# A CONFORMAL SYMPLECTIC WEINSTEIN CONJECTURE

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Abstract. We introduce a direct generalization of the Weinstein conjecture to closed, Lichnerowicz exact, locally conformally symplectic manifolds, (for short lcs manifolds). This conjectures existence of certain 2-curves in the manifold, which we call Reeb 2-curves. The conjecture readily holds for all closed exact lcs surfaces. In higher dimensions, we give partial verifications of this conjecture, based on certain extended ($\mathbb{Q} \sqcup \{\pm\infty\}$ valued) Gromov-Witten, elliptic curve counts in lcs manifolds. As a basic application we get some novel results in classical Reeb dynamics. The most basic such result gives sufficient conditions for a strict contactomorphism to fix the image of some closed Reeb orbit on a closed contact manifold. Along the way we give a Gromov-Witten theoretic construction of the classical dynamical Fuller index (for Reeb vector field), which among other things explains its rationality.

Contents



*Key words and phrases.* locally conformally symplectic manifolds, conformal symplectic non-squeezing, Gromov-Witten theory, virtual fundamental class, Fuller index, Seifert conjecture, Weinstein conjecture.

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## 1. Introduction: Overview

A locally conformally symplectic manifold or lcs manifold for short, is a natural direct generalization of both symplectic and contact manifolds. The main goal here is to study an lcs variant of a very influential conjecture in contact geometry and dynamics: the Weinstein conjecture. The latter conjectures existence of closed orbits of the Reeb flow for any contact form on a closed manifold. This is now proved for contact three manifolds by Taubes [31].

**Definition 1.1.** *A locally conformally symplectic manifold or just* lcs *manifold, is a smooth $2n$-fold $M$ with an* lcs *structure: which is a non-degenerate 2-form $\omega$, with the property that for every $p \in M$ there is an open $U \ni p$ such that $\omega|_U = f_U \cdot \omega_U$, for some symplectic form $\omega_U$ defined on $U$ and some smooth positive function $f_U$ on $U$.*

These kinds of structures were originally considered by Lee in [14], arising naturally as part of an abstract study of "a kind of even dimensional Riemannian geometry", and then further studied by a number of authors see for instance, [2] and [32]. This is a fascinating object, an lcs manifold admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and Murphy show that if a closed almost complex $2n$-fold $M$ has $H^1(M, \mathbb{R}) \neq 0$ then it admits a lcs structure, [8]. Another result of Apostolov, Dloussky [1] is that any complex surface with an odd first Betti number admits an lcs structure, which tames the complex structure.

We will state an analogue of the Weinstein conjecture for exact lcs manifolds: "a conformal symplectic Weinstein conjecture", or just CSW conjecture for short. We shall then give partial verifications using suitably extended ($\mathbb{Q} \sqcup \{\pm\infty\}$ valued) Gromov-Witten theory, counting certain elliptic curves, and which has some connections to the Fuller index in dynamical systems theory. Finally, we shall develop some immediate applications in Reeb dynamics.

Let $(C, \lambda)$ be a contact manifold with $\lambda$ a contact form, denote by $R^\lambda$ the Reeb vector field satisfying:

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

Recall that a ***closed $\lambda$-Reeb orbit*** (or just Reeb orbit when $\lambda$ is implicit) is a smooth map $o : S^1 \to C$ such that

$$o'(t) = cR^\lambda(o(t)),$$

with $o'(t)$ denoting the time derivative, for some $c > 0$ called period.

**Definition 1.2.** *Let $\phi : (C, \lambda) \to (C, \lambda)$ be strict contactomorphism of a contact manifold. Then a* **fixed Reeb string of** *$\phi$ is a closed $\lambda$-Reeb orbit $o$, fixed by $\phi$ up to reparametrization. That is*

$$\text{image}(o) = \phi(\text{image}(o)).$$

First we have the following dichotomy, referring the reader to the definition of sky catastrophes for families of vector fields, in Definition A.4.

**Theorem 1.3.** *Either:*

(1) *There exist (at all) sky catastrophes for families of Reeb vector fields on a closed manifold, and in fact exist on $S^{2k+1}$, and are moreover stable.*

(2) *Or for any contact form $\lambda$, for the standard contact structure on $S^{2k+1}$, any strict contactomorphism $\phi$ of $(S^{2k+1}, \lambda)$, homotopic via strict contactomorphisms to the id, has a fixed Reeb string.*

We note that at present it is unknown if sky catastrophes for Reeb vector fields on closed manifolds exist. Or indeed, if even they exist for geodesible vector fields, see however Savelyev [27] for a discussion of one basic conjecture on the subject. The basic point being that $C^\infty$ aperiodic flows on $S^{2k+1}$, which exist Kuperberg [13], can be translated to constructions of $C^\infty$ sky catastrophes via the main result of [27]. And this is partly reversible, that is construction of sky catastrophes is step one to construction of an aperiodic flow. On the other hand we know that there are no Reeb aperiodic flows on $S^{2k+1}$, this being one of the early results of symplectic geometry due to Viterbo [33]. Although, there is no mathematical implication that there are no Reeb sky catastrophes, given state of the art, the overall framework makes it somewhat likely. As far as existence of fixed Reeb strings, there appears to be no prior results of the sort, however the Sandon conjecture on translated points is at least partly related, and we discuss this further ahead.

Here is a more concrete example of the above. For more details on the Fuller index and the terminology see Appendix A. Let $\lambda$ be a contact form on a closed manifold $C$ and let $i(N, R^\lambda, \beta) \in \mathbb{Q}$ denote the Fuller index of some ***good set*** (Appendix A) $N \subset \mathcal{O}(R^\lambda)$ of closed unparametrized orbits of the Reeb vector field $R^\lambda$, in class $\beta \in \pi_1(C)$. For example, for the standard contact form $\lambda_{st}$ on $S^{2n+1}$ any connected component $N$ of the orbit space $\mathcal{O}(R^{\lambda_{st}})$ is good and we have $i(N, R^{\lambda_{st}}, 0) \neq 0$. Then as a partial corollary of Theorem 2.17 and Theorem 2.14 we get:

**Theorem 1.4.** *Let $(C, \lambda)$ be a contact manifold satisfying $i(N, R^\lambda, \beta) \neq 0$, for some good $N$, and some $\beta$. Then there is an $\epsilon > 0$ s.t any strict contactomorphism $\phi : (C, \lambda) \to (C, \lambda)$, $C^\infty$ $\epsilon$-close to id, has a fixed Reeb string.*

This theorem is true for $(S^{2k+1}, \lambda_{st})$, and for any strict contactomorphism $\phi$ isotopic to the *id*, by more elementary considerations. To see this note that the unparametrized orbit space $\mathcal{O}(R^{\lambda_{st}})$ has connected components $N$ diffeomorphic to $\mathbb{CP}^k$, and $\phi$ induces a topological endomorphism $\widetilde{\phi}$ of each of these components, with non-zero Lefschetz number by the condition that $\phi$ is isotopic to the *id*. (The endomorphism is moreover a symplectomorphism, where the symplectic form comes from symplectic reduction, but we don't need this). So that for the standard contact form the result follows for $\phi$ as above by the Lefschetz fixed point theorem.

However, in general, even with the $\epsilon$ condition, the above is not intuitively clear. For example, the components $N$ of $\mathcal{O}(R^\lambda)$ can be far from manifolds, so we can't easily look for a fixed point condition. Moreover, any fixed point conditions like the Lefschetz number of $\widetilde{\phi}$ cannot be related to the Fuller index above, as the latter is associated purely to $(\mathbb{R}^\lambda, N)$, it has no dependence on $\phi$. Of course, the argument of this paper has a very different approach. We use partial verifications of the CSW conjecture, which are based on Gromov-Witten theory.

The Gromov-Witten theory for lcs manifolds has some intriguing new difficulties. The first problem that occurs is that a priori energy bounds are gone, as since $\omega$ is not necessarily closed, the $L^2$-energy can now be unbounded on the moduli spaces of $J$-holomorphic curves in such a $(M, \omega)$. A more acute problem is the potential presence of holomorphic sky catastrophes - given a smooth family $\{J_t\}$,

$t \in [0,1]$, of $\{\omega_t\}$-compatible almost complex structures, we may have a continuous family $\{u_t\}$ of $J_t$-holomorphic curves s.t. energy$(u_t) \mapsto \infty$ as $t \mapsto a \in (0,1)$ and s.t. there are no $J$-holomorphic curves for $t \geq a$. This potential phenomenon is an analogue for holomorphic curves, of (blue) sky catastrophes discovered by Fuller [12], for closed orbits of dynamical systems. And which we already mentioned above in the context of Reeb vector fields.

*Remark* 1.5. One way to sidestep the above difficulties in Gromov-Witten theory is to construct an lcs homology theory. Most geometric ingredients for this are already contained in this paper. For example generators should be Reeb 2-curves/elliptic Reeb 2-curves, which are part of the statement of the CSW conjecture.

## 2. Introduction: Background and Statements

To see the connection with the first cohomology group $H^1(M, \mathbb{R})$, mentioned above, let us point out right away the most basic invariant of a lcs structure $\omega$, when $M$ has dimension at least 4. This is the Lee class, $\alpha = \alpha_\omega \in H^1(M, \mathbb{R})$. This class has the property that on the associated $\alpha$-covering space (see proof of Theorem 3.5) $\widetilde{M}$, the lift $\widetilde{\omega}$ is globally conformally symplectic. Thus, an lcs form is globally conformally symplectic, that is diffeomorphic to $f \cdot \omega'$, with $\omega'$ symplectic, $f > 0$, iff its Lee class vanishes.

Again assuming $M$ has dimension at least 4, the Lee class $\alpha$ has a natural differential form representative, called the Lee form, which is defined as follows. We take a cover of $M$ by open sets $U_a$ in which $\omega = f_a \cdot \omega_a$ for $\omega_a$ symplectic, and $f_a$ a positive smooth function. Then we have 1-forms $d(\ln f_a)$ in each $U_a$, which glue to a well-defined closed 1-form on $M$, as shown by Lee. We may denote this 1-form and its cohomology class both by $\alpha$. It is moreover immediate that for an lcs form $\omega$,

$$d\omega = \alpha \wedge \omega,$$

for $\alpha$ the Lee form as defined above.

As we mentioned lcs manifolds can also be understood to generalize contact manifolds. This works as follows. First we have a class of explicit examples of lcs manifolds, obtained by starting with a symplectic cobordism (see [8]) of a closed contact manifold $C$ to itself, arranging for the contact forms at the two ends of the cobordism to be proportional and then gluing the boundary components, (after a global conformal rescaling of the form on the cobordism, to match the boundary conditions).

**Terminology 1.** For us a contact manifold is a pair $(C, \lambda)$ where $C$ is a closed manifold and $\lambda$ a contact form: $\forall p \in C : \lambda \wedge \lambda^{2n}(p) \neq 0$. This is not a completely common terminology as classically it is the equivalence class of $(C, \lambda)$ that is called a contact manifold, where $(C, \lambda) \sim (C, \lambda')$ if $\lambda = f\lambda'$ for $f$ a positive function. (Given that the contact structure, in the classical sense, is co-oriented.) A ***contactomorphism*** between $(C_1, \lambda_1)$, $(C_2, \lambda_2)$ is a diffeomorphism $\phi : C_1 \to C_2$ s.t. $\phi^* \lambda_2 = f\lambda_1$ for some $f > 0$. It is called ***strict*** if $\phi^* \lambda_2 = \lambda_1$.

Another basic example, which can be understood as a special case of the above cobordism construction, is the following.

*Example* 1 (Banyaga). Let $(C, \lambda)$ be a contact manifold, $S^1 = \mathbb{R}/\mathbb{Z}$, $d\theta$ the standard non-degenerate 1-form on $S^1$ satisfying $\int_{S^1} d\theta = 1$. And take $M = C \times S^1$ with the 2-form

$$\omega_\lambda = d_\alpha \lambda := d\lambda - \alpha \wedge \lambda,$$

for $\alpha := pr_{S^1}^* d\theta$, $pr_{S^1} : C \times S^1 \to S^1$ the projection, and $\lambda$ likewise the pull-back of $\lambda$ by the projection $C \times S^1 \to C$. We call $(M, \omega_\lambda)$ as above the ***lcs-fication*** of $(C, \lambda)$. This is also a basic example of a first kind lcs manifold, as in Definition 2.3 ahead.

The operator

$$d_\alpha : \Omega^k(M) \to \Omega^{k+1}(M) \tag{2.1}$$

is called the Lichnerowicz differential with respect to a closed 1-form $\alpha$, and it satisfies $d_\alpha \circ d_\alpha = 0$ so that we have an associated Lichnerowicz chain complex.

We assume from now on, unless explicitly stated otherwise, that our lcs manifolds have dimension at least 4.

**Definition 2.2.** *An* **exact lcs form** *on $M$ is an lcs 2-form s.t. there exists a pair of one forms $(\lambda, \alpha)$ with $\alpha$ a closed 1-form, s.t. $\omega = d_\alpha \lambda$ is non-degenerate. In the case above we also call the pair $(\lambda, \alpha)$* **an exact lcs structure**. *The triple $(M, \lambda, \alpha)$ will be called an* **exact lcs manifold**, *but we may also call $(M, \omega)$ an exact lcs manifold when $(\lambda, \alpha)$ are implicit.*

An exact lcs structure determines a generalized distribution $\mathcal{V}_\lambda$ on $M$:

$$\mathcal{V}_\lambda(p) = \{v \in T_pM | d\lambda(v, \cdot) = 0\},$$

which we call the ***vanishing distribution***. We also define a generalized distribution $\xi_\lambda$ that is the $\omega$-orthogonal complement to $\mathcal{V}_\lambda$, which we call ***co-vanishing distribution***. For each $p \in M$, $\mathcal{V}_\lambda(p)$ has dimension at most 2 since $d\lambda - \alpha \wedge \lambda$ is non-degenerate. If $M^{2n}$ is closed $\mathcal{V}_\lambda$ cannot identically vanish since $(d\lambda)^n$ cannot be non-degenerate by Stokes theorem.

**Definition 2.3.** *Let $(\lambda, \alpha)$ be an exact lcs structure on $M$. We call $\alpha$ integral, rational or irrational if its periods are integral, respectively rational, respectively irrational. We call the structure $(\lambda, \alpha)$* **scale integral**, *if $c\alpha$ is integral for some $0 \neq c \in \mathbb{R}$. Otherwise we call the structure* **scale irrational**. *If $\mathcal{V}$ is non-zero at each point of $M$, in particular is a smooth 2-distribution, then such a structure is called* **first kind**. *If $\omega$ is an exact lcs form then we call $\omega$ integral, rational, irrational, first kind if the exists $\lambda, \alpha$ s.t. $\omega = d_\alpha \lambda$ and $(\lambda, \alpha)$ is integral, respectively irrational, respectively first kind. Similarly define, scale integral, scale irrational $\omega$.*

**Definition 2.4.** *A* **conformal symplectomorphism** *of lcs manifolds $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ is a diffeomorphism $\phi$ s.t. $\phi^*\omega_2 = e^f \omega_1$ where $f$ is some function on $M$. Note that in this case we have an induced relation:*

$$\phi^*\alpha_1 = \alpha_0 + df,$$

*where $\alpha_1$ is the Lee form of $\omega_1$ and $\alpha_0$ is the Lee form of $\omega_0$. If $f = 0$ we call $\phi$ a* **symplectomorphism**. *A (conformal) symplectomorphism of exact lcs structures $(\lambda_1, \alpha_1)$, $(\lambda_2, \alpha_2)$ on $M_1$ respectively $M_2$ is a (conformal) symplectomorphism of the corresponding lcs 2-forms. If a diffeomorphism $\phi : M_1 \to M_2$ satisfies $\phi^*\lambda_2 = \lambda_1$ and $\phi^*\alpha_2 = \alpha_1$ we call it an* **isomorphism** *of the exact lcs structures. This is analogous to a strict contactomorphism of contact manifolds.*

To summarize, with the above notions we have the following basic points whose proof is left to the reader:

1. An isomorphism of exact lcs structures $(\lambda_1, \alpha_1)$, $(\lambda_2, \alpha_2)$ preserves the first kind condition, and moreover preserves the corresponding vanishing distributions.
2. A (conformal) symplectomorphism of exact lcs structures generally does not preserve the first kind condition.
3. A (conformal) symplectomorphism of first kind lcs structures generally does not preserve the vanishing distributions. (Similar to a contactomorphism not preserving Reeb distributions.)
4. A symplectomorphism of lcs forms preserves the first kind condition. (Almost tautologically.)
5. A conformal symplectomorphism of lcs forms and exact lcs structures preserves the rationality, integrality, scale integrality conditions.

