GROMOV-WITTEN THEORY OF LOCALLY CONFORMALLY SYMPLECTIC MANIFOLDS

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Abstract. We study here Gromov-Witten theory of almost complex and locally conformally symplectic manifolds or l. c. s. manifolds for short. The main new phenomenon that one has to deal with in the general almost complex case, (aside from unbounded energy) is the existence of holomorphic sky catastrophes, an analogue of sky catastrophes in dynamical systems originally discovered by Fuller. We are able to rule these out in some specific geometric situations, particularly for certain l. c. s. 4-folds, and as one application we show that in dimension 4 the classical Gromov non-squeezing theorem has certain C^0 rigidity or persistence with respect to l. c. s. deformations, in particular this shows that there are non-obvious global invariants of the l. c. s. structure itself, a first result of its kind. Whenever Gromov non-squeezing is non-persistent as above we show that there must be a sky catastrophe. In a different direction we study Gromov-Witten theory of the l. c. s. m. $C \times S^1$ induced by a contact manifold (C, λ), and show that the Gromov-Witten invariant (as defined here) counting certain elliptic curves in $C \times S^1$ is identified with the classical Fuller index of the Reeb vector field R^λ. This has some non-classical applications, and based on the story we develop, we give a kind of "holomorphic Seifert/Weinstein conjecture" which is a direct extension for some types of l. c. s. m.'s of the classical Seifert/Weinstein conjecture. This is proved for l. c. s. structures C^∞ nearby to the Hopf l. c. s. structure on $S^{2k+1} \times S^1$.

Contents

1. Introduction

The theory of pseudo-holomorphic curves in symplectic manifolds as initiated by Gromov and Floer has revolutionized the study of symplectic and contact manifolds. What the symplectic form gives that is missing for a general almost complex manifold is apriori compactness for moduli spaces. On the other hand there is a very natural structure which directly generalizes both symplectic and contact manifolds, called locally conformally symplectic structure of l. c. s. for short. A locally conformally symplectic manifold is a smooth $2n$-fold M with a non-degenerate 2-form ω, which is locally diffeomorphic to $e^f \omega_0$, for some (non-fixed) function f, with ω_{st} the standard symplectic form on $\mathbb{R}^{2n}$. It is very

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natural to try to do Gromov-Witten theory for such manifolds. The first problem that occurs is that apriori compactness is gone as since ω is not necessarily closed, the L^2-energy can now be unbounded on the moduli spaces J-holomorphic curves in such a (M, ω). Strangely a more accute problem is potential presence of sky catastrophes: given a smooth family $\{J_t\}$ of $\{\omega_t\}$, $t \in [0,1]$, compatible almost complex structures, we may have a continuous family $\{u_t\}$ of J_t-holomorphic curves s.t. energy$(u_t) \mapsto \infty$ as $t \mapsto a \in (0,1)$ and s.t. there are no holomorphic curves for $t \geq a$. These are analogues of sky catastrophes discovered by Fuller [8].

We are able to tame these problems in certain situations, for example for some 4-d l. c. s. m.'s, and this is how we arrive at a version of Gromov non-squeezing theorem for such l. c. s. m.'s. Even when it is impossible to tame these problems we show that there is still a potentially interesting theory which is analogous to the theory of Fuller index in dynamical systems. For example we show that there is a direct generalization of Seifert, Weinstein conjectures for certain l. c. s. m.'s, and which postulates existence of certain elliptic curves in these l. c. s. m.'s. We prove this conjecture for l. c. s. structures C^∞ nearby to the Hopf l. c. s. structure on $S^{2k+1} \times S^1$.

We begin with the well known observation:

Theorem 1.1. *[[22]] Let (M, J) be a compact almost complex manifold, Σ a closed Riemann surface, and $u : \Sigma \to M$ be J-holomorphic map. Given a Riemannian metric g on M, there is an $\hbar > 0$ s.t. if* energy$_g(u) < \hbar$ *then u is constant, where* energy *is the L^2 energy.*

Using this we get the following extension of Gromov compactness to this setting. Let

$$\mathcal{M}_{g,n}(J, A) = \mathcal{M}_{g,n}(M, J, A)$$

denote the moduli space of isomorphism classes of class A, J-holomorphic curves in M, with domain a genus g closed Riemann surface, with n marked labeled points. Here an isomorphism between $u_1 : \Sigma_1 \to M$, and $u_2 : \Sigma_2 \to M$ is a biholomorphism of marked Riemann surfaces $\phi : \Sigma_1 \to \Sigma_2$ s.t. $u_2 \circ \phi = u_1$.

Theorem 1.2. *Let (M, J) be an almost complex manifold. Then $\mathcal{M}_{g,n}(J, A)$ has a completion*

$$\overline{\mathcal{M}}_{g,n}(J, A),$$

by Kontsevich stable maps. The completion here is in the sense of the natural metrizable Gromov topology see for instance [15], for genus 0 case. Moreover given $E > 0$, the subspace $\overline{\mathcal{M}}_{g,n}(J, A)_E \subset \overline{\mathcal{M}}_{g,n}(J, A)$ consisting of elements u with energy$(u) \leq E$ *is compact. In other words* energy *is a proper and bounded from below function.*

Thus the most basic situation where we can talk about Gromov-Witten "invariants" of (M, J) is when the energy function is bounded on $\overline{\mathcal{M}}_{g,n}(J, A)$, and we shall say that J is ***bounded*** (in class A). In this case $\overline{\mathcal{M}}_{g,n}(J, A)$ is compact, and has a virtual moduli cycle following for example [16]. So we may define functionals:

$$GW_{g,n}(\omega, A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}. \tag{1.3}$$

Of course symplectic manifolds with any tame almost complex structure is one class of examples, another class of examples comes from some locally conformally symplectic manifolds.

1.1. **Locally conformally symplectic manifolds and Gromov non-squeezing.** Let us give a bit of background on l. c. s. m.'s. These were originally considered by Lee in [11], arising naturally as part of an abstract study of "even" dimensional Riemannian geometry, and then further studied by a number of authors see for instance, [1] and [20]. This is a fascinating object, an l. c. s. admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and Murphy show that if a closed almost complex $2n$-fold M has $H^1(M, \mathbb{R}) \neq 0$ then it admits a l. c. s. structure, [5], see also [2].

As we mentioned l. c. s. m.'s can also be understood to generalize contact manifolds. This works as follows. First we have a natural class of explicit examples of l. c. s. m.'s, obtained by starting with a symplectic cobordism of a closed contact manifold C to itself, arranging for the contact forms at the two ends of the cobordism to be proportional (which can always be done) and then gluing together the boundary components. As a particular case of this we get Banyaga's basic example.

Example 1 (Banyaga). Let (C, ξ) be a contact manifold with a contact form λ and take $M = C \times S^1$ with 2-form $\omega = d^{\alpha}\lambda := d\lambda - \alpha \wedge \lambda$, for α the pull-back of the volume form on S^1 to $C \times S^1$ under the projection.

Using above we may then faithfully embedd the category of contact manifolds, and contactomorphism into the category of l. c. s. m.'s, and ***loose*** l. c. s. m. morphisms. These can be defined as diffeomorphisms $f : (M_1, \omega_1) \to (M_2, \omega_2)$ s.t. $f^*\omega_2$ is deformation equivalent through l. c. s. structures to ω_1. Note that when ω_i are symplectic this is just a global conformal symplectomorphism by Moser's trick.

