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# SMOOTH SIMPLICIAL SETS AND UNIVERSAL CHERN-WEIL HOMOMORPHISM

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**ABSTRACT.** We start by introducing and developing a basic geometric-categorical notion of a smooth simplicial set. Loosely, this is to diffeological spaces of Chen-Souriau as simplicial sets are to spaces. Given a Frechet Lie group  $G$ , we construct abstract classifying spaces  $BG^{\mathcal{U}}$  as smooth Kan complexes. Here the index  $\mathcal{U}$  is a chosen Grothendieck universe of a certain type. When  $G$  in addition has the homotopy type of a CW complex, there is a homotopy equivalence  $BG \simeq |BG^{\mathcal{U}}|$ , where  $BG$  is the usual Milnor classifying space. This leads to our main application that for  $G$  a Frechet Lie group, having the homotopy type of a CW complex, there is a universal Chern-Weil homomorphism:

$$\mathbb{R}[\mathfrak{g}]^G \rightarrow H^*(BG, \mathbb{R}),$$

generalizing the classical Chern-Weil homomorphism for compact Lie groups. We give an example in the form of an application to the study of the infinite dimensional symplectomorphism group of a symplectic manifold.

## CONTENTS

1. Introduction	2
1.1. Acknowledgements	3
2. Preliminaries and notation	3
2.1. Geometric realization	5
3. Smooth simplicial sets	5
3.1. Simplex category of a smooth simplicial set	7
3.2. Products	8
3.3. Smooth homotopy	8
4. Differential forms on smooth simplicial sets and DeRham theorem	8
4.1. Homology and cohomology of a simplicial set	9
4.2. Integration	10
4.3. Pull-back	10
4.4. Relation with ordinary homology and cohomology	11
5. Simplicial $G$ -bundles	12
5.1. Pull-backs of simplicial bundles	17
6. Connections on simplicial $G$ -bundles	17
7. Chern-Weil homomorphism	18
7.1. The classical case	18
7.2. Chern-Weil homomorphism for smooth simplicial bundles	18
8. The universal simplicial $G$ -bundle	20
8.1. The classifying spaces $BG^{\mathcal{U}}$	20
8.2. The universal smooth simplicial $G$ -bundle $EG^{\mathcal{U}}$	21
9. The universal Chern-Weil homomorphism	27
10. Universal Chern-Weil theory for the group of symplectomorphisms	29
References	30

## 1. INTRODUCTION

We introduce the notion of a smooth simplicial set, which is loosely an analogue in simplicial sets of diffeological spaces of Chen-Souriau, with the latter perhaps a most basic notion of a “smooth space”. The language of smooth simplicial sets turn out to be a powerful tool to resolve the problem of the construction of the universal Chern-Weil homomorphism for Frechet Lie groups.

One problem of topology is the construction of a “smooth structure” on the classifying space  $BG$  of a Frechet Lie group  $G$ . There are specific requirements for what such a notion of a smooth structure should entail. At the very least we hope to be able to carry out Chern-Weil theory universally on  $BG$ . That is we want a differential geometric construction of the Chern-Weil homomorphism:

$$\mathbb{R}[\mathfrak{g}]^G \rightarrow H^*(BG, \mathbb{R}),$$

with  $\mathbb{R}[\mathfrak{g}]^G$  denoting  $Ad_G$  invariant polynomials on the Lie algebra  $\mathfrak{g}$  of  $G$ . When  $G$  is compact  $BG$  can be written as a colimit of smooth manifolds and so in that case the existence of the universal Chern-Weil homomorphism is classical.

One candidate for a smooth structure on  $BG$  is some kind of diffeology. For example Magnot and Watts [7] construct a natural diffeology on the Milnor classifying space  $BG$ . However, this and related diffeologies do not appear to be sufficient to carry out Chern-Weil theory directly. A further specific possible requirement for the above discussed “smooth structures”, is that the simplicial set  $BG_\bullet$ , of smooth maps  $\Delta^d \rightarrow BG$ , should have a geometric realization weakly homotopy equivalent to  $BG$ . See for instance [5] for one approach to this particular problem in the context of diffeologies. This kind of requirement is crucial for instance in [13], which may be understood as a kind of “quantum Chern-Weil theory” on  $BHam(M, \omega)$  for  $Ham(M, \omega)$  the group of Hamiltonian symplectomorphisms of a symplectic manifold. In the language of smooth simplicial sets, the analogue of this latter requirement is always trivially satisfied. The specific content of this is Proposition 3.6.

The structure of a smooth simplicial set is initially more flexible than a space with diffeology, but with further conditions, like the Kan condition, can become forcing. Given a Frechet Lie group  $G$ , we construct, for each choice of a particular kind of Grothendieck universe  $\mathcal{U}$ , a smooth simplicial set  $BG^{\mathcal{U}}$  with a specific classifying property, analogous to the classifying property of  $BG$ . The simplicial set  $BG^{\mathcal{U}}$  is moreover a Kan complex, and so is a basic example of a smooth Kan complex. We then show that if  $G$  in addition has the homotopy type of a CW complex then the geometric realization  $|BG^{\mathcal{U}}|$  is homotopy equivalent to  $BG$ .

All of the dreams of “smoothness” mentioned above then in some sense hold true for  $BG^{\mathcal{U}}$  via its smooth Kan complex structure. In particular, as one immediate application we get:

**Theorem 1.1.** *Let  $G$  be a Frechet Lie group having the homotopy type of a CW complex, then there is a universal Chern-Weil algebra homomorphism:*

$$\mathbb{R}[\mathfrak{g}]^G \rightarrow H^*(BG, \mathbb{R}).$$

Here is one concrete example, using Reznikov’s polynomials [11] on the Lie algebra of the group of Hamiltonian symplectomorphisms. Let  $Symp(\mathbb{CP}^k)$  denote the group of symplectomorphisms of  $\mathbb{CP}^k$ , that is diffeomorphisms  $\phi : \mathbb{CP}^k \rightarrow \mathbb{CP}^k$  s.t.  $\phi^*\omega_0 = \omega_0$  for  $\omega_0$  the Fubini-Study symplectic 2-form on  $\mathbb{CP}^k$ .

**Theorem 1.2.** *The natural map*

$$i : BPU(n) \rightarrow BSymp(\mathbb{CP}^{n-1})$$

*is injective on real homology for all  $n \geq 2$ .*

More history and background surrounding these theorems is in Sections 9 and 10. We end this introduction with some natural open questions.

**Question 1.3.** *Our argument is formalized in Grothendieck's extension of ZFC, where ZFC is Zermelo-Fraenkel set theory plus axiom of choice. Does theorem 1.1 have a proof in ZFC?*

Probably the answer is yes, on the other hand as communicated to me by Dennis Sullivan there are known set theoretical (ZFC) issues with some questions on universal characteristic and secondary characteristic classes. So that the answer of no may be possible.

**Question 1.4.** *Is there a full dictionary between smooth simplicial sets and diffeological spaces?*

As will be explained, given a diffeological space we naturally get a smooth simplicial set. However the other direction seems to be rather complicated, as the naive geometric realization does not remember much of the smooth simplicial set structure.

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## 2. PRELIMINARIES AND NOTATION

We denote by  $\Delta$  the simplex category, i.e. the category with objects finite sets:  $[n] = \{0, 1, \dots, n\}$ , with  $\text{hom}_\Delta([n], [m])$  non-decreasing maps. A simplicial set  $X$  is a functor

$$X : \Delta^{op} \rightarrow \text{Set}.$$

We usually write  $X(n)$  instead of  $X([n])$ , and this is called the set of  $n$ -simplices of  $X$ .  $\Delta_{\text{simp}}^d$  will denote the standard representable  $d$ -simplex:

$$\Delta_{\text{simp}}^d(n) = \text{hom}_\Delta([n], [d]).$$

The element of  $\Delta_{\text{simp}}^d(0)$  corresponding to the map  $i_k : [0] \rightarrow [d]$ ,  $i_k(0) = k$  will usually be denoted by just  $k$ .

Let  $\Delta^d$  be the topological  $d$ -simplex, i.e.

$$\Delta^d := \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 + \dots + x_d \leq 1, \text{ and } \forall i : x_i \geq 0\}.$$

The vertices of  $\Delta^d$  will be assumed ordered in the standard way.

**Definition 2.1.** *Let  $X$  be a smooth manifold with corners. We say that a map  $\sigma : \Delta^n \rightarrow X$  is smooth if it is smooth as a map of manifolds with corners. In particular  $\sigma : \Delta^n \rightarrow \Delta^d$  is smooth iff it has an extension to a smooth map from a neighborhood in  $\mathbb{R}^n$  of  $\Delta^n$  into a neighborhood of  $\Delta^d$  in  $\mathbb{R}^d$ . We say that a smooth  $\sigma : \Delta^n \rightarrow X$  is **collared** if there is a neighborhood  $U \supset \partial\Delta^n$  in  $\Delta^n$ , such that  $\Sigma|_U = \Sigma \circ \text{ret}$  for  $\text{ret} : U \rightarrow \partial\Delta^n$  some smooth retraction. Here smooth means that  $\text{ret}$  has an extension to a smooth map  $V \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ , with  $V \supset \partial\Delta^n$  open in  $\mathbb{R}^d$ .*

We denote by  $\Delta_{\bullet}^d$  the simplicial set of all smooth simplices in  $\Delta^d$ :  $\Delta_{\bullet}^d(k)$  is the set of smooth maps

$$\sigma : \Delta^k \rightarrow \Delta^d.$$

We call an affine map  $\Delta^k \rightarrow \Delta^d$  taking vertices to vertices in a order preserving way **simplicial**. And we denote by

$$\Delta_{simp}^d \subset \Delta_{\bullet}^d$$

the subset consisting of simplicial maps. Note that  $\Delta_{simp}^d$  is naturally isomorphic to the standard representable  $d$ -simplex  $\Delta_{simp}^d$  as previously defined. Thus we may also understand  $\Delta$  as the category with objects topological simplices  $\Delta^d$ ,  $d \geq 0$  and morphisms simplicial maps.

**Notation 2.2.** A morphism  $m \in \text{hom}_{\Delta}([n], [k])$  uniquely corresponds to a simplicial map  $\Delta_{simp}^n \rightarrow \Delta_{simp}^k$ , which uniquely corresponds to a simplicial map in the above sense  $\Delta^n \rightarrow \Delta^k$ . The correspondence is by taking the maps  $\Delta_{simp}^n \rightarrow \Delta_{simp}^k$ ,  $\Delta^n \rightarrow \Delta^k$ , to be determined by the map of the vertices corresponding to  $m$ . We will not notationally distinguish these corresponding morphisms. So that  $m$  may will simultaneously refer to all of the above morphisms.

By a  $d$ -simplex  $\Sigma$  of a simplicial set  $X$ , we may mean, interchangeably, either the element in  $X(d)$  or the morphism (natural transformation):

$$\Sigma : \Delta_{simp}^d \rightarrow X,$$

uniquely corresponding to  $\Sigma$  via the Yoneda lemma. If we write  $\Sigma^d$  for a simplex of  $X$ , it is implied that it is a  $d$ -simplex.

**Definition 2.3.** For  $X$  a simplicial set,  $\Delta(X)$  will denote the over category of simplices of  $X$ , explicitly the category whose objects are simplices

$$\Sigma : \Delta_{simp}^d \rightarrow X, \quad d \geq 0$$

and morphisms  $f : \Sigma_1 \rightarrow \Sigma_2$ , commutative diagrams:

$$(2.1) \quad \begin{array}{ccc} \Delta_{simp}^d & \xrightarrow{\tilde{f}} & \Delta_{simp}^n \\ & \searrow \Sigma_1 & \downarrow \Sigma_2 \\ & & X \end{array}$$

with top arrow a simplicial map, which we denote by  $\tilde{f}$ . An object  $\Sigma : \Delta_{simp}^d \rightarrow X$  is likewise called a  $d$ -simplex, and such a  $\Sigma$  may be said to have degree  $d$ . As noted in the paragraph before the definition, the degree  $d$  may also be specified by a superscript.

**Definition 2.4.** We say that  $\Sigma^n \in \Delta(X)$  is **non-degenerate** if there is no morphism  $f : \Sigma^n \rightarrow \Sigma^m$  in  $\Delta(X)$  s.t.  $m < n$ .

There is a forgetful functor

$$T : \Delta(X) \rightarrow \Delta,$$

$T(\Sigma^d) = \Delta_{simp}^d$ ,  $T(f) = \tilde{f}$ . We denote by  $\Delta^{inj}(X) \subset \Delta(X)$  the sub-category with same objects, and morphisms  $f$  such that  $\tilde{f}$  are monomorphisms, i.e. are face inclusions.

**2.1. Geometric realization.** Let  $Top$  be the category of topological spaces. Let  $X$  be a simplicial set, then define as usual the *geometric realization* of  $X$  by the colimit in  $Top$ :

$$|X| := \operatorname{colim}_{\Delta(X)} T,$$

for  $T : \Delta(X) \rightarrow \Delta \subset Top$  as above, understanding  $\Delta$  as a subcategory of  $Top$  as previously explained.

### 3. SMOOTH SIMPLICIAL SETS

If

$$\sigma : \Delta^d \rightarrow \Delta^n$$

is a smooth map we then have an induced map of simplicial sets

$$(3.1) \quad \sigma_{\bullet} : \Delta_{\bullet}^d \rightarrow \Delta_{\bullet}^n,$$

defined by

$$\sigma_{\bullet}(\rho) = \sigma \circ \rho.$$

We give a pair of equivalent definitions. The first is more concrete but the second is more concise and conceptual.

**Definition 3.1.** *A smooth simplicial set is a data consisting of:*

- (1) *A simplicial set  $X$ .*
- (2) *For each  $\Sigma : \Delta_{simp}^n \rightarrow X$  an  $n$ -simplex, there is an assigned map of simplicial sets*

$$g(\Sigma) : \Delta_{\bullet}^n \rightarrow X,$$

*which satisfies:*

$$(3.2) \quad g(\Sigma)|_{\Delta_{simp}^n} = \Sigma.$$

*We abbreviate  $g(\Sigma)$  by  $\Sigma_*$ , when there is no need to disambiguate which structure  $g$  is meant.*

- (3) *The following property will be called **push-forward functoriality**:*

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_{\bullet}$$

*where  $\sigma : \Delta^k \rightarrow \Delta^d$  is a  $k$ -simplex of  $\Delta_{\bullet}^d$ , and where  $\Sigma$  as before is a  $d$ -simplex of  $X$ .*

Thus, formally a smooth simplicial set is a 2-tuple  $(X, g)$ , satisfying the axioms above. When there is no need to disambiguate we omit specifying  $g$ .

