GLOBAL FUKAYA CATEGORY II

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Abstract. We make an explicit calculation with the global Fukaya category introduced in Part I. This calculation uses a regularization technique based on Hofer geometry, together with combinatorial algebraic topology via the theory of Kan complexes. In particular we show that the map

$$BHam(S^2) \to (\mathcal{S}, NFuk(S^2)$$

induces a group injection

$$(\mathbb{Z} = \pi_4(BHam(S^2), id) \to \pi_4(\mathcal{S}, NFuk(S^2)).$$

This may be read as an application of geometry to algebraic topology, as the right hand side is not known. On the way we also construct a higher dimensional version of the relative Seidel morphism studied by Hu and Lalonde, compute this in a particular case and discuss an application to Hofer geometry of the space of Lagrangian equators in S^2.

1. Introduction

Given a Hamiltonian bundle $M \hookrightarrow P \to X$ in part I, we have constructed a (co)-Cartesian fibration over X, with fiber modelled on the A_∞ nerve of the Fukaya category of M. We called this the global Fukaya category $Fuk_\infty(P)$ of P. Here we show that for P a non-trivial Hamiltonian S^2 fibrations over S^4, the maximal Kan sub-fibration of $Fuk_\infty(P)$, which is just a combinatorial analogue of a Serre fibration, is non-trivial. In particular $Fuk_\infty(P)$ is non-trivial as a (co)-Cartesian fibration and so has a non homotopically trivial classifying map to $\mathcal{S}$ - the space of ∞-categories in the component of $NFuk(S^2)$, for $NFuk(S^2)$ the A_∞ nerve of the A_∞ category $Fuk(S^2)$.

This gives in particular:

Theorem 1.1. *The natural homomorphism as constructed in Part I,*

$$\mathbb{Z} = \pi_4(BHam(S^2, id) \xrightarrow{k} \pi_4(\mathcal{S}, NFuk(S^2)) = HH^{-2}_{geom}(Fuk(S^2)),$$

is injective.

The right hand side of the injection is not known, so this already gives an application of geometry to algebraic topology. Morally this is possible because geometry forces a priori A_∞-associativity of certain structures, which then has formal consequences in algebraic topology.

The calculation is performed by carefully constructing perturbation data, so that we are reduced to calculation of a certain higher product in an associated A_∞ category. Note that this is an actual chain level calculation. To perform it we construct a higher relative Seidel element - a higher dimensional analogue of the relative Seidel element in [2]. The calculation of this higher Seidel element uses a regularization technique based on "virtual Morse theory" for the Hofer length functional, of the author [**?**].

It is likely that k is surjective. Surjectivity is in a sense the statement that up to equivalence there are no exotic (co)-Cartesian fibrations over S^4, with fiber equivalent to $N(Fuk(S^2))$ - they all come from Hamiltonian S^2 fibrations, via the global Fukaya category.

Other than the topological/algebraic application above the calculation also yields an application in Hofer geometry, which we state here:

Theorem 1.2. *Let $L_0 \subset M$ be the equator. And let $f : S^2 \to \Omega_{L_0} Lag(S^2)$, represent the generator of π_2, where $Lag(S^2)$ denotes the space of Lagrangians Hamiltonian isotopic to L_0. Then*

$$\min_{f', [f']=[f]} \max_{s \in S^2} L^+(f'(s)) = 1/2 \cdot \mathrm{area}(S^2, \omega),$$

where L^+ denotes the positive Hofer length functional.

The geometrically interesting fact here is the existence of a lower bound on the minimax, as at the moment we have extremely poor understanding of "Hofer small" balls in the group of Hamiltonian symplectomorphisms and the spaces of Lagrangians, for any symplectic manifold. This is proved in section 9. On the way in Section 7 we construct a higher dimensional version of the relative Seidel morphism [2] in the monotone context, and show its non triviality in Section 8. The Sections 9 and 7 are logically independent of the ∞-categorical and even the A_∞ setup and may be read independently.

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Contents

3. Conventions and notations

We use notation Δ^n to denote the standard topological n-simplex. For the standard representable n-simplex as a simplicial set we use the notation $\Delta^n_\bullet$, and in general the under-bullet notation implies we are dealing with a simplicial set. For a topological space X and singular simplex $\Sigma : \Delta^n \to X$ we may denote its image just by Σ.

Although we follow Fukaya-Oh-Ono-Ohta for some things we use Seidel's notation μ^k for composition operations in the A_∞ categories as opposed to m_k. Mostly because the letter m seems better used for naming morphisms in our quasi-categories.

4. Preliminaries on coupling forms

We refer the reader to [4, Chapter 6] for more details on what follows. A Hamiltonian fibration is a smooth fiber bundle

$$M \hookrightarrow P \to X,$$

with structure group $\mathrm{Ham}(M, \omega)$. A *coupling form*, originally appearing in [1], for a Hamiltonian fibration $M \hookrightarrow P \xrightarrow{p} X$, is a closed 2-form $\widetilde{\Omega}$ whose restriction to fibers coincides with ω and which has the property:

$$\int_M \widetilde{\Omega}^{n+1} = 0 \in \Omega^2(X),$$

with integration being integration over the fiber. Such a 2-form determines a Hamiltonian connection $\mathcal{A}_{\widetilde{\Omega}}$, by declaring horizontal spaces to be $\widetilde{\Omega}$-orthogonal spaces to the vertical tangent spaces. A coupling form generating a given connection $\mathcal{A}$ is unique. A Hamiltonian connection $\mathcal{A}$ in turn determines a coupling form $\widetilde{\Omega}_{\mathcal{A}}$ as follows. First we ask that $\widetilde{\Omega}_{\mathcal{A}}$ generates the connection $\mathcal{A}$ as above. This determines $\widetilde{\Omega}_{\mathcal{A}}$, up to values on $\mathcal{A}$ horizontal lifts $\widetilde{v}, \widetilde{w} \in T_pP$ of $v, w \in T_xX$. We specify these values by the formula

$$\widetilde{\Omega}_{\mathcal{A}}(\widetilde{v}, \widetilde{w}) = R_{\mathcal{A}}(v, w)(p), \tag{4.1}$$

where $R_{\mathcal{A}}|_x$ is the curvature 2-form with values in $C^\infty_{norm}(p^{-1}(x))$ - the space of 0-mean normalized smooth functions.

5. Setup

In what follows when we say Part I we will mean [9]. A Hamiltonian S^2 fibration over S^4 is classified by an element

$$[g] \in \pi_3(\mathrm{Ham}(S^2), id) \simeq \pi_3(SO(3), id) \simeq \mathbb{Z}.$$

Such an element determines a fibration P_g over S^4 via the clutching construction:

$$P_g = D^4_- \times S^2 \sqcup D^4_+ \times S^2 \sim,$$

with D^4_-, D^4_+ being 2 different names for the standard closed 4-ball D^4, and the equivalence relation $\sim$ is $(d, x) \sim \widetilde{g}(d, x)$,

$$\widetilde{g} : \partial D^4 \times S^2 \to \partial D^4 \times S^2 \quad \widetilde{g}(d, x) = (d, g^{-1}(x)).$$

From now on P_g will denote such a fibration for a non-trivial class $[g]$. Note that the fiber of P_g over the base point (chosen for definition of the homotopy group $\pi_3(Ham(S^2), id)$) $x_0 \in S^3 \subset D^4_\pm$ has a distinguished, by the construction, identification with S^2.

A bit of possibly non-standard terminology: we say that A is a *model* for B in some category if there is a morphism $mod : A \to B$ which is an (weak)-equivalence, in an appropriate sense that will be clear from context. The map *mod* will be called a *modelling map*. In our context the modeling map *mod* always turns out to be a monomorphism, but this is not always essential.

5.1. **A model for the maximal Kan subcomplex of** $N(Fuk(S^2))$**.** Let $Fuk(S^2)$ denote the $\mathbb{Z}_2$-graded A_∞ category over $\mathbb{Q}$, with objects oriented spin Lagrangian submanifolds Hamiltonian isotopic to the equator. Our particular construction of this category is presented in Part I.

Let us denote by $Fuk^{eq}(S^2)$ the sub-category of $Fuk(S^2)$ obtained by restricting our objects to be great circles in S^2, and taking our perturbation data named $\mathcal{D}_{pt}$ so that the associated Hamiltonian connections $\mathcal{A}(L_0, L_1)$ are all $SO(3)$-connections. We call such data $\mathcal{D}_{pt}$ ***equatorial***. The associated Donaldson-Fukaya category $DFuk^{eq}(S^2)$ is equivalent as a linear category over $\mathbb{Q}$ to $FH(L_0, L_0)$ (considered as a linear category with one object) for $L_0 \in Fuk^{eq}(S^2)$.

It is easily verified that a morphism (1-edge) f is an isomorphism in $NFuk^{eq}(S^2)$, see Part I for definitions, if and only if it is the image by N of a morphism in $Fuk^{eq}(S^2)$ that induces an isomorphism in $DFuk^{eq}(S^2)$. Such a morphism will be called a *c-isomorphism*.

Consequently the maximal Kan subcomplex of $NFuk^{eq}(S^2)$ is characterized as the maximal subcomplex of this nerve with 1-simplices the images by N of c-isomorphisms in $Fuk^{eq}(S^2)$.

Notation 5.1. *We denote the maximal Kan sub-complex of* $N(Fuk^{eq}(S^2))$ *by* $K(S^2)$.

Remark 5.2. *It would be most interesting to identify the geometric realization of* $K(S^2)$ *as a space, up to homotopy.*

5.2. **A model for the maximal Kan-subcomplex of** $Fuk_\infty(P_g)$**.** The strategy is then as follows. First we construct an analogous maximal Kan subcomplex $K(P_g)$ for a quasi-category named $Fuk^{eq}_\infty(P_g)$, itself modeling $Fuk_\infty(P_g)$, so that there is a Kan fibration

$$K(S^2) \hookrightarrow K(P_g) \to S^4_\bullet,$$

which will be shown to be non-trivial. It will follow that $Fuk_\infty(P_g)$ is also non-trivial as a (co)-Cartesian fibration, add this from which our results will follow.

$Fuk^{eq}_\infty(P_g)$ is constructed as follows, where for convenience we rename P_g by P. Assume that the structure group of P is reduced to $SO(3)$. Then if $P_x \simeq S^2$ denotes the fiber of P over x, we can define great circles in P_x to be S^1-submanifolds which pull-back to great circles in S^2 by any $SO(3)$-trivialization map $S^2 \to P_x$. By $SO(3)$-trivialization map we just mean a trivialization that is part of an atlas of trivializations of a bundle with $SO(3)$ structure group.

We may then construct a functor named F^{eq},

$$F^{eq} : \Delta/S^4_\bullet \to A_\infty - Cat,$$

as in part I, but taking the objects of $F^{eq}(x)$ to be great circles in P_x and taking all connections in the perturbation data $\mathcal{D}$ to be $SO(3)$-connections. We then define

$$Fuk^{eq}_\infty(P) = \operatorname{colim}_{\Delta/S^{4,mod}_\bullet} F^{eq},$$

where $S^{4,mod}_\bullet$ is a model for the singular set $S^4_\bullet$, which recall is a Kan complex with n-simplices smooth maps

$$\Sigma : \Delta^n \to S^4.$$

Restricting to a model for the singular set of S^4 is justified by Proposition 4.5 of Part I.

