

# A CONFORMAL SYMPLECTIC WEINSTEIN CONJECTURE

YASHA SAVELYEV

ABSTRACT. We introduce a direct generalization of the Weinstein conjecture to closed, exact, locally conformally symplectic manifolds, (for short lcs manifolds). This conjectures existence of certain, 2-curves in the manifold, we call Reeb 2-curves. The conjecture holds immediately for all surfaces. In higher dimensions, we give partial verifications of this conjecture, based on certain extended ( $\mathbb{Q} \sqcup \{\pm\infty\}$  valued) Gromov-Witten, elliptic curve counts in lcs manifolds. A special case of this conjecture implies a new phenomenon in classical Reeb dynamics, and our partial verifications give examples for this.

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## 1. INTRODUCTION: OVERVIEW

A locally conformally symplectic manifold or lcs manifold for short, is a natural direct generalization of both symplectic and contact manifolds. The main goal here is to study an lcs variant of a very

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influential conjecture in contact geometry: the Weinstein conjecture. The latter conjectures existence of closed orbits of the Reeb flow for any contact form on a closed manifold. This is now proved for contact three manifolds by Taubes [26].

**Definition 1.1.** *A locally conformally symplectic manifold or just lcs manifold, is a smooth  $2n$ -fold  $M$  with an lcs structure: which is a non-degenerate 2-form  $\omega$ , with the property that for every  $p \in M$  there is an open  $U \ni p$  such that  $\omega|_U = f_U \cdot \omega_U$ , for some symplectic form  $\omega_U$  defined on  $U$  and some smooth positive function  $f_U$  on  $U$ .*

These kinds of structures were originally considered by Lee in [11], arising naturally as part of an abstract study of “a kind of even dimensional Riemannian geometry”, and then further studied by a number of authors see for instance, [2] and [27]. This is a fascinating object, an lcs manifold admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and Murphy show that if a closed almost complex  $2n$ -fold  $M$  has  $H^1(M, \mathbb{R}) \neq 0$  then it admits a lcs structure, [6]. As constructed, these lcs structures are Lichnerowicz exact (see (2.1)). Another result of Apostolov, Dloussky [1] is that any complex surface with an odd first Betti number admits an lcs structure, which tames the complex structure. In this case the corresponding lcs structures are generally non-exact.

We will state an analogue of the Weinstein conjecture for exact lcs manifolds: “a conformal symplectic Weinstein conjecture”. We shall then give partial verifications using suitably extended  $(\mathbb{Q} \sqcup \{\pm\infty\})$  valued) Gromov-Witten theory, counting certain elliptic curves. And then we shall discuss a new consequent phenomenon in classical Reeb dynamics.

It is indeed natural to try to do Gromov-Witten theory for lcs manifolds. The first problem that occurs is that a priori energy bounds are gone, as since  $\omega$  is not necessarily closed, the  $L^2$ -energy can now be unbounded on the moduli spaces of  $J$ -holomorphic curves in such a  $(M, \omega)$ . A more acute problem is potential presence of holomorphic sky catastrophes - given a smooth family  $\{J_t\}$ ,  $t \in [0, 1]$ , of  $\{\omega_t\}$ -compatible almost complex structures, we may have a continuous family  $\{u_t\}$  of  $J_t$ -holomorphic curves s.t.  $\text{energy}(u_t) \mapsto \infty$  as  $t \mapsto a \in (0, 1)$  and s.t. there are no holomorphic curves for  $t \geq a$ . These are analogues of sky catastrophes discovered by Fuller [10] for closed orbits of dynamical systems.

Even when it is impossible to tame these problems we show that there can still be an extended Gromov-Witten type theory which is analogous to the theory of extended Fuller index in dynamical systems, [23]. In a very particular situation the relationship with the Fuller index becomes perfect as one of the results of this paper obtains the (extended) Fuller index for Reeb vector fields on a contact manifold  $C$  as a certain (extended) genus 1 Gromov-Witten invariant of the Banyaga lcs manifold  $C \times S^1$ , cf. Example 1. The latter also gives a conceptual interpretation for why the Fuller index is rational, as it is reinterpreted as an (virtual) orbifold Euler number.

## 2. INTRODUCTION: BACKGROUND AND STATEMENTS

To see the connection with the first cohomology group  $H^1(M, \mathbb{R})$ , mentioned above, let us point out right away the most basic invariant of a lcs structure  $\omega$ , when  $M$  has dimension at least 4. This is the Lee class,  $\alpha = \alpha_\omega \in H^1(M, \mathbb{R})$ . This class has the property that on the associated  $\alpha$ -covering space (see proof of Theorem 3.4)  $\widetilde{M}$ , the lift  $\widetilde{\omega}$  is globally conformally symplectic. Thus, an lcs form is globally conformally symplectic, that is diffeomorphic  $f \cdot \omega'$ , with  $\omega'$  symplectic, iff its Lee class vanishes.

Again assuming  $M$  has dimension at least 4, the Lee class  $\alpha$  has a natural differential form representative, called the Lee form and defined as follows. We take a cover of  $M$  by open sets  $U_a$  in which  $\omega = f_a \cdot \omega_a$  for  $\omega_a$  symplectic, and  $f_a$  a positive smooth function. Then we have 1-forms  $d(\ln f_a)$  in each  $U_a$  which glue to a well-defined closed 1-form on  $M$ , as shown by Lee. We may denote this 1-form and its cohomology class both by  $\alpha$ . It is moreover immediate that for an lcs form  $\omega$ ,

$$d\omega = \alpha \wedge \omega,$$

for  $\alpha$  the Lee form as defined above.

As we mentioned lcs manifolds can also be understood to generalize contact manifolds. This works as follows. First we have a class of explicit examples of lcs manifolds, obtained by starting with a symplectic cobordism (see [6]) of a closed contact manifold  $C$  to itself, arranging for the contact forms at the two ends of the cobordism to be proportional (which can always be done) and then appropriately gluing together the boundary components. As a particular case of this we get Banyaga's basic example below.

**Notation 1.** For us a contact manifold is a pair  $(C, \lambda)$  where  $C$  is a closed manifold and  $\lambda$  a contact form:  $\forall p \in C : \lambda \wedge \lambda^{2n}(p) \neq 0$ . This is not a completely common terminology as often it is the equivalence class of  $(C, \lambda)$  that is called a contact manifold, where  $(C, \lambda) \sim (C, \lambda')$  if  $\lambda = f\lambda'$  for  $f$  a positive function.

*Example 1* (Banyaga). Let  $(C, \lambda)$  be a contact  $(2n+1)$ -manifold and take  $M = C \times S^1$  with 2-form

$$\omega_\lambda = d_\alpha \lambda := d\lambda - \alpha \wedge \lambda,$$

for  $\alpha := pr_{S^1}^* d\theta$ ,  $pr_{S^1} : C \times S^1 \rightarrow S^1$  the projection, and  $\lambda$  likewise the pull-back of  $\lambda$  by the projection  $C \times S^1 \rightarrow C$ . We call  $(M, \omega_\lambda)$  as above the **lcs-fiction** of  $(C, \lambda)$ . This is also a basic example of a first kind lcs manifold, see Definition 2.5.

The operator

$$(2.1) \quad d_\alpha : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

is called the Lichnerowicz differential with respect to a closed 1-form  $\alpha$ , and satisfies  $d_\alpha \circ d_\alpha = 0$  so that we have an associated Lichnerowicz chain complex.

We assume from now on, unless explicitly stated otherwise, that our lcs manifolds have dimension at least 4.

**Definition 2.2.** An exact lcs structure on  $M$  is a pair  $(\lambda, \alpha)$  with  $\alpha$  a closed 1-form, s.t.  $\omega = d_\alpha \lambda$  is non-degenerate.

Such a structure determines a generalized distribution  $\mathcal{V}_\lambda$  on  $M$ :

$$\mathcal{V}_\lambda(p) = \{v \in T_p M \mid d\lambda(v, \cdot) = 0\},$$

which we call the **vanishing distribution**. We also define a generalized distribution  $\xi_\lambda$  that is the  $\omega$ -orthogonal complement to  $\mathcal{V}_\lambda$ , which we call **co-vanishing distribution**. For each  $p \in M$ ,  $\mathcal{V}_\lambda(p)$  has dimension at most 2 since  $d\lambda - \alpha \wedge \lambda$  is non-degenerate. If  $M^{2n}$  is closed  $\mathcal{V}_\lambda$  cannot identically vanish since  $(d\lambda)^n$  cannot be non-degenerate by Stokes theorem.

**Definition 2.3.** A conformal symplectomorphism of exact lcs manifolds  $\phi : (M_1, \lambda_1, \alpha_1) \rightarrow (M_2, \lambda_2, \alpha_2)$  is a diffeomorphism  $\phi$  s.t.  $\phi^* \omega_2 = f \omega_1$  where  $\omega_2 = d_{\alpha_2} \lambda_2$  and  $\omega_1 = d_{\alpha_1} \lambda_1$ , and where  $f$  is a positive function on  $M$ .

*Remark 2.4.* It is important to note that for us the pair  $(\lambda, \alpha)$  is the structure not its conformal symplectomorphism class, as in for example [2]. In other words conformally symplectomorphic structures on a given manifold are distinct but isomorphic objects of our category, whose objects are lcs manifolds and morphisms conformal symplectomorphisms.

**Definition 2.5.** Let  $(M, \lambda, \alpha)$  be an exact lcs manifold. If  $\alpha$  is rational, respectively irrational we call the lcs structure **rational**, respectively **irrational**. If  $\mathcal{V}$  is non-zero at each point of  $M$ , in particular is a smooth 2-distribution, then such an lcs manifold is called **first kind**.

A typical example of an lcs structure of the first kind is the mapping torus of a strict contactomorphism, see Banyaga [2]. If  $(M, \lambda, \alpha)$  is closed, first kind and rational, then it is conformally symplectomorphic to a mapping torus as above, see [3]. Note that of course a conformal symplectomorphism does not preserve the vanishing and covanishing distributions, just as a contactomorphism will not preserve Reeb distributions. So that the above equivalence is of limited use, particularly in the context of conformal symplectic Weinstein conjecture below.

**2.1. Conformal symplectic Weinstein conjecture.** As previously mentioned lcs manifolds can be understood to generalize both symplectic and contact manifolds. There are two very influential conjectures in these two respective areas: the Arnold conjecture and the Weinstein conjecture. The statement of Arnold conjecture on fixed points of a symplectomorphisms can be directly generalized to lcs manifolds, but it is very easy to construct counterexamples using Banyaga's example above: there are Hamiltonian conformal symplectomorphisms of the lcs-fication of the standard contact  $S^3$ , with no fixed points. (We leave this to the reader as an exercise.) For one deeper discussion of the Arnold conjecture in this context see [4]. We are instead interested here in extending the Weinstein conjecture.

**Definition 2.6.** Let  $(M, \lambda, \alpha)$  be an exact lcs structure and  $\omega = d_\alpha \lambda$ . Define  $X_\lambda$  by  $\omega(X_\lambda, \cdot) = \lambda$  and  $X_\alpha$  by  $\omega(X_\alpha, \cdot) = \alpha$ . Let  $\mathcal{D}$  denote the (generalized) distribution spanned by  $X_\alpha, X_\lambda$ , meaning  $\mathcal{D}(p) := \text{span}(X_\alpha(p), X_\lambda(p))$ . This will be called the **canonical distribution**.

The distribution  $\mathcal{D}$  is one analogue for exact lcs manifolds of the Reeb distribution on contact manifolds. A Reeb 2-curve, as defined ahead, will be a certain kind of singular leaf of  $\mathcal{D}$ , and so is a kind of 2-dimensional analogue of a Reeb orbit.

**Definition 2.7.** Let  $\Sigma$  be a nodal Riemann surface (the set of nodes is empty or not). Let  $u : \Sigma \rightarrow M$  be a smooth map and let  $\tilde{u} : \tilde{\Sigma} \rightarrow M$  be its normalization (cf. Definition 6.3). We say that  $u$  is a **Reeb 2-curve** in  $(M, \lambda, \alpha)$ , if the following is satisfied:

- (1) For each  $z \in \tilde{\Sigma}$ ,  $\tilde{u}_*(T_z \tilde{\Sigma}) = \mathcal{D}(\tilde{u}(z))$ , whenever  $d\tilde{u}(z) : T_z \tilde{\Sigma} \rightarrow T_{\tilde{u}(z)} M$  is non-singular, and  $\dim \mathcal{D}(\tilde{u}(z)) = 2$ .
- (2)  $0 \neq [u^* \alpha] \in H^1(\Sigma, \mathbb{R})$ .
- (3) The set of critical points of  $\tilde{u}$  is finite.

**Conjecture 1** (CSW conjecture). A closed exact lcs manifold  $(M, \lambda, \alpha)$  has a Reeb 2-curve.

**Theorem 2.8.** Conjecture 1 implies the Weinstein conjecture: every closed contact manifold  $(C, \lambda)$  has a closed Reeb orbit, i.e. there is a smooth map  $o : S^1 \rightarrow C$  such that

$$\dot{o}(t) = cR^\lambda(o(t)),$$

for some  $c > 0$  called period, where  $R^\lambda$  is the Reeb vector field satisfying

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

The following is immediate from definitions.

**Theorem 2.9.** CSW conjecture holds for closed surfaces.

To give more examples, we quickly introduce the relevant spaces of lcs structures.

**Definition 2.10.** Define the space of exact lcs structures on  $M$ ,  $\mathcal{L}(M)$ , to be:

$$\mathcal{L}(M) = \{(\beta, \gamma) \in \Omega^1(M) \times \Omega^1(M) \mid \gamma \text{ is closed, } d_\gamma \beta \text{ is non-degenerate}\}.$$

Define  $\mathcal{F}(M) \subset \mathcal{L}(M)$  to be subspace of (possibly irrational) first kind lcs structures.

In what follows we use the following  $C^k$  metric on  $\mathcal{L}(M)$ . For  $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in \mathcal{L}(M)$  define:

$$(2.11) \quad d_k((\lambda_1, \alpha_1), (\lambda_2, \alpha_2)) = d_{C^k}(\lambda_1, \lambda_2) + d_{C^k}(\alpha_1, \alpha_2),$$

where  $d_{C^k}$  on the right side is the usual  $C^k$  metric.

