

# GROMOV-WITTEN INVARIANTS OF RIEMANN-FINSLER MANIFOLDS

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**ABSTRACT.** We define a  $\mathbb{Q}$ -valued deformation invariant of certain complete Riemann-Finsler manifolds. We use this to prove (possibly non-compact but complete) fibration generalizations of Preissman's theorem on non-existence of negative sectional curvature metrics on compact products. In addition we prove that every non-positive rational number is the value of this invariant for some Riemannian manifold. We also prove that sky catastrophes of smooth dynamical systems are not geodesible by a certain class of forward complete Riemann-Finsler metrics, in particular by Riemannian metrics with non-positive sectional curvature. This partially answers a question of Fuller and gives important examples for our theory here. In a sister paper [9], we study a direct generalization of this metric invariant, by lifting the count of geodesics to a Gromov-Witten count of elliptic curves in an associated locally conformally symplectic manifold.

## 1. INTRODUCTION

We will define certain rational number valued metric invariants for certain complete Riemann-Finsler manifolds, in particular for complete Riemannian metrics with non-positive sectional curvature. These invariants can be directly interpreted as a part of certain elliptic Gromov-Witten invariants in an associated lcs manifold, [9]. However, in the more basic setting here, we can reduce the invariants to counts of closed geodesic strings (equivalence classes of closed geodesics up to reparametrization  $S^1$  action), via Fuller index [4]. And so this self contained more elementary story is developed separately here.

**Terminology 1.** All our metrics are Riemann-Finsler metrics unless specified otherwise, and usually denoted by just  $g$ . Completeness, always means forward completeness. Curvature always means sectional curvature in the Riemannian case and flag curvature in the Finsler case. Thus we will usually just say complete metric  $g$ , for a forward complete Riemann-Finsler metric. A reader may certainly choose to interpret all metrics as Riemannian metrics, completeness as standard completeness, and curvature as sectional curvature.

In what follows  $\pi_1(X)$  denotes the set of free homotopy classes of continuous maps  $o : S^1 \rightarrow X$ .

**Definition 1.1.** Let  $X$  be a smooth manifold. Fix an exhaustion by nested compact sets  $\bigcup_{i \in \mathbb{N}} K_i = X$ ,  $K_i \supset K_{i-1}$  for all  $i \geq 1$ . We say that a class  $\beta \in \pi_1(X)$  is boundary compressible if  $\beta$  is in the image of

$$inc_* : \pi_1(X - K_i) \rightarrow \pi_1(X)$$

for all  $i$ , where  $inc : X - K_i \rightarrow X$  is the inclusion map. We say that  $\beta$  is **boundary incompressible** if it is not boundary compressible.

Let  $\pi_1^{inc}(X)$  denote the set of such boundary incompressible classes. When  $X$  is compact, we set  $\pi_1^{inc}(X) := \pi_1(X) - const$ , where  $const$  denotes the set of homotopy classes of constant loops.

It is easily seen that the above is well defined and moreover any homeomorphism  $X_1 \rightarrow X_2$  of a pair of manifolds induces a set isomorphism  $\pi_1^{inc}(X_1) \rightarrow \pi_1^{inc}(X_2)$ . Denote by  $L_\beta X$  the class  $\beta \in \pi_1^{inc}(X)$

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component of the free loop space of  $X$ , with its compact open topology. Let  $g$  be a complete metric on  $X$ , and let  $S(g, \beta) \subset L_\beta X$  denote the subspace of all unit speed parametrized, smooth, closed  $g$ -geodesics in class  $\beta$ .

**Definition 1.2.** *We say that a complete metric  $g$  on  $X$  is  $\beta$ -**taut** if  $S(g, \beta)$  is compact. We will say that  $g$  is **taut** if it is  $\beta$ -taut for each  $\beta \in \pi_1^{inc}(X)$ .*

A complete metric all of whose boundary incompressible closed geodesics are minimizing in their homotopy class is taut, see Lemma 5.1. Thus, by the Cartan-Hadamard theorem [1], a basic example of a complete taut metric is a complete metric with non-positive curvature.

**Definition 1.3.** *Let  $g_0, g_1$  be a pair of taut complete metrics on  $X$ , and  $\beta \in \pi_1^{inc}(X)$ . A  $\beta$ -**taut homotopy or deformation** between  $g_0, g_1$ , is a continuous (in the topology of  $C^0$  convergence on compact sets) family  $\{g_t\}$ ,  $t \in [0, 1]$  of complete, taut metrics on  $X$ , s.t.*

$$S(\{g_t\}, \beta) := \{(o, t) \in L_\beta X \times [0, 1] \mid o \in S(g_t, \beta)\}$$

*is compact. We say that  $\{g_t\}$  is a **taut homotopy** if it is  $\beta$ -taut for each  $\beta \in \pi_1^{inc}(X)$ . The above definitions of tautness is extended naturally to the case of a smooth fibration  $X \hookrightarrow P \rightarrow [0, 1]$ , with a smooth fiber-wise family of metrics.*

A useful criterion for  $\beta$ -tautness is the following.

**Theorem 1.4.** *Let  $\{g_t\}_{t \in [0, 1]}$ , be a continuous family of complete, taut metrics on  $X$ . Suppose that:*

$$\sup_t \left| \max_{o \in S(g_t, \beta)} l_{g_t}(o) - \min_{o \in S(g_t, \beta)} l_{g_t}(o) \right| < \infty,$$

*where  $l_{g_t}$  is the length functional with respect to  $g_t$ , then  $\{g_t\}$  is  $\beta$ -taut. It follow that sky catastrophes of vector fields on closed manifolds are not geodesible by metrics all of whose geodesics are minimal, Appendix A.1.*

For example, the hypothesis is trivially satisfied if  $g_t$  have the property that all their class  $\beta$  geodesics are minimal. In particular if  $g_t$  have non-positive curvature then  $\{g_t\}$  is taut, again by the Cartan-Hadamard theorem.

Fuller at the end of [4] has asked for any metric conditions on vector fields to rule out sky catastrophes, see Appendix A.1. By the above, non-positivity of curvature is one such condition.

