A REMARK ON DEFORMATION OF GROMOV NON-SQUEEZING

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Abstract. We prove that in dimension 4 the Gromov non-squeezing phenomenon is persistent with respect to C^0 symplectic perturbations of the symplectic form on the range. This motivates an intriguing question on further deforming non-squeezing to general nearby forms. Our methods consist of a certain trap idea for holomorphic curves, analogous to traps in dynamical systems, and Hofer-Wysocki-Zehnder polyfold regularization in Gromov-Witten theory, especially as recently worked out in this present context by the team of Franziska Beckschulte, Ipsita Datta, Irene Seifert, Anna-Maria Vocke, and Katrin Wehrheim.

1. Introduction

One of the most important to this day results in symplectic geometry is the so called Gromov non-squeezing theorem, appearing in the seminal paper of Gromov [3]. Let $\omega_{st} = \sum_{i=1}^n dp_i \wedge dq_i$ denote the standard symplectic form on $\mathbb{R}^{2n}$. Gromov's theorem then says that there does not exist a symplectic embedding

$$(B_R, \omega_{st}) \hookrightarrow (S^2 \times \mathbb{R}^{2n-2}, \omega_{\pi r^2} \oplus \omega_{st}),$$

for $R > r$, with B_R the standard closed radius R ball in $\mathbb{R}^{2n}$ centered at 0, and $\omega_{\pi r^2}$ a symplectic form on S^2 with area πr^2.

We show that in dimension 4 Gromov's non-squeezing is C^0 persistent in the following sense.

Theorem 1.1. *Let $R > r > 0$ be given, and let $\omega_{\pi r^2} \oplus \omega_{st}$ be the symplectic form on $M = S^2 \times \mathbb{R}^2$ as above. Then there is an $\epsilon > 0$ s.t. for any symplectic form ω' on M, C^0 ϵ-close to ω, there is no symplectic embedding $\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega')$, meaning that $\phi^* \omega' = \omega_{st}$.*

It is not clear if the dimension 4 restriction is essential. A suitable holomorphic trap (Definition 2.1) is certainly much more difficult to construct in higher dimensions.

It is natural to ask if the above theorem continues to hold for general nearby forms. Or formally this translates to:

Question 1. Let $R > r > 0$ be given, and let $\omega = \omega_{\pi r^2} \oplus \omega_{st}$ be the symplectic form on $M = S^2 \times \mathbb{R}^{2n-2}$, as above. For every $\epsilon > 0$ is there a (necessarily *globally* non-closed, for very small ϵ) 2-form ω' on M, C^0 ϵ-close to ω, such that there is a symplectic embedding $\phi : B_R \hookrightarrow M$, i.e. s.t. $\phi^* \omega' = \omega_{st}$?

We cannot readily reduce this question to just applying Theorem 1.1 (in dimension 4). This is because, while a symplectic form on a subdomain of the form $\phi(B_R) \subset M$ extends to a symplectic form, by a classical theorem of Gromov [2], the extension may not be C^0 close. Indeed, this appears to be rather unlikely to happen.

The above question seems to be difficult. My opinion is that the answer is 'yes', in part because it is difficult to imagine any obstruction, for example we no longer have Gromov-Witten theory for general ω'. On the other, my attempts to construct an example failed, so that 'no' is certainly very possible.

A work of Müller [6] explores a different kind of question, by instead relaxing the condition of the map being symplectic. This is a very different idea, and there is no direct connection to our problem, as pull-backs by diffeomorphisms of nearby forms may not be nearby. Hence, there is no way to go from nearby embeddings that we work with to ϵ-symplectic embeddings of Müller.

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2. A trap for holomorphic curves

For basic notions of J-holomorphic curves we refer the reader to [5].

Definition 2.1. *Let (M, J) be an almost complex manifold, $A \in H_2(M)$ fixed. Let $K \subset M$ be a closed subset. Suppose that for every $x \in \partial K$ (the topological boundary) there is a J-holomorphic (real codimension 2) compact hyperplane H_x through x satisfying:*

- *$H_x \subset K$.*
- *$A \cdot H_x \leq 0$, where the left-hand side is the homological intersection number.*

*We call such a K a **J-holomorphic trap** (for class A curves).*

Lemma 2.2. *Let M, J and A be as above, and K be a J-holomorphic trap for class A curves. Let $u : \Sigma \to M$ be a J-holomorphic class A curve u with Σ a connected Riemann surface. Then*

$$\operatorname{image} u \cap K \neq \emptyset \implies \operatorname{image} u \subset K.$$

Proof. Suppose that u intersects ∂K, otherwise we already have $\operatorname{image} u \subset \mathit{interior}(K)$, since $\operatorname{image} u$ is connected (and by elementary topology). Then u intersects H_x as in the definition of a holomorphic trap, for some x. Consequently, as $A \cdot H_x \leq 0$, by positivity of intersections [5, Section 2.6], $\operatorname{image} u \subset H_x \subset K$. □

3. Proof of Theorem 1.1

Definition 3.1. *We say that a pair (ω, J) of a 2-form ω on M and an almost complex structure J on M are **compatible** if $\omega(\cdot, J\cdot)$ defines a J-invariant inner product on M.*

Suppose by contradiction that for every $\epsilon > 0$ there is an ω_1 s.t. $d_{C^0}(\omega, \omega_1) < \epsilon$ and such that there exists a symplectic embedding

$$\phi : B_R \hookrightarrow (M, \omega_1).$$

Our specific C^0 distance d_{C^0}, on the spaces of forms, will be with respect to the metric g induced as in Definition 3.1 by (ω, J) for J the standard product complex structure.