We now give some specifics. First we describe $S^{4,mod}_\bullet$. We model $D^4_\bullet$ as follows. Take the standard representable 3-simplex $\Delta^3_\bullet$, and the standard representable 0-simplex $\Delta^0_\bullet$. Then collapse all faces of $\Delta^3_\bullet$ to a point, that is take the colimit of the following diagram:

$$\begin{tikzcd}
& & \Delta^0_\bullet & \\
\Delta^2_\bullet \arrow[urr] \arrow[drr, "i_0"'] & \Delta^2_\bullet \arrow[ur] \arrow[dr, "i_1"] & \Delta^2_\bullet \arrow[u] \arrow[d, "i_2"] & \Delta^2_\bullet \arrow[ul] \arrow[dl, "i_3"] \\
& & \Delta^3_\bullet &
\end{tikzcd}$$

Here i_j are the inclusion maps of the non-degenerate 2-faces. This gives a Kan complex $S^{3,mod}_\bullet$ modelling the singular set of S^3. Now take the cone on $S^{3,mod}_\bullet$, denoted by $C(S^{3,mod}_\bullet)$, and collapse the one non-degenerate 1-edge. The resulting Kan complex $D^{4,mod}_\bullet$ is our model for $D^4_\bullet$.

Take a pair of copies $D^{4,mod}_{\bullet,\pm}$ of $D^{4,mod}_\bullet$ and identify them along $S^{3,mod}_\bullet$. This gives a simplicial set $S_\bullet$, so that there is a natural embedding map

$$mod : S_\bullet \to S^4_\bullet.$$

We can then take our model $S^{4,mod}_\bullet$ for $S^4_\bullet$ to be the Kan closure of the image $mod(S_\bullet) \subset S^4_\bullet$, that is minimal Kan subcomplex of $S^4_\bullet$ containing $mod(S_\bullet)$. However, concretely we will only be needing the simplices of $mod(S_\bullet)$ for our calculation. We will then from now on identify simplicial sets $S^{4,mod}_\bullet, D^{4,mod}_{\bullet,\pm}$ as subsets of $S^4_\bullet$ without referring to the modeling map.

The perturbation data $\mathcal{D}$. Suppose that g' is homotopic to g and maps into $SO(3)$, rename $P := P_{g'}$. Let $x_{0,\pm} \in D^4_\pm$ correspond to the 0-simplex of $D^{4,mod}_{\bullet,\pm}$ by the modelling map $D^{4,mod}_{\bullet,\pm} \to D^4_{\pm\bullet}$, and let x_0 be the image of $x_{0,\pm}$ in $S^4_\bullet$ by *mod*. By construction of P we have a distinguished $SO(3)$-trivialization of $P_+ := P|_{D^4_+}$, and so under this trivialization a distinguished identification of the fiber

$$P_{+,x_0} := P_+|_{x_0}$$

with S^2.

Denote by $\Sigma^0_\bullet$ the image of the map

$$\Delta^0_\bullet \to S^{4,mod}_\bullet,$$

induced by the inclusion of the 0-simplex x_0. Let $\mathcal{D}^{ex}_{pt}$ denote the extended perturbation data, induced by $\mathcal{D}_{pt}$, for simplices of $\Sigma^0_\bullet$. (Recall from Part I that given perturbation data for a non-degenerate simplex, we assigned extended perturbation data for all degeneracies of this simplex.)

Define the perturbation data for the trivial bundle P_+ to be the extended pullback perturbation data:

$$\mathcal{D}_+ = c^* \mathcal{D}^{ex}_{pt},$$

for

$$c : D^4_+ \to x_{0,+}$$

the constant map.

Now define the perturbation data $\mathcal{D}_-$ for $P_-|_{\partial D^4_-}$, $P_- := P|_{D^4_-}$ by naturally pulling it back from the data $\mathcal{D}_+$ by the previously defined $SO(3)$-bundle diffeomorphism

$$\widetilde{g} : (\partial D^4_- \simeq S^3) \times S^2 \to (\partial D^4_+ \simeq S^3) \times S^2.$$

For the moment extend this perturbation data over D^4_- in any way using $SO(3)$-connections. This defines our provisional data $\mathcal{D}$.

5.2.1. *Connection with simplicial homotopy groups.* By construction

$$Fuk_\infty(P_+, \mathcal{D}_+) := \operatorname{colim}_{\Delta/D^{4,mod}_\bullet} F^{eq}_{\mathcal{D}_+}$$

can be naturally identified with the product $D^{4,mod}_{\bullet,+} \times N(F^{eq}(x_0))$ by Theorem **??** from Part I. Let $const_{L_0}$ denote the constant section of this product, corresponding to some object $L_0 \in Fuk^{eq}(S^2)$.

Also by construction we have a transition isomorphism of (co)-Cartesian fibrations.

$$Fuk_\infty(P_-|_{\partial D^{4,mod}_{\bullet,-}}, \mathcal{D}_-) \to Fuk_\infty(P_+|_{\partial D^{4,mod}_{\bullet,+}}, \mathcal{D}_+).$$

Abusing notation, we will denote this map by $\widetilde{g}$, but the reader should be wary that this $\widetilde{g}$ is now acting on simplicial sets. Set

$$sec = \widetilde{g}^{-1} \circ const_{L_0}|_{\partial D^{4,mod}_{\bullet,-}}.$$

Define $K(P)$ to be the maximal Kan sub-fibration of

$$Fuk^{eq}_\infty(P, \mathcal{D}) := \operatorname{colim}_{\Delta/S^{4,mod}_\bullet} F^{eq}_{\mathcal{D}}.$$

By the above discussion

$$K(P_+) := K(P)|_{D^{4,mod}_{\bullet,-}}$$

is naturally identified with $D^{4,mod}_{\bullet,+} \times K(S^2)$. We also set

$$K(P_-) := K(P)|_{D^{4,mod}_{\bullet,-}}$$

Theorem 5.3. *Suppose that $g : S^3 \to SO(3)$ represents a non-trivial class in $\pi_3(SO(3), id)$, then the class $[sec]$ is non-vanishing in*

$$\pi_3(K(P_-)) \simeq \pi_3(K(S^2)).$$

In particular $K(P_g)$ is a non-trivial Kan fibration over $S^4_\bullet$ and so $Fuk^\infty(P_g)$ is a non-trivial (co)-Cartesian fibration over $S^4_\bullet$.

For g the generator, this class $[sec]$ in $\pi_3(K(S^2))$ can be thought of as a "quantum" analogue of the class of the classical Hopf map.

Proof of Theorem 1.1. This follows by Theorem 5.3 as the map

$$\pi_{k-1}(Ham(M,\omega)) \to HH^{2-k}_{geom}(Fuk(M,\omega))$$

constructed in Part I is a group homomorphism. □

6. Proof of Theorem 5.3 Part I

As indicated 2 sections will be dedicated to the argument, so that we may better subdivide it.

6.1. **Outline of the argument.** In our simplical set $D^{4,mod}_{\bullet,-}$ we have a single non-degenerate 4-simplex Σ^4. It is the image of the non-degenerate 4-simplex of $C(\Delta^3_\bullet) \simeq \Delta^4_\bullet$ for the natural composition

$$C(\Delta^3_\bullet) \to C(S^{3,mod}_\bullet) \to D^{4,mod}_{\bullet,-}.$$

From now on Σ^4 always refers to this simplex. We also have a single 0-simplex, which we denote by Σ^0, and as before the image of $\Sigma^0 : \Delta^0_\bullet \to S^{4,mod}_\bullet$ will be denoted by $\Sigma^0_\bullet$. The fiber of $K(P)$ over $\Sigma^0_\bullet$ by construction has a distinguished identification with $K(S^2)$, since the fiber of P over x_0 has a distinguished identification with S^2. We will denote by $L_{0,\bullet}$ the image of the map

$$\Delta^0_\bullet \to K(P_-),$$

induced by the inclusion of L_0 into $K(S^2)$ as a 0-simplex, which serves as the base point of our homotopy groups.

If *sec* is null-homotopic in the Kan complex $K(P_-)$ then by definition of homotopy groups of a Kan complex there would be a diagram:

$$\begin{tikzcd}
\Delta^3_\bullet \arrow[d, "i_0"] \arrow[dr, "sec"] & \\
\Delta^3_\bullet \times I \arrow[r, "H"] & K(P_-) \\
\Delta^3_\bullet. \arrow[u, "i_1"] \arrow[ur, "null"] &
\end{tikzcd}$$

Here $\partial\Delta^3_\bullet \times I$ maps by H to $L_{0,\bullet} \subset K(P_-)$ and the map *null* is the unique simplicial map onto $L_{0,\bullet} \subset K(P_-)$.

Since H is a null-homotopy we may factor it as:

$$\Delta^3_\bullet \times I \to C(\Delta^3_\bullet) \simeq \Delta^4_\bullet \xrightarrow{T} K(P_-), \tag{6.1}$$

for a certain induced T. Note that T must lie over Σ^4 in $D^{4,mod}_{\bullet,-}$ as the latter is the only non-degenerate 4-simplex.

One of the 3-faces of the 4-simplex T is *sec* and all the other faces are totally degenerate with image $L_{0,\bullet}$. By discussion above, the simplex T is in the image of the natural map

$$K(\Sigma^4) \to K(P_-),$$

where $K(\Sigma^4)$ denotes the maximal Kan sub-complex of $NF^{eq}(\Sigma^4)$. (Note that we have not yet completely specified the perturbation data $\mathcal{D}$ over D^4_-.) Let us denote by $\gamma \in \hom_{F(x_0)}(L_0, L_0)$ a "fundamental chain" which projects to the identity in $DFuk^{eq}(L_0, L_0)$. Let us denote by L_0^i the image of L_0 by the embedding $F(x_0) \to F(\Sigma^4)$ corresponding to the i'th vertex inclusion into Δ^4, $i = 0, \ldots, 4$. Let $\Sigma^1_{i,i+1}$ denote the 1-simplex obtained by restriction of Σ^4 to the edge between $i, i+1$, and Σ^0_i the 0-simplex obtained by restriction of Σ^4 to the i'th vertex. For each L_0^i, L_0^{i+1}, we have an isomorphism $\gamma_i : L_0^i \to L_0^{i+1}$ which corresponds to γ, that is the fully-faithful projection $F(\Sigma^1_{i,i+1}) \to F(x_0)$ corresponding to the degeneracy morphism $\Sigma^1_{i,i+1} \to \Sigma^0_i$ takes γ_i to γ. We will denote by $\gamma_{i,j}$ the analogous morphisms $L_0^i \to L_0^j$.

We will now setup up our perturbation data $\mathcal{D}$ and show that T cannot exist for this data.