Banyaga type l. c. s. m.'s give immediate examples of almost complex manifolds where the energy function is unbounded, as well as where null-homologous J-holomorphic curves can be non-constant. We show that it is still possible to extract a variant of Gromov-Witten theory here. The story is closely analogous to that of the Fuller index in dynamical systems, which is concerned with certain rational counts of periodic orbits. In that case sky catastrophes prevent us from obtaining a completely well defined invariant, but Fuller constructs certain partial invariants which give dynamical information. In a very particular situation the relationship with the Fuller index becomes perfect as one of the results of this paper obtains the classical Fuller index for Reeb vector fields on a contact manifold C as a certain genus 1 Gromov-Witten invariant of the l. c. s. m. $C \times S^1$. The latter also gives a conceptual interpretation for why the Fuller index is rational, as it is reinterpreted as an (virtual) orbifold Euler number.

1.1.1. *Non-squeezing.* One of the most fascinating early results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [10]. The most well known formulation of this is that there does not exist a symplectic embedding $B_R = B^{2n}(R) \to D^2(r) \times \mathbb{R}^{2n-2}$ for $R > r$, with $B^{2n}(R)$ the standard radius R ball centered at 0 in $\mathbb{R}^{2n}$. We shall give a certain generalization of the above for some four dimensional l. c. s. m.'s.

Definition 1.4. *Given a pair of* l. c. s. m.*'s (M_i, ω_i), $i = 0, 1$, we say that $f : M_1 \to M_2$ is a* **morphism***, if $f^*\omega_2 = \omega_1$. A morphism is called an* l. c. s. *embedding if it is injective.*

A pair (ω, J) for ω l. c. s. and J compatible will be called a compatible l. c. s. pair, or just a compatible pair, where there is no confusion. The following tells us that the non-squeezing problem for l. c. s. m. can be is closely tied to pseudo-holomorphic curve theory as in the symplectic case. Note that in what follows we take the C^0 norm on the space of l. c. s. structures that is stronger then the natural norm (with respect to a metric) on the space of forms.

Theorem 1.5. *The following alternative holds. Let ω be the standard product symplectic form on $M = S^2 \times T^{2n}$, with $\langle \omega, A = [S^2 \times \{pt\}] \rangle = \pi r^2$. Let $R > r$, then either there is an $\epsilon > 0$ s.t. if ω_1 is an* l. c. s. *on M C^0 ϵ-close to ω, then there is no* l. c. s. *embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1),$$

or there is a compatible l. c. s. *family $(\{\omega_t\}, \{J_t\})$ on $S^2 \times T^{2n}$ with a sky catastrophe (the definition is given in the section just below).*

Note that the pair of hypersurfaces $\Sigma_1 = S^2 \times S^1 \times \{pt\} \subset S^2 \times T^2$, $\Sigma_2 = S^2 \times \{pt\} \times S^1 \subset S^2 \times T^2$ are naturally foliated by symplectic spheres, we denote by $T^{fol}\Sigma$ the sub-bundle of the tangent bundle consisting of vectors tangent to the foliation. The following theorem says that it is impossible to have certain "nearby" l. c. s. embeddings.

Theorem 1.6. *Let ω be the standard product symplectic form on $M = S^2 \times T^2$, and A as above with $\langle \omega, A \rangle = \pi r^2$. Let $R > r$, then there is an $\epsilon > 0$ s.t. if ω_1 is an* l. c. s. *on M C^0 ϵ-close to ω, then there is no* l. c. s. *embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1),$$

s.t $\phi_* j$ preserves the $T^{fol}\Sigma_i$.

For a nearby symplectic manifold the above follows by Gromov's argument, without restrictions. We note that the image of the embedding ϕ would be of course a symplectic submanifold of (M, ω_1). However it could be highly distorted, so that it might be impossible to complete $\phi_* \omega_{st}$ to a symplectic form on M nearby to ω, (and perhaps impossible to complete to any symplectic form). We also note that it is certainly possibly to have a nearby volume preserving as opposed to l. c. s. embedding which satisfies all other conditions. Take $\omega = \omega_1$, then if the symplectic form on T^2 has enough volume, we can find a volume preserving map $\phi : B_R \to M$ s.t. $\phi_* j$ preserves $T^{fol}\Sigma_i$. (This is just the squeeze map.) So the above is indeed a kind of geometric rigidity phenomenon of l. c. s. structures.

1.1.2. *Toward non-squeezing for loose morphisms.* In some ways loose morphisms of l. c. s. m.'s are more natural, particularly when we thing about them from the contact angle. So what about non-squeezing for loose morphisms as defined above? We can try a direct generalization of contact non-squeezing of Eliashberg-Polterovich [4]. Specifically let $R^{2n} \times S^1$ be the prequantization space of R^{2n}, or in other words the contact manifold with the contact form $dt - \alpha$, for $\alpha = \frac{1}{2}(ydx - xdy)$. Let B_R now denote the open radius R ball in $\mathbb{R}^{2n}$. The following is an analogue of contact non-squeezing.

Conjecture 1. *If $R \geq 1$ is an integer then there is no compactly supported, loose endomorphism of the* l. c. s. m. $\mathbb{R}^{2n} \times S^1 \times S^1$ *which takes the closure of $U := B_R \times S^1 \times S^1$ into U.*

Our methods here are not well adapted for this conjecture, as we likely have to extend contact homology rather the Gromov-Witten theory as we do here.

1.2. **Sky catastrophes.** Given a continuous family $\{J_t\}$, $t \in [0, 1]$ we denote by $\overline{\mathcal{M}}_g(\{J_t\}, A)$ the space of pairs (u, t), $u \in \overline{\mathcal{M}}_g(J_t, A)$.

Definition 1.7. *We say that a continuous family $\{J_t\}$ on a compact manifold M has a* **sky catastrophe** *in class A if there is an element $u \in \overline{\mathcal{M}}_g(J_i, A)$, $i = 0, 1$ which does not belong to any open compact (equivalently energy bounded) subset of $\overline{\mathcal{M}}_g(\{J_t\}, A)$.*

Let us slightly expand this definition. If the connected components of $\overline{\mathcal{M}}_g(\{J_t\}, A)$ are open subspaces of this space, then we have a sky catastrophe in the sense above if and only if there is a $u \in \overline{\mathcal{M}}_g(J_i, A)$ which has a non-compact connected component in $\overline{\mathcal{M}}_g(\{J_t\}, A)$.

Proposition 1.8. *Let M be a closed manifold, and suppose that $\{J_t\}$, $t \in [0, 1]$ has no sky catastrophes, then if J_i, $i = 0, 1$ are bounded:*

$$GW_{g,n}(A, J_0) = GW_{g,n}(A, J_1),$$

if $A \neq 0$. If only J_0 is bounded then there is at least one class A J_1-holomorphic curve in M.

The assumption on A is for simplicity in this case. At this point in time there are no known examples of families $\{J_t\}$ with sky catastrophes, cf. [8].

Question 1. Do sky catastrophes exist?

Really what we are interested in is whether they exist generically. The author's opinion is that they do appear even generically. However for locally conformally symplectic deformations $\{(\omega_t, J_t)\}$ as previously defined, it might be possible that sky catastrophes cannot exist generically, for example it looks very unlikely that an example can be constructed with Reeb tori (see the following section), cf. [18].

