

INSTABILITY OF GROMOV NON-SQUEEZING

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ABSTRACT. We show that the Gromov non-squeezing phenomenon disappears after an arbitrarily small, general (non-symplectic) C^∞ perturbation of the symplectic form on the range. In particular the lcs non-squeezing theorem in [2] is sharp, (in the sense that the lcs condition cannot be relaxed to just non-degeneracy.)

One of the most fascinating early results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [1]. The original formulation of this is that there does not exist a symplectic embedding

$$B_R \rightarrow D^2(r) \times \mathbb{R}^{2n-2},$$

for $R > r$, with B_R the standard closed radius R ball in \mathbb{R}^{2n} centered at 0. Gromov's non-squeezing is C^0 persistent in the following sense. We attribute this persistence to Gromov since it is just a simple generalization of his argument.

We say that a symplectic form ω on $M \times N$ is *split* if $\omega = \omega_1 \oplus \omega_2$ for symplectic forms ω_1, ω_2 on M respectively N .

Theorem 0.1 (Gromov). *Given $R > r$, there is an $\epsilon > 0$ s.t. for any pair of symplectic forms ω, ω' on $M = S^2 \times T^{2n-2}$ with ω split, and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt],$$

if $d_{C^0}(\omega, \omega') < \epsilon$ then there is no symplectic embedding

$$\phi : B_R \hookrightarrow (S^2 \times T^{2n-2}, \omega').$$

This theorem is generalized in author's [2] to lcs forms ω' . We show here that this persistence disappears if we take a completely general ω' . In particular the theorem of [2] is a truly lcs phenomenon.

Theorem 0.2. *Let $R > r, \epsilon > 0$, be given. Then there is a 2-form ω' on $M = S^2 \times T^{2n-2}$, and a split symplectic form ω on M , satisfying*

$$\langle \omega, A \rangle = \pi r^2,$$

for A as above, satisfying $d_{C^\infty}(\omega, \omega') < \epsilon$, s.t. there is an embedding

$$\phi : B_R \hookrightarrow M,$$

with

$$\phi^* \omega' = \omega_{st}.$$

*We call such an embedding **symplectic** in analogy with the classical symplectic case. Moreover, ϕ can be chosen so that*

$$\phi(B_R) \subset (M - \bigcup_i \Sigma_i),$$

where Σ_i are certain hypersurfaces explained in the proof.

Proof. Let R, r, ϵ be given. For simplicity suppose $r = 1, R = 2$, with general case following by the same argument. Let

$$M' = [0, r]^2 \times \mathbb{R}^{2n-2}.$$

We first construct a 2-form ω'' on M' , C^∞ -nearby to the standard symplectic form ω and a symplectic embedding $\phi : \text{Cube}(R) \rightarrow M'$, where $\text{Cube}(R)$ denotes the closed cube in \mathbb{R}^{2n} with side R .

For simplicity we take in what follows $n = 2$, with construction obviously generalizing to any n . Let p, q be the coordinates on

$$sq = [0, r]^2 \subset \mathbb{R}^2.$$

Let (p, q, s, t) be the natural coordinates on

$$M' = sq \times \mathbb{R}^2,$$

and g the standard Euclidean metric.

Let $\{f_l\}_{l \in \mathbb{N}}$ be a collection of smooth functions, further specified in Figure 1, and satisfying:

- (1) $\forall l : f_l(p) = 0$ for p in a neighborhood of $0, r$.
- (2) $\forall l : |f_l|_{C^0} < 2^9 R$, (the multiplier 2^9 , which of course is not optimally chosen, changes if r, R are chosen differently, which in our case are $r = 1, R = 2$).
- (3) $\forall l : \text{length}_g(\text{graph}(f_l)) \geq 2^{11} R \cdot l$, and g being the standard Euclidean metric again.

The specific functions f_l we have in mind are saw shaped as in Figure 1, with the number of teeth l .

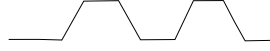


FIGURE 1. The corners are meant to be smoothed, so that this function is smooth. All obtuse angles are $\pi - \pi/4$. The saw is meant to be with uniform size teeth and uniform gaps between side edges of the teeth, these sides have g -length $2^{10} R$. The trailing edges have arbitrary non-zero length.

Define the following surface S_0^l in M' :

$$S_0^l = \{(p, q, f_l(p), 0) \in \mathbb{R}^4 \mid (p, q) \in sq\},$$

Then S_0^l is a ω -symplectic surface whose ω -orthogonal spaces are spanned by $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$. Define

$$S_{s,t}^l := S_0^l + (0, 0, s, t), \quad 0 \leq s \leq R, 0 \leq t \leq R.$$

Then

$$C = C^l := \cup_{s,t} S_{s,t}^l$$

is a domain in M'' that is diffeomorphic to the standard closed cube in \mathbb{R}^4 , foliated by the surfaces $S_{s,t}^l$. Let $\mathcal{F} \subset TC^l$ denote the 2-dimensional distribution corresponding to this foliation, that is $\mathcal{F}(z)$ is the sub-space of vectors tangent to the leaf through $z = (p, q, s, t)$. And let $V \subset TC$ denote the ω -orthogonal distribution, that is the distribution with

$$V(p, q, s, t) = \text{span}\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right),$$

by the observations above.