Lemma 6.1. *There exists perturbation data $\mathcal{D}_0$ for P satisfying the previous restrictions for $\mathcal{D}$, in particular extending $\mathcal{D}_+$ above, so that the simplex T exists if and only if*

$$\mu^4_{\Sigma^4, \mathcal{D}_0}(\gamma_1, \ldots, \gamma_4)$$

is exact.

This is proved further below. We will show that $\mu^4_{\Sigma^4, \mathcal{D}_0}(\gamma_1, \ldots, \gamma_4)$ does not vanish in homology, which will finish the argument. However the calculation will require significant setup.

6.2. **The data $\mathcal{D}_0$.** To forewarn, we use here some special notation (particularly for moduli spaces) from Sections 4 and 5 in Part I.

By assumption $\mathcal{D}_0$ should extend $\mathcal{D}_+$ over D^4_+, however $\mathcal{D}_+$ is only defined up to the choice of perturbation data over $x_{0,+} \in D^4_+$. We partially specify it by first asking that all the connections $\mathcal{A}(L, L')$ for $L, L' \subset P_{x_0}$ objects (equators with spin structure), are $SO(3)$-connections, s.t. for $L = L'$ $\mathcal{A}(L, L')$ is nearly trivial, that is generated by a path p_κ in $SO(3)$ with Hofer length κ nearly 0. And s.t. for $L \neq L'$ the $SO(3)$ connection $\mathcal{A}(L, L')$ is the trivial connection. Furthermore to make some geometry cleaner, for γ as above we ask that for the chosen perturbation data

$$\mu^d_{x_{0,+}}(\gamma, \ldots, \gamma) = 0, \text{for } 2 < d < 5, \tag{6.2}$$

(the upper bound can be made arbitrarily large by taking κ to be sufficiently small) Note that in any case $\mu^2_{x_{0,+}}(\gamma, \gamma) = \gamma$, as our complex must be perfect. In the future $DFuk^{eq}(S^2)$ will denote the Donaldson-Fukaya category for the above (partially specified) perturbation data.

Let us explain why this can be done. Fix a complex structure j_0 on $P_{x_0,+}$. Let $\{\mathcal{A}_r^d\}$ be a family of connections on $S^2 \times \mathcal{S}_r$, $r \in \overline{R}_d$ admissible with respect to $L_0, \ldots, L_d$, $L_i = L_0$ in the sense of Part I, Definition 4.3. This means in particular that $\mathcal{A}_r^d$ preserves the constant Lagrangian subbundle over the boundary with fiber L_0, and that in the strip coordinate charts at each end $\mathcal{A}_r^d$ has the form of flat $\mathbb{R}$-translation invariant extension of $\mathcal{A}(L_0, L_0)$. Suppose in addition that:

$$\operatorname{area}(\mathcal{A}_r^d) < 1/2 \operatorname{area}(S^2, \omega), \tag{6.3}$$

for each r, where area is as in (6.15). We can assure the latter by taking κ to be sufficiently small. Let $\{J_r\}$ be the family of complex structures induced by $\{\mathcal{A}_r^d\}$, j_0.

Lemma 6.2. *Whenever A is such that*

$$\overline{\mathcal{M}}(\gamma, \ldots, \gamma; \gamma_0, x_0, \{J_r\}, A),$$

has virtual dimension 0, and the number of inputs γ is more then 2, but less then 5, the moduli space is empty.

Proof. For a fixed r, let $\mathcal{M}(A)$ be the moduli space of finite vertical L^2 energy class A holomorphic sections of $\widetilde{\mathcal{S}}_r = S^2 \times \mathcal{S}_r$ with boundary in the sub-bundle $\mathcal{L}_r$ over the boundary of $\mathcal{S}_r$ which in this case is just the constant sub-bundle with fiber L_0. Let us point out that having finite vertical energy is equivalent to the condition that the section is asymptotic to some Hamiltonian cords (flat sections of the associated connection) at the ends. By Riemann-Roch (Appendix B) we get that the expected dimension of $\mathcal{M}(A)$ is

$$1 + Maslov^{vert}(A).$$

Consequently, when $\gamma_0 = \gamma$ the expected dimension of

$$\mathcal{M}(\{\gamma_i\}_{i \in J}; \gamma, \Sigma^4, \mathcal{F}, A)$$

is:

$$1 + Maslov^{vert}(A) - 1 + (\dim \mathcal{R}_d = d - 2). \tag{6.4}$$

We need the expected dimension to be 0, and $d \geq 3$, so $Maslov^{vert}(A) \leq -1$. But $Maslov^{vert}(A) = -1$ is impossible as the minimal Maslov number is 2. And $Maslov^{vert}(A) < -1$ is impossible, by monotonicity, by Lemma 6.10 slightly ahead, and by (6.3). When γ_0 is the Poincare dual generator the argument is similar. $\square$

Pulling back the perturbation data $\mathcal{D}_+$ over Σ_+^3 by the gluing map we get induced perturbation data $\mathcal{D}_-$ over Σ_-^3. Extend $\mathcal{D}_-$ over $\Sigma^4 = \Sigma_-^4$ in any way. In totality this gives our data $\mathcal{D}_0$. By (6.2), (**??**) it follows that with respect to $\mathcal{D}_0$:

$$\mu^2_{\Sigma^4}(\gamma_i, \gamma_{i+1}) = \gamma_{i,i+2} \tag{6.5}$$

$$\mu^3_{\Sigma^4}(\{\gamma_i\}_{i \in J}) = 0, \tag{6.6}$$

for J a cardinality 3 subset of $\{0, \ldots, 4\}$ of consecutive numbers.

Proof of Lemma 6.1. The following argument will be over $\mathbb{F}_2$ as opposed to $\mathbb{Q}$ as the signs will not matter. We use the perturbation system $\mathcal{D}_0$ above. Recall that all positive codimension faces of T are uniquely determined, the question is what could be the 4-face. Let $\{f_j\}$, $j : [n_j] \to [4]$, a monomorphism, (equivalently cardinality n_{j+1} subset of $[4] = \{0, \ldots, 4\}$) be as in the definition of the A_∞ nerve in Part

I, corresponding to the various (arbitrary positive codimension) faces of T. If the 4-simplex T exists then there is an $f_{[4]} \in hom_{F^{eq}(\Sigma^4)}(L_0^0, L_0^4)$ so that

$$\mu^1_{\Sigma^4} f_{[4]} = \sum_{1<i<4} f_{[4]-i} + \sum_s \sum_{(j_1,\ldots j_s)\in decomp_s} \mu^s_{\Sigma^4}(f_{j_1}, \ldots, f_{j_s}). \tag{6.7}$$

By (6.5), (6.6) we must have $f_{j'} = 0$ whenever $n_{j'} = 2$, and $f_{[4]-i} = 0$, $0 \leq i \leq 4$. ($\leq$ is intended.) Given this (6.7) holds if and only if $\mu^4_{\Sigma^4}(\gamma_1, \ldots, \gamma_4)$ is exact. □

Let m_i correspond to γ_i, and let us abbreviate $pr_1\mathcal{F}(m_1, \ldots m_4, \Sigma^4, r)$ given by $\mathcal{D}_0$ above as $\mathcal{A}_r$, and we assume that

$$\text{area}(\mathcal{A}_r) < 1/2\, \text{area}(S^2, \omega), \tag{6.8}$$

is satisfied for r near the boundary. We now need to study the moduli space

$$\overline{\mathcal{M}}(\gamma_1, \ldots, \gamma_4; \gamma_{0,4}, \Sigma^4, \{\mathcal{A}_r\}, A). \tag{6.9}$$

By the dimension formula (6.4), since we need the expected dimension to be zero, the class A must have vertical Maslov number -2.

Notation 6.3. *From now on A_0 refers to this Maslov number -2 class.*

To perform the calculation of the above space we shall need to do a deformation of $\{\mathcal{A}_r\}$ to a certain special geometric form. Let us call a family $\{\mathcal{A}'_r\}$ which restricts on the boundary of $\overline{\mathcal{R}}_4$ to the family $\{\mathcal{A}_r\}$ *admissible*. Note that for any (not necessarily regular) admissible family $\{\mathcal{A}'_r\}$, elements

$$(u, r) \in \overline{\mathcal{M}}(\gamma_1, \ldots \gamma_4; \gamma_{0,4}, \Sigma^4, \{\mathcal{A}'_r\}, A_0)$$

must have the r parameter stay away from the boundary of $\overline{\mathcal{R}}_4$, by (6.8). We shall see shortly that $\mu^4_{\Sigma^4}(\gamma_1, \ldots, \gamma_4)$ is well defined in homology $HF(L_0^0, L_0^4)$ for any choice of a regular $\{\mathcal{A}'_r\}$ as above. And we shall compute this class by relating it to the higher Seidel morphism.

6.3. **Connection between the product $\mu^4_{\Sigma^4}(\gamma_1, \ldots, \gamma_4)$ and the higher Seidel morphism.** Let $D^2_\epsilon \subset \overline{\mathcal{R}}_4$ be embedded so that ∂D^2_ϵ is in a ϵ-neighborhood of $\partial \overline{\mathcal{R}}_4$, where ϵ is as in the first naturality property for $\mathcal{U}$.

Definition 6.4. *Given $\{\mathcal{A}'_r\}$ as above, we say that ϵ above is $(\{\mathcal{A}'_r\}, A_0)$-***compatible** *or just compatible, if there are no A_0 class $J(\mathcal{A}'_r)$-holomorphic curves for r outside D^2_ϵ.*

Given a compatible ϵ as above, for each $r \in D^2_\epsilon$, we may close the open ends $\{e_i\}$, $i \neq 0$ of $\mathcal{S}_r$, by gluing at the ends with copies of the surface $\mathcal{D}$ which is topologically $D^2 - z_0$, $z_0 \in \partial D^2$, endowed with a choice of a strip chart at the end. More explicitly cut of $[0,1] \times (t, \infty)$, for $t > 0$ in the strip coordinate charts at the ends $\{e_i\}$, of $\mathcal{S}_r$ and likewise with $\mathcal{D}$ and then glue along the new boundary component. Let us denote the closed off surface by $\mathcal{S}^\wedge_r \simeq \mathcal{D}$. Since $\widetilde{\mathcal{S}}_r$ is naturally trivialized at the ends, we may similarly close off $\widetilde{\mathcal{S}}_r$ by "gluing" with bundles $S^2 \times \mathcal{D}$ at the ends obtaining an S^2 bundle $\widetilde{\mathcal{S}}^\wedge_r$ over $\mathcal{S}^\wedge_r$. Likewise close up the trivial Lagrangian subbundle $\mathcal{L}_r$ over $\partial \mathcal{S}_r$, to a trivial Lagrangian subbundle $\mathcal{L}^\wedge_r$ over $\partial \mathcal{D}$.

