Direct link to author's version October 28, 2021

GLOBAL FUKAYA CATEGORY II: SINGULAR CONNECTIONS, QUANTUM OBSTRUCTION THEORY AND OTHER APPLICATIONS

YASHA SAVELYEV

Abstract. We perform computations and give applications of the theory developed in Part I of the homotopy natural classifying map

$$BHam(M, \omega) \to (\mathbb{S}, NFuk(M, \omega)),$$

with the right hand side the space (∞-groupoid) of infinity categories in the component of the A_∞ nerve of the Fukaya category of a monotone symplectic manifold (M, ω). For example we find certain curvature lower bounds for certain singular $PU(2)$ and $Ham(S^2)$ connections on principal bundles over S^4, the former being of critical interest in physical Yang-Mills theory. For $PU(2)$, this phenomenon is invisible to Chern-Weil theory, and is inaccessible to known Yang-Mills theory techniques, which are the sharpest known "classical" techniques in this setting. So this can be understood as one application of Floer theory and the theory of ∞-categories in basic differential geometry. We also introduce new integer invariants of smooth manifolds and Hamiltonian bundles, which we call the first quantum obstruction, and we use our computation to show non-triviality of these invariants. On the way we also construct what we call quantum Maslov classes, which are higher dimensional versions of the relative Seidel morphism studied by Hu and Lalonde. We compute this in a particular case and discuss an application to Hofer geometry of the space of Lagrangian equators in S^2.

1. Introduction

Let $Ham(M, \omega)$ denote the group of Hamiltonian symplectomorphisms of a symplectic manifold (M, ω). A *Hamiltonian bundle* is a smooth fiber bundle

$$M \hookrightarrow P \to X,$$

with structure group $\mathrm{Ham}(M, \omega)$ with its C^∞ Frechet topology. Given such a bundle P with M monotone, in Part I [19] we have constructed a classifying map

$$f_P : X \to (\mathbb{S}, NFuk(M))$$

where $\mathbb{S}$ denotes the space of ∞-categories, concretely quasi-categories, in the component of the A_∞ nerve $NFuk(M)$ of the Fukaya category of M, with this nerve having the structure of an ∞-category. This extends to the universal level so that there is a classifying map:

$$B\mathrm{Ham}(M, \omega) \to (\mathbb{S}, NFuk(M)).$$

From here on we just refer to [19] as Part I.

The construction also induces a kind simplicial fibration over X called (co)-Cartesian fibration, with fiber modelled on $NFuk(M)$. We called this the global Fukaya category $Fuk_\infty(P)$ of P. As first computational step, we show that for

---

2000 *Mathematics Subject Classification.* 53D37, 55U35, 53C21.

Key words and phrases. Fukaya category, infinity categories, curvature constraints.

P a non-trivial Hamiltonian S^2 fibration over S^4, the maximal Kan sub-fibration of $Fuk_\infty(P)$, which is just a combinatorial analogue of a Serre fibration, is non-trivial. In particular $Fuk_\infty(P)$ is non-trivial as a (co)-Cartesian fibration and so f_P is homotopically non-trivial. This gives in particular:

Theorem 1.1. *The natural homomorphism as constructed in Part I,*

$$\mathbb{Z} = \pi_4(BHam(S^2), id) \xrightarrow{k} \pi_4(\mathbb{S}, NFuk(S^2)),$$

is injective.

None of the homotopy groups of $\mathbb{S}$ are known, so this is in a sense an application of geometry to algebraic topology. Morally, such an application is possible because geometry forces a priori A_∞-associativity of certain structures, which may then have formal consequences in algebraic topology.

The calculation is performed by constructing perturbation data in such a way that we are reduced to calculation of a certain higher product in a certain Fukaya type A_∞ category. Note that this is an actual chain level calculation. To perform this calculation we relate it to the computation of a certain quantum Maslov, which are certain higher dimensional analogues of the relative Seidel element in [9]. The calculation of this quantum Maslov class uses a regularization technique based on "virtual Morse theory" for the Hofer length functional [17].

It is likely that k is surjective. Surjectivity is in a sense the statement that up to equivalence there are no exotic (co)-Cartesian fibrations over S^4, with fiber equivalent to $N(Fuk(S^2))$ - they all come from Hamiltonian S^2 fibrations, via the global Fukaya category.

1.1. Application in basic Riemannian geometry.

As one less expected application, we can use the computation of Theorem 1.1 to obtain lower bounds for curvature of certain types singular connections.

Definition 1.2. *Let $G \hookrightarrow P \to X$ be a principal G bundle, where G is a Frechet Lie group. A* **singular G-connection** *on P is a closed subset $C \subset X$, and a smooth Ehresmann G-connection $\mathcal{A}$ on $P|_{X-C}$.*

The above definition is basic, as one often puts additional conditions, see for instance [7], [23].

1.1.1. *A non-metric measure of curvature.*

Let G as above be a Frechet Lie group, we denote by $\operatorname{lie} G$ its Lie algebra and let

$$\mathfrak{n} : \operatorname{lie} G \to \mathbb{R}$$

be an Ad invariant Finsler norm. For a principal G-bundle P over a Riemann surface (S, j), and given a G connection $\mathcal{A}$ on P define a 2-form $\alpha_\mathcal{A}$ on S by:

$$\alpha_\mathcal{A}(v, jv) = \mathfrak{n}(R_\mathcal{A}(v, jv)),$$

where $R_\mathcal{A}$ is the curvature 2-form of $\mathcal{A}$, with

$$R_\mathcal{A}(v, w) \in \operatorname{lie} \operatorname{Aut} P_z,$$

for $z \in S$, $v, w \in T_zS$, P_z fiber over z, and $\operatorname{Aut} P_z \simeq G$ the group of G-torsor automorphisms of P_z, where $\simeq$ means non-canonical group isomorphism.

And define

$$\operatorname{area}_\mathfrak{n}(\mathcal{A}) = \int_S \alpha_\mathcal{A}. \tag{1.1}$$

In the case $\mathcal{A}$ is singular with singular set C, $\alpha_{\mathcal{A}}$ is defined on $S - C$ so we define

$$\mathrm{area}_{\mathfrak{n}}(\mathcal{A}) = \int_{S-C} \alpha_{\mathcal{A}},$$

with the right hand side now being an extended integral. This area is a non-metric measurement meaning that no Riemannian metric on S is needed.

It is possible to extend the functional above to a functional on the space $\mathcal{C}$ of G-connections on principal G bundles $P \to \Delta^n$. It may seem that Δ^n has no connection to Riemann surfaces but in fact there is an intriguing connection by way of existence of certain axiomatized systems of maps:

$$u : \overline{\mathcal{S}}_d^\circ \to \Delta^n,$$

where $\overline{\mathcal{S}}_d$ denotes the universal curve over $\overline{\mathcal{R}}_d$ - the moduli space of complex structures on the disk with $d+1$ punctures on the boundary, and $\overline{\mathcal{S}}_d^\circ$ is $\overline{\mathcal{S}}_d$ with nodal points of the fibers removed. Such a chosen system is referred to as $\mathcal{U}$. This was developed in Part I and plays a principal role there.

There is then a functional:

$$\mathrm{area}_{\mathcal{U}} : \mathcal{C} \to \mathbb{R}_{\geq 0}, \tag{1.2}$$

defined with respect to a choice of $\mathcal{U}$. When $n = 2$ it is just the area functional as previously defined.

1.1.2. *Abstract resolutions of singular connections.* Avoiding generality, suppose that $\mathcal{A}$ is a singular G-connection on a principal G-bundle $P \to S^n$, with a single singularity x_0. We will show that it is possible to control the curvature of the singular connection $\mathcal{A}$ if we impose a certain structure on the singularity of $\mathcal{A}$. The simplest way to do this is to ask for existence of a certain kind of abstract resolution. First a simplicial G-connection on $\mathcal{D}$ on P, as defined in Section 13.1, is basically a functorial assignment of a smooth G connection $\mathcal{D}_\Sigma$ on $\Sigma^* P$ for each smooth

$$\Sigma : \Delta^n \to S^n.$$

Definition 1.3. *For $\mathcal{A}, P$ as above a* **simplicial resolution** *of $\mathcal{A}$ is a simplicial G connection $\mathcal{A}^{res}$ on P, with the following properties. Let $\Sigma_0 : \Delta^n \to S^n$ represent $\pi_n(S^n, x_0)$, then*

$$\Sigma_0^*|_{interior\,\Delta^n} \mathcal{A} = \mathcal{A}^{res}_{\Sigma_0}|_{interior\,\Delta^n}.$$

In the following theorem $G = PU(2), n = 4$ and the norm $\mathfrak{n}$ on lie $PU(2)$ will be taken to be the operator norm normalized so that the Finsler length of the shortest one parameter subgroup from id to $-id$ is $\frac{1}{2}$. We will omit $\mathfrak{n}$ in notation. We also impose an additional constraint on $\mathcal{A}^{res}$, so that the curvature at "∞" is bounded by a threshold, which means the following. Let

$$\Sigma_\infty : \Delta^4 \to x_0$$

be the constant map, suppose that:

$$\mathrm{area}_{\mathcal{U}}\, \mathcal{A}^{res}_{\Sigma_\infty} < 1/2,$$

and suppose for simplicity that $\mathcal{A}^{res}_{\Sigma_\infty}$ is trivial along the edges of Δ^4, later on this condition is generalized, see Proposition 13.3. We say in this case that $\mathcal{A}^{res}$ is a ***sub-quantum resolution***. The following is proved in Section 13.

Theorem 1.4. *Let $P \to S^4$ be a non-trivial principal $PU(2)$ bundle. Let $\mathcal{A}$ be a singular $PU(2)$-connection on P with a single singularity at x_0. Then for any sub-quantum resolution $\mathcal{A}^{res}$ of $\mathcal{A}$ and for any $\mathcal{U}$ as above*

$$\operatorname{area}_{\mathcal{U}}(\mathcal{A}^{res}_{\Sigma_0}) \geq 1/2.$$

The theorem has certain extensions to Hamiltonian singular connections $\mathcal{A}$, understanding P as a principal $Ham(S^2)$ bundle, Section 13.

It is trivial to construct examples of singular connections $\mathcal{A}$ with sub-quantum resolution. Since simplicial connections can be constructed simplex by simplex arbitrarily, provided we have matching along faces, we may start by constructing a suitable simplicial connection and then "push it forward" to such a singular connection. Here is one basic class of examples.

Example 1.5. *Let P be as above, and $\mathcal{A}'$ be an ordinary smooth $PU(2)$ connection on P. Express S^4 as a union of sub-balls $D^4_\pm \subset S^4$, intersecting only in the boundary. Suppose that that we have the property that $\operatorname{area}_{\mathcal{U}}(\mathcal{A}')\,|_{D^4_-} < \frac{1}{2}$. Let $\mathcal{A}$ be the singular connection on P obtained as the push-forward of $\mathcal{A}'$ by the bundle map $\widetilde{q}: P \to P$ over the singular smooth map $q: S^4 \to S^4$ taking D^4_- to a single point $\infty \in S^4$, with $q|_{interior\, D^4_-}$ an immersion. Then $\mathcal{A}$ has a sub-quantum resolution $\mathcal{A}^{res}$ essentially by construction. Explicitly, it is partially defined by the formula $\mathcal{A}^{res}_{\Sigma_0} = \Sigma_+^* \mathcal{A}'$, where $\Sigma_+ : \Delta^4 \to S^4$ is the 4-simplex naturally corresponding to the embedding $D^4_+ \to S^4$, (as in Section 4.1). And $\mathcal{A}^{res}_{\Sigma_\infty} = \Sigma_-^* \mathcal{A}'$, where Σ_- is likewise corresponding to the embedding $D^4_- \to S^4$. $\mathcal{A}^{res}$ is then extended arbitrarily to other simplices. There are in general no obstructions to existence of extension of such data to the data of a simplicial connection. In this case, the theorem above simply yields that $\operatorname{area}_{\mathcal{U}}(\mathcal{A}'_{D^4_+}) \geq 1/2$.*

Let us summarize the above example as the following basic differential geometric result. The proof of this traverses the entirety of the theory here, and it quite non-obvious how it might be simplified, even for $PU(2)$.

Corollary 1.6 (Of Theorem 1.4 and of Theorem 13.3)**.** *Let P and $D^4_\pm$ be as above and let $\mathcal{A}$ be a smooth $PU(2)$ or $Ham(S^2)$ connection on P. Suppose that $\operatorname{area}_{\mathcal{U}}(\mathcal{A})\,|_{D^4_-} < \frac{1}{2}$, then $\operatorname{area}_{\mathcal{U}}(\mathcal{A})\,|_{D^4_+} \geq \frac{1}{2}$.*

The corollary above is certainly intuitively natural. We may use the same idea as in the example above to "push forward" simplicial, (not just smooth) connections to singular connections with more complicated singularities, in such a way that we again by construction would have sub-quantum resolutions (which would now be no longer related to any ordinary smooth connections as $\mathcal{A}'$ above).
Physical Interpretation. There are possible physical interpretations for singular connections, as appearing in the context here. A $PU(2) = PSU(2)$ connection $\mathcal{A}$ on P in physical terms represents a Yang-Mills field on the space-time S^4. When the space-time has a black hole singularity, the fields solving the Einstein-Yang-Mills equations (mathematically connections as above) likewise develop singularities. There is a wealth of physics literature on this subject, and I don't know what has the highest priority, but here is one reference [14]. Note that in the above example we, in a sense, collapse a region in the "space-time" S^4, also forming a kind of "black-hole". As quantum gravity is often related to simplicial ideas, it is

not inconceivable that the mathematical sub-quantum resolution condition above also has a physical interpretation.

The theorem tells us that it is impossible to use the singularity to manipulate the curvature of $\mathcal{A}$ arbitrarily, when $\mathcal{A}$ has a sub-quantum resolution. Naturally Gauss-Bonnet/Chern-Weil theory on surfaces also gives curvature constraints for singular metric connections. However, Chern-Weil theory cannot give the constraint of the theorem above. This is simply because $c_1(P) = 0$, so that the curvature integrals will not see the curvature of $\mathcal{A}$ over a Riemann surface in Δ^4, and $\text{area}_{\mathcal{U}}(\mathcal{A}^{res}_{\Sigma_0})$ is a kind of norm of this curvature, for some such surface.

The mathematical Yang-Mills theory is more promising, for example in author's [21] similar curvature bounds are obtained for smooth connections. In the case above we have a singularity, so that (mathematical) Yang-Mills theory must be somehow extended. If we want to mimic the argument presented in this paper, then we should first extend Yang-Mills theory to work with G-bundles over surfaces with corners and holonomy constraints over boundary. This might be possible but beyond this things are unclear since we also use certain abstract algebraic topology to glue the data, and it is not clear how this would work for Yang-Mills theory. Given a sub-quantum resolution of $\mathcal{A}$ it should be possible to "renormalize" the singular connection $\mathcal{A}$ obtaining a sequence of smooth connections $\mathcal{A}_n$ with (in some sense) singular limit $\mathcal{A}$. This may give a different, analytic approach to the above, but also likely very difficult.

At this point the reader may be curious why Theorem 1.1 has something to do with Theorem 1.4. We cannot give the full story, but the idea is that the simplicial fibration $Fuk_\infty(P)$ only sees the principal bundle P (and its curvature) by the behavior of certain holomorphic curves. When one has the sub-quantum condition on the curvature of $\mathcal{A}^{res}_{\Sigma_\infty}$, certain holomorphic curves are ruled out so that from the view point of $Fuk_\infty(P)$, $\mathcal{A}^{res}_{\Sigma_\infty}$ is the trivial connection, (its curvature is undetectable) but $Fuk_\infty(P)$ is non-trivial as a fibration so that the aforementioned holomorphic curves and consequently curvature must appear elsewhere.

1.2. **First quantum obstruction and smooth invariants.** It is very tempting to use the theory of the global Fukaya category to find new invariants of smooth manifolds. One such invariant is already discussed in Part I, as the homotopy class of the classifying map $X \to \mathbb{S}$ of the projectivized, complexified tangent bundle of a smooth manifold X. This by itself is not a very practical invariant, but we may try to extract more manageable invariants from this. We present here a construction of an integer invariant which is based on our theory. This is probably just the beginning of the story for invariants of smooth manifolds based on Floer-Fukaya theory.

Let $P \to X$ be a Hamiltonian M-bundle, as previously. Let

$$Fuk_\infty(P) \to X$$

be the associated (co)-Cartesian fibration, and let

$$K(P) \to X$$

be its maximal Kan sub-fibration as in Lemma 3.2, which we may understand as just a usual Serre topological fibration. Define

$$\text{q-obs}(P) \in \mathbb{N} \sqcup \{\infty\},$$

to be the degree of the first obstruction to a section of $K(P)$. That is q-obs(P) is the smallest integer n such that there is no section of $K(P)$ over the n skeleton of X. over the $n-1$ skeleton of X does not extend to a section over the n-skeleton. When no such n exists we set q-obs$(P) = \infty$.

Theorem 1.7. *Let $S^2 \hookrightarrow P \to S^4$ be a non-trivial Hamiltonian fibration then:*

$$\text{q-obs}(P) = 4.$$

Indeed the proof of Theorem 1.1 can be understood as showing that the associated obstruction class in

$$H^4(S^4, \pi_3(NFuk(S^2)))$$

is non-trivial.

1.2.1. *First quantum obstruction as a manifold invariant.* Let X be a smooth manifold, and let $P(X)$ denote the fiber-wise projectivization of $TX \otimes \mathbb{C}$. We then define

$$\text{q-obs}(X) := \text{q-obs}(P(X)) \in \mathbb{N} \sqcup \{\infty\},$$

which is then an invariant of the smooth manifold X. Either this invariant is expressible in terms of classical invariants, which would be fascinating since the construction is in terms pseudo-holomorphic curves or this invariant is new, that is not expressible in classical terms, which would also be interesting. There are of course gauge theory based invariants of smooth (3,4)-folds, like Donaldson and Seiberg-Witten invariants. I do not see any connections of the above to these invariants at the moment, even in dimension 4. It should be noted that this “first quantum obstruction” invariant is only sensitive to the tangent bundle, whereas for example Donaldson invariants can see finer aspects of the smooth structure. In fact the “quantum Novikov conjecture” of Part I would immediately imply that the first quantum obstruction is only a topological invariant of X. So that the above mentioned conjecture is a sense very pessimistic, as its negation means new smooth structure sensitive invariants of manifolds.