*Remark* 2.5. It is important to note that for us the form $\omega$ is the structure not its conformal equivalence class, as for some authors. Where $\omega_0$ is conformally equivalent to $\omega_1$ if $\omega_1 = f\omega_0$ for some $f > 0$ (i.e. the identity map is a conformal symplectomorphism $id : (M, \omega_0) \to (M, \omega_1)$.) In other words conformally equivalent structures on a given manifold determine distinct but isomorphic objects of the category, whose objects are lcs manifolds and morphisms conformal symplectomorphisms.

One example of an lcs structure of the first kind is a mapping torus of a strict contactomorphism, see Banyaga [2]. The mapping tori $M_{\phi,c}$ of a strict contactomorphism $\phi$ of $(C, \lambda)$ fiber over $S^1$,

$$\pi : C \hookrightarrow M_\phi \to S^1,$$

with Lee form of the type $\alpha = c\pi^*(d\theta)$, for some $0 \neq c \in \mathbb{R}$. In particular, these are scale integral first kind lcs structures. Moreover we have:

**Theorem 2.6** (Reformulating [3])**.** *A first kind lcs structure $(\lambda, \alpha)$ on a closed manifold $M$ is isomorphic to a mapping torus of a strict contactomorphism if and only if it is scale integral.*

The (scaled) integrality condition is of course necessary since the Lee form of a mapping torus a strict contactomorphism will have this property. Thus we may understand scale irrational first kind lcs structures as first (and rather dramatic) departures from the world of contact manifolds into a brave new lcs world.

*Remark* 2.7. Note that scale irrational first kind structures certainly exist. A simple example is given by taking $\lambda, \alpha$ to be closed scale irrational 1-forms on $T^2$ with transverse kernels. Then $\omega = \lambda \wedge \alpha$ is a scale irrational first kind structure on $T^2$. In particular $(\lambda, \alpha)$ cannot be a mapping torus of a strict contactomorphism even up to a conformal symplectomorphism. In general, on a closed manifold we may always perturb a (first kind) integral lcs structure to a (first kind) scale irrational one. It appears that scale irrational first kind structures have not been much studied, (and which we also only touch on).

2.1. **Conformal symplectic Weinstein conjecture.** As previously mentioned lcs manifolds can be understood to generalize both symplectic and contact manifolds. There are two very influential conjectures in these two respective areas: the Arnold conjecture and the Weinstein conjecture. The statement of Arnold conjecture on fixed points of a symplectomorphisms can be directly generalized to lcs manifolds, but it is very easy to construct counterexamples using Banyaga's example above: there are Hamiltonian conformal symplectomorphisms of the lcs-fication of the standard contact $S^3$, with no fixed points. (We leave this to the reader as an exercise.) For one further discussion of the Arnold conjecture in this context see [5]. We are instead interested here in extending the Weinstein conjecture.

**Definition 2.8.** *Let $(M, \lambda, \alpha)$ be an exact* lcs *structure and $\omega = d_\alpha \lambda$. Define $X_\lambda$ by $\omega(X_\lambda, \cdot) = \lambda$ and $X_\alpha$ by $\omega(X_\alpha, \cdot) = \alpha$. Let $\mathcal{D}$ denote the (generalized) distribution spanned by $X_\alpha, X_\lambda$, meaning $\mathcal{D}(p) := \operatorname{span}(X_\alpha(p), X_\lambda(p))$. This will be called the* **canonical distribution**.

The (generalized) distribution $\mathcal{D}$ is one analogue for exact lcs manifolds of the Reeb distribution on contact manifolds. A Reeb 2-curve, as defined ahead, will be a certain kind of singular leaf of $\mathcal{D}$, and so is a kind of 2-dimensional analogue of a Reeb orbit.

*Example* 2. The simplest example of a Reeb 2-curve, in the case $\mathcal{D}$ is a true 2-dimensional distribution (for example if $(\lambda, \alpha)$ is first kind), is a closed immersed surface $u : \Sigma \to M$ tangent to $\mathcal{D}$. However, it will be necessary to consider more generalized curves.

**Definition 2.9.** *Let $\Sigma$ be a closed nodal Riemann surface (the set of nodes is empty or not). Let $u : \Sigma \to M$ be a smooth map and let $\widetilde{u} : \widetilde{\Sigma} \to M$ be its normalization (see Definition 6.2). We say that $u$ is a* **Reeb 2-curve** *in $(M, \lambda, \alpha)$, if the following is satisfied:*

(1) *For each $z \in \widetilde{\Sigma}$, $\widetilde{u}_*(T_z\widetilde{\Sigma}) = \mathcal{D}(\widetilde{u}(z))$, whenever $d\widetilde{u}(z) : T_z\Sigma \to T_{\widetilde{u}(z)}M$ is non-singular, and* $\dim \mathcal{D}(\widetilde{u}(z)) = 2$.

(2) $0 \neq [u^*\alpha] \in H^1(\Sigma, \mathbb{R})$.

(3) *The set of critical points of $\widetilde{u}$ is finite.*

**Conjecture 1** (CSW conjecture)**.** *A closed exact lcs manifold $(M, \lambda, \alpha)$ has a Reeb 2-curve.*

**Theorem 2.10.** *Conjecture 1 implies the Weinstein conjecture: every closed contact manifold $(C, \lambda)$ has a closed Reeb orbit.*

The following is very elementary, see Section 6.

**Theorem 2.11.** *CSW conjecture holds for closed surfaces.*

To give more examples, we quickly introduce the relevant spaces of lcs structures.

**Definition 2.12.** *Define* **the set $\mathcal{L}(M)$ of exact** lcs **structures on** *$M$, to be:*

$$\mathcal{L}(M) = \{(\beta, \gamma) \in \Omega^1(M) \times \Omega^1(M) \,|\, \gamma \text{ is closed, } d_\gamma\beta \text{ is non-degenerate}\}.$$

*Define $\mathcal{F}(M) \subset \mathcal{L}(M)$ to be subset of (possibly irrational) first kind lcs structures.*

In what follows we use the following $C^k$ metric on $\mathcal{L}(M)$. For $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in \mathcal{L}(M)$ define:

$$d_k((\lambda_1, \alpha_1), (\lambda_2, \alpha_2)) = d_{C^k}(\lambda_1, \lambda_2) + d_{C^k}(\alpha_1, \alpha_2), \tag{2.13}$$

where $d_{C^k}$ on the right side is the usual $C^k$ metric.

The following theorems are proved in Section 6, based on the theory of elliptic pseudo-holomorphic curves in $M$.

**Theorem 2.14.** *Let $(C, \lambda)$ be a contact manifold, with at least one non-degenerate Reeb orbit. Or more generally satisfying $i(N, R^\lambda, \beta) \neq 0$ where the latter is the Fuller index of some good set $N$ of orbits of $R^\lambda$, in class $\beta$ see Appendix A. Then we have the following:*

(1) *Conjecture 1 holds for some $d_3$ neighborhood $U$ of the* lcs*-fication $(\lambda, \alpha)$ of the space $\mathcal{F}(M = C \times S^1)$.*

(2) *For $(\lambda, \alpha) \in U$, the corresponding Reeb 2-curve $u : \Sigma \to M$ can be assumed to be* **elliptic** *meaning that $\Sigma$ is elliptic (more specifically: a nodal, topological genus 1, closed, connected Riemann surface).*

(3) *$u$ can also be assumed to be charge 1 (see Definition 3.3).*

(4) *If $M$ has dimension 4 then $u$ can be assumed to be embedded and normal (the set of nodes is empty). And so, in particular, represents a closed, $(\omega = d_\alpha\lambda)$-symplectic torus hypersurface.*

**Definition 2.15.** *We say that a pair of exact lcs forms $(\lambda_0, \alpha_0)$, $(\lambda_1, \alpha_1)$ on $M$ are* **formally homotopic** *if there is a $C^0$ continuous family of non-degenerate 2-forms $\{\omega_t\}$, $t \in [0, 1]$, with $\omega_0 = d_{\alpha_0}\lambda_0$, $\omega_1 = d_{\alpha_1}\lambda_1$.*

**Theorem 2.16.** *Let $(C, \lambda)$ be a contact manifold s.t. the extended Fuller index of the Reeb vector field satisfies*

$$i(R^\lambda, \beta) \neq 0 \in \mathbb{Q} \sqcup \{\pm\infty\},$$

*for some class $\beta$, (see Appendix A.) Let $(M, \lambda, \alpha)$ be the lcs-fication of $(C, \lambda)$. Then either there is an elliptic $\alpha$-charge 1 Reeb 2-curve for every first kind* lcs *structure on $M$, formally homotopic to $\omega_0 = d_\alpha \lambda$, or stable holomorphic sky catastrophes exist, (the latter are further discussed in Section 3.3). In addition, in the case of the first alternative, if $M$ has dimension 4 then the elliptic Reeb 2-curve can be assumed to be normal and embedded.*

*Example* 3. Take $C = S^{2k+1}$ and $\lambda = \lambda_H$ the standard, or as we call it Hopf, contact form. We call its lcs-fication the ***standard or Hopf lcs structure*** on $C \times S^{2k+1}$. Then $i(R^{\lambda_H}, 0) = \pm\infty$, (sign depends on $k$), [27]. Or take $C$ to be unit cotangent bundle of a hyperbolic Riemannian manifold $(X, g)$, $\lambda$ the associated Liouville form, and $(\lambda, \alpha)$ the lcs-fication. In this case $i(R^\lambda, \beta) = \pm 1$ for every $\beta \neq 0$.

The above examples motivate us to state a special version of this conjecture for first kind lcs manifolds.

**Conjecture 2.** *Let $(M, \lambda, \alpha)$ be a closed first kind lcs manifold, then there is an elliptic Reeb 2-curve in $M$. In addition if $M$ has dimension 4 then the elliptic Reeb 2-curve can be assumed to be normal and embedded.*

Note that in the latter case, we in particular get that $M$ admits a torus symplectic hypersurface, (with additional properties). In symplectic context there are various existence results for symplectic hypersurfaces, the most basic is the theorem of Donaldson, saying that every closed integral symplectic manifold admits a closed symplectic hypersurface.

We will call the conjecture above: ***elliptic Weinstein conjecture***, and we will discuss some consequences of this in Section 2.3 and Section 2.4.

2.2. **Fuller vs Gromov-Witten.** As indicated by the above, there is a connection of the classical Fuller index with the Gromov Witten theory. In a very particular situation this relationship becomes perfect as, Theorem 5.5 equates the (extended) Fuller index for Reeb vector fields on a contact manifold $C$, to a certain (extended) genus 1 Gromov-Witten invariant of the Banyaga lcs manifold $C \times S^1$, see Example 1. The latter also gives a conceptual interpretation for why the Fuller index is rational, as it is reinterpreted as an (virtual) orbifold Euler number.

2.3. **Applications to Reeb dynamics.**

**Theorem 2.17.** *Assume elliptic Weinstein conjecture for rational first kind lcs manifolds. Let $\phi : (C, \lambda) \to (C, \lambda)$ be a strict contactomorphism of a closed contact manifold. Then there is an $n > 0$, s.t. $\phi^n$ has a fixed Reeb string. In other words, a closed $\lambda$-Reeb orbit $o$ s.t.:*

$$\operatorname{image} \phi^n(o) = \operatorname{image} o.$$

*Remark* 2.18. It is not hard to see that allowing $n \neq 1$ is necessary. Take $\lambda$ to be the perturbation of the Hopf contact form on $S^3$, with two geometrically distinct simple Reeb orbits. If the perturbation is well chosen there is a strict contactomorphism taking one orbit to the other, so that we need $n = 2$ in this case.

We can't say anything at the moment in the case the contactomorphism is not strict. However, in light of the Sandon conjecture on translated points of contactomorphisms, [26], the theorem is not obviously false in non-strict case. Since Sandon's conjecture in particular implies that any contactomorphism of a closed contact manifold fixes some possibly non-closed Reeb orbit. We further note however that the general form of Sandon's conjecture has counterexamples on $S^{2k+1}$, Cant [4].

We have already discussed, in the first part of the introduction, one more concrete example of this theorem, that is Theorem 1.4. We now give a more general version of Theorem 1.3.

**Definition 2.19.** *We say that a pair of strict contactomorphisms $\phi_1, \phi_2 : (C, \lambda_1) \to (C, \lambda_2)$ are* **formally homotopic** *if the exact lcs structures on the mapping tori $M_{\phi_1}, M_{\phi_2}$ (see proof of Theorem 2.17) are formally homotopic.*

**Theorem 2.20.** *Let $(C, \lambda)$ be a closed contact manifold. Suppose that the extended Fuller index satisfies:*

$$i(R^\lambda, \beta) \neq 0 \in \mathbb{Q} \sqcup \{\pm\infty\},$$

*for some class $\beta$. Then for any strict contactomorphism $\phi : (C, \lambda) \to (C, \lambda)$ formally homotopic to the identity, either $\phi$ has a fixed Reeb string (in class $\beta$) or stable holomorphic sky catastrophes exist.*

### 2.4. Reeb 1-curves.

**Definition 2.21.** *We say that a smooth map $o : S^1 \to M$ is a* **Reeb 1-curve** *in an exact lcs manifold $(M, \lambda, \alpha)$ if*

$$\forall t \in S^1 : (\lambda(o'(t)) > 0) \wedge (o'(t) \in \mathcal{D}).$$

The CSW conjecture has implications for existence of Reeb 1-curves.

**Theorem 2.22.** *Suppose that $(M, \lambda, \alpha)$ is a closed exact lcs manifold satisfying the "*Reeb condition*" $\lambda(X_\alpha) > 0$. If $(M, \lambda, \alpha)$ has an immersed Reeb 2-curve then it also has a Reeb 1-curve. Furthermore, if it has an immersed elliptic Reeb 2-curve, then this curve is normal.*

We have an immediate corollary of Theorem 2.14 and Theorem 2.22.

**Corollary 2.23.** *Let $\lambda$ be a contact form, on closed 3-manifold $C$, with at least one non-degenerate Reeb orbit, or more generally satisfying $i(N, R^\lambda, \beta) \neq 0$. Then there is a $d_3$ neighborhood $U$ of the* lcs-*fication $(\lambda, \alpha)$ in the space $\mathcal{F}(M = C \times S^1)$, s.t. for each $(\lambda', \alpha') \in U$ there is a Reeb 1-curve.*

As we have seen, the ideas behind the CSW conjecture already give some dynamical style applications. One future problem of more geometric character we have in mind is to develop non-squeezing in lcs geometry, cf. [29] and also Oh-Savelyev [22]. The contact non-squeezing [7] is closely connected to the theory or Reeb orbits, so that it is likely that in the lcs analogue we will need something like Reeb 2-curves, whose theory is developed here.

## 3. Pseudoholomorphic curves in lcs manifolds

First kind lcs manifolds give immediate examples of almost complex manifolds where the $L^2$ energy functional is unbounded on the moduli spaces of fixed class $J$-holomorphic curves, as well as where null-homologous $J$-holomorphic curves can be non-constant. We are going to see this shortly after developing a more general theory.

