# CONFORMAL SYMPLECTIC WEINSTEIN CONJECTURE AND NON-SQUEEZING

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ABSTRACT. We initiate here, from the Gromov-Witten theory point of view, the study of aspects of rigidity of locally conformally symplectic manifolds, or lcs manifolds for short, which are a natural generalization of both contact and symplectic manifolds. As a first step we show that the classical Gromov non-squeezing theorem has a certain $C^0$ rigidity or persistence with respect to lcs deformations. This is one version of lcs non-squeezing, another possible version of non-squeezing related to contact non-squeezing is also discussed. In a different direction we study Gromov-Witten theory of the lcs manifold $C \times S^1$ induced by a contact form $\lambda$ on $C$, and show that the extended Gromov-Witten invariant counting certain elliptic curves in $C \times S^1$ is identified with the extended classical Fuller index of the Reeb vector field $R^\lambda$, by extended we mean that these invariants can be $\pm\infty$-valued. Partly inspired by this, we conjecture existence of certain 2-dimensional curves we call Reeb curves in certain lcs manifolds, which we call conformal symplectic Weinstein conjecture, and this is a direct extension of the classical Weinstein conjecture. Also using Gromov-Witten theory, we show that the CSW conjecture holds for a $C^0$- neighborhood of the Banyaga lcs form on $C \times S^1$, for $C$ a contact three fold with contact form whose Reeb flow has non-zero extended Fuller index, e.g. $S^3$ with standard contact form. We show that this can be globalized provided there do not exist sky catastrophes for families of holomorphic curves in a lcs manifold. The latter phenomenon is analogous to sky catastrophes in dynamical systems discovered by Fuller.

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## 1. Introduction

The theory of pseudo-holomorphic curves in symplectic manifolds as initiated by Gromov and Floer has revolutionized the study of symplectic and contact manifolds. What the symplectic form gives that is missing for a general almost complex manifold is a priori energy bounds for pseudo-holomorphic curves a fixed class. On the other hand there is a natural structure which directly generalizes both symplectic and contact manifolds, called locally conformally symplectic structure or lcs structure for short. A locally conformally symplectic manifold or lcsm is a smooth $2n$-fold $M$ with an lcs structure: which is a non-degenerate 2-form $\omega$, which is locally diffeomorphic to $f \cdot \omega_{st}$, for some (non-fixed) positive smooth function $f$, with $\omega_{st}$ the standard symplectic form on $\mathbb{R}^{2n}$. It is natural to try to do Gromov-Witten theory for such manifolds. The first problem that occurs is that a priori energy bounds are gone, as since $\omega$ is not necessarily closed, the $L^2$-energy can now be unbounded on the moduli spaces of $J$-holomorphic curves in such a $(M, \omega)$. Strangely a more acute problem is potential presence of holomorphic sky catastrophes - given a smooth family $\{J_t\}$, $t \in [0,1]$, of $\{\omega_t\}$-compatible almost complex structures, we may have a continuous family $\{u_t\}$ of $J_t$-holomorphic curves s.t. energy$(u_t) \mapsto \infty$ as $t \mapsto a \in (0,1)$ and s.t. there are no holomorphic curves for $t \geq a$. These are analogues of sky catastrophes discovered by Fuller [10] for closed orbits of dynamical systems.

We can tame these problems in certain situations and this is how we arrive at a certain lcs extension of Gromov non-squeezing. Even when it is impossible to tame these problems we show that there can still be an extended Gromov-Witten type theory which is analogous to the theory of extended Fuller index in dynamical systems, [21]. In a very particular situation the relationship with the Fuller index becomes perfect as one of the results of this paper obtains the (extended) Fuller index for Reeb vector fields on a contact manifold $C$ as a certain (extended) genus 1 Gromov-Witten invariant of the Banyaga lcsm $C \times S^1$, see Example 1. The latter also gives a conceptual interpretation for why the Fuller index is rational, as it is reinterpreted as an (virtual) orbifold Euler number.

Inspired by this, we conjecture that certain lcsm's must poses certain 2-d curves that we call Reeb curves, and this is a direct generalization of the Weinstein conjecture, we may call this conformal symplectic Weinstein conjecture. We prove this CSW conjecture for certain lcs structures $C^0$ nearby to Banyaga type lcs structures on $C \times S^1$, for $C$ a three-fold. This partly uses the above mentioned connection of Gromov-Witten theory of $C \times S^1$ with the classical Fuller index. Note that Seifert [23] was likewise initially motivated by a $C^0$ neighborhood version of the Seifert conjecture for $S^{2k+1}$, which he proved. We could say that in our case there is more evidence for globalizing, since the original Weinstein conjecture is already proved, Taubes [27], for $C$ a closed contact three-fold. In addition to the $C^0$ neighborhood version, we also prove a stronger result that relates the CSW conjecture to existence of holomorphic sky catastrophes.

Finally, we should exclaim that the Gromov-Witten theory in this story plays a local role, unless addition global geometric control is obtained. This is analogous to what happens with Fuller index in dynamical systems. A global lcs invariant, which takes the form of a homology theory, is under development by the author, but many ingredients for this are already present here.

