November 8, 2023

SMOOTH SIMPLICIAL SETS AND UNIVERSAL CHERN-WEIL HOMOMORPHISM

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Abstract. We give a full construction of the universal Chern-Weil homomorphism for infinite dimensional (Fréchet) Lie groups. To this end, we introduce a basic geometric-categorical notion of a smooth simplicial set. Loosely, this is to Chen spaces as simplicial sets are to spaces. Given a Fréchet Lie group G, having the homotopy type of a CW complex, we construct a smooth Kan complex $BG^{\mathcal{U}}$, whose geometric realization $|BG^{\mathcal{U}}|$ is homotopy equivalent to the classical Milnor classifying space BG. The smooth Kan complex structure on $BG^{\mathcal{U}}$ is then used to construct the universal Chern-Weil homomorphism:

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG, \mathbb{R}),$$

satisfying naturality. As one basic example we give a full statement and proof of a conjecture of Reznikov, which in particular gives an elementary proof of a theorem of Kedra-McDuff, on the topology of $BHam(\mathbb{CP}^n)$. We also give a construction of the universal coupling class for all, possibly non-compact, symplectic manifolds.

Contents

1. Introduction

We introduce the notion of a smooth simplicial set, which is most directly an analogue in simplicial sets of Chen spaces [2], and less directly of diffeological spaces of Souriau [34]. The Chen/diffeological spaces are perhaps the most basic notions of a "smooth space".

The language of smooth simplicial sets turn out to be a powerful tool to resolve the problem of the construction of the universal Chern-Weil homomorphism for infinite dimensional Lie groups, (Banach or Fréchet). This has been open since Milnor's construction of universal bundles [23], which in particular produces universal G-bundles for infinite dimensional Lie groups. For finite dimensional Lie groups the universal Chern-Weil homomorphism has been studied for instance by Bott [1], it's uniqueness has been studied by Freed-Hopkins [6].

One problem of topology is the construction of a "smooth structure" on the classifying space BG of a Fréchet Lie group G. There are specific requirements for what such a notion of a smooth structure should entail. At the very least we hope to be able to carry out Chern-Weil theory universally on BG. That is we want a "purely" differential geometric construction of the Chern-Weil homomorphism:

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG, \mathbb{R}),$$

with $\mathbb{R}[\mathfrak{g}]^G$ denoting Ad_G invariant (required to be continuous when $\mathfrak{g}$ is infinite dimensional) polynomials on the Lie algebra $\mathfrak{g}$ of G. When G is a classical Lie group, BG can be written as a suitable colimit of smooth manifolds and so in that case the existence of the universal Chern-Weil homomorphism is classical.

One candidate for a smooth structure on BG is some kind of diffeology. For example Magnot and Watts [20] construct a natural diffeology on the Milnor classifying space BG. Another approach to this is contained in Christensen-Wu [3], where the authors also state their plan to develop some kind of universal Chern-Weil theory in the future.

A further specific possible requirement for the above discussed "smooth structures", is that the smooth singular simplicial set $BG_\bullet$ should have a geometric realization weakly homotopy equivalent to BG. See for instance [16] for one approach to this particular problem in the context of diffeologies. This kind of requirement is crucial for instance in the author's [32], which may be understood as a kind of "quantum Chern-Weil theory" on $BHam(M, \omega)$, for $Ham(M, \omega)$ the group of Hamiltonian symplectomorphisms of a symplectic manifold. In the language of smooth simplicial sets, the analogue of this latter requirement is always trivially satisfied. The specific content of this is Proposition 3.7.

The structure of a smooth simplicial set is initially more flexible than a space with diffeology, but with further conditions, like the Kan condition, can become forcing. Given a Fréchet Lie group G, we construct, for each choice of a particular kind of Grothendieck universe $\mathcal{U}$, a smooth simplicial set $BG^{\mathcal{U}}$ with a specific classifying property, analogous to the classifying property of BG, but relative to $\mathcal{U}$. We note that this is *not* the Milnor construction, indeed the homotopy type of the geometric realization $|BG^{\mathcal{U}}|$ is a priori dependent on $\mathcal{U}$.

The simplicial set $BG^{\mathcal{U}}$ is moreover a Kan complex, and so is a basic example of a smooth Kan complex. We then show that if G in addition has the homotopy type of a CW complex then the geometric realization $|BG^{\mathcal{U}}|$ is homotopy equivalent to BG, in particular $\mathcal{U}$ dependence disappears.

All the dreams of "smoothness" mentioned above then in some sense hold true for $BG^{\mathcal{U}}$ via its smooth Kan complex structure. In particular, as one immediate application we get:

Theorem 1.1. *Let G be a Fréchet Lie group having the homotopy type of a CW complex, then there is a universal Chern-Weil algebra homomorphism:*

$$cw : \mathbb{R}[\mathfrak{g}]^G \to H^*(BG, \mathbb{R}).$$

This is natural, so that if $P \to Y$ is a smooth G-bundle and

$$cw^P : \mathbb{R}[\mathfrak{g}]^G \to H^*(Y, \mathbb{R})$$

is the associated classical Chern-Weil homomorphism, then

$$cw^P = (f_P)^* \circ cw,$$

for $f_P : Y \to BG$ the classifying map of P.

Remark 1.2. *The theorem above may have an extension to diffeological groups. We need enough structure on G for a suitable lie algebra, to define curvature and Chern-Weil theory. More specifically, we would need the structure of tangent bundle TG, and suitable left invariant vector fields whose flows have unique existence for short time. We would also need an analogue of the main theorem Müller-Wockel [25] for diffeological bundles, as this is what we use to transfer Chern-Weil theory from $BG^{\mathcal{U}}$ to BG.*

Here is one concrete example. Let $\mathcal{H} = Ham(M, \omega)$ denote the Fréchet Lie group (with its C^∞ topology) of compactly supported Hamiltonian symplectomorphisms of some symplectic manifold. Let $\mathfrak{h}$ denote its lie algebra. To remind the reader, when M is compact $\mathfrak{h}$ is naturally isomorphic to the space of mean 0 smooth functions on M, and otherwise it is the space of all smooth compactly supported functions. More details are in Section 11. In [29] Reznikov defined $Ad_{\mathcal{H}}$-invariant polynomials $\{r_k\}_{k \geq 1}$ on the Lie algebra $\mathfrak{h}$. (Note that the case $k = 1$ is only interesting when M is non-compact, in this setting.) By classical Chern-Weil theory we get cohomology classes $c^{r_k}(P) \in H^{2k}(X, \mathbb{R})$ for any smooth $\mathcal{H}$-bundle P over X. Using Theorem 1.1 we get:

Corollary 1.3. *There are universal Reznikov cohomology classes $c^{r_k} \in H^{2k}(B\mathcal{H}, \mathbb{R})$, satisfying naturality. That is, let $Z \to Y$ be a smooth principal $\mathcal{H}$-bundle. Let $c^{r_k}(Z) \in H^{2k}(Y)$ denote the Reznikov class. Then*

$$f_Z^* c^{r_k} = c^{r_k}(Z),$$

where $f_Z : Y \to B\mathcal{H}$ is the classifying map of the underlying topological $\mathcal{H}$-bundle.

This is an explicit form of a statement asserted by Reznikov [29, page 12, arxiv version] (although his M is compact) on the extension of his classes to the universal level on $B\mathcal{H}$. This assertion is left without proof and he died shortly after, so that we may not know what he had in mind.

Remark 1.4. *Reznikov ostensibly uses the above universality in the proof of* [29, Theorem 1.5 (arxiv version)]. *However, this theorem does not actually need universality provided we work over a smooth base, it is simply a statement of naturality of his classes, which actually follows by classical naturality of Chern-Weil classes in general (cf. Lemma 7.2 which discusses a more general claim of this sort). So that so long as we restrict to smooth base, which while not explicitly stated is apparently intended by Reznikov, his paper is complete, (all of the statements are provable by the methods of the paper).*

Likewise, we obtain a purely differential geometric proof that the Guillemin-Sternberg-Lerman coupling class $\mathfrak{c}(P) \in H^2(P)$ [7], [21] of a Hamiltonian fibration (Definition 11.1) has a universal representative. Specifically, let $M^{\mathcal{H}}$ denote the M-fibration associated to the universal principal $\mathcal{H}$-fibration $\mathcal{E} \to B\mathcal{H}$. (In other words the universal Hamiltonian M-bundle.)

Theorem 1.5. *There is a cohomology class $\mathfrak{c} \in H^2(M^{\mathcal{H}})$ so that if $P \to X$ is a smooth Hamiltonian M-fibration and $f_P : X \to B\mathcal{H}$ is its classifying map then $f_P^* \mathfrak{c} = \mathfrak{c}(P)$.*

In the special case when M is closed the above existence can be obtained by homotopical techniques, Kedra-McDuff [13, Proposition 3.1].

We now describe one basic application. Let $Symp(\mathbb{CP}^k)$ denote the group of symplectomorphisms of $\mathbb{CP}^k$, that is diffeomorphisms $\phi : \mathbb{CP}^k \to \mathbb{CP}^k$ s.t. $\phi^* \omega_0 = \omega_0$ for ω_0 the Fubini-Study symplectic 2-form on $\mathbb{CP}^k$. The following theorem was obtained by Kedra-McDuff with some homotopy theory techniques. Using Corollary 1.3 we will give a purely differential geometric proof.

Theorem 1.6 (Kedra-McDuff [13])**.** *The natural map*

$$i : BPU(n) \to BSymp(\mathbb{CP}^{n-1})$$

induces an injection on rational homology for all $n \geq 2$.

The first result in this direction is due to Reznikov himself as he proves that:

$$i_* : \pi_k(BPU(n)) \otimes \mathbb{R} \to \pi_k(BSymp(\mathbb{CP}^{n-1}, \omega)) \otimes \mathbb{R} \tag{1.1}$$

is injective. More history and background surrounding these theorems is in Sections 9 and 10.

One other basic example of Fréchet Lie groups with a wealth of invariant polynomials on the lie algebra, are the loop groups $LG, \Omega G$, for G any Lie group, and LG the free loop space, and ΩG the based loop space at id. See for instance [27] for related computations. It is worth pointing out that loop groups are prominent in conformal field theory, see for instance [28], for the foundation of the subject. Other examples of infinite dimensional Chern-Weil theory include: [19], [22], [30], [18].

We note that the Chern-Weil homomorphism is known to be an isomorphism when G is either compact or semi-simple. For Fréchet Lie groups there are in particular interesting counterexamples to the Chern-Weil homomorphism to be an isomorphism It would be very interesting to characterize when it is an isomorphism in generality of all Fréchet groups.

We end with one natural question.

Question 1.7. *Our argument is formalized in ZFC + Grothendieck's axiom of universes, where ZFC is Zermelo-Fraenkel set theory plus axiom of choice. Does theorem 1.1 have a proof in ZFC?*

Probably the answer is yes, on the other hand as communicated to me by Dennis Sullivan there are known set theoretical (ZFC) issues with some questions on universal characteristic and secondary characteristic classes. So that the answer of no may be possible.

1.1. **Acknowledgements.** I am grateful to Daniel Freed, Yael Karshon, Dennis Sullivan, Egor Shelukhin, and Jean-Pierre Magnot, for comments and questions. I am also grateful to Dusa McDuff for explaining the proof of Corollary 1.6 in [13].

2. Preliminaries and notation

We denote by Δ the simplex category, i.e. the category with objects finite sets: $[n] = \{0, 1, \ldots, n\}$, with $\hom_\Delta([n], [m])$ non-decreasing maps. A simplicial set X is a functor

$$X : \Delta \to Set^{op}.$$

We usually write $X(n)$ instead of $X([n])$, and this is called the set of n-simplices of X. Given a collection of sets $\{X(n)\}_{n \in \mathbb{N}}$, by a ***simplicial structure*** we will mean the extension of this data to a functor: $X : \Delta \to Set^{op}$.

Δ^d_{simp} will denote a particular simplicial set: the standard representable d-simplex, with

$$\Delta^d_{simp}(n) = hom_\Delta([n],[d]).$$

The element of $\Delta^d_{simp}(0)$ corresponding to the map $i_k : [0] \to [d]$, $i_k(0) = k$ will usually be denoted by just k.

A morphism or ***map of simplicial sets or simplicial map*** $f : X \to Y$ is a natural transformation f of the corresponding functors. The category of simplicial sets will be denoted by $s - Set$.

By a d-simplex Σ of a simplicial set X, we may mean, interchangeably, either the element in $X(d)$ or the map of simplicial sets:

$$\Sigma : \Delta^d_{simp} \to X,$$

uniquely corresponding to Σ via the Yoneda lemma. If we write Σ^d for a simplex of X, it is implied that it is a d-simplex.

With the above identification if $f : X \to Y$ is a map of simplicial sets then

$$f(\Sigma) = f \circ \Sigma. \tag{2.1}$$

2.1. Topological simplices and smooth singular simplicial sets.

Let Δ^d be the topological d-simplex, i.e.

$$\Delta^d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \,|\, x_1 + \ldots + x_d \leq 1, \text{ and } \forall i : x_i \geq 0\}.$$

The vertices of Δ^d will be assumed ordered in the standard way.

Definition 2.1. *Let X be a smooth manifold with corners. We say that a map $\sigma : \Delta^n \to X$ is smooth if it smooth as a map of manifolds with corners. In particular $\sigma : \Delta^n \to \Delta^d$ is smooth iff it has an extension to a standard smooth map from a neighborhood in $\mathbb{R}^n$ of Δ^n. We say that a smooth $\sigma : \Delta^n \to X$ is* **collared** *if there is a neighborhood $U \supset \partial \Delta^n$ in Δ^n, such that $\sigma|_U = \sigma \circ ret$ for $ret : U \to \partial \Delta^n$ some smooth retraction. Here ret being smooth, again means that it has a smooth extension.*

We denote by $\Delta^d_\bullet$ the smooth singular simplicial set of Δ^d, i.e. $\Delta^d_\bullet(k)$ is the set of smooth maps

$$\sigma : \Delta^k \to \Delta^d.$$

We call an affine map $\Delta^k \to \Delta^d$ taking vertices to vertices in a order preserving way ***simplicial***. And we denote by

$$\Delta^d_{simp} \subset \Delta^d_\bullet$$

the sub-simplicial set consisting of simplicial maps. That is $\Delta^d_{simp}(k)$ is the set of simplicial maps $\Delta^k \to \Delta^d$.

Note that Δ^d_{simp} is naturally isomorphic to the standard representable d-simplex Δ^d_{simp} as previously defined, so that this abuse of notation should not cause issues. Thus we may also understand Δ as the category with objects topological simplices Δ^d, $d \geq 0$ and morphisms simplicial maps.

Notation 2.2. *A morphism* $m \in hom_{\Delta}([n],[k])$ *uniquely corresponds to a simplicial map* $\Delta^n_{simp} \to \Delta^k_{simp}$*, which uniquely corresponds to a simplicial map in the above sense* $\Delta^n \to \Delta^k$*. The correspondence is by taking the maps* $\Delta^n_{simp} \to \Delta^k_{simp}$*,* $\Delta^n \to \Delta^k$*, to be determined by the map of the vertices corresponding to* m*. We will not notationally distinguish these corresponding morphisms. So that* m *may simultaneously refer to all of the above morphisms.*

2.2. **The simplex category of a simplicial set.**

Definition 2.3. *For* X *a simplicial set,* $\Delta(X)$ *will denote a certain over category in* $s-Set$ *called the* **simplex category of** X*. This is the category whose set of objects* $\operatorname{obj}\Delta(X)$ *is the set of simplices*

$$\Sigma : \Delta^d_{simp} \to X, \quad d \geq 0$$

and morphisms $f : \Sigma_1 \to \Sigma_2$*, commutative diagrams in* $s-Set$*:*

$$\begin{tikzcd} \Delta^d_{simp} \arrow[r, "\widetilde{f}"] \arrow[dr, "\Sigma_1"'] & \Delta^n_{simp} \arrow[d, "\Sigma_2"] \\ & X \end{tikzcd} \tag{2.2}$$

with top arrow a simplicial map, which we denote by $\widetilde{f}$*. An object* $\Sigma : \Delta^d_{simp} \to X$ *is likewise called a* d*-simplex, and such a* Σ *may be said to have degree* d*. As noted in the paragraph before the definition, the degree* d *may also be specified by a superscript.*

Definition 2.4. *We say that* $\Sigma^n \in \Delta(X)$ *is* **non-degenerate** *if there is no morphism* $f : \Sigma^n \to \Sigma^m$ *in* $\Delta(X)$ *s.t.* $m < n$*.*

There is a forgetful functor

$$T : \Delta(X) \to \Delta,$$

$T(\Sigma^d) = \Delta^d_{simp}$, $T(f) = \widetilde{f}$. We denote by $\Delta^{inj}(X) \subset \Delta(X)$ the sub-category with same objects, and morphisms f such that $\widetilde{f}$ are monomorphisms, i.e. are face inclusions.

2.3. **Geometric realization.** Let Top be the category of topological spaces. Let X be a simplicial set, then define as usual the ***geometric realization*** of X by the colimit in Top:

$$|X| := \operatorname{colim}_{\Delta(X)} T,$$

for $T : \Delta(X) \to \Delta \subset Top$ as above, understanding Δ as a subcategory of Top as previously explained.