1.3. **Hamiltonian rigidity vs flexibility.** By way of the calculation we also obtain an application in Hofer geometry. It can be understood as a relative analogue of a result in [18]. Let $Lag(M, L_0)$ denote the space of oriented Lagrangian submanifolds of a symplectic manifold (M, ω), Hamiltonian isotopic to L_0, we may also just write $Lag(M)$. Let $\Omega_{L_0} Lag(M)$ denote the space of based smooth loops in $Lag(M)$, constant near end points, and let $\Omega^{taut}_{L_0} Lag(M) \subset \Omega_{L_0} Lag(M)$ be the subspace of loops taut concordant to the constant loop at L_0. The notion of taut concordance is defined in more generality in Definition 6.5. In the case above, two loops

$$p_1, p_2 \in \Omega_{L_0} Lag(M)$$

are said to be *taut concordant* if the following holds. We have a Lagrangian subfibration

$$\mathcal{L} \subset Cyl \times M, \quad Cyl = S^1 \times [0, 1],$$

so that $\mathcal{L}$ over the boundary circles corresponds, in the natural sense, to the pair p_1, p_2, so that there is a Hamiltonian connection $\mathcal{A}$ on $M \times [0, 1]$ preserving $\mathcal{L}$, and so that the coupling form $\Omega_{\mathcal{A}}$ of $\mathcal{A}$ vanishes on $\mathcal{L}$, see Section 6.1 for the definition of coupling forms.

Note that of course $Lag(S^2)$ is homotopy equivalent to $Lag^{eq}(S^2) \simeq S^2$ where $Lag^{eq}(S^2)$ denotes the space of oriented equators in S^2. Moreover, there is an embedding

$$i : \Omega S^2 \hookrightarrow \Omega^{taut}_{L_0} Lag(S^2),$$

this is because two loops $p_1, p_2 \in \Omega Lag^{eq}(S^2) \simeq \Omega S^2$ are taut concordant iff they are homotopic in $Lag^{eq}(S^2)$, see Lemma 10.4.

Theorem 1.8. *Let $L_0 \subset S^2$ be the equator. And let*

$$f : S^2 \to \Omega^{taut}_{L_0} Lag(S^2),$$

represent $i_* g$, for g the generator of

$$\pi_2(\Omega(S^2)) \simeq \pi_3(S^2)) \simeq \mathbb{Z},$$

and i as above. Then we have identity for the systole with respect to L^+:

$$\min_{f', [f'] = [f]} \max_{s \in S^2} L^+(f'(s)) = 1/2 \cdot \text{area}(S^2, \omega),$$

where L^+ denotes the positive Hofer length functional, as defined in Section 10.1.1. The minimum is attained on a cycle of equators in S^2.

Even though everything is now smooth, this is not obvious. For by contrast if we measure a related quantity of the "girth" (infimum of the diameter of a representative) of the generator $[g]$ of $\pi_2 Lag(S^2)$, as in [15], then there is an upper bound for this girth which is smaller then the lower bound for girth considered in the subspace of $Lag(S^2)$ consisting of equators. So moving from equators to general oriented S^1 Lagrangians in S^2 we may reduce the girth to less than the classically expected girth. By "classical" we mean for the classical objects: great circles. Indeed, it may be that girth of the generator

$$[g] \in \pi_2 Lag(S^2)$$

is actually 0. (This would rather astonishing however.) On the other hand, our theorem says that this kind of non-classical squeezing does not happen at all for the systole we consider. In other words whereas our systole exhibits Hamiltonian rigidity, the girth in [15] while closely related, exhibits flexibility.

Theorem 1.8 is proved in section 12. On the way in Section 9.1 we construct the quantum Maslov classes. We show their non-triviality in Section 10. The Sections 12, 9.1, 10 are mostly logically independent of the ∞-categorical and even the A_∞ setup and may be read independently. Theorems 1.1, 1.7 are proved in Section 4.2, they are basic consequences of the main technical lemma.

2. Acknowledgements

I am grateful to RIMS institute at Kyoto university and Kaoru Ono for the invitation, financial assistance and a number of discussions which took place there. Much thanks also ICMAT Madrid and Fran Presas for providing financial assistance, and a lovely research environment during my stay there.

Contents

3. Outline

Notation 3.1. *We use notation Δ^n to denote the standard topological n-simplex. For the standard representable n-simplex as a simplicial set we use the notation $\Delta^n_\bullet$, in general when X is a topological space $X_\bullet$ will mean the singular simplicial set of X. Likewise if $p : X \to Y$ is map of spaces $p_\bullet : X_\bullet \to Y_\bullet$ will mean the induced simplicial map. However the notation $Y_\bullet$ could also mean an abstract simplicial set.*

In what follows when we say Part I we will mean [19]. Let us briefly review what we do in Part I. Let $M \hookrightarrow P \xrightarrow{p} X$ be a Hamiltonian fibration. Denote by $\Delta(X) := \Delta/X_\bullet$ the smooth simplex category of X, with objects smooth maps

$\Sigma : \Delta^n \to X$ and morphisms commutative diagrams:

$$\begin{tikzcd} \Delta^n \arrow[r, "mor"] \arrow[dr, "\Sigma_0"'] & \Delta^m \arrow[d, "\Sigma_1"] \\ & X, \end{tikzcd}$$

where $mor : \Delta^n \to \Delta^m$ is a simplicial map, that is a linear map taking vertices to vertices, preserving the order. The distinction with $Simp(X)$ is that in $\Delta(X)$ we allow mor to be degenerate.

Then given certain auxiliary perturbation data $\mathcal{D}$, which in particular involves a choice of a natural system of maps from the universal curves to Δ^n and certain choices of Hamiltonian connections, we construct in Part I a functor

$$F : \Delta(X) \to A_\infty - Cat,$$

where $A_\infty - Cat$ denotes the category of A_∞ categories. The properties of this functor are such that we may algebraically, naturally construct from this a functor

$$F^{unit} : \Delta(X) \to A_\infty - Cat^{unit},$$

with $A_\infty - Cat^{unit}$ denoting the category of unital A_∞ categories, by taking unital replacements. In what follows we rename F by F^{raw} and F^{unit} by F, as this will visually simplify notation.

We then define

$$Fuk_\infty(P) = \operatorname{colim}_{\Delta(X)} NF,$$

which is shown to be an ∞-category whose equivalence class (under concordance) is independent of all choices. This also has the structure of a co-Cartesian fibration:

$$NFuk(M) \hookrightarrow Fuk_\infty(P) \to X_\bullet,$$

where $NFuk(M)$ is the A_∞ nerve of the Fukaya category of M and $X_\bullet$ denotes the simplicial set of singular simplices in X. The notion of co-Cartesian fibration corresponds to, in categorical language, a relaxation of the notion of Serre/Kan fibrations. Indeed what will do here is extract from the above data a Kan fibration and work with that, since then we can just use standard tools of topology. To this end we have the following elementary lemma.

Lemma 3.2. *Suppose we have a (co)-Cartesian fibration $p : Y \to X$, where X is a Kan complex. Let $K(Y)$ denote the maximal Kan sub-complex of Y then $p : K(Y) \to X$ is a Kan fibration.*

The proof is given in Appendix A. In particular by the above lemma $K(P) := K(Fuk_\infty(P))$ is a Kan fibration over $X_\bullet$.

Definition 3.3. *We say that a Kan fibration or a (co)-Cartesian fibration P over a Kan complex X is* **non-trivial** *if it is not null-concordant. Here P is* **null-concordant** *means that there is a Kan respectively (co)-Cartesian fibration*

$$Y \to X \times \Delta^1_\bullet,$$

whose pull-back by $i_0 : X \to X \times \Delta^1_\bullet$ is trivial and by $i_1 : X \to X \times \Delta^1_\bullet$ is P. Here the two maps i_0, i_1 correspond to the two vertex inclusions $\Delta^0_\bullet \to \Delta^1_\bullet$.

Theorem 3.4. *Suppose that $P \to S^4$ is a non-trivial Hamiltonian S^2 fibration then $p_\bullet : K(P) \to S^4_\bullet$ does not admit a section. In particular $K(P)$ is a non-trivial Kan fibration over $S^4_\bullet$ and so $Fuk_\infty(P)$ is a non-trivial (co)-Cartesian fibration over $S^4_\bullet$.*

This is the main technical result of the paper. Although in a sense we just are just deducing existence of certain holomorphic curve, for this deduction we need a global compatibility condition involving multiple moduli spaces, involved in multiple local datum's of Fukaya categories, so that this computation is not straightforward.

The proof will be aided by constructing suitable perturbation data, and will be split into a number of sections. As previously indicated the arguments are quiet general, however for concreteness and to simplify an already rather complex framework we focus on a special case.

4. Qualitative description of the perturbation data

A bit of possibly non-standard terminology: we say that A is a *model* for B in some category, with weak equivalences, if there is a morphism $mod: A \to B$ which is a weak-equivalence. The map mod will be called a *modelling map*. In our context the modeling map mod always turns out to be a monomorphism.

Let $Fuk(S^2)$ denote the $\mathbb{Z}_2$-graded A_∞ category over $\mathbb{Q}$, with objects oriented spin Lagrangian submanifolds Hamiltonian isotopic to the equator. Our particular construction of $Fuk(M)$ is presented in Part I.

Denote by $Fuk^{eq}(S^2) \subset Fuk(S^2)$ the full sub-category obtained by restricting our objects to be equators in S^2. We take our perturbation data $\mathcal{D}_{pt}$ for construction of $Fuk(S^2)$ so that the following is satisfied. All the connections $\mathcal{A}(L, L')$ for $L, L' \in Fuk^{eq}(S^2)$ are $PU(2)$-connections. For L intersecting L' transversally the $PU(2)$ connection $\mathcal{A}(L, L')$ is the trivial flat connection. For $L = L'$ the corresponding connection is generated by an autonomous Hamiltonian.

The associated cohomological Donaldson-Fukaya category $DF(S^2)$ is equivalent as a linear category over $\mathbb{Q}$ to $FH(L_0, L_0)$ (considered as a linear category with one object) for $L_0 \in Fuk(S^2)$.

It is easily verified that a morphism (1-edge) f is an isomorphism in the nerve $NFuk(S^2)$, see Part I for definitions, if and only if it corresponds, under the nerve construction N, to a morphism in $Fuk(S^2)$ that induces an isomorphism in $DF(S^2)$. Such a morphism will be called a *c-isomorphism*.

Consequently the maximal Kan subcomplex $K(S^2)$ of $NFuk(S^2)$ is characterized as the maximal subcomplex of this nerve with 1-simplices the images by N of c-isomorphisms in $Fuk(S^2)$.

Remark 4.1. *It would be interesting (and likely elementary) to identify the homotopy type of $K(S^2)$.*

4.1. Extending $\mathcal{D}_{pt}$ to higher dimensional simplices.

Let us model $D^4_\bullet$ and $S^3_\bullet$ as follows. Take the standard representable 3-simplex $\Delta^3_\bullet$, and the standard representable 0-simplex $\Delta^0_\bullet$. Then collapse all faces of $\Delta^3_\bullet$ to a point, that is take the colimit of the following diagram:

$$\begin{tikzcd}
 & & \Delta^0_\bullet & \\
\Delta^2_\bullet \arrow[urr] \arrow[drr, "i_0"'] & \Delta^2_\bullet \arrow[ur] \arrow[dr, "i_1"] & \Delta^2_\bullet \arrow[u] \arrow[d, "i_2"] & \Delta^2_\bullet \arrow[ul] \arrow[dl, "i_3"] \\
 & & \Delta^3_\bullet &
\end{tikzcd} \tag{4.1}$$

Here i_j are the inclusion maps of the non-degenerate 2-faces. This gives a simplicial set $S^{3,mod}_\bullet$ modelling the simplicial set $S^3_\bullet$, in other words there is a natural a weak-equivalence

$$S^{3,mod}_\bullet \to S^3_\bullet.$$

Now take the cone on $S^{3,mod}_\bullet$, denoted by $C(S^{3,mod}_\bullet)$, and collapse the one non-degenerate 1-edge. The resulting simplicial set $D^{4,mod}_\bullet$ is our model for $D^4_\bullet$, it may be identified with a subcomplex of $D^4_\bullet$ so that the inclusion map $mod : D^{4,mod}_\bullet \to D^4_\bullet$ induces a weak homotopy equivalence of pairs

$$(D^{4,mod}_\bullet, S^{3,mod}_\bullet) \to (D^4_\bullet, S^3_\bullet). \tag{4.2}$$

We set $b_0 \in D^4_\bullet$ to be the vertex which is the image by mod of the unique 0-vertex in $D^{4,mod}_\bullet$.

Suppose we have a commutative diagram:

$$\begin{tikzcd}
D^4 \arrow[r, "h_+"] & S^4 & D^4 \arrow[l, "h_-"'] \\
 & S^3 \arrow[ul, "i"] \arrow[ur, "i"'] &
\end{tikzcd}$$

where $i : S^3 \to D^4$ is the natural boundary inclusion, s.t. $h_\pm : D^4 \to S^4$ are smooth, with their images covering S^4, s.t.

$$h_+(D^4) \cap h_-(D^4)$$

is contained in the image E of

$$h_\pm \circ i : S^3 \to S^4,$$

and so that $h_\pm$ takes b_0 to x_0. For example we may just let h_- to represent the generator of $\pi_4(S^4, x_0)$ and for h_+ to be constant map to x_0. We call such a pair $h_\pm$ a *complementary pair*.

We set

$$D_\pm := h_\pm(D^{4,mod}_\bullet) \subset S^4_\bullet$$

and we set $\Sigma_\pm \in S^4_\bullet$ to be the image by $h_\pm$ of the sole non-degenerate 4-simplex of $D^{4,mod}_\bullet$. We also set

$$\partial D_\pm := h_\pm(\partial D^{4,mod}_\bullet),$$

where $\partial D^{4,mod}_\bullet)$ is the image of the natural inclusion $S^{3,mod}_\bullet \to D^{4,mod}_\bullet$.

Fix a Hamiltonian frame for the fiber P_{x_0} of P over x_0, in other words a diffeomorphism $S^2 \to P_{x_0}$ that is in the maximal atlas of trivializations of P as a

$Ham(S^2)$ structure group bundle. In particular this allows us to identify $Fuk(S^2)$ with $F^{raw}(x_0)$. Denote by $x_{0,\bullet}$ the image of the map

$$\Delta^0_\bullet \to S^{4,mod}_\bullet,$$

induced by the inclusion of the 0-simplex x_0. Recall from Part I that given perturbation data for a non-degenerate simplex, we assigned extended perturbation data for all degeneracies of this simplex. So our data $\mathcal{D}_{pt}$ for x_0 induces perturbation data for all degeneracies x_0, that is for all simplices of $x_{0,\bullet}$, this data will again be denoted by $\mathcal{D}_{pt}$.

Fix an object $L_0 \in Fuk^{eq}(S^2) \subset F^{raw}(x_0)$. Denote by $\gamma \in \hom_{F^{raw}(x_0)}(L_0, L_0)$ the generator of $FH_1(L_0, L_0)$, i.e. the fundamental chain, so that it corresponds to the identity in $DF(L_0, L_0)$. This γ is uniquely determined by our conditions and corresponds to a single geometric section. Denote by L^i_0 the image of L_0 by the embedding

$$F^{raw}(x_0) \to F^{raw}(\Sigma_+)$$

corresponding to the i'th vertex inclusion into Δ^4, $i = 0, \ldots, 4$.

Let m_i be the edge between $i-1, i$ vertices and set

$$\overline{m}_i := \Sigma_+ \circ m_i.$$

Let Σ^0_i denote the 0-simplex obtained by restriction of Σ^4 to the i'th vertex. Note that each $\overline{m}_i$ is degenerate by construction, so we have a induced morphisms

$$F^{raw}(pr) : F^{raw}(\overline{m}_i) \to F^{raw}(x_0)$$

corresponding (F^{raw} is a functor) to the degeneracy morphism in $\Delta(S^4)$:

$$pr : \overline{m}_i \to \Sigma^0_i.$$

Finally, for each L^{i-1}_0, L^i_0 we have a c-isomorphism

$$\gamma_i : L^{i-1}_0 \to L^i_0$$

in $F^{raw}(\overline{m}_i) \subset F^{raw}(\Sigma_+)$ which corresponds to γ, meaning that the fully-faithful projection $F^{raw}(pr)$ takes γ_i to γ. We will denote by $\gamma_{i,j}$ the analogous morphisms $L^i_0 \to L^j_0$.

Notation 4.2. *Let us abbreviate from now the morphism spaces $hom_{F^{raw}(\Sigma_\pm)}$ by $hom_{\Sigma_\pm}$, and $\mu^d_{F^{raw}(\Sigma_\pm)}$ by $\mu^d_{\Sigma_\pm}$.*

Definition 4.3. *We call perturbation data $\mathcal{D}$ for P* **small** *if it is extends the data $\mathcal{D}_{pt}$ as above, and if with respect to $\mathcal{D}$*

$$\mu^d_{\Sigma_+}(\gamma^1, \ldots, \gamma^d) = 0, \text{ for } 2 < d < 4, \tag{4.3}$$

where $(\gamma^1, \ldots, \gamma^d)$ is a composable chain, and each γ^i is of the form $\gamma_{i,j}$ as above.

We will see further on how to construct such small data, assume for now that it is constructed.

Let $\{f_J\}$, corresponding to an n-simplex, be as in the definition of the A_∞ nerve in Part I, where J is a subset of $[n] = \{0, \ldots, n\}$.

Lemma 4.4. *Let $\mathcal{D}$ be small as above, then there is a a 4-simplex $\sigma \in NF^{raw}(\Sigma_+)$ with faces determined by the conditions:*

- *$f_J = 0$, for J any subset of $[4]$ with at least 3 elements.*
- *$f_{\{i-1,i\}} = \gamma_i$ for γ_i as before.*

Proof. This follows by (4.3) and by the identity $\mu^2_{\Sigma_+}(\gamma, \gamma) = \gamma$. □

If we take our unital replacements so that γ corresponds to the unit then σ induces a section of $K(P_+) \to D_+$, where $K(P_\pm)$ will be shorthand for $K(P)$ restricted over $D_\pm$.

Let

$$i : \left(K(P_+)|_{\partial D_+} := p_\bullet^{-1}(\partial D_+)\right) \to K(P_-),$$

be the natural inclusion map. Set

$$sec = i \circ \sigma \circ h_+|_{\partial D^{4,mod}_\bullet}.$$

4.2. **The main lemma and immediate consequences.**

Lemma 4.5. *Suppose that P is a non-trivial Hamiltonian fibration and $\mathcal{D}$ small data for P as above, then sec as above does not extend to a section of $K(P_-)$.*

The proof of this lemma which will be broken up in parts, will follow shortly.

Proof of Theorem 1.7. Clearly q-obs$(P) \geq 4$, since the 3-skeleton of S^4 is trivial. By Lemma 4.5 above $K(P)$ does not have a section over the 4-cell representing the generator of $\pi_4(S^4)$. □

Remark 4.6. *When P is obtained by clutching with a generator of $\pi_3(PU(2))$, and when $h_\pm$ are embeddings, the class $[sec]$ in $\pi_3(K(P_-)) \simeq \pi_3(K(S^2))$ can be thought of as "quantum" analogue of the class of the classical Hopf map.*

Proof of Theorem 3.4. If we take any small perturbation data $\mathcal{D}$ for P, then the first part follows immediately by Lemma 4.5. So $K(P)$ is non-trivial as a Kan fibration. This then implies that $Fuk_\infty(P)$ is non-trivial as a (co)-Cartesian fibration, which means specifically that its classifying map

$$f_P : S^4 \to (\mathcal{S}, NFuk(S^2))$$

is not null-homotopic.

