LOCALLY CONFORMALLY SYMPLECTIC DEFORMATION OF GROMOV NON-SQUEEZING

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Abstract. We prove one deformation theoretic extension of the Gromov non-squeezing phenomenon to lcs structures, or locally conformally symplectic structures, which suitably generalize both symplectic and contact structures. We also conjecture an analogue in lcs geometry of contact non-squeezing of Eliashberg-Polterovich and discuss other related questions.

1. Introduction

We study here some analogues of Gromov non-squeezing for locally conformally symplectic manifolds, which generalize both symplectic and contact manifolds. Let us recall the definition.

Definition 1.1. *A* **locally conformally symplectic manifold** *or* **lcs manifold** *is a smooth $2n$-fold M, with a* lcs *structure: a non-degenerate 2-form ω, with the property that for every $p \in M$ there is an open $U \ni p$ such that $\omega|_U = f_U \cdot \omega_U$, for some symplectic form ω_U defined on U and some smooth positive function f_U on U. In the case of our paper we always have $n \geq 2$, as in case $n = 1$ there are other candidates for what should be an* lcs *structure.*

These structures have recently come into focus, for example we have a fascinating recent theorem of Apostolov-Dloussky [1] that every complex surface with an odd first Betti number admits a natural compatible lcs structure. Without compatibility, a more general existence result of this form is in Eliashberg-Murphy [4].

A basic invariant of a lcs structure ω is the Lee class,

$$\alpha = \alpha_\omega \in H^1(M, \mathbb{R}),$$

which we now briefly describe. The class α has the following differential form representative, called the Lee form and also denoted by α for simplicity. If U is an open set so that $\omega|_U = f_U \cdot \omega_U$ for ω_U symplectic, and f_U a positive smooth function, then $\alpha = d(\ln f_U)$ on U. By a simple calculation this can be seen to give well-defined 1-form α, see also Lee [8]. The class α has the property that on the associated α-covering space $\widetilde{M}$, the lift $\widetilde{\omega}$ is globally conformally symplectic, that is $\widetilde{\omega} = f \cdot \omega_0$ with ω_0 symplectic and $f > 0$. By α-covering space we mean the covering space associated to the normal subgroup $\ker\langle\alpha, \cdot\rangle \subset \pi_1(M, x)$, where $\langle\alpha, \cdot\rangle : \pi_1(M, x) \to \mathbb{R}$ is the homomorphism

$$[\gamma] \mapsto \langle\alpha, [\gamma]\rangle = \int_{S^1} \gamma^*\alpha.$$

It is moreover immediate that for a lcs form ω

$$d\omega = \alpha \wedge \omega,$$

for α the Lee form as defined above. For some authors, the pair (ω, α) with α closed s.t. $d\omega = \alpha \wedge \omega$ is the definition of a lcs structure. This has the advantage of being interesting even in dimension 2, but in dimension at least 4 the Lee form is uniquely determined, so that there is no difference of our definition with this second definition.

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Let α be a closed 1-form on a smooth manifold M. The operator

$$d^\alpha : \Omega^k(M) \to \Omega^{k+1}(M),$$
$$d^\alpha(\eta) = d\eta - \alpha \wedge \eta$$

is called the Lichnerowicz differential. It satisfies

$$d^\alpha \circ d^\alpha = 0$$

so that we have an associated chain complex called the ***Lichnerowicz chain complex***. The following is one basic example of an lcs manifold.

Example 1 (Banyaga). Let (C, λ) be a contact $(2n+1)$-manifold where λ is a contact form:

$$\forall p \in C : \lambda \wedge \lambda^{2n}(p) \neq 0.$$

Take $M = C \times S^1$ with the 2-form

$$\omega_\lambda = d^\alpha \lambda$$

for $\alpha := pr_{S^1}^* d\theta$, $pr_{S^1} : C \times S^1 \to S^1$ the projection, and λ likewise the pull-back of λ by the projection $C \times S^1 \to C$. We call (M, ω_λ) as above the ***lcs-fication*** of (C, λ).

1.1. **Symplectic and** lcs **non-squeezing.** One of the most important to this day results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [7]. The most well known formulation of this is that there does not exist a symplectic embedding

$$B_R \hookrightarrow D_r^2 \times \mathbb{R}^{2n-2}$$

for $R > r$, with B_R the standard closed radius R ball in $\mathbb{R}^{2n}$ centered at 0, and D^2_r the radius r closed disk in $\mathbb{R}^2$. Gromov's non-squeezing is C^0 persistent in the following sense. The proof of this is subsumed by the proof of Theorem 4.1 which follows, but is more elementary. It is nevertheless still a new theorem, and its proof necessitates the use of the virtual moduli cycle. Our particular approach to the latter follows Pardon [11].

