SMOOTH SIMPLICIAL SETS AND UNIVERSAL CHERN-WEIL HOMOMORPHISM

YASHA SAVELYEV

ABSTRACT. We start by developing a basic geometric-categorical notion of a smooth simplicial set. Loosely, this is to diffeological spaces of Chen-Souriau as simplicial sets are to spaces. Given a Frechet group G, we construct abstract classifying spaces $BG^{\mathcal{U}}$ as smooth Kan complexes. Here the index $\mathcal{U}$ is a chosen Grothendieck universe of a certain type. When G in addition has the homotopy type of a CW complex, there is a homotopy equivalence $BG \simeq |BG^{\mathcal{U}}|$, where BG is the usual Milnor classifying space. This leads to our main application that for G a Frechet Lie group, having the homotopy type of a CW complex, there is a universal Chern-Weil homomorphism:

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG, \mathbb{R}).$$

This has immediate applications to the study of the infinite dimensional symplectomorphism group of a symplectic manifold.

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1. INTRODUCTION

This note is based around the notion of a smooth simplicial set, which is an analogue in simplicial sets of diffeological spaces of Chen-Souriau, with the latter

perhaps a most basic notion of a "smooth space". The language of smooth simplicial sets turn out to be a powerful tool to resolve the problem of the construction of the universal Chern-Weil homomorphism.

One problem of topology is the construction of a "smooth structure" on the classifying space BG of a Frechet Lie group G. There are specific requirements for what such a notion of a smooth structure should entail. At the very least we hope to be able to carry out Chern-Weil theory universally on BG. That is we want a differential geometric construction of the Chern-Weil homomorphism:

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG, \mathbb{R}),$$

with $\mathbb{R}[\mathfrak{g}]^G$ denoting Ad_G invariant polynomials on the Lie algebra $\mathfrak{g}$ of G. When G is compact BG can be written as a colimit of smooth manifolds and so in that case the existence of the universal Chern-Weil homomorphism is classical.

One candidate for a smooth structure on BG is some kind of diffeology. For example Magnot and Watts [6] construct a natural diffeology on the Milnor classifying space BG. However, this diffeology does not appear to be sufficient to carry out Chern-Weil theory directly. A further specific possible requirement for the above discussed "smooth structures", is that the simplicial set $BG_\bullet$, of smooth maps $\Delta^d \to BG$, should have a geometric realization weakly homotopy equivalent to BG. See for instance [4] for one approach to this particular problem in the context of diffeologies. This kind of requirement is crucial for instance in [11], which may be understood as a kind of "quantum Chern-Weil theory" on $BHam(M, \omega)$ for $Ham(M, \omega)$ the group of Hamiltonian symplectomorphisms of a symplectic manifold. In the language of smooth simplicial sets, the analogue of this latter requirement is always trivially satisfied. The specific content of this is Corollary 3.5.

The structure of a smooth simplicial set is initially more flexible than a space with diffeology, but with further conditions, like the Kan condition, can become forcing. Given a Frechet Lie group G, we construct, for each choice of a particular kind of Grothendieck universe $\mathcal{U}$, a smooth simplicial set $BG^{\mathcal{U}}$ with a specific classifying property, analogous to the classifying property of BG. The simplicial set $BG^{\mathcal{U}}$ is moreover a Kan complex, and so is a basic example of a smooth Kan complex. We then show that if G in addition has the homotopy type of a CW complex then the geometric realization $|BG^{\mathcal{U}}|$ is homotopy equivalent to BG.

All of the dreams of "smoothness" mentioned above then in some sense hold true for $BG^{\mathcal{U}}$ via its smooth Kan complex structure. In particular, as one immediate application we get:

Theorem 1.1. *Let G be a Frechet Lie group having the homotopy type of a CW complex, then there is a universal Chern-Weil algebra homomorphism:*

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG, \mathbb{R}).$$

Here is one concrete example, using Reznikov's polynomials [10] on the Lie algebra of the group of Hamiltonian symplectomorphisms. Let $Symp(\mathbb{CP}^k)$ denote the group of symplectomorphisms of $\mathbb{CP}^k$, that is diffeomorphisms $\phi : \mathbb{CP}^k \to \mathbb{CP}^k$ s.t. $\phi^* \omega_0 = \omega_0$ for ω_0 the Fubini-Study symplectic 2-form on $\mathbb{CP}^k$.

Theorem 1.2. *The natural map*

$$i : BPU(n) \to BSymp(\mathbb{CP}^{n-1})$$

is injective on real homology for all $n \geq 2$.

More history and background surrounding these theorems is in Sections 8 and 9.

2. PRELIMINARIES AND NOTATION

We denote by Δ the simplex category, i.e. the category with objects finite sets: $[n] = \{0, 1, \dots, n\}$, with $\hom_\Delta([n], [m])$ non-decreasing maps. A simplicial set X is a functor

$$X : \Delta \to Set^{op}.$$

We usually write $X(n)$ instead of $X([n])$. Δ^d_{simp} will denote the standard representable d-simplex:

$$\Delta^d_{simp}(n) = \hom_\Delta([n], [d]).$$

The element of $\Delta^d_{simp}(0)$ corresponding to the map $i_k : [0] \to [d]$, $i_k(0) = k$ will usually be denoted by just k.

Let Δ^d be the topological d-simplex, i.e.

$$\Delta^d := \{(x_1, \dots, x_n) \in \mathbb{R}^d \, | \, x_1 + \dots + x_n \leq 1, \text{ and } \forall i : x_i \geq 0\}.$$

The vertices of Δ^d are assumed ordered in the standard way. We denote by $\Delta^d_\bullet$ the simplicial set of all smooth simplices in Δ^d. That is $\Delta^d_\bullet(k)$ is the set of maps

$$\sigma : \Delta^k \to \Delta^d,$$

that have a smooth extension to a map from a neighborhood in $\mathbb{R}^k$ of Δ^k into a neighborhood of Δ^d in $\mathbb{R}^d$.

We call a linear map $\Delta^k \to \Delta^d$ taking vertices to vertices in a order preserving way ***simplicial***. And we denote by

$$\Delta^d_{simp} \subset \Delta^d_\bullet$$

the subset consisting of simplicial maps. Note that Δ^d_{simp} is naturally isomorphic to the standard representable d-simplex Δ^d_{simp} as previously defined. Thus we may also understand Δ as the category with objects topological simplices Δ^d, $d \geq 0$ and morphisms simplicial maps.

Notation 2.1. *A morphism $m \in \hom_\Delta([n], [k])$ uniquely corresponds to a simplicial map $\Delta^n_{simp} \to \Delta^k_{simp}$, which uniquely corresponds to a simplicial map in the above sense $\Delta^n \to \Delta^k$. The correspondence is by taking the maps $\Delta^n_{simp} \to \Delta^k_{simp}$, $\Delta^n \to \Delta^k$, to be determined by the map of the vertices corresponding to m. We will not notationally distinguish these corresponding morphisms. So that m may will simultaneously refer to all of the above morphisms.*

By a d-simplex Σ of a simplicial set X, we may mean, interchangeably, either the element in $X(d)$ or the morphism (natural transformation):

$$\Sigma : \Delta^d_{simp} \to X,$$

uniquely corresponding to Σ via the Yoneda lemma. If we write Σ^d for a simplex of X, it is implied that it is a d-simplex.

Definition 2.2. $\Delta(X)$ *will denote the over category of simplices of X, explicitly the category whose objects are simplices*

$$\Sigma : \Delta^d_{simp} \to X, \quad d \geq 0$$

and morphisms f, commutative diagrams:

$$(2.1) \qquad \begin{array}{ccc} \Delta^d_{simp} & \xrightarrow{\widetilde{f}} & \Delta^n_{simp} \\ & \searrow{\scriptstyle \Sigma_1} & \downarrow{\scriptstyle \Sigma_2} \\ & & X \end{array}$$

with top arrow a simplicial map, which we denote by $\widetilde{f}$.

There is a forgetful functor

$$T : \Delta(X) \to \Delta,$$

$T(\Sigma^d) = \Delta^d_{simp}$, $T(f) = \widetilde{f}$. We denote by $Simp(X) \subset \Delta(X)$ the sub-category with same objects, and morphisms f such that $\widetilde{f}$ are monomorphisms, i.e. are face inclusions.

2.1. **Geometric realization.** Let Top be the category of topological spaces. Let X be a simplicial set, then define as usual the **geometric realization** of X by the colimit in Top:

$$|X| := \operatorname{colim}_{\Delta(X)} T,$$

for $T : \Delta(X) \to \Delta \subset Top$ as above, understanding Δ as a subcategory of Top as previously explained.

3. Smooth simplicial sets

If

$$\sigma : \Delta^d \to \Delta^n$$

is a smooth map we then have an induced map of simplicial sets

$$(3.1) \qquad\qquad \sigma_\bullet : \Delta^d_\bullet \to \Delta^n_\bullet,$$

defined by

$$\sigma_\bullet(\rho) = \sigma \circ \rho.$$

Definition 3.1. *A* **a smooth simplicial set** *is a data consisting of:*

(1) *A simplicial set X.*
(2) *For each $\Sigma : \Delta^n_{simp} \to X$ an n-simplex, there is an assigned map of simplicial sets*

$$g(\Sigma) : \Delta^n_\bullet \to X,$$

which satisfies:

$$(3.2) \qquad\qquad g(\Sigma)|_{\Delta^n_{simp}} = \Sigma.$$

We abbreviate $g(\Sigma)$ by Σ_*, when there is no possible confusion which structure g is meant.

(3) *The following property will be called* **push-forward functoriality**:

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_\bullet$$

where $\sigma : \Delta^k \to \Delta^d$ is a k-simplex of $\Delta^d_\bullet$, and where Σ as before is a d-simplex of X.

Thus, formally a smooth simplicial set is a 2-tuple (X, g), satisfying the axioms above. When there is no need to disambiguate we omit specifying g.

Definition 3.2. *A* **smooth** *map between smooth simplicial sets*

$$(X_1, g_1), (X_2, g_2)$$

is a simplicial map

$$f : X_1 \to X_2,$$

which satisfies the condition:

(3.3) $$g_2(f(\Sigma)) = f \circ g_1(\Sigma),$$

or more compactly:

$$f(\Sigma)_* = f \circ \Sigma_*.$$

A ***diffeomorphism*** between smooth simplicial sets is defined to be a smooth map, with a smooth inverse.