**Definition 3.1.** *Let $(M, \lambda, \alpha)$ be an exact lcs manifold, satisfying the* **Reeb condition***: $\omega(X_\lambda, X_\alpha) = \lambda(X_\alpha) > 0$, where $\omega = d_\lambda \alpha$. In this case, $\mathcal{D}$ is a 2-dimensional distribution, and we say that an $\omega$-compatible $J$ is $\omega$*-**admissible** *if:*

- *$J$ preserves the canonical distribution $\mathcal{D}$ and preserves the $\omega$-orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$. That is $J(V) \subset \mathcal{D}$ and $J(\mathcal{D}^\perp) \subset \mathcal{D}^\perp$.*

- *$d\lambda$ tames $J$ on $\mathcal{D}^\perp$.*

*Admissible $J$ exist by classical symplectic geometry, and the space of such $J$ is contractible see [19]. We call $(\lambda, \alpha, J)$ as above a* **tamed exact** lcs **structure***, and $(\omega, J)$ is called a tamed exact* lcs *structure if $\omega = d_\alpha \lambda$, for $(\lambda, \alpha, J)$ a tamed exact* lcs *structure. In this case $(M, \omega, J)$, $(M, \lambda, \alpha, J)$ will be called a* **tamed exact** lcs **manifold***.*

*Example* 4. If $(M, \lambda, \alpha)$ is first kind then $\omega(X_\lambda, X_\alpha) = 1$ everywhere. In particular, we may find a $J$ such that $(\lambda, \alpha, J)$ is a tamed exact lcs structure, and the space of such $J$ is contractible. We will call $(M, \lambda, \alpha, J)$ a ***tamed first kind*** lcs manifold.

**Lemma 3.2.** *Let $(M, \lambda, \alpha, J)$ be a tamed first kind lcs manifold. Then given a smooth $u : \Sigma \to M$, where $\Sigma$ is a closed (nodal) Riemann surface, $u$ is $J$-holomorphic only if*

$$\operatorname{image} d\widetilde{u}(z) \subset \mathcal{V}_\lambda(\widetilde{u}(z))$$

*for all $z \in \widetilde{\Sigma}$, where $\widetilde{u} : \widetilde{\Sigma} \to M$ is the normalization of $u$ (see Definition 6.2). In particular $\widetilde{u}^* d\lambda = 0$.*

*Proof.* As previously observed, by the first kind condition, $\mathcal{V}_\lambda$ is the span of $X_\lambda, X_\alpha$ and hence

$$V := \mathcal{V}_\lambda = \mathcal{D}_\lambda.$$

Let $u$ be $J$-holomorphic, so that $\widetilde{u}$ is $J$-holomorphic (by definition of a $J$-holomorphic nodal map). We have

$$\int_\Sigma \widetilde{u}^* d\lambda = 0$$

by Stokes theorem. Let $proj(p) : T_pM \to V^\perp(p)$ be the projection induced by the splitting $TM = V \oplus V^\perp$. Then if for some $z \in \widetilde{\Sigma}$, $proj \circ d\widetilde{u}(z) \neq 0$, since $J$ is tamed by $d\lambda$ on $V^\perp$ and since $J$ preserves the splitting, we would have $\int_{\widetilde{\Sigma}} \widetilde{u}^* d\lambda > 0$. Thus,

$$\forall z \in \widetilde{\Sigma} : proj \circ d\widetilde{u}(z) = 0,$$

so

$$\forall z \in \widetilde{\Sigma} : \operatorname{image} d\widetilde{u}(z) \subset \mathcal{V}_\lambda(\widetilde{u}(z)).$$

□

*Example* 5. Let $(C \times S^1, \lambda, \alpha)$ be the lcs-fication of a contact manifold $(C, \lambda)$. In this case

$$X_\alpha = (R^\lambda, 0),$$

where $R^\lambda$ is the Reeb vector field and

$$X_\lambda = (0, \frac{d}{d\theta})$$

is the vector field generating the natural action of $S^1$ on $C \times S^1$.

If we denote by $\xi \subset T(C \times S^1)$ the distribution $\xi(p) = \ker \lambda(p)$, then in this case $\xi = V^\perp$ in the notation above.

We take $J$ to be an almost complex structure on $\xi$, which is $S^1$ invariant, and compatible with $d\lambda$. The latter means that

$$g_J(\cdot, \cdot) := d\lambda|_\xi(\cdot, J\cdot)$$

is a $J$ invariant Riemannian metric on the distribution $\xi$.

There is an induced almost complex structure $J^\lambda$ on $C \times S^1$, which is $S^1$-invariant, coincides with $J$ on $\xi$ and which satisfies:

$$J^\lambda(X_\alpha) = X_\lambda.$$

Then $(C \times S^1, \lambda, \alpha, J^\lambda)$ is a tamed first kind lcs manifold.

3.1. **Moduli of Pseudo-holomorphic curves in an lcs manifold.** We now study moduli spaces of elliptic curves, constrained to have a certain charge, in a lcs manifold. There is a somewhat related notion of charge appearing in Oh-Wang [23], in the context of contact instantons. Partly, the reason for introduction of "charge" for curves is that it is now possible for non-constant holomorphic curves to be null-homologous, so we need additional control. Here is a simple example: take $S^3 \times S^1$ with $J = J^\lambda$, for the $\lambda$ the standard contact form, then all the Reeb holomorphic tori (as defined further below) are null-homologous. In some cases we can just work with homology classes $A \neq 0$, and ignore charge conditions, but in many of our examples $A = 0$.

Let $\Sigma$ be a complex torus with a chosen marked point $z \in \Sigma$, i.e. an elliptic curve over $\mathbb{C}$. An isomorphism $\phi : (\Sigma_1, z_1) \to (\Sigma_2, z_2)$ is a biholomorphism s.t. $\phi(z_1) = z_2$. The set of isomorphism classes forms a smooth orbifold $M_{1,1}$. This has a natural compactification - the Deligne-Mumford compactification $\overline{M}_{1,1}$, by adding a point at infinity, corresponding to a nodal genus 1 curve with one node.

The notion of charge can be defined in a general setting.

**Definition 3.3.** *Let $M$ be a manifold endowed with a closed 1-form $\alpha$. Let $u : T^2 \to M$ be a continuous map. Let $\gamma, \rho$ be a basis of $H_1(T^2, \mathbb{Z})$ with $\gamma \cdot \rho = 1$, where $\cdot$ is the intersection pairing with respect to the standard complex orientation on $T^2$. Suppose in addition:*

$$\langle \gamma, u^* \alpha \rangle = 0,$$

*where $\langle , \rangle$ is the natural pairing of homology and cohomology. Then we call*

$$n = |\langle \rho, u^* \alpha \rangle| \in \mathbb{R}_{\geq 0},$$

*the $\alpha$-**charge of** $u$, or just the charge of $u$ when $\alpha$ is implicit.*

It is easy to see that charge is always defined and is independent of choices above. We may extend the definition of charge to curves $u : \Sigma \to M$, with $\Sigma$ a nodal elliptic curve. There are a number of equivalent ways of doing this, for example we may use the standard gluing construction at the node, and define the charge of $u$ to be the charge of the (Gromov metric topology) nearby smooth curve $u : \Sigma' \to M$, with $\Sigma'$ a non-nodal elliptic curve. Equivalently, we may find piecewise smooth transverse curves $\gamma : S^1 \to \Sigma$, $\rho : S^1 \to \Sigma$, with $\rho$ representing generator of $H_1(\Sigma, \mathbb{Z})$ and satisfying $\gamma \cdot \rho = 1$ (the geometric intersection number). In this case, define charge of $u$ to be $|\langle \rho, u^* \alpha \rangle|$.

In either case, it is obvious that the charge condition is preserved under Gromov convergence of stable maps.

Let $(M, J)$ be an almost complex manifold and $\alpha$ a closed 1-form on $M$ non vanishing in cohomology, then we call $(M, J, \alpha)$ a ***Lee manifold***. Suppose for the moment that there are no non-constant $J$-holomorphic maps $(S^2, j) \to (M, J)$ (otherwise we need stable maps), then we define:

$$\overline{\mathcal{M}}^n_{1,1}(J, A)$$

as the set of equivalence classes of tuples $(u, S)$, for $S = (\Sigma, z)$ a possibly nodal elliptic curve and $u : \Sigma \to M$ a charge $n$ $J$-holomorphic map. The equivalence relation is $(u_1, S_1) \sim (u_2, S_2)$ if there is an isomorphism $\phi : S_1 \to S_2$ s.t. $u_2 \circ \phi = u_1$. As such an isomorphism of course preserves charge, the charge is well defined on equivalence classes.

By slight abuse we may just denote such an equivalence class above simply by $u$, so we may write $u \in \overline{\mathcal{M}}^n_{1,1}(J, A)$, with $S$ implicit.

3.2. **Reeb holomorphic tori in** $(C \times S^1, J^\lambda)$**.** Let $(C, \lambda)$ be a contact manifold and let $\alpha$ and $J^\lambda$ be as in Example 5. So that in particular we get a Lee manifold $(C \times S^1, J^\lambda, \alpha)$.

In this case we have one natural class of charge 1 $J^\lambda$-holomorphic tori in $C \times S^1$. Let $o$ be a period $c$, closed Reeb orbit $o$ of $R^\lambda$. A ***Reeb torus*** $u_o$ for $o$, is the map

$$u_o : (S^1 \times S^1 = T^2) \to C \times S^1$$
$$u_o(s,t) = (o(s), t).$$

A Reeb torus is $J^\lambda$-holomorphic for a uniquely determined holomorphic structure $j$ on $T^2$ defined by:

$$j(\frac{\partial}{\partial s}) = c\frac{\partial}{\partial t}.$$

Let $\mathcal{O}(R^\lambda)$ as before denote the space of general period, unparametrized closed $\lambda$-Reeb orbits. We have a map:

$$\mathcal{P} : \mathcal{O}(R^\lambda) \to \overline{\mathcal{M}}^1_{1,1}(J^\lambda, A), \quad \mathcal{P}(o) = u_o.$$

**Proposition 3.4.** *The map $\mathcal{P}$ is a bijection.* [1] *(Note that there is an analogous bijection $\mathcal{O}(R^\lambda) \to \overline{\mathcal{M}}^n_{1,1}(J^\lambda, A)$, for $n > 1$.)*

So in the particular case of $J^\lambda$, as above, the domains of elliptic curves in $C \times S^1$ are "rectangular". That is, they are quotients of the complex plane by a rectangular lattice. For more general almost complex structures we cannot expect this, the domain almost complex structure on our curves can in principle be arbitrary, in particular we might have nodal degenerations. Also note that the expected dimension of $\overline{\mathcal{M}}^1_{1,1}(J^\lambda, A)$ is 0. It is given by the Fredholm index of the operator (5.2) which is 2, minus the dimension of the reparametrization group (for non-nodal curves) which is 2. That is given an elliptic curve $S = (\Sigma, z)$, let $\mathcal{G}(\Sigma)$ be the 2-dimensional group of biholomorphisms $\phi$ of $\Sigma$. Then given a $J$-holomorphic map $u : \Sigma \to M$, $(\Sigma, z, u)$ is equivalent to $(\Sigma, \phi(z), u \circ \phi)$ in $\overline{\mathcal{M}}^1_{1,1}(J^\lambda, A)$, for $\phi \in \mathcal{G}(\Sigma)$.

The following is an elementary result, to be proved in Section 5. It will be a crucial ingredient for us.

**Theorem 3.5.** *Let $(M, \lambda, \alpha, J)$ be a tamed first kind* lcs *manifold with $M$ closed. Then every non-constant (nodal) $J$-holomorphic curve $u : \Sigma \to M$ is a Reeb 2-curve.*

3.2.1. *Connection with the extended Fuller index.* Another important ingredient is a connection of the extended Fuller index with certain extended Gromov-Witten invariants. If $\beta$ is a free homotopy class of a loop in $C$ set

$$A_\beta = [\beta] \otimes [S^1] \in H_2(C \times S^1).$$

**Theorem 3.6.** *Suppose that $\lambda$ is a contact form on a closed manifold $C$, such that its Reeb flow is definite type, see Appendix A, then*

$$GW^1_{1,1}(A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(R^\lambda, \beta),$$

*where the left hand side is a certain extended genus 1 Gromov-Witten invariant of charge 1 curves, and the right hand side extended Fuller index. Both sides are extended rational numbers $\mathbb{Q} \sqcup \{\pm\infty\}$. So that in particular if either side does not vanish then there are $\lambda$-Reeb orbits in class $\beta$.*

[1] It is in fact an equivalence of the corresponding topological action groupoids, but we do not need this explicitly.

What about higher genus invariants of $C \times S^1$? Following the proof of Proposition 3.4, it is not hard to see that all $J^\lambda$-holomorphic curves must be branched covers of Reeb tori. If one can show that these branched covers are regular when the underlying tori are regular, the calculation of invariants would be fairly automatic from this data. See [36], [34] where these kinds of regularity calculation are made.

3.3. **Holomorphic sky catastrophes.** The following is well known.

**Theorem 3.7.** *[[20, Proposition 4.1.4]] Let $(M, J)$ be a compact almost complex manifold, and $u : (S^2, j) \to M$ a $J$-holomorphic map. Given a Riemannian metric $g$ on $M$, there is an $\hbar = \hbar(g, J) > 0$ s.t. if $e_g(u) < \hbar$ then $u$ is constant, where $e_g$ is the $L^2$-energy functional,*

$$e_g(u) = \text{energy}_g(u) = \int_{S^2} |du|^2 dvol.$$

Using this we get the following (trivial) extension of Gromov compactness. Let

$$\mathcal{M}_{g,n}(J, A) = \mathcal{M}_{g,n}(M, J, A)$$

denote the moduli space of isomorphism classes of class $A$, $J$-holomorphic curves in $M$, with domain a genus $g$ closed Riemann surface, with $n$ marked labeled points. Here an isomorphism between $u_1 : \Sigma_1 \to M$, and $u_2 : \Sigma_2 \to M$ is a biholomorphism of marked Riemann surfaces $\phi : \Sigma_1 \to \Sigma_2$ s.t. $u_2 \circ \phi = u_1$.

**Notation 1.** We will often say ***$J$-curve*** in place of $J$-holomorphic curve.

The following is proved by the same argument as [19, Theorem 5.6.6].

**Theorem 3.8.** *Let $(M, J)$ be an almost complex manifold. Then $\mathcal{M}_{g,n}(J, A)$ has a pre-compactification*

$$\overline{\mathcal{M}}_{g,n}(J, A),$$

*by Kontsevich stable maps, with respect to the natural metrizable Gromov topology see for instance [19, Chapter 5.6], for genus 0 case, [24] for general case. Moreover, given $E > 0$, the subspace $\overline{\mathcal{M}}_{g,n}(J, A)_E \subset \overline{\mathcal{M}}_{g,n}(J, A)$ consisting of elements $u$ with $e(u) \leq E$ is compact, where $e$ is the $L^2$ energy with respect to an auxiliary metric. In other words $e$ is a proper function.*

Thus, the most basic situation where we can talk about Gromov-Witten "invariants" of $(M, J)$ is when the energy function is bounded on $\overline{\mathcal{M}}_{g,n}(J, A)$, and we shall say that $J$ is ***bounded*** (in class $A$), later on we generalize this in terms of what we call ***finite type***. In this case $\overline{\mathcal{M}}_{g,n}(J, A)$ is compact, and has a virtual moduli cycle as in the original approach of Fukaya-Ono [11], or the more algebraic approach [24]. So we may define functionals:

$$GW_{g,n}(A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}, \tag{3.9}$$

where $\overline{M}_{g,n}$ denotes the compactified moduli space of Riemann surfaces. Of course symplectic manifolds with any tame almost complex structure is one class of examples, another class of examples comes from some locally conformally symplectic manifolds. (We can take for instance the lcs-fication of $(C, \lambda)$ with the latter the unit cotangent bundle of a hyperbolic manifold, with $\lambda$ the canonical Liouville form, and $J = J^\lambda$ as in Example 5).

Given a continuous in the $C^\infty$ topology family $\{J_t\}$, $t \in [0, 1]$ we denote by $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ the space of pairs $(u, t)$, $u \in \overline{\mathcal{M}}_{g,n}(J_t, A)$.

