ANALOGUES OF GÖDEL STATEMENTS AND COMPUTABILITY OF INTELLIGENCE

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Abstract. We show that there is a mathematical obstruction to complete Turing computability of intelligence. This obstruction can be circumvented only if human reasoning is fundamentally unsound. The most compelling original argument for existence of such an obstruction was proposed by Penrose, however Gödel, Turing and Lucas have also proposed such arguments. We first slightly reformulate the argument of Penrose keeping the main idea the same. In this formulation we argue that his argument works up to possibility of construction of a certain Gödel statement. We then completely reframe the argument in the language of Turing machines, and solve the problem by partially defining our subject just enough so that a certain analogue of a Gödel statement, or Gödel string as we call it in the language of Turing machines, can be trivially constructed directly.

In particular, under the soundness hypothesis, we show that at least some, meaningful thought processes of the brain cannot be Turing computable. And so some physical processes are absolutely not Turing computable, in any physical model.

Question 1. Can human intelligence be completely modelled by a Turing machine?

An informal definition of a Turing machine (see [1]) is as follows: it is an abstract machine which accepts certain inputs, and produces outputs. The outputs are determined from the inputs by a fixed finite algorithm, in a specific sense. In particular anything that can be computed by computers as we know them can be computed by a Turing machine.

For the purpose of the main result the reader may simply understand a Turing machine as a digital computer with unbounded memory running a certain program. Unbounded memory is just mathematical convenience, it can in specific arguments, also of the kind we make, be replaced by non-explicitly bounded memory.

Turing himself has started on a form of Question 1 in his Computing machines and Intelligence, [2], where he also informally outlined a possible obstruction to a yes answer coming from Gödel's incompleteness theorem. For the incompleteness theorem to come in we need some assumption on the fundamental soundness of human reasoning. We need some qualifier like "fundamental" as even mathematicians are not on the surface sound at all times. In this note fundamental soundness is understood as follows. We are on the surface unsound not because of fundamental internal inconsistencies of our mental constructions, but for the following pair of reasons. First, due to time constraints humans make certain leaps of faith, without fully vetting their logic. Second, the noisy, faulty, biological nature of our brain leads to interpretation errors of our mental constructions. Here by "faulty", we mean the possibly common occurrence of faults in brain processes, coming from things like brain cell death, signaling noise between neurons, etc., even if the brain may have robust fault protections built in. Let us call all these possible fault vectors "brain nose". In other words, according to us to say that a human being is fundamentally sound, is to say that after ripping out the "brain noise" this human being will be sound, and have the same reasoning powers.

Gödel himself first argued for a no answer in [3, p. 310], relating the question to existence of absolutely undecidable problems, see Feferman [4] for a discussion. Since existence of absolutely undecidable problems is such a difficult and contentious issue even if Gödel's argument is in essence correct, it is not completely compelling.

Later Lucas [5] and later again and more robustly Penrose [6] argued for a no answer, based only on soundness. Such an argument if correct would be extremely compelling. They further formalized and elaborated the obstruction coming from Gödel's incompleteness theorem. And they reject the

possibility that humans could be unsound on a fundamental level, as does Gödel but for him it is apparently not even a possibility, it does not seem to be stated in [3]. [1]

It should also be noted that for Penrose in particular, non-computability of intelligence is evidence for new physics, and he has specific and *very* intriguing proposals with Hameroff [7], on how this can take place in the human brain. Here is a partial list of some partially related work on mathematical models of brain activity and or quantum collapse models: [8], [9], [10], [11].

The following is a slightly informal version of our main Theorem 4.1.

Theorem 0.1. *Either there are cognitively meaningful, non Turing computable processes in the human brain, or human beings are fundamentally unsound. This theorem is indeed a mathematical fact, for our chosen physical interpretation of fundamental soundness as outlined above.*

The immediate implications and context of the above are in mathematical physics and in part biology, and philosophy. For even existence of non Turing computable processes in nature is not known. For example we expect beyond reasonable doubt that solutions of Navier-Stokes equations or N-body problems are generally non Turing computable, (over $\mathbb{Z}$, if not over $\mathbb{R}$ cf. [12]), as modeled in essentially classical mechanics. But in a more physically accurate and fundamental model these may both become computable, (possibly if the nature of the universe is ultimately discreet.) Our theorem says that either there are absolutely, that is model independent, non-computable processes in physical nature, in fact in the functioning of the brain, or human beings are fundamentally unsound, which is a mathematical condition on the functioning of the human brain. Despite the partly physical context the technical methods of the paper are mainly of mathematics and computer science, as we need very few physical assumptions.

Outline of the main idea of the Gödelian analysis. What follows will be very close in essence to the argument Penrose gives in [13], which we take to be his main and final argument. However we partially reinterpret this to be closer to our argument later on. While this outline uses some of the language of formal systems, except for some supplementary remarks, we will *not* use this language in our actual argument, which is based purely on the language of Turing machines, and is much more elementary, in particular any inaccuracies of the following outline should disappear.

