RESEARCH STATEMENT

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Broadly I am working in differential geometry, particularly symplectic, dynamical systems, and algebraic topology. I also recently became interested in computer science. My most recent work is in a number of research directions. For example, I am researching rigidity phenomena in locally conformally symplectic geometry as partly initiated by my recent work [17], and thinking about the holomorphic Weinstein conjecture that appears there. This is a relatively new subject, but has the potential to become big, as it has a number of connections to other branches of differential geometry and dynamical systems. I am also working towards the conjecture on non-existence, or C^0-nearby non-existence of sky catastrophes for homotopies of Reeb vector fields. The latter conjecture appears in my [18] and its resolution would likely be one of the most exciting possibilities that is presented in this research statement, as the are a number of applications, in particular new existence results for Reeb orbits which traditionally needed techniques of elliptic pde's. I now describe in more detail the pair of research directions mentioned above, and their relationship, as this is where most of my most recent activity concentrated.

1. Sky catastrophes, Reeb vector fields, and the no Reeb sky catastrophe conjecture

The original Seifert conjecture [20] asked if a non-singular vector field on S^3 must have a periodic orbit. In this formulation the answer was shown to be no for C^1 vector fields by Schweitzer [19], for C^2 vector fields by Harrison [10] and later for C^∞ vector fields by Kuperberg [15]. A C^1 volume preserving counter-example is given by Kuperberg in [14]. For a vector field X C^0 close to the Hopf vector field it was shown to hold by Seifert and later by Fuller [7] in his 1967 paper, using his Fuller index, which is a kind fixed point index for orbits. Part of the importance of the C^0 condition for Fuller is that it rules out "sky catastrophes" for an appropriate homotopy of non-singular vector fields connecting X to the Hopf vector field. The latter "sky catastrophes" are the last discovered kind of bifurcations originally constructed by Fuller himself [6]. He constructs a smooth family $\{X_t\}, t \in [0,1]$ of vector fields on a solid torus, for which there is a continuous (and isolated) family of $\{X_t\}$ periodic orbits $\{o_t\}$, with the period of o_t going to infinity as $t \mapsto 1$, and so that for $t = 1$ the orbit disappears. Let us give the following a bit more general but still incomplete definition here, a full definition (according to us) appears in [18].

Definition 1.1 (Incomplete)**.** *A* **sky catastrophe** *for a smooth family $\{X_t\}$, $t \in [0,1]$, of vector fields on a manifold M is a continuous family of closed orbits $\tau \mapsto o_{t_\tau}$, o_{t_τ} is a non-constant periodic orbit of X_{t_τ}, $\tau \in [0,\infty)$, such that the period of o_{t_τ} unbounded from above.*

These sky catastrophes (and their more robust analogues called blue sky catastrophes) turned out to be common in many kinds of systems appearing in nature and have been studied on their own, see for instance Shilnikov-Turaev [21].

However since the time of Fuller's original papers it has not been understood if this the only thing that can go wrong. That is if without existence of a "sky catastrophe" in an appropriate general sense, the time 1 limit of a homotopy of smooth non-singular vector fields on S^{2n+1} starting at the Hopf vector field must have a periodic orbit. The difficulty in answering this is that although our orbits cannot "disappear into the sky", as there are infinitely many of them they may "cancel each other out", even if the Fuller index is "locally positive" - that is the index of isolated components in the orbit space is positive. In the C^0 nearby case this cancellation is prevented as orbits from isolated components of the orbit space may not interact. The reader may think of trying to make sense of the infinite sum

$$(5-1)+(5-1)+\ldots+(5-1)+\ldots.$$

While generally meaningless it has some meaning if we are not allowed to move the terms out of the parentheses. So one has to develop a version of Fuller's index which precludes such total cancellation in general. We do this in [18] and using this prove as a particular case:

Theorem 1. *Let $X = X_1$ be a smooth non-singular vector field on S^{2k+1} homotopic to the Hopf vector field $H = X_0$ through homotopy $\{X_t\}$ of smooth non-singular vector fields. Suppose that $\{X_t\}$ has no sky catastrophes then X has periodic orbits.*

Can we use Theorem 1 and its general analogues to show existence of orbits? The most promising case where this should be possible is for Reeb vector fields.

1.1. Reeb vector fields and sky catastrophes.

As a first step we may ask if a homotopy of Reeb vector fields $\{X_t\}$ on a closed manifold is necessarily free of sky catastrophes. Our following elementary theorem puts a very strong restriction on the kinds of sky catastrophes that can happen, for general contact manifolds. It is likely, that if they exist, they must be pathological, and very hard to construct.

