

## INSTABILITY OF GROMOV NON-SQUEEZING

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**ABSTRACT.** We show that the Gromov non-squeezing phenomenon can disappear after an arbitrarily small, general (non-symplectic)  $C^0$  perturbation of the symplectic form on the range. In particular the lcs (short for locally conformally symplectic) non-squeezing theorem in [2] is sharp, (in the sense that the lcs condition cannot be relaxed to just non-degeneracy.) We also suggest that in contrast to the  $C^0$  case, Gromov non-squeezing is generally  $C^\infty$  persistent.

One of the most fascinating early results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [1]. The original formulation of this is that there does not exist a symplectic embedding

$$B_R \hookrightarrow D^2(r) \times \mathbb{R}^{2n-2},$$

for  $R > r$ , with  $B_R$  the standard closed radius  $R$  ball in  $\mathbb{R}^{2n}$  centered at 0. Gromov's non-squeezing is partially  $C^0$  persistent in the following sense. We attribute this persistence to Gromov since it is just a simple generalization of his argument.

We say that a symplectic form  $\omega$  on  $M \times N$  is *split* if  $\omega = \omega_1 \oplus \omega_2$  for symplectic forms  $\omega_1, \omega_2$  on  $M$  respectively  $N$ .

**Theorem 0.1** (Gromov). *Given  $R > r$ , there is an  $\epsilon > 0$  s.t. for any pair of symplectic forms  $\omega, \omega'$  on  $M = S^2 \times T^{2n-2}$  with  $\omega$  split, and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt],$$

*if  $d_{C^0}(\omega, \omega') < \epsilon$  then there is no symplectic embedding*

$$\phi : B_R \hookrightarrow (M, \omega').$$

This theorem is generalized in author's [2] to lcs forms  $\omega'$ . We show here that this persistence disappears with respect to general  $C^0$  perturbations. In particular the theorem of [2] is a truly lcs phenomenon.

**Theorem 0.2.** *Let  $R > r$ ,  $\epsilon > 0$ , be given. Then there is a 2-form  $\omega'$  on  $M = S^2 \times T^{2n-2}$ , and a split symplectic form  $\omega$  on  $M$ , satisfying*

$$\langle \omega, A \rangle = \pi r^2,$$

*for  $A$  as above, satisfying  $d_{C^0}(\omega, \omega') < \epsilon$ , s.t. there is an embedding*

$$\phi : B_R \hookrightarrow M,$$

*with*

$$\phi^* \omega' = \omega_{st}.$$

*We call such an embedding **symplectic** in analogy with the classical symplectic case. Moreover,  $\phi$  can be chosen so that*

$$\phi(B_R) \subset (M - \bigcup_i \Sigma_i).$$

*Here  $\Sigma_i$  are the following hypersurfaces:*

$$\Sigma_i = S^2 \times (S^1 \times \dots \times S^1 \times \{pt\} \times S^1 \times \dots \times S^1) \subset M,$$

*where the singleton  $\{pt\} \subset S^1$  replaces the  $i$ 'th factor of  $T^{2n-2} = S^1 \times \dots \times S^1$ .*

*Proof.* Let  $R, r, \epsilon$  be given. Let

$$M' = [0, r]^2 \times \mathbb{R}^{2n-2}.$$

We first construct a 2-form  $\omega''$  on  $M'$ ,  $C^0$ -nearby to the standard symplectic form  $\omega$  and a symplectic embedding  $\phi : \text{Cube}(R) \rightarrow M'$ , where  $\text{Cube}(R)$  denotes the closed cube in  $\mathbb{R}^{2n}$  with side  $R$ .

For simplicity we take in what follows  $n = 2$ , with construction obviously generalizing to any  $n$ . Let  $p, q$  be the coordinates on

$$sq = [0, r]^2 \subset \mathbb{R}^2.$$

Let  $(p, q, s, t)$  be the natural coordinates on

$$M' = sq \times \mathbb{R}^2,$$

and  $g$  the standard Euclidean metric.

