LOCALLY CONFORMALLY SYMPLECTIC DEFORMATION OF GROMOV NON-SQUEEZING

YASHA SAVELYEV

ABSTRACT. We prove one deformation theoretic extension of the Gromov non-squeezing phenomenon to lcs structures, or locally conformally symplectic structures, which suitably generalize both symplectic and contact structures. We also propose an analogue in lcs geometry of contact non-squeezing of Eliashberg-Polterovich and some other related questions.

1. INTRODUCTION

Definition 1.1. *A locally conformally symplectic manifold is a smooth $2n$-fold M, (in the present case of dimension at least 4) with an* lcs *structure: which is a non-degenerate 2-form ω, which is locally diffeomorphic to $f \cdot \omega_{st}$, for some (non-fixed) positive smooth function f, with ω_{st} the standard symplectic form on $\mathbb{R}^{2n}$. Explicitly, for every $p \in M$ there is a smooth chart*

$$\phi : V \subset \mathbb{R}^{2n} \to M,$$

so that $\phi(V) \ni p$, and $\phi^* \omega = f \cdot \omega_{st}$, for some smooth positive f.

These structures have recently come into focus, for example we have a fascinating recent theorem of Apostolov, Dloussky [1] that every complex surface with an odd first Beti number admits a natural compatible lcs structure. This suggests that lcs structures are absolutely fundamental.

A basic invariant of a lcs structure ω when M has dimension at least 4: is the Lee class, $\alpha = \alpha_\omega \in H^1(M, \mathbb{R})$. Again assuming M has dimension at least 4, the Lee class α has a natural differential form representative, called the Lee form and defined as follows. We take a cover of M by open sets U_i in which $\omega = f_i \cdot \omega_i$ for ω_i symplectic, and f_i a positive smooth function. Then we have 1-forms $d(\ln f_i)$ in each U_i which glue to a well defined closed 1-form on M, as shown by Lee. By slight abuse, we denote this 1-form and its cohomology class all by α. The class α has the property that on the associated α-covering space $\widetilde{M}$, the lift $\widetilde{\omega}$ is globally conformally symplectic. By α-covering space we mean the covering space associated to the normal sub-group $\ker\langle \alpha, \cdot \rangle \subset \pi_1(M, x)$, where $\langle \alpha, \cdot \rangle : \pi_1(M, x) \to \mathbb{R}$ is the homomorphism $[\gamma] \mapsto \langle \alpha, [\gamma] \rangle$.

It is moreover immediate that for an lcs form ω

$$d\omega = \alpha \wedge \omega,$$

for α the Lee form as defined above. For some authors, the pair (ω, α) with α closed s.t. $d\omega = \alpha \wedge \omega$ is the lcs structure, this has the advantage of being interesting even in dimension 2.

Introduce the operator

$$d^\alpha : \Omega^k(M) \to \Omega^{k+1}(M),$$
$$d^\alpha(\omega) = d\omega - \alpha \wedge \omega.$$

This is called the Lichnerowicz differential with respect to a closed 1-form α, and it satisfies

$$d^\alpha \circ d^\alpha = 0$$

so that we have an associated ***Lichnerowicz chain complex***. The following is one basic example of an lcs manifold.

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Example 1 (Banyaga). Let (C, λ) be a contact $(2n+1)$-manifold where λ is a contact form,

$$\forall p \in C : \lambda \wedge \lambda^{2n}(p) \neq 0,$$

and take $M = C \times S^1$ with 2-form

$$\omega_\lambda = d^\alpha \lambda$$

for $\alpha := pr_{S^1}^* d\theta$, $pr_{S^1} : C \times S^1 \to S^1$ the projection, and λ likewise the pull-back of λ by the projection $C \times S^1 \to C$. We call (M, ω_λ) as above the ***lcs-fication*** of (C, λ).