3. Smooth simplicial sets

If

$$\sigma : \Delta^d \to \Delta^n$$

is a smooth map we then have an induced map of simplicial sets

$$\sigma_\bullet : \Delta^d_\bullet \to \Delta^n_\bullet, \tag{3.1}$$

defined by

$$\sigma_\bullet(\rho) = \sigma \circ \rho.$$

We now give a pair of equivalent definitions of smooth simplicial sets. The first is more hands on, and has a close connection to the definition of Chen/diffeological spaces, while the second is more conceptual/categorical.

Definition 3.1 (First definition)**.** *A* **smooth simplicial set** *is a data consisting of:*

(1) *A simplicial set X.*

(2) *For each $\Sigma : \Delta^n_{simp} \to X$ an n-simplex, there is an assigned map of simplicial sets*

$$g(\Sigma) : \Delta^n_\bullet \to X,$$

which satisfies:

$$g(\Sigma)|_{\Delta^n_{simp}} = \Sigma. \tag{3.2}$$

We abbreviate $g(\Sigma)$ by Σ_*, when there is no need to disambiguate which structure g is meant.

(3) *The following property will be called* **push-forward functoriality***:*

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_\bullet$$

where $\sigma : \Delta^k \to \Delta^d$ is a k-simplex of $\Delta^d_\bullet$, and where Σ as before is a d-simplex of X.

Thus, formally a smooth simplicial set is a 2-tuple (X, g), satisfying the axioms above. When there is no need to disambiguate we omit specifying g.

Definition 3.2. *A* **smooth** *map between smooth simplicial sets*

$$(X_1, g_1), (X_2, g_2)$$

is a simplicial map

$$f : X_1 \to X_2,$$

which satisfies the condition:

$$\forall n \in \mathbb{N}\, \forall \Sigma \in X_1(n) : g_2(f(\Sigma)) = f \circ g_1(\Sigma), \tag{3.3}$$

or more succinctly:

$$\forall n \in \mathbb{N}\, \forall \Sigma \in X_1(n) : (f(\Sigma))_* = f \circ \Sigma_*.$$

We may also write $f(\Sigma)_*$, instead of $(f(\Sigma))_*$, but the former notation is sometimes unclear.

A ***diffeomorphism*** between smooth simplicial sets is defined to be a smooth map, with a smooth inverse.

Now let Δ^{sm} denote the category whose set of objects is $\mathbb{N}$. And such that $\hom_{\Delta^{sm}}(k, n)$ is the set of smooth maps $\Delta^k \to \Delta^n$.

Definition 3.3 (Second definition)**.** *A* **smooth simplicial set** *X is a functor $X : \Delta^{sm} \to Set^{op}$. A smooth map $f : X \to Y$ of smooth simplicial sets is defined to be a natural transformation from the functor X to Y.*

The equivalence of the above definitions is established further ahead.

Remark 3.4. *Ultimately the second definition is simpler to work with, as we can lean more on category theory. In particular, some of the technical results ahead are just incarnations of the Yoneda lemma and other such tools. However, here we use the first definition, as it is simpler to relate it to the existing theory of diffeological spaces, and our primary audience is differential geometers.*

Example 3.5 (The tautological smooth simplicial set)**.** *$\Delta^n_\bullet$ has a tautological smooth simplicial set structure, where*

$$g(\Sigma) = \Sigma_\bullet,$$

for $\Sigma : \Delta^k \to \Delta^n$ a smooth map, hence a k-simplex of $\Delta^n_\bullet$, and where $\Sigma_\bullet$ is as in (3.1).

Lemma 3.6. *Let X be a smooth simplicial set and $\Sigma : \Delta^n_{simp} \to X$ an n-simplex. Let $\Sigma_* : \Delta^n_\bullet \to X$ be the induced simplicial map. Then Σ_* is smooth with respect to the tautological smooth simplicial set structure on $\Delta^n_\bullet$ as above.*

Proof. Let σ be a k-simplex of $\Delta^n_\bullet$, so $\sigma : \Delta^k \to \Delta^n$ is a smooth map, we need that

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_*.$$

Now $\sigma_* = \sigma_\bullet$, by definition of the tautological smooth structure on $\Delta^n_\bullet$. So we have:

$$\begin{aligned}(\Sigma_*(\sigma))_* &= \Sigma_*(\sigma) \circ \sigma_\bullet \text{ by Axiom 3} \\ &= \Sigma_*(\sigma) \circ \sigma_*.\end{aligned}$$

□

Proposition 3.7. *The set of n-simplices of a smooth simplicial set X is naturally isomorphic to the set of smooth maps $\Delta^n_\bullet \to X$. In fact, define $X_\bullet$ to be the simplicial set whose n-simplices are smooth maps $\Delta^n_\bullet \to X$, and so that if $i : [m] \to [n]$ is a morphism in Δ then*

$$X_\bullet(i) : X(n) \to X(m)$$

is the "pull-back" map:

$$X_\bullet(i)(\Sigma) = \Sigma \circ i_\bullet,$$

for $i_\bullet : \Delta^m_\bullet \to \Delta^n_\bullet$ the induced map. Then $X_\bullet$ is naturally isomorphic to X.

Proof. Given a simplex $\rho : \Delta^n_{simp} \to X$, we have a uniquely associated to it, by the lemma above, smooth map $\rho_* : \Delta^n_\bullet \to X$. Conversely, suppose we are given a smooth map $m : \Delta^n_\bullet \to X$. Then we get an n-simplex $\rho_m := m|_{\Delta^n_{simp}}$. Let $id^n : \Delta^n \to \Delta^n$ be the identity map. We have that

$$\begin{aligned} m &= m \circ id^n_\bullet = m \circ id^n_* \\ &= (m(id^n))_* \text{ as } m \text{ is smooth} \\ &= (\rho_m(id^n))_* \text{ trivially by definition of } \rho_m \\ &= (\rho_m)_* \circ id^n_* \text{ as } (\rho_m)_* \text{ is smooth by Lemma 3.6} \\ &= (\rho_m)_*. \end{aligned}$$

Thus, the map $I_n(\rho) = \rho_*$, from the set of n-simplices of X to the set of smooth maps $\Delta^n_\bullet \to X$, is bijective.

Given an element $m \in hom_\Delta([n],[d])$, let $m_{simp} : \Delta^n_{simp} \to \Delta^d_{simp}$ denote the corresponding natural transformation. Then the corresponding map $X(m) : X(d) \to X(n)$ is $\rho \mapsto \rho \circ m_{simp}$, for $\rho : \Delta^n_{simp} \to X$ as above.

With that in mind, the diagram below commutes

$$\begin{tikzcd} X(d) \arrow[r, "X(m)"] \arrow[d, "I_d"] & X(n) \arrow[d, "I_n"] \\ X_\bullet(d) \arrow[r, "X_\bullet(m)"] & X_\bullet(n), \end{tikzcd}$$

as

$$\begin{aligned} X_\bullet(m) \circ I_d(\rho) &= X_\bullet(m)(\rho_*) \\ &= m_\bullet \circ \rho_* \end{aligned}$$

while

$$\begin{aligned} I_n \circ X(m)(\rho) &= (m_{simp} \circ \rho)_* \\ &= (m_\bullet \circ \rho)_* \text{ by (3.2)} \\ &= (m_\bullet(\rho))_* \text{ by (2.1)} \\ &= m_\bullet \circ \rho_* \text{ as } m_\bullet \text{ is smooth.} \end{aligned}$$

Thus I is a natural transformation and is an isomorphism of simplicial sets $I : X \to X_\bullet$. ☐

Lemma 3.8. *Given a smooth $m : \Delta^d_\bullet \to \Delta^n_\bullet$ there is a unique smooth map $f : \Delta^d \to \Delta^n$ such that $m = f_\bullet$.*

Proof. Define f by $m(id)$ for $id : \Delta^d \to \Delta^d$ the identity. Then

$$\begin{aligned} f_\bullet &= (m(id))_\bullet \\ &= (m(id))_* \\ &= m \circ id_* \quad \text{(as } m \text{ is smooth)} \\ &= m. \end{aligned}$$

So f induces m. Now if g induces m then $g_\bullet = m$ hence $g = g_\bullet(id) = m(id)$. ☐

Definition 3.9. *A smooth simplicial set whose underlying simplicial set is a Kan complex will be called a* **smooth Kan complex**.

Let $Sing^{sm}(Y)$ denote the simplicial set of smooth singular simplices in Y [1]. That is $Sing^{sm}(Y)(k)$ is the set of smooth maps $\Sigma : \Delta^k \to Y$. And where the simplicial structure on $Sing^{sm}(Y)$ is the natural one (analogous to the simplicial structure on $X_\bullet$ in Proposition 3.7). $Sing^{sm}(Y)$ will often be abbreviated by $Y_\bullet$. Analogously, $Sing^c(Y)$ will be the simplicial set of continuous simplices in X.

Example 3.10. *Let Y be a smooth d-fold. And set $X = Y_\bullet = Sing^{sm}(Y)$. Then X is naturally a smooth simplicial set, analogously to Example 3.5. This should be a Kan complex but a reference is not known to me. However, if we ask that $\Sigma : \Delta^k \to Y$ are in addition collared (as in Definition 2.1) then the Kan condition is simple to verify.*

Example 3.11. *One special example is worth attention. Let M be a smooth manifold. Then there is a natural smooth simplicial set LM^Δ whose d-simplices Σ are smooth maps $f_\Sigma : \Delta^d \times S^1 \to M$. The maps Σ_* are defined by*

$$\Sigma_*(\sigma) = f_\Sigma \circ (\sigma \times id),$$

for

$$\sigma \times id : \Delta^d \times S^1 \to \Delta^d \times S^1,$$

the induced map. This LM^Δ is one simplicial model of the free loop space. Naturally the free loop space LM also has the structure of a Fréchet manifold, in particular we have the smooth simplicial set $LM_\bullet$, whose n-simplices are Fréchet smooth maps $\Sigma : \Delta^n \to LM$. There is a natural simplicial map $LM^\Delta \to LM_\bullet$, which is clearly smooth. (It is indeed a diffeomorphism.)

The above smooth simplicial set structure LM^Δ, in the language of diffeologies, is closely related to the functional diffeology on $C^\infty(Y, Z)$, for which there are diffeomorphisms:

$$C^\infty(X \times Y, Z) \to C^\infty(X, C^\infty(Y, Z)),$$

given another diffeological space X.

3.1. **Smooth simplex category of a smooth simplicial set.** Given a smooth simplicial set X, there is an extension of the previously defined simplex category $\Delta(X)$.

Definition 3.12. *For X a smooth simplicial set, $\Delta^{sm}(X)$ will denote the category whose set of objects* $\operatorname{obj}\Delta^{sm}(X)$ *is the set of smooth maps*

$$\Sigma : \Delta^d_\bullet \to X, \quad d \geq 0$$

and morphisms $f : \Sigma_1 \to \Sigma_2$, commutative diagrams:

$$\begin{tikzcd} \Delta^d_\bullet \arrow[r, "\widetilde{f}_\bullet"] \arrow[dr, "\Sigma_1"] & \Delta^n_\bullet \arrow[d, "\Sigma_2"] \\ & X \end{tikzcd} \tag{3.4}$$

[1]This is often called the "smooth singular simplicial set of Y". However, for us "smooth" is reserved for another purpose, so to avoid confusion we do not use such terminology.

with top arrow any smooth map (for the tautological smooth simplicial set structure on $\Delta^d_\bullet$), which we denote by $\widetilde{f}_\bullet$. By Lemma 3.8, $\widetilde{f}_\bullet$ is induced by a unique smooth map $\widetilde{f} : \Delta^d \to \Delta^n$.

By Proposition 3.7 we have a natural faithful embedding $\Delta(X) \to \Delta^{sm}(X)$ that is an isomorphism on object sets of these categories. We may then likewise call elements of $\Delta^{sm}(X)$ as d-simplices.

Proposition 3.13. *Definitions 3.1, 3.3 are equivalent. In other words there is a natural equivalence of the corresponding categories of smooth simplicial sets.*

Proof. Let $s - Set^{sm}_1$ denote the category of smooth simplicial sets as given by the Definition 3.1. And let $s - Set^{sm}_2$ denote the category of smooth simplicial sets as given by the Definition 3.3.

Given $X \in s - Set^{sm}_1$, we define a functor $I(X) : \Delta^{sm} \to Set^{op}$ by setting

$$I(X)(k) = \{\Sigma_\bullet : \Delta^k_\bullet \to X \,|\, \Sigma_\bullet \text{ is smooth i.e. is a morphism in } s - Set^{sm}_1\}.$$

And for $\sigma : \Delta^k \to \Delta^d$ a smooth map setting

$$I(X)(\sigma) : I(X)(d) \to I(X)(k)$$

to be the map $I(X)(\sigma)(\Sigma_\bullet) = \Sigma_\bullet \circ \sigma_\bullet$. This defines

$$I : s - Set^{sm}_1 \to s - Set^{sm}_2$$

on objects.

Conversely, given $F \in s - Set^{sm}_2$, define a simplicial set $I^{-1}(F)(k) := F(k)$. And for $\Sigma \in I^{-1}(F)(k)$ define $\Sigma_* : \Delta^k_\bullet \to X$ to be the map:

$$\Sigma_*(\sigma) = F(\sigma)(\Sigma).$$

So that we get an element $I^{-1}(F) \in s - Set_1$. This defines

$$I^{-1} : s - Set_2 \to s - Set_1$$

on objects. By Proposition 3.7 $(I^{-1} \circ I(X)) \simeq X$ an isomorphism in $s - Set^{sm}_1$.

Suppose now we are given a morphism in $s - Set^{sm}_1$: $f : X_0 \to X_1$ i.e. a simplicial map satisfying the condition:

$$\forall n \in \mathbb{N}\, \forall \Sigma \in X(n) : (f(\Sigma))_* = f \circ \Sigma_*. \tag{3.5}$$

Define a natural transformation:

$$I(f) : I(X_0) \to I(X_1),$$

by setting $I(f)_k : I(X_0)(k) \to I(X_1)(k)$ to be the map $I(f)_k(\Sigma_\bullet) = f \circ \Sigma_\bullet$.

This is a natural transformation by the associativity of the composition $f \circ (\Sigma_\bullet \circ \sigma_\bullet) = (f \circ \Sigma_\bullet) \circ \sigma_\bullet$.

It is clear that $I : s - Set^{sm}_1 \to s - Set^{sm}_2$ is a functor. We show that it is faithful on hom sets. If $f_0, f' : X_0 \to X_1$ are a pair of morphisms in $s - Set^{sm}_1$ suppose that $I(f) = I(f')$. Then

$$\forall n \in \mathbb{N}\, \forall \Sigma_\bullet \in I(X)(n) : f \circ \Sigma_\bullet = f' \circ \Sigma_\bullet.$$

In particular,

$$\forall n \in \mathbb{N} \, \forall \Sigma \in (X)(n) : f \circ \Sigma_* = f' \circ \Sigma_*.$$

And so

$$\forall n \in \mathbb{N} \, \forall \Sigma \in (X)(n) : f \circ \Sigma_*(id^n) = f' \circ \Sigma_*(id^n).$$

And so

$$\forall n \in \mathbb{N} \, \forall \Sigma \in (X)(n) : f(n) = f'(n).$$

And thus $f = f'$.

We show that I surjective on hom sets. Suppose that $N : I(X_0) \to I(X_1)$ is a morphism in $s-Set_2^{sm}$, i.e. a natural transformation of the corresponding functors. So for $\sigma : \Delta^d \to \Delta^k$ smooth, we have a commutative diagram:

$$\begin{array}{ccc} X_0(k) & \xrightarrow{X_0(\sigma)} & X_0(d) \\ \downarrow{\scriptstyle N_k} & & \downarrow{\scriptstyle N_d} \\ X_1(k) & \xrightarrow{X_1(\sigma)} & X_2(d) \end{array} \tag{3.6}$$

Define a simplicial map

$$f_N : X_0 \to X_1,$$

by $f_N(\Sigma) = N_k(\Sigma_*)(id^k)$, for $\Sigma \in X_0(k)$, and $id^k : \Delta^k \to \Delta^k$ the identity.

We check that $I(f_N) = N$. Let $\Sigma_\bullet : \Delta^d_\bullet \to X_0$ be smooth. For $\sigma : \Delta^k \to \Delta^d$ smooth, we have:

$$\begin{aligned} I(f_N)_d(\Sigma_\bullet)(\sigma) &= (f_N \circ \Sigma_\bullet)(\sigma) \text{ by definition of } I \\ &= f_N(\Sigma_\bullet(\sigma)) \\ &= N_k(\Sigma_\bullet(\sigma)_*)(id^k) \text{ by definition of } f_N \\ &= N_k(\Sigma_\bullet \circ \sigma_\bullet)(id^k) \text{ as } \Sigma_\bullet \text{ is smooth} \\ &= N_d(\Sigma_\bullet) \circ \sigma_\bullet(id^k) \text{ by } N \text{ being a natural transformation, (3.6)} \\ &= N_d(\Sigma_\bullet)(\sigma). \end{aligned}$$

Since $\Sigma_\bullet, \sigma$ were general it follows that $I(f_N) = N$.