**Definition 3.10.** *We say that a continuous family $\{J_t\}$, $t \in [0, 1]$ on a compact manifold $M$ has a* **holomorphic sky catastrophe** *in class $A$ if there is an element $u \in \overline{\mathcal{M}}_{g,n}(J_i, A)$, $i = 0, 1$ which does not belong to any open compact (equivalently energy bounded) subset of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$. We say* *that the sky catastrophe is* **stable** *if the same is true for any sufficiently $C^0$ nearby, $C^\infty$ family $\{J'_t\}$ satisfying $J'_0 = J_0$ and $J'_1 = J_1$.*

Let us slightly expand this definition. If $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ is locally connected, so that the connected components are open, then we have a sky catastrophe in the sense above if and only if there is a $u \in \overline{\mathcal{M}}_{g,n}(J_i, A)$ which has a non-compact connected component in $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$. At this point in time there are no known examples of families $\{J_t\}$ with sky catastrophes (even unstable).

*Question* 1. Do ***stable holomorphic sky catastrophes*** exist?

As a corollary of Theorem 2.16 and Example 3 we get:

**Corollary 3.11.** *Assume that stable holomorphic sky catastrophes do not exist, then the Elliptic Weinstein conjecture holds for every first kind* lcs *structures on $S^{2k+1} \times S^1$ formally homotopic to the Hopf lcs structure.*

The author's opinion is that stable holomorphic sky catastrophes exist. However, if we ask that each $J_t$ is tamed by an lcs structure, then the question becomes much more subtle, see also [27] for a related discussion on possible obstructions to sky catastrophes. We also partly discussed this in Part I of the Introduction. See also [29] for non-trivial examples where holomorphic sky catastrophes can be ruled out.

## 4. Elements of Gromov-Witten theory of an lcs manifold

Suppose $(M, J)$ is a compact almost complex manifold, where the almost complex structures $J$ are assumed throughout the paper to be $C^\infty$, and let $N \subset \overline{\mathcal{M}}_{g,k}(J, A)$ be an open compact subset with energy positive on $N$. The latter condition is only relevant when $A = 0$. We shall primarily refer in what follows to work of Pardon in [24], being more familiar to the author. But we should mention that the latter is a followup to a theory that is originally created by Fukaya-Ono [11], and later expanded with Oh-Ohta [10].

The construction in [24] of an implicit atlas, on the moduli space $\mathcal{M}$ of $J$-curves in a symplectic manifold, only needs a neighborhood of $\mathcal{M}$ in the space of all curves. So more generally, if we have an almost complex manifold and an *open* compact component $N$ as above, this will likewise have a natural implicit atlas, or a Kuranishi structure in the setup of [11]. And so such an $N$ will have a virtual fundamental class in the sense of Pardon [24], (or in any other approach to virtual fundamental cycle, particularly the original approach of Fukaya-Oh-Ohta-Ono). This understanding will be used in other parts of the paper, following Pardon for the explicit setup. We may thus define functionals:

$$GW_{g,n}(N, A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M) \to \mathbb{Q}. \tag{4.1}$$

In our more specific context we must in addition restrict the charge, which is defined at the moment for genus 1 curves, so that we further simplify our discussion to this setting. So supposing $(M, J, \alpha)$ is a Lee manifold we may likewise define functionals:

$$GW^k_{1,1}(N, A, J) : H_*(\overline{M}_{1,1}) \otimes H_*(M) \to \mathbb{Q}, \tag{4.2}$$

meaning that we restrict to the space of $\alpha$-charge $k$ curves, $\overline{\mathcal{M}}^k_{1,1}(J, A)$, with $N$ an open compact subset of the latter.

How do these functionals depend on $N, J$?

**Lemma 4.3.** *Let $\{J_t\}$, $t \in [0,1]$ be a Frechet smooth family of almost complex structures on $M$. Suppose that $\widetilde{N}$ is an open compact subset of the cobordism moduli space $\overline{\mathcal{M}}^k_{1,1}(\{J_t\}, A)$, with $k > 0$. Let*

$$N_i = \widetilde{N} \cap \left(\overline{\mathcal{M}}^k_{1,1}(J_i, A)\right),$$

*then*

$$GW^k_{1,1}(N_0, A, J_0) = GW^k_{1,1}(N_1, A, J_1).$$

*In particular if $GW^k_{1,1}(N_0, A, J_0) \neq 0$, there is a class $A$ $J_1$-holomorphic, stable, charge $k$ curve in $M$.*

*Proof of Lemma 4.3.* We may construct exactly as in [24] a natural implicit atlas on $\widetilde{N}$, with boundary $N_0^{op} \sqcup N_1$, ($op$ denoting opposite orientation). Note that the condition that $k > 0$ is essential as otherwise we may have boundary components corresponding to degenerations to constant curves.

*Remark* 4.4. In the case the manifold is closed, degenerations to constant curves are impossible, when each $J_t$ is tamed by a $t$-continuous family $\omega_t$ of symplectic or lcs forms, [28]. They are also impossible for general closed almost complex manifolds, for rational curves, by energy quantization. But as far as I know, such degenerations might happen for genus 1 curves in general closed almost complex manifolds.

And so

$$GW^k_{1,1}(N_0, A, J_0) = GW^k_{1,1}(N_1, A, J_1),$$

as functionals. □

We now state a basic technical lemma, following some standard definitions.

**Definition 4.5.** *An* **almost symplectic pair** *on $M$ is a tuple $(\omega, J)$, where $\omega$ is a non-degenerate 2-form on $M$, and $J$ is $\omega$-compatible, meaning that $\omega(\cdot, J\cdot)$ defines $J$-invariant Riemannian metric. When $\omega$ is lcs we call such a pair an* **lcs pair**.

**Definition 4.6.** *We say that a pair of almost symplectic pairs $(\omega_i, J_i)$ are* ***$\delta$-close****, if $\{\omega_i\}$ are $C^0$ $\delta$-close, and $\{J_i\}$ are $C^2$ $\delta$-close, $i = 0, 1$. Define a similar metric on pairs $(g, J)$ for $g$ a Riemannian metric and $J$ any almost complex structure.*

**Definition 4.7.** *For an almost symplectic pair $(\omega, J)$ on $M$, and a smooth map $u : \Sigma \to M$ define:*

$$e_\omega(u) = \int_\Sigma u^*\omega.$$

By an elementary calculation, this coincides with the $L^2$ $g_J$-energy of $u$, for $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$. That is $e_\omega(u) = e_{g_J}(u)$. In what follows by $f^{-1}(a, b)$, with $f$ a function, we mean the preimage by $f$ of the open set $(a, b)$.

**Lemma 4.8.** *Let $(M, g, J, \alpha)$ be as above. Suppose that $N \subset \overline{\mathcal{M}}^k_{1,1}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = e_g^{-1}(E^0, E^1)\right) \subset \left(V = e_g^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

*with $\epsilon > 0$, $E_0 > 0$, and with $V \cap \overline{\mathcal{M}}^k_{1,1}(J, A) = N$. Then there is a $\delta > 0$ s.t. whenever $(g', J')$ is $\delta$-close to $(g, J)$ if $u \in \overline{\mathcal{M}}^k_{1,1}(J', A)$ and*

$$E^0 - \epsilon < e_{g'}(u) < E^1 + \epsilon$$

*then*

$$E^0 < e_{g'}(u) < E^1.$$

*Proof.* Suppose otherwise, then there is a sequence $\{(g_k, J_k)\}$ converging to $(g, J)$, and a sequence $\{u_k\}$ of $J_k$-holomorphic stable maps satisfying:

$$E^0 - \epsilon < e_{g_k}(u_k) \leq E^0$$

or

$$E^1 \leq e_{g_k}(u_k) < E^1 + \epsilon.$$

By Gromov compactness, specifically theorems [20, B.41, B.42], we may find a Gromov convergent subsequence $\{u_{k_j}\}$ to a $J$-holomorphic stable map $u$, with

$$E^0 - \epsilon \leq e_g(u) \leq E^0$$

or

$$E^1 \leq e_g(u) \leq E^1 + \epsilon.$$

But by our assumptions such a $u$ does not exist. □

**Lemma 4.9.** *Let $M$ be compact, let $(\omega, J), \alpha$ be as above, and such that $N \subset \overline{\mathcal{M}}_{1,1}^k(J, A)$ is exactly as in the lemma above, with respect to some $\epsilon > 0$. Then, there is a $\delta > 0$ s.t. the following is satisfied. Let $(\omega', J')$ be $\delta$-close to $(\omega, J)$, then there is a continuous in the $C^\infty$ topology family of almost symplectic pairs $\{(\omega_t, J_t)\}$, $(\omega_0, J_0) = (g, J)$, $(\omega_1, J_1) = (g', J')$ s.t. there is an open compact subset*

$$\widetilde{N} \subset \overline{\mathcal{M}}_{1,1}^k(\{J_t\}, A),$$

*satisfying: if $(u, t) \in \widetilde{N}$ then*

$$E^0 < e_{g_t}(u) < E^1,$$

*for $g_t = \omega_t(\cdot, J_t)$.*

*Proof.* For $\epsilon$ as in the hypothesis, let $\delta$ be as in Lemma 4.8. We then need:

**Lemma 4.10.** *Given a $\delta > 0$ there is a $\delta' > 0$ s.t. if $(\omega', J')$ is $\delta'$-near $(\omega, J)$ then there is a continuous in the $C^\infty$ topology family $\{(\omega_t, J_t)\}$ satisfying:*

- *$(\omega_t, J_t)$ is $\delta$-close to $(\omega, J)$ for each $t$.*
- *$(\omega_0, J_0) = (\omega, J)$ and $(\omega_1, J_1) = (\omega', J')$.*

*Proof.* Let $\{g_t\}$ be the family of metrics on $M$ given by the convex linear combination of $g = g_{\omega_J}, g' = g_{\omega', J'}$. Clearly $g_t$ is $\delta'$-close to $g_0$ for each $t$. Likewise, the family of 2 forms $\{\omega_t\}$ given by the convex linear combination of $\omega$, $\omega'$ is non-degenerate for each $t$ if $\delta'$ was chosen to be sufficiently small and is $\delta'$-close to $\omega_0 = \omega_{g,J}$ for each moment.

Let

$$ret : Met(M) \times \Omega(M) \to \mathcal{J}(M)$$

be the "retraction map" (it can be understood as a retraction followed by projection) as defined in [19, Prop 2.50], where $Met(M)$ is space of metrics on $M$, $\Omega(M)$ the space of 2-forms on $M$, and $\mathcal{J}(M)$ the space of almost complex structures. This map has the property that the almost complex structure $ret(g, \omega)$ is compatible with $\omega$, and that $ret(g_J, \omega) = J$ for $g_J = \omega(\cdot, J\cdot)$. Then $\{(\omega_t, ret(g_t, \omega_t)\}$ is a compatible family. As $ret$ is continuous in $C^2$-topology, $\delta'$ can be chosen such that $\{ret_t(g_t, \omega_t\}$ are $C^2$ $\delta$-nearby. □

Returning to the proof of the main lemma. Let $\delta' < \delta$ be chosen as in the above lemma and let $\{(\omega_t, J_t)\}$ be the corresponding family. Let $\widetilde{N}$ consist of all elements $(u, t) \in \overline{\mathcal{M}}_{1,1}^k(\{J_t\}, A)$ s.t.

$$E^0 - \epsilon < e_{\omega_t}(u) < E^1 + \epsilon.$$

Then by Lemma 4.8 for each $(u,t) \in \widetilde{N}$, we have:

$$E^0 < e_{\omega_t}(u) < E^1.$$

In particular $\widetilde{N}$ must be closed, it is also clearly open, and is compact as the energy $e$ is a proper function, as discussed. Renaming $\delta := \delta'$ we are then done. $\square$

**Proposition 4.11.** *Given a compact $M$ and a triple $(\omega, J, \alpha)$ on $M$, as above, suppose that $N \subset \overline{\mathcal{M}}^k_{1,1}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = e_\omega^{-1}(E^0, E^1)\right) \subset \left(V = e_\omega^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

*with $\epsilon > 0$, $E^0 > 0$, and with $V \cap \overline{\mathcal{M}}^k_{1,1}(J, A) = N$. Suppose also that $GW^k_{1,1}(N, J, A) \neq 0$. Then there is a $\delta > 0$ s.t. whenever $(\omega', J')$ is a compatible almost symplectic pair $\delta$-close to $(\omega, J)$, there exists $u \in \overline{\mathcal{M}}^k_{1,1}(J', A)$, satisfying*

$$E^0 < e_{\omega'}(u) < E^1.$$

*Proof.* For $N, \epsilon$ as in the hypothesis, let $\delta, \widetilde{N}$ be as in Lemma 4.9, then by Lemma 4.3

$$GW^k_{1,1}(N_1, J', A) = GW^k_{1,1}(N, J, A) \neq 0,$$

where $N_1 = \widetilde{N} \cap \overline{\mathcal{M}}^k_{1,1}(J_1, A)$. $\square$

*Proof of Theorem 3.8.* (Outline, as the argument is standard.) Suppose that we have a sequence $u^k$ of $J$-holomorphic maps with $L^2$-energy $\leq E$. By [19, 4.1.1], a sequence $u^k$ of $J$-holomorphic curves has a convergent subsequence if $sup_k ||du^k||_{L^\infty} < \infty$. On the other hand, when this condition does not hold, rescaling argument tells us that a holomorphic sphere bubbles off. The quantization Theorem 3.7, tells us that the energy of a non-constant $J$-holomorphic map of $\mathbb{CP}^1$ is at least $\hbar > 0$. So if the energy of the maps $u^k$ is bounded from above by $E$, only finitely many bubbles may appear, so that a subsequence of $u^k$ must converge in the Gromov topology to a Kontsevich stable map. $\square$

## 5. Elliptic curves in the lcs-fication of a contact manifold and the Fuller index

*Proof of Proposition 3.4.* Suppose we have a curve $u \in \overline{\mathcal{M}}^1_{1,1}(J^\lambda, A)$, represented by $u : \Sigma \to M = C \times S^1$. Then $u$ has no spherical components, as otherwise we would have a non-constant $J^\lambda$-holomorphic sphere. And so by Theorem 3.5, we would have a Reeb 2-curve $u' : \mathbb{CP}^1 \to M$, which is impossible by property 2 of the definition.

By Theorem 3.5 $u$ is a Reeb 2-curve. By Lemma 6.6 it's normalization $\widetilde{u}$ is also a Reeb 2-curve. If $u$ is not normal then $\widetilde{u}$ is a Reeb 2-curve with domain $\mathbb{CP}^1$, which is impossible by the argument above. Hence $u$ is normal.

By the charge 1 condition $pr_{S^1} \circ u$ is surjective, where $pr_{S^1} : C \times S^1 \to S^1$ is the projection. By the Sard theorem we have a regular value $t_0 \in S^1$, so that $u^{-1} \circ pr_{S^1}^{-1}(t_0)$ contains an embedded circle $S_0 \subset \Sigma$.

Now $d(pr_{S^1} \circ u)$ is surjective onto $T_{t_0}S^1$ along $T\Sigma|_{S_0}$. And so since $u$ is $J^\lambda$-holomorphic, $o = pr_C \circ u|_{S_0}$ has non-vanishing differential $d(o)$. By the first part of the condition in Definition 2.9, $o$ is tangent to $\ker d\lambda$. It follows that $o$ is an unparametrized $\lambda$-Reeb orbit. Also, the image of $d(pr_C \circ u)$ is in $\ker d\lambda$ from which it follows that $\operatorname{image} d(pr_C \circ u) = \operatorname{image} d(o)$. It follows that $u$ is an elliptic charge 1 curve with image contained in the image of the Reeb torus $u_o$. Consequently, because of the charge 1 condition, up to parametrization, $u$ is the Reeb torus $u_{\widetilde{o}}$, for some covering map $\widetilde{o}$ of $o$. By $J^\lambda$-holomorphicity, $u$ then must be the Reeb torus $u_{\widetilde{o}}$ up to equivalence. $\square$

**Proposition 5.1.** *Let $(C, \xi)$ be a general contact manifold. If $\lambda$ is a non-degenerate contact 1-form for $\xi$ then all the elements of $\overline{\mathcal{M}}_{1,1}^1(J^\lambda, A)$ are regular curves. Moreover, if $\lambda$ is degenerate then for a period $c$ Reeb orbit $o$, the kernel of the associated real linear Cauchy-Riemann operator for the Reeb torus $u_o$ is naturally identified with the 1-eigenspace of $\phi^\lambda_{c,*}$ - the time $c$ linearized return map $\xi(o(0)) \to \xi(o(0))$ induced by the $R^\lambda$ Reeb flow.*

*Proof.* We already know by Proposition 3.4 that all $u \in \overline{\mathcal{M}}_{1,1}^1(J^\lambda, A)$ are equivalent to Reeb tori. In particular, such curves have representation by a $J^\lambda$-holomorphic map

$$u : (T^2, j) \to (Y = C \times S^1, J^\lambda).$$

Since each $u$ is immersed we may naturally get a splitting $u^*T(Y) \simeq N \times T(T^2)$, using the $g_J$ metric, where $N \to T^2$ denotes the pull-back, of the $g_J$-normal bundle to image $u$, and which is identified with the pullback of the distribution $\xi_\lambda$ on $Y$, (which we also call the co-vanishing distribution).