If sky catastrophes do not exist then the theory of Gromov-Witten invariants of an l. c. s. m. would be very different. In this direction we have the following. For a l. c. s. m. (M, ω) we have a uniquely associated class (the Lee class) $\alpha = \alpha_\omega \in H^1(M, \mathbb{R})$ s.t. on the associated covering space $\widetilde{M}$, $\widetilde{\omega}$ is globally conformally symplectic. The class α is the Cech 1-cocycle, given as follows. Let $\phi_{\alpha,\beta}$ be the transition map for l. c. s. charts ϕ_α, ϕ_β of (M, ω). Then $\phi_{\alpha,\beta}^* \omega_{st} = g_{\alpha,\beta} \cdot \omega_{st}$ for a positive real constant $g_{\alpha,\beta}$ and $\{\ln g_{\alpha,\beta}\}$ gives our 1-cocycle.

Theorem 1.9. *Let M be a closed 4-fold and $\{(\omega_t, J_t)\}$, $t \in [0,1]$, a continuous family of compatible* l. c. s. *pairs. Let $\Sigma_i \subset M$, $i = 0, \ldots, m$ be a collection of hypersurfaces s.t. $PD(\alpha_t) = \sum_i a_{i,t}[\Sigma_i]$ for each t. Let $\{J_t\}$ be such that for each t there is a folliation of Σ_i by J_t-holomorphic class B curves, then $\{J_t\}$ has no sky catastrophes in every class A s.t. $A \cdot B \leq 0$.*

2. Introduction part II, beyond bounded case and the Fuller index

Suppose (M, J) is a compact almost complex manifold, let $N \subset \overline{\mathcal{M}}_{g,k}(J, A)$ be an open compact subset with energy positive on N. The latter condition is only relevant when $A = 0$. Since the construction in [16] of implicit atlas on the moduli space $\mathcal{M}$ of curves in a symplectic manifold only needs a neighborhood of $\mathcal{M}$ in the space of all curves, and since N is open, the construction obviously generalizes to give N a natural implicit atlas structure, and so a virtual fundamental class in the sense of Pardon [16], (or in any other approach to virtual fundamental cycle.) This understanding will be used in other parts of the paper. We may thus define functionals:

$$GW_{g,n}(N, A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}.$$

The first question is: how do these functionals depend on N, J?

Lemma 2.1. *Let $\{J_t\}$, $t \in [0,1]$ be a continuous in the C^∞ topology homotopy. Suppose that $\widetilde{N}$ is an open compact subset of the cobordism moduli space $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$. Let*

$$N_i = \widetilde{N} \cap \left(\overline{\mathcal{M}}_{g,n}(J_i, A)\right),$$

with energy *positive on N_i, then*

$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1).$$

In particular if $GW_{g,n}(N_0, A, J_0) \neq 0$, there is a class A J_1-holomorphic stable map in M.

The most basic lemma in this setting is the following, and we shall use it in the following section.

Definition 2.2. *An* **almost symplectic pair** *on M is a tuple (M, ω, J), where ω is a non-degenerate 2-form on M, and J is ω-compatible.*

Definition 2.3. *We say that a pair of almost symplectic pairs (ω_i, J_i) are δ**-close**, if $\{\omega_i\}$, and $\{J_i\}$ are C^∞ δ-close, $i = 0, 1$.*

Lemma 2.4. *Given a compact M and an almost symplectic tuple (ω, J) on M, suppose that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = \text{energy}_\omega^{-1}(E^0, E^1)\right) \subset \left(V = \text{energy}_\omega^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

with $\epsilon > 0$, $E^0 > 0$ and with $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$. Suppose also that $GW_{g,n}(N, J, A) \neq 0$. Then there is a $\delta > 0$ s.t. whenever (ω', J') is a compatible almost symplectic pair δ-close to (ω, J), there exists $u \in \overline{\mathcal{M}}_{g,n}(J', A) \neq \emptyset$, with

$$E^0 < \text{energy}_{\omega'}(u) < E^1.$$

2.1. Gromov-Witten theory of the l. c. s. m. $C \times S^1$.

Let (C, λ) be a closed contact manifold with contact form λ. Then $T = S^1$ acts on $C \times S^1$ by rotation in the S^1 coordinate. Let J be an almost complex structure on the contact distribution, compatible with $d\lambda$. There is an induced almost complex structure J^λ on $C \times S^1$, which is T-invariant, coincides with J on the contact distribution

$$\xi \subset TC \oplus \{\theta\} \subset T(C \times S^1),$$

for each θ and which maps the Reeb vector field

$$R^\lambda \in TC \oplus 0 \subset T(C \times S^1)$$

to

$$\frac{d}{d\theta} \in \{0\} \oplus TS^1 \subset T(C \times S^1),$$

for $\theta \in [0, 2\pi]$ the global angular coordinate on S^1. This almost complex structure is compatible with $d^\alpha \lambda$.

We shall be looking below at the moduli space of marked holomorphic tori, (elliptic curves) in $C \times S^1$, in a certain class A. Our notation for this is $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$, where A is a class of the maps, (to be explained). The elements are equivalence classes of pairs (u, Σ): u a J^λ-holomorphic map of a stable genus 1, elliptic curve Σ into $C \times S^1$. So Σ is a nodal curve with principal component an elliptic curve, and other components spherical. So the principal component determines an element of $M_{1,1}$ the compactified moduli space of elliptic curves, which is understood as an orbifold. The equivalence relation is $(u, \Sigma) \sim (u', \Sigma')$ if there is an isomorphism of marked elliptic curves $\phi : \Sigma \to \Sigma'$ s.t. $u' \circ \phi = u$. When Σ is smooth, we may write $[u, j]$ for an equivalence class where j is understood as a complex structure on the sole principal component of the domain, and u the map. Or we may just write $[u]$, or even just u keeping track of j, and of the fact that we are dealing with equivalence classes, implicitly.

Let us explain what class A means. We need to be careful because it is now possible for non-constant holomorphic curves to be null-homologous. Here is a simple example take $S^3 \times S^1$ with J determined by the Hopf contact form as above, then all the Reeb tori are null-homologous. In general we can just work with homology classes $A \neq 0$, and this will remove some headache, but in the above specific situation this is inadequate.

Given $u \in \overline{\mathcal{M}}_{1,1}(J^\lambda, A)$ we may compose $\Sigma \xrightarrow{u} C \times S^1 \xrightarrow{pr} S^1$, for Σ the nodal domain of u.

Definition 2.5. *In the setting above we say that u is in class A, if $(pr \circ u)^* d\theta$ can be completed to an integral basis of $H^1(\Sigma, \mathbb{Z})$, and if the homology class of u is A, possibly zero.*

It is easy to see that the above notion of class is preserved under Gromov convergence, and that a class A J-holomorphic map cannot be constant for any A, in particular by Theorem 1.1 a class A map has energy bounded from below by a positive constant, depending on (ω, J). And this holds for any l. c. s. pair (ω, J) on $C \times S^1$.