The following theorems are proved in Section 6, based on the theory of elliptic pseudo-holomorphic curves in  $M$ .

**Theorem 2.12.** Let  $(C, \lambda)$  be a contact manifold, with at least one non-degenerate Reeb orbit. Then Conjecture 1 holds for a  $d_3$  neighborhood of the lcs-fication  $(\lambda, \alpha)$  in the space  $\mathcal{F}(M = C \times S^1)$ . Moreover, in this case the Reeb 2-curve  $u : \Sigma \rightarrow M$  can be assumed to be **elliptic** meaning that  $\Sigma$  is elliptic (more specifically: a nodal, topological genus 1, closed, connected Riemann surface). If  $C$  in addition has dimension 3 then the elliptic Reeb 2-curve can be assumed to be embedded and normal (the set of nodes is empty).

**Theorem 2.13.** *Let  $C$  be a closed contact manifold with contact form  $\lambda$ , with  $i(R^\lambda, \beta) \neq 0$ , for some class  $\beta$ , where the latter is the extended Fuller index, as described in Appendix A. Let  $(\lambda, \alpha)$  be the associated exact lcs-structure on  $M = C \times S^1$ , the lcs-fication. Then either there is an elliptic Reeb 2-curve for every first kind lcs structure on  $M$ , homotopic through non-degenerate 2-forms to  $\omega_0 = d_\alpha \lambda$ , or holomorphic sky catastrophes exist, (the latter are further discussed in Section 3.3).*

*Example 2.* Take  $C = S^{2k+1}$  and  $\lambda = \lambda_H$  the standard contact structure. We call its lcs-fication the **standard lcs structure** on  $C \times S^{2k+1}$ . Then  $i(R^\lambda, 0) = \pm\infty$ , (sign depends on  $k$ ), [23]. Or take  $C$  to be unit cotangent bundle of a hyperbolic manifold  $(X, g)$ ,  $\lambda$  the associated Liouville form, and  $(\lambda, \alpha)$  the associated Banyaga lcs structure, in this case  $i(R^\lambda, \beta) = \pm 1$  for every  $\beta \neq 0$ .

The above examples motivate us to state a special version of this conjecture for first kind lcs manifolds.

**Conjecture 2.** *Let  $(M, \lambda, \alpha)$  be a closed first kind lcs manifold, then there is an elliptic Reeb 2-curve in  $M$ .*

We will call this elliptic Weinstein conjecture.

**2.2. Application to Reeb dynamics.** The above conjecture has a novel dynamical consequence. Suppose that we have an  $S^1$ -family  $\{\lambda_t\}$ ,  $t \in S^1 = \mathbb{R}/\mathbb{Z}$ , of contact forms on a closed manifold  $C$ . We may ask: when does there exist a smooth map  $u : S^1 \times S^1 \rightarrow C$ , s.t. for all  $t$  the image of  $u|_{S^1 \times \{t\}}$  is the image of a  $\lambda_t$ -Reeb orbit? We call this a **smooth Reeb torus** for  $\{\lambda_t\}$ . Even if the Weinstein conjecture holds, in general there may not be such a  $u$ . In fact there may not even be a  $u$  s.t. the above condition is satisfied for all but finitely many  $t$ . See the following example.

*Example 3.* Let  $C = S^3$  and  $\lambda$  the standard contact form. The latter form is Morse-Bott. We set  $\lambda_0 = \lambda$ ,  $\lambda_{\frac{1}{2}} = \lambda$ , then extend to a family  $\{\lambda_t\}$ , so that  $\lambda_t$  is non-degenerate for  $t \in (0, \frac{1}{2})$ , and for  $t \in (\frac{1}{2}, 1)$ . If the extension is appropriately chosen there will be no smooth Reeb tori (or even continuous). Moreover, the family  $\{\lambda_t\}$  can be chosen to be  $C^\infty$  close to the constant family  $\{\lambda_t = \lambda\}$ .

On the other hand we have an elementary observation:

**Theorem 2.14.** *Assume elliptic Weinstein conjecture. Let  $\{\lambda_t\}$ ,  $t \in S^1$  be as above a circular family of contact forms on a closed manifold. Suppose that  $\{\lambda_t\}$  extends to a first kind lcs structure  $(\lambda, \alpha)$  on  $M = C \times S^1$ , meaning that  $\lambda|_{M_t} = \lambda_t$ , where  $M_t = C \times \{t\}$ . Then there is a smooth Reeb torus for  $\{\lambda_t\}$ . (In particular, this applies in special cases where CSW conjecture holds given by Theorems 2.12, 2.13).*

### 2.3. Reeb 1-curves.

**Definition 2.15.** *We say that a smooth map  $o : S^1 \rightarrow M$  is a **Reeb 1-curve** in  $(M, \lambda, \alpha)$  if for each  $t \in S^1$*

$$\lambda(\dot{o}(t)) > 0$$

*and if  $\dot{o}(t)$  is in  $\mathcal{D}$ .*

The CSW conjecture has implications for existence of Reeb 1-curves.

**Theorem 2.16.** *Suppose that  $(M, \lambda, \alpha)$  is a closed exact lcs manifold satisfying the, as we call it, Reeb condition  $\lambda(X_\alpha) > 0$ . If  $(M, \lambda, \alpha)$  has an immersed Reeb 2-curve then it also has a Reeb 1-curve. Furthermore, if it has an immersed elliptic Reeb 2-curve, then it is normal (the set of its nodes is empty).*

We have an immediate corollary of Theorem 2.12 and Theorem 2.16.

**Corollary 2.17.** *Let  $\lambda$  be a contact form, on closed 3-manifold  $C$ , with at least one non-degenerate Reeb orbit. Then there is a  $d_3$  neighborhood  $U$  of the lcs-fication  $(\lambda, \alpha)$  in the space  $\mathcal{F}(M = C \times S^1)$ , s.t. for each  $(\lambda', \alpha') \in U$  there is a Reeb 1-curve.*

The above conjectures are not just a curiosity. In contact geometry, rigidity is based on existence phenomena of closed Reeb orbits, and as explained lcs manifolds generalize contact manifolds. To attack rigidity questions in lcs geometry, it would be of great help to have an analogue of closed Reeb orbits. Here we have a pair of proposals: Reeb 2-curves, and Reeb 1-curves. In the future we plan to use this in investigations of non-squeezing in the lcs world.

### 3. PSEUDOHOLOMORPHIC CURVES IN LCS MANIFOLDS

First kind lcs manifolds give immediate examples of almost complex manifolds where the  $L^2$  energy functional is unbounded on the moduli spaces of fixed class  $J$ -holomorphic curves, as well as where null-homologous  $J$ -holomorphic curves can be non-constant. We are going to see this shortly after developing a more general theory.

**Definition 3.1.** Let  $(M, \lambda, \alpha)$  be an exact lcs manifold, satisfying the **Reeb condition**:  $\omega(X_\lambda, X_\alpha) = \lambda(X_\alpha) > 0$ , where  $\omega = d_\lambda \alpha$ . We say that an  $\omega$ -compatible  $J$  is **admissible** if

- $J$  preserves the span  $V$  of  $X_\lambda, X_\alpha$  and preserves the  $\omega$  orthogonal complement  $V^\perp$  of  $V$ . That is  $J(V) \subset V$  and  $J(V^\perp) \subset V^\perp$ .
- $d\lambda$  tames  $J$  on  $V^\perp$ .

Admissible  $J$  exist by classical symplectic geometry, and the space of such  $J$  is contractible cf. [16]. We call  $(\lambda, \alpha, J)$  as above a **tamed exact lcs structure**, and  $(\omega, J)$  is called a **tamed exact lcs structure** if  $\omega = d_\alpha \lambda$ , for  $(\lambda, \alpha, J)$  a **tamed exact lcs structure**. In this case  $(M, \omega, J)$ ,  $(M, \lambda, \alpha, J)$  will be called a **tamed exact lcs manifold**.

*Example 4.* If  $(M, \lambda, \alpha)$  is first kind then  $\omega(X_\lambda, X_\alpha) = 1$  everywhere. In particular, we may find a  $J$  so that  $(\lambda, \alpha, J)$  is a **tamed exact lcs structure**, and the space of such  $J$  is contractible. We will call  $(M, \lambda, \alpha, J)$  a **tamed first kind lcs manifold**.

**Lemma 3.2.** Let  $(M, \lambda, \alpha, J)$  be a **tamed first kind lcs manifold**. Then given a smooth  $u : \Sigma \rightarrow M$ , where  $\Sigma$  is a closed (nodal) Riemann surface,  $u$  is  $J$ -holomorphic only if

$$\text{image } d\tilde{u}(z) \subset \mathcal{V}_\lambda(\tilde{u}(z))$$

for all  $z \in \tilde{\Sigma}$ , where  $\tilde{u} : \tilde{\Sigma} \rightarrow M$  is the normalization of  $u$  (see Definition 6.3). In particular  $\tilde{u}^* d\lambda = 0$ .

*Proof.* As previously observed, by the first kind condition,  $\mathcal{V}_\lambda$  is the span of  $X_\lambda, X_\alpha$  and hence is  $\mathcal{V}_\lambda = V$  in the notation above. Let  $u$  be  $J$ -holomorphic, so that  $\tilde{u}$  is  $J$ -holomorphic (by definition of a  $J$ -holomorphic nodal map). We have

$$\int_{\Sigma} \tilde{u}^* d\lambda = 0$$

by Stokes theorem. Let  $\text{proj}(p) : T_p M \rightarrow V^\perp(p)$  be the projection induced by the splitting  $TM = V \oplus V^\perp$ . Then if for some  $z \in \tilde{\Sigma}$ ,  $\text{proj} \circ d\tilde{u}(z) \neq 0$ , since  $J$  is tamed by  $d\lambda$  on  $V^\perp$  and since  $J$  preserves the splitting, we would have  $\int_{\tilde{\Sigma}} \tilde{u}^* d\lambda > 0$ . Thus,

$$\forall z \in \tilde{\Sigma} : \text{proj} \circ d\tilde{u}(z) = 0,$$

so

$$\forall z \in \tilde{\Sigma} : \text{image } d\tilde{u}(z) \subset \mathcal{V}_\lambda(\tilde{u}(z)).$$

□

*Example 5.* Let  $(C \times S^1, \lambda, \alpha)$  be the lcs-fication of a contact manifold  $(C, \lambda)$ . In this case

$$X_\alpha = (R^\lambda, 0),$$

where  $R^\lambda$  is the Reeb vector field and

$$X_\lambda = (0, \frac{d}{d\theta})$$

is the vector field generating the natural action of  $S^1$  on  $C \times S^1$ .

If we denote by  $\xi \subset T(C \times S^1)$  the distribution  $\xi(p) = \ker \lambda(p)$ , then in this case  $\xi = V^\perp$  in the notation above.

We take  $J$  to be an almost complex structure on  $\xi$ , which is  $S^1$  invariant, and compatible with  $d\lambda$ . The latter means that

$$g_J(\cdot, \cdot) := d\lambda|_{\xi}(\cdot, J\cdot)$$

is a  $J$  invariant Riemannian metric on the distribution  $\xi$ .

There is an induced almost complex structure  $J^\lambda$  on  $C \times S^1$ , which is  $S^1$ -invariant, coincides with  $J$  on  $\xi$  and which satisfies:

$$J^\lambda(X_\alpha) = X_\lambda.$$

Then  $(C \times S^1, \lambda, \alpha, J^\lambda)$  is a tamed first kind lcs manifold.

**3.1. Moduli of Pseudo-holomorphic curves in an lcs manifold.** We now consider a moduli space of holomorphic tori in  $C \times S^1$ , which have a certain charge, an analogue of this charge condition is also studied Oh-Wang [20], and its name was suggested to me by Yong-Geun Oh. Partly the reason for introduction of “charge” is that it is now possible for non-constant holomorphic curves to be null-homologous, so we need additional control. Here is a simple example take  $S^3 \times S^1$  with  $J = J^\lambda$ , for the  $\lambda$  the standard contact form, then all the Reeb holomorphic tori (as defined further below) are null-homologous. In many cases we can just work with homology classes  $A \neq 0$ , but this is inadequate for our setup for conformal symplectic Weinstein conjecture.

Let  $\Sigma$  be a complex torus with a chosen marked point  $z \in \Sigma$ , i.e. an elliptic curve over  $\mathbb{C}$ . An isomorphism  $\phi : (\Sigma_1, z_1) \rightarrow (\Sigma_2, z_2)$  is a biholomorphism s.t.  $\phi(z_1) = z_2$ . The set of isomorphism classes forms a smooth orbifold  $M_{1,1}$ , with a natural compactification, the Deligne-Mumford compactification  $\overline{M}_{1,1}$ , by adding a point at infinity corresponding to a nodal genus 1 curve with one node.

Suppose then that  $(M, \omega)$  is an lcs manifold,  $J$   $\omega$ -compatible almost complex structure, and  $\alpha$  the Lee class corresponding to  $\omega$ . Assuming for simplicity, at the moment, (otherwise take stable maps) that  $(M, J)$  does not admit non-constant  $J$ -holomorphic maps  $(S^2, j) \rightarrow (M, J)$ , we define:

$$\overline{\mathcal{M}}_{1,1}^{1,0}(J, A)$$

as a set of equivalence classes of tuples  $(u, S)$ , for  $S = (\Sigma, z) \in \overline{M}_{1,1}$ , and  $u : \Sigma \rightarrow M$  a  $J$ -holomorphic map satisfying the **charge (1,0) condition**: there exists a pair of generators  $\rho, \gamma$  for  $H_1(\Sigma, \mathbb{Z})$ , such that

$$\begin{aligned} \langle \rho, u_* \alpha \rangle &= 1 \\ \langle \gamma, u_* \alpha \rangle &= 0, \end{aligned}$$

and with  $[u] = A$ . The equivalence relation is  $(u_1, S_1) \sim (u_2, S_2)$  if there is an isomorphism  $\phi : S_1 \rightarrow S_2$  s.t.  $u_2 \circ \phi = u_1$ .

