots, m_s)$ denote the least dimension of a non-degenerate subsimplex of Δ^n, which contains the edges corresponding to $\{m_i\}$. Now suppose that we have chosen maps

$$u(m_1, \ldots, m_s, n), \tag{4.6}$$

for all $s \geq 2$ and all $n \leq N$ and every composable chain $(m_1, \ldots, m_s)$, with $D(m_1, \ldots, m_s) = s$ so that axioms 1,2,4 are satisfied and so that

$$u(m_1, \ldots, m_s, n) = \sigma^{-1} \circ u(\sigma(m_1), \ldots, \sigma(m_s), n)$$

for $\sigma : \Delta^n \to \Delta^n$ a simplicial homeomorphism. That is we satisfy axiom 3 only partially, and only for restricted $(m_1, \ldots, m_s)$ at the moment. We first construct maps $u(m_1, \ldots, m_s, n)$ for all $s \geq 2$, $n \leq N + 1$ and all composable chains $(m_1, \ldots, m_s)$ with $D(m_1, \ldots, m_s) = s$ so that axioms 1,2,4 are satisfied and so that

$$u(m_1, \ldots, m_s, n) = \sigma^{-1} \circ u(\sigma(m_1), \ldots, \sigma(m_s), n), \tag{4.7}$$

for σ a simplicial homeomorphism as before. Then by induction we will have maps $u(m_1, \ldots, m_s, n)$, with $D(m_1, \ldots, m_s) = s$ for all n, and we will construct from these our natural system.

Note that the Axiom 2 and the maps (4.6) uniquely determine

$$u(m_1, \ldots, m_s, N + 1), \tag{4.8}$$

for all $(s, (m_1, \ldots, m_s))$, with $D(m_1, \ldots, m_s) = s \leq N$. We need an extension in the case $D(m_1, \ldots, m_s) = s = N + 1$.

Assume that $N + 1 > 3$ as the cases $N + 1 = 2, 3$ are special and geometrically trivial. Let then $(m^0_1, \ldots, m^0_{N+1})$, $N + 1 > 3$ be a chosen composable sequence with $D(m^0_1, \ldots, m^0_{N+1}) = N + 1$.

Then gluing as in the axiom 1 of naturality and the maps (4.8), naturally determine a map

$$u = u(m^0_1, \ldots, m^0_{N+1}, N + 1) : Sub_{N+1} \to \Delta^{N+1}, \tag{4.9}$$

where $Sub_{N+1} = \rho^{-1}(\partial\overline{\mathcal{R}}_{N+1})$,

$$\rho : \overline{\mathcal{S}}^{\circ}_{N+1} \to \overline{\mathcal{R}}_{N+1}.$$

Extend u in any way to $\rho^{-1}(U)$ for U a normal gluing neighborhood of $\partial\overline{\mathcal{R}}_{N+1}$, so that Axiom (1) of naturality is satisfied. We then need to further extend u to $\overline{\mathcal{S}}^{\circ}_{N+1}$ so that the final axiom of naturality is satisfied.

Let $S^{N-2}_0 \subset U$, be an embedded sphere in U not intersecting $\partial \overline{\mathcal{R}}_{N+1}$. Then u induces a map of a pair

$$g : (S^{N-2}_0 \times D^2, S^{N-2}_0 \times \partial D^2) \to (\partial \Delta^{N+1} \simeq S^N, \mathit{loop}), \tag{4.10}$$

where *loop* is a topologically embedded S^1 in $\partial \Delta^{N+1}$, which is the image of the loop $(m^0_1 \cdot \ldots \cdot m^0_{N+1} \cdot m^{-1}_{s(m^0_1), t(m^0_{N+1})})$, where $\cdot$ is concatenation of paths and the order of composition is diagrammatic. This is constructed analogously to the map (4.5), with D^2 a homeomorphic model of each $\mathcal{S}_r - \mathit{ends}$.

Lemma 4.4. *The map g is homological degree 1.*

Proof. As $N > 2$, *loop* has codimension greater than 1, so that the meaning of homological degree is unambiguous, as the pair (S^N, loop) has a well defined fundamental class, by the homology long exact sequence for a pair. Moreover approximating g by a smooth map we may compute the homological degree via the smooth degree, (denote the approximation still by g). That is let f be a N-face of Δ^{N+1}, and $p \in \mathit{interior}(f)$ a regular image point of g. The homological degree of g is then the count of elements of $g^{-1}(p)$ with signs given by whether dg_k, $k \in g^{-1}(p)$ is orientation preserving or reversing. Suppose without loss of generality that the vertices of f are $0, \ldots, N$.

As the degree of g is clearly independent of the choice of S^{N-2}_0 we may assume that S^{N-2}_0 is chosen so that for some $\epsilon > 0$:

$$St_1 : \left(R^{N-2}_0 \subset \overline{\mathcal{R}}_N\right) \times \overline{\mathcal{R}}_2 \times \{\epsilon\} \to \overline{\mathcal{R}}_{N+1}, \tag{4.11}$$

is an embedding into S^{N-2}_0, and that moreover $S^{N-2}_0 - T$ is covered by such embeddings corresponding to the various other faces of $\overline{\mathcal{R}}_N$, where the region $T \subset S^{N-2}_0$, is such that g maps $\rho^{-1}(T)$ into the union of $(N-1)$-faces of Δ^{N+1}. The image of the map (4.11) will be denote by V.

Then by the naturality axiom 4 the face f is covered by the image of

$$\kappa = u(m^0_1, \ldots, m^0_N, N+1) \star_1 u(m^0_1 \cdot \ldots \cdot m^0_N, m^0_{N+1}, N+1)_1|_{\widetilde{V}},$$

where

$$\widetilde{V} = \rho^{-1}(V).$$

By construction the smooth degree of $g|_{\widetilde{V}}$ is the smooth degree of κ. But then by naturality axiom 4 κ is smooth degree one. And again by naturality and the assumption on the form of S^{N-2}_0 above, nothing else in $S^{N-2}_0 \times D^2$ can hit p by the map g. It follows that g is smooth degree one and so is homological degree one. □

As g is degree 1, we may find a degree one extension:

$$\widetilde{g} : (R^{N-1}_0 \times D^2, S^{N-2}_0 \times D^2 \sqcup R^{N-1}_0 \times \partial D^2) \to (\Delta^{N+1}, \partial \Delta^{N+1}), \tag{4.12}$$

with $\widetilde{g}(R^{N-1}_0 \times \partial D^2) = \mathit{loop}$. Given this $\widetilde{g}$ we may readily construct our extension $u : \overline{\mathcal{S}}^{\circ}_{N+1} \to \Delta^{N+1}$, so that

$$u \in \mathcal{T}(m^0_1, \ldots, m^0_{N+1}, N+1)$$

and so that the last naturality axiom is satisfied.

Now given any other composable sequence $(m_1, \ldots, m_{N+1})$ with $D(m_1, \ldots, m_{N+1}) = N+1$, let

$$\sigma : \Delta^{N+1} \to \Delta^{N+1}$$

be the unique simplicial homeomorphism mapping $(m_1, \ldots, m_{N+1})$ to $(m^0_1, \ldots, m^0_{N+1})$, (it is unique because the action is determined by the action on the vertices.) And we define
$$u(m_1, \ldots, m_{N+1}, N+1) : \overline{\mathcal{S}}^\circ_{N+1} \to \Delta^{N+1},$$
by
$$u(m_1, \ldots, m_{N+1}, N+1) = \sigma^{-1} \circ u.$$
It remains only to check that Axiom 1 and 2 are satisfied for this resulting partial system of maps
$$u(m_1, \ldots, m_{s'}, n)_{(m_1,\ldots,m_{s'}), n \leq N+1, D(m_1,\ldots,m_{s'})=s'},$$
but this is immediate by construction and the inductive hypothesis. Consequently we complete the induction step. It remains to extend the partial system above to a full natural system.

Given $(m_1, \ldots, m_s)$, a composable sequence in $\Pi(\Delta^n)$ with $s > n$, we may write $m_i = pr \circ \widetilde{m}_i$ for $(\widetilde{m}_1, \ldots, \widetilde{m}_s)$ a composable sequence in Δ^s s.t. $D(\widetilde{m}_1, \ldots, \widetilde{m}_s) = s$, for $pr : \Delta^s \to \Delta^n$ surjective simplicial projection. Moreover $(\widetilde{m}_1, \ldots, \widetilde{m}_s)$ is clearly unique up to an action of a simplicial homeomorphism $\sigma : \Delta^s \to \Delta^s$, fixing the image $i(\Delta^n)$ for $i : \Delta^n \to \Delta^s$ inclusion of face s.t. $pr \circ i = id$. We then define
$$u(m_1, \ldots, m_s, n) := pr \circ u(\widetilde{m}_1, \ldots, \widetilde{m}_s, s).$$

Let $(\widetilde{m}'_1, \ldots, \widetilde{m}'_s)$ be another choice of a composable sequence with $pr(\widetilde{m}'_i) = m_i$, and $\sigma : \Delta^s \to \Delta^s$ a simplicial homeomorphism with $\sigma(\widetilde{m}_i) = \widetilde{m}'_i$, fixing the image of $i(\Delta^n)$ as above. Then we have
$$pr \circ u(\widetilde{m}'_1, \ldots, \widetilde{m}'_s, s) = pr \circ \sigma \circ u(\widetilde{m}_1, \ldots, \widetilde{m}_s, s).$$
But $pr \circ \sigma = pr$ since σ fixes $i(\Delta^n)$. So we obtain that
$$pr \circ u(\widetilde{m}'_1, \ldots, \widetilde{m}'_s, s) = pr \circ u(\widetilde{m}_1, \ldots, \widetilde{m}_s, s),$$
so that $u(m_1, \ldots, m_s, n)$ is well defined. So we have constructed our system of maps satisfying all the axioms of naturality.

To prove uniqueness up to homotopy, note that by our axioms, a system of natural maps is completely determined by all the maps $u(m_1, \ldots, m_s, n)$ with
$$D(m_1, \ldots, m_s) = s = n.$$
We then again proceed by induction. Suppose that we have a pair of natural systems $\mathcal{U}_1, \mathcal{U}_2$ and suppose we have a continuous family of maps $u_t(m_1, \ldots, m_s, n)$, $t \in [0, 1]$, for all $n \leq N > 0$, $(m_1, \ldots, m_s)$, with
$$D(m_1, \ldots, m_s) = s = n,$$
so that
$$u_{t=i}(m_1, \ldots, m_s, n) = u_i(m_1, \ldots, m_s, n)$$
for $i = 0, 1$, and so that for each t, $u_t(m_1, \ldots, m_s, n)$ satisfy naturality axioms.

Then given some composable sequence $(m_1, \ldots, m_s)$ in $\Pi(\Delta^{N+1})$, with
$$D(m_1, \ldots, m_s) = s = N+1,$$
we have continuous in t families of induced (as before) maps:
$$u_t(m_1, \ldots, m_s, N+1) : Sub_s \to \Delta^{N+1},$$
with Sub_s as before, and so that
$$u_{t=1}(m_1, \ldots, m_s, N+1) : Sub_s \to \Delta^{N+1},$$

coincides with the map

$$u_1(m_1, \ldots, m_s, N+1) : Sub_s \to \Delta^{N+1}.$$

Use the homotopy extension property to get a homotopy:

$$\widetilde{u}_t(m_1, \ldots, m_{N+1}, N+1) : \overline{\mathcal{S}}^\circ_d \to \Delta^{N+1},$$

of the map $u_0(m_1, \ldots, m_{N+1}, N+1)$. Now

$$\widetilde{u}_1(m_1, \ldots, m_{N+1}, N+1)$$

and

$$u_1(m_1, \ldots, m_{N+1}, N+1)$$

do not necessarily coincide on $\overline{\mathcal{S}}^\circ_d$ but they coincide on Sub_s, since we used homotopy extension, and both these maps are "degree one" (that is they both satisfy the last naturality axiom). We may then construct a homotopy relative to Sub_s, between the pair $\widetilde{u}_1(m_1, \ldots, m_{N+1}, N+1)$ and $u_1(m_1, \ldots, m_{N+1}, N+1)$. Taking the composite homotopy we complete the induction step. □

4.1. **Outline of an explicit construction.** This section is not logically necessary, but in order to give the reader more intuition we now give a partial explicit construction of the maps that would satisfy the axioms. This construction could be in principle extended to all generality but at the cost of much complexity.

Fix a geometric model for $\overline{\mathcal{R}}_d$, for example as the Stasheff associahedra. When $d = 4$ this is a pentagon. Recall that to each corner of $\overline{\mathcal{R}}_4$ we have a uniquely associated nodal Riemann surface, with 3 components, and 5 marked points, one of which is called the root. Recall that we label the root component by ω, the next component by β and the component furthest from root by α. (With respect to the linear ordering described earlier.) Denote by M_α the collection of marked points, different from the root e_0, on α, likewise with β, ω. This determines a sub-composable sequence $mor(S_\alpha)$ of a composable sequence $(m_1, \ldots, m_4)$, and likewise with β, ω, (note that M_ω, M_β could be empty).