1.0.1. *Transformations of* lcs *manifolds.* One type of transformations of lcs manifolds which is often considered is a ***conformal symplectomorphism***. That is a diffeomorphism $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ satisfying $\phi^* \omega_2 = f \omega_1$ for some smooth positive function f. To see why this kind of transformation is natural it is perhaps best to think in terms Hamiltonian dynamics on lcs manifolds. This is studied for instance in the work of Banyaga [2]. Using this notion, we may then faithfully embed the category of contact manifolds, and contactomorphisms into the category we call LCS^c of lcsm's, and conformal symplectomorphisms. However, with this type of transformation the natural embedding of categories

$$i : Symp \to LCS^c,$$

where $Symp$ is the category of symplectic manifolds, is far from full, it is not even injective on the sets of isomorphism equivalence classes.

Here is one fix of the problem that will be conceptually useful for us.

Definition 1.2. *An* lcs **map** $\phi : (M_0, \omega_0) \to (M_1, \omega_1)$ *of* lcs *manifolds is a smooth map so that there is a smooth family* $\{\omega'_t\}_{t=0}^{t=1}$, $\omega'_1 = \phi^* \omega_1$, $\omega'_0 = \omega_0$, *of lcs forms on* M_0 *in the same* d^α *Lichnerowicz cohomology class, for* α *the Lee form of* ω_0*. In other words,*

$$\forall t \in [0,1] : (\omega'_t - \omega_0) \text{ is } d^\alpha \text{ exact.}$$

An lcs *map, whose inverse is an* lcs *map will be called an* lcs **diffeomorphism**. *We denote by* LCS *the category of* lcs *manifolds with morphisms* lcs *morphisms above, in particular* lcs *diffeomorphisms are just isomorphisms in* LCS*.*

Thus, applying Moser's principle, an lcs map from a closed symplectic manifold is a symplectomorphism if we understand a symplectic structure as an equivalence class of isotopic symplectic structures.

Lemma 1.3. *There is a natural faithful functor*

$$emb : Cont \to LCS,$$

with $Cont$ *denoting the category of contact manifolds made explicit in the proof.*

If $Symp$ *denotes the category of closed symplectic manifolds then the canonical functor*

$$Symp \to LCS$$

is injective on the sets of isomorphism equivalence classes. More specifically, if closed symplectic manifolds $(M, \omega_0), (M, \omega_1)$ *are* lcs *diffeomorphic they are symplectomorphic. If a closed* lcs *manifold* (M', ω') *is* lcs *diffeomorphic to a symplectic manifold* (M, ω) *then* (M', ω') *is symplectic. In particular in this case* $(M, \omega), (M', \omega')$ *are symplectomorphic by the above.*

1.1. **Symplectic and** lcs **non-squeezing.** One of the most important to this day results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [6]. The most well known formulation of this is that there does not exist a symplectic embedding

$$B_R \hookrightarrow D_r^2 \times \mathbb{R}^{2n-2}$$

for $R > r$, with B_R the standard closed radius R ball in $\mathbb{R}^{2n}$ centered at 0, and D_r^2 the radius r closed disk in $\mathbb{R}^2$. Gromov's non-squeezing is C^0 persistent in the following sense. The proof of this is subsumed by the proof of Theorem 1.5 which follows, but is much more elementary.

We say that a symplectic form ω on $M \times N$ is *split* if $\omega = \omega_1 \oplus \omega_2$ for symplectic forms ω_1, ω_2 on M respectively N.

Theorem 1.4. *Given $R > r$, there is an $\epsilon > 0$ s.t. for any symplectic form ω' on $M = S^2 \times T^{2n-2}$ C^0-close to a split symplectic form ω and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt] \in H_2(M),$$

there is no symplectic embedding $\phi : B_R \hookrightarrow (M, \omega')$.

On the other hand it is natural to ask:

Question 1. Given $R > r$ and every $\epsilon > 0$ is there a (necessarily non-closed by above) 2-form ω' on $S^2 \times T^{2n-2}$ C^0 or even C^∞ ϵ-close to a split symplectic form ω, satisfying $\langle \omega, A \rangle = \pi r^2$, and such that there is an embedding $\phi : B_R \hookrightarrow S^2 \times T^{2n-2}$, with $\phi^* \omega' = \omega_{st}$? We may call such an embedding symplectic by analogy with classical symplectic maps.

