

LOCALLY CONFORMALLY SYMPLECTIC DEFORMATION OF GROMOV NON-SQUEEZING

YASHA SAVELYEV

ABSTRACT. We prove one deformation theoretic extension of the Gromov non-squeezing phenomenon to lcs structures, or locally conformally symplectic structures, which suitably generalize both symplectic and contact structures. We also propose an analogue in lcs geometry of contact non-squeezing of Eliashberg-Polterovich and some other related questions.

1. INTRODUCTION

We study here some Gromov non-squeezing inspired global properties of locally conformally symplectic manifolds, which generalize both symplectic and contact manifolds. Let us recall the definition.

Definition 1.1. *A locally conformally symplectic manifold or lcs manifold is a smooth $2n$ -fold M , with an lcs structure: a non-degenerate 2-form ω , with the property that for every $p \in M$ there is an open $U \ni p$ such that $\omega|_U = f_U \cdot \omega_U$, for some symplectic form ω_U defined on U and some smooth positive function f_U on U . In the present case $n \geq 2$, as in dimension $n = 2$ there are other candidates for what should be an lcs structure.*

These structures have recently come into focus, for example we have a fascinating recent theorem of Apostolov, Dloussky [1] that every complex surface with an odd first Betti number admits a natural compatible lcs structure. This suggests that lcs structures are absolutely fundamental.

A basic invariant of a lcs structure ω when M has dimension at least 4: is the Lee class,

$$\alpha = \alpha_\omega \in H^1(M, \mathbb{R}).$$

Again assuming M has dimension at least 4, the Lee class α has a natural differential form representative, called the Lee form and also denoted by α for simplicity. If U is an open set so that $\omega|_U = f_U \cdot \omega_U$ for ω_U symplectic, and f_U a positive smooth function, then $\alpha = d(\ln f_U)$ on U . By a simple calculation this can be seen to give well defined 1-form α , see also Lee [7]. The class α has the property that on the associated α -covering space \tilde{M} , the lift $\tilde{\omega}$ is globally conformally symplectic, that is $\tilde{\omega} = f \cdot \omega_0$ with ω_0 symplectic and $f > 0$. By α -covering space we mean the covering space associated to the normal subgroup $\ker \langle \alpha, \cdot \rangle \subset \pi_1(M, x)$, where $\langle \alpha, \cdot \rangle : \pi_1(M, x) \rightarrow \mathbb{R}$ is the homomorphism $[\gamma] \mapsto \langle \alpha, [\gamma] \rangle$.

It is moreover immediate that for an lcs form ω

$$d\omega = \alpha \wedge \omega,$$

for α the Lee form as defined above. For some authors, the pair (ω, α) with α closed s.t. $d\omega = \alpha \wedge \omega$ is the definition of an lcs structure. This has the advantage of being interesting even in dimension 2, but in dimension at least 4 is the same notion as given here, since then the Lee form is uniquely determined.

Introduce the operators

$$\begin{aligned} d^\alpha : \Omega^k(M) &\rightarrow \Omega^{k+1}(M), \\ \forall \eta \in \Omega^k(M) : d^\alpha(\eta) &= d\eta - \alpha \wedge \eta. \end{aligned}$$

Key words and phrases. locally conformally symplectic manifolds, conformal symplectic non-squeezing, Gromov-Witten theory, virtual fundamental class, Fuller index, Weinstein conjecture.

Partially supported by PRODEP grant.

This is called the Lichnerowicz differential with respect to a closed 1-form α , and it satisfies

$$d^\alpha \circ d^\alpha = 0$$

so that we have an associated **Lichnerowicz chain complex**. The following is one basic example of an lcs manifold.

Example 1 (Banyaga). Let (C, λ) be a contact $(2n + 1)$ -manifold where λ is a contact form,

$$\forall p \in C : \lambda \wedge \lambda^{2n}(p) \neq 0,$$

and take $M = C \times S^1$ with 2-form

$$\omega_\lambda = d^\alpha \lambda$$

for $\alpha := pr_{S^1}^* d\theta$, $pr_{S^1} : C \times S^1 \rightarrow S^1$ the projection, and λ likewise the pull-back of λ by the projection $C \times S^1 \rightarrow C$. We call (M, ω_λ) as above the **lcs-fication** of (C, λ) .