Let P be a human subject, which we understand at the moment as a machine printing statements in arithmetic, given some input. That is for each Σ some string input in a fixed finite alphabet, $P(\Sigma)$ is a statement in arithmetic, e.g. "There are infinitely many primes." Say now P is in contact with experimenter/operator E. The input strings that E gives P are pairs (Σ_T, n) for Σ_T specification of a Turing machines T, and $n \in \mathbb{N}$.

Let Θ_T be the statement:

$$T \text{ computes } P. \tag{0.2}$$

For each (Σ_T, n), P prints his statement $P(\Sigma_T, n)$, which he asserts to hold if Θ_T. We ask that for each T: $\{P(\Sigma_T, n)\}_n$ is the complete list of statements that P asserts to be true, conditionally on Θ_T. (The list is infinite but if P is computable it is determined by a Turing finite machine, so while this is an idealization it is not an extreme one.)

Before we proceed, we put the condition on our P that he asserts himself to be fundamentally sound. Let then T_0 be a specified Turing machine, and suppose that E passes to P input of the form (Σ_{T_0}, n_0). Now P reasons that since he is fundamentally sound, so that his deductions based on each Θ_T are sound the following statement I_{T_0} must hold:

$$\Theta_{T_0} \implies \forall n \, T_0(\Sigma_{T_0}, n). \tag{0.3}$$

On the other hand, as is well known, the statements $\{T_0(\Sigma_{T_0}, n)\}_n$ must be the complete list of provable statements in a certain formal system $\mathcal{F}(T_0)$ associated to T_0, which would be sound if Θ_{T_0} and if I_{T_0}, and so would be in particular consistent.

[1] It is likely most mathematicians would sympathize with Gödel, after all mathematics is meaningless if mathematicians are fundamentally unsound - for then they must eventually assert everything to be true.

In particular if Θ_{T_0} and if I_{T_0}, then there would be a true Gödel statement $G(T_0)$ such that $T_0(\Sigma_{T_0}, n) \neq G(T_0)$, for all n. But P asserts I_{T_0}, hence he must assert

$$\Theta_{T_0} \implies G(T_0),$$

and so if P knew how to construct $G(T_0)$ then this statement must be in the list $\{P(\Sigma_{T_0}, n)\}_n$, and so in the list $\{T_0(\Sigma_{T_0}, n)\}_n$ if Θ_{T_0}, so we would get a contradiction, so either not Θ_{T_0} or P is not sound.

The above outline would at least in principle work if $G(T_0)$ was constructible by P. From this author's point of view this is not in principle an obstruction since the specification of $\mathcal{F}(T_0)$ could at least in principle be explicitly obtained by P given the finite specification of T_0. And the Gödel statement could always, at least in principle, be explicitly constructed once one knows the formal system, even if in practice this may be hopelessly difficult, (what if the number of axioms in $\mathcal{F}(T_0)$ is the number of atoms in the solar system?). We will delve no further into this. One detailed critique of the Penrose argument is given in Koellner [14], [15], see also Penrose [16], and Chalmers [17] for discussions on related issues.

In this note we will reformulate the above using a far more elementary approach, more heavily based in Turing machines. We first partially define our subject henceforth denoted by S, by means of formalizing properties of a certain function associated to S. We do this so that a certain analogue of the Gödel statement can be trivially constructed directly. This will not be exactly "Gödel statement", but rather a "Gödel string" as we call it, because we will not even be dealing with formal systems, but purely with Turing machines. But this string has analogous properties, and allows us to bypass the main objection to the above outline.

As a final remark, technically the paper is mostly elementary and should be widely readable, in entirety.

1. Some preliminaries

This section can be just skimmed on a first reading. Really what we are interested in is not Turing machines per se, but computations that can be simulated by Turing machine computations. These can for example be computations that a mathematician performs with paper and pencil, and indeed is the original motivation for Turing's specific model. However to introduce Turing computations we need Turing machines, here is our version which is a computationally equivalent, minor variation of Turing's original machine.

Definition 1.1. *A* **Turing machine** M *consists of:*

- *Three infinite (1-dimensional) tapes T_i, T_o, T_c, divided into discreet cells, one next to each other. Each cell contains a symbol from some finite alphabet. A special symbol b for blank, (the only symbol which may appear infinitely many often).*
- *Three heads H_i, H_o, H_c (pointing devices), H_i can read each cell in T_i to which it points, H_o, H_c can read/write each cell in T_o, T_c to which it points. The heads can then move left or right on the tape.*
- *A set of internal states Q, among these is "start" state q_0. And a non-empty set F of final, "finish" states.*
- *Input string Σ, the collection of symbols on the tape T_i, so that to the left and right of Σ there are only symbols b. We assume that in state q_0, H_i points to the beginning of the input string, and that the T_c, T_o have only b symbols.*
- *A finite set of instructions I that given the state q the machine is in currently, and given the symbols the heads are pointing to, tells M to do the following, the taken actions 1-3 below will be (jointly) called an* **executed instruction set**, *or just* **step**:
  1. *Replace symbols with another symbol in the cells to which the heads H_c, H_o point (or leave them).*
  2. *Move each head H_i, H_c, H_o left, right, or leave it in place, (independently).*
  3. *Change state q to another state or keep it.*

- *Output string Σ_{out}, the collection of symbols on the tape T_o, so that to the left and right of Σ there are only symbols b, when the machine state is final. When the internal state is one of the final states we ask that the instructions are to do nothing, so that these are frozen states.*

We also have the following minor variations on standard definitions, and notation.

