NOTES ON LCS HOMOLOGY

We can try a direct generalization of contact non-squeezing of Eliashberg-Polterovich [1], and Fraser in [2]. Specifically let $R^{2n} \times S^1$ be the prequantization space of R^{2n}, or in other words the contact manifold with the contact form $d\theta - \lambda$, for $\lambda = \frac{1}{2}(ydx - xdy)$. Let B_R now denote the open radius R ball in $\mathbb{R}^{2n}$.

Question 1. If $R \geq 1$ is there a compactly supported, *lcs* endomorphism of the l. c. s. m. $\mathbb{R}^{2n} \times S^1 \times S^1$ which takes the closure of $U := B_R \times S^1 \times S^1$ into U?

1. An l. c. s.-homology theory

For general l. c. s. manifolds M we need to develop an analogue of contact homology, denoted by $CSH(M)$ for example. Indeed for the Banyaga l. c. s. structure $\omega_\lambda = d\lambda + \lambda \wedge d\theta$ on $M = C \times S^1$ with (C, λ) contact, for an appropriate almost complex structure J_λ all J_λ-holomorphic tori, are in one to one correspondence with Reeb orbits of (C, λ). They are just products of Reeb orbits by the S^1 factor of $M \times S^1$. But these Reeb tori as we call them have an additional structure: the form $d\lambda$ vanishes on them identically, we say that they are ***calibrated*** by $d\lambda$.

We first generalize the above to a Lichnerowitz exact l. c. s. structure ω on $M = C^{2n-1} \times S^1$, with C closed, i.e. $\omega = d\lambda + \lambda \wedge d\theta$, for λ a general 1-form on M, s.t. ω is non-degenerate. This might be enough for the applications we have in mind.

Lemma 1.1. *There is a class $\mathcal{J}(\omega)$ of ω compatible almost complex structures on M, s.t. for $J \in \mathcal{J}(\omega)$, every non-constant closed pseudo-holomorphic curve u satisfies $u^* d\lambda = 0$.*

Proof. Let $\mathcal{V}$ denote the vanishing distribution of $d\lambda$. That is $v \in \mathcal{V}_p \subset T_pM$ iff $\omega(v, \cdot) = 0$. Then $\mathcal{V}$ is a 2-dimensional distribution: $\mathcal{V}_p$ has dimension at least 2 since $d\lambda$ cannot be symplectic since M is closed, and has dimension at most 2 since $d\lambda + \lambda \wedge d\theta$ is non-degenerate. Let ξ denote the co-vanishing distribution that is ξ_p is the ω-orthogonal complement to $\mathcal{V}_p$. We define $\mathcal{J}(\omega)$ to be the set of ω-compatible complex structures J which preserve both ξ and $\mathcal{V}$. This extends the type of J used in symplectizations. Then an elementary calculation shows that for every u as in the hypothesis and for $J \in \mathcal{J}(\omega)$ $u^* d\lambda = 0$. □

The condition $u^* d\lambda = 0$, will be called ***calibration condition***. We define l. c. s.-homology $CSH(M)$ over $\mathbb{Z}_2$ to have generators non-constant J-holomorphic elliptic curves u in M, for $J \in \mathcal{J}(\omega)$ suitably generic.

Here generators are like in contact homology algebra, so really we must take certain words in generators. But I won't make it explicit yet. Also when $C = S^{2n-1}$ we should be able to work with honest homology groups, like in the case of contact homology of C.

To actually define the homology we need instantons. There are taken to be J-holomorphic maps $u : S^1 \times \mathbb{R} \to M$ with $\int u^* d\lambda < \infty$. Such instantons are necessarily asymptotic at the ends to generators. In other words:

Lemma 1.2. *Given an instant on u as above, the images of the maps $u_{r,+} = u|_{S^1 \times \mathbb{R}_{\geq r}}$ Hausdorff converge as $r \mapsto \infty$ to a fixed J-holomorphic elliptic curve u_+ in M. Likewise the images of the maps $u_{r,-} = u|_{S^1 \times \mathbb{R}_{\leq r}}$ Hausdorff converge as $r \mapsto \infty$ to a fixed J-holomorphic elliptic curve u_- in M.*