Note that the charge condition directly makes sense for nodal curves. And it is easy to see that the charge condition is preserved under Gromov convergence, and obviously a charge (1,0)  $J$ -holomorphic map cannot be constant for any  $A$ .

By slight abuse we may just denote such an equivalence class above by  $u$ , so we may write  $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(J, A)$ , with  $S$  implicit.

**3.2. Reeb holomorphic tori in  $(C \times S^1, J^\lambda)$ .** For the almost complex structure  $J^\lambda$  as above we have one natural class of charge (1,0) holomorphic tori in  $C \times S^1$ . Let  $o$  be a period  $c$  closed Reeb orbit  $o$  of  $R^\lambda$ , that is a map:

$$\begin{aligned} o : S^1 &\rightarrow C, \\ \dot{o}(s) &= c \cdot R^\lambda(o(s)), c > 0. \end{aligned}$$

A **Reeb torus**  $u_o$  for  $o$ , is the map

$$u_o(s, t) = (o(s), t),$$



$s, t \in S^1$ . A Reeb torus is  $J^\lambda$ -holomorphic for a uniquely determined holomorphic structure  $j$  on  $T^2$  defined by:

$$j\left(\frac{\partial}{\partial s}\right) = c \frac{\partial}{\partial t}.$$

Let  $\tilde{S}(\lambda)$  denote the space of general period, closed  $\lambda$ -Reeb orbits. There is an  $S^1$  action on this space, with  $\theta \cdot o$  the orbit

$$\theta \cdot o(s) = o(s + \theta).$$

Let  $S(\lambda) := \tilde{S}(\lambda)/S^1$  denote the quotient by this action. We have a map:

$$R : S(\lambda) \rightarrow \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A), \quad R(o) = u_o.$$

**Proposition 3.3.** *The map  $R$  is a bijection.*<sup>1</sup>

So in the particular case of  $J^\lambda$ , as above, the domains of elliptic curves in  $C \times S^1$  are “rectangular”, that is are quotients of the complex plane by a rectangular lattice, however for a more general almost complex structure on  $C \times S^1$ , tamed by more general lcs forms as we soon consider, the domain almost complex structure on our curves can in principle be arbitrary, in particular we might have nodal degenerations. Also note that the expected dimension of  $\overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$  is 0. It is given by the Fredholm index of the operator (5.2) which is 2, minus the dimension of the reparametrization group (for non-nodal curves) which is 2. That is given an elliptic curve  $S = (\Sigma, z)$ , let  $\mathcal{G}(\Sigma)$  be the 2-dimensional group of biholomorphisms  $\phi$  of  $\Sigma$ . Then given a  $J$ -holomorphic map  $u : \Sigma \rightarrow M$ ,  $(\Sigma, z, u)$  is equivalent to  $(\Sigma, \phi(z), u \circ \phi)$  in  $\overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$ , for  $\phi \in \mathcal{G}(\Sigma)$ .

The following is to be proved in Section 5, and is one of the principal ingredients for our main theorems.

**Theorem 3.4.** *Let  $(M, \lambda, \alpha, J)$  be a tamed first kind lcs manifold with  $M$  closed. Then every non-constant (nodal)  $J$ -holomorphic curve  $u : \Sigma \rightarrow M$  is a Reeb 2-curve.*

3.2.1. *Connection with the extended Fuller index.* Another important ingredient is a connection of the extended Fuller index with certain extended Gromov-Witten invariants. If  $\beta$  is a free homotopy class of a loop in  $C$  set

$$A_\beta = [\beta] \otimes [S^1] \in H_2(C \times S^1).$$

Then we have:

**Theorem 3.5.** *Suppose that  $\lambda$  is a contact form on a closed manifold  $C$ , so that its Reeb flow is definite type, see Appendix A, then*

$$GW_{1,1}(A_\beta, J^\lambda)([\overline{\mathcal{M}}_{1,1}] \otimes [C \times S^1]) = i(R^\lambda, \beta),$$

where both sides are certain extended rational numbers  $\mathbb{Q} \sqcup \{\pm\infty\}$  valued invariants, so that in particular if either side does not vanish then there are  $\lambda$ -Reeb orbits in class  $\beta$ .

What about higher genus invariants of  $C \times S^1$ ? Following the proof of Proposition 3.3, it is not hard to see that all  $J^\lambda$ -holomorphic curves must be branched covers of Reeb tori. If one can show that these branched covers are regular when the underlying tori are regular, the calculation of invariants would be fairly automatic from this data. See [31], [29] where these kinds of regularity calculation are made.

**3.3. Sky catastrophes.** The following is well known.

**Theorem 3.6.** *[17, Proposition 4.1.4], [28]] Let  $(M, J)$  be a compact almost complex manifold, and  $u : (S^2, j) \rightarrow M$  a  $J$ -holomorphic map. Given a Riemannian metric  $g$  on  $M$ , there is an  $\hbar = \hbar(g, J) > 0$  s.t. if  $e_g(u) < \hbar$  then  $u$  is constant, where  $e_g$  is the  $L^2$ -energy functional,*

$$e_g(u) = \text{energy}_g(u) = \int_{S^2} |du|^2 d\text{vol}.$$

---

<sup>1</sup>It is in fact an equivalence of the corresponding topological action groupoids, but we do not need this explicitly.



Using this we get the following (trivial) extension of Gromov compactness. Let

$$\mathcal{M}_{g,n}(J, A) = \mathcal{M}_{g,n}(M, J, A)$$

denote the moduli space of isomorphism classes of class  $A$ ,  $J$ -holomorphic curves in  $M$ , with domain a genus  $g$  closed Riemann surface, with  $n$  marked labeled points. Here an isomorphism between  $u_1 : \Sigma_1 \rightarrow M$ , and  $u_2 : \Sigma_2 \rightarrow M$  is a biholomorphism of marked Riemann surfaces  $\phi : \Sigma_1 \rightarrow \Sigma_2$  s.t.  $u_2 \circ \phi = u_1$ .

**Notation 2.** We will often say  *$J$ -curve* in place of  $J$ -holomorphic curve.

The following is proved by the same argument as [16, Theorem 5.6.6]. We claim no originality.

**Theorem 3.7.** *Let  $(M, J)$  be an almost complex manifold. Then  $\mathcal{M}_{g,n}(J, A)$  has a pre-compactification*

$$\overline{\mathcal{M}}_{g,n}(J, A),$$

*by Kontsevich stable maps, with respect to the natural metrizable Gromov topology see for instance [16, Chapter 5.6], for genus 0 case, [21] for general case. Moreover, given  $E > 0$ , the subspace  $\overline{\mathcal{M}}_{g,n}(J, A)_E \subset \overline{\mathcal{M}}_{g,n}(J, A)$  consisting of elements  $u$  with  $e(u) \leq E$  is compact, where  $e$  is the  $L^2$  energy with respect to an auxiliary metric. In other words  $e$  is a proper function.*

Thus, the most basic situation where we can talk about Gromov-Witten “invariants” of  $(M, J)$  is when the energy function is bounded on  $\overline{\mathcal{M}}_{g,n}(J, A)$ , and we shall say that  $J$  is **bounded** (in class  $A$ ), later on we generalize this in terms of what we call **finite type**. In this case  $\overline{\mathcal{M}}_{g,n}(J, A)$  is compact, and has a virtual moduli cycle as in the original approach of Fukaya-Ono [9], or the more algebraic approach [21]. So we may define functionals:

$$(3.8) \quad GW_{g,n}(A, J) : H_*(\overline{\mathcal{M}}_{g,n}) \otimes H_*(M^n) \rightarrow \mathbb{Q},$$

where  $\overline{\mathcal{M}}_{g,n}$  denotes the compactified moduli space of Riemann surfaces. Of course symplectic manifolds with any tame almost complex structure is one class of examples, another class of examples comes from some locally conformally symplectic manifolds. (We can take for instance the lcs-ification of  $(C, \lambda)$  with the latter the unit cotangent bundle of a hyperbolic manifold, with  $\lambda$  the canonical Liouville form, and  $J$  as in Section 5).

Given a continuous in the  $C^\infty$  topology family  $\{J_t\}$ ,  $t \in [0, 1]$  we denote by  $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$  the space of pairs  $(u, t)$ ,  $u \in \mathcal{M}_{g,n}(J_t, A)$ .

**Definition 3.9.** *We say that a continuous family  $\{J_t\}$ ,  $t \in [0, 1]$  on a compact manifold  $M$  has a **holomorphic sky catastrophe** in class  $A$  if there is an element  $u \in \overline{\mathcal{M}}_{g,n}(J_i, A)$ ,  $i = 0, 1$  which does not belong to any open compact (equivalently energy bounded) subset of  $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ .*

Let us slightly expand this definition. If  $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$  is locally connected, so that the connected components are open, then we have a sky catastrophe in the sense above if and only if there is a  $u \in \overline{\mathcal{M}}_{g,n}(J_i, A)$  which has a non-compact connected component in  $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ . At this point in time there are no known examples of families  $\{J_t\}$  with sky catastrophes.

*Question 1.* Do holomorphic sky catastrophes generically exist? That is given a family  $\{J_t\}$  can it be  $C^\infty$  perturbed to a family  $\{J'_t\}$ ,  $J_0 = J'_0$ ,  $J_1 = J'_1$  so that  $\{J'_t\}$  has no sky catastrophe?

As a corollary of Theorem 2.13 and Example 2 we get:

**Corollary 3.10.** *Assume that generically sky catastrophes do not exist, then CSW conjecture holds for every first kind lcs structures on  $S^{2k+1} \times S^1$  formally homotopic to the standard lcs structure.*

The author’s opinion is that they may appear even generically. However, if we ask that each  $J_t$  is tamed by an exact lcs structure, then the question becomes much more subtle, see also [23] for a related discussion on possible obstructions to sky catastrophes. See also [24] for non-trivial examples where sky catastrophes can be ruled out.

If holomorphic sky catastrophes are discovered, this would be a very interesting discovery. The original discovery by Fuller [10] of sky catastrophes in dynamical systems is one of the most important in dynamical systems, see also [25] for an overview.

## 4. ELEMENTS OF GROMOV-WITTEN THEORY OF AN lcs MANIFOLD

Suppose  $(M, J)$  is a compact almost complex manifold, where the almost complex structures  $J$  are assumed throughout the paper to be  $C^\infty$ , and let  $N \subset \overline{\mathcal{M}}_{g,k}(J, A)$  be an open compact subset with energy positive on  $N$ . The latter condition is only relevant when  $A = 0$ . We shall primarily refer in what follows to work of Pardon in [21], only because this is what is more familiar to the author, due to greater comfort with algebraic topology. But we should mention that the latter is a followup to a profound theory that is originally created by Fukaya-Ono [9], and later expanded with Oh-Ohta [8].

The construction in [21] of implicit atlas, on the moduli space  $\mathcal{M}$  of curves in a symplectic manifold, only needs a neighborhood of  $\mathcal{M}$  in the space of all curves. So more generally if we have an almost complex manifold and an *open* compact component  $N$  as above, this will likewise have a natural implicit atlas, or a Kuranishi structure in the setup of [9]. And so such an  $N$  will have a virtual fundamental class in the sense of Pardon [21], (or in any other approach to virtual fundamental cycle, particularly the original approach of Fukaya-Oh-Ohta-Ono). This understanding will be used in other parts of the paper, following Pardon for the explicit setup. We may thus define functionals:

$$(4.1) \quad GW_{g,n}(N, A, J) : H_*(\overline{\mathcal{M}}_{g,n}) \otimes H_*(M^n) \rightarrow \mathbb{Q}.$$

How do these functionals depend on  $N, J$ ?

**Lemma 4.2.** *Let  $\{J_t\}$ ,  $t \in [0, 1]$  be a Frechet smooth family. Suppose that  $\tilde{N}$  is an open compact subset of the cobordism moduli space  $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$  and that the energy function is positive on  $\tilde{N}$ , (the latter only relevant when  $A = 0$ ). Let*

$$N_i = \tilde{N} \cap (\overline{\mathcal{M}}_{g,n}(J_i, A)),$$

then

$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1).$$

In particular if  $GW_{g,n}(N_0, A, J_0) \neq 0$ , there is a class  $A$   $J_1$ -holomorphic stable map in  $M$ .

*Proof of Lemma 4.2.* We may construct exactly as in [21] a natural implicit atlas on  $\tilde{N}$ , with boundary  $N_0^{op} \sqcup N_1$ , (*op* denoting opposite orientation). And so

$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1),$$

as functionals. □

The most basic lemma in this setting is the following, and we shall use it in the following section.

**Definition 4.3.** *An almost symplectic pair on  $M$  is a tuple  $(M, \omega, J)$ , where  $\omega$  is a non-degenerate 2-form on  $M$ , and  $J$  is  $\omega$ -compatible, meaning that  $\omega(\cdot, J\cdot)$  defines  $J$ -invariant Riemannian metric. When  $\omega$  is lcs we call such a pair an **lcs pair**.*

**Definition 4.4.** *We say that a pair of almost symplectic pairs  $(\omega_i, J_i)$  are  $\delta$ -close, if  $\{\omega_i\}$  are  $C^0$   $\delta$ -close, and  $\{J_i\}$  are  $C^2$   $\delta$ -close,  $i = 0, 1$ . Define this similarly for a pair  $(g_i, J_i)$  for  $g$  a Riemannian metric and  $J$  an almost complex structure.*

**Definition 4.5.** *For an almost symplectic pair  $(\omega, J)$  on  $M$ , and a smooth map  $u : \Sigma \rightarrow M$  define:*

$$e_\omega(u) = \int_\Sigma u^* \omega.$$

By an elementary calculation this coincides with the  $L^2$   $g_J$ -energy of  $u$ , for  $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$ . That is  $e_\omega(u) = e_{g_J}(u)$ . In what follows by  $f^{-1}(a, b)$ , with  $f$  a function, we mean the preimage by  $f$  of the open set  $(a, b)$ .