This appears to be a very difficult question, my opinion is that at least in the C^0 case the answer is yes, in part because it is difficult to imagine any obstruction, for example we no longer have Gromov-Witten theory for such a general ω'. In the following Theorem 1.5 we show that if ω' is lcs then the answer is no under some additional conditions. One may think that recent work of Müller [9] may be related to the present discussion. But there seems to be no obvious such relation as pull-backs by diffeomorphisms of nearby forms may not be nearby. Hence there is no way to go from nearby embeddings that we work with to ϵ-symplectic embeddings of Müller.

Let again $M = S^2 \times T^{2n-2}$, with ω a split symplectic form on M. Note that in what follows we take a certain natural metric C^0 topology $\mathcal{T}^0$ on the space of general lcs forms, defined in Section 2, which is finer than the standard C^0 metric topology on the space of forms, cf. [2, Section 6]. The corresponding metric is denoted d_0.

We have a real codimension 1 hypersurfaces

$$\Sigma_i = S^2 \times (S^1 \times \ldots \times S^1 \times \{pt\} \times S^1 \times \ldots \times S^1) \subset M,$$

where the singleton $\{pt\} \subset S^1$ replaces the i'th factor of $T^{2n-2} = S^1 \times \ldots \times S^1$. The hypersurfaces Σ_i are naturally folliated by symplectic submanifolds

$$M_\theta = S^2 \times (S^1 \times \ldots \times S^1 \times \{pt\} \times \{\theta\} \times S^1 \times \ldots \times S^1) \simeq S^2 \times T^{2n-2},$$

$\theta \in S^1$. We denote by $T^{fol}\Sigma_i \subset TM$, the distribution of vectors tangent to the leaves of the above mentioned folliation. In other words

$$T^{fol}\Sigma_i = \cup_\theta i_* TM_\theta,$$

where $i : M_\theta \to M$ are the inclusion maps.

Theorem 1.5. *Let ω be a split symplectic form on $M = S^2 \times T^{2n-2}$, let A be as above with $\langle \omega, A \rangle = \pi r^2$. Let $R > r$, then there is an $\epsilon > 0$ (depending only on R, r, ω) s.t. if $\{\omega_t\}$, $t \in [0, 1]$, $\omega_0 = \omega$ is a $\mathcal{T}^0$-continuous family of* lcs *forms on M, with $d_0(\omega_t, \omega_0) < \epsilon$ for all t, then there is no symplectic embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M - \bigcup_i \Sigma_i, \omega_1),$$

meaning an embedding ϕ such that $\phi^* \omega_1 = \omega_{st}$.

Note that the latter is a full-volume subspace diffeomorphic to $S^2 \times \mathbb{R}^{2n-2}$. More generally there is no symplectic embedding

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1),$$

s.t $\phi_* j$ preserves the bundles $T^{fol}\Sigma_i$, for j the standard almost complex structure on B_R.

Smooth lcs deformations ω_t of our symplectic form ω, with Lee forms α_t likewise smooth in t, are obstructed unless α_t are exact, as pointed out to me by Kevin Sackel. But our families are not required to be smooth so that non-trivial lcs deformations of a symplectic form may exist.

Question 2. Do there exist non-trivial lcs deformations of the standard product symplectic form on $S^2 \times T^{2n-2}$?

We hope that the theorem above helps to motivate this question.

