RELATING MEAN CURVATURE AND DIAMETER IN GENERAL RIEMANNIAN MANIFOLDS

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Abstract. Using a theorem of Topping we give a simple relation between mean curvature and intrinsic diameter for closed submanifolds of general compact Riemannian manifolds.

In [1] Topping gave via a concise but sophisticated argument based on ideas of Ricci flows a simple relation between intrinsic diameter and mean curvature for immersed submanifold of $\mathbb{R}^n$. Let us state it here:

Theorem 0.1 ([1]). *For Σ a smoothly immersed closed submanifold of $\mathbb{R}^m$ we have:*

$$\operatorname{diam}(\Sigma) \leq Const(m) \int_\Sigma |\boldsymbol{H}|^{m-1} \, dvol,$$

for $\boldsymbol{H}$ the mean curvature vector field along Σ, vol the volume measure induced by the standard ambient metric, and diam *the intrinsic diameter:* $\max_{x,y \in \Sigma} dist_{(\Sigma, g_{st})}(x, y)$.

Here we use the above and Nash embedding theorem to give via an otherwise elementary argument the following (mostly) simple relation between diameter and mean curvature of closed submanifolds of general compact Riemannian manifolds. In plain words it is the following: if the volume of a given closed immersed submanifold of a compact Riemannian manifold is "small" but diameter "large" then the mean curvature must be somewhere large. Here is the more precise statement.

Theorem 0.2. *Consider the set $S = S(C)$ of immersed $\Sigma \subset X$, for Σ a closed smooth m-manifold and (X, g) a fixed compact Riemannian manifold, with the magnitude of the mean curvature of Σ bounded from above by $C > 0$. Let $Vol(\Sigma)$ denote the g-volume, and* $\operatorname{diam}(\Sigma)$ *the (intrinsic) diameter in X, g. Then for all $\Sigma \in S$*

$$\operatorname{diam}(\Sigma) \leq F(g, C, m) \operatorname{vol}(\Sigma),$$

for F some function.

Proof. Pick an isometric Nash embedding N of (M, g) into $\mathbb{R}^n$, where n is large enough.

Lemma 0.3. *For all $\Sigma \in S(C)$ the magnitude of the mean curvature vector field along $N(\Sigma)$ in $\mathbb{R}^n$ is bounded from above by some C'.*

Proof. In what follows we conflate the notation for Σ and its images in $M, \mathbb{R}^n$. In other words we just think in terms of subspaces $\Sigma \subset M \subset \mathbb{R}^n$. Let h be the second fundamental form on T_pM:

$$h(v, w) = \tilde{\tilde{\nabla}}_v w - \nabla_v w,$$

where $\tilde{\tilde{\nabla}}$ is the Levi-Civita connection of $(\mathbb{R}^n, g_{st})$, ∇ is the Levi-Civita connection of (Σ, g) and where we locally extend $v, w \in T_p\Sigma$ to vector fields tangent to Σ. If $\dim \Sigma = m$, the mean curvature vector of Σ in $\mathbb{R}^n$ at p is given by:

$$\mathbf{H}(p) = \frac{1}{m} \sum_i h(e_i, e_i),$$

1991 *Mathematics Subject Classification.* 53C42.
Partially supported by PRODEP grant of SEP, Mexico.

where $\{e_i\}$ is an orthonormal basis for $T_p\Sigma$. Likewise $\widetilde{\nabla}$ will denote in what follows the Levi-Civita connection of (M, g). So we have:

$$\begin{aligned} m|\mathbf{H}(p)| = |\sum_i(\widetilde{\widetilde{\nabla}}_{e_i}e_i - \nabla_{e_i}e_i)| &= |\sum_i(\widetilde{\widetilde{\nabla}}_{e_i}e_i - \widetilde{\nabla}_{e_i}e_i + \widetilde{\nabla}_{e_i}e_i - \nabla_{e_i}e_i)| \\ &\leq |\sum_i(\widetilde{\nabla}_{e_i}e_i - \nabla_{e_i}e_i)| + mB \\ &\leq mC + mB, \end{aligned}$$

where $B = \sup_{e \in TM, |e|=1} |\widetilde{\widetilde{\nabla}}_e e - \widetilde{\nabla}_e e|$. □

Since Σ is closed we get by Topping's theorem:

$$\mathrm{diam}(N\Sigma) \leq Const(m) \int_\Sigma |\mathbf{H}_{\mathbb{R}^n}|^{m-1} \, dVol.$$

By the lemma above the function $|\mathbf{H}_{\mathbb{R}^n}|$ on Σ is universally (independently of f) bounded from above by some C'. So we get:

$$\mathrm{diam}(\Sigma) = \mathrm{diam}(N\Sigma) \leq Const(m) \cdot (C')^{m-1} \cdot \mathrm{vol}(\Sigma),$$

So we get the required inequality. □

Acknowledgements. I am grateful to Peter Topping for comments on earlier ideas.

References

[1] P. Topping, *Relating diameter and mean curvature for submanifolds of Euclidean space.*, Comment. Math. Helv., 83 (2008), pp. 539–546.

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