To see this, suppose otherwise that we have a null-homotopy H of f_P, then this gives a (co)-Cartesian fibration

$$\widetilde{P} \to S^4_\bullet \times I_\bullet,$$

restricting to $Fuk_\infty(P)$ over $S^4_\bullet \times 0_\bullet$ and to $NFuk(S^2) \times S^4_\bullet$ over the other end $S^4_\bullet \times 1_\bullet$. Here $0_\bullet$, $1_\bullet$ are notatation for simplicial set isomorphic Δ^0 corresponding to the boundary of $I_\bullet$. If we take the maximal Kan sub-fibration of $\widetilde{P}$, then by Lemma 3.2 we would obtain a trivialization of $K(P)$ which is a contradiction. □

Proof of Theorem 1.1. Theorem 3.4 implies that the group homomorphism

$$\mathbb{Z} \simeq \pi_4(BHam(S^2, id) \to \pi_4(\mathcal{S}, NFuk(S^2)),$$

has vanishing kernel, so that the result follows. □

5. Towards the proof of Lemma 4.5

We will denote by $L_{0,\bullet}$ the image of the map $\Delta^0_\bullet \to K(P_-)$, induced by the inclusion of L_0 into $K(S^2)$ as a 0-simplex. Suppose that *sec* extends to a section of $K(P_-)$, so we have map

$$e : D^{4,mod}_\bullet \to K(P_-)$$

extending *sec* over $\partial D^{4,mod}_\bullet$. We may assume that e lies over h_-, meaning

$$p_\bullet \circ e = h_-.$$

Since it can be homotoped to have this property. To see this, first take a relative homotopy of

$$p_\bullet \circ e : (D^{4,mod}_\bullet, \partial D^{4,mod}_\bullet) \to (D_-, \partial D_-)$$

to h_-, using that we have a homotopy equivalence of pairs (4.2), and then lift the homotopy to a relative homotopy upstairs using the defining lifting property of Kan fibrations.

And so we have a 4-simplex

$$T = e(\Sigma^4) \in K(P_-)$$

projecting to $\Sigma_- \in D_-$ by $p_\bullet$. Since T is in the image of e, all but one 3-faces of T are totally degenerate with image in $L_{0,\bullet}$, and with this exceptional 3-face being the sole non-degenerate 3-face of *sec*, (of $sec(\partial D^{4,mod}_\bullet)$).

Then if $m_{i,j}, \gamma_{i,j}$ are as in the previous section but corresponding now to Σ_- rather then Σ_+, by the boundary condition on e, the edges of T correspond under the nerve construction to the $\gamma_{i,j}$, since this is the corresponding condition for the edges of *sec*.

Lemma 5.1. *For $\mathcal{D}$ small as above, and for the unital replacement F of F^{raw} as above, the simplex T exists if and only if $\mu^4_{\Sigma_-}(\gamma_1, \ldots, \gamma_4)$ is exact.*

Proof. The following argument will be over $\mathbb{F}_2$ as opposed to $\mathbb{Q}$ as the signs will not matter. Recall that we take the unital replacement so that $\gamma \in hom_{F^{raw}(P_{x_0})}(L_0, L_0)$ corresponds to the unit in the unital replacement.

Now if $T \in K(P_-)$ as above exists, then it corresponds under unital replacement to a 4-simplex $T' \in NF^{raw}(\Sigma_-)$ satisfying the following, by the nerve construction, condition on its 4-face $f_{[4]} \in hom_{\Sigma_-}(L^0_0, L^4_0)$:

$$\mu^1_{\Sigma_-} f_{[4]} = \sum_{1<i<4} f_{[4]-i} + \sum_s \sum_{(J_1, \ldots J_s) \in decomp_s} \mu^s_{\Sigma_-}(f_{J_1}, \ldots, f_{J_s}). \tag{5.1}$$

By our conditions on the boundary of T, by the condition on the unital replacement, and by the conditions in Lemma 4.4, we must have $f_J = 0$, for every proper subset $J \subset [4]$, in some length s decomposition of $[4]$, unless $J = \{i, j\}$ in which case $f_{i,j} = \gamma_{i,j}$. Given this (5.1) holds if and only if $\mu^4_{\Sigma_-}(\gamma_1, \ldots, \gamma_4)$ is exact.

□

We are going to show that for a certain small $\mathcal{D}_0$, $\mu^4_{\Sigma_-}(\gamma_1, \ldots, \gamma_4)$ does not vanish in homology, which will finish the argument. However the calculation will require significant setup.

6. Hamiltonian fibrations and taut structures, holomorphic sections and area bounds

We collect here some preliminaries on moduli spaces of holomorphic sections of fibrations with Lagrangian boundary constraints, and the closely related curvature bounds. There is a likely new discussion involving taut Hamiltonian structures, but much of this material has previously appeared elsewhere, perhaps in less generality. We will eventually need all that is presented in this section, but the reader may only skim on the first reading.

6.1. **Coupling forms.** We refer the reader to [11, Chapter 6] for more details on what follows. A Hamiltonian fibration is a smooth fiber bundle

$$M \hookrightarrow P \to X,$$

with structure group $\mathrm{Ham}(M, \omega)$ with its C^∞ Frechet topology. A ***Hamiltonian connection*** is just an Ehresmann connection for a Hamiltonian fibration.

A *coupling form*, originally appearing in [6], for a Hamiltonian fibration $M \hookrightarrow P \xrightarrow{p} X$, is a closed 2-form Ω on P whose restriction to fibers coincides with ω and which has the property:

$$\int_M \Omega^{n+1} = 0 \in \Omega^2(X),$$

with integration being integration over the fiber operation. Such a 2-form determines a Hamiltonian connection $\mathcal{A}_\Omega$, by declaring horizontal spaces to be Ω-orthogonal spaces to the vertical tangent spaces. A coupling form generating a given connection $\mathcal{A}$ is unique. A Hamiltonian connection $\mathcal{A}$ in turn determines a coupling form $\Omega_\mathcal{A}$ as follows. First we ask that $\Omega_\mathcal{A}$ induces the connection $\mathcal{A}$ as above. This determines $\Omega_\mathcal{A}$ up to values on $\mathcal{A}$-horizontal lifts $\widetilde{v}, \widetilde{w} \in T_pP$ of $v, w \in T_xX$. We specify these values by the formula

$$\Omega_\mathcal{A}(\widetilde{v}, \widetilde{w}) = R_\mathcal{A}(v, w)(p), \tag{6.1}$$

where $R_\mathcal{A}$ is the lie algebra valued curvature 2-form of $\mathcal{A}$. Specifically, for each x, $R_\mathcal{A}|_x$ is a 2-form valued in $C^\infty_{norm}(p^{-1}(x))$ - the space of 0-mean normalized smooth functions on $p^{-1}(x)$.

6.2. **Hamiltonian structures on fibrations.** Let S be a Riemann surface with boundary, with punctures in the boundary, and a fixed structure of strip charts at ends, positive or negative, as in Part I. Let $M \hookrightarrow \widetilde{S} \xrightarrow{pr} S$ be a Hamiltonian fiber bundle, with model fiber monotone symplectic manifold (M, ω), with distinguished Hamiltonian bundle trivializations

$$[0, 1] \times (0, \infty) \times M \to \widetilde{S}$$

at the positive ends, and with distinguished Hamiltonian bundle trivializations

$$[0, 1] \times (-\infty, 0) \times M \to \widetilde{S},$$

at the negative ends. These are collectively called called ***strip charts***. Given the structure of such bundle trivializations we say that $\widetilde{S}$ has ***end structure***.

Definition 6.1. *Let*

$$\mathcal{L} \subset (\widetilde{S}|_{\partial S} = pr^{-1}(\partial S)) \to \partial S$$

be a Lagrangian sub-bundle, with model fiber an object, in the sense of Part I, (in particular a spin oriented Lagrangian submanifold) so that $\mathcal{L}$ is a constant sub-bundle in the strip chart trivializations above. We say in this case that $\mathcal{L}$ **respects the end structure**. *In the strip chart coordinates at the end e_i, let L_i^j be the fibers of $\mathcal{L}$ over*

$$\{j\} \times \{t\},\ j = 0, 1, \text{ by assumption } t \text{ independent.}$$

For $\mathcal{L}$ as above, we say that a Hamiltonian connection $\mathcal{A}$ on $\widetilde{S}$ is ***compatible*** with the connections $\{\mathcal{A}_i\}$ on $[0,1] \times M$ for each end e_i, if in the strip coordinate chart at the e_i end, $\mathcal{A}$ is flat and $\mathbb{R}$-translation invariant and has the form $\overline{\mathcal{A}}_i$ where $\overline{\mathcal{A}}_i$ denotes its $\mathbb{R}$-translation invariant extension $\mathcal{A}_i$ to $(0, \pm\infty) \times \mathbb{R} \times M$, depending on whether the end is positive or negative. We say that a Hamiltonian connection $\mathcal{A}$ on $\widetilde{S}$ as above, is ***$\mathcal{L}$-exact*** if $\mathcal{A}$ preserves $\mathcal{L}$ (this means that the horizontal spaces of $\mathcal{A}$ are tangent to $\mathcal{L}$).

For $\mathcal{A}$ compatible with $\{\mathcal{A}_i\}$ as above, a family $\{j_z\}$ of fiber wise ω-compatible almost complex structures on $\widetilde{S}$ will be said to ***respect the end structure*** if at each end e_i, in the strip coordinate chart above, the family $\{j_z\}$ is $\mathbb{R}$-translation invariant and is admissible with respect to $\mathcal{A}_i$, in the sense of Part I, Definition 5.3. The data $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A}, \{j_z\})$, with $\mathcal{A}$ compatible with $\{\mathcal{A}_i\}$, $\{j_z\}$, respecting the end structure, will be called ***Hamiltonian structure***.

We will normally suppress $\{j_z\}$ in the notation and elsewhere for simplicity, as it will be purely in the background in what follows, (we do not need to manipulate it explicitly).

Definition 6.2. *Let $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be a Hamiltonian structure, we say that a smooth section σ of $\widetilde{S} \to S$ is* **asymptotically flat** *if at each end e_i of S, σ C^1-converges to an $\mathcal{A}$-flat section. Specifically, in the strip coordinates for a positive end, this means that there is a $\mathcal{A}$-flat section*

$$\widetilde{\sigma} : [0,1] \times (0, \infty) \to [0,1] \times (0, \infty) \times M,$$

so that for every $\epsilon > 0$ there is a $t > 0$ so that

$$d_{C^1}(\widetilde{\sigma}, \sigma|_{[0,1]\times[t,\infty)}) < \epsilon.$$

Similarly for a negative end. Given a pair of asymptotically flat sections σ_1, σ_2, with boundary in $\mathcal{L}$, we say that they have the same **relative class** *if they are asymptotic to the same flat sections at each end, and are homologous, relative boundary condition and relative the asymptotic constraints at the ends. The set of relative classes will be denoted by $H_2^{sec}(\widetilde{S}, \mathcal{L})$.*

6.2.1. *Families of Hamiltonian structures.*

Definition 6.3. *A* **family Hamiltonian structure** *or henceforth just Hamiltonian structure, consists of the following:*

1. *A smooth, connected, compact, oriented manifold $\mathcal{K}$ with boundary, (or corners).*
2. *For each $r \in \mathcal{K}$ a Hamiltonian structure $(\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$, so that there are smooth fibrations*

$$\widetilde{S} \hookrightarrow \widetilde{\boldsymbol{S}} \xrightarrow{p_1} \mathcal{K}, S \hookrightarrow \boldsymbol{S} \xrightarrow{p} \mathcal{K},$$

and $\{\widetilde{S}_r\}, \{S_r\}$ correspond to the fibers of the first and second fibration, respectively, where the second fibration has fiber a Riemann surface, so that

$\{S_r\} = \{p^{-1}r\}$. The first fibration is a fibration whose fibers $p_1^{-1}(r)$ are themselves the total spaces of a smooth Hamiltonian fibration $\widetilde{S}_r \to S_r$, so that the structure group of $\widetilde{\boldsymbol{S}} \xrightarrow{p_1} \mathcal{K}$ can be reduced to smooth Hamiltonian bundle maps of the fiber. To elaborate further, let

$$M \hookrightarrow \widetilde{S} \to S$$

be a Hamiltonian M-fibration over a Riemann surface S. Let Aut denote the group of Hamiltonian M-bundle automorphisms of $\widetilde{S}$. Then $\widetilde{\boldsymbol{S}} \xrightarrow{p_1} \mathcal{K}$ is the associated bundle $P \times_{Aut} \widetilde{S}$ for some principal Aut bundle P over $\mathcal{K}$.

(3) *The charts*

$$e_{i,r} : [0,1] \times (0,\infty) \times M \to \widetilde{S}_r,$$

for the positive ends, fit into a Hamiltonian M-bundle diffeomorphism onto the image:

$$\widetilde{e}_i : [0,1] \times (0,\infty) \times \mathcal{K} \times M \to \widetilde{\boldsymbol{S}}, \tag{6.2}$$

similarly for the negative ends.

(4) *We then have an induced smooth r-family of connections $\{e_{i,r}^*\mathcal{A}_r\}$ on $[0,1] \times (0,\pm\infty) \times M$, and an induced smooth r-family of Lagrangian subfibrations $\{e_{i,r}^{-1}\mathcal{L}_r\}$ over $\partial[0,1] \times (0,\pm\infty)$. We ask that*

$$\forall r : \{e_{i,r}^{-1}\mathcal{L}_r\} = \{0\} \times (0,\pm\infty) \times L_i^0 \cup \{1\} \times (0,\pm\infty) \times L_i^1,$$

where L_i^j are as in Definition 6.1 and we ask that

$$\forall r : \{e_{i,r}^*\mathcal{A}_r\} = \overline{\mathcal{A}}_i$$

for $\mathcal{A}_i, \overline{\mathcal{A}}_i$ as previously.

(5) *There is a Hamiltonian connection $\mathcal{A}$ on $\widetilde{\boldsymbol{S}} \to \boldsymbol{S}$ that extends all the connections $\mathcal{A}_r$ (in the natural sense), and preserving $\boldsymbol{L} := \cup_r \mathcal{L}_r$.*

We will write $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ for this data, $\mathcal{K}$ may be omitted from notation when it is implicit.

In the notation above, if there exists a Hamiltonian connection $\mathcal{A}$ on $\widetilde{\mathbf{S}} \to \mathbf{S}$ as in the last point, so that $\Omega_{\mathcal{A}}$ vanishes on $\mathbf{L}$, we will say that $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is a ***hyper taut Hamiltonian structure***.

6.2.2. *Moduli spaces of sections of Hamiltonian structures.* Let $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be a Hamiltonian structure. For a section σ of $\widetilde{S}$ define its vertical L^2 energy by

$$e(\sigma) := \int_S |\pi_{vert} \circ d\sigma|^2,$$

$$\pi_{vert} : T\widetilde{S} \to T^{vert}\widetilde{S}$$

is the $\mathcal{A}$-projection, for $T^{vert}\widetilde{S}$ the vertical tangent bundle of $\widetilde{S}$, that is the kernel of the projection $T\widetilde{S} \to TS$. Define $\overline{\mathcal{M}}(\Theta)$ to be the Gromov-Floer compactification of the space of $J(\mathcal{A})$-holomorphic sections σ of $\widetilde{\mathcal{S}}$, with finite vertical L^2 energy, (also called Floer energy), and with boundary on $\mathcal{L}$. Note that for any $J_{\mathcal{A}}$-holomorphic σ we have an identity:

$$e(\sigma) = \int_S \sigma^*\Omega_{\mathcal{A}},$$

and $\Omega_{\mathcal{A}}$ vanishes on $\mathcal{L}$ by the condition that $\mathcal{A}$ preserves $\mathcal{L}$, so that the standard energy controls apply, to deduce the standard Gromov-Floer compactification structure.

More generally, if $\{\Theta_r\} = \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is a Hamiltonian structure, let

$$\overline{\mathcal{M}}(\{\Theta_r\})$$

be the Gromov-Floer compactification of the space of pairs (σ, r), $r \in \mathcal{K}$ with σ a $J(\mathcal{A}_r)$-holomorphic, finite vertical L^2 energy section of $\widetilde{\mathcal{S}}_r$, with boundary on $\mathcal{L}_r$.

We also denote by

$$\overline{\mathcal{M}}(\{\Theta_r\}, A) \subset \overline{\mathcal{M}}(\{\Theta_r\})$$

the subset corresponding to relative class A curves, where the latter is as defined above.

Let $\{\Theta_r = (\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)\}$ be a Hamiltonian structure, then for each end e_i of S_r we have a Floer chain complex

$$CF(L^0_i, L^1_i, \mathcal{A}_i, \{j_z\}),$$

(independent of r by part 4 of Definition 6.3) generated by $\mathcal{A}_i$-flat sections of $[0,1] \times M$, with boundary on L^0_0, L^0_1. This is defined as in Section 6.1 of Part I. We may also abbreviate $CF(L^0_i, L^1_i, \mathcal{A}_i, \{j_z\})$ by just $CF(L^0_i, L^1_i)$.

Definition 6.4. *We say that $\{\Theta_r\}$ is A-**regular** if the pairs $(\mathcal{A}_i, \{j_z\})$ are regular so that the Floer chain complexes $CF(\mathcal{A}_i, \{j_z\})$ are defined, and if $\mathcal{M}(\{\Theta_r\}, A)$ is regular, (transversely cut out). And $\{\Theta_r\}$ is **regular** if it is A-regular for all A. We say that it is A-**admissible** if there are no elements*

$$(\sigma, r) \in \overline{\mathcal{M}}(\{\Theta_r\}, A),$$

for r in a neighborhood of the boundary of $\mathcal{K}$.

Definition 6.5. *Given a pair $\{\Theta^i_r\} = \{\widetilde{S}^0_r, S^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}_{\mathcal{K}}$, $i = 1, 2$, of Hamiltonian structures we say that they are **concordant** if there is a Hamiltonian structure*

$$\{\mathcal{T}_r\} = \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times [0,1]},$$

with an oriented diffeomorphism (in the natural sense, preserving all structure)

$$\{\widetilde{S}^0_r, S^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}_{\mathcal{K}^{op}} \sqcup \{\widetilde{S}^1_r, S^1_r, \mathcal{L}^1_r, \mathcal{A}^1_r\}_{\mathcal{K}} \to \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times \partial I},$$

where op denotes the opposite orientation for $\mathcal{K}$.