1.0.1. *Transformations of lcs manifolds.* One type of transformations of lcs manifolds which is often considered is a **conformal symplectomorphism**. That is a diffeomorphism $\phi : (M_1, \omega_1) \rightarrow (M_2, \omega_2)$ satisfying $\phi^* \omega_2 = f \omega_1$ for some smooth positive function f . To see why this kind of transformation is natural it is perhaps best to think in terms Hamiltonian dynamics on lcs manifolds. This is studied for instance in the work of Banyaga [2]. Using this notion, we may then faithfully embed the category of contact manifolds, and contactomorphisms into the category we call LCS^c of lcsm's, and conformal symplectomorphisms. However, with this type of transformation the natural embedding of categories

$$i : \text{Symp} \rightarrow LCS^c,$$

where Symp is the category of symplectic manifolds, is far from full, it is not even injective on the sets of isomorphism equivalence classes.

Here is one fix of the problem that will be conceptually useful for us.

Definition 1.2. An lcs **map** $\phi : (M_0, \omega_0) \rightarrow (M_1, \omega_1)$ of lcs manifolds is a smooth map so that there is a smooth family $\{\omega'_t\}_{t=0}^1$, $\omega'_1 = \phi^* \omega_1$, $\omega'_0 = \omega_0$, of lcs forms on M_0 in the same d^α Lichnerowicz cohomology class, for α the Lee form of ω_0 . In other words,

$$\forall t \in [0, 1] : (\omega'_t - \omega_0) \text{ is } d^\alpha \text{ exact.}$$

An lcs map, whose inverse is an lcs map will be called an lcs **diffeomorphism**. We denote by LCS the category of lcs manifolds with morphisms lcs morphisms above, in particular lcs diffeomorphisms are just isomorphisms in LCS .

Thus, applying Moser's principle, an lcs map from a closed symplectic manifold is a symplectomorphism if we understand a symplectic structure as an equivalence class of isotopic symplectic structures.

Lemma 1.3. *There is a natural faithful functor*

$$\text{emb} : \text{Cont} \rightarrow LCS,$$

with Cont denoting the category of contact manifolds made explicit in the proof.

If Symp denotes the category of closed symplectic manifolds then the canonical functor

$$\text{Symp} \rightarrow LCS$$

is injective on the sets of isomorphism equivalence classes. More specifically, if closed symplectic manifolds $(M, \omega_0), (M, \omega_1)$ are lcs diffeomorphic they are symplectomorphic. If a closed lcs manifold (M', ω') is lcs diffeomorphic to a symplectic manifold (M, ω) then (M', ω') is symplectic. In particular in this case $(M, \omega), (M', \omega')$ are symplectomorphic by the above.

1.1. Symplectic and lcs non-squeezing. One of the most important to this day results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [6]. The most well known formulation of this is that there does not exist a symplectic embedding

$$B_R \hookrightarrow D_r^2 \times \mathbb{R}^{2n-2}$$

for $R > r$, with B_R the standard closed radius R ball in \mathbb{R}^{2n} centered at 0, and D_r^2 the radius r closed disk in \mathbb{R}^2 . Gromov's non-squeezing is C^0 persistent in the following sense. The proof of this is subsumed by the proof of Theorem 1.5 which follows, but is much more elementary.

We say that a symplectic form ω on $M \times N$ is *split* if $\omega = \omega_1 \oplus \omega_2$ for symplectic forms ω_1, ω_2 on M respectively N .

Theorem 1.4. *Given $R > r$, there is an $\epsilon > 0$ s.t. for any symplectic form ω' on $M = S^2 \times T^{2n-2}$ C^0 -close to a split symplectic form ω and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt] \in H_2(M),$$

there is no symplectic embedding $\phi : B_R \hookrightarrow (M, \omega')$.

On the other hand it is natural to ask:

Question 1. Given $R > r$ and every $\epsilon > 0$ is there a (necessarily non-closed by above) 2-form ω' on $S^2 \times T^{2n-2}$ C^0 or even C^∞ ϵ -close to a split symplectic form ω , satisfying $\langle \omega, A \rangle = \pi r^2$, and such that there is an embedding $\phi : B_R \hookrightarrow S^2 \times T^{2n-2}$, with $\phi^* \omega' = \omega_{st}$? We likewise call such a map ϕ **symplectic embedding**.