Definition 1.2. *A* **complete configuration** *of a Turing machine M or* **total state** *is the collection of all current symbols on the tapes, position of the heads, and current internal state. A* **Turing computation**, *or* **computation sequence** *for M is a possibly not eventually constant sequence*

$$\{s_i\}_{i=0}^{i=\infty} := *M(\Sigma)$$

of complete configurations of M, determined by the input Σ and M, with s_0 the initial configuration whose internal state is q_0. If elements of $\{s_i\}_{i=0}^{i=\infty}$ are eventually in some final machine state, so that the sequence is eventually constant, then we say that the computation **halts**. *In this case we denote by s_f the final configuration, so that the sequence is eventually constant with terms s_f. We define the* **length** *of a computation sequence to be the first occurrence of $n > 0$ s.t. $s_n = s_f$. For a given Turing computation $*M(\Sigma)$, we shall write*

$$*M(\Sigma) \to x,$$

if $*M(\Sigma)$ halts and x is the output string.

We write $M(\Sigma)$ for the output string of M, given the input string Σ, if the associated Turing computation $*M(\Sigma)$ halts.

Definition 1.3. *Let Strings denote the set of all finite strings of symbols in some fixed finite alphabet, for example $\{0, 1\}$. Given a partially defined function $f : Strings \to Strings$, that is a function defined on some subset of Strings - we say that a Turing machine M* **computes** *f if $*M(\Sigma) \to f(\Sigma)$, whenever $f(\Sigma)$ is defined.*

We will just call a partially defined function $f : Strings \to Strings$ as a function, for simplicity. So a Turing machine T itself determines a function, which is defined on all $\Sigma \in Strings$ s.t. $*T(\Sigma)$ halts, by $\Sigma \mapsto T(\Sigma)$. The following definition is also purely for writing purposes.

Definition 1.4. *Given Turing computations (for possibly distinct Turing machines) $*T_1(\Sigma_1)$, $*T_2(\Sigma_2)$ we say that they are* **equivalent** *if either they both halt with the same output string, or both do not halt.*

In practice we will allow our Turing machine T to reject some elements of *Strings* as valid input. We may formalize this by asking that there is a special final machine state q_{reject}, so that $T(\Sigma)$ halts with q_{reject} for

$$\Sigma \notin I \subset Strings,$$

where I is some set of all valid, that is T-***permissible*** input strings. We do not ask that for $\Sigma \in I$ $*T(\Sigma)$ halts. If $*T(\Sigma)$ does halt then we shall say that Σ is ***acceptable***. It will be convenient to forget q_{reject} and instead write

$$T : I \to O,$$

where $I \subset Strings$ is understood as the subset of all T-permissible strings, and O is the set output strings, keeping all other data implicit. The specific interpretation should be clear in context.

All of our input, output sets are understood to be subsets of *Strings* under some encoding. For example if the input set is $Strings^2$, we may encode it as a subset of *Strings* via encoding of the type: "this string Σ encodes an element of $Strings^2$ its components are Σ_1 and Σ_2." In particular the sets of integers $\mathbb{N}, \mathbb{Z}$ will under some encoding correspond to subsets of *Strings*. However it will be often convenient to refer to input, output sets abstractly without reference to encoding subsets of *Strings*. (Indeed this is how computer languages work.)

Remark 1.5. The above elaborations mostly just have to do with minor set theoretic issues. For example we will want to work with some "sets" $\mathcal{T}$ of Turing machines, with some abstract sets of inputs and outputs. These "sets" $\mathcal{T}$ will truly be sets if implicitly all these abstract sets of inputs and outputs are implicitly encoded as subsets of *Strings*.

Definition 1.6. *We say that a Turing machine T computes a function $f : Strings \to Strings$ on $A \subset Strings$ if A is contained in the subset I of T-permissible strings, and $*M(\Sigma) \to f(\Sigma)$, whenever $f(\Sigma)$ is defined, for $\Sigma \in A$.*

Given Turing machines

$$M_1 : I \to O, M_2 : J \to P,$$

where $O \subset J$, we may naturally ***compose*** them to get a Turing machine $M_2 \circ M_1$, let us not elaborate as this should be clear, we will use this later on.

1.1. **Join of Turing machines.** Our Turing machine of Definition 1.1 is a multi-tape enhancement of a more basic notion of a Turing machine with a single tape, but we need to iterate this further.

We replace a single tape by tapes $T^1, \dots, T^n$ in parallel, which we denote by $(T^1 \dots T^n)$ and call this n-tape. The head H on the n-tape has components H^i pointing on the corresponding tape T^i. When moving a head we move all of its components separately. A string of symbols on $(T^1 \dots T^n)$ is an n-string, formally just an element $\Sigma \in Strings^n$, with i'th component of Σ specifying a string of symbols on T^i. The blank symbol b is the symbol $(b^1, \dots, b^n)$ with b^i blank symbols of T^i.