Now let  $\Delta^{sm}$  denote the category whose objects are the topological simplices  $\Delta^k$ ,  $k \geq 0$ . And  $\operatorname{hom}_{\Delta^{sm}}(\Delta^k, \Delta^n)$  is the set of smooth maps  $\Delta^k \rightarrow \Delta^n$ . Then

**Definition 3.2.** *A smooth simplicial set  $X$  is a functor  $X : \Delta^{sm} \rightarrow Set^{op}$ .*

In what follows we mostly use the first definition, to avoid too much abstraction in this first paper.

**Definition 3.3.** *A smooth map between smooth simplicial sets*

$$(X_1, g_1), (X_2, g_2)$$

*is a simplicial map*

$$f : X_1 \rightarrow X_2,$$

*which satisfies the condition:*

$$(3.3) \quad g_2(f(\Sigma)) = f \circ g_1(\Sigma),$$

or more compactly:

$$f(\Sigma)_* = f \circ \Sigma_*.$$

A **diffeomorphism** between smooth simplicial sets is defined to be a smooth map, with a smooth inverse.

**Example 3.4** (The tautological smooth simplicial set).  $\Delta_\bullet^n$  has a tautological smooth simplicial set structure, where

$$g(\Sigma) = \Sigma_\bullet,$$

for  $\Sigma : \Delta^k \rightarrow \Delta^n$  a smooth map, hence a  $k$ -simplex of  $\Delta_\bullet^n$ , and where  $\Sigma_\bullet$  is as in (3.1).

**Lemma 3.5.** Let  $X$  be a smooth simplicial set and  $\Sigma : \Delta_{simp}^n \rightarrow X$  an  $n$ -simplex. Let  $\Sigma_* : \Delta_\bullet^n \rightarrow X$  be the induced simplicial map. Then  $\Sigma_*$  is smooth with respect to the tautological smooth simplicial set structure on  $\Delta_\bullet^n$  as above.

*Proof.* Let  $\sigma$  be a  $k$ -simplex of  $\Delta_\bullet^n$ , so  $\sigma : \Delta^k \rightarrow \Delta^n$  is a smooth map, we need that

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_*.$$

Now  $\sigma_* = \sigma_\bullet$ , by definition of the tautological smooth structure on  $\Delta_\bullet^n$ . So we have:

$$\begin{aligned} (\Sigma_*(\sigma))_* &= \Sigma_*(\sigma) \circ \sigma_\bullet \text{ by Axiom 3} \\ &= \Sigma_*(\sigma) \circ \sigma_*. \end{aligned}$$

□

**Proposition 3.6.** The set of  $n$ -simplices of a smooth simplicial set  $X$  is naturally isomorphic to the set of smooth maps  $\Delta_\bullet^n \rightarrow X$ . In fact, define  $X_\bullet$  to be the simplicial set whose  $n$ -simplices are smooth maps  $\Delta_\bullet^n \rightarrow X$ , with  $X_\bullet$  given the obvious simplicial structure. Then  $X_\bullet$  is naturally isomorphic to  $X$ .

*Proof.* Given a simplex  $\rho : \Delta_{simp}^n \rightarrow X$ , we have a uniquely associated to it, by the lemma above, smooth map  $\rho_* : \Delta_\bullet^n \rightarrow X$ . Conversely, suppose we are given a smooth map  $m : \Delta_\bullet^n \rightarrow X$ . Then we get an  $n$ -simplex  $\rho_m := m|_{\Delta_{simp}^n}$ . Let  $id^n : \Delta^n \rightarrow \Delta^n$  be the identity map. We have that

$$\begin{aligned} m &= m \circ id_\bullet^n = m \circ id_*^n \\ &= m(id^n)_* \text{ as } m \text{ is smooth} \\ &= \rho_m(id^n)_* \text{ trivially by definition of } \rho_m \\ &= \rho_{m,*} \circ id_*^n \text{ as } \rho_{m,*} \text{ is smooth by Lemma 3.5} \\ &= \rho_{m,*}. \end{aligned}$$

Thus the map  $\rho \mapsto \rho_*$ , from the set of  $n$ -simplices of  $X$  to the set of smooth maps  $\Delta_\bullet^n \rightarrow X$ , is bijective.

The proof of the second part of the proposition is straightforward from the first part and is omitted. □

**Lemma 3.7.** Given a smooth  $m : \Delta^d \rightarrow \Delta_\bullet^n$  there is a unique smooth map  $f : \Delta^d \rightarrow \Delta^n$  such that  $m = f_\bullet$ .

*Proof.* This is very trivial. Define  $f$  by  $m(id)$  for  $id : \Delta^d \rightarrow \Delta^d$  the identity. Then

$$\begin{aligned} f_\bullet &= m(id)_\bullet \\ &= m(id)_* \\ &= m \circ id_* \text{ (as } m \text{ is smooth )} \\ &= m. \end{aligned}$$

So  $f$  induces  $m$ . Now if  $g$  induces  $m$  then  $g_\bullet = m$  hence  $g_\bullet(id) = m(id)$  and so  $g = m(id)$ . Hence  $f$  is the unique map inducing  $m$ .  $\square$

add equivalence of the definitions

**Definition 3.8.** A smooth simplicial set whose underlying simplicial set is a Kan complex will be called a **smooth Kan complex**.

Let  $Sing^{sm}(Y)$ <sup>1</sup> denote the simplicial set of smooth simplices in  $X$ . That is  $Sing^{sm}(Y)(k)$  is the set of smooth maps  $\Sigma : \Delta^k \rightarrow Y$ . And where the simplicial structure on  $Sing^{sm}(Y)$  is the natural one.  $Sing^{sm}(Y)$  will often be abbreviated by  $Y_\bullet$ . Analogously,  $Sing^c(Y)$  will be the simplicial set of continuous simplices in  $X$ .

**Example 3.9.** Let  $Y$  be a smooth  $d$ -fold. And set  $X = Y_\bullet = Sing^{sm}(Y)$ . Then  $X$  is naturally a smooth simplicial set, analogously to Example 3.4. This should be a Kan complex but a reference is not known to me. However, if we ask that  $\Sigma : \Delta^k \rightarrow Y$  are in addition collared (as in Definition 2.1) then the Kan condition is simple to verify. More generally, we may clearly take  $Y$  to be a manifold with boundary or with corners, an orbifold or any diffeological space.

**Example 3.10.** One special example is worth attention. Let  $M$  be a smooth manifold. Then there is a natural smooth simplicial set  $LM^\Delta$  whose  $d$ -simplices  $\Sigma$  are smooth maps  $f_\Sigma : \Delta^d \times S^1 \rightarrow M$ . The maps  $\Sigma_*$  are defined by

$$\Sigma_*(\sigma) = f_\Sigma \circ (\sigma \times id),$$

for

$$\sigma \times id : \Delta^d \times S^1 \rightarrow \Delta^d \times S^1.$$

This  $LM^\Delta$  is one simplicial model of the free loop space. Naturally the free loop space  $LM$  also has the structure of a Frechet manifold, in particular we have the smooth simplicial set  $LM_\bullet$ , whose  $n$ -simplices are Frechet smooth maps  $\Sigma : \Delta^n \rightarrow LM$ . There is a natural simplicial map  $LM^\Delta \rightarrow LM_\bullet$ , which is clearly smooth.

**3.1. Simplex category of a smooth simplicial set.** Given a smooth simplicial set  $X$ , there is an extension of the previously defined category  $\Delta(X)$ .

**Definition 3.11.** For  $X$  a smooth simplicial set,  $\Delta^{sm}(X)$  will denote the category whose objects are smooth maps

$$\Sigma : \Delta_\bullet^d \rightarrow X, \quad d \geq 0$$

and morphisms  $f : \Sigma_1 \rightarrow \Sigma_2$ , commutative diagrams:

$$(3.4) \quad \begin{array}{ccc} \Delta_\bullet^d & \xrightarrow{\tilde{f}_\bullet} & \Delta_\bullet^n \\ & \searrow \Sigma_1 & \downarrow \Sigma_2 \\ & & X \end{array}$$

<sup>1</sup>This is often called the “smooth singular set of  $Y$ ”. However, for us “smooth” is reserved for another purpose, so to avoid confusion we do not use such terminology.

with top arrow any smooth map (for the tautological smooth simplicial set structure on  $\Delta_{\bullet}^d$ ), which we denote by  $\tilde{f}_{\bullet}$ . By Lemma 3.7,  $\tilde{f}_{\bullet}$  is induced by a unique smooth map  $\tilde{f} : \Delta^d \rightarrow \Delta^n$ .

By Proposition 3.6 we have a faithful embedding  $\Delta(X) \rightarrow \Delta^{sm}(X)$  which is an isomorphism on object sets.

**3.2. Products.** Given a pair of smooth simplicial sets  $(X_1, g_1), (X_2, g_2)$ , the product  $X_1 \times X_2$  of the underlying simplicial sets, has the structure of a smooth simplicial set

$$(X_1 \times X_2, g_1 \times g_2),$$

constructed as follows. Denote by  $\pi_i : X_1 \times X_2 \rightarrow X_i$  the simplicial projection maps. Then for each  $\Sigma \in X_1 \times X_2(d)$ ,

$$g_1 \times g_2(\Sigma) : \Delta_{\bullet}^d \rightarrow X_1 \times X_2$$

is defined by:

$$g_1 \times g_2(\Sigma)(\sigma) := (g_1(\pi_1(\Sigma))(\sigma), g_2(\pi_2(\Sigma))(\sigma)).$$

### 3.3. Smooth homotopy.

**Definition 3.12.** Let  $X, Y$  be smooth simplicial sets. Set  $I := \Delta_{\bullet}^1$  and let  $0_{\bullet}, 1_{\bullet} \subset I$  be the images of the pair of inclusions  $\Delta_{\bullet}^0 \rightarrow I$  corresponding to the pair of endpoints. A pair of smooth maps  $f, g : X \rightarrow Y$  are called **smoothly homotopic** if there exists a smooth map

$$H : X \times I \rightarrow Y$$

such that  $H|_{X \times 0_{\bullet}} = f$  and  $H|_{X \times 1_{\bullet}} = g$ .

The following notion will be useful later on.

**Definition 3.13.** Let  $X$  be a smooth simplicial set. We define  $\pi_k^{sm}(X)$  to be the set of smooth homotopy equivalence classes of smooth maps  $f : S_{\bullet}^k \rightarrow X$ .

## 4. DIFFERENTIAL FORMS ON SMOOTH SIMPLICIAL SETS AND DERHAM THEOREM

**Definition 4.1.** Let  $X$  be a smooth simplicial set. A **simplicial differential  $k$ -form**  $\omega$ , or just differential form where there is no possibility of confusion, is for each  $d$ -simplex  $\Sigma$  of  $X$  a smooth differential  $k$ -form  $\omega_{\Sigma}$  on  $\Delta^d$ , such that

$$(4.1) \quad i^* \Omega_{\Sigma_2} = \Omega_{\Sigma_1},$$

for every morphism  $i : \Sigma_1 \rightarrow \Sigma_2$  in  $\Delta^{inj}(X)$ , (see Section 2). If in addition:

$$(4.2) \quad \omega_{g(\Sigma)(\sigma)} = \sigma^* \omega_{\Sigma},$$

for every  $\sigma \in \Delta_{\bullet}^d$ , and every  $d$ -simplex  $\Sigma$ , then we say that  $\omega$  is **coherent**.

**Example 4.2.** If  $X = Y_{\bullet}$  for  $Y$  a smooth  $d$ -fold, and if  $\omega$  is a differential  $k$ -form on  $Y$ , then  $\{\omega_{\Sigma} = \Sigma^* \omega\}_{\Sigma}$  is a coherent differential  $k$ -form on  $X$  called the **induced simplicial differential form**.

**Example 4.3.** Let  $LM^{\Delta}$  be the smooth Kan complex of Example 3.10. Then Chen's iterated integrals [1] naturally give coherent differential forms on  $LM^{\Delta}$ .



The above coherence condition is often unnecessary, hence is not part of the basic definition here. This is already one difference with differential forms on diffeological or Chen spaces, where coherence is ostensibly forced.

Let  $X$  be a smooth simplicial set. We denote by  $\Omega^k(X)$  the  $\mathbb{R}$ -vector space of differential  $k$ -forms on  $X$ . Define

$$d : \Omega^k(X) \rightarrow \Omega^{k+1}(X)$$

by

$$d(\{\omega_\Sigma\}) := d\{\omega_\Sigma\} = \{d\omega_\Sigma\}.$$

Clearly we have

$$d^2 = 0.$$

A  $k$ -form  $\omega$  is said to be **closed** if  $d\omega = 0$ , and **exact** if for some  $(k-1)$ -form  $\eta$ ,  $\omega = d\eta$ .

**Definition 4.4.** *The wedge product on*

$$\Omega^\bullet(X) = \bigoplus_{k \geq 0} \Omega^k(X)$$

is defined by

$$\omega \wedge \eta = \{\omega_\Sigma \wedge \eta_\Sigma\}_\Sigma.$$

Then  $\Omega^\bullet(X)$  has the structure of a differential graded  $\mathbb{R}$ -algebra with respect to  $\wedge$ .

We then, as usual, define the **De Rham cohomology** of  $X$ :

$$H_{DR}^k(X) = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}},$$

which is a graded commutative  $\mathbb{R}$ -algebra.

The simplicial De Rham complex above is certainly not a new invention, versions of it have been used by Whitney and perhaps most famously by Sullivan [15].