Example 3.3 (The tautological smooth simplicial set)**.** *The Kan complex $\Delta^n_\bullet$ has a tautological smooth simplicial set structure, where*

$$g(\Sigma) = \Sigma_\bullet,$$

for $\Sigma : \Delta^k \to \Delta^n$ a smooth map, hence a k-simplex of $\Delta^n_\bullet$, and where $\Sigma_\bullet$ is as in (3.1)*.*

Lemma 3.4. *Let X be a smooth simplicial set and $\Sigma : \Delta^n_{simp} \to X$ an n-simplex. Let $\Sigma_* : \Delta^n_\bullet \to X$ be the induced simplicial map. Then Σ_* is smooth with respect to the tautological smooth simplicial set structure on $\Delta^n_\bullet$ as above.*

Proof. Let σ be a k-simplex of $\Delta^n_\bullet$, so $\sigma : \Delta^k \to \Delta^n$ is a smooth map, we need that

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_*.$$

But $\sigma_* = \sigma_\bullet$, by definition of the tautological smooth structure on $\Delta^n_\bullet$, so that by the Axiom 3 $\Sigma_* \circ \sigma_* = (\Sigma_*(\sigma))_*$. $\qquad \square$

Corollary 3.5. *The set of n-simplices of a smooth simplicial set X is naturally isomorphic to the set of smooth maps $\Delta^n_\bullet \to X$. In fact, define $X_\bullet$ to be the simplicial set whose n-simplices are smooth maps $\Delta^n_\bullet \to X$, with $X_\bullet$ given the obvious simplicial structure. Then $X_\bullet$ is naturally isomorphic to X.*

Proof. Given a simplex $\rho : \Delta^n_{simp} \to X$, we have a uniquely associated to it, by the lemma above, smooth map $\rho_* : \Delta^n_\bullet \to X$. Conversely, suppose we are given a smooth map $m : \Delta^n_\bullet \to X$. Then we get an n-simplex $\rho_m := m|_{\Delta^n_{simp}}$. Let $id^n : \Delta^n \to \Delta^n$ be the identity map. We have that

$$
\begin{aligned}
m &= m \circ id^n_\bullet = m \circ id^n_* \\
&= m(id^n)_* \text{ as } m \text{ is smooth} \\
&= \rho_m(id^n)_* \text{ trivially} \\
&= \rho_{m,*} \circ id^n_* \text{ as } \rho_{m,*} \text{ is smooth} \\
&= \rho_{m,*}.
\end{aligned}
$$

Thus the map $\rho \mapsto \rho_*$, from the set of n-simplices of X to the set of smooth maps $\Delta^n_\bullet \to X$, is bijective.

The proof of the second part of the corollary is straightforward from the first part and is omitted. $\qquad \square$

Definition 3.6. *A smooth simplicial set whose underlying simplicial set is a Kan complex will be called a* **smooth Kan complex***.*

Example 3.7. *Let Y be a smooth d-fold. And set $X = Y_\bullet$ to be the simplicial set of smooth simplices in X, i.e. smooth maps $\Sigma : \Delta^k \to Y$. Then X is naturally a smooth Kan complex, analogously to Example 3.3. More generally we may take Y to be a manifold with boundary or with corners, an orbifold or any diffeological space.*

Example 3.8. *One special example is worth attention. Let M be a smooth manifold. Then there is a natural smooth Kan complex LM^Δ whose d-simplices Σ are smooth maps $f_\Sigma : \Delta^d \times S^1 \to M$. The maps Σ_* are defined by*

$$\Sigma_*(\sigma) = f_\Sigma \circ (\sigma \times id),$$

for

$$\sigma \times id : \Delta^d \times S^1 \to \Delta^d \times S^1.$$

This LM^Δ can be understand as a smooth Kan complex model of the free loop space. Naturally the free loop space LM also has the structure of a Frechet manifold, in particular we have the smooth Kan complex $LM_\bullet$, whose n-simplices are Frechet smooth maps $\Sigma : \Delta^n \to LM$. There is a natural simplicial map $LM^\Delta \to LM_\bullet$, which is clearly smooth, and is an isomorphism on homotopy groups, and so is a homotopy equivalence of these Kan complexes.

3.1. **Products.** Given a pair of smooth simplicial sets $(X_1, g_1), (X_2, g_2)$, the product $X_1 \times X_2$ of the underlying simplicial sets, has the structure of a smooth simplicial set

$$(X_1 \times X_2, g_1 \times g_2),$$

constructed as follows. Denote by $\pi_i : X_1 \times X_2 \to X_i$ the simplicial projection maps. Then for each $\Sigma \in X_1 \times X_2(d)$,

$$g_1 \times g_2(\Sigma) : \Delta^d_\bullet \to X_1 \times X_2$$

is defined by:

$$g_1 \times g_2(\Sigma)(\sigma) := (g_1(\pi_1(\Sigma))(\sigma), g_2(\pi_2(\Sigma))(\sigma)).$$

Definition 3.9. *Let X, Y be smooth simplicial sets. Set $I := \Delta^1_\bullet$ and let $0_\bullet, 1_\bullet \subset I$ be the images of the pair of inclusions $\Delta^0_\bullet \to I$ corresponding to the pair of endpoints. A pair of smooth maps $f, g : X \to Y$ are called* **smoothly homotopic** *if there exists a smooth map*

$$H : X \times I \to Y$$

such that $H|_{X \times 0_\bullet} = f$ and $H|_{X \times 1_\bullet} = g$.

The following notion will be useful later on.

Definition 3.10. *Let X be a smooth simplicial set. We define $\pi_k^{sm}(X)$ to be the set of smooth homotopy equivalence classes of smooth maps $f : S^k_\bullet \to X$.*

4. Differential forms on smooth simplicial sets and DeRham theorem

Definition 4.1. *Let X be a smooth simplicial set. A **simplicial differential k-form** ω, or just differential form where there is no possibility of confusion, is for each d-simplex Σ of X a smooth differential k-form ω_Σ on Δ^d, such that*

$$(4.1) \qquad i^*\Omega_{\Sigma_2} = \Omega_{\Sigma_1},$$

for every morphism $i : \Sigma_1 \to \Sigma_2$ in $Simp(X)$, (see Section 2). If in addition:

$$(4.2) \qquad \omega_{g(\Sigma)(\sigma)} = \sigma^*\omega_\Sigma,$$

*for every $\sigma \in \Delta^d_\bullet$, and every d-simplex Σ, then we say that ω is **coherent**.*

Example 4.2. *If $X = Y_\bullet$ for Y a smooth d-fold, and if ω is a differential k-form on Y, then $\{\omega_\Sigma = \Sigma^*\omega\}_\Sigma$ is a coherent differential k-form on X called the **induced simplicial differential form**.*

Example 4.3. *Let LM^Δ be the smooth Kan complex of Example 3.8. Then Chen's iterated integrals [1] naturally give coherent differential forms on LM^Δ.*

The above coherence condition is often unnecessary, hence is not part of the basic definition here. This is already one difference with differential forms on diffeological or Chen spaces, where coherence is ostensibly forced.

Let X be a smooth simplicial set. We denote by $\Omega^k(X)$ the $\mathbb{R}$-vector space of differential k-forms on X. Define

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

by

$$d(\{\omega_\Sigma\}) := d\{\omega_\Sigma\} = \{d\omega_\Sigma\}.$$

Clearly we have

$$d^2 = 0.$$

A form ω is said to be **closed** if $d\omega = 0$, and **exact** if for some $k-1$ form η, $\omega = d\eta$.

Definition 4.4. *The **wedge product** on*

$$\Omega^\bullet(X) = \bigoplus_{k \geq 0} \Omega^k(X)$$

is defined by

$$\omega \wedge \eta = \{\omega_\Sigma \wedge \eta_\Sigma\}_\Sigma.$$

Then $\Omega^\bullet(X)$ has the structure of a differential graded $\mathbb{R}$-algebra with respect to $\wedge$.

We then, as usual, define the **De Rham cohomology** of X:

$$H^k_{DR}(X) = \frac{\text{closed k-forms}}{\text{exact k-forms}},$$

which is a graded commutative $\mathbb{R}$-algebra.

The simplicial De Rham complex above is certainly not a new invention, versions of it have been used by Whitney and perhaps most famously by Sullivan [13].

4.1. **Homology and cohomology of a smooth simplicial set.** For a smooth simplicial set X, we define an abelian group

$$C_k(X, \mathbb{Z})$$

as for a general simplicial set, as the free abelian group generated by k-simplices of X. Elements of $C_k(X, \mathbb{Z})$ are called k-***chains***. The boundary operator:

$$\partial : C_k(X, \mathbb{Z}) \to C_{k-1}(X, \mathbb{Z}),$$

is defined on a k-simplex σ as classically by

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i d_i \sigma,$$

where d_i are the face maps, this is then extended by linearity to general chains. As classically, $\partial^2 = 0$. The homology of this complex is denoted by $H_k(X, \mathbb{Z})$, called integral homology. The integral cohomology is defined analogously to the classical topology setting, using dual chain groups $C^k(X, \mathbb{Z}) = hom(C_k(X, \mathbb{Z}), \mathbb{Z})$. The corresponding coboundary operator is denoted by d as usual:

$$d : C^k(X, \mathbb{Z}) \to C^{k+1}(X, \mathbb{Z}).$$

Homology and cohomology with other ring coefficients (or modules) are likewise defined analogously.

4.2. **Integration.** Let X be a smooth simplicial set. Given a chain

$$\sigma = \sum_{i} a_i \Sigma_i \in C_k(X, \mathbb{Z})$$

and a smooth differential form ω, we define:

$$\int_\sigma \omega = \sum_i a_i \int_{\Delta^k} \omega_{\Sigma_i}$$

where the integrals on the right are the classical integrals of a differential form. Thus we obtain a homomorphism:

$$\int : \Omega^k(X) \to C^k(X, \mathbb{R}),$$

$\int(\omega)$ is the k-cochain defined by:

$$\int(\omega)(\sigma) := \int_\sigma \omega,$$

where σ is a k-chain. We will abbreviate $\int(\omega) = \int \omega$.