Clearly we may put a $\mathcal{L}_r$-exact Hamiltonian connection, $\mathcal{A}^\wedge_r$ on $\widetilde{\mathcal{S}}^\wedge_r$ (see Definition 6.5), with

$$\text{area}(\mathcal{A}^\wedge_r) \simeq \text{area}(\mathcal{A}_r) + 4\kappa,$$

with $\simeq$ meaning arbitrarily close. In particular if κ is chosen to be sufficiently small then:

$$\text{area}(\mathcal{A}_r^\wedge) < 1/2\,\text{area}(S^2, \omega) \tag{6.10}$$

for r near $\partial\overline{\mathcal{R}}_4$.

By (6.10) there exists an $(\{\mathcal{A}_r^\wedge\}, A_0)$-***compatible*** ϵ. When $\{\mathcal{A}_r^\wedge\}$ is regular we get an element $ev_0 \in CF(L_0, L_0)$, defined by correlation:

$$\langle ev_0, \gamma_0\rangle = \#\mathcal{M}(\{\widetilde{\mathcal{S}}_r^\wedge, S_r^\wedge, \mathcal{L}_r^\wedge, \mathcal{A}_r^\wedge\}, A, \gamma_0), \tag{6.11}$$

where

$$\mathcal{M}(\{\widetilde{\mathcal{S}}_r^\wedge, S_r^\wedge, \mathcal{L}_r^\wedge, \mathcal{A}_r^\wedge\}, A, \gamma_0), \tag{6.12}$$

consists of pairs (σ, r), $r \in D_\epsilon^2$ with σ a $J(\mathcal{A}_r^\wedge)$-holomorphic, finite vertical L^2 energy, class A section of $\widetilde{\mathcal{S}}_r^\wedge$ with boundary on $\mathcal{L}_r^\wedge$, asymptotic at the e_0 end to γ_0 - a geometric generator of $CF(L_0, L_0)$.

We shall relate ev_0 to our product for μ^4, but before we do that let us explain why ev_0 is a cycle, and formalize conditions for invariance of its homology class. The latter unfortunately requires some lengthy definitions, which will be necessary in most of their generality.

Definition 6.5. *Let $M \hookrightarrow \widetilde{S} \to S$ be a symplectic fiber bundle, with model fiber monotone symplectic manifold (M, ω), over a Riemann surface with punctures, and distinguished "strip charts" $M \times [0,1] \times (0, \infty) \to \widetilde{S}$ at the ends. Say then that $\widetilde{S}$ has* **end structure***.*

Let $\mathcal{L}$ be a Lagrangian sub-bundle of $\widetilde{S}$ as above, with model fiber an object in the sense of part I, over the boundary of S, which is constant at the ends, in these trivializations, say then that $\mathcal{L}$ **respects the end structure***.*

*In the coordinates at the end e_i let L_j^i be the fibers of $\mathcal{L}$ over $\{j\} \times \{t\} \in [0,1] \times (0, \infty)$, $j = 0, 1$, for $t > 0$. We say that a Hamiltonian connection $\mathcal{A}$ on $\widetilde{S}$ as above, is $\mathcal{L}$-***exact***, for $\mathcal{L}$ as above, if it preserves $\mathcal{L}$ and if, in the strip coordinate chart at the e_i end, $\mathcal{A}$ is flat and $\mathbb{R}$-translation invariant and so has the form of a flat, $\mathbb{R}$-translation invariant extension of a connection $\mathcal{A}(L_0^i, L_1^i)$ on $M \times [0,1]$.*

A family $\{j_z\}$ of fiber wise ω-compatible almost complex structures on $\widetilde{S}$ will be said to respect the end structure *if at each end e_i in the strip coordinate chart above the family $\{j_z\}$ is $\mathbb{R}$-translation invariant and is admissible with respect to $\mathcal{A}(L_0^i, L_1^i)$, in the sense of Part I, Section 4.1. The data $(\widetilde{S}, S, \mathcal{L}, \mathcal{A}, \{j_z\})$ as above will be called* **admissible***.*

We shall normally suppress $\{j_z\}$ in the notation, and elsewhere for simplicity, as it will be purely in the background in what follows, (we do not need to manipulate it explicitly).

Definition 6.6. *Let $\mathcal{K}$ be a smooth oriented manifold with boundary, and for each $r \in \mathcal{K}$ suppose we are given admissible data $(\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$, for $S_r \simeq \mathcal{D}$ so that $(\widetilde{S}_r, S_r, \mathcal{L}_r)$ fits into a smooth fibration $\widetilde{S} \to \mathcal{K}$ in the natural sense. We shall write $\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r, \mathcal{K}\}$, for this* **fibration data***. Note that everything is assumed to vary smoothly in r. In particular, the charts*

$$e_{i,r} : M \times [0,1] \times (0, \infty) \to \widetilde{\mathcal{S}}_r,$$

for the ends, fit into a smooth map

$$\widetilde{e}_i : M \times [0,1] \times (0,\infty) \times \mathcal{K} \to \widetilde{S}, \tag{6.13}$$

and we have an induced smooth r-family of connections $\{e_{i,r}^* \mathcal{A}_r\}$ *on* $M \times [0,1] \times (0,\infty)$*, and an induced smooth r-family of Lagrangian subfibrations* $\{e_{i,r}^{-1} \mathcal{L}_r\}$ *over* $\partial[0,1] \times (0,\infty)$*. We ask that* $\{e_{i,r}^* \mathcal{A}_r\}$*, and* $\{e_{i,r}^{-1} \mathcal{L}_r\}$ *are r-invariant. We say that* $\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r, \mathcal{K}\}$ *is A-**admissible**, if there are no elements*

$$(\sigma, r) \in \overline{\mathcal{M}}(\{\widetilde{\mathcal{S}}_r, S_r, \mathcal{L}_r, \mathcal{A}_r, \mathcal{K}\}, A),$$

for r near the boundary of $\mathcal{K}$*, where this moduli space is as in* (6.12) *(but without specified constraints).*

Definition 6.7. *Given an A-admissible pair* $\{\widetilde{\mathcal{S}}_r^i, \mathcal{S}_r^i, \mathcal{L}_r^i, \mathcal{A}_r^i, \mathcal{K}\}$*,* $i = 1, 2$*, we say that they are A-**admissibly concordant** if there is an A-admissible fibration data*

$$\{\widetilde{\mathcal{T}}_r, \mathcal{T}_r, \mathcal{L}'_r, \mathcal{A}'_r, \mathcal{K} \times [0,1]\},$$

with an oriented diffeomorphism (in the natural sense, preserving all structure)

$$\{\widetilde{\mathcal{S}}_r^0, \mathcal{S}_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0, \mathcal{K}^{op}\} \sqcup \{\widetilde{\mathcal{S}}_r^1, \mathcal{S}_r^1, \mathcal{L}_r^1, \mathcal{A}_r^1, \mathcal{K}\} \to \{\widetilde{\mathcal{T}}_r, \mathcal{T}_r, \mathcal{L}'_r, \mathcal{A}'_r, \mathcal{K} \times \partial I\},$$

where op denotes the opposite orientation for $\mathcal{K}$*.*

In what follows we may omit $\mathcal{K}$ from notation.

Lemma 6.8. *Let* M, L *be as above. And let* $\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r\}$ *be regular and A-admissible, with* S_r *having one distinguished end* e_0*. Define*

$$ev = ev(\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r\}, A) \in CF(L, L)$$

as before by:

$$\langle ev(\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r\}, A), \gamma_0 \rangle = \#\mathcal{M}(\{\widetilde{\mathcal{S}}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}, A, \gamma_0),$$

for γ_0 *constraint at the* e_0 *end. Then* $ev(\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r\}, A) \in CF(L, L)$ *is a cycle, and its homology class depends only on the A-admissible concordance class of* $\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r\}$*. Moreover for the family as in* (6.11) *the homology class of* $ev_0 = ev(\{\widetilde{\mathcal{S}}_r^{\wedge}, S_r^{\wedge}, \mathcal{L}_r^{\wedge}, \mathcal{A}_r^{\wedge}\}, A_0) \in CF(L_0, L_0)$ *coincides with the homology class of*

$$\mu^4_{\Sigma^4}(\gamma_1, \ldots, \gamma_4).$$

Proof. The first and second part of the lemma, follow by the same arguments as given in the construction of the relative Seidel morphism in [2], keeping in mind that we are now dealing with families of surfaces and the parameter family has boundary. Let us only indicate the argument, as this is standard. To show that $ev \in CF(L, L)$ is closed suppose that $\langle \mu^1(ev), \gamma' \rangle \neq 0$ for a geometric generator γ', then $\mathcal{M}(\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r\}, A, \gamma')$ is a compact 1-dimensional manifold with boundary. By our various assumptions, the elements of its boundary can only correspond to Floer degenerations. And by gluing any Floer degeneration that could happen does happen. So the signed count of boundary points of this moduli space which must be 0, is in correspondence with $\langle \mu^1(ev), \gamma' \rangle$, and this is a contradiction.

Similarly, given an A-admissible concordance (which we may assume to be regular)

$$\{\widetilde{\mathcal{T}}_r, \mathcal{T}_r, \mathcal{L}'_r, \mathcal{A}'_r, \mathcal{K} \times [0,1]\},$$

between $\{\widetilde{\mathcal{S}}^0_r, \mathcal{S}^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}$ and $\{\widetilde{\mathcal{S}}^1_r, \mathcal{S}^1_r, \mathcal{L}^1_r, \mathcal{A}^1_r\}$ will get a chain c,

$$\langle c, \gamma_0 \rangle = \#\mathcal{M}(\{\widetilde{\mathcal{T}}_r, \mathcal{T}_r, \mathcal{L}'_r, \mathcal{A}'_r\}, A, \gamma_0),$$

so that $\mu^1(c) = ev_1 - ev_0$, for

$$ev_0 = ev(\{\widetilde{\mathcal{S}}^0_r, \mathcal{S}^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}), \text{ and } ev_1 = ev(\{\widetilde{\mathcal{S}}^1_r, \mathcal{S}^1_r, \mathcal{L}^1_r, \mathcal{A}^1_r\}).$$

The last part is immediate from standard gluing arguments. □

The cycle ev_0 is closely related to a relative form of the higher Seidel morphism, which in its most basic form is a group homomorphism:

$$\Psi : \pi_{k-1}\Omega_L Lag(M) \simeq \pi_k(Lag(M, L)) \to FH(L, L) \quad k > 1$$

with $Lag(M)$ denoting the space whose components are Hamiltonian isotopic Lagrangian submanifolds of (M, ω), and $Lag(M, L)$ denoting the component of L. This generalizes the idea behind relative Seidel morphism. We shall present this construction in the next section, (the reader may also safely read that section first) and will show that:

$$\Psi([lag]) = [ev_0], \tag{6.14}$$

for a certain map

$$lag : S^2 \to \Omega_{L_0} Lag(S^2).$$

6.3.1. *Constructing lag.* We deform our data as follows. Fix the construction of the fibration $P_g \to S^4$ as before, with respect to $D^4_-, D^4_+ \subset S^4$. Let us continuously homotop the map $\Sigma^4 : \Delta^4 \to S^4$, through maps Σ^4_t, $t \in [0, 1]$ taking all the 2-faces of Δ^4, and all but one 3-face of Δ^4 to x_0, to the map Σ^4_1, taking all faces to x_0. This induces a homotopy of families of maps

$$\widetilde{u}_{r,t} = \widetilde{u}_t(m_1, \ldots, m_4, r) = \Sigma^4_t \circ u(m_1, \ldots, m_4, \Delta^4, r).$$

And this induces a concordance $\{\widetilde{\mathcal{S}}^\wedge_{r,t}, S^\wedge_{r,t}, \mathcal{L}^\wedge_{r,t}, \mathcal{A}^\wedge_{r,t}\}$, for $S^\wedge_{r,t} = S_r$,

$$\widetilde{\mathcal{S}}^\wedge_{r,t} = (\widetilde{u}^\wedge_{r,t})^* P,$$

with $\widetilde{u}^\wedge_{r,t}$ the map of $S^\wedge_r$ induced by $\widetilde{u}_{r,t}$,

$$\mathcal{L}^\wedge_{r,t} = (\widetilde{u}^\wedge_{r,t})|^*_{\partial S^\wedge_r} L_0,$$

and for $\mathcal{A}^\wedge_{r,t}$ at the moment an arbitrary extension. Let us rename $\{\widetilde{\mathcal{S}}^\wedge_{r,1}, S^\wedge_{r,1}, \mathcal{L}^\wedge_{r,1}, \mathcal{A}^\wedge_{r,1}\}$, by $\{\widetilde{\mathcal{S}}'_r, S^\wedge_r, \mathcal{L}'_r, \mathcal{A}'_r\}$.