1.1.1. *Invariance of the* lcs *non-squeezing Theorem 1.5 under transformations.* Since we use a seemingly rigid notion of a morphism of lcs manifolds in the statement of lcs non-squeezing, a natural additional question is: how invariant is the lcs non-squeezing result with respect to more general lcs transformations? It is certainly not invariant under transformations in LCS^c, which is not surprising since the original Gromov non-squeezing is not invariant this way. For if $\omega_1 = C\omega_0$, $C > 0$ some constant, ω_1 is conformally symplectomorphic to ω_0 and we may certainly have a symplectic embedding of B_R into (M, ω_1) even if there is no such embedding into (M, ω_0). However LCS also appears to an interesting category to study, and in this category the above non-squeezing result is invariant in the following sense.

Corollary 1.6. *Let ω be an* lcs *form on M,* lcs *diffeomorphic to ω'' (and so also symplectomorphic), where ω'' on $M = S^2 \times T^{2n-2}$, is a split symplectic form. Let A be as above, and suppose that $\langle \omega, A \rangle = \pi r^2$. Then there exists a collection of hypersurfaces $\{\Sigma'_i\}_{i=1}^{2n-2}$ in M so that for all $R > r$ there is an $\epsilon > 0$ (depending on R, r, ω, ω'') s.t. if $\{\omega_t\}$, $t \in [0,1]$, $\omega_0 = \omega$ is a $\mathcal{T}^0$-continuous family of* lcs *forms on M, with $d_0(\omega_t, \omega_0) < \epsilon$ for all t, then there is no symplectic embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M - \bigcup_i \Sigma'_i, \omega_1).$$

1.1.2. *Toward direct generalization of contact non-squeezing.* We can also try a direct generalization of contact non-squeezing of Eliashberg-Polterovich [3], and Fraser in [4]. Specifically let $R^{2n} \times S^1$ be the prequantization space of R^{2n}, or in other words the contact manifold with the contact form $d\theta - \lambda$, for $\lambda = \frac{1}{2}(ydx - xdy)$. Let B_R now denote the open radius R ball in $\mathbb{R}^{2n}$. A Hamiltonian conformal symplectomorphism of an lcs manifold (M, ω), which we just abbreviate by the short name: ***Hamiltonian map***, is a conformal symplectomorphism ϕ_H generated as usual by $H : M \times [0,1] \to \mathbb{R}$, by defining the time dependent vector field X_t

$$\omega(X_t, \cdot) = d^\alpha H_t,$$

for α the Lee form, and then taking ϕ_H to be the time 1 flow map of $\{X_t\}$. For example if $\omega = d^\alpha \lambda$ on $C \times S^1$ is the lcs-fication of the contact form λ on C, and if $\forall t : H_t = -1$, then $d^\alpha(H_t) = -\alpha$ and clearly

$$X_t = (R^\lambda \oplus 0) \subset TC \oplus TS^1,$$

with R^λ the λ-Reeb vector field. Thus in this case the associated flow is naturally induced by the Reeb flow.

Question 3. If $R \geq 1$ is there a compactly supported, Hamiltonian lcs map

$$\phi : \mathbb{R}^{2n} \times S^1 \times S^1 \to \mathbb{R}^{2n} \times S^1 \times S^1,$$

so that $\phi(\overline{U}) \subset U$, for $U := B_R \times S^1 \times S^1$ and $\overline{U}$ the topological closure? Instead of Hamiltonian lcs maps we may try to take lcs maps as in Definition 1.2. The relationship between these notions is not very well understood at the moment, and is somewhat beyond our scope here.

2. Proofs

Let M be a closed smooth manifold of dimension at least 4. The C^0 metric topology $\mathcal{T}^0$ on the set $LCS(M)$ of smooth lcs 2-forms on M will be defined with respect to the following metric.

Definition 2.1. *Fix a Riemannian metric g on M. For $\omega_1, \omega_2 \in LCS(M)$ define*

$$d_0(\omega_1, \omega_2) = d_{C^0}(\omega_1, \omega_2) + d_{C^0}(\alpha_1, \alpha_2),$$

for α_i the Lee forms of ω_i and d_{C^0} the usual C^0 metrics induced by g.

The following characterization of convergence will be helpful.