Lemma 4.5. *For a smooth simplicial set X, the homomorphism $\int$ commutes with d, and so induces a homomorphism:*

$$\int : H^k_{DR}(X) \to H^k(X, \mathbb{R}).$$

Proof. We need that

$$\int d\omega = d \int \omega.$$

Let $\Sigma : \Delta^k_{simp} \to X$ be a k-simplex. Then

$$\int d\omega(\Sigma) = \int_{\Delta^k} d\omega_\Sigma \text{ by definition}$$

$$= \int_{\partial \Delta^k} \omega_\Sigma \text{ by Stokes theorem}$$

$$= d(\int \omega)(\Sigma) \text{ by the definition of } d \text{ on co-chains.}$$

$\square$

4.3. Pull-back. Given a smooth map $f : X_1 \to X_2$ of smooth simplicial sets, we define

$$f^* : \Omega^k(X_2) \to \Omega^k(X_1)$$

naturally by

$$f^*\omega = \{(f^*\omega)_\Sigma\}_\Sigma := \{\omega_{f(\Sigma)}\}_\Sigma.$$

Clearly f^* commutes with d so that we have an induced differential graded $\mathbb{R}$-algebra homomorphism:

$$f^* : \Omega^\bullet(X_2) \to \Omega^\bullet(X_1).$$

And in particular an induced $\mathbb{R}$-algebra homomorphism:

$$f^* : H^\bullet_{DR}(X_2) \to H^\bullet_{DR}(X_1).$$

4.4. Relation with ordinary homology and cohomology. Let $s - Set$ denote the category of simplicial sets and Top the category of topological spaces. Let

$$|\cdot| : s - Set \to Top$$

be the geometric realization functor as defined in Section 2.1. Let X be a (smooth) simplicial set. Then for any ring K we have natural chain maps

$$CR_* : C_\bullet(X, K) \to C_\bullet(|X|, K),$$

$$CR^* : C^\bullet(|X|, K) \to C^\bullet(X, K),$$

as a d-simplex $\Sigma : \Delta^d_{simp} \to X$, by construction of $|X|$ uniquely induces a map $\Delta^d \to |X|$, which is set to be $CR_*(\Sigma)$. When X is a Kan complex, the natural map $X \to |X|_\bullet$ is a simplicial homotopy equivalence. Here, given a topological space Y, for the moment the notation $Y_\bullet$ means the simplicial set of continuous simplices: $\Delta^d \to Y$. (Elsewhere it usually refers to the simplicial set of smooth simplices.)

And so the induced maps in homology/cohomology:

(4.3) $$R_* : H_\bullet(X, K) \to H_\bullet(|X|_\bullet, K) = H_\bullet(|X|, K),$$

(4.4) $$R^* : H^\bullet(|X|, K) = H^\bullet(|X|_\bullet, K) \to H^\bullet(X, K),$$

are isomorphisms. The following will be used later on.

Notation 4.6. *Given a Kan complex X and a cohomology class $c \in H^\bullet(X, K)$, we will denote by $|c| \in H^\bullet(|X|, K)$ the class $(R^*)^{-1}(c)$.*

5. SIMPLICIAL G-BUNDLES AND CONNECTIONS

Part of our motivation is the construction of the universal Chern-Weil homomorphisms for Frechet Lie groups. A *Frechet Lie group G* is Lie group whose underlying manifold is a possibly infinite dimensional smooth manifold locally modelled on a Frechet space, that is a locally convex, complete Hausdorff vector space. Later on it will also be important that G have the homotopy type of a CW complex. By Milnor [8], a prototypical example of such a Lie group is the group of diffeomorphisms $\text{Diff}(M)$ of a smooth manifold. Another very interesting example for us is the group of Hamiltonian symplectomorphisms $Ham(M, \omega)$ of a symplectic manifold. Particularly because its Lie algebra admits natural bi-invariant polynomials, so that it is possible to define interesting Chern-Weil theory for this group.

We now introduce the basic building blocks for simplicial G-bundles.

Definition 5.1. *Let G be a Frechet Lie group. A **smooth G-bundle** P over Δ^n is a topological principal G-bundle over $\Delta^n \subset \mathbb{R}^n$, together with a choice of an open neighborhood V of Δ^n in $\mathbb{R}^n$ and a choice of a smooth G-bundle $\widetilde{P} \to V$ such that $i^* \widetilde{P} = P$ for $i : \Delta^n \to V$ the inclusion map. We may thus write $(P, \widetilde{P}, V)$ for this structure, but often this is abbreviated as just P, or $P \to \Delta^n$ when the specifics of $V, \widetilde{P}$ are not needed.*

Definition 5.2. *Let $(P_1, \widetilde{P}_1, V_1)$, $(P_2, \widetilde{P}_2, V_2)$ be smooth G-bundles over Δ^n. A **smooth G-bundle map** of tuples $(P_1, \widetilde{P}_1, V_1)$, $(P_2, \widetilde{P}_2, V_2)$, is a topological G-bundle map*

$$f : P_1 \to P_2,$$

*over the $id : \Delta^n \to \Delta^n$ with the following property. There is an open $U \subset V_1 \cap V_2$ and a smooth G-bundle map $\widetilde{f} : \widetilde{P}_1|_U \to \widetilde{P}_2|_U$ extending f. If f has an inverse which is also a smooth G-bundle map, then f is said to be an **isomorphism** of the structures $(P_1, \widetilde{P}_1, V_1)$, $(P_2, \widetilde{P}_2, V_2)$.*

If $P \to \Delta^d$ is a smooth G-bundle, given a smooth map $\sigma : \Delta^k \to \Delta^d$, clearly the pull-back $\sigma^* P$ has a smooth G-bundle structure uniquely determined up to natural G-bundle isomorphism.

To warn, at this point our terminology may partially clash with common terminology, in particular a simplicial G-bundle will *not* be a pre-sheaf on Δ with values in the category of smooth G-bundles. It will instead be a direct analogue of a simplicial differential form. The latter pre-sheafs will not appear in the paper so that this should not cause confusion.

In the definition of simplicial differential forms we omitted coherence. In the case of simplicial G-bundles, the analogous condition turns out to be necessary if we want universal simplicial G-bundles with expected behavior.

Definition 5.3. *Let G be a Frechet Lie group and X a smooth simplicial set. A **smooth simplicial G-bundle** is a data consisting of:*

- *For every d-simplex, $d \geq 0$, Σ of X, a smooth G-bundle:*

$$P_\Sigma \to \Delta^d.$$

- *The assignment $\Sigma \mapsto P_\Sigma$ is coherent, meaning that for all d, Σ as above if $\sigma \in \Delta_\bullet^d(k)$, then*

$$\sigma^* P_\Sigma = P_{\Sigma_*(\sigma)},$$

where the equality means the following. $\sigma^* P_\Sigma = P_{\Sigma_*(\sigma)}$ as sets and the identity map $id : \sigma^* P_\Sigma \to P_{\Sigma_*(\sigma)}$ is a smooth G-bundle isomorphism.

We will only deal with smooth simplicial G-bundles, and so will usually just say **simplicial G-bundle**, omitting the qualifier 'smooth'.

Example 5.4. *If X is a smooth simplicial set and G is as above, we denote by $X \times G$ the simplicial G-bundle,*

$$\forall n \in \mathbb{N}, \forall \Sigma^n \in \Delta(X) : (X \times G)_{\Sigma^n} = \Delta^n \times G,$$

*with $\Delta^n \times G \to \Delta^n$ the trivial projection. This is called the **trivial simplicial G-bundle over** X.*

Example 5.5. *Let $N \to Y$ be a smooth G-bundle over a smooth manifold Y. Then we have a simplicial G-bundle $N_\bullet$ over $Y_\bullet$ defined by*

$$N_\bullet(\Sigma) = \Sigma^* N.$$

Definition 5.6. *If $X = Y_\bullet$ for Y a smooth manifold, we say that a simplicial G-bundle P over X is **induced by a smooth G-bundle** $N \to Y$ if there is a simplicial G-bundle isomorphism $N_\bullet \to P$.*

Definition 5.7. *Let $P_1 \to X_1, P_2 \to X_2$ be a pair of simplicial G-bundles. A **smooth simplicial G-bundle map** or just simplicial G-bundle map, is a smooth map*

$$h : X_1 \to X_2$$

and for each $\Sigma \in X_1$ a smooth G-bundle map:

$$\widetilde{h}_\Sigma : P_{1,\Sigma} \to P_{2,h(\Sigma)}.$$

*These are required to have the following **coherence** condition. If Σ is a d-simplex and $\sigma \in \Delta_\bullet^d(k)$ then we have the cone map in the corresponding pull-back diagram:*

$$u : (P_{1,\Sigma_*\sigma} = \sigma^* P_{1,\Sigma}) \to P_{1,\Sigma}.$$

Now $\widetilde{h}_\Sigma \circ u$ is a smooth G-bundle map

$$P_{1,\Sigma_*\sigma} \to P_{2,h(\Sigma)},$$

which is over the map $\sigma : \Delta^k \to \Delta^d$. Hence, by universality of the pull-back there is a uniquely induced G-bundle map

$$k : P_{1,\Sigma_*\sigma} \to (\sigma^* P_{2,h(\Sigma)} = P_{2,h(\Sigma_*(\sigma))}),$$

and as the final property we require that

$$k = \widetilde{h}_{\Sigma_*(\sigma)}.$$

Definition 5.8. *Let P_1, P_2 be simplicial G-bundles over X_1, X_2 respectively. A **simplicial G-bundle isomorphism** is a simplicial G-bundle map*

$$\widetilde{h} : P_1 \to P_2$$

s.t. there is a simplicial G-bundle map

$$\widetilde{h}^{-1} : P_2 \to P_1$$

with

$$\widetilde{h}^{-1} \circ \widetilde{h} = id.$$

This is clearly the same as asking that each

$$\widetilde{h}_\Sigma : (P_1)_\Sigma \to (P_2)_\Sigma$$

be a G-bundle isomorphism. Usually, $X_1 = X_2$ and in this case, unless specified otherwise, it is assumed $h = id$. A simplicial G-bundle isomorphic to the trivial simplicial G-bundle is called **trivializeable***.*

The following will be one of the crucial ingredients later on.

Theorem 5.9. *Let G be as above and let $P \to Y_\bullet$ be a simplicial G-bundle, for Y a smooth d-manifold. Then P is induced by some smooth G-bundle $N \to Y$.*

Proof. We need to introduce an auxiliary notion. Let Z be a smooth d-manifold (or a simplex Δ^d for some d) and let $\mathcal{D}(Z)$ denote the category whose objects are smooth embeddings

$$\Sigma : \Delta^d \to Z$$

and morphisms f commutative diagrams:

(5.1)

$$\begin{array}{ccc} \Delta^d & \xrightarrow{\widetilde{f}} & \Delta^d \\ & {\scriptstyle \Sigma_1} \searrow & \downarrow {\scriptstyle \Sigma_2} \\ & & Z. \end{array}$$

Going back to our Y. Let $\{O_i\}_{i \in I}$ be a locally finite open cover of Y, closed under intersections, with each O_i diffeomorphic to an open ball. Such a cover is often called a good cover of a manifold. Let $\mathcal{G}$ denote the category of smooth G-bundles, with morphisms smooth G-bundle maps, which are fiberwise isomorphisms. Let $\mathcal{O}$ denote the category with the set of objects $\{O_i\}$ and with morphisms set inclusions. Set $C_i = \mathcal{D}(O_i)$. For each i, we have a functor

$$F_i : C_i \to \mathcal{G}$$

defined by $F_i(\Sigma) = P_\Sigma$.