The full associated real linear Cauchy-Riemann operator takes the form:

$$D^J_u : \Omega^0(N \oplus T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2), N \oplus T(T^2)). \tag{5.2}$$

This is an index 2 Fredholm operator (after standard Sobolev completions), whose restriction to $\Omega^0(N \oplus T(T^2))$ preserves the splitting, that is the restricted operator splits as

$$D \oplus D' : \Omega^0(N) \oplus \Omega^0(T(T^2)) \to \Omega^{0,1}(T(T^2), N) \oplus \Omega^{0,1}(T(T^2), T(T^2)).$$

On the other hand the restricted Fredholm index 2 operator

$$\Omega^0(T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2)),$$

is surjective by classical Teichmuller theory, see also [35, Lemma 3.3] for a precise argument in this setting. It follows that $D^J_u$ will be surjective if the restricted Fredholm index 0 operator

$$D : \Omega^0(N) \to \Omega^{0,1}(N),$$

has no kernel.

The bundle $N$ is symplectic with symplectic form on the fibers given by restriction of $u^*d\lambda$, and together with $J^\lambda$ this gives a Hermitian structure $(g_\lambda, j_\lambda)$ on $N$. We have a linear symplectic connection $\mathcal{A}$ on $N$, which over the slices $S^1 \times \{t\} \subset T^2$ is induced by the pullback by $u$ of the linearized $R^\lambda$ Reeb flow. Specifically the $\mathcal{A}$-transport map from the fiber $N_{(s_0,t)}$ to the fiber $N_{(s_1,t)}$ over the path $[s_0, s_1] \times \{t\} \subset T^2$, is given by

$$(u_*|_{N_{(s_1,t)}})^{-1} \circ (\phi^\lambda_{c(s_1-s_0)})_* \circ u_*|_{N_{(s_0,t)}},$$

where $\phi^\lambda_{c(s_1-s_0)}$ is the time $c \cdot (s_1 - s_0)$ map for the $R^\lambda$ Reeb flow, where $c$ is the period of the Reeb orbit $o_u$, and where $u_* : N \to TY$ denotes the natural map, (it is the universal map in the pull-back diagram.)

The connection $\mathcal{A}$ is defined to be trivial in the $\theta_2$ direction, where trivial means that the parallel transport maps are the $id$ maps over $\theta_2$ rays. In particular the curvature $R_\mathcal{A}$, understood as a lie algebra valued 2-form, of this connection vanishes. The connection $\mathcal{A}$ determines a real linear CR operator $D_\mathcal{A}$ on $N$ in the standard way, take the complex anti-linear part of the vertical differential of a section. Explicitly,

$$D_\mathcal{A} : \Omega^0(N) \to \Omega^{0,1}(N),$$

is defined by

$$D_\mathcal{A}(\mu)(p) = j_\lambda \circ \pi^{vert}(\mu(p)) \circ d\mu(p) - \pi^{vert}(\mu(p)) \circ d\mu(p) \circ j,$$

where

$$\pi^{vert}(\mu(p)) : T_{\mu(p)}N \to T^{vert}_{\mu(p)}N \simeq N$$

is the $\mathcal{A}$-projection, and where $T^{vert}_{\mu(p)}N$ is the kernel of the projection $T_{\mu(p)}N \to T_p\Sigma$. It is elementary to verify from the definitions that this operator is exactly $D$. See also [22, Section 10.1] for a computation of this kind in much greater generality.

We have a differential 2-form $\Omega$ on the total space of $N$ defined as follows. On the fibers $T^{vert}N$, $\Omega = u_*\omega$, for $\omega = d_\alpha\lambda$, and for $T^{vert}N \subset TN$ denoting the vertical tangent space, or subspace of vectors $v$ with $\pi_* v = 0$, for $\pi : N \to T^2$ the projection. While on the $\mathcal{A}$-horizontal distribution $\Omega$ is defined to vanish. The 2-form $\Omega$ is closed, which we may check explicitly by using that $R_{\mathcal{A}}$ vanishes to obtain local symplectic trivializations of $N$ in which $\mathcal{A}$ is trivial. Clearly $\Omega$ must vanish on the 0-section since it is a $\mathcal{A}$-flat section. But any section is homotopic to the 0-section and so in particular if $\mu \in \ker D$ then $\Omega$ vanishes on $\mu$.

Since $\mu \in \ker D$, and so its vertical differential is complex linear, it follows that the vertical differential vanishes. To see this note that $\Omega(v, J^\lambda v) > 0$, for $0 \neq v \in T^{vert}N$ and so if the vertical differential did not vanish we would have $\int_\mu \Omega > 0$. So $\mu$ is $\mathcal{A}$-flat, in particular the restriction of $\mu$ over all slices $S^1 \times \{t\}$ is identified with a period $c$ orbit of the linearized at $o$ $R^\lambda$ Reeb flow, and which does not depend on $t$ as $\mathcal{A}$ is trivial in the $t$ variable. So the kernel of $D$ is identified with the vector space of period $c$ orbits of the linearized at $o$ $R^\lambda$ Reeb flow, as needed. □

**Proposition 5.3.** *Let $\lambda$ be a contact form on a $(2n+1)$-fold $C$, and $o$ a non-degenerate, period $c$, $\lambda$-Reeb orbit, then the orientation of $[u_o]$ induced by the determinant line bundle orientation of $\overline{\mathcal{M}}^1_{1,1}(J^\lambda, A)$, is $(-1)^{CZ(o)-n}$, which is*

$$\operatorname{sign}\operatorname{Det}(\operatorname{Id}|_{\xi(o(0))} - \phi^\lambda_{c,*}|_{\xi(o(0))}).$$

*Proof of Proposition 5.3.* Abbreviate $u_o$ by $u$. Let $N \to T^2$ be the vector bundle associated to $u$ as in the proof of Proposition 5.1. Fix a trivialization $\phi$ of $N$ induced by any trivialization of the contact distribution $\xi$ along $o$ in the obvious sense: $N$ is the pullback of $\xi$ along the composition

$$T^2 \to S^1 \xrightarrow{o} C.$$

Let the symplectic connection $\mathcal{A}$ on $N$ be defined as before. Then the pullback connection $\mathcal{A}' := \phi^*\mathcal{A}$ on $T^2 \times \mathbb{R}^{2n}$ is a connection whose parallel transport paths $p_t : [0,1] \to \operatorname{Symp}(\mathbb{R}^{2n})$, along the closed loops $S^1 \times \{t\}$, are paths starting at 1, and are $t$ independent. And so the parallel transport path of $\mathcal{A}'$ along $\{s\} \times S^1$ is constant, that is $\mathcal{A}'$ is trivial in the $t$ variable. We shall call such a connection $\mathcal{A}'$ on $T^2 \times \mathbb{R}^{2n}$ *induced by $p$*.

By non-degeneracy assumption on $o$, the map $p(1)$ has no 1-eigenvalues. Let $p'' : [0,1] \to \operatorname{Symp}(\mathbb{R}^{2n})$ be a path from $p(1)$ to a unitary map $p''(1)$, with $p''(1)$ having no 1-eigenvalues, and s.t. $p''$ has only simple crossings with the Maslov cycle. Let $p'$ be the concatenation of $p$ and $p''$. We then get

$$CZ(p') - \frac{1}{2}\operatorname{sign}\Gamma(p', 0) \equiv CZ(p') - n \equiv 0 \mod 2,$$

since $p'$ is homotopic relative end points to a unitary geodesic path $h$ starting at $id$, having regular crossings, and since the number of negative, positive eigenvalues is even at each regular crossing of $h$ by unitarity. Here $\operatorname{sign}\Gamma(p', 0)$ is the index of the crossing form of the path $p'$ at time 0, in the notation of [25]. Consequently,

$$CZ(p'') \equiv CZ(p) - n \mod 2, \tag{5.4}$$

by additivity of the Conley-Zehnder index.

Let us then define a free homotopy $\{p_t\}$ of $p$ to $p'$, $p_t$ is the concatenation of $p$ with $p''|_{[0,t]}$, reparametrized to have domain $[0,1]$ at each moment $t$. This determines a homotopy $\{\mathcal{A}'_t\}$ of connections induced by $\{p_t\}$. By the proof of Proposition 5.1, the CR operator $D_t$ determined by each $\mathcal{A}'_t$ is surjective except at some finite collection of times $t_i \in (0,1)$, $i \in N$ determined by the crossing times of $p''$ with the

Maslov cycle, and the dimension of the kernel of $D_{t_i}$ is the 1-eigenspace of $p''(t_i)$, which is 1 by the assumption that the crossings of $p''$ are simple.

The operator $D_1$ is not complex linear. To fix this we concatenate the homotopy $\{D_t\}$ with the homotopy $\{\widetilde{D}_t\}$ defined as follows. Let $\{\widetilde{\mathcal{A}}_t\}$ be a homotopy of $\mathcal{A}'_1$ to a unitary connection $\widetilde{\mathcal{A}}_1$, where the homotopy $\{\widetilde{\mathcal{A}}_t\}$ is through connections induced by paths $\{\widetilde{p}_t\}$, giving a path homotopy of $p' = \widetilde{p}_0$ to $h$. Then $\{\widetilde{D}_t\}$ is defined to be induced by $\{\widetilde{\mathcal{A}}_t\}$.

Let us denote by $\{D'_t\}$ the concatenation of $\{D_t\}$ with $\{\widetilde{D}_t\}$. By construction, in the second half of the homotopy $\{D'_t\}$, $D'_t$ is surjective. And $D'_1$ is induced by a unitary connection, since it is induced by unitary path $\widetilde{p}_1$. Consequently, $D'_1$ is complex linear. By the above construction, for the homotopy $\{D'_t\}$, $D'_t$ is surjective except for $N$ times in $(0,1)$, where the kernel has dimension one. In particular the sign of $[u]$ by the definition via the determinant line bundle is exactly

$$-1^N = -1^{CZ(p)-n},$$

by (5.4), which was what to be proved. □

**Theorem 5.5.**

$$GW^1_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(\mathcal{P}^{-1}(N), R^\lambda, \beta),$$

*where $N \subset \overline{\mathcal{M}}^1_{1,1}(J^\lambda, A_\beta)$ is an open compact set (where $\mathcal{P}$ is as in Proposition 3.4), $i(\mathcal{P}^{-1}(N), R^\lambda, \beta)$ is the Fuller index as described in the appendix below, and where the left-hand side of the equation is the functional as in (4.2).*

*Proof.* Suppose that $N \subset \overline{\mathcal{M}}^1_{1,1}(J^\lambda, A_\beta)$ is open-compact and consists of isolated regular Reeb tori $\{u_i\}$, corresponding to orbits $\{o_i\}$. Denote by $mult(o_i)$ the multiplicity of the orbits as in Appendix A. Then we have:

$$GW^1_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = \sum_i \frac{(-1)^{CZ(o_i)-n}}{mult(o_i)},$$

where the denominator $mult(o_i)$ appears because our moduli space is understood as a non-effective orbifold, and $mult(o_i)$ is the order of the corresponding isotropy group, see Appendix B.

The expression on the right is exactly the Fuller index $i(\mathcal{P}^{-1}(N), R^\lambda, \beta)$. Thus, the theorem follows for $N$ as above. However, in general if $N$ is open and compact then perturbing slightly we obtain a smooth family $\{R^{\lambda_t}\}$, $\lambda_0 = \lambda$, s.t. $\lambda_1$ is non-degenerate, that is has non-degenerate orbits. And such that there is an open-compact subset $\widetilde{N}$ of $\overline{\mathcal{M}}^1_{1,1}(\{J^{\lambda_t}\}, A_\beta)$ with $(\widetilde{N} \cap \overline{\mathcal{M}}^1_{1,1}(J^\lambda, A_\beta)) = N$, see Lemma 4.9. Then by Lemma 4.3 if

$$N_1 = (\widetilde{N} \cap \overline{\mathcal{M}}^1_{1,1}(J^{\lambda_1}, A_\beta))$$

we get

$$GW^1_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = GW^1_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}_{1,1}] \otimes [C \times S^1]).$$

By the previous discussion

$$GW^1_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(N_1, R^{\lambda_1}, \beta),$$

but by the invariance of Fuller index (see Appendix A),

$$i(N_1, R^{\lambda_1}, \beta) = i(N, R^\lambda, \beta).$$

□

## 6. Proofs of main theorems

**Lemma 6.1.** *Let $(M, \lambda, \alpha)$ be an exact lcs manifold with $M$ closed then $0 \neq [\alpha] \in H^1(M, \mathbb{R})$.*

*Proof.* Suppose by contradiction that $\alpha$ is exact and let $g$ be its primitive. Then computing we get: $d_\alpha \lambda = \frac{1}{f} d(f\lambda)$ with $f = e^g$. Consequently, $d(f\lambda)$ is non-degenerate on $M$ which contradicts Stokes theorem. $\square$

*Proof of Theorem 2.11.* Let $(M, \lambda, \alpha)$ be a closed exact lcs 2-manifold. By Lemma 6.1 the cohomology class of $\alpha$ is non-zero in $H^1(M, \mathbb{R})$. So we may take $\Sigma = M$ and $u : \Sigma \to M$, to be the identity. Clearly it is a Reeb 2-curve. $\square$

To set notation and terminology we review the basic definition of a nodal curve.

**Definition 6.2.** *A* **nodal Riemann surface** *(without boundary) is a pair $\Sigma = (\widetilde{\Sigma}, \mathcal{N})$ where $\widetilde{\Sigma}$ is a Riemann surface, and $\mathcal{N}$ a set of pairs of points of $\widetilde{\Sigma} : \mathcal{N} = \{(z_0^0, z_0^1), \ldots, (z_n^0, z_n^1)\}$, $n_i^j \neq n_k^l$ for $i \neq k$ and all $j, l$. By slight abuse, we may also denote by $\Sigma$ the quotient space $\widetilde{\Sigma}/\sim$, where the equivalence relation is generated by $n_i^0 \sim n_i^1$. Let $q_\Sigma : \widetilde{\Sigma} \to (\widetilde{\Sigma}/\sim)$ denote the quotient map. The elements $q_\Sigma(\{z_i^0, z_i^1\}) \in \widetilde{\Sigma}/\sim$, are called* **nodes***. Let $M$ be a smooth manifold. By a map $u : \Sigma \to M$ of a nodal Riemann surface $\Sigma$, we mean a set map $u : (\widetilde{\Sigma}/\sim) \to M$. $u$ is called smooth or immersion or $J$-holomorphic (when $M$ is almost complex) if the map $\widetilde{u} = u \circ q_\Sigma$ is smooth or respectively immersion or respectively $J$-holomorphic. We call $\widetilde{u}$* **normalization of** *$u$. $u$ is called an embedding if $u$ is a topological embedding and its normalization is an immersion. The cohomology groups of $\Sigma$ are defined as $H^\bullet(\Sigma) := H^\bullet(\widetilde{\Sigma}/\sim)$, likewise with homology. The genus of $\Sigma$ is the topological genus of $\widetilde{\Sigma}/\sim$.*

We shall say that $(\widetilde{\Sigma}, \mathcal{N})$ is normal if $\mathcal{N} = \emptyset$. Similarly, $u : \Sigma \to M$, $\Sigma = (\widetilde{\Sigma}, \mathcal{N})$ is called ***normal*** if $\mathcal{N} = \emptyset$. The normalization of $u$ is the map of the nodal Riemann surface $\widetilde{u} : \widetilde{\Sigma} \to M$, $\widetilde{\Sigma} = (\widetilde{\Sigma}, \emptyset)$. Note that if $u$ is a Reeb 2-curve, its normalization $\widetilde{u}$ may not be a Reeb 2-curve (the second condition may fail).