### 1.1. **Locally conformally symplectic manifolds.**

Let us give a bit of background on lcsm's. These were originally considered by Lee in [14], arising naturally as part of an abstract study of "a kind of even dimensional Riemannian geometry", and then further studied by a number of authors see for instance, [1] and [28]. This is a fascinating object, a lcsm admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and Murphy show that if a closed almost complex $2n$-fold $M$ has $H^1(M, \mathbb{R}) \neq 0$ then it admits a lcs structure, [5], see also [2].

To see the connection with the first cohomology group, let us point out right away the most basic invariant of a lcs structure $\omega$: the Lee class, $\alpha = \alpha_\omega \in H^1(M, \mathbb{R})$. This has the property that on the associated $\alpha$-covering space $\widetilde{M}$, $\widetilde{\omega}$ is globally conformally symplectic. The class $\alpha$ may be defined as the following Cech 1-cocycle. Let $\phi_{a,b}$ be the transition map for lcs charts $\phi_a, \phi_b$ of $(M, \omega)$. Then $\phi_{a,b}^* \omega_{st} = g_{a,b} \cdot \omega_{st}$ for a positive real constant $g_{a,b}$ and $\{\ln g_{a,b}\}$ gives our 1-cocycle. Thus an lcs form is globally conformally symplectic iff its Lee class vanishes.

The Lee class $\alpha$ has a natural differential form representative, called the Lee form and defined as follows. We take a cover of $M$ by open sets $U_a$ in which $\omega = f_a \cdot \omega_a$ for $\omega_a$ symplectic, and $f_a$ a positive smooth function. Then we have 1-forms $d(\ln f_a)$ in each $U_a$ which glue to a well defined closed 1-form on $M$. By slight abuse, we denote this 1-form, its cohomology class and the Cech 1-cocycle from before all by $\alpha$. It is moreover immediate that for an lcs form $\omega$

$$d\omega = \alpha \wedge \omega.$$

As we mentioned lcsm's can also be understood to generalize contact manifolds. This works as follows. First we have a natural class of explicit examples of lcsm's, obtained by starting with a symplectic cobordism (see [5]) of a closed contact manifold $C$ to itself, arranging for the contact forms at the two ends of the cobordism to be proportional (which can always be done) and then gluing together the boundary components. As a particular case of this we get Banyaga's basic example.

*Example* 1 (Banyaga). Let $(C, \xi)$ be a contact manifold with a contact form $\lambda$ and take $M = C \times S^1$ with 2-form $\omega = d^\alpha \lambda := d\lambda - \alpha \wedge \lambda$, for $\alpha$ the pull-back of the volume form on $S^1$ to $C \times S^1$ under the projection, and $\lambda$ likewise the pull-back of $\lambda$ by the projection $C \times S^1 \to C$.

The operator $d^\alpha : \Omega^k(M) \to \Omega^{k+1}(M)$ is called the Lichnerowicz differential with respect to a closed 1-form $\alpha$, and satisfies $d^\alpha \circ d^\alpha = 0$ so that we have an associated Lichnerowicz complex.

Using above we may then faithfully embed the category of contact manifolds, and contactomorphism into the category of lcsm's, and certain lcs morphisms as defined below.

**Definition 1.1.** *A diffeomorphism $\phi : (M_0, \omega_0) \to (M_1, \omega_1)$ is said to be an* **lcs map** *if $\phi^* \omega_1$ is homotopic through lcs forms $\{\omega_t\}$, in the same $d^\alpha$ Lichnerowicz cohomology class, to $\omega_0$, where $\alpha$ is the Lee form of $\omega_0$ as before. In other words, for each $t_0 \in [0, 1]$,*

$$d^\alpha (\frac{d}{dt}|_{t=t_0} \omega_t) = 0.$$

We also define, following Banyaga, ***conformal symplectomorphisms*** $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ to be diffeomorphisms satisfying $\phi^* \omega_2 = f \omega_1$ for a smooth positive function $f$.

1.2. **Conformal symplectic Weinstein conjecture.** We state this conjecture immediately and then motivate it. An ***exact lcs structure*** on $M$ is a pair $(\lambda, \alpha)$ with $\alpha$ a closed 1-form, s.t. $d^\alpha \lambda$ is non-degenerate.

**Definition 1.2.** *Let $(M, \lambda, \alpha)$ be an exact* lcs *structure. For $\Sigma$ a closed (at the moment possibly nodal) Riemann surface, we say that a smooth map $u : \Sigma \to M$ is a* **Reeb curve** *if it a branched cover of a smoothly immersed curve $\Sigma' \to M$, if $u^* d\lambda = 0$ and if*

$$0 \neq [u^* \alpha] \in H^1_{DR}(\Sigma).$$

We then have the following "conformal symplectic Weinstein conjecture".