2.1.1. *Reeb tori.* For the almost complex structure J^λ as above we have one natural class of holomorphic tori in $C \times S^1$ that we call *Reeb tori.* Given a closed orbit o of R^λ, a Reeb torus u_o for o, is the map

$$u_o(\theta_1, \theta_2) = (o(\theta_1), \theta_2),$$

$\theta_1, \theta_2 \in S^1$ A Reeb torus is J^λ-holomorphic for a uniquely determined holomorphic structure j on T^2. If

$$D_t o(t) = c \cdot R^\lambda(o(t)),$$

then

$$j(\frac{\partial}{\partial \theta_1}) = c \frac{\partial}{\partial \theta_2}.$$

Proposition 2.6. *Let (C, λ) be as above. Let A be a class in the sense above, and J^λ be as above. Then the entire moduli space $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$ consists of Reeb tori.*

Note that the formal dimension of $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$ is 0, for A as in the proposition above. It is given by the Fredholm index of the operator (4.2) which is 2, minus the dimension of the reparametrization group (for smooth curves) which is 2. We shall relate the count of these curves to the classical Fuller index, which is reviewed in the Appendix A.

If β is a free homotopy class of a loop in C denote by A_β the induced homology class of a Reeb torus in $C \times S^1$. We show:

Theorem 2.7.

$$GW_{1,1}(N, A_\beta, J^\lambda)([M_{1,1}] \otimes [C \times S^1]) = i(\widetilde{N}, R^\lambda, \beta),$$

where $N \subset \overline{\mathcal{M}}_1(J^\lambda, A)$ is open-compact as before, $\widetilde{N}$ is the corresponding subset of periodic orbits of R^λ, β, A_β as above, and where $i(\widetilde{N}, R^\lambda, \beta)$ is the Fuller index.

What about higher genus invariants of $C \times S^1$? Following Proposition 2.6, it is not hard to see that all J^λ-holomorphic curves must be branched covers of Reeb tori. However as they may not be regular even when the underlying tori are regular, the calculation of invariants is not automatic from this data. We will not consider higher genus invariants of $C \times S^1$ here.

What follows is one non-classical application of the above theory.

Theorem 2.8. *Let $(S^{2k+1} \times S^1, d^\alpha \lambda_{st})$ be the* l. c. s. m. *associated to a contact manifold (S^{2k+1}, λ_{st}) for λ_{st} the standard contact form. There exists a $\delta > 0$ s.t. for any* l. c. s. *pair (ω, J) δ-close to $(d^\alpha \lambda, J^\lambda)$, there exists an elliptic, class A, J-holomorphic curve in $C \times S^1$. (Where A is as in the discussion above.)*

It is natural to conjecture that the δ-nearby condition can be removed. Indeed if $\omega = d^\alpha \lambda$ for λ the standard contact form on S^{2k+1}, or any contact structure on a threefold, and $J = J^\lambda$ then we known there are J-holomorphic class A tori, since we know there are λ-Reeb orbits, as the Weinstein conjecture is known to hold in these cases, [21], [19] and hence there are Reeb tori. Thus the following is a kind of "holomorphic Seifert/Weinstein conjecture".

Conjecture 2. *For any* l. c. s. *pair (ω, J) on $S^{2k+1} \times S^1$ or on $C \times S^1$ for C a threefold, there is an elliptic, class A, J-holomorphic curve.*

We note that it implies the Weinstein conjecture for S^{2k+1} and for C a contact three-fold by Proposition 2.6.

3. Basic results, and non-squeezing

Proof of Theorem 1.2. (Outline, as the argument is standard.) Suppose that we have a sequence u^k of J-holomorphic maps with L^2-energy $\leq E$. By [15, 4.1.1], a sequence u^k of J-holomorphic curves has a convergent subsequence if $sup_k ||du^k||_{L^\infty} < \infty$. On the other hand when this condition does not hold rescaling argument tells us that a holomorphic sphere bubbles off. The quantization Theorem 1.1, then tells us that these bubbles have some minimal energy, so if the total energy is capped by E, only finitely many bubbles may appear, so that a subsequence of u^k must converge in the Gromov topology to a Kontsevich stable map. □

Proof of Lemma 2.1. Let $\widetilde{N}$ be as in the hypothesis. By Theorem 1.1 there is an $\epsilon > 0$ s.t.

$$\text{energy}^{-1}([0, \epsilon)) \subset \overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$$

consists of constant curves. (Only relevant when $A = 0$.) Thus $\widetilde{N}' = \widetilde{N} - \text{energy}^{-1}([0, \epsilon))$ is an open-closed subset of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ and is compact, as $\widetilde{N}$ is compact.

We may then construct exactly as in [16] a natural implicit atlas on $\widetilde{N}'$, with boundary $N_0^{op} \sqcup N_1$, (op denoting opposite orientation). And so

$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1).$$

□

Proof of Theorem 2.4.

Lemma 3.1. *Given a Riemannian manifold (M, g), and J an almost complex structure, suppose that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = \text{energy}_g^{-1}(E^0, E^1)\right) \subset \left(V = \text{energy}_g^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

with $\epsilon > 0$, $E_0 > 0$, and with $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$. Then there is a $\delta > 0$ s.t. whenever (g', J') is C^∞ δ-close to (g, J) if $u \in \overline{\mathcal{M}}_{g,n}(J', A)$ and

$$E^0 - \epsilon < \text{energy}_{g'}(u) < E^1 + \epsilon$$

then

$$E^0 < \text{energy}_{g'}(u) < E^1.$$

Proof of Lemma 3.1. Suppose otherwise then there is a sequence $\{(g_k, J_k)\}$ C^∞ converging to (g, J), and a sequence $\{u_k\}$ of J_k-holomorphic stable maps satisfying

$$E^0 - \epsilon < \mathrm{energy}_{g_k}(u_k) \leq E^0$$

or

$$E^1 \leq \mathrm{energy}_{g_k}(u_k) < E^1 + \epsilon.$$

By Gromov compactness we may find a Gromov convergent subsequence $\{u_{k_j}\}$ to a J-holomorphic stable map u, with

$$E^0 - \epsilon < \mathrm{energy}_g(u) \leq E^0$$

or

$$E^1 \leq \mathrm{energy}_g(u) < E^1 + \epsilon.$$

But by our assumptions such a u does not exist. □

Lemma 3.2. *Given a compact almost symplectic compatible triple (M, ω, J), so that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is exactly as in the lemma above. There is a $\delta' > 0$ s.t. the following is satisfied. Let (ω', J') be δ'-close to (ω, J), then there is a smooth family of almost symplectic pairs $\{(\omega_t, J_t)\}$, $(\omega_0, J_0) = (g, J)$, $(\omega_1, J_1) = (g', J')$ s.t. there is open compact subset*

$$\widetilde{N} \subset \overline{\mathcal{M}}_{g,n}(\{J_t\}, A),$$

and with

$$\widetilde{N} \cap \overline{\mathcal{M}}(J, A) = N.$$

Moreover if $(u, t) \in \widetilde{N}$ then

$$E^0 < \mathrm{energy}_{g_t}(u) < E^1.$$

Proof. Given ϵ as in the hypothesis let δ be as in Lemma 3.1.

Lemma 3.3. *Given a $\delta > 0$ there is a $\delta' > 0$ s.t. if (ω', J') is δ'-near (ω, J) there is an interpolating family $\{(\omega_t, J_t)\}$ with (ω_t, J_t) δ-close to (ω, J) for each t.*

Proof. Let $\{g_t\}$ be the family of metrics on M given by the convex linear combination of $g = g_{\omega_J}, g' = g_{\omega', J'}$. Clearly g_t is δ'-close to g_0 for each t. Likewise the family of 2 forms $\{\omega_t\}$ given by the convex linear combination of ω, ω' is non-degenerate for each t if δ' was chosen to be sufficiently small and is δ'-close to $\omega_0 = \omega_{g,J}$ for each moment.