This appears to be a very difficult question, my opinion is that at least in the C^0 case the answer is yes, in part because it is difficult to imagine any obstruction, for example we no longer have Gromov-Witten theory for such a general ω' . In the following Theorem 1.5 we show that if ω' is lcs then the answer is no under some additional conditions. One may think that recent work of Müller [10] may be related to the present discussion. But there seems to be no obvious such relation as pull-backs by diffeomorphisms of nearby forms may not be nearby. Hence there is no way to go from nearby embeddings that we work with to ϵ -symplectic embeddings of Müller.

Let again $M = S^2 \times T^{2n-2}$, with ω a split symplectic form on M . Note that in what follows we take a certain natural metric C^0 topology \mathcal{T}^0 on the space of general lcs forms, defined in Section 2, which is finer than the standard C^0 metric topology on the space of forms, cf. [2, Section 6]. The corresponding metric is denoted d_0 .

We have a real codimension 1 hypersurfaces

$$\Sigma_i = S^2 \times (S^1 \times \dots \times S^1 \times \{pt\} \times S^1 \times \dots \times S^1) \subset M,$$

where the singleton $\{pt\} \subset S^1$ replaces the i 'th factor of $T^{2n-2} = S^1 \times \dots \times S^1$. The hypersurfaces Σ_i are naturally foliated by symplectic submanifolds

$$M_\theta = S^2 \times (S^1 \times \dots \times S^1 \times \{pt\} \times \{\theta\} \times S^1 \times \dots \times S^1) \simeq S^2 \times T^{2n-2},$$

$\theta \in S^1$. We denote by $T^{fol} \Sigma_i \subset TM$, the distribution of vectors tangent to the leaves of the above mentioned foliation. In other words

$$T^{fol} \Sigma_i = \cup_\theta i_* TM_\theta,$$

where $i : M_\theta \rightarrow M$ are the inclusion maps.

Theorem 1.5. *Let ω be a split symplectic form on $M = S^2 \times T^{2n-2}$, let A be as above with $\langle \omega, A \rangle = \pi r^2$. Let $R > r$, then there is an $\epsilon > 0$ (depending only on R, r, ω) s.t. if $\{\omega_t\}$, $t \in [0, 1]$, $\omega_0 = \omega$ is a \mathcal{T}^0 -continuous family of lcs forms on M , with $d_0(\omega_t, \omega_0) < \epsilon$ for all t , then there is no symplectic embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M - \bigcup_i \Sigma_i, \omega_1),$$

meaning an embedding ϕ such that $\phi^ \omega_1 = \omega_{st}$.*

Note that the latter is a full-volume subspace diffeomorphic to $S^2 \times \mathbb{R}^{2n-2}$. More generally there is no symplectic embedding

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1),$$

s.t. $\phi_* j$ preserves the bundles $T^{fol} \Sigma_i$, for j the standard almost complex structure on B_R .

Smooth lcs deformations ω_t of our symplectic form ω , with Lee forms α_t likewise smooth in t , are obstructed unless α_t are DeRham exact, as pointed out to me by Kevin Sackel. This can be verified by an elementary calculation by taking the t derivative at 0 of the equation:

$$d^\alpha \omega_t = \alpha_t \wedge \omega_t.$$

But our families are not required to be smooth so that non-trivial lcs deformations of a symplectic form may exist.

Question 2. Do there exist non-trivial lcs deformations of the standard product symplectic form on $S^2 \times T^{2n-2}$?

We hope that the theorem above helps to motivate this question.

1.1.1. *Invariance of the lcs non-squeezing Theorem 1.5 under transformations.* Since we use a seemingly rigid notion of a morphism of lcs manifolds in the statement of lcs non-squeezing, a natural additional question is: how invariant is the lcs non-squeezing result with respect to more general lcs transformations? It is certainly not invariant under transformations in LCS^c , which is not surprising since the original Gromov non-squeezing is not invariant this way. For if $\omega_1 = C\omega_0$, $C > 0$ some constant, ω_1 is conformally symplectomorphic to ω_0 and we may certainly have a symplectic embedding of B_R into (M, ω_1) even if there is no such embedding into (M, ω_0) . However LCS also appears to an interesting category to study, and in this category the above non-squeezing result is invariant in the following sense.