Definition 6.6. *We say that a Hamiltonian structure $\{\Theta_r\} = \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is **taut** if for any pair $r_1, r_2 \in \mathcal{K}$, Θ_{r_1} is concordant to Θ_{r_2} by a concordance $\{\widetilde{T}_\tau, T_\tau, \mathcal{L}'_\tau, \mathcal{A}'_\tau\}_{[0,1]}$ which is a hyper taut Hamiltonian structure.*

Definition 6.7. *Given an A-admissible pair $\{\Theta^i_r\}$, $i = 1, 2$, of Hamiltonian structures, we say that they are A-**admissibly concordant** if there is an A-admissible Hamiltonian structure*

$$\{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times [0,1]},$$

*which furnishes a concordance. If this concordance is in addition a taut Hamiltonian structure, then we say that these pairs are A-**admissibly taut concordant**.*

Lemma 6.8. *Let $\{\Theta_r\} = \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}$ be A-regular and A-admissible, with S_r having one distinguished negative end e_0, and let γ_0 be the asymptotic constraint at e_0 of the class A. For L^j_0, $j = 0, 1$ as above. Define*

$$ev_A = ev(\{\Theta_r\}, A) := \#\mathcal{M}(\{\Theta_r\}, A)\gamma_0 \in CF(L^0_0, L^1_0),$$

where $\#\mathcal{M}(\{\Theta_r\}, A, \gamma_0)$ means signed count of elements when the dimension is 0, and is otherwise set to be zero. Suppose that $CF(L^0_0, L^1_0)$ is defined with respect to the connection $\mathcal{A}_0$ and that this chain complex is perfect. Then ev_A is a cycle since $CF(L^0_0, L^1_0)$ is perfect, and its homology class depends only on the A-admissible concordance class of $\{\Theta_r\}$.

Proof. Suppose we are given an A-admissible concordance (which we may assume to be regular)

$$\mathcal{T} = \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K}\times[0,1]}$$

between Hamiltonian structures $\{\Theta^0_r\}$, and $\{\Theta^1_r\}$. Then we get a one dimensional compact moduli space $\mathcal{M}(\{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}, A)$. By assumption on the perfection of $CF(L^0_0, L^0_1)$, boundary contributions from Floer degenerations cancel out, so that the boundary is:

$$\partial\mathcal{M}(\{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}, A) = \mathcal{M}(\{\Theta_r\}^{op} \sqcup \mathcal{M}(\{\Theta_r\},$$

where *op* denotes opposite orientation. From which the result follows. □

6.3. **Area of fibrations.**

Definition 6.9. *For a Hamiltonian connection $\mathcal{A}$ on a bundle $M \hookrightarrow \widetilde{S} \to S$, with S a Riemann surface, define*

$$\text{area}(\mathcal{A}) = \inf_\alpha\{\int_S \alpha \,|\Omega_\mathcal{A} + \pi^*(\alpha) \textit{ is nearly symplectic}\}, \tag{6.3}$$

where $\Omega_\mathcal{A}$ is the coupling form of $\mathcal{A}$, α is a 2-form on S, and where $\Omega_\mathcal{A} + \pi^*(\alpha)$ nearly symplectic means that

$$(\Omega_\mathcal{A} + \pi^*(\alpha))(\widetilde{v}, \widetilde{jv}) \geq 0,$$

for $\widetilde{v}, \widetilde{jv}$ horizontal lifts with respect to $\Omega_\mathcal{A}$, of $v, jv \in T_zS$, for all $z \in S$.

Note that area$(\mathcal{A})$ could be infinite if there are no constraints on $\mathcal{A}$ at the ends. However, when the infimum above is finite it is attained on the 2-form determined by:

$$\alpha_\mathcal{A}(v, jv) := |R_\mathcal{A}(v, jv)|_+, \tag{6.4}$$

where $v \in T_zS$, $R_\mathcal{A}(v, w)$ as before identified with zero mean smooth function on the fiber $\widetilde{S}_z$ over z and $|\cdot|_+$ is operator: $|H|_+ = \max_{\widetilde{S}_z} H$.

Lemma 6.10. *Let $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be Hamiltonian data. For $\sigma \in \overline{\mathcal{M}}(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ we have*

$$-\int_S \sigma^*\Omega_\mathcal{A} \leq \text{area}(\mathcal{A}).$$

Proof. We have

$$\int_S \sigma^*(\Omega_\mathcal{A} + \pi^*\alpha) \geq 0,$$

whenever $\Omega_\mathcal{A} + \pi^*(\alpha)$ is nearly symplectic, by the defining properties of $J_\mathcal{A}$ and by σ being $J_\mathcal{A}$-holomorphic. From which our conclusion follows. □

Lemma 6.11. *Let $\{(\widetilde{S}_t, S_t, \mathcal{L}_t, \mathcal{A}_t)\}_{[0,1]}$ be a taut concordance. Let σ_j, $j = 0, 1$ be asymptotically flat sections of $\widetilde{S}_j$ in relative class A. Then*

$$-\int_{S_1} \sigma_1^* \Omega_{\mathcal{A}_1} = -\int_{S_0} \sigma_0^* \Omega_{\mathcal{A}_0},$$

whenever both integrals are finite. In particular, for a Hamiltonian structure $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ $\int_S \sigma^* \Omega_{\mathcal{A}}$ depends only on the relative class of A, whenever the integral is finite.

Proof. By the hypothesis, there is a connection $\mathcal{A}$ on $\widetilde{\mathbf{S}}$, extending each $\mathcal{A}_t$ and so that $\Omega_{\mathcal{A}}$ vanishes on $\mathbf{L} \subset \widetilde{\mathbf{S}}$. The first part then follows by Stokes theorem. Here are the details. For σ_j as above and for each end e_i, cut off the part of the section σ_j lying over $[0,1] \times (t_{\delta_1,\delta_2}, \infty)$ in the corresponding strip chart at the end. Here t_{δ_1,δ_2} is such that $\sigma_0|_{[0,1]\times\{t\}}$ is C^1 δ_1-close to $\sigma_1|_{[0,1]\times\{t\}}$ for all $t > t_{\delta_1,\delta_2}$ and for each end, and is such that

$$\int_{[0,1]\times(t_{\delta_1,\delta_2},\infty)} \sigma_j^*|_{[0,1]\times(t_{\delta_1,\delta_2},\infty)} \Omega_{\mathcal{A}_j} < \delta_2,\ j = 1, 2,$$

for each end e_i. Call the sections with the ends cut off as above by $\sigma_j^{\delta_1,\delta_2}$, they are sections over the compact surfaces S_j^{cut}, with ends correspondingly cut off. Then by Stokes theorem, using that $\Omega_{\mathcal{A}}$ is closed and using the vanishing of $\Omega_{\mathcal{A}}$ on $\mathbf{L}$: for each ϵ there exists δ_1, δ_2 such that

$$\int_{S_1^{cut}} (\sigma_1^{\delta_1,\delta_2})^* \Omega_{\mathcal{A}} - \int_{S_0^{cut}} (\sigma_0^{\delta_1,\delta_2})^* \Omega_{\mathcal{A}} < \epsilon,$$

and

$$\int_{S_j^{cut}} (\sigma_j^{\delta_1,\delta_2})^* \Omega_{\mathcal{A}_j} - \int_{S_j} \sigma_j^* \Omega_{\mathcal{A}_j} < \epsilon,\ j = 1, 2.$$

The last part of the lemma follows, as $\mathcal{A}$ preserving $\mathcal{L}$ immediately implies that $\Omega_{\mathcal{A}}$ vanishes on $\mathcal{L}$, so that a constant concordance

$$\{(\widetilde{S}_t, S_t, \mathcal{L}_t, \mathcal{A}_t)\}_{[0,1]}$$

is taut. ☐

Definition 6.12. *For σ a relative class A section of $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ let us call:*

$$-\int_S \sigma^* \Omega_{\mathcal{A}},$$

*the $\mathcal{A}$-**coupling area** of σ, denoted by $carea(\Theta, \sigma)$, we may also write $carea(\Theta, A)$ for the same quantity. By the lemma above this is an invariant of the taut concordance class of Θ.*

Definition 6.13. *Given Hamiltonian structure $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ we will say that Θ is A-**small** if*

$$\text{area}(\Theta) < carea(\Theta, A).$$

Similarly, given a taut Hamiltonian structure $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ we say that it is is A-**small near boundary** *if $(\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$ is A-small for r in a neighborhood of the $\partial\mathcal{K}$.*

Lemma 6.14. *Suppose that $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ is A-small then $\overline{\mathcal{M}}(\Theta, A)$ is empty. Or as a contrapositive, if $\overline{\mathcal{M}}(\Theta, A)$ is non-empty then:*

$$\text{carea}(\Theta, A) \leq \text{area}(\Theta).$$

Proof. This is just a reformulation of Lemma 6.10. □

Lemma 6.15. *Let $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ be a taut Hamiltonian structure with $\mathcal{K}$ connected, so that in particular, for each r, $\Theta_r = (\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$ is taut concordant to a fixed Θ. Suppose that $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is A-small near boundary then $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is A-admissible for all A such that carea$(\Theta, A) > 0$.*

Proof. Follows immediately by the lemma above. □

6.4. **Gluing Hamiltonian structures.** A Hamiltonian connection $\mathcal{A}$ on $[0,1]\times M$ is determined by a choice of a function $H : [0,1]\times M \to \mathbb{R}$, normalized to have mean zero at each moment. The holonomy path of $\mathcal{A}$ is a path $\phi_{\mathcal{A}} : [0,1] \to Ham(M,\omega)$, generated by the Hamiltonian H. Given $L_0 \in Lag(M)$ we get a path $\widetilde{\phi}_{\mathcal{A}} : [0,1] \to Lag(M)$ starting at L_0, defined by $\widetilde{\phi}_{\mathcal{A}}(t) = \phi_{\mathcal{A}}(t)(L_0)$. We will say that these paths are ***generated by $\mathcal{A}$ or by H***, with the latter as above.

Let $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be a Hamiltonian structure. Where at each end e_i the corresponding connection $\mathcal{A}_i$ is determined by $L^{\pm}$ length κ_i, where $L^{\pm}$ is as in Section 10.1.1.

Let $\mathcal{D}$ denote Riemann the surface which is topologically $D^2 - z_0$, $z_0 \in \partial D^2$, endowed with a choice of a strip chart at the end (positive or negative depending on context). The complex structure j here is as induced from $\mathbb{C}$ under the assumed embedding $D^2 \subset \mathbb{C}$.

We may then cap off some of the open ends $\{e_i\}_{i=0}^n$ of S by gluing at the ends copies of $\mathcal{D}$ with oppositely signed end. More explicitly, in the strip coordinate charts at some, say positive, end e_i of S, excise $[0,1] \times (t,\infty)$ for some $t > 0$, call the resulting surface $S - e_i$. Likewise excise the negative end of $\mathcal{D}$, call this surface $\mathcal{D} - end$. Then glue $S - e_i$ with $\mathcal{D} - end$, along their new smooth boundary components. Let us denote the capped off surface by $S^{/}$. Since $\widetilde{S}$ is naturally trivialized at the ends, we may similarly cap off $\widetilde{\mathcal{S}}_r$ over the e_i end by gluing with a bundle $\mathcal{D} \times M$ at the end obtaining a Hamiltonian M bundle $\widetilde{S}^{/}$ over $S^{/}$.

More generally we have a certain gluing operation of Hamiltonian structures. In the case of "capping off" as above we glue $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ with the Hamiltonian structure $\Theta' = (\mathcal{D} \times M, \mathcal{D}, \mathcal{L}', \mathcal{A}')$ at the e_i end, provided $\mathcal{A}'$ is compatible with the connection $\mathcal{A}_i$, in the sense of Section 6.2, and provided $\mathcal{L}$ is compatible with $\mathcal{L}'$. The latter means that $L_i^j = {L'}_i^j$ where these are Lagrangians corresponding to the strip chart trivialization of $\mathcal{L}, \mathcal{L}'$ at the corresponding ends, as in Definition 6.1.

Let us name the result of this capping off $\Theta \#_i \Theta'$. The following is immediate:

Lemma 6.16. *Suppose that $\{\Theta_r\}_{\mathcal{K}}$, $\{\Theta'_r\}_{\mathcal{K}}$ with $\Theta'_r = (\mathcal{D} \times M, \mathcal{D}, \mathcal{L}'_r, \mathcal{A}'_r)$ are taut Hamiltonian structures. Then:*

$$\{\Theta_r \#_i \Theta'_r\}_{\mathcal{K}}$$

is taut, whenever the gluing operation is well defined, that is whenever we have compatibility of connections and Lagrangian sub-fibrations at the corresponding end.

Definition 6.17. *Let $\pi : \mathbb{R} \to [0,1]$ denote the retraction map, sending $(-\infty, 0]$ to 0, and sending $[1,\infty)$ to 1. Fix a parametrization $\zeta : \mathbb{R} \to \partial\mathcal{D}$, which satisfies*

$\zeta(t) \in \{0\} \times (0, \pm\infty)$ for $t \in (-\infty, 0]$, and $\zeta(t) \in \{1\} \times (0, \pm\infty)$ for $t \in \sqcup[1, \infty)$, where we are using the coordinates of the strip chart $e_0 : [0,1] \times (0, \pm\infty) \to \mathcal{D}$. Given a smooth path

$$p : [0,1] \to Lag(M)$$

constant near $0, 1$, let $\mathcal{L}_p \subset \partial\mathcal{D} \times M$ denote the Lagrangian subfibration over $\partial\mathcal{D}$, with fiber over $r \in \partial\mathcal{D}$ given by $p \circ \pi(r)$. We say that a Lagrangian subfibration $\mathcal{L}$ as above is **determined by** *p if $\mathcal{L} = \mathcal{L}_p$, after a fixed choice of parametrization of boundary of $\mathcal{D}$ by $\mathbb{R}$.*

Lemma 6.18. *Let p and $\mathcal{L}_p \subset \partial\mathcal{D} \times M$ be as in Definition above with $L^{\pm}(\widetilde{p}) = \rho$, where $\widetilde{p}$ is some lift of p to $Ham(M, \omega)$, that is $p(t) = \widetilde{p}(t)(p(0))$. Let $\mathcal{A}_0$ be a Hamiltonian connection on $[0,1] \times M$, generated by a Hamiltonian H with $L^{\pm}$ length κ, constant near the end points. Then there is a Hamiltonian connection $\widetilde{\mathcal{A}}_0^p$ on $\mathcal{D} \times M$, preserving $\mathcal{L}_p$, and compatible with respect to $\mathcal{A}_0$, with $area(\widetilde{\mathcal{A}}_0) \leq \kappa + \rho$. The construction is natural in the sense that $(\widetilde{p}, \mathcal{A}_0) \mapsto \widetilde{\mathcal{A}}_0$ can be made into a smooth map (of Frechet spaces).*

Proof. Let $q : [0,1] \to Ham(M, \omega)$ be the holonomy path of $\mathcal{A}_0$, $q(0) = id$, generated by H. Let $\widetilde{p} \cdot q$ be the concatenation in diagramatic order, and H' be its generating Hamiltonian.

Define a coupling form Ω' on $D^2 \times M$:

$$\Omega' = \omega - d(\eta(rad) \cdot H' d\theta),$$

for (rad, θ) the modified angular coordinates on D^2, $\theta \in [0,1]$, $0 \leq rad \leq 1$, and $\eta : [0,1] \to [0,1]$ is a smooth function satisfying

$$0 \leq \eta'(rad),$$

and

$$\eta(rad) = \begin{cases} 1 & \text{if } 1 - \delta \leq rad \leq 1, \\ rad^2 & \text{if } rad \leq 1 - 2\delta, \end{cases} \tag{6.5}$$

for a small $\delta > 0$. By elementary calculation

$$\text{area}(\mathcal{A}') = L^+(p \cdot q) = L^+(p) + L^+(q),$$

where $\mathcal{A}'$ is the connection induced by Ω'. Set $arc = \{(1, \theta)\}$, $0 \leq \theta \leq 1/2$. Let arc^c denote the complement of arc in ∂D^2. Fix a smooth embedding $i : D^2 \hookrightarrow \mathcal{D}$ so that the image of the embedding contains $\partial\mathcal{D} - end$ where end is the image of the distinguished (say positive) end strip chart

$$[0,1] \times (0, \infty) \to \mathcal{D},$$

so that $i(arc) \subset end^c$, and so that $i(arc^c) \subset end$ as illustrated in the Figure 1.

Next fix a deformation retraction ret of $\mathcal{D}$ onto $i(D^2)$, so that in the strip chart above ret, for $r \geq 1$, is the composition $i \circ param \circ pr$, for

$$pr : [0,1] \times (0, \infty) \to [0,1]$$

the projection and for

$$param : [0,1] \to arc^c \subset D^2$$

a diffeomorphism. Finally set $\Omega = ret^* \Omega'$ on $\mathcal{D} \times S^2$, and set $\widetilde{\mathcal{A}}_0^p$ to be the induced Hamiltonian connection. As constructed $\widetilde{\mathcal{A}}_0$ will be compatible with $\mathcal{A}_0$, when

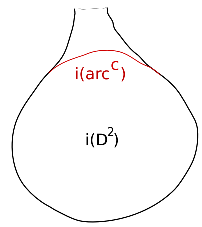

FIGURE 1.

the end of $\mathcal{D}$ is positive. When the end is negative we take the reverse paths p^{-1}, q^{-1}. $\square$

Using the above lemma, we may then put a $\mathcal{L}^{/}$-exact Hamiltonian connection, $\mathcal{A}^{/}$ on $\widetilde{S}^{/}$ (see Definition 6.2), with

$$\operatorname{area}(\mathcal{A}^{/}) \leq \operatorname{area}(\mathcal{A}) + \kappa_i. \tag{6.6}$$

This connection is obtained by gluing with the Hamiltonian structure $(\mathcal{D}\times M, \widetilde{\mathcal{A}}_0^{p=const})$ as obtained in the Lemma 6.18 above.

Lemma 6.19. *Let $L_0 \subset M$ be a monotone Lagrangian submanifold with monotonicity constant $const > 0$: $\omega(A) = const \cdot \mu(A)$, μ the Maslov number. Let*

$$\Theta := \{\Theta_r\} := \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$$

be a Hamiltonian structure with $\mathcal{K}$ connected. And let $\Theta_r^{/} = (\widetilde{S}_r^{/}, S_r^{/}, \mathcal{L}_r^{/}, \mathcal{A}_r^{/})$ be obtained from Θ_r by capping off each end e_i, so that (6.6) is satisfied. Suppose that the Floer chain complex $CF(L_i^0, L_i^1, \mathcal{A}_i)$ is perfect for each i and that $\mathcal{A}_i$ is generated by a time dependent Hamiltonian H_i with $L^{\pm}$ length κ_i.