*Remark 1.5.* Note that if sky catastrophes were never geodesible then the geodesible Seifert conjecture would follow, by the main result of [8]. Hence, this is a subtle situation and it now seems to be plausible that sky catastrophes for geodesible and Reeb families of vector fields do exist. The qualitative structure of such potential geodesible or Reeb sky catastrophes is somewhat understood by results in [8], and [8, Theorem 1.10] in particular. But this does not greatly aid constructing potential examples, which must be topologically very complex, (there are necessarily infinitely many suitably synchronized bifurcation events). No results prior to the theorem above are known to me aside from those mentioned by Fuller himself in [4].

Let  $\mathcal{G}(X)$  be the set of equivalence classes of complete taut metrics  $g$ , where  $g_0$  is equivalent to  $g_1$  whenever there is a taut homotopy between them. We may denote an equivalence class by its representative  $g$  by a slight abuse of notation.

**Theorem 1.6.** *For each manifold  $X$ , there is a natural, functional*

$$F : \mathcal{G}(X) \times \pi_1^{inc}(X) \rightarrow \mathbb{Q}.$$

The value  $F(g, \beta)$  can be interpreted as a count of the set of closed  $g$ -geodesic strings in class  $\beta$ . But one must take care of exactly how to count, as in general this set should be understood as an orbifold or rather a Kuranishi space of a certain kind, hence this is why  $F$  is  $\mathbb{Q}$  valued.

*Example 1.* Let  $g$  be a Riemannian metric with negative sectional curvature on a closed manifold  $X$  and  $\beta \in \pi_1(X)$  a class represented by a multiplicity  $n$  closed geodesic, then

$$(1.7) \quad F(g, \beta) = \frac{1}{n}.$$

In particular, if  $\beta$  is indivisible as a class in the fundamental group then  $F(g, \beta) = 1$ . More generally, (1.7) holds whenever  $g$  has a unique and non-degenerate closed geodesic string in class  $\beta$ . Here and throughout the paper, a closed geodesic string is **non-degenerate** if the corresponding  $S^1$  family of closed geodesics is Morse-Bott non-degenerate.

**Theorem 1.8.** *Every non-positive rational number has the form  $F(g, \beta)$  for some  $\beta$ -taut Riemannian  $g$  on some compact manifold  $X$  and for some  $\beta \in \pi_1^{\text{inc}}(X)$ .*

**Theorem 1.9.** *Suppose that  $\beta \in \pi_1^{\text{inc}}(X)$  is indivisible, and  $X$  admits a  $\beta$ -taut metric, then the  $S^1$  equivariant homology  $H_*^{S^1}(L_\beta X, \mathbb{Z})$  is finite dimensional. Denote by  $\chi^{S^1}(L_\beta X)$  the Euler characteristic of this homology. Then for any  $\beta$ -taut metric  $g$  on  $X$ :*

$$F(g, \beta) = \chi^{S^1}(L_\beta X).$$

Example for the theorem above can be found by the proof of Theorem 1.8, for this type of examples  $F(g, \beta)$  will be a negative integer.

*Remark 1.10.* If  $\beta$  is divisible, the above idea breaks down, as the  $S^1$ -equivariant homology of  $L_\beta X$  may then be infinite dimensional even if  $X$  admits a  $\beta$ -taut  $g$ , in particular the connection with the Euler characteristic breaks down. It is thus an interesting open problem if the functional  $F$  remains topological. (This looks unlikely.)

A related question follows the proof of Corollary 1.14.

A celebrated theorem of Preissman [6] says that there are no negative sectional curvature metrics on compact products. Fibration counterexamples to Preissman's product theorem certainly exist. In fact, every closed 3-manifold  $X^3$ , for which there is no injection  $\mathbb{Z}^2 \rightarrow \pi_1(X, x_0)$ , and which fibers over a circle has a hyperbolic structure  $g_h$ , Thurston [10]. We are going to give a certain generalization of Preissman's theorem to fibrations, with possibly non-compact fibers, also replacing the negative sectional curvature condition by much weaker tautness conditions.

**Definition 1.11.** *Let  $Z \hookrightarrow X \xrightarrow{p} Y$  be a smooth fiber bundle with  $X$  having a complete,  $\beta$ -taut Riemannian metric  $g$ , for  $\beta \in \pi_1^{\text{inc}}(X)$ , and let  $g_Y$  be a metric on  $Y$ . Suppose that*

- (1) *The fibers  $Z_y = p^{-1}(y)$  are totally  $g$ -geodesic, for closed geodesics in class  $\beta$ . We denote by  $g_y$  the metric  $g$  restricted to  $Z_y$ .*
- (2) *The fibers are parallel (the distribution  $T^{\text{vert}}X = \ker p_*$  is parallel along any smooth curve in  $X$  with respect to the Levi-Civita connection of  $g$ ).*
- (3) *For any pair of fibers  $(Z_{y_0}, g_{y_0})$ ,  $(Z_{y_1}, g_{y_1})$ , and a path  $\gamma : [0, 1]$  from  $y_0$  to  $y_1$  the fiber family  $\{(Z_{\gamma(t)}, g_{\gamma(t)})\}$  furnishes a taut homotopy.*
- (4)  *$p$  projects  $g$ -geodesics to geodesics of  $Y, g_Y$ .*

We then call  $p : X \rightarrow Y$  a  **$\beta$ -taut submersion**, with the metrics  $g, g_Y$  and  $g_Z$  all possibly implicit.

**Definition 1.12.** For  $Z \hookrightarrow X \rightarrow Y$  as above, we say that  $\beta \in \pi_1(X)$  is a **fiber class** if it is in the image of the inclusion  $i_Z : \pi_1(Z) \rightarrow \pi_1(X)$ . We say that a class  $\beta \in \pi_1(X, x_0)$  is at **most a  $k$ -power**, if whenever  $\beta = \alpha^n$ , then  $n \leq k$ . Similarly,  $\beta \in \pi_1(X)$  is at most a  $k$ -power if as a class in the fundamental group it is at most a  $k$ -power.