Let r be in the normal gluing neighborhood of some corner, corresponding to non-zero gluing parameters $d_{\alpha,\beta}$, $d_{\beta,\omega}$. We now construct a map

$$f_r = f_r(m_1, \ldots, m_4) : [0, 4] \times [0, 1] \to \Delta^4$$

In what follows by *concatenation* of a collection of paths we mean their product in the Moore path category of Δ^4, the notation for composition will be assumed to be diagrammatic. That is the category with objects points, and morphisms from x_0 to x_1 continuous paths $[0, T] \to \Delta^4$, $T > 0$, between x_0, x_1, with composition the natural concatenation of paths. For a morphism m in $\Pi(\Delta^4)$, let $s(m)$, and $t(m)$ denote the source respectively target of m. Let $H^m : \Delta^4 \times [0, 1] \to \Delta^4$ denote the natural deformation retraction of Δ^4 onto the edge determined by $s(m), t(m)$, with time 1 map the orthogonal linear projection onto this edge (for the standard metric on Δ^n). Set $H^m_\tau = H^m|_{\Delta^n \times \{\tau\}}$. Next for a general piece-wise linear path $p : [0, T] \to \Delta^4$, with end points $s(m), t(m)$ set $D(p, \tau)$, $\tau \in [0, 1]$ to be the concatenation of the homotopy $H^m_\tau \circ p$, $\tau \in [0, 1]$ from p to a path $\widetilde{p} : [0, T] \to \Delta^4$, with image in the edge determined by m, with the homotopy G_τ of paths with fixed end points, from $\widetilde{p}$ to the map $\widetilde{m} : [0, T] \to \Delta^4$ linearly parametrizing the edge determined by m. This second homotopy G_τ, $\tau \in [0, 1]$ can be chosen in a way that depends only on $\widetilde{p}$. This can be done explicitly, using piece-wise linearity of p.

The map f_r^t from the $y = t$ slice $[0,4] \times \{t\}$ is constructed as follows. Set $I_\alpha = (1 - d_{\alpha,\beta})/2$, set $f_{\alpha,r}$ to be the concatenation of the morphisms in $mor(M_\alpha)$, that is if $M_\alpha = (m_1^\alpha, \ldots, m_k^\alpha)$ then $f_{\alpha,r} = m_1^\alpha \cdot \ldots \cdot m_k^\alpha$. Then for $t \in [0, I_\alpha]$, set $f_{\alpha,r}^t = D(f_{\alpha,r}, 2t)$. Then set f_r^t to be the concatenation of morphisms in $mor(M_\beta)$, $mor(M_\gamma)$ and of $f_{\alpha,r}^t$, in that order, although note that the order of the morphisms in the concatenation is uniquely determined by the end point conditions, this holds further on as well.

Next set $I_\beta = I_\alpha + (1 - I_\alpha)(1 - d_\beta)/2$. If α and β components have a nodal point in common we set $f_{\beta,r} : [0,4] \times \{t\} \to \Delta^4$ to be the concatenation of $f_{\alpha,r}^{I_\alpha}$ with morphisms in $mor(M_\beta)$, and for $t \in [I_\alpha, I_\beta]$ we set

$$f_{\beta,r}^t = D(f_{\beta,r}, \frac{2(t - I_\alpha)}{1 - I_\alpha}).$$

And then for $t \in [I_\alpha, I_\beta]$ set f_r^t to be the concatenation of morphisms in $mor(M_\omega)$ and of $f_{\beta,r}^t$.

Finally set $f_{\omega,r}$ to be the concatenation of $f_{\beta,r}^{I_\beta}$ with morphisms in $mor(M_\omega)$, and for $t \in [I_\beta, 1]$ set

$$f_r^t = D(f_{\omega,r}, \frac{2(t - I_\beta)}{1 - I_\beta}).$$

When α has a nodal point in common with the ω component, set $f_{\beta,r}$ to be the concatenation of morphisms in $mor(M_\beta)$, and for $t \in [I_\alpha, I_\beta]$ set

$$f_{\beta,r}^t = D(f_{\beta,r}, \frac{2(t - I_\alpha)}{1 - I_\alpha}).$$

Then for $t \in [I_\alpha, I_\beta]$ set f_r^t to be the concatenation of morphisms $f_r^{I_\alpha}$ and $f_{\beta,r}^t$, and $mor(M_\omega)$ (although $mor(M_\omega)$ in this particular case is empty, we add this so that the degenerate case $M_\alpha = \emptyset$, $M_\beta = \emptyset$ makes sense, see the discussion below). Finally for $t \in [I_\beta, 1]$ set $f_r^t = D(f_r^{I_\beta}, \frac{2(t - I_\beta)}{1 - I_\beta})$.

When $r \in \overline{\mathcal{R}}_4$ is in the gluing neighborhood of a face but not of a corner the construction of $f_r : [0,4] \times [0,1] \to \Delta^4$ is similar, in fact we can think of it as a special case of the above by setting $d_\beta = 1$, $M_\beta = \emptyset$. When $r \in \overline{\mathcal{R}}_4$ is not in the gluing neighborhood of the boundary, we can also think of this as a special case of the above with $M_\alpha = \emptyset, M_\beta = \emptyset$, $d_\alpha = 1$, $d_\beta = 1$ in the above construction.

4.1.1. *Flattening* $\{f_r\}$. We now slightly rig our family of maps. Fix an embedding $i : \overline{\mathcal{R}}_4 \to \overline{\mathcal{R}}_4' \simeq \overline{\mathcal{R}}_4$, so that the boundary of embedded domain is contained in the ϵ-normal neighborhood of the boundary of target $\overline{\mathcal{R}}_4'$, where $0 < \epsilon < 1$, and set $g : \overline{\mathcal{R}}_4' \to \overline{\mathcal{R}}_4'$ to be the smooth retraction onto $i(\overline{\mathcal{R}}_4)$. The family of maps $\{f_r\}$, $r \in \overline{\mathcal{R}}_4$ then gives us a family $\{f_{i^{-1}g(r)}\}$, $r \in \overline{\mathcal{R}}_4'$. Let us identify $\overline{\mathcal{R}}_4'$ back with $\overline{\mathcal{R}}_4$ and rename $\{f_{i^{-1}g(r)}\}$ by $\{f_r\}$. This procedure is just meant to flatten out the family of maps $\{f_r\}$ near the boundary of $\overline{\mathcal{R}}_4$, so that axiom (1) will be satisfied.

4.1.2. *Retracting $\mathcal{S}_r$ onto* $[0,4] \times [0,1]$. We now construct a smooth r-family of maps $ret_r : \mathcal{S}_r \to [0,4] \times [0,1]$, $r \in \mathcal{R}_4$, suitably compatible with the maps $f_r : [0,4] \times [0,1] \to \Delta^4$. In figure 3 (a), (b), (c) represent cases where (c): r is not within gluing normal neighborhood of boundary, (b): r is in a gluing neighborhood of a side but not a corner, and (a): r is within gluing neighborhood of a corner, (we picked a particular corner and side for these diagrams). The color shading will be

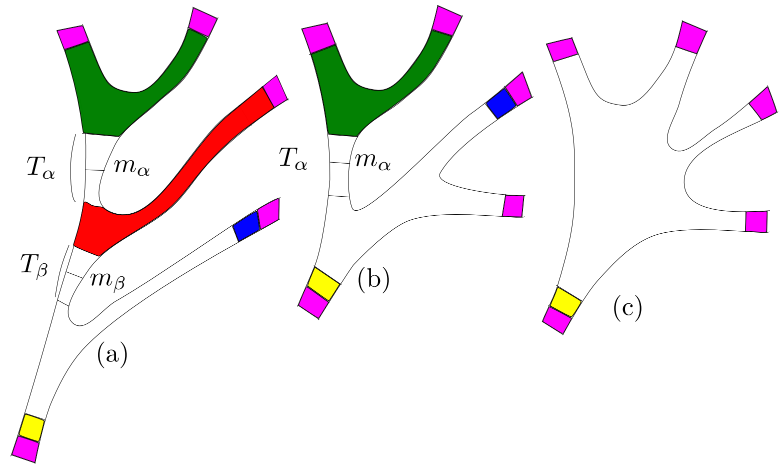

FIGURE 3. The uncolored enclosed regions labeled T_α, T_β surrounding segments m_α, m_β are "thin".

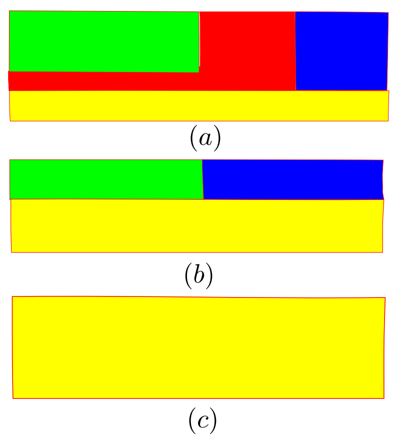

FIGURE 4. Diagram for S_d. Solid black border is boundary, while dashed red lines are open ends. The connection $\mathcal{A}(r, \{L_i\})$ preserves Lagrangians L_i over boundary components labeled L_i.

explained in a moment. In each case $(a), (b), (c)$ we first color shade $[0,4] \times [0,1]$ as in figure 4, the green region is the domain of $f^t_{\alpha,r}$ contained in $[0,4] \times [0, I_\alpha]$, in the blue regions the map f_r is vertically constant, the red region is the domain of $f^t_{\beta,r}$ contained in $[0,4] \times [I_\alpha, I_\beta]$ and yellow region is the rest of the domain of f_r. The maps $ret_r : \mathcal{S}_r \to [0,4] \times [0,1]$ are defined for each r by taking color shaded areas to color shaded areas, so that the following holds.

(1) The ends $e_0, e_1, \ldots, e_4$ of $\mathcal{S}_r$, colored in purple, are identified in strip coordinate charts as $[1, \infty) \times [0,1]$ and in these coordinates ret_r is the composition of the projection $[1, \infty) \times [0,1] \to [0,1]$ with the map to the boundary of $[0,4] \times [0,1]$, so that composition with f_r parametrizes the morphism $m_0 = m_1 \cdot \ldots \cdot m_4, m_1, \ldots, m_4$ in $\Pi(\Delta^4)$ respectively.

(2) The boundary of $\mathcal{S}_r$ goes either to the boundary of $[0,4] \times [0,1]$ or to the vertical boundary lines between colored regions.
(3) The unshaded "thin" regions labeled T_α, T_β come from the gluing construction and are identified with $[0,1] \times [-\phi(\tau_\alpha), \phi(\tau_\alpha)]$, respectively $[0,1] \times [-\phi(\tau_\beta), \phi(\tau_\beta)]$. In these coordinates ret_r on T_α, T_β is the projection to $[0,1]$ composed with a diffeomorphism onto the lower edge of green, respectively red region, (linear in respective natural coordinates).
(4) The unshaded part of $\mathcal{S}_r$ is collapsed onto the horizontal line bounding yellow region of $[0,4] \times [0,1]$.
(5) Blue shaded regions are identified in strip coordinate charts $[0,\infty] \times [0,1] \to \Sigma_r$, as $[0,1] \times [0,1]$, and are mapped to the corresponding blue regions in $[0,4] \times [0,1]$.

(The above prescription naturally extends to the boundary $\overline{\mathcal{R}}_4$.)

4.2. **Auxiliary data $\mathcal{D}$.** Let

$$M \hookrightarrow P \xrightarrow{\pi} X$$

be a Hamiltonian fiber bundle with model fiber (M^{2n}, ω), which we shall assume here to be a closed, monotone:

$$\omega = const \cdot 2c_1(TM),$$

$const \geq 0$, symplectic manifold. The data $\mathcal{D}$ consists of the following.

We say that a Lagrangian submanifold $L \subset M$ is *monotone* if the homomorphisms given by symplectic area and Maslov class

$$[\omega] : H_2(M,L) \to \mathbb{R}, \quad \mu : H_2(M,L) \to \mathbb{Z}$$

are proportional:

$$[\omega] = const \cdot \mu.$$

Let L be a spin, monotone Lagrangian submanifold L in $P_x = x^*P \simeq M$ with minimal Maslov number at least 2, so that the inclusion map $\pi_1(L) \to \pi_1(M)$ vanishes. For an almost complex structure j compactible with ω, let $\mathcal{M}(L,j)$ denote the moduli space of Maslov number 2 j-holomorphic discs, with one marked point on the boundary, with boundary of the disk going to L. It is well known, c.f. [13] that for a generic such j, $\mathcal{M}(L,j)$ is regular, that is transversely cut out n-dimensional manifold, and is compact. Then we have an evaluation map at the marked point:

$$ev : \mathcal{M}(L,j) \to L,$$

and we define $\omega(L) \in \mathbb{Z}$ as the degree of ev.