**4.1. Homology and cohomology of a simplicial set.** We go over this mostly to establish notation. For a simplicial set  $X$ , we define an abelian group

$$C_k(X, \mathbb{Z}),$$

as the free abelian group generated by  $k$ -simplices of  $X$ . Elements of  $C_k(X, \mathbb{Z})$  are called  **$k$ -chains**. The boundary operator:

$$\partial : C_k(X, \mathbb{Z}) \rightarrow C_{k-1}(X, \mathbb{Z}),$$

is defined on a  $k$ -simplex  $\sigma$  as classically by

$$\partial\sigma = \sum_{i=0}^n (-1)^i d_i\sigma,$$

where  $d_i$  are the face maps, this is then extended by linearity to general chains. As classically,  $\partial^2 = 0$ . The homology of this complex is denoted by  $H_k(X, \mathbb{Z})$ , called integral homology. The integral cohomology is defined analogously to the classical topology setting, using dual chain groups  $C^k(X, \mathbb{Z}) = \text{hom}(C_k(X, \mathbb{Z}), \mathbb{Z})$ . The corresponding coboundary operator is denoted by  $d$  as usual:

$$d : C^k(X, \mathbb{Z}) \rightarrow C^{k+1}(X, \mathbb{Z}).$$

Homology and cohomology with other ring coefficients (or modules) are likewise defined analogously. Given a simplicial map  $f : X \rightarrow Y$  there are natural induced

chain maps  $f^* : C^k(Y, \mathbb{Z}) \rightarrow C^k(X, \mathbb{Z})$ , and  $f_* : C_k(X, \mathbb{Z}) \rightarrow C_k(X, \mathbb{Z})$ . We say that  $f, g : X \rightarrow Y$  are homotopic if there a simplicial map  $H : X \times \Delta_{simp}^1 \rightarrow Y$  so that  $f = H \circ i_0$ ,  $g = H \circ i_1$  for  $i_0, i_1 : X \rightarrow X \times \Delta_{simp}^1$  corresponding to the pair of end point inclusions  $\Delta_{simp}^0 \rightarrow \Delta_{simp}^1$ .

As is well known if  $f, g$  are homotopic then  $f^*, g^*$  and  $f_*, g_*$  are chain homotopic.

**4.2. Integration.** Let  $X$  be a smooth simplicial set. Given a chain

$$\sigma = \sum_i a_i \Sigma_i \in C_k(X, \mathbb{Z})$$

and a smooth differential form  $\omega$ , we define:

$$\int_\sigma \omega = \sum_i a_i \int_{\Delta^k} \omega_{\Sigma_i}$$

where the integrals on the right are the classical integrals of a differential form. Thus we obtain a homomorphism:

$$\int : \Omega^k(X) \rightarrow C^k(X, \mathbb{R}),$$

$\int(\omega)$  is the  $k$ -cochain defined by:

$$\int(\omega)(\sigma) := \int_\sigma \omega,$$

where  $\sigma$  is a  $k$ -chain. We will abbreviate  $\int(\omega) = \int \omega$ .

**Lemma 4.5.** *For a smooth simplicial set  $X$ , the homomorphism  $\int$  commutes with  $d$ , and so induces a homomorphism:*

$$\int : H_{DR}^k(X) \rightarrow H^k(X, \mathbb{R}).$$

*Proof.* We need that

$$\int d\omega = d \int \omega.$$

Let  $\Sigma : \Delta_{simp}^k \rightarrow X$  be a  $k$ -simplex. Then

$$\begin{aligned} \int d\omega(\Sigma) &= \int_{\Delta^k} d\omega_\Sigma \text{ by definition} \\ &= \int_{\partial \Delta^k} \omega_\Sigma \text{ by Stokes theorem} \\ &= d\left(\int \omega\right)(\Sigma) \text{ by the definition of } d \text{ on co-chains.} \end{aligned}$$

□

**4.3. Pull-back.** Given a smooth map  $f : X_1 \rightarrow X_2$  of smooth simplicial sets, we define

$$f^* : \Omega^k(X_2) \rightarrow \Omega^k(X_1)$$

naturally by

$$f^*(\omega) = f^*\omega := \{(f^*\omega)_\Sigma\}_\Sigma := \{\omega_{f(\Sigma)}\}_\Sigma.$$

Clearly  $f^*$  commutes with  $d$  so that we have an induced differential graded  $\mathbb{R}$ -algebra homomorphism:

$$f^* : \Omega^\bullet(X_2) \rightarrow \Omega^\bullet(X_1).$$

And in particular an induced  $\mathbb{R}$ -algebra homomorphism:

$$f^* : H_{DR}^\bullet(X_2) \rightarrow H_{DR}^\bullet(X_1).$$

**4.4. Relation with ordinary homology and cohomology.** Let  $s\text{-Set}$  denote the category of simplicial sets and  $Top$  the category of topological spaces. Let

$$|\cdot| : s\text{-Set} \rightarrow Top$$

be the geometric realization functor as defined in Section 2.1. Let  $X$  be a (smooth) simplicial set. Then for any ring  $K$  we have natural chain maps

$$CR : C_k(X, K) \rightarrow C_k(|X|, K),$$

$$CR^c : C^k(|X|, K) \rightarrow C^k(X, K),$$

as a  $d$ -simplex  $\Sigma : \Delta_{simp}^d \rightarrow X$ , by construction of  $|X|$  uniquely induces a continuous map  $\Delta^d \rightarrow |X|$ , which is set to be  $CR(\Sigma)$ .

When  $X$  is a Kan complex, the natural map  $X \rightarrow Sing^c(|X|)$  is a simplicial homotopy equivalence. And so, in this case, the induced maps in homology/cohomology:

$$(4.3) \quad R : H_k(X, K) \rightarrow H_k(|X|, K),$$

$$(4.4) \quad R^c : H^k(|X|, K) \rightarrow H^k(X, K),$$

are isomorphisms.

Now let  $Y$  be a smooth manifold and  $X = Y_\bullet = Sing^{sm}(Y)$ . As mentioned this is expected to be a Kan complex, but no reference is known to me. However in this case we still have isomorphisms:

$$R : H_k(Y_\bullet, K) \rightarrow H_k(|Y_\bullet|, K),$$

$$R^c : H^k(|Y_\bullet|, K) \rightarrow H^k(Y_\bullet, K) \text{ if } K \text{ is a field of characteristic } 0.$$

First note that we have a natural homotopy equivalence  $|Y_\bullet| \simeq Y$ . This is just because the natural map  $|Y_\bullet| \rightarrow Y$  is a weak homotopy equivalence, (by homotopy approximating continuous maps by smooth maps), and so is a homotopy equivalence, by Whitehead theorem. Let us denote by

$$(4.5) \quad n : Y \rightarrow |Y_\bullet|,$$

its homotopy inverse. Then factor  $R$  and  $R^c$  as:

$$(4.6) \quad H_k(Y_\bullet, K) \xrightarrow{I} H_k(Y, K) \xrightarrow{n_*} H_k(|Y_\bullet|, K),$$

$$(4.7) \quad H^k(|Y_\bullet|, K) \xrightarrow{n^*} H^*(Y, K) \xrightarrow{I^c} H^k(Y_\bullet, K)$$

The map  $I$  is induced by the chain map  $CI$  sending the generator of  $C_k(Y_\bullet, K)$  corresponding to a simplex  $\Sigma \in Y_\bullet(k)$  to the element of  $C_k(Y)$  given by the smooth map  $\Sigma : \Delta^d \rightarrow Y$  (as  $\Sigma$  by definition corresponds to such a smooth map).  $I$  is of course injective, it is surjective because any cycle in  $C_k(Y, K)$  is homologous to a smooth cycle (meaning  $\sum_{i=1}^n c_i \Sigma_i$  with  $\Sigma_i : \Delta^k \rightarrow Y$  smooth). Likewise  $I^c$  is induced by the cochain map sending a cochain  $\alpha$  to the cochain  $\alpha'$  defined by

$$\alpha'(\sigma) := \alpha(CI(\sigma)),$$

where  $\sigma \in C_k(Y_\bullet, K)$ .  $I^c$  is of course surjective. Now suppose that  $I^c(\alpha) = 0$ , then the functional

$$\langle I^c(\alpha), \cdot \rangle : H_k(Y_\bullet, K) \rightarrow K$$

is 0. Since  $I$  is an isomorphism, the functional

$$\langle \alpha, \cdot \rangle : H_k(Y, K) \rightarrow K$$

is 0. Hence, if  $K$  is a field of 0 characteristic we get that  $\alpha = 0$ , and  $I^c$  is injective.

**Notation 4.6.** *In the case of the paper  $K = \mathbb{R}$ . Then given a Kan complex  $X$ , or  $X = Y_\bullet$  for a smooth manifold  $Y$ , and given a cohomology class  $\alpha \in H^k(X, K)$ , we will denote by  $|\alpha| \in H^k(|X|, K)$  the class  $(R^c)^{-1}(\alpha)$ .*

Given a map of simplicial sets  $f : X_1 \rightarrow X_2$  we let  $|f| : |X_1| \rightarrow |X_2|$  denote the induced map of geometric realizations. The following is immediate from definitions.

**Lemma 4.7.** *Let  $f : X_1 \rightarrow X_2$  be a simplicial map with  $X_i$  Kan complexes, or with  $X_i = Y_{i,\bullet}$  for  $Y_i$  smooth manifolds. Let  $f^* : H^k(X_2, K) \rightarrow H^k(X_1, K)$  be the induced homomorphism then:*

$$|f^*(\alpha)| = |f|^*(|\alpha|).$$

## 5. SIMPLICIAL $G$ -BUNDLES

Part of our motivation is the construction of the universal Chern-Weil homomorphisms for Frechet Lie groups. A *Frechet Lie group*  $G$  is Lie group whose underlying manifold is a possibly infinite dimensional smooth manifold locally modelled on a Frechet space, that is a locally convex, complete Hausdorff vector space. Later on it will also be important that  $G$  have the homotopy type of a CW complex. By Milnor [9], a prototypical example of such a Lie group is the group of diffeomorphisms  $\text{Diff}(M)$  of a smooth manifold. Another very interesting example for us is the group of Hamiltonian symplectomorphisms  $\text{Ham}(M, \omega)$  of a symplectic manifold. Particularly because its Lie algebra admits natural bi-invariant polynomials, so that it is possible to define interesting Chern-Weil theory for this group.

In what follows  $G$  is always assumed to be a Frechet Lie group. We now introduce the basic building blocks for simplicial  $G$ -bundles.

**Definition 5.1.** *A smooth  $G$ -bundle  $P$  over  $\Delta^n$  is a topological principal  $G$ -bundle over  $\Delta^n \subset \mathbb{R}^n$ , together with a choice of an open neighborhood  $V$  of  $\Delta^n$  in  $\mathbb{R}^n$  and a choice of a smooth  $G$ -bundle  $\tilde{P} \rightarrow V$  such that  $i^* \tilde{P} \simeq P$  for  $i : \Delta^n \rightarrow V$  the inclusion map, and for  $\simeq$  isomorphism of topological  $G$ -bundles (it will be implicit). We may thus write  $(P, \tilde{P}, V)$  for this structure, but often this is abbreviated as just  $P$ , or  $P \rightarrow \Delta^n$  when the specifics of  $V, \tilde{P}$  are not needed.*

As the reader may observe the above structure is equivalent to the structure of a smooth  $G$ -bundle over  $\Delta^n$  with the latter understood as a specific smooth manifold with corners. However the above concrete definition will be helpful.

**Definition 5.2.** *Let  $(P_1, \tilde{P}_1, V_1), (P_2, \tilde{P}_2, V_2)$  be smooth  $G$ -bundles over  $\Delta^n$ . A smooth  $G$ -bundle map of tuples  $(P_1, \tilde{P}_1, V_1), (P_2, \tilde{P}_2, V_2)$ , is a topological  $G$ -bundle map*

$$f : P_1 \rightarrow P_2,$$

*over the id :  $\Delta^n \rightarrow \Delta^n$  with the following property. There is an open  $U \subset V_1 \cap V_2$  and a smooth  $G$ -bundle map  $\tilde{f} : \tilde{P}_1|_U \rightarrow \tilde{P}_2|_U$  extending  $f$ . If  $f$  has an inverse which is also a smooth  $G$ -bundle map, then  $f$  is said to be an **isomorphism** of the structures  $(P_1, \tilde{P}_1, V_1), (P_2, \tilde{P}_2, V_2)$ .*

If  $P \rightarrow \Delta^d$  is a smooth  $G$ -bundle, given a smooth map  $\sigma : \Delta^k \rightarrow \Delta^d$ , clearly the pull-back  $\sigma^*P$  has a smooth  $G$ -bundle structure uniquely determined up to natural  $G$ -bundle isomorphism.

To warn, at this point our terminology may partially clash with common terminology, in particular a simplicial  $G$ -bundle will *not* be a pre-sheaf on  $\Delta$  with values in the category of smooth  $G$ -bundles. Instead, a simplicial  $G$ -bundle is to smooth  $G$ -bundles as simplicial differential forms are to smooth forms. The latter pre-sheaves will not appear in the paper so that this should not cause confusion.

In the definition of simplicial differential forms we omitted coherence. In the case of simplicial  $G$ -bundles, the analogous condition (full functoriality on  $\Delta^{sm}(X)$ ) turns out to be necessary if we want universal simplicial  $G$ -bundles with expected behavior.

**Notation 5.3.** *Given a Frechet Lie group  $G$ , let  $\mathcal{G}$  denote the category of smooth  $G$ -bundles over manifolds with corners, with morphisms smooth  $G$ -bundle maps, which are fiberwise isomorphisms.*

**Definition 5.4.** *Let  $G$  be a Frechet Lie group and  $X$  a smooth simplicial set. A smooth simplicial  $G$ -bundle  $P$  over  $X$  is the following data:*

- *A functor  $P : \Delta^{sm}(X) \rightarrow \mathcal{G}$ , which assigns to each smooth  $\Sigma : \Delta^\bullet \rightarrow X$  a smooth  $G$ -bundle over  $\Delta^d$ .*
- *If  $f : \Sigma_1^k \rightarrow \Sigma_2^d$  is a morphism then  $P(f) : P(\Sigma_1^k) \rightarrow P(\Sigma_2^d)$  is a bundle map over some smooth map  $\underline{f} : \Delta^k \rightarrow \Delta^d$ . On the hand by Lemma 3.7  $\tilde{f}_\bullet$  is uniquely induced by a smooth map we name  $\tilde{f} : \Delta^k \rightarrow \Delta^d$  and we ask that  $\tilde{f} = \underline{f}$ . We call this condition **compatibility**. (This can also be stated in terms of categorical fibrations, but this extra abstraction will not be useful at the moment.)*

We will only deal with smooth simplicial  $G$ -bundles, and so will usually just say **simplicial  $G$ -bundle**, omitting the qualifier ‘smooth’. We often use notation  $P_\Sigma$  for  $P(\Sigma)$ .