In the above definition of a taut submersion and the following theorem we need the auxiliary metric  $g$  on  $X$  to be Riemannian, and there is no obvious extension of the theorem to the Riemann-Finsler case. However, the  $\beta$ -taut deformations in the theorem are free to be through Riemann-Finsler metrics.

**Theorem 1.13.** Let  $p : (X, g) \rightarrow (Y, g_Y)$  be a  $\beta$ -taut submersion, where  $\beta \in \pi_1^{\text{inc}}(X)$  is a fiber class. Suppose further that  $Y$  is connected and closed, and is such that all smooth closed contractible  $g_Y$  geodesics in  $Y$  are constant, then the following holds.

- (1) If  $\beta$  is at most a  $k$ -power, and if  $|\chi(Y)| > \text{lcm}(1, \dots, k)$ , where  $\text{lcm}(1, \dots, k)$  denotes the least common multiple of  $1, \dots, k$ , then  $g$  cannot be  $\beta$ -taut deformed to a Riemann-Finsler metric  $g'$  on  $X$  with a unique non-degenerate geodesic string in class  $\beta$ , in particular to a complete metric with negative curvature.
- (2) If  $\chi(Y) = 0$  then for any  $\beta \in \pi_1^{\text{inc}}(X)$ , and any  $\beta$ -taut  $g$ :  $F(g, \beta) = 0$ . If in addition  $\emptyset \neq \pi_1^{\text{inc}}(X)$ , then  $X$  does not admit a complete Riemann-Finsler metric with negative curvature. More generally,  $X$  does not admit a Riemann-Finsler metric with a unique and non-degenerate geodesic string in any indivisible class  $\beta \in \pi_1^{\text{inc}}(X)$ .

As an immediate corollary we get a very close analogue of Preissman's theorem, a stronger form of this corollary is further ahead.

**Corollary 1.14.** Let  $X = Z \times S^1$  where  $Z$  admits a complete metric of non-positive curvature. Then  $X$  does not admit a metric with a unique and non-degenerate geodesic string in any indivisible class  $\beta \in \pi_1^{\text{inc}}(X)$ . In particular, if  $X$  satisfies  $\emptyset \neq \pi_1^{\text{inc}}(X)$ , then  $X$  does not admit a metric of negative curvature.

*Proof.* Let  $g_Z$  be a complete metric on  $Z$  with non-positive curvature. Take the product metric  $g = g_Z \times g_{S^1}$  on  $X = Z \times S^1$ . By the Cartan-Hadamard theorem,  $g$  is  $\beta$ -taut for all fiber class  $\beta \in \pi_1^{\text{inc}}(X)$ . Moreover, the natural projection  $X \rightarrow S^1$  is automatically a  $\beta$ -taut submersion satisfying conditions of the theorem above. Hence, the first part follows by Part 2 of the theorem above.

The second part follows by the first part. If  $\emptyset \neq \pi_1^{\text{inc}}(X)$  then in particular, we may find an indivisible class  $\beta \in \pi_1^{\text{inc}}(X)$ . As previously observed, a complete metric with negative curvature is  $\beta$ -taut, and has a unique and non-degenerate class  $\beta$  string. So by the first part such a metric cannot exist.  $\square$

*Question 1.* Do there exist a pair of taut metrics  $g_1, g_2$  on a manifold  $X$  which are not taut homotopic?

Probably both possibilities are interesting. If the answer is yes then in Part 1 of the above theorem we may replace the conclusion on non existence of  $\beta$ -taut homotopy between  $g, g'$  to just non-existence of  $g'$ , with the same conditions. On the other hand if the answer is no then the previous Theorem 1.6 becomes far more intriguing.

A basic set of examples for the theorem is obtained by starting with any homomorphism

$$(1.15) \quad \phi : \pi_1(Y, y_0) \rightarrow \text{Isom}(Z, g_Z), \quad (\text{the group of all isometries}).$$

where  $g_Z$  is a complete taut metric, and there is a class  $\beta_Z \in \pi_1^{\text{inc}}(Z)$ . Suppose further that  $(Y, g_Y)$  satisfies:

- $Y$  is closed and connected.

- All contractible  $g_Y$  geodesics in  $Y$  are constant.

We have the obvious induced diagonal action

$$\pi_1(Y, y_0) \rightarrow \text{Diff}(Z \times \tilde{Y}), \text{ (the group of all diffeomorphisms),}$$

$$\gamma \mapsto ((z, y) \mapsto (\phi(\gamma)(z), \gamma \cdot y)), \text{ (anonymous function notation)}$$

for  $\tilde{Y}$  the universal cover of  $Y$ . Taking the quotient of  $Z \times \tilde{Y}$  by this action, we get an associated “flat” bundle  $Z \hookrightarrow X_\phi \xrightarrow{p} Y$ , with a metric  $g_\phi$  induced from the product metric  $\tilde{g} = g_Z \oplus g_Y$ , on the covering space  $q : Z \times \tilde{Y} \rightarrow Z \times Y$ .

**Lemma 1.16.** *Let  $p : (X_\phi, g_\phi) \rightarrow (Y, g_Y)$  be as above, then this is a  $\beta$ -taut submersion, where  $\beta = i_*(\beta_Z)$ , for  $i_* : \pi_1^{inc}(Z) \rightarrow \pi_1^{inc}(X_\phi)$  induced by inclusion.*

By the lemma above,  $p : (X_\phi, g_\phi) \rightarrow (Y, g_Y)$  satisfies the hypothesis of the theorem above. Yet more concretely:

*Example 2.* Suppose we have  $\beta_Z \in \pi_1^{inc}(Z)$ , and let  $\phi : Z \rightarrow Z$  be an isometry of a complete taut metric  $g_Z$ . Then by the construction above, the mapping torus  $(Z, g_Z) \hookrightarrow (X_\phi, g_\phi) \xrightarrow{\pi} S^1$  has the structure of a  $\beta$ -taut submersion, satisfying the hypothesis of the theorem, for  $\beta = i_*(\beta_Z)$  as above.

The next corollary of Theorem 1.13 is proved analogously to Corollary 1.14.