We say that a symplectic form ω on $M \times N$ is *split* if $\omega = \omega_1 \oplus \omega_2$ for symplectic forms ω_1, ω_2 on M respectively N.

Theorem 1.2. *Given $R > r$, there is an $\epsilon > 0$ s.t. for any symplectic form ω' on $M = S^2 \times T^{2n-2}$ C^0-close to a split symplectic form ω and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt] \in H_2(M),$$

(for $\langle , \rangle$ the usual pairing of homology and cohomology classes) there is no symplectic embedding $\phi : B_R \hookrightarrow (M, \omega')$.

On the other hand it is natural to ask if the above theorem continues to hold for general nearby forms. Or formally this translates to:

Question 1. Given $R > r$ and every $\epsilon > 0$ is there a (necessarily non-closed by above) 2-form ω' on $S^2 \times T^{2n-2}$ C^0 or even C^∞ ϵ-close to a split symplectic form ω, satisfying $\langle \omega, A \rangle = \pi r^2$, and such that there is an embedding $\phi : B_R \hookrightarrow S^2 \times T^{2n-2}$, with $\phi^* \omega' = \omega_{st}$? We likewise call such a map ϕ ***symplectic embedding***.

We cannot reduce this question to just applying Theorem 1.2. This is because:

(1) A symplectic form on a subdomain of the form $\phi(B_R) \subset M$ may not extend to a symplectic form on M (even if M has a symplectic form!).
(2) When an extension to a symplectic form on M does exist, it may not be C^0-close to a split form ω of the form above.

This appears to be a very difficult question, my opinion is that at least in the C^0 case the answer is yes, in part because it is difficult to imagine any obstruction, for example we no longer have Gromov-Witten theory for such a general ω'.

In Theorem 4.1 we show that if ω' is lcs then the answer to the above question is no, under a mild additional condition. One may think that recent work of Müller [12] may be related to the present discussion. But there seems to be no obvious such relation as pull-backs by diffeomorphisms of nearby forms may not be nearby. Hence, there is no way to go from nearby embeddings that we work with to ϵ-symplectic embeddings of Müller.

The following theorem is a more elementary precursor to Theorem 4.1. This is a theorem about rigidity of lcs structures relative to general non-degenerate 2-forms, rather than rigidity of lcs maps relative to say volume preserving maps. This is in contrast to the original Gromov non-squeezing which is first and foremost about rigidity of symplectic maps. (However, as stated in Theorem 1.2 Gromov non-squeezing can be extended to a statement about rigidity of symplectic structures relative to general non-degenerate 2-forms.)

Theorem 1.3. *Let ω be a split symplectic form on $M = S^2 \times T^{2n-2}$. Let A be as above with $\langle \omega, A \rangle = \pi r^2$. There is a full volume open subspace $U \subset M$, meaning that $vol_g(U) = vol_g(M)$ with respect to any Riemannian metric g, such that the following holds. Let $R > r$, then there is an $\epsilon > 0$ (depending only on R, r, ω) s.t. if $\{\omega_t\}$, $t \in [0,1]$, $\omega_0 = \omega$ is a C^1-continuous family of* lcs *forms on M, with $d_0(\omega_t, \omega_0) < \epsilon$ for all t, then there is no symplectic embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow U,$$

meaning an embedding ϕ such that $\phi^* \omega_1 = \omega_{st}$. Here $\omega_{st} = \sum_{i=1}^n dp_i \wedge dq_i$ is the standard symplectic form on $\mathbb{R}^{2n}$, with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$.

We shall see in Theorem 4.1 that U can be taken to be M, provided ϕ satisfies a certain mild complex linearity condition on its differential, whenever it intersects a fixed hypersurface in M, of a certain kind. The C^1 continuity is used to establish energy controls for certain pseudo-holomorphic curves, as Gromov-Witten theory behaves very differently in lcs setting. This is relaxed in Theorem 4.1 to certain $\mathcal{T}^0$-continuity, close to C^0-continuity. Relaxing this further to C^0 continuity would probably require substantially new ideas.