**Lemma 4.6.** *Given a Riemannian manifold  $(M, g)$ , and  $J$  an almost complex structure, suppose that  $N \subset \overline{\mathcal{M}}_{d,n}(J, A)$  is a compact and open component which is energy isolated meaning that*

$$N \subset (U = e_g^{-1}(E^0, E^1)) \subset (V = e_g^{-1}(E^0 - \epsilon, E^1 + \epsilon)),$$

with  $\epsilon > 0$ ,  $E_0 > 0$ , and with  $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$ . Then there is a  $\delta > 0$  s.t. whenever  $(g', J')$  is  $\delta$ -close to  $(g, J)$  if  $u \in \overline{\mathcal{M}}_{g,n}(J', A)$  and

$$E^0 - \epsilon < e_{g'}(u) < E^1 + \epsilon$$

then

$$E^0 < e_{g'}(u) < E^1.$$

*Proof.* Suppose otherwise then there is a sequence  $\{(g_k, J_k)\}$  converging to  $(g, J)$ , and a sequence  $\{u_k\}$  of  $J_k$ -holomorphic stable maps satisfying

$$E^0 - \epsilon < e_{g_k}(u_k) \leq E^0$$

or

$$E^1 \leq e_{g_k}(u_k) < E^1 + \epsilon.$$

By Gromov compactness, specifically theorems [17, B.41, B.42], we may find a Gromov convergent subsequence  $\{u_{k_j}\}$  to a  $J$ -holomorphic stable map  $u$ , with

$$E^0 - \epsilon \leq e_g(u) \leq E^0$$

or

$$E^1 \leq e_g(u) \leq E^1 + \epsilon.$$

But by our assumptions such a  $u$  does not exist.  $\square$

**Lemma 4.7.** *Let  $M$  be compact, and let  $(M, \omega, J)$  be an almost symplectic triple, so that  $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$  is exactly as in the lemma above with respect to some  $\epsilon > 0$ . Then, there is a  $\delta > 0$  s.t. the following is satisfied. Let  $(\omega', J')$  be  $\delta$ -close to  $(\omega, J)$ , then there is a continuous in the  $C^\infty$  topology family of almost symplectic pairs  $\{(\omega_t, J_t)\}$ ,  $(\omega_0, J_0) = (g, J)$ ,  $(\omega_1, J_1) = (g', J')$  s.t. there is open compact subset*

$$\tilde{N} \subset \overline{\mathcal{M}}_{g,n}(\{J_t\}, A),$$

and with

$$\tilde{N} \cap \overline{\mathcal{M}}(J, A) = N.$$

Moreover, if  $(u, t) \in \tilde{N}$  then

$$E^0 < e_{g_t}(u) < E^1.$$

*Proof.* For  $\epsilon$  as in the hypothesis, let  $\delta$  be as in Lemma 4.6.

**Lemma 4.8.** *Given a  $\delta > 0$  there is a  $\delta' > 0$  s.t. if  $(\omega', J')$  is  $\delta'$ -near  $(\omega, J)$  there is an interpolating, continuous in  $C^\infty$  topology family  $\{(\omega_t, J_t)\}$  with  $(\omega_t, J_t)$   $\delta$ -close to  $(\omega, J)$  for each  $t$ .*

*Proof.* Let  $\{g_t\}$  be the family of metrics on  $M$  given by the convex linear combination of  $g = g_{\omega, J}$ ,  $g' = g_{\omega', J'}$ . Clearly  $g_t$  is  $\delta'$ -close to  $g_0$  for each  $t$ . Likewise, the family of 2 forms  $\{\omega_t\}$  given by the convex linear combination of  $\omega$ ,  $\omega'$  is non-degenerate for each  $t$  if  $\delta'$  was chosen to be sufficiently small and is  $\delta'$ -close to  $\omega_0 = \omega_{g, J}$  for each moment.

Let

$$ret : Met(M) \times \Omega(M) \rightarrow \mathcal{J}(M)$$

be the “retraction map” (it can be understood as a retraction followed by projection) as defined in [16, Prop 2.50], where  $Met(M)$  is space of metrics on  $M$ ,  $\Omega(M)$  the space of 2-forms on  $M$ , and  $\mathcal{J}(M)$  the space of almost complex structures. This map has the property that the almost complex structure  $ret(g, \omega)$  is compatible with  $\omega$ , and that  $ret(g_J, \omega) = J$  for  $g_J = \omega(\cdot, J\cdot)$ . Then  $\{(\omega_t, ret(g_t, \omega_t))\}$  is a compatible family. As  $ret$  is continuous in  $C^2$ -topology,  $\delta'$  can be chosen so that  $\{ret_t(g_t, \omega_t)\}$  are  $\delta$ -nearby.  $\square$

Returning to the proof of the main lemma. Let  $\delta' < \delta$  be chosen as in the above lemma and let  $\{(\omega_t, J_t)\}$  be the corresponding family. Let  $\tilde{N}$  consist of all elements  $(u, t) \in \overline{\mathcal{M}}(\{J_t\}, A)$  s.t.

$$E^0 - \epsilon < e_{\omega_t}(u) < E^1 + \epsilon.$$

Then by Lemma 4.6 for each  $(u, t) \in \tilde{N}$ , we have:

$$E^0 < e_{\omega_t}(u) < E^1.$$

In particular  $\tilde{N}$  must be closed, it is also clearly open, and is compact as the energy  $e$  is a proper function, as discussed. Renaming  $\delta := \delta'$  we are then done.  $\square$

**Proposition 4.9.** *Given a compact  $M$  and an almost symplectic pair  $(\omega, J)$  on  $M$ , suppose that  $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$  is a compact and open component which is energy isolated meaning that*

$$N \subset (U = e_{\omega}^{-1}(E^0, E^1)) \subset (V = e_{\omega}^{-1}(E^0 - \epsilon, E^1 + \epsilon)),$$

*with  $\epsilon > 0$ ,  $E^0 > 0$  and with  $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$ . Suppose also that  $GW_{g,n}(N, J, A) \neq 0$ . Then there is a  $\delta > 0$  s.t. whenever  $(\omega', J')$  is a compatible almost symplectic pair  $\delta$ -close to  $(\omega, J)$ , there exists  $u \in \overline{\mathcal{M}}_{g,n}(J', A) \neq \emptyset$ , with*

$$E^0 < e_{\omega'}(u) < E^1.$$

*Proof.* For  $N, \epsilon$  as in the hypothesis, let  $\delta, \tilde{N}$  be as in Lemma 4.7, then by Lemma 4.2

$$GW_{g,n}(N_1, J', A) = GW_{g,n}(N, J, A) \neq 0,$$

where  $N_1 = \tilde{N} \cap \overline{\mathcal{M}}_{g,n}(J_1, A)$ .  $\square$

While not having sky catastrophes gives us a certain compactness control, the above proposition is not immediate because we can still in principle have total cancellation of the infinitely many components of the moduli space  $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$ . In other words a virtual 0-dimension Kuranishi space  $\overline{\mathcal{M}}^{1,0}(J^\lambda, A)$ , with an infinite number of compact connected components, can certainly be null-cobordant, by a cobordism all of whose components are compact. So we need a certain additional algebraic and geometric control to preclude such a total cancellation.

*Proof of Theorem 3.7.* (Outline, as the argument is standard.) Suppose that we have a sequence  $u^k$  of  $J$ -holomorphic maps with  $L^2$ -energy  $\leq E$ . By [16, 4.1.1], a sequence  $u^k$  of  $J$ -holomorphic curves has a convergent subsequence if  $\sup_k \|du^k\|_{L^\infty} < \infty$ . On the other hand, when this condition does not hold, rescaling argument tells us that a holomorphic sphere bubbles off. The quantization Theorem 3.6, tells us that the energy of a non-constant  $J$ -holomorphic map of  $\mathbb{CP}^1$  is at least  $\hbar > 0$ . So if the energy of the maps  $u^k$  is bounded from above by  $E$ , only finitely many bubbles may appear, so that a subsequence of  $u^k$  must converge in the Gromov topology to a Kontsevich stable map.  $\square$

## 5. ELLIPTIC CURVES IN THE LCS-FICATION OF A CONTACT MANIFOLD AND THE FULLER INDEX

*Proof of Proposition 3.3.* Suppose we have a curve  $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$ , represented by  $u : \Sigma \rightarrow M = C \times S^1$ . Then  $u$  has no spherical components, as otherwise we would have a non-constant  $J^\lambda$ -holomorphic sphere. And so by Theorem 3.4, we would have a Reeb 2-curve  $u : \mathbb{CP}^1 \rightarrow M$ , which is impossible by property 2 of the definition.

By Theorem 3.4  $u$  is a Reeb 2-curve. By Lemma 6.6 its normalization  $\tilde{u}$  is also a Reeb 2-curve. By the charge  $(1, 0)$  condition  $pr_{S^1} \circ u$  is surjective, where  $pr_{S^1} : C \times S^1 \rightarrow S^1$  is the projection. And so  $pr_{S^1} \circ \tilde{u}$  is surjective. Let us rename  $\tilde{u}$  by  $u$  in what follows.

By the Sard theorem we have a regular value  $t_0 \in S^1$ , so that  $u^{-1} \circ pr_{S^1}^{-1}(t_0)$  contains an embedded circle  $S_0 \subset \Sigma$ . Now  $d(pr_{S^1} \circ u)$  is surjective onto  $T_{t_0} S^1$  along  $T(\Sigma)|_{S_0}$ . And so since  $u$  is  $J^\lambda$ -holomorphic,  $o = pr_C \circ u|_{S_0}$  has non-vanishing differential. By the first part of the condition in Definition 2.7,  $o$  is an unparametrized Reeb curve. Also, the image of  $d(pr_C \circ u)$  is in  $\ker d\lambda$  from which it follows that  $\text{image } d(pr_C \circ u) = \text{image } d(o)$ . It follows that  $u$  is an elliptic charge  $(1, 0)$  curve with image contained in the image of the Reeb torus  $u_o$ . Unique extension theorem for pseudo-holomorphic curves implies

that up to parametrization,  $u$  is the Reeb torus  $u_{o'}$  for some  $o'$  ( $o'$  is up to parametrization some covering map of  $o$ ).

□

**Proposition 5.1.** *Let  $(C, \xi)$  be a general contact manifold. If  $\lambda$  is a non-degenerate contact 1-form for  $\xi$  then all the elements of  $\overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$  are regular curves. Moreover, if  $\lambda$  is degenerate then for a period  $c$  Reeb orbit  $o$  the kernel of the associated real linear Cauchy-Riemann operator for the Reeb torus  $u_o$  is naturally identified with the 1-eigenspace of  $\phi_{c,*}^\lambda$  - the time  $c$  linearized return map  $\xi(o(0)) \rightarrow \xi(o(0))$  induced by the  $R^\lambda$  Reeb flow.*

*Proof.* We already know by Proposition 3.3 that all  $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$  are equivalent to Reeb tori. In particular, such curves have representation by a  $J^\lambda$ -holomorphic map

$$u : (T^2, j) \rightarrow (Y = C \times S^1, J^\lambda).$$

Since each  $u$  is immersed we may naturally get a splitting  $u^*T(Y) \simeq N \times T(T^2)$ , using the  $g_J$  metric, where  $N \rightarrow T^2$  denotes the pull-back, of the  $g_J$ -normal bundle to image  $u$ , and which is identified with the pullback of the distribution  $\xi_\lambda$  on  $Y$ , (which we also call the co-vanishing distribution).

The full associated real linear Cauchy-Riemann operator takes the form:

$$(5.2) \quad D_u^J : \Omega^0(N \oplus T(T^2)) \oplus T_j M_{1,1} \rightarrow \Omega^{0,1}(T(T^2), N \oplus T(T^2)).$$

This is an index 2 Fredholm operator (after standard Sobolev completions), whose restriction to  $\Omega^0(N \oplus T(T^2))$  preserves the splitting, that is the restricted operator splits as

$$D \oplus D' : \Omega^0(N) \oplus \Omega^0(T(T^2)) \rightarrow \Omega^{0,1}(T(T^2), N) \oplus \Omega^{0,1}(T(T^2), T(T^2)).$$

On the other hand the restricted Fredholm index 2 operator

$$\Omega^0(T(T^2)) \oplus T_j M_{1,1} \rightarrow \Omega^{0,1}(T(T^2)),$$

is surjective by classical Teichmüller theory, see also [30, Lemma 3.3] for a precise argument in this setting. It follows that  $D_u^J$  will be surjective if the restricted Fredholm index 0 operator

$$D : \Omega^0(N) \rightarrow \Omega^{0,1}(N),$$

has no kernel.

The bundle  $N$  is symplectic with symplectic form on the fibers given by restriction of  $u^*d\lambda$ , and together with  $J^\lambda$  this gives a Hermitian structure  $(g_\lambda, j_\lambda)$  on  $N$ . We have a linear symplectic connection  $\mathcal{A}$  on  $N$ , which over the slices  $S^1 \times \{t\} \subset T^2$  is induced by the pullback by  $u$  of the linearized  $R^\lambda$  Reeb flow. Specifically the  $\mathcal{A}$ -transport map from the fiber  $N_{(s_0,t)}$  to the fiber  $N_{(s_1,t)}$  over the path  $[s_0, s_1] \times \{t\} \subset T^2$ , is given by

$$(u_*|_{N_{(s_1,t)}})^{-1} \circ (\phi_{c(s_1-s_0)}^\lambda)_* \circ u_*|_{N_{(s_0,t)}},$$

where  $\phi_{c(s_1-s_0)}^\lambda$  is the time  $c \cdot (s_1 - s_0)$  map for the  $R^\lambda$  Reeb flow, where  $c$  is the period of the Reeb orbit  $o_u$ , and where  $u_* : N \rightarrow TY$  denotes the natural map, (it is the universal map in the pull-back diagram.)

The connection  $\mathcal{A}$  is defined to be trivial in the  $\theta_2$  direction, where trivial means that the parallel transport maps are the *id* maps over  $\theta_2$  rays. In particular the curvature  $R_{\mathcal{A}}$ , understood as a Lie algebra valued 2-form, of this connection vanishes. The connection  $\mathcal{A}$  determines a real linear CR operator  $D_{\mathcal{A}}$  on  $N$  in the standard way, take the complex anti-linear part of the vertical differential of a section. Explicitly,

$$D_{\mathcal{A}} : \Omega^0(N) \rightarrow \Omega^{0,1}(N),$$

is defined by

$$D_{\mathcal{A}}(\mu)(p) = j_\lambda \circ \pi^{vert}(\mu(p)) \circ d\mu(p) - \pi^{vert}(\mu(p)) \circ d\mu(p) \circ j,$$

where

$$\pi^{vert}(\mu(p)) : T_{\mu(p)}N \rightarrow T_{\mu(p)}^{vert}N \simeq N$$

is the  $\mathcal{A}$ -projection, and where  $T_{\mu(p)}^{vert}N$  is the kernel of the projection  $T_{\mu(p)}N \rightarrow T_p\Sigma$ . It is elementary to verify from the definitions that this operator is exactly  $D$ . See also [19, Section 10.1] for a computation of this kind in much greater generality.