Lemma 6.9. *We may choose $\{\mathcal{A}'_r\}$ and the interpolating family of connections so that the above concordance is A_0 admissible.*

Proof. We show this using area positivity, which already came up before, and will have to come in again later.

Lemma 6.10. *Let $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be admissible as above. A (nodal, broken) finite vertical L^2 energy, $J_{\mathcal{A}}$-holomorphic class A section σ of $\widetilde{S}$, with boundary in $\mathcal{L}$, gives a lower bound*

$$-\int_S \sigma^* \widetilde{\Omega}_{\mathcal{A}} \leq \text{area}(\mathcal{A}),$$

where

$$\text{area}(\mathcal{A}) = \inf_\alpha \{\int_S \alpha \,|\widetilde{\Omega}_{\mathcal{A}} + \pi^*(\alpha) \textit{ is nearly symplectic}\}, \tag{6.15}$$

where α is a 2-form on S, $\widetilde{\Omega}_{\mathcal{A}}$ the coupling form and nearly symplectic means that

$$(\widetilde{\Omega}_{\mathcal{A}} + \pi^*(\alpha))(\widetilde{v}, \widetilde{jv}) \geq 0,$$

for $\widetilde{v}, \widetilde{jv}$ horizontal lifts with respect to $\widetilde{\Omega}_{\mathcal{A}}$, of $v, jv \in T_z S$, for all $z \in S$.

Proof. This follows by the classical symplectic area positivity for J-holomorphic curves, with J compatible with the symplectic form. □

Lemma 6.11. *Let $\{(\widetilde{S}_t, S_t, \mathcal{L}_t, \mathcal{A}_t)\}$ be a concordance of admissible data as above. And let $\{\sigma_t\}$, σ_t a finite vertical L^2 energy section of $\widetilde{S}_t$ with boundary on $\mathcal{L}_t$, be a continuous homotopy. Here continuous means as a map into the total space $\cup_t \partial S_t$ of the concordance*

$$\{(\widetilde{S}_t, S_t, \mathcal{L}_t\}.$$

Then the pairing

$$-\int_{S_t} \sigma_t^* \widetilde{\Omega}_{\mathcal{A}},$$

is t independent.

Proof. Given the above we have a Lagrangian subfibration $\cup_t \mathcal{L}_t$ of $\cup_t \widetilde{S}_t$ over $\cup_t \partial S_t$. We may construct a Hamiltonian connection $\widetilde{\mathcal{A}}$ on $\cup_t \widetilde{S}_t$ restricting to $\mathcal{A}_t$ over each $\widetilde{S}_t$, and trivial in the t variable. $\widetilde{\mathcal{A}}$ then preserves $\cup_t \mathcal{L}_t$. There is an induced closed coupling form $\widetilde{\Omega}_{\widetilde{\mathcal{A}}}$ on $\cup_t \widetilde{S}_t$ extending $\widetilde{\Omega}_{\mathcal{A}_i}$ over $\widetilde{S}_i$, for $i = 0$ or $i = 1$. Invariance then readily follows by Stokes theorem as $\widetilde{\Omega}_{\widetilde{\mathcal{A}}}$ vanishes on $\cup_t \mathcal{L}_t$. □

Let us call

$$-\int_S \sigma^* \widetilde{\Omega}_{\mathcal{A}},$$

the *coupling area* of σ, (with respect to $\mathcal{A}$). Given admissible data $(\widetilde{S}, S, \mathcal{L})$ we shall say that a $\mathcal{L}$-exact Hamiltonian connection $\mathcal{A}$, on $\widetilde{S} \to S$ is *small* if $\operatorname{area}(\mathcal{A}) < \hbar$ for $\hbar$ the minimal coupling area of a $J_{\mathcal{A}}$-holomorphic section of $\widetilde{S}$ with boundary on $\mathcal{L}$. In in the example of our calculation $\hbar = 1/2 \operatorname{area}(S^2, \omega)$. Given data $\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r, \mathcal{K}\}$ we say that $\{\mathcal{A}_r\}$ is *small near boundary* if $\mathcal{A}_r$ is small near $\partial\mathcal{K}$. To finish the proof of the lemma we note that by construction $\{\widetilde{\mathcal{S}}_{r,t}\}$, is naturally trivialized for r near $\partial\overline{R}_4$, so that in this trivialiation $\{\mathcal{L}_{r,t}\}$ is just the trivial sub-bundle with fiber L_0. And so we can clearly choose our family $\{\mathcal{A}'_r\}$ and the interpolating family to be small near boundary, by which our lemma follows. □

We now construct a follow up concordance. By the last naturality axiom for the system $\mathcal{U}$, the maps $\{\widetilde{u}_{r,1}\}$, $r \in D^2_\epsilon$ may be perturbed to induce a singular foliation of S^4, with one singular point, by surfaces diffeomorphic to disks whose boundary is mapped to x_0. The intersection of this foliation with $S^3 \simeq \partial D^4_-$ is the image of the family of loops $\{c(r)\}$, and itself must be a singular foliation of S^3 by loops based at x_0. Deform the family $\{\mathcal{A}'_r\}$ through families of small near boundary connections $\{\mathcal{A}_{r,\tau}\}$, $\tau \in [0,1]$, so that for each r $\mathcal{A}_{r,1}$ is a trivial connection on $\widetilde{\mathcal{S}}'_r$ over the region R^+ in $\mathcal{S}^\wedge_r$ mapped by $\widetilde{u}^\wedge_{r,1}$ to D^4_+. Here trivial means with respect to the distinguished trivialization of $\widetilde{\mathcal{S}}'_r$ over R^+ determined by the distinguished trivialization of $P|_{D^4_+}$. Then $(\widetilde{u}^\wedge_{r,1})^{-1}(c_r)$ is an embedded submanifold $C_r \simeq \mathbb{R}$ in $\mathcal{S}^\wedge_r$ with ends going into the boundary of e_0. Over C_r, $\widetilde{\mathcal{S}}'_r$ has a distinguished trivialization determined by the distinguished trivialization $P|_{D^4_-}$, and with respect to this trivialization we have a Lagrangian subfibration $\mathcal{L}_r$ over C_r with fiber over $p \in C_r$ given by $g(\widetilde{u}^{\wedge}_{r,1}(p))(L_0)$. By construction $\mathcal{A}_{r,1}$ leaves $\mathcal{L}_r$ invariant, and we may choose a parametrization $\mathbb{R} \to C_r$, so that $\mathcal{L}_r|_{\mathbb{R}-(0,1)} = L_0$. So we get an induced loop

$$\gamma_r : S^1 \to Lag^{eq}(S^2) \simeq S^2,$$

based at L_0, and a map

$$lag : (D^2_\epsilon/\partial D^2_\epsilon \simeq S^2) \to \Omega_{L_0} Lag(S^2), \quad lag(r) = \gamma_r.$$

Definition 6.12. *Given a smooth*

$$\gamma : [0,1] \to Lag(M, L)$$

constant near $0, 1$, $\pi : \mathbb{R} \to [0,1]$ *the retraction map, and given a parametrization by* $\mathbb{R}$ *of the boundary of* $\mathcal{D}$*, let* $\mathcal{L}_\gamma$ *denote the Lagrangian subfibration of* $M \times \mathcal{D} \to \mathcal{D}$ *over* $\partial\mathcal{D}$*, with fiber over* $r \in \partial\mathcal{D}$ *given by* $\gamma \circ \pi(r)$*. We say that a Lagrangian subfibration* $\mathcal{L}$ *as above is* **determined by** γ *as above if* $\mathcal{L} = \mathcal{L}_\gamma$*, after a choice of parametrization of boundary of* $\mathcal{D}$ *by* $\mathbb{R}$*.*

Then with the above definition $\mathcal{L}_r = \mathcal{L}_{lag(r)}$.

Lemma 6.13. *The class* $a = [lag] \in \pi_2(\Omega_{L_0} Lag^{eq}(S^2), L_0) \simeq \mathbb{Z}$ *is independent of all choices, and coincides with the generator.*

Proof. Independence of all choices is obvious. The second assertion follows immediately by the construction. □

Lemma 6.14. *The* A_0*-admissible family* $\{\widetilde{\mathcal{S}}^{\wedge}_r, S^{\wedge}_r, \mathcal{L}^{\wedge}_r, \mathcal{A}^{\wedge}_r\}$ *is* A_0*-admissibly concordant to* $\{S^2 \times \mathcal{D}, \mathcal{D}, \mathcal{L}_{lag(r)}, \mathcal{A}'_{r,1}\}$*, for a certain* $\{\mathcal{A}'_{r,1}\}$*.*

Proof. Let $R^- = R^-_r$ be the region in $\mathcal{S}^{\wedge}_r$ mapped by $\widetilde{u}^{\wedge}_{r,1}$ to D^4_- this region by construction is diffeomorphic to $\mathcal{D}$. The intersection $R^+_r \cap R^-_r = C_r$. Since $\mathcal{A}_{r,1}$ is trivial over R^+, fixing a deformation retraction

$$ret_r : \mathcal{S}^{\wedge}_r \times I \to \mathcal{S}^{\wedge}_r,$$

$\mathcal{S}^{\wedge}_r$ onto R^-, smoothly in r, the pull-back by ret_r of $\{\widetilde{\mathcal{S}}'_r, S^{\wedge}_r, \mathcal{L}'_r, \mathcal{A}_{r,1}\}$ then gives a concordance between $\{\widetilde{\mathcal{S}}'_r, S^{\wedge}_r, \mathcal{L}'_r, \mathcal{A}_{r,1}\}$ and $\{S^2 \times \mathcal{D}, \mathcal{D}, \mathcal{L}_{lag(r)}, \mathcal{A}'_{r,1}\}$, once we use smooth Riemann mapping theorem to identify each R^-_r with its induced complex structure j_r with $(\mathcal{D}, j_{st})$, smoothly in r. This concordance is A_0-admissible as $\{\mathcal{A}_{r,1}\}$ is small near boundary and so the interpolating family of connections will be small near boundary. □

7. Higher relative Seidel morphism

The relative Seidel morphism appears in Seidel's [11] in the exact case and further developed in [2] in the monotone case. Let $Lag(M)$ denote the space whose components are Hamiltonian isotopic Lagrangian submanifolds of M, we may also denote the component of L by $Lag(M, L)$. Then the ungraded relative Seidel morphism, is a homomorphism

$$S : \pi_1(Lag(M, L), L) \to FH(L, L),$$

defined for L an object in the previous sense.