By assumption that each O_i is diffeomorphic to an open ball, O_i has an exhaustion by embedded d-simplices. Meaning that there is a sequence of smooth embeddings $\Sigma_j : \Delta^d \to O_i$ with image$(\Sigma_{j+1}) \supset$ image(Σ_j) for each j. And so that $\bigcup_j$ image$(\Sigma_j) = O_i$. In particular, for each i, the colimit in $\mathcal{G}$:

$$P_i := \text{colim}_{C_i} F_i$$

is naturally a smooth G-bundle over O_i, with G-bundle charts defined as follows. Take the collection of maps

$$\{\phi^i_{\Sigma,j}\}_{\Sigma \in C_i, j \in J^\Sigma},$$

satisfying the following.

- Each $\phi^i_{\Sigma,j}$ is the composition map

$$V^i_{\Sigma,j} \times G \xrightarrow{\xi_{ij}} P_\Sigma \xrightarrow{c_\Sigma} P_i$$

  where $V^i_{\Sigma,j} \subset (\Delta^d)^\circ$ is open, for $(\Delta^d)^\circ$ the topological interior of the subspace $\Delta^d \subset \mathbb{R}^d$. And where c_Σ is the corresponding map in the co-cone diagram.

- The collection

$$\{\xi_{i,j}\}_{j\in J^\Sigma}$$

  forms an atlas of smooth G-bundle charts for $P_\Sigma|_{(\Delta^d)^\circ}$.

The collection $\{\phi^i_{\Sigma,j}\}$ then forms a smooth G-bundle atlas for P_i.

So we obtain a functor

$$D : \mathcal{O} \to \mathcal{G},$$

defined by

$$D(O_i) = P_i.$$

Let $t : \mathcal{O} \to Top$ denote the tautological functor, so that $Y = \mathrm{colim}_\mathcal{O}\, t$, where for simplicity we write equality for natural isomorphisms here and further on in this proof. Now,

$$N = \mathrm{colim}_\mathcal{O}\, D,$$

is naturally a topological G-bundle over $\mathrm{colim}_\mathcal{O}\, t = Y$. Let $c_i : P_i \to N$ denote the natural co-cone maps. The collection of charts $\{c_i \circ \phi^i_{\Sigma,j}\}_{i,j,\Sigma}$ forms a smooth atlas on N, giving it a structure of a smooth G-bundle.

We now prove that P is induced by N. Let Σ be a d-simplex of $X := Y_\bullet$, then $\{V_i := \Sigma^{-1}(O_i)\}_{i\in I}$ is a locally finite open cover of Δ^d closed under finite intersections. Let $N_\Sigma = \Sigma^* N$, and let N'_i likewise denote the restriction of N_Σ over V_i. Each V_i is paracompact, and so we may express V_i as a locally finite union of open subsets:

$$V_i = \cup_{j\in J} W^i_j,$$

having the following properties.

- The collection $\{W^i_j\}_{j\in J}$, is closed under intersections.
- Each W^i_j has an exhaustion by nested images of embedded simplices, that is

$$W^i_j = \bigcup_{k\in\mathbb{N}} \mathrm{image}\,\sigma^{i,j}_k$$

  for $\sigma^{i,j}_k : \Delta^d \to W^i_j$ smooth and embedded, with $\mathrm{image}\,\sigma^{i,j}_k \subset \mathrm{image}\,\sigma^{i,j}_{k+1}$ for each k.

Let C be the small category with objects $I \times J \times \mathbb{N}$, so that there is exactly one morphism from $a = (i,j,k)$ to $b = (i',j',k')$ whenever $\mathrm{image}\,\sigma^{i,j}_k \subset \mathrm{image}\,\sigma^{i',j'}_{k'}$, and no morphisms otherwise. Let

$$F : C \to \mathcal{D}(\Delta^d)$$

be the functor $F(a) = \sigma^{i,j}_k$ for $a = (i,j,k)$. For brevity, we then reset $\sigma_a := F(a)$.

If $\mathcal{O}(Y)$ denotes the category of topological subspaces of Y with morphisms inclusions. Then there is a forgetful functor

$$T : \mathcal{D}(Y) \to \mathcal{O}(Y)$$

which takes f to $\mathrm{image}(\widetilde{f})$. With all this in place, we obviously have a colimit in Top:

$$\Delta^d = \mathrm{colim}_C\, T \circ F,$$

Now by construction, for each $a \in C$ we may express:

(5.2) $$\Sigma \circ \sigma_a = \Sigma_a \circ \sigma_a,$$

for some i and some $\Sigma_a : \Delta^d \to U_i \subset Y$ a smooth embedded d-simplex. Then for all $a \in C$:

$$(5.3) \qquad\qquad \sigma_a^* P_\Sigma = \sigma_a^* P_{\Sigma_a} \quad \text{by coherence}$$

$$(5.4) \qquad\qquad\qquad = \sigma_a^* N_{\Sigma_a} \quad \text{by construction of } N$$

$$(5.5) \qquad\qquad\qquad = \sigma_a^* N_\Sigma \quad \text{by (5.2)}.$$

The second equality above follows since by construction of N, there is the natural co-cone map $P_{\Sigma_a} \to N$ over Σ_a. And so there is the induced universal map

$$N_{\Sigma_a} \to P_{\Sigma_a},$$

which is a G-bundle isomorphism.

Now we have a natural functor $F_\Sigma : \mathcal{D}(\Delta^d) \to \mathcal{G}$, given by $F_\Sigma(\sigma) = \sigma^* P_\Sigma$, and

$$(5.6) \qquad\qquad\qquad P_\Sigma = \mathrm{colim}_C\, F_\Sigma \circ F.$$

Similarly,

$$(5.7) \qquad\qquad\qquad N_\Sigma = \mathrm{colim}_C\, F'_\Sigma \circ F$$

where $F'(\sigma) = \sigma^* N_\Sigma$. Now (5.5) gives that

$$F'_\Sigma \circ F = F_\Sigma \circ F.$$

Then the conjunction of this with (5.6), (5.7) yields that

$$P_\Sigma = N_\Sigma.$$

More explicitly, for every simplex Σ of X, we have constructed an isomorphism $h_\Sigma : P_\Sigma \to N_\Sigma$ natural in Σ. By construction $h = \{h_\Sigma\}$ will satisfy the coherence property of the Definition 5.7, so that h is a G-bundle isomorphism. $\qquad\square$

5.1. Connections.

Definition 5.10. *Let G be a Frechet Lie group. A **simplicial G-connection** D on a simplicial G-bundle P over a smooth simplicial set X is for each d-simplex Σ of X, a smooth G-invariant Ehresmann G-connection D_Σ on P_Σ. This data is required to satisfy: if $f : \Sigma_1 \to \Sigma_2$ is a morphism in $\Delta(X)$ then*

$$\widetilde{f}^* D_{\Sigma_2} = D_{\Sigma_1},$$

*keeping in mind our conventions as in Notation 2.1. We say that D is **coherent** if for any d-simplex Σ,*

$$\sigma^* D_\Sigma = D_{\Sigma_*(\sigma)},$$

for all $\sigma \in \Delta_\bullet^d$. Will will often just say G-connection instead of simplicial G-connection, where there is no need to disambiguate.

As with differential forms the coherence condition is very restrictive, and is not part of the basic definition.

Lemma 5.11. *G-connections on simplicial G-bundles exist and any pair of G-connections D_1, D_2 on a simplicial G-bundle P are **concordant**. The latter means that there is a G-connection on $\widetilde{D}$ on $P \times I$,*

$$I := [0,1]_\bullet$$

which restricts to D_1, D_2 on $P \times I_0$, respectively on $P \times I_1$, for $I_0, I_1 \subset I$ denoting the images of the two end point inclusions $\Delta_\bullet^0 \to I$.

Proof. If $\Sigma : \Delta_{simp}^d \to X$ is a degeneracy of a 0-simplex $\Sigma_0 : \Delta_{simp}^0 \to X$, then $P_\Sigma = \Delta^d \times P_{\Sigma_0}$ and we fix the corresponding trivial connection D_Σ on P_Σ. We then proceed inductively. If Σ is a non-degenerate 1-simplex, fix D_Σ arbitrarily. This defines D_Σ for all 1-simplices. If Σ' is a m-simplex that is a degeneracy of a 1-simplex Σ^1, then $P_{\Sigma'} = pr^*P_{\Sigma^1}$ for a certain determined simplicial projection $pr : \Delta^m \to \Delta^1$, and we define $D_{\Sigma'} = pr^*D_{\Sigma^1}$. If Σ is a non-degenerate 2-simplex, then D_Σ is already determined on the boundary of Δ^2, then let D_Σ over all Δ^2 to be any extension. If Σ' is a m-simplex that is a degeneracy of a 2-simplex Σ^2, then $P_{\Sigma'} = pr^*P_{\Sigma^2}$ for a certain determined simplicial projection $pr : \Delta^m \to \Delta^2$, and we define $D_\Sigma = pr^*D_{\Sigma^2}$.

Suppose we have constructed connections D_Σ for all degeneracies of n-simplices, $n \geq 2$. We extend to all degeneracies of $(n + 1)$-simplices as follows. If Σ is a non-degenerate $(n + 1)$-simplex then D_Σ is already determined over the boundary of Δ^{n+1}, extend D_Σ over all of Δ^{n+1} arbitrarily. Finally, if Σ' is a m-simplex that is a degeneracy of a $(n + 1)$-simplex Σ^{n+1}, then $P_{\Sigma'} = pr^*P_{\Sigma^{n+1}}$ for a certain determined simplicial projection $pr : \Delta^m \to \Delta^{n+1}$, and we define $D_\Sigma = pr^*D_{\Sigma^{n+1}}$.

The second part of the lemma follows by an analogous argument, since we may just extend D_1, D_2 to a concordance connection $\widetilde{D}$, using the above inductive procedure.