For a given $A \in H_2^{sec}(\widetilde{S}, \mathcal{L})$, if

$$\forall r : \operatorname{area}(\mathcal{A}_r) < carea(\Theta_r^{/}, A^{/}) - \sum_i \kappa_i,$$

where $A^{/}$ is the capping off of A is described in the proof, then $\overline{\mathcal{M}}(\{\Theta_r\}, A)$ is empty. Moreover, if $\mathcal{L}_r$ is the trivial bundle with fiber L_0 for each r and $\mathcal{A}_r$ is likewise trivial over the boundary of S_r then

$$\forall r : carea(\Theta_r^{/}, A^{/}) = -const \cdot Maslov^{vert}(A^{/}),$$

where $Maslov^{vert}$ is as in Appendix B.

Proof. Suppose otherwise that we have an element $(\sigma_0, r_0) \in \overline{\mathcal{M}}(\{\Theta_r\}, A)$. Suppose for the moment that $\overline{\mathcal{M}}(\{\Theta_r\}, A)$ is regular.

There is a PSS isomorphism

$$QH(L) \to FH(L, L),$$

where the right hand side is defined using our construction in terms of flat sections, and the left hand side is interpreted for example as the Pearl complex complex homology, [3]. We won't give the full construction of this map, as it just reformulation of well known constructions, as presented in [3] for instance. But we can quickly say how to get $PSS([L])$. Let

$$\Theta_- = (\mathcal{D} \times M, \mathcal{D}, \mathcal{L}, \mathcal{A}_-),$$

be the Hamiltonian structure with e_0 being a negative end, $\mathcal{L}$ trivial with fiber L (which is an object as before), and

$$\mathcal{A}_- := \widetilde{\mathcal{A}}_0^{p=const},$$

with right hand side as in Lemma 6.18, for p being the constant path. Suppose that Θ_- is regular. Define $PSS([L])$ as the homology class of the chain c determined by:

$$\langle c, \gamma \rangle = \sum_A \#ev(\Theta_-, A),$$

where the sum is over all classes $A \in H_2^{sec}(\mathcal{D} \times M, \mathcal{L})$, which have asymptotic constraint γ, for γ a geometric generator of $CF(L, L, \mathcal{A}_0)$. As the PSS map is an isomorphism in our monotone context, as is well known, and since $CF(L_i^0, L_i^1, \mathcal{A}_i)$ is perfect for each i, the asymptotic constraint γ_i of σ_0 at each e_i end must be hit by the PSS map, (note that at the negative ends γ_i are interpreted in the dual complex $CF(L_i^0, L_i^1, \mathcal{A}_i^r)$, where $\mathcal{A}_i^r$ is the reverse of $\mathcal{A}_i$).

By gluing with the holomorphic sections of $M \times \mathcal{D}$ coming from the construction of the PSS maps for each end e_i, we obtain a $J(\mathcal{A}'_{r_0})$-holomorphic, class A' section σ'_0 of Θ'_{r_0}.

By Lemma 6.14:

$$carea(\Theta'_{r_0}, A') \leq \text{area}(\mathcal{A}'_{r_0}) \leq \text{area}(\mathcal{A}_{r_0}) + \sum_i \kappa_i,$$

so

$$carea(\Theta'_{r_0}, A') - \sum_i \kappa_i \leq \text{area}(\mathcal{A}_{r_0}),$$

so that we contradict the hypothesis and in the case $\overline{\mathcal{M}}(\{\Theta_r\}, A)$ is regular we are done with the first part of the lemma. When it is not regular instead of gluing just pre-glue to get a holomorphic building σ'_0, and the conclusion follows by the same argument.

To prove the last part of the lemma, note that in this case each Θ'_r is taut concordant to

$$\Theta_0 = (D^2 \times M, D^2, \mathcal{L}, \mathcal{A}^{tr}),$$

with $\mathcal{L}$ trivial with fiber L_0, and for $\mathcal{A}^{tr}$ the trivial connection. And

$$carea(\Theta_0, \cdot) = -const \cdot Maslov^{vert}(\cdot)$$

as functionals on $H_2^{sec}(D^2 \times M, \mathcal{L})$ with $const > 0$. It follows by Lemma 6.11 that

$$carea(\Theta', \sigma'_0) = carea(\Theta_0, \sigma'_0) = -const \cdot Maslov^{vert}(\sigma'_0) = -const \cdot Maslov(A').$$

□

7. Construction of small data

To forewarn, we use here notation and notions from Part I, especially from Sections 4, 5 in Part I. Let m_i denote the edges of Δ^4 as before.

Let $\{m^k\}_{k=1}^{k=d}$ be a composable sequence in $\Pi(\Delta^n)$, which we recall means that the target of m_{i-1} is the source of m_i for each i. Recall from Part I that the perturbation data $\mathcal{D}$ in particular specifies for each n and for each such composable sequence certain maps

$$u(\{m^k\}, n) : \mathcal{E}_d^\circ \to \Delta^n,$$

where $\mathcal{E}_d$ is the universal curve over $\overline{\mathcal{R}}_d$, and $\mathcal{E}_d^\circ$ denotes $\mathcal{E}^d$ with nodal points of the fibers removed. Collective such a chosen collection of maps satisfying certain axioms is denoted $\mathcal{U}$. And we have already mentioned this in the introduction. The restriction of $u(\{m^k\}, n)$ to the fiber $\mathcal{S}_r$ of $\mathcal{E}_r^\circ$ over r, is denoted by $u(\{m^k\}, n, r)$, which may also be abbreviated by u_r.

$\mathcal{D}$ also specifies for each $\Sigma : \Delta^n \to X$, each r, each composable chain $(m_1, \ldots, m_d)$ in $\Pi(\Delta^n)$, and for each chain of objects $L_0', \ldots, L_d'$ with $L_i' \subset P|_{\overline{m}_i(1)}$, $i \geq 1$, $L_0' \subset P|_{\overline{m}_1(0)}$, a Hamiltonian connection on

$$\widetilde{\mathcal{S}}_r := (\Sigma \circ u(\{m^k\}, n, r))^* P \to \mathcal{S}_r. \tag{7.1}$$

Here as before $\overline{m}_i = \Sigma \circ m_i$. When $n = 4$, $\Sigma = \Sigma_+$, and $\forall i : L_i' = L_0$ with the latter as before we write these connections as $\mathcal{A}_r^+(\{m^k\})$, further abbreviated by $\mathcal{A}_r^+$ as $\{m^k\}$ will be implicit in what follows.

Suppose that $\mathcal{D}$ extends $\mathcal{D}_{pt}$ from before. If $\mathcal{L}_r \subset \widetilde{\mathcal{S}}_r|_{\partial \mathcal{S}_r}$ denotes the trivial Lagrangian sub-bundle with fiber L_0, then we obtain Hamiltonian structure (for each composable d-chain $\{m^k\}$) $\Theta^+ = \{\Theta_r^+\} = \{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r^+\}_{\overline{\mathcal{R}}_d}$, which were discussed in the previous section. By the properties of these connections, necessitated by $\mathcal{D}$, at each end e_i of $\mathcal{S}_r$, $\mathcal{A}_r^+$ is compatible with the connection $\mathcal{A}_i = \mathcal{A}(L_0, L_0)$, where $\mathcal{A}(L_0, L_0)$ is the connection on $[0,1] \times S^2$ part of our Floer data $\mathcal{D}$. Then Θ^+ is trivially taut since for each r $\mathcal{L}_r$ is naturally trivial and $\mathcal{A}_r$ is likewise trivial over $\partial \mathcal{S}_r$, for each r, by assumed properties of these connections.

Set

$$\hbar := \frac{1}{2} \operatorname{area}(S^2, \omega).$$

Let κ denote the $L^\pm$ length of the holonomy path in $Ham(S^2)$ of $\mathcal{A}_0 = \mathcal{A}(L_0, L_0)$. We may suppose that

$$\forall r : \operatorname{area}(\mathcal{A}_r^+) < \hbar - 5\kappa, \tag{7.2}$$

is satisfied after taking κ to be sufficiently small. So that in particular $\mathcal{A}_r^+$ is small for each r, with small as defined in the previous section.

Fix a complex structure j_0 on M, and let $\{J_r\}$ be the family of complex structures on $\{\widetilde{\mathcal{S}}_r\}$ induced by $(\{\mathcal{A}_r^+\}, j_0)$.

Lemma 7.1. *As in Part I, let*

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}(\gamma^1, \ldots, \gamma^d; \gamma^0, \Sigma_+, \{J_r\}, A),$$

denote the set of elements of $\overline{\mathcal{M}}(\Theta^+, A)$ with asymptotic constraints γ^i at each e_i end. Here each γ^k, $k \neq 0$, is of the form $\gamma_{i,j}$ where this is as before. Then whenever the class A is such that $\mathcal{M}$ has virtual dimension 0, and d satisfies $2 < d \leq 4$, $\overline{\mathcal{M}}$ is empty.

Proof. Let

$$\Theta^/ := (\Theta^+)^/,$$

and $A^/$ be as in Section 6.4. For a fixed r, by the Riemann-Roch (Appendix B) we get that the expected dimension of $\mathcal{M}(\Theta^/, A^/)$ is

$$1 + Maslov^{vert}(A^/).$$

Consequently, when $\gamma^0 = \gamma$, the expected dimension of $\mathcal{M}$ is:

$$1 + Maslov^{vert}(A^/) - 1 + (\dim \mathcal{R}_d = d - 2). \tag{7.3}$$

We need the expected dimension of $\mathcal{M}$ to be 0, and $d \geq 3$, so $Maslov^{vert}(A^/) \leq -1$. But $Maslov^{vert}(A^/) = -1$ is impossible as the minimal positive Maslov number is 2. Consequently, the result follows by Lemma 6.19 and by the property (7.2).

When γ^0 is the Poincare dual to γ, we would get $Maslov^{vert}(A^/) \leq -2$ so for the same reason the conclusion follows. □

So if we choose our data $\mathcal{D}$ so that the hypothesis of the lemma above are satisfied, then with respect to $\mathcal{D}$:

$$\mu^2_{\Sigma^4_\pm}(\gamma_{i,j}, \gamma_{j,k}) = \gamma_{i,k} \tag{7.4}$$

$$\mu^3_{\Sigma^4_\pm}(\gamma^1, \ldots, \gamma^3) = 0, \text{for } \gamma^i \text{ as above} \tag{7.5}$$

$$\mu^4_{\Sigma^4_+}(\gamma_1, \ldots, \gamma_4) = 0. \tag{7.6}$$

In particular this $\mathcal{D}$ is small.

8. The product $\mu^4_{\Sigma^4_-}(\gamma_1, \ldots, \gamma_4)$ and the quantum Maslov classes

The product

$$\mu^4_{\Sigma^4_-}(\gamma_1, \ldots, \gamma_4)$$

a priori depends on various choices, like the choices of $h_\pm$, and then choice of data $\mathcal{D}_0$. However by Lemma 6.8, so long as there is a homotopy of the choices, together with a homotopy of associated perturbation data $\{\mathcal{D}_t\}$, so that $\mathcal{D}_t$ is small for all t, the above product is t-invariant. Thus, in particular for the purpose of computation we may take h_+ to be the constant map to x_0 and

$$h_- : (D^4, \partial D^4) \to (S^4, x_0)$$

the complementary map, that is representing the generator of $\pi_4(S^4, x_0) \simeq \mathbb{Z}$, we further suppose that h_- is an embedding in the interior of D^4.

Let Σ_- be the corresponding 4-simplex of $S^4_\bullet$ as before. We need to study the moduli spaces

$$\overline{\mathcal{M}}(\gamma_1, \ldots, \gamma_4; \gamma^0, \Sigma_-, \{\mathcal{A}_r\}, A), \tag{8.1}$$

where $\mathcal{A}_r$ now denotes the connections on

$$\widetilde{\mathcal{S}}_r := (\Sigma_- \circ u(m_1, \ldots, m_4, 4, r))^* P \to \mathcal{S}_r, \tag{8.2}$$

part of some small data $\mathcal{D}_0$ as above. We abbreviate $u(m_1, \ldots, m_4, 4, r)$ by u_r in what follows.

By the dimension formula (7.3), since we need the expected dimension of (8.1) to be zero, the class $A^/$ satisfies:

$$Maslov^{vert}(A^/) = -2,$$

and we must have $\gamma^0 = \gamma_{0,4}$, in other words the latter morphism corresponds to the fundamental chain.

Notation 8.1. *From now on, by slight abuse, A_0 refers to various section classes of various Hamiltonian structures such that the associated class $A_0^/$ satisfies:*

$$Maslov^{vert}(A_0^/) = -2.$$

8.1. **Constructing suitable $\{\mathcal{A}_r\}$.** To get a handle on (8.1) we want to construct very special small data $\mathcal{D}_0$.

A Hamiltonian S^2 fibration over S^4 is classified by an element

$$[g] \in \pi_3(\mathrm{Ham}(S^2), id) \simeq \pi_3(PU(2), id) \simeq \mathbb{Z}.$$

Such an element determines a fibration P_g over S^4 via the clutching construction:

$$P_g = D^4_- \times S^2 \sqcup D^4_+ \times S^2 \sim,$$

with D^4_-, D^4_+ being 2 different names for the standard closed 4-ball D^4, and the equivalence relation $\sim$ is $(d,x) \sim \widetilde{g}(d,x)$,

$$\widetilde{g} : \partial D^4_- \times S^2 \to \partial D^4_+ \times S^2, \quad \widetilde{g}(d,x) = (d, g(d)^{-1}(x)).$$

We suppose that $x_0 = h_\pm(b_0)$ that previously appeared, is a point common to $D^4_\pm \subset S^4$.

From now on P_g will denote such a fibration for a non-trivial class $[g]$. Note that the fiber of P_g over the base point $x_0 \in S^3 \subset D^4_\pm$ (chosen for definition of the homotopy group $\pi_3(Ham(S^2), id)$) has a distinguished, by the construction, identification with S^2. Take $\mathcal{A}$ to be a connection on $P \simeq P_g$ which is trivial in the distinguished trivialization over D^4_+. This gives connections

$$\mathcal{A}'_r := (\widetilde{u}_r)^* \mathcal{A}$$

on $\widetilde{\mathcal{S}}_r$,

$$\widetilde{u}_r = \Sigma_- \circ u_r.$$

By the last axiom for the system $\mathcal{U}$ introduced in Part I, we may choose $\{u_r\}$ so that the family $\{\widetilde{u}_r(\mathcal{S}_r)\}$ induces a singular foliation of S^4, that is smooth outside x_0, with x_0 being the image by $\widetilde{u}_r$ of the ends (images of e_i) and the boundary of each $\mathcal{S}_r$, and so that each $\widetilde{u}_r$ is an embedding on the complement of $\widetilde{u}_r^{-1}(x_0)$. Denote by E the subset $S^3 \subset S^4$ bounding $D^4_\pm$. We may in addition suppose that each $\widetilde{u}_r$ intersects E transversally, again on the complement of $\widetilde{u}_r^{-1}(x_0)$.

By the above, the preimage by $\widetilde{u}_r$ of E contains a smoothly embedded curve c_r as in Figure 2, and $\widetilde{u}_r$ takes c_r into E. This c_r not uniquely determined but we may fix a family $r \mapsto c_r$, with parametrizations

$$c_r : \mathbb{R} \to \mathcal{S}_r$$

so that c_r maps $(-\infty, 0)$ diffeomorphically onto the image by e_0 of $\{0\} \times (-\infty, 0)$, so that likewise c_r maps $(1, \infty)$ diffeomorphically onto the image by e_0 of $\{1\} \times (-\infty, 0)$, and so that $\{c_r\}$ is a C^0 continuous family. We set:

$$\widetilde{c}_r := \widetilde{u}_r \circ c_r.$$

In Figure 2, the regions $R_\pm$ are the preimages by $\widetilde{u}_r$ of $D^4_\pm \subset S^4$, and c_r bounds R_-. It follows that $\{\widetilde{c}_r\}$ likewise induces a singular foliation of the equator $E \simeq S^3$ that is smooth outside x_0.

So each $\mathcal{A}'_r$ is flat in the region R_+, in fact is trivial in the distinguished trivialization of $\widetilde{\mathcal{S}}_r$ over R_+, corresponding to the distinguished trivialization of P over D^4_+. Likewise we have a distinguished trivialization of $\widetilde{\mathcal{S}}_r$ over R_-, corresponding to the distinguished trivialization of P over D^4_-. In this latter trivialization let

$$\phi_r : \mathbb{R} \to Ham(S^2, \omega)$$

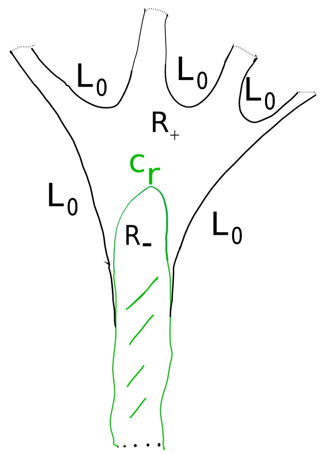

FIGURE 2. The labels L_0 indicate that the Lagrangian subbundle is constant with corresponding fiber L_0. The curve c_r bounds R_-.

be the holonomy path of $\mathcal{A}'_r$ over c_r. Then by construction,

$$\phi_r|_{(-\infty,0]\sqcup[1,\infty)} = id,$$

so that we may define

$$f(r) \in \Omega_{L_0} Lag(M, L_0),$$

by

$$f(r)(t) = \phi_r(t)(L_0), \quad t \in [0,1]$$

where the right hand side means apply an element of $Ham(S^2)$ to L_0 to get a new Lagrangian. We will say that $f(r)$ is ***generated by*** $\mathcal{A}'_r$.

Note that by construction

$$\phi_r(t) = g(\widetilde{c}_r(t)), \quad t \in [0,1], \tag{8.3}$$

if we identify $\widetilde{c}_r(t)$ with an element of S^3.

Let $D^2_0 \subset \overline{\mathcal{R}}_4$ be an embedded closed disk D^2, not intersecting the boundary $\partial\overline{\mathcal{R}}_4$, so that ∂D^2_0 is in the normal gluing neighborhood N of $\partial\overline{\mathcal{R}}_4$, where N is as in Part I.

So we have a continuous map

$$f : D^2_0 \to \Omega_{L_0} Lag(S^2).$$

And $f(\partial D^2_0) = L_0$, with the right hand side denoting the constant loop at L_0. Then by construction and (8.3) in particular, $f \simeq lag$, where $\simeq$ is a homotopy equivalence, and where

$$lag : S^2 \to \Omega_{L_0} Lag(S^2) \tag{8.4}$$

is the composition

$$S^2 \xrightarrow{g'} \Omega_{id} PU(2) \to \Omega_{L_0} Lag^{eq}(S^2),$$

for g' naturally induced by g, and for the second map naturally induced by the map

$$PU(2) \to Lag^{eq}(S^2), \quad \phi \mapsto \phi(L_0).$$

We then deform each $\mathcal{A}'_r$ to a connection $\mathcal{A}_r$, which is as follows. In the region R_+ $\mathcal{A}_r$ is still flat, but at each end e_i, $\mathcal{A}_r$ is compatible with $\mathcal{A}(L_0, L_0)$, where this is as in Section 6.2, and so that $\mathcal{A}_r$ is still trivial over the boundary of $\mathcal{S}_r$.

