MEAN CURVATURE VERSUS DIAMETER AND ENERGY QUANTIZATION

YASHA SAVELYEV

ABSTRACT. We first partially extend a theorem of Topping, on the relation between mean curvature and intrinsic diameter, from immersed submanifolds of $\mathbb{R}^n$ to almost everywhere immersed, closed submanifolds of a compact Riemannian manifold. We use this to prove quantization of energy for pseudo-holomorphic closed curves of all genus in a compact locally conformally symplectic manifold.

Resume. Tout d'abord, nous étendons partiellement un théorème de Topping, sur la relation entre courbure moyenne et diamètre intrinsèque, des sous-variétés immergées de $\mathbb{R}^n$ aux sous-variétés fermées, immergées presque partout d'une variété compacte de Riemannian. Nous l'utilisons pour prouver la quantification de l'énergie pour les courbes fermées pseudo-holomorphes de tous les genres dans une variété compacte localement symplectiquement conforme.

1. INTRODUCTION

For a compact almost complex manifold (M, J) with metric g, energy quantization is a statement that there is a $\hbar(M, g, J) > 0$ so that any non-constant pseudo-holomorphic map of a sphere into M has energy at least $\hbar(M, g, J)$.

In order to carry out Gromov-Witten theory for locally conformally symplectic manifolds, or l. c. s. manifolds for short, we need in general a stronger form of energy quantization of pseudo-holomorphic curves, because it may be necessary to work with trivial homology classes (cf. [1]). In this case, if we are working with a higher genus curve, the energy of the curve a priori can collapse to 0, which would break compactness and invariance arguments. This is the "black earth" analogue of the "blue sky catastrophe" cf. [2], [3], [4]. We show that this cannot happen.

The classical energy quantization is proved using mean value inequalities for pseudo-holomorphic maps, [5], [6]. To prove quantization of energy for higher genus curves in a l. c. s. manifold, we need more sophisticated differential geometry. In this note we first generalize a theorem of Topping relating diameter and mean curvature of immersed submanifolds of $\mathbb{R}^n$, and then use this to prove our quantization result.

1.1. **Mean curvature vs diameter.** In [7] Topping gave via a concise but sophisticated argument, partially based on ideas of Ricci flows, a simple relation between intrinsic diameter and mean curvature, for immersed submanifold of $\mathbb{R}^n$. Let us state it here:

Theorem 1.1 ([7]). *For Σ^m a smoothly immersed closed submanifold of $\mathbb{R}^n$ we have:*

$$\operatorname{diam}(\Sigma) \leq Const(m) \int_\Sigma |\boldsymbol{H}|^{m-1} \, dvol,$$

for $\boldsymbol{H}$ the mean curvature vector field along Σ, vol the volume measure induced by the standard ambient metric, and diam *the intrinsic diameter:*

$$\max_{x,y \in \Sigma} dist_{(\Sigma, g_{st})}(x, y),$$

for g_{st} the metric on Σ induced by the standard metric on $\mathbb{R}^n$.

We generalize this to almost everywhere immersed submanifolds of general compact Riemannian manifolds, using Nash embedding theorem. It is indeed not obvious what the right generalization should be. Ultimately our hand is forced by our intended application, but our generalization does seem natural from at least one point of view.

In plain words the statement is the following: if the volume of a given closed immersed submanifold of a compact Riemannian manifold is "small" but diameter "large" then the mean curvature must be somewhere large. However we suitably extend this to non-immersed submanifolds, so the actual formulation is more involved.

Let (M, g) be a smooth Riemannian manifold. Denote by $S(M, \Sigma)$ the space of smooth maps u of a closed Riemannian m-fold Σ into (M, J). Define:

$$Vol(u) = \int_\Sigma |du|_g \, dvol_\Sigma,$$

to be the volume functional, for $dvol_\Sigma$ the volume measure on Σ for a fixed auxiliary metric g_{aux} on Σ, and $|du|_g$ the operator norm of du with respect to g_{aux} and g.

For $u \in S(M, \Sigma)$ we say that it is **non-singular** at $p \in \Sigma$ if $du(p)$ is non-singular. Let

$$S(C) = S(C, M, \Sigma) \subset S(M, \Sigma)$$

consist of elements u, non-singular on a subset of full $dvol_\Sigma$ measure and satisfying:

$$|\mathbf{H}(u(p))| < C,$$

for all p s.t. u is non-singular at p, where $\mathbf{H}(u(p))$ is the mean curvature vector at $u(p)$. In this case Vol is continuously differentiable at u.