Lemma 2.2. *Let M be as above and let $\{\omega_k\} \subset LCS(M)$ be a sequence $\mathcal{T}^0$ converging to a symplectic form ω. Denote by $\{\widetilde{\omega}_k\}$ the lift sequence on the universal cover $\widetilde{M}$. Then there is a sequence $\{\widetilde{\omega}_k^{symp}\}$ of symplectic forms on $\widetilde{M}$, and a sequence $\{f_k\}$ of positive functions pointwise converging to 1, such that $\widetilde{\omega}_k = f_k \widetilde{\omega}_k^{symp}$.*

Proof. We may assume that M is connected. Let α_k be the Lee form of ω_k, and g_k functions on $\widetilde{M}$ defined by $g_k([p]) = \int_{[0,1]} p^* \alpha_k$, where the universal cover $\widetilde{M}$ is understood as the set equivalence classes of paths p starting at $x_0 \in M$, with a pair p_1, p_2 equivalent if $p_1(1) = p_2(1)$ and $p_2^{-1} \cdot p_1$ is null-homotopic, where $\cdot$ is the path concatenation.

Then we get:

$$d\widetilde{\omega}_k = dg_k \wedge \widetilde{\omega}_k,$$

so that if we set $f_k := e^{g_k}$ then

$$d(f_k^{-1}\widetilde{\omega}_k) = 0.$$

Since by assumption $|\alpha_k|_{C^0} \to 0$, then pointwise $g_k \to 0$ and pointwise $f_k \to 1$, so that if we set

$$\widetilde{\omega}_k^{symp} := f_k^{-1}\widetilde{\omega}_k$$

then we are done. □

Definition 2.3. *We say that a pair (ω, J) of an* lcs *form ω on M and an almost complex structure J on M are* **compatible** *if $\omega(\cdot, J\cdot)$ defines a J-invariant inner product on M.*

Theorem 2.4. *Let M be as above, $A \in H_2(M)$ fixed, and $\{\omega_t\}$, $t \in [0,1]$, a $\mathcal{T}^0$-continuous family of* lcs *forms on M. Let $\{J_t\}$ be a Frechet smooth family of almost complex structures, with J_t compatible with ω_t for each t. Let $D \subset \widetilde{M}$, with $\pi : \widetilde{M} \to M$ the universal cover of M, be a fundamental domain, and $K := \overline{D}$ its topological closure. Suppose that for each t, and for every $x \in \partial K$ (the topological boundary) there is a $\widetilde{J}_t$-holomorphic hyperplane H_x through x, with $H_x \subset K$, such that $\pi(H_x) \subset M$ is a closed submanifold and such that $A \cdot \pi_*([H_x]) \leq 0$. Define:*

$$E_t(u) := \int_{S^2} u^* \omega_t.$$

Then

$$\sup_{u,t} E_t(u) < \infty,$$

where the supremum is over all pairs (u, t), u is a J_t-holomorphic class A genus 0 curve in M.

Proof.

Lemma 2.5. *Let M, A be as above, let $D \subset \widetilde{M}$, with $\pi : \widetilde{M} \to M$ the universal cover of M, be a fundamental domain, and $K := \overline{D}$ its topological closure. Let (ω, J) be a compatible* lcs *pair on M such that for every $x \in \partial K$ there is a $\widetilde{J}$-holomorphic (real codimension 2) hyperplane $H_x \subset K \subset \widetilde{M}$ through x, such that $\pi(H_x) \subset M$ is a closed submanifold and such that $A \cdot [\pi(H_x)] \leq 0$. Then any genus 0, J-holomorphic class A curve u in M has a lift $\widetilde{u}$ with image in K.*