Let

$$ret : Met(M) \times \Omega(M) \to \mathcal{J}(M)$$

be the retraction map, as defined in [14, Prop 2.50], where $Met(M)$ is space of metrics on M, $\Omega(M)$ the space of 2-forms on M, and $\mathcal{J}(M)$ the space of almost complex structures. Then $\{(\omega_t, ret(g_t, \omega_t)\}$ is a compatible family. As ret is continuous in the C^∞-topology, δ' can be chosen so that $\{ret_t(g_t, \omega_t\}$ are δ-nearby. □

Let δ' be chosen with respect to δ as in the above lemma and $\{(\omega_t, J_t)\}$ be the corresponding family. Let $\widetilde{N}$ consist of all elements $(u, t) \in \overline{\mathcal{M}}(\{J_t\}, A)$ s.t.

$$E^0 - \epsilon < \mathrm{energy}_{\omega_t}(u) < E^1 + \epsilon.$$

Then by Lemma 3.1 for each $(u, t) \in \widetilde{N}$, we have:

$$E^0 < \mathrm{energy}_{\omega_t}(u) < E^1.$$

In particular $\widetilde{N}$ must be closed, it is also clearly open, and is compact as energy is a proper function. □

To finish the proof of the theorem, let N be as in the hypothesis, δ' as in Lemma 3.2, and $\widetilde{N}$ as in the conclusion to Lemma 3.2, then by Lemma 2.1

$$GW_{g,n}(N_1, J', A) = GW_{g,n}(N, J, A) \neq 0,$$

where $N_1 = \widetilde{N} \cap \overline{\mathcal{M}}_{g,n}(J_1, A)$. So the conclusion follows. □

Proof of Proposition 1.8. For each $u \in \overline{\mathcal{M}}_{g,n}(J_i, A)$, $i = 0, 1$ fix an open-compact subset V_u of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ containing u. We can do this by the hypothesis that there are no sky catastrophes. Since $\overline{\mathcal{M}}_{g,n}(J_i, A)$ are compact we may find a finite subcover

$$\{V_{u_i}\} \cap (\overline{\mathcal{M}}_{g,n}(J_0, A) \cup \overline{\mathcal{M}}_{g,n}(J_1, A))$$

of $\overline{\mathcal{M}}_{g,n}(J_0, A) \cup \overline{\mathcal{M}}_{g,n}(J_1, A)$, considering $\overline{\mathcal{M}}_{g,n}(J_0, A) \cup \overline{\mathcal{M}}_{g,n}(J_1, A)$ as a subset of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ naturally. Then $V = \bigcup_i V_{u_i}$ is an open compact subset of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$, s.t.

$$V \cap \overline{\mathcal{M}}_{g,n}(J_i, A) = \overline{\mathcal{M}}_{g,n}(J_i, A).$$

Now apply Lemma 2.1.

Likewise if only J_0 is bounded, for each $u \in \overline{\mathcal{M}}_{g,n}(J_0, A)$, fix an open-compact subset V_u of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ containing u. Since $\overline{\mathcal{M}}_{g,n}(J_0, A)$ is compact we may find a finite subcover

$$\{V_{u_i}\} \cap (\overline{\mathcal{M}}_{g,n}(J_0, A)$$

of $\overline{\mathcal{M}}_{g,n}(J_0, A)$. Then $V = \bigcup_i V_{u_i}$ is an open compact subset of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$, s.t.

$$V \cap \overline{\mathcal{M}}_{g,n}(J_i, A) = \overline{\mathcal{M}}_{g,n}(J_i, A).$$

Again apply Lemma 2.1. □

Proof of Theorem. 1.9 We shall actually prove a stronger statement that there is a universal energy bound from above for class A, J_t-holomorphic curves. Let $\{u_k\}$ be a sequence J_{t_k}-holomorphic class A curves. We may assume that t_k is convergent to $t_0 \in [0, 1]$. Let $\{\widetilde{u}_k\}$ be a lift of the curves to the covering space $\widetilde{M} \xrightarrow{\pi} M$ determined by the class α as described prior to the formulation of the theorem. If the image of $\{\widetilde{u}_k\}$ is contained (for a specific choice of lifts) in a compact $K \subset \widetilde{M}$ then we have:

$$\text{energy}_{t_k}(\widetilde{u}_k) \simeq \text{energy}_{t_0}(\widetilde{u}_k) \le C\langle \widetilde{\omega}_{t_0}^{symp}, A\rangle,$$

where $\widetilde{\omega}_{t_0} = f\widetilde{\omega}^{symp}$ for $\widetilde{\omega}^{symp}$ symplectic on M, $f > 0$ and $C = sup_K f$. Hence energy would be universally bounded for all $\{u_k\}$, and we would be done.

Suppose there is no such K. Let $\{u_k\}$ be the corresponding sequence. We may suppose that image u_k does not intersect Σ_i for all k and i, since otherwise u_k must be a branched covering map of a leaf of the J_{t_k}-holomorphic foliation of Σ_i by the positivity of intersections, and consequently all such u_k have lifts contained in a compact subset of $\widetilde{M}$.

By our hypothesis that $PD(\alpha) = \sum_i a_i[\Sigma_i]$ we have that $\pi^{-1}(M - \bigcup_i \Sigma_i)$ is a disjoint union $\sqcup_i K_i$ of bounded subsets. Then for some k' sufficiently large, the image of some lift $\widetilde{u}_{k'}$ intersects more then one of the K_i, and so $u_{k'}$ intersects some Σ_i, a contradiction.

□

Proof of Theorem 1.6. We need to say what is our C^0 norm on the space of l. c. s. forms. The class α has a natural differential form representative, called the Lee form and defined as follows. We take a cover of M by open sets U_a in which $\omega = f_a \cdot \omega^{symp}$ for ω^{symp} symplectic, and f_a a positive smooth function. Then we have 1-forms $d(\ln f_a)$ in each U_a which glue to a well defined closed 1-form on M. We denote this 1-form by α still.

Definition 3.4. *The C^0 norm on the space of* l. c. s. *2-forms on M, is defined with respect to a fixed Riemannian metric g on M, and is given by*

$$||\omega|| = ||\omega(v, w)||_{mass} + ||\alpha||_{mass},$$

for $||\cdot||_{mass}$ the co-mass norms with respect to g on differential k-forms. That is $||\eta||_{mass}$ is the supremum over orthonormal k-vectors v of $|\eta(v)|$.

Explicitly this means the following: a sequence of l. c. s. forms $\{\omega_k\}$ converges to ω iff the lift sequence $\{\widetilde{\omega}_k\}$ on the associated (to α) cover $\widetilde{M}$ converges to ω on compact sets and $\widetilde{\omega_k} = f_k\widetilde{\omega_k}^{symp}$, with ω_k^{symp} symplectic and $\{f_k\}$ converging to 1 on compact sets. Fix an $\epsilon > 0$ s.t. for any l. c. s.

ω_1 on M, C^0 ϵ-close to ω, the convex linear combination of ω and ω_1 is non-degenerate and hence conformally symplectic, and s.t. for any ω_1 ϵ-close to ω_0, the foliation of Σ_i stays symplectic.