Note that Frechet smooth lcs deformations $\{\omega_t\}$ of our symplectic form ω, with Lee forms α_t likewise smoothly varying in t, are obstructed unless α_t are DeRham exact, as pointed out to me by Kevin Sackel. This can be verified by an elementary calculation by taking the t derivative at 0 of the equation:

$$d^\alpha \omega_t = \alpha_t \wedge \omega_t.$$

But our families are not required to be smooth so that non-trivial lcs deformations of a symplectic form may exist. This motivates the question:

Question 2. Do there exist (continuous) lcs deformations $\{(\omega_t, \alpha_t)\}$ of the standard product symplectic form on $S^2 \times T^{2n-2}$, α_t the Lee form of ω_t, so that α_t are not DeRham exact?

1.1.1. *Toward direct generalization of contact non-squeezing.* The Eliashberg-Kim-Polterovich contact non-squeezing theorem as stated by Fraser [5] has the following form. Let $C = R^{2(n-1)} \times S^1$, $S^1 = \mathbb{R}/\mathbb{Z}$, be the prequantization space of R^{2n-2}, or in other words the contact manifold with the contact form $d\theta - \lambda$, for $\lambda = \frac{1}{2}(ydx - xdy)$. Let B_R denote the open radius R ball in $\mathbb{R}^{2n-2}$, and $\overline{B}_R$ its topological closure.

Theorem 1.4 (Eliashberg-Kim-Polterovich [3], Fraser [5], Chiu [2])**.** *For $R \geq 1$ there is no contactomorphism $\phi : C \to C$, isotopic to the identity, so that $\phi(\overline{B}_R \times S^1) \subset B_R \times S^1$.*

A Hamiltonian conformal symplectomorphism of an lcs manifold (M, ω), which we just abbreviate by the short name: ***Hamiltonian lcs map***, is a lcs diffeomorphism ϕ_H generated analogously to the symplectic case by a smooth function $H : M \times [0,1] \to \mathbb{R}$. Specifically, we define the time dependent vector field X_t by:

$$\omega(X_t, \cdot) = d^\alpha H_t,$$

for α the Lee form, and then taking ϕ_H to be the time 1 flow map of $\{X_t\}$. For example, let $(C \times S^1, \omega_\lambda)$ be the lcs-fication of a contact manifold (C, λ) as above.

If $\forall t : H_t = -1$ then $d^\alpha(H_t) = \alpha$ and clearly

$$X_t = (R^\lambda \oplus 0),$$

as a section of $TC \oplus TS^1$ with R^λ the λ-Reeb vector field. The latter is the vector field defined by:

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

Thus in this case the associated flow is naturally induced by the Reeb flow. More generally, given a smooth contact isotopy $\{\phi_t\}$, $\phi_t : C \to C$ contactomorphism of a closed contact manifold C, s.t. $\phi_0 = id$, there is a similarly induced Hamiltonian isotopy $\{\widetilde{\phi}_t\}$ on the lcs-fication $C \times S^1$, s.t. $\{pr_C \circ \widetilde{\phi}_t\} = \{\phi_t\}$, for $pr_C : C \times S^1 \to C$ the projection. This is left as an exercise for the reader. Thus, the following conjecture is a direct generalization of the contact non-squeezing Theorem 1.4.

Conjecture 1 (see also Oh-Savelyev [10]). *If $R \geq 1$ there is no compactly supported, Hamiltonian lcs map*

$$\phi : \mathbb{R}^{2n} \times S^1 \times S^1 \to \mathbb{R}^{2n} \times S^1 \times S^1,$$

so that $\phi(\overline{U}) \subset U$, for $U := B_R \times S^1 \times S^1$ and $\overline{U}$ the topological closure.

2. Topology on the space of lcs forms and J-holomorphic curves

Theorem 1.3 is stated for the standard C^1 topology on the space of differential forms. However, this can be relaxed to use a certain natural C^0 style topology $\mathcal{T}_0$, specific to lcs forms. We will now discuss this. Let M be a closed smooth manifold of dimension at least 4. The metric topology $\mathcal{T}^0$ on the set $LCS(M)$ of smooth lcs 2-forms on M will be defined with respect to the following metric.

Definition 2.1. *Fix a Riemannian metric g on M. For $\omega_1, \omega_2 \in LCS(M)$ define*

$$d_0(\omega_1, \omega_2) = d_{C^0}(\omega_1, \omega_2) + d_{C^0}(\alpha_1, \alpha_2),$$

for α_i the Lee forms of ω_i and d_{C^0} the usual C^0 metric induced by g. In general d_{C^k} will denote the usual C^k metric. Recall that for $M = \mathbb{R}^n$ the C^k norm on the space $\Omega^m_c(\mathbb{R}^n)$ of compactly supported forms is defined as follows. Expand $\omega \in \Omega^m_c(\mathbb{R}^n)$ as $\omega = \Sigma_\alpha f_\alpha dx^\alpha$, so that $\{dx^\alpha(p)\}$ is the standard basis of $\Lambda^m(T^*_p\mathbb{R}^n)$ for each p. Then take the sum of the C^k norms of the coefficient functions f_α. For a general compact manifold M, a (class of equivalent) C^k metric on the space of forms, is analogously defined by fixing a Riemmanian metric g on M.