**Example 5.5.** *If  $X$  is a smooth simplicial set and  $G$  is as above, we denote by  $X \times G$  the simplicial  $G$ -bundle,*

$$\forall n \in \mathbb{N}, \forall \Sigma^n \in \Delta(X) : (X \times G)_{\Sigma^n} = \Delta^n \times G,$$

*with  $\Delta^n \times G \rightarrow \Delta^n$  the trivial projection. This is called the **trivial simplicial  $G$ -bundle over  $X$** .*

**Example 5.6.** *Let  $Z \rightarrow Y$  be a smooth  $G$ -bundle over a smooth manifold  $Y$ . Then we have a simplicial  $G$ -bundle  $Z_\bullet$  over  $Y_\bullet$  defined by*

$$Z_\bullet(\Sigma) = \Sigma^*Z.$$

*And where for  $f : \Sigma_1 \rightarrow \Sigma_2$  a morphism, the bundle map*

$$N(f) : (Z_\bullet(\Sigma_1) = \Sigma_1^*Z) \rightarrow (Z_\bullet(\Sigma_2) = \Sigma_2^*Z)$$

*is just the natural map  $f^*\Sigma_2^*Z \rightarrow \Sigma_2^*Z$  in the pull-back square.*

**Definition 5.7.** *Let  $P_1 \rightarrow X_1, P_2 \rightarrow X_2$  be a pair of simplicial  $G$ -bundles. A smooth simplicial  $G$ -bundle map or just simplicial  $G$ -bundle map, is a smooth map*

$$h : X_1 \rightarrow X_2$$

and a natural transformation of functors:

$$\tilde{h} : P_1 \rightarrow P_1 \circ h.$$

**Definition 5.8.** Let  $P_1, P_2$  be simplicial  $G$ -bundles over  $X_1, X_2$  respectively. A **simplicial  $G$ -bundle isomorphism** is a simplicial  $G$ -bundle map

$$\tilde{h} : P_1 \rightarrow P_2$$

s.t. there is a simplicial  $G$ -bundle map

$$\tilde{h}^{-1} : P_2 \rightarrow P_1$$

with

$$\tilde{h}^{-1} \circ \tilde{h} = id.$$

This is clearly the same as asking that  $\tilde{h}$  be a natural isomorphism of the corresponding functors. Usually,  $X_1 = X_2$  and in this case, unless specified otherwise, it is assumed  $h = id$ . A simplicial  $G$ -bundle isomorphic to the trivial simplicial  $G$ -bundle is called **trivializable**.

**Definition 5.9.** If  $X = Y_\bullet$  for  $Y$  a smooth manifold, we say that a simplicial  $G$ -bundle  $P$  over  $X$  is **induced by a smooth  $G$ -bundle**  $N \rightarrow Y$  if there is a simplicial  $G$ -bundle isomorphism  $N_\bullet \rightarrow P$ .

The following will be one of the crucial ingredients later on.

**Theorem 5.10.** Let  $G$  be as above and let  $P \rightarrow Y_\bullet$  be a simplicial  $G$ -bundle, for  $Y$  a smooth  $d$ -manifold. Then  $P$  is induced by some smooth  $G$ -bundle  $N \rightarrow Y$ .

*Proof.* We need to introduce an auxiliary notion. Let  $Z$  be a smooth  $d$ -manifold with corners. And let  $\mathcal{D}(Z)$  denote the category whose objects are smooth embeddings  $\Sigma : \Delta^d \rightarrow Z$ , (for the same fixed  $d$ ) and so that a morphism  $f \in \text{hom}_{\mathcal{D}(Z)}(\Sigma_1, \Sigma_2)$  is a commutative diagrams:

$$(5.1) \quad \begin{array}{ccc} \Delta^d & \xrightarrow{\tilde{f}} & \Delta^d \\ & \searrow \Sigma_1 & \downarrow \Sigma_2 \\ & & Z. \end{array}$$

Note that the map  $\tilde{f}$  is unique, when such a diagram exists as  $\Sigma_i$  are embeddings. Thus  $\text{hom}_{\mathcal{D}(Z)}(\Sigma_1, \Sigma_2)$  is either empty or consists of a single element.

Going back to our  $Y$ . Let  $\{O_i\}_{i \in I}$  be a locally finite open cover of  $Y$ , closed under intersections, with each  $O_i$  diffeomorphic to an open ball. Such a cover is often called a good cover of a manifold, and the existence of such a cover is folklore theorem, but a proof can be found in [3, Prop A1]. Let  $\mathcal{O}$  denote the category with the set of objects  $\{O_i\}$  and with morphisms set inclusions. Set  $C_i = \mathcal{D}(O_i)$ , then we naturally have  $C_i \subset \Delta^{sm}(Y_\bullet)$ . For each  $i$ , we have the functor

$$F_i = P|_{C_i} : C_i \rightarrow \mathcal{G}.$$

By assumption that each  $O_i$  is diffeomorphic to an open ball,  $O_i$  has an exhaustion by embedded  $d$ -simplices. Meaning that there is a sequence of smooth embeddings  $\Sigma_j : \Delta^d \rightarrow O_i$  with  $\text{image}(\Sigma_{j+1}) \supset \text{image}(\Sigma_j)$  for each  $j$ . And so that  $\bigcup_j \text{image}(\Sigma_j) = O_i$ . In particular, for each  $i$ , the colimit in  $\mathcal{G}$ :

$$(5.2) \quad P_i := \text{colim}_{C_i} F_i$$

is naturally a smooth  $G$ -bundle over  $O_i$ , with  $G$ -bundle charts defined as follows. Take the collection of maps

$$\{\phi_{\Sigma,j}^i\}_{\Sigma \in C_i, j \in J^\Sigma},$$

satisfying the following.

- Each  $\phi_{\Sigma,j}^i$  is the composition map

$$V_{\Sigma,j}^i \times G \xrightarrow{\xi_{ij}} P_\Sigma \xrightarrow{c_\Sigma} P_i$$

where  $V_{\Sigma,j}^i \subset (\Delta^d)^\circ$  is open, for  $(\Delta^d)^\circ$  the topological interior of the subspace  $\Delta^d \subset \mathbb{R}^d$ . And where  $c_\Sigma : (P_\Sigma = F_i(\Sigma)) \rightarrow P_i$  is the natural map in the colimit diagram of (5.2).

- The collection

$$\{\xi_{i,j}\}_{j \in J^\Sigma}$$

forms an atlas of smooth  $G$ -bundle charts for  $P_\Sigma|_{(\Delta^d)^\circ}$ .

The collection  $\{\phi_{\Sigma,j}^i\}$  then forms a smooth  $G$ -bundle atlas for  $P_i$ .

So we obtain a functor

$$D : \mathcal{O} \rightarrow \mathcal{G},$$

defined by

$$D(O_i) = P_i,$$

and defined naturally on morphisms. Specifically, a morphism  $O_{i_1} \rightarrow O_{i_2}$  induces a functor  $C_{i_1} \rightarrow C_{i_2}$  and hence a map  $P_{i_1} \rightarrow P_{i_2}$ .

Let  $t : \mathcal{O} \rightarrow \text{Top}$  denote the tautological functor, so that  $Y = \text{colim}_{\mathcal{O}} t$ , where for simplicity we write equality for natural isomorphisms here and further on in this proof. Now,

$$(5.3) \quad N = \text{colim}_{\mathcal{O}} D,$$

is naturally a topological  $G$ -bundle over  $\text{colim}_{\mathcal{O}} t = Y$ . Let  $c_i : P_i \rightarrow N$  denote the natural maps in the colimit diagram of (5.3). The collection of charts  $\{c_i \circ \phi_{\Sigma,j}^i\}_{i,j,\Sigma}$  forms a smooth atlas on  $N$ , giving it a structure of a smooth  $G$ -bundle.

We now prove that  $P$  is induced by  $N$ . Let  $\Sigma$  be a  $d$ -simplex of  $X := Y_\bullet$ , then  $\{V_i := \Sigma^{-1}(O_i)\}_{i \in I}$  is a locally finite open cover of  $\Delta^d$  closed under finite intersections. Let  $N_\Sigma = \Sigma^* N$ , and let  $N'_i$  likewise denote the restriction of  $N_\Sigma$  over  $V_i$ . As  $\Delta^d$  is a convex subset of  $\mathbb{R}^d$ , the open metric balls in  $\Delta^d$ , for the induced metric, are convex as subsets of  $\mathbb{R}^d$ . Consequently, as each  $V_i \subset \Delta^d$  is open, it has a basis of convex (as subsets of  $\mathbb{R}^d$ ) metric balls, with respect to the induced metric. By Rudin [12] there is then a locally finite cover of  $V_i$  by elements of this basis. In fact, as Rudin shows any open cover of  $V_i$  has a locally finite refinement by elements of such a basis.

Let  $\{W_j^i\}$  consist of elements of this cover and all intersections of its elements, (which must then be finite intersections). So  $W_j^i \subset V_i$  are open convex subsets and  $\{W_j^i\}$  is a locally finite open cover of  $V_i$ . In particular,  $W_j^i$  have an exhaustion by nested images of embedded simplices, that is

$$W_j^i = \bigcup_{k \in \mathbb{N}} \text{image } \sigma_k^{i,j}$$

for  $\sigma_k^{i,j} : \Delta^d \rightarrow W_j^i$  smooth and embedded, with  $\text{image } \sigma_k^{i,j} \subset \text{image } \sigma_{k+1}^{i,j}$  for each  $k$ . Alternatively, we can use that each  $V_i$  is a manifold with corners, and then take a good cover, however the above is more elementary.

Let  $C$  be the small category with objects  $I \times J \times \mathbb{N}$ , so that there is exactly one morphism from  $a = (i, j, k)$  to  $b = (i', j', k')$  whenever  $\text{image } \sigma_k^{i,j} \subset \text{image } \sigma_{k'}^{i',j'}$ , and no morphisms otherwise. Let

$$F : C \rightarrow \mathcal{D}(\Delta^d)$$

be the functor  $F(a) = \sigma_k^{i,j}$  for  $a = (i, j, k)$ , (the definition on morphisms is forced). For brevity, we then reset  $\sigma_a := F(a)$ .

If  $\mathcal{O}(Y)$  denotes the category of topological subspaces of  $Y$  with morphisms inclusions, then there is a forgetful functor

$$T : \mathcal{D}(Y) \rightarrow \mathcal{O}(Y)$$

which takes  $f$  to  $\text{image}(f)$ . With all this in place, we obviously have a colimit in  $\text{Top}$ :

$$\Delta^d = \text{colim}_C T \circ F,$$

Now by construction, for each  $a \in C$  we may express:

$$(5.4) \quad \Sigma \circ \sigma_a = \Sigma_a \circ \sigma_a,$$

for some  $i$  and some  $\Sigma_a : \Delta^d \rightarrow U_i \subset Y$  a smooth embedded  $d$ -simplex. Then for all  $a \in C$ :

$$(5.5) \quad \sigma_a^* P_\Sigma = P_{\Sigma \circ \sigma_a} = P_{\Sigma_a \circ \sigma_a}$$

$$(5.6) \quad = N_{\Sigma_a \circ \sigma_a} \quad \text{by construction of } N$$

$$(5.7) \quad = \sigma_a^* N_{\Sigma_a}$$

$$(5.8) \quad = \sigma_a^* N_\Sigma \quad \text{by (5.4).}$$

To better explain the equality (5.6), (a natural isomorphism) note that we have a composition of natural bundle maps:

$$P_{\Sigma_a} \rightarrow P_i \rightarrow N,$$

with the first map the bundle map in the colimit diagram of (5.2), and the second map the bundle map in the colimit diagram of (5.3). This composition is over  $\Sigma_a$ . And so there is the induced (natural) universal map

$$N_{\Sigma_a} \rightarrow P_{\Sigma_a},$$

which is a  $G$ -bundle isomorphism.

Now we have a natural functor  $F_\Sigma : \mathcal{D}(\Delta^d) \rightarrow \mathcal{G}$ , given by  $F_\Sigma(\sigma) = \sigma^* P_\Sigma$ , and

$$(5.9) \quad P_\Sigma = \text{colim}_C F_\Sigma \circ F.$$

Similarly,

$$(5.10) \quad N_\Sigma = \text{colim}_C F'_\Sigma \circ F$$

where  $F'(\sigma) = \sigma^* N_\Sigma$ . Now (5.8) gives that

$$F'_\Sigma \circ F = F_\Sigma \circ F.$$

Then the conjunction of this with (5.9), (5.10) yields that

$$P_\Sigma = N_\Sigma.$$

More explicitly, for every simplex  $\Sigma$  of  $X$ , we have constructed a natural isomorphism  $h_\Sigma : P_\Sigma \rightarrow N_\Sigma$ . And by construction this clearly determines the natural transformation we are looking for.  $\square$



**5.1. Pull-backs of simplicial bundles.** Let  $P \rightarrow X$  be a simplicial  $G$ -bundle over a smooth simplicial set  $X$ . And let  $f : Y \rightarrow X$  be smooth. We define the pull-back simplicial  $G$ -bundle  $f^*P \rightarrow Y$  by the functor  $f^*P = P \circ f$ .

## 6. CONNECTIONS ON SIMPLICIAL $G$ -BUNDLES

**Definition 6.1.** Let  $G$  be a Frechet Lie group. A **simplicial  $G$ -connection**  $D$  on a simplicial  $G$ -bundle  $P$  over a smooth simplicial set  $X$  is for each  $d$ -simplex  $\Sigma$  of  $X$ , a smooth  $G$ -invariant Ehresmann  $G$ -connection  $D_\Sigma$  on  $P_\Sigma$ . This data is required to satisfy: if  $f : \Sigma_1 \rightarrow \Sigma_2$  is a morphism in  $\Delta(X)$  then

$$P(f)^* D_{\Sigma_2} = D_{\Sigma_1}.$$

We say that  $D$  is **coherent** if the same holds for all morphisms  $f : \Sigma_1 \rightarrow \Sigma_2$  in  $\Delta^{sm}(X)$ . Will will often just say  $G$ -connection instead of simplicial  $G$ -connection, where there is no need to disambiguate.

As with differential forms the coherence condition is very restrictive, and is not part of the basic definition.

**Lemma 6.2.**  $G$ -connections on simplicial  $G$ -bundles exist and any pair of  $G$ -connections  $D_1, D_2$  on a simplicial  $G$ -bundle  $P$  are **concordant**. The latter means that there is a  $G$ -connection on  $\tilde{D}$  on  $P \times I$ ,

$$I := [0, 1]_\bullet$$

which restricts to  $D_1, D_2$  on  $P \times I_0$ , respectively on  $P \times I_1$ , for  $I_0, I_1 \subset I$  denoting the images of the two end point inclusions  $\Delta_\bullet^0 \rightarrow I$ .