We have a differential 2-form  $\Omega$  on the total space of  $N$  defined as follows. On the fibers  $T^{vert}N$ ,  $\Omega = u_*\omega$ , for  $\omega = d_\alpha\lambda$ , and for  $T^{vert}N \subset TN$  denoting the vertical tangent space, or subspace of vectors  $v$  with  $\pi_*v = 0$ , for  $\pi : N \rightarrow T^2$  the projection. While on the  $\mathcal{A}$ -horizontal distribution  $\Omega$  is defined to vanish. The 2-form  $\Omega$  is closed, which we may check explicitly by using that  $R_{\mathcal{A}}$  vanishes to obtain local symplectic trivializations of  $N$  in which  $\mathcal{A}$  is trivial. Clearly  $\Omega$  must vanish on the 0-section since it is a  $\mathcal{A}$ -flat section. But any section is homotopic to the 0-section and so in particular if  $\mu \in \ker D$  then  $\Omega$  vanishes on  $\mu$ .

Since  $\mu \in \ker D$ , and so its vertical differential is complex linear, it follows that the vertical differential vanishes. To see this note that  $\Omega(v, J^\lambda v) > 0$ , for  $0 \neq v \in T^{vert}N$  and so if the vertical differential did not vanish we would have  $\int_\mu \Omega > 0$ . So  $\mu$  is  $\mathcal{A}$ -flat, in particular the restriction of  $\mu$  over all slices  $S^1 \times \{t\}$  is identified with a period  $c$  orbit of the linearized at  $o$   $R^\lambda$  Reeb flow, and which does not depend on  $t$  as  $\mathcal{A}$  is trivial in the  $t$  variable. So the kernel of  $D$  is identified with the vector space of period  $c$  orbits of the linearized at  $o$   $R^\lambda$  Reeb flow, as needed.  $\square$

**Proposition 5.3.** *Let  $\lambda$  be a contact form on a  $(2n+1)$ -fold  $C$ , and  $o$  a non-degenerate, period  $c$ ,  $R^\lambda$ -Reeb orbit, then the orientation of  $[u_o]$  induced by the determinant line bundle orientation of  $\overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$ , is  $(-1)^{CZ(o)-n}$ , which is*

$$\text{sign Det}(\text{Id}|_{\xi(o(0))} - \phi_{c,*}^\lambda|_{\xi(o(0))}).$$

*Proof of Proposition 5.3.* Abbreviate  $u_o$  by  $u$ . Let  $N \rightarrow T^2$  be associated to  $u$  as in the proof of Proposition 5.1. Fix a trivialization  $\phi$  of  $N$  induced by any trivialization of the contact distribution  $\xi$  along  $o$  in the obvious sense:  $N$  is the pullback of  $\xi$  along the composition

$$T^2 \rightarrow S^1 \xrightarrow{o} C.$$

Let the symplectic connection  $\mathcal{A}$  on  $N$  be defined as before. Then the pullback connection  $\mathcal{A}' := \phi^*\mathcal{A}$  on  $T^2 \times \mathbb{R}^{2n}$  is a connection whose parallel transport paths  $p_t : [0, 1] \rightarrow \text{Symp}(\mathbb{R}^{2n})$ , along the closed loops  $S^1 \times \{t\}$ , are paths starting at 1, and are  $t$  independent. And so the parallel transport path of  $\mathcal{A}'$  along  $\{s\} \times S^1$  is constant, that is  $\mathcal{A}'$  is trivial in the  $t$  variable. We shall call such a connection  $\mathcal{A}'$  on  $T^2 \times \mathbb{R}^{2n}$  induced by  $p$ .

By non-degeneracy assumption on  $o$ , the map  $p(1)$  has no 1-eigenvalues. Let  $p'' : [0, 1] \rightarrow \text{Symp}(\mathbb{R}^{2n})$  be a path from  $p(1)$  to a unitary map  $p''(1)$ , with  $p''(1)$  having no 1-eigenvalues, and s.t.  $p''$  has only simple crossings with the Maslov cycle. Let  $p'$  be the concatenation of  $p$  and  $p''$ . We then get

$$CZ(p') - \frac{1}{2} \text{sign } \Gamma(p', 0) \equiv CZ(p') - n \equiv 0 \pmod{2},$$

since  $p'$  is homotopic relative end points to a unitary geodesic path  $h$  starting at  $id$ , having regular crossings, and since the number of negative, positive eigenvalues is even at each regular crossing of  $h$  by unitarity. Here  $\text{sign } \Gamma(p', 0)$  is the index of the crossing form of the path  $p'$  at time 0, in the notation of [22]. Consequently,

$$(5.4) \quad CZ(p'') \equiv CZ(p) - n \pmod{2},$$

by additivity of the Conley-Zehnder index.

Let us then define a free homotopy  $\{p_t\}$  of  $p$  to  $p'$ ,  $p_t$  is the concatenation of  $p$  with  $p''|_{[0,t]}$ , reparametrized to have domain  $[0, 1]$  at each moment  $t$ . This determines a homotopy  $\{\mathcal{A}'_t\}$  of connections induced by  $\{p_t\}$ . By the proof of Proposition 5.1, the CR operator  $D_t$  determined by each  $\mathcal{A}'_t$  is surjective except at some finite collection of times  $t_i \in (0, 1)$ ,  $i \in N$  determined by the crossing times of  $p''$  with the Maslov cycle, and the dimension of the kernel of  $D_{t_i}$  is the 1-eigenspace of  $p''(t_i)$ , which is 1 by the assumption that the crossings of  $p''$  are simple.

The operator  $D_1$  is not complex linear. To fix this we concatenate the homotopy  $\{D_t\}$  with the homotopy  $\{\tilde{D}_t\}$  defined as follows. Let  $\{\tilde{\mathcal{A}}_t\}$  be a homotopy of  $\mathcal{A}'_1$  to a unitary connection  $\tilde{\mathcal{A}}_1$ , where



the homotopy  $\{\tilde{\mathcal{A}}_t\}$  is through connections induced by paths  $\{\tilde{p}_t\}$ , giving a path homotopy of  $p' = \tilde{p}_0$  to  $h$ . Then  $\{\tilde{D}_t\}$  is defined to be induced by  $\{\tilde{\mathcal{A}}_t\}$ .

Let us denote by  $\{D'_t\}$  the concatenation of  $\{D_t\}$  with  $\{\tilde{D}_t\}$ . By construction in the second half of the homotopy  $\{D'_t\}$ ,  $D'_t$  is surjective. And  $D'_1$  is induced by a unitary connection, since it is induced by unitary path  $\tilde{p}_1$ . Consequently,  $D'_1$  is complex linear. By the above construction, for the homotopy  $\{D'_t\}$ ,  $D'_t$  is surjective except for  $N$  times in  $(0, 1)$ , where the kernel has dimension one. In particular the sign of  $[u]$  by the definition via the determinant line bundle is exactly

$$-1^N = -1^{CZ(p)-n},$$

by (5.4), which was what to be proved.  $\square$

**Theorem 5.5.**

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(\tilde{N}, R^\lambda, \beta),$$

where  $N \subset \overline{M}_{1,1}^{1,0}(J^\lambda, A_\beta)$  is an open compact set,  $\tilde{N}$  the corresponding under  $R$  subset of periodic orbits of  $R^\lambda$  (where  $R$  is as in Proposition 3.3),  $i(\tilde{N}, R^\lambda, \beta)$  is the Fuller index as described in the appendix below, and where the left-hand side of the equation is the functional as in (4.1).

*Proof.* If  $N \subset \overline{M}_{1,1}^{1,0}(J^\lambda, A_\beta)$  is open-compact and consists of isolated regular Reeb tori  $\{u_i\}$ , corresponding to orbits  $\{o_i\}$  we have:

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = \sum_i \frac{(-1)^{CZ(o_i)-n}}{\text{mult}(o_i)},$$

where the denominator  $\text{mult}(o_i)$  is there because our moduli space is understood as a non-effective orbifold, see Appendix B.

The expression on the right is exactly the Fuller index  $i(\tilde{N}, R^\lambda, \beta)$ . Thus, the theorem follows for  $N$  as above. However, in general if  $N$  is open and compact then perturbing slightly we obtain a smooth family  $\{R^{\lambda_t}\}$ ,  $\lambda_0 = \lambda$ , s.t.  $\lambda_1$  is non-degenerate, that is has non-degenerate orbits. And such that there is an open-compact subset  $\tilde{N}$  of  $\overline{M}_{1,1}^{1,0}(\{J^{\lambda_t}\}, A_\beta)$  with  $(\tilde{N} \cap \overline{M}_{1,1}^{1,0}(J^\lambda, A_\beta)) = N$ , cf. Lemma 4.7. Then by Lemma 4.2 if

$$N_1 = (\tilde{N} \cap \overline{M}_{1,1}^{1,0}(J^{\lambda_1}, A_\beta))$$

we get

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}_{1,1}] \otimes [C \times S^1]).$$

By the previous discussion

$$GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(N_1, R^{\lambda_1}, \beta),$$

but by the invariance of Fuller index (see Appendix A),

$$i(N_1, R^{\lambda_1}, \beta) = i(N, R^\lambda, \beta).$$

$\square$

## 6. PROOFS OF MAIN THEOREMS

**Lemma 6.1.** *Let  $(M, \lambda, \alpha)$  be an exact lcs manifold with  $M$  closed then  $0 \neq [\alpha] \in H^1(M, \mathbb{R})$ .*

*Proof.* Suppose that  $\alpha$  is exact and let  $g$  be its primitive. Then  $d_\alpha \lambda = \frac{1}{f} d(f\lambda)$  with  $f = e^g$ . Consequently,  $d(f\lambda)$  is non-degenerate on  $M$  which contradicts Stokes theorem.  $\square$

*Proof of Theorem 2.9.* By Lemma 6.1 the cohomology class of  $\alpha$  is non-zero in  $H^1(\Sigma, \mathbb{R})$ . So we may take  $u = id : \Sigma \rightarrow \Sigma$ , clearly it is a Reeb 2-curve.  $\square$

**Definition 6.2.** *Let  $\alpha$  be a closed rational 1-form on a closed smooth manifold  $M$ . The **classifying map**  $p : M \rightarrow S^1$  of  $\alpha$  is a smooth fibration s.t.  $\alpha = c \cdot p^* d\theta$  for  $c \in \mathbb{Q}$ .*

For completeness, we review the basic definition of a nodal curve.



**Definition 6.3.** A **nodal Riemann surface** (without boundary) is a pair  $\Sigma = (\tilde{\Sigma}, \mathcal{N})$  where  $\tilde{\Sigma}$  is a Riemann surface, and  $\mathcal{N}$  a set of pairs of points of  $\tilde{\Sigma} : \mathcal{N} = \{(z_0^0, z_0^1), \dots, (z_n^0, z_n^1)\}$ ,  $n_i^j \neq n_k^l$  for  $i \neq k$  and all  $j, l$ . By slight abuse, we may also denote by  $\Sigma$  the quotient space  $(\tilde{\Sigma}/\sim)$ , where the equivalence relation is generated by  $n_i^0 \sim n_i^1$ . Let  $\phi : \tilde{\Sigma} \rightarrow (\tilde{\Sigma}/\sim)$  be the quotient map. The elements  $\phi(\{z_i^0, z_i^1\}) \in \tilde{\Sigma}/\sim$ , are called **nodes**. Let  $M$  be a smooth manifold. By a map  $u : \Sigma \rightarrow M$  of a nodal Riemann surface  $\Sigma$ , we mean the set map  $u : (\tilde{\Sigma}/\sim) \rightarrow M$ . It is called *smooth* or *immersion* or *J-holomorphic* (when  $M$  is almost complex) if the map  $\tilde{u} = u \circ \phi$  is smooth or immersion or J-holomorphic. We call  $\tilde{u}$  **normalization** of  $u$ .  $u$  is called an *embedding* if  $u$  is a topological embedding and its normalization is an immersion. The cohomology groups of  $\Sigma$  are defined as  $H^\bullet(\Sigma) := H^\bullet(\tilde{\Sigma}/\sim)$ , likewise with homology. The genus of  $\Sigma$  is the topological genus of  $\tilde{\Sigma}/\sim$ .

We shall say that  $(\tilde{\Sigma}, \mathcal{N})$  is normal if  $\mathcal{N} = \emptyset$ . Similarly, a Reeb 2-curve  $u : \Sigma \rightarrow M$ ,  $\Sigma = (\tilde{\Sigma}, \mathcal{N})$  is **normal** if  $\mathcal{N} = \emptyset$ . The normalization of  $u$  is the map of the nodal Riemann surface  $\tilde{u} : \tilde{\Sigma} \rightarrow M$ ,  $\tilde{\Sigma} = (\tilde{\Sigma}, \emptyset)$ . Note that if  $u$  is a Reeb 2-curve, its normalization  $\tilde{u}$  may not be a Reeb 2-curve (the second condition may fail).

*Proof of Theorem 2.8.* Let  $(M = C \times S^1, \lambda, \alpha)$  be the lcs-fication of a closed contact manifold  $(C, \lambda)$ . In this case  $\mathcal{V}_\lambda = \mathcal{D}$  and is spanned by  $X_\lambda = (0, \frac{\partial}{\partial \theta})$ ,  $X_\alpha = (R^\lambda, 0)$  for  $R^\lambda$  the  $\lambda$ -Reeb vector field.