We note however that in the proof of [13, Theorem 1.19] a kind of partial Reeb plug is constructed, which is missing the matching condition, see for instance [15] for terminology of plugs, also see Kerman [12], Ginzburg [8], where plugs are utilized in Hamiltonian context. If one had a plug with all conditions, then it is simple to construct a sky catastrophe. For we may deform such a plug through partial plugs satisfying all conditions except the trapping condition (condition 3 in [15]) to a trivial plug. This deformation then readily gives a sky catastrophe corresponding to the trapped orbit. Without matching, this argument does not obviously work.

Theorem 2. *Let $\{X_t\}$, $t \in [0,1]$ be a smooth homotopy through Reeb vector fields on a contact manifold M. (Informally) let S denote the space of orbits of the family $\{X_t\}$, where period is allowed to vary. Then there is no (period) unbounded from above locally Lipschitz continuous $p : [0,\infty) \to S$ whose composition with the projection $\pi : S \to [0,1]$ has finite length, with π the projection to the time coordinate t.*

The above rules out for example Fuller's sky catastrophe that appears in [6], and described in the beginning of our paper.

Conjecture 1. *Let $\{X_t\}$, $t \in [0,1]$ be a smooth homotopy through Reeb vector fields on a compact contact manifold M. Then there is a C^0 nearby smooth family $\{X'_t\}$, $t \in [0,1]$, $X'_i = X_i$, $i = 0,1$, such that $\{X'_t\}$ has no sky catastrophes.*

Given this conjecture we may readily apply general analogues in [17] of Theorem 1 to get applications to existence of Reeb orbits. Of course one exciting thing here is that it will be without so called "hard" elliptic pde techniques of pseudo-holomorphic curves. And that moreover we would obtain new qualitative dynamical-topological information about Reeb vector fields.

2. Locally conformally symplectic geometry and rigidity

A locally conformally symplectic manifold or l. c. s. m. is a smooth $2n$-fold M with an l. c. s. structure: which is a non-degenerate 2-form ω, which is locally diffeomorphic to $f \cdot \omega_{st}$, for some (non-fixed) positive smooth function f, with ω_{st} the standard symplectic form on $\mathbb{R}^{2n}$. These were originally considered by Lee in [16], arising naturally as part of an abstract study of "a kind of even dimensional Riemannian geometry", and then further studied by a number of authors see for instance, [1] and [23]. This is a fascinating object, an l. c. s. m. admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and Murphy show that if a closed almost complex $2n$-fold M has $H^1(M,\mathbb{R}) \neq 0$ then it admits a l. c. s. structure, [4], see also [2].

l. c. s. m.'s can also be understood to generalize contact manifolds. This works as follows. First we have a natural class of explicit examples of l. c. s. m.'s, obtained by starting with a symplectic cobordism (see [4]) of a closed contact manifold C to itself, arranging for the contact forms at the two ends of the cobordism to be proportional (which can always be done) and then gluing together the boundary components. As a particular case of this we get Banyaga's basic example.

Example 1 (Banyaga). Let (C, ξ) be a contact manifold with a contact form λ and take $M = C \times S^1$ with 2-form $\omega = d^\alpha \lambda := d\lambda - \alpha \wedge \lambda$, for α the pull-back of the volume form on S^1 to $C \times S^1$ under the projection.

Using above we may then faithfully embed the category of contact manifolds, and contactomorphism into the category of l. c. s. m.'s, and ***loose*** l. c. s. morphisms. These can be defined as diffeomorphisms $f : (M_1, \omega_1) \to (M_2, \omega_2)$ s.t. $f^*\omega_2$ is deformation equivalent through l. c. s. structures to ω_1. Note that when ω_i are symplectic this is just a global conformal symplectomorphism by Moser's trick.

Banyaga type l. c. s. m.'s give immediate examples of almost complex manifolds where the energy function is unbounded on the moduli spaces of fixed class pseudo-holomorphic curves, as well as where null-homologous J-holomorphic curves can be non-constant. We show in [17] that it is still possible to extract a variant of Gromov-Witten theory for l. c. s. m.'s. The story is closely analogous to that of the Fuller index in dynamical systems, which is concerned with certain rational counts of periodic orbits. In that case sky catastrophes prevent us from obtaining a completely well defined invariant, but Fuller constructs certain partial invariants which give dynamical information. In a very particular situation the relationship with the Fuller index becomes perfect as one of the results of [17] obtains the classical Fuller index for Reeb vector fields on a contact manifold C as a certain genus 1 Gromov-Witten invariant of the l. c. s. m. $C \times S^1$. The latter also gives a conceptual interpretation for why the Fuller index is rational, as it is reinterpreted as an (virtual) orbifold Euler number.