To a loop o in $Lag(M)$ based at L we have an associated Lagrangian subbundle $\mathcal{L}_o$ of $M \times \mathcal{D}$ over the boundary, as in Definition 6.12. Recall that to define $FH(L, L)$

we fix a generic Hamiltonian connection $\mathcal{A}(L, L)$, on $M \times [0,1]$. Pick an $\mathcal{L}_o$-exact Hamiltonian connection $\mathcal{A}$ on

$$M \times \mathcal{D} \to \partial \mathcal{D},$$

which at end in the strip chart, has the form of the flat $\mathbb{R}$-translation invariant extension of the connection $\mathcal{A}(L, L)$. We shall say in this case that $\mathcal{A}$ is ***compatible*** with $\mathcal{A}(L, L)$. Fix a family $\{j_z\}$ of fiber wise almost complex structures on $M \times \mathcal{D}$ giving admissible data

$$(M \times \mathcal{D}, \mathcal{D}, \mathcal{L}_o, \mathcal{A}, \{j_z\}),$$

where admissible is as before. We have the moduli spaces $\mathcal{M}(\mathcal{L}_o, \mathcal{A}, \{j_z\}, \gamma_0, A)$ whose elements are class A, $J_{\mathcal{A}}$-holomorphic sections of $M \times \mathcal{D}$ with boundary on $\mathcal{L}_o$, asymptotic to γ_0. Here $J_{\mathcal{A}}$ is induced as previously by $\mathcal{A}, \{j_z\}$. Assuming that $\mathcal{A}$ is regular, we define

$$\langle S([o]), \gamma_0 \rangle = \sum_A \# \mathcal{M}(\mathcal{L}_o, \mathcal{A}, \{j_z\}, \gamma_0, A),$$

the sum is over all section classes A such that $\mathcal{M}(\mathcal{L}_o, \mathcal{A}, \{j_z\}, \gamma_0, A)$ has expected dimension 0, and is finite by monotonicity.

There is a natural extension of S to an algebra homomorphism

$$\Psi : H_*(\Omega_L Lag(M, L), \mathbb{Q}) \to FH(L, L), \tag{7.1}$$

working exactly like the author's extension [7] of the Seidel homomorphism. The algebra structure on the left is the Pontryagin algebra structure and the algebra structure on the right is with respect to quantum multiplication. If L is an object as before, then Ψ will be $\mathbb{Z}_2$ graded.

The map Ψ is defined as follows. To a smooth cycle

$$f : B \to \Omega_L Lag(M),$$

for B a smooth closed oriented manifold, we may construct an associated fibration data (suppressing the choice of family of almost complex structures)

$$\{M \times \mathcal{D}, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b, B\},$$

$b \in B$, $\mathcal{L}_b$ a Lagrangian subbundle of $M \times \mathcal{D}$ over $\partial \mathcal{D}$ determined by $f(b)$. If $\{\mathcal{A}_b\}$ is compatible with $\mathcal{A}(L, L)$ and is regular, we define:

$$\mathcal{M}(\{\mathcal{L}_b\}, \{\mathcal{A}_b\}, \gamma_0, A),$$

to be the space whose elements are pairs: (σ, b) for σ a $J(\mathcal{A}_b)$-holomorphic class A section of $M \times \mathcal{D}$ with boundary on $\mathcal{L}_b$, asymptotic to γ_0. We may then define as previously:

$$\langle \Psi([f]), \gamma_0 \rangle = \sum_A \# \mathcal{M}(\{\mathcal{L}_b\}, \{\mathcal{A}_b\}, \gamma_0, A),$$

where the sum is over all A s.t. $\mathcal{M}(\{\mathcal{L}_b\}, \{\mathcal{A}_b\}, \gamma_0, A)$, has expected dimension 0, as before the sum is finite by monotonicity.

Given the above definition, it is then clear that (6.14) holds as A_0 is the only class that can contribute to $\Psi([lag])$ for dimension reasons.

7.1. **Higher relative Seidel morphism path space version.** Let M, L be as before, and let $\mathcal{P}(L_0, L_1)$ denote the space of smooth paths in $Lag(M, L)$ from L_0 to L_1, which are assumed to be constant near end points. There is then an additive group homomorphism:

$$\Psi : H_*(\mathcal{P}(L_0, L_1), \mathbb{Q}) \to FH(L_0, L_1) \tag{7.2}$$

defined analogously as above. Let us elaborate for the sake of completeness. To a smooth cycle

$$f : B \to \mathcal{P}(L_0, L_1)$$

for B a smooth closed oriented manifold, we may associate fibration data

$$\{M \times \mathcal{D}, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b, B\},$$

$b \in B$, $\mathcal{L}_b$ a Lagrangian subbundle of $M \times \mathcal{D}$ over $\partial \mathcal{D}$ determined by $f(b)$.

Let $\mathcal{A}(L_0, L_1)$ denote the generic Hamiltonian connection on $M \times [0, 1]$ which is part of the perturbation data, for the definition of $FH(L_0, L_1)$. If $\{\mathcal{A}_b\}$ is compatible with $\mathcal{A}(L_0, L_1)$ and is regular, we define

$$\mathcal{M}(\{\mathcal{L}_b\}, \{\mathcal{A}_b\}, \gamma_0, A),$$

to be the space whose elements are pairs: (σ, b) for σ a $J(\mathcal{A}_b)$-holomorphic class A section of $M \times \mathcal{D}$ with boundary on $\mathcal{L}_b$, asymptotic to γ_0.

We may then define as previously:

$$\langle \Psi([f]), \gamma_0 \rangle = \sum_A \# \mathcal{M}(\{\mathcal{L}_b\}, \{\mathcal{A}_b\}, \gamma_0, A),$$

where the sum is over all A s.t. $\mathcal{M}(\{\mathcal{L}_b\}, \{\mathcal{A}_b\}, \gamma_0, A)$, has expected dimension 0.

7.2. **Higher Seidel morphism Pontryagin product and functoriality.** Let $Vect^{\mathbb{Q}}$ denotes the category of $\mathbb{Q}$ vector spaces, and let $H_* \mathcal{P}Lag(M, L)$ denote the $Vect^{\mathbb{Q}}$ enriched (that is $\mathbb{Q}$ linear) category of cycles of paths in $Lag(M, L)$, for M, L as above. More explicitly for a pair $L_0, L_1 \in Lag(M, L)$, define

$$hom_{H_* \mathcal{P}Lag(M,L)}(L_0, L_1)$$

to be the rational homology of the space of maps $[0, 1] \to Lag(M, L)$, with endpoints L_0, L_1. The composition map in $H_* \mathcal{P}Lag(M, L)$ is the Pontryagin product, denoted by $\star$.

Proposition 7.1. *The morphism Ψ in (7.2) extends to a $Vect^{\mathbb{Q}}$ enriched functor*

$$\Psi : H_* \mathcal{P}(Lag(M, L)) \to DFuk(M). \tag{7.3}$$

Proof. The proof is completely analogous to the proof that Ψ in (7.1) is an algebra morphism, which is as in [7]. □

8. Proof of Theorem 5.3 Part II (computation of the higher Seidel element)

8.1. **Computation of $\Psi([lag])$ via Morse theory for the Hofer length functional.** We shall now compute $\Psi(a) = \Psi([lag])$ by constructing special perturbation data, and using the functoriality above. Since Ψ is a group homomorphism we may restrict for simplicity to the case where $g : S^3 \to S^3$ and so a represent generators of the respective fundamental groups.

Under certain conditions the spaces of perturbation data for certain problems in Gromov-Witten theory admit a Hofer like functional. Although these spaces of perturbations are usually contractible, there may be a gauge group in the background that we have to respect, the reader may think of the situation in classical Yang-Mills theory. Without elaborating, the basic idea of regularization that we now do consists of pushing the perturbation data as far down as possible (in the sense of the functional) to obtain a mini-max (for the functional) data, which turns out to be especially nice and amenable to calculation. This idea first appeared in the author's [?]. We define the positive Hofer length functional

$$L^+ : \mathcal{P}Lag^{eq}(S^2) \to \mathbb{R},$$

$$L^+(\gamma) = \inf_{H^\gamma} \int_0^1 \max_{\gamma(t)} H_t^\gamma dt,$$

$\gamma(0) = L$ and where $H^\gamma : S^2 \times [0,1] \to \mathbb{R}$ is a function normalized to have zero mean at each moment, generating a lift of γ to $SO(3)$ starting at id. (That is H^γ generates a path in $SO(3)$, which moves L_0 along γ.) And where $\mathcal{P}Lag^{eq}(S^2)$ denotes the path space with some fixed end points. (Which we may later prescribe.) Note that $Lag^{eq}(S^2)$ is naturally diffeomorphic to S^2 and moreover it is easy to see that the functional L^+ is proportional to the Riemannian length functional L_{met} on the path space of S^2, with its standard round metric met. The idea of the computation is then this: perturb lag to be transverse to the (infinite dimensional) stable manifolds for the Riemannian length functional on $\Omega_{L_0} Lag^{eq}(S^2)$, push it down by the "infinite time" negative gradient flow for this functional, and use the resulting representative to compute $\Psi(a)$. Unfortunately the above length functional and even the energy functional is degenerate on the based path space giving rise to an unnecessarily complicated picture for the limiting representative, (its image is a pinched sphere in the loop space). To fix this we shall first perturb the end points so that we are doing Morse theory on the path space rather than loop space. Also we shall arrange details so as to (mostly) avoid dealing with infinite dimensional differential topology.

8.1.1. *The "energy" minimizing perturbation data.* Let us fix a pair of non conjugate for the standard round metric, points $L_0, L_1 \in Lag^{eq}(S^2) = S^2$, which in particular intersect transversely as Lagrangian submanifolds. Given a path from L_0 to L_1, say the minimal geodesic $geod$, the class a naturally induces

$$a' \in \pi_2(\mathcal{P}_{L_0,L_1}(S^2), geod).$$

And so a also gives an element $a' \in hom_{H_*\mathcal{P}Lag^{eq}(S^2)}(L_0, L_1)$. Clearly

$$a = [geod]^{-1} \star a' \in hom_{H_*\mathcal{P}Lag^{eq}(S^2)}(L_0, L_0),$$

and since $[geod]^{-1}$ is invertible in $hom_{H_*\mathcal{P}Lag^{eq}(S^2)}(L_1, L_0)$ it follows that $\Psi(a)$ is non-zero if $\Psi(a') \in HF(L_0, L_1)$ is non-zero, since $\Psi([geod]) \neq 0$ and is invertible by functoriality, or more concretely because it is the image of the fundamental class by the PSS map. We shall now construct suitable perturbation data for computation of $\Psi(a')$.