For an $x : pt \to X$ by an *object* we shall mean a spin monotone Lagrangian submanifold L in $P_x = x^*P \simeq M$ with minimal Maslov number at least 2, so that the inclusion map $\pi_1(L) \to \pi_1(M)$ vanishes. Given a non-degenerate simplex $\Sigma : \Delta^n \to X$, let $m : [0,1] \to \Delta^n$ be the edge between i,j corners of Δ^n and set $\overline{m} = \Sigma \circ m$. The data $\mathcal{D}$ gives for every pair of objects $L_0 \subset P_{x_i}, L_1 \subset P_{x_j}$, (including $i = j$) with $\omega(L_0) = \omega(L_1)$ a Hamiltonian connection $\mathcal{A}(L_0, L_1) = \mathcal{A}(L_0, L_1, \overline{m})$ on $\overline{m}^*P$. Denote by $\mathcal{A}(L_0,L_1)(L_0)$ the image in P_{x_j} of the $\mathcal{A}(L_0,L_1)$-parallel transport of L_0 over $[0,1]$. Then we require that $\mathcal{A}(L_0,L_1)(L_0)$ is transverse to L_1. We also fix for every L_0, L_1 and m as above a family $\{j_t\} = \{j_t(L_0, L_1, \overline{m})\}$ of fiber-wise almost complex structures on $\overline{m}^*P$ so that:

- For each t, Chern number 1 j_t-holomorphic spheres in $P_{\overline{m}(t)} \subset \overline{m}^* P$ do not intersect any of the elements of $S(L_0, L_1) \simeq L_i$, which denotes the space of $\mathcal{A}(L_0, L_1)$-flat sections, with boundary on L_0, L_1. Here $P_{\overline{m}(t)}$ denotes the fiber of $\overline{m}^* P$ over $\overline{m}(t)$.
- The moduli spaces $\mathcal{M}(L_0, j_0)$, $\mathcal{M}(L_1, j_1)$ are regular, and the evaluation map
$$ev_0 : \mathcal{M}(L_0, j_0) \to L,$$
does not intersect the set of starting positions of elements of $S(L_0, L_1)$. Likewise the evaluation map
$$ev_1 : \mathcal{M}(L_1, j_1) \to L,$$
does not intersect the set of ending positions of elements of $S(L_0, L_1)$.

Such a family $\{j_t\}$ is easily seen to exist, cf [13]. Let us call such a $\{j_t(L_0, L_1, \overline{m})\}$ *admissible with respect to* $\mathcal{A}(L_0, L_1)$. Next $\mathcal{D}$ makes a choice of a natural system $\mathcal{U}$ as previously described. Finally $\mathcal{D}$ specifies a certain natural system of Hamiltonian connections, and a system of complex structures, which we now describe. This is a bit notationally complicated but ultimately trivial, given the main geometric input of the system $\mathcal{U}$. It is just a system of compatible of perturbations in the sense of Seidel, but relative to $\mathcal{U}$.

Given a composable chain $(m_1, \ldots, m_d)$ and a map
$$u(m_1, \ldots, m_d, n) : \overline{\mathcal{S}}_d^\circ \to \Delta^n, \text{ which is part of a natural system } \mathcal{U},$$
we have an induced fibration
$$M \hookrightarrow \widetilde{S}(m_1, \ldots, m_d, \Sigma^n) \to \overline{\mathcal{S}}_d^\circ$$
by pulling back $M \hookrightarrow P \to X$ first by $\Sigma^n : \Delta^n \to X$ and then by $u(m_1, \ldots, m_d, n)$. We have a natural projection $\widetilde{S}(m_1, \ldots, m_d, \Sigma_n) \to \overline{\mathcal{R}}_d$ and we denote the fiber over $r \in \overline{\mathcal{R}}_d$ by $\widetilde{S}(m_1, \ldots, m_d, \Sigma_n, r)$, or simply by $\widetilde{\mathcal{S}}_r$ where there can be no confusion. So $\widetilde{\mathcal{S}}_r$ is an M fibration over the surface $\mathcal{S}_r$, smooth over smooth components. Suppose now that we have chosen labels by objects L_i for the sides between punctures $i, i+1$, $0 \le i \le d-1$ and L_d for the side between $d, 0$ of $\mathcal{S}_r$, $r \in \mathcal{R}_d$, where $L_i \in Fuk(P_{s(m_{i+1})})$ for $0 \le i \le d-1$, and $L_d \in Fuk(P_{t(m_d)})$. We suppose in addition that $\omega(L_i) = \omega(L_d)$ for all i. Extend the labeling naturally to nodal elements of $\overline{\mathcal{R}}_d$, so as to be naturally compatible with gluing.

Definition 4.5. *We say that a Hamiltonian connection $\mathcal{A}$ on $\widetilde{\mathcal{S}}(m_1, \ldots, m_d, \Sigma^n, r) \to \mathcal{S}_r$ is* **admissible** *with respect to $L_0, \ldots, L_d$ if:*

- *Parallel transport by $\mathcal{A}$ over the boundary component labeled L_i preserves Lagrangian L_i. This condition is unambiguous, as by construction over this boundary component, $\widetilde{\mathcal{S}}_r$ is naturally trivialized as $P_{s(m_{i+1})} \times [0,1]$, $0 \le i \le d-1$, or as $P_{t(m_d)} \times [0,1]$ in the case of L_d.*
- *At the i'th end of $\mathcal{S}_r$, in the strip coordinate chart $e_i^1 : [0,1] \times [1, \infty] \to \mathcal{S}_r$, $\mathcal{A}$ has the form of the canonical, flat, $\mathbb{R}$-translation invariant extension of the connection $\mathcal{A}(L_{i-1}, L_i, \overline{m}_i)$.*

We also have a Lagrangian sub-fibration of
$$\widetilde{\mathcal{S}}_r \to \mathcal{S}_r$$
over the boundary of $\mathcal{S}_r$, whose fiber over an element of the boundary component labeled L_i is L_i, with respect to the natural trivializations mentioned above.

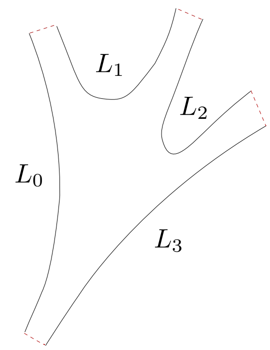

FIGURE 5. The connection $\mathcal{A}$ preserves Lagrangians L_i over boundary components labeled L_i.

We name this sub-fibration by

$$\mathcal{L}(\mathcal{U}, L_0, \ldots, L_d, r), \tag{4.13}$$

so that if $\mathcal{A}$ is admissible as above it preserves this sub-fibration. Denote by $\mathcal{T}(L_0, \ldots, L_s, \Sigma^n, r)$ the space of Hamiltonian connections on

$$\widetilde{\mathcal{S}}(m_1, \ldots, m_s, \Sigma^n, r)$$

admissible with respect to $L_0, \ldots, L_s$. Given an element $\mathcal{A}$ in $\mathcal{T}(L_0, \ldots, L_{s_1}, \Sigma^n, r)$ and an element

$$\mathcal{A}' \in \mathcal{T}(L'_0, \ldots, L'_{i-1}, L'_i, L'_{i+1}, \ldots, L'_{s_2}, \Sigma^n, r'),$$

with $L'_{i-1} = L_0, L'_i = L_{s_1}$ we have a naturally induced, by the gluing maps St_i, connection denoted by

$$St_i(\mathcal{A}, \mathcal{A}', 0) \in \mathcal{T}(L'_0, \ldots, L'_{i-2}, L_0, \ldots, L_{s_1}, L'_{i+2}, \ldots, L'_{s_2}, \Sigma^n, St_i(r, r', 0)).$$

Such a pair $\mathcal{A}, \mathcal{A}'$ will be called *composable*.

Using the second property in the Definition 4.5, we can naturally extend this to a family of connections $\{St_i(\mathcal{A}, \mathcal{A}', \tau)\}_\tau$, so that

$$St_i(\mathcal{A}, \mathcal{A}', \tau) \in \mathcal{T}(L'_0, \ldots, L'_{i-2}, L_0, \ldots, L_{s_1}, L'_{i+2}, \ldots, L'_{s_2}, \Sigma^n, St_i(r, r', \tau)),$$

for $0 \leq \tau < 1$, and so that in addition we have the following. Outside of the thin region $thin_{\tau,i}$ of $\mathcal{S}_{St_i(r,r',\tau)}$ the connection $St_i(\mathcal{A}, \mathcal{A}', \tau)$ is naturally identified with $St_i(\mathcal{A}, \mathcal{A}', 0)$. On the other hand over $thin_{\tau,i}$, for $\tau > 0$, $\widetilde{\mathcal{S}}_{St_i(r,r',\tau)}$ is naturally isomorphic to $(\overline{m}')^*_i P \times [-\phi(\tau), \phi(\tau)]$, by the Axiom 1 of the natural system $\mathcal{U}$. And over $thin_{\tau,i}$, in the above trivialization, $St_i(\mathcal{A}, \mathcal{A}', \tau)$ is the trivial in the second variable extension of the connection $\mathcal{A}(L_{i-1}, L_i, \overline{m}_i)$.

Definition 4.6. *We say that a family $\{j_z\}$ of fiber-wise, $\{\omega_z\}$-compatible, almost complex structures on $\widetilde{\mathcal{S}}(m_1, \ldots, m_d, \Sigma^n, r) \to \mathcal{S}_r$ is* **admissible** *with respect to $L_0, \ldots, L_d$ if:*

- *At the i'th end of $\mathcal{S}_r$, in the coordinate chart $[0,1] \times [1,\infty] \times \overline{m}^* P \to \widetilde{\mathcal{S}}(m_1,\ldots,m_d,\Sigma^n,r)$, induced by the strip chart e_i^1, the family $\{j_z\}$ coincides with the canonical $\mathbb{R}$-translation invariant extension of the family $\{j_t(L_{i-1},L_i,\overline{m}_i)\}$.*

Denote by $\mathcal{J}(L_0,\ldots,L_s,\Sigma^n,r)$ the space of families of fiberwise almost complex structures $\{j_z\}$ on

$$\widetilde{\mathcal{S}}(m_1,\ldots,m_s,\Sigma^n,r)$$

admissible with respect to $L_0,\ldots,L_s$. Given an element $\{j_z\}$ in $\mathcal{J}(L_0,\ldots,L_{s_1},\Sigma^n,r)$ and an element

$$\{j'_z\} \in \mathcal{J}(L'_0,\ldots,L'_{i-2},L_0,L_{s_1},L'_{i+1},\ldots,L'_{s_2},\Sigma^n,r'),$$

we have an induced element:

$$St_i(\{j_z\},\{j'_z\},\tau) \in \mathcal{J}(L'_0,\ldots,L'_{i-2},L_0,\ldots,L_{s_1},L'_{i+2},\ldots,L'_{s_2},\Sigma^n,St_i(r,r',\tau)),$$

for each $0 \le \tau < 1$. Such a pair $\{j_z\},\{j'_z\}$ will be called *composable.*

Definition 4.7. *A **system** $\mathcal{F}$: of connections, and almost complex structures compatible with a system of maps $\mathcal{U}$ is an element of*

$$\prod_{r\in\overline{\mathcal{R}}_s}\ \prod_{\{(L_0,\ldots,L_s)|L_i\in F(\Sigma^n)\}}\ \prod_{s\geq 2}\prod_{\Sigma^n}\mathcal{T}(L_0,\ldots,L_s,\Sigma^n,r)\times\mathcal{J}(L_0,\ldots,L_s,\Sigma^n,r)$$

The projection of $\mathcal{F}$ onto the $(r,(L_0,\ldots,L_s),\Sigma^n,s)$ component will be denoted by $\mathcal{F}(L_0,\ldots,L_s,\Sigma^n,r)$. And for shorthand we say that a Hamiltonian connection $\mathcal{A} \in \mathcal{F}$, if it is of the form $pr_1\mathcal{F}(L_0,\ldots,L_s,\Sigma^n,r)$, for pr_i, $i = 1,2$ the pair of projections of $\mathcal{F}(L_0,\ldots,L_s,\Sigma^n,r)$ onto the component of $\mathcal{T}(L_0,\ldots,L_s,\Sigma^n,r)$, respectively $\mathcal{J}(L_0,\ldots,L_s,\Sigma,r)$.

We say that $\mathcal{F}$ is **natural** *if:*

- *For a composable pair $\mathcal{A},\mathcal{A}' \in \mathcal{F}$ as above the connection $St_i(\mathcal{A},\mathcal{A}',0)$ coincides with*

$$pr_1\mathcal{F}(L'_0,\ldots,L'_{i-2},L_0,\ldots,L_{s_1},L'_{i+1},\ldots,L'_{s_2},\Sigma^n,St_i(r,r',0)).$$

- *The pair of connections above also agree for a sufficiently small non zero ϵ on the "thin part" of $\mathcal{S}_{St_i(r,r',\epsilon))}$.*
- *Given a face map $f : \Delta^{n-1} \to \Delta^n$, by the second naturality property of maps $u(m_1,\ldots,m_d,n)$ determined by $\mathcal{U}$, there is bundle map*

$$p : \widetilde{\mathcal{S}}(m_1,\ldots,m_d,\Sigma_f^{n-1} = \Sigma^n\circ f,r) \to \widetilde{\mathcal{S}}(f(m_1),\ldots,f(m_d),\Sigma^n,r),$$

and we ask that the pullback map takes $pr_1\mathcal{F}(p(L_0),\ldots,p(L_d),\Sigma^n,r)$ to

$$pr_1\mathcal{F}(L_0,\ldots,L_d,\Sigma_f^{n-1},r).$$

- *There are analogous conditions on the families of almost complex structures $pr_2\mathcal{F}(L_0,\ldots,L_s,\Sigma^n,r)$, which we will not state to keep things a bit simpler.*

Notation 4.8. *We shall sometimes write by abuse of notation $\mathcal{F}(\ldots)$, for either the connection $pr_1\mathcal{F}(\ldots)$, or the family of almost complex structures $pr_2\mathcal{F}(\ldots)$, since there usually can be no confusion.*

Lemma 4.9. *A natural system $\mathcal{F}$ compatible with a given $\mathcal{U}$ exists, and is unique up to homotopy.*

Proof. When $n = 0$, this is the classical Fukaya category case, and the proof of existence of a natural system is given in [11], in the language of what Seidel calls compatible system of perturbations. Now suppose that we have chosen an element

$$\mathcal{F} \in \prod_{r \in \overline{\mathcal{R}}_s} \prod_{\{(L_0,\ldots,L_s) | L_i \in F(\Sigma^n)\}} \prod_{s \geq 2} \prod_{\{\Sigma^n | n \leq N\}} \mathcal{T}(L_0, \ldots, L_s, \Sigma^n, r) \times \mathcal{J}(L_0, \ldots, L_s, \Sigma^n, r)$$

satisfying naturality. We need to extend this to an element of

$$\prod_{r \in \overline{\mathcal{R}}_s} \prod_{\{(L_0,\ldots,L_s) | L_i \in F(\Sigma^n)\}} \prod_{s \geq 2} \prod_{\{\Sigma^n | n \leq N+1\}} \mathcal{T}(L_0, \ldots, L_s, \Sigma^n, r) \times \mathcal{J}(L_0, \ldots, L_s, \Sigma^n, r),$$

also satisfying naturality.