**Corollary 1.17.** *Let  $X_\phi, g_\phi$  be the mapping torus as above, for  $Z, g_Z$  having non-positive curvature, and with a class  $\beta_Z \in \pi_1^{inc}(Z)$ . Then  $X_\phi$  does not admit a complete metric with a single non-degenerate class  $\beta$  geodesic string, for  $\beta = i_*(\beta_Z)$  as above. In particular,  $X_\phi$  does not admit a complete Riemann-Finsler metric with negative curvature.*

*Remark 1.18.* In the Riemannian case, the conclusions on non-existence of negative sectional curvature metrics can likely be deduced with some work from Preissman’s theorem, previously mentioned. (We need to do some extensions to non-compact setting.) In special cases we can obtain similar conclusions using  $S^1$ -equivariant Morse theory. Specifically, we need at least that the energy function on the free loop space  $LX$ ,

$$(1.19) \quad e(o) = \int_{S^1} \langle \dot{o}(t), \dot{o}(t) \rangle_g dt$$

is Morse-Bott. Take for instance  $X = T^n$  with the product metric. It would of course be very interesting to understand the relationship of the functional  $F$  with  $S^1$ -equivariant Morse homology in general. This relationship is very complex as the latter equivariant Morse homology may be infinite dimensional even when  $X$  admits a taut metric  $g$ .

## 2. PROOF OF THEOREM 1.4

The first part of the theorem clearly follows by the second part. So let  $\{g_t\}$ ,  $t \in [0, 1]$  be as in the hypothesis, with

$$\sup_t \left| \max_{o \in S(g_t, \beta)} l_{g_t}(o) - \min_{o \in S(g_t, \beta)} l_{g_t}(o) \right| < c,$$

and suppose that

$$\sup_{(o, t) \in \mathcal{O}(\{g_t\}, \beta)} l_{g_t}(o) = \infty.$$

Then we have a sequence  $\{o_k\}$ ,  $k \in \mathbb{N}$ , of closed class  $\beta$   $g_{t_k}$ -geodesics in  $X$ , satisfying:

- (1)  $\lim_{k \rightarrow \infty} t_k = t_\infty \in [0, 1]$ .
- (2)  $\lim_{k \rightarrow \infty} l_{g_{t_k}}(o_k) = \infty$ , where  $l_{g_{t_k}}(o_{t_k})$  is the length with respect to  $g_{t_k}$ .

Let  $o_\infty$  be a minimal, class  $\beta$ ,  $g_\infty = g_{t_\infty}$  geodesic in  $X$ , with  $g_\infty$  length  $L$ . Let  $g_{aux}$  be a fixed auxiliary metric on  $X$ , and let  $L_{aux}$  be the  $g_{aux}$  length of  $o_\infty$ .

Define a pseudo-metric on the space of metrics on  $X$  as follows. Let  $K \subset X$  be a fixed compact set containing image  $o_\infty$ , and set

$$V \subset TX = \{v \in TX \mid \pi(v) \in K \text{ for } \pi : TX \rightarrow X \text{ the canonical projection, and } |v|_{aux} = 1\},$$

where  $|v|_{aux}$  is the norm taken with respect to  $g_{aux}$ .

Then define:

$$d_{C^0}(g_1, g_2) = \sup_{v \in V} ||v|_{g_1} - |v|_{g_2}|.$$

By properties 1 and 2 we may find a  $k > 0$  such that

$$(2.1) \quad d_{C^0}(g_{t_k}, g_{t_\infty}) < \epsilon$$

and

$$l_{g_{t_k}}(o_k) > c + L + L_{aux} \cdot \epsilon.$$

As  $L + L_{aux} \cdot \epsilon \geq l_{g_{t_k}}(o_\infty)$  by (2.1), we have:

$$l_{g_{t_k}}(o_k) > l_{g_{t_k}}(o_\infty) + c.$$

Since we may find a closed  $g_{t_k}$ -geodesic  $o'$  satisfying  $l_{g_{t_k}}(o') \leq l_{g_{t_k}}(o_\infty)$ , we get that

$$|\max_{o \in S(g_{t_k}, \beta)} l_{g_{t_k}}(o) - \min_{o \in S(g_{t_k}, \beta)} l_{g_{t_k}}(o)| > c,$$

and so we are in contradiction.

Thus,

$$\sup_{(o, t) \in \mathcal{O}(\{g_t\}, \beta)} l_{g_t}(o) < \infty.$$

By an analogue of Lemma 5.2, the images of all elements  $o \in \mathcal{O}(\{g_t\}, \beta)$  are contained in a fixed compact  $K \subset X$ . Compactness of  $\mathcal{O}(\{g_t\}, \beta)$  then follows by the Arzella-Ascoli theorem.  $\square$

### 3. PROOF OF LEMMA 1.16

This is elementary so we limit ourselves to a sketch. Let  $\phi_* : \pi_1(Y, y_0) \rightarrow \text{Aut}(\pi_1^{inc}(Z))$  be the natural induced action, where  $\text{Aut}(\pi_1^{inc}(Z))$  denotes the group of set isomorphisms. The orbit

$$O := \bigcup_{\gamma \in \pi_1(Y, y_0)} \phi_*(\gamma)(\beta_Z)$$

is readily seen to be finite by the assumption that the action  $\phi$  is by isometries, and by the fact that  $\beta_Z \in \pi_1^{inc}(Z)$ . In fact, only finitely many distinct classes in  $\pi_1^{inc}(Z)$  have representatives by  $g_Z$ -geodesics of fixed length, as otherwise we may find a sequence of boundary incompressible  $g_Z$ -geodesics, of fixed length, with no convergent subsequence.