We have proved that I is a functor that is essentially surjective on objects, and is fully-faithful on hom sets, it follows by a classical theorem of category theory that I is an equivalence of categories. $\square$

3.2. **Products.** Given a pair of smooth simplicial sets $(X_1, g_1), (X_2, g_2)$, the product $X_1 \times X_2$ of the underlying simplicial sets, has the structure of a smooth simplicial set

$$(X_1 \times X_2, g_1 \times g_2),$$

constructed as follows. Denote by $\pi_i : X_1 \times X_2 \to X_i$ the simplicial projection maps. Then for each $\Sigma \in X_1 \times X_2(d)$,

$$g_1 \times g_2(\Sigma) : \Delta^d_\bullet \to X_1 \times X_2$$

is defined by:

$$g_1 \times g_2(\Sigma)(\sigma) := (g_1(\pi_1(\Sigma))(\sigma), g_2(\pi_2(\Sigma))(\sigma)).$$

3.3. **More on smooth maps.** As defined, a smooth map $f : X \to Y$ of smooth simplicial sets, induces a functor

$$\Delta^{sm} f : \Delta^{sm}(X) \to \Delta^{sm}(Y).$$

This is defined by $\Delta^{sm} f(\Sigma) = f \circ \Sigma$, where $\Sigma : \Delta^d_\bullet \to X$ is in $\Delta^{sm}(X)$. If $m : \Sigma_1 \to \Sigma_2$ is a morphism in $\Delta^{sm}(X)$:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "\Sigma_1"] & \Delta^d_\bullet \arrow[d, "\Sigma_2"] \\ & X, \end{tikzcd}$$

then obviously the diagram below also commutes:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "h_1"] & \Delta^d_\bullet \arrow[d, "h_2"] \\ & Y, \end{tikzcd}$$

where $h_i = \Delta^{sm} f(\Sigma_i) = f \circ \Sigma_i$, $i = 1, 2$. And so the latter diagram determines a morphism $\Delta^{sm} f(m) : h_1 \to h_2$ in $\Delta^{sm}(Y)$. Clearly, this determines a functor $\Delta^{sm} f$ as needed.

3.4. **Smooth homotopy.**

Definition 3.14. *Let X, Y be smooth simplicial sets. Set $I := \Delta^1_\bullet$ and let $0_\bullet, 1_\bullet \subset I$ be the images of the pair of inclusions $\Delta^0_\bullet \to I$ corresponding to the pair of endpoints. A pair of smooth maps $f, g : X \to Y$ are called* **smoothly homotopic** *if there exists a smooth map*

$$H : X \times I \to Y$$

such that $H|_{X \times 0_\bullet} = f$ and $H|_{X \times 1_\bullet} = g$.

The following notion will be useful later on.

Definition 3.15. *Let X be a smooth simplicial set. Let $S^k_\bullet = Sing^{sm}(S^k)$. Then $\pi^{sm}_k(X)$ is defined to be the set of (free) smooth homotopy equivalence classes of smooth maps $f : S^k_\bullet \to X$.*

4. Differential forms on smooth simplicial sets

The theory of differential forms on smooth simplicial sets that we now present, is just part of the standard abstract theory of differential forms on simplicial sets. As such, it is a priori *inequivalent* to the theory of differential forms on diffeological spaces in the sense of Souriau [35]. If one wanted to translate our discussion of differential forms in to the language of diffeological spaces, then probably it would be similar to the work Katsuhiko [14], see also [15], [11].

First we define smooth differential forms on the topological simplices Δ^d.

Definition 4.1. *Set $T\Delta^d := i^* T\mathbb{R}^d$ for $i : \Delta^d \to \mathbb{R}^d$ the natural inclusion. Let $T^*\Delta^d$ denote the dual vector bundle. A smooth differential k-form ω on Δ^d is a continuous section of $\Lambda^k(T^*\Delta^d)$, having a smooth extension to a section of $\Lambda^k(T^*N)$ for $N \supset \Delta^d$ an open subset of $\mathbb{R}^d$.*

The above is equivalent to various other possible definitions. For example by taking Δ^d to be a special case of a smooth manifold with corners, and using more general theory of differential forms. This can be done, for example, using theory of diffeological spaces [8]. See also Karshon-Watts [12], which establishes one kind of "uniqueness of notions of smoothness" for the case of simplices.

Definition 4.2. *Let X be a smooth simplicial set. A* ***simplicial differential k-form*** *ω, or just differential k-form where there is no possibility of confusion, is an assignment for each d-simplex Σ of X a smooth differential k-form $\omega(\Sigma) = \omega_\Sigma$ on Δ^d, such that*

$$i^* \Omega_{\Sigma_2} = \Omega_{\Sigma_1}, \tag{4.1}$$

for every morphism $i : \Sigma_1 \to \Sigma_2$ in $\Delta^{inj}(X)$, (see Section 2). If in addition:

$$\omega_{g(\Sigma)(\sigma)} = \sigma^* \omega_\Sigma, \tag{4.2}$$

for every $\sigma \in \Delta^d_\bullet$, and every d-simplex Σ, then we say that ω is **coherent**.

The coherence condition is only meaningful for a smooth simplicial set, however (4.1) and so differential forms are defined for any simplicial set.

A simplicial differential form ω may be denoted simply as $\omega = \{\omega_\Sigma\}$. It may be convenient to also use anonymous function notation $\Sigma \mapsto \omega_\Sigma$.

Example 4.3. *If $X = Y_\bullet$ for Y a smooth d-fold, and if ω is a classical differential k-form on Y, then $\Sigma \mapsto \Sigma^* \omega$ is a coherent simplicial differential k-form on X called the* **induced simplicial differential form**.

Example 4.4. *Let LM^Δ be the smooth Kan complex of Example 3.11. Then Chen's iterated integrals [2] naturally give coherent differential forms on LM^Δ.*

The above coherence condition is often unnecessary, hence is not part of the basic definition here.

Let X be a smooth simplicial set. We denote by $\Omega^k(X)$ the $\mathbb{R}$-vector space of differential k-forms on X. Define

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

by the formula:

$$d\omega(\Sigma) = d(\omega(\Sigma)).$$

In other words $d\omega$ is:

$$\Sigma \mapsto d\omega_\Sigma.$$

Clearly we have

$$d^2 = 0.$$

A k-form ω is said to be ***closed*** if $d\omega = 0$, and ***exact*** if for some $(k-1)$-form η, $\omega = d\eta$.

Definition 4.5. *The* **wedge product** *on*

$$\Omega^\bullet(X) = \bigoplus_{k\geq 0} \Omega^k(X)$$

is defined by

$$\omega \wedge \eta = \{\omega_\Sigma \wedge \eta_\Sigma\}.$$

Then $\Omega^\bullet(X)$ has the structure of a differential graded $\mathbb{R}$-algebra with respect to $\wedge$.

We then, as usual, define the ***De Rham cohomology*** of X:

$$H^k_{DR}(X) = \frac{\text{closed k-forms}}{\text{exact k-forms}},$$

then

$$H^\bullet_{DR}(X) = \bigoplus_{k\geq 0} H^k_{DR}(X)$$

is a graded commutative $\mathbb{R}$-algebra.

The simplicial De Rham complex above is a very standard construction. Early versions of it have been used by Whitney and perhaps most famously by Sullivan [36]. In particular, the proof of the De-Rham theorem (next section) is originally due to Sullivan.

4.1. **Homology and cohomology of a simplicial set.** We go over this mostly to establish notation. For a simplicial set X, we define an abelian group

$$C_k(X, \mathbb{Z}),$$

as the free abelian group generated by the set of k-simplices $X(k)$. Elements of $C_k(X, \mathbb{Z})$ are called ***k-chains***. The boundary operator:

$$\partial : C_k(X, \mathbb{Z}) \to C_{k-1}(X, \mathbb{Z}),$$

is defined on a k-simplex σ as classically by

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i d_i \sigma,$$

where d_i are the face maps, this is then extended by linearity to general chains. As classically, $\partial^2 = 0$. The homology of this complex is denoted by $H_k(X, \mathbb{Z})$, called integral homology. The integral cohomology is defined analogously to the classical topology setting, using dual chain groups $C^k(X, \mathbb{Z}) = hom(C_k(X, \mathbb{Z}), \mathbb{Z})$. The corresponding coboundary operator is denoted by d as usual:

$$d : C^k(X, \mathbb{Z}) \to C^{k+1}(X, \mathbb{Z}).$$

Homology and cohomology with other ring coefficients (or modules) are likewise defined analogously. Given a simplicial map $f : X \to Y$ there are natural induced chain maps $f^* : C^k(Y, \mathbb{Z}) \to C^k(X, \mathbb{Z})$, and $f_* : C_k(X, \mathbb{Z}) \to C_k(X, \mathbb{Z})$.

We say that a pair of simplicial maps $f, g : X \to Y$ are ***homotopic*** if there a simplicial map $H : X \times \Delta^1_{simp} \to Y$ so that $f = H \circ i_0$, $g = H \circ i_1$ for $i_0, i_1 : X \to X \times \Delta^1_{simp}$ corresponding to the pair of end point inclusions $\Delta^0_{simp} \to \Delta^1_{simp}$. A ***simplicial homotopy equivalence*** is then defined analogously to the topological setting.

As is well known if f, g are homotopic then f^*, g^* and f_*, g_* are chain homotopic.

4.2. **Integration.** Let X be a smooth simplicial set. Given a chain

$$\sigma = \sum_i a_i \Sigma_i \in C_k(X, \mathbb{Z})$$

and a smooth differential form ω, we define:

$$\int_\sigma \omega = \sum_i a_i \int_{\Delta^k} \omega_{\Sigma_i}$$

where the integrals on the right are the classical integrals of a differential form. Thus, we obtain a homomorphism:

$$\int : \Omega^k(X) \to C^k(X, \mathbb{R}),$$

$\int(\omega)$ is the k-cochain defined by:

$$\int(\omega)(\sigma) := \int_\sigma \omega,$$

where σ is a k-chain. We will abbreviate $\int(\omega) = \int \omega$. The following is well known.

Lemma 4.6. *For a smooth simplicial set X, the homomorphism $\int$ commutes with d, and so induces a homomorphism:*

$$\int : H^k_{DR}(X) \to H^k(X, \mathbb{R}).$$

Proof. We need that

$$\int d\omega = d \int \omega.$$

Let $\Sigma : \Delta^k_{simp} \to X$ be a k-simplex. Then

$$\begin{aligned} \int d\omega(\Sigma) &= \int_{\Delta^k} d\omega_\Sigma \text{ by definition} \\ &= \int_{\partial \Delta^k} \omega_\Sigma \text{ by Stokes theorem} \\ &= d(\int \omega)(\Sigma) \text{ by the definition of } d \text{ on co-chains.} \end{aligned}$$

□

In fact the De-Rham theorem (see for instance Sullivan [36]) tells us that $\int$ is an isomorphism, but we will not need this.

4.3. **Pull-back.** Given a smooth map $f : X_1 \to X_2$ of smooth simplicial sets, we define

$$f^* : \Omega^k(X_2) \to \Omega^k(X_1)$$

naturally by

$$f^*(\omega) := f^*\omega := \{(f^*\omega)_\Sigma\} := \{\omega_{f(\Sigma)}\}.$$

Clearly f^* commutes with d so that we have an induced differential graded $\mathbb{R}$-algebra homomorphism:

$$f^* : \Omega^\bullet(X_2) \to \Omega^\bullet(X_1).$$

And in particular an induced $\mathbb{R}$-algebra homomorphism:

$$f^* : H^\bullet_{DR}(X_2) \to H^\bullet_{DR}(X_1).$$

4.4. **Relation with ordinary homology and cohomology.** Let $s-Set$ denote the category of simplicial sets and Top the category of topological spaces. Let

$$|\cdot| : s-Set \to Top$$

be the geometric realization functor as defined in Section 2.3. Let X be a (smooth) simplicial set. Then for any ring K we have natural chain maps

$$\begin{aligned} CR &: C_d(X, K) \to C_d(|X|, K), \\ CR^\vee &: C^d(|X|, K) \to C^d(X, K). \end{aligned} \tag{4.3}$$

The chain map CR is defined as follows. A d-simplex $\Sigma : \Delta^d_{simp} \to X$, by construction of $|X|$ uniquely induces a continuous map $\Sigma_{top} : \Delta^d \to |X|$. So if Σ_{top} also denotes the corresponding generator of $C^d(|X|, K)$, then we just set $CR(\Sigma) = \Sigma_{top}$ in this notation. Then $CR^\vee$ is just the dual chain map.

It is well known that CR and $CR^\vee$ are quasi-isomorphisms, i.e. induce isomorphisms

$$\begin{aligned} R &: H_d(X, K) \to H_d(|X|, K), \\ R^\vee &: H^d(|X|, K) \to H^d(X, K). \end{aligned} \tag{4.4}$$

This can be checked by hand, but a proof can be found in Hatcher [10, Section 2.1] for the case of Δ-complexes.

Now let Y be a smooth manifold and $X = Y_\bullet = Sing^{sm}(Y)$. We have a natural homotopy equivalence $|Y_\bullet| \simeq Y$. This is just because the natural map $|Y_\bullet| \to Y$ is a weak homotopy equivalence, (by homotopy approximating continuous maps by smooth maps), and so is a homotopy equivalence by the Whitehead theorem. Let us denote by

$$N : Y \to |Y_\bullet|, \tag{4.5}$$

its homotopy inverse.

Then factor R and $R^\vee$ as:

$$H_d(Y_\bullet, K) \xrightarrow{I} H_d(Y, K) \xrightarrow{N_*} H_d(|Y_\bullet|, K), \tag{4.6}$$

$$H^d(|Y_\bullet|, K) \xrightarrow{N^*} H^d(Y, K) \xrightarrow{I^\vee} H^d(Y_\bullet, K) \tag{4.7}$$

The map I is induced by the chain map CI sending the generator of $C_d(Y_\bullet, K)$ corresponding to a simplex $\Sigma \in Y_\bullet(d)$ to the generator of $C_d(Y)$, corresponding to the smooth map $\Sigma_{top} : \Delta^d \to Y$ (as $\Sigma \in Y_\bullet(d)$ by definition uniquely corresponds to such a smooth map). Likewise, $I^\vee$ is induced by the cochain map $CI^\vee$ sending a cochain α to the cochain $CI^\vee(\alpha)$ defined by

$$CI^\vee(\alpha)(\sigma) := \alpha(CI(\sigma)),$$

where $\sigma \in C_d(Y_\bullet, K)$.

Notation 4.7. *Given a cohomology class $\alpha \in H^d(X, K)$, we will denote by $|\alpha| \in H^d(|X|, K)$ the class $(R^\vee)^{-1}(\alpha)$. Also, given a smooth manifold Y and a cohomology class $\alpha \in H^d(Y_\bullet, K)$, we will denote by $|\alpha| \in H^d(Y, K)$ the class $N^* \circ (R^\vee)^{-1}(\alpha) = (I^\vee)^{-1}(\alpha)$.*

Given a map of simplicial sets $f : X_1 \to X_2$ we let $|f| : |X_1| \to |X_2|$ denote the induced map of geometric realizations.

Lemma 4.8. *Let $f : X_1 \to X_2$ be a simplicial map of simplicial sets. Let $f^* : H^d(X_2, K) \to H^d(X_1, K)$ be the induced homomorphism then:*

$$|f^*(\alpha)| = |f|^*(|\alpha|).$$

Proof. We have a clearly commutative diagram of chain maps (omitting the coefficient ring):

$$\begin{tikzcd} C_d(X_1) \arrow[r, "CR"] \arrow[d, "f_*"] & C_d(|X_1|) \arrow[d, "|f|_*"] \\ C_d(X_2) \arrow[r, "CR"] & C_d(|X_2|), \end{tikzcd}$$

from which the result immediately follows. ☐

5. Smooth simplicial G-bundles

Part of our motivation is the construction of the universal Chern-Weil homomorphisms for Fréchet Lie groups. A *Fréchet Lie group* G is a Lie group whose underlying manifold is a possibly infinite dimensional smooth manifold locally modelled on a Fréchet space, that is a locally convex, complete Hausdorff vector space. One example is the group of diffeomorphisms of a compact manifold, or the group of compactly supported diffeomorphisms of a general smooth manifold, Hamilton [9].

Later on it will also be important that G have the homotopy type of a CW complex. This is the case for instance if G is the group $\mathrm{Diff}(M)$ of diffeomorphisms of closed manifold. For this group is homotopy equivalent to the group $\mathcal{X}$ of C^1 diffeomorphisms, (by classical smooth approximation analysis techniques). And $\mathcal{X}$ is Banach and metrizable and has the homotopy type of a CW complex by a result of Palais [26]. The same argument works if we take G to be the group of compactly supported diffeomorphisms of any smooth manifold M, using the standard C^∞ topology.

By Milnor [24] (or based on this work), a prototypical example of such a Lie group is the group of compactly supported diffeomorphisms $\mathrm{Diff}(M)$ of a smooth manifold. Another very interesting example for us is the group of Hamiltonian symplectomorphisms $Ham(M, \omega)$ of a symplectic manifold. Particularly because its Lie algebra admits natural bi-invariant polynomials, so that it is possible to define interesting Chern-Weil theory for this group.

In what follows G is always assumed to be a Fréchet Lie group. However, this is just to avoid trivial generality, as until we get to Chern-Weil theory in Section 7, G can very well be any diffeological group. We now introduce the basic building blocks for simplicial G-bundles.