Denote by $D(L_0, \ldots, L_s)$ the least dimension of a subsimplex of Δ^{N+1}, with vertices determined by the $\{L_i\}$. There is a unique extension of $\mathcal{F}$ to an element

$$\begin{aligned} \mathcal{F} \in \prod_{r \in \overline{\mathcal{R}}_s} \prod_{\{(L_0,\ldots,L_s) | \, N \geq D(L_0,\ldots,L_s)} \prod_{s \geq 2} \prod_{\{\Sigma^n | n \leq N+1\}} \\ \mathcal{T}(L_0, \ldots, L_s, \Sigma^n, r) \times \mathcal{J}(L_0, \ldots, L_s, \Sigma^n, r), \end{aligned} \tag{4.14}$$

satisfying the naturality condition. We need to extend to the case $N + 1 = D(L_0, \ldots, L_s)$, and so that naturality is satisfied. For all $(L_0, \ldots, L_s)$ with

$$D(L_0, \ldots, L_s) = N + 1,$$

and given Σ^{N+1} the naturality condition and $\mathcal{F}$ from (4.14) determine

$$\mathcal{F}(L_0, \ldots, L_s, \Sigma^{N+1}, r)$$

for r in the boundary of $\overline{\mathcal{R}}_s$, see the discussion following (4.9). Now

$$\{\mathcal{T}(L_0, \ldots, L_s, \Sigma^n, r) \times \mathcal{J}(L_0, \ldots, L_s, \Sigma^n, r)\}$$

forms a a Serre fibration over $\overline{\mathcal{R}}_s$, with non-empty contractible fibers. Thus, we may simply pick an extension of the family

$$\{\mathcal{F}(L_0, \ldots, L_s, \Sigma^{N+1}, r)\}$$

for all $r \in \overline{\mathcal{R}}_s$, such that the second naturality condition is satisfied. The other naturality conditions then follow by construction. Uniqueness up to homotopy is obvious from the argument above. □

5. The functor F

Let $A_\infty - Cat$ denote the category of small $\mathbb{Z}_2$ graded A_∞ categories over $\mathbb{Q}$, with morphisms fully-faithful embeddings as defined below, which are in addition quasi-equivalences.

Definition 5.1. *We say that an A_∞ functor F is a* **fully-faithful embedding**, *if F has vanishing higher order components, is injective on objects and if the first component map on hom spaces is an isomorphism of chain complexes. In other words F above is just an identification map of a full A_∞ sub-category.*

We now describe the construction of the functor

$$F_{P,\mathcal{D}} : Simp(X) \to A_\infty - Cat,$$

associated to a Hamiltonian fibration P, and the chosen auxiliary data $\mathcal{D}$, described in the previous section. In what follows we usually drop $\mathcal{D}$ and P from notation.

5.1. F **on a point.** For $x : pt \to X$, $F(x)$ is defined to be the Fukaya A_∞ category $Fuk(x^*P)$, whose objects are the objects as described in Section 4.2. For a pair L_0, L_1 of objects with $\omega(L_0) \neq \omega(L_1)$ we set $hom(L_0, L_1) = 0$, (to avoid dealing with curved A_∞ categories), otherwise we set

$$hom(L_0, L_1) = CF(L_0, L_1, \mathcal{D}),$$

where the latter is a $\mathbb{Z}_2$ graded Floer chain complex over $\mathbb{Q}$, which is defined as follows.

Let $\mathcal{A}(L_0, L_1)$ be the Hamiltonian connection on $P_x \times [0, 1]$, determined by the chosen data $\mathcal{D}$, and likewise let $j(L_0, L_1)$ to be the family of almost complex structures determined by $\mathcal{D}$. Then $CF(L_0, L_1, \mathcal{D})$ is generated over $\mathbb{Q}$ by $\mathcal{A}(L_0, L_1)$-flat sections γ of $P_x \times [0, 1]$, with boundary on the pair of Langrangians $L_0 \subset P_x \times \{0\}$, $L_1 \subset P_x \times \{1\}$, these are called geometric generators. The $\mathbb{Z}_2$ grading is given by the sign of the corresponding intersection of $\mathcal{A}(L_0, L_1)(L_0)$ with L_1, where $\mathcal{A}(L_0, L_1)(L_0)$ as before denotes the image of L_0 by the parallel transport map over $[0, 1]$.

5.1.1. *Differential on* $CF(L_0, L_1, \mathcal{D})$. For γ_0, γ_1 geometric generators of

$$CF(L_0, L_1, \mathcal{D}),$$

let $\overline{\mathcal{M}}(\gamma_0; \gamma_1)$ denote the natural Floer compactification of the quotient by the translation $\mathbb{R}$ action of the space of holomorphic sections of $P_x \times ([0, 1] \times \mathbb{R})$, with boundary on the Lagrangian sub-bundles $L_0 \times \mathbb{R}, L_1 \times \mathbb{R}$, and asymptotic to γ_i, at the ∞, respectively $-\infty$ ends.

Terminology 5.2. *Here and elsewhere the term* **holomorphic section** *of various Hamiltonian fibrations over 2-d Riemann surfaces will mean the following. Our Hamiltonian fibrations $\widetilde{S}$ always come with choices of a Hamiltonian connection $\mathcal{A}$, and a family of fiber-wise almost complex structures $\{j_z\}$, determined by the perturbation data $\mathcal{D}$. This gives an induced almost complex structure $J(\mathcal{A}, \{j_z\})$ on $\widetilde{S}$ restricting to $\{j_z\}$ on the fibers, having a holomorphic projection map to the base, and preserving the horizontal distribution of $\mathcal{A}$. Holomorphic then means that the section has $J(\mathcal{A}, \{j_z\})$-complex linear differential.*

In the above case, "holomorphic" is with respect to the almost complex structure induced by the flat, $\mathbb{R}$-translation invariant extension of $\mathcal{A}(L_0, L_1)$ to $P_x \times [0, 1] \times \mathbb{R}$, and likewise by the translation invariant extension of $j(L_0, L_1)$ to $P_x \times [0, 1] \times \mathbb{R}$.

For a generic pair $\mathcal{A}(L_0, L_1), j(L_0, L_1)$, all the moduli spaces $\overline{\mathcal{M}}(\gamma_0; \gamma_1)$ are transversely cut out for all γ_i. The differential $\mu^1 : CF(L_0, L_1, \mathcal{D}) \to CF(L_0, L_1, \mathcal{D})$ is defined as usual by

$$\mu^1(\gamma_j) = \sum_i \#\mathcal{M}(\gamma_i; \gamma_j)\gamma_j,$$

for $\{\gamma_i\}$ a basis of geometric generators for $CF(L_0, L_1, \mathcal{D})$, when $\mathcal{M}(\gamma_i; \gamma_j)$ has dimension 0, and hence is compact in the monotone case as index bounds give energy bounds.

5.1.2. *Higher multiplication maps.* The multiplication maps

$$\begin{aligned} \mu^d : hom(L_0, L_1) \otimes hom(L_1, L_2) \otimes \ldots \\ \otimes hom(L_{d-1}, L_d) \to hom(L_0, L_d), \end{aligned} \tag{5.1}$$

$d > 1$ are defined as follows.

Let $\mathcal{U}, \mathcal{F}$ be the natural systems determined by $\mathcal{D}$. Then we may define the moduli space

$$\mathcal{M}(\{\gamma^i\}; \gamma_0, x, \mathcal{F}, A), \tag{5.2}$$

whose elements are class A $\mathcal{F}(\{L_i\}, x, r)$-holomorphic sections (σ, r), $r \in \overline{\mathcal{R}}_d$ of

$$P_x \times \mathcal{S}_r.$$

s.t.

- The boundary of (σ, r) is in the subfibration $\mathcal{L}(\mathcal{U}, L_0, \ldots, L_d, r)$, see (4.13), of $P_x \times \mathcal{S}_r$ over the boundary of $\mathcal{S}_r$.
- By assumptions, at the e_i end, of $\mathcal{S}_r$, in the strip coordinate charts $e_i^1 : [0,1] \times [1, \infty] \to \mathcal{S}_r$, the data $\mathcal{F}(\{L_i\}, x, r)$ is $\mathbb{R}$-translation invariant, and when $i \neq 0$ we ask that σ is asymptotic to γ^i, a geometric generator of $\hom_{F(x)}(L_{i-1}, L_i)$, or at the e_0 asymptotic to γ_0 a geometric generator of $\hom_{F(x)}(L_0, L_d)$.

Given basis of geometric generators $\{\gamma^i_j\}$ for $\hom_{F(x)}(L_{i-1}, L_i)$, $i \neq 0$, and a basis $\{\gamma^0_j\}$ for $\hom_{F(x)}(L_0, L_d)$, $i = 0$, for $d \geq 2$, and assuming that $\mathcal{F}(\{L_i\}, x, r)$ is regular we define

$$\langle \mu^d(\gamma^1_{j_1}, \ldots, \gamma^d_{j_d}), \gamma^0_{j_0} \rangle = \sum_A \#\mathcal{M}(\gamma^1_{j_1}, \ldots, \gamma^d_{j_d}; \gamma^0_{j_0}, x, \mathcal{F}, A), \tag{5.3}$$

when the above moduli spaces have dimension 0, for $\langle , \rangle$ the natural pairing induced by our choice of basis. The sum is finite by monotonicity.

5.1.3. *Compactification regularity, and associativity.* The moduli spaces

$$\mathcal{M}(\{\gamma^i\}; \gamma_0, x, \mathcal{F}, A)$$

are identical to the moduli spaces in Sheridan [13] with respect to a system of Hamiltonian perturbations, and a domain dependent family of almost complex structures on (in this case) trivial bundles, determined by $\mathcal{F}$. To be more explicit, a Hamiltonian connection on a trivial M bundle over a surface S is the same as the data of a 1-form on S with values in $C^\infty_0(M)$: mean 0 smooth functions, which is the data of a Hamiltonian perturbation. So in our case we just have a language change, the reason for which will be obvious when we shall construct the value of F on higher dimensional simplices of X. Consequently the compactification and regularity story is word for word identical to Sheridan [13].

5.1.4. A_∞ *associativity.* The maps $\mu^d_{F(\Sigma)}$ satisfy the A_∞-associativity equations (stated over $\mathbb{F}_2$ for simplicity)

$$\sum_{n,m} \mu_\Sigma^{d-m+1}(\gamma_1, \ldots, \gamma_n, \mu^m_\Sigma(\gamma_{n+1}, \ldots, \gamma_{m+n}), \gamma_{n+m+1}, \ldots, \gamma_d) = 0, \tag{5.4}$$

This is shown as usual by considering boundary of the one dimensional moduli spaces, $\overline{\mathcal{M}}(\{\gamma^i\}; \gamma_0, x, \mathcal{F}, A)$.

5.2. *F* **on higher dimensional simplices.** We will consider for the moment the case of non-degenerate simplices $\Sigma : \Delta^n \to X$. The category $F(\Sigma)$, has objects $\bigsqcup_i \operatorname{obj} F(x_i))$, where $x_i : pt \to X$ is the composition of the vertex inclusion $\Delta^0 \to \Delta^n$ corresponding to the i'th corner, with the map Σ. Abusing notation we may also write x_i for $x_i(pt) \in X$.

Let $m : [0,1] \to \Delta^n$ be the edge between i, j corners of Δ^n and set $\overline{m} = \Sigma \circ m$. Given a pair of objects $L_0 \subset P_{x_i}, L_1 \subset P_{x_j}$, (including $i = j$) and the Hamiltonian connection $\mathcal{A}(L_0, L_1) = \mathcal{A}(L_0, L_1, \overline{m})$ on $\overline{m}^*P$, determined by $\mathcal{D}$, we define as before $hom_{F(\Sigma)}(L_0, L_1)$ to be the $\mathbb{Z}_2$ graded chain complex over $\mathbb{Q}$ generated by flat sections of $(\overline{m}^*P, \mathcal{A}(L_0, L_1))$, with boundary on the Lagrangian submanifolds $L_0 \in \overline{m}^*P|_0$, $L_1 \in \overline{m}^*P|_1$. The differential μ^1 is defined identically to the differential on morphism spaces of categories $Fuk(P_x)$. The only difference is that $\overline{m}^*P$ may no longer be naturally trivialized.