Classical Morse theory [6] tells us that the functional L_{met} on $\mathcal{P}_{L_0,L_1}(S^2)$ is Morse non-degenerate with a single critical point in each degree. Consequently a' has a representative in the 2-skeleton of $\mathcal{P}_{L_0,L_1}(S^2)$, for the Morse cell decomposition induced by L_{met}. Furthermore since $\pi_2(S^1) = 0$ such a representative cannot entirely lie in the 1-skeleton. It follows since we have a single Morse 2-cell that there is a representative $f' : S^2 \to \mathcal{P}_{L_0,L_1}(S^2)$, for a' s.t. the function ${f'}^* L_g$ is Morse with a unique maximizer max, (necessarily of index 2), and s.t. $\gamma_0 = f'(max)$ is the index 2 geodesic. (In principle there maybe more than one such maximizer, but recall that we assumed that *lag* represents the generator, in which case by further deformation we may insure that there is only one maximizer as the "degree" of f' is the intersection number of f' with the (infinite dimensional) stable manifold of the geodesic γ_0). The representative f' can also fairly easy be constructed by hand.

We shall also need that each path $f'(r)$ is constant for time near $0, 1$, $r \in S^2$, the necessary amount will be specified further on.

8.1.2. *Construction of the distinguished data* $\{\mathcal{A}'_r\}$. We construct $\{\mathcal{A}'_r\}$, $r \in S^2$ - a family of connections on

$$S^2 \times \mathcal{D},$$

s.t. $\mathcal{A}'_r$ is $\mathcal{L}_r$-exact, where $\mathcal{L}_r = \mathcal{L}_{f'(r)}$ is the Lagrangian sub-bundle over the boundary of $\mathcal{D}$ induced by $f'(r)$. Moreover $\mathcal{A}'_r$ will be $\mathcal{A}(L_0, L_1)$ compatible which by our assumptions on the perturbation data, since L_0, L_1 intersect transversally this just means that $\mathcal{A}'_r$ is the trivial connection in the strip coordinate chart at the end.

Let $SO(3) \to S^2$ be the principal S^1 bundle, whose fiber over $L \in S^2$ is the space of $g \in SO(3)$ which take L_0 into L. Fix an S^1 connection on this bundle, which by averaging may be assumed to be $SO(3)$ invariant. Let α be the associated lie algebra lie $S^1 \simeq i\mathbb{R}$ valued connection 1-form on $SO(3)$. Decoding the above, we see that this is just the standard contact form on $\mathbb{RP}^3$ (given the canonical isomorphism lie $S^1 \simeq \mathbb{R}$.) Note that an α-flat lift of a geodesic for the round metric is a geodesic in $SO(3)$, for its natural (induced by the round metric on S^3) bi-invariant metric, hence a one parameter subgroup (and hence is a geodesic for the pull-back Hofer metric).

Given a path $p : [0, 1] \to S^2$ starting at L, we denote by $\widetilde{p}$ the α-flat lift of p starting at id. We now set H^r to be the zero mean normalized smooth function generating the path $\widetilde{f'(r)}$ for each r. For each $r \in S^2$ we define the coupling form $\widetilde{\Omega}_r$ on $S^2 \times D^2$:

$$\widetilde{\Omega}_r = \omega - d(\eta(rad) \cdot H^r d\theta),$$

for (rad, θ) the modified angular coordinates on D^2, $\theta \in [0, 1]$ and $\eta : [0, 1] \to [0, 1]$ is a smooth function satisfying

$$0 \leq \eta'(rad),$$

and

$$\eta(rad) = \begin{cases} 1 & \text{if } 1 - \delta \leq rad \leq 1, \\ rad^2 & \text{if } rad \leq 1 - 2\delta, \end{cases} \tag{8.1}$$

for a small $\delta > 0$. Fix an embedding $i : D^2 \hookrightarrow \mathcal{D}$ so that the image of the embedding contains $\partial\mathcal{D} - \partial\, end$ where *end* is the image of the distinguished strip chart $[0, 1] \times (0, \infty) \to \mathcal{D}$. Next fix a deformation retraction *ret* of $\mathcal{D}$ onto $i(D^2)$ as above. And set $\widetilde{\Omega}'_r = ret^* \widetilde{\Omega}_r$ on $S^2 \times \mathcal{D}$, whose induced connection is our $\mathcal{A}'_r$, if we insure that $f'(r)$ is constant outside of $i^{-1}(\partial(i(D^2) \cap \mathcal{D}))$ as a map of $S^1 = \partial D^2$, which we can do by adjusting f'.

8.1.3. *The properties of* $\{\mathcal{A}'_r\}$. Let $\mathcal{C}(L_0, L_1)$ be the space of coupling forms $\widetilde{\Omega}$ on $S^2 \times \mathcal{D}$ s.t. for each such $\widetilde{\Omega}$ the associated connection is $\mathcal{L}_p$-exact, for some

$$p \in \mathcal{P}_{L_0,L_1} Lag^{eq}(S^2).$$

Define

$$\begin{aligned}&\text{area} : \mathcal{C}(L_0, L_1) \to \mathbb{R}\\ &\text{area}(\widetilde{\Omega}) = \inf_{\alpha} \int_{\mathcal{D}} \alpha \,|\widetilde{\Omega} + \pi^*(\alpha) \text{ is nearly symplectic}\},\end{aligned}$$

where nearly symplectic is as before. Then by elementary calculation we have:

$$\text{area}(\widetilde{\Omega}'_r) = L^+(f'(r)). \tag{8.2}$$

To verify (8.2) first check that the infimum is attained on the uniquely defined 2-form $\alpha_{\widetilde{\Omega}}$:

$$\alpha_{\widetilde{\Omega}}(v, w) = \max_{\mathcal{D}} R_{\widetilde{\Omega}}(v, w), \tag{8.3}$$

where $R_{\widetilde{\Omega}}$ is the Lie algebra valued curvature 2-form of (the connection induced by) $\widetilde{\Omega}$, and we are using the isomorphism $lie\text{Ham}(S^2, \omega) \simeq C^{\infty}_{norm}(S^2)$. The following then readily follows.

Lemma 8.1. *The function area* $: r \mapsto \text{area}(\widetilde{\Omega}'_r)$ *has a unique maximizer, coinciding with the maximizer* max *of* ${f'}^* L_{met}$ *and* area *is Morse at* max *with index* 2.

8.1.4. *Finding class* A_0 *holomorphic sections for the data.* As $f'(\max)$ is a closed geodesic for met, and so a geodesic for L^+ there is a point

$$x_{\max} \in \bigcap_t (L_t = f'(\max)(t))$$

maximizing $H^{\max}_t = H^{\max}|_{S^2 \times \{t\}}$ at each moment, c.f. [3] moreover since H^r generates a non-constant path in $SO(3)$ this point is unique. Define

$$\sigma_{\max} : \mathcal{D} \to S^2 \times \mathcal{D},$$

to be the pull-back by the retraction ret above, of the constant section

$$z \mapsto x_{\max},$$

of $S^2 \times D^2$. Then $\sigma_{\max}$ is a flat section for $\mathcal{A}'_{\max}$, with boundary on $\mathcal{L}_{f'(\max)}$, and is consequently holomorphic. Let $\gamma_0 \in CF(L_0, L_1)$ be the generator corresponding to $x_{\max}$, then $\sigma_{\max}$ will be an element of $\mathcal{M}(\{\mathcal{A}_b\}, \gamma_0, A)$, if it is in class A that is if it has vertical Maslov number -2. We now check this. Let p' be the path in

$$S^1 \simeq Lag(T_{x_{\max}} S^2) \simeq Lag(\mathbb{R}^2)$$

obtained by pulling back the vertical tangent bundle of $\mathcal{L}_r$ by $\sigma_{\max}$. By our conventions for the Hamiltonian vector field:

$$\omega(X_H, \cdot) = -dH(\cdot),$$

p' is a clockwise path from $T_{x_{\max}} L_0$ to $T_{x_{\max}} L_1$. By the Morse index theorem [6] the condition that the Morse index of $f'(\max)$ has Morse index 2, the concatenation of p' with the counter-clockwise path from $T_{x_{\max}} L_1$ back to $T_{x_{\max}} L_0$ is a degree -1 loop, if $S^1 \simeq Lag(\mathbb{R}^2)$ is given the counter-clockwise orientation. Consequently $\sigma_{\max}$ has Maslov number -2, by the definition Appendix B.

Proposition 8.2. $\sigma_{\max}$ *is the sole element of*

$$\overline{\mathcal{M}}(\{\mathcal{L}_r\}, \{\mathcal{A}'_r\}, \gamma_0, A_0).$$

Proof. By direct calculation:

$$-\int_{\mathcal{D}} \sigma^*_{\max} \widetilde{\Omega}'_{\max} = L^+(f'(\max)), \tag{8.4}$$

so by (8.2) and by Lemmas 6.10, 6.11 we have:

$$L^+(f'(\max)) \leq \operatorname{area}(\widetilde{\Omega}^-_r) = L^+(f'(r)),$$

whenever there is an element

$$(\sigma, r) \in \overline{\mathcal{M}}(\{\mathcal{L}_r\}, \{\mathcal{A}'_r\}, \gamma_0, A_0).$$

But clearly this is impossible unless $r = \max$, since $L^+(f'(r)) < L^+(f'(\max))$ for $r \neq \max$. So to finish the proof of the proposition we just need:

Lemma 8.3. *There are no elements* $(\sigma, \max)$ *other than* $(\sigma_{\max}, \max)$ *of the moduli space*

$$\overline{\mathcal{M}}(\{\mathcal{L}_{\max}\}, \{\mathcal{A}'_{\max}\}, \gamma_0, A_0).$$

Proof. We have by (8.4), and by (8.2)

$$0 = \langle [\widetilde{\Omega}'_{\max} + \alpha_{\widetilde{\Omega}'_{\max}}], [\sigma_{\max}] \rangle,$$

and so given another element $(\sigma, \max)$ by invariance we have:

$$0 = \langle [\widetilde{\Omega}'_{\max} + \alpha_{\widetilde{\Omega}'_{\max}}], [\sigma] \rangle.$$

It follows that σ is necessarily $\widetilde{\Omega}'_{\max}$-horizontal, since

$$(\widetilde{\Omega}'_{\max} + \alpha_{\widetilde{\Omega}'_{\max}})(v, J_{\widetilde{\Omega}'_{\max}} w) \geq 0,$$

and is strictly positive for v in the vertical tangent bundle of

$$S^2 \hookrightarrow S^2 \times \mathcal{D} \to \mathcal{D}.$$

But then $\sigma = \sigma_{\max}$ since $\sigma_{\max}$ is the only flat section asymptotic to γ_0. □

□

8.1.5. *Regularity.* It will follow that

$$\Psi(a') = \pm[\gamma_0],$$

if we knew that $(\sigma_{\max}, \max)$ was a regular element of

$$\overline{\mathcal{M}}(\{\mathcal{L}_r\}, \{\mathcal{A}'_r\}, \gamma_0, A_0).$$

We won't answer directly if $(\sigma_{\max}, \max)$ is regular, although it likely is. But it is regular after a suitably small Hamiltonian perturbation of the family $\{\mathcal{A}'_r\}$ vanishing at $\mathcal{A}'_{\max}$. This is the essential regularity mentioned earlier.

