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INSTABILITY OF GROMOV NON-SQUEEZING

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ABSTRACT. We show that the Gromov non-squeezing phenomenon disappears after an arbitrarily small, general (non-symplectic) C^0 perturbation of the symplectic form on the range. In particular the lcs (short for locally conformally symplectic) non-squeezing theorem in [2] is sharp, (in the sense that the lcs condition cannot be relaxed to just non-degeneracy.)

One of the most fascinating early results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [1]. The original formulation of this is that there does not exist a symplectic embedding

$$B_R \hookrightarrow D^2(r) \times \mathbb{R}^{2n-2},$$

for $R > r$, with B_R the standard closed radius R ball in $\mathbb{R}^{2n}$ centered at 0. Gromov's non-squeezing is C^0 persistent in the following sense. We attribute this persistence to Gromov since it is just a simple generalization of his argument.

We say that a symplectic form ω on $M \times N$ is *split* if $\omega = \omega_1 \oplus \omega_2$ for symplectic forms ω_1, ω_2 on M respectively N.

Theorem 0.1 (Gromov)**.** *Given $R > r$, there is an $\epsilon > 0$ s.t. for any pair of symplectic forms ω, ω' on $M = S^2 \times T^{2n-2}$ with ω split, and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt],$$

if $d_{C^0}(\omega, \omega') < \epsilon$ then there is no symplectic embedding

$$\phi : B_R \hookrightarrow (M, \omega').$$

This theorem is generalized in author's [2] to lcs forms ω'. We show here that this persistence disappears with respect to general perturbations. In particular the theorem of [2] is a truly lcs phenomenon.

Theorem 0.2. *Let $R > r$, $\epsilon > 0$, be given. Then there is a 2-form ω' on $M = S^2 \times T^{2n-2}$, and a split symplectic form ω on M, satisfying*

$$\langle \omega, A \rangle = \pi r^2,$$

for A as above, satisfying $d_{C^0}(\omega, \omega') < \epsilon$, s.t. there is an embedding

$$\phi : B_R \hookrightarrow M,$$

with

$$\phi^* \omega' = \omega_{st}.$$

We call such an embedding **symplectic** *in analogy with the classical symplectic case. Moreover, ϕ can be chosen so that*

$$\phi(B_R) \subset (M - \bigcup_i \Sigma_i),$$

where Σ_i are certain hypersurfaces explained in the proof.

Proof. Let R, r, ϵ be given. Let

$$M' = [0, r]^2 \times \mathbb{R}^{2n-2}.$$

We first construct a 2-form ω'' on M', C^0-nearby to the standard symplectic form ω and a symplectic embedding $\phi : Cube(R) \to M'$, where $Cube(R)$ denotes the closed cube in $\mathbb{R}^{2n}$ with side R.

For simplicity we take in what follows $n = 2$, with construction obviously generalizing to any n. Let p, q be the coordinates on

$$sq = [0, r]^2 \subset \mathbb{R}^2.$$

Let (p, q, s, t) be the natural coordinates on

$$M' = sq \times \mathbb{R}^2,$$

and g the standard Euclidean metric.

Let $\{f_l\}_{l \in \mathbb{N}}$ be a collection of smooth functions, further specified in Figure 1, and satisfying:

(1) $\forall l : f_l(p) = 0$ for p in a neighborhood of $0, r$.
(2) $\forall l : |f_l|_{C^0} < R$.
(3) $\forall l : length_g(graph(f_l)) \geq 2 \cdot R \cdot l$, and g being the standard Euclidean metric again.

The specific functions f_l we have in mind are saw shaped as in Figure 1, with the number of teeth l.

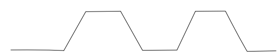

FIGURE 1. The corners are meant to be smoothed, so that this function is smooth. The saw is meant to be with uniform size teeth and uniforms gaps between side edges of the teeth, these sides have g-length R. The trailing edges have arbitrary non-zero length.

Define the following surface S^l_0 in M':

$$S^l_0 = \{(p, q, f_l(p), 0) \in \mathbb{R}^4 \mid (p, q) \in sq\},$$

Then S^l_0 is a ω-symplectic surface whose ω-orthogonal spaces are spanned by $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$. Define

$$S^l_{s,t} := S^l_0 + (0, 0, s, t), \quad 0 \leq s \leq R, 0 \leq t \leq R.$$

Then

$$C = C^l := \cup_{s,t} S^l_{s,t}$$

is a domain in M' that is diffeomorphic to the standard closed cube in $\mathbb{R}^4$, folliated by the surfaces $S^l_{s,t}$. Let $\mathcal{F} \subset TC^l$ denote the 2-dimensional distribution corresponding to this folliation, that is $\mathcal{F}(z)$ is the sub-space of vectors tangent to the leaf through $z = (p, q, s, t)$. And let $V \subset TC$ denote the ω-orthogonal distribution, that is the distribution with

$$V(p, q, s, t) = span(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}),$$

by the observations above.