This completely describes all objects and morphisms of $F(\Sigma)$. We now need to describe the A_∞ structure. Given $\{L_{\rho(k)} \in F(x_{\rho(k)})\}_{k=0}^{k=d}$,

$$\rho : \{0, \ldots, d\} \to \{x_i\}_{i=0}^{i=n},$$

with $\omega(L_{\rho(k)}) = n \in \mathbb{Z}$, we define the higher composition maps

$$\mu^d_\Sigma : hom(L_{\rho(0)}, L_{\rho(1)}) \otimes \ldots \otimes hom(L_{\rho(d-1)}, L_{\rho(d)}) \to hom(L_{\rho(0)}, L_{\rho(d)}). \tag{5.5}$$

Note that by construction, to each morphism of $F(\Sigma)$ naturally corresponds either an edge or a vertex of Δ^n, in either case we may naturally associate to these a morphism in the category $\Pi(\Delta^n)$. The collection $\{x_{\rho(k)}\}$ then clearly determines a *composable chain* $(m_1, \ldots, m_d)$ of morphisms in $\Pi(\Delta^n)$. Given the map

$$u(m_1, \ldots, m_d, n) : \overline{\mathcal{S}}^\circ_d \to \Delta^n, \text{ which is part of a natural system } \mathcal{U},$$

and the system $\mathcal{F}$ determined by $\mathcal{D}$, we define the moduli space $\mathcal{M}(\{\gamma^i\}; \gamma_0, \Sigma^n, \mathcal{F}, A)$ analogously to (5.2). Explicitly, its elements are class A $\mathcal{F}(\{L_i\}, \Sigma, r)$-holomorphic sections (σ, r), $r \in \overline{\mathcal{R}}_d$ of

$$\widetilde{S}_r = \widetilde{S}(m_1, \ldots, m_d, \Sigma, r),$$

satisfying:

- The boundary of (σ, r) is in the sub-fibration $\mathcal{L}(\mathcal{U}, L_0, \ldots, L_d, r)$, see (4.13), of $\widetilde{S}_r$ over the boundary of $\mathcal{S}_r$.
- By assumptions, at the i'th end of $\mathcal{S}_r$, in the strip coordinate chart $e^1_i : [0,1] \times [1, \infty] \to \mathcal{S}_r$, the data $\mathcal{F}(\{L_i\}, \Sigma, r)$ is $\mathbb{R}$-translation invariant and when $i \neq 0$ we ask that σ is asymptotic to γ^i a geometric generator of $\hom_{F(\Sigma)}(L_{i-1}, L_i)$, or when $i = 0$ asymptotic in forward time to γ_0 a geometric generator of $\hom_{F(\Sigma)}(L_0, L_d)$.

5.2.1. *Compactness and regularity.* We do not need to reinvent the wheel proving compactness and regularity results for the above moduli spaces. (Although it obviously works the same way.) Instead pick a Hamiltonian trivialization of $M \times \Delta^n \xrightarrow{tr} \Sigma^*P$, then we may pullback the systems of connections $\mathcal{F}(\{L_i\}, \Sigma, r)$, and the systems of almost complex structures $\mathcal{J}(\{L_i\}, \Sigma)$ to $M \times \Delta^n$. Then in the coordinates of $M \times \Delta^n$, these systems of connections and complex structures are essentially equivalent to systems of compatible perturbations, in the sense of Seidel [11], for the Lagrangian boundary conditions given by $\{\pi \circ tr^{-1}(L_i)\}$ for $\pi : M \times \Delta^n \to M$ the projection. (Only "essentially" because we now allow extra copies of the same object.) Consequently compactness and regularity works the same way as described in Section 5.1.3.

5.2.2. *Composition maps.* Then as before given a basis of geometric generators $\{\gamma^i_j\}$ for $\hom_{F(x)}(L_{i-1}, L_i)$, $i \neq 0$, and $\{\gamma^0_j\}$ for $\hom_{F(x)}(L_0, L_d)$, $i = 0$, for $d \geq 2$, and assuming that $\mathcal{F}(\{L_i\}, \Sigma, r)$ is regular we define

$$\langle \mu^d_{F(\Sigma)}(\gamma^1_{j_1}, \ldots, \gamma^d_{j_d}), \gamma^0_{j_0} \rangle = \sum_A \# \mathcal{M}(\gamma^1_{j_1}, \ldots, \gamma^d_{j_d}; \gamma^0_{j_0}, \Sigma, \mathcal{F}, A), \tag{5.6}$$

when the above moduli spaces are of dimension 0, for $\langle , \rangle$ as before.

5.2.3. *Associativity.* This works as before.

Lemma 5.3. *The assignment $\Sigma \mapsto F(\Sigma)$ extends to a natural functor*

$$F : Simp(X) \to A_\infty - Cat.$$

Proof. Given a face map $f : \Delta^{n-1} \to \Delta^n$, and $\Sigma^n : \Delta^n \to X$, by the third naturality property of our connections there is a canonical functor $F(\Sigma^n \circ f) \to F(\Sigma^n)$, which is by construction a fully-faithful embedding. It follows via iteration, that a morphism $m : \Sigma^k \to \Sigma^l$, with $\Sigma^k, \Sigma^l \in Simp(X)$, $k < l$ induces a fully-faithful embedding:

$$F(m) : F(\Sigma^k) \to F(\Sigma^l),$$

and this assignment is clearly functorial. Note that $F(m)$ is essentially surjective on the cohomological level, which follows by a classical continuation argument, cf. [11, Section 10a], and so each $F(m)$ is a quasi-equivalence. □

Let us call the functor $F_{P,\mathcal{D}} : Simp(X) \to A_\infty - Cat$ as constructed geometrically in this section a *geometric functor* to emphasize the origin.

5.3. **Unital replacement of F.** Let $A_\infty - Cat^{unit}$ denote the subcategory of $A_\infty - Cat$ consisting of unital A_∞ categories and unital functors. By *unital replacement* for $F : Simp(X) \to A_\infty - Cat$ we mean a functor $F^{unit} : Simp(X) \to A_\infty - Cat^{unit}$, together with a natural transformation $N : F \to F^{unit}$, which is object-wise quasi-equivalence.

Lemma 5.4. *Any functor $F : Simp(X) \to A_\infty - Cat$ has a unital replacement.*

Proof. To obtain this we proceed inductively: for each 0-simplex $x \in Simp(X)$, since each $F(x)$ is c-unital we may fix a formal diffeomorphism $\Phi_x : F(x) \to F(x)$, with first component maps Φ^1_x the identity maps, such that induced A_∞-structure $F^{unit}(x) = \Phi_*(F(x))$ is strictly unital, [11, Lemma 2.1]. Let $N_x : F(x) \to F^{unit}(x)$ denote the induced A_∞ functor. Let F_k denote the restriction F to $Simp^{\leq k}(X)$ with $Simp^{\leq k}(X)$ denoting the sub-category of $Simp(X)$, consisting of simplices whose degree is at most k. And suppose that the maps N_x can be extended to a natural transformation N_k of functors

$$F_k : Simp^{\leq k}(X) \to A_\infty - Cat,$$

$$F^{unit}_k : Simp^{\leq k}(X) \to A_\infty - Cat^{unit},$$

where for each Σ $N_k(\Sigma) : F(\Sigma) \to F^{unit}(\Sigma)$ is induced by a formal diffeomorphism $\Phi_\Sigma : F(\Sigma) \to F(\Sigma)$, whose first component maps are the identity maps.

We construct an extension N_{k+1}. Since for each given $\Sigma^{k+1} : \Delta^{k+1} \to X$, and $i : \Sigma^k \to \Sigma^{k+1}$ a morphism in $Simp(X)$, $F(i)$ is a fully-faithful embedding by assumption, and identifying $F(\Sigma^k)$ with a full subcategory of $F(\Sigma^{k+1})$, we may clearly construct as in the proof of [11, Lemma 2.1] a formal diffeomorphism

$$\Phi_{\Sigma^{k+1}} : F(\Sigma^{k+1}) \to F(\Sigma^{k+1}),$$

with $\Phi^*_{\Sigma^{k+1}}(\Sigma^{k+1})$ unital, so that its restriction to $F(\Sigma^k)$ coincides with the formal diffeomorphisms $\{\Phi_{\Sigma^{k+1}\circ i}\}$, for each $i : \Sigma^k \to \Sigma^{k+1}$. The result then follows. □

Let us write F^{unit} for the particular unital replacement of F as constructed in the proof of the lemma above.

5.4. **Naturality.** Given a smooth embedding $f : Y \to X$ and $M \hookrightarrow P \to X$ a Hamiltonian bundle as before, there is an induced functor

$$f_* : Simp(Y) \to Simp(X).$$

And consequently there is an associated pullback functor

$$f^* F_{P,\mathcal{D}} : Simp(Y) \to A_\infty - Cat,$$

where $F_{P,\mathcal{D}} : Simp(X) \to A_\infty - Cat$ is the geometric functor as above. On the other hand we may pullback by f the bundle as well as the perturbation data $\mathcal{D}$, to get another functor $F_{f^*P, f^*\mathcal{D}} : Simp(Y) \to A_\infty - Cat$. The following is immediate from construction.

Lemma 5.5. $F_{f^*P,f^*\mathcal{D}} = f^* F_{P,\mathcal{D}}$.

5.5. **Concordance classes of functors $F : Simp(X) \to A_\infty - Cat$.** We say that a pair of functors $F_0, F_1 : Simp(X) \to A_\infty - Cat$ are *concordant* if there is a functor

$$T : Simp(X \times I) \to A_\infty - Cat,$$

restricting to F_0, F_1 over $Simp(X \times \{0\})$, respectively over $Simp(X \times \{1\})$. Note that by the proof of Lemma 5.4 if F_1, F_2 are concordant then so are F_1^{unit}, F_2^{unit}.

Theorem 5.6. *For a given pair of data $\mathcal{D}_1, \mathcal{D}_2$, the functors*

$$F_{P,\mathcal{D}_1} : Simp(X) \to A_\infty - Cat,$$

$$F_{P,\mathcal{D}_2} : Simp(X) \to A_\infty - Cat$$

are concordant.

Proof. A given pair of choices $\mathcal{D}_1, \mathcal{D}_2$ of auxiliary data, are homotopic in the natural sense through perturbation data $\{\mathcal{D}_t\} = (\{\mathcal{U}_t\}, \{\mathcal{F}_t\})$, $t \in I = [0,1]$, by Theorem 4.3. Let $\widetilde{P} \to X \times I$ be the Hamiltonian fibration given by pull-back of P by the projection map $X \times I \to X$.

We have partial perturbation data $\{\mathcal{D}_t\}$, for $\widetilde{P}$, which is defined for any

$$\Sigma : \Delta^n \to X \times \{t\} \subset X \times I,$$

by the data $\mathcal{D}_t$. Then extend the perturbation data $\{\mathcal{D}_t\}$ in any way to all $Simp(X \times I)$, call this data $\mathcal{H}$. The extension $\mathcal{H}$ clearly exists once one has an extension for the natural system of maps $\{\mathcal{U}_t\}$, since there are no obstructions at all for extending $\{\mathcal{F}_t\}$. On the other hand extension of $\{\mathcal{U}_t\}$ clearly exists by the same inductive argument as in the construction of a natural system $\mathcal{U}$.

Consequently we may define the functor T giving a concordance between $F_{P,\mathcal{D}_1}, F_{P,\mathcal{D}_2}$ to be the geometric functor $F_{\widetilde{P},\mathcal{H}} : Simp(X \times I) \to A_\infty - Cat$.

□

Remark 5.7. *Concordance relation is an equivalence relation (in the special case above). Although we will not show this here. The concordance class of the functor $F_{P,\mathcal{D}}$ is then the most fundamental invariant that is constructed in this paper, however calculating with it maybe very difficult.*

6. Global Fukaya category

Concordance "classes" of functors $F_{P,\mathcal{D}}$ constructed as above, have a wealth of information, but we will cut down on this information by assigning a more geometric object to it. We will assign to the concordance classes above, equivalence classes of certain simplicial fibrations over $X_\bullet$, which are certain analogues of Serre fibrations in the topological category.

The first necessary ingredient for this story is the notion of a quasi-category, which is a simplicial set with an additional property, relaxing the notion of Kan complex. The latter are fibrant objects in the Quillen model structure on the category $sSet$ of simplicial sets, and play the same role in the category of simplicial sets as CW complexes play in the category of topological spaces: they are the fibrant objects in the corresponding Quillen equivalent model categories. Quasi-categories or alternatively called $\infty - categories$, are in turn the fibrant objects for a different non Quillen equivalent model structure on $sSet$ called the Joyal model structure, which will play a background role in this paper. For reader's convenience we will review some of this theory of simplicial sets in the Appendix.

6.1. **The A_∞-nerve.** The A_∞-nerve is an analogue for A_∞ categories of the classical nerve functor from the category of small categories to the category of simplicial sets, (in-fact quasi-categories). From now A_∞-nerve will be just "nerve": N, where there can be no confusion. This construction is due to Lurie [8, Construction 1.3.1.6], and the output is an ∞-category, or in the more specific model in this paper a quasi-category. Thus N is a functor from the category of all (strictly-unital) A_∞ categories, with morphisms A_∞ unital functors to the category of quasi-categories $\infty - \mathcal{C}at$. More precisely Lurie discusses the case of dg-categories, and only indicates the case of A_∞ categories. A complete description of the nerve construction for A_∞ categories is contained in the thesis of Tanaka, [9], where it plays a central role, and is also carefully worked out in Faonte [10], where a number of properties are proved. We will reproduce it here for the reader's convenience, a bit further on.