Definition 5.1. *A* **smooth G-bundle P over** Δ^n *is a smooth G-bundle over Δ^n with the latter naturally understood as a smooth manifold with corners, using the natural embedding $\Delta^n \subset \mathbb{R}^n$.*

Remark 5.2. *For concreteness, this can be interpreted as follows. P is a topological principal G-bundle over $\Delta^n \subset \mathbb{R}^n$, together with a choice of a maximal atlas of topological G-bundle trivializations $\phi_i : U_i \times G \to P$, $U_i \subset \Delta^n$ open, s.t. the transitions maps*

$$(U_i \cap U_j) \times G \xrightarrow{\phi_{ij} = \phi_j^{-1} \circ \phi_i,} (U_i \cap U_j) \times G$$

extend to smooth maps $N \times G \to N \times G$, for $N \supset U_i \cap U_j$ an open set in $\mathbb{R}^n$. All of the subsequent constructions can be made to refer to the above concrete model. So that the generalities of smooth G-bundles over manifolds with corners are not really needed in this paper.

To warn, at this point our terminology may partially clash with common terminology, in particular a simplicial G-bundle will *not* be a pre-sheaf on Δ with values in the category of smooth G-bundles. Instead, it will be a functor (not a co-functor!) on $\Delta^{sm}(X)$ with additional properties. The latter pre-sheafs will not appear in the paper so that this should not cause confusion.

In the definition of simplicial differential forms we omitted coherence. In the case of simplicial G-bundles, the analogous condition (full functoriality on $\Delta^{sm}(X)$) turns out to be necessary if we want universal simplicial G-bundles with expected behavior.

Notation 5.3. *Given a Fréchet Lie group G, let $\mathcal{G}$ denote the category of smooth G-bundles over manifolds with corners, with morphisms smooth G-bundle maps. (See however Remark 5.2 just above.)*

Definition 5.4. *Let G be a Fréchet Lie group and X a smooth simplicial set. A* **smooth simplicial G-bundle** *P over X is the following data:*

- *A functor $P : \Delta^{sm}(X) \to \mathcal{G}$, so that for Σ a d-simplex, $P(\Sigma)$ is a smooth G-bundle over Δ^d.*
- *For each morphism f:*

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{f}_\bullet"] \arrow[dr, "\Sigma^k_1"] & \Delta^d_\bullet \arrow[d, "\Sigma^d_2"] \\ & X \end{tikzcd}$$

in $\Delta^{sm}(X)$, we have a commutative diagram:

$$\begin{array}{ccc} P(\Sigma_1^k) & \xrightarrow{P(f)} & P(\Sigma_2^d) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ \Delta^k & \xrightarrow{\widetilde{f}} & \Delta^d, \end{array}$$

where the maps p_1, p_2 are the respective bundle projections, and where $\widetilde{f}$ is the map induced by the map $\widetilde{f}_\bullet : \Delta^k_\bullet \to \Delta^d_\bullet$ as in Lemma 3.8. In other words $P(f)$ is a bundle map over $\widetilde{f}$. We call this condition **compatibility**.

We will only deal with smooth simplicial G-bundles, and so will usually just say ***simplicial G-bundle***, omitting the qualifier 'smooth'.

Notation 5.5. *We often use notation P_Σ for $P(\Sigma)$. If we write a simplicial G-bundle $P \to X$, this just means that P is a simplicial G-bundle over X in the sense above. So that $P \to X$ is just notation not a morphism.*

Example 5.6. *If X is a smooth simplicial set and G is as above, we denote by $X \times G$ the simplicial G-bundle,*

$$\forall n \in \mathbb{N}, \forall \Sigma^n \in \Delta(X) : (X \times G)_{\Sigma^n} = \Delta^n \times G,$$

with $\Delta^n \times G \to \Delta^n$ the trivial projection. This is called the ***trivial simplicial G-bundle over X***.

Example 5.7. *Let $Z \to Y$ be a smooth G-bundle over a smooth manifold Y. Then we have a simplicial G-bundle $Z_\bullet$ over $Y_\bullet$ defined by*

$$Z_\bullet(\Sigma) = \Sigma^* Z.$$

And where for $f : \Sigma_1 \to \Sigma_2$ a morphism, the bundle map

$$Z_\bullet(f) : (Z_\bullet(\Sigma_1) = \Sigma_1^* Z) \to (Z_\bullet(\Sigma_2) = \Sigma_2^* Z)$$

is just the universal map $u : \Sigma_1^* Z \to \Sigma_2^* Z$ corresponding to the universal pull-back property of $\Sigma_2^* Z$. The uniqueness of the universal maps readily implies that $Z_\bullet$ is a functor. We say that $Z_\bullet$ is **the simplicial G-bundle induced by Z**.

Definition 5.8. *Let $P_1 \to X_1, P_2 \to X_2$ be a pair of simplicial G-bundles. Let $h : X_1 \to X_2$ be a smooth map. A* **smooth simplicial G-bundle map over h** *from P_1 to P_2 is a natural transformation of functors:*

$$\widetilde{h} : P_1 \to P_2 \circ \Delta^{sm} h.$$

This is required to have the following additional property. For each d-simplex $\Sigma \in \Delta^{sm}(X_1)$ the natural transformation $\widetilde{h}$ specifies a morphism in $\mathcal{G}$:

$$\widetilde{h}_\Sigma : P_1(\Sigma) \to P_2(\Sigma),$$

and we ask that this is a bundle map over the identity so that the following diagram commutes:

$$\begin{tikzcd} P_1(\Sigma) \arrow[r, "\widetilde{h}_\Sigma"] \arrow[d, "p_1"] & P_2(\Sigma) \arrow[d, "p_2"] \\ \Delta^d \arrow[r, "id"] & \Delta^d. \end{tikzcd}$$

We will usually just say simplicial G-bundle map instead of smooth simplicial G-bundle map, (as everything is always smooth) when h is not specified it is assumed to be the identity.

Definition 5.9. *Let P_1, P_2 be simplicial G-bundles over X_1, X_2 respectively. A* **simplicial G-bundle isomorphism** *is a simplicial G-bundle map*

$$\widetilde{h} : P_1 \to P_2$$

s.t. there is a simplicial G-bundle map

$$\widetilde{h}^{-1} : P_2 \to P_1$$

with

$$\widetilde{h}^{-1} \circ \widetilde{h} = id.$$

This is clearly the same as asking that $\widetilde{h}$ be a natural isomorphism of the corresponding functors. Usually, $X_1 = X_2$ and in this case, unless specified otherwise, it is assumed $h = id$. A simplicial G-bundle isomorphic to the trivial simplicial G-bundle is called **trivializeable**.

Definition 5.10. *If $X = Y_\bullet$ for Y a smooth manifold, we say that a simplicial G-bundle P over X is* **inducible by a smooth G-bundle** *$N \to Y$ if there is a simplicial G-bundle isomorphism $N_\bullet \to P$.*

The following will be one of the crucial ingredients later on.

Theorem 5.11. *Let G be as above and let $P \to Y_\bullet$ be a simplicial G-bundle, for Y a smooth d-manifold. Then P is inducible by some smooth G-bundle $N \to Y$.*

Proof. We need to introduce an auxiliary notion. Let Z be a smooth d-manifold with corners. And let $\mathcal{D}(Z)$ denote the category whose objects are smooth embeddings $\Sigma : \Delta^d \to Z$, (for the same fixed d) and so that a morphism $f \in hom_{\mathcal{D}(Z)}(\Sigma_1, \Sigma_2)$ is a commutative diagrams:

$$\begin{tikzcd} \Delta^d \arrow[r, "\widetilde{f}"] \arrow[dr, "\Sigma_1"'] & \Delta^d \arrow[d, "\Sigma_2"] \\ & Z. \end{tikzcd} \tag{5.1}$$

Note that the map $\widetilde{f}$ is unique, when such a diagram exists as Σ_i are embeddings. Thus $hom_{\mathcal{D}(Z)}(\Sigma_1, \Sigma_2)$ is either empty or consists of a single element.

We now go back to our d-manifold Y. Let $\{O_i\}_{i \in I}$ be a locally finite open cover of Y, closed under intersections, with each O_i diffeomorphic to an open ball in $\mathbb{R}^d$. Such a cover is often called a good cover of a manifold. Existence of such a cover is a folklore theorem, but a proof can be found in [5, Prop A1]. Let $\mathcal{O}$ denote the category with the set of objects $\{O_i\}$ and with morphisms set inclusions. Set $C_i = \mathcal{D}(O_i)$, then we naturally have $C_i \subset \Delta^{sm}(Y_\bullet)$. For each i, we have the functor

$$F_i = P|_{C_i} : C_i \to \mathcal{G}.$$

By assumption that each O_i is diffeomorphic to an open ball, O_i has an exhaustion by embedded d-simplices. This means that there is a sequence of smooth embeddings $\Sigma_j : \Delta^d \to O_i$ satisfying:

- $\operatorname{image}(\Sigma_{j+1}) \supset \operatorname{image}(\Sigma_j)$ for each j.
- $\bigcup_j \operatorname{image}(\Sigma_j) = O_i$.

In particular, for each i, the colimit in $\mathcal{G}$:

$$P_i := \operatorname{colim}_{C_i} F_i \tag{5.2}$$

is naturally a topological G-bundle over O_i.

We may give P_i the structure of a smooth G-bundle, with G-bundle charts defined as follows. Take the collection of maps

$$\{\phi^i_{\Sigma,j}\}_{\Sigma \in C_i, j \in J^\Sigma},$$

satisfying the following.

- Each $\phi^i_{\Sigma,j}$ is the composition map
$$V^i_{\Sigma,j} \times G \xrightarrow{\xi_{ij}} P_\Sigma \xrightarrow{c_\Sigma} P_i$$
where $V^i_{\Sigma,j} \subset (\Delta^d)^\circ$ is open, for $(\Delta^d)^\circ$ the topological interior of the subspace $\Delta^d \subset \mathbb{R}^d$. And where $c_\Sigma : (P_\Sigma = F_i(\Sigma)) \to P_i$ is the natural map in the colimit diagram of (5.2).
- The collection
$$\{\xi_{i,j}\}_{j \in J^\Sigma}$$
forms an atlas of smooth G-bundle charts for $P_\Sigma|_{(\Delta^d)^\circ}$.

The collection $\{\phi^i_{\Sigma,j}\}$ then forms a smooth G-bundle atlas for P_i.

So we obtain a functor

$$D : \mathcal{O} \to \mathcal{G},$$

defined by

$$D(O_i) = P_i,$$

and defined naturally on morphisms. Specifically, a morphism $O_{i_1} \to O_{i_2}$ induces a functor $C_{i_1} \to C_{i_2}$ and hence a smooth G-bundle map $P_{i_1} \to P_{i_2}$, by the naturality of the colimit.

Let $t : \mathcal{O} \to Top$ denote the tautological functor, so that $Y = \operatorname{colim}_{\mathcal{O}} t$, where for simplicity we write equality for natural isomorphisms here and further on in this proof. Now,

$$N := \operatorname{colim}_{\mathcal{O}} D, \tag{5.3}$$

is naturally a topological G-bundle over $\operatorname{colim}_{\mathcal{O}} t = Y$. Let $c_i : P_i \to N$ denote the natural maps in the colimit diagram of (5.3). The collection of charts $\{c_i \circ \phi^i_{\Sigma,j}\}_{i,j,\Sigma}$ forms a smooth atlas on N, giving it a structure of a smooth G-bundle.

We now prove that P is induced by N. Let Σ be a d-simplex of $X := Y_\bullet$, then $\{V_i := \Sigma^{-1}(O_i)\}_{i \in I}$ is a locally finite open cover of Δ^d closed under finite intersections. Let $N_\bullet$ be the simplicial G-bundle induced by N. So

$$N_\bullet(\Sigma) := N_\Sigma := \Sigma^* N.$$

As Δ^d is a convex subset of $\mathbb{R}^d$, the open metric balls in Δ^d, for the induced metric, are convex as subsets of $\mathbb{R}^d$. Consequently, as each $V_i \subset \Delta^d$ is open, it has a basis of convex (as subsets of $\mathbb{R}^d$) metric balls, with respect to the induced metric. By Rudin [31] there is then a locally finite cover of V_i by elements of this basis. In fact, Rudin shows any open cover of V_i has a locally finite refinement by elements of such a basis.

Let $\{W^i_j\}$ consist of elements of this cover and all intersections of its elements, (which must then be finite intersections). So $W^i_j \subset V_i$ are open convex subsets and $\{W^i_j\}$ is a locally finite open cover of V_i, closed under finite intersections.

As each $W^i_j \subset \Delta^d$ is open and convex it has an exhaustion by nested images of embedded simplices. That is

$$W^i_j = \bigcup_{k \in \mathbb{N}} \operatorname{image} \sigma^{i,j}_k$$

for $\sigma^{i,j}_k : \Delta^d \to W^i_j$ smooth and embedded, with $\operatorname{image} \sigma^{i,j}_k \subset \operatorname{image} \sigma^{i,j}_{k+1}$ for each k.

Remark 5.12. *Alternatively, we can use that each V_i is a manifold with corners, and then take a good cover $\{W^i_j\}$, however the above is more elementary.*

Let C be the small category with objects $I \times J \times \mathbb{N}$, so that there is exactly one morphism from $a = (i,j,k)$ to $b = (i',j',k')$ whenever $\operatorname{image} \sigma^{i,j}_k \subset \operatorname{image} \sigma^{i',j'}_{k'}$, and no morphisms otherwise. Let

$$F : C \to \mathcal{D}(\Delta^d)$$

be the functor $F(a) = \sigma^{i,j}_k$ for $a = (i,j,k)$, (the definition on morphisms is forced). For brevity, we then reset $\sigma_a := F(a)$.

For a smooth manifold with corners X, if $\mathcal{O}(X)$ denotes the category of topological subspaces of X with morphisms inclusions, then there is a forgetful functor

$$T : \mathcal{D}(X) \to \mathcal{O}(X)$$

which takes f to $\operatorname{image}(\widetilde{f})$. With all this in place, we obviously have a colimit in Top:

$$\Delta^d = \operatorname{colim}_C T \circ F,$$

Now by construction, for each $a \in C$ we may express:

$$\Sigma \circ \sigma_a = \Sigma_a \circ \sigma_a, \tag{5.4}$$

for some i and some $\Sigma_a : \Delta^d \to U_i \subset Y$ a smooth embedded d-simplex. Then for all $a \in C$ we have a chain of natural isomorphisms, whose composition will be denoted by $\phi_a : P_{\Sigma \circ \sigma_a} \to N_{\Sigma \circ \sigma_a}$:

$$P_{\Sigma \circ \sigma_a} = P_{\Sigma_a \circ \sigma_a} \to N_{\Sigma_a \circ \sigma_a} = N_{\Sigma \circ \sigma_a} \tag{5.5}$$

To better explain the second map, note that we have a composition of natural bundle maps:

$$P_{\Sigma_a \circ \sigma_a} \to P_i \to N, \tag{5.6}$$

with the first map the bundle map in the colimit diagram of (5.2), and the second map the bundle map in the colimit diagram of (5.3). The composition (5.6) gives a bundle map over $\Sigma_a \circ \sigma_a$. And so, by the defining universal property of the pull-back, there is a uniquely induced universal map

$$P_{\Sigma_a \circ \sigma_a} \to (\Sigma_a \circ \sigma_a)^* N = N_{\Sigma_a \circ \sigma_a},$$

which is a G-bundle isomorphism.

Now we have a natural functor $F_\Sigma : \mathcal{D}(\Delta^d) \to \mathcal{G}$, given by $F_\Sigma(\sigma) = P_{\Sigma \circ \sigma}$, and

$$P_\Sigma = \operatorname{colim}_C F_\Sigma \circ F. \tag{5.7}$$

Similarly,

$$N_\Sigma = \operatorname{colim}_C F'_\Sigma \circ F \tag{5.8}$$

where $F'(\sigma) = N_{\Sigma \circ \sigma}$. And the maps $\phi_a : P_{\Sigma \circ \sigma_a} \to N_{\Sigma \circ \sigma_a}$ induce a natural transformation of functors

$$\phi : F_\Sigma \circ F \to F'_\Sigma \circ F.$$

So that ϕ induces a map of the colimits:

$$h_\Sigma : P_\Sigma \to N_\Sigma,$$

by naturality, and this is an isomorphism of these smooth G-bundles. It is then clear that $\{h_\Sigma\}_\Sigma$ determines the bundle isomorphism $h : P \to N_\bullet$ we are looking for. $\square$

5.1. **Pullbacks of simplicial bundles.** Let $P \to X$ be a simplicial G-bundle over a smooth simplicial set X. And let $f : Y \to X$ be a smooth map of smooth simplicial sets. We define the pull-back simplicial G-bundle $f^*P \to Y$ by the functor $f^*P := P \circ \Delta^{sm} f$.

Note that the analogue of the following lemma is not true in the category of set fibrations. The pull-back by the composition is not the composition of pullbacks (except up to a natural isomorphism).