Proposition 2.2. *The metric d_0 on $LCS(M)$ is continuous with respect to the usual C^1 metric.*

Proof. The following argument was suggested to me by Vestislav Apostolov. Let $\Lambda(TM)$ be the vector bundle over M with fiber $\Lambda(TM)_p$ over p, the alternating tensor algebra $\Lambda(T_pM)$. Let $\Lambda^2(TM)$ denote the sub-bundle of degree 2 elements. Let $\Phi^2(M) = \Omega(\Lambda^2(TM))$ denote the space of C^∞ sections of $\Lambda^2(TM)$ with C^0 topology. Likewise, $\Lambda(T^*M)$ will denote the bundle whose fiber over p is the alternating tensor algebra $\Lambda(T^*_pM)$.

Let $\Theta^2(M)$ denote the space of non-degenerate C^∞ differential 2-forms on M with C^0 topology. We first construct a continuous map:

$$\phi : \Theta^2(M) \to \Phi^2(M).$$

Let ω be a non-degenerate 2-form, so that for each $p \in M$ we get an isomorphism $i_\omega : T_pM \to T^*_pM$, $i_\omega = \omega(v, \cdot)$. Let i^{-1}_ω denote the inverse of this map. Then for each $p \in M$ we have a bi-linear form ω^{-1}_p on $T^*_p(M)$ defined by $\omega^{-1}_p(\eta, \mu) = \eta(i^{-1}(\mu))$. This is readily seen to be skew-symmetric. Hence determines a section $\omega^{-1} \in \Phi^2(M)$. We then set $\phi(\omega) = \omega^{-1}$, so that ϕ is continuous by construction.

Now for $\omega \in LCS(M)$ define the one-form η on M as follows. Let $v \in T_pM$ then

$$\eta_p(v) = (d\omega)_p(v \wedge \phi(\omega)_p),$$

so that $v \wedge \phi(\omega)_p \in \Lambda^3(T_pM)$ and $(d\omega)_p \in \Lambda^3(T^*_pM)$ identified with a functional in $(\Lambda^3(T_pM))^*$. Taking a basis for T_pM so that ω_p in this basis is the the standard symplectic form, it is easily verified that

$$\forall p \in M : \eta_p = (n-1)\alpha_p,$$

for α the Lee form satisfying $d\omega = \alpha \wedge \omega$, and where $2n$ is the dimension of M. We have thus obtained a map $LCS(M) \to \Omega(T^*M)$, which takes an lcs form and produces its Lee form, and which is continuous with respect to the C^1 topology on $LCS(M)$ and the C^0 topology on the space of 1-forms. Clearly the result follows. $\square$

The following characterization of convergence will be helpful.

Lemma 2.3. *Let M be as above and let $\{\omega_k\} \subset LCS(M)$ be a sequence $\mathcal{T}^0$ converging to a symplectic form ω. Denote by $\{\widetilde{\omega}_k\}$ the lift sequence on the universal cover $\widetilde{M}$. Then there is a sequence $\{\widetilde{\omega}_k^{symp}\}$ of symplectic forms on $\widetilde{M}$, and a sequence $\{f_k\}$ of positive functions pointwise converging to 1, such that $\widetilde{\omega}_k = f_k \widetilde{\omega}_k^{symp}$.*

Proof. We may assume that M is connected. Let α_k be the Lee form of ω_k, and g_k functions on $\widetilde{M}$ defined by $g_k([p]) = \int_{[0,1]} p^* \alpha_k$, where the universal cover $\widetilde{M}$ is understood as the set of equivalence classes of paths p starting at a fixed $x_0 \in M$, with a pair p_1, p_2 equivalent if $p_1(1) = p_2(1)$ and $p_2^{-1} \cdot p_1$ is null-homotopic, where $\cdot$ is the path concatenation.