Since $\widetilde{\mathcal{S}}_r$ and $\mathcal{A}'_r$ are trivial for $r \in \overline{\mathcal{R}}_d - D^2_0$, with trivialization induced by the trivialization of P_+, and since the condition (7.2) holds, we may insure that

$$\text{area}(\mathcal{A}_r) < \hbar - 5\kappa, \tag{8.5}$$

for r in the complement of D^2_0. In other words $\{\mathcal{A}_r\}$ extends to a system of connections corresponding to small data $\mathcal{D}_0$ for P, as intended.

8.2. **Restructuring the data $\{\mathcal{A}_r\}$.** Applying Lemma 6.19 we see that the resulting Hamiltonian structure $\mathcal{H} := \{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r\}$ is A_0-admissible. We now further mold this data for the purposes of computation.

First cap off the ends e_i, $i \neq 0$, of $\mathcal{S}_r$ as in Section 6.4. This gives a Hamiltonian structure

$$\mathcal{H}^\wedge := \{\widetilde{S}^\wedge_r, S^\wedge_r, \mathcal{L}^\wedge_r, \mathcal{A}^\wedge_r\}_{\mathcal{K}=D^2_0},$$

satisfying

$$\text{area}(\mathcal{A}^\wedge_r) + \kappa < \hbar,$$

for each r. Again by Lemma 6.19 $\mathcal{H}^\wedge$ is A_0-admissible.

By the now classical gluing of holomorphic curves it follows that

$$[\mu^4_{\Sigma^4}(\gamma_1, \ldots, \gamma_4)] = [ev(\mathcal{H}^\wedge, A_0)]. \tag{8.6}$$

It remains to compute the right hand side, to this end we further restructure.

Let $p_1 : [0,1] \to Lag(S^2, L_0)$ be the path generated by $\mathcal{A}(L_0, L_0)$, with p_1 starting at L_0, and where generated has the same meaning as in the previous section. Suppose we have defined p_{i-1}, set $L_{i-1} := p_{i-1}(1)$ and define p_i to be the path in $Lag(S^2, L_0)$ starting at L_{i-1}, generated by $\mathcal{A}(L_0, L_0)$. Set $p_0 := p_1 \cdot \ldots \cdot p_d$, where $\cdot$ is path concatenation in diagrammatic order. We may assume that L_0 is transverse to $L_4 = p_0(1)$ by adjusting the connection $\mathcal{A}(L_0, L_0)$ if necessary. Then deform $\mathcal{L}^\wedge_r$ in the Hamiltonian data $\{\widetilde{S}^\wedge_r, S^\wedge_r, \mathcal{L}^\wedge_r, \mathcal{A}^\wedge_r\}$ as in Figure 3. The resulting Lagrangian subbundle over $\partial\mathcal{S}_r$ will be denoted $\mathcal{L}^{\mathfrak{n}}_r$, $\mathfrak{n}$ here stands for 'new'.

We simultaneously deform $\mathcal{A}^\wedge_r$ to an $\mathcal{L}^{\mathfrak{n}}_r$ exact Hamiltonian connection $\mathcal{A}^{\mathfrak{n}}_r$ which satisfies the following conditions. $\mathcal{A}^{\mathfrak{n}}_r$ is flat in the entire region R_+ (which includes the red shaded finger regions). Along the dotted line (which is contained in the strip chart at the e_0 end) $\mathcal{A}^{\mathfrak{n}}_r$ is the trivial connection in the distinguished trivialization at the end, and such that at the e_0 end, which is down in the Figure 3, the connection is unchanged over $[0,1] \times (t, \infty)$, for t large. In order to get such a deformation, we introduce curvature in the blue stripped region of Figure 3. We name this new Hamiltonian data by

$$\mathcal{H}^{\mathfrak{n}} := \{\widetilde{S}^\wedge_r, S^\wedge_r, \mathcal{L}^{\mathfrak{n}}_r, \mathcal{A}^{\mathfrak{n}}_r\}.$$

As $L^\pm(p_i)$, for each p_i, can be arranged to be arbitrarily small, it is clear that we may choose the deformation from $\mathcal{H}^\wedge$ to $\mathcal{H}^{\mathfrak{n}}$ to be small near boundary (Definition 6.13) and hence be an A_0-admissible concordance. More specifically, we may choose a concordance from $\mathcal{H}^\wedge$ to $\mathcal{H}^{\mathfrak{n}}$ so that for the associated family of connections $\{\mathcal{A}_{r,t}\}$,

$$\mathcal{A}_{r,0} = \mathcal{A}^\wedge_r, \mathcal{A}_{r,1} = \mathcal{A}^{\mathfrak{n}}_r,$$

so that for each r,

$$\frac{d}{dt}|R_{\mathcal{A}_{r,t}}(v, jv)|_+ < 0, \quad v \in T_z\mathcal{S}_r$$

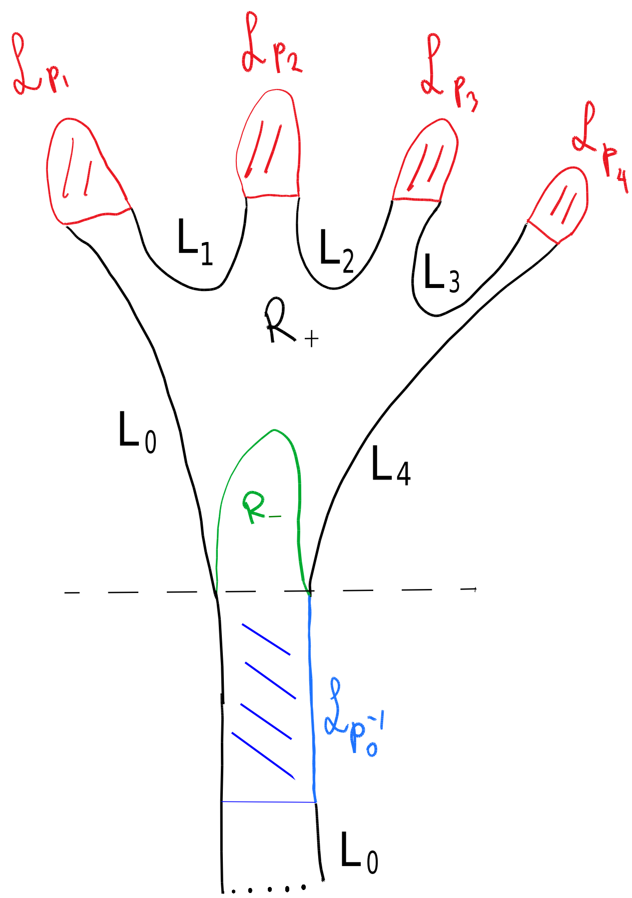

FIGURE 3. Over the boundary components with black labels L_i the Lagrangian subbundle $\mathcal{L}_r^{\mathfrak{n}}$ is constant with corresponding fiber L_i. Over the i'th red boundary component the Lagrangian subbundle corresponds to the path of Lagrangians p_i, analogously to Definition 6.17 further below. Likewise over the blue boundary component the Lagrangian subbundle corresponds to the path of Lagrangians p_0^{-1}. In the red striped regions we have removed the curvature of the connection, the blue striped regions we have added it.

for each $z \in \mathcal{S}_r$, except for z in the region which is blue striped in Figure 3. However, the area increase in this blue region is bounded from above by $L^+(p_0^{-1})$, so that

$$\forall t : |\operatorname{area}(\mathcal{A}_{r,t}) - \operatorname{area}(\mathcal{A}_r^{\wedge})| \leq L^+(p_0^{-1}).$$

In fact we can arrange that

$$\frac{d}{dt}\operatorname{area}(\mathcal{A}_{r,t}) = 0,$$

since the gain of area in the blue striped region is exactly equal to the loss of area in the red striped regions, but this extra precision is not necessary.

Of course:

$$[ev(\mathcal{H}^{\wedge}, A_0)] = [ev(\mathcal{H}^{\mathfrak{n}}, A_0)]$$

since the corresponding Hamiltonian data are A_0-admissibly concordant. This finishes our restructuring.

8.3. **Computing** $[ev(\mathcal{H}^{\mathfrak{n}}, A_0)]$**.** If we stretch the neck along the dashed line in Figure 3, the upper half of the resulting building gives us new Hamiltonian data

$$\mathcal{H}^0 = \{\widetilde{S}^0_r, S^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}.$$

By the classical theory of continuation maps in Floer homology we clearly have that

$$[ev_{\mathfrak{n}}] := [ev(\mathcal{H}^{\mathfrak{n}}, A_0)] \in FH(L_0, L_0)$$

is non-zero iff

$$[ev_0] := [ev(\mathcal{H}^0, A_0] \in FH(L_0, L_4)$$

is non-zero.

Let $\mathcal{P}_{L_0,L_4}Lag(M)$ denote the space of smooth paths in $Lag(M)$ from L_0 to L_4. Let

$$f' : D^2_0 \to \mathcal{P}_{L_0,L_4}Lag(S^2),$$

be like f but defined with respect to $\{\mathcal{A}^0_r\}$, so that

$$f'(t) = g(\widetilde{c}_r(t))(p_0(t)), \tag{8.7}$$

if we suppose that the holonomy path of $\mathcal{A}^0_r$ over c_r in the trivialization over R_+ generates p_0 (which can be insured by adjusting $\{\mathcal{A}^0_r\}$ or the parametrizations $\{c_r\}$). Here the right hand side of (8.7) means as before: apply an element of $Ham(S^2)$ to a Lagrangian to get a new Lagrangian. In this case f' takes ∂D^2_0 to $p_0 \in \mathcal{P}_{L_0,L_4}Lag(M)$. In particular, f' represents a class $a \in \pi_2(\mathcal{P}_{L_0,L_4}Lag(M), p_0)$.

In what follows we omit specifying the parameter space D^2_0 for r, since it will be the same everywhere. Let $\mathcal{L}_p$ be as in the Definition 6.17.

Lemma 8.2. *The A_0-admissible Hamiltonian structure $\mathcal{H}_0 = \{\widetilde{\mathcal{S}}^0_r, \mathcal{S}^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}$ is A_0-admissibly concordant to*

$$\Theta' = \{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f'}(r), \mathcal{B}_r\},$$

for certain Hamiltonian connections $\{\mathcal{B}_r\}$ (which are not explicitly relevant yet).

Proof. Let $R_\pm \subset \mathcal{S}^\wedge_r$ be as before. Fix a deformation retraction

$$ret_r : \mathcal{S}^0_r \times I \to \mathcal{S}^0_r,$$

of $\mathcal{S}^0_r$ onto R^-, smoothly in r. Since $\mathcal{A}^0_r$ is flat over R^+, the pull-back by ret_r of the data $\mathcal{H}_0$ then induces an A_0-admissible concordance between $\mathcal{H}_0$ and

$$\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f'}(r), \mathcal{B}_r = ret^*_r \mathcal{A}^0_r\},$$

once we use smooth Riemann mapping theorem to identify each $R^- \subset \mathcal{S}^0_r$ with its induced complex structure j_r with $(\mathcal{D}, j_{st})$, smoothly in r. ☐

9. Quantum Maslov classes

The class of $ev(\Theta', A_0)$ is related to what we christen as quantum Maslov classes. These are relative analogues of the quantum characteristic classes [16]. We chose the name quantum Maslov class, because the classical Maslov numbers are relative analogues of Chern numbers, while quantum characteristic classes are directly related (via semi-classical approximation) to Chern classes, [18].

We will not give extensive detail here since we don't need the full theory, we present it because it gives extra perspective. The ordinary relative Seidel morphism appears in Seidel's [22] in the exact case and further developed in [9] in the monotone case. Let $Lag(M)$ denote the space whose components are objects of $Fuk(M)$ in the previous sense, so in particular oriented, spin, Hamiltonian isotopic Lagrangian submanifolds of M. We may also denote the component of L by $Lag(M, L)$. Then the relative Seidel morphism is a functor

$$S : \Pi Lag(M) \to DF(M),$$

where $\Pi Lag(M)$ is the fundamental groupoid of $Lag(M)$ and $DF(M)$ is the Donaldson-Fukaya category of M, see also [5], [4] which can be understood as an extension.

We sketch how this works. To a path p in $Lag(M)$ from L_0 to L_1 we have have an associated Lagrangian subbundle $\mathcal{L}_p$ of $\mathcal{D} \times M$ over the boundary, as in Definition 6.17. Extend this to a Hamiltonian structure

$$\Theta_p = (\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_p, \mathcal{A}_0^p)$$

where $\mathcal{A}_0^p$ is as in Lemma 6.18. Assuming Θ_p is regular, we define $S([p]) \in DF(L_0, L_1)$ by

$$S([p]) = \sum_A [ev(\Theta_p, A)],$$

where by monotonicity only finitely many A can have non-zero contribution.

9.1. Definition of the quantum Maslov classes.

Let M be as before, and let $\mathcal{P}(L_0, L_1)$ denote the space of smooth paths in $Lag(M)$ from L_0 to L_1, constant in $[0, \epsilon] \cup [1-\epsilon, 1]$ for some $0 < \epsilon < 1$. There is then an additive group homomorphism:

$$\Psi : H_*(\mathcal{P}(L_0, L_1), \mathbb{Q}) \to FH(L_0, L_1) \tag{9.1}$$

defined analogously to above and to [16] in non-relative context. Although formally we will only need the restriction of Ψ to spherical classes.

This works as follows. To a smooth cycle

$$f : B \to \mathcal{P}(L_0, L_1)$$

for B a smooth closed oriented manifold, we may associate a Hamiltonian structure

$$\{\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_b\}_B,$$

$\mathcal{L}_b := \mathcal{L}_{f(b)}$ a Lagrangian subbundle of $M \times \mathcal{D}$ over $\partial\mathcal{D}$ determined by $f(b)$ as before. The end of $\mathcal{D}$ here is negative. Now let $\mathcal{A}_0$ be a Hamiltonian connection on $[0, 1] \times M$, so that $\mathcal{A}_0(L_0)$ is transverse to L_1 where $\mathcal{A}_0(L_0) \subset \{1\} \times M$ denotes the $\mathcal{A}_0$-transport over $[0, 1]$ of $L_0 \subset \{0\} \times M$.

For each b the space of Hamiltonian connections $\mathcal{L}_b$-exact with respect to $\mathcal{A}_0$, (as in Section 6.2) is contractible, c.f. [1]. So we get an induced Hamiltonian structure:

$$\Theta_f = \{\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}$$

well defined up to concordance.

We may then define $\Psi([f])$ by:

$$\Psi([f]) = \sum_A [ev(\Theta_f, A)],$$

where again by monotonicity only finitely many A can give non-zero contribution. It is immediate that Ψ is an additive group homomorphism.

Remark 9.1. *We should mention that the morphism Ψ extends to a certain functor to $DF(M)$, see [4] for a related discussion.*

Given the definition above,

$$[ev(\Theta', A_0)] = \Psi(a)$$

clearly holds, as A_0 is the only class that can contribute to $\Psi(a)$, since by the dimension formula (B.1) only a class A with $Maslov^{vert}(A^/) = -2$ can contribute.

10. Computation of the quantum Maslov class $\Psi(a)$

10.1. **Morse theory for the Hofer length functional.** Under certain conditions the spaces of perturbation data for certain problems in Gromov-Witten theory admit a Hofer like functional. Although these spaces of perturbations are usually contractible, there may be a gauge group in the background that we have to respect, so that working equivariantly there is topology. The reader may think of the analogous situation in Yang-Mills theory [2].

Without elaborating too much, the basic idea of the computation that we will perform consists of cooling the perturbation data as much as possible (in the sense of the functional) to obtain a mini-max (for the functional) data, using which we may write down our moduli spaces explicitly. This idea was first used in [17].

10.1.1. *Hofer length.* For $p : [0, 1] \to Ham(M, \omega)$ a smooth path, define

$$L^+(p) := \int_0^1 \max_M H_t^p dt,$$
$$L^-(p) := \int_0^1 \max_M (-H_t^p) dt,$$
$$L^{\pm}(p) := \max\{L^+(p), L^-(p)\},$$

where $H^p : M \times [0, 1] \to \mathbb{R}$ generates p normalized by the condition that for each t, $H_t^p := H^p|_{M\times\{t\}}$ has mean 0, that is $\int_M H_t^p dvol_\omega = 0$. Also define

$$L^+_{lag} : \mathcal{P}Lag(M) \to \mathbb{R},$$

$$L^+_{lag}(p) := \int_0^1 \max_{p(t)} H_t^p dt,$$

$p(0) = L$ and where $H^p : M \times [0, 1] \to \mathbb{R}$ is normalized as above and generates a lift $\widetilde{p}$ of p to $Ham(M)$ starting at id. By lift we mean that $p(t) = \widetilde{p}(t)(p(0))$. (That is H^p generates a path in $Ham(M)$, which moves L_0 along p.) Some theory of this latter functional is developed in [10]. We may however omit the subscript lag from notation, as usually there can be no confusion which functional we mean.

Note that $Lag^{eq}(S^2)$ is naturally diffeomorphic to S^2 and moreover it is easy to see that the functional $L^+|_{Lag^{eq}(S^2)}$ is proportional to the Riemannian length functional L_{met} on the path space of S^2, with its standard round metric *met*.

Let now $L_0, L_1 \in Lag^{eq}(S^2)$ be any transverse pair, and

$$f' : S^2 \to \mathcal{P}(L_0, L_1) := \mathcal{P}_{L_0,L_1} Lag^{eq}(S^2),$$

be the generator of the group $H_2(\mathcal{P}(L_0, L_1), \mathbb{Z})$. The idea of the computation is then this: perturb f' to be transverse to the (infinite dimensional) stable manifolds for the Riemannian length functional on

$$\mathcal{P}(L_0, L_1) := \mathcal{P}_{L_0,L_1} Lag^{eq}(S^2),$$

push the cycle down by the "infinite time" negative gradient flow for this functional, and use the resulting representative to compute $\Psi(a = [f'])$. Although, we will not actually need infinite dimensional topology.

10.1.2. *The "energy" minimizing perturbation data.* Classical Morse theory [13] tells us that the energy functional

$$E(p) = \int_{[0,1]} \langle \dot{p}(t), \dot{p}(t) \rangle_{met} \, dt$$

on $\mathcal{P}(L_0, L_1)$ is Morse non-degenerate with a single critical point in each degree. Consequently a (as a homology class) has a representative in the 2-skeleton of $\mathcal{P}(L_0, L_1)$, for the Morse cell decomposition induced by E. This follows by Whitehead's compression lemma which is as follows.