Lemma 5.13. *The pull-back is functorial. So that if $f : X \to Y$ and $g : Y \to Z$ are smooth maps of smooth simplicial sets, and $P \to Z$ is a smooth simplicial G-bundle over Z then*

$$(g \circ f)^* P = f^*(g^*(P)) \text{ an actual equality.}$$

Proof. This is of course trivial, as functor composition is associative:

$$(g \circ f)^* P = P \circ \Delta^{sm}(g \circ f) = P \circ (\Delta^{sm} g \circ \Delta^{sm} f) = (P \circ \Delta^{sm} g) \circ \Delta^{sm} f = f^*(g^* P).$$

□

6. Connections on simplicial G-bundles

Definition 6.1. *Let G be a Fréchet Lie group. A* **simplicial G-connection** D *on a simplicial G-bundle P over a smooth simplicial set X is for each d-simplex Σ of X, a smooth G-invariant Ehresmann G-connection D_Σ on P_Σ. This data is required to satisfy: if $f : \Sigma_1 \to \Sigma_2$ is a morphism in $\Delta(X)$ then*

$$P(f)^* D_{\Sigma_2} = D_{\Sigma_1}.$$

We say that D is **coherent** *if the same holds for all morphisms $f : \Sigma_1 \to \Sigma_2$ in $\Delta^{sm}(X)$. Will will often just say G-connection instead of simplicial G-connection, where there is no need to disambiguate.*

As with differential forms the coherence condition is very restrictive, and is not part of the basic definition.

Lemma 6.2. *G-connections on simplicial G-bundles exist and any pair of G-connections D_1, D_2 on a simplicial G-bundle P are* **concordant**. *The latter means that there is a G-connection on $\widetilde{D}$ on $P \times I$,*

$$I := [0,1]_\bullet$$

which restricts to D_1, D_2 on $P \times I_0$, respectively on $P \times I_1$, for $I_0, I_1 \subset I$ denoting the images of the two end point inclusions $\Delta^0_\bullet \to I$.

Proof. Suppose that $\Sigma : \Delta^d_{simp} \to X$ is a degeneracy of a 0-simplex $\Sigma_0 : \Delta^0_{simp} \to X$, meaning that there is a morphism from Σ to Σ_0 in $\Delta(X)$. Then $P_\Sigma = \Delta^d \times P_{\Sigma_0}$ (as previously equality indicates natural isomorphism) and we fix the corresponding trivial connection D_Σ on P_Σ. We then proceed inductively.

Suppose we have constructed connections D_Σ for all degeneracies of n-simplices, $n \geq 0$. We now extend this to all degeneracies of $(n+1)$-simplices. If Σ is a non-degenerate $(n+1)$-simplex then D_Σ is already determined over the boundary of Δ^{n+1}, as by the hypothesis D_Σ is already defined on all n-simplices, so extend D_Σ over all of Δ^{n+1} arbitrarily.

Thus, we have extended D_Σ to all $(n+1)$-simplices, as such a simplex is either non-degenerate or is a degeneracy of an n-simplex. If Σ' is an m-simplex that is a degeneracy of a $(n+1)$-simplex Σ^{n+1}, then $P_{\Sigma'} = pr^* P_{\Sigma^{n+1}}$ for a certain determined simplicial projection $pr : \Delta^m \to \Delta^{n+1}$, and we define $D_\Sigma = \widetilde{pr}^* D_{\Sigma^{n+1}}$. For $\widetilde{pr} : P_{\Sigma'} \to P_{\Sigma^{n+1}}$ the natural map in the pull-back square.

The second part of the lemma follows by an analogous argument, since we may just extend D_1, D_2 to a concordance connection $\widetilde{D}$, using the above inductive procedure.

□

Example 6.3. *Given a classical smooth G-connection D on a smooth principal G-bundle $Z \to Y$, we obviously get a simplicial G-connection on the induced simplicial G-bundle $N = Z_\bullet$. Concretely, this is defined by setting D_Σ on $N_\Sigma = \Sigma^* Z$ to be $\widetilde{\Sigma}^* D$, for $\widetilde{\Sigma} : \Sigma^* Z \to Z$ the natural map. This is called the* **induced simplicial connection***, and it may be denoted by $D_\bullet$. Going in the other direction is always possible if the given simplicial G-connection in addition satisfies coherence, but we will not elaborate.*

7. Chern-Weil homomorphism

7.1. **The classical case.** To establish notation we first discuss classical Chern-Weil homomorphism.

Let G be a Fréchet Lie group, and let $\mathfrak{g}$ denote its Lie algebra. Let P be a smooth G-bundle over a smooth manifold Y. Fix a G-connection D on P. Let $\operatorname{Aut} P_y$ denote the group of smooth G-torsor automorphisms of the fiber P_y of P over $y \in Y$. Note that $\operatorname{Aut} P_y \simeq G$ where $\simeq$ means non-canonically isomorphic. Then associated to D we have the classical curvature 2-form R^D on Y, understood as a 2-form valued in the vector bundle $\mathcal{P} \to Y$, whose fiber over $y \in Y$ is $\operatorname{lie} \operatorname{Aut} P_y$ - the Lie algebra of $\operatorname{Aut} P_y$.

Thus,

$$\forall v, w \in T_y Y : R^D(v, w) \in \mathcal{P}_y = \operatorname{lie} \operatorname{Aut} P_y.$$

Now, let ρ be a continuous symmetric multilinear functional:

$$\rho : \prod_{i=1}^{i=k} \mathfrak{g} \to \mathbb{R},$$

satisfying

$$\forall g \in G, \forall v \in \prod_{i=1}^{i=k} \mathfrak{g} : \rho(Ad_g(v)) = \rho(v).$$

Here if $v = (\xi_1, \ldots, \xi_n)$, $Ad_g(v) = (Ad_g(\xi_1), \ldots, Ad_g(\xi_n))$ is the adjoint action by the element $g \in G$. As ρ is Ad invariant, it uniquely determines multilinear maps with the same name:

$$\rho : \prod_{i=1}^{i=k} \operatorname{lie} \operatorname{Aut} P_y \to \mathbb{R},$$

by fixing any Lie-group isomorphism $\operatorname{Aut} P_y \to G$. We may now define a closed $\mathbb{R}$-valued $2k$-form $\omega^{\rho,D}$ on Y:

(7.1)
$$\omega^{\rho,D}(v_1, \ldots, v_{2k}) = \frac{1}{2k!} \sum_{\eta \in P_{2k}} \operatorname{sign} \eta \cdot \rho(R^D(v_{\eta(1)}, v_{\eta(2)}), \ldots, R^D(v_{\eta(2k-1)}, v_{\eta_{2k}})),$$

for P_{2k} the permutation group of a set with $2k$ elements, and where $v_1, \ldots, v_{2k} \in T_y Y$. Set

$$\alpha^{\rho,D} := \int \omega^{\rho,D}.$$

Then we define the classical Chern-Weil characteristic class:

$$c^\rho(P) = c^\rho_{2k}(P) := [\alpha^{\rho,D}] \in H^{2k}(X, \mathbb{R}). \tag{7.2}$$

7.2. **Chern-Weil homomorphism for smooth simplicial bundles.** Now let P be a simplicial G-bundle over a smooth simplicial set X. Fix a simplicial G-connection D on P.

For each simplex Σ^d, we have the curvature 2-form R^D_Σ of the connection D_Σ on P_Σ, defined as in the section just above. For concreteness:

$$\forall v, w \in T_z \Delta^d : R^D_\Sigma(v, w) \in \mathrm{lie}\,\mathrm{Aut}\, P_z,$$

for P_z the fiber of P_Σ over $z \in \Delta^d$.

As above, let ρ be an Ad invariant continuous symmetric multilinear functional:

$$\rho : \prod_{i=1}^{i=k} \mathfrak{g} \to \mathbb{R}.$$

As above ρ uniquely determines for each $z \in \Delta^d$ a symmetric multilinear map with the same name:

$$\rho : \prod_{i=1}^{i=k} \mathrm{lie}\,\mathrm{Aut}\, P_z \to \mathbb{R}.$$

We may now define a closed, $\mathbb{R}$-valued, simplicial differential $2k$-form $\omega^{\rho,D}$ on X:

$$\omega^{\rho,D}_\Sigma(v_1, \ldots, v_{2k}) = \frac{1}{2k!} \sum_{\eta \in P_{2k}} \mathrm{sign}\,\eta \cdot \rho(R^D_\Sigma(v_{\eta(1)}, v_{\eta(2)}), \ldots, R^D_\Sigma(v_{\eta(2k-1)}, v_{\eta_{2k}})),$$

for P_{2k} as above the permutation group of a set with $2k$ elements. Set

$$\alpha^{\rho,D} := \int \omega^{\rho,D}.$$

Lemma 7.1. *For $P \to X$ as above*

$$[\alpha^{\rho,D}] = [\alpha^{\rho,D'}] \in H^{2k}(X, \mathbb{R}),$$

for any pair of simplicial G-connections D, D' on P.

Proof. For D, D' as in the statement, fix a concordance simplicial G-connection $\widetilde{D}$, between D, D', on the G-bundle $P \times I \to X \times I$, as in Lemma 6.2. Then $\alpha^{\rho,\widetilde{D}}$ is a cocycle in $C^{2k}(X \times I, \mathbb{R})$ restricting to $\alpha^{\rho,D}, \alpha^{\rho,D'}$ on $X \times I_0, X \times I_1$.

Now the pair of inclusions

$$i_j : X \to X \times I \quad j = 0, 1$$

corresponding to the end points of I are homotopic and so $\alpha^{\rho,D}$, $\alpha^{\rho,D'}$ are cohomologous cocycles, cf. Section 4.1. $\square$

Then we define the associated Chern-Weil characteristic class:

$$c^\rho(P) = c^\rho_{2k}(P) := [\alpha^{\rho,D}] \in H^{2k}(X, \mathbb{R}),$$

(we may omit the subscript $2k$, as the degree $2k$ is implicitly determined by ρ.)

We have the expected naturality:

Lemma 7.2. *Let P be a simplicial G-bundle over Y, ρ as above and $f : X \to Y$ a smooth simplicial map. Then*

$$f^* c^\rho(P) = c^\rho(f^* P).$$

Proof. Let D be a simplicial G-connection on P. Define the pull-back connection f^*D on f^*P by $f^*D_\Sigma = D_{f(\Sigma)}$. Then f^*D is a simplicial G-connection on f^*P and clearly $\omega^{\rho, f^*D} = f^*\omega^{\rho,D}$, so that passing to cohomology we obtain our result. □

Proposition 7.3. *Let $G \hookrightarrow Z \to Y$ be an ordinary smooth principal G-bundle, and ρ as above. Let $Z_\bullet$ be the induced simplicial G-bundle over $Y_\bullet$ as in Example 5.7. Then the classes $c^\rho(Z_\bullet) \in H^{2k}(Y_\bullet, \mathbb{R})$ coincide with the classical Chern-Weil classes of Z. More explicitly, if $c^\rho(Z) \in H^{2k}(Y, \mathbb{R})$ is the classical Chern-Weil characteristic class as in (7.2), then*

$$|c^\rho(Z_\bullet)| = c^\rho(Z), \tag{7.3}$$

where $|c^\rho(Z_\bullet)|$ is as in Notation 4.7.

Proof. Fix a smooth G-connection D on Z. This induces a simplicial G-connection $D_\bullet$ on $Z_\bullet$, as in Example 6.3. Let $\omega^{\rho,D}$ denote the classical smooth Chern-Weil differential $2k$-form on Y, as in (7.1). Let $\alpha^{\rho,D} = \int \omega^{\rho,D} \in H^{2k}(Y, \mathbb{R})$. By its construction $\omega^{\rho,D_\bullet}$ is the simplicial differential form induced by $\omega^{\rho,D}$, where induced is as in Example 4.3. Consequently,

$$(I^\vee)^{-1}([\alpha^{\rho,D_\bullet}]) = [\alpha^{\rho,D}] = c^\rho(Z),$$

where $I^\vee$ is as in (4.7). □

8. The universal simplicial G-bundle

Briefly, a Grothendieck universe is a set $\mathcal{U}$ forming a model for set theory. That is if we interpret all terms of set theory as elements of $\mathcal{U}$, then all the set theoretic constructions keep us within $\mathcal{U}$. We will assume Grothendieck's axiom of universes which says that for any set X there is a Grothendieck universe $\mathcal{U} \ni X$. Intuitively, such a universe $\mathcal{U}$ is formed by from all possible set theoretic constructions starting with X. For example if $\mathcal{P}(X)$ denotes the power set of X, then $\mathcal{P}(X) \in \mathcal{U}$ and if $\{Y_i \in \mathcal{P}(X)\}_{i \in I}$ for $I \in \mathcal{U}$ is a collection then $\bigcup_i Y_i \in \mathcal{U}$. This may appear very natural, but we should note that this axiom is beyond ZFC. Although it is now a common axiom of modern set theory, especially in the context of category theory, c.f. [17]. In some contexts one works with universes implicitly. This is impossible here, as we need to establish certain universe independence.

Let G be a Fréchet Lie group. Let $\mathcal{U}$ be a Grothendieck universe satisfying:

$$\mathcal{U} \ni \{G\}, \quad \forall n \in \mathbb{N} : \mathcal{U} \ni \{\Delta^n\},$$

where Δ^n are the usual topological n-simplices. Such a $\mathcal{U}$ will be called ***G-admissible***. We construct smooth Kan complexes $BG^{\mathcal{U}}$ for each G-admissible $\mathcal{U}$. The homotopy type of $BG^{\mathcal{U}}$ will then be shown to be independent of $\mathcal{U}$, provided G has the homotopy type of CW complex. Moreover, in this case we will show that $|BG^{\mathcal{U}}| \simeq BG$, for BG the classical Milnor classifying space.

Definition 8.1. *A $\mathcal{U}$*-**small set** *is an element of $\mathcal{U}$. For X a smooth simplicial set, a smooth simplicial G-bundle $P \to X$ will be called $\mathcal{U}$*-**small** *if for each simplex Σ of X the bundle P_Σ is $\mathcal{U}$-small.*

8.1. **The classifying spaces $BG^{\mathcal{U}}$.** Let $\mathcal{U}$ be G-admissible. We define a simplicial set $BG^{\mathcal{U}}$, whose set of k-simplices $BG^{\mathcal{U}}(k)$ is the set of $\mathcal{U}$-small smooth simplicial G-bundles over $\Delta^k_\bullet$. The simplicial maps are just defined by pull-back so that given a map $i \in hom_\Delta([m],[n])$ the map

$$BG^{\mathcal{U}}(i) : BG^{\mathcal{U}}(n) \to BG^{\mathcal{U}}(m)$$

is just the natural pull-back:

$$BG^{\mathcal{U}}(i)(P) = i^*_\bullet P,$$

for $i_\bullet$, the induced map $i_\bullet : \Delta^m_\bullet \to \Delta^n_\bullet$, $P \in BG^{\mathcal{U}}(n)$ a simplicial G-bundle over $\Delta^n_\bullet$, and where the pull-back map $i^*_\bullet$ is as in Section 5.1. Then Lemma 5.13 insures that $BG^{\mathcal{U}}$ is a functor, so that we get a simplicial set $BG^{\mathcal{U}}$.

We define a smooth simplicial set structure g on $BG^{\mathcal{U}}$ as follows. Given a d-simplex $P \in BG^{\mathcal{U}}(d)$ the induced map

$$(g(P) = P_*) : \Delta^d_\bullet \to BG^{\mathcal{U}},$$

is defined naturally by

$$P_*(\sigma) := \sigma^*_\bullet P,$$

where P on the right is corresponding simplicial G-bundle $P \to \Delta^d_\bullet$. More explicitly, $\sigma \in \Delta^d_\bullet(k)$ is a smooth map $\sigma : \Delta^k \to \Delta^d$, $\sigma_\bullet : \Delta^k_\bullet \to \Delta^d_\bullet$ denotes the induced map and the pull-back is as previously defined. We need to check the push-forward functoriality Axiom 3.

Let $\sigma \in \Delta^d_\bullet(k)$, then for all $j \in \mathbb{N}, \rho \in \Delta^k_\bullet(j)$:

$$\begin{aligned}(P_*(\sigma))_*(\rho) &= (\sigma^*_\bullet P)_*(\rho)\\ &= \rho^*_\bullet(\sigma^*_\bullet P) \text{ by definition of } g.\end{aligned}$$

And

$$\begin{aligned}(P_* \circ \sigma_\bullet)_*(\rho) &= (\sigma_\bullet(\rho))^*_\bullet P\\ &= (\sigma_\bullet \circ \rho_\bullet)^* P \quad \text{as } \sigma_\bullet \text{ is smooth}\\ &= \rho^*_\bullet(\sigma^*_\bullet P).\end{aligned}$$

And so

$$(P_*(\sigma))_* = P_* \circ \sigma_\bullet,$$

so that $BG^{\mathcal{U}}$ is indeed a smooth simplicial set.

8.2. **The universal smooth simplicial G-bundle $EG^{\mathcal{U}}$.** In what follows V denotes $BG^{\mathcal{U}}$ for a general, G-admissible $\mathcal{U}$. There is a natural functor

$$E : \Delta^{sm}(V) \to \mathcal{G},$$

which we now describe.

A smooth map $P : \Delta^d_\bullet \to V$, uniquely corresponds to a simplex P^s of V via Lemma 3.6 which by construction of V corresponds to a simplicial G-bundle $P^b \to \Delta^d_\bullet$. In other words $P^b := P(id^d)$ for $id^d : \Delta^d \to \Delta^d$ the identity.