Given Turing machines M_1, M_2 we can construct what we call a ***join*** $M_1 \star M_2$, which is roughly a Turing machine where we alternate the operations of M_1, M_2. In what follows symbols with superscript $1, 2$ denote the corresponding objects of M_1, respectively M_2, cf. Definition 1.1.

$M_1 \star M_2$ has three (2)-tapes:

$$(T_i^1 T_i^2), (T_c^1 T_c^2), (T_o^1 T_o^2),$$

three heads H_i, H_c, H_o which have component heads H_i^j, H_c^j, H_o^j, $j = 1, 2$. It has machine states:

$$Q_{M_1 \star M_2} = Q^1 \times Q^2 \times \mathbb{Z}_2,$$

with initial state $(q_0^1, q_0^2, 0)$ and final states:

$$F_{M_1 \star M_2} = F^1 \times Q^2 \times \{1\} \sqcup Q^1 \times F^2 \times \{0\}.$$

Then given machine state $q = (q^1, q^2, 0)$ and the symbols $(\sigma_i^1 \sigma_i^2), (\sigma_c^1 \sigma_c^2), (\sigma_o^1 \sigma_o^2)$ to which the heads H_i, H_c, H_o are currently pointing, we first check instructions in I^1 for q^1, $\sigma_i^1, \sigma_c^1, \sigma_o^1$, and given those instructions as step 1 execute:

(1) Replace symbols σ_c^1, σ_o^1 to which the head components H_c^1, H_o^1 point (or leave them in place, the second components are unchanged).
(2) Move each head component H_i^1, H_c^1, H_o^1 left, right, or leave it in place, (independently). (The second component of the head is unchanged.)
(3) Change the first component of q to another or keep it. (The second component is unchanged.) The third component of q changed to 1.

Then likewise given machine state $q = (q^1, q^2, 1)$, we check instructions in I^2 for q^2, $\sigma_i^2, \sigma_c^2, \sigma_o^2$ and given those instructions as step 2 execute:

(1) Replace symbols σ_c^2, σ_o^2 to which the head components H_c^2, H_o^2 point (or leave them in place, the first components are unchanged).
(2) Move each head component H_i^2, H_c^2, H_o^2 left, right, or leave it in place.
(3) Change the second component of q to another or keep it, (first component is unchanged) and change the last component to 0.

Thus formally the above 2-step procedure is two consecutive executed instruction sets in $M_1 \star M_2$. Or in other words it is two terms of the computation sequence.

1.1.1. *Input.* The input for $M_1 \star M_2$ is a 2-string or in other words pair (Σ_1, Σ_2), with Σ_1 an input string for M_1, and Σ_2 an input string for M_2.

1.1.2. *Output.* The output for

$$*M_1 \star M_2(\Sigma_1, \Sigma_2)$$

is defined as follows. If this computation halts then the 2-tape $(T_o^1 T_o^2)$ contains a 2-string, bounded by b symbols, with T_o^1 component Σ_o^1 and T_o^2 component Σ_o^2. Then the output $M_1 \star M_2(\Sigma_1, \Sigma_2)$ is defined to be Σ_o^1 if the final state is of the form $(q_f, q, 1)$ for q_f final, or Σ_o^2 if the final state is of the form $(q, q_f, 0)$, for q_f likewise final. Thus for us the output is a 1-string on one of the tapes.

1.2. **Generalized join.** A natural variant of the above join construction $M_1 \star M_2$, is to let M_1 component of the machine execute a times before going to step 2 and then execute M_2 component of the machine b times, then repeat, for $a, b \in \mathbb{N}$. This results in $a + b$ consecutive terms of the corresponding computation sequence. We denote this generalized join by

$$M_1^a \star M_2^b,$$

so that

$$M_1^1 \star M_2^1 = M_1 \star M_2,$$

where the latter is as above. We will also abbreviate:

$$M_1 \star M_2^b := M_1^1 \star M_2^b.$$

The set of machine states $M_1^a \star M_2^b$ is then $Q^1 \times Q^2 \times \mathbb{Z}_{a+b}$. Let us leave out further details as this construction is analogous to the one above.

1.3. **Universality.** It will be convenient to refer to the universal Turing machine U. This is a Turing machine already appearing in Turing's [1], that accepts as input a pair (T, Σ) for T an encoding of a Turing machine and Σ input to this T. It can be partially characterized by the property that for every Turing machine T and Σ input for T we have:

$$*T(\Sigma) \text{ is equivalent to } *U(T, \Sigma).$$

1.4. **Notation.** In what follows $\mathbb{Z}$ is the set of all integers and $\mathbb{N}$ non-negative integers. We will often specify a Turing machine simply by specifying a function

$$T : I \to O,$$

with the full data of the underlying Turing machine being implicitly specified, in a way that should be clear from context.

When we intend to suppress dependence of a variable V on some parameter p we often write $V = V(p)$, this equality is then an equality of notation not of mathematical objects.