It should be noted that in the Lagrangian cobordism approach to Fukaya category in Nadler-Tanaka [14] a stable quasi-category $\mathcal{Z}$ is constructed directly. The category $\mathcal{Z}$ is expected to be closely related to the nerve of the triangulated envelope of the Fukaya category.

6.1.1. *Outline of the (A_∞)-nerve construction.* The first step in the construction of N is as follows. Let C be a strictly unital A_∞ category. The 2-skeleton of the nerve $N(C)$, has objects of C as 0-simplices, morphisms of C as 1-simplices and the 2-simplices consist of a triple of objects X, Y, Z, a triple of morphisms

$$f \in hom_C(X), g \in hom_C(Y, Z), h \in hom_C(X, Z),$$

a morphism $e \in \hom_C(X, Z)_1$, with $de = h - f \circ g$. We will describe the full nerve construction in the Appendix A following Tanaka [9].

6.2. **Global Fukaya category.** Suppose we are given a geometric functor $F^{unit} = F^{unit}_{P,\mathcal{D}} : Simp(X) \to A_\infty - Cat^{unit}$. We shall see in Section 7 that there is a canonical extension of F to a functor $F^{unit} : \Delta/X_\bullet \to A_\infty - Cat^{unit}$, that is to the entire category of simplicies of $X_\bullet$.

Definition 6.1. *And we define:*

$$Fuk_\infty(P, \mathcal{D}) := colim_{\Delta/X_\bullet} N \circ F^{unit} \in sSet.$$

An explicit construction of the colimit is given in Lemma 7.2. In principle the above definition could be very impractical since general objects in $sSet$ are difficult to deal with, while taking fibrant replacements for the Joyal model category structure could obfuscate all the original geometry contained in the Fukaya category. Thankfully none of this is necessary as we have a couple of miracles coming from the underlying geometry to save us. The proof of the following will be given in Section 7.

Theorem 6.2. *As defined $Fuk_\infty(P, \mathcal{D}) \in \infty - \mathcal{C}at$, i.e. is a quasi-category, moreover there is a natural (co)-Cartesian fibration*

$$N(Fuk(M)) \hookrightarrow Fuk_\infty(P, \mathcal{D}) \to X_\bullet,$$

whose equivalence class in the over category $sSet/X_\bullet$ is independent of the choice of $\mathcal{D}$.

6.3. **Universal construction.** Let $P_U \to BHam(M, \omega)$ be the associated Hamiltonian M-bundle to the universal principal $Ham(M, \omega)$ bundle. $BHam(M, \omega)$ or the classifying space of any smooth Lie group, based on Milnor's construction [15] admits a well defined notion of smooth maps into it from smooth manifolds. To be precise it has a natural diffeology, see Magnot-Watts [16] and likewise the universal M-bundle $E_M \to BHam(M, \omega)$ has a natural diffeology, so that for a diffeological map $f : B \to BHam(M, \omega)$ the pull-back bundle f^*E_M is naturally diffeological. If B is in a addition a smooth dimension k manifold then f^*E_M is a diffeological space locally (diffeologically) diffeomorphic to $(U \subset B) \times M$, but the latter is clearly locally diffeomorphic to $\mathbb{R}^{k+2n}$, for $2n$ the dimension of M. Thus f^*E_M is a diffeological space locally diffeomorphic to $\mathbb{R}^{2n}$ and hence is a smooth manifold. More formally, f^*E_M is contained in the full subcategory of the category of diffeological spaces corresponding to smooth manifolds.

In the case $B = \Delta^k$ is the k-simplex, and given an open U with $\Delta^k \subset U \subset \mathbb{R}^k$, by a diffeological map $f : \Delta^k \to BHam(M, \omega)$ we mean a map with a diffeological extension $\widetilde{f} : U \to BHam(M, \omega)$, with U given the canonical diffeology. Then we may conclude as above that $\widetilde{f}^*E_M$ is naturally a smooth bundle. Now, formally all our constructions are based on these smooth bundle. So that the construction of $Fuk_\infty(P_U)$ proceeds exactly the same way, if we define the category of smooth simplices in $BHam(M, \omega)$ to be the category of diffeological simplices.

Proof of Theorems 1.3, 1.5. By the above, Theorem 6.2 and Lurie's straightening Corollary A.5 give a homotopy class of a "classifying" map

$$cl = cl(Fuk_\infty(P_U)) : BHam(M, \omega)_\bullet \to \mathcal{S}.$$

And so a group homomorphism

$$cl_* : \pi_i BHam(M, \omega) \to \pi_i(\mathcal{S}, NFuk(M)).$$

□

The following theorem shows only the topological type of the Hamiltonian fibration is detected by the associated (co)-Cartesian fibration.

Theorem 6.3. *Let $M \hookrightarrow P_1 \to X$, $M \hookrightarrow P_2 \to X$ be as before. Suppose that there is a topological Hamiltonian bundle map $P_1 \to P_2$. Then*

$$[cl(P_1)] = [cl(P_2)].$$

Proof. By the main result of [16], P_i are classified by diffeological smooth maps $f_i : X \to BHam(M, \omega)$, it then follows by Lemma 5.5, (and discussion above) that $Fuk_\infty(P_i)$ are equivalent to pull-backs by f_i of $Fuk_\infty(P_U)$. But f_i are homotopic, so the conclusion follows. □

6.4. **Global Fukaya category and unital replacement.** In the construction of $Fuk_\infty(P)$ we had to take a unital replacement for the functor $F : Simp(X) \to A_\infty - Cat$. One may worry then that this algebraic step will obfuscate the "geometry" of simplices of $Fuk_\infty(P)$. This is not really the case. First the A_∞ nerve NC of a non-unital A_∞ category C still exists as a semi-simplicial set, that is as a co-functor $\Delta^{inj} \to Set$, with Δ^{inj} the subcategory of Δ consisting of injective morphisms. For a unital replacement equivalence $C \to C^{unit}$ of C, constructed as in Section 5.3, we then have an induced morphism of semi-simplicial sets $NC \to NC^{unit}$, which by construction induces a bijection $NC([n]) \to NC^{unit}([n])$, for each $[n]$. So we may think without loss of geometric information, of simplices of NC^{unit} in terms of simplices of NC. (The former just have an extra formal algebraic structure.)

7. Algebraic considerations

In this section by equivalence of quasi-categories we always mean categorical equivalence. This and other categorical preliminaries needed for this section are discussed in the Appendix A.

We will prove here Theorem 6.2.

7.1. **Extending F to degenerate simplices.** We first construct an abstract algebraic extension, in our geometric setting this extension also has a geometric interpretation that we later describe. Let

$$F : Simp(X_\bullet) \to A_\infty - Cat^{unit}$$

be a functor. We extend this to a functor:

$$F^{ext} : \Delta/X_\bullet \to A_\infty - Cat^{unit}, \tag{7.1}$$

although later on we use the same name F for F^{ext}.

We need to say what to do with degenerate simplices. Suppose we are given a diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "j+1"] \arrow[dr, "j_*"'] & \Delta^{n+1} \arrow[r, "pr_j"] \arrow[d, "s_j(\Sigma)"] & \Delta^n \arrow[dl, "\Sigma"] \\ & X, & \end{tikzcd}$$

where

$$pr_j : \Delta^{n+1} \to \Delta^n, \quad j \in [n]$$

is induced by the unique surjection $[n+1] \to [n]$, hitting j twice, and where Σ is non-degenerate. Here $j_* = \Sigma \circ j$, and j also denotes the map $pt \to \Delta^n$ corresponding to this vertex.

Then

$$F(s_j(\Sigma)) \tag{7.2}$$

is defined to be the A_∞ category with objects

$$\operatorname{obj} F(\Sigma)) \sqcup \operatorname{obj} F(j_*).$$

Note that there are then two embeddings $\operatorname{obj} F(j_*) \to \operatorname{obj} F(s_j(\Sigma))$, one given by

$$F(inc_j) : \operatorname{obj} F(j_*) \to \operatorname{obj} F(\Sigma),$$

$inc_j : j_* \to \Sigma$ the map in $Simp(X)$ corresponding to the vertex inclusion map of j, and the other just being the tautological map τ to the summand $\operatorname{obj} F(j_*)$.

The *hom* sets are defined by the conditions:

(1) There are tautological strict, full embeddings of A_∞ categories

$$F(\Sigma) \to F(s_j(\Sigma)), \quad F(j_*) \to F(s_j(\Sigma)),$$

corresponding to the natural set embeddings:

$$\operatorname{obj} F(\Sigma)) \to \operatorname{obj} F(s_j(\Sigma))$$
$$\operatorname{obj} F(j_*) \to \operatorname{obj} F(s_j(\Sigma)).$$

(2)

$$hom_{F(s_j(\Sigma))}(L', \tau(L)) := hom_{F(\Sigma)}(L', F(inc_j)(L)),$$

for $L' \in \operatorname{obj} F(\Sigma)$, and $L \in \operatorname{obj} F(j_*)$.

The composition operations $\mu^d_{F_{s_j(\Sigma)}}$ are then defined so that the tautological projection

$$F(d_j) : \operatorname{obj} F(s_j(\Sigma)) \to \operatorname{obj} F(\Sigma)$$

extends to a strict, fully-faithful A_∞ functor, where $d_j : s_j\Sigma \to \Sigma$ the morphism induced by pr_j.

If we have a higher degeneracy in the form of a commutative diagram:

$$\begin{tikzcd} \Delta^{n+k} \arrow[r] \arrow[d] & \Delta^n \arrow[dl, "\Sigma"] \\ X, & \end{tikzcd}$$

we may rewrite it as a composition of simple degeneracies as above, and define the extension inductively using the above prescription.

7.2. **Extension for geometric functors $F_{\mathcal{D}} : Simp(X) \to A_\infty - Cat^{unit}$.** When $F_{\mathcal{D}}$ is a geometric functor defined via perturbation data $\mathcal{D}$, it is possible to make sense of the above algebraic extension geometrically, but this necessitates slightly extending the notion of perturbation data as follows. This extension will be denoted by $\mathcal{D}^{ext}$ in what follows.

Suppose as above that we are given a commutative diagram:

$$\begin{tikzcd} \Delta^{n+k} \arrow[r, "pr"] \arrow[d, "\widetilde{\Sigma}"] & \Delta^n \arrow[dl, "\Sigma"] \\ X, & \end{tikzcd}$$

where Σ is non-degenerate, and $k > 0$. The objects of $F(\widetilde{\Sigma}) = F_{\mathcal{D}^{ext}}(\widetilde{\Sigma})$ are defined as before for a non-degenerate Σ. So pr clearly induces a map of sets of objects

$$pr_* : \mathrm{obj}\, F(\widetilde{\Sigma}) \to \mathrm{obj}\, F(\Sigma).$$

Given a pair of vertices i, j of Δ^{n+k} and objects $L \in F(x_i), L' \in F(x_j)$ for $x_i = \widetilde{\Sigma}(i), x_j = \widetilde{\Sigma}(j)$ we set

$$\mathcal{A}(L, L') = (pr \circ m_{i,j})^* \mathcal{A}(pr_* L, pr_* L'),$$

where the latter connection is determined by the given $\mathcal{D}$, and $m_{i,j} \in hom_{\Pi(\Delta^{n+k})}(i, j)$. Likewise given objects $L_0, \ldots, L_s \in F(\widetilde{\Sigma})$ we set

$$\mathcal{F}(L_0, \ldots, L_s, \widetilde{\Sigma}, r) = (pr \circ u(m_1, \ldots, m_s, r, n)^* \mathcal{F}(pr_* L_0, \ldots, pr_* L_s, \Sigma^n, r).$$

The pull-back operation $(pr \circ u(m_1, \ldots, m_s, r, \Delta^n))^*$ is induced be the natural pull-back operation of connections, and families of fiberwise almost complex structures, and the asked for equality makes sense by the axiom 3 of naturality of $\mathcal{U}$.

We then define the A_∞ category $F_{\mathcal{D}^{ext}}(\widetilde{\Sigma})$ using the data $\{\mathcal{F}(L_0, \ldots, L_s, \widetilde{\Sigma}, r)\}$ constructed as above, and it is clear that

$$F_{\mathcal{D}^{ext}}(\widetilde{\Sigma}) = F^{ext}(\widetilde{\Sigma}), \tag{7.3}$$

for F^{ext} the name of the abstract extension of F to degenerate simplices constructed as above.