As  $g_Z$  is taut,  $S(g_Z, \phi_*(\gamma)(\beta_Z))$  is compact for each  $\gamma$ , where  $S(g_Z, \phi_*(\gamma)(\beta_Z))$  is the space of geodesics as in Definition 1.2. By the condition on contractible geodesics of  $g_Y$ , we get:

$$\begin{aligned} S(g_\phi, \beta) &= q_*(S(g_Z \oplus g_Y, \beta)) \\ &= \bigcup_{\beta \in O} q_*(S(g_Z, \beta) \times \tilde{Y}), \end{aligned}$$

for  $q_* : L(Z \times \tilde{Y}) \rightarrow L(Z \times Y)$  induced by the quotient map  $q : Z \times \tilde{Y} \rightarrow Z \times Y$  as above, and where  $S(g_Z, \gamma) \times \tilde{Y}$  is identified with the corresponding subset of  $L(Z \times \tilde{Y})$ . Given that  $O$  is finite, this then readily implies our claim.  $\square$

## 4. PRELIMINARIES ON REEB FLOW

Let  $(C^{2n+1}, \lambda)$  be a contact manifold with  $\lambda$  a contact form, that is a one form s.t.  $\lambda \wedge (d\lambda)^n \neq 0$ . Denote by  $R^\lambda$  the Reeb vector field satisfying:

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

Recall that a **closed  $\lambda$ -Reeb orbit** (or just Reeb orbit when  $\lambda$  is implicit) is a smooth map

$$o : (S^1 = \mathbb{R}/\mathbb{Z}) \rightarrow C$$

such that

$$\dot{o}(t) = cR^\lambda(o(t)),$$

with  $\dot{o}(t)$  denoting the time derivative, for some  $c > 0$  called period. Let  $S(R^\lambda, \beta)$  denote the set of all  $R^\lambda$ -Reeb orbits in free homotopy class  $\beta$  and set

$$\mathcal{O}(R^\lambda, \beta) = S(R^\lambda, \beta)/S^1,$$

where  $S^1 = \mathbb{R}/\mathbb{Z}$  acts by reparametrization  $t \cdot o(\tau) = o(t + \tau)$ .

## 5. DEFINITION OF THE FUNCTIONAL F

Let  $X$  be a manifold with a complete taut metric  $g$ . Let  $C$  be the unit cotangent bundle of  $X$ , with its Liouville contact 1-form  $\lambda_g$ . If  $o : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow X$  is a unit speed closed geodesic, it has a canonical lift  $\tilde{o} : S^1 \rightarrow C$ . If  $\beta \in \pi_1^{inc}(X)$ , let  $\tilde{\beta} \in \pi_1(C)$  denote class  $[\tilde{o}] \in \pi_1(C)$ , where  $o$  is a unit speed closed geodesic representing  $\beta$ .

Let  $S(R^{\lambda_g}, \tilde{\beta})$  be the orbit space as in Section 4, for the Reeb flow of the contact form  $\lambda_g$ . And set

$$\mathcal{O}_{g,\beta} = \mathcal{O}(R^{\lambda_g}, \tilde{\beta}) = S(R^{\lambda_g}, \tilde{\beta})/S^1.$$

By the tautness assumptions  $\mathcal{O}_{g,\beta}$  is compact.

We then define

$$F(g, \beta) = i(\mathcal{O}_{g,\beta}, R^{\lambda_g}, \tilde{\beta}) \in \mathbb{Q}$$

where the right hand side is the Fuller index of  $R^{\lambda_g}$  in class  $\beta$ . As a basic example we have:

**Lemma 5.1.** *Suppose that  $g$  is a complete metric on  $X$ , all of whose boundary incompressible geodesics are minimal, then  $g$  is taut.*

*Proof.* First we state a more basic lemma.

**Lemma 5.2.** *Suppose that  $g$  is a complete metric on  $X$ ,  $\beta \in \pi_1^{inc}(X)$  and let  $S \subset L_\beta X$  be a subset on which the  $g$ -length functional is bounded. Then the images in  $X$  of elements of  $S$  are contained in a fixed compact subset of  $X$ .*

*Proof.* Suppose otherwise. Fix an exhaustion by nested compact sets

$$\bigcup_{i \in \mathbb{N}} K_i = X, \quad K_i \supset K_{i-1}.$$

Then either there is sequence  $\{o_i\}_{i \in \mathbb{N}}$ ,  $o_i \in S$  s.t.  $o_i \in K_i^c$ , for  $K_i^c$  the complement of  $K_i$ , which contradicts the fact that  $\beta$  is incompressible. Or there is a sequence  $\{o_k\}_{k \in \mathbb{N}}$ ,  $o_k \in S$  s.t.:

- (1) Each  $o_k$  intersects  $K_{i_0}$  for some  $i_0$  fixed.
- (2) For each  $i \in \mathbb{N}$  there is a  $k_i > i$  s.t.  $o_{k_i}$  is not contained in  $K_i$ .

Now if  $\text{diam}(o_k)$  is bounded in  $k$ , then condition 1 implies that  $o_k$  are contained in a set of bounded diameter. (Here  $\text{diam}(o_k)$  denotes the diameter of image  $o_k$ .) Consequently, by Hopf-Rinow theorem [1],  $o_k$  are contained in a compact set. But this contradicts condition 2, and the fact that  $K_i$  form an exhaustion of  $X$ .

Thus, we conclude that  $\text{diam}(o_k)$  is unbounded, but this contradicts the hypothesis.  $\square$

Returning to the proof of the main lemma. By assumption, closed, class  $\beta \in \pi_1^{\text{inc}}(X)$  geodesics are  $g$ -minimizing in their homotopy class and in particular have fixed length. By the lemma above there is a fixed  $K \subset X$  s.t. every class  $\beta$  closed geodesic has image contained in  $K$ . Then compactness of  $S(g, \beta)$  follows by Arzella-Ascoli theorem.  $\square$

*Proof Theorem 1.6.* Let  $\beta \in \pi_1^{\text{inc}}(X)$ , be given and let  $g$  be  $\beta$ -taut. We just need to prove that  $F(g, \beta)$  is invariant under a  $\beta$ -taut deformation of  $g$ . So let  $\{g_t\}$ ,  $t \in [0, 1]$  be a  $\beta$ -taut deformation of metrics on a compact manifold  $X$ . Let  $R^{\lambda_{g_t}}$  be the geodesic flow on the  $g_t$  unit cotangent bundle  $C_t$ . Trivializing the family  $\{C_t\}$  we get a family  $\{R_t\}$  of flows on  $C \simeq C_t$ , with  $R_t$  conjugate to  $R^{\lambda_{g_t}}$ .