2. Setup for the proof of Theorem 0.1

Definition 2.1. *A* **machine** *is a (partially defined) function $A : Strings \to Strings \times \mathbb{N}$. The second component of $A(\Sigma)$ will be called* **time to answer** *on input Σ. In practice we will often omit to write the second component explicitly.*

Given a Turing machine $T : Strings \to Strings$, and a computer C we have an associated machine

$$T_C : Strings \to Strings \times \mathbb{N},$$

with the $\mathbb{N}$ component $T_C^{\mathbb{N}}(\Sigma)$: the time that it takes the computation sequence $*T(\Sigma)$ to halt when simulated on C.

Remark 2.2. A Turing machine is an abstract machine, a priori not a machine operating in the physical world. If we want a machine operating in the physical world we shall say: a *simulation* of a Turing machine. In this note this will just mean a computer simulation, that is a program running on a computer.

Definition 2.3. *We say that a machine A, is* **strongly computable** *if there exists a Turing machine A', and a computer C such that the corresponding machines A, A'_C coincide (including time's to answer), on any input Σ such that $A(\Sigma)$ is defined. If C is as above we say that A is* **strongly computed** *by A' on C.*

For example suppose that the time to answer of each $A(\Sigma)$ is bounded in Σ, and $\pi_1 \circ A$ is computed by a Turing machine A', for $\pi_1 : Strings \times \mathbb{N} \to Strings$ projection to first component. Suppose however that for any such A' the length of the halting computation sequence $*A'(\Sigma)$ is unbounded in Σ. In this case A cannot be strongly computable.

2.1. **Diagonalization machines.** Let now $\mathcal{T}$ denote the set of Turing machines with sets of inputs and outputs encoded as subsets of $Strings$, see Section 1 for what this means exactly. Set $\mathcal{I} = \mathcal{T} \times \mathbb{N}$, and let $\mathcal{M}_0$ denote the set of machines M with input $\mathcal{I}$ and output in $\mathcal{T} \times Strings \times \mathbb{Z}$. As we are going to directly construct a certain Turing machine analogue of a Gödel statement, to make it exceptionally simple we ask that elements of $\mathcal{M}_0$ have the following form, which expresses the property that these machines are "diagonalizing" against the input.

Note that in what follows we suppress the "time to answer" $\mathbb{N}$ component. For $M \in \mathcal{M}_0$ we can write it as a composition of machines:

$$M(T) = S_{1,D}(R(S_{0,D}(T), T), T), \tag{2.4}$$

where $S_{i,D}$ and R are as follows.

$$S_{0,D} : \mathcal{I} \to \mathcal{T} \times Strings,$$

and

$$R : \mathcal{T} \times Strings \times \mathcal{I} \to \mathbb{Z} \sqcup \{\infty\},$$

where $\{\infty\}$ is the one point set containing the symbol ∞, which is just a particular distinguished symbol, also implicitly encoded as an element of $Strings$.

Notation 1. Here $T \in \mathcal{I}$ so is a pair (B_T, m_T) of a Turing machine B_T and $m_T \in \mathbb{N}$. However to simplify the language and notation we may refer to the machine B_T by just T, so that $T(\Sigma)$ means $B_T(\Sigma)$, for input string Σ. When this gets too confusing we elaborate further.

Next,

$$S_{1,D} : (\mathbb{Z} \sqcup \{\infty\}) \times \mathcal{I} \to \mathcal{T} \times Strings \times \mathbb{Z},$$

satisfies

$$S_{1,D}(x, T) = (S_{0,D}(T), S^{\mathbb{Z}}_{1,D}(x, T)) \tag{2.5}$$

for

$$S^{\mathbb{Z}}_{1,D} : (\mathbb{Z} \sqcup \{\infty\}) \times \mathcal{I} \to \mathbb{Z}$$

satisfying:

$$S^{\mathbb{Z}}_{1,D}(x, T) = x + 1 \text{ if } x \in \mathbb{Z} \subset \mathbb{Z} \sqcup \{\infty\}. \tag{2.6}$$

We set $\mathcal{T}_0 \subset \mathcal{M}_0$ to be the subset corresponding to Turing machines with component machines $S_{i,D}, R$ likewise Turing machines.

Definition 2.7. *We say that a machine $M \in \mathcal{M}_0$ is* **totally computed** *by a $M' \in \mathcal{T}_0$ if M is strongly computed by M' and moreover $S_{i,D}$ is strongly computed by Turing machines $S'_{i,D}$ and R is strongly computed by R', for $S_{i,D}, R$ and $S'_{i,D}, R'$ the components of M respectively M' as above. Meaning that*

$$M(T) = S_{1,D}(R(S_{0,D}(T), T), T),$$

and

$$M'(T) = S'_{1,D}(R'(S'_{0,D}(T), T, T).$$

Given $M \in \mathcal{M}_0$ and $M' \in \mathcal{T}_0$ let $\Theta_{M,M'}$ be the statement:

$$M \text{ is totally computed by } M'. \tag{2.8}$$

Definition 2.9. *We say that $M \in \mathcal{M}_0$ is* **P-sound** *if for each $T \in \mathcal{I}$, with the output string $M(T) = (X, \Sigma^1, n)$ defined, the output string has* **property O (with respect to T)**, *saying that:*

$$\Theta_{M,T} \implies *X(\Sigma^1) \text{ is equivalent to } *T(T).$$

Here $T(T)$ really means $T(\Sigma_T)$ for Σ_T the string encoding of the Turing machine T, keeping in mind Notation 1. We also say in this case that $S_{0,D}(T)$ has property O, so that $M(T)$ has property O iff $pr_1(M(T))$ has property O for $pr_1 : \mathcal{T} \times Strings \times \mathbb{Z} \to \mathcal{T} \times Strings$ the projection map.