Suppose first  $u : \Sigma \rightarrow M$  is a normal Reeb 2-curve. By definitions  $u^*\alpha$  is a rational 1-form non-vanishing in  $H_{DR}^1(\Sigma)$ . Then let  $p : \Sigma \rightarrow S^1$  be its classifying map, and let  $o : S^1 \rightarrow \Sigma$  smoothly parametrize a component of a regular fiber of  $p$ , not intersecting the singular set of  $u$ . By the first condition of the definition of a Reeb 2-curve,  $u^*\alpha, u^*\lambda$  are non-vanishing along  $S = o(S^1)$ . Since  $u^*\alpha$  vanishes on  $TS \subset T\Sigma$  it follows that  $u^*\lambda$  is non-vanishing on  $TS$ . Also by the first condition,  $u \circ o$  is tangent to  $\mathcal{V}_\lambda$ . And so  $pr_C \circ u \circ o$  is tangent to  $\ker d\lambda$ , and  $\lambda((pr_C \circ u \circ o)') > 0$ . It follows that  $pr_C \circ u \circ o$  is a Reeb orbit up to parametrization.

If  $u$  is not normal, then since  $(M, \lambda, \alpha)$  is rational and first kind, by Lemma 6.6 the normalization  $\tilde{u}$  of  $u$  is a normal Reeb 2-curve. (The proof of that lemma is based on the first part of the argument above). Then apply the argument above to  $\tilde{u}$ , and we are done.  $\square$

*Proof of Theorem 3.4.* Let  $u : \Sigma \rightarrow M$  be a non-constant, nodal J-curve. Since singularities of  $u$  are isolated, by Lemma 3.2 it is enough to show that  $[u^*\alpha] \neq 0$ . Let  $\tilde{M}$  denote the  $\alpha$ -covering space of  $M$ , that is the space of equivalence classes of paths  $p$  starting at  $x_0 \in M$ , with a pair  $p_1, p_2$  equivalent if  $p_1(1) = p_2(1)$  and

$$\int_{[0,1]} p_1^* \alpha = \int_{[0,1]} p_2^* \alpha.$$

Then the lift of  $\omega$  to  $\tilde{M}$  is

$$\tilde{\omega} = \frac{1}{f} d(f\lambda),$$

where  $f = e^g$  and where  $g$  is a primitive for the lift  $\tilde{\alpha}$  of  $\alpha$  to  $\tilde{M}$ , that is  $\tilde{\alpha} = dg$ . In particular  $\tilde{\omega}$  is conformally symplectomorphic to an exact symplectic form on  $\tilde{M}$ . So if  $\tilde{J}$  denotes the lift of  $J$ , any closed  $\tilde{J}$ -curve is constant by Stokes theorem. Now if  $[u^*\alpha] = 0$  then  $u$  has a lift to a  $\tilde{J}$ -holomorphic map  $v : \Sigma \rightarrow \tilde{M}$ . Since  $\Sigma$  is closed, it follows by the above that  $v$  is constant, so that  $u$  is constant, which is impossible.  $\square$

**Proposition 6.4.** Let  $(C, \lambda)$  be a closed contact manifold with  $\lambda$  having at least one closed non-degenerate Reeb orbit  $o$ . Then there exists an  $\epsilon > 0$  s.t. for any tamed exact lcs structure  $(\lambda', \alpha', J)$  on  $M = C \times S^1$ , with  $(d_{\alpha'}\lambda', J)$   $\epsilon$ -close to  $(d_\alpha\lambda, J^\lambda)$  (as in Definition 4.4), there exists an elliptic, charge  $(1, 0)$ , J-holomorphic curve  $u$  in  $M$ . In addition, if  $(M, \lambda', \alpha')$  is first kind and has dimension 4 then  $u$  may be assumed to be embedded and normal.

*Proof.* Let  $N \subset \overline{\mathcal{M}}_{1,1}^{1,0}(A, J^\lambda)$ , be the subspace consisting of a single point, corresponding to  $o$  (under the correspondence of Proposition 3.3). Clearly  $N$  is open and compact by the non-degeneracy assumption. By Theorem 5.5  $GW_{1,1}(N, J^\lambda, A) \neq 0$ , since the Fuller index of  $o$  is non-zero by non-degeneracy.  $N$  is also clearly energy isolated, where this is as in Proposition 4.9. The proposition then follows by Proposition 4.9.

We now verify the second part. Suppose that  $M$  has dimension 4. Let  $U$  be an  $\epsilon$ -neighborhood of  $(\lambda, \alpha, J^\lambda)$  as given in the first part, and let  $(\lambda, \alpha, J') \in U$ . Suppose that  $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(A, J')$ . Let  $\underline{u}$  be a simple  $J'$ -holomorphic curve covered by  $u$ , (see for instance [17, Section 2.5].

For convenience, we now recall the adjunction inequality.

**Theorem 6.5** (McDuff-Micallef-White [18], [13]). *Let  $(M, J)$  be an almost complex 4-manifold and  $A \in H_2(M)$  be a homology class that is represented by a simple  $J$ -holomorphic curve  $u : \Sigma \rightarrow M$ . Let  $\delta(u)$  denote the number of self-intersections of  $u$ , then*

$$2\delta(u) - \chi(\Sigma) \leq A \cdot A - c_1(A),$$

*with equality if and only if  $u$  is an immersion with only transverse self-intersections.*

In the theorem above  $\Sigma$  is just a Riemann surface. But the theorem easily extends to the case  $\Sigma$  is nodal by basic topology. In our case  $A = 0$ ,  $\chi(\Sigma) = \chi(T^2) = 0$ , or  $\chi(\Sigma) = 1$  if  $\Sigma$  is nodal, so that  $\delta(\underline{u}) = 0$ , and so  $\underline{u}$  is an embedding, (meaning  $\underline{u}$  is a topological embedding and its normalization is a smooth immersion). Finally, if  $u$  is not normal its normalization is of the form  $\tilde{u} : \mathbb{CP}^1 \rightarrow M$  with a single self intersection and with  $0 = [\tilde{u}] \in H_2(M)$ , but this contradicts positivity of intersections. (As well as the above adjunction inequality.) So  $u$  is normal.  $\square$

*Proof of Theorem 2.12.* Let

$$U \ni (\omega_0 := d_\alpha \lambda, J_0 := J^\lambda)$$

be a set of pairs  $(\omega, J)$  satisfying the following:

- $\omega$  is a first kind lcs structure.
- For each  $(\omega, J) \in U$ ,  $J$  is  $\omega$ -compatible and admissible.
- Let  $\epsilon$  be chosen as in the first part of Theorem 6.4. Then each  $(\omega, J) \in U$ , is  $\epsilon$ -close to  $(\omega_0, J_0)$ , (as in Definition 4.4).

To prove the theorem we need to construct a map  $E : V \rightarrow \mathcal{J}(M)$ , where  $V$  is some  $d_3$  neighborhood of  $\omega_0$  in the space  $\mathcal{F}(M)$  (see Definition 2.10) and where

$$\forall \omega \in V : (\omega, E(\omega)) \in U.$$

As then Proposition 6.4 tells us that for each  $\omega \in V$ , there is a class  $A$ ,  $E(\omega)$ -holomorphic, elliptic curve  $u$  in  $M$ . Using Theorem 3.4 we would then conclude that there is an elliptic Reeb 2-curve in  $(M, \omega)$  and so we will be done.

Define a metric  $\rho_0$  measuring the distance between subspaces  $W_1, W_2$ , of same dimension, of an inner product space  $(T, g)$  as follows.

$$\rho_0(W_1, W_2) := |P_{W_1} - P_{W_2}|,$$

for  $|\cdot|$  the  $g$ -operator norm, and  $P_{W_i}$   $g$ -projection operators onto  $W_i$ . We may of course generalize this to a  $C^2$  metric  $\rho_2$  again in terms of these projection operators.

Let  $\delta > 0$  be given. Suppose that  $\omega = d^{\alpha'} \lambda'$  is a first kind lcs structure  $\delta$ -close to  $\omega_0$  for the  $C^3$  metric  $d_3$  as in the statement of the theorem. Then  $\mathcal{V}_{\lambda'}, \xi_{\lambda'}$  are smooth distributions by the assumption that  $(\alpha', \lambda')$  is a lcs structure of the first kind and  $TM = \mathcal{V}_{\lambda'} \oplus \xi_{\lambda'}$ . Moreover, for each  $p \in M$ ,

$$\rho_2(\mathcal{V}_{\lambda'}(p), \mathcal{V}_\lambda(p)) < \epsilon_\delta$$

and

$$\rho_2(\xi_{\lambda'}(p), \xi_\lambda(p)) < \epsilon_\delta$$

where  $\epsilon_\delta \rightarrow 0$  as  $\delta \rightarrow 0$ , and where  $\rho_2$  is the metric as defined above for subspaces of the inner product space  $(T_p M, g)$ .

Then choosing  $\delta$  to be suitably small, for each  $p \in M$  we have an isomorphism

$$\phi(p) : T_p M \rightarrow T_p M,$$

$\phi_p := P_1 \oplus P_2$ , for  $P_1 : \mathcal{V}_{\lambda_0}(p) \rightarrow \mathcal{V}_{\lambda'}(p)$ ,  $P_2 : \xi_{\lambda_0}(p) \rightarrow \xi_{\lambda'}(p)$  the  $g$ -projection operators. Define  $E(\omega)(p) := \phi(p)_* J_0$ . Then clearly, if  $\delta$  was chosen to be sufficiently small,  $E$ , defined on the  $d_3$   $\delta$ -neighborhood  $V$ , has the needed property.  $\square$

*Proof of Theorem 2.13.* Let  $\{\omega_t\}$ ,  $t \in [0, 1]$ , be a continuous in the usual  $C^\infty$  topology homotopy of non-degenerate 2-forms on  $M = C \times S^1$ , with  $\omega_0 = d_\alpha \lambda$  as in the hypothesis and with  $\omega_1$  a first kind lcs structure. Fix an almost complex structure  $J_1$  on  $M$  admissible with respect to  $(\alpha', \lambda')$ , so that  $(M, \lambda', \alpha', J_1)$  is a tamed first kind lcs manifold. And let  $J_0$  be the almost complex structure  $J^\lambda$ , as in Section 5. Extend  $J_0, J_1$  to a smooth family  $\{J_t\}$  of almost complex structures on  $M$ , so that  $J_t$  is  $\omega_t$ -compatible for each  $t$ . Then in the absence of holomorphic sky catastrophes, by Theorem 7.11, there is a non-constant elliptic  $J_1$ -holomorphic curve in  $M$ , so that the result follows by Theorem 3.4.  $\square$

**Lemma 6.6.** *Let  $u : \Sigma \rightarrow M$  be a Reeb 2-curve in a closed rational first kind lcs manifold  $(M, \lambda, \alpha)$ , then its normalization  $\tilde{u} : \tilde{\Sigma} \rightarrow M$  is a Reeb 2-curve.*

*Proof.* By Lemma 6.1 we have a surjective classifying map  $p : M \rightarrow S^1$  of  $\alpha$ . Note that the fibers of  $p$ ,  $M_t$ ,  $t \in S^1$ , are contact with contact form  $\lambda_t = \lambda|_{C_t}$ , as  $0 \neq \omega^n = \alpha \wedge \lambda \wedge d\lambda^{n-1}$  and  $\alpha = 0$  on  $M_t$ .

Let  $\tilde{u} : \tilde{\Sigma} \rightarrow M$  be the normalization of  $u$ . Suppose it is not a Reeb 2-curve, which by definitions just means that  $0 = [\tilde{u}^* \alpha] \in H^1(\tilde{\Sigma}, \mathbb{R})$ . Since  $0 \neq [u^* \alpha] \in H^1(\Sigma, \mathbb{R})$ , some node  $z_0$  of  $\Sigma$  lies on closed loop  $o : S^1 \rightarrow \Sigma$  with  $\langle [o], [u^* \alpha] \rangle \neq 0$ . In this case we may find a smooth map  $\rho : D \subset \tilde{\Sigma} \rightarrow M$ , where  $D \simeq D^2$  and  $\partial D$  is a component of a regular fiber  $C_t$  of the classifying map  $p' : \tilde{\Sigma} \rightarrow S^1$  of  $\tilde{u}^* \alpha$ . ( $D$  is a certain disk containing an element of  $\phi^{-1}(z_0)$ .)

Then analogously to the proof of the first part of Theorem 2.8,  $\rho|_{\partial D}$  is a (unparametrized)  $\lambda_t$ -Reeb orbit in  $M_t$ . And in particular  $\int_{\partial D} \tilde{u}^* \lambda \neq 0$ . But the first condition of being a Reeb 2-curve implies that  $\int_D d\tilde{u}^* \lambda = 0$ , since  $\ker d\lambda$  on  $M$  is spanned by  $X_\lambda, X_\alpha$ . So we have a contradiction to Stokes theorem. Thus,  $\tilde{u}$  must be a Reeb 2-curve.  $\square$

*Proof of Theorem 2.14.* Let  $(M, \lambda, \alpha)$  be the extension as in the hypothesis, which must then be a rational first kind lcs structure. And let  $u : \Sigma \rightarrow M$  be an elliptic Reeb 2-curve, which exists by the hypothesis. By Lemma 6.6 its normalization  $\tilde{u} : \tilde{\Sigma} \rightarrow M$  is a Reeb 2-curve. If  $\Sigma$  is not normal, then all components of  $\tilde{\Sigma}$  are biholomorphic to  $\mathbb{CP}^1$ . But then  $\tilde{u}$  cannot be a Reeb 2-curve since the second condition in Definition 2.7 fails, a contradiction.

So  $\Sigma$  is normal. Let  $p : \Sigma \rightarrow S^1$  be the classifying map of  $u^* \alpha$ . Following the proof of Theorem 2.8, we get that a fiber  $S_t \simeq S^1$  of  $p$ ,  $t \in S^1$ , s.t.  $S_t$  does not contain a critical point of  $u$ , is mapped by  $u$  to an unparametrized  $\lambda_t$ -Reeb orbit in  $M_t$  (the fiber of the classifying map  $\tilde{p} : M \rightarrow S^1$  of  $\alpha$ , taking  $p = \tilde{p} \circ u$ .)