Proof. For u as in the statement, let $\widetilde{u}$ be a lift intersecting the fundamental domain D, (as in the statement of main theorem). Suppose that $\widetilde{u}$ intersects ∂K, otherwise we already have image $\widetilde{u} \subset K^\circ$, for K° the interior, since image $\widetilde{u}$ is connected (and by elementary topology). Then $\widetilde{u}$ intersects u_x as in the statement, for some x. So u is a J-holomorphic map intersecting the closed hyperplane $\pi(H_x)$ with $A \cdot [\pi(H_x)] \leq 0$. By positivity of intersections [7, Section 2.6], which in this case is just a simple exercise, image $u \subset \pi(H_x)$, and so image $\widetilde{u} \subset H_x$. And so image $\widetilde{u} \subset \partial K$. □

Now, let u be a J_t-holomorphic class A curve. By the lemma above u has a lift $\widetilde{u}$ contained in the compact $K \subset \widetilde{M}$. Then we have:

$$E_t(u) = \int_{S^2} \widetilde{u}^* \widetilde{\omega}_t \leq C_t \langle \widetilde{\omega}_t^{symp}, A \rangle,$$

where $\widetilde{\omega}_t = f_t \widetilde{\omega}_t^{symp}$, for $\widetilde{\omega}_t^{symp}$ symplectic on $\widetilde{M}$, and $f_t : \widetilde{M} \to \mathbb{R}$ positive function constructed as in the proof of Lemma 2.2, and where $C_t = \max_K f_t$. Since $\{\omega_t\}$ is continuous in $\mathcal{T}_0$, $\{f_t\}$, $\{\widetilde{\omega}_t^{symp}\}$ are C_0 continuous. In particular

$$C = \sup_t \max_K f_t$$

and

$$D = \sup_t \langle \widetilde{\omega}_{t'}^{symp}, A \rangle$$

are finite. And so

$$\sup_{(u,t)} E_t(u) \leq C \cdot D,$$

where the supremum is over all pairs (u, t), u is J_t-holomorphic curve in M. □

2.1. Quick review of genus 0 Gromov-Witten theory.

Let

$$\mathcal{M}_{g,0}(J, A) = \mathcal{M}_{g,0}(M, J, A)$$

denote the moduli space of isomorphism classes of class A, J-holomorphic curves in M, with domain the Riemann sphere, with n marked labeled points. Here an isomorphism between $u_1 : \Sigma_1 \to M$, and $u_2 : \Sigma_2 \to M$ is a biholomorphism of marked Riemann surfaces $\phi : \Sigma_1 \to \Sigma_2$ s.t. $u_2 \circ \phi = u_1$.

The following is well known and follows by the same argument as [7, Theorem 5.6.6].

Theorem 2.6. *Let (M, J) be an almost complex manifold. Then $\mathcal{M}_{g,0}(J, A)$ has a pre-compactification*

$$\overline{\mathcal{M}}_{g,0}(J, A),$$

by Kontsevich stable maps, with respect to the natural metrizable Gromov topology [7, Chapter 5.6]. *Moreover given $E > 0$, the subspace $\overline{\mathcal{M}}_{g,0}(J, A)_E \subset \overline{\mathcal{M}}_{g,0}(J, A)$ consisting of elements u with $e(u) \leq E$ is compact, where e is the L^2 energy with respect to an auxiliary metric. In other words e is a proper function.*

Thus the most basic situation where we can talk about Gromov-Witten "invariants" of (M, J) is when the energy function is bounded on $\overline{\mathcal{M}}_{g,0}(J, A)$. In this case $\overline{\mathcal{M}}_{g,n}(J, A)$ is compact, and has a virtual moduli cycle as in the original approach of Fukaya-Ono [5], or the more algebraic approach [8]. So we may define functionals:

$$GW_{g,n}(A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}, \tag{2.7}$$

where $\overline{M}_{g,n}$ denotes the compactified moduli space of Riemann surfaces. Of course symplectic manifolds with any tame almost complex structure is one class of examples.

These functionals will not in general be J-invariant but they are invariant for a smooth family $\{J_t\}$, $t \in [0, 1]$ such that the corresponding cobordism moduli space $\overline{\mathcal{M}}_{g,0}(\{J_t\}, A)$, is compact, where $\overline{\mathcal{M}}_{g,0}(\{J_t\}, A)$ is the space of pairs (u, t), $u \in \overline{\mathcal{M}}_{g,0}(J_t, A)$.