Suppose by contradiction that for every $\epsilon > 0$ there exists an l. c. s. embedding

$$\phi : B_R \hookrightarrow (M, \omega_1),$$

satisfying the conditions. Extend $\phi_* j$ to an almost complex structure J_1 on M, preserving $T^{fol}\Sigma_i$. We may then extend this to a family $\{J_t\}$ of almost complex structures M, s.t. J_t is ω_t compatible for each t, and such that J_t preserves $T^{fol}\Sigma$ for each i, since by construction the foliation of Σ_i is ω_t-symplectic for each t. (For construction of $\{J_t\}$ use for example the map *ret* from Lemma 3.3). Then the family $\{(\omega_t, J_t)\}$ satisfies the hypothesis of Theorem 1.9, and so has no sky catastrophes in class A. Consequently by Lemma 2.1 there is a class A J_1-holomorphic curve u passing through $\phi(0)$.

By the proof of Theorem 1.9 we may choose a lift to $\widetilde{M}$ for each such curve u so that it is contained in a compact set $K \subset \widetilde{M}$, (independent of ϵ and all other choices). Now by definition of our C^0-norm for every δ we may find an ϵ so that if ω_1 is ϵ-close to ω then $\widetilde{\omega}^{symp}$ is δ-close to $\widetilde{\omega}_1^{symp}$ on K. Since $\langle \widetilde{\omega}^{symp}, [\widetilde{u}] \rangle = \pi r^2$, if δ above is chosen to be sufficiently small then

$$|\max_K f_1 \langle \widetilde{\omega}_1^{symp}, [\widetilde{u}] \rangle - \pi \cdot r^2| < \pi R^2 - \pi r^2,$$

since

$$|\langle \widetilde{\omega}_1^{symp}, [\widetilde{u}] \rangle - \pi \cdot r^2| \simeq |\langle \widetilde{\omega}^{symp}, [\widetilde{u}] \rangle - \pi \cdot r^2| = 0,$$

for δ small enough, and $\max_K f_1 \simeq 1$ for δ small enough, where $\simeq$ denotes approximate equality. In particular we get that ω_1-area of u is less then πR^2.

We may then proceed as in the classical proof Gromov [10] of the non-squeezing theorem to get a contradiction and finish the proof. More specifically $\phi^{-1}(\mathrm{image}\,\phi \cap \mathrm{image}\,u)$ is a (nodal) minimal surface in B_R, with boundary on the boundary of B_R, and passing through $0 \in B_R$. By construction it has area strictly less then πR^2 which is impossible by a classical result of differential geometry, (the monotonicity theorem.) □

Proof of Theorem 1.5. Fix an $\epsilon > 0$ s.t. for any l. c. s. ω_1 on M, C^0 ϵ-close to ω, the convex linear combination $\{\omega_t\}$ of ω and ω_1 is non-degenerate and hence conformally symplectic, and s.t. for any ω_1 ϵ-close to $\omega = \omega_0$, the foliation of Σ_i stays symplectic.

Suppose by contradiction that for every $\epsilon > 0$ there exists an l. c. s. embedding

$$\phi : B_R \hookrightarrow (M, \omega_1).$$

Extend $\phi_* j$ to any ω_1-compatible almost complex structure J_1 on M. We may then extend this to a family $\{J_t\}$ of almost complex structures M, s.t. J_t is ω_t compatible for each t. If $\{(\omega_t, J_t)\}$ has no sky catastrophes in class A, then by Lemma 2.1 there is a class A J_1-holomorphic curve u passing through $\phi(0)$. Now proceed as in the proof of Theorem 1.6. □

4. Proof of Theorem 2.7 and Theorem 2.8

Proof of Proposition 2.6. Suppose we a have a curve without spherical nodal components $u \in \overline{\mathcal{M}}_{1,1}(J^\lambda, A)$. We claim that $(pr_C \circ u)_*$, has rank everywhere ≤ 1, for $pr_C : C \times S^1 \to C$ the projection. Suppose otherwise than it is immediate by construction of J^λ, that $\int_\Sigma u^* d\lambda > 0$, for Σ domain of u, but $d\lambda$ is exact so that that this is impossible. It clearly follows from this that Σ must be smooth, (non-nodal).

Next observe that when the rank of $(pr_C \circ u)_*$ is 1, its image is in the Reeb line sub-bundle of TC, for otherwise the image has a contact component, but this is J^λ invariant and so again we get that $\int_\Sigma u^* d\lambda > 0$. We now show that the image of $pr_C \circ u$ is in fact the image of some Reeb orbit.

Pick an identification of the domain Σ of u with a marked Riemann surface (T^2, j), T^2 the standard torus. We shall use throughout coordinates (θ_1, θ_2) on T^2 $\theta_1, \theta_2 \in S^1$, with S^1 unit complex numbers. Then by assumption on the class A (and WLOG) $\theta \mapsto pr_{S^1} \circ u(\{\theta_0^1\} \times \{\theta\})$, is a degree 1 curve, where $pr_{S^1} : C \times S^1 \to S^1$ is the projection. And so by the Sard theorem we have a regular value θ_0, so that $S_0 = u^{-1} \circ pr_{S^1}^{-1}(\theta_0)$ is an embedded circle in T^2. Now $d(pr_{S^1} \circ u)$ is surjective along $T(T^2)|_{S_0}$, which means, since u is J^λ-holomorphic that $pr_C \circ u|_{S_0}$ has non-vanishing differential. From this and the discussion above it follows that image of $pr_C \circ u$ is the image of some embedded Reeb orbit o_u. Consequently the image of u is contained in the image of the Reeb torus of o_u, and so (again by the assumption on A) u is a itself a Reeb torus map for some o covering o_u.

The statement of the lemma follows when u has no spherical nodal components. On the other hand non-constant holomorphic spheres are impossible also by the previous argument. So there are no nodal elements in $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$ which completes the argument. □

Proposition 4.1. *Let (C, ξ) be a general contact manifold. If λ is non-degenerate then all the elements of $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$, are regular curves. Moreover if λ is degenerate then for a period P Reeb orbit o the kernel of the associated real linear Cauchy-Riemann operator for the Reeb torus of o is naturally identified with the 1-eigenspace of $\phi^\lambda_{P,*}$ - the time P linearized return map $\xi(o(0)) \to \xi(o(0))$ induced by the R^λ Reeb flow.*

Proof. We have previously shown that all $[u, j] \in \overline{\mathcal{M}}_{1,1}(J^\lambda, A)$, are represented by smooth immersed curves, (covering maps of Reeb tori). Since each u is immersed we may naturally get a splitting $u^*T(C \times S^1) \simeq N_u \times T(T^2)$, using g_J metric, where N_u denotes the pull-back normal bundle, which is identified with the pullback along the projection of $C \times S^1 \to C$ of the distribution ξ.

The full associated real linear Cauchy-Riemann operator takes the form:

$$D^J_u : \Omega^0(N_u \oplus T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2), N_u \oplus T(T^2)). \tag{4.2}$$

This is an index 2 Fredholm operator (after standard Sobolev completions), whose restriction to $\Omega^0(N_u \oplus T(T^2))$ preserves the splitting, that is the restricted operator splits as

$$D \oplus D' : \Omega^0(N_u) \oplus \Omega^0(T(T^2)) \to \Omega^{0,1}(T(T^2), N_u) \oplus \Omega^{0,1}(T(T^2), T(T^2)).$$

On the other hand the restricted Fredholm index 2 operator

$$\Omega^0(T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2)),$$

is surjective by classical algebraic geometry. It follows that D^J_u will be surjective if the restricted Fredholm index 0 operator

$$D : \Omega^0(N_u) \to \Omega^{0,1}(N_u),$$

has no kernel.