$\square$

Example 5.12. *Given a classical smooth G-connection D on a smooth principal G-bundle $Z \to Y$, we obviously get a simplicial G-connection on the induced simplicial G-bundle $Z_\bullet$, simply by defining $D_\Sigma = \Sigma^*D$, for $\Sigma : \Delta^d \to Y$ a d-simplex. This is called the* **induced simplicial connection***. Going in the other direction is always possible if the given simplicial G-connection in addition satisfies coherence, but we will not elaborate.*

6. Chern-Weil homomorphism

Let G be a Frechet Lie group, and let $\mathfrak{g}$ denote its Lie algebra. Let P be a simplicial G-bundle over X. Fix a G-connection D on P. Let $\operatorname{Aut} P_z$ denote the group of smooth G-torsor automorphisms of the fiber P_z of P over $z \in \Delta^d$. Note that $\operatorname{Aut} P_z \simeq G$ where $\simeq$ means non-canonically isomorphic. Then for each simplex Σ^d, we have the classical curvature 2-form R_Σ^D of the connection D_Σ on P_Σ. This is interpreted as a 2-form R_Σ^D on Δ^d so that:

$$\forall v, w \in T_z\Delta^d : R_\Sigma^D(v, w) \in \operatorname{lie} \operatorname{Aut} P_z,$$

with $\operatorname{lie} \operatorname{Aut} P_z$ denoting the Lie algebra of $\operatorname{Aut} P_z$.

Now, let ρ be a symmetric multi-linear functional:

$$\rho : (V = \prod_{i=1}^{i=k} \mathfrak{g}) \to \mathbb{R},$$

satisfying

$$\forall g \in G, \forall v \in V : \rho(Ad_g(v)) = \rho(v).$$

Here if $v = (\xi_1, \ldots, \xi_n)$, $Ad_g(v) = (Ad_g(\xi_1), \ldots, Ad_g(\xi_n))$ is the adjoint action by the element $g \in G$. As ρ is Ad invariant, it uniquely determines for each $z \in \Delta^d$ a

symmetric multi-linear map with the same name:

$$\rho : (V = \prod_{i=1}^{i=k} \text{lie Aut } P_z) \to \mathbb{R},$$

by fixing any Lie-group isomorphism $Aut P_z \to G$. We may now define a closed (simplicial) $\mathbb{R}$-valued $2k$-form $\omega^{\rho,D}$ on X:

$$\omega_\Sigma^{\rho,D}(v_1,\ldots,v_{2k}) = \frac{1}{2k!} \sum_{\eta \in P_{2k}} \text{sign } \eta \cdot \rho(R_\Sigma^D(v_{\eta(1)}, v_{\eta(2)}),\ldots, R_\Sigma^D(v_{\eta(2k-1)}, v_{\eta_{2k}})),$$

for P_{2k} the permutation group of a set with $2k$ elements. Set

$$\alpha^{\rho,D} := \int \omega^{\rho,D}.$$

Lemma 6.1. *For $P \to X$ as above*

$$[\alpha^{\rho,D}] = [\alpha^{\rho,D'}] \in H^{2k}(X,\mathbb{R}),$$

for any pair of G-connections D, D' on P.

Proof. For D, D' as in the statement, fix a concordance G-connection $\widetilde{D}$, between D, D', on the G-bundle $P \times I \to X \times I$, as in Lemma 5.11. Then $\alpha^{\rho,\widetilde{D}}$ is a closed $2k$ cocycle on $X \times I$ restricting to $\alpha^{\rho,D}, \alpha^{\rho,D'}$ on $X \times I_0, X \times I_1$. Since the pair of inclusions

$$i_j : X \to X \times I \quad j = 0, 1$$

corresponding to the end points of I are homotopic, $\alpha^{\rho,D}$, $\alpha^{\rho,D'}$ are cohomologous cocycles. $\square$

Then we define the associated Chern-Weil characteristic class:

$$c^\rho(P) = [\alpha(D)] \in H^{2k}(X,\mathbb{R}).$$

We have the expected naturality:

Lemma 6.2. *Let P be a simplicial G-bundle over Y, ρ as above and $f : X \to Y$ a smooth simplicial map. Then*

$$f^* c^\rho(P) = c^\rho(f^* P).$$

Proof. Let D be a simplicial G-connection on P then $f^* D$ is a simplicial G-connection on $f^* P$ and clearly $\omega^{\rho,f^*D} = f^* \omega^{\rho,D}$, so that passing to cohomology we obtain our result. $\square$

Proposition 6.3. *For $G \hookrightarrow Z \to Y$ a smooth principal G-bundle, and ρ as above, the classes $c_{2k}^\rho(Z_\bullet) \in H^{2k}(Y_\bullet,\mathbb{R})$ coincide with the classical Chern-Weil classes of Z. More explicitly, if $c_{2k}^\rho(Z) \in H^{2k}(Y,\mathbb{R})$ is the classical Chern-Weil characteristic class, then*

(6.1) $$|c_{2k}^\rho(Z_\bullet)| = c_{2k}^\rho(Z),$$

where $|c_{2k}^\rho(Z_\bullet)|$ is as in Notation 4.6

Proof. Fix a smooth G-connection D' on Z. This induces a simplicial G-connection D on $Z_\bullet$, as in Example 5.12. Then by construction $\rho(D)$ is the simplicial differential form induced by $\rho(D')$, where induced is as in Example 4.2. Consequently,

$$R^*([\rho(D')]) = [\rho(D)],$$

where R^* is as in (4.3), and where $[\rho(D')]$ is identified with a cohomology class of $|Y_\bullet|$ via the homotopy equivalence $|Y_\bullet| \simeq Y$. So

$$[\rho(D')] = (R^*)^{-1}([\rho(D)]) := |[\rho(D)]|.$$

$\square$

7. The universal simplicial G-bundle

Briefly, a Grothendieck universe is a set $\mathcal{U}$ forming a model for set theory. That is if we interpret all terms of set theory as elements of $\mathcal{U}$, then all the set theoretic constructions keep us within $\mathcal{U}$. We will assume Grothendieck's axiom of universes which says that for any set X there is a Grothendieck universe $\mathcal{U} \ni X$. Intuitively, such a universe $\mathcal{U}$ is formed by from all possible set theoretic constructions starting with X. For example if $\mathcal{P}(X)$ denotes the power set of X, then $\mathcal{P}(X) \in \mathcal{U}$ and if $\{Y_i \in \mathcal{P}(X)\}_{i \in I}$ for $I \in \mathcal{U}$ is a collection then $\bigcup_i Y_i \in \mathcal{U}$. We should note that this axiom is beyond ZFC, but is now a common axiom of modern set theory, especially in the context of category theory, c.f. [5]. In some contexts one works with universes implicitly. This is impossible here, as a priori our constructions are universe dependent. They become universe independent once we pass to the homotopy world.

Let G be a Frechet Lie group. Let $\mathcal{U}$ be a Grothendieck universe satisfying:

$$\mathcal{U} \ni \{G\}, \quad \forall n \in \mathbb{N} : \mathcal{U} \ni \{\Delta^n\},$$

where Δ^n are the usual topological n-simplices. These conditions are of course partly redundant by intent. Such a $\mathcal{U}$ will be called G-***admissible***. We construct smooth Kan complexes $BG^{\mathcal{U}}$ for each G-admissible $\mathcal{U}$. The homotopy type of $BG^{\mathcal{U}}$ will then be shown to be independent of $\mathcal{U}$. Moreover we will show that $BG^{\mathcal{U}} \simeq BG$, for BG the classical classifying space, provided G has the homotopy type of a CW complex.

Definition 7.1. *A $\mathcal{U}$-small set is an element of $\mathcal{U}$. For X a smooth simplicial set, a smooth simplicial G-bundle $P \to X$ will be called $\mathcal{U}$-small if for each simplex Σ of X the bundles P_Σ are $\mathcal{U}$-small.*

7.1. **The classifying spaces $BG^{\mathcal{U}}$.** Let $\mathcal{U}$ be G-admissible. We define a simplicial set $BG^{\mathcal{U}}$, whose set of k-simplices $BG^{\mathcal{U}}(k)$ is the set of $\mathcal{U}$-small smooth G-bundles over Δ^k. For a given k-simplex Σ of $BG^{\mathcal{U}}$ we denote by $S(\Sigma)$ the associated G-bundle over Δ^k. Given a map $i \in hom_\Delta([m], [n])$ the map

$$BG^{\mathcal{U}}(i) : BG^{\mathcal{U}}(n) \to BG^{\mathcal{U}}(m)$$

is just the natural pull-back map:

$$BG^{\mathcal{U}}(i)(\Sigma) := i^* S(\Sigma),$$

for i, understood as the simplicial map $i : \Delta^m \to \Delta^n$.

This defines a simplicial set $BG^{\mathcal{U}}$. By construction we also obtain a functor

$$S : \Delta(BG^{\mathcal{U}}) \to \mathcal{G},$$

where $\mathcal{G}$ is the previously defined category of G-bundles, and where S sends a simplex $\Sigma : \Delta^d_{simp} \to BG^{\mathcal{U}}$ to the smooth G-bundle $S(\Sigma)$ over Δ^d. We define a smooth simplicial set structure on $BG^{\mathcal{U}}$ as follows. Given a d-simplex Σ the induced map

$$\Sigma_* : \Delta^d_\bullet \to BG^{\mathcal{U}},$$

is defined naturally by

$$\Sigma_*(\sigma) := \sigma^* S(\Sigma).$$

In what follows V denotes $BG^{\mathcal{U}}$ for a general unspecified G-admissible $\mathcal{U}$.

Proposition 7.2. *V is a Kan complex.*

Proof. Let

$$S : \Delta(V) \to \mathcal{G}$$

be the functor as previously. Recall that $\Lambda_k^n \subset \Delta_{simp}^n$, denotes the sub-simplicial set corresponding to the "boundary" of Δ^n with the k'th face removed, where by k'th face we mean the face opposite to k'th vertex. Let $h : \Lambda_k^n \to V$, $0 \le k \le n$, be a simplicial map, this is also called a horn. We need to construct an extension of h to Δ_{simp}^n. For simplicity we assume $n = 2$, the general case is identical. Let

$$\Delta(h) : \Delta(\Lambda_k^n) \to \Delta(V)$$

be the induced functor. Set $P = S \circ \Delta(h)$.