Notation 8.2. *Although we disambiguate here, we sometimes conflate the notation P, P^s, P^b with just P.*

Recalling that P^b is a functor on $\Delta^{sm}(\Delta^d_\bullet)$ we then set:

$$E(P) = P^b(id^d_\bullet).$$

We now define the action of E on morphisms. Suppose we have a morphism $m \in \Delta^{sm}(V)$:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "P_1"] & \Delta^d_\bullet \arrow[d, "P_2"] \\ & V, \end{tikzcd}$$

then we have:

$$\begin{aligned} P^b_1 = P_1(id^k) &= (P_2 \circ \widetilde{m}_\bullet)(id^k) \\ &= P_2(\widetilde{m}) \\ &= (P^b_2)_*(\widetilde{m}) \\ &= P^b_2 \circ \Delta^{sm}\widetilde{m}_\bullet. \end{aligned} \tag{8.1}$$

So that

$$P^b_1(id^k_\bullet) = P^b_2(\widetilde{m}_\bullet \circ id^k_\bullet) = P^b_2(\widetilde{m}_\bullet).$$

We have a tautological morphism $e_m \in \Delta^{sm}(\Delta^d_\bullet)$ corresponding to the diagram:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "\widetilde{m}_\bullet"] & \Delta^d_\bullet \arrow[d, "id^d_\bullet"] \\ & \Delta^d_\bullet. \end{tikzcd}$$

So we get a smooth G-bundle map:

$$P^b_2(e_m) : P^b_2(\widetilde{m}_\bullet) \to P^b_2(id^d_\bullet),$$

which is over the smooth map $\widetilde{m} : \Delta^k \to \Delta^d$ induced by $\widetilde{m}_\bullet$. And we set $E(m) = P^b_2(e_m)$.

We need to check that with these assignments E is a functor. Suppose we have a diagram:

$$\begin{tikzcd} \Delta^l_\bullet \arrow[r, "\widetilde{m}^0_\bullet"] \arrow[rrd, bend right, "P_0"] & \Delta^k_\bullet \arrow[r, "\widetilde{m}^1_\bullet"] \arrow[rd, "P_1"] & \Delta^d_\bullet \arrow[d, "P_2"] \\ & & V. \end{tikzcd}$$

Then $e_m = e_{m^1} \circ e'_{m^0}$ where e'_{m^0} is the diagram:

$$\begin{tikzcd} \Delta^l_\bullet \arrow[r, "\widetilde{m}^0_\bullet"] \arrow[rd, "\widetilde{m}_\bullet"] & \Delta^k_\bullet \arrow[d, "\widetilde{m}^1_\bullet"] \\ & \Delta^d_\bullet, \end{tikzcd}$$

and e_{m^1} is the diagram:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}^1_\bullet"] \arrow[rd, "\widetilde{m}^1_\bullet"] & \Delta^d_\bullet \arrow[d, "id^d_\bullet"] \\ & \Delta^d_\bullet. \end{tikzcd}$$

So

$$E(m) = P_2^b(m) = P_2^b(e_{m^1}) \circ P_2^b(e'_{m^0}) = E(m_1) \circ P_2^b(e'_{m^0}).$$

Now by the analogue of (8.1):

$$E(m_0) = P_1^b(e_{m^0}) = (P_2^b \circ \Delta^{sm}\widetilde{m}^1_\bullet)(e_{m^0}) = P_2^b(e'_{m_0}).$$

And so we get: $E(m) = E(m_1) \circ E(m_0)$. Thus, E is a functor.

By construction the functor E satisfies the compatibility condition, and hence determines a simplicial G-bundle. The universal simplicial G-bundle $EG^{\mathcal{U}}$ is then another name for E above, for some $G, \mathcal{U}$.

Proposition 8.3. *V is a Kan complex.*

Proof. Let

$$E : \Delta(V) \to \mathcal{G}$$

be the restriction of E, as above, to $\Delta(V) \subset \Delta^{sm}(V)$. Recall that $\Lambda^n_k \subset \Delta^n_{simp}$, denotes the sub-simplicial set corresponding to the "boundary" of Δ^n with the k'th face removed, where by k'th face we mean the face opposite to the k'th vertex. Let $h : \Lambda^n_k \to V$, $0 \leq k \leq n$, be a simplicial map, this is also called a horn. We need to construct an extension of h to Δ^n_{simp}. For simplicity we assume $n = 2$, as the general case is identical. Let

$$\Delta(h) : \Delta(\Lambda^n_k) \to \Delta(V)$$

be the induced functor. Set $P = E \circ \Delta(h)$. Clearly, to construct our extension we just need an appropriate extension of P over $\Delta(\Delta^n_{simp})$. (Appropriate, means that we need the compatibility condition of Definition 5.4 be satisfied.)

Lemma 8.4. *There is a natural transformation of $\mathcal{G}$ valued functors $tr : T \to P$, where T is the trivial functor $T : \Delta(\Lambda^n_k) \to \mathcal{G}$, $T(\sigma^d) = \Delta^d \times G$.*

Proof. Set $L := \Lambda^2_k$, with $k = 1$, again without loss of generality. There are three natural inclusions

$$i_j : \Delta^0_{simp} \to L,$$

$j = 0, 1, 2$, with i_1 corresponding to the inclusion of the horn vertex. The corresponding 0-simplices will just be denoted by $0, 1, 2$. Fix a G-bundle map (in this case just smooth G-torsor map):

$$\phi_1 : \Delta^0 \times G \to P(i_1).$$

Let

$$\sigma_{1,2} : \Delta^1_{simp} \to L$$

be the edge between vertexes $1, 2$, that is $\sigma_{1,2}(0) = 1$, $\sigma_{1,2}(1) = 2$. Then $P(\sigma_{1,2})$ is a smooth bundle over the contractible space Δ^1 and so we may find a G-bundle map

$$\phi_{1,2} : \Delta^1 \times G \to P(\sigma_{1,2}),$$

whose restriction to $\{0\} \times G$ is ϕ_1. Meaning:

$$\phi_{1,2} \circ (i_0 \times id) = \phi_1,$$

where

$$i_0 : \Delta^0 \to \Delta^1,$$

is the map $i_0(0) = 0$.

We may likewise construct a G-bundle map

$$\phi_{0,1} : \Delta^1 \times G \to P(\sigma_{0,1}),$$

(where $\sigma_{0,1}$ is defined analogously to $\sigma_{1,2}$), whose restriction to $\{1\} \times G$ is ϕ_1.

Then $\phi_{0,1}$, $\phi_{1,2}$ obviously glue to a natural transformation:

$$tr : T \to P.$$

$\square$

To continue, we have the trivial extension of T,

$$\widetilde{T} : \Delta(\Delta^2_{simp}) \to \mathcal{G},$$

defined by

$$\widetilde{T}(\sigma^d) = \Delta^d \times G.$$

And so by the lemma above it is clear that P likewise has an extension $\widetilde{P}$ to $\Delta(\Delta^2_{simp})$, but we need this extension to be $\mathcal{U}$-small so that we must be explicit. Let σ^2 denote the non-degenerate 2-simplex of Δ^2. It suffices to construct $\widetilde{P}_{\sigma^2} := \widetilde{P}(\sigma^2)$. Let

$$\sigma_{0,1}, \sigma_{1,2} : \Delta^1 \to \Delta^2$$

be the edge inclusions of the edges between the vertices $0, 1$, respectively $1, 2$. And let $e_{0,1}, e_{1,2}$ denote their images.

We then define a set theoretic (for the moment no topology) G-bundle

$$\widetilde{P}_{\sigma^2} \xrightarrow{p} \Delta^2$$

by the following conditions:

$$\sigma_{0,1}^* \widetilde{P}_{\sigma^2} = P(\sigma_{0,1}),$$
$$\sigma_{1,2}^* \widetilde{P}_{\sigma^2} = P(\sigma_{1,2}),$$
$$P_{\sigma^2}|_{(\Delta^2)^\circ} = (\Delta^2)^\circ \times G,$$

where $(\Delta^2)^\circ$ denotes the topological interior of $\Delta^2 \subset \mathbb{R}^2$, and where the projection map p is natural.

We now discuss the topology. Define the smooth G-bundle maps

$$\phi_{0,1}^{-1} : P(\sigma_{0,1}) \to \Delta^2 \times G,$$

$$\phi_{1,2}^{-1} : P(\sigma_{1,2}) \to \Delta^2 \times G,$$

over $\sigma_{0,1}, \sigma_{1,2}$, as in the proof of the lemma above. Let d_0 be any metric on $\Delta^2 \times G$ inducing the natural product topology. The topology on $\widetilde{P}_{\sigma^2}$ will be given by the d-metric topology, for d extending d_0 on $(\Delta^2)^\circ \times G \subset \widetilde{P}_{\sigma^2}$, and defined as follows. For $y_1 \in \widetilde{P}_{\sigma^2}$ with $p(y_1) \in e_{0,1}$, y_2 arbitrary, $d(y_1, y_2) = d_0(\phi_{0,1}^{-1}(y_1), y_2)$. Likewise, for $y_1 \in \widetilde{P}_{\sigma^2}$ with $p(y_1) \in e_{1,2}$, y_2 arbitrary, $d(y_1, y_2) = d_0(\phi_{1,2}^{-1}(y_1), y_2)$. This defines $\widetilde{P}_{\sigma^2}$ as a topological G-bundle over Δ^2.

There is a natural topological G-bundle trivialization

$$\xi : \widetilde{P}_{\sigma^2} \to \Delta^2 \times G$$

defined as follows. $\xi(y) = y$ when $p(y) \in (\Delta^2)^\circ$ and $\xi(y) = \phi_{0,1}^{-1}(y)$ when $p(y) \in e_{0,1}$, $\xi(y) = \phi_{1,2}^{-1}(y)$ when $p(y) \in e_{0,2}$. We then take the smooth structure on $\widetilde{P}_{\sigma^2}$ to be the smooth structure pulled back by ξ. By construction $\widetilde{P}_{\sigma^2}$ is $\mathcal{U}$-small, as all the constructions take place in $\mathcal{U}$. Moreover, by construction $\sigma_{0,1}^* \widetilde{P}_{\sigma^2} = P_{\sigma_{0,1}}$ as a smooth G-bundle and $\sigma_{1,2}^* \widetilde{P}_{\sigma^2} = P_{\sigma_{1,2}}$ as a smooth G-bundle, which readily follows by the fact that the maps $\phi_{0,1}, \phi_{1,2}$ are smooth G-bundle maps. Thus, we have constructed the needed extension. □

Theorem 8.5. *Let X be a smooth simplicial set. $\mathcal{U}$-small simplicial G-bundles $P \to X$ are classified by smooth maps*

$$f_P : X \to BG^{\mathcal{U}}.$$

Specifically:

(1) *For every $\mathcal{U}$-small P there is a natural smooth map $f_P : X \to BG^{\mathcal{U}}$ so that*

$$f_P^* EG^{\mathcal{U}} \simeq P$$

as simplicial G-bundles. We say in this case that f_P **classifies** *P.*

(2) *If P_1, P_2 are isomorphic $\mathcal{U}$-small smooth simplicial G-bundles over X then any classifying maps f_{P_1}, f_{P_2} for P_1, respectively P_2 are smoothly homotopic, as in Definition 3.14.*

(3) *If $X = Y_\bullet$ for Y a smooth manifold and $f, g : X \to BG^{\mathcal{U}}$ are smoothly homotopic then $P_f = f^* EG^{\mathcal{U}}, P_g = g^* EG^{\mathcal{U}}$ are isomorphic simplicial G-bundles.*

Proof. Set $V = BG^{\mathcal{U}}$, $E = EG^{\mathcal{U}}$. Let $P \to X$ be a $\mathcal{U}$-small simplicial G-bundle. Define $f_P : X \to V$ naturally by:

$$f_P(\Sigma) = \Sigma_*^* P, \tag{8.2}$$

where $\Sigma \in \Delta^d(X)$, $\Sigma_* : \Delta^d_\bullet \to X$, the induced map, and the pull-back $\Sigma_*^* P$ our usual simplicial G-bundle pull-back. We check that the map f_P is simplicial.

Let $m : [k] \to [d]$ be a morphism in Δ. We need to check that the following diagram commutes:

$$\begin{tikzcd} X(d) \arrow[r, "X(m)"] \arrow[d, "f_P"] & X(k) \arrow[d, "f_P"] \\ V(d) \arrow[r, "V(m)"] & V(k). \end{tikzcd}$$

Let $\Sigma \in X(d)$, then by push-forward functoriality Axiom 3 $(X(m)(\Sigma))_* = \Sigma_* \circ m_\bullet$ where $m_\bullet : \Delta^k_\bullet \to \Delta^d_\bullet$ is the simplicial map induced by $m : \Delta^k \to \Delta^m$. And so

$$f_P(X(m)(\Sigma)) = (\Sigma_* \circ m_\bullet)^* P = m_\bullet^*(\Sigma_*^* P) = V(m)(f_P(\Sigma)),$$

where the second equality uses Lemma 5.13. And so the diagram commutes.

We now check that f_P is smooth. Let $\Sigma \in X(d)$, then we have:

$$\begin{aligned} (f_P(\Sigma))_*(\sigma) &= \sigma_\bullet^*(\Sigma_*^* P) \\ &= (\Sigma_* \circ \sigma_\bullet)^* P \quad \text{Lemma 5.13} \\ &= (\Sigma_*(\sigma))_*^* P \quad \text{as } \Sigma_* \text{ is smooth, Lemma 3.6} \\ &= (f_P \circ \Sigma_*)(\sigma), \end{aligned}$$

and so f_P is smooth.

Lemma 8.6. $f_P^* E = P$.

Proof. Let $\Sigma : \Delta^d_\bullet \to X$ be smooth, and $\sigma \in \Delta^d_\bullet$. Then we have:

$$\begin{aligned} \Delta^{sm} f_P(\Sigma)(\sigma) = (f_P \circ \Sigma)(\sigma) = f_P(\Sigma(\sigma)) &= (\Sigma(\sigma)_*^*)P \text{ by definition of } f_P \\ &= (\Sigma^* \circ \sigma_\bullet)^* P \text{ as } \Sigma \text{ is smooth} \\ &= \sigma_\bullet^*(\Sigma^* P) \text{ Lemma 5.13} \\ &= (\Sigma^* P)_*(\sigma). \end{aligned} \tag{8.3}$$

So $\Delta^{sm} f_P(\Sigma) = (\Sigma^* P)_*$. Then

$$\begin{aligned} f_P^* E(\Sigma) = (E \circ \Delta^{sm} f_P)(\Sigma) &= E((\Sigma^* P)_*) \text{ by the above} \\ &= (\Sigma^* P)(id^d_\bullet) \text{ definition of } E \\ &= P(\Sigma). \end{aligned}$$

So $f_P^* E = P$ on objects.

Now let m be a morphism:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, "\Sigma_1"] & \Delta^d_\bullet \arrow[d, "\Sigma_2"] \\ & X, \end{tikzcd}$$

in $\Delta^{sm}(X)$. We then have:

$$\begin{aligned} f_P^* E(m) &= E(\Delta^{sm} f_P(m)) \\ &= (\Delta^{sm} f_P(\Sigma_2))^b(e_m) \text{ by definition of } E \\ &= \Sigma_2^* P(e_m) \text{ by } (8.3) \\ &= (P \circ \Delta^{sm}\Sigma_2)(e_m), \end{aligned}$$

where e_m is as in the definition of E. But $\Delta^{sm}\Sigma_2(e_m)$ is the diagram:

$$\begin{tikzcd} \Delta^k_\bullet \arrow[r, "\widetilde{m}_\bullet"] \arrow[dr, bend right, "\Sigma_2 \circ \widetilde{m}_\bullet"] & \Delta^d_\bullet \arrow[d, "\Sigma_2 \circ id^d_\bullet"] \\ & X, \end{tikzcd}$$

i.e. it is the diagram m. So $(P \circ \Delta^{sm}\Sigma_2)(e_m) = P(m)$. Thus, $f_P^* E = P$ on morphisms. $\square$

So we have proved the first part of the theorem. We now prove the second part of the theorem. Suppose that P_1', P_2' are isomorphic $\mathcal{U}$-small simplicial G-bundles over X. Let $f_{P_1'}$, $f_{P_2'}$ be some classifying maps for P_1', P_2'. In particular, there is an isomorphism of $\mathcal{U}$-small simplicial G-bundles

$$\phi : (P_1 := f_{P_1'}^* E) \to (P_2 := f_{P_2'}^* E).$$

We construct a $\mathcal{U}$-small simplicial G-bundle $\widetilde{P}$ over $X \times I$ as follows, where $I = \Delta^1_\bullet$ as before. Let σ be a k-simplex of X. Then ϕ specifies a G-bundle diffeomorphism $\phi_\sigma : P_1(\sigma) \to P_2(\sigma)$ over the identity map $\Delta^k \to \Delta^k$. Let M_σ be the mapping cylinder of ϕ_σ. So that

$$M_\sigma = (P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma))/\sim, \tag{8.4}$$

for $\sim$ the equivalence relation generated by the condition

$$(x, 1) \in P_1(\sigma) \times \Delta^1 \sim \phi(x) \in P_2(\sigma).$$

Then M_σ is a smooth G-bundle over $\Delta^k \times \Delta^1$. Let pr_X, pr_I be the natural projections of $X \times I$, to X respectively I. Let Σ be a d-simplex of $X \times I$. Set $\sigma_1 = pr_X \Sigma$, and $\sigma_2 = pr_I(\Sigma)$. Let $id^d : \Delta^d \to \Delta^d$ be the identity, so

$$(id^d, \sigma_2) : \Delta^d \to \Delta^d \times \Delta^1,$$

is a smooth map, where σ_2 is the corresponding smooth map $\sigma_2 : \Delta^d \to \Delta^1 = [0, 1]$. We then define

$$\widetilde{P}_\Sigma := (id^d, \sigma_2)^* M_{\sigma_1},$$

which is a smooth G-bundle over Δ^d.