Now suppose that  $S_0 := S_{t^0}$  contains a critical point of  $u$ . Let  $\{t_i\} \in S^1$  be a sequence converging to  $t^0$ , with  $S_{t_i}$  not containing a critical point of  $u$ . Let

$$A^t : LM_t \rightarrow \mathbb{R}$$

be the functional

$$A(o) = \int_{S^1} o^* \lambda_t.$$

Then clearly  $dA^{t_i} \rightarrow dA^{t^0}$ , (as differential one forms) after identifying  $M_t$  using some trivialization of  $\tilde{p} : M \rightarrow S^1$  over  $U \supset \{t_i\}$ . Let  $o_t := u \circ \tilde{o}_t$  where  $\tilde{o}_t$  is any degree one parametrization of  $S_t$  and set  $o := o_{t^0}$ .

Now for each  $i$   $dA^{t_i}(o_{t_i}) = 0$  as  $o_{t_i}$  are Reeb orbits up to parametrization and hence are critical points of the functional  $A^{t_i}$ . On the other hand clearly  $dA^{t_i}(o_{t_i}) \rightarrow dA^{t^0}(o)$  and so  $dA^{t^0}(o) = 0$  and

hence  $o$  is tangent to the Reeb vector field  $R^\lambda$ ,  $\lambda = \lambda_{t^0}$ . Moreover,

$$\forall \tau \in S^1 : \lambda(o'(\tau)) \geq 0$$

since

$$\forall i \forall \tau \in S^1 : \lambda_{t_i}(o'(\tau)) > 0.$$

Moreover, the set of  $\tau$  s.t.  $o(\tau) = 0$  is finite by the above, and the assumption that the critical points of  $u$  are isolated. Hence, we may perturb  $o$  to a map  $o_{pert}$  with the same image so that  $\forall \tau \in S^1 : \lambda(o'_{pert}(\tau)) > 0$ , and so that  $o_{pert}$  is tangent to the Reeb vector field. Thus  $o_{pert}$  is a Reeb orbit up to parametrization and so we are done.  $\square$

*Proof of Theorem 2.16.* Suppose that  $u : \Sigma \rightarrow M$  is an immersed Reeb 2-curve, we then show that  $M$  also has a Reeb 1-curve. Let  $\tilde{u} : \tilde{\Sigma} \rightarrow M$  be the normalization of  $u$ , so that  $\tilde{u}$  is an immersion. We have a pair of transverse 1-distributions  $D_1 = \tilde{u}^*\mathbb{R}\langle X_\alpha \rangle$ ,  $D_2 = \tilde{u}^*\mathbb{R}\langle X_\lambda \rangle$  on  $\tilde{\Sigma}$ . We may then find an embedded path  $\gamma : [0, 1] \rightarrow \tilde{\Sigma}$ , tangent to  $D_1$  s.t.  $\lambda(\gamma'(t)) > 0$ ,  $\forall t \in [0, 1]$ , and s.t.  $\gamma(0)$  and  $\gamma(1)$  are on a leaf of  $D_2$ . It is then simple to obtain from this a Reeb 1-curve  $o$ , by joining the end points of  $\gamma$  by an embedded path tangent to  $D_2$ , and perturbing, see Figure 1.

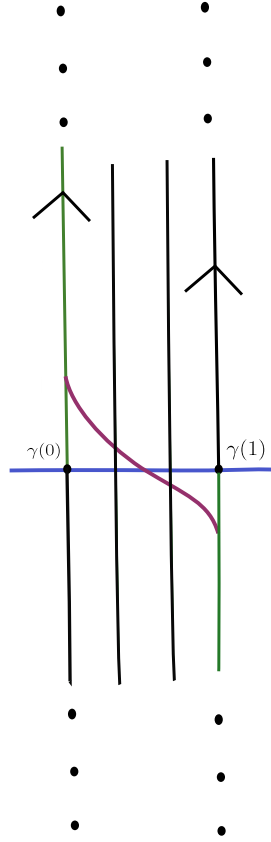


FIGURE 1. The green shaded path is  $\gamma$ , the indicated orientation is given by  $u^*\lambda$ , the  $D_1$  foliation is shaded in black, the  $D_2$  foliation is shaded in blue. The purple segment is part of the loop  $o : S^1 \rightarrow \Sigma$ , which is smooth and satisfies  $\lambda(o'(t)) > 0$  for all  $t$ .

Now suppose that  $u : \Sigma \rightarrow M$  is an immersed elliptic Reeb 2-curve. Suppose that  $u$  is not normal. Let  $\tilde{u} : \tilde{\Sigma} \rightarrow M$  be its normalization. Then  $\tilde{\Sigma}$  has a genus 0 component  $\mathcal{S}$ . So that  $\tilde{u} : \mathcal{S} \simeq \mathbb{CP}^1 \rightarrow M$  is immersed. The distribution  $D_1 = \tilde{u}^*\mathbb{R}\langle X_\alpha \rangle$ , as appearing above, is then a  $\tilde{u}^*\lambda$ -oriented 1-dimensional distribution on  $\mathbb{CP}^1$  which is impossible.  $\square$

## 7. EXTENDED GROMOV-WITTEN INVARIANTS AND THE EXTENDED FULLER INDEX

In what follows  $M$  is a closed oriented  $2n$ -fold,  $n \geq 2$ , and  $J$  an almost complex structure on  $M$ . Much of the following discussion extends to general moduli spaces  $\mathcal{M}_{g,n}(J, A, a_1, \dots, a_n)$  with  $a_1, \dots, a_n$  homological constraints in  $M$ . We shall however restrict for simplicity to the case  $(\omega, J)$  is a compatible lcs pair on  $M$ ,  $g = 1, n = 1$ , the homological constraint is  $[M]$ , as this is the main interest in this paper. Moreover, we restrict our moduli space to consist of non-zero charge pair (for example  $(1, 0)$ ) curves, with charge defined with respect to the Lee form  $\alpha$  of  $\omega$  as in Section 3, and this will be implicit, so that we no longer specify this in notation.

In what follows, for an almost symplectic pair  $(\omega, J)$ , and  $u : \Sigma \rightarrow M$  a smooth map,  $e(u) := e_{g_J}(u)$  the  $L^2$  energy with respect to  $g_J$ , where the latter is as previously.

**Definition 7.1.** Let  $h = \{(\omega_t, J_t)\}$  be a homotopy of almost symplectic pairs on  $M$ , so that  $\{J_t\}$  is Frechet smooth, and  $\{\omega_t\}$   $C^0$  continuous. We say that it is **partially admissible for  $A$**  if every element of

$$\overline{\mathcal{M}}_{1,1}(M, J_0, A)$$

is contained in a compact open subset of  $\overline{\mathcal{M}}_{1,1}(M, \{J_t\}, A)$ . We say that  $h$  is **admissible for  $A$**  if every element of

$$\overline{\mathcal{M}}_{1,1}(M, J_i, A),$$

$i = 0, 1$  is contained in a compact open subset of  $\overline{\mathcal{M}}_{1,1}(M, \{J_t\}, A)$ .

Thus, in the above definition, a homotopy is partially admissible if there are no sky catastrophes going one way, and admissible if there are no sky catastrophes going either way. Partly to simplify notation, we denote by a capital  $X$  a general almost symplectic pair  $(\omega, J)$  on a smooth manifold  $M$ . Then we introduce the following simplified notation.

$$\begin{aligned} S(X, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(X, A)\} \\ S(X, a, A) &= \{u \in S(X, A) \mid e(u) \leq a\} \\ S(h, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(h, A)\}, \text{ for } h = \{(\omega_t, J_t)\} \text{ a homotopy as above} \\ S(h, a, A) &= \{u \in S(h, A) \mid e(u) \leq a\} \end{aligned} \tag{7.2}$$

**Definition 7.3.** For an isolated element  $u$  of  $S(X, A)$ , which means that  $\{u\}$  is open as a subset, we set  $gw(u) \in \mathbb{Q}$  to be the local Gromov-Witten invariant of  $u$ . This is defined as:

$$gw(u) = GW_{1,1}(\{u\}, A, J)([\overline{\mathcal{M}}_{1,1}] \otimes [M]),$$

with the right-hand side as in (4.1).

Denote by  $\mathcal{S}(M, A)$  the set of equivalence classes of all smooth stable maps  $\Sigma \rightarrow M$ , in class  $A$ , for  $\Sigma$  an (non-fixed) elliptic curve, and where equivalence has the same meaning as in Section 3.

**Definition 7.4.** Suppose that  $S(X, A)$  has open connected components. And suppose that we have a collection of almost symplectic pairs

$$\{X^a = (M, \omega^a, J^a)\}, a \in \mathbb{R}_+$$

satisfying the following:

- $S(X^a, a, A)$  consists of isolated curves for each  $a$ .
- 

$$S(X^a, a, A) = S(X^b, a, A),$$

(equality of subsets of  $\mathcal{S}(M, A)$ ) if  $b > a$ ,

- For  $b > a$ , and for each  $u \in S(X^a, a, A) = S(X^b, a, A)$ :

$$GW_{1,1}(\{u\}, A, J^a) = GW_{1,1}(\{u\}, A, J^b),$$

thus we may just write  $gw(u)$  for the common number.

- There is a prescribed homotopy  $h^a = \{X_t^a\}$  of each  $X^a$  to  $X$ , called **structure homotopy**, with the following property. For every

$$y \in S(X_0^a, A)$$

there is an open compact subset  $C_y \subset S(h^a, A)$ ,  $y \in C_y$ , which is **non-branching**, where the latter means that

$$C_y \cap S(X_i^a, A),$$

$i = 0, 1$  are connected.

•

$$S(h^a, a, A) = S(h^b, a, A),$$

(similarly equality of subsets) if  $b > a$  is sufficiently large.

We will then say that

$$\mathcal{P}(A) = \{(X^a, h^a)\}$$

is a **perturbation system** for  $X$  in the class  $A$ .

We shall see shortly that, given a contact  $(C, \lambda)$ , the associated Banyaga lcs structure on  $C \times S^1$  always admits a perturbation system for the moduli spaces of charge  $(1, 0)$  curves in any class, if  $\lambda$  is Morse-Bott.

**Definition 7.5.** Suppose that  $X$  admits a perturbation system  $\mathcal{P}(A)$  so that there exists an  $E = E(\mathcal{P}(A))$  with the property that

$$S(X^a, a, A) = S(X^E, a, A)$$

for all  $a > E$ , where this as before is equality of subsets, and the local Gromov-Witten invariants of the identified elements are also identified. Then we say that  $X$  is **finite type** and set:

$$GW(X, A) = \sum_{u \in S(X^E, A)} gw(u).$$

**Definition 7.6.** Suppose that  $X$  admits a perturbation system  $\mathcal{P}(A)$  and there is an  $E = E(\mathcal{P}(A)) > 0$  so that  $gw(u) > 0$  for all

$$\{u \in S(X^a, A) \mid E \leq e(u) \leq a\}$$

respectively  $gw(u) < 0$  for all

$$\{u \in S(X^a, A) \mid E \leq e(u) \leq a\},$$

and every  $a > E$ . Suppose in addition that

$$\lim_{a \rightarrow \infty} \sum_{u \in S(X, a, A)} gw(u) = \infty, \text{ respectively } \lim_{a \rightarrow \infty} \sum_{u \in S(X, a, \beta)} gw(u) = -\infty.$$

Then we say that  $X$  is **positive infinite type**, respectively **negative infinite type** and set

$$GW(X, A) = \infty,$$

respectively  $GW(X, A) = -\infty$ . These are meant to be interpreted as extended Gromov-Witten invariants, counting elliptic curves in class  $A$ . We say that  $X$  is **infinite type** if it is one or the other.

**Definition 7.7.** We say that  $X$  is **definite type** if it admits a perturbation system and is infinite type or finite type.

With the above definitions

$$GW(X, A) \in \mathbb{Q} \sqcup \infty \sqcup -\infty,$$

when it is defined.

*Proof of Theorem 3.5.* Given the definitions above, and the definition of the extended Fuller index in [23], this follows by the same argument as the proof of Theorem 5.5.  $\square$

### 7.0.1. Perturbation systems for Morse-Bott Reeb vector fields.

**Definition 7.8.** A contact form  $\lambda$  on  $M$ , and its associated flow  $R^\lambda$  are called Morse-Bott if the  $\lambda$  action spectrum  $\sigma(\lambda)$  - that is the space of critical values of  $o \mapsto \int_{S^1} o^* \lambda$ ,  $o : S^1 \rightarrow M$ , is discrete and if for every  $a \in \sigma(\lambda)$ , the space

$$N_a := \{x \in M \mid F_a(x) = x\},$$

$F_a$  the time  $a$  flow map for  $R^\lambda$  - is a closed smooth manifold such that  $\text{rank } d\lambda|_{N_a}$  is locally constant and  $T_x N_a = \ker(dF_a - I)_x$ .

**Proposition 7.9.** Let  $\lambda$  be a contact form of Morse-Bott type, on a closed contact manifold  $C$ . Then the corresponding lcs pair  $X_\lambda = (C \times S^1, d_\alpha \lambda, J^\lambda)$  admits a perturbation system  $\mathcal{P}(A)$ , for moduli spaces of charge  $(1, 0)$  curves for every class  $A$ .

*Proof.* This follows immediately by [23, Proposition 2.12], and by Proposition 3.3.  $\square$

**Lemma 7.10.** The Hopf lcs pair  $(S^{2k+1} \times S^1, d_\alpha \lambda_H, J^{\lambda_H})$ , for  $\lambda_H$  the standard contact structure on  $S^{2k+1}$  is infinite type.

*Proof.* This follows immediately by [23, Lemma 2.13], and by Proposition 3.3.  $\square$

**Theorem 7.11.** Let  $(C, \lambda)$  be a closed contact manifold so that  $R^\lambda$  has definite type, and suppose that  $i(R^\lambda, \beta) \neq 0$ . Let  $\omega_0 = d_\alpha \lambda$  be the Banyaga structure, and suppose we have a partially admissible homotopy  $h = \{(\omega_t, J_t)\}$ , for class  $A_\beta$ , then there is an element  $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(J_1, A_\beta)$ .

The proof of this will follow.