Then we get:

$$d\widetilde{\omega}_k = dg_k \wedge \widetilde{\omega}_k,$$

so that if we set $f_k := e^{g_k}$ then

$$d(f_k^{-1} \widetilde{\omega}_k) = 0.$$

Since by assumption $|\alpha_k|_{C^0} \to 0$, then pointwise $g_k \to 0$ and pointwise $f_k \to 1$, so that if we set

$$\widetilde{\omega}_k^{symp} := f_k^{-1} \widetilde{\omega}_k$$

then we are done. $\square$

Definition 2.4. *We say that a pair (ω, J) of an* lcs *form ω on M and an almost complex structure J on M are* **compatible** *if $\omega(\cdot, J\cdot)$ defines a J-invariant inner product on M. For other basic notions of J-holomorphic curves we refer the reader to* [9].

Theorem 2.5. *Let M be as above, $A \in H_2(M)$ fixed, and $\{\omega_t\}$, $t \in [0,1]$, a $\mathcal{T}^0$-continuous family of* lcs *forms on M. Let $\{J_t\}$ be a Frechet smooth family of almost complex structures, with J_t compatible with ω_t for each t. Let $D \subset \widetilde{M}$, with $\pi : \widetilde{M} \to M$ the universal cover of M, be a fundamental domain, and $K := \overline{D}$ its topological closure. Suppose that for each t, and for every $x \in \partial K$ (the topological boundary) there is a $\widetilde{J}_t$-holomorphic hyperplane H_x through x, with $H_x \subset K$, such that $\pi(H_x) \subset M$ is a closed submanifold and such that $A \cdot \pi_*([H_x]) \leq 0$. Define:*

$$e_t(u) := \int_{\mathbb{CP}^1} u^* \omega_t.$$

Then

$$\sup_{u,t} e_t(u) < \infty,$$

where the supremum is over all pairs (u,t), $u : \mathbb{CP}^1 \to M$ is J_t-holomorphic and in class A.

Proof.

Lemma 2.6. *Let M, A be as above, let $D \subset \widetilde{M}$, with $\pi : \widetilde{M} \to M$ the universal cover of M, be a fundamental domain, and $K := \overline{D}$ its topological closure. Let (ω, J) be a compatible* lcs *pair on M such that for every $x \in \partial K$ there is a $\widetilde{J}$-holomorphic (real codimension 2) hyperplane $H_x \subset K \subset \widetilde{M}$ through x, such that $\pi(H_x) \subset M$ is a closed submanifold and such that $A \cdot [\pi(H_x)] \leq 0$. Then any genus 0, J-holomorphic class A curve u in M has a lift $\widetilde{u}$ with image in K.*

Proof. For u as in the statement, let $\widetilde{u}$ be a lift intersecting the fundamental domain D, (as in the statement of main theorem). Suppose that $\widetilde{u}$ intersects ∂K, otherwise we already have $\operatorname{image} \widetilde{u} \subset K^\circ$, for K° the interior, since $\operatorname{image} \widetilde{u}$ is connected (and by elementary topology). Then $\widetilde{u}$ intersects u_x as in the statement, for some x. So u is a J-holomorphic map intersecting the closed hyperplane $\pi(H_x)$ with $A \cdot [\pi(H_x)] \leq 0$. By positivity of intersections [9, Section 2.6], which in this case is just a simple exercise, $\operatorname{image} u \subset \pi(H_x)$, and so $\operatorname{image} \widetilde{u} \subset H_x$. And so $\operatorname{image} \widetilde{u} \subset \partial K$. $\square$

Now, let $u : \mathbb{CP}^1 \to M$ be a J_t-holomorphic class A curve. By the lemma above u has a lift $\widetilde{u}$ contained in the compact $K \subset \widetilde{M}$. Then we have:

$$e_t(u) = \int_{\mathbb{CP}^1} \widetilde{u}^* \widetilde{\omega}_t \leq C_t \langle \widetilde{\omega}_t^{symp}, A \rangle,$$

where $\widetilde{\omega}_t = f_t \widetilde{\omega}_t^{symp}$, for $\widetilde{\omega}_t^{symp}$ symplectic on $\widetilde{M}$, and $f_t : \widetilde{M} \to \mathbb{R}$ positive function constructed as in the proof of Lemma 2.3, and where $C_t = \max_K f_t$. Since $\{\omega_t\}$ is continuous in $\mathcal{T}_0$, we have that $\{f_t\}$, $\{\widetilde{\omega}_t^{symp}\}$ are C_0 continuous families in t. In particular

$$C = \sup_t \max_K f_t$$

and

$$D = \sup_t \langle \widetilde{\omega}_{t'}^{symp}, A \rangle$$

are finite. And so

$$\sup_{(u,t)} e_t(u) \leq C \cdot D,$$

where the supremum is over all pairs (u, t), u is J_t-holomorphic, class A, curve in M as above. □

3. Quick review of genus 0 Gromov-Witten theory

Let M be a compact smooth manifold with a pair (ω, J) for ω a non-degenerate smooth 2-form and J an almost complex structure. We assume that $\omega(\cdot, J\cdot)$ is a J-invariant inner product on M. We will call the above data (M, ω, J) an ***almost symplectic triple***.