Theorem 1.2. *Let* $\operatorname{diam}(u)$ *denote the (intrinsic) diameter in* (M, g) *of* $\operatorname{image}(u)$. *Then for all* $u \in S(C)$

$$\operatorname{diam}(u) \leq F(g, C, m)Vol(u),$$

for some function F.

1.2. **Application to energy quantization.** A locally conformally symplectic manifold or l. c. s. manifold, is a smooth $2n$-fold M, with a non-degenerate 2-form ω, which is locally diffeomorphic to $e^f \omega_0$ for some functions f, and ω_0 the standard symplectic form on $\mathbb{R}^{2n}$. There has been some interest recently in developing Gromov-Witten theory for l. c. s. manifolds, [1], [8]. This is partly impeded by our lack of understanding of how badly holomorphic curves can behave in such a manifold. For instance, in the context of holomorphic curves in symplectic manifolds, we have the following energy quantization phenomenon:

Theorem 1.3. *Given a closed symplectic* (M, ω), J *an* ω-*tamed almost complex structure, meaning* $\omega(\cdot, J\cdot) > 0$, *there exists a constant* $\hbar = \hbar(\omega, J)$, *s.t. for any* $u : (\Sigma, j) \to M$ *a non-constant J-holomorphic map of a closed Riemann surface* (Σ, j), *the energy of the map satisfies:*

$$e(u) = \int_\Sigma |du|_{g_J}^2 \, dvol_\Sigma = \int_\Sigma u^* \omega \geq \hbar,$$

for g_J the metric $\omega(\cdot, J\cdot)$.

For $\Sigma = S^2$ this holds via a generalized mean value inequality, [5], [6] and the symplectic condition on M can be loosened to just almost complex. For more general Σ, but with M symplectic this is not previously known, but we give here a very simple, given state of the art, argument via geometric measure theory.

We then directly generalize this to l. c. s. manifolds.

Theorem 1.4. *Given a closed* l. c. s. (M, ω), J *an* ω-*compatible almost complex structure there exists a constant* $\hbar = \hbar(\omega, J)$, *s.t. for any* $u : (\Sigma, j) \to M$ *a non-constant J-holomorphic map of a closed Riemann surface* (Σ, j), *the energy of the map satisfies:*

$$e(u) = \int_\Sigma |du|^2 dvol_\Sigma = \int_\Sigma u^* \omega \geq \hbar,$$

for $|du|$ the operator norm with respect g_{aux} and $g_J = \omega(\cdot, J\cdot)$.

2. Proofs

Proof of Theorem 1.3. Suppose otherwise, then we have a sequence $\{u_i\}$ of J-holomorphic curves with $e(u_i) \to 0$ as $i \to \infty$. In particular, $|u_i| \to 0$, where $|\cdot|$ is the mass norm, and u_i are understood as integral 2-currents, Federer [9]. By the main compactness theorem for currents, $\{u_i\}$ has a convergent subsequence $\{u_{i_k}\}$ to an integral 2-current with mass necessarily 0, and hence 0 in the vector space of closed integral 2-currents $\mathcal{I}_2(M)$, since mass norm is a norm. Next it is proved in [9] that the space $\mathcal{H}_k(M)$ of closed integral k-currents modulo exact integral k-currents, is discreet with respect to the topology induced by the mass norm, and isomorphic to singular integral k-homology. Moreover the natural map

$$q_k : \mathcal{I}_k(M) \to \mathcal{H}_k(M)$$

is continuous. Thus $\{q_2(u_{i_k})\}$ is eventually constant, which means that u_{i_k} are eventually in class 0, which is a contradiction, since all u_i have positive symplectic area. $\qquad\square$

Proof of Theorem 1.2. For $u \in S(C, M, \Sigma)$ let $dVol(u)$ denote the differential of Vol at u, and $|dVol(u)|$ its operator norm, for the induced by g "Riemannian metric" on $S(M, \Sigma)$, which is understood as a Frechet manifold. Explicitly the "Riemannian metric" is:

$$\langle V_1, V_2 \rangle = \int_\Sigma \langle V_1(p), V_2(p) \rangle_g \, dvol_\Sigma,$$

for $V_i \in T_u S(M, \Sigma)$, i.e. V_i are smooth maps

$$V_i : \Sigma \to TM$$

satisfying

$$V_i(p) \in T_{u(p)}M,$$

we call such a V_i a **vector field along** u.

Given $u \in S(C, M, \Sigma)$, non-singular on $U \subset \Sigma$, and $V = fN$ for N the normal unit vector field along U, and f a smooth function with support in U, we have:

$$|dVol(u)(V)| = \int_\Sigma mf\mathbf{H} \, dvol_\Sigma,$$

by classical differential geometry.