*Proof.* Suppose that  $\Sigma : \Delta_{simp}^d \rightarrow X$  is a degeneracy of a 0-simplex  $\Sigma_0 : \Delta_{simp}^0 \rightarrow X$ , meaning that there is a morphism from  $\Sigma$  to  $\Sigma_0$  in  $\Delta(X)$ . Then  $P_\Sigma = \Delta^d \times P_{\Sigma_0}$  (as previously equality indicates natural isomorphism) and we fix the corresponding trivial connection  $D_\Sigma$  on  $P_\Sigma$ . We then proceed inductively.

Suppose we have constructed connections  $D_\Sigma$  for all degeneracies of  $n$ -simplices,  $n \geq 0$ . We now extend this to all degeneracies of  $(n+1)$ -simplices. If  $\Sigma$  is a non-degenerate  $(n+1)$ -simplex then  $D_\Sigma$  is already determined over the boundary of  $\Delta^{n+1}$ , as by the hypothesis  $D_\Sigma$  is already defined on all  $n$ -simplices, so extend  $D_\Sigma$  over all of  $\Delta^{n+1}$  arbitrarily. Thus we have extended  $D_\Sigma$  to all  $(n+1)$ -simplices, as such a simplex is either non-degenerate or is a degeneracy of a  $n$ -simplex. If  $\Sigma'$  is a  $m$ -simplex that is a degeneracy of a  $(n+1)$ -simplex  $\Sigma^{n+1}$ , then  $P_{\Sigma'} = pr^* P_{\Sigma^{n+1}}$  for a certain determined simplicial projection  $pr : \Delta^m \rightarrow \Delta^{n+1}$ , and we define  $D_\Sigma = \tilde{pr}^* D_{\Sigma^{n+1}}$ . For  $\tilde{pr} : P_{\Sigma'} \rightarrow P_{\Sigma^{n+1}}$  the natural map in the pull-back square. The second part of the lemma follows by an analogous argument, since we may just extend  $D_1, D_2$  to a concordance connection  $\tilde{D}$ , using the above inductive procedure.  $\square$

**Example 6.3.** Given a classical smooth  $G$ -connection  $D$  on a smooth principal  $G$ -bundle  $Z \rightarrow Y$ , we obviously get a simplicial  $G$ -connection on the induced simplicial  $G$ -bundle  $N = Z_\bullet$ . Concretely, this is defined by setting  $D_\Sigma$  on  $N_\Sigma = \Sigma^* Z$  to be  $\tilde{\Sigma}^* D$ , for  $\tilde{\Sigma} : \Sigma^* Z \rightarrow Z$  the natural map. This is called the **induced simplicial connection**, and may be denoted by  $D_\bullet$ . Going in the other direction is always possible if the given simplicial  $G$ -connection in addition satisfies coherence, but we will not elaborate.

## 7. CHERN-WEIL HOMOMORPHISM

**7.1. The classical case.** To establish notation we first discuss classical Chern-Weil homomorphism.

Let  $G$  be a Frechet Lie group, and let  $\mathfrak{g}$  denote its Lie algebra. Let  $P$  be a smooth  $G$ -bundle over a smooth manifold  $Y$ . Fix a  $G$ -connection  $D$  on  $P$ . Let  $\text{Aut } P_y$  denote the group of smooth  $G$ -torsor automorphisms of the fiber  $P_y$  of  $P$  over  $y \in Y$ . Note that  $\text{Aut } P_y \simeq G$  where  $\simeq$  means non-canonically isomorphic. Then associated to  $D$  we have the classical curvature 2-form  $R^D$  on  $Y$ , understood as a 2-form valued in the vector bundle  $\mathcal{P} \rightarrow Y$ , whose fiber over  $Y$  is  $\text{lie Aut } P_y$  - the Lie algebra of  $\text{Aut } P_y$ .

Thus,

$$\forall v, w \in T_y Y : R^D(v, w) \in \text{lie Aut } P_y.$$

Now, let  $\rho$  be a symmetric multi-linear functional:

$$\rho : (V = \prod_{i=1}^{i=k} \mathfrak{g}) \rightarrow \mathbb{R},$$

satisfying

$$\forall g \in G, \forall v \in V : \rho(\text{Ad}_g(v)) = \rho(v).$$

Here if  $v = (\xi_1, \dots, \xi_n)$ ,  $\text{Ad}_g(v) = (\text{Ad}_g(\xi_1), \dots, \text{Ad}_g(\xi_n))$  is the adjoint action by the element  $g \in G$ . As  $\rho$  is  $\text{Ad}$  invariant, it uniquely determines multi-linear maps with the same name:

$$\rho : (V = \prod_{i=1}^{i=k} \text{lie Aut } P_z) \rightarrow \mathbb{R},$$

by fixing any Lie-group isomorphism  $\text{Aut } P_z \rightarrow G$ . We may now define a closed  $\mathbb{R}$ -valued  $2k$ -form  $\omega^{\rho, D}$  on  $Y$ :

$$(7.1) \quad \omega^{\rho, D}(v_1, \dots, v_{2k}) = \frac{1}{2k!} \sum_{\eta \in P_{2k}} \text{sign } \eta \cdot \rho(R^D(v_{\eta(1)}, v_{\eta(2)}), \dots, R^D(v_{\eta(2k-1)}, v_{\eta(2k)})),$$

for  $P_{2k}$  the permutation group of a set with  $2k$  elements. Set

$$\alpha^{\rho, D} := \int \omega^{\rho, D}.$$

Then we define the classical Chern-Weil characteristic class:

$$(7.2) \quad c^\rho(P) = c_{2k}^\rho(P) := [\alpha^{\rho, D}] \in H^{2k}(X, \mathbb{R}).$$

**7.2. Chern-Weil homomorphism for smooth simplicial bundles.** Now let  $P$  be a simplicial  $G$ -bundle over a smooth simplicial set  $X$ . Fix a simplicial  $G$ -connection  $D$  on  $P$ .

For each simplex  $\Sigma^d$ , we have the 2-form  $R_\Sigma^D$  of the connection  $D_\Sigma$  on  $P_\Sigma$ , defined as in Section 7.1 above. For concreteness:

$$\forall v, w \in T_z \Delta^d : R_\Sigma^D(v, w) \in \text{lie Aut } P_z,$$

for  $P_z$  the fiber of  $P_\Sigma$  over  $z \in \Delta^d$ .

As above, let  $\rho$  be a  $\text{Ad}$  invariant symmetric multi-linear functional:

$$\rho : (V = \prod_{i=1}^{i=k} \mathfrak{g}) \rightarrow \mathbb{R}.$$

Thus  $\rho$  uniquely determines for each  $z \in \Delta^d$  a symmetric multi-linear map with the same name:

$$\rho : (V = \prod_{i=1}^{i=k} \text{lie Aut } P_z) \rightarrow \mathbb{R},$$

by fixing any Lie-group isomorphism  $\text{Aut } P_z \rightarrow G$ . We may now define a closed (simplicial)  $\mathbb{R}$ -valued  $2k$ -form  $\omega^{\rho,D}$  on  $X$ :

$$\omega_{\Sigma}^{\rho,D}(v_1, \dots, v_{2k}) = \frac{1}{2k!} \sum_{\eta \in P_{2k}} \text{sign } \eta \cdot \rho(R_{\Sigma}^D(v_{\eta(1)}, v_{\eta(2)}), \dots, R_{\Sigma}^D(v_{\eta(2k-1)}, v_{\eta(2k)})),$$

for  $P_{2k}$  as above the permutation group of a set with  $2k$  elements. Set

$$\alpha^{\rho,D} := \int \omega^{\rho,D}.$$

**Lemma 7.1.** *For  $P \rightarrow X$  as above*

$$[\alpha^{\rho,D}] = [\alpha^{\rho,D'}] \in H^{2k}(X, \mathbb{R}),$$

for any pair of  $G$ -connections  $D, D'$  on  $P$ .

*Proof.* For  $D, D'$  as in the statement, fix a concordance  $G$ -connection  $\tilde{D}$ , between  $D, D'$ , on the  $G$ -bundle  $P \times I \rightarrow X \times I$ , as in Lemma 6.2. Then  $\alpha^{\rho, \tilde{D}}$  is a  $2k$  cocycle on  $X \times I$  restricting to  $\alpha^{\rho,D}, \alpha^{\rho,D'}$  on  $X \times I_0, X \times I_1$ .

Now the pair of inclusions

$$i_j : X \rightarrow X \times I \quad j = 0, 1$$

corresponding to the end points of  $I$  are homotopic and so  $\alpha^{\rho,D}, \alpha^{\rho,D'}$  are cohomologous cocycles, cf. Section 4.1.  $\square$

Then we define the associated Chern-Weil characteristic class:

$$c^{\rho}(P) = c_{2k}^{\rho}(P) := [\alpha^{\rho,D}] \in H^{2k}(X, \mathbb{R}),$$

(the subscript  $2k$  is implicitly given by  $\rho$ .) We have the expected naturality:

**Lemma 7.2.** *Let  $P$  be a simplicial  $G$ -bundle over  $Y$ ,  $\rho$  as above and  $f : X \rightarrow Y$  a smooth simplicial map. Then*

$$f^* c^{\rho}(P) = c^{\rho}(f^* P).$$

*Proof.* Let  $D$  be a simplicial  $G$ -connection on  $P$  then  $f^* D$  is a simplicial  $G$ -connection on  $f^* P$  and clearly  $\omega^{\rho, f^* D} = f^* \omega^{\rho, D}$ , so that passing to cohomology we obtain our result.  $\square$

**Proposition 7.3.** *Let  $G \hookrightarrow Z \rightarrow Y$  be an ordinary smooth principal  $G$ -bundle, and  $\rho$  as above. Let  $Z_{\bullet}$  be the induced simplicial  $G$ -bundle over  $Y_{\bullet}$  as in Example 5.6. Then the classes  $c^{\rho}(Z_{\bullet}) \in H^{2k}(Y_{\bullet}, \mathbb{R})$  coincide with the classical Chern-Weil classes of  $Z$ . More explicitly, if  $c^{\rho}(Z) \in H^{2k}(Y, \mathbb{R})$  is the classical Chern-Weil characteristic class as in (7.2), then*

$$(7.3) \quad n^* (|c^{\rho}(Z_{\bullet})|) = c^{\rho}(Z),$$

where  $|c^{\rho}(Z_{\bullet})|$  is as in Notation 4.6, and  $n$  is as in (4.5).

*Proof.* Fix a smooth  $G$ -connection  $D$  on  $Z$ . This induces a simplicial  $G$ -connection  $D_\bullet$  on  $Z_\bullet$ , as in Example 6.3. Let  $\omega^{\rho,D}$  denote the classical smooth Chern-Weil differential  $2k$ -form on  $Y$ , as in (7.1). Let  $\alpha^{\rho,D} = \int \omega^{\rho,D} \in H^{2k}(Y, \mathbb{R})$ . By its construction  $\omega^{\rho,D_\bullet}$  is the simplicial differential form induced by  $\omega^{\rho,D}$ , where induced is as in Example 4.2. Consequently,

$$(I^c)^{-1}([\alpha^{\rho,D_\bullet}]) = [\alpha^{\rho,D}] = c^\rho(Z),$$

where  $I^c$  is as in (4.7). But

$$(I^c)^{-1}([\alpha^{\rho,D_\bullet}]) = n^*([\alpha^{\rho,D_\bullet}]),$$

and so

$$n^*(|c^\rho(Z_\bullet)|) = c^\rho(Z).$$

□

## 8. THE UNIVERSAL SIMPLICIAL $G$ -BUNDLE

Briefly, a Grothendieck universe is a set  $\mathcal{U}$  forming a model for set theory. That is if we interpret all terms of set theory as elements of  $\mathcal{U}$ , then all the set theoretic constructions keep us within  $\mathcal{U}$ . We will assume Grothendieck's axiom of universes which says that for any set  $X$  there is a Grothendieck universe  $\mathcal{U} \ni X$ . Intuitively, such a universe  $\mathcal{U}$  is formed by from all possible set theoretic constructions starting with  $X$ . For example if  $\mathcal{P}(X)$  denotes the power set of  $X$ , then  $\mathcal{P}(X) \in \mathcal{U}$  and if  $\{Y_i \in \mathcal{P}(X)\}_{i \in I}$  for  $I \in \mathcal{U}$  is a collection then  $\bigcup_i Y_i \in \mathcal{U}$ . We should note that this axiom is beyond  $ZFC$ , but is now a common axiom of modern set theory, especially in the context of category theory, c.f. [6]. In some contexts one works with universes implicitly. This is impossible here, as a priori our constructions are universe dependent. They become universe independent once we pass to the homotopy world.

Let  $G$  be a Frechet Lie group. Let  $\mathcal{U}$  be a Grothendieck universe satisfying:

$$\mathcal{U} \ni \{G\}, \quad \forall n \in \mathbb{N} : \mathcal{U} \ni \{\Delta^n\},$$

where  $\Delta^n$  are the usual topological  $n$ -simplices. These conditions are of course partly redundant by intent. Such a  $\mathcal{U}$  will be called  **$G$ -admissible**. We construct smooth Kan complexes  $BG^\mathcal{U}$  for each  $G$ -admissible  $\mathcal{U}$ . The homotopy type of  $BG^\mathcal{U}$  will then be shown to be independent of  $\mathcal{U}$ , provided  $G$  has the homotopy type of a CW complex. Moreover, in this case we will show that  $BG^\mathcal{U} \simeq BG$ , for  $BG$  the classical classifying space.

**Definition 8.1.** A  $\mathcal{U}$ -small set is an element of  $\mathcal{U}$ . For  $X$  a smooth simplicial set, a smooth simplicial  $G$ -bundle  $P \rightarrow X$  will be called  $\mathcal{U}$ -small if for each simplex  $\Sigma$  of  $X$  the bundles  $P_\Sigma$  are  $\mathcal{U}$ -small.