Let

$$\mathcal{M}_{0,n}(J, A) = \mathcal{M}_{0,n}(M, J, A)$$

denote the moduli space of isomorphism classes of class A, J-holomorphic curves in M, with domain the Riemann sphere, with n marked labeled points $\{x_1, \ldots x_n\}$. In other words, $\mathcal{M}_{0,n}(J, A)$ is the set of isomorphism classes of tuples $(u, \{x_1, \ldots, x_n\})$, where $u : \mathbb{CP}^1 \to M$ is a J-holomorphic map. Here an isomorphism between $(u_1, \{x_1, \ldots, x_n\})$ and $(u_2, \{x'_1, \ldots, x'_n\})$ is a biholomorphism $\phi : \mathbb{CP}^1 \to \mathbb{CP}^1$, s.t. $\phi(x_i) = x'_i$ and s.t. $u_2 \circ \phi = u_1$. Let

$$e_\omega : \mathcal{M}_{0,n}(J, A) \to \mathbb{R},$$

be the energy:

$$e_\omega([u]) := e_\omega(u) := \int_{\mathbb{CP}^1} u^* \omega,$$

where we take any representative u of the class $[u]$. (Note that this (up to a factor) is the L^2 energy of the map u with respect to appropriate inner products, see [9, Section 2.2]).

Notation 1. In what follows we usually neglect to distinguish classes and representatives. As this should be clear from context. So from now on we just write u.

Let $\{(M, \omega_t, J_t)\}$, $t \in [0, 1]$, be a family of almost symplectic triples with $\{(\omega_t, J_t)\}$ varying smoothly in t. We will say that $\{(M, \omega_t, J_t)\}$ is a ***smooth family of almost symplectic triples***. Given a smooth family of almost symplectic triples $\{(M, \omega_t, J_t)\}$, $t \in [0, 1]$, we denote by

$$\mathcal{M}_{0,n}(\{J_t\}, A)$$

the space of pairs (u, t), $u \in \mathcal{M}_{0,n}(J_t, A)$. (Dropping the marked points from the notation.)

The following is well known and follows by the same argument as [9, Theorem 5.6.6].

Theorem 3.1. *Let (M, ω, J) be as above. Then $\mathcal{M}_{0,n}(M, J, A)$ has a pre-compactification*

$$\overline{\mathcal{M}}_{0,n}(M, J, A),$$

by Kontsevich stable maps, with respect to the natural metrizable Gromov topology [9, Chapter 5.6]. *Moreover given $E > 0$, the subspace $\overline{\mathcal{M}}_{g,0}(J, A)_E \subset \overline{\mathcal{M}}_{g,0}(J, A)$ consisting of elements u with $e_\omega(u) \leq E$*

is compact. In other words $e = e_\omega$ is a proper function on $\overline{\mathcal{M}}_{g,0}(J, A)$. Similarly, if $\{(M, \omega_t, J_t)\}$ is a smooth family of almost symplectic triples, and we define

$$e : \overline{\mathcal{M}}_{0,n}(\{J_t\}, A) \to \mathbb{R}$$

by

$$e(u, t) = e_{\omega_t}(u),$$

then e is a proper function.

Thus the most basic situation where we can talk about Gromov-Witten "invariants" of (M, J) is when the energy function is bounded on $\overline{\mathcal{M}}_{g,0}(J, A)$. In this case $\overline{\mathcal{M}}_{g,n}(J, A)$ is compact, and has a virtual moduli cycle as in the original approach of Fukaya-Ono [6], or the more algebraic approach of Pardon [11]. So we may define, as usual, functionals called the Gromov-Witten invariants:

$$GW_{g,n}(A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}, \tag{3.2}$$

where $\overline{M}_{g,n}$ denotes the compactified moduli space of Riemann surfaces. Of course closed symplectic manifolds with any tame almost complex structure is one class of examples, where these functionals are defined, as in that case we have a priori bounds on the energy of holomorphic curves in a fixed class.

Even when defined, these functionals will not in general be J-invariant, but it is immediate, again by Pardon [11], that they are invariant for a smooth family $\{J_t\}$, $t \in [0, 1]$ such that the corresponding "cobordism moduli space": $\overline{\mathcal{M}}_{g,0}(\{J_t\}, A)$, is compact.