Corollary 1.6. *Let ω be an lcs form on M , lcs diffeomorphic to ω'' (and so also symplectomorphic), where ω'' on $M = S^2 \times T^{2n-2}$, is a split symplectic form. Let A be as above, and suppose that $\langle \omega, A \rangle = \pi r^2$. Then there exists a collection of hypersurfaces $\{\Sigma'_i\}_{i=1}^{2n-2}$ in M so that for all $R > r$ there is an $\epsilon > 0$ (depending on R, r, ω, ω'') s.t. if $\{\omega_t\}$, $t \in [0, 1]$, $\omega_0 = \omega$ is a \mathcal{T}^0 -continuous family of lcs forms on M , with $d_0(\omega_t, \omega_0) < \epsilon$ for all t , then there is no symplectic embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M - \bigcup_i \Sigma'_i, \omega_1).$$

1.1.2. *Toward direct generalization of contact non-squeezing.* We can also try a direct generalization of contact non-squeezing of Eliashberg-Polterovich [3], and Fraser in [4]. Specifically let $R^{2n} \times S^1$ be the prequantization space of R^{2n} , or in other words the contact manifold with the contact form $d\theta - \lambda$, for $\lambda = \frac{1}{2}(ydx - xdy)$. Let B_R now denote the open radius R ball in \mathbb{R}^{2n} . A Hamiltonian conformal symplectomorphism of an lcs manifold (M, ω) , which we just abbreviate by the short name: **Hamiltonian map**, is a conformal symplectomorphism ϕ_H generated as usual by $H : M \times [0, 1] \rightarrow \mathbb{R}$, by defining the time dependent vector field X_t

$$\omega(X_t, \cdot) = d^\alpha H_t,$$

for α the Lee form, and then taking ϕ_H to be the time 1 flow map of $\{X_t\}$. For example if $\omega = d^\alpha \lambda$ on $C \times S^1$ is the lcs-ification of the contact form λ on C , and if $\forall t : H_t = -1$, then $d^\alpha(H_t) = -\alpha$ and clearly

$$X_t = (R^\lambda \oplus 0),$$

as a section of $TC \oplus TS^1$ with R^λ the λ -Reeb vector field. Thus in this case the associated flow is naturally induced by the Reeb flow.

Question 3. If $R \geq 1$ is there a compactly supported, Hamiltonian lcs map

$$\phi : \mathbb{R}^{2n} \times S^1 \times S^1 \rightarrow \mathbb{R}^{2n} \times S^1 \times S^1,$$

so that $\phi(\overline{U}) \subset U$, for $U := B_R \times S^1 \times S^1$ and \overline{U} the topological closure? Instead of Hamiltonian lcs maps we may try to take lcs maps as in Definition 1.2. The relationship between these notions is not very well understood at the moment, and is somewhat beyond our scope here.

2. PROOFS

Let M be a closed smooth manifold of dimension at least 4. The C^0 metric topology \mathcal{T}^0 on the set $LCS(M)$ of smooth lcs 2-forms on M will be defined with respect to the following metric.

Definition 2.1. Fix a Riemannian metric g on M . For $\omega_1, \omega_2 \in LCS(M)$ define

$$d_0(\omega_1, \omega_2) = d_{C^0}(\omega_1, \omega_2) + d_{C^0}(\alpha_1, \alpha_2),$$

for α_i the Lee forms of ω_i and d_{C^0} the usual C^0 metrics induced by g .

The following characterization of convergence will be helpful.

Lemma 2.2. Let M be as above and let $\{\omega_k\} \subset LCS(M)$ be a sequence \mathcal{T}^0 converging to a symplectic form ω . Denote by $\{\tilde{\omega}_k\}$ the lift sequence on the universal cover \tilde{M} . Then there is a sequence $\{\tilde{\omega}_k^{symp}\}$ of symplectic forms on \tilde{M} , and a sequence $\{f_k\}$ of positive functions pointwise converging to 1, such that $\tilde{\omega}_k = f_k \tilde{\omega}_k^{symp}$.