*Proof of Theorem 2.10.* Let $(M = C \times S^1, \lambda, \alpha)$ be the lcs-fication of a closed contact manifold $(C, \lambda)$. In this case $\mathcal{V}_\lambda = \mathcal{D}$ and is spanned by $X_\lambda = (0, \frac{\partial}{\partial \theta}), X_\alpha = (R^\lambda, 0)$ for $R^\lambda$ the $\lambda$-Reeb vector field.

Suppose first $u : \Sigma \to M$ is a normal Reeb 2-curve. By definitions $u^*\alpha$ is an integral 1-form non-vanishing in $H^1_{DR}(\Sigma)$. Then let $p : \Sigma \to S^1$ be its classifying map, and let $o : S^1 \to \Sigma$ smoothly parametrize a component of a regular fiber of $p$, not intersecting the singular set of $u$. By the first condition of the definition of a Reeb 2-curve, $u^*\alpha, u^*\lambda$ are non-vanishing along $S = o(S^1)$. Let $TS$ denote the intrinsic tangent bundle. Since $u^*\alpha$ vanishes on $TS \subset T\Sigma$ it follows that $u^*\lambda$ is non-vanishing on $TS$. Also by the first condition, $u \circ o$ is tangent to $\mathcal{V}_\lambda$. And so $pr_C \circ u \circ o$ is tangent to $\ker d\lambda$, and $\lambda((pr_C \circ u \circ o)') > 0$. It follows that $pr_C \circ u \circ o$ is a Reeb orbit up to parametrization.

If $u$ is not normal, then since $(M, \lambda, \alpha)$ is integral and first kind, by Lemma 6.6 the normalization $\widetilde{u}$ of $u$ is a normal Reeb 2-curve. (The proof of that lemma is based on the first part of the argument above). Then apply the argument above to $\widetilde{u}$, and we are done. $\square$

*Proof of Theorem 3.5.* Let $u : \Sigma \to M$ be a non-constant, nodal $J$-curve. Since singularities of $u$ are isolated, by Lemma 3.2 it is enough to show that $[u^*\alpha] \neq 0$. Let $\widetilde{M}$ denote the $\alpha$-covering space of $M$, that is the space of equivalence classes of paths $p$ starting at $x_0 \in M$, with a pair $p_1, p_2$ equivalent if $p_1(1) = p_2(1)$ and

$$\int_{[0,1]} p_1^*\alpha = \int_{[0,1]} p_2^*\alpha.$$

Then the lift of $\omega$ to $\widetilde{M}$ is

$$\widetilde{\omega} = \frac{1}{f} d(f\lambda),$$

where $f = e^g$ and where $g$ is a primitive for the lift $\widetilde{\alpha}$ of $\alpha$ to $\widetilde{M}$, that is $\widetilde{\alpha} = dg$. In particular $\widetilde{\omega}$ is conformally symplectomorphic to an exact symplectic form on $\widetilde{M}$. So if $\widetilde{J}$ denotes the lift of $J$, any closed $\widetilde{J}$-curve is constant by Stokes theorem. Now if $[u^*\alpha] = 0$ then $u$ has a lift to a $\widetilde{J}$-holomorphic map $v : \Sigma \to \widetilde{M}$. Since $\Sigma$ is closed, it follows by the above that $v$ is constant, so that $u$ is constant, which is impossible. □

**Proposition 6.3.** *Let $(C, \lambda)$ be a closed contact manifold with $\lambda$ having at least one closed non-degenerate Reeb orbit $o$, or more generally satisfying $i(N, R^\lambda, \beta) \neq 0$ for some good $N$ and some $\beta$. Then:*

(1) *There exists an $\epsilon > 0$ s.t. for any tamed exact lcs structure $(\lambda', \alpha', J)$ on $M = C \times S^1$, with $(d_{\alpha'}\lambda', J)$ $\epsilon$-close to $(d_\alpha \lambda, J^\lambda)$ (as in Definition 4.6), there exists an elliptic, $J$-holomorphic $\alpha$-charge 1 curve $u$ in $M$.*

(2) *In addition, if $(M, \lambda', \alpha')$ is first kind and has dimension 4 then $u$ may be assumed to be normal and embedded.*

*Proof.* If we have a closed non-degenerate $\lambda$-Reeb orbit $o$ then we also have an open compact subset $N \subset S_\lambda$ consisting of one point corresponding to $o$, and which is then good. Thus, it suffices to prove the proposition for a general good $N \subset S_\lambda$ as in the hypothesis. Then set

$$(\widetilde{N} := \mathcal{P}(N)) \subset \overline{\mathcal{M}}^1_{1,1}(A_\beta, J^\lambda),$$

which is an open, compact set and energy isolated by the assumption that $N$ is good. By Theorem 5.5, and by the assumption that $i(N, R_\lambda, \beta) \neq 0$

$$GW^1_{1,1}(N, J^\lambda, A_\beta) \neq 0.$$

The first part of the proposition then follows by Proposition 4.11.

We now verify the second part. Suppose that $M$ has dimension 4. Let $U$ be an $\epsilon$-neighborhood of $(\lambda, \alpha, J^\lambda)$ as given in the first part, and let $(\lambda', \alpha', J) \in U$. Suppose that $u \in \overline{\mathcal{M}}^1_{1,1}(A_\beta, J)$. Let $\underline{u}$ be a simple $J$-holomorphic curve covered by $u$, (see for instance [20, Section 2.5].

For convenience, we now recall the adjunction inequality.

**Theorem 6.4** (McDuff-Micallef-White [21], [16])**.** *Let $(M, J)$ be an almost complex 4-manifold and $A \in H_2(M)$ be a homology class that is represented by a simple $J$-holomorphic curve $u : \Sigma \to M$. Let $\delta(u)$ denote the number of self-intersections of $u$, then*

$$2\delta(u) - \chi(\Sigma) \leq A \cdot A - c_1(A),$$

*with equality if and only if $u$ is an immersion with only transverse self-intersections.*

In our case $A = A_\beta$ so that $c_1(A) = 0$ and $A \cdot A = 0$. If $u$ is not normal its normalization is of the form $\widetilde{u} : \mathbb{CP}^1 \to M$ with at least one self intersection and with $0 = [\widetilde{u}] \in H_2(M)$, but this contradicts positivity of intersections. So $u$ and hence $\underline{u}$ are normal. Moreover, the domain $\Sigma'$ of $\underline{u}$ satisfies: $\chi(\Sigma') = \chi(T^2) = 0$, so that $\delta(\underline{u}) = 0$, and the above inequality is an equality. In particular $\underline{u}$ is an embedding, which of course implies our claim.

□

*Proof of Theorem 2.14.* Let
$$U \ni (\omega_0 := d_\alpha \lambda, J_0 := J^\lambda)$$
be a set of pairs $(\omega, J)$ satisfying the following:

- $\omega$ is a first kind lcs structure.
- For each $(\omega, J) \in U$, $J$ is $\omega$-compatible and admissible.
- Let $\epsilon$ be chosen as in the first part of Theorem 6.3. Then each $(\omega, J) \in U$, is $\epsilon$-close to $(\omega_0, J_0)$, (as in Definition 4.6).

To prove the theorem we need to construct a map $E : V \to \mathcal{J}(M)$, where $V$ is some $d_3$ neighborhood of $\omega_0$ in the space $\mathcal{F}(M)$ (see Definition 2.12) and where
$$\forall \omega \in V : (\omega, E(\omega)) \in \mathcal{U}.$$
As then Proposition 6.3 tells us that for each $\omega \in V$, there is a class $A$, $E(\omega)$-holomorphic, elliptic curve $u$ in $M$. Using Theorem 3.5 we would then conclude that there is an elliptic Reeb 2-curve $u$ in $(M, \omega)$. If $M$ has dimension 4 then in addition $u$ may be assumed to be normal and embedded. If $\omega$ is integral, by Proposition 6.3, $u$ may be assumed to be charge 1. And so we will be done.

Define a metric $\rho_0$ measuring the distance between subspaces $W_1, W_2$, of same dimension, of an inner product space $(T, g)$ as follows.
$$\rho_0(W_1, W_2) := |P_{W_1} - P_{W_2}|,$$
for $|\cdot|$ the $g$-operator norm, and $P_{W_i}$ $g$-projection operators onto $W_i$. We may of course generalize this to a $C^2$ metric $\rho_2$ again in terms of these projection operators.

Let $\delta > 0$ be given. Suppose that $\omega = d^{\alpha'} \lambda'$ is a first kind lcs structure $\delta$-close to $\omega_0$ for the $C^3$ metric $d_3$ as in the statement of the theorem. Then $\mathcal{V}_{\lambda'}, \xi_{\lambda'}$ are smooth distributions by the assumption that $(\alpha', \lambda')$ is a lcs structure of the first kind and $TM = \mathcal{V}_{\lambda'} \oplus \xi_{\lambda'}$. Moreover, for each $p \in M$,
$$\rho_2(\mathcal{V}_{\lambda'}(p), \mathcal{V}_\lambda(p)) < \epsilon_\delta$$
and
$$\rho_2(\xi_{\lambda'}(p), \xi_\lambda(p)) < \epsilon_\delta$$
where $\epsilon_\delta \to 0$ as $\delta \to 0$, and where $\rho_2$ is the metric as defined above for subspaces of the inner product space $(T_pM, g)$.

Then choosing $\delta$ to be suitably be small, for each $p \in M$ we have an isomorphism
$$\phi(p) : T_pM \to T_pM,$$
$\phi_p := P_1 \oplus P_2$, for $P_1 : \mathcal{V}_{\lambda_0}(p) \to \mathcal{V}_{\lambda'}(p)$, $P_2 : \xi_{\lambda_0}(p) \to \xi_{\lambda'}(p)$ the $g$-projection operators. Define $E(\omega)(p) := \phi(p)_* J_0$. Then clearly, if $\delta$ was chosen to be sufficiently small, $E$, defined on the $d_3$ $\delta$-neighborhood $V$, has the needed property. $\square$

*Proof of Theorem 2.16.* Let $\{\omega_t\}$, $t \in [0, 1]$, be a continuous in the usual $C^\infty$ topology homotopy of non-degenerate 2-forms on $M = C \times S^1$, with $\omega_0 = d_\alpha \lambda$ as in the hypothesis and with $\omega_1$ a first kind lcs structure. Fix an almost complex structure $J_1$ on $M$ admissible with respect to $(\alpha', \lambda')$, so that $(M, \lambda', \alpha', J_1)$ is a tamed first kind lcs manifold. And let $J_0$ be the almost complex structure $J^\lambda$, as in Section 5. Extend $J_0, J_1$ to a smooth family $\{J_t\}$ of almost complex structures on $M$, so that $J_t$ is $\omega_t$-compatible for each $t$. Then in the absence of stable holomorphic sky catastrophes, by Theorem 7.11, there is a non-constant charge 1 elliptic $J_1$-holomorphic curve $u$ in $M$. If $M$ has dimension 4 then by the proof of Proposition 6.3 $u$ may be assumed to be normal and embedded. So that the theorem follows by Theorem 3.5. $\square$

**Definition 6.5.** *Let $\alpha$ be a scale integral closed 1-form on a closed smooth manifold $M$. Let $c \in \mathbb{R}$ be the minimal positive real such that $c\alpha$ is integral. A* **classifying map** *$p : M \to S^1$ of $\alpha$ is a smooth map s.t. $c\alpha = p^* d\theta$. A map $p$ with these properties is of course not unique.*

**Lemma 6.6.** *Let $u : \Sigma \to M$ be a Reeb 2-curve in a closed, scale integral, first kind lcs manifold $(M, \lambda, \alpha)$, then its normalization $\widetilde{u} : \widetilde{\Sigma} \to M$ is a Reeb 2-curve.*

*Proof.* By Lemma 6.1 we have a surjective classifying map $p : M \to S^1$ of $\alpha$. Note that the fibers of $p$, $M_t$, $t \in S^1$, are contact with contact form $\lambda_t = \lambda|_{C_t}$, as $0 \neq \omega^n = \alpha \wedge \lambda \wedge d\lambda^{n-1}$ and $c \cdot \alpha = 0$ on $M_t$, where $c$ is as in the definition of $p$.

Let $\widetilde{u} : \widetilde{\Sigma} \to M$ be the normalization of $u$. Suppose it is not a Reeb 2-curve, which by definitions just means that $0 = [\widetilde{u}^* \alpha] \in H^1(\widetilde{\Sigma}, \mathbb{R})$. Since $0 \neq [u^*\alpha] \in H^1(\Sigma, \mathbb{R})$, some node $z_0$ of $\Sigma$ lies on closed loop $o : S^1 \to \Sigma$ with $\langle [o], [u^*\alpha] \rangle \neq 0$.

Let $q_\Sigma : \widetilde{\Sigma} \to \Sigma$ be the quotient map as previously appearing. In this case, we may find a smooth embedding $\eta : D^2 \to \widetilde{\Sigma}$, s.t. $q_\Sigma \circ \eta(D^2)|_{\partial D^2}$ is a component of a regular fiber $C_t$, of the classifying map $p' : \Sigma \to S^1$ of $u^*\alpha$. See Figure 1, $\eta(D^2)$ is a certain disk in $\widetilde{\Sigma}$, whose interior contains an element of $\phi^{-1}(z_0)$.

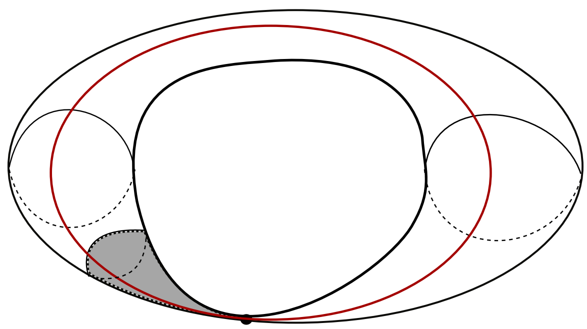


FIGURE 1. The figure for $\Sigma$. The gray shaded area is the image $q_\Sigma \circ \eta(D^2)$. The red shaded curve is the image of the closed loop $o$ as above.

Then analogously to the proof of the first part of Theorem 2.10, $\widetilde{u} \circ \eta|_{\partial D^2}$ is a (unparametrized) $\lambda_t$-Reeb orbit in $M_t$. (The classifying maps can be arranged, such that $u(C_t) \subset M_t$.) And in particular $\int_{\partial D} \widetilde{u}^*\lambda \neq 0$. Now $u$ (not $\widetilde{u}$) is a Reeb 2-curve, and the first condition of this implies that $\int_D d\widetilde{u}^*\lambda = 0$, since $\ker d\lambda$ on $M$ is spanned by $X_\lambda, X_\alpha$. So we have a contradiction to Stokes theorem. Thus, $\widetilde{u}$ must be a Reeb 2-curve.

□

*Proof of Theorem 2.17.* Let $(C, \lambda)$ and $\phi$ be as in the hypothesis and let $M = (M_{\phi,1}, \lambda_\phi, \alpha)$ denote the mapping torus of $\phi$, as also appearing in Theorem 2.6. More specifically, $M = C \times [0, 1]/\sim$, where the equivalence $\sim$ is generated by $(x, 1) \sim (\phi(x), 0)$, for more details on the corresponding lcs structure see for instance [3]. Then $(M, \lambda_\phi, \alpha)$ is an integral first kind lcs manifold. By the hypothesis there is an elliptic Reeb 2-curve in $M$.