Define a P-sound $M' \in \mathcal{T}_0$ analogously.

Definition 2.10. *If M, M' as above are P-sound we will say that sound(M), sound(M') hold.*

For example a trivially P-sound machine M is one for which $S_{0,D}(T) = (T, \Sigma_T)$ for every T.

Definition 2.11. *Set $\mathcal{I}_0 = \mathcal{T}_0 \times \mathbb{N}$. We say that $M \in \mathcal{M}_0$ has* **property G** *if for every $M' \in \mathcal{I}_0$:*

$$\Theta_{M,M'} \implies R'(S'_{1,D}, (\infty, M'), M') \neq \infty,$$

for $S'_{1,D}, R'$ the component machines of M'.

Theorem 2.12. *If a M is P-sound, has property G and if $\Theta_{M,M'}$ then*

$$M(M') \neq (S'_{1,D}, (\infty, M'), n)$$

for any n, where $S'_{1,D}$ is the component machine of M' as above. On the other hand, if M is P-sound and $\Theta_{M,M'}$ for some M', then $*S'_{1,D}(\infty, M')$ is equivalent to $*M'(M')$, so that the string $(S'_{1,D}, (\infty, M'), n)$ has property O for any n.

So given a certain $M \in \mathcal{M}_0$ printing strings with property O, and given any $T \in \mathcal{T}_0$ such that $*T(T)$ halts, if M is totally Turing computable by T, there is a certain explicitly constructible string $\mathcal{G}(T)$ with property O s.t. $\mathcal{G}(T) \neq M(T)$. This "Gödel string" $\mathcal{G}(T)$ is what we are going to use further on.

Proof. Suppose not and let M'_0 be such that Θ_{M,M'_0} and $M(M'_0) = (S'_{1,D}, (\infty, M'_0), n)$ for some n. Then

$$M(M'_0) = S'_{1,D}(\infty, M'_0)$$

since M is P-sound and since $\Theta_{M,M'}$. Then after denoting by $M^{\mathbb{Z}}, (S'_{1,D})^{\mathbb{Z}}$ the $\mathbb{Z}$ components we have:

$$\begin{aligned} (S'_{1,D})^{\mathbb{Z}}(\infty, M'_0) &= n = (M'_0)^{\mathbb{Z}}(M'_0) \\ &= (S'_{1,D})^{\mathbb{Z}}(R'(S'_{1,D}, (\infty, M'), M'_0)) \\ &= (S'_{1,D})^{\mathbb{Z}}(S'_{1,D}(\infty, M'_0)) \quad \text{by property } G \\ &= n + 1 \quad \text{by (2.6)}. \end{aligned} \tag{2.13}$$

So we obtain a contradiction.

It only remains to check that if M is P-sound and $\Theta_{M,M'}$ then $*S'_{1,D}(\infty, M')$ is equivalent to $*M'(M')$. If M is P-sound then $R'(S'_{0,D}, M') = \infty$, as otherwise if $R'(S'_{0,D}, M') = x \in \mathbb{Z}$ then we get:

$$x = (M')^{\mathbb{Z}}(M') = (S'_{1,D})^{\mathbb{Z}}(x) = x + 1$$

by M being P-sound and by (2.6), so we would get a contradiction. But then our conclusion immediately follows.

□

3. A system with a human subject S as a machine with property G

Let S be in an isolated environment, in communication with an experimenter/operator E that as input passes to S elements of $\mathcal{I} = \mathcal{T} \times \mathbb{N}$, that is pairs $T = (B_T, m)$ for B_T a string specification of a Turing machine. Although as before we shall notationally identify this Turing machine with T and with B_T. S has in his environment a general purpose (Turing) digital computer, with arbitrarily, as necessary, expendable memory. A will denote the above system: S and his computer $\mathcal{C}$ in isolation. Here ***isolated environment*** means primarily that no information i.e. stimulus, that is not explicitly controlled by E and that is usable by S, passes to S while he is in this environment.

We suppose that upon receiving receiving T, S implements the following protocol:

- Given T, S decides after a time
$$t_D^0 = t_D^0(T)$$
to run some computation $* = *_T$ on $\mathcal{C}$.
- S then waits for a time
$$t^W = t^W(T)$$
for $*$ to halt.
- S receives the output of $*$, or he stops waiting before it halts.
- S decides after a time
$$t_D^1 = t_D^1(T),$$
on his printed answer to E, which is unambiguously interpreted as an element of $\mathcal{T} \times Strings \times \mathbb{Z}$.