Lemma 8.4. *There is a family* $\{\mathcal{A}^{reg}_r\}$ *arbitrarily* C^∞*-close to* $\{\mathcal{A}'_r\}$ *with* $\mathcal{A}^{reg}_{\max} = \mathcal{A}'_{\max}$ *and such that*

$$\mathcal{M}(\{\mathcal{L}_r\}, \{\mathcal{A}^{reg}_r\}, \gamma_0, A_0), \tag{8.5}$$

is regular, with $(\sigma_{\max}, \max)$ *its sole element.*

Proof. The associated real linear Cauchy-Riemann operator

$$D_{\sigma_{\max}} : \Omega^0(\sigma^*_{\max} T^{vert} \widetilde{\mathcal{S}}_{\max}) \to \Omega^{0,1}(\sigma^*_{\max} T^{vert} \widetilde{\mathcal{S}}_{\max}),$$

has no kernel, by Riemann-Roch [5, Appendix C], as the vertical Maslov number of $[\sigma_{\max}]$ is -2. And the Fredholm index of $(\sigma_{\max}, \max)$ which is -2, is -1 times the Morse index of the function area at max, by Lemma 8.1. Given this our lemma follows by a direct analogue of [10, Theorem 1.20], itself elaborating on the argument in [**?**]. $\square$

To summarize we have shown the following:

Theorem 8.5. *Let $L_0 \subset S^2$ be the equator, with a given spin structure. And let $a \in \pi_2 \Omega_{L_0} Lag(S^2, L_0)$ be the generator. Then*

$$0 \neq \Psi(a) \in HF(L_0, L_0).$$

Proof. This follows by multiplicativity

$$\Psi(a) = \Psi([geod])^{-1} * \Psi(a').$$

$\square$

This finishes the section and the proof of the theorem.

9. Application to Hofer geometry

We give here a proof of Theorem 1.2. This theorem is a relative analogue of the theorem given in the author's thesis [8].

Proof of Theorem 1.2. Let $f_0(r) = geod^{-1} \cdot f'(r)$ where $\cdot$ is concatenation of paths. Then $eq = f_0(\max)$ is a simple great circle in $S^2 = Lag^{eq}(S^2)$. And so

$$\max_{s \in S^2} L^+(f_0(r)) = L^+(eq) = 1/2 \operatorname{area}(S^2, \omega),$$

$[f_0] \in a$ and we will show that f_0 is minimizing.

Let $H^{\max}$ be a function generating eq, and let A_0 be the Maslov index -2 class of the constant section $z \mapsto x_{\max}$, $x_{\max}$ the maximizer of $H^{\max}$ on L_0 at each moment. Given any $f \in o$ let $\{\mathcal{A}_r\}$, $r \in S^2$ be the family of connections on $S^2 \times \mathcal{D}$, with $\mathcal{A}_r$ $f_0(r)$-admissible, s.t. $\operatorname{area}(\mathcal{A}_r) = L^+(f(r))$. This family is constructed analogously to the family $\{\mathcal{A}'_r\}$. By Lemma 6.10, Lemma 6.11 and by the fact that the coupling area of a class A_0 section of $S^2 \times \mathcal{D}$ is $1/2 \operatorname{area}(S^2, \omega)$ we get that any element $(\sigma, r) \in \mathcal{M}(\{\mathcal{A}_r\}, A_0)$, gives a lower bound:

$$1/2 \operatorname{area}(S^2, \omega) \leq L^+(f(s)),$$

where $\mathcal{M}(\{\mathcal{A}_r\}, A_0)$ denotes the space of class A finite energy holomorphic sections. And $\mathcal{M}(\{\mathcal{A}_r\}, A_0)$ is non-empty by Theorem 8.5. Consequently we get:

$$\min_{f \in a} \max_{r \in S^2} L^+(f(r)) \geq 1/2 \cdot \operatorname{area}(S^2, \omega).$$

And we have seen that the bound is sharp with the minimum attained on f_0.

$\square$

$\square$

Appendix A. Homotopy groups of Kan complexes

Given a pointed Kan complex $(X_\bullet, x)$ and $n \geq 1$ the *n'th simplicial homotopy group* of $(X_\bullet, x)$: $\pi_n(X_\bullet, x)$ is defined to be the set of equivalence classes of morphisms

$$\Sigma : \Delta^n_\bullet \to X_\bullet,$$

for $\Delta^n_\bullet(k) = hom_\Delta([k],[n])$, for Δ the simplicial category, such that Σ takes $\partial\Delta^n_\bullet$ to x. Since for us $X_\bullet$ is often the singular set associated to a topological space X, we note that such morphisms are in complete correspondence with maps:

$$\Sigma : \Delta^n \to X,$$

taking the topological boundary of Δ^n to x.

Two such maps are equivalent if there is a diagram (simplicial homotopy):

$$\begin{tikzcd}
\Delta^n_\bullet \arrow[d, "i_0"] \arrow[dr, "\Sigma_1"] & \\
\Delta^n_\bullet \times I_\bullet \arrow[r, "H"] & X_\bullet \\
\Delta^n_\bullet . \arrow[u, "i_1"] \arrow[ur, "\Sigma_2"'] &
\end{tikzcd}$$

such that $\partial\Delta^n_\bullet \times I_\bullet$ is taken by H to $x_\bullet$, with the latter denoting the image of $\Delta^0_\bullet \to X$, induced by the vertex inclusion $x \to X_\bullet$. The simplicial homotopy groups of a Kan complex $(X_\bullet, x)$ coincide with the classical homotopy groups of the geometric realization $(|X_\bullet, x|)$. But the power of the above definition is that if we know our Kan complex well, (like in the example of the present paper) the simplicial homotopy groups are very computable since they are completely combinatorial in nature.

Appendix B. On the Maslov number

Let S be obtained from a compact connected Riemann surface S' with boundary, by removing a finite number of points $\{e_i\}$ removed from the boundary of S'.

Let $\mathcal{V} \to S$ be a rank r complex vector bundle, trivialized at the open ends $\{e_i\}$, that is so we have distinguished bundle charts $\mathbb{C}^r \times [0,1] \times [0,\infty) \to \mathcal{V}$ at the ends.

Let

$$\Xi \to \partial S \subset S$$

be a totally real rank r subbundle of $\mathcal{V}$, which is constant in the coordinates

$$\mathbb{C}^r \times [0,1] \times [0,\infty),$$

at the ends. For each end e_i and its distinguished chart $e_i : [0,1] \times [0,\infty) \to S$ let $b_i^j : [0,\infty) \to \partial S$, $j = 0,1$ be the restrictions of e_i to $\{i\} \times [0,\infty)$.

We then have a pair of real vector spaces

$$\Xi_i^j = \lim_{\tau \mapsto \infty} \Xi|_{b_i^j(\tau)}.$$

There is a Maslov number $Maslov(\mathcal{V}, \Xi, \{\Xi_i^j\})$ associated to this data coinciding with the boundary Maslov index in the sense of [5, Appendix C3], in the case $\Xi_i^0 = \Xi_i^1$, for the modified pair $(\mathcal{V}^{/}, \Xi^{/})$ obtained from $(\mathcal{V}, \Xi, \{\Xi_i^j\})$ by naturally closing off each e_i end of $\mathcal{V} \to S$. When Ξ_i^0 is transverse to Ξ_i^1 $Maslov(\mathcal{V}, \Xi, \{\Xi_i^j\})$

is obtained as the Maslov index for the modified pair $(\mathcal{V}^/, \Xi^/)$ by again closing off the ends e_i via gluing (at each end e_i) with

$$(\mathbb{C}^r \times \mathcal{D}, \widetilde{\Xi}, \{\widetilde{\Xi}_0^j\}),$$

where $\mathcal{D}$ as before is diffeomorphic to D^2 with a point e_0 on the boundary removed. Here $\widetilde{\Xi}_i^0 = \Xi_0^1$ and $\widetilde{\Xi}_i^1 = \Xi_0^0$, while $\widetilde{\Xi}$ over the boundary of $\mathcal{D}$ is determined by the "shortest path" from $\widetilde{\Xi}_0^0$ to $\widetilde{\Xi}_0^1$, meaning that since these are a pair of transverse totally real subspaces up to a complex isomorphism of $\mathbb{C}^r$ (whose choice will not matter) we may identify them with the subspaces $\mathbb{R}^r$, and $i\mathbb{R}^r$ after this identification our shortest path is just $e^{i\theta}\mathbb{R}^r$, $\theta \in [0, \pi_2]$.

For a real linear Cauchy-Riemann operator on $\mathcal{V}$, with suitable asymptotics, the Fredholm index is given by:

$$r \cdot \chi(S) + Maslov(\mathcal{V}, \Xi, \{\Xi_i\}).$$

The proof of this is analogous to [5, Appendix C], we can also reduce it to that statement via a gluing argument. (This kind of argument appears for instance in [11])

B.1. **Dimension formula for moduli space of sections.** Suppose we are given a Hamiltonian fiber bundle $M^{2r} \hookrightarrow \widetilde{S} \to S$, with end structure and S as above. Let $\mathcal{L}$ be a Lagrangian sub-bundle of $\widetilde{S}$ over the boundary of S, compatible with the end structure, and such that the Lagrangian submanifolds

$$L_i^j = \lim_{\tau \mapsto \infty} \mathcal{L}|_{b_i^j(\tau)},$$

intersect transversally for b_i^j as above, or coincide.

Given an $\mathcal{L}$-exact Hamiltonian connection $\mathcal{A}$, on $\widetilde{S}$, (see Definition 6.5) which is additionally assumed to be trivial in the strip coordinate charts at the ends, and a choice of a family $\{j_z\}$ of compatible almost complex structures on the fibers of $\widetilde{S}$, set $\mathcal{M}(A)$ to be the moduli space of (relative) class A finite vertical L^2 energy holomorphic sections of $\widetilde{S} \to S$ with boundary in $\mathcal{L}$. Define

$$Maslov^{vert}(A)$$

to be the Maslov number of the triple $(\mathcal{V}, \Xi, \{\Xi_i\})$ determined by the pullback by $\sigma \in \mathcal{M}(A)$ of the vertical tangent bundle of $\widetilde{S}$, $\mathcal{L}$. Then the expected dimension of $\mathcal{M}(A)$ is:

$$r \cdot \chi(S) + Maslov^{vert}(A).$$

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