**8.1. The classifying spaces  $BG^\mathcal{U}$ .** Let  $\mathcal{U}$  be  $G$ -admissible. We define a simplicial set  $BG^\mathcal{U}$ , whose set of  $k$ -simplices  $BG^\mathcal{U}(k)$  is the set of  $\mathcal{U}$ -small smooth simplicial  $G$ -bundles over  $\Delta_\bullet^k$ . The simplicial maps are just defined by pull-back so that given a map  $i \in \text{hom}_\Delta([m], [n])$  the map

$$BG^\mathcal{U}(i) : BG^\mathcal{U}(n) \rightarrow BG^\mathcal{U}(m)$$

is just the natural pull-back:

$$BG^\mathcal{U}(i) := i_\bullet^* BG^\mathcal{U},$$

for  $i_\bullet$ , the induced map  $i_\bullet : \Delta_\bullet^m \rightarrow \Delta_\bullet^n$ , and where pull-back is as in Section 5.1. This defines a simplicial set  $BG^\mathcal{U}$ . We define a smooth simplicial set structure on  $BG^\mathcal{U}$  as follows. Given a  $d$ -simplex  $\Sigma$  the induced map

$$\Sigma_* : \Delta_\bullet^d \rightarrow BG^\mathcal{U},$$

is defined naturally by

$$\Sigma_*(\sigma) := \sigma_*^* BG^\mathcal{U}(\Sigma).$$

More explicitly,  $\sigma \in \Delta_\bullet^d$  is a smooth map  $\sigma : \Delta^k \rightarrow \Delta^d$ ,  $\sigma_\bullet : \Delta_\bullet^k \rightarrow \Delta_\bullet^d$  denotes the induced map and the pull-back is a previously.

**8.2. The universal smooth simplicial  $G$ -bundle  $EG^\mathcal{U}$ .** By construction we have a natural functor

$$E : \Delta^{sm}(BG^\mathcal{U}) \rightarrow \mathcal{G}.$$

Mainly, a simplex  $\Sigma \in BG^\mathcal{U}(d)$ , by assumption corresponds to a simplicial  $G$ -bundle  $\Sigma : \Delta_\bullet^d \rightarrow \mathcal{G}$ . And we then set:  $E(\Sigma) = \Sigma(id)$ , for  $id \in \Delta_\bullet^d$ ,  $id : \Delta^d \rightarrow \Delta^d$  the identity map.

By construction the functor  $E$  satisfies the compatibility condition, and hence determines a simplicial  $G$ -bundle. The universal simplicial  $G$ -bundle  $EG^\mathcal{U}$  is then another name for  $E$  above, for specified  $G, \mathcal{U}$ .

In what follows  $V$  denotes  $BG^\mathcal{U}$  for a general unspecified  $G$ -admissible  $\mathcal{U}$ .

**Proposition 8.2.**  *$V$  is a Kan complex.*

*Proof.* Let

$$E : \Delta(V) \rightarrow \mathcal{G}$$

be the restriction of  $E$  above. Recall that  $\Lambda_k^n \subset \Delta_{simp}^n$ , denotes the sub-simplicial set corresponding to the “boundary” of  $\Delta^n$  with the  $k$ ’th face removed, where by  $k$ ’th face we mean the face opposite to the  $k$ ’th vertex. Let  $h : \Lambda_k^n \rightarrow V$ ,  $0 \leq k \leq n$ , be a simplicial map, this is also called a horn. We need to construct an extension of  $h$  to  $\Delta_{simp}^n$ . For simplicity we assume  $n = 2$ , the general case is identical. Let

$$\Delta(h) : \Delta(\Lambda_k^n) \rightarrow \Delta(V)$$

be the induced functor. Set  $P = E \circ \Delta(h)$ . Clearly, to construct our extension  $h$  we just need an extension of  $P$  over  $\Delta(\Delta_{simp}^n)$ .

**Lemma 8.3.** *There is a natural transformation of  $\mathcal{G}$  valued functors  $T \rightarrow P$ , where  $T$  is the trivial functor  $T : \Delta(\Lambda_k^n) \rightarrow \mathcal{G}$ ,  $T(\sigma^d) = \Delta^d \times G$ .*

*Proof.* Set  $L := \Lambda_k^2$ , with  $k = 1$ , again without loss of generality. There are three natural inclusions

$$i_j : \Delta_{simp}^0 \rightarrow L,$$

$j = 0, 1, 2$ , with  $i_1$  corresponding to the inclusion of the horn vertex. The corresponding 0-simplices will just be denoted by  $0, 1, 2$ . Fix a  $G$ -bundle map (in this case just smooth  $G$ -torsor map):

$$\phi_1 : \Delta^0 \times G \rightarrow P(i_1).$$

Let

$$\sigma_{1,2} : \Delta_{simp}^1 \rightarrow L$$

be the edge between vertexes 1, 2, that is  $\sigma_{1,2}(0) = 1$ ,  $\sigma_{1,2}(1) = 2$ . Then  $P(\sigma_{1,2})$  is a smooth bundle over the contractible space  $\Delta^1$  and so we may find a  $G$ -bundle map

$$\phi_{1,2} : \Delta^1 \times G \rightarrow P(\sigma_{1,2}),$$

whose restriction to  $\{0\} \times G$  is  $\phi_1$ . Meaning:

$$\phi_{1,2} \circ (i_0 \times id) = \phi_1,$$

where

$$i_0 : \Delta^0 \rightarrow \Delta^1,$$

is the map  $i_0(0) = 0$ .

We may likewise construct a  $G$ -bundle map

$$\phi_{0,1} : \Delta^1 \times G \rightarrow P(\sigma_{0,1}),$$

(where  $\sigma_{0,1}$  is defined analogously to  $\sigma_{1,2}$ ), whose restriction to  $\{1\} \times G$  is  $\phi_1$ .

Then  $\phi_{0,1}$ ,  $\phi_{1,2}$  obviously glue to a natural transformation:

$$tr : T \rightarrow P.$$

□

We have the trivial extension of  $T$ ,

$$\tilde{T} : \Delta(\Delta_{simp}^2) \rightarrow \mathcal{G},$$

defined by

$$\tilde{T}(\sigma^d) = \Delta^d \times G.$$

And so it is clear that  $P$  likewise has an extension  $\tilde{P}$  to  $\Delta(\Delta_{simp}^2)$ , but we need this extension to be  $\mathcal{U}$ -small so that we must be explicit. Let  $\sigma^2$  denote the non-degenerate 2-simplex of  $\Delta^2$ . It suffices to construct  $\tilde{P}_{\sigma^2} := \tilde{P}(\sigma^2)$ . Let

$$\sigma_{0,1}, \sigma_{1,2} : \Delta^1 \rightarrow \Delta^2$$

be the edge inclusions of the edges between the vertices 0, 1, respectively 1, 2. And let  $e_{0,1}, e_{1,2}$  denote their images.

We then define a set theoretic (for the moment no topology)  $G$ -bundle

$$\tilde{P}_{\sigma^2} \xrightarrow{p} \Delta^2$$

by the following conditions:

$$\begin{aligned} \sigma_{0,1}^* \tilde{P}_{\sigma^2} &= P(\sigma_{0,1}), \\ \sigma_{1,2}^* \tilde{P}_{\sigma^2} &= P(\sigma_{1,2}), \\ P_{\sigma^2}|_{(\Delta^2)^\circ} &= (\Delta^2)^\circ \times G, \end{aligned}$$

where  $(\Delta^2)^\circ$  denotes the topological interior of  $\Delta^2 \subset \mathbb{R}^2$ , and where the projection map  $p$  is natural.

We now discuss the topology. We have the smooth  $G$ -bundle maps

$$\begin{aligned} \phi_{0,1}^{-1} : P(\sigma_{0,1}) &\rightarrow \Delta^2 \times G, \\ \phi_{1,2}^{-1} : P(\sigma_{1,2}) &\rightarrow \Delta^2 \times G, \end{aligned}$$

over  $\sigma_{0,1}, \sigma_{1,2}$ , as in the above Lemma. Let  $d_0$  be any metric on  $\Delta^2 \times G$  inducing the natural product topology. The topology on  $\tilde{P}_{\sigma^2}$  will be given by the  $d$ -metric topology, for  $d$  extending  $d_0$  on  $(\Delta^2)^\circ \times G \subset \tilde{P}_{\sigma^2}$ , and defined as follows. For

$y_1 \in \tilde{P}_{\sigma^2}$  with  $p(y_1) \in e_{0,1}$ ,  $y_2$  arbitrary,  $d(y_1, y_2) = d_0(\phi_{0,1}^{-1}(y_1), y_2)$ . Likewise, for  $y_1 \in \tilde{P}_{\sigma^2}$  with  $p(y_1) \in e_{1,2}$ ,  $y_2$  arbitrary,  $d(y_1, y_2) = d_0(\phi_{1,2}^{-1}(y_1), y_2)$ . This defines  $\tilde{P}_{\sigma^2}$  as a topological  $G$ -bundle over  $\Delta^2$ .

There is a natural topological  $G$ -bundle trivialization

$$\xi : \tilde{P}_{\sigma^2} \rightarrow \Delta^2 \times G$$

defined as follows.  $\xi(y) = y$  when  $p(y) \in (\Delta^2)^\circ$  and  $\xi(y) = \phi_{0,1}^{-1}(y)$  when  $p(y) \in e_{0,1}$ ,  $\xi(y) = \phi_{1,2}^{-1}(y)$  when  $p(y) \in e_{0,2}$ . We then take the smooth structure on  $\tilde{P}_{\sigma^2}$  to be the smooth structure pulled back by  $\xi$ . By construction  $\tilde{P}_{\sigma^2}$  is  $\mathcal{U}$ -small, as all of the constructions take place in  $\mathcal{U}$ . Moreover, by construction  $\sigma_{0,1}^* \tilde{P}_{\sigma^2} = P_{\sigma_{0,1}}$  as a smooth  $G$ -bundle and  $\sigma_{1,2}^* \tilde{P}_{\sigma^2} = P_{\sigma_{1,2}}$  as a smooth  $G$ -bundle, which readily follows by the fact that the maps  $\phi_{0,1}, \phi_{1,2}$  are smooth  $G$ -bundle maps. Thus, we have constructed the needed extension.  $\square$

**Theorem 8.4.** *Let  $X$  be a smooth simplicial set.  $\mathcal{U}$ -small simplicial  $G$ -bundles  $P \rightarrow X$  are classified by smooth maps*

$$f_P : X \rightarrow BG^{\mathcal{U}}.$$

*Specifically:*

- (1) *For every  $\mathcal{U}$ -small  $P$  there is a smooth map  $f_P : X \rightarrow BG^{\mathcal{U}}$  so that*

$$f_P^* EG^{\mathcal{U}} \simeq P$$

*as simplicial  $G$ -bundles.*

- (2) *If  $P_1, P_2$  are isomorphic  $\mathcal{U}$ -small smooth simplicial  $G$ -bundles over  $X$  then  $f_{P_1}, f_{P_2}$  are smoothly homotopic, as in Definition 3.12.*
- (3) *If  $X = Y_\bullet$  for  $Y$  a smooth manifold and  $f, g : X \rightarrow BG^{\mathcal{U}}$  are smoothly homotopic then  $P_f = f^* EG^{\mathcal{U}}, P_g = g^* EG^{\mathcal{U}}$  are isomorphic simplicial  $G$ -bundles.*

*Proof.* Set  $V = BG^{\mathcal{U}}$ ,  $E = EG^{\mathcal{U}}$ . Let  $P \rightarrow X$  be a  $\mathcal{U}$ -small simplicial  $G$ -bundle. Define  $f_P : X \rightarrow V$  by:

$$f_P(\Sigma) = \Sigma^* P,$$

where the simplex  $\Sigma \in X(k)$  is understood as a smooth map  $\Sigma : \Delta_\bullet^k \rightarrow X$  by Proposition 3.6. Then clearly  $f_P$  is a smooth simplicial map. And clearly  $f_P^* E \simeq P$ .

We now prove the second part of the theorem. Let  $\phi : P_1 \rightarrow P_2$  be an isomorphism of  $\mathcal{U}$ -small simplicial  $G$ -bundles over  $X$ . We construct a simplicial  $G$ -bundle  $\tilde{P}$  over  $X \times I$  as follows, where  $I = \Delta_\bullet^1$  as before. Let  $\sigma$  be a  $k$ -simplex of  $X$ . Then  $\phi$  specifies a  $G$ -bundle diffeomorphism  $\phi_\sigma : P_{1,\sigma} \rightarrow P_{2,\sigma}$  over the identity map  $\Delta^k \rightarrow \Delta^k$ . The mapping cylinder of  $\phi_\sigma$ ,  $M_\sigma$ , is a smooth  $G$ -bundle over  $\Delta^k \times \Delta^1$ . Let  $pr_X, pr_I$  be the natural projections of  $X \times I$ , to  $X$  respectively  $I$ . Let  $\Sigma$  be a  $d$ -simplex of  $X \times I$ . Let  $\sigma_1 = pr_X \Sigma$ , and  $\sigma_2 = pr_I(\Sigma)$ . Let  $id^d : \Delta^d \rightarrow \Delta^d$  be the identity, so

$$(id^d, \sigma_2) : \Delta^d \rightarrow \Delta^d \times \Delta^1,$$

is a smooth map, where  $\sigma_2$  is the corresponding smooth map  $\sigma_2 : \Delta^d \rightarrow \Delta^1 = [0, 1]$ . We then define

$$\tilde{P}_\Sigma := (id^d, \sigma_2)^* M_{\sigma_1},$$

which is a smooth  $G$ -bundle over  $\Delta^d$ .

Suppose that  $\rho : \sigma' \rightarrow \sigma$  is a morphism in  $\Delta^{sm}(X)$ , for  $\Sigma$  as above. As  $\phi$  is a simplicial  $G$ -bundle map we have an induced bundle map:

$$(8.1) \quad g_\rho : M_{\sigma'} \rightarrow M_\sigma.$$

Now suppose we have a morphism  $m : \Sigma' \rightarrow \Sigma$  in  $\Delta^{sm}(X \times I)$ . Then

$$\tilde{P}_{\Sigma'} = (id^k, pr_I(\Sigma'))^* M_{pr_X(\Sigma')} = \tilde{m}^* \circ (id^d, pr_I(\Sigma))^* M_{pr_X(\Sigma)}$$

and

$$\tilde{P}_\Sigma = (id^d, pr_I(\Sigma))^* M_{pr_X(\Sigma)}.$$

Then we have the natural map:

$$\tilde{m}^* \circ (id^d, pr_I(\Sigma))^* M_{pr_X(\Sigma')} \rightarrow \tilde{m}^* \circ (id^d, pr_I(\Sigma))^* M_{pr_X(\Sigma)},$$

induced by  $g_{pr_X(m)} : M_{pr_X(\Sigma')} \rightarrow M_{pr_X(\Sigma)}$ . And we have the natural map in the pull-back square:

$$\tilde{m}^* \circ (id^d, pr_I(\Sigma))^* M_{pr_X(\Sigma)} \rightarrow (id^d, pr_I(\Sigma))^* M_{pr_X(\Sigma)}.$$

The composition of these maps gives a natural map:

$$\tilde{P}(m) : \tilde{P}_{\Sigma'} \rightarrow \tilde{P}_\Sigma.$$

This constructs the needed functor  $\tilde{P}$ , which determines our smooth simplicial  $G$ -bundle  $\tilde{P} \rightarrow X \times I$ . By the first part of the theorem, we have an induced smooth classifying map  $f_{\tilde{P}} : X \times I \rightarrow V$ . By construction it is a homotopy between  $f_{P_1}, f_{P_2}$ .