It follows since u is non-singular on a set of full measure, and $\mathbf{H}$ being bounded that $dVol$ is continuously differentiable at u and

$$(2.1) \qquad\qquad\qquad |dVol(u)| \leq m|\mathbf{H}|,$$

where the right hand side is the supremum over $p \in \Sigma$ s.t. u is non-singular at p, of the magnitude of the mean curvature $|\mathbf{H}(u(p))|$.

Pick an isometric Nash embedding N of (M, g) into $\mathbb{R}^n$, where n is large enough.

Lemma 2.2. *For all $u \in S(C, M, \Sigma)$:*

$$|dVol(N \circ u)| < C',$$

for some C', independent of u.

Proof. Let $U \subset \Sigma$ be an open set on which u is non-singular. In what follows we conflate the notation for U and its images $u(U)$, $N \circ u(U)$. In other words we just think in terms of subspaces $U \subset M \subset \mathbb{R}^n$. Let h be the second fundamental form on T_pU:

$$h(v, w) = \widetilde{\widetilde{\nabla}}_v w - \nabla_v w,$$

where $\widetilde{\widetilde{\nabla}}$ is the Levi-Civita connection of $(\mathbb{R}^n, g_{st})$, ∇ is the Levi-Civita connection of the submanifold $U \subset \mathbb{R}^n$ and where we locally extend $v, w \in T_p U$ to vector fields tangent to U. The mean curvature vector at $p \in U \subset \mathbb{R}^n$ is given by:

$$\mathbf{H}_{\mathbb{R}^n}(p) = \frac{1}{m} \sum_i h(e_i, e_i),$$

where $\{e_i\}$ is an orthonormal basis for $T_p U$. Likewise $\widetilde{\nabla}$ will denote in what follows the Levi-Civita connection of (M, g). So we have:

$$m|\mathbf{H}_{\mathbb{R}^n}(p)| = \sum_i h(e_i, e_i)(p)| = |\sum_i (\widetilde{\widetilde{\nabla}}_{e_i} e_i - \nabla_{e_i} e_i)| = |\sum_i (\widetilde{\widetilde{\nabla}}_{e_i} e_i - \widetilde{\nabla}_{e_i} e_i + \widetilde{\nabla}_{e_i} e_i - \nabla_{e_i} e_i)|$$

$$\leq |\sum_i (\widetilde{\nabla}_{e_i} e_i - \nabla_{e_i} e_i)| + mB$$

$$\leq mC + mB,$$

where

$$B = \sup_{e \in TM, |e|=1} |\widetilde{\widetilde{\nabla}}_e e - \widetilde{\nabla}_e e|.$$

The lemma then follows by (2.1). $\qquad \square$

Now let u' be C^∞ δ-close to $N \circ u$ s.t. u' is immersed, for some $\delta > 0$ sufficiently small so that $|dVol(u')| < C'$ for C' as in the lemma above. We can find such a u' since Vol is continuously differentiable at $N \circ u$. By (2.1)

$$|\mathbf{H}_{\mathbb{R}^n}(p)| < \frac{C'}{m},$$

for $\mathbf{H}_{\mathbb{R}^n}$ the mean curvature vector field along u', in $\mathbb{R}^n$.

Since Σ is closed we get by Topping's theorem:

$$\operatorname{diam}(u'(\Sigma)) \leq Const(m) \int_\Sigma |\mathbf{H}_{\mathbb{R}^n}|^{m-1} \, dvol_\Sigma.$$

By the lemma above the function $|\mathbf{H}_{\mathbb{R}^n}|$ on Σ is universally (independently of u) bounded from above by some C'. So we get:

$$\operatorname{diam}(u'(\Sigma)) \leq Const(m) \cdot (\frac{C'}{m})^{m-1} \cdot \operatorname{vol}(\Sigma),$$

passing to the limit as $u' \to N \circ u$ we get the required inequality. $\qquad \square$

Proof of Theorem 1.4. First we need the following.

Theorem 2.3. *For $u : \Sigma \to M$ a non-constant J-holomorphic map of a Riemann surface into an* l. c. s. *(M, ω), as above, with J ω-compatible, and with* $\operatorname{area}(u) < D$:

$$|dVol(u)| < C(\omega, J, D),$$

for a constant $C(\omega, J, D) > 0$, independent of u.