Proof. We may assume that M is connected. Let α_k be the Lee form of ω_k , and g_k functions on \tilde{M} defined by $g_k([p]) = \int_{[0,1]} p^* \alpha_k$, where the universal cover \tilde{M} is understood as the set equivalence classes of paths p starting at a fixed $x_0 \in M$, with a pair p_1, p_2 equivalent if $p_1(1) = p_2(1)$ and $p_2^{-1} \cdot p_1$ is null-homotopic, where \cdot is the path concatenation.

Then we get:

$$d\tilde{\omega}_k = dg_k \wedge \tilde{\omega}_k,$$

so that if we set $f_k := e^{g_k}$ then

$$d(f_k^{-1} \tilde{\omega}_k) = 0.$$

Since by assumption $|\alpha_k|_{C^0} \rightarrow 0$, then pointwise $g_k \rightarrow 0$ and pointwise $f_k \rightarrow 1$, so that if we set

$$\tilde{\omega}_k^{symp} := f_k^{-1} \tilde{\omega}_k$$

then we are done. \square

Definition 2.3. We say that a pair (ω, J) of an lcs form ω on M and an almost complex structure J on M are **compatible** if $\omega(\cdot, J\cdot)$ defines a J -invariant inner product on M . For other basic notions of J -holomorphic curves we refer the reader to [8].

Theorem 2.4. Let M be as above, $A \in H_2(M)$ fixed, and $\{\omega_t\}$, $t \in [0, 1]$, a \mathcal{T}^0 -continuous family of lcs forms on M . Let $\{J_t\}$ be a Frechet smooth family of almost complex structures, with J_t compatible with ω_t for each t . Let $D \subset \tilde{M}$, with $\pi : \tilde{M} \rightarrow M$ the universal cover of M , be a fundamental domain, and $K := \overline{D}$ its topological closure. Suppose that for each t , and for every $x \in \partial K$ (the topological boundary) there is a \tilde{J}_t -holomorphic hyperplane H_x through x , with $H_x \subset K$, such that $\pi(H_x) \subset M$ is a closed submanifold and such that $A \cdot \pi_*([H_x]) \leq 0$. Define:

$$E_t(u) := \int_{S^2} u^* \omega_t.$$

Then

$$\sup_{u,t} E_t(u) < \infty,$$

where the supremum is over all pairs (u, t) , u is a J_t -holomorphic class A genus 0 curve in M .

Proof.

Lemma 2.5. *Let M, A be as above, let $D \subset \widetilde{M}$, with $\pi : \widetilde{M} \rightarrow M$ the universal cover of M , be a fundamental domain, and $K := \overline{D}$ its topological closure. Let (ω, J) be a compatible lcs pair on M such that for every $x \in \partial K$ there is a \widetilde{J} -holomorphic (real codimension 2) hyperplane $H_x \subset K \subset \widetilde{M}$ through x , such that $\pi(H_x) \subset M$ is a closed submanifold and such that $A \cdot [\pi(H_x)] \leq 0$. Then any genus 0, J -holomorphic class A curve u in M has a lift \tilde{u} with image in K .*

Proof. For u as in the statement, let \tilde{u} be a lift intersecting the fundamental domain D , (as in the statement of main theorem). Suppose that \tilde{u} intersects ∂K , otherwise we already have image $\tilde{u} \subset K^\circ$, for K° the interior, since image \tilde{u} is connected (and by elementary topology). Then \tilde{u} intersects u_x as in the statement, for some x . So u is a J -holomorphic map intersecting the closed hyperplane $\pi(H_x)$ with $A \cdot [\pi(H_x)] \leq 0$. By positivity of intersections [8, Section 2.6], which in this case is just a simple exercise, image $u \subset \pi(H_x)$, and so image $\tilde{u} \subset H_x$. And so image $\tilde{u} \subset \partial K$. \square

Now, let u be a J_t -holomorphic class A curve. By the lemma above u has a lift \tilde{u} contained in the compact $K \subset \widetilde{M}$. Then we have:

$$E_t(u) = \int_{S^2} \tilde{u}^* \tilde{\omega}_t \leq C_t \langle \tilde{\omega}_t^{symp}, A \rangle,$$

where $\tilde{\omega}_t = f_t \tilde{\omega}_t^{symp}$, for $\tilde{\omega}_t^{symp}$ symplectic on \widetilde{M} , and $f_t : \widetilde{M} \rightarrow \mathbb{R}$ positive function constructed as in the proof of Lemma 2.2, and where $C_t = \max_K f_t$. Since $\{\omega_t\}$ is continuous in \mathcal{T}_0 , $\{f_t\}$, $\{\tilde{\omega}_t^{symp}\}$ are C_0 continuous. In particular

$$C = \sup_t \max_K f_t$$

and

$$D = \sup_t \langle \tilde{\omega}_t^{symp}, A \rangle$$

are finite. And so

$$\sup_{(u,t)} E_t(u) \leq C \cdot D,$$

where the supremum is over all pairs (u, t) , u is J_t -holomorphic curve in M . \square