We now prove the third part of the theorem. Suppose that  $f, g : X \rightarrow V$  are smoothly homotopic, and let  $H : X \times I \rightarrow V$  be the corresponding smooth homotopy. By Lemma 5.10, the bundles  $P_f, P_g$  are induced by smooth  $G$ -bundles  $P'_f, P'_g$  over  $Y$ . Now  $P_H = H^*E$  is a simplicial  $G$ -bundle over  $X \times I = (Y \times [0, 1])_\bullet$ , and hence by Lemma 5.10  $P_H$  is also induced by a smooth  $G$ -bundle  $P'_H$  over  $Y \times [0, 1]$ . We may clearly in addition arrange that  $P'_H$  restricts to  $P'_f \sqcup P'_g$  over  $Y \times \partial[0, 1]$ . It follows that  $P'_f, P'_g$  are smoothly concordant and hence isomorphic smooth  $G$ -bundles, and so  $P_f, P_g$  are isomorphic simplicial  $G$ -bundles.  $\square$

We now study the dependence on a Grothendieck universe  $\mathcal{U}$ .

**Theorem 8.5.** *Let  $G$  be a Frechet Lie group having the homotopy type of a CW complex. Let  $\mathcal{U}$  be a  $G$ -admissible universe, let  $|BG^\mathcal{U}|$  denote the geometric realization of  $BG^\mathcal{U}$  and let  $BG^{top}$  denote the classical classifying space of  $G$  as defined by the Milnor construction [8]. Then there is a homotopy equivalence*

$$e^\mathcal{U} : |BG^\mathcal{U}| \rightarrow BG^{top},$$

which is natural in the sense that if  $\mathcal{U} \ni \mathcal{U}'$  then

$$(8.2) \quad [e^{\mathcal{U}'} \circ i^{\mathcal{U}, \mathcal{U}'}] = [e^\mathcal{U}],$$

where  $i^{\mathcal{U}, \mathcal{U}'} : |BG^\mathcal{U}| \rightarrow |BG^{\mathcal{U}'}|$  is the map of geometric realizations, induced by the natural inclusion  $i^{\mathcal{U}, \mathcal{U}'} : BG^\mathcal{U} \rightarrow BG^{\mathcal{U}'}$  and where  $[\cdot]$  denotes the homotopy class. In particular, for  $G$  as above the homotopy type of  $BG^\mathcal{U}$  is independent of the choice of  $G$ -admissible  $\mathcal{U}$ .

*Proof.* For  $\mathcal{U}$   $G$ -admissible let  $\mathcal{U}'$  be a universe enlargement of  $\mathcal{U}$ , that is  $\mathcal{U}'$  is a universe with  $\mathcal{U}' \ni \mathcal{U}$ . Set  $V := BG^\mathcal{U}$ ,  $V' := BG^{\mathcal{U}'}$ ,  $E := EG^\mathcal{U}$ ,  $E' := EG^{\mathcal{U}'}$ . There is a natural inclusion map  $i = i^{\mathcal{U}, \mathcal{U}'} : V \rightarrow V'$ , and  $i^*E' = E$ .



**Lemma 8.6.** *Let  $G$  be any Frechet Lie group and  $V$  as above.*

$$i_* : \pi_k^{sm}(V) \rightarrow \pi_k^{sm}(V')$$

*is a set isomorphism for all  $k \in \mathbb{N}$ , where  $\pi_k^{sm}$  are as in Definition 3.13.*

*Proof.* We show that  $i_*$  is injective. Let  $f, g : S_\bullet^k \rightarrow V$  be a pair of smooth maps. Let  $P_f, P_g$  denote the smooth bundles over  $S_\bullet^k$  induced via Lemma 5.10 by  $f^*E, g^*E$ . Set  $f' = i \circ f, g' = i \circ g$  and suppose that  $F : S_\bullet^k \times I \rightarrow V'$  is a smooth homotopy between  $f', g'$ . By Lemma 5.10 the simplicial bundle  $F^*E'$  is induced by a smooth bundle  $P_F \rightarrow S_\bullet^k \times I$ . In particular  $P_f, P_g$  are classically isomorphic smooth  $\mathcal{U}$ -small  $G$ -bundles. Taking the mapping cylinder for the corresponding  $G$ -bundle isomorphism gives us a smooth  $G$ -bundle  $P' \rightarrow S_\bullet^k \times I$  that is  $\mathcal{U}$ -small by construction. Finally,  $P'$  induces a smooth simplicial  $G$ -bundle  $H$  over  $S_\bullet^k \times I$  that by construction is  $\mathcal{U}$ -small. The classifying map  $f_H : S_\bullet^k \times I \rightarrow V$  then gives a smooth homotopy between  $f, g$ .

We now show surjectivity of  $i_*$ . Let  $f : S_\bullet^k \rightarrow V'$  be smooth. By Lemma 5.10 the simplicial  $G$ -bundle  $f^*E'$  is induced by a smooth  $G$ -bundle  $P' \rightarrow S_\bullet^k$ . Any such bundle is obtained by the clutching construction, that is  $P'$  is isomorphic as a smooth  $G$ -bundle to the bundle:

$$C = D_-^k \times G \sqcup D_+^k \times G / \sim,$$

where  $D_+^k, D_-^k$  are two copies of the standard closed  $k$ -ball in  $\mathbb{R}^k$ , and  $\sim$  is the following equivalence relation: for

$$(d, g) \in D_-^k \times G$$

$$(d, g) \sim \tilde{f}(d, g) \in D_+^k \times G,$$

where

$$\tilde{f} : \partial D_-^k \times G \rightarrow \partial D_+^k \times G, \quad \tilde{f}(d, x) = (d, f(d)^{-1} \cdot x),$$

for some smooth  $f : S^{k-1} \rightarrow G$ .

Then  $C$  is  $\mathcal{U}$ -small, since this gluing construction is carried out in  $\mathcal{U}$ . Let

$$C_\bullet \rightarrow S_\bullet^k$$

denote the induced  $\mathcal{U}$ -small smooth simplicial  $G$ -bundle. Now  $C_\bullet$  and  $f^*E'$  are induced by isomorphic  $\mathcal{U}'$ -small smooth  $G$ -bundles, hence are isomorphic  $\mathcal{U}'$ -small simplicial  $G$ -bundles. And so by Theorem 8.4, the classifying map  $f_{C_\bullet} \rightarrow V'$  is smoothly homotopic to  $f$ .

Since  $C_\bullet$  is  $\mathcal{U}$ -small it is also classified by a smooth map  $f' : S_\bullet^k \rightarrow V$ . It is immediate that  $[i \circ f'] = [f_{C_\bullet}]$ , since  $i^*E' = E$ , and so  $i_*([f']) = [f]$ .  $\square$

**Corollary 8.7.** *Let  $G$  be any Frechet Lie group, and  $V$  as above. Simplicial  $G$ -bundles over  $S_\bullet^k$ , up to smooth isomorphism, are classified by smooth homotopy classes of maps  $f : S_\bullet^k \rightarrow V$ . That is the mapping  $c_V$ :*

$$[f] \mapsto [P_f := f^*E]$$

*is a set bijection from the set of smooth homotopy classes of maps  $f : S_\bullet^k \rightarrow V$  to the set of isomorphism classes of simplicial  $G$ -bundles over  $S_\bullet^k$ .*

*Proof.*  $c_V$  is well defined by the third part of Theorem 8.4. It is injective by the second part Theorem 8.4. Let  $P$  be a simplicial  $G$ -bundle over  $S_\bullet^k$ , then  $P$  is  $\mathcal{U}'$  small for some  $G$ -admissible universe  $\mathcal{U}' \supset \mathcal{U}$ . So by the first part of Theorem 8.4,  $P$  is classified by some map:

$$f' : S_\bullet^k \rightarrow BG^{\mathcal{U}'}$$

By the preceding lemma there is a smooth map  $f_P : S_\bullet^k \rightarrow V$  so that  $[i \circ f_P] = [f']$ , where  $i : V \rightarrow BG^{\mathcal{U}'}$  is the inclusion. In particular by the second part of Theorem 8.4  $f_P^*E$  is isomorphic to  $P$  as a simplicial  $G$ -bundle. Thus  $c_V$  is surjective.  $\square$

We now show the second part of the theorem. Set as before  $V := BG^{\mathcal{U}}$ ,  $E := EG^{\mathcal{U}}$  and set

$$|E| := \operatorname{colim}_{\Delta(V)} E$$

where  $E : \Delta(V) \rightarrow \mathcal{G}$  is as previously discussed, and where the colimit is understood to be in the category of topological  $G$ -bundles. Let  $|V|$  be the geometric realization as previously defined. Then we have a topological  $G$ -fibration

$$|E| \rightarrow |V|,$$

which is classified by some

$$e = e^{\mathcal{U}} : |V| \rightarrow BG^{top},$$

uniquely determined up to homotopy. In particular,

$$(8.3) \quad |E| \simeq e^* EG^{top},$$

where  $EG^{top}$  is the universal  $G$ -bundle over  $BG^{top}$  and where  $\simeq$  in this argument will always mean  $G$ -bundle isomorphism. We will show that  $e$  induces an isomorphism of all homotopy groups. At this point we will use the assumption that  $G$  has the homotopy type of a CW complex, so that  $BG^{top}$  has the homotopy type of a CW complex, and so  $e$  must then be a homotopy equivalence by Whitehead theorem, which will finish the proof.

Let  $f : S_\bullet^k \rightarrow BG^{top}$  be continuous. By Müller-Wockel [10], main result, the bundle  $P_f := f^* EG^{top}$  is topologically isomorphic to a smooth  $G$ -bundle  $P' \rightarrow S_\bullet^k$ . By the axiom of universes  $P'$  is  $\mathcal{U}_0$ -small for some  $G$ -admissible  $\mathcal{U}_0$ . So we obtain a  $\mathcal{U}_0$ -small simplicial  $G$ -bundle  $P'_\bullet \rightarrow S_\bullet^k$ .

By Lemma 8.6  $P'_\bullet \simeq g^*E$  for some

$$g : S_\bullet^k \rightarrow V,$$

where  $\simeq$  is an isomorphism of simplicial  $G$ -bundles. Let  $|P'_\bullet|$  denote the colimit

$$\operatorname{colim}_{\Delta(S_\bullet^k)} P'_\bullet,$$

where  $P'_\bullet : \Delta(S_\bullet^k) \rightarrow \mathcal{G}$  is functor determined by  $P'_\bullet$ . And where as before the colimit is understood to be in the category of topological  $G$ -bundles. Of course  $|P'_\bullet| \simeq P'$  as topological  $G$ -bundles. Then  $|P'_\bullet| \rightarrow |S_\bullet^k|$  is a topological  $G$ -bundle classified by  $e \circ |g|$ , for

$$|g| : |S_\bullet^k| \rightarrow |V|,$$

the naturally induced topological map.

By construction, as a topological  $G$ -bundle,  $P_f$  is isomorphic to  $h^*|P'_\bullet|$ , where

$$h : S_\bullet^k \rightarrow |S_\bullet^k|$$

represents the generator of  $\pi_k(|S_\bullet^k|)$ . Thus  $e \circ |g| \circ h$  represents the homotopy class  $[f]$  and so  $e_* : \pi_k(V) \rightarrow \pi_k(BG^{top})$  is surjective. Here,  $\pi_k(Y)$  denotes the set of free homotopy classes of maps  $S^k \rightarrow Y$ .

We prove injectivity. Let  $f : S^k \rightarrow |V|$  be continuous. Let  $P \rightarrow S^k$  be a smooth  $G$ -bundle topologically isomorphic to  $f^*|E|$ . Again  $P$  exists by [10]. By Corollary 8.7,  $P_\bullet$  is classified by a smooth map:

$$g : S_\bullet^k \rightarrow V.$$

As before we then represent the class  $[f]$ , by  $|g| \circ h$  for  $h : S^k \rightarrow |S_\bullet^k|$  as above. Now suppose that  $e \circ f$  is null-homotopic. Then by [10]  $P$  is smoothly isomorphic to the trivial  $G$ -bundle. Thus by Corollary 8.7  $g$  is smoothly null-homotopic, so that  $|g|$  is null-homotopic and so  $[f] = [|g| \circ h]$  is the trivial class. So  $e_* : \pi_k(V) \rightarrow \pi_k(BG^{top})$  is injective. At this point, we use a simple lemma to get that  $e_*$  is an isomorphism of all homotopy groups.

**Lemma 8.8.** *Let  $f : X \rightarrow Y$  induce an isomorphism  $f_* : \pi_k(X) \rightarrow \pi_k(Y)$ , then  $f_* : \pi_k(X, x_0) \rightarrow \pi_k(Y, f_{x_0})$  is a group isomorphism.*

Finally, we show naturality. Let

$$|i^{\mathcal{U}, \mathcal{U}'}| : |V| \rightarrow |V'|$$

denote the map induced by the inclusion  $i^{\mathcal{U}, \mathcal{U}'}$ . Since  $E \simeq (i^{\mathcal{U}, \mathcal{U}'})^* E'$ , naturally, we have that

$$|E| \simeq |i^{\mathcal{U}, \mathcal{U}'}|^* |E'|$$

naturally and so

$$|E| \simeq |i^{\mathcal{U}, \mathcal{U}'}|^* \circ (e^{\mathcal{U}'})^* EG^{top},$$

by (8.3), from which the conclusion immediately follows.  $\square$

## 9. THE UNIVERSAL CHERN-WEIL HOMOMORPHISM

Let  $G$  be a Frechet Lie group. Pick any simplicial  $G$ -connection  $D$  on  $EG^{\mathcal{U}} \rightarrow BG^{\mathcal{U}}$ . Then given any  $Ad$  invariant symmetric multi-linear functional:

$$\rho : (V = \prod_{i=1}^{i=k} \mathfrak{g}) \rightarrow \mathbb{R},$$

applying the theory of Section 7 we obtain the simplicial Chern-Weil differential  $2k$ -form  $\omega^{\rho, D}$  on  $BG^{\mathcal{U}}$ . And we obtain an associated cohomology class  $c^{\rho, \mathcal{U}} \in H^{2k}(BG^{\mathcal{U}}, \mathbb{R})$ . We thus first arrive at an abstract form of the universal Chern-Weil homomorphism.