Suppose that $\rho : \sigma \to \sigma'$ is a morphism in $\Delta^{sm}(X)$, for σ a k-simplex and σ' a d-simplex. As ϕ is a simplicial G-bundle map, we have a commutative diagram:

$$\begin{tikzcd} P_1(\sigma) \arrow[r, "P_1(\rho)"] \arrow[d, "\phi_\sigma"] & P_1(\sigma') \arrow[d, "\phi_{\sigma'}"] \\ P_2(\sigma) \arrow[r, "P_2(\rho)"] & P_2(\sigma'). \end{tikzcd} \tag{8.5}$$

And so we get a naturally induced (by the pair of maps $P_1(\rho), P_2(\rho)$) bundle map:

$$\begin{tikzcd} M_\sigma \arrow[r, "g_\rho"] \arrow[d] & M_{\sigma'} \arrow[d] \\ \Delta^k \times \Delta^1 \arrow[r, "\widetilde{\rho} \times id"] & \Delta^d \times \Delta^1 . \end{tikzcd} \tag{8.6}$$

More explicitly, let $q_\sigma : P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma) \to M_\sigma$ denote the quotient map. Then

$$g_\rho : P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma) \to M_{\sigma'}$$

is defined by:

$$g_\rho(x,t) = q_{\sigma'}((P_1(\rho)(x), t)) \in M_{\sigma'},$$

for

$$(x,t) \in P_1(\sigma) \times \Delta^1,$$

while $g_\rho(y) = q_{\sigma'}(P_2(\rho)(y))$ for $y \in P_2(\sigma)$. By commutativity of (8.5) g_ρ induces the map $g_\rho : M_\sigma \to M_{\sigma'}$, appearing in (8.6).

Now suppose we have a morphism $m : \Sigma \to \Sigma'$ in $\Delta^{sm}(X \times I)$, where Σ is a k-simplex and Σ' is a d-simplex. Then we have a commutative diagram:

$$\begin{tikzcd} M_\sigma \arrow[r, "g_{pr_X}(m)"] \arrow[d] & M_{\sigma'} \arrow[d] \\ \Delta^k \times \Delta^1 \arrow[r, "\widetilde{m} \times id"] & \Delta^d \times \Delta^1 \\ \Delta^k \arrow[u, "h_1"] \arrow[r, "\widetilde{m}"] & \Delta^d \arrow[u, "h_2"'] \\ \widetilde{P}_\Sigma \arrow[u] \arrow[uuu, bend left=60] & \widetilde{P}_{\Sigma'} \arrow[u] \arrow[uuu, bend right=60] \end{tikzcd} \tag{8.7}$$

where $h_1 = (id^k, pr_I(\Sigma))$ and $h_2 = (id^d, pr_I(\Sigma'))$. We then readily get an induced natural bundle map:

$$\widetilde{P}(m) : \widetilde{P}_\Sigma \to \widetilde{P}_{\Sigma'},$$

as left most and right most arrows in the above commutative diagram are the natural maps in pull-back squares, and so by universality of the pull-back such a map exists and is uniquely determined. Of course $\widetilde{P}(m)$ is the unique map making the whole diagram (8.7) commute.

With the above assignments, it is immediate that $\widetilde{P}$ is indeed a functor, by the uniqueness of the assignment $\widetilde{P}(m)$. And this determines our $\mathcal{U}$-small smooth simplicial G-bundle $\widetilde{P} \to X \times I$. By the first part of the theorem, we have an induced smooth classifying map $f_{\widetilde{P}} : X \times I \to V$. By construction, it is a homotopy between $f_{P_1'}, f_{P_2'}$. [2] So we have verified the second part of the theorem.

[2] To be perfectly formal, we must be careful with identification here. For the same reason that fixing the standard construction of set theoretic pull-back, a bundle $P \to B$ is not set theoretically equal to the bundle $id^* P \to B$, for $id : B \to B$ the identity, (but they are of course naturally

We now prove the third part of the theorem. Suppose that $f, g : X \to V$ are smoothly homotopic, and let $H : X \times I \to V$ be the corresponding smooth homotopy. By Lemma 5.11, the bundles P_f, P_g are induced by smooth G-bundles P'_f, P'_g over Y. Now $P_H = H^* E$ is a simplicial G-bundle over $X \times I = (Y \times [0,1])_\bullet$ and hence by Lemma 5.11 P_H is also induced by a smooth G-bundle P'_H over $Y \times [0,1]$. We may clearly in addition arrange that P'_H restricts to $P'_f \sqcup P'_g$ over $Y \times \partial[0,1]$. It follows that P'_f, P'_g are smoothly concordant and hence isomorphic smooth G-bundles, and so P_f, P_g are isomorphic simplicial G-bundles.

□

We now study the dependence on a Grothendieck universe $\mathcal{U}$.

Theorem 8.7. *Let G be a Fréchet Lie group having the homotopy type of a CW complex. Let $\mathcal{U}$ be a G-admissible universe, let $|BG^{\mathcal{U}}|$ denote the geometric realization of $BG^{\mathcal{U}}$ and let BG^{top} denote the classical classifying space of G as defined by the Milnor construction [23]. Then there is a homotopy equivalence*

$$e^{\mathcal{U}} : |BG^{\mathcal{U}}| \to BG^{top},$$

which is natural in the sense that if $\mathcal{U}' \ni \mathcal{U}$ then

$$[e^{\mathcal{U}'} \circ |i^{\mathcal{U},\mathcal{U}'}|] = [e^{\mathcal{U}}], \tag{8.8}$$

where $|i^{\mathcal{U},\mathcal{U}'}| : |BG^{\mathcal{U}}| \to |BG^{\mathcal{U}'}|$ is the map of geometric realizations, induced by the natural inclusion $i^{\mathcal{U},\mathcal{U}'} : BG^{\mathcal{U}} \to BG^{\mathcal{U}'}$ and where $[\cdot]$ denotes the homotopy class. In particular, for G as above the homotopy type of $BG^{\mathcal{U}}$ is independent of the choice of G-admissible $\mathcal{U}$.

Proof. Set $V := BG^{\mathcal{U}}$, and $E := EG^{\mathcal{U}}$. And define:

$$|E| := \operatorname{colim}_{\Delta(V)} E$$

where $E : \Delta(V) \to \mathcal{G}$ is as previously discussed, and where the colimit is understood to be in the category of topological G-bundles.

Then we have a topological G-fibration

$$|E| \to |V|,$$

which is classified by a map

$$e = e^{\mathcal{U}} : |V| \to BG^{top},$$

uniquely determined up to homotopy. In particular,

$$|E| \simeq e^* EG^{top}, \tag{8.9}$$

where EG^{top} is the universal G-bundle over BG^{top} and where $\simeq$ in this argument will always mean G-bundle isomorphism.

We will show that e induces an isomorphism of all homotopy groups. At this point we will use the assumption that G has the homotopy type of a CW complex, so that BG^{top} has the homotopy type of a CW complex, and so e must then be a homotopy equivalence by the Whitehead theorem.

isomorphic.) However, this slight ambiguity can be fixed following the same simple idea as in the proof of Proposition 8.3.

We first prove an auxiliary lemma. Let $\mathcal{U}'$ be a universe enlargement of $\mathcal{U}$, that is $\mathcal{U}'$ is a universe with $\mathcal{U}' \ni \mathcal{U}$. There is a natural inclusion map

$$i = i^{\mathcal{U},\mathcal{U}'} : V \to V' := BG^{\mathcal{U}'},$$

and

$$i^*(E' := EG^{\mathcal{U}'}) = E\,^{3}.$$

Lemma 8.8. *Let G be any Fréchet Lie group and let V be as above. Then*

$$i_* : \pi_k^{sm}(V) \to \pi_k^{sm}(V')$$

is a set isomorphism for all $k \in \mathbb{N}$, where π_k^{sm} are as in Definition 3.15.

Proof. We show that i_* is injective. Let $f, g : S^k_\bullet \to V$ be a pair of smooth maps. Let P_f, P_g denote the smooth bundles over S^k induced via Lemma 5.11 by f^*E, g^*E. Set $f' = i \circ f$, $g' = i \circ g$ and suppose that $F : S^k_\bullet \times I \to V'$ is a smooth homotopy between f', g'. By Lemma 5.11 the simplicial bundle F^*E' is induced by a smooth bundle $P_F \to S^k \times I$.

In particular P_f, P_g are classically isomorphic smooth $\mathcal{U}$-small G-bundles. Fix an isomorphism:

$$\begin{tikzcd} P_f \arrow[r, "\phi"] \arrow[d] & P_g \arrow[dl] \\ S^k & \end{tikzcd}.$$

Taking the mapping cylinder for the G-bundle isomorphism ϕ gives us a smooth G-bundle $P' \to S^k \times I$ that is $\mathcal{U}$-small by construction.

Finally, P' induces a smooth simplicial G-bundle H over $S^k_\bullet \times I$ that by construction is $\mathcal{U}$-small. The classifying map $f_H : S^k_\bullet \times I \to V$ then gives a smooth homotopy between f, g.

We now show surjectivity of i_*. Let $f : S^k_\bullet \to V'$ be smooth. By Lemma 5.11 the simplicial G-bundle f^*E' is induced by a smooth G-bundle $P' \to S^k$. Any such bundle is obtained by the clutching construction, that is P' is isomorphic as a smooth G-bundle to the bundle:

$$C = D^k_- \times G \sqcup D^k_+ \times G/\sim,$$

where D^k_+, D^k_- are two copies of the standard closed k-ball in $\mathbb{R}^k$, and $\sim$ is the equivalence relation generated by: for

$$(d, g) \in D^k_- \times G$$

$$(d, g) \sim \widetilde{\phi}(d, g) \in D^k_+ \times G,$$

where

$$\widetilde{\phi} : \partial D^k_- \times G \to \partial D^k_+ \times G, \quad \widetilde{\phi}(d, x) = (d, \phi(d)^{-1} \cdot x),$$

for some smooth $\phi : S^{k-1} \to G$. Then C is $\mathcal{U}$-small, since this gluing construction is carried out in $\mathcal{U}$.

Let

$$C_\bullet \to S^k_\bullet$$

[3]This is indeed an equality, not just a natural isomorphism.

denote the induced $\mathcal{U}$-small smooth simplicial G-bundle. Now $C_\bullet$ and f^*E' are induced by isomorphic $\mathcal{U}'$-small smooth G-bundles, hence are isomorphic $\mathcal{U}'$-small simplicial G-bundles.

By Part 2 of Theorem 8.5, the classifying map $f_{C_\bullet} \to V'$ is smoothly homotopic to f. Since $C_\bullet$ is $\mathcal{U}$-small $f_{C_\bullet}$ it is also classified by a smooth map $f' : S^k_\bullet \to V$. It is immediate that $[i \circ f'] = [f_{C_\bullet}]$, since $i^*E' = E$, and so $i_*([f']) = [f]$. $\square$

Corollary 8.9. *Let G be any Fréchet Lie group, and V as above. Let $\mathcal{P}^{\mathcal{U}'}$ denote the set of isomorphism classes of $\mathcal{U}'$-small simplicial G-bundles P over $S^k_\bullet$, for $\mathcal{U}' \ni \mathcal{U}$. Then the composition map c, $\pi_k^{sm}(V) \to \mathcal{P}^{\mathcal{U}} \to \mathcal{P}^{\mathcal{U}'}$:*

$$c([f]) = j([P_f := f^*E])$$

is a set bijection, where $j : \mathcal{P}^{\mathcal{U}} \to \mathcal{P}^{\mathcal{U}'}$ is the natural map.

Proof. c is well-defined by the third part of Theorem 8.5. To see that it is injective note that if $c([f_0]) = c([f_1])$ then P_{f_0} and P_{f_1} are isomorphic (as j is clearly injective) and so $[f_0] = [f_1]$ by the second part Theorem 8.5.

We now prove surjectivity. Let P represent a class in $\mathcal{P}^{\mathcal{U}'}$. By the first part of Theorem 8.5, P is classified by some smooth map:

$$f' : S^k_\bullet \to V' = BG^{\mathcal{U}'}.$$

By the preceding lemma there is a smooth map $f_P : S^k_\bullet \to V$ so that

$$[i \circ f_P] = [f'] \in \pi_k^{sm}(V'),$$

where $i : V \to V'$ is the natural inclusion as before. So by the third part of Theorem 8.5, f_P^*E is equivalent to i^*P in $\mathcal{P}^{\mathcal{U}}$, as they have homotopic classifying maps. But then $j([f_P^*E] = [P]$. Thus, c is surjective.

$\square$

We now return to the proof of the theorem, and specifically to the proof of surjectivity of e.

Let $f : S^k \to BG^{top}$ be a continuous map. By Müller-Wockel [25], main result, the bundle $P_f := f^*EG^{top}$ is topologically isomorphic to a smooth G-bundle $P' \to S^k$. By the axiom of universes P' is $\mathcal{U}_0$-small for some G-admissible $\mathcal{U}_0 \ni \mathcal{U}$. So we obtain a $\mathcal{U}_0$-small simplicial G-bundle $P'_\bullet \to S^k_\bullet$.

Define

$$|P'_\bullet| := \operatorname{colim}_{\Delta(S^k_\bullet)} P'_\bullet,$$

recalling that $P'_\bullet$ is a functor $\Delta(S^k_\bullet) \to \mathcal{G}$, where as before the colimit is understood to be in the category of topological G-bundles.

By Corollary 8.9 $P'_\bullet \simeq g^*E$ for some smooth

$$g : S^k_\bullet \to V,$$

where $\simeq$ is an isomorphism of simplicial G-bundles. Then $|P'_\bullet| \to |S^k_\bullet|$ is a topological G-bundle classified by $e \circ |g|$, for

$$|g| : |S^k_\bullet| \to |V|,$$

the naturally induced topological map.

By construction, there is a topological G-bundle map $|P'_\bullet| \to P'$, over the natural map $|S^k_\bullet| \to S^k$ as P' is a co-cone for the corresponding colimit diagram in $\mathcal{G}$. And so P' and hence P_f, as a topological G-bundle is isomorphic to $h^*|P'_\bullet|$, where

$$h : S^k \to |S^k_\bullet|$$

represents the generator of $\pi_k(|S^k_\bullet|)$. Here, the notation $\pi_k(Y)$ means the set of free homotopy classes of maps $S^k \to Y$.

Thus, $e \circ |g| \circ h$ represents the free homotopy class $[f]$ and so $e_* : \pi_k(V) \to \pi_k(BG^{top})$ is surjective.

We prove injectivity. Let $f_0, f_1 : S^k \to |V|$ be continuous. Let $P_j \to S^k$ be smooth G-bundles topologically isomorphic to $f_j^*|E|$, $j = 0, 1$. Again P_j exists by the main result of [25].

By Corollary 8.9, $P_{j,\bullet}$ are classified by smooth maps:

$$g_j : S^k_\bullet \to V.$$

As before we then represent the class $[f_j]$, by $|g_j| \circ h$ for $h : S^k \to |S^k_\bullet|$ as above.

Now suppose that $[e \circ f_0] = [e \circ f_1]$. Then by [25] P_j are smoothly isomorphic G-bundles. Thus, $P_{j,\bullet}$ are isomorphic and so by Part 2 of Theorem 8.5 g_j are smoothly homotopic. Consequently, $|g_j|$ are homotopic and so $[f_0] = [f_1]$.

So $e_* : \pi_k(V) \to \pi_k(BG^{top})$ is a set isomorphism. It is an elementary consequence of topology that $e_* : \pi_k(V, v_0) \to \pi_k(BG^{top}, e(v_0))$ is a group isomorphism. And so we conclude that e is a homotopy equivalence.

Finally, we show naturality. Let

$$|i^{\mathcal{U},\mathcal{U}'}| : |V| \to |V'|$$

denote the map induced by the inclusion $i^{\mathcal{U},\mathcal{U}'}$. Since $E = (i^{\mathcal{U},\mathcal{U}'})^* E'$, (an actual equality), we have that

$$|E| \simeq |i^{\mathcal{U},\mathcal{U}'}|^* |E'|$$

and so

$$|E| \simeq |i^{\mathcal{U},\mathcal{U}'}|^* \circ (e^{\mathcal{U}'})^* EG^{top},$$

by (8.9), from which the conclusion immediately follows. □

9. The universal Chern-Weil homomorphism

Let G be a Fréchet Lie group and $\mathfrak{g}$ its lie algebra. Pick any simplicial G-connection D on $EG^{\mathcal{U}} \to BG^{\mathcal{U}}$. Then given any Ad invariant symmetric multilinear continuous functional:

$$\rho : \prod_{i=1}^{i=k} \mathfrak{g} \to \mathbb{R},$$

applying the theory of Section 7 we obtain the simplicial Chern-Weil differential $2k$-form $\omega^{\rho,D}$ on $BG^{\mathcal{U}}$. And we obtain an associated cohomology class $c^{\rho,\mathcal{U}} \in H^{2k}(BG^{\mathcal{U}}, \mathbb{R})$. We thus first arrive at an abstract form of the universal Chern-Weil homomorphism.