Lemma 7.3. *There is a natural transformation of $\mathcal{G}$ valued functors $T \to P$, where T is the trivial functor $T : \Delta(\Lambda_k^n) \to \mathcal{G}$, $T(\sigma^d) = \Delta^d \times G$.*

Proof. Set $L := \Lambda_k^2$, with $k = 1$, again without loss of generality. There are three natural inclusions

$$i_j : \Delta_{simp}^0 \to L,$$

$j = 0, 1, 2$, with i_1 corresponding to the inclusion of the horn vertex. The corresponding 0-simplices will just be denoted by $0, 1, 2$. Fix a G-bundle map (in this case just smooth G-torsor map):

$$\phi_1 : \Delta^0 \times G \to P(i_1),$$

after taking any Lie group isomorphism $\phi : G \to S(x_0)$. Let

$$\sigma_{1,2} : \Delta_{simp}^1 \to L$$

be the edge between vertexes $1, 2$, that is $\sigma_{1,2}(0) = 1$, $\sigma_{1,2}(1) = 2$. . $P(\sigma_{1,2})$ is a smooth bundle over the contractible space Δ^1 and so we may find a G-bundle map

$$\phi_{1,2} : \Delta^1 \times G \to P(\sigma_{1,2}),$$

whose restriction to $\{0\} \times G$ is ϕ_1. Meaning:

$$\phi_{1,2} \circ (i_0 \times id) = \phi_1,$$

where

$$i_0 : \Delta^0 \to \Delta^1,$$

is the map $i_0(0) = 0$.

We may likewise construct a G-bundle map

$$\phi_{0,1} : \Delta^1 \times G \to P(\sigma_{0,1}),$$

(defined analogously to $\sigma_{1,2}$), whose restriction to $\{1\} \times G$ is ϕ_1.

Then $\phi_{0,1}, \phi_{1,2}$ obviously glue to a natural transformation:

$$tr : T \to P.$$

$\square$

We have the trivial extension of T,

$$T^{ext} : \Delta(\Delta^2_{simp}) \to \mathcal{G},$$

defined by

$$T^{ext}(\sigma^d) = \Delta^d \times G.$$

And so it is clear that P likewise has an extension $\widetilde{P}$ to $\Delta(\Delta^2_{simp})$, but we need this extension to be $\mathcal{U}$-small so that we must be explicit. Let σ^2 denote the non-degenerate 2-simplex of Δ^2. We need to construct $\widetilde{P}_{\sigma^2} := \widetilde{P}(\sigma^2)$. Let

$$\sigma_{0,1}, \sigma_{1,2} : \Delta^1 \to \Delta^2$$

be the edge inclusions of the edges between the vertices $0, 1$, respectively $1, 2$. And let $e_{0,1}, e_{1,2}$ denote their images.

We then define a set theoretic (for the moment no topology) G-bundle

$$\widetilde{P}_{\sigma^2} \xrightarrow{p} \Delta^2$$

by the following conditions:

$$\sigma^*_{0,1}\widetilde{P}_{\sigma^2} = P(\sigma_{0,1}),$$
$$\sigma^*_{1,2}\widetilde{P}_{\sigma^2} = P(\sigma_{1,2}),$$
$$P_{\sigma^2}|_{(\Delta^2)^\circ} = (\Delta^2)^\circ \times G,$$

where $(\Delta^2)^\circ$ denotes the topological interior of $\Delta^2 \subset \mathbb{R}^2$, and where the projection map p is natural.

We now discuss the topology. We have the smooth G-bundle maps

$$\phi^{-1}_{0,1} : P(\sigma_{0,1}) \to \Delta^2 \times G,$$
$$\phi^{-1}_{1,2} : P(\sigma_{1,2}) \to \Delta^2 \times G,$$

over $\sigma_{0,1}, \sigma_{1,2}$, as in the above Lemma. Let d_0 be any metric on $\Delta^2 \times G$ inducing the natural product topology. The topology on $\widetilde{P}_{\sigma^2}$ will be given by the d-metric topology, for d extending d_0 on $(\Delta^2)^\circ \times G \subset \widetilde{P}_{\sigma^2}$, and defined as follows. For $y_1 \in \widetilde{P}_{\sigma^2}$ with $p(y_1) \in e_{0,1}$, y_2 arbitrary, $d(y_1, y_2) = d_0(\phi^{-1}_{0,1}(y_1), y_2)$. Likewise, for $y_1 \in \widetilde{P}_{\sigma^2}$ with $p(y_1) \in e_{1,2}$, y_2 arbitrary, $d(y_1, y_2) = d_0(\phi^{-1}_{1,2}(y_1), y_2)$. This defines P_{σ^2} as a topological G-bundle over Δ^2.

There is a natural topological G-bundle trivialization

$$\xi : P_{\sigma^2} \to \Delta^2 \times G$$

defined as follows. $\xi(y) = y$ when $p(y) \in (\Delta^2)^\circ$ and $\xi(y) = \phi^{-1}_{0,1}(y)$ when $p(y) \in e_{0,1}$, $\xi(y) = \phi^{-1}_{1,2}(y)$ when $p(y) \in e_{0,2}$. We then take the smooth structure on P_{σ^2} to be the smooth structure pulled back by ξ. By construction P_{σ^2} is $\mathcal{U}$-small, as all of the constructions take place in $\mathcal{U}$. Moreover, by construction, $\sigma^*_{0,1}\widetilde{P}_{\sigma^2} = P_{\sigma_{0,1}}$, as a smooth G-bundle and $\sigma^*_{1,2}\widetilde{P}_{\sigma^2} = P_{\sigma_{1,2}}$ as a smooth G-bundle, which readily follows by the fact that the maps $\phi_{0,1}, \phi_{1,2}$ are smooth G-bundle maps. Thus we have constructed a $\mathcal{U}$-small simplicial G-bundle as needed, and we are done.

$\square$

Theorem 7.4. *Let X be a smooth simplicial set. $\mathcal{U}$-small simplicial G-bundles $P \to X$ are classified by smooth maps*

$$f_P : X \to BG^{\mathcal{U}}.$$

Specifically:

(1) *For every $\mathcal{U}$-small P there is a smooth map $f_P : X \to BG^{\mathcal{U}}$ so that*
$$f_P^* EG^{\mathcal{U}} = P$$
as simplicial G-bundles.

(2) *If P_1, P_2 are isomorphic $\mathcal{U}$-small smooth simplicial G-bundles over X then f_{P_1}, f_{P_2} are smoothly homotopic, as in Definition 3.9.*

(3) *If $X = Y_\bullet$ for Y a smooth manifold and $f, g : X \to BG^{\mathcal{U}}$ are smoothly homotopic then $P_f = f^* EG^{\mathcal{U}}, P_g = g^* EG^{\mathcal{U}}$ are isomorphic simplicial G-bundles.*

Proof. Set $V = BG^{\mathcal{U}}$, $E = EG^{\mathcal{U}}$. Let $P \to X$ be a $\mathcal{U}$-small simplicial G-bundle. Define $f_P : X \to V$ as follows. Given a d-simplex Σ of X, we have the associated $\mathcal{U}$-small, smooth G-bundle $P_\Sigma \to \Delta^d$. By the definition of V this uniquely determines a d-simplex $\alpha_{P_\Sigma} \in V$. So define
$$f_P(\Sigma) := \alpha_{P_\Sigma},$$
then clearly f_P is a simplicial map and is smooth.

We now prove the second part of the theorem. Let $\phi : P_1 \to P_2$ be an isomorphism of $\mathcal{U}$-small simplicial G-bundles over X. We construct a simplicial G-bundle $\widetilde{P}$ over $X \times I$ as follows, where $I = \Delta^1_\bullet$ as before. Let σ be a k-simplex of X. Then ϕ specifies a G-bundle diffeomorphism $\phi_\sigma : P_{1,\sigma} \to P_{2,\sigma}$ over the identity map $\Delta^k \to \Delta^k$. The mapping cylinder of ϕ_σ, M_σ, is a smooth G-bundle over $\Delta^k \times \Delta^1$. Let pr_X, pr_I be the natural projections of $X \times I$, to X respectively I. Let Σ be a d-simplex of $X \times I$. Let $\sigma_1 = pr_X \Sigma$, and $\sigma_2 = pr_I(\Sigma)$. Let $id^d : \Delta^d \to \Delta^d$ be the identity, so
$$(id^d, \sigma_2) : \Delta^d \to \Delta^d \times \Delta^1,$$
is a smooth map, where σ_2 is the corresponding smooth map $\sigma_2 : \Delta^d \to \Delta^1 = [0,1]$. We then define
$$\widetilde{P}_\Sigma := (id^d, \sigma_2)^* M_{\sigma_1},$$
which is a smooth G-bundle over Δ^d.

We need to check the coherence condition. Suppose that $\rho \in \Delta^d_\bullet(k)$, and Σ is as above. As f is a simplicial G-bundle map, by the coherence condition in the Definition 5.7 of a simplicial G-bundle map:

(7.1)
$$M_{pr_X(\Sigma_* \rho)} = (\rho, id)^* M_{pr_X(\Sigma)},$$

for $(\rho, id) : \Delta^k \times \Delta^1 \to \Delta^d \times \Delta^1$, with $id : \Delta^1 \to \Delta^1$ the identity.

So we have:
$$\begin{aligned}
\widetilde{P}_{\Sigma_*(\rho)} &= (id^k, pr_I(\Sigma_*(\rho)))^* M_{pr_X(\Sigma_*(\rho))} \text{ by definition} \\
&= (id^k, pr_I(\Sigma_*(\rho)))^* (\rho, id)^* M_{pr_X(\Sigma)} \text{ by (7.1)} \\
&= (\rho, pr_I(\Sigma_* \rho))^* M_{pr_X(\Sigma)} \\
&= \rho^* (id^d, pr_I(\Sigma))^* M_{pr_X(\Sigma)} \\
&= \rho^* \widetilde{P}_\Sigma \text{ by definition.}
\end{aligned}$$

Thus $\widetilde{P}$ is a smooth simplicial bundle. By the first part of the theorem, we have an induced smooth classifying map $f_{\widetilde{P}} : X \times I \to V$. By construction it is a homotopy between f_{P_1}, f_{P_2}.

We now prove the third part of the theorem. Suppose that $f, g : X \to V$ are smoothly homotopic, and let $H : X \times I \to V$ be the corresponding smooth homotopy. By Lemma 5.9, the bundles P_f, P_g are induced by smooth G-bundles P'_f, P'_g over Y. Now $P_H = H^* E$ is a simplicial G-bundle over $X \times I = (Y \times [0,1])_\bullet$ and hence by Lemma 5.9 P_H is also induced by a smooth G-bundle P'_H over $Y \times [0,1]$. We may clearly in addition arrange that P'_H restricts to $P'_f \sqcup P'_g$ over $Y \times \partial[0,1]$. It follows that P'_f, P'_g are smoothly concordant and hence isomorphic smooth G-bundles, and so P_f, P_g are isomorphic simplicial G-bundles.