**Proposition 9.1.** *Let  $G$  be a Frechet Lie group and  $\mathcal{U}$  a  $G$ -admissible Grothendieck universe. There is an algebra homomorphism:*

$$\mathbb{R}[\mathfrak{g}]^G \rightarrow H^*(BG^{\mathcal{U}}, \mathbb{R}),$$

sending  $\rho$  as above to  $c^{\rho, \mathcal{U}}$  and satisfying the following. Let  $G \hookrightarrow Z \rightarrow Y$  be a  $\mathcal{U}$ -small smooth principal  $G$ -bundle. Let  $c^\rho(Z_\bullet) \in H^{2k}(Y_\bullet)$  denote the Chern-Weil class associated to  $\rho$ . Then

$$f_{Z_\bullet}^* c^{\rho, \mathcal{U}} = c^\rho(Z_\bullet),$$

where  $f_{Z_\bullet} : Y \rightarrow BG^{\mathcal{U}}$  is the classifying map of  $Z_\bullet$ .

*Proof.* This follows immediately by Lemma 7.2.  $\square$

Suppose now that  $G$  has the homotopy type of a CW complex. Let  $e^{\mathcal{U}}$  be as in Theorem 8.5. We define the associated cohomology class

$$c^{\rho} := e^{\mathcal{U}}_*(|c^{\rho, \mathcal{U}}|) \in H^{2k}(BG^{top}, \mathbb{R}),$$

where the  $G$ -admissible universe  $\mathcal{U}$  is chosen arbitrarily, where the pushforward means pull-back by the homotopy inverse, and where  $|c^{\rho, \mathcal{U}}| \in H^{2k}(|BG^{\mathcal{U}}|, \mathbb{R})$  is as in Notation 4.6.

**Lemma 9.2.** *The cohomology class  $c^{\rho}$  is well defined.*

*Proof.* Given another choice of a  $G$ -admissible universe  $\mathcal{U}'$ , let  $\mathcal{U}'' \supset \{\mathcal{U}, \mathcal{U}'\}$  be a common universe enlargement. By Lemma 7.2 and Lemma 4.7

$$|i^{\mathcal{U}, \mathcal{U}''}|_*(|c^{\rho, \mathcal{U}''}|) = |c^{\rho, \mathcal{U}}|.$$

Since  $|i^{\mathcal{U}, \mathcal{U}''}|$  is a homotopy equivalence we conclude that

$$|i^{\mathcal{U}, \mathcal{U}''}|_*(|c^{\rho, \mathcal{U}}|) = |c^{\rho, \mathcal{U}''}|,$$

where  $|i^{\mathcal{U}, \mathcal{U}''}|_*$  denotes the pull-back by the homotopy inverse. Consequently, by the naturality part of Theorem 8.5 and the equation above, we have

$$e^{\mathcal{U}}_*(|c^{\rho, \mathcal{U}}|) = e^{\mathcal{U}''}_* \circ |i^{\mathcal{U}, \mathcal{U}''}|_*(|c^{\rho, \mathcal{U}}|) = e^{\mathcal{U}''}_*(|c^{\rho, \mathcal{U}''}|).$$

In the same way we have:

$$e^{\mathcal{U}'}_*(|c^{\rho, \mathcal{U}'}|) = e^{\mathcal{U}''}_*(|c^{\rho, \mathcal{U}''}|).$$

So

$$e^{\mathcal{U}}_*(|c^{\rho, \mathcal{U}}|) = e^{\mathcal{U}'}_*(|c^{\rho, \mathcal{U}'}|),$$

and so we are done.  $\square$

We call  $c^{\rho} \in H^{2k}(BG^{top}, \mathbb{R})$  **the Chern-Weil characteristic class associated to  $\rho$** .

Let  $\mathbb{R}[\mathfrak{g}]$  denote the algebra of polynomial functions on  $\mathfrak{g}$ . And let  $\mathbb{R}[\mathfrak{g}]^G$  denote the sub-algebra of fixed points by the adjoint  $G$  action. By classical algebra, degree  $k$  homogeneous elements of  $\mathbb{R}[\mathfrak{g}]^G$  are in correspondence with symmetric  $G$ -invariant multi-linear functionals  $\Pi_{i=1}^k \mathfrak{g} \rightarrow \mathbb{R}$ . Then to summarize we have the following theorem purely about the classical classifying space  $BG^{top}$ :

**Theorem 9.3.** *Let  $G$  be a Frechet Lie group having the homotopy type of a CW complex. There is an algebra homomorphism:*

$$\mathbb{R}[\mathfrak{g}]^G \rightarrow H^*(BG^{top}, \mathbb{R}),$$

*sending  $\rho$  as above to  $c^{\rho}$  as above and satisfying the following. Let  $G \hookrightarrow Z \rightarrow Y$  be a smooth principal  $G$ -bundle. Let  $c^{\rho}(Z) \in H^{2k}(Y)$  denote the classical Chern-Weil class associated to  $\rho$ . Then*

$$f_Z^* c^{\rho} = c^{\rho}(Z),$$

*where  $f_Z : Y \rightarrow BG^{top}$  is the classifying map of the underlying topological  $G$ -bundle.*

*Proof.* Let  $\mathcal{U}_0 \ni Z$  be a  $G$ -admissible Grothendieck universe. By Lemma 7.2

$$c^{\rho}(Z_{\bullet}) = f_{Z_{\bullet}}^*(c^{\rho, \mathcal{U}_0}).$$

And by Proposition 7.3,  $n^*(|c^\rho(Z_\bullet)|) = c^\rho(Z)$ . So we have

$$\begin{aligned} c^\rho(Z) &= n^*(|c^\rho(Z_\bullet)|) \\ &= n^*(|f_{Z_\bullet}^*(c^{\rho, \mathcal{U}_0})|) \\ &= n^*(|f_{Z_\bullet}|^*(|c^{\rho, \mathcal{U}_0}|)) \text{ by Lemma 4.7} \\ &= n^* \circ |f_{Z_\bullet}|^* \circ (e^{\mathcal{U}_0})^* c^\rho, \text{ by definition of } c^\rho. \end{aligned}$$

Now  $e^{\mathcal{U}_0} \circ |f_{Z_\bullet}| \circ n \simeq f_Z$  as by construction  $e^{\mathcal{U}} \circ |f_{Z_\bullet}| \circ n$  classifies the topological  $G$ -bundle  $Z$ . So that

$$c^\rho(Z) = f_Z^* c^\rho,$$

and we are done.  $\square$

In other words we have constructed the universal Chern-Weil homomorphism for Frechet Lie groups with homotopy type of CW complexes. Another, related approach to the universal Chern-Weil homomorphism is contained in the book of Dupont [2]. Dupont only states the theorem above for compact Lie groups. Like us Dupont makes heavy use of simplicial techniques, for example the simplicial DeRham complex. However, the main thrust of his argument appears to be rather different, essentially arguing that all of the necessary differential geometry can be indirectly carried out on the Milnor classifying bundle  $EG \rightarrow BG$ , without endowing it with extra structure, beyond the tautological structures inherent in the Milnor construction. On the other hand we need the extra structure of a smooth simplicial set, and so work with the smooth Kan complexes  $BG^{\mathcal{U}}$  to do our differential geometry, and then transfer the cohomological data to  $BG$  using technical ideas like [10]. So we have a more conceptually involved space, with a certain “smooth structure”, but our differential geometry is rendered trivial, and in Dupont’s case the space is the ordinary  $BG$  but the differential geometry is more involved.

## 10. UNIVERSAL CHERN-WEIL THEORY FOR THE GROUP OF SYMPLECTOMORPHISMS

Let  $(M, \omega)$  be a symplectic manifold, so that  $\omega$  is a closed non-degenerate 2-form on  $M$ . Let  $\mathcal{G} = \text{Ham}(M, \omega)$  denote the group of its Hamiltonian symplectomorphisms, and  $\mathfrak{h}$  its Lie algebra. When  $M$  is simply connected this is just the group  $\text{Symp}(M, \omega)$  of diffeomorphisms  $\phi : M \rightarrow M$  s.t.  $\phi^*\omega = \omega$ . For example  $M = \mathbb{CP}^{n-1}$  with its Fubini-Study symplectic 2-form  $\omega_{st}$ . In this case the natural action of  $PU(n)$  is Hamiltonian.

In [11] Reznikov constructs polynomials

$$\{\rho_k\}_{k>1} \subset \mathbb{R}[\mathfrak{h}]^{\mathcal{G}},$$

each  $\rho_k$  homogeneous of degree  $k$ . In particular given a principal bundle  $\mathcal{G} \hookrightarrow P \rightarrow X$  for  $X$  a smooth manifold we obtain characteristic classes

$$c^{\rho_k}(P) \in H^{2k}(X, \mathbb{R}), \quad k > 1,$$

which were already used by Reznikov in [11] to great effect.

The group  $\text{Ham}(M, \omega)$  is a Frechet Lie group having the homotopy type of a CW complex by Milnor [9]. In particular, Theorem 9.3 immediately tells us that there are induced cohomology classes

$$(10.1) \quad c^{\rho_k} \in H^{2k}(B\text{Ham}(M, \omega), \mathbb{R}).$$

As mentioned, the group  $PU(n)$  naturally acts on  $\mathbb{CP}^{n-1}$  by Hamiltonian symplectomorphisms. So we have an induced map

$$i : BPU(n) \rightarrow BHam(\mathbb{CP}^{n-1}, \omega_0).$$

Then as one application we have:

**Theorem 10.1.**

$$i^* : H^k(BHam(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}) \rightarrow H^k(BPU(n), \mathbb{R})$$

is surjective for all  $n \geq 2$ ,  $k \geq 0$  and so

$$i_* : H_k(BPU(n), \mathbb{R}) \rightarrow H_k(BHam(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}),$$

is injective for all  $n \geq 2$ ,  $k \geq 0$ .

*Proof.* Let  $\mathfrak{g}$  denote the Lie algebra of  $PU(n)$ . Let  $j : \mathfrak{g} \rightarrow \mathfrak{h}$  denote the natural Lie algebra map induced by the homomorphism  $PU(n) \rightarrow Ham(\mathbb{CP}^{n-1}, \omega_0)$ . Reznikov [11] shows that  $\{j^* \rho_k\}_{k \geq 1}$  are the Chern polynomials. Specifically, the classes

$$c^{j^* \rho_k} \in H^{2k}(BPU(n), \mathbb{R}),$$

are the Chern classes  $\{c_k\}_{k \geq 1}$ , which generate real cohomology of  $BPU(n)$ , as is well known. But  $c^{j^* \rho_k} = i^* c^{\rho_k}$ , for  $c^{\rho_k}$  as in (10.1), and so the result immediately follows.  $\square$

There are previous related results, for example Reznikov [11] proves that

$$(10.2) \quad i_* : \pi_k(BPU(n), \mathbb{R}) \rightarrow \pi_k(BHam(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}),$$

is an injection for all  $k$ . See also Kedra-McDuff [4]. In Savelyev-Shelukhin [14] there are results about induced maps in (twisted)  $K$ -theory.

Theorem 10.1 surely extends to completely general compact semi-simple Lie groups  $G$ , with  $\mathbb{CP}^n$  replaced by co-adjoint orbits  $M$  of  $G$ . We just need to compute the associated Reznikov polynomials in  $\mathbb{R}[\mathfrak{h}]^G$  and their pull-backs to  $\mathfrak{g}$  as above. We can no longer expect injection in general. But the failure to be injective should be solely due to effects of classical representation theory, rather than transcendental effects of extending the structure group to  $Ham(M, \omega)$ , from a compact Lie group.

## REFERENCES

- [1] K.-T. CHEN, *Iterated path integrals*, Bull. Am. Math. Soc., 83 (1977), pp. 831–879.
- [2] J. L. DUPONT, *Curvature and characteristic classes*, vol. 640, Springer, Cham, 1978.
- [3] D. FIORENZA, U. SCHREIBER, AND J. STASHEFF, *Čech cocycles for differential characteristic classes: an  $\infty$ -Lie theoretic construction*, Adv. Theor. Math. Phys., 16 (2012), pp. 149–250.
- [4] J. KEDRA AND D. MCDUFF, *Homotopy properties of Hamiltonian group actions*, Geom. Topol., 9 (2005), pp. 121–162.
- [5] H. LEE TANAKA AND Y.-G. OH, *Smooth constructions of homotopy-coherent actions*, To appear in Algebraic and Geometric Topology, arXiv:2003.06033.
- [6] J. LURIE, *Higher topos theory*, Annals of Mathematics Studies 170. Princeton, NJ: Princeton University Press., 2009.
- [7] J.-P. MAGNOT AND J. WATTS, *The diffeology of Milnor’s classifying space*, Topology Appl., 232 (2017), pp. 189–213.
- [8] J. W. MILNOR, *Construction of universal bundles. II.*, Ann. Math. (2), 63 (1956), pp. 430–436.
- [9] ———, *On spaces having the homotopy of a CW-complex*, Trans. Am. Math. Soc., 90 (1959), pp. 272–280.
- [10] C. MÜLLER AND C. WOCKEL, *Equivalences of smooth and continuous principal bundles with infinite-dimensional structure group*, Adv. Geom., 9 (2009), pp. 605–626.

- [11] A. G. REZNIKOV, *Characteristic classes in symplectic topology*, Selecta Math. (N.S.), 3 (1997), pp. 601–642. Appendix D by L. Katzarkov.
- [12] M. E. RUDIN, *A new proof that metric spaces are paracompact*, Proc. Am. Math. Soc., 20 (1969), p. 603.
- [13] Y. SAVELYEV, *Global Fukaya category I: quantum Novikov conjecture*, To appear in IMRN, arXiv:1307.3991v5.
- [14] Y. SAVELYEV AND E. SHELUKHIN, *K-theoretic invariants of Hamiltonian fibrations*, J. Symplectic Geom., 18 (2020), pp. 251–289.
- [15] D. SULLIVAN, *Infinitesimal computations in topology.*, Publ. Math., Inst. Hautes Étud. Sci., (1977).

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