Let  $\mathcal{O}(\{R_t\}, \tilde{\beta})$  be the cobordism as in (A.2), where  $\tilde{\beta} \in \pi_1(C)$  is as above. Then  $\mathcal{O}(\{R_t\}, \tilde{\beta})$  is compact as  $\mathcal{O}(\{g_t\}, \beta)$  is compact by assumption.

Basic invariance of the Fuller index immediately yields:  $F(g_0, \beta) = F(g_1, \beta)$ .  $\square$

*Proof of Theorem 1.9.* This is an application of Morse homology theory. As  $g$  is  $\beta$ -taut, and so  $S(g, \beta)$  is compact, we may find a  $C^\infty$ -nearby metric  $g'$ , s.t.  $g'$  has finitely many class  $\beta$  closed geodesic strings, all of which are non-degenerate. Now the  $S^1$  action on  $L_\beta X$  is free, by the condition that  $\beta$  is indivisible. Then  $S^1$ -equivariant Morse homology models, for example [2], readily yield that the  $S^1$ -equivariant Morse homology for the  $g'$ -energy function  $e$  on  $L_\beta X$ , see (1.19), is finite dimensional. Furthermore, it is immediate that the Euler characteristic of this homology is given by:

$$(5.3) \quad \sum_o -1^{\text{morse}(o)} = F(g', \beta) = F(g, \beta),$$

where the sum is over all  $g'$ -geodesic strings  $o$  in class  $\beta$ , and where the second equality is by local invariance of the Fuller index.

*Remark 5.4.* Note that because the action is free, the above result also readily follows by classical Morse theory if we could find  $S^1$ -equivariant perturbations for the Morse complex of  $e$ . This is a problem that can be solved using virtual moduli cycle methods, as in Pardon [5] and this gives a different approach.  $\square$

## 6. PROOF OF THEOREM 1.13

We first prove:

**Theorem 6.1.** *Let  $p : X \rightarrow Y$  be a  $\beta$ -taut submersion as in the statement of Theorem 1.13 and  $\beta \in \pi_1^{\text{inc}}(X)$  a fiber class. Then*

$$F(g, \beta) = \text{card} \cdot \chi(Y) \cdot F(g_Z, \beta_Z),$$

where  $\text{card} \in \mathbb{N}$  is the cardinality of a certain orbit of the holonomy group (as explained in the proof), and where  $\beta_Z$  is as in Lemma 1.16.



*Proof.* We have a natural subset of  $\mathcal{O}' \subset \mathcal{O}g, \beta$ , consisting of all vertical geodesics, that is  $g$ -geodesics contained in fibers  $p^{-1}(y) = Z_y$ . In fact,  $\mathcal{O}' = \mathcal{O}g, \beta$ , for if  $o$  is any class  $\beta$  geodesic, the projection  $p(o)$  is a contractible  $g_Y$  geodesic, and by assumptions is constant.

In particular there a natural continuous projection  $\tilde{p} : \mathcal{O}g, \beta \rightarrow Y$ ,  $\tilde{p}(o) = y$  where  $y$  is determined by the condition that  $Z_y \supset \text{image } o$ . We will use this to construct a suitable (in a sense abstract i.e. not Reeb) perturbation of the vector field  $R^{\lambda_g}$ , using which we can calculate the invariant  $F(g, \beta)$ .

Fix a Morse function on  $f$  on  $Y$ , let  $C = S^*X$  denote the  $g$ -unit cotangent bundle of  $X$ . For  $v \in T_x X$  let  $\langle v|$  denote the functional  $T_x X \rightarrow \mathbb{R}$ ,  $w \mapsto \langle v, w \rangle_g$ . Define  $\tilde{f} : C \rightarrow \mathbb{R}$  by  $\tilde{f}(\langle v|) := f(p(v))$ , also define  $P : C \rightarrow \mathbb{R}$  by  $P(\langle v|) := |P^{vert}(v)|_g^2$ , where  $P^{vert}(v)$  denotes the  $g$ -orthogonal projection of  $v$  onto the  $T_x^{vert} X \subset T_x X$ , for  $T^{vert} X$  the vertical tangent bundle of  $X$ , i.e. the kernel of the map  $p_* : TX \rightarrow TY$ . And define  $F : C \rightarrow \mathbb{R}$  by:

$$F(\langle v|) := P(\langle v|) + \tilde{f}(\langle v|).$$

Set

$$V_t = R^{\lambda_g} - t \text{grad}_{g_S} F,$$

where the gradient is taken with respect to the Sasaki metric  $g_S$  on  $C$  [7] induced by  $g$ . The latter Sasaki metric is the natural metric for which we have an orthogonal splitting  $TC = T^{vert}C \oplus T^{hor}C$ , where  $T^{vert}C$  is the kernel of  $pr_* : TC \rightarrow TX$ , induced by the natural projection  $pr : C \rightarrow X$ , and where  $T^{hor}C$  is the  $g$  Levi-Civita horizontal sub-bundle.

Set  $\mathcal{O}_t = \mathcal{O}(V_t, \tilde{\beta})$ , where  $\tilde{\beta}$  is as in Section 5.

**Lemma 6.2.** (1) For all  $t \in [0, 1]$ ,  $N_t := \mathcal{O}_t \cap \mathcal{O}_{g, \beta}$  is open and closed in  $\mathcal{O}_t$ .

(2) For all  $t \in (0, 1]$ ,  $N_t = \cup_{y \in \text{crit}(f)} \tilde{p}^{-1}(y)$ , where  $\text{crit}(f)$  is the set of critical points of  $f$ .