2.1. Quick review of genus 0 Gromov-Witten theory. Let

$$\mathcal{M}_{g,0}(J, A) = \mathcal{M}_{g,0}(M, J, A)$$

denote the moduli space of isomorphism classes of class A , J -holomorphic curves in M , with domain the Riemann sphere, with n marked labeled points. Here an isomorphism between $u_1 : \Sigma_1 \rightarrow M$, and $u_2 : \Sigma_2 \rightarrow M$ is a biholomorphism of marked Riemann surfaces $\phi : \Sigma_1 \rightarrow \Sigma_2$ s.t. $u_2 \circ \phi = u_1$.

The following is well known and follows by the same argument as [8, Theorem 5.6.6].

Theorem 2.6. *Let (M, J) be an almost complex manifold. Then $\mathcal{M}_{g,0}(J, A)$ has a pre-compactification*

$$\overline{\mathcal{M}}_{g,0}(J, A),$$

by Kontsevich stable maps, with respect to the natural metrizable Gromov topology [8, Chapter 5.6]. Moreover given $E > 0$, the subspace $\overline{\mathcal{M}}_{g,0}(J, A)_E \subset \overline{\mathcal{M}}_{g,0}(J, A)$ consisting of elements u with $e(u) \leq E$ is compact, where e is the L^2 energy with respect to an auxiliary metric. In other words e is a proper function.

Thus the most basic situation where we can talk about Gromov-Witten “invariants” of (M, J) is when the energy function is bounded on $\overline{\mathcal{M}}_{g,0}(J, A)$. In this case $\overline{\mathcal{M}}_{g,n}(J, A)$ is compact, and has a virtual moduli cycle as in the original approach of Fukaya-Ono [5], or the more algebraic approach [9]. So we may define functionals:

$$(2.7) \quad GW_{g,n}(A, J) : H_*(\overline{\mathcal{M}}_{g,n}) \otimes H_*(M^n) \rightarrow \mathbb{Q},$$

where $\overline{\mathcal{M}}_{g,n}$ denotes the compactified moduli space of Riemann surfaces. Of course symplectic manifolds with any tame almost complex structure is one class of examples.

These functionals will not in general be J -invariant but they are invariant for a smooth family $\{J_t\}$, $t \in [0, 1]$ such that the corresponding “cobordism moduli space”: $\overline{\mathcal{M}}_{g,0}(\{J_t\}, A)$, is compact, where $\overline{\mathcal{M}}_{g,0}(\{J_t\}, A)$ is the space of pairs (u, t) , $u \in \overline{\mathcal{M}}_{g,0}(J_t, A)$.

Proof of Theorem 1.5. Fix an $\epsilon' > 0$ s.t. any 2-form ω_1 on M , C^0 ϵ' -close to ω , is non-degenerate and is non-degenerate on the leaves of the foliation of each Σ_i , discussed prior to the formulation of the theorem. Suppose by contradiction that for every $\epsilon > 0$ there is a homotopy $\{\omega_t\}$ of lcs forms, with $\omega_0 = \omega$, such that $\forall t : d_0(\omega_t, \omega) < \epsilon$ and such that there exists a symplectic embedding

$$\phi : B_R \hookrightarrow (M, \omega_1),$$

satisfying conditions of the statement of the theorem. Take $\epsilon < \epsilon'$, and let $\{\omega_t\}$ be as in the hypothesis above. In particular ω_t is an lcs form for each t , and is non-degenerate on Σ_i . Extend ϕ_*j to an ω_1 -compatible almost complex structure J_1 on M , preserving $T^{fol}\Sigma_i$ for each i . We may then extend this to a family $\{J_t\}$ of almost complex structures on M , s.t. J_t is ω_t -compatible for each t , with J_0 is the standard split complex structure on M and such that J_t preserves $T^{fol}\Sigma_i$ for each t, i . The latter condition can be satisfied since Σ_i are ω_t -symplectic for each t . When $\phi(B_R)$ does not intersect $\cup_i \Sigma_i$ these conditions can be trivially satisfied, first find an extension J_1 of ϕ_*j preserving $T^{fol}\Sigma_i$ for each i . Then extend to a family $\{J_t\}$.