□

We now study the dependence on a Grothendieck universe $\mathcal{U}$.

Theorem 7.5. *Let G be a Frechet Lie group having the homotopy type of a CW complex. Let $\mathcal{U}$ be a G-admissible universe, let $|BG^{\mathcal{U}}|$ denote the geometric realization of $BG^{\mathcal{U}}$ and let BG^{top} denote the classical classifying space of G as defined by the Milnor construction [7]. Then there is a homotopy equivalence*

$$e^{\mathcal{U}} : |BG^{\mathcal{U}}| \to BG^{top},$$

which is natural in the sense that if $\mathcal{U} \ni \mathcal{U}'$ then

$$(7.2) \qquad e^{\mathcal{U}'} \circ |i^{\mathcal{U},\mathcal{U}'}| \simeq e^{\mathcal{U}},$$

where $|i^{\mathcal{U},\mathcal{U}'}| : |BG^{\mathcal{U}}| \to |BG^{\mathcal{U}'}|$ is the map of geometric realizations, induced by the natural inclusion $i^{\mathcal{U},\mathcal{U}'} : BG^{\mathcal{U}} \to BG^{\mathcal{U}'}$ and where $\simeq$ is the relation of being homotopic. In particular, for G as above the homotopy type of $BG^{\mathcal{U}}$ is independent of the choice of G-admissible $\mathcal{U}$.

Proof. For $\mathcal{U}$ G-admissible let $\mathcal{U}'$ be a universe enlargement of $\mathcal{U}$, that is $\mathcal{U}'$ is a universe with $\mathcal{U}' \ni \mathcal{U}$. Set $V := BG^{\mathcal{U}}$, $V' := BG^{\mathcal{U}'}$, $E := EG^{\mathcal{U}}$, $E' := EG^{\mathcal{U}'}$. There is a natural inclusion map $i : V \to V'$, and $i^* E' = E$.

Lemma 7.6.

$$i_* : \pi_k^{sm}(V) \to \pi_k^{sm}(V')$$

is a set isomorphism for all $k \in \mathbb{N}$, where π_k^{sm} are as in Definition 3.10.

Proof. We show that i_* is injective. Let $f, g : S_\bullet^k \to V$ be a pair of smooth maps. Let P_f, P_g denote the smooth bundles over S^k induced via Lemma 5.9 by $f^* E, g^* E$. Set $f' = i \circ f$, $g' = i \circ g$ and suppose that $F : S_\bullet^k \times I \to V'$ is a smooth homotopy between f', g'. By Lemma 5.9 the simplicial bundle $F^* E'$ is induced by a smooth bundle $P_F \to S^k \times I$. In particular P_f, P_g are classically isomorphic smooth $\mathcal{U}$-small G-bundles. Taking the mapping cylinder for the corresponding G-bundle isomorphism gives us a smooth G-bundle $P' \to S^k \times I$ that is $\mathcal{U}$-small by construction. Finally, P' induces a smooth simplicial G-bundle H over $S_\bullet^k \times I$ that by construction is $\mathcal{U}$-small. The classifying map $f_H : S_\bullet^k \times I \to V$ then gives a smooth homotopy between f, g.

We now show surjectivity of i_*. Let $f : S_\bullet^k \to V'$ be smooth. By Lemma 5.9 the simplicial G-bundle $f^* E'$ is induced by a smooth G-bundle $P' \to S^k$. Any such bundle is obtained by the clutching construction, that is P' is isomorphic as a smooth G-bundle to the bundle:

$$C = D_-^k \times G \sqcup D_+^k \times G / \sim,$$

where D_+^k, D_-^k are two copies of the standard closed k-ball in $\mathbb{R}^k$, and $\sim$ is the following equivalence relation: for

$$(d, g) \in D_-^k \times G$$

$$(d, g) \sim \widetilde{f}(d, g) \in D_+^k \times G,$$

where

$$\widetilde{f} : \partial D_-^k \times G \to \partial D_+^k \times G, \quad \widetilde{f}(d, x) = (d, f(d)^{-1} \cdot x),$$

for some smooth $f : S^{k-1} \to G$.

Then C is $\mathcal{U}$-small, since this gluing construction is carried out in $\mathcal{U}$. Let

$$C_\bullet \to S_\bullet^k$$

denote the induced $\mathcal{U}$-small smooth simplicial G-bundle. Now $C_\bullet$ and f^*E' are induced by isomorphic $\mathcal{U}'$-small smooth G-bundles, hence are isomorphic $\mathcal{U}'$-small simplicial G-bundles. And so by Theorem 7.4, the classifying map $f_{C_\bullet} \to V'$ is smoothly homotopic to f.

Since $C_\bullet$ is $\mathcal{U}$-small it is also classified by a smooth map $f' : S_\bullet^k \to V$. It is immediate that $i \circ f' = f_{C_\bullet}$ and so $i_*([f']) = [f]$. □

Corollary 7.7. *Simplicial G-bundles over $S_\bullet^k$, up to smooth isomorphism, are classified by smooth homotopy classes of maps $f : S_\bullet^k \to V$. That is the mapping c_V:*

$$[f] \mapsto [P_f := f^*E]$$

is a set bijection from the set of smooth homotopy classes of maps $f : S_\bullet^k \to V$ to the set of isomorphism classes of simplicial G-bundles over $S_\bullet^k$.

Proof. c_V is well defined by the third part of Theorem 7.4. It is injective by the second part Theorem 7.4. Let P be a simplicial G-bundle over $S_\bullet^k$, then P is $\mathcal{U}'$ small for some G-admissible universe $\mathcal{U}' \supset \mathcal{U}$. So by the first part of Theorem 7.4, P is classified by some map:

$$f' : S_\bullet^k \to BG^{\mathcal{U}'}.$$

By the preceding lemma there is a smooth map $f_P : S_\bullet^k \to V$ so that $[i \circ f_P] = [f']$, where $i : V \to BG^{\mathcal{U}'}$ is the inclusion. In particular by the second part of Theorem 7.4 f_P^*E is isomorphic to P as a simplicial G-bundle. Thus c_V is surjective.

□

We now show the second part of the theorem. Set as before $V := BG^{\mathcal{U}}$, $E := EG^{\mathcal{U}}$ and set

$$|E| := \operatorname{colim}_{\Delta(V)} S$$

where $S : \Delta(V) \to \mathcal{G}$ is the previously discussed functor, and where the colimit is understood to be in the category of topological G-bundles. Let $|V|$ be the geometric realization as previously defined. Then we have a topological G-fibration

$$|E| \to |V|,$$

which is classified by

$$e = e^{\mathcal{U}} : |V| \to BG^{top},$$

uniquely determined up to homotopy. In particular, $|E| \simeq e^*EG^{top}$, where EG^{top} is the universal G-bundle over BG^{top}. We show that e induces an isomorphism of all homotopy groups. As G has the homotopy type of a CW complex, BG^{top} has the

homotopy type of a CW complex, and so e must then be a homotopy equivalence by Whitehead theorem, which will finish the proof.

Let $f : S^k \to BG^{top}$ be continuous. By Müller-Wockel [9], main result, the bundle $P_f := f^* EG^{top}$ is topologically isomorphic to a smooth G-bundle $P' \to S^k$. By the axiom of universes P' is $\mathcal{U}_0$-small for some G-admissible $\mathcal{U}_0$. So we obtain a $\mathcal{U}_0$-small simplicial G-bundle $P'_\bullet \to S^k_\bullet$.

By Lemma 7.6 $P'_\bullet \simeq g^* E$ for some

$$g : S^k_\bullet \to V,$$

where $\simeq$ is an isomorphism of simplicial G-bundles. Let $|P'_\bullet|$ denote the colimit

$$\mathrm{colim}_{\Delta(S^k_\bullet)} S,$$

where $S : \Delta(S^k_\bullet) \to \mathcal{G}$ is the functor $S(\Sigma) = (P'_\bullet)_\Sigma$, and where as before the colimit is understood to be in the category of topological G-bundles. Of course $|P'_\bullet| \simeq P'$ as topological G-bundles. Then $|P'_\bullet| \to |S^k_\bullet|$ is a topological G-bundle classified by $e \circ |g|$, for

$$|g| : |S^k_\bullet| \to |V|,$$

the naturally induced topological map.

By construction, as a topological G-bundle, P_f is isomorphic to $h^*|P'_\bullet|$, where

$$h : S^k \to |S^k_\bullet|$$

represents the generator of $\pi_k(|S^k_\bullet|)$. Thus $e \circ |g| \circ h$ represents the homotopy class $[f]$ and so c_* is surjective on homotopy groups.

We prove injectivity. Let $f : S^k \to |V|$ be continuous. Let $P \to S^k$ be a smooth G-bundle topologically isomorphic to $f^*|E|$. Again P exists by [9]. By Corollary 7.7, $P_\bullet$ is classified by a smooth map:

$$g : S^k_\bullet \to V.$$

As before we then represent the class $[f]$, by $|g| \circ h$ for $h : S^k \to |S^k_\bullet|$ as above. Now suppose that $e \circ f$ is null-homotopic. Then by [9] P is smoothly isomorphic to the trivial G-bundle. Thus by Corollary 7.7 g is smoothly null-homotopic, so that $|g|$ is null-homotopic and so $[f] = [|g| \circ h]$ is the trivial class. So e_* is injective on homotopy groups.

Finally, we show naturality. Let

$$|i^{\mathcal{U},\mathcal{U}'}| : |BG^{\mathcal{U}}| \to |BG^{\mathcal{U}'}|$$

denote the map induced by the inclusion $i^{\mathcal{U},\mathcal{U}'}$. Observe that

$$|EG^{\mathcal{U}}| \simeq |i^{\mathcal{U},\mathcal{U}'}|^* |EG^{\mathcal{U}'}|$$

tautologically and so

$$|EG^{\mathcal{U}}| \simeq |i^{\mathcal{U},\mathcal{U}'}|^* \circ (e^{\mathcal{U}'})^* EG^{top}.$$

From which the conclusion follows. $\qquad\square$

8. The universal Chern-Weil homomorphism

Let G be a Frechet Lie group. Pick any simplicial G-connection D on $EG^{\mathcal{U}} \to BG^{\mathcal{U}}$. Then given any Ad invariant symmetric multi-linear functional:

$$\rho : (V = \prod_{i=1}^{i=2k} \mathfrak{g}) \to \mathbb{R},$$

by Section 6 we obtain the Chern-Weil differential form:

$$\omega^{\rho,D} \in H_{DR}^{2k}(BG^{\mathcal{U}}),$$

whose cohomology class, denoted $c^{\rho,\mathcal{U}}$, is independent of the choice of D by Lemma 6.1. We thus first obtain an abstract form of the universal Chern-Weil homomorphism.