7.3. **Colimit of F.** For $F : \Delta/X_\bullet \to A_\infty - Cat^{unit}$ as above, let

$$Fuk_\infty(F) = colim_{\Delta/X_\bullet} NF.$$

Proposition 7.1. *There is a natural projection of simplicial sets $p : Fuk_\infty(F) \to X_\bullet$, and this is a (co)-Cartesian fibration.*

Proof. Recall that a given $\Sigma : \Delta^n \to X$, could equally be thought of as an element of $X_\bullet([n])$, or of $\Delta/X_\bullet$ that is as a natural transformation $\Delta^n_\bullet \to X_\bullet$. Let us first give a more easily conceptualized presentation of $Fuk_\infty(F)$. Define a partial order $<$ on the set of pairs (f, Σ), $f \in F(\Sigma)$, $\Sigma \in \Delta/X_\bullet$ by $(f, \Sigma) < (f', \Sigma')$ if there is a morphism $i : \Sigma \to \Sigma'$ in $Simp(X)$ induced by $[n] \to [m]$ with $n \le m$, i.e. a *face morphism*, s.t. $F(i)(f) = f'$. Clearly for every (f, Σ) there is a unique least pair $(f_{min}, \Sigma_{\min}) < (f, \Sigma)$. Let $\widetilde{L}$ be the set of minimal pairs. Define an equivalence relation on $\widetilde{L}$ by $(f, \Sigma) \sim (f', \Sigma')$ if there exists a morphism $d : \Sigma \to \Sigma'$ induced by $[m] \to [n]$ with $m > n$, (i.e. it is a degeneracy morphism) with $F(d)(f) = f'$. Denote the equivalence class of (f, Σ) by $[f, \Sigma]$. Then $L = \widetilde{L}/\sim$ is naturally a simplicial set, with

$$L([k]) = \{[f, \Sigma] \in L \mid f \in NF(\Sigma)([k])\}.$$

For example $L([0])$ is just

$$\sqcup_{x \in X} Obj\, F(x).$$

Lemma 7.2. $L = Fuk_\infty(F)$.

Proof. Note first that L is a co-cone on the diagram NF. Indeed, for each Σ define $\phi_\Sigma : NF(\Sigma) \to L$ by

$$\phi_\Sigma(f) = [f_{min}, \Sigma_{\min}].$$

It is easy to see that for a face morphism $i : \Sigma \to \Sigma'$ we have that the composition

$$NF(\Sigma) \xrightarrow{NF(i)} NF(\Sigma') \xrightarrow{\phi_{\Sigma'}} L,$$

coincides with ϕ_Σ. Likewise for a degeneracy morphism $d : \Sigma \to \Sigma'$ we have that the composition

$$NF(\Sigma) \xrightarrow{NF(d)} NF(\Sigma') \xrightarrow{\phi_{\Sigma'}} L,$$

coincides with ϕ_Σ, because of the equivalence relation $\sim$.

The universal property is also easy to verify, for given another co-cone L' with maps $\rho_\Sigma : NF(\Sigma) \to L'$, $\Sigma \in Simp(X)$ we can naturally define $U : L \to L'$ by

$$U([f, \Sigma]) = \rho_\Sigma(f).$$

Then U is clearly well defined, as L' is by assumption a co-cone, and for a given $f \in NF(\Sigma)$, we have

$$U\phi_\Sigma(f) = \rho_{\Sigma_{\min}}(f_{\min}) = \rho_\Sigma f,$$

where the last equality holds since $L', \{\rho_\Sigma\}$ is a co-cone, and since by construction there is a morphism $i : NF(\Sigma_{\min}) \to NF(\Sigma)$, with $F(i)f_{\min} = f$. ☐

We have a natural simplicial map

$$p_\Sigma : NF(\Sigma) \to \Sigma_*(\Delta^n_\bullet) \subset X_\bullet,$$

defined as follows. On the vertices of $NF(\Sigma)$, p_Σ is just the obvious projection. Now a k-simplex f in $NF(\Sigma)$ by definition determines a composable chain $(f_1, \ldots, f_k)$ in $NF(\Sigma)$, and hence determines a sequence of vertices $e_0, \ldots, e_k$ s.t. the source/target of f_i is e_{i-1} respectively e_i. This in turn determines a sequence of vertices $\{p_\Sigma(e_i)\}$, and we set $p_\Sigma(f)$ to be the unique (degenerate) k-simplex of $\Sigma_*(\Delta^n_\bullet)$ with these vertices. We shall omit the verification that p_Σ is simplicial. The simplicial projection

$$p : L \to X_\bullet$$

is then: send $[f, \Sigma]$ to $p_\Sigma f$, which is readily seen to be well defined.

It is immediate from definitions that p is an inner-fibration if and only if the pre-image of every simplex $\Sigma : \Delta^n_\bullet \to X_\bullet$ by p is a quasi-category, where "pre-image" $p^{-1}(\Sigma)$ is the preimage by p of the simplicial subset $\Sigma(\Delta^n_\bullet)$. In our case this follows by construction as the preimage of Σ is clearly identified $NF(\Sigma)$. Let m be an edge in $X_\bullet$ from x_0 to x_1 and let $L_m = p^{-1}m$ and let $L_i = p^{-1}x_i$, see definition of pre-image above. To show that p is a (co)-Cartesian fibration, by [17, Proposition 2.4.1.5] it is enough to show that for every such m and $a \in L_1$ there is an equivalence $e_a \in L_m$ with target a, with $p(e_a) = m$.

Lemma 7.3. *The functor $N : A_\infty - Cat^{unit} \to \mathcal{C}at - \infty$, takes quasi-equivalences to weak equivalences in the Joyal model structure, i.e. categorical equivalences.*

Proof. The proof of this is contained in the proof of proposition 1.3.1.20, Lurie [8]. We can also prove this directly by first recalling that quasi-equivalences of A_∞-categories A, B are invertible, up to homotopy, and then via the nerve construction translate this to a categorical equivalence of $N(A), N(B)$. ☐

Then by construction of L the inclusion of L_i into L_m are equivalences of quasi-categories, and so e_a as above must exist. ☐

Proof of Theorem 6.2, 1.1. By the discussion above we have a (co)-Cartesian fibration

$$Fuk_\infty(P, \mathcal{D}) \to X_\bullet.$$

The first part of the theorem follows by the following general fact: for an inner fibration of simplicial sets $p : P_\bullet \to X_\bullet$, if $X_\bullet$ is a quasi-category then $P_\bullet$ is a quasi-category. Let us prove this elementary point. Suppose we are given $\rho : \Lambda^n_k \to P_\bullet$, for $0 < k < n$. As $X_\bullet$ is a quasi-category there a simplex $\widetilde{\rho} : \Delta^n_\bullet \to X_\bullet$ extending $p \circ \rho$. But then ρ maps into the quasi-category $p^{-1}(\widetilde{\rho})$, consequently there is an extension of ρ, c.f. Proposition A.3.

The final part of the theorem follows by the following.

Lemma 7.4. *For the geometric functor $F_{P,\mathcal{D}}$, the equivalence class of the fibration $p : Fuk_\infty(P, \mathcal{D}) \to X_\bullet$, in the over category $sSet/X_\bullet$ is independent of the choice of $\mathcal{D}$.*

Proof. By Theorem 5.6 and previous discussion for any pair $\mathcal{D}_i$ of perturbation data there exists a (co)-Cartesian fibration:

$$\mathcal{T} \to X_\bullet \times I_\bullet,$$

whose restriction over $X_\bullet \times \partial I_\bullet$ coincides with

$$Fuk_\infty(P, \mathcal{D}_1) \sqcup Fuk_\infty(P, \mathcal{D}_2),$$

The lemma then follows by Lurie's straightening theorem A.4, or more simply by Corollary A.5. (Strictly speaking by its proof, as we need the straightening functor for $\mathcal{T}$ to restrict to straightening functor for $Fuk_\infty(P, \mathcal{D}_1) \sqcup Fuk_\infty(P, \mathcal{D}_2)$ over $\partial X_\bullet \times I_\bullet$.) ☐

☐

7.4. **Naturality.** We may expect if our constructions are really natural that $Fuk_\infty(P, \mathcal{D})$ is functorial with respect to pull-back and this is indeed the case. Let $f : X \to Y$ be a smooth map, $P \to Y$ a smooth Hamiltonian fibration and $\mathcal{D}$ extended perturbation data. We may then naturally define pull-back extended perturbation data $f^*\mathcal{D}$, for f^*P over X. Specifically if $\Sigma \in X_\bullet$, then we set

$$\mathcal{F}(L_0, \ldots, L_s, \Sigma, r) = \mathcal{F}(L_0, \ldots, L_s, \widetilde{\Sigma}, r),$$

for $\widetilde{\Sigma} = f \circ \Sigma$.

Let $\widetilde{f} : X_\bullet \to Y_\bullet$ be the induced map of singular sets.

Theorem 7.5. $Fuk_\infty(f^*P, f^*\mathcal{D}) = \widetilde{f}^* Fuk_\infty(P, f^*\mathcal{D})$, *where $f^*\mathcal{D}$ is as above.*

Proof. The proof is immediate given (7.3). ☐

Appendix A. Quasi-categories and Joyal model structure

A very good concise reference for much of this material is Riehl [18], which we will mostly follow. The material on co-Cartesian fibrations is taken from Lurie [17, Section 2.4]. First let us recall the notion of a Kan complex, which maybe thought of as formalizing the property of a simplicial set to be like the singular set of a topological space.

Let Δ^n be the standard representable n-simplex: $\Delta([i]) = \Delta([i], [n])$. Previously we denoted this by $\Delta^n_\bullet$ but as there are no topological simplices in this section we simplify the notation, which is also consistent with above references. Let $\Lambda^n_k \subset \Delta^n$ denote the sub-simplicial set corresponding to the "boundary" of Δ^n with the k'th face removed, $0 \leq k \leq n$. By $k'th$ face we mean the face opposite to k'th vertex.

A simplicial set $S_\bullet$ is said to be a *Kan complex* if for all n, k given a diagram with solid arrows

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & S_\bullet \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd},$$

there is a dotted arrow making the diagram commute.

A *quasi-category* is a simplicial set $S_\bullet$ for which the above extension property is only required to hold for inner horns Λ^n_k, i.e. those horns with $0 < k < n$. A morphism between quasi-categories is just a simplicial map. We will denote quasi-categories by calligraphic letters e.g. $\mathcal{B}$. The full-subcategory of *sSet* with objects quasi-categories will be denoted by $\infty - \mathcal{C}at$.

A.1. **Categorical equivalences, morphisms and equivalences.** We have a natural functor $\tau : sSet \to Cat$, $\tau(S_\bullet)$ is the category with objects 0-simplices of $S_\bullet$, 1-simplices as morphisms, degenerate 1-simplices as identities and freely generated composition subject to the relation $g = f \circ h$ if there is a 2-simplex e with 0-face h, 2-face f and 1-face g. (Remembering our diagrammatic order for composition.) We then have a functor $\tau_0 : sSet \to Set$ by sending $A_\bullet$ to the set of isomorphism classes of objects in $\tau A_\bullet$. If $S_\bullet = \mathcal{X}$ is a quasi-category an edge $e \in \mathcal{X}$ is said to be an *equivalence* if it is an isomorphism in $\tau\mathcal{X}$. The *maximal Kan subcomplex* of a quasi-category $\mathcal{X}$ is the sub-complex obtained by removing edges which are not equivalences.

Notation A.1. *The category $\infty - \mathcal{C}at$ naturally has a structure of a quasi-category itself see [17, Chapter 3], we shall denote its maximal Kan subcomplex by $\mathcal{S}$.*

We define $sSet^{\tau_0}$ to be the category with the same objects as $sSet$ but with the morphisms given by $sSet^{\tau_0}(A_\bullet, B_\bullet) = \tau^0(B_\bullet^{A_\bullet})$. A map of simplicial sets $u : A_\bullet \to B_\bullet$ is said to be a *categorical equivalence* if the induced map in $sSet^{\tau_0}$ is an equivalence. It is said to be a weak equivalence if the pull-back map

$$sSet^{\tau_0}(B_\bullet, X_\bullet) \to sSet^{\tau_0}(A_\bullet, X_\bullet)$$

induced by u is an equivalence for all $X_\bullet$. A categorical equivalence is necessarily a weak categorical equivalence. Conversely a weak categorical equivalence between quasi-categories is necessarily a categorical equivalence. We will say that a pair of quasi-categories are *equivalent* if there is categorical equivalence between them.

As we are following Riehl [18], we refer the reader there for the following:

Theorem A.2 (Joyal, Lurie, Riehl)**.** *There is a model structure on sSet, with weak equivalences weak categorical equivalences and cofibrations monomorphisms. Moreover the fibrant objects are quasi-categories.*

A.2. **Inner fibrations.** A map $p : \mathcal{A} \to \mathcal{B}$ is said to be an *inner fibration* if it has the lifting property with respect to all inner horn inclusions. More specifically if for $0 < k < n$ whenever we are given a commutative diagram with solid arrows,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathcal{A} \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{B}, \end{tikzcd} \tag{A.1}$$

there exists a dashed arrow as in the diagram. For reference p is said to be an *Kan fibration* if the above extension property holds for all horns. A Kan fibration is an analogue in the simplicial world of Serre fibrations of topological spaces. The following is immediate from definitions.

Proposition A.3. *A map $p : A_\bullet \to B_\bullet$ is an inner fibration, if and only if the pre-image of every simplex of $B_\bullet$ is a quasi-category.*

A.3. **(co)-Cartesian fibrations.** These are the analogue in the quasi-categories world of Grothendieck fibrations. We will explain the co-Cartesian version, as the other is just dual to it.