Let  $\{f_l\}_{l \in \mathbb{N}}$  be a collection of smooth functions, further specified as saw functions with  $l$  teeth in Figure 1, and satisfying:

- (1)  $\forall l : f_l(p) = 0$  for  $p$  in a neighborhood of  $0, r$ .
- (2)  $\forall l : |f_l|_{C^0} < R$ ,
- (3) The  $g$ -length of the sides of the saw teeth are  $R$ ,  $g$  being the standard Euclidean metric again.

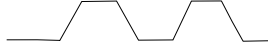


FIGURE 1. The corners are meant to be smoothed, so that this function is smooth. The saw is meant to be with uniform size teeth and uniform gaps between side edges of the teeth, these side edges have  $g$ -length  $R$ . The trailing edges have arbitrary non-zero length.

Define the following surface  $S_0^l$  in  $M'$ :

$$S_0^l = \{(p, q, f_l(p), 0) \in \mathbb{R}^4 \mid (p, q) \in sq\},$$

Then  $S_0^l$  is a  $\omega$ -symplectic surface whose  $\omega$ -orthogonal spaces are spanned by  $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ . Define

$$S_{s,t}^l := S_0^l + (0, 0, s, t), \quad 0 \leq s \leq R, 0 \leq t \leq R.$$

Then

$$C = C^l := \cup_{s,t} S_{s,t}^l$$

is a domain in  $M'$  that is diffeomorphic to the standard closed cube in  $\mathbb{R}^4$ , foliated by the surfaces  $S_{s,t}^l$ . Let  $\mathcal{F} \subset TC$  denote the 2-dimensional distribution corresponding to this foliation, that is  $\mathcal{F}(z)$  is the sub-space of vectors tangent to the leaf through  $z = (p, q, s, t)$ . And let  $V \subset TC$  denote the  $\omega$ -orthogonal distribution, that is the distribution with

$$V(p, q, s, t) = \text{span}\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right),$$

by the observations above.

Let  $\{h_0^l\}$ ,  $l$  as above, be a collection of smooth non-negative functions:

$$h_0^l : (S_0^l \subset \mathbb{R}^4) \rightarrow \mathbb{R},$$

satisfying:

- (1)  $|h_0^l|_{C^0} < \epsilon$ .
- (2)  $\int_{S_0^l} h_0^l dA_g \geq r \cdot R \cdot l \cdot \epsilon$ ,  $dA_g$  the area form on  $S_0^l$  induced by the restriction of  $g$  to  $S_0^l$ , with orientation coinciding with  $\omega$ -orientation.

More specifically, in terms of our saw functions, we may readily construct such an  $h_0^l$  by taking its support as in Figure 2. (The support condition is for simplicity.)

Thus if  $l$  is taken to be:

$$l_0 = \text{ceiling}\left(\frac{R}{r \cdot \epsilon}\right),$$

then

$$(0.3) \quad \int_{S_0^{l_0}} h_0^{l_0} dA_g > R^2.$$

We then assume that  $l = l_0$  is taken as above and it will no longer appear in notation.



FIGURE 2. A point  $z \in S_0^l$  is in the support of  $h_0^l$ , only if its image by the projection  $\mathbb{R}^4 \rightarrow \mathbb{R}^2, (p, q, s, t) \mapsto (p, s)$  is in the red regions of the figure, i.e. the sides of the teeth.

Define

$$h : C \rightarrow \mathbb{R}, \quad h(p, q, s, t) = h_0(p', q', s', t'),$$

where  $(p', q', s', t')$  is the unique point on  $S_0$  defined by the condition: if

$$(p, q, s, t) \in S_{s'', t''} \subset C$$

then

$$(p', q', s', t') = (p, q, s, t) - (0, 0, s'', t'').$$

Since  $C$  is smoothly foliated by  $\{S_{s,t}\}$  this defines a smooth function on  $C$ .