4. Main argument

We will first state and prove a more general result, from which Theorem 1.3 will be deduced.

Let $M = S^2 \times T^{2n-2}$. We have real codimension 1 hypersurfaces

$$\Sigma_i = S^2 \times (S^1 \times \ldots \times S^1 \times \{pt\} \times S^1 \times \ldots \times S^1) \subset M,$$

where the singleton $\{pt\} \subset S^1$ replaces the i'th factor of $T^{2n-2} = S^1 \times \ldots \times S^1$. The hypersurfaces Σ_i are naturally folliated by the symplectic submanifolds

$$M_\theta = S^2 \times (S^1 \times \ldots \times S^1 \times \{pt\} \times \{\theta\} \times S^1 \times \ldots \times S^1) \simeq S^2 \times T^{2n-2},$$

$\theta \in S^1$. We denote by $T^{fol}\Sigma_i \subset TM$, the distribution of vectors tangent to the leaves of the above mentioned folliation. In other words

$$T^{fol}\Sigma_i = \cup_\theta i_* TM_\theta,$$

where $i : M_\theta \to M$ are the inclusion maps. Set $\Sigma = \bigcup_i \Sigma_i$, and $U = M - \Sigma$.

Theorem 4.1. *Let ω be a split symplectic form on $M = S^2 \times T^{2n-2}$, let A be as above with $\langle \omega, A \rangle = \pi r^2$. Let $\{\omega_t\}$, $t \in [0, 1]$, $\omega_0 = \omega$ be a $\mathcal{T}^0$-continuous family of* lcs *forms on M. Set $R > r$, then there is an $\epsilon > 0$ (depending only on R, r, ω) s.t. if $d_0(\omega_t, \omega_0) < \epsilon$ for all t, then there is no symplectic embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow U,$$

meaning an embedding ϕ such that $\phi^*\omega_1 = \omega_{st}$. (Note that U is a full ω_1-volume subspace diffeomorphic to $S^2 \times \mathbb{R}^{2n-2}$. That is $vol_{\omega_1}(U) = vol_{\omega_1}(M)$.)

More generally, there is no symplectic embedding

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1),$$

s.t $\phi_* j$ preserves the bundle $T^{fol}\Sigma_i$, for j the standard almost complex structure on B_R, whenever $\phi(x) \in \Sigma_i$. In other words,

$$\phi_* j(T^{fol}\Sigma_i) \subset T^{fol}\Sigma_i \subset TM, \tag{4.2}$$

whenever $\phi(x) \in \Sigma_i$.

The second part of the theorem is in some ways more intriguing. Note that for any $\epsilon > 0$ there are certainly many volume preserving maps $\phi : (B_R, \omega_{st}) \to (M, \omega_1)$, with ω_1 as in the hypothesis, with image ϕ non-trivially intersecting $\Sigma = \bigcup_i \Sigma_i$ and satisfying the condition (4.2). (For example maps of the form $(z_1, z_2) \mapsto (\frac{1}{k} z_1, k z_2)$, in the complex coordinates. Assuming the ω_1-volume of T^{2n-2} is taken to be sufficiently large.)

Proof of Theorem 4.1. The second part of the theorem vacuously implies the first, and we proceed with the proof of the second part. Fix an $\epsilon' > 0$ s.t. any 2-form ω_1 on M, C^0 ϵ'-close to ω, is non-degenerate and is non-degenerate on the leaves of the folliation of each Σ_i, discussed prior to the formulation of the theorem. Suppose by contradiction that for every $\epsilon > 0$ there is a $\mathcal{T}^0$-continuous homotopy $\{\omega_t\}$ of lcs forms, with $\omega_0 = \omega$, such that $\forall t : d_0(\omega_t, \omega) < \epsilon$ and such that there exists a symplectic embedding

$$\phi : B_R \hookrightarrow (M, \omega_1),$$

s.t

$$\phi_* j(T^{fol}\Sigma_i) \subset T^{fol}\Sigma_i \subset TM,$$

whenever $\phi(x) \in \Sigma_i$.