Then the family $\{(\omega_t, J_t)\}$ satisfies the hypothesis of Theorem 2.4 for the class $A = [S^2] \otimes [pt]$ as in the statement of the theorem we are proving. And so since we have an energy bound

$$C = \overline{\mathcal{M}}_{0,1}(\{J_t\}, A)$$

is compact by Theorem 2.6.

Now we have the classical Gromov-Witten invariant counting class A , J_0 -holomorphic, genus 0 curves passing through a fixed point:

$$GW_{0,1}(A, J_0)([pt]) = 1,$$

whose calculation already appears in [6]. Then by compactness of C , and the discussion preceding the proof

$$GW_{0,1}(A, J_1)([pt]) = 1.$$

In particular there is a class A J_1 -holomorphic curve u passing through $\phi(0)$.

By Lemma 2.5 we may choose a lift \tilde{u} of u to \tilde{M} , with homology class $[\tilde{u}]$ also denoted by A so that the image of \tilde{u} is contained in a compact set $K \subset \tilde{M}$, (independent of the choice of $\{\omega_t\}, \{J_t\}$ satisfying above conditions). Let $\tilde{\omega}_t^{symp}$ and f_t be as in Lemma 2.2, then by this lemma for every $\delta > 0$ we may find an $\epsilon > 0$ so that if $d_0(\omega_1, \omega) < \epsilon$ then $d_{C^0}(\tilde{\omega}^{symp}, \tilde{\omega}_1^{symp}) < \delta$ on K , and $\sup_K |f_1 - 1| < \delta$.

Let δ as above be chosen, and let ϵ correspond to this δ . Then if this δ is sufficiently small we get

$$\left| \int_{S^2} u^* \omega_1 - \pi r^2 \right| \leq \left| \max_K f_1 \langle \tilde{\omega}_1^{symp}, A \rangle - \pi \cdot r^2 \right| < \pi R^2 - \pi r^2,$$

since

$$|\langle \tilde{\omega}_1^{symp}, A \rangle - \pi \cdot r^2| = |\langle \tilde{\omega}_1^{symp}, A \rangle - \langle \tilde{\omega}^{symp}, A \rangle| \leq \delta \pi \cdot r^2,$$

for $\langle \tilde{\omega}^{symp}, A \rangle = \pi r^2$ and $d_{C^0}(\tilde{\omega}^{symp}, \tilde{\omega}_1^{symp}) < \delta$ on K , and since

$$\max_K f_1 \leq 1 + \delta.$$

So if δ, ϵ are chosen appropriately as above, we get that $\int_{S^2} u^* \omega_1 < \pi R^2$.

We may then proceed exactly as in the now classical proof of Gromov [6] of the non-squeezing theorem to get a contradiction and finish the proof. A bit more specifically, $\phi^{-1}(\text{image } \phi \cap \text{image } u)$ is a minimal surface in B_R , with boundary on the boundary of B_R , and passing through $0 \in B_R$. By construction it has area strictly less than πR^2 which is impossible by the classical monotonicity theorem of differential geometry. \square

Proof of Lemma 1.3. Specifically we define $Cont$ to be the category with objects (C, λ) where λ is a contact form and morphisms

$$\phi : (C_1, \lambda_1) \rightarrow (C_2, \lambda_2)$$

contactomorphisms, so that

$$\phi^*(\lambda_2) = f\lambda_1$$

for a positive function f . Define

$$emb : Cont \rightarrow LCS$$

by

$$emb(C, \lambda) = (C \times S^1, d^\alpha \lambda),$$

$\alpha = d\theta$, in other words the lcs-fication as usual. For a contactomorphism $\phi : (C_1, \lambda_1) \rightarrow (C_2, \lambda_2)$ define $emb(\phi) = (\phi \times id)$. Then

$$emb(\phi)^* d^\alpha \lambda_2 = d^\alpha f \lambda_1$$

is homotopic through the lcs forms

$$\{d^\alpha f_t \lambda_1\},$$

for $\{f_t\}$ a smooth homotopy of positive functions, $f_1 = f$, $f_0 = 1$. And so $emb(\phi)$ is an lcs map. It is obvious that emb is functorial.