*Proof.* It is easy to see that  $V_t$  is complete and without zeros. Suppose that  $t > 0$ . Let  $\langle v_\tau|$ ,  $\tau \in \mathbb{R}$  be the flow line of  $V_t$ , through  $\langle v_0|$ , i.e.  $\langle v_\tau| = \phi_\tau(\langle v_0|)$ , for  $\phi_\tau$  the time  $\tau$  flow map of  $V_t$ . By the fact that the fibers of  $p$  are assumed to be parallel, we have that

$$R^{\lambda_g}(P) = 0, \quad \text{using the derivation notation.}$$

Also,

$$\text{grad}_{g_S} \tilde{f}(P) = 0,$$

which readily follows by the conjunction of  $g_S$  being Sasaki and the fibers of  $p$  being parallel. Consequently, the function

$$\tau \mapsto P(\langle v_\tau|) = |P^{vert}(v_\tau)|_g^2$$

is monotonically decreasing unless either:

- (1)  $v_0$  is tangent to  $T^{vert}X$ , in which case for all  $\tau$ ,  $v_\tau$  are tangent to  $T^{vert}X$  and  $|P^{vert}(v_\tau)|_g^2 = 1$ .
- (2) For all  $\tau$ ,  $|P^{vert}(v_\tau)|_g^2 = 0$ .

In particular, the closed orbits of  $V_t$  split into two types.

- (1) Closed orbits  $o(\tau) = \langle v_\tau|$  with  $v_\tau$  always tangent to  $T^{vert}X$ . In this case we may immediately, conclude that  $o$  is a lift to  $C$  of a closed  $g$ -geodesic in contained in the fiber over a critical point of  $f$ .
- (2) Closed orbits  $o(\tau) = \langle v_\tau|$  for which  $v_\tau$  is always  $g$ -orthogonal to  $T^{vert}X$ .

Clearly, the conclusion follows.  $\square$

*Remark 6.3.* It would be very fruitful to remove the condition on the fibers of  $p$  being parallel. But our argument would need to substantially change.

We return to the proof of the theorem. Set

$$\tilde{N} = \{(o, t) \in L_{\tilde{\beta}}C \times [0, \epsilon] \mid o \in N_t\},$$

where  $L_{\tilde{\beta}}C$  denotes the  $\tilde{\beta}$  component of the free loop space as previously. By part I of Lemma 6.2, this is an open compact subset of  $\mathcal{O}(\{V_t\}, \tilde{\beta})$  s.t.

$$\tilde{N} \cap (L_{\tilde{\beta}}C \times \{0\}) = \mathcal{O}(R^{\lambda_g}, \tilde{\beta}),$$

(equalities throughout are up to natural set theoretic identifications.)

By definitions:

$$N_t = \tilde{N} \cap (L_{\tilde{\beta}}C \times \{t\}).$$

Now the invariance of the Fuller index gives:

$$i(N_0, R^{\lambda_g}, \tilde{\beta}) = i(N_1, V_1, \tilde{\beta}).$$

We proceed to compute the right hand side. Fix any smooth Ehresmann connection  $\mathcal{A}$  on the fiber bundle  $p : X \rightarrow Y$ . This induces a holonomy homomorphism:

$$\text{hol}_y : \pi_1(Y, y) \rightarrow \text{Aut } \pi_1(Z_y) \text{ (the right-hand side is the group of set automorphisms),}$$

with image denoted  $\mathcal{H}_y \subset \text{Aut } \pi_1(Z_y)$ .

Let  $\beta_Z$  denote a class in  $\pi_1(Z_y)$  s.t.  $(i_{Z_y})_*(\beta_Z) = \beta$ , for  $i_{Z_y} : Z_y \rightarrow X$  the inclusion map. Set

$$S_y := \bigcup_{g \in \mathcal{H}_y} g(\beta_Z) \subset \pi_1(Z_y).$$

Then for another  $y' \in Y$ ,

$$(6.4) \quad h_* : S_{y'} \rightarrow S_y,$$

is an isomorphism, where  $h : Z_{y'} \rightarrow Z_y$  is a smooth map given by the  $\mathcal{A}$ -holonomy map determined by some path from  $y$  to  $y'$ , and where  $h_*$  is the naturally induced map.

Denoting by  $g_y$  the restriction of  $g$  to the fiber  $Z_y$ , let  $R^y$  denote the  $\lambda_{g_y}$  Reeb vector field on the  $g_{Z_y}$ -unit cotangent bundle  $C_y$  of  $Z_y$ . The cardinality  $\text{card}$  of  $S_y$  is finite, as otherwise we get a contradiction to the compactness of  $S(g, \beta)$ . Now

$$\tilde{p}^{-1}(y) = \bigcup_{\alpha \in S_y} \mathcal{O}(R^{\lambda_y}, \alpha).$$

From part 2 of Lemma 6.2 and by straightforward index computations we get:

$$i(N_1, V_1, \tilde{\beta}) = \sum_{y \in \text{crit}(f)} (-1)^{\text{morse}(y)} \cdot i(\tilde{p}^{-1}(y), R^{\lambda_y}, \tilde{\beta}),$$

where  $\text{morse}(y)$  denote the Morse index of  $y$ . Now

$$\begin{aligned} i(\tilde{p}^{-1}(y), R^{\lambda_y}, \tilde{\beta}) &= \sum_{\alpha \in S_y} i(\mathcal{O}(R^{\lambda_y}), R^{\lambda_y}, \tilde{\alpha}) \\ &= \sum_{\alpha \in S_y} F(g_y, \alpha). \\ &= \text{card} \cdot F(g_Z, \beta_Z), \end{aligned}$$

where the last equality follows by (6.4), and by the condition 3 in the Definition 1.2. And so the result follows.  $\square$

Now returning to the proof of the main theorem. If  $\beta$  is at most a  $k$ -power then elements of  $\mathcal{O}(R^{\lambda_{g_Z}}, \beta_Z)$  have multiplicity at most  $k$ . It follows that either

$$|\text{lcm}(1, \dots, k) \cdot F(g_Z, \beta_Z)| \geq 1,$$

or  $F(g_Z, \beta_Z) = 0$ . So if  $|\chi(X)| > \text{lcm}(1, \dots, k)$ , by the theorem above  $|F(g, \beta)| \geq 1$ , or  $F(g, \beta) = 0$ . In either case the first conclusion follows by Theorem 1.6 and by Example 1.