Take $\epsilon < \epsilon'$, and let $\{\omega_t\}$ be as in the hypothesis above. In particular ω_t is an lcs form for each t, and is non-degenerate on Σ_i. Extend $\phi_* j$ to an ω_1-compatible almost complex structure J_1 on M, preserving $T^{fol}\Sigma_i$ for each i. We may then extend this to a family $\{J_t\}$ of almost complex structures on M, s.t. J_t is ω_t-compatible for each t, with J_0 is the standard split complex structure on M and such that J_t preserves $T^{fol}\Sigma_i$ for each t, i. The latter condition can be satisfied since Σ_i are ω_t-symplectic for each t. When $\phi(B_R)$ does not intersect Σ these conditions can be trivially satisfied, first find an extension J_1 of $\phi_* j$ preserving $T^{fol}\Sigma_i$ for each i. Then extend to a family $\{J_t\}$.

Then the family $\{(\omega_t, J_t)\}$ satisfies the hypothesis of Theorem 2.5 for the class $A = [S^2] \otimes [pt]$ as in the statement of the theorem we are proving. Then by Theorem 2.5 L^2 energy e is bounded on

$$C = \overline{\mathcal{M}}_{0,1}(\{J_t\}, A)$$

and hence C is compact by Theorem 3.1.

Now we have the classical Gromov-Witten invariant counting class A, J_0-holomorphic, genus 0 curves passing through a fixed point:

$$GW_{0,1}(A, J_0)([pt]) = 1,$$

whose calculation already appears in [7]. Then by compactness of C, and the discussion preceding the proof:

$$GW_{0,1}(A, J_1)([pt]) = 1.$$

In particular there is a class A J_1-holomorphic curve $u : \mathbb{CP}^1 \to M$ passing through $\phi(0)$.

By Lemma 2.6 we may choose a lift $\widetilde{u}$ of u to $\widetilde{M}$, with homology class $[\widetilde{u}]$ also denoted by A so that the image of $\widetilde{u}$ is contained in a compact set $K \subset \widetilde{M}$, (independent of the choice of $\{\omega_t\}, \{J_t\}$ satisfying above conditions). Let $\widetilde{\omega}_t^{symp}$ and f_t be as in Lemma 2.3, then by this lemma for every $\delta > 0$ we may find an $\epsilon > 0$ so that if $d_0(\omega_1, \omega) < \epsilon$ then $d_{C^0}(\widetilde{\omega}^{symp}, \widetilde{\omega}_1^{symp}) < \delta$ on K, and $\sup_K |f_1 - 1| < \delta$.

Let δ as above be chosen, and let ϵ correspond to this δ. Now we have:

$$|\langle \widetilde{\omega}_1^{symp}, A \rangle - \pi \cdot r^2| = |\langle \widetilde{\omega}_1^{symp}, A \rangle - \langle \widetilde{\omega}^{symp}, A \rangle| \leq \delta \pi \cdot r^2,$$

as $\langle \widetilde{\omega}^{symp}, A \rangle = \pi r^2$, and as $d_{C^0}(\widetilde{\omega}^{symp}, \widetilde{\omega}_1^{symp}) < \delta$. And we have

$$\max_K f_1 \leq 1 + \delta.$$

So choosing ϵ, δ appropriately we get

$$|\int_{\mathbb{CP}^1} u^* \omega_1 - \pi r^2| \leq |\max_K f_1 \langle \widetilde{\omega}_1^{symp}, A \rangle - \pi \cdot r^2| < \pi R^2 - \pi r^2.$$

Consequently,

$$\int_{\mathbb{CP}^1} u^* \omega_1 < \pi R^2.$$

We may then proceed exactly as in the now classical proof of Gromov [7] of the non-squeezing theorem to get a contradiction and finish the proof. A bit more specifically, $\phi^{-1}(\operatorname{image}\phi \cap \operatorname{image} u)$ is a minimal surface in B_R, with boundary on the boundary of B_R, and passing through $0 \in B_R$. By construction it has area strictly less then πR^2 which is impossible by the classical monotonicity theorem of differential geometry. □

Proof of Theorem 1.3. Set $U = M - \bigcup_i \Sigma_i$. Let ϵ be as given by the Theorem 4.1. By Proposition 2.2 there is a ϵ' s.t. whenever $\omega_0, \omega_1 \in LCS(M)$ are C^1 ϵ'-close, they are $\mathcal{T}_0$ ϵ-close.

Let $\{\omega_t\}$ be given as in the hypothesis, and such that $d_{C^1}(\omega_0, \omega_t) < \epsilon'$ for all t. By Proposition 2.2. $\{\omega_t\}$ is $\mathcal{T}^0$ continuous, and by the discussion above

$$\forall t : d_0(\omega_0, \omega_t) < \epsilon.$$

So applying Theorem 4.1 we obtain that there is no symplectic embedding $B_R \hookrightarrow (U, \omega_1)$. And so we are done. □

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