Lemma 10.1 (Whitehead, see [8])**.** *Let (X, A) be a CW pair and let (Y, B) be any pair with $B \neq \emptyset$. For each n such that $X - A$ has cells of dimension n, assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f : (X, A) \to (Y, B)$ is homotopic relative to A to a map $X \to B$.*

Suppose that a has a representative $f' : S^2 \to \mathcal{P}_{L_0,L_1}(S^2)$ mapping into the n-skeleton B^n for the Morse cell decomposition for E, $n > 2$. Apply the lemma above with $(X, A) = (S^2, pt)$, $Y = B^n$ and $B = B^{n-1}$ as above. Then the quotient B^n/B^{n-1} is a wedge of n-spheres and since $\pi_2(S^n) = 0$ for $n > 2$, f can be homotoped into B^{n-1} by the Whitehead lemma. Descend this way until we get a representative mapping into B^2.

Furthermore since $\pi_2(S^1) = 0$ such a representative cannot entirely lie in the 1-skeleton. It follows, since we have a single Morse 2-cell that there is a representative $f : S^2 \to \mathcal{P}_{L_0,L_1}(S^2)$, for a, s.t. the function f^*E is Morse with a maximizer max, of index 2, and s.t. $\gamma_0 = f(max)$ is the index 2 geodesic. We call such a representative ***minimizing***.

Remark 10.2. *In principle there maybe more than one such maximizer* max*, but recall that we assumed that a is the generator, so by further deformation we may insure that there is only one maximizer. The relevant representative f, with a single maximizer* max *as above, can also be constructed by hand.*

It follows that max is likewise the unique index 2 maximizer of the function f^*L_{met} by the classical relation between the energy functional and length functional. And so max is the index 2 maximizer of f^*L^+.

10.1.3. *The corresponding minimizing data.*

Lemma 10.3. *There is a minimizing representative f_0 for the class a and a taut Hamiltonian structure*

$$\Theta_{f_0} = \{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f_0(b)}, (\mathcal{A}_b = \mathcal{A}_0^{f_0(b)})\},$$

where $\mathcal{A}_0^{f_0(b)}$ is again as in Lemma 6.18, satisfying:

$$\forall b : \operatorname{area}(\mathcal{A}_b) = L^+(f_0(b)). \tag{10.1}$$

Proof. Note that a geodesic segment $p : [0,1] \to S^2$ for the round metric *met* on S^2 has a unique lift

$$\widetilde{p} : [0,1] \to PU(2) \simeq SO(3),$$

$\widetilde{p}(0) = id$ with $\widetilde{p}$ a segment of a one parameter subgroup, and in this case

$$L^+_{lag}(p) = L^+(\widetilde{p}).$$

It then follows that for a piecewise geodesic path p in S^2, there is likewise a unique lift $\widetilde{p}$ satisfying

$$L^+_{lag}(p) = L^+(\widetilde{p}).$$

Now if f is a minimizing representative of a, we may homotop it to a likewise minimizing representative f_0 so that for all b $f_0(b)$ is piecewise geodesic, this follows by the piecewise geodesic approximation theorem Milnor [13, Theorem 16.2] of the loop space.

Let $\mathcal{A}_0$ be the trivial Hamiltonian connection on $[0,1] \times M$. Use the construction of Lemma 6.18, to get a family of Hamiltonian connections $\{\mathcal{A}_0^{f_0(b)}\}$. In this case since $\mathcal{A}_0$ is trivial

$$\operatorname{area}(\mathcal{A}_0^{f_0(b)}) = L^+(f_0(b)).$$

It remains to verify that $\Theta_{f_0} = \{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f_0(b)}, \mathcal{A}_b\}$, is taut. This follows by the following more general lemma.

Lemma 10.4. *Let $Lag^{eq}(\mathbb{CP}^n)$ denote the space of oriented Lagrangian submanifolds Hamiltonian isotopic to $\mathbb{RP}^n$, then two loops $p_1, p_2 : S^1 \to Lag^{eq}(\mathbb{CP}^n)$ are taut concordant as defined in Section 1.3 iff they are homotopic.*

Proof. Let $\mathcal{L}$ be a sub-fibration of $Cyl \times M$ as in the definition of taut concordance of loops. Let $\mathcal{A}$ be any $PU(n)$ connection on $Cyl \times \mathbb{CP}^n$ which preserves $\mathcal{L}$ (there are no obstructions to constructing this). Then R_A is a lie $PU(n)$ valued 2-form, such that for all $v, w \in T_z Cyl$ the vector field $X = R_{\mathcal{A}}(z)(v,w)$ is tangent to $\mathcal{L}_z$. In particular if H_X is the Hamiltonian generating X, then since X is an infinitesimal unitary isometry preserving $\mathcal{L}_z$, H_X vanishes on $\mathcal{L}_z$. It follows by the definition of $\Omega_{\mathcal{A}}$, that it vanishes on $\mathcal{L}$ and so we are done. □

□

So given $\{\mathcal{A}_b\}$ as in the lemma above, since

$$\forall b : \operatorname{area}(\mathcal{A}_b) = L^+(f_0(b)),$$

we immediately deduce:

Lemma 10.5. *The function area : $b \mapsto \operatorname{area}(\mathcal{A}_b)$ has a unique maximizer, coinciding with the maximizer* max *of $f_0^* L_{met}$ and* area *is Morse at* max *with index* 2.

10.1.4. *Finding class A_0 holomorphic sections for the data.* Let us now rename f_0 by f, $\mathcal{L}_{f_0(b)}$ by $\mathcal{L}_b$, and Θ_{f_0} by $\Theta = \{\Theta_b\}$.

As $f(\max)$ is a geodesic for *met*, its lift $\widetilde{f}(\max)$ to $SO(3)$ is a rotation around an axis intersecting $L_0 = f(\max)(0)$ in a pair of points, in particular there there is a unique point

$$x_{\max} \in \bigcap_t (L_t = f(\max)(t))$$

maximizing $H_t^{\max}$ for each t. In our case this follows by elementary geometry but there is a more general phenomenon of this form c.f. [10].

Define

$$\sigma_{\max} : \mathcal{D} \to \mathcal{D} \times S^2$$

to be the constant section $z \mapsto x_{\max}$. Then $\sigma_{\max}$ is a $\mathcal{A}_{\max}$-flat section with boundary on $\mathcal{L}_{\max}$, and is consequently $J(\mathcal{A}_{\max})$-holomorphic.

Lemma 10.6.

$$[\sigma_{\max}] = A_0.$$

Proof. Set

$$T_z^{vert}\mathcal{L}_{\max} := \{v \in T\mathcal{L} \subset T_z(\mathcal{D} \times S^2) \,|\, pr_* v = 0\}$$

where $pr : \mathcal{D} \times S^2 \to \mathcal{D}$ is the projection. Denote by

$$Lag(T_{x_{\max}} S^2 \simeq Lag(\mathbb{R}^2) \simeq S^1$$

the space of oriented linear Lagrangian subspaces of $T_{x_{\max}} S^2$. Let ρ be the path in $Lag(T_{x_{\max}} S^2)$ defined by

$$\rho(t) = T^{vert}_{(\zeta(t), x_{\max})}\mathcal{L}_{\max}, \quad t \in [0,1]$$

where $\zeta : \mathbb{R} \to \partial\mathcal{D}$ is a fixed parametrization as in Definition 6.17.

By our conventions for the Hamiltonian vector field:

$$\omega(X_H, \cdot) = -dH(\cdot),$$

ρ is a clockwise oriented path from

$$T_{x_{\max}} L_0 := T^{vert}_{(\zeta(0), x_{\max})}\mathcal{L}_{\max}$$

to

$$T_{x_{\max}} L_1 := T^{vert}_{(\zeta(1), x_{\max})}\mathcal{L}_{\max}$$

for the orientation induced by the complex orientation on $T_{x_{\max}} S^2$.

By the Morse index theorem in Riemannian geometry [13] and by the condition that $f(\max)$ has Morse index 2, ρ visits initial point $\rho(0)$ exactly twice if we count the start, as this corresponds to the geodesic $f(\max)$ passing through two conjugate points in S^2. So the concatenation of ρ with the minimal counter-clockwise path from $T_{x_{\max}} L_1$ back to $T_{x_{\max}} L_0$ is a degree -1 loop, if $S^1 \simeq Lag(\mathbb{R}^2)$ is given the counter-clockwise orientation. Consequently

$$Maslov^{vert}(\sigma'_{\max}) = -2,$$

cf. Appendix B, in other words $[\sigma_{\max}] = A_0$. ☐

Proposition 10.7. $(\sigma_{\max}, \max)$ *is the sole element of* $\overline{\mathcal{M}}(\Theta, A_0)$.

Proof. By Stokes theorem, since ω vanishes on $\sigma_{\max}$, it is immediate:

$$carea(\Theta_{\max}, A_0) = -\int_{\mathcal{D}} \sigma^*_{\max} \widetilde{\Omega}_{\max} = L^+(f(\max)). \tag{10.2}$$

Moreover, since $\Theta = \{\Theta_b\}$ is taut $carea(\Theta_b, A_0) = L^+(f(\max))$. So by (10.1) and by Lemmas 6.10, 6.11 we have:

$$L^+(f(\max)) \leq \mathrm{area}(\mathcal{A}_b) = L^+(f(b)),$$

whenever there is an element

$$(\sigma, b) \in \overline{\mathcal{M}}(\{\Theta_b\}, A_0).$$

But clearly this is impossible unless $b = \max$, since $L^+(f(b)) < L^+(f(\max))$ for $b \neq \max$. So to finish the proof of the proposition we just need:

Lemma 10.8. *There are no elements σ other than $\sigma_{\max}$ of the moduli space*

$$\overline{\mathcal{M}}(\Theta_{\max}, A_0).$$

Proof. We have by (10.2), and by (10.1)

$$0 = \langle [\widetilde{\Omega}_{\max} + \alpha_{\widetilde{\Omega}_{\max}}], [\sigma_{\max}] \rangle,$$

and so given another element σ we have:

$$0 = \langle [\widetilde{\Omega}_{\max} + \alpha_{\widetilde{\Omega}_{\max}}], [\sigma] \rangle.$$

It follows that σ is necessarily $\widetilde{\Omega}_{\max}$-horizontal, since

$$(\widetilde{\Omega}_{\max} + \alpha_{\widetilde{\Omega}_{\max}})(v, J_{\widetilde{\Omega}_{\max}} v) \geq 0.$$

Since $J_{\widetilde{\Omega}_{\max}}$ by assumptions preserves the vertical and $\mathcal{A}_{\max}$-horizontal subspaces of $T(\mathcal{D} \times S^2)$, and since the inequality is strict for v in the vertical tangent bundle of

$$S^2 \hookrightarrow \mathcal{D} \times S^2 \to \mathcal{D},$$

the above inequality is strict whenever v is not horizontal. So σ must be $\mathcal{A}_{\max}$-horizontal. But then $\sigma = \sigma_{\max}$ since $\sigma_{\max}$ is the only flat section asymptotic to γ_0. □

□

10.1.5. *Regularity.* It will follow that

$$\Psi(a) = \pm[\gamma_0]$$

if we knew that $(\sigma_{\max}, \max)$ be a regular element of

$$\overline{\mathcal{M}}(\{\Theta_b\}, A_0).$$

We won't answer directly if $(\sigma_{\max}, \max)$ is regular, although it likely is. But it is regular after a suitably small Hamiltonian perturbation of the family $\{\mathcal{A}_r\}$ vanishing at $\mathcal{A}_{\max}$. We call this essentially automatic regularity.

Lemma 10.9. *There is a family $\{\mathcal{A}^{reg}_b\}$ arbitrarily C^∞-close to $\{\mathcal{A}_b\}$ with $\mathcal{A}^{reg}_{\max} = \mathcal{A}_{\max}$ and such that*

$$\overline{\mathcal{M}}(\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}^{reg}_b\}, A_0), \tag{10.3}$$

is regular, with $(\sigma_{\max}, \max)$ its sole element. In particular

$$\Psi(a) = \pm[\gamma_0].$$

Proof. The associated real linear Cauchy-Riemann operator

$$D_{\sigma_{\max}} : \Omega^0(\sigma_{\max}^* T^{vert}\mathcal{D} \times S^2_{\max}) \to \Omega^{0,1}(\sigma_{\max}^* T^{vert}\mathcal{D} \times S^2_{\max}),$$

has no kernel, by Riemann-Roch [12, Appendix C], as the vertical Maslov number of $[\sigma_{\max}]$ is -2. And the Fredholm index of $(\sigma_{\max}, \max)$ which is -2, is -1 times the Morse index of the function area at max, by Lemma 10.5. Given this, our lemma follows by a direct translation of [20, Theorem 1.20], itself elaborating on the argument in [17]. □

To summarize:

Theorem 10.10. *For* $0 \neq a \in H_2(\mathcal{P}_{L_0,L_1} Lag(S^2), \mathbb{Z})$,

$$0 \neq \Psi(a) \in HF(L_0, L_1).$$

Proof. We have shown that $0 \neq \Psi(a) \in HF(L_0, L_1)$, for a the generator of the group $H_2(\mathcal{P}_{L_0,L_1} Lag(S^2), \mathbb{Z})$. Since Ψ is an additive group homomorphism the conclusion follows. □

11. Finishing up the proof of Lemma 4.5

Starting with (8.6) we showed that $[\mu^4_{\Sigma^4}(\gamma_1, \ldots, \gamma_4)]$ is non-vanishing in Floer homology iff

$$[ev(\mathcal{H}_0, A_0)] \in HF(L_0, L_4),$$

is non-vanishing. We then use Lemma 8.2 to identify $[ev(\mathcal{H}_0, A_0)]$ with $[ev(\Theta', A_0)]$, which is also identified with $\Psi(a)$, for a certain spherical 2-class a. Finally, in Section 10 we compute $\Psi(a)$ and show that it is non-zero. This together with Lemma 5.1 imply Lemma 4.5. □

12. Proof of Theorem 1.8

Suppose otherwise, so that

$$\min_{f, [f]=a'} \max_{b \in S^2} L^+(f(b)) = U < \hbar,$$

for $a' = i_* g$ as in the statement of the theorem. Fix $L_1 \in Lag^{eq}(S^2)$ so that L_0 intersects L_1 transversally, and so that there is a geodesic path $p_0 \in \mathcal{P}Lag^{eq}(L_0, L_1)$ with

$$\kappa := L^{\pm}(\widetilde{p}_0) < \epsilon = (\hbar - U)/2.$$

Here $\widetilde{p}_0$ is the geodesic lift to $PU(2)$ starting at id. Then concatenating f with p_0 we obtain a smooth family of paths

$$g : S^2 \to \mathcal{P}(L_0, L_1)$$
$$g(0) = p_0,$$

and g represents the previously appearing class a, that is the generator of the corresponding π_2 group. Let

$$\{\Theta_b\} = \{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}_{\mathcal{K}=S^2},$$

be the corresponding Hamiltonian structure, where $\mathcal{A}_b = \mathcal{A}_0^{g(b)}$ is as in Lemma 10.3, for $\mathcal{A}_0$ the trivial connection, and where $\mathcal{L}_b := \mathcal{L}_{g(b)}$. In particular $\{\Theta_b\}$ is taut and satisfies:

$$\forall b \in S^2 : \text{area}(\mathcal{A}_b) = L^+(g(b)) < \hbar - \kappa. \tag{12.1}$$

By assumption that each $f(b)$ is taut concordant to the constant loop at L_0, each Θ_b is taut concordant to

$$\mathcal{H} = (\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_0, \mathcal{A}_0),$$

where $\mathcal{L}_0 = \mathcal{L}_{p_0}$, $\mathcal{A}_0 = \mathcal{A}_0^{p_0}$, where the latter is again as in Lemma 6.18. And so for each b $\Theta_b^{/}$ is taut concordant to $\mathcal{H}^{/}$ by Lemma 6.16.

On the other hand by Lemma 10.4 $\mathcal{H}^{/}$ is taut concordant to the trivial Hamiltonian structure $(D^2 \times S^2, D^2, \mathcal{L}_{tr}, \mathcal{A}_{tr})$, $\mathcal{L}_{tr}$ the trivial bundle with fiber L_0 and $\mathcal{A}_{tr}$ the trivial Hamiltonian connection. So for each b:

$$\operatorname{carea}(\Theta_b^{/}, A_0) = \operatorname{carea}(\mathcal{H}^{/}, A_0^{/}) = \hbar. \tag{12.2}$$

Now by Theorem 10.10

$$ev(\{\Theta_b\}, A_0) = \Psi(a) \neq 0.$$

And so:

$$\overline{\mathcal{M}}(\{\Theta_b\}, A_0) \neq \emptyset,$$

but this contradicts the conjunction of (12.1), (12.2), and Lemma 6.19.

□

13. Singular and simplicial connections and curvature bounds

Let $\mathcal{A}$ be a G connection on a principal G bundle $P \to \Delta^n$, and the Finsler norm $\mathfrak{n}$ on $\operatorname{lie} G$ be as in Section 1.1.1 of the introduction. A given system $\mathcal{U}$ in particular specifies maps:

$$u(m_1, \ldots, m_n, r, n) : \mathcal{S}_r \to \Delta^n,$$

where $r \in \overline{\mathcal{R}}_n$, $\mathcal{S}_r$ is the fiber of $\overline{\mathcal{S}}_n^{\circ}$ over r, and where $(m_1, \ldots, m_n)$ is the composable chain of morphisms in $\Pi(\Delta^n)$, m_i being the edge morphism from the vertex $i-1$ to i. Then define

$$\operatorname{area}_{\mathcal{U}}(\mathcal{A}) = \sup_r \operatorname{area}_{\mathfrak{n}}(u(m_1, \ldots, m_n, r, n)^* \mathcal{A}), \tag{13.1}$$

where $\operatorname{area}_{\mathfrak{n}}$ on the right hand side is as defined in equation (1.1). In the case $G = Ham(M, \omega)$ we take

$$\mathfrak{n} : \operatorname{lie} Ham(M, \omega) \to \mathbb{R}$$

to be

$$\mathfrak{n}(H) = |H|_+ = \max_M H.$$

Let ω be the area 1 Fubini-Study symplectic 2-form on $M = \mathbb{CP}^1$. Then the pull-back by the natural map

$$\operatorname{lie} h : \operatorname{lie} PU(2) \to \operatorname{lie} Ham(\mathbb{CP}^1, \omega) \simeq C_0^{\infty}(\mathbb{CP}^1)$$

of the semi-norm: $|H|_+ = \max_M H$ is the operator norm on $PU(2)$, up to normalization. This will be used to get the specific form of Theorem 1.4, from the more general form here.