If  $\chi(X) = 0$ , then by the theorem above for all  $\beta$  as in the statement:  $F(g, \beta) = 0$ . So if we take  $\beta$  to be indivisible, then by Theorem 1.9 we have

$$0 = F(g, \beta) = \chi^{S^1}(L_\beta X) = F(g', \beta) = 1,$$

if  $g'$  has a unique and non-degenerate class  $\beta$  geodesic string. But this is impossible, so that the second part of the theorem readily follows.  $\square$

*Proof of Theorem 1.8.* By Theorem 6.1 it is clearly enough to prove that any negative rational number is the value of the invariant. Let  $n = -\frac{p}{q} \in \mathbb{Q}$  be given with  $p, q \in \mathbb{N} - 0$ . Let  $Y$  be a closed surface of genus  $(p+1) > 1$  with a hyperbolic metric  $g_Y$ , let  $Z$  be the genus 2 closed surface with a hyperbolic metric  $g_Z$  and let  $\beta \in \pi_1^{\text{inc}}(Z)$  be the class represented by a  $2 \cdot q$ -fold covering of a simple loop representing a generator of the fundamental group of  $Z$ .

Let  $X = Y \times Z$  with the product metric  $g = g_Y \times g_Z$  and  $p : X \rightarrow Y$  the canonical projection. By Theorem 6.1

$$F(g, \beta) = \chi(Y) \cdot F(g_Z, \beta_Z) = (-2p) \cdot \frac{1}{2q} = -\frac{p}{q}.$$

$\square$

#### A. FULLER INDEX AND SKY CATASTROPHES

Let  $X$  be a complete vector field without zeros on a manifold  $M$ . Set

$$(A.1) \quad S(X, \beta) = \{o \in L_\beta M \mid \exists p \in (0, \infty), \ o : \mathbb{R}/\mathbb{Z} \rightarrow M \text{ is a periodic orbit of } pX\}.$$

The above  $p$  is uniquely determined and we denote it by  $p(o)$  called the period of  $o$ .

There is a natural  $S^1$  reparametrization action on  $S(X, \beta)$ :  $t \cdot o$  is the loop  $t \cdot o(\tau) = o(t + \tau)$ . The elements of  $\mathcal{O}(X, \beta) := S(X, \beta)/S^1$  will be called **orbit strings** or just closed orbits. Slightly abusing notation we just write  $o$  for the equivalence class of  $o$ .

The multiplicity  $m(o)$  of an orbit string is the ratio  $p(o)/l$  for  $l > 0$  the period of a simple orbit string covered by  $o$ .

We want a kind of fixed point index which counts orbit strings  $o$  with certain weights. Assume for simplicity that  $N \subset \mathcal{O}(X, \beta)$  is finite. (Otherwise, for a general open compact  $N \subset \mathcal{O}(X, \beta)$ , we need to perturb.) Then to such an  $(N, X, \beta)$  Fuller associates an index:

$$i(N, X, \beta) = \sum_{o \in N} \frac{1}{m(o)} i(o),$$

where  $i(o)$  is the fixed point index of the time  $p(o)$  return map of the flow of  $X$  with respect to a local surface of section in  $M$  transverse to the image of  $o$ .

Fuller then shows that  $i(N, X, \beta)$  has the following invariance property. For a continuous homotopy  $\{X_t\}$ ,  $t \in [0, 1]$  set

$$S(\{X_t\}, \beta) = \{(o, t) \in L_\beta M \times [0, 1] \mid o \in S(X_t)\}.$$

And given a continuous homotopy  $\{X_t\}$ ,  $X_0 = X$ ,  $t \in [0, 1]$ , suppose that  $\tilde{N}$  is an open compact subset of

$$(A.2) \quad \mathcal{O}(\{X_t\}, \beta) := S(\{X_t\}, \beta)/S^1,$$

such that

$$\tilde{N} \cap (L_\beta M \times \{0\}) / S^1 = N.$$

Then if

$$N_1 = \tilde{N} \cap (L_\beta M \times \{1\}) / S^1$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta).$$

### A.1. Blue sky catastrophes.

**Definition A.3** (Preliminary). *A **sky catastrophe** for a smooth family  $\{X_t\}$ ,  $t \in [0, 1]$ , of non-vanishing vector fields on a closed manifold  $M$  is a continuous family of closed orbit strings  $\tau \mapsto o_{t_\tau}$ ,  $o_{t_\tau}$  is an orbit string of  $X_{t_\tau}$ ,  $\tau \in [0, \infty)$ , such that the period of  $o_{t_\tau}$  is unbounded from above.*

A sky catastrophe as above was initially constructed by Fuller [3]. Or rather his construction essentially contained this phenomenon. A more general definition appears in [8], we slightly extend it here to the case of non-compact manifolds. All these definitions become equivalent given certain regularity conditions on the family  $\{X_t\}$  and assuming  $M$  is compact.

**Definition A.4.** *Let  $\{X_t\}$ ,  $t \in [0, 1]$  be a continuous family of non-zero, complete smooth vector fields on a manifold  $M$  and  $\beta \in \pi_1^{\text{inc}}(X)$ .*

*We say that  $\{X_t\}$  has a **catastrophe in class  $\beta$** , if there is an element*

$$y \in \mathcal{O}(X_0, \beta) \sqcup \mathcal{O}(X_1, \beta) \subset \mathcal{O}(\{X_t\}, \beta)$$

*such that there is no open compact subset of  $\mathcal{O}(\{X_t\}, \beta)$  containing  $y$ .*

A vector field  $X$  on  $M$  is **geodesible** if there exists a metric  $g$  on  $M$  s.t. every flow line of  $X$  is a unit speed  $g$ -geodesic. A family  $\{X_t\}$  is **geodesible** if there is a continuous family  $\{g_t\}$  of metrics, with  $X_t$  geodesible with respect to  $g_t$  for each  $t$ . A family  $\{X_t\}$  is **geodesible** if there is a continuous family  $\{g_t\}$  of metrics with  $X_t$  geodesible with respect to  $g_t$  for each  $t$ . A **geodesible sky catastrophe** is a geodesible family  $\{X_t\}$  with a sky catastrophe. A **Reeb sky catastrophe** is a family of Reeb vector fields  $\{X_t\}$  with a sky catastrophe.

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