Proposition 8.1. *Let G be a Frechet Lie group and $\mathcal{U}$ a G-admissible Grothendieck universe. There is an algebra homomorphism:*

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG^{\mathcal{U}}, \mathbb{R}),$$

sending ρ as above to $c^{\rho,\mathcal{U}}$ and satisfying the following. Let $G \hookrightarrow Z \to Y$ be a $\mathcal{U}$-small smooth principal G-bundle. Let $c^\rho(Z_\bullet) \in H^{2k}(Y_\bullet)$ denote the Chern-Weil class associated to ρ. Then

$$f_{Z_\bullet}^* c^{\rho,\mathcal{U}} = c^\rho(Z_\bullet),$$

where $f_{Z_\bullet} : Y \to BG^{\mathcal{U}}$ is the classifying map of $Z_\bullet$.

Proof. This follows immediately by Lemma 6.2. $\qquad\square$

Suppose now that G has the homotopy type of a CW complex. Let $e^{\mathcal{U}}$ be as in Theorem 7.5. We define the associated cohomology class

$$c^\rho := e_*^{\mathcal{U}}(|c^{\rho,\mathcal{U}}|) \in H^{2k}(BG^{top}),$$

where the G-admissible universe $\mathcal{U}$ is chosen arbitrarily, where the pushforward means pull-back by the homotopy inverse, and where $|c^{\rho,\mathcal{U}}|$ is as in Notation 4.6.

Lemma 8.2. *The cohomology class c^ρ is well defined.*

Proof. Given another choice of a G-admissible universe $\mathcal{U}'$, let $\mathcal{U}'' \ni \{\mathcal{U}, \mathcal{U}'\}$ be a common universe enlargement. By Lemma 6.2

$$|i^{\mathcal{U},\mathcal{U}''}|^* |c^{\rho,\mathcal{U}''}| = |c^{\rho,\mathcal{U}}|.$$

Since $|i^{\mathcal{U},\mathcal{U}''}|$ is a homotopy equivalence we conclude that

$$|i^{\mathcal{U},\mathcal{U}''}|_* |c^{\rho,\mathcal{U}}| = |c^{\rho,\mathcal{U}''}|,$$

where $|i^{\mathcal{U},\mathcal{U}''}|_*$ denotes the pull-back by the homotopy inverse. Consequently, by the naturality part of Theorem 7.5 and the equation above, we have

$$e_*^{\mathcal{U}}(|c^{\rho,\mathcal{U}}|) = e_*^{\mathcal{U}''} \circ |i^{\mathcal{U},\mathcal{U}''}|_*(|c^{\rho,\mathcal{U}}|) = e_*^{\mathcal{U}''}(|c^{\rho,\mathcal{U}''}|).$$

In the same way we have:

$$e_*^{\mathcal{U}'}(|c^{\rho,\mathcal{U}'}|) = e_*^{\mathcal{U}''}(|c^{\rho,\mathcal{U}''}|).$$

So

$$e_*^{\mathcal{U}}(|c^{\rho,\mathcal{U}}|) = e_*^{\mathcal{U}'}(|c^{\rho,\mathcal{U}'}|),$$

and so we are done.

$$\square$$

We call $c^\rho \in H^{2k}(BG^{top})$ **the Chern-Weil characteristic class associated to** ρ.

Let $\mathbb{R}[\mathfrak{g}]$ denote the algebra of polynomial functions on $\mathfrak{g}$. And let $\mathbb{R}[\mathfrak{g}]^G$ denote the sub-algebra of fixed points by the adjoint G action. By classical algebra, degree k homogeneous elements of $\mathbb{R}[\mathfrak{g}]^G$ are in correspondence with symmetric G-invariant multi-linear functionals $\Pi_{i=1}^k \mathfrak{g} \to \mathbb{R}$. Then to summarize we have the following theorem purely about the classical classifying space BG^{top}:

Theorem 8.3. *Let G be a Frechet Lie group having the homotopy type of a CW complex. There is an algebra homomorphism:*

$$\mathbb{R}[\mathfrak{g}]^G \to H^*(BG^{top}, \mathbb{R}),$$

sending ρ as above to c^ρ as above and satisfying the following. Let $G \hookrightarrow Z \to Y$ be a smooth principal G-bundle. Let $c^\rho(Z) \in H^{2k}(Y)$ denote the classical Chern-Weil class associated to ρ. Then

$$f_Z^* c^\rho = c^\rho(Z),$$

where $f_Z : Y \to BG^{top}$ is the classifying map of the underlying topological G-bundle.

Proof. Let $\mathcal{U}_0$ be a G-admissible Grothendieck universe containing Z. By Lemma 6.2

$$c^\rho(Z_\bullet) = f_{Z_\bullet}^*(c^{\rho, \mathcal{U}_0}.)$$

And by Proposition 6.3, $|c^\rho(Z_\bullet)| = c^\rho(Z)$. So we have

$$c^\rho(Z) = |c^\rho(Z_\bullet)| = |f_{Z_\bullet}^*(c^{\rho, \mathcal{U}_0})| = |f_{Z_\bullet}|^* \circ (e^{\mathcal{U}_0})^* c^\rho,$$

with the last equality by definition of c^ρ. Now $e^{\mathcal{U}_0} \circ |f_{Z_\bullet}| \simeq f_Z$ as by construction $e^{\mathcal{U}} \circ |f_{Z_\bullet}|$ classifies the topological G-bundle Z. So that

$$c^\rho(Z) = f_Z^* c^\rho,$$

and we are done. □

In other words we have constructed the universal Chern-Weil homomorphism for Frechet Lie groups with homotopy type of CW complexes. Another, related approach to the universal Chern-Weil homomorphism is contained in the book of Dupont [2]. Dupont only states the theorem above for compact Lie groups. Like us Dupont makes heavy use of simplicial techniques, for example the simplicial DeRham complex. However, the main thrust of his argument appears to be rather different, essentially arguing that all of the necessary differential geometry can be indirectly carried out on the Milnor classifying bundle $EG \to BG$, without endowing it with extra structure, beyond the tautological structures inherent in the Milnor construction. On the other hand we need the extra structure of a smooth simplicial set, and so work with the smooth Kan complexes $BG^{\mathcal{U}}$ to do our differential geometry, and then transfer the cohomological data to BG using technical ideas like [9]. So we have a more conceptually involved space, with a certain "smooth structure", but our differential geometry is rendered trivial, and in Dupont's case the space is the ordinary BG but the differential geometry is more involved.

9. Universal Chern-Weil theory for the group of symplectomorphisms

Let (M, ω) be a symplectic manifold, so that ω is a closed non-degenerate 2-form on M. Let $\mathcal{G} = Ham(M, \omega)$ denote the group of its Hamiltonian symplectomorphisms, and $\mathfrak{h}$ its Lie algebra. When M is simply connected this is just the group $Symp(M, \omega)$ of diffeomorphisms $\phi : M \to M$ s.t. $\phi^* \omega = \omega$. For example $M = \mathbb{CP}^{n-1}$ with its Fubini-Study symplectic 2-form ω_{st}. In this case the natural action of $PU(n)$ is Hamiltonian.

In [10] Reznikov constructs polynomials

$$\{\rho_k\}_{k>1} \subset \mathbb{R}[\mathfrak{h}]^{\mathcal{G}},$$

each ρ_k homogeneous of degree k. In particular given a principal bundle $\mathcal{G} \hookrightarrow P \to X$ for X a smooth manifold we obtain characteristic classes

$$c^{\rho_k}(P) \in H^{2k}(X, \mathbb{R}), \quad k > 1,$$

which were already used by Reznikov in [10] to great effect.

The group $Ham(M, \omega)$ is a Frechet Lie group having the homotopy type of a CW complex by Milnor [8]. In particular, Theorem 8.3 immediately tells us that there are induced cohomology classes

$$(9.1) \qquad\qquad c^{\rho_k} \in H^{2k}(BHam(M, \omega), \mathbb{R}).$$

As mentioned, the group $PU(n)$ naturally acts on $\mathbb{CP}^{n-1}$ by Hamiltonian symplectomorphisms. So we have an induced map

$$i : BPU(n) \to BHam(\mathbb{CP}^{n-1}, \omega_0).$$

Then as one application we have:

Theorem 9.1.

$$i^* : H^\bullet(BHam(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}) \to H^\bullet(BPU(n), \mathbb{R})$$

is surjective for all $n \geq 2$, and so

$$i_* : H_\bullet(BPU(n), \mathbb{R}) \to H_\bullet(BHam(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}),$$

is injective for all $n \geq 2$.

Proof. Let $\mathfrak{g}$ denote the Lie algebra of $PU(n)$. Let $j : \mathfrak{g} \to \mathfrak{h}$ denote the natural Lie algebra map induced by the homomorphism $PU(n) \to Ham(\mathbb{CP}^{n-1}, \omega_0)$. Reznikov [10] shows that $\{j^* \rho_k\}_{k>1}$ are the Chern polynomials. Specifically, the classes

$$c^{j^* \rho_k} \in H^{2k}(BPU(n), \mathbb{R}),$$

are the Chern classes $\{c_k\}_{k>1}$, which generate real cohomology of $BPU(n)$, as is well known. But $c^{j^* \rho_k} = i^* c^{\rho_k}$, for c^{ρ_k} as in (9.1), and so the result immediately follows. $\square$

There are previous related results, for example Reznikov [10] proves that

$$(9.2) \qquad\qquad i_* : \pi_\bullet(BPU(n), \mathbb{R}) \to \pi_\bullet(BHam(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}),$$

is an injection. See also Kedra-McDuff [3]. In [12] there are results about induced maps in (twisted) K-theory.

Theorem 9.1 surely extends to completely general compact semi-simple Lie groups G, with $\mathbb{CP}^n$ replaced by co-adjoint orbits M of G. We just need to compute the

associated Reznikov polynomials in $\mathbb{R}[\mathfrak{h}]^{\mathcal{G}}$ and their pull-backs to $\mathfrak{g}$ as above. We can no longer expect injection in general. But the failure to be injective should be solely due to effects of classical representation theory, rather than transcendental effects of extending the structure group to $Ham(M, \omega)$, from a compact Lie group.

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University of Colima, CUICBAS

Email address: yasha.savelyev@gmail.com