The bundle N_u is symplectic with symplectic form on the fibers given by restriction of $u^*d\lambda$, and together with J^λ this gives a Hermitian structure on N_u. We have a linear symplectic connection A on N_u, which over the slices $S^1 \times \{\theta_2'\} \subset T^2$ is induced by the pullback by u of the linearized R^λ Reeb flow. Specifically the A-transport map from $N|_{(\theta_1', \theta_2')}$ to $N|_{(\theta_1'', \theta_2')}$ over $[\theta_1', \theta_2''] \times \{\theta_2'\} \subset T^2$, $0 \leq \theta_1' \leq \theta_2'' \leq 2\pi$ is given by

$$(u_*|_{N|_{(\theta_1'', \theta_2')}})^{-1} \circ \phi^R_{mult \cdot (\theta_1'' - \theta_1')} \circ u_*|_{N|_{(\theta_1', \theta_2')}},$$

where $mult$ is the multiplicity of o and where $\phi^R_{mult \cdot (\theta_1'' - \theta_1')}$ is the time $mult \cdot (\theta_1'' - \theta_1')$ map for the R^λ Reeb flow.

The connection A is defined to be trivial in the θ_2 direction, where trivial means that the parallel transport maps are the id maps over θ_2 rays. In particular the curvature R_A of this connection vanishes. The connection A determines a real linear CR operator on N_u in the standard way (take the complex anti-linear part of the vertical differential of a section). It is easy to verify that this operator is exactly D. We have a differential 2-form Ω on the N_u which in the fibers of N_u is just the fiber symplectic form and which is defined to vanish on the horizontal distribution. This 2-form is closed, which we may check explicitly by using that R_A vanishes to obtain local symplectic trivializations of N_u in which A is trivial. Clearly Ω must vanish on the 0-section since it is a A-flat section. But any section is homotopic to the 0-section and so in particular if $\mu \in \ker D$ then Ω vanishes on μ. But then since $\mu \in \ker D$, and so its vertical differential is complex linear, it must follow that the vertical differential vanishes, since $\Omega(v, J^\lambda v) > 0$, for $0 \neq v \in T^{vert} N_u$ and so otherwise we would have $\int_\mu \Omega > 0$. So μ is A-flat, in particular the restriction of μ over all slices $S^1 \times \{\theta_2'\}$ is identified with a period P orbit of the linearized at o R^λ Reeb flow, which does not depend on θ_2' as A is trivial in the θ_2 variable. So the kernel of D is identified with the vector space of period P orbits of the linearized at o R^λ Reeb flow, as needed. □

Proposition 4.3. *Let (C, λ) be a contact $2n+1$-fold, and o a non-degenerate, period P, R^λ-Reeb orbit, then the orientation of $[u_o]$ induced by the determinant line bundle orientation of $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$, is $(-1)^{CZ(o)-n}$, which is*

$$\operatorname{sign}\operatorname{Det}(\operatorname{Id}|_{\xi(o(0))} - \phi^\lambda_{P,*}|_{\xi(o(0))}).$$

Proof of Proposition 4.3. Abbreviate u_o by u. Fix a trivialization ϕ of N_u induced by a trivialization of the contact distribution ξ along o in the obvious sense: N_u is the pullback of ξ along the composition

$$T^2 \to S^1 \xrightarrow{o} C.$$

Then the pullback A' of A to $T^2 \times \mathbb{R}^{2n}$ is a connection whose parallel transport path p along $S^1 \times \{\theta_2\}$ is θ independent and so that the parallel transport path of A' along $\{\theta_1\} \times S^1$ is constant, that is A' is trivial in the θ_2 variable. We shall call such a connection A' on $T^2 \times \mathbb{R}^{2n}$ *induced by* p. The path p is a smooth path $p : [0,1] \to \operatorname{Symp}(\mathbb{R}^{2n})$, starting at 1, and by non-degeneracy assumption on o, the end point map $p(1)$ has no 1-eigenvalues. Let $p'' : [0,1] \to \operatorname{Symp}(\mathbb{R}^{2n})$ be a path from $p(1)$ to a unitary map $p''(1)$, with $p''(1)$ having no 1-eigenvalues, s.t. p'' has only simple crossings with the Maslov cycle. Let p' be the concatenation of p and p''. We then get

$$CZ(p') - \frac{1}{2}\operatorname{sign}\Gamma(p', 0) \equiv CZ(p') - n \equiv 0 \mod 2,$$

since p' is homotopic relative end points to a unitary geodesic path h starting at id, having regular crossings, and since the number of negative, positive eigenvalues is even at each regular crossing of h by unitarity. Here $\operatorname{sign}\Gamma(p', 0)$ is the index of the crossing form of the path p' at time 0, in the notation of [17]. Consequently

$$CZ(p'') \equiv CZ(p') - n \mod 2$$

by additivity of the Conley-Zehnder index. Let us then define a free homotopy $\{p_t\}$ of p to p', p_t is the concatenation of p with $p''|_{[0,t]}$, reparametrized to have domain $[0,1]$ at each moment t. This determines a homotopy $\{A'_t\}$ of connections induced by $\{p_t\}$. By the proof of Proposition 4.1, the CR operator D_t determined by each A'_t is surjective except at some finite collection of times $t_i \in (0,1)$, $i \in N$ determined by the crossing times of p'' with the Maslov cycle, and the dimension of the kernel of D_{t_i} is the 1-eigenspace of $p''(t_i)$, which is 1 by the assumption that the crossings of p'' are simple. The operator D_1 is not complex linear so we concatenate the homotopy $\{D_t\}$ with the homotopy $\{\widetilde{D}_t\}$ induced by the homotopy $\{\widetilde{A}_t\}$ of A'_1 to a unitary connection $\widetilde{A}_1$, where the homotopy $\{\widetilde{A}_t\}$, is through connections induced by paths $\{\widetilde{p}_t\}$, giving a homotopy of $p' = \widetilde{p}_0$ to a unitary path $\widetilde{p}_1$. Let us denote by $\{D'_t\}$ the concatenation of $\{D_t\}$ with $\{\widetilde{D}_t\}$. By construction in the second half of the homotopy $\{D'_t\}$, D'_t is surjective. And D'_1 is induced by a unitary connection, since it is induced by unitary path $\widetilde{p}_1$. Consequently D'_1 is complex linear. By the above construction, for the homotopy $\{D'_t\}$, D'_t is surjective except for N times in $(0,1)$, where the kernel has dimension one. In particular the sign of $[u]$ by the definition via the determinant line bundle is exactly

$$-1^N = -1^{CZ(p')-n},$$

which was what to be proved. □

Thus if $N \subset \overline{\mathcal{M}}_{1,1}(J^\lambda, A_\beta)$ is open-compact and consists of isolated regular Reeb tori $\{u_i\}$, corresponding to orbits $\{o_i\}$ we have:

$$GW_{1,1}(N, A_\beta, J^\lambda)([M_{1,1}] \otimes [C \times S^1]) = \sum_i \frac{(-1)^{CZ(o_i)-n}}{mult(o_i)},$$

where the denominator $mult(o_i)$ is there because our moduli space is understood as a non-effective orbifold, see Appendix B.