Let $\{h_0^l\}$, l as above, be a collection of smooth non-negative functions:

$$h_0^l : (S_0^l \subset \mathbb{R}^4) \rightarrow \mathbb{R},$$

satisfying:

- (1) $|h_0^l|_{C^\infty} < \epsilon$.
- (2) $\int_{S_0^l} h_0^l dA_g \geq r \cdot 2^{10} R \cdot l \cdot \epsilon$, dA_g the area form on S_0^l induced by the restriction of g to S_0^l , with orientation coinciding with ω -orientation.

More specifically, in terms of our saw functions, we may construct such an h_0^l by taking its support as in Figure 2. Thus if l is taken to be:

$$l_0 = \text{ceiling}\left(\frac{R}{2^{10} \cdot r \cdot \epsilon}\right),$$

then

$$(0.3) \quad \int_{S_0^{l_0}} h_0^{l_0} dA_g > R^2,$$

while the gap gap between teeth (minimal g -distance between the sides of the teeth in the graph of f^l) is then

$$gap \geq \frac{r}{2l_0}.$$

So that assuming $\epsilon < 2^9$ and remembering that $r = 1, R = 2$ we get:

$$(0.4) \quad gap \geq 2^7 \epsilon.$$

We then assume that $l = l_0$ is taken as above and it will no longer appear in notation.



FIGURE 2. A point $z \in S_0^l$ is in the support of h_0^l , only if its image by the projection $\mathbb{R}^4 \rightarrow \mathbb{R}^2, (p, q, s, t) \mapsto (p, s)$ is in the red regions of the figure, i.e. the sides of the teeth.

Define

$$h : C \rightarrow \mathbb{R}, \quad h(p, q, s, t) = h_0(p', q', s', t'),$$

where (p', q', s', t') is the unique point on S_0 defined by the condition: if

$$(p, q, s, t) \in S_{s'', t''}^l \subset C$$

then

$$(p', q', s', t') = (p, q, s, t) - (0, 0, s'', t'').$$

Then h is a smooth function on C .

Let ω_ϵ be the 2-form on C , so that splitting

$$TC \simeq \mathcal{F} \oplus V,$$

stays ω_ϵ -orthogonal, and such that:

$$\forall z \in C, \forall v, w \in V(z) \subset T_z C : \omega_\epsilon(v, w) = \omega(v, w),$$

and

$$\forall z \in C, \forall v, w \in \mathcal{F}(z) \subset T_z C : \omega_\epsilon(v, w) = \omega(v, w) + h(z) \cdot \omega_g(v, w),$$

where ω_g is the g -area 2-form (previously also dA_g) on the corresponding leaf, with same orientation as ω .

By (0.3) the ω_ϵ -area of each leaf $S_{s,t}$ is at least R^2 . By the gap condition (0.4) we clearly have

$$d_{C^\infty}(\omega, \omega_\epsilon) < \epsilon$$

on C . Now, by construction (specifically properties of h , and property 1 of each f^l) ω_ϵ extends to a 2-form ω'' on M' coinciding with ω outside $N_\epsilon(C)$, the open ϵ -neighborhood in M' of C , and satisfying:

$$d_{C^\infty}(\omega'', \omega) < \epsilon.$$

Also by construction of f_l and property 2 in particular, if $\epsilon < 1$, $N_\epsilon(C) \subset K$ where $K \subset M'$ is a fixed compact (in particular independent of ϵ). We can in fact take

$$K = sq \times [0, 1 + R + 2^{10}R]^2.$$

Now fix a symplectic embedding

$$\phi_0 : [0, R]^2 \rightarrow (S_0, \omega''|_{S_0}),$$

(recall that the ω area of S_0 is by construction at least R^2) and define

$$\phi : \text{Cube}(R) \rightarrow C$$

by

$$\phi(p, q, s, t) = \phi_0(p, q) + (0, 0, s, t).$$

Then by construction

$$\phi^* \omega'' = \omega_{st}.$$

Now since $\omega'' = \omega$ outside K , we obviously get an induced 2-form ω' , on M , C^∞ ϵ -nearby to a split (in fact standard) symplectic form, s.t. there is a symplectic embedding:

$$\phi : (Cube(R), \omega_{st}) \rightarrow (M, \omega').$$

Moreover, by construction we may insure that

$$\text{image}(\phi) \subset M - \bigcup_i \Sigma_i,$$

where

$$\Sigma_i = S^2 \times (S^1 \times \dots \times S^1 \times \{pt\} \times S^1 \times \dots \times S^1) \subset M,$$

where the singleton $\{pt\} \subset S^1$ replaces the i 'th factor of $T^{2n-2} = S^1 \times \dots \times S^1$. And so we are done. \square

REFERENCES

- [1] M. GROMOV, *Pseudo holomorphic curves in symplectic manifolds.*, Invent. Math., 82 (1985), pp. 307–347.
- [2] Y. SAVELYEV, *Conformal symplectic Weinstein conjecture and non-squeezing*, arXiv, (2016).

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