We now show that up to parametrization all Reeb 2-curves in $M$ must be of a certain type. Let $o : S^1 \to C$ be a $\lambda$-Reeb orbit and let $\widetilde{o} : S^1 \times I \to C \times I$ be the map $\widetilde{o}(t, \tau) = (o(t), \tau)$. Suppose that image $\phi^n(o) = \text{image}(o)$, for some $n > 0$, then clearly there is a charge $n$ Reeb 2-curve in $M$ with image the image of $q \circ \widetilde{o}$ for $q : C \times [0, 1] \to M_\phi$ the quotient map.

This Reeb 2-curve $u^n_o$ is unique up reparametrization and we call it the *charge $n$ generalized Reeb torus of $o$*. It is not hard to see that, up to parametrization, every Reeb 2-curve in $M$ is a charge $n$

generalized Reeb torus of some $o$ and for some $n$. The formal proof of this is completely analogous to the proof of Theorem 3.4. In particular, up to parametrization $u = u_o^n$ for some $o, n$ and hence image $\phi^n(o) = \text{image}(o)$, so that the conclusion follows. □

*Proof of Theorem 1.4.* Let $(C, \lambda)$ and $\phi$ be as in the hypothesis and let $(M = M_{\phi,1}, \lambda_\phi, \alpha_\phi)$ be the mapping torus of $\phi$ as above. Then $(M, \lambda_\phi, \alpha_\phi)$ is an integral first kind lcs manifold. If $\phi$ is sufficiently $C^\infty$ small then $(\lambda_\phi, \alpha_\phi)$ is $d_3$ $\epsilon$-close to the lcs-fication $(\lambda, \alpha)$ of $\lambda$. Hence, by Theorem 2.14 there is an elliptic charge 1 Reeb 2-curve $u$ in $(M, \lambda_\phi, \alpha_\phi)$. Then by the proof of Theorem 2.17, just above, we conclude that $u$ is the charge 1 Reeb torus of some orbit $o$, and so image $\phi(o) = \text{image}(o)$. □

*Proof of Theorem 2.20.* Suppose we have a formal homotopy of $\phi$ to the identity as in the hypothesis. Then assuming that $i(R^\lambda, \beta) \neq 0$ and assuming that stable holomorphic sky catastrophes do not exist, by Corollary 3.11 we get that the mapping torus $(M_{\phi,1}, \lambda_\phi, \alpha_\phi)$ has a charge 1 Reeb 2-curve $u$. Then as above, we conclude that $u$ is charge 1 Reeb torus of some orbit $o$, and so image $\phi(o) = \text{image}(o)$. □

*Proof of Theorem 1.3.* Let $(C, \lambda)$ and $\phi$ be as in the hypothesis. Let $\lambda_H$ be the Hopf contact form and $\lambda_t$, $t \in [0,1]$, a homotopy of contact forms with $\lambda_0 = \lambda_H$ and $\lambda_1 = \lambda$. We then get a family $(\lambda_t, \alpha)$ of first kind integral lcs structures on $C \times S^1$.

Now let $\phi_t$, $t \in [0,1]$ be a homotopy of strict contactomorphisms of $(C, \lambda)$ with $\phi_0 = id$ and $\phi_1 = \phi$. This gives a smooth fibration over $\widetilde{M} \to [0,1]$, with fiber over $t \in [0,1]$: $M_{\phi_t,1}$, which is moreover endowed with a first kind lcs structure (the mapping torus structure). Let $tr : \widetilde{M} \to C \times S^1 \times [0,1]$ be a smooth trivialization, restricting to the identity $C \times S^1 \to C \times S^1$ over 0. Pushing forward by the bundle map $tr$, the above mentioned family of lcs structures, we get another smooth family $\{(\lambda''_t, \alpha)\}$, $t \in [0,1]$, of first kind integral lcs structures on $C \times S^1$. In addition each $(\lambda''_t, \alpha)$ is isomorphic to some mapping torus structure, as in the Proof of Theorem 2.17.

Concatenating the above families, we get a continuous family of first kind integral lcs structures $\{\omega_t = d_\alpha \lambda'_t\}$, $t \in [0,1]$, with $\lambda'_0 = \lambda_H$ and with $\lambda'_1$ isomorphic to $(\lambda_\phi, \alpha)$. Let $\{J'_t\}$, $t \in [0,1]$ be a family of almost complex structures on $M = C \times S^1$ s.t. $J'_t$ is $\omega_t$-admissible for each $t$.

By the Proof of Theorem 2.16 we get that either $(\lambda_\phi, \alpha)$ has a Reeb 2-curve $u$, and hence by the proof of Theorem 2.20 $\phi$ has a fixed Reeb string, or the family $\{J'_t\}$, $t \in [0,1]$ has a stable holomorphic sky catastrophy, of charge 1, class $0 = A_\beta$ curves.

Suppose that the latter holds. By the admissibility condition and by Theorem 3.5, each element $(u_t, t) \in \mathcal{M}^1_{1,1}(\{J^{\lambda'_t}\}, A_\beta)$ is such that $u_t$ is a charge 1 Reeb 2-curve. And so, by the Proof of Theorem 2.17, up to parametrization $u_t$ is a charge 1 (generalized) Reeb torus in the first kind lcs manifold $(C \times S^1, \lambda'_t, \alpha)$. It immediately follows that the family of vector fields $\{R^{\lambda'_t}\}$, $t \in [0,1]$ has a stable sky catastrophy in class $\beta = 0$, using the correspondence of the (generalized) Reeb tori with orbits. □

*Proof of Theorem 2.22.* Suppose that $u : \Sigma \to M$ is an immersed Reeb 2-curve, we then show that $M$ also has a Reeb 1-curve. Let $\widetilde{u} : \widetilde{\Sigma} \to M$ be the normalization of $u$, so that $\widetilde{u}$ is an immersion. We have a pair of transverse 1-distributions $D_1 = \widetilde{u}^*\mathbb{R}\langle X_\alpha \rangle$, $D_2 = \widetilde{u}^*\mathbb{R}\langle X_\lambda \rangle$ on $\widetilde{\Sigma}$. We may then find an embedded path $\gamma : [0,1] \to \widetilde{\Sigma}$, tangent to $D_1$ s.t. $\lambda(\gamma'(t)) > 0$, $\forall t \in [0,1]$, and s.t. $\gamma(0)$ and $\gamma(1)$ are on a leaf of $D_2$. It is then simple to obtain from this a Reeb 1-curve $o$, by joining the end points of $\gamma$ by an embedded path tangent to $D_2$, and perturbing, see Figure 2. This proves the first part of the theorem.

To prove the second part, suppose that $u : \Sigma \to M$ is an immersed elliptic Reeb 2-curve. Suppose that $u$ is not normal. Let $\widetilde{u} : \widetilde{\Sigma} \to M$ be its normalization. Then $\widetilde{\Sigma}$ has a genus 0 component $\mathcal{S}$. So that $\widetilde{u} : \mathcal{S} \simeq \mathbb{CP}^1 \to M$ is immersed. The distribution $D_1 = \widetilde{u}^*\mathbb{R}\langle X_\alpha \rangle$, as appearing above, is then a $\widetilde{u}^*\lambda$-oriented 1-dimensional distribution on $\mathbb{CP}^1$ which is impossible. □

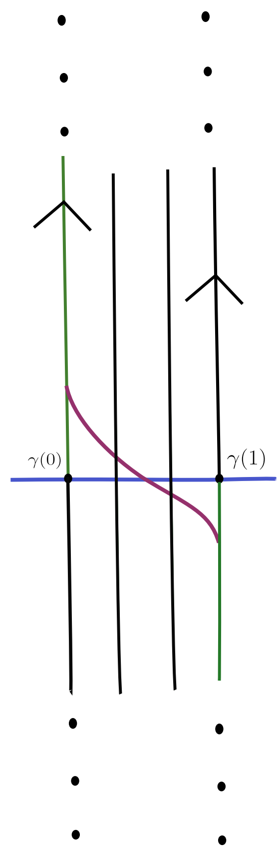


FIGURE 2. The green shaded path is $\gamma$, the indicated orientation is given by $u^*\lambda$, the $D_1$ folliation is shaded in black, the $D_2$ folliation is shaded in blue. The purple segment is part of the loop $o: S^1 \to \Sigma$, which is is smooth and satisfies $\lambda(o'(t)) > 0$ for all $t$.

## 7. Extended Gromov-Witten invariants and the extended Fuller index

Let $(M, \omega, J, \alpha)$ be such that $\omega$ a non-degenerate 2-form on $M$ compatible with the almost complex structure $J$ and $\alpha$ a closed 1-form non-vanishing in cohomology. Recall the definition of charge from Section 3. We fix an $n_0 \in \mathbb{N}$. To slightly simplify the notation, the charge $n_0$ of our curves will be mostly implicit, so that we will not specify this in notation. Likewise $M$ will be implicit. Thus $\overline{\mathcal{M}}_{1,1}(J, A)$ will be shorthand for $\overline{\mathcal{M}}_{1,1}^{n_0}(M, \omega, \alpha, J, A)$, with the latter denoting the space $\alpha$-charge $n_0$, class $A$, $J$-curves in $M$ as previously.

In what follows, for an almost symplectic pair $(\omega, J)$, and $u : \Sigma \to M$ a stable $J$-holomorphic map,

$$e(u) := e_\omega := \int_\Sigma u^*\omega.$$

**Definition 7.1.** *Let $h = \{(\omega_t, J_t)\}$ be a homotopy of almost symplectic pairs on $M$, such that $\{J_t\}$ is Frechet smooth, and such that $\{\omega_t\}$ $C^0$ continuous. We say that this homotopy is* **partially admissible for** *$A$ if every element of*

$$\overline{\mathcal{M}}_{1,1}(J_0, A)$$

*is contained in a compact open subset of $\overline{\mathcal{M}}_{1,1}(\{J_t\}, A)$. We say that $h$ is* **admissible for** *$A$ if every element of*

$$\overline{\mathcal{M}}_{1,1}(J_i, A),$$

$i = 0, 1$ *is contained in a compact open subset of* $\overline{\mathcal{M}}_{1,1}(\{J_t\}, A)$.

Thus, in the above definition, a homotopy is partially admissible if there are no sky catastrophes going one way, and admissible if there are no sky catastrophes going either way. To simplify notation, we denote by a capital $X$ a pair $(\omega, J)$ as above. Then we introduce the following simplified notation.

$$
\begin{aligned}
S(X, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(J, A)\} \\
S(X, a, A) &= \{u \in S(X, A) \mid e(u) \leq a\} \\
S(h, A) &= \{(u, t) \in \overline{\mathcal{M}}_{1,1}(\{J_t\}, A)\}, \text{ for } h = \{(\omega_t, J_t)\} \text{ a homotopy as above} \\
S(h, a, A) &= \{(u, t) \in S(h, A) \mid e_{\omega_t}(u) \leq a\}
\end{aligned} \tag{7.2}
$$

**Definition 7.3.** *For an isolated element* $u$ *of* $S(X, A)$*, which means that* $\{u\}$ *is open as a subset, we set* $gw(u) \in \mathbb{Q}$ *to be the local Gromov-Witten invariant of* $u$*. This is defined as:*

$$gw(u) = GW_{1,1}(\{u\}, A, J)([\overline{M}_{1,1}] \otimes [M]),$$

*with the right-hand side as in* (4.2)*, (with charge again implicit.)*

Denote by $\mathcal{SM}(A)$ the set of equivalence classes of all smooth stable maps $\Sigma \to M$, in class $A$, for $\Sigma$ an (non-fixed) elliptic curve, and where equivalence has the same meaning as in Section 3.

**Definition 7.4.** *Suppose that* $S(X, A)$ *has open connected components. And suppose that we have a collection of almost symplectic pairs on* $M$

$$\{X^a = (\omega^a, J^a)\}, \, a \in \mathbb{R}_+$$

*satisfying the following:*

- $S(X^a, a, A)$ *consists of isolated curves for each* $a$*.*
- $$S(X^a, a, A) = S(X^b, a, A),$$
  *(equality of subsets of* $\mathcal{SM}(A)$*) if* $b > a$*,*
- *For* $b > a$*, and for each* $u \in S(X^a, a, A) = S(X^b, a, A)$*:*
  $$GW_{1,1}(\{u\}, A, J^a) = GW_{1,1}(\{u\}, A, J^b),$$
  *thus we may just write* $gw(u)$ *for the common number.*
- *There is a prescribed homotopy (as in Definition 7.1)* $h^a = \{X^a_t\}$ *of each* $X^a$ *to* $X$*, called* **structure homotopy***, with the following property. For every*
  $$y \in S(X^a_0, A)$$
  *there is an open compact subset* $\mathcal{C}_y \subset S(h^a, A)$*,* $y \in C_y$*, which is* **non-branching***, where the latter means that*
  $$\mathcal{C}_y \cap S(X^a_i, A),$$
  $i = 0, 1$ *are connected.*
- $$S(h^a, a, A) = S(h^b, a, A),$$
  *(similarly equality of subsets) if* $b > a$ *is sufficiently large.*

*We will then say that*

$$\mathcal{P}(A) = \{(X^a, h^a)\}$$

*is a* **perturbation system** *for* $X$ *in the class* $A$*.*

We shall see shortly that, given a contact $(C,\lambda)$, with $\lambda$ Morse-Bott, its lcs-fication $(C\times S^1,\lambda,\alpha)$ always admits a perturbation system for the moduli spaces of charge 1 (or $n$) curves in any class.

**Definition 7.5.** *Suppose that $X$ admits a perturbation system $\mathcal{P}(A)$ such that there exists an $E = E(\mathcal{P}(A))$ with the property that*
$$S(X^a,a,A) = S(X^E,a,A)$$
*for all $a > E$, where this as before is equality of subsets (as previously), and the local Gromov-Witten invariants of the identified elements are also identified. Then we say that $X$ is* **finite type** *and set:*
$$GW(X,A) = \sum_{u\in S(X^E,A)} gw(u).$$

**Definition 7.6.** *Suppose that $X$ admits a perturbation system $\mathcal{P}(A)$ and there is an $E = E(\mathcal{P}(A)) > 0$ such that $gw(u) > 0$ for all*
$$\{u \in S(X^a,A) \,|\, E \le e(u) \le a\}$$
*respectively $gw(u) < 0$ for all*
$$\{u \in S(X^a,A) \,|\, E \le e(u) \le a\},$$
*and every $a > E$. Suppose in addition that*
$$\lim_{a\mapsto\infty} \sum_{u\in S(X,a,A)} gw(u) = \infty, \text{ respectively } \lim_{a\mapsto\infty} \sum_{u\in S(X,a,\beta)} gw(u) = -\infty.$$
*Then we say that $X$ is* **positive infinite type***, respectively* **negative infinite type** *and set*
$$GW(X,A) = \infty,$$
*respectively $GW(X,A) = -\infty$. These are meant to be interpreted as extended Gromov-Witten invariants, counting elliptic curves in class $A$. We say that $X$ is* **infinite type** *if it is one or the other.*

**Definition 7.7.** *We say that $X$ is* **definite** *type if it admits a perturbation system and it is infinite type or finite type.*

With the above definitions
$$GW(X,A) \in \mathbb{Q} \sqcup \infty \sqcup -\infty,$$
when it is defined.