### 7.1. Preliminaries on admissible homotopies.

**Definition 7.12.** Let  $h = \{X_t\}$  be a smooth homotopy of almost symplectic pairs. For  $b > a > 0$  we say that  $h$  is **partially**  $a, b$ -admissible, respectively  $a, b$ -admissible (in class  $A$ ) if for each

$$y \in S(X_0, a, A)$$

there is a compact open subset  $\mathcal{C}_y \subset S(h, A)$ ,  $y \in \mathcal{C}_y$  with  $e(u) < b$ , for all  $u \in \mathcal{C}_y$ . Respectively, if for each

$$y \in S(X_i, a, A),$$

$i = 0, 1$  there is a compact open subset  $\mathcal{C}_y \ni y$  of  $S(h, A)$  with  $e(u) < b$ , for all  $u \in \mathcal{C}_y$ .

**Lemma 7.13.** Suppose that  $X_0$  has a perturbation system  $\mathcal{P}(A)$ , and  $\{X_t\}$  is partially admissible, then for every  $a$  there is a  $b > a$  so that  $\{\tilde{X}_t^b\} = \{X_t\} \cdot \{X_t^b\}$  is partially  $a, b$ -admissible, where  $\{X_t\} \cdot \{X_t^b\}$  is the (reparametrized to have  $t$  domain  $[0, 1]$ ) concatenation of the homotopies  $\{X_t\}$ ,  $\{X_t^b\}$ , and where  $\{X_t^b\}$  is the structure homotopy from  $X^b$  to  $X_0$ .

*Proof.* This is a matter of pure topology, and the proof is completely analogous to the proof of [23, Lemma 3.8].  $\square$

The analogue of Lemma 7.13 in the admissible case is the following:

**Lemma 7.14.** Suppose that  $X_0, X_1$  and  $\{X_t\}$  are admissible, then for every  $a$  there is a  $b > a$  so that

$$(7.15) \quad \{\tilde{X}_t^b\} = \{X_{1,t}^b\}^{-1} \cdot \{X_t\} \cdot \{X_{0,t}^b\}$$

is  $a, b$ -admissible, where  $\{X_{i,t}^b\}$  are the structure homotopies from  $X_i^b$  to  $X_i$ .



## 7.2. Invariance.

**Theorem 7.16.** *Suppose  $X_0$  is definite type, with  $GW(X_0, A) \neq 0$ , and suppose it is joined to  $X_1$  by a partially admissible homotopy  $\{X_t\}$ , then  $X_1$  has non-constant elliptic class  $A$  curves.*

*Proof of Theorem 7.11.* This follows by Theorem 7.16 and by Theorem 3.5.  $\square$

We also have a more precise result.

**Theorem 7.17.** *If  $X_0, X_1$  are definite type pairs and  $\{X_t\}$  is admissible then  $GW(X_0, A) = GW(X_1, A)$ .*

*Proof of Theorem 7.16.* Suppose that  $X_0$  is definite type with  $GW(X_0, A) \neq 0$ ,  $\{X_t\}$  is partially admissible and  $\overline{\mathcal{M}}_{1,1}(X_1, A) = \emptyset$ . Let  $a$  be given and  $b$  determined so that  $\tilde{h}^b = \{\tilde{X}_t^b\}$  is a partially  $(a, b)$ -admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\tilde{h}^b, A),$$

for  $y \in S(X_0^b, a, A)$ . Here we use a natural identification of  $S(X^b, a, A) = S(\tilde{X}_0^b, a, A)$  as a subset of  $S(\tilde{h}^b, A)$  by its construction. Then  $S_a$  is an open-compact subset of  $S(h, A)$  and so admits an implicit atlas (Kuranishi structure) with boundary, (with virtual dimension 1) s.t:

$$\partial S_a = S(X^b, a, A) + Q_a,$$

where  $Q_a$  as a set is some subset (possibly empty), of elements  $u \in S(X^b, b, A)$  with  $e(u) \geq a$ . So we have for all  $a$ :

$$(7.18) \quad \sum_{u \in Q_a} gw(u) + \sum_{u \in S(X^b, a, A)} gw(u) = 0.$$

**7.3. Case I,  $X_0$  is finite type.** Let  $E = E(\mathcal{P})$  be the corresponding cutoff value in the definition of finite type, and take any  $a > E$ . Then  $Q_a = \emptyset$  and by definition of  $E$  we have that the left side is

$$\sum_{u \in S(X^b, E, A)} gw(u) \neq 0.$$

Clearly this gives a contradiction to (7.18).

**7.4. Case II,  $X_0$  is infinite type.** We may assume that  $GW(X_0, A) = \infty$ , and take  $a > E$ , where  $E = E(\mathcal{P}(A))$  is the corresponding cutoff value in the definition of infinite type. Then

$$\sum_{u \in Q_a} gw(u) \geq 0,$$

as  $a > E(\mathcal{P}(A))$ . On the other hand,

$$\lim_{a \rightarrow \infty} \sum_{u \in S(X^b, a, A)} gw(u) = \infty,$$

as  $GW(X_0, A) = \infty$ . This also contradicts (7.18).  $\square$

*Proof of Theorem 7.17.* This is somewhat analogous to the proof of Theorem 7.16. Suppose that  $X_i, \{X_t\}$  are definite type as in the hypothesis. Let  $a$  be given and  $b$  determined so that  $\tilde{h}^b = \{\tilde{X}_t^b\}$ , see (7.15) is an  $(a, b)$ -admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\tilde{h}^b, A)$$

for  $y \in S(X_i^b, a, A)$ . Then  $S_a$  is an open-compact subset of  $S(h, A)$  and so has admits an implicit atlas (Kuranishi structure) with boundary, (with virtual dimension 1) and satisfies the following.

$$\partial S_a = (S(X_0^b, a, A) + Q_{a,0})^{op} + S(X_1^b, a, A) + Q_{a,1},$$

with  $op$  denoting the opposite orientation, where  $Q_{a,i}$  as sets are some subsets (possibly empty), of elements  $u \in S(X_i^b, b, A)$  with  $e(u) \geq a$ . So we have for all  $a$ :

$$(7.19) \quad \sum_{u \in Q_{a,0}} gw(u) + \sum_{u \in S(X_0^b, a, A)} gw(u) = \sum_{u \in Q_{a,1}} gw(u) + \sum_{u \in S(X_1^b, a, A)} gw(u).$$

**7.5. Case I,  $X_0$  is finite type and  $X_1$  is infinite type.** Suppose in addition  $GW(X_1, A) = \infty$  and let  $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$ , for  $\mathcal{P}_i$ , the perturbation systems of  $X_i$ . Take any  $a > E$ . Then  $Q_{a,0} = \emptyset$  and the left-hand side of (7.19) is

$$\sum_{u \in S(X_0^b, E, A)} gw(u) < \infty.$$

The right-hand side tends to  $\infty$  as  $a$  tends to infinity since,

$$\sum_{u \in Q_{a,1}} gw(u) \geq 0,$$

as  $a > E(\mathcal{P}_1(A))$ , and since

$$\lim_{a \rightarrow \infty} \sum_{u \in S(X_1^b, a, A)} gw(u) = \infty.$$

Clearly this gives a contradiction to (7.19).

**7.6. Case II,  $X_i$  are infinite type.** Suppose in addition  $GW(X_0, A) = -\infty$ ,  $GW(X_1, A) = \infty$  and let  $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$ , for  $\mathcal{P}_i$ , the perturbation systems of  $X_i$ . Take any  $a > E$ . Then  $\sum_{u \in Q_{a,0}} gw(u) \leq 0$ , and  $\sum_{u \in Q_{a,1}} gw(u) \geq 0$ . So by definition of  $GW(X_i, A)$  the left hand side of (7.18) tends to  $-\infty$  as  $a$  tends to  $\infty$ , and the right hand side tends to  $\infty$ . Clearly this gives a contradiction to (7.19).

**7.7. Case III,  $X_i$  are finite type.** The argument is analogous. □

## A. FULLER INDEX

Let  $X$  be a vector field on  $M$ . Set

$$S(X) = S(X, \beta) = \{(o, p) \in L_\beta M \times (0, \infty) \mid o : \mathbb{R}/\mathbb{Z} \rightarrow M \text{ is a periodic orbit of } pX\},$$

where  $L_\beta M$  denotes the free homotopy class  $\beta$  component of the free loop space. Elements of  $S(X)$  will be called orbits. There is a natural  $S^1$  reparametrization action on  $S(X)$ , and elements of  $S(X)/S^1$  will be called *unparametrized orbits*, or just orbits. Slightly abusing notation we write  $(o, p)$  for the equivalence class of  $(o, p)$ . The multiplicity  $m(o, p)$  of a periodic orbit is the ratio  $p/l$  for  $l > 0$  the least period of  $o$ . We want a kind of fixed point index which counts orbits  $(o, p)$  with certain weights - however in general to get invariance we must have period bounds. This is due to potential existence of sky catastrophes as described in the introduction.

Let  $N \subset S(X)$  be a compact open set. Assume for simplicity that elements  $(o, p) \in N$  are isolated. (Otherwise we need to perturb.) Then to such an  $(N, X, \beta)$  Fuller associates an index:

$$i(N, X, \beta) = \sum_{(o,p) \in N/S^1} \frac{1}{m(o,p)} i(o,p),$$

where  $i(o, p)$  is the fixed point index of the time  $p$  return map of the flow of  $X$  with respect to a local surface of section in  $M$  transverse to the image of  $o$ . Fuller then shows that  $i(N, X, \beta)$  has the following invariance property. Given a continuous homotopy  $\{X_t\}$ ,  $t \in [0, 1]$  let

$$S(\{X_t\}, \beta) = \{(o, p, t) \in L_\beta M \times (0, \infty) \times [0, 1] \mid o : \mathbb{R}/\mathbb{Z} \rightarrow M \text{ is a periodic orbit of } pX_t\}.$$

Given a continuous homotopy  $\{X_t\}$ ,  $X_0 = X$ ,  $t \in [0, 1]$ , suppose that  $\tilde{N}$  is an open compact subset of  $S(\{X_t\})$ , such that

$$\tilde{N} \cap (LM \times \mathbb{R}_+ \times \{0\}) = N.$$

Then if

$$N_1 = \tilde{N} \cap (LM \times \mathbb{R}_+ \times \{1\})$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta).$$

In the case where  $X$  is the  $R^\lambda$ -Reeb vector field on a contact manifold  $(C^{2n+1}, \xi)$ , and if  $(o, p)$  is non-degenerate, we have:

$$(A.1) \quad i(o, p) = \text{sign Det}(\text{Id}|_{\xi(x)} - F_{p,*}^\lambda|_{\xi(x)}) = (-1)^{CZ(o)-n},$$

where  $F_{p,*}^\lambda$  is the differential at  $x$  of the time  $p$  flow map of  $R^\lambda$ , and where  $CZ(o)$  is the Conley-Zehnder index, (which is a special kind of Maslov index) see [22].

There is also an extended Fuller index  $i(X, \beta) \in \mathbb{Q} \sqcup \{\pm\infty\}$ , for certain  $X$  having definite type. This is constructed in [23], and is conceptually completely analogous to the extended Gromov-Witten invariant constructed in this paper.

## B. REMARK ON MULTIPLICITY

This is a small note on how one deals with curves having non-trivial isotropy groups, in the virtual fundamental class technology. We primarily need this for the proof of Theorem 5.5. Given a closed oriented orbifold  $X$ , with an orbibundle  $E$  over  $X$  Fukaya-Ono [9] show how to construct using multi-sections its rational homology Euler class, which when  $X$  represents the moduli space of some stable curves, is the virtual moduli cycle  $[X]^{vir}$ . When this is in degree 0, the corresponding Gromov-Witten invariant is  $\int_{[X]^{vir}} 1$ . However, they assume that their orbifolds are effective. This assumption is not really necessary for the purpose of construction of the Euler class but is convenient for other technical reasons. A different approach to the virtual fundamental class which emphasizes branched manifolds is used by McDuff-Wehrheim, see for example McDuff [12], [15] which does not have the effectivity assumption, a similar use of branched manifolds appears in [5]. In the case of a non-effective orbibundle  $E \rightarrow X$  McDuff [14], constructs a homological Euler class  $e(E)$  using multi-sections, which extends the construction [9]. McDuff shows that this class  $e(E)$  is Poincare dual to the completely formally natural cohomological Euler class of  $E$ , constructed by other authors. In other words there is a natural notion of a homological Euler class of a possibly non-effective orbibundle. We shall assume the following black box property of the virtual fundamental class technology.

**Axiom B.1.** *Suppose that the moduli space of stable maps is cleanly cut out, which means that it is represented by a (non-effective) orbifold  $X$  with an orbifold obstruction bundle  $E$ , that is the bundle over  $X$  of cokernel spaces of the linearized CR operators. Then the virtual fundamental class  $[X]^{vir}$  coincides with  $e(E)$ .*

Given this axiom it does not matter to us which virtual moduli cycle technique we use. It is satisfied automatically by the construction of McDuff-Wehrheim, (at the moment in genus 0, but surely extending). It can be shown to be satisfied in the approach of John Pardon [21]. And it is satisfied by the construction of Fukaya-Ono-Ohta [7], the latter is communicated to me by Kaoru Ono. When  $X$  is 0-dimensional this does follow immediately by the construction in [9], taking any effective Kuranishi neighborhood at the isolated points of  $X$ , (this actually suffices for our paper.)

As a special case most relevant to us here, suppose we have a moduli space of elliptic curves in  $X$ , which is regular with expected dimension 0. Then its underlying space is a collection of oriented points. However, as some curves are multiply covered, and so have isotropy groups, we must treat this as a non-effective 0 dimensional oriented orbifold. The contribution of each curve  $[u]$  to the Gromov-Witten invariant  $\int_{[X]^{vir}} 1$  is  $\frac{\pm 1}{|\Gamma([u])|}$ , where  $|\Gamma([u])|$  is the order of the isotropy group  $\Gamma([u])$  of  $[u]$ , in the McDuff-Wehrheim setup this is explained in [12, Section 5]. In the setup of Fukaya-Ono [9] we may readily calculate to get the same thing taking any effective Kuranishi neighborhood at the isolated points of  $X$ .

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*Email address:* `yasha.savelyev@gmail.com`

UNIVERSITY OF COLIMA, CUICBAS