Let $\{h^l_0\}$, l as above, be a collection of smooth non-negative functions:

$$h^l_0 : (S^l_0 \subset \mathbb{R}^4) \to \mathbb{R},$$

satisfying:

(1) $|h^l_0|_{C^0} < \epsilon$.
(2) $\int_{S^l_0} h^l_0 \, dA_g \geq r \cdot R \cdot l \cdot \epsilon$, dA_g the area form on S^l_0 induced by the restriction of g to S^l_0, with orientation coinciding with ω-orientation.

More specifically, in terms of our saw functions, we may readily construct such an h^l_0 by taking its support as in Figure 2. (The support condition is for simplicity.)

Thus if l is taken to be:

$$l_0 = ceiling(\frac{R}{r \cdot \epsilon}),$$

then

$$\int_{S^l_0} h^{l_0}_0 \, dA_g > R^2. \tag{0.3}$$

We then assume that $l = l_0$ is taken as above and it will no longer appear in notation.

Define

$$h : C \to \mathbb{R}, \quad h(p, q, s, t) = h_0(p', q', s', t'),$$

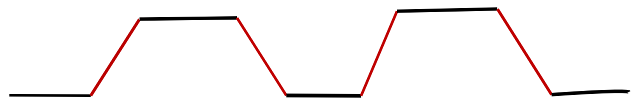

FIGURE 2. A point $z \in S_0^l$ is in the support of h_0^l, only if its image by the projection $\mathbb{R}^4 \to \mathbb{R}^2, (p,q,s,t) \mapsto (p,s)$ is in the red regions of the figure, i.e. the sides of the teeth.

where (p', q', s', t') is the unique point on S_0 defined by the condition: if

$$(p,q,s,t) \in S_{s'',t''} \subset C$$

then

$$(p', q', s', t') = (p,q,s,t) - (0,0,s'',t'').$$

Since C is smoothly folliated by $\{S_{s,t}\}$ this defines a smooth function on C.

Let ω_ϵ be the 2-form on C, so that splitting

$$TC \simeq \mathcal{F} \oplus V,$$

stays ω_ϵ-orthogonal, and such that:

$$\forall z \in C, \forall v, w \in V(z) \subset T_zC : \omega_\epsilon(v,w) = \omega(v,w),$$

and

$$\forall z \in C, \forall v, w \in \mathcal{F}(z) \subset T_zC : \omega_\epsilon(v,w) = \omega(v,w) + h(z) \cdot \omega_g(v,w),$$

where ω_g is the g-area 2-form (previously also dA_g) on the corresponding leaf, with same orientation as ω.

By (0.3) the ω_ϵ-area of each leaf $S_{s,t}$ is at least R^2. Now, by construction (specifically properties of h, and property 1 of each f^l) ω_ϵ extends to a 2-form ω'' on M' coinciding with ω outside $N_\epsilon(C)$, the open ϵ-neighborhood in M' of C, and satisfying:

$$d_{C^0}(\omega'', \omega) < \epsilon.$$

Also by construction of f_l and property 2 in particular, if $\epsilon < 1$, $N_\epsilon(C) \subset K$ where $K \subset M'$ is a fixed compact (in particular independent of ϵ). We can in fact take

$$K = sq \times [0, 1 + 3 \cdot R]^2.$$

Now fix a symplectic embedding

$$\phi_0 : [0,R]^2 \to (S_0, \omega''|_{S_0}),$$

(recall that the ω_ϵ area of S_0 is by construction at least R^2) and define

$$\phi : Cube(R) \to C$$

by

$$\phi(p,q,s,t) = \phi_0(p,q) + (0,0,s,t).$$

Then by construction

$$\phi^* \omega'' = \omega_{st}.$$

Now since $\omega'' = \omega$ outside K, we obviously get an induced 2-form ω', on M, C^0 ϵ-nearby to a split (in fact standard) symplectic form, s.t. there is a symplectic embedding:

$$\phi : (Cube(R), \omega_{st}) \to (M, \omega'),$$

and moreover

$$\text{image}(\phi) \subset M - \bigcup_i \Sigma_i,$$

where

$$\Sigma_i = S^2 \times (S^1 \times \ldots \times S^1 \times \{pt\} \times S^1 \times \ldots \times S^1) \subset M,$$

where the singleton $\{pt\} \subset S^1$ replaces the i'th factor of $T^{2n-2} = S^1 \times \ldots \times S^1$. And so we are done. $\square$

We end the note with a conjecture:

Conjecture 1. *Gromov non-sqeezing is generally C^∞ persistent. That is given $R > r$, there is an $\epsilon > 0$ s.t. for any pair of forms ω, ω' on $M = S^2 \times T^{2n-2}$ with ω split and symplectic, and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt],$$

if $d_{C^\infty}(\omega, \omega') < \epsilon$ then there is no symplectic embedding

$$\phi : B_R \hookrightarrow (S^2 \times T^{2n-2}, \omega').$$

The only evidence for this at the moment is that trying to extend the proof above to C^∞, (even C^1) topology seems to run into insurmountable difficulty. Furthermore, it might be possible to recover just enough Gromov-Witten theory for general forms C^∞-close to symplectic forms, to prove this conjecture.

References

[1] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds.*, Invent. Math., 82 (1985), pp. 307–347.

[2] Y. Savelyev, *Conformal symplectic Weinstein conjecture and non-squeezing*, arXiv, (2021).

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