Proof. The map u, being J-holomorphic, is non-singular outside of a set of finitely many points, McDuff-Salamon [5]. We may fix a finite cover of M by charts $\phi_i : U_i \subset \mathbb{R}^{2n} \to M$, $\phi_i^* \omega = e^{f_i} \omega_0$ with ω_0 the standard symplectic form, and with U_i contractible. Then $\phi_i^{-1} \circ u$ is a $\phi_i^* J$-holomorphic map defined on $u^{-1}(\phi_i(U_i))$ into $\mathbb{R}^{2n}$, and $\phi_i^* J$ is compatible with $f_i \omega_0$ and hence with ω_0. Fix a cover of Σ by disk domains $\{V_{i_j}\}$, with each $V_{i_j} \subset u^{-1}(\phi_i(U_i))$ for some i. Then $u_{i,j} = \phi_i^{-1} \circ u|_{V_{i_j}}$ is a non-constant $\phi_i^* J$-holomorphic curve in $\mathbb{R}^{2n}$.

Since $(\omega_0, \phi_i^* J)$ is an almost Kähler manifold,

$$dVol(u_{i,j})(V) = 0$$

for every vector field V, with $|V| = 1$, along $u_{i,j}$, vanishing on a neighborhood of the boundary of V_{i_j}, and near the singularities, since ω_0 is then a calibration, Harvey-Lawson [10]. Since the cover $\{U_i\}$ is

finite, the C^∞ norm of the functions f_i is universally bounded: $|f_i| < B$, for some some B and all i. It readily follows that

$$|dVol(u_{i,j})(V)| < C(\omega, J, D),$$

for V, as before but with Vol computed with respect to the metric $(e^{f_i}\omega_0, \phi_i^* J)$, and some $C(\omega, J, D)$. This is because the distortion of derivative $dVol$ corresponding to the bounded conformal distortion e^{f_i} can be easily bounded in terms of B, D. (By essentially product rule.)

The result readily follows from this. $\qquad\square$

Now let ϵ be the Lebesgue covering number of $\{U_i\}$ with respect to the metric (ω, J). Combining Theorem 2.3, Theorem 1.2 and (2.1) we get that for u non-constant as in the hypothesis if area$(u) < \hbar$ then diam$(u) < \epsilon$, for some $\hbar$ independent of u. Consequently the image of u is contained in some U_i, and so $\phi_i^{-1} \circ u$ is a $\phi_i^* J$-holomorphic map of a sphere into the almost Kahler contractible manifold $(U_i, \omega_0, \phi_i^* J)$ and so must be constant, which is a contradiction. $\qquad\square$

References

[1] Yasha Savelyev. "Gromov Witten theory of a locally conformally symplectic manifold and the Fuller index". In: *arXiv* (2016). URL: http://front.math.ucdavis.edu/1609.08991.

[2] Andrey Shilnikov, Leonid Shilnikov, and Dmitry Turaev. "Blue-sky catastrophe in singularly perturbed systems." English. In: *Mosc. Math. J.* 5.1 (2005), pp. 269–282. ISSN: 1609-3321.

[3] F.B. Fuller. "Note on trajectories in a solid torus." English. In: *Ann. Math. (2)* 56 (1952), pp. 438–439. ISSN: 0003-486X; 1939-8980/e.

[4] Yasha Savelyev. "Extended Fuller index, sky catastrophes and the Seifert conjecture". In: *International Journal of mathematics, to appear* ().

[5] Dusa McDuff and Dietmar Salamon. *J–holomorphic curves and symplectic topology.* American Math. Society Colloquium Publ. 52. Amer. Math. Soc., 2004, pp. xii+669. ISBN: 0-8218-3485-1.

[6] Katrin Wehrheim. "Energy quantization and mean value inequalities for nonlinear boundary value problems." English. In: *J. Eur. Math. Soc. (JEMS)* 7.3 (2005), pp. 305–318. ISSN: 1435-9855; 1435-9863/e.

[7] Peter Topping. "Relating diameter and mean curvature for submanifolds of Euclidean space." English. In: *Comment. Math. Helv.* 83.3 (2008), pp. 539–546. ISSN: 0010-2571; 1420-8946/e.

[8] Chantraine Baptiste and Amy Murphy. "Conformal symplectic geometry of cotangent bundles". In: *arXiv* (2016).

[9] Herbert Federer. *Geometric measure theory. Repr. of the 1969 ed.* English. Repr. of the 1969 ed. Berlin: Springer-Verlag, 1996, pp. xvi + 680. ISBN: 3-540-60656-4.

[10] Reese Harvey and H. Blaine Lawson. "Calibrated geometries." English. In: *Acta Math.* 148 (1982), pp. 47–157. ISSN: 0001-5962; 1871-2509/e.

Email address: yasha.savelyev@gmail.com

University of Colima, CUICBAS