2.0.1. *Non-squeezing and rigidity.* Of course what we are really interested in is what kind of rigidity phenomenon can appear in l. c. s. geometry. As a first attempt what can be said about non-squeezing? Recall that one of the most fascinating early results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [9]. The most well known formulation of this is that there does not exist a symplectic embedding $B_R \to D^2(r) \times \mathbb{R}^{2n-2}$ for $R > r$, with B_R the standard closed radius R ball in $\mathbb{R}^{2n}$ centered at 0. Gromov's non-squeezing is C^0 persistent in the following sense.

Theorem 3. *Given $R > r$, there is an $\epsilon > 0$ s.t. for any symplectic form ω' on $S^2 \times T^{2n-2}$ C^0 close to a split symplectic form ω and satisfying*

$$\langle \omega, (A = [S^2] \otimes [pt]) \rangle = \pi r^2,$$

there is no symplectic embedding $\phi : B_R \hookrightarrow (S^2 \times T^{2n-2}, \omega')$.

On the other hand it is natural to ask:

Question 1. Given $R > r$ and every $\epsilon > 0$ is there a 2-form ω' on $S^2 \times T^{2n-2}$ C^0 or even C^∞ ϵ-close to a split symplectic form ω, satisfying $\langle \omega, A \rangle = \pi r^2$, and s.t. there is an embedding $\phi : B_R \hookrightarrow S^2 \times T^{2n-2}$, with $\phi^* \omega' = \omega_{st}$?

The above theorem follows immediately by Gromov's argument in [9], we give give in [17] a certain extension of this theorem for l. c. s. structures. One may think that recent work of Müller [22] may be related to the question above and our theorem below. But there seems to be no obvious such relation as pull-backs by diffeomorphisms of nearby forms may not be nearby. Hence there is no way to go from nearby embeddings that we work with to ϵ-symplectic embeddings of Müller.

Definition 2.1. *Given a pair of* l. c. s. m.*'s (M_i, ω_i), $i = 0, 1$, we say that $f : M_1 \to M_2$ is a* **morphism***, if $f^*\omega_2 = \omega_1$. A morphism is called an* l. c. s. *embedding if it is injective.*

A pair (ω, J) for ω l. c. s. and J compatible will be called a compatible l. c. s. pair, or just a compatible pair, where there is no confusion. Note that the pair of hypersurfaces $\Sigma_1 = S^2 \times S^1 \times \{pt\} \subset S^2 \times T^2$,

$\Sigma_2 = S^2 \times \{pt\} \times S^1 \subset S^2 \times T^2$ are naturally foliated by symplectic spheres, we denote by $T^{fol}\Sigma_i$ the sub-bundle of the tangent bundle consisting of vectors tangent to the foliation. The following theorem says that it is impossible to have certain "nearby" l. c. s. embeddings, which means that we have a first rigidity phenomenon for l. c. s. structures. There is a small caveat here that in what follows we take the C^0 norm on the space of l. c. s. structures that is (likely) stronger then the natural C^0 norm (with respect to a metric) on the space of forms.

Theorem 4. *Let ω be a split symplectic form on $M = S^2 \times T^2$, and A as above with $\langle \omega, A \rangle = \pi r^2$. Let $R > r$, then there is an $\epsilon > 0$ s.t. if ω_1 is an* l. c. s. *on M C^0 ϵ-close to ω, then there is no* l. c. s. *embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1),$$

s.t $\phi_* j$ preserves the bundles $T^{fol}\Sigma_i$, for j the standard almost complex structure.

We note that the image of the embedding ϕ would be of course a symplectic submanifold of (M, ω_1). However it could be highly distorted, so that it might be impossible to complete $\phi_*\omega_{st}$ to a symplectic form on M nearby to ω. We also note that it is certainly possible to have a nearby volume preserving as opposed to l. c. s. embedding which satisfies all other conditions. Take $\omega = \omega_1$, then if the symplectic form on T^2 has enough volume, we can find a volume preserving map $\phi : B_R \to M$ s.t. $\phi_* j$ preserves $T^{fol}\Sigma_i$. This is just the squeeze map, which as a map $\mathbb{C}^2 \to \mathbb{C}^2$ is $(z_1, z_2) \mapsto (\frac{z_1}{a}, a \cdot z_2)$. In fact we can just take any volume preserving map ϕ, which doesn't hit Σ_i.