Proof. First a construction of Eliashberg-Murphy [] shows that in this case M fibers over S^1 with contact fibers, with contact distributions restrictions of ξ above. Let $(M_\theta, \lambda_\theta)$ denote the corresponding contact fibers. In this case analogously to the Banyaga example a non-constant elliptic curve in M must be foliated by $\{\lambda_\theta\}$-Reeb closed orbits, by the calibration condition. Now given an instanton u, at the ends $u^* d\lambda$ is asymptotically vanishing which means that u is asymptotically a "Reeb cylinder":

an smooth s-family of $\lambda_{f(s)}$-Reeb orbits, for $s \in \mathbb{R}_+$ and $f(s) \in S^1$ for f determined by u. To finish the proof we need to show that given any Reeb cylinder as above, with finite energy, it must be a Reeb torus. Let u_s denote the slice of a Reeb cylinder u over $f(s) \in S^1$, that is u_s is a $\lambda_{f(s)}$-Reeb orbit. Let s_0 be fixed, and suppose that there is no $s > s_0$ with

$$f(s) = f(s_0) = \theta_0$$

such that $u_s = u_{s_0}$. Then by the finite energy condition we obtain a non-constant sequence $\{\gamma_n = u_{s_n}\}$ of λ_{θ_0}-Reeb orbits with bounded period, which must have a convergent subsequence $\{\gamma_{n_k}\}$ by Azrella-Ascolli. If we assume that λ_{θ_0} is Reeb non-degenerate then this sequence must eventually be constant and we are done. □

Question 2. Why do we need the l. c. s. condition on ω? This rules out bubbling of J-holomorphic spheres for a sequence of instantons. Since any J-holomorphic sphere lifts to a $\widetilde{J}$-holomorphic sphere in the covering space $\widetilde{M} = C \times \mathbb{R}$. And on $\widetilde{M}$ the lift $\widetilde{\omega}$ of ω is globally conformally symplectic. In particular $\widetilde{J}$ is compatible with a symplectic form, and hence there are no non-constant $\widetilde{J}$-holomorphic spheres in $\widetilde{M}$.

For a generic J, elliptic curves in M up to equivalence are isolated. Then given the lemma above the space of all instantons in M breaks up into finite dimensional components $\mathcal{M}(u_-, u_+)$ for u_-, u_+ some elliptic curves. That is $u \in \mathcal{M}(u_-, u_+)$ is asymptotic at the ends to u_-, u_+. The lemma above can be strengthened to certain C^∞ convergence but this takes more care to state, and we don't need this yet. Each $\mathcal{M}(u_-, u_+)$ is compact after adding broken instantons.

1.0.1. *Problem 1.* Can we define relative $\mathbb{Z}$-grading? Spectral flow? In other words how to compute dimensions of moduli spaces of instantons $\mathcal{M}(u_-, u_+)$? I think this is probably a straightforward generalization of contact homology case.

1.0.2. *Problem 2.* Can we define absolute $\mathbb{Z}$-grading, analogous to Conley-Zehnder index. Actually if we can then it is clear what it must be, it is the Conley-Zehnder index of any of the slices of the Reeb torus, as the CZ index does not depend on the slice.

1.0.3. *Problem 3.* Show that $CSH(M) \simeq CH(C)$. Assuming Problem 3, we get an immediate application:

Theorem 1. *Let $f : S^1 \to Cont(C)$ be a smooth family, for C as above, with $Cont(C)$ the space of contact forms on C, i.e. 1-forms λ such that $\lambda \wedge \lambda^{2k} \neq 0$. Suppose that there a 1-form λ on $C \times S^1$, s.t. $\lambda|_{C_\theta} = f(\theta)$, for C_θ the fiber over θ and s.t. $\omega_\lambda = d\lambda + \lambda \wedge d\theta \neq 0$ (Does this condition always hold?). Suppose that $CH(C) \neq 0$ then there is an S^1 family of Reeb orbits for f, meaning a continuous map $R : S^1 \to LC$ s.t. $R(\theta)$ is a Reeb orbit of $f(\rho(\theta))$, for LC the free loop space of C, and $\rho : S^1 \to S^1$ some covering map.*

Proof. Assuming Problem 3 we get that $CSH(M) \neq 0$, in particular there must be a non-constant J_λ-holomorphic elliptic curve u in M, where $J_\lambda \in \mathcal{J}(\omega_\lambda)$, defined as above. On the other hand by Lemma 1.1 image $u \cap C_\theta$ must be an image of a $f(\theta)$-Reeb orbit. □

References

[1] Y. Eliashberg, S. S. Kim, and L. Polterovich, *Geometry of contact transformations and domains: orderability versus squeezing.*, Geom. Topol., 10 (2006), pp. 1635–1748.

[2] M. Fraser, *Contact non-squeezing at large scale in $\mathbb{R}^{2n} \times S^1$.*, Int. J. Math., 27 (2016), p. 25.

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