Proof of Theorem 1.5. Fix an $\epsilon' > 0$ s.t. any 2-form ω_1 on M, C^0 ϵ'-close to ω, is non-degenerate and is non-degenerate on the leaves of the folliation of each Σ_i, discussed prior to the formulation of the theorem. Suppose by contradiction that for every $\epsilon > 0$ there is a homotopy $\{\omega_t\}$ of lcs forms, with $\omega_0 = \omega$, such that $\forall t : d_0(\omega_t, \omega) < \epsilon$ and such that there exists a symplectic embedding

$$\phi : B_R \hookrightarrow (M, \omega_1),$$

satisfying conditions of the statement of the theorem. Take $\epsilon < \epsilon'$, and let $\{\omega_t\}$ be as in the hypothesis above. In particular ω_t is an lcs form for each t, and is non-degenerate on Σ_i. Extend $\phi_* j$ to an ω_1-compatible almost complex structure J_1 on M, preserving $T^{fol}\Sigma_i$ for each i. We may then extend this to a family $\{J_t\}$ of almost complex structures on M, s.t. J_t is ω_t-compatible for each t, with J_0 is the standard split complex structure on M and such that J_t preserves $T^{fol}\Sigma_i$ for each t, i. The latter condition can be satisfied since Σ_i are ω_t-symplectic for each t. When $\phi(B_R)$ does not intersect $\cup_i \Sigma_i$ these conditions can be trivially satisfied, first find an extension J_1 of $\phi_* j$ preserving $T^{fol}\Sigma_i$ for each i. Then extend to a family $\{J_t\}$.

Then the family $\{(\omega_t, J_t)\}$ satisfies the hypothesis of Theorem 2.4 for the class $A = [S^2] \otimes [pt]$ as in the statement of the theorem we are proving. And so since we have an energy bound

$$C = \overline{\mathcal{M}}_{0,1}(\{J_t\}, A)$$

is compact by Theorem 2.6.

Now we have the classical Gromov-Witten invariant counting class A, J_0-holomorphic, genus 0 curves passing through a fixed point:

$$GW_{0,1}(A, J_0)([pt]) = 1,$$

whose calculation already appears in [6]. Then by compactness of C, and the discussion preceding the proof

$$GW_{0,1}(A, J_1)([pt]) = 1.$$

In particular there is a class A J_1-holomorphic curve u passing through $\phi(0)$.

By Lemma 2.5 we may choose a lift $\widetilde{u}$ of u to $\widetilde{M}$, with homology class $[\widetilde{u}]$ also denoted by A so that the image of $\widetilde{u}$ is contained in a compact set $K \subset \widetilde{M}$, (independent of the choice of $\{\omega_t\}, \{J_t\}$ satisfying above conditions). Let $\widetilde{\omega}_t^{symp}$ and f_t be as in Lemma 2.2, then by this lemma for every $\delta > 0$ we may find an $\epsilon > 0$ so that if $d_0(\omega_1, \omega) < \epsilon$ then $d_{C^0}(\widetilde{\omega}^{symp}, \widetilde{\omega}_1^{symp}) < \delta$ on K, and $\sup_K |f_1 - 1| < \delta$.

Let δ as above be chosen, and let ϵ correspond to this δ. Then if this δ is sufficiently small we get

$$|\int_{S^2} u^*\omega_1 - \pi r^2| \leq |\max_K f_1 \langle \widetilde{\omega}_1^{symp}, A\rangle - \pi \cdot r^2| < \pi R^2 - \pi r^2,$$

since

$$|\langle \widetilde{\omega}_1^{symp}, A\rangle - \pi \cdot r^2| = |\langle \widetilde{\omega}_1^{symp}, A\rangle - \langle \widetilde{\omega}^{symp}, A\rangle| \leq \delta\pi \cdot r^2,$$

for $\langle \widetilde{\omega}^{symp}, A\rangle = \pi r^2$ and $d_{C^0}(\widetilde{\omega}^{symp}, \widetilde{\omega}_1^{symp}) < \delta$ on K, and since

$$\max_K f_1 \leq 1 + \delta.$$

So if δ, ϵ are chosen appropriately as above, we get that $\int_{S^2} u^*\omega_1 < \pi R^2$.

