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INSTABILITY OF GROMOV NON-SQUEEZING

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ABSTRACT. We show that the Gromov non-squeezing phenomenon disappears after an arbitrarily small, general (non-symplectic) C^∞ perturbation of the symplectic form on the range. In particular the lcs non-squeezing theorem in [2] is sharp, (in the sense that the lcs condition cannot be removed.)

One of the most fascinating early results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [1]. The most well known formulation of this is that there does not exist a symplectic embedding $B_R \to D^2(r) \times \mathbb{R}^{2n-2}$ for $R > r$, with B_R the standard closed radius R ball in $\mathbb{R}^{2n}$ centered at 0. Gromov's non-squeezing is C^0 persistent in the following sense.

We say that a symplectic form ω on $M \times N$ is *split* if $\omega = \omega_1 \oplus \omega_2$ for symplectic forms ω_1, ω_2 on M respectively N.

Theorem 0.1. *Given $R > r$, there is an $\epsilon > 0$ s.t. for any symplectic form ω' on $S^2 \times T^{2n-2}$ C^0-close to a split symplectic form ω and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt],$$

there is no symplectic embedding $\phi : B_R \hookrightarrow (S^2 \times T^{2n-2}, \omega')$.

This theorem is generalized in [2] to lcs forms ω'. We show here that this persistence disappears if we take a completely general ω'. In particular the theorem of [2] is a truly lcs phenomenon.

Theorem 0.2. *Given $R > r$ and every $\epsilon > 0$ there is a (necessarily non-closed by above) 2-form ω' on $S^2 \times T^{2n-2}$ C^∞ ϵ-close to a split symplectic form ω, satisfying $\langle \omega, A \rangle = \pi r^2$, and such that there is an embedding $\phi : B_R \hookrightarrow S^2 \times T^{2n-2}$, with $\phi^* \omega' = \omega_{st}$? We call such an embedding* **symplectic** *in analogy with the classical symplectic case. Moreover, ϕ can be chosen so that*

$$\phi(B_R) \subset (S^2 \times T^{2n-2} - \bigcup_i \Sigma_i),$$

where Σ_i are certain hypersurfaces explained in the proof.

Proof. Let R, r, ϵ be given. Let

$$M' = [0, r]^2 \times \mathbb{R}^{2n-2}.$$

We first construct a 2-form ω'' on M', C^∞-nearby to the standard symplectic form ω and a symplectic embedding $\phi : Cube(R) \to M'$, where $Cube(R)$ denotes the closed cube in $\mathbb{R}^{2n}$ with side R.

For simplicity we take in what follows $n = 2$, with construction obviously generalizing to any n. Let p, q be the coordinates on $sq = [0, r]^2 \subset \mathbb{R}^2$. Let (p, q, s, t) be the natural coordinates on M', and let g be the standard Euclidean metric on M'.

Define the following surface S_0 in M':

$$S_0 = \{(p, q, f_l(p), 0) \,|\, (p, q) \in sq\},$$

where

$$f_l : [0, r] \to \mathbb{R}$$

is a smooth function satisfying:

$$f(0) = 0, \forall p : f'(p) \geq 0,$$

f is constant near 0 and r and the g-length of the graph of f is l, (g being the standard Eucledian metric). Then S_0 is a ω-symplectic surface whose ω-orthogonal spaces are spanned by $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$. Define

$$S_{s,t} := S_0 + (0, 0, s, t), \quad 0 \leq s \leq R, 0 \leq t \leq R.$$

Then

$$C := \cup_{s,t} S_{s,t}$$

is a domain in M' that is diffeomorphic to the standard closed cube in $\mathbb{R}^4$, folliated by the surfaces $S_{s,t}$. Let $\mathcal{F} \subset TC$ denote the 2-dimensional distribution corresponding to this folliation, that is $\mathcal{F}(z)$ is the sub-space of vectors tangent to the leaf through $z = (p, q, s, t)$. And let $V \subset TC$ denote the ω-orthogonal distribution, that is the distribution with

$$V(p, q, s, t) = span(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}),$$

by the observations above.

Let $h : C \to \mathbb{R}$, be a smooth function (that is a function with a smooth extension to a neighborhood of C) satisfying:

- $h \geq 0$.
- $d_{C^\infty}(h) < \frac{\epsilon}{2}$.
- $h = \frac{\epsilon}{2}$ on a open (in $\mathbb{R}^4$) subset $L_l \subset C$, chosen so that the g-volume of L_l tends to ∞ as l tends to ∞.
- $h = 0$ on a neighborhood of ∂C, with the latter the topological boundary.

Let ω_ϵ be the 2-form on C, preserving the splitting

$$TC \simeq \mathcal{F} \oplus V,$$

and such that:

$$\forall z \in C, \forall v, w \in V(z) \subset T_z C : \omega_\epsilon(v, w) = \omega(v, w),$$

and

$$\forall z \in C, \forall v, w \in \mathcal{F}(z) : \omega_\epsilon(v, w) = \omega(v, w) + h(z) \cdot \omega_g(v, w),$$

Where ω_g is the g-area 2-form on the corresponding leaf, with same orientation as ω. Since the g-area of each leaf $S_{s,t}$ can be made arbitrarily large, by taking l to be sufficiently large, the ω_ϵ-area of each leaf $S_{s,t}$ can be assumed to be R^2. Moreover we clearly have

$$d_{C^0}(\omega, \omega_\epsilon) = \epsilon/2$$

on C.

By construction (specifically properties of h) ω_ϵ extends to a 2-form ω'' on M' coinciding with ω outside a compact set, and satisfying:

$$d_{C^\infty}(\omega'', \omega) < \epsilon.$$

Now fix a symplectomorphism

$$\phi_0 : [0, R]^2 \to (S_0, \omega''|_{S_0}),$$

and define

$$\phi : Cube(R) \to C$$

by

$$\phi(p, q, s, t) = \phi_0(p, q) + (0, 0, s, t),$$

by construction $\phi^* \omega'' = \omega_{st}$.

Now since $\omega'' = \omega$ outside a compact set K, we obviously get an induced 2-form ω', on M, C^∞ ϵ-nearby to a split symplectic form, s.t. there is a symplectic embedding:

$$\phi : (Cube(R), \omega_{st}) \to (M, \omega').$$

Moreover, by construction we may insure that

$$\text{image}(\phi) \subset M - \bigcup_i \Sigma_i,$$

where

$$\Sigma_i = S^2 \times (S^1 \times \ldots \times S^1 \times \{pt\} \times S^1 \times \ldots \times S^1) \subset M,$$

where the singleton $\{pt\} \subset S^1$ replaces the i'th factor of $T^{2n-2} = S^1 \times \ldots \times S^1$. And so we are done. □

References

[1] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds.*, Invent. Math., 82 (1985), pp. 307–347.

[2] Y. Savelyev, *Gromov Witten theory of a locally conformally symplectic manifold and the Fuller index*, arXiv, (2016).

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