Let $\epsilon' > 0$ be s.t. any symplectic form ω_1 on M, C^0 ϵ'-close to ω satisfies:

- $\omega_t = (1-t)\omega + t\omega_1$ is non-degenerate, for each $t \in [0, 1]$.
- For each $t \in [0, 1]$, ω_t is non-degenerate on all the fibers of the natural projection $p : (M = S^2 \times \mathbb{R}^2) \to \mathbb{R}^2$. In what follows we just call them ***fibers***.

For $\epsilon < \epsilon'$ as above, let ω_1 and $\phi : B_R \to (M, \omega_1)$ be as in our hypothesis. Set $B := \phi(B_R)$ and let $D \supset B$ be an open domain, with compact closure K, s.t. K is the product $S^2 \times D^2$ for $D^2 \subset \mathbb{R}^2$ a closed disk. In particular, ∂K is smoothly folliated by the fibers. We denote by $T^{vert}\partial K \subset TM$, the sub-bundle of vectors tangent to the leaves of the above-mentioned foliation.

We may extend $\phi_* j$ to an ω_1-compatible almost complex structure J_1 on M, preserving $T^{vert}\partial K$ using:

- $\operatorname{image} \phi$ does not intersect ∂K.
- The non-degeneracy of ω_1 on the fibers, which follows by the defining condition of ϵ.
- The well known existence/flexibility results for compatible almost complex structures on symplectic vector bundles.

We may then extend J_1 to a family $\{J_t\}$, $t \in [0, 1]$, of almost complex structures on M, s.t. J_t is ω_t-compatible for each t, with $J_0 = J$ the standard complex structure on M, and such that J_t preserves $T^{vert}\partial K$ for each t. The latter condition can be satisfied by similar reasoning as above, using that ω_t is non-degenerate on the fibers for each t.

So the fibers above are J_t-holomorphic for each t. And so for each t, ∂K is folliated by J_t-holomorphic closed hyperplanes (the fibers). Moreover, if $A = [S^2] \otimes [pt]$ is as in the statement, then $A \cdot p^{-1}(z) = 0$, for $z \in \mathbb{R}^2$. And so K is a compact J_t-holomorphic trap for class A curves, for each t.

Set $x_0 := \phi(0)$. Denote by $\mathcal{M}_t$ the space of equivalence classes of maps $u : \mathbb{CP}^1 \to M$, where u is a J_t-holomorphic, class A curve passing through x_0. The equivalence relation is by the usual reparametrization group action. Then $\mathcal{M} = \cup_t \mathcal{M}_t$ is compact by energy minimality of A (which rules out bubbling), by Lemma 2.2, and by compactness of K.

As explained in [1, Section 3.5], in a essentially identical situation, we may embed $\mathcal{M}$ into a natural polyfold setup of Hofer-Wysocki-Zehnder [4]. That is we express $\mathcal{M}$ as the zero set of an sc-Fredholm section of a suitable (tame, strong) M-polyfold bundle. The only difference with their setup is that they compactify M to $S^2 \times T^2$. We of course cannot compactify, and so we have to use the holomorphic trap idea, to force compactness of $\mathcal{M}$. Again as in [1], we take the M-polyfold regularization of $\mathcal{M}$. This gives a one dimensional compact cobordism $\mathcal{M}^{reg}$ between $\mathcal{M}_0^{reg}$ and $\mathcal{M}_1^{reg}$.

Now $\mathcal{M}_0^{reg}$ is a point: corresponding to the unique $(J = J_0)$-holomorphic class A, curve $u : \mathbb{CP}^1 \to M$ passing through x_0. Consequently, $\mathcal{M}_1^{reg}$ is non-empty, that is there is a J_1-holomorphic class A curve $u_0 : \mathbb{CP}^1 \to M$ passing through x_0.

Remark 3.2. It is certainly possible that more classical, geometric perturbation style arguments may be adopted to the present problem. There are however difficulties: it is important for us to work with curves constrained to pass through a specific point, instead of doing homological intersection of an unconstrained evaluation cycle, with a point (as in the classical proof of Gromov non-squeezing). For without the specific constraint our moduli space is not even compact, and hence the homological intersection theory makes no sense. Such a constraint may not neatly fit into classical analytical framework of McDuff-Salamon [5].

Now we have:

$$|\langle \omega_1, A\rangle - \pi \cdot r^2| = |\langle \omega_1, A\rangle - \langle \omega, A\rangle| \leq \epsilon\pi \cdot r^2,$$

as $\langle \omega, A\rangle = \pi r^2$, and as $d_{C^0}(\omega, \omega_1) < \epsilon$, (also using that we can find a representative for A whose g-area is πr^2). So choosing ϵ appropriately we get

$$|\int_{\mathbb{CP}^1} u_0^* \omega_1 - \pi r^2| < \pi R^2 - \pi r^2,$$

And consequently,

$$\int_{\mathbb{CP}^1} u_0^* \omega_1 < \pi R^2.$$

We may then proceed exactly as in the now classical proof of Gromov [3] of the non-squeezing theorem to get a contradiction and finish the proof. A bit more specifically, $\phi^{-1}(\mathrm{image}\,\phi \cap \mathrm{image}\,u_0)$ is a minimal surface in B_R, with boundary on the boundary of B_R, and passing through $0 \in B_R$. By construction it has area strictly less than πR^2, which is impossible by the classical monotonicity theorem of differential geometry. See also [1] where the monotonicity theorem is suitably generalized, to better fit the present context. □

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