2.0.2. *Toward non-squeezing for loose morphisms.* In some ways loose morphisms of l. c. s. m.'s are more natural, particularly when we think about l. c. s. m.'s from the contact angle. So what about non-squeezing for loose morphisms as defined above? We can try a direct generalization of contact non-squeezing of Eliashberg-Polterovich [3], and Fraser in [5]. Specifically let $R^{2n} \times S^1$ be the pre-quantization space of R^{2n}, or in other words the contact manifold with the contact form $d\theta - \lambda$, for $\lambda = \frac{1}{2}(ydx - xdy)$. Let B_R now denote the open radius R ball in $\mathbb{R}^{2n}$.

Question 2. If $R \geq 1$ is there a compactly supported, loose endomorphism of the l. c. s. m. $\mathbb{R}^{2n} \times S^1 \times S^1$ which takes the closure of $U := B_R \times S^1 \times S^1$ into U?

Naturally we expect the answer is no. We can prove this assuming the following l. c. s. analogue of the Weinstein conjecture.

2.1. Holomorphic Weinstein conjecture.

In contact geometry rigidity phenomena circulate around existence phenomena of Reeb orbits. The most important conjecture concerning Reeb orbits is the Weinstein conjecture, which says that a closed contact manifold always has a Reeb orbit. If l. c. s. m.'s are generalizations of contact manifolds, what is the analogue of this conjecture in l. c. s. geometry? To start we propose the following:

Conjecture 2. *For any compatible pair (ω, J), for ω* l. c. s. *form on $C \times S^1$ for C a threefold or S^{2k+1}, there is an elliptic, non-constant J-holomorphic curve in $C \times S^1$.*

In [17] we prove this conjecture for ω C^∞ nearby to the Hopf l. c. s. structure on $S^{2k+1} \times S^1$, by exploiting the connection with Fuller index, to which we already alluded in our discussion of sky catastrophes. We also show there that this conjecture implies the Weinstein conjecture for S^{2k+1} and for C a contact three-fold.

Of course it is natural to ask, other then curiosity what is the significance of the above conjecture? Well as we mentioned contact geometry rigidity is linked to existence of Reeb orbits, for example there are certain capacities called ECH capacities, and related constructions [11] that come from the machine of embedded contact homology of Huchings-Taubes, (and built using Reeb orbits) constructed partly for the proof of the three dimensional Weinstein conjecture. There is an analogous story in the l. c. s. setting, and using it we will get new rigidity phenomena in l. c. s. geometry that we are looking for, in particular Question 2 on loose l. c. s. non-squeezing will be answered.

2.2. **Outline of the proof of Conjecture 2 when C is a 3-fold.** In this section we shall assume that the reader knows some basic language of symplectic field theory, Seiberg-Witten theory and embedded contact homology. When C is a 3-fold, and the Seiberg-Witten invariant of $M = C \times S^1$ is non-vanishing the path to the proof of Conjecture 2 is almost obvious. First we take a separating contact hypersurface Σ in M, and perform neck-stretching on the pair (ω, J) at Σ. In the end we brake up M into a pair of pieces M_i with cylindrical ends, with l. c. s. forms ω_i which are in fact globally conformally symplectic, at least when M_i are simply connected - for a general C we have to make additional assumptions on the Lee class of ω, and or the hypersurface Σ. (Lee class is a certain invariant of an l. c. s. form which vanishes when it is symplectic, see my [17] for instance.)

Globally conformally symplectic is as good as symplectic for the purpose of Gromov-Witten or SFT analysis. We may then consider the count (in the total homology class A determined by the spin-c structure induced by ω) of holomorphic buildings consisting of a pair of holomorphic curves with ECH index 0 in each part M_i with the same asymptotic constraints at the cylindrical end. This count is an invariant of the l. c. s. (M, ω), even though the count of J-holomorphic curves in M by itself is not a priori an invariant, and we claim that this invariant is the Seiberg-Witten invariant of M, for the spin-c structure determined by ω. This of course builds on the foundational work of Hutchings and Taubes. Luckily the technical work for this correspondence in a setup closely analogous to the above is worked out in the thesis of Chris Gerig, which generalizes work of Taubes on GW-SW correspondence to general smooth 4-folds.

So if the Seiberg-Witten invariant of M is non-vanishing then a gluing argument gives an existence of a non-constant, class A, J-holomorphic curve in M. This is enough for the applications that we have in mind (e.g. Question 2), however a more careful analysis should yield that this curve is actually an elliptic curve.

The above assumes non-vanishing of the SW invariant of M, it is tempting to guess that this always happens, analogously with the symplectic case. This would then be the goal of further research, towards complete resolution of the conjecture above.

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