13.1. **Simplicial connections.** We now introduce a certain abstraction of singular connections, which can partly be understood as simplicial resolutions of singular connections. Let $G \hookrightarrow P \to X$ be a principal G bundle, where G is a Frechet Lie group. Denote by $X_\bullet$ the simplicial set whose set of n-simplices, $X_\bullet(n)$, consists smooth maps $\Sigma : \Delta^n \to X$, with Δ^n standard topological n-simplex with vertices ordered $0, \ldots, n$. And denote by $Simp(X_\bullet)$ the category with objects $\cup_n X_\bullet(n)$ and with $hom(\Sigma_0, \Sigma_1)$ commutative diagrams:

$$\begin{tikzcd} \Delta^n \arrow[r, "mor"] \arrow[dr, "\Sigma_0"'] & \Delta^m \arrow[d, "\Sigma_1"] \\ & X, \end{tikzcd}$$

for mor a simplicial face map, that is an injective linear map preserving order of the vertices.

Definition 13.1. *Define a* **singular simplicial G-connection** $\mathcal{A}$ *or hereby just simplicial G-connection on P to be the following data:*

- *For each $\Sigma : \Delta^n \to X$ in $X_\bullet(n)$ a smooth G-connection $\mathcal{A}_\Sigma$ on $\Sigma^* P \to \Delta^n$, (a usual Ehresmann G-connection.)*
- *For a morphism $mor : \Sigma_0 \to \Sigma_1$ in $Simp(X_\bullet)$, we ask that $mor^* \mathcal{A}_{\Sigma_1} = \mathcal{A}_{\Sigma_0}$.*

Example 13.2. *If $\mathcal{A}$ if a smooth G-connection on P define a simplicial connection by $\mathcal{A}_\Sigma = \Sigma^* \mathcal{A}$ for every simplex $\Sigma \in X_\bullet$.*

If we try to "push forward" a simplicial connection to get a "classical" connection on P over X, then we get a kind of multi-valued singular connection. Multi-valued because each $x \in X$ may be in the image of a number of $\Sigma : \Delta^n \to X$ and Σ itself may not be injective, and singular because each Σ is in general singular so that the naive push-forward may have blow up singularities. We will call the above the naive pushforward of a simplicial connection.

Proof of Theorem 1.4. We will prove this by way of a stronger result. Let P be a Hamiltonian fibration $S^2 \hookrightarrow P \to S^4$. And let $\mathcal{A}$ a simplicial $Ham(S^2)$ connection on P. Let $\sigma^1_0 \in S^4_\bullet$ be the degenerate 1-simplex at x_0, in other words the constant map: $\sigma^1_0 : [0,1] \to x_0$. Let κ be the $L^\pm$ length of the holonomy path of $\mathcal{A}_{\sigma^1_0}$ over $[0,1]$.

Let $\Sigma_- \in X_\bullet(4)$ represent the generator of $\pi_4(S^4_\bullet, x_0)$. Let $\Sigma_+ : \Delta^4 \to S^4$ be the constant map to x_0. Using $\mathcal{A}$, perturbing if necessary, by inductive procedure as in Part I, we may find perturbation data $\mathcal{D}$ for P so that for this data

$$\forall r : pr_1 \mathcal{F}(L^0_0, \ldots, L^n_0, \Sigma_\pm, r) \simeq_\delta u(m_1, \ldots, m_s, r, n)^* \mathcal{A}_{\Sigma_\pm}, \tag{13.2}$$

$$\mathcal{A}(L_0, L_0) \simeq_\delta \mathcal{A}_{\sigma^1_0}, \tag{13.3}$$

where L^i_0 are the objects as before, where $\simeq_\delta$ means δ-close in the metrized C^∞ topology, and δ is as small as we like. Here we are using notation of Part I as before. Set

$$\widetilde{u}_r := \Sigma_- \circ u(m_1, \ldots, m_4, r, 4),$$

so $\widetilde{u}_r : \mathcal{S}_r \to S^4$. Set $\widetilde{\mathcal{S}}_r := \widetilde{u}^*_r P$, set $\mathcal{A}'_r := pr_1 \mathcal{F}(L^0_0, \ldots, L^n_0, \Sigma_-, r)$ and set

$$\{\Theta_r\} := \{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}'_r\}.$$

We suppose that $\mathcal{A}$ satisfies the condition that for every δ sufficiently small there exists $\mathcal{A}(L_0, L_0)$ as above so that the corresponding Floer chain complex is perfect. Let's call such an $\mathcal{A}$ ***perfect***.

Proposition 13.3. *Let $\mathcal{A}$ be a simplicial Hamiltonian connections on P, which is perfect as defined above, suppose that δ above is chosen to be sufficiently small then*

$$(\mathrm{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_+}) \geq \hbar - 5\kappa) \vee (\mathrm{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_-}) \geq \hbar - 5\kappa),$$

if P is non-trivial as Hamiltonian bundle.

Proof. Suppose

$$\mathrm{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_+}) < \hbar - 5\kappa, \tag{13.4}$$

then by (13.2), (13.2) and by Lemma 6.19 $\mathcal{D}$ as defined above can be assumed to be small provided δ is chosen to be sufficiently small. Take the unital replacement as in Lemma 5.1. Since we know that $K(P)$ does not admit a section by Theorem 3.4, the simplex T of the Lemma 5.1 does not exist. Hence again by this lemma

$$ev(\{\Theta_r\}, A_0) = [\mu^4_\Sigma(\gamma_1, \ldots, \gamma_4)] \neq 0.$$

In particular

$$\overline{\mathcal{M}}(\{\Theta_r\}, A_0) \neq \emptyset.$$

So by Lemma 6.19 there exists an r_0 so that

$$\mathrm{area}(\mathcal{A}'_{r_0}) \geq \hbar - 5\kappa'. \tag{13.5}$$

where κ' denotes the $L^\pm$ length of the holonomy path in $Ham(S^2)$ of $\mathcal{A}(L_0, L_0)$. By (13.3) $\kappa' \to \kappa$ as $\delta \to 0$. By (13.2), passing to the limit as $\delta \to 0$ we get:

$$\mathrm{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_-}) \geq \hbar - 5\kappa.$$

□

To finish the proof of the theorem note that a simplicial $PU(2)$ connection $\mathcal{A}$ on a principal $PU(2)$ bundle $PU(2) \hookrightarrow P' \to S^4$ is automatically perfect, when understood as a Hamiltonian connection on the associated bundle $S^2 \hookrightarrow P \to S^4$. So that the result follows by the proposition above. □

Appendix A. Homotopy groups of Kan complexes

For convenience let us quickly review Kan complexes just to set notation. This notation is also used in Part I. Let

$$\Delta^n_\bullet(k) := hom_\Delta([k], [n]),$$

be the standard representable n-simplex, where Δ is the simplicial category with objects ordered finite sets $[n] = \{1, \ldots, n\}$ and morphisms order preserving set maps.

Let $\Lambda^n_k \subset \Delta^n_\bullet$ denote the sub-simplicial set corresponding to the "boundary" of $\Delta^n_\bullet$ with the k'th face removed, $0 \leq k \leq n$. By $k'th$ face we mean the face opposite to the k'th vertex. A simplicial map

$$h : \Lambda^n_k \subset \Delta^n_\bullet \to X_\bullet$$

will be called a ***horn***. A simplicial set $S_\bullet$ is said to be a ***Kan complex*** if for all $n, k \in \mathbb{N}$ given a diagram with solid arrows

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "h"] \arrow[d, "i"] & S_\bullet \\ \Delta^n_\bullet \arrow[ur, dotted, "\widetilde{h}"] & \end{tikzcd},$$

there is a dotted arrow making the diagram commute. The map $\widetilde{h}$ will be called ***the Kan filling*** of the horn h. The k'th face of $\widetilde{h}$ will be called ***Kan filled face along*** h.

Given a pointed Kan complex $(X_\bullet, x)$ and $n \geq 1$ the *n'th simplicial homotopy group* of $(X_\bullet, x)$: $\pi_n(X_\bullet, x)$ is defined to be the set of equivalence classes of maps

$$\Sigma : \Delta^n_\bullet \to X_\bullet,$$

such that Σ takes $\partial \Delta^n_\bullet$ to $x_\bullet$, with the latter denoting the image of $\Delta^0_\bullet \to X$, induced by the vertex inclusion $x \to X_\bullet$.

More precisely, we have a commutative diagram:

$$\begin{tikzcd} \Delta^n_\bullet \arrow[r] \arrow[dr, "\Sigma"] & \Delta^0_\bullet \arrow[d, "x"] \\ & X_\bullet. \end{tikzcd}$$

Example A.1. *When $X_\bullet$ is the singular set associated to a topological space X, the maps above are in complete correspondence with maps:*

$$\Sigma : \Delta^n \to X,$$

taking the topological boundary of Δ^n to x.

For $X_\bullet$ general simplicial set, a pair of maps $\Sigma_1 : \Delta^n_\bullet \to X_\bullet, \Sigma_2 : \Delta^n_\bullet \to X_\bullet$, are equivalent if there is a diagram, called simplicial homotopy:

$$\begin{tikzcd} \Delta^n_\bullet \arrow[d, "i_0"] \arrow[dr, "\Sigma_1"] & \\ \Delta^n_\bullet \times I_\bullet \arrow[r] & X_\bullet \\ \Delta^n_\bullet. \arrow[u, "i_1"] \arrow[ur, "\Sigma_2"] & \end{tikzcd}$$

such that $\partial \Delta^n_\bullet \times I_\bullet$ is taken by H to $x_\bullet$. The simplicial homotopy groups of a Kan complex $(X_\bullet, x)$ coincide with the classical homotopy groups of the geometric realization $(|X_\bullet, x|)$. But the power of the above definition is that if we know our Kan complex well, (like in the example of the present paper) the simplicial homotopy groups are very computable since they are completely combinatorial in nature.

Proof of Lemma 3.2. We refer the reader to Part I, Appendix A.2, for more details on the notions here. We prove a stronger claim.

Lemma A.2. *Let $p : Y \to X$ be an inner fibration of simplicial sets Y, X, with X a Kan complex. And let $K(Y) \subset Y$ denote the maximal Kan subcomplex. Then $p : K(Y) \to X$ is a Kan fibration.*

The above is surely well known, but it is simple to just provide the proof for convenience.

Proof. Thus, whenever we are given a commutative diagram with solid arrows:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, hook] & K(Y) \arrow[r, hook] & Y \arrow[dl, "p"] \\ \Delta^n \arrow[r] \arrow[urr, dashed, "\Sigma"] & X, & \end{tikzcd} \tag{A.1}$$

there exists a dashed arrow Σ as indicated, making the whole diagram commutative. On the other hand the edges of Σ are all invertible in Y, as Σ extends σ, and all edges of σ are invertible by definition. It follows that Σ maps into $K(Y) \subset Y$. Since the starting diagram was arbitrary, we just proved that $p : K(Y) \to X$ is an inner fibration. In particular the pre-images $p^{-1}(\Sigma(\Delta^n)) \subset K(Y)$ are quasi-categories, for all n, where $\Sigma : \Delta^n \to X$ is any n-simplex, see Part I, Appendix A.2. But $K(Y)$ is a Kan complex, so that also the above pre-images $p^{-1}(\Sigma(\Delta^n))$ are Kan complexes. It readily follows from this that $p : K(Y) \to X$ is a Kan fibration. □

The main lemma then follows, since if $p : Y \to X$ is a (co)-Cartesian fibration, it is in particular an inner fibration. □

Appendix B. On the Maslov number

Let S be obtained from a compact connected Riemann surface S' with boundary, by removing a finite number of points $\{e_i\}$ removed from the boundary of S'.

Let $\mathcal{V} \to S$ be a rank r complex vector bundle, trivialized at the open ends $\{e_i\}$, that is so we have distinguished bundle charts $\mathbb{C}^r \times [0,1] \times [0,\infty) \to \mathcal{V}$ at the ends.

Let

$$\Xi \to \partial S \subset S$$

be a totally real rank r subbundle of $\mathcal{V}$, which is constant in the coordinates

$$\mathbb{C}^r \times [0,1] \times [0,\infty),$$

at the ends. For each end e_i and its distinguished chart $e_i : [0,1] \times [0,\infty) \to S$ let $b_i^j : [0,\infty) \to \partial S$, $j = 0,1$ be the restrictions of e_i to $\{i\} \times [0,\infty)$.

We then have a pair of real vector spaces

$$\Xi_i^j = \lim_{\tau \mapsto \infty} \Xi|_{b_i^j(\tau)}.$$

There is a Maslov number $Maslov(\mathcal{V}, \Xi, \{\Xi_i^j\})$ associated to this data coinciding with the boundary Maslov index in the sense of [12, Appendix C3], in the case $\Xi_i^0 = \Xi_i^1$, for the modified pair $(\mathcal{V}^/, \Xi^/)$ obtained from $(\mathcal{V}, \Xi, \{\Xi_i^j\})$ by naturally closing off each e_i end of $\mathcal{V} \to S$.

When Ξ_i^0 is transverse to Ξ_i^1 for each i, $Maslov(\mathcal{V}, \Xi, \{\Xi_i^j\})$ is obtained as the Maslov index for the modified pair $(\mathcal{V}^/, \Xi^/)$ by again closing off the ends e_i via gluing (at each end e_i) with

$$(\mathbb{C}^r \times \mathcal{D}, \widetilde{\Xi}, \{\widetilde{\Xi}_0^j\}),$$

where $\mathcal{D}$ as before is diffeomorphic to D^2 with a point e_0 on the boundary removed. Here $\widetilde{\Xi}^0_i = \Xi^1_0$ and $\widetilde{\Xi}^1_i = \Xi^0_0$, while $\widetilde{\Xi}$ over the boundary of $\mathcal{D}$ is determined by the "shortest path" from $\widetilde{\Xi}^0_0$ to $\widetilde{\Xi}^1_0$, meaning that since these are a pair of transverse totally real subspaces up to a complex isomorphism of $\mathbb{C}^r$ (whose choice will not matter) we may identify them with the subspaces $\mathbb{R}^r$, and $i\mathbb{R}^r$ after this identification our shortest path is just $e^{i\theta}\mathbb{R}^r$, $\theta \in [0, \pi_2]$.

For a real linear Cauchy-Riemann operator on $\mathcal{V}$, with suitable asymptotics, the Fredholm index is given by:

$$r \cdot \chi(S) + Maslov(\mathcal{V}, \Xi, \{\Xi_i\}).$$

The proof of this is analogous to [12, Appendix C], we can also reduce it to that statement via a gluing argument. (This kind of argument appears for instance in [22])

B.1. **Dimension formula for moduli space of sections.** Suppose we are given a Hamiltonian fiber bundle $M^{2r} \hookrightarrow \widetilde{S} \to S$, with end structure and S as above. Let $\mathcal{L}$ be a Lagrangian sub-bundle of $\widetilde{S}$ over the boundary of S, compatible with the end structure, and such that the Lagrangian submanifolds

$$L^j_i = \lim_{\tau \mapsto \infty} \mathcal{L}|_{b^j_i(\tau)},$$

intersect transversally (identifying the corresponding fibers) or coincide.

Given an $\mathcal{L}$-exact Hamiltonian connection $\mathcal{A}$, on $\widetilde{S}$, (see Definition 6.2) which is additionally assumed to be trivial in the strip coordinate charts at the ends, and a choice of a family $\{j_z\}$ of compatible almost complex structures on the fibers of $\widetilde{S}$, set $\mathcal{M}(A)$ to be the moduli space of (relative) class A finite vertical L^2 energy holomorphic sections of $\widetilde{S} \to S$ with boundary in $\mathcal{L}$. Define

$$Maslov^{vert}(A)$$

to be the Maslov number of the triple $(\mathcal{V}, \Xi, \{\Xi_i\})$ determined by the pullback by $\sigma \in \mathcal{M}(A)$ of the vertical tangent bundle of $\widetilde{S}$, $\mathcal{L}$. Then the expected dimension of $\mathcal{M}(A)$ is:

$$r \cdot \chi(S) + Maslov^{vert}(A). \tag{B.1}$$

References

[1] M. Akveld and D. Salamon, *Loops of Lagrangian submanifolds and pseudoholomorphic discs.*, Geom. Funct. Anal., 11 (2001), pp. 609–650.

[2] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces.*, Philos. Trans. R. Soc. Lond., A, 308 (1983), pp. 523–615.

[3] P. Biran and O. Cornea, *A Lagrangian quantum homology*, (2009), pp. 1–44.

[4] ———, *Lagrangian cobordism and Fukaya categories*, Geom. Funct. Anal., 24 (2014), pp. 1731–1830.

[5] O. Cornea and P. Biran, *Lagrangian Cobordism*, arXiv:1109.4984.

[6] V. Guillemin, E. Lerman, and S. Sternberg, *Symplectic fibrations and multiplicity diagrams*, Cambridge University Press, Cambridge, 1996.

[7] R. Harvey and H. B. jun. Lawson, *A theory of characteristic currents associated with a singular connection*, Bull. Am. Math. Soc., New Ser., 31 (1994), pp. 54–63.

[8] A. T. Hatcher, *Algebraic topology.*, Cambridge: Cambridge University Press, 2002.

[9] S. Hu and F. Lalonde, *A relative Seidel morphism and the Albers map.*, Trans. Am. Math. Soc., 362 (2010), pp. 1135–1168.

[10] H. Iriyeh and T. Otofuji, *Geodesics of Hofer's metric on the space of Lagrangian submanifolds.*, Manuscr. Math., 122 (2007), pp. 391–406.

[11] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford Math. Monographs, The Clarendon Oxford University Press, New York, second ed., 1998.

[12] ———, *J–holomorphic curves and symplectic topology*, no. 52 in American Math. Society Colloquium Publ., Amer. Math. Soc., 2004.

[13] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963.

[14] F. Naeimipour, B. Mirza, and F. Jahromi, *Yang–mills black holes in quasitopological gravity*, The European Physical Journal C, 81 (2021).

[15] I. R. Rauch, *On The Hofer girth of the sphere of great circles*, https://arxiv.org/pdf/2009.05256.pdf.

[16] Y. Savelyev, *Quantum characteristic classes and the Hofer metric.*, Geom. Topol., 12 (2008), pp. 2277–2326.

[17] ———, *Virtual Morse theory on* $\Omega Ham(M, \omega)$., J. Differ. Geom., 84 (2010), pp. 409–425.

[18] ———, *Bott periodicity and stable quantum classes.*, Sel. Math., New Ser., 19 (2013), pp. 439–460.

[19] ———, *Global Fukaya category I: quantum Novikov conjecture*, arXiv:1307.3991, (2014).

[20] ———, *Morse theory for the Hofer length functional*, Journal of Topology and Analysis, 06, Issue No. 2 (2014).

[21] ———, *Yang-Mills theory and jumping curves*, International J. of Math., 26:5 (2015).

[22] P. Seidel, *Fukaya categories and Picard-Lefschetz theory.*, Zurich Lectures in Advanced Mathematics. Zürich: European Mathematical Society (EMS). vi, 326 p. EUR 46.00 , 2008.

[23] L. M. Sibner and R. J. Sibner, *Classification of singular Sobolev connections by their holonomy*, Commun. Math. Phys., 144 (1992), pp. 337–350.

University of Colima, CUICBAS

Email address: yasha.savelyev@gmail.com