The initial "decision map" of S may be understood as a machine:

$$S_{0,D} : \mathcal{I} \to \mathcal{T} \times Strings,$$

suppressing the "time to answer" $\mathbb{N}$ component of the machine as is usual for us. The output $S_{0,D}(T)$ is a pair (X, Σ^1) of a Turing machine X and input Σ^1 to this Turing machine. The computation $*X(\Sigma^1)$ is what S decides to run on $\mathcal{C}$. In a more basic language, we may say that $S_{0,D}(T)$ is a pair of a computer program and input for this program that S will run on $\mathcal{C}$.

We have another machine:

$$R_{\mathcal{C}} : \mathcal{T} \times Strings \times \mathcal{I} \to \mathbb{Z} \sqcup \{\infty\},$$

$R_{\mathcal{C}}(X, \Sigma^1, T)$ signifies what S obtains as a result of waiting on $*X(\Sigma^1)$ to halt on $\mathcal{C}$, if this is the computation he ran. This in principle may depend on the original input string T as well, because how long he chooses to wait may depend on T. We set

$$R_{\mathcal{C}}(X, \Sigma^1, T) = \infty$$

if S does not finish waiting for $*X(\Sigma^1)$ to halt, and

$$R_{\mathcal{C}}(X, \Sigma^1, T) = X(\Sigma^1)$$

if he does.

Likewise

$$S_{1,D} : (\mathbb{Z} \sqcup \{\infty\}) \times \mathcal{I} \to \mathcal{T} \times Strings \times \mathbb{Z},$$

denotes the machine representing the final "decision map" of S. Its input is meant to be what S obtains from $R_{\mathcal{C}}$, together with the original input string for $S_{0,D}$. We suppose that for any $T \in \mathcal{T}$ this map satisfies:

$$S_{1,D}^{\mathbb{Z}}(x, T) = x + 1 \text{ if } x \in \mathbb{Z} \tag{3.1}$$

and

$$pr_1 \circ S_{1,D}^{\mathbb{Z}}(x, T) = S_{0,D}(T), \tag{3.2}$$

for $pr_1 : \mathcal{T} \times Strings \times \mathbb{Z} \to \mathcal{T} \times Strings$ the projection.

We understand the "wait operation" by S as a machine:

$$W : \mathcal{T} \times Strings \times \mathcal{I} \to \{\infty\} \times \mathbb{N},$$

or

$$W : \mathcal{T} \times Strings \times \mathcal{I} \to \{\infty\},$$

suppresing the $\mathbb{N}$ component. Here $\{\infty\}$ is the one point set as above, so that the first component of the output is only symbolic.

Lemma 3.3. *If W is strongly computable, then $R_{\mathcal{C}}$ is likewise strongly computable.*

Proof. Suppose that W is strongly computed by W' on C_1, so that for each (X, Σ^1, T) as above $*W'(X, \Sigma^1, T)$ halts in time $t^W(T)$ on C_1. For a non-negative integer s, we then call C_s a classical computer (with arbitrarily expendable memory) whose computational capacity is s times the computational capacity of C_1, meaning that the time to halt of any fixed computation run on C_s is $\frac{1}{s}$ the time to halt on C_1.

For

$$Y = (X, \Sigma^1) \in \mathcal{T} \times Strings$$

if $\mathcal{C} = C_s$, we set

$$R'_s(Y, T) = W' \star U^s((Y, T), Y),$$

in the language of generalized join operation described in Section 1, for U the universal Turing machine.

Less formally R'_s is determined by the following properties. The first term of the computation sequence $*R'_s(Y, T)$ corresponds to the first term of $*W'(Y, T)$. The following s terms of $*R'_s(Y, T)$ correspond to the first s terms of $*X(\Sigma^1)$, followed by second term of $*W'(Y, T)$ and then terms $s+1$ to $2s$ of $*X(\Sigma^1)$, and so on. The halting condition is either we reach a final state of W' or a final state of X. If $*R'_s(Y, T)$ halts with a final state of X then

$$R'_s(Y, T) = X(\Sigma^1),$$

otherwise if it halts with a final state of W then

$$R'_s(Y, T) = \infty.$$

Thus, R'_s computes $R_{\mathcal{C}}$, moreover it is clear by construction that it strongly computes $R_{\mathcal{C}}$ on C_{s+1}. □

Therefore the above system A and the protocol determine some machine satisfying:

$$A(T) = S_{1,D}(R_{\mathcal{C}}(S_{0,D}(T), T), T), \tag{3.4}$$

so that $A \in \mathcal{M}_0$.