We may then proceed exactly as in the now classical proof of Gromov [6] of the non-squeezing theorem to get a contradiction and finish the proof. A bit more specifically, $\phi^{-1}(\operatorname{image}\phi \cap \operatorname{image} u)$ is a minimal surface in B_R, with boundary on the boundary of B_R, and passing through $0 \in B_R$. By construction it has area strictly less then πR^2 which is impossible by the classical monotonicity theorem of differential geometry. $\square$

Proof of Lemma 1.3. Specifically we define $Cont$ to be the category with objects (C, λ) where λ is a contact form and morphisms

$$\phi : (C_1, \lambda_1) \to (C_2, \lambda_2)$$

contactomorphisms, so that

$$\phi^*(\lambda_2) = f\lambda_1$$

for a positive function f. Define

$$emb : Cont \to LCS$$

by

$$emb(C, \lambda) = (C \times S^1, d^\alpha \lambda),$$

$\alpha = d\theta$, in other words the lcs-fication as usual. For a contactomorphism $\phi : (C_1, \lambda_1) \to (C_2, \lambda_2)$ define $emb(\phi) = (\phi \times id)$. Then

$$emb(\phi)^* d^\alpha \lambda_2 = d^\alpha f\lambda_1$$

is homotopic through the lcs forms

$$\{d^\alpha f_t \lambda_1\},$$

for $\{f_t\}$ a smooth homotopy of positive functions, $f_1 = f$, $f_0 = 1$. And so $emb(\phi)$ is an lcs map. It is obvious that emb is functorial.

Now let $(M, \omega), (M', \omega')$ be closed symplectic manifolds. Let

$$\phi : (M, \omega) \to (M', \omega')$$

be a lcs diffeomorphism. Then since the Lee form α of ω is 0, by definition we have that $\phi^*\omega'$ is homotopic through symplectic forms in the same cohomology class to ω. So by Moser's lemma, $(M, \omega), (M', \omega')$ are symplectomorphic.

Finally let (M, ω) be a closed symplectic manifold, (M', ω') be an lcs manifold and let

$$\phi : (M, \omega) \to (M', \omega')$$

be an lcs diffeomorphism. Then by the same point as above $\phi^* \omega'$ is homotopic through symplectic forms to ω. In particular ω' is closed, so (M', ω') is symplectic. $\square$

Proof of the Corollary 1.6. Let

$$\rho : (M, \omega) \to (M, \omega'')$$

be an lcs diffeomorphism. By Lemma 1.3 ω is then symplectic and there is an induced symplectomorphism

$$\rho' : (M, \omega) \to (M, \omega'').$$

Let ϵ' be chosen with respect to ω'', R, r as in the statement of Theorem 1.5. And let ϵ be taken so that:

$$d_0(\omega, \omega') < \epsilon \implies d_0(\omega'', \rho'_* \omega') < \epsilon', \tag{2.8}$$

for any lcs-form ω'. Define $\Sigma'_i := (\rho')^{-1}(\Sigma_i)$. Given $\{\omega_t\}$ as in the statement, suppose otherwise that we have a symplectic embedding:

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M - \bigcup_i \Sigma'_i, \omega_1).$$

Then

$$\rho' \circ \phi : (B_R, \omega_{st}) \hookrightarrow (M - \bigcup_i \Sigma_i, \rho'_*(\omega_1))$$

is a symplectic embedding. But this contradicts the conjunction of Theorem 1.5 and (2.8). $\square$

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Email address: yasha.savelyev@gmail.com

University of Colima, CUICBAS