The expression on the right is exactly the Fuller index $i(\widetilde{N}, R^\lambda, \beta)$. Thus the theorem follows for N as above. However in general if N is open and compact then perturbing slightly we obtain a smooth family $\{R^{\lambda_t}\}$, $\lambda_0 = \lambda$, s.t. λ_1 is non-degenerate, that is has non-degenerate orbits. And such that there is an open-compact subset $\widetilde{N}$ of $\overline{\mathcal{M}}_{1,1}(\{J^{\lambda_t}\}, A_\beta)$ with $(\widetilde{N} \cap \overline{\mathcal{M}}_{1,1}(J^\lambda, A_\beta)) = N$, cf. Lemma 3.2. Then by Lemma 2.1 if

$$N_1 = (\widetilde{N} \cap \overline{\mathcal{M}}_{1,1}(J^{\lambda_1}, A_\beta))$$

we get

$$GW_{1,1}(N, A_\beta, J^\lambda)([M_{1,1}] \otimes [C \times S^1]) = GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([M_{1,1}] \otimes [C \times S^1]).$$

By previous discussion

$$GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([M_{1,1}] \otimes [C \times S^1]) = i(N_1, R^{\lambda_1}, \beta),$$

but by the invariance of Fuller index (c.f. Appendix A),

$$i(N_1, R^{\lambda_1}, \beta) = i(N, R^\lambda, \beta).$$

This finishes the proof of Theorem 2.7 □

Proof of Theorem 2.8. Let $N \subset \overline{\mathcal{M}}_{1,1}(A, J^\lambda)$, be the subspace corresponding to the subspace $\widetilde{N}$ of all period 2π R^λ-orbits. (Under the Reeb tori correspondence.) It is easy to compute see for instance [9]

$$i(\widetilde{N}, R^\lambda) = \pm\chi(\mathbb{CP}^k) \neq 0.$$

By Theorem 2.7 $GW_{1,1}(N, J^\lambda, A) \neq 0$. This result then follows by Theorem 2.4. □

A. Fuller index

Let X be a vector field on M. Set

$$S(X) = S(X, \beta) = \{(o, p) \in L_\beta M \times (0, \infty) \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\},$$

where $L_\beta M$ denotes the free homotopy class β component of the free loop space. Elements of $S(X)$ will be called orbits. There is a natural S^1 reparametrization action on $S(X)$, and elements of $S(X)/S^1$ will be called *unparametrized orbits*, or just orbits. Slightly abusing notation we write (o, p) for the equivalence class of (o, p). The multiplicity $m(o, p)$ of a periodic orbit is the ratio p/l for $l > 0$ the least period of o. We want a kind of fixed point index which counts orbits (o, p) with certain weights - however in general to get invariance we must have period bounds. This is due to potential existence of sky catastrophes as described in the introduction.

Let $N \subset S(X)$ be a compact open set. Assume for simplicity that elements $(o, p) \in N$ are isolated. (Otherwise we need to perturb.) Then to such an (N, X, β) Fuller associates an index:

$$i(N, X, \beta) = \sum_{(o,p) \in N/S^1} \frac{1}{m(o, p)} i(o, p),$$

where $i(o, p)$ is the fixed point index of the time p return map of the flow of X with respect to a local surface of section in M transverse to the image of o. Fuller then shows that $i(N, X, \beta)$ has the following invariance property. Given a continuous homotopy $\{X_t\}$, $t \in [0, 1]$ let

$$S(\{X_t\}, \beta) = \{(o, p, t) \in L_\beta M \times (0, \infty) \times [0, 1] \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX_t\}.$$

Given a continuous homotopy $\{X_t\}$, $X_0 = X$, $t \in [0, 1]$, suppose that $\widetilde{N}$ is an open compact subset of $S(\{X_t\})$, such that

$$\widetilde{N} \cap (LM \times \mathbb{R}_+ \times \{0\}) = N.$$

Then if

$$N_1 = \widetilde{N} \cap (LM \times \mathbb{R}_+ \times \{1\})$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta).$$

In the case where X is the R^λ-Reeb vector field on a contact manifold (C^{2n+1}, ξ), and if (o, p) is non-degenerate, we have:

$$\text{(A.1)} \qquad i(o, p) = \operatorname{sign} \operatorname{Det}(\operatorname{Id}|_{\xi(x)} - F^\lambda_{p,*}|_{\xi(x)}) = (-1)^{CZ(o) - n},$$

where $F^{\lambda}_{p,*}$ is the differential at x of the time p flow map of R^{λ}, and where $CZ(o)$ is the Conley-Zehnder index, (which is a special kind of Maslov index) see [17].

B. Virtual fundamental class

This is a small note on how one deals with curves having non-trivial isotropy groups, in the virtual fundamental class technology. We primarily need this for the proof of Theorem 2.7. Given a closed oriented orbifold X, with an orbibundle E over X Fukaya-Ono [7] show how to construct using multi-sections its rational homology Euler class, which when X represents the moduli space of some stable curves, is the virtual moduli cycle $[X]^{vir}$. (Note that the story of the Euler class is older than the work of Fukaya-Ono, and there is possibly prior work in this direction.) When this is in degree 0, the corresponding Gromov-Witten invariant is $\int_{[X]^{vir}} 1$. However they assume that their orbifolds are effective. This assumption is not really necessary for the purpose of construction of the Euler class but is convenient for other technical reasons. A different approach to the virtual fundamental class which emphasizes branched manifolds is used by McDuff-Wehrheim, see for example McDuff [12], which does not have the effectivity assumption, a similar use of branched manifolds appears in [3]. In the case of a non-effective orbibundle $E \to X$ McDuff [13], constructs a homological Euler class $e(E)$ using multi-sections, which extends the construction [7]. McDuff shows that this class $e(E)$ is Poincare dual to the completely formally natural cohomological Euler class of E, constructed by other authors. In other words there is a natural notion of a homological Euler class of a possibly non-effective orbibundle. We shall assume the following black box property of the virtual fundamental class technology.

Axiom B.1. *Suppose that the moduli space of stable maps is cleanly cut out, which means that it is represented by a (non-effective) orbifold X with an orbifold obstruction bundle E, that is the bundle over X of cokernel spaces of the linearized CR operators. Then the virtual fundamental class $[X]^{vir}$ coincides with $e(E)$.*

Given this axiom it does not matter to us which virtual moduli cycle technique we use. It is satisfied automatically by the construction of McDuff-Wehrheim, (at the moment in genus 0, but surely extending). It can be shown to be satisfied in the approach of John Pardon [16]. And it is satisfied by the construction of Fukaya-Oh-Ono-Ohta [6], although not quiet immediately. This is also communicated to me by Kaoru Ono. When X is 0-dimensional this does follow immediately by the construction in [7], taking any effective Kuranishi neighborhood at the isolated points of X, (this suffices for our paper.)

As a special case most relevant to us here, suppose we have a moduli space of elliptic curves in X, which is regular with expected dimension 0. Then its underlying space is a collection of oriented points. However as some curves are multiply covered, and so have isotropy groups, we must treat this is a non-effective 0 dimensional oriented orbifold. The contribution of each curve $[u]$ to the Gromov-Witten invariant $\int_{[X]^{vir}} 1$ is $\frac{\pm 1}{[\Gamma([u])]}$, where $[\Gamma([u])]$ is the order of the isotropy group $\Gamma([u])$ of $[u]$, in the McDuff-Wehrheim setup this is explained in [12, Section 5]. In the setup of Fukaya-Ono [7] we may readily calculate to get the same thing taking any effective Kuranishi neighborhood at the isolated points of X.

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