Now let $(M, \omega), (M', \omega')$ be closed symplectic manifolds. Let

$$\phi : (M, \omega) \rightarrow (M', \omega')$$

be a lcs diffeomorphism. Then since the Lee form α of ω is 0, by definition we have that $\phi^*\omega'$ is homotopic through symplectic forms in the same cohomology class to ω . So by Moser's lemma, $(M, \omega), (M', \omega')$ are symplectomorphic.

Finally let (M, ω) be a closed symplectic manifold, (M', ω') be an lcs manifold and let

$$\phi : (M, \omega) \rightarrow (M', \omega')$$

be an lcs diffeomorphism. Then by the same point as above $\phi^*\omega'$ is homotopic through symplectic forms to ω . In particular ω' is closed, so (M', ω') is symplectic. \square

Proof of the Corollary 1.6. Let

$$\rho : (M, \omega) \rightarrow (M, \omega'')$$

be an lcs diffeomorphism. By Lemma 1.3 ω is then symplectic and there is an induced symplectomorphism

$$\rho' : (M, \omega) \rightarrow (M, \omega'').$$

Let ϵ' be chosen with respect to ω'', R, r as in the statement of Theorem 1.5. And let ϵ be taken so that:

$$(2.8) \quad d_0(\omega, \omega') < \epsilon \implies d_0(\omega'', \rho'_*\omega') < \epsilon',$$

for any lcs-form ω' . Define $\Sigma'_i := (\rho')^{-1}(\Sigma_i)$. Given $\{\omega_t\}$ as in the statement, suppose otherwise that we have a symplectic embedding:

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M - \bigcup_i \Sigma'_i, \omega_1).$$

Then

$$\rho' \circ \phi : (B_R, \omega_{st}) \hookrightarrow (M - \bigcup_i \Sigma_i, \rho'_*(\omega_1))$$

is a symplectic embedding. But this contradicts the conjunction of Theorem 1.5 and (2.8). \square

3. ACKNOWLEDGEMENTS

I am grateful to Baptiste Chantraine and Kevin Sackel for discussions.

REFERENCES

- [1] V. APOSTOLOV AND G. DLOUSSKY, *Locally conformally symplectic structures on compact non-Kähler complex surfaces*, Int. Math. Res. Not., 2016 (2016), pp. 2717–2747.
- [2] A. BANYAGA, *Some properties of locally conformal symplectic structures.*, Comment. Math. Helv., 77 (2002), pp. 383–398.
- [3] Y. ELIASHBERG, S. S. KIM, AND L. POLTEROVICH, *Geometry of contact transformations and domains: orderability versus squeezing.*, Geom. Topol., 10 (2006), pp. 1635–1748.
- [4] M. FRASER, *Contact non-squeezing at large scale in $\mathbb{R}^{2n} \times S^1$.*, Int. J. Math., 27 (2016), p. 25.
- [5] K. FUKAYA AND K. ONO, *Arnold Conjecture and Gromov–Witten invariant*, Topology, 38 (1999), pp. 933 – 1048.
- [6] M. GROMOV, *Pseudo holomorphic curves in symplectic manifolds.*, Invent. Math., 82 (1985), pp. 307–347.
- [7] H.-C. LEE, *A kind of even-dimensional differential geometry and its application to exterior calculus.*, Am. J. Math., 65 (1943), pp. 433–438.
- [8] D. McDUFF AND D. SALAMON, *J-holomorphic curves and symplectic topology*, no. 52 in American Math. Society Colloquium Publ., Amer. Math. Soc., 2004.
- [9] J. PARDON, *An algebraic approach to virtual fundamental cycles on moduli spaces of J-holomorphic curves*, Geometry and Topology.
- [10] STEFAN MÜLLER, *Epsilon-non-squeezing and C_0 -rigidity of epsilon-symplectic embeddings*, arXiv:1805.01390, (2018).
Email address: yasha.savelyev@gmail.com

UNIVERSITY OF COLIMA, CUICBAS