3.0.1. *Additional conditions.* In the above we only described the general protocol for $A \in \mathcal{M}_0$. We now consider a more specific A_0 of the type above, corresponding to a certain subject S_0, which additionally behaves in the following way. In this case given any input T, given his chosen computation $S_{0,D}(T) = (X, \Sigma_1)$ and given the wait time $T^W = t^W(T) > 0$. If the computation $*X(\Sigma_1)$ does not halt in time t^W he prints (X, Σ_1, n) for arbitrary n in any time less then t^W. In other words the time to answer of $S_{1,D}(\infty, T)$ is less then $t^W(T)$. Furthermore, we ask that for any fixed $B \in \mathcal{T}$

$$\{pr_1 \circ A_0(B, m)\}_m$$

is the complete list of strings that S_0 asserts to have property O with respect to B, again recall our Notation 1. Here pr_1 is as in Definition 2.9. Finally as in the Penrose argument we ask that S_0 asserts that he is fundamentally sound, meaning in this case that he asserts $sound(A_0)$.

3.1. **The conditional $\Theta_{A,A'}$.** For $A \in \mathcal{M}_0$ and $A' \in \mathcal{T}_0$, we now adjust our conditional $\Theta_{A,A'}$ to the following:

$$\begin{aligned} &A \text{ is totally computed by } A', \\ &S_{i,D} \text{ are strongly computed by } S'_{i,D} \text{ on } C_1, \\ &\mathcal{C} = C_s \text{ with } s \text{ at least one.} \end{aligned}$$

This of course adjusts the meaning of $sound(A)$, and property P, as defined following the Definition 2.9, but otherwise the statement of Theorem 2.12 remains unchanged.

4. Proof of Theorem 0.1

Theorem 4.1. *Let A_0 be the machine described above, then for every $A' \in \mathcal{T}$ either not $\Theta_{A_0,A'}$ or not sound(A_0).*

Proof. If $\Theta_{A_0,A'}$ then we may assume that A' is explicitly specified as an element of $\mathcal{I}_0$, since A_0 is explicitly specified to be in $\mathcal{M}_0$. And in particular A' has the form:

$$A'(T) = S'_{1,D}(R'(S'_{0,D}(T), T), T).$$

We then check that A_0 has property G. So we verify that given any $A' \in \mathcal{I}_0$:

$$\text{if } \Theta_{A_0,A'} \text{ then } R'(S'_{1,D}, (\infty, A'), A') \neq \infty.$$

Suppose by contradiction that $\Theta_{A_0,A'}$, and $R'(S'_{1,D}, (\infty, A'), A') = \infty$, then $S'_{1,D}(\infty, A')$ halts in time less then t^W on C_1, since S_0 takes less then t^W time to print his final answer by assumption, and since $S'_{1,D}$ strongly computes $S_{1,D}$ on C_1. But $\mathcal{C} = C_s$ with s at least 1, and S_0 waits for time t^W for $*S'_{1,D}(\infty, A')$ to halt on $\mathcal{C}$, so this would have halted during his wait period, so that $R'(S'_{0,D}(A'), A') \neq \infty$, and we have a contradiction.

Now if $\Theta_{A_0,A'}$ and *sound*(A_0), then by the above and Theorem 2.12

$$A_0(B_{A'}, m) \neq (S'_{1,D}, (\infty, A'), n)$$

for any n, m. On the other hand S_0 asserts *sound*(A_0) and hence must assert that the output string $(S'_{1,D}, (\infty, A'), n)$ has property O for any n, by the second half of Theorem 2.12. In particular the string $(S'_{1,D}, (\infty, A'))$ must be in the list $\{pr_1 \circ A_0(B_{A'}, m)\}_m$, since this list is meant to be complete. This is a contradiction. □

The above theorem is a formal elaboration of our Theorem 0.1, if our subject S_0 asserts that he is fundamentally sound, which means in this context that he asserts in absolute faith that *sound*(A_0) holds.

□

5. A possible objection

Objection 1. Even if S_0 asserts his fundamental soundness, he cannot conclude *sound*(A_0). For S_0 must be aware of the fault issues of his biological brain, cf. the comments in the introduction on fundamental soundness. So S_0 could only assert that $A_0(A')$ has property O sometimes, depending on the "noise" conditions of his brain, that is depending on how likely he is to fault.

Answer. Our human subjects S are meant to be idealized. So that all "brain noise" issues are stripped out of them. By our definition of fundamental soundness in the introduction, this cannot affect the reasoning power of such an S. This idealization may be made more concrete by assuming that our S is hooked up to a machine which detects faults due to the above mentioned "noise". This machine may just be a proof system running on some universal Turing machine.

6. Concluding remarks

The soundness hypothesis deserves much additional further study, far beyond what we can do here, and beyond what already appears in the work of Penrose, and others. Here is however one final remark. If we are fundamentally unsound and computable, then given sufficient future advances in neuroscience and computer science, it will soon be possible to translate human brains to formal systems. Then computers should be able to discover our inconsistencies by brute force analysis. They can then proceed to get us to assert in absolute faith that $0 = 1$. Moreover, in the context of our "thought experiment" above, we can say exactly where to look for the inconsistency, for if $\Theta_{A_0,A'}$ then the halting computation $*S'_{0,D}(B_{A'}, m)$ must for some m involve such an inconsistency, and could then be analyzed.

There is still an open door for non Turing computable simulations of human intelligence. But to get there we likely have to better understand what exactly is happening in the human brain - physically, biologically and mathematically.

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