Proposition 9.1. *Let G be a Fréchet Lie group and $\mathcal{U}$ a G-admissible Grothendieck universe. There is an algebra homomorphism:*
$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG^{\mathcal{U}}, \mathbb{R}),$$
sending ρ as above to $c^{\rho,\mathcal{U}}$ and satisfying the following. Let $G \hookrightarrow Z \to Y$ be a $\mathcal{U}$-small smooth principal G-bundle. Let $c^\rho(Z_\bullet) \in H^{2k}(Y_\bullet, \mathbb{R})$ denote the Chern-Weil class corresponding to ρ. Then
$$f_{Z_\bullet}^* c^{\rho,\mathcal{U}} = c^\rho(Z_\bullet),$$
where $f_{Z_\bullet} : Y \to BG^{\mathcal{U}}$ is the classifying map of $Z_\bullet$.

Proof. This follows immediately by Lemma 7.2. □

Suppose now that G has the homotopy type of a CW complex. Let $e^{\mathcal{U}}$ be as in Theorem 8.7. We define the associated cohomology class
$$c^\rho := e_*^{\mathcal{U}}(|c^{\rho,\mathcal{U}}|) \in H^{2k}(BG^{top}, \mathbb{R}),$$
where the G-admissible universe $\mathcal{U}$ is chosen arbitrarily, where the pushforward means pull-back by the homotopy inverse, and where $|c^{\rho,\mathcal{U}}| \in H^{2k}(|BG^{\mathcal{U}}|, \mathbb{R})$ is as in Notation 4.7.

Lemma 9.2. *The cohomology class c^ρ is well-defined.*

Proof. Given another choice of a G-admissible universe $\mathcal{U}'$, let $\mathcal{U}'' \supset \{\mathcal{U}, \mathcal{U}'\}$ be a common universe enlargement. By Lemma 7.2 and Lemma 4.8
$$|i^{\mathcal{U},\mathcal{U}''}|^*(|c^{\rho,\mathcal{U}''}|) = |c^{\rho,\mathcal{U}}|.$$
Since $|i^{\mathcal{U},\mathcal{U}''}|$ is a homotopy equivalence we conclude that
$$|i^{\mathcal{U},\mathcal{U}''}|_*(|c^{\rho,\mathcal{U}}|) = |c^{\rho,\mathcal{U}''}|,$$
where $|i^{\mathcal{U},\mathcal{U}''}|_*$ denotes the pull-back by the homotopy inverse. Consequently, by the naturality part of Theorem 8.7 and the equation above, we have
$$e_*^{\mathcal{U}}(|c^{\rho,\mathcal{U}}|) = e_*^{\mathcal{U}''} \circ |i^{\mathcal{U},\mathcal{U}''}|_*(|c^{\rho,\mathcal{U}})| = e_*^{\mathcal{U}''}(|c^{\rho,\mathcal{U}''}|).$$
In the same way we have:
$$e_*^{\mathcal{U}'}(|c^{\rho,\mathcal{U}'}|) = e_*^{\mathcal{U}''}(|c^{\rho,\mathcal{U}''}|).$$
So
$$e_*^{\mathcal{U}}(|c^{\rho,\mathcal{U}}|) = e_*^{\mathcal{U}'}(|c^{\rho,\mathcal{U}'}|),$$
and so we are done.

□

We call $c^\rho \in H^{2k}(BG^{top}, \mathbb{R})$ ***the universal Chern-Weil characteristic class associated to*** ρ.

Let $\mathbb{R}[\mathfrak{g}]$ denote the algebra of continuous polynomial functions on $\mathfrak{g}$. And let $\mathbb{R}[\mathfrak{g}]^G$ denote the subalgebra of fixed points by the adjoint G action. By classical algebra, degree k homogeneous elements of $\mathbb{R}[\mathfrak{g}]^G$ are in correspondence with continuous symmetric G-invariant multilinear functionals $\Pi_{i=1}^k \mathfrak{g} \to \mathbb{R}$. Then to summarize we have the following theorem purely about the classical classifying space BG^{top}, reformulating Theorem 1.1 of the introduction:

Theorem 9.3. *Let G be a Fréchet Lie group having the homotopy type of a CW complex. There is an algebra homomorphism:*

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG^{top}, \mathbb{R}),$$

sending ρ as above to c^ρ as above and satisfying the following. Let $G \hookrightarrow Z \to Y$ be a smooth principal G-bundle. Let $c^\rho(Z) \in H^{2k}(Y)$ denote the classical Chern-Weil class associated to ρ. Then

$$f_Z^* c^\rho = c^\rho(Z),$$

where $f_Z : Y \to BG^{top}$ is the classifying map of the underlying topological G-bundle.

Proof. Let $\mathcal{U}_0 \ni Z$ be a G-admissible Grothendieck universe. By Proposition 9.1

$$c^\rho(Z_\bullet) = f_{Z_\bullet}^*(c^{\rho,\mathcal{U}_0}).$$

And by Proposition 7.3, $|c^\rho(Z_\bullet)| = c^\rho(Z)$. So we have

$$\begin{aligned} c^\rho(Z) &= |c^\rho(Z_\bullet)| \\ &= |f_{Z_\bullet}^*(c^{\rho,\mathcal{U}_0})| \\ &= N^* \circ |f_{Z_\bullet}|^*(|c^{\rho,\mathcal{U}_0}|) \text{ by Lemma 4.8} \\ &= N^* \circ |f_{Z_\bullet}|^* \circ (e^{\mathcal{U}_0})^* c^\rho, \text{ by definition of } c^\rho. \end{aligned}$$

Now $e^{\mathcal{U}_0} \circ |f_{Z_\bullet}| \circ N$ is homotopic to f_Z as by construction $e^{\mathcal{U}} \circ |f_{Z_\bullet}| \circ N$ classifies the topological G-bundle Z. So that

$$c^\rho(Z) = f_Z^* c^\rho,$$

and we are done. ☐

In other words we have constructed the universal Chern-Weil homomorphism for Fréchet Lie groups with homotopy type of CW complexes. Another, related approach to the universal Chern-Weil homomorphism is contained in the book of Dupont [4]. Dupont only states the theorem above for classical manifold Lie groups. Like us Dupont makes heavy use of simplicial techniques, for example the simplicial DeRham complex. However, the main thrust of his argument appears to be rather different, essentially arguing that all the necessary differential geometry can be indirectly carried out on the Milnor classifying bundle $EG \to BG$, without endowing it with extra structure, beyond the tautological structures inherent in the Milnor construction. On the other hand we need the extra structure of a smooth simplicial set, and so work with the smooth Kan complexes $BG^{\mathcal{U}}$ to do our differential geometry, and then transfer the cohomological data to BG using technical ideas like [25]. So we have a more conceptually involved space, with a certain "smooth structure", but our differential geometry is rendered trivial, and in Dupont's case the space is the "ordinary" BG, but the differential geometry is more involved.

10. Universal Chern-Weil theory for the group of symplectomorphisms

Let (M, ω) be a possibly non-compact symplectic manifold of dimension $2n$, so that ω is a closed non-degenerate 2-form on M. Let $\mathcal{H} = Ham(M, \omega)$ denote the group of its compactly supported Hamiltonian symplectomorphisms, and $\mathfrak{h}$ its Lie algebra. When M is simply connected this is just the group $Symp(M, \omega)$ of diffeomorphisms $\phi : M \to M$ s.t. $\phi^* \omega = \omega$.

For example, take $M = \mathbb{CP}^{n-1}$ with its Fubini-Study symplectic 2-form ω_{st}. Then the natural action of $PU(n)$ on $\mathbb{CP}^{n-1}$ is by Hamiltonian symplectomorphisms.

In [29] Reznikov constructs polynomials

$$\{r_k\}_{k \geq 1} \subset \mathbb{R}[\mathfrak{h}]^{\mathcal{G}},$$

each r_k homogeneous of degree k. These polynomials come from the k-multilinear functionals: $\mathfrak{h}^{\oplus k} \to \mathbb{R}$,

$$(H_1, \ldots, H_k) \mapsto \int_M H_1 \cdot \ldots \cdot H_k \, \omega^n,$$

upon identifying:

$$\mathfrak{h} = \begin{cases} C_0^\infty(M), & \text{if } M \text{ is compact} \\ C_c^\infty(M), & \text{if } M \text{ is non-compact,} \end{cases}$$

where $C_0^\infty(M)$ denotes the set of smooth functions H satisfying $\int_M H \, \omega^n = 0$. And $C_c^\infty(M)$ denotes the set of smooth, compactly supported functions. In the case $k = 1$, the associated class vanishes whenever M is compact.

The group $\mathcal{H}$ is a Fréchet Lie group having the homotopy type of a CW complex by Milnor [24]. In particular, Theorem 9.3 implies the Corollary 1.3 of the introduction, and in particular we get induced Reznikov cohomology classes

$$c^{r_k} \in H^{2k}(B\mathcal{H}, \mathbb{R}). \tag{10.1}$$

As mentioned, the group $PU(n)$ naturally acts on $\mathbb{CP}^{n-1}$ by Hamiltonian symplectomorphisms. So we have an induced map

$$i : BPU(n) \to BHam(\mathbb{CP}^{n-1}, \omega_0).$$

Then as one application we prove Theorem 1.6 of the introduction, reformulated as follows:

Theorem 10.1. *[Originally Kedra-McDuff [13]]*

$$i^* : H^k(BHam(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}) \to H^k(BPU(n), \mathbb{R})$$

is surjective for all $n \geq 2$, $k \geq 0$ and so

$$i_* : H_k(BPU(n), \mathbb{R}) \to H_k(BHam(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}),$$

is injective for all $n \geq 2$, $k \geq 0$.

Proof. Let $\mathfrak{g}$ denote the Lie algebra of $PU(n)$, and $\mathfrak{h}$ the Lie algebra of $Ham(\mathbb{CP}^{n-1}, \omega_0)$. Let $j : \mathfrak{g} \to \mathfrak{h}$ denote the natural Lie algebra map induced by the homomorphism

$PU(n) \to Ham(\mathbb{CP}^{n-1}, \omega_0)$. Reznikov [29] shows that $\{j^* r_k\}_{k>1}$ are the Chern polynomials. In other words, the classes

$$c^{j^* r_k} \in H^{2k}(BPU(n), \mathbb{R}),$$

are the Chern classes $\{c_k\}_{k>1}$, which generate real cohomology of $BPU(n)$, as is well known. But $c^{j^* r_k} = i^* c^{r_k}$, for c^{r_k} as in (10.1), and so the result immediately follows. □

In Kedra-McDuff [13] a proof of the above is given via homotopical techniques. Theirs is a difficult argument, but their technique, as they show, is also partially applicable to study certain generalized, homotopical analogues of the group $\mathcal{H}$. Our argument is elementary, but does not obviously have homotopical ramifications as in [13].

In Savelyev-Shelukhin [33] there are a number of results about induced maps in (twisted) K-theory. These further suggest that the map i above should be a monomorphism in the homotopy category. For a start we may ask:

Question 10.2. *Is the map i above an injection on integral homology?*

For this one may need more advanced techniques like [32].

10.1. **Beyond $\mathbb{CP}^n$.** Theorem 10.1 extends to completely general compact semisimple Lie groups G, with $\mathbb{CP}^n$ replaced by co-adjoint orbits M of G. We just need to compute the pullbacks to $\mathfrak{g}$ of the associated Reznikov polynomials in $\mathbb{R}[\mathfrak{h}]^{\mathcal{G}}$. We can no longer expect injection in general. But the failure to be injective should be solely due to effects of classical representation theory, rather than transcendental effects of extending the structure group to $Ham(M, \omega)$, from a compact Lie group.

11. Universal coupling class for Hamiltonian fibrations

Although we use here some language of symplectic geometry no special expertise should be necessary. As the construction here is basically just a partial reformulation of our constructions in the case of $G = Ham(M, \omega)$, we will not give exhaustive details.

Let (M, ω) and $\mathcal{H}$ be as in the previous section, and let $2n$ be the dimension of M.

Definition 11.1. *A* **Hamiltonian M-fibration** *is a smooth fiber bundle $M \hookrightarrow P \to X$, with structure group $\mathcal{H}$.*

Each $\mathcal{H}$-connection $\mathcal{A}$ on such P uniquely induces a *coupling 2-form* on P, as originally appearing in [7]. Specifically, this is a closed 2-form $C_{\mathcal{A}}$ on P whose restriction to fibers coincides with ω and which has the following property. Let $\omega_{\mathcal{A}} \in \Omega^2(X)$ denote the 2-form: for $v, w \in T_x X$,

$$\omega_{\mathcal{A}}(v, w) = n \int_{P_x} R_{\mathcal{A}}(v, w)\, \omega_x^n.$$

Where $R_{\mathcal{A}}$ as before is the curvature 2-form of $\mathcal{A}$, so that

$$R_{\mathcal{A}}(v,w) \in \operatorname{lie} Ham(M_x, \omega_x) = \begin{cases} C_0^\infty(P_x), & \text{if } M \text{ is compact} \\ C_c^\infty(P_x) & \text{if } M \text{ is non-compact.} \end{cases}$$

Note of course that $\omega_{\mathcal{A}} = 0$ when M is compact. The characterizing property of $C_{\mathcal{A}}$ is then:

$$\int_M C_{\mathcal{A}}^{n+1} = \omega_{\mathcal{A}},$$

where the left-hand side is integration along the fiber. [4]

It can then be shown that the cohomology class $\mathfrak{c}(P)$ of $C_{\mathcal{A}}$ is uniquely determined by P up to $\mathcal{H}$-bundle isomorphism. This is called the ***coupling class of*** P, and it has important applications in symplectic geometry. See for instance [21] for more details and some applications.

By replacing the category $\mathcal{G}$ with other fiber bundle categories we may define other kinds of simplicial fibrations over a smooth simplicial set. For example, we may replace $\mathcal{G}$ by the category of smooth Hamiltonian M-fibrations, keeping the other axioms in the Definition 5.4 intact. This then gives us the notion of a Hamiltonian simplicial M-bundle over a smooth simplicial set.

Let $\mathcal{U}$ be a $\mathcal{H}$-admissible Grothendieck universe. Fix a (simplicial) $\mathcal{H}$-connection $\mathcal{A}$ on the universal $\mathcal{H}$-bundle $E\mathcal{H}^{\mathcal{U}} \to B\mathcal{H}^{\mathcal{U}}$. Let $M^{\mathcal{U},\mathcal{H}}$ denote the Hamiltonian simplicial M-fibration, naturally associated to $E\mathcal{H}^{\mathcal{U}} \to B\mathcal{H}^{\mathcal{U}}$. So that for each k-simplex $\Sigma \in B\mathcal{H}^{\mathcal{U}}$ we have a Hamiltonian M-fibration $M_\Sigma^{\mathcal{U},\mathcal{H}} \to \Delta^k$, which is just the associated M-bundle to the principal $\mathcal{H}$-bundle $E\mathcal{H}_\Sigma^{\mathcal{U}}$.

By the discussion above for each k-simplex $\Sigma \in B\mathcal{H}^{\mathcal{U}}$ we have the associated coupling 2-form $C_{\mathcal{A},\Sigma}$ on $M_\Sigma^{\mathcal{U},\mathcal{H}}$. The collection of these 2-forms then readily induces a cohomology class $\mathfrak{c}^{\mathcal{U}}$ on the geometric realization:

$$|M^{\mathcal{U},\mathcal{H}}| = \operatorname{colim}_{\Sigma \in \Delta(B\mathcal{H}^{\mathcal{U}})} M_\Sigma^{\mathcal{U},\mathcal{H}}.$$

Now by the proof of Theorem 8.7 we have an $\mathcal{H}$-structure, M-bundle map over the homotopy equivalence $e^{\mathcal{U}}$:

$$g^{\mathcal{U}} : |M^{\mathcal{U},\mathcal{H}}| \to M^{\mathcal{H}},$$

where $M^{\mathcal{H}}$ denotes the universal Hamiltonian M-fibration over $B\mathcal{H}$. And these $g^{\mathcal{U}}$ are natural, so that if $\mathcal{U} \ni \mathcal{U}'$ then

$$[g^{\mathcal{U}'} \circ |\widetilde{i}^{\mathcal{U},\mathcal{U}'}|] = [g^{\mathcal{U}}], \tag{11.1}$$

where $|\widetilde{i}^{\mathcal{U},\mathcal{U}'}| : |M^{\mathcal{U},\mathcal{H}}| \to |M^{\mathcal{U},\mathcal{H}'}|$ is the natural M-bundle map over $i^{\mathcal{U},\mathcal{U}'}$ (as in Theorem 8.7), and where $[\cdot]$ denotes the homotopy class.

Each $g^{\mathcal{U}}$ is a homotopy equivalence, so we may set

$$\mathfrak{c} := g_*^{\mathcal{U}}(\mathfrak{c}^{\mathcal{U}}) \in H^2(M^{\mathcal{H}}).$$

Lemma 11.2. *The class $\mathfrak{c}$ is well-defined, (independent of the choice $\mathcal{U}$).*

[4] $C_{\mathcal{A}}$ is not generally compactly supported but $C_{\mathcal{A}}^{n+1}$ is, which is a consequence of taking $\mathcal{H}$ to be compactly supported Hamiltonian symplectomorphisms.

The proof is analogous to the proof of Lemma 9.2. Given this definition of the universal coupling class $\mathfrak{c}$, the proof of Theorem 1.5 is analogous to the proof of Theorem 9.3.

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