Given $p : A_\bullet \to B_\bullet$, an edge $f : \Delta^1 \to A_\bullet$ is said to be *co-Cartesian* if whenever we are given a diagram with solid arrows:

$$\begin{tikzcd} \Delta^n_{0,1} \arrow[d, hook] \arrow[dr, "f"] & \\ \Lambda^n_0 \arrow[r] \arrow[d] & A_\bullet \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & B_\bullet, \end{tikzcd} \tag{A.2}$$

there is a dashed arrow making the diagram commutative. Here $\Delta^n_{0,1}$ denotes the "edge" (sub-simplical complex) joining the vertexes $0, 1$.

The map p is said to be a *co-Cartesian fibration*, if it is an inner fibration and if for every edge $e : \Delta^1 \to B_\bullet$, with co-domain b, and every $\widetilde{b}$ lifting b there is a co-Cartesian lift $\widetilde{e} : \Delta^1 \to A_\bullet$, with co-domain $\widetilde{b}$.

Denote by $coCFib(\mathcal{B})$ the quasi-category of (co)-Cartesian fibrations over $\mathcal{B}$, which by definition is the full-subcategory of the over-category $\mathcal{C}at_\infty/\mathcal{B}$, with objects (co)-Cartesian fibrations.

Theorem A.4. *[17, Theorem 3.2.01]* ***Straightening theorem****. There is equivalence of quasi-categories $Fun(\mathcal{B}, \mathcal{C}at_\infty) \simeq coCFib(\mathcal{B})$.*

Stated more properly this combines [17, Theorem 3.2.01] and [17, Proposition 3.1.5.3], both of which are statements on the level of model categories. When $\mathcal{B}$ is a Kan complex, the notions of Cartesian and co-Cartesian fibrations over $\mathcal{B}$ coincide and the model category presenting $coCFib(\mathcal{B})$ is just the over category $sSet/\mathcal{B}$ with the induced Joyal model structure, the fibrant objects in this model structure are the (co)-Cartesian fibrations over $\mathcal{B}$. The term "presents" here means that the underlying quasi-category is the simplicial nerve of the Dwyer-Kan [19] simplicial localization of the model category. Thus we simplify the above as follows for our needs in this paper. We say that a pair of co-Cartesian fibrations $\mathcal{P}_1 \to \mathcal{B}$, $\mathcal{P}_2 \to \mathcal{B}$

are ***equivalent*** if there is an (categorical) equivalence of quasi-categories $\mathcal{P}_1 \to \mathcal{P}_2$ over $\mathcal{B}$. The set of such equivalence classes is formally $\tau_0 CFib(\mathcal{B})$.

Corollary A.5. *For a Kan complex $X_\bullet$, the set $\tau_0 CFib(X_\bullet)$ is naturally identified with the set of homotopy classes of maps $X_\bullet \to \mathcal{S}$, where $\mathcal{S}$ denotes the maximal Kan subcomplex of the quasi-category $\infty - \mathcal{C}at$.*

A.4. **Semi-locality.** Suppose that $S_\bullet \subset X_\bullet$ is a Kan sub-complex, whose inclusion map is a weak equivalence.

Lemma A.6. *The restriction functor*

$$\tau CFib(X_\bullet) \to \tau CFib(S_\bullet),$$

is an isomorphism, and so the restriction map

$$\tau_0 CFib(X_\bullet) \to \tau_0 CFib(S_\bullet),$$

is a set-isomorphism.

Proof. By Lurie's straightening theorem A.4, this is equivalent to the restriction functor

$$\tau Func(X_\bullet, \infty - \mathcal{C}at) \to \tau Func(S_\bullet, \infty - \mathcal{C}at), \tag{A.3}$$

being an isomorphism. But $X_\bullet$ and $S_\bullet$ are Kan complexes and so

$$\tau Func(X_\bullet, \infty - \mathcal{C}at) \simeq \tau Func(X_\bullet, \mathcal{S}) \simeq \operatorname{ho} Top(|X_\bullet|, |\mathcal{S}|),$$

where $|\cdot|$ denotes geometric realization functor and where $\operatorname{ho} Top$ denotes the homotopy category of topological spaces. The last equivalence is due to the following. $Func(X_\bullet, \mathcal{S})$ is a Kan complex as it is the mapping space of Kan complexes, then observe that for Kan complexes τ_0 is just the functor of connected components, next use that the geometric realization $|\cdot|$, and singular set functors induce a derived Quillen equivalence between $\operatorname{ho} Top$ and $\operatorname{ho} sSet$.

Similarly

$$\tau Func(S_\bullet, \infty - \mathcal{C}at) \simeq \tau Func(S_\bullet, \mathcal{S}) \simeq \operatorname{ho} Top(|S_\bullet|, |\mathcal{S}|).$$

The inclusion $|S_\bullet| \to |X_\bullet| \simeq X$ is a homotopy equivalence, as the inclusion $S_\bullet \subset X_\bullet$ is a weak equivalence by assumption and since geometric realization has image in CW complexes. It follows that (A.3) is an isomorphism. ∎

Using this we may compute the class of global Fukaya category of $M \hookrightarrow P \to X$ in $\tau_0 CFib(X_\bullet)$, by restricting F to a sub-category $\Delta/S_\bullet$, with $S_\bullet \subset X_\bullet$ minimal Kan sub-complex generated by the sub-simplicial set of $X_\bullet$ corresponding to some smooth triangulation of X. This is in principle finite local data if X is compact, and in general locally finite. This plays a role in the calculation in part II.

A.5. A_∞**-nerve.** This is a natural analogue for A_∞ categories of the dg-nerve construction of Lurie, [8], which is a functor

$$N : dg - Cat \to \infty - \mathcal{C}at.$$

This section follows Tanaka [9, p. 2.3], except that for us everything will be ungraded, and for simplicity with $\mathbb{F}_2$-coefficients.

For $[n] \in \Delta$, a *length s wedge decomposition* of $[n]$, is a collection of monomorphisms in Δ

$$j_i : [n_i] \to [n], \quad i = 1, \ldots, s, \quad |n_i + 1| \geq 2,$$

such that the fiber product

$$[n_i] \times_{[n]} [n_{i+1}] \simeq [0]$$

and the canonical projection

$$[n_i] \times_{[n]} [n_{i+1}] \to [n_i],$$

is the map $[0] \to [n_i]$ sending $\{0\}$ to $\{n_i\} \subset [n_i]$. Here we are thinking of $[n]$ as the totally ordered finite set $\{0, \ldots, n\}$. We denote the set of all decompositions of $[n]$ by $D[n]$.

Definition A.7. *For A a small unital A_∞ category its nerve $N(A)$ is a simplicial set with the set of vertices the set of objects of $\mathcal{A}$. A n-simplex f of $N(A)$ consists of the following data:*

- *A map $[n] \to Ob\mathcal{A}$. We denote the corresponding objects $X_0, \ldots, X_n$.*
- *For each mono-morphism $j : [n_j] \to [n]$ with $|n_j| \geq 2$ an element*

$$f_j \in hom_A(X_{j(0)}, X_{j(n_j)}).$$

  We may completely characterize each such j by its image set, and will sometimes write j for the corresponding set and vice versa, thus $f_{[n]}$ corresponds to the identity $j : [n] \to [n]$.
- *For a given $j : [n_j] \to \Delta^n$, denote by $j - \{i\} : [n_j - 1] \to [n]$ the morphism with image set $j - \{i\}$. Then the collection of these f_j is required to satisfy the following equation:*

$$\mu^1(f_j) = \sum_{0<i<n_j} f_{j-\{i\}} + \sum_s \sum_{decomp_s \in D[n_j]} \mu^s(f_{j_1}, \ldots, f_{j_s}), \tag{A.4}$$

  with $decomp_s \in D[n_j]$ denoting a length s decomposition and $\{f_{j_i}\}$ its elements.

The simplicial maps are as follows. Given an injection $k : [m] \to [n]$, an n simplex f, for $j : [l] \to [m]$ an injection, define an m-simplex f' by $\{f'_j = f_{k \circ j}\}$.

On the other hand given $s_i : [n+1] \to [n]$, $s_i(i+1) = s_i(i)$, and an n-simplex f, define an $(n+1)$-simplex f' by setting

$$f_j = \begin{cases} e_{X_i} & \text{if } j = \{i, i+1\} \text{ ;} \\ f_{s_i \circ j} & \text{if } s_i|_j \text{ is injective.} \\ 0 & \text{otherwise,} \end{cases}$$

for $j : [l] \to [n+1]$ an injection. It is straightforward but tedious to verify that the latter is indeed a face and that simplicial relations are satisfied.

Proposition A.8. *[9, p. 2.3.2], [10] For A a unital A_∞ category its nerve $\mathcal{A} = N(A)$ is a quasi-category.*

For the reader's convenience we outline the proof here.

Proof. Suppose we have an inner horn $\rho_k : \Lambda^n_k \to \mathcal{A}$, in particular this determines morphisms in A: f_j, for all j except $j = [n] - \{k\}$ as part of the structure of the face simplices. Set $f_{[n]} = 0$, and set

$$f_{[n]-\{k\}} = \sum_{0<i<n; i\neq k} f_{[n]-\{i\}} + \sum_s \sum_{decomp_s \in D[n]} \mu^s(f_{j_1}, \ldots, f_{j_s}),$$

then by construction (A.4) is satisfied for the collection of maps $\{f_j\}$. Only thing left to check is that as defined $f_{[n]-\{k\}}$ actually determines the k'th face of our simplex. A direct calculation for this is long but straightforward, using the A_∞ associativity equations. For $n = 2$ this is automatic and for $n = 3$ this can be checked in a few lines. □

For $F : A \to B$ an A_∞ functor we define $NF : NA \to NB$ via the assignment:

$$f_j \mapsto \sum_{decomp_s \in D[n_j]} F^s(f_{j_1}, \ldots, f_{j_s}).$$

Lemma A.9. *[9], [10] The assignment $A \mapsto NA$, and $F \mapsto NF$ as above, determines a functor*

$$N : A_\infty - Cat^{unit} \to \infty - Cat.$$

The details on why this constitutes a functor N are omitted.

References

[1] P. Seidel. "π_1 of symplectic automorphism groups and invertibles in quantum homology rings". In: *Geom. Funct. Anal.* 7.6 (1997), pp. 1046–1095. ISSN: 1016-443X.

[2] Yasha Savelyev. "Quantum characteristic classes and the Hofer metric." English. In: *Geom. Topol.* 12.4 (2008), pp. 2277–2326.

[3] Michael Hutchings. "Floer homology of families. I." English. In: *Algebr. Geom. Topol.* 8.1 (2008), pp. 435–492.

[4] Hông-Vân Lê and Kaoru Ono. "Parametrized Gromov-Witten invariants and topology of symplectomorphism groups." English. In: (2008), pp. 51–75.

[5] Olguţa Buşe. "Deformations of Whitehead products, symplectomorphism groups, and Gromov-Witten invariants." English. In: *Int. Math. Res. Not.* 2010.17 (2010), pp. 3303–3340. ISSN: 1073-7928; 1687-0247/e.

[6] Kevin Costello. "Topological conformal field theories and Calabi-Yau categories." English. In: *Adv. Math.* 210.1 (2007), pp. 165–214.

[7] Bertrand Toen. "The homotopy theory of dg-categories and derived Morita theory." English. In: *Invent. Math.* 167.3 (2007), pp. 615–667.

[8] Jacob Lurie. "Higher Algebra". In: *http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf* ().

[9] Hiro Lee Tanaka. "A functor from Lagrangian cobordisms to the Fukaya category". In: *thesis* (2013).

[10] Giovanni Faonte. "Simplicial nerve of an $\mathcal{A}_\infty$-category." English. In: *Theory Appl. Categ.* 32 (2017), pp. 31–52. ISSN: 1201-561X/e.

[11] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory.* English. Zurich Lectures in Advanced Mathematics. Zürich: European Mathematical Society (EMS). vi, 326 p. EUR 46.00, 2008.

[12] Victor Guillemin, Eugene Lerman, and Shlomo Sternberg. *Symplectic fibrations and multiplicity diagrams*. Cambridge: Cambridge University Press, 1996, pp. xiv+222. ISBN: 0-521-44323-7.

[13] Nick Sheridan. "On the Fukaya category of a Fano hypersurface in projective space." English. In: *Publ. Math., Inst. Hautes Étud. Sci.* 124 (2016), pp. 165–317. ISSN: 0073-8301; 1618-1913/e.

[14] David Nadler and Hiro Lee Tanaka. "A stable infinity-category of Lagrangian cobordisms". In: *http://arXiv.org/abs/1109.4835* (). URL: http://arXiv.org/abs/1109.4835.

[15] John W. Milnor. "Construction of universal bundles. II." English. In: *Ann. Math. (2)* 63 (1956), pp. 430–436. ISSN: 0003-486X; 1939-8980.

[16] Jean-Pierre Magnot and Jordan Watts. "The diffeology of Milnor's classifying space". In: *arXiv:1606.06680* (2016).

[17] Jacob Lurie. *Higher topos theory.* English. Annals of Mathematics Studies 170. Princeton, NJ: Princeton University Press., 2009.

[18] Emily Riehl. "A model structure for quasi-categories". In: *http://www.math.harvard.edu/~eriehl/topic.pdf* (). URL: http://www.math.harvard.edu/~eriehl/topic.pdf.

[19] W.G. Dwyer and D.M. Kan. "Simplicial localization of Categories". In: *Journal of Pure and Applied algebra* (1980).

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