Let  $\omega_\epsilon$  be the 2-form on  $C$ , so that splitting

$$TC \simeq \mathcal{F} \oplus V,$$

stays  $\omega_\epsilon$ -orthogonal, and such that:

$$\forall z \in C, \forall v, w \in V(z) \subset T_z C : \omega_\epsilon(v, w) = \omega(v, w),$$

and

$$\forall z \in C, \forall v, w \in \mathcal{F}(z) \subset T_z C : \omega_\epsilon(v, w) = \omega(v, w) + h(z) \cdot \omega_g(v, w),$$

where  $\omega_g$  is the  $g$ -area 2-form (previously also  $dA_g$ ) on the corresponding leaf, with same orientation as  $\omega$ .

By (0.3) the  $\omega_\epsilon$ -area of each leaf  $S_{s,t}$  is at least  $R^2$ . Now, by construction (specifically properties of  $h$ , and property 1 of each  $f^l$ )  $\omega_\epsilon$  extends to a 2-form  $\omega''$  on  $M'$  coinciding with  $\omega$  outside  $N_\epsilon(C)$ , the open  $\epsilon$ -neighborhood in  $M'$  of  $C$ , and satisfying:

$$d_{C^0}(\omega'', \omega) < \epsilon.$$

Also by construction of  $f_l$  and property 2 in particular, if  $\epsilon < 1$ ,  $N_\epsilon(C) \subset K$  where  $K \subset M'$  is a fixed compact (in particular independent of  $\epsilon$ ). We can in fact take

$$K = sq \times [0, 1 + 3 \cdot R]^2.$$

Now fix a symplectic embedding

$$\phi_0 : [0, R]^2 \rightarrow (S_0, \omega''|_{S_0}),$$

(recall that the  $\omega_\epsilon$  area of  $S_0$  is by construction at least  $R^2$ ) and define

$$\phi : \text{Cube}(R) \rightarrow C$$

by

$$\phi(p, q, s, t) = \phi_0(p, q) + (0, 0, s, t).$$

Then by construction

$$\phi^* \omega'' = \omega_{st}.$$

Now since  $\omega'' = \omega$  outside  $K$ , we obviously get an induced 2-form  $\omega'$ , on  $M$ ,  $C^0$   $\epsilon$ -nearby to a split (in fact standard) symplectic form, s.t. there is a symplectic embedding:

$$\phi : (Cube(R), \omega_{st}) \rightarrow (M, \omega'),$$

and moreover

$$\text{image}(\phi) \subset M - \bigcup_i \Sigma_i.$$

And so we are done. □

We end the note with a conjecture:

**Conjecture 1.** *In contrast to  $C^0$  instability, Gromov non-squeezing is generally  $C^\infty$  persistent. That is given  $R > r$ , there is an  $\epsilon > 0$  s.t. for any pair of forms  $\omega, \omega'$  on  $M = S^2 \times T^{2n-2}$  with  $\omega$  split and symplectic, and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt],$$

*if  $d_{C^\infty}(\omega, \omega') < \epsilon$  then there is no symplectic embedding*

$$\phi : B_R \hookrightarrow (S^2 \times T^{2n-2}, \omega').$$

The only evidence for this at the moment is that trying to extend the proof above to  $C^\infty$ , (even  $C^1$ ) topology seems to run into insurmountable difficulty. Furthermore, it might be possible to recover just enough Gromov-Witten theory for general forms  $C^\infty$ -close to symplectic forms, to prove this conjecture.

#### REFERENCES

- [1] M. GROMOV, *Pseudo holomorphic curves in symplectic manifolds.*, Invent. Math., 82 (1985), pp. 307–347.
- [2] Y. SAVELYEV, *Conformal symplectic Weinstein conjecture and non-squeezing*, arXiv, (2021).

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