*Proof of Theorem 3.6.* Given the definitions above, and the definition of the extended Fuller index in [27], this follows by the same argument as the proof of Theorem 5.5. □

7.0.1. *Perturbation systems for the lcs-fication of Morse-Bott $(C,\lambda)$.*

**Definition 7.8.** *A contact form $\lambda$ on $M$, and its associated flow $R^\lambda$ are called* Morse-Bott *if the $\lambda$ action spectrum $\sigma(\lambda)$ - that is the space of critical values of $o \mapsto \int_{S^1} o^*\lambda$, $o: S^1 \to M$, is discreet and if for every $a \in \sigma(\lambda)$, the space*
$$N_a := \{x \in M |\, F_a(x) = x\},$$
*$F_a$ the time $a$ flow map for $R^\lambda$ - is a closed smooth manifold such that rank $d\lambda|_{N_a}$ is locally constant and $T_xN_a = \ker(dF_a - I)_x$.*

**Proposition 7.9.** *Let $\lambda$ be a contact form of Morse-Bott type, on a closed contact manifold $C$. Then the corresponding (lcs-fication) structure $X_\lambda = (d_\alpha\lambda, J^\lambda)$ on $C\times S^1$, admits a perturbation system $\mathcal{P}(A)$, for every class $A$ and every charge $n_0$, computed with respect to $\alpha$.*

*Proof.* This follows immediately by [27, Proposition 2.12 ], and by Proposition 3.4. □

**Lemma 7.10.** *Let $\lambda_H$ be the Hopf contact form on $S^{2k+1}$ as previously. Then the almost symplectic pair $X = (d_\alpha \lambda_H, J^{\lambda_H})$ on $S^{2k+1} \times S^1$ is infinite type.*

*Proof.* This follows immediately by [27, Lemma 2.13], and by Proposition 3.4. □

**Theorem 7.11.** *Let $(C, \lambda)$ be a closed contact manifold such that $R^\lambda$ has definite type, and such that $i(R^\lambda, \beta) \neq 0$. Let $\omega_0 = d_\alpha \lambda$ be the lcs-fication, and suppose we have a partially admissible homotopy $h = \{(\omega_{t,}, J_t\}$, for class $A_\beta$, then there in an element $u \in \overline{\mathcal{M}}^1_{1,1}(J_1, A_\beta)$.*

The proof of this will follow.

## 7.1. Preliminaries on admissible homotopies.

**Definition 7.12.** *Let $h = \{X_t\}$ be a homotopy as in Definition 7.1. For $b > a > 0$ we say that $h$ is* **partially $a, b$-admissible**, *respectively* **$a, b$-admissible** *(in class $A$) if for each*

$$y \in S(X_0, a, A)$$

*there is a compact open subset $\mathcal{C}_y \subset S(h, A)$, $y \in \mathcal{C}_y$ with $e(u) < b$, for all $u \in \mathcal{C}_y$. Respectively, if for each*

$$y \in S(X_i, a, A),$$

*$i = 0, 1$ there is a compact open subset $\mathcal{C}_y \ni y$ of $S(h, A)$ with $e(u) < b$, for all $u \in \mathcal{C}_y$.*

**Lemma 7.13.** *Suppose that $X_0$ has a perturbation system $\mathcal{P}(A)$, and $\{X_t\}$ is partially admissible, then for every $a$ there is a $b > a$ so that $\{\widetilde{X}^b_t\} = \{X_t\} \cdot \{X^b_t\}$ is partially $a, b$-admissible, where $\{X_t\} \cdot \{X^b_t\}$ is the (reparametrized to have $t$ domain $[0, 1]$) concatenation of the homotopies $\{X_t\}, \{X^b_t\}$, and where $\{X^b_t\}$ is the structure homotopy from $X^b$ to $X_0$.*

*Proof.* This is a matter of pure topology, and the proof is completely analogous to the proof of [27, Lemma 3.8]. □

The analogue of Lemma 7.13 in the admissible case is the following:

**Lemma 7.14.** *Suppose that $X_0, X_1$ and $\{X_t\}$ are admissible, then for every $a$ there is a $b > a$ so that*

$$\{\widetilde{X}^b_t\} = \{X^b_{1,t}\}^{-1} \cdot \{X_t\} \cdot \{X^b_{0,t}\} \tag{7.15}$$

*is $a, b$-admissible, where $\{X^b_{i,t}\}$ are the structure homotopies from $X^b_i$ to $X_i$.*

## 7.2. Invariance.

**Theorem 7.16.** *Suppose $X_0$ is definite type, with $GW(X_0, A) \neq 0$, and suppose it is joined to $X_1$ by a partially admissible homotopy $\{X_t\}$, then $X_1$ has non-constant elliptic class $A$ curves.*

*Proof of Theorem 7.11.* This follows by Theorem 7.16 and by Theorem 3.6. □

We also have a more precise result.

**Theorem 7.17.** *If $X_0, X_1$ are definite type pairs and $\{X_t\}$ is admissible then $GW(X_0, A) = GW(X_1, A)$.*

*Proof of Theorem 7.16.* Suppose that $X_0$ is definite type with $GW(X_0, A) \neq 0$, $\{X_t\}$ is partially admissible and $\overline{\mathcal{M}}_{1,1}(X_1, A) = \emptyset$. Let $a$ be given and $b$ determined so that $\widetilde{h}^b = \{\widetilde{X}_t^b\}$ is a partially $(a, b)$-admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\widetilde{h}^b, A),$$

for $y \in S(X_0^b, a, A)$. Here we use a natural identification of $S(X^b, a, A) = S(\widetilde{X}_0^b, a, A)$ as a subset of $S(\widetilde{h}^b, A)$ by its construction. Then $S_a$ is an open-compact subset of $S(h, A)$ and so admits an implicit atlas with boundary, (with virtual dimension 1) s.t:

$$\partial S_a = S(X^b, a, A) + Q_a,$$

where $Q_a$ as a set is some subset (possibly empty), of elements $u \in S(X^b, b, A)$ with $e(u) \geq a$. So we have for all $a$:

$$\sum_{u \in Q_a} gw(u) + \sum_{u \in S(X^b, a, A)} gw(u) = 0. \tag{7.18}$$

7.3. **Case I, $X_0$ is finite type.** Let $E = E(\mathcal{P})$ be the corresponding cutoff value in the definition of finite type, and take any $a > E$. Then $Q_a = \emptyset$ and by definition of $E$ we have that the left side is

$$\sum_{u \in S(X^b, E, A)} gw(u) \neq 0.$$

Clearly this gives a contradiction to (7.18).

7.4. **Case II, $X_0$ is infinite type.** We may assume that $GW(X_0, A) = \infty$, and take $a > E$, where $E = E(\mathcal{P}(A))$ is the corresponding cutoff value in the definition of infinite type. Then

$$\sum_{u \in Q_a} gw(u) \geq 0,$$

as $a > E(\mathcal{P}(A))$. On the other hand,

$$\lim_{a \mapsto \infty} \sum_{u \in S(X^b, a, A)} gw(u) = \infty,$$

as $GW(X_0, A) = \infty$. This also contradicts (7.18). $\square$

*Proof of Theorem 7.17.* This is somewhat analogous to the proof of Theorem 7.16. Suppose that $X_i, \{X_t\}$ are definite type as in the hypothesis. Let $a$ be given and $b$ determined so that $\widetilde{h}^b = \{\widetilde{X}_t^b\}$, see (7.15) is an $(a, b)$-admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\widetilde{h}^b, A)$$

for $y \in S(X_i^b, a, A)$. Then $S_a$ is an open-compact subset of $S(h, A)$ and so has admits an implicit atlas with boundary, (with virtual dimension 1) and satisfies the following.

$$\partial S_a = (S(X_0^b, a, A) + Q_{a,0})^{op} + S(X_1^b, a, A) + Q_{a,1},$$

with *op* denoting the opposite orientation, where $Q_{a,i}$ as sets are some subsets (possibly empty), of elements $u \in S(X_i^b, b, A)$ with $e(u) \geq a$. So we have for all $a$:

$$\sum_{u \in Q_{a,0}} gw(u) + \sum_{u \in S(X_0^b, a, A)} gw(u) = \sum_{u \in Q_{a,1}} gw(u) + \sum_{u \in S(X_1^b, a, A)} gw(u). \tag{7.19}$$

7.5. **Case I, $X_0$ is finite type and $X_1$ is infinite type.** Suppose in addition $GW(X_1, A) = \infty$ and let $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$, for $\mathcal{P}_i$, the perturbation systems of $X_i$. Take any $a > E$. Then $Q_{a,0} = \emptyset$ and the left-hand side of (7.19) is

$$\sum_{u \in S(X_0^b, E, A)} gw(u) < \infty.$$

The right-hand side tends to $\infty$ as $a$ tends to infinity since,

$$\sum_{u \in Q_{a,1}} gw(u) \geq 0,$$

as $a > E(\mathcal{P}_1(A))$, and since

$$\lim_{a \mapsto \infty} \sum_{u \in S(X_1^b, a, A)} gw(u) = \infty.$$

Clearly this gives a contradiction to (7.19).

7.6. **Case II, $X_i$ are infinite type.** Suppose that $GW(X_0, A) = -\infty$, $GW(X_1, A) = \infty$ and let $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$, for $\mathcal{P}_i$, the perturbation systems of $X_i$. Take any $a > E$. Then $\sum_{u \in Q_{a,0}} gw(u) \leq 0$, and $\sum_{u \in Q_{a,1}} gw(u) \geq 0$. So by definition of $GW(X_i, A)$ the left hand side of (7.18) tends to $-\infty$ as $a$ tends to $\infty$, and the right hand side tends to $\infty$. Clearly this gives a contradiction to (7.19).

7.7. **Case III, $X_i$ are finite type.** The argument is analogous. $\square$

## A. Fuller index

Let $X$ be a vector field on $M$. Set

$$S(X) = S(X, \beta) = \{o \in L_\beta M \mid \exists p \in (0, \infty),\ o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\}, \tag{A.1}$$

where $L_\beta M$ denotes the free homotopy class $\beta$ component of the free loop space

$$LM = \{o : S^1 \to M \mid o \text{ is smooth}\}.$$

And where recall that $S^1 = \mathbb{R}/\mathbb{Z}$. The above $p$ is uniquely determined and we denote it by $p(o)$ called the period of $o$.

There is a natural $S^1$ reparametrization action on $S(X)$: $t \cdot o$ is the loop $t \cdot o(\tau) = o(t + \tau)$. The elements of $\mathcal{O}(X) := S(X)/S^1$ will be called *unparametrized orbits*, or just orbits. Slightly abusing notation we just write $o$ for the equivalence class of $o$.

The multiplicity $m(o)$ of a periodic orbit is the ratio $p(o)/l$ for $l > 0$ the period of a simple orbit covered by $o$.

We want a kind of fixed point index which counts orbits $o$ with certain weights. Assume for simplicity that $N \subset \mathcal{O}(X)$ is discreet. (Otherwise we need to perturb.) Then to such an $(N, X, \beta)$ Fuller associates an index:

$$i(N, X, \beta) = \sum_{o \in N} \frac{1}{m(o)} i(o),$$

where $i(o)$ is the fixed point index of the time $p(o)$ return map of the flow of $X$ with respect to a local surface of section in $M$ transverse to the image of $o$.

Fuller then shows that $i(N, X, \beta)$ has the following invariance property. For a continuous homotopy $\{X_t\}$, $t \in [0, 1]$ set

$$S(\{X_t\}) = S(\{X_t\}, \beta) = \{(o, t) \in L_\beta M \times [0, 1] \mid o \in S(X_t)\}.$$

And given a continuous homotopy $\{X_t\}$, $X_0 = X$, $t \in [0,1]$, suppose that $\widetilde{N}$ is an open compact subset of $S(\{X_t\})/S^1$, such that

$$\widetilde{N} \cap (L_\beta M \times \{0\})/S^1 = N.$$

Then if

$$N_1 = \widetilde{N} \cap (L_\beta M \times \{1\})/S^1$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta).$$

In the case where $X$ is the $R^\lambda$-Reeb vector field on a contact manifold $(C^{2n+1}, \lambda)$, and if $o$ is non-degenerate, we have:

$$i(o) = \operatorname{sign} \operatorname{Det}(\mathrm{Id}|_{\xi(x)} - F^\lambda_{p(o),*}|_{\xi(x)}) = (-1)^{CZ(o)-n}, \tag{A.2}$$

where $F^\lambda_{p(o),*}$ is the differential at $x$ of the time $p(o)$ flow map of $R^\lambda$, and where $CZ(o)$ is the Conley-Zehnder index, see [25].

There is also an extended Fuller index $i(X, \beta) \in \mathbb{Q} \sqcup \{\pm\infty\}$, for certain $X$ having definite type. This is constructed in [27], and is conceptually analogous to the extended Gromov-Witten invariant described in this paper.

The following technical notion is useful in connection with Gromov-Witten theory and is used in some of the statements appearing in the introduction. Let $p : \mathcal{O}(X) \to \mathbb{R}_+$ be as above.

**Definition A.3.** *We say that $N \subset \mathcal{O}(X)$ is* **good** *if it is open, compact and if it is energy isolated meaning:*

$$\exists E_0 > 0\, \exists E_1\, \exists \epsilon > 0 : N \subset \left(U = p^{-1}(E^0, E^1)\right) \subset \left(V = p^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

*with $V \cap \mathcal{O}(X) = N$.*

The following is the definition of sky catastrophes first appearing in Savelyev [27], generalizing a notion commonly called a "blue sky catastrophe", see Shilnikov-Turaev [30].

**Definition A.4.** *Let $\{X_t\}$, $t \in [0,1]$ be a continuous family of non-zero, smooth vector fields on a closed manifold $M$, and $\beta$ free homotopy class of a loop as above. And let $S(\{X_t\})$ be as above. We say that $\{X_t\}$ has a* **sky catastrophe in class** *$\beta$, if there is an element*

$$y \in S(X_0) \subset S(\{X_t\})$$

*so that there is no open compact subset of $S(\{X_t\})$ containing $y$. We say that the sky catastrophe is* **stable** *if the same holds for any $C^0$ nearby $C^\infty$ family $\{X'_t\}$ satisfying $X'_0 = X_0$ and $X'_1 = X_1$.*

## B. Remark on multiplicity

This is a small note on how one deals with curves having non-trivial isotropy groups, in the virtual fundamental class technology. We primarily need this for the proof of Theorem 5.5.

Given a closed oriented orbifold $X$, with an orbibundle $E$ over $X$ Fukaya-Ono [11] show how to construct using multi-sections its rational homology Euler class, which when $X$ represents the moduli space of some stable curves, is the virtual moduli cycle $[X]^{vir}$. When this is in degree 0, the corresponding Gromov-Witten invariant is $\int_{[X]^{vir}} 1$. However, they assume that their orbifolds are effective. This assumption is not really necessary for the purpose of construction of the Euler class but is convenient for other technical reasons. A different approach to the virtual fundamental class which emphasizes branched manifolds is used by McDuff-Wehrheim, see for example McDuff [15], [18] which does not have the effectivity assumption, a similar use of branched manifolds appears in [6]. In the case of a non-effective orbibundle $E \to X$ McDuff [17], constructs a homological Euler class $e(E)$ using multi-sections, which extends the construction [11]. McDuff shows that this class $e(E)$ is Poincare dual to the completely formally natural cohomological Euler class of $E$, constructed by other authors. In other words there is a natural notion of a homological Euler class of a possibly non-effective orbibundle. We shall assume the following black box property of the virtual fundamental class technology.

**Axiom B.1.** *Suppose that the moduli space of stable maps is cleanly cut out, which means that it is represented by a (non-effective) orbifold $X$ with an orbifold obstruction bundle $E$, that is the bundle over $X$ of cokernel spaces of the linearized CR operators. Then the virtual fundamental class $[X]^{vir}$ coincides with $e(E)$.*

Given this axiom it does not matter to us which virtual moduli cycle technique we use. It is satisfied automatically by the construction of McDuff-Wehrheim, (at the moment in genus 0, but surely extending). It can be shown to be satisfied in the approach of John Pardon [24]. And it is satisfied by the construction of Fukaya-Oh-Ono-Ohta [9], the latter is communicated to me by Kaoru Ono. When $X$ is 0-dimensional this does follow immediately by the construction in [11], taking any effective Kuranishi neighborhood at the isolated points of $X$, (this actually suffices for our paper.)

As a special case most relevant to us here, suppose we have a moduli space of elliptic curves in $X$, which is regular with expected dimension 0. Then its underlying space is a collection of oriented points. However, as some curves are multiply covered, and so have isotropy groups, we must treat this is a non-effective 0 dimensional oriented orbifold. The contribution of each curve $[u]$ to the Gromov-Witten invariant $\int_{[X]^{vir}} 1$ is $\frac{\pm 1}{[\Gamma([u])]}$, where $[\Gamma([u])]$ is the order of the isotropy group $\Gamma([u])$ of $[u]$, in the McDuff-Wehrheim setup this is explained in [15, Section 5]. In the setup of Fukaya-Ono [11] we may readily calculate to get the same thing taking any effective Kuranishi neighborhood at the isolated points of $X$.

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