**Conjecture 1.** *Let $M$ be closed, and $\omega$ be an exact* lcs *form on $M$, so that the cohomology class of the Lee form $\alpha$ of $\omega$ is rational, then there is an elliptic Reeb curve $u : \Sigma \to M$, meaning that the domain $\Sigma$ is an elliptic curve* [1].

The following is proved in Section 4.

**Theorem 1.3.** *Conjecture 1 implies the Weinstein conjecture.*

In what follows we use the following $C^0$ metric on the space $\mathcal{L}(M)$ of exact lcs structures on $M$. For $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in \mathcal{L}(M)$ define:

$$d((\lambda_1, \alpha_1), (\lambda_2, \alpha_2)) = d_{mass}(\lambda_1, \lambda_2) + d_{mass}(\alpha_1, \alpha_2),$$

where $d_{mass}$ is the usual co-mass distance as defined in Section 3. We say that an exact lcs structure $(M^{2n}, \lambda, \alpha)$ is ***regular*** if the set:

$$V(M, \lambda) := \{p \in M | (d\lambda)^n(p) = 0\},$$

[1] One may further conjecture that it must be non-nodal.

is a smooth submanifold of $M$. A ***regular neighborhood*** of an lcs structure is then a neighborhood with respect to $d$ intersected with the subset of all regular lcs structures.

**Theorem 1.4.** *Conjecture 1 holds for a regular neighborhood of the Hopf* lcs *structure $\omega_H$ on $C^3 \times S^1$.*

This is proved in Section 4. Note that Seifert [23] initially found a similar existence phenomenon of orbits on $S^{2k+1}$ for a vector field $C^0$-nearby to the Hopf vector field. And he asked if the nearby condition can be removed, this became known as the Seifert conjecture. This turned out not to be quite true [13]. Likewise it is natural for us to conjecture that the nearby condition can be removed and this is the CSW conjecture. In our case this has some additional evidence that we discuss below.

Directly extending Theorem 1.4 we have the following.

**Theorem 1.5.** *Let $C$ be a closed 3-fold. Let $(\lambda, \alpha)$ be a Banyaga lcs structure on $M = C \times S^1$, with $i(R^\lambda, \beta) \neq 0$, for some $\beta$, where the latter is the extended Fuller index, as described in Appendix A. Then either the conformal symplectic Weinstein conjecture holds for any regular exact* lcs *structure $(\lambda', \alpha')$ on $M$, so that $\omega_1 = d^{\alpha'}\lambda'$ is homotopic through (general)* lcs *forms $\{\omega_t\}$ to $\omega_0 = d^\alpha \lambda$ or holomorphic sky catastrophes exist.*

*Example* 2. Take $C = S^3$ and $\lambda = \lambda_H$, then $i(R^\lambda, 0) = \infty$, [21]. Or take $C$ to be unit cotangent bundle of a hyperbolic surface $(\Sigma, g)$, $\lambda$ the associated Louiville form, and $(\lambda, \alpha)$ the associated Banyaga lcs structure, in this case $i(R^\lambda, \beta) = 1$ for every $\beta \neq 0$.

To motivate the above conjecture we need pseudo-holomorphic curves in lcs manifolds.

1.2.1. *Pseudo-holomorphic curves in the* lcsm $C \times S^1$. Banyaga type lcsm's give immediate examples of almost complex manifolds where the energy function is unbounded on the moduli spaces of fixed class pseudo-holomorphic curves, as well as where null-homologous $J$-holomorphic curves can be non-constant. We are going to see this shortly.

Let $(C, \lambda)$ be a closed contact $(2n+1)$-fold with a contact form $\lambda$, that is a 1-form satisfying $\lambda \wedge d\lambda^n \neq 0$. The Reeb vector field $R^\lambda$ on $C$ is a vector field satisfying $d\lambda(R^\lambda, \cdot) = 0$ and $\lambda(R^\lambda) = 1$. We also denote by $\lambda$ the pull-back of $\lambda$ by the projection $C \times S^1 \to C$, and by $\xi \subset T(C \times S^1)$ the distribution $\xi(p) = \ker d\lambda(p)$.

Identifying $T = S^1 = \mathbb{R}/\mathbb{Z}$, $T$ acts on $C \times S^1$ by $s \cdot (x, \theta) = (x, \theta + s)$. We take $J$ to be an almost complex structure on $\xi$, which is $T$ invariant, and compatible with $d\lambda$. The latter means that $g_J(\cdot, \cdot) := d\lambda|_\xi(\cdot, J\cdot)$ is a $J$ invariant Riemannian metric on the distribution $\xi$.

There is an induced almost complex structure $J^\lambda$ on $C \times S^1$, which is $T$-invariant, coincides with $J$ on $\xi$ and which satisfies:

$$J^\lambda(R^\lambda \oplus \{0\}(p)) = \frac{d}{d\theta}(p),$$

where $R^\lambda \oplus \{0\}$ is the section of $T(C \times S^1) \simeq TC \oplus \mathbb{R}$, corresponding to $R^\lambda$, and where $\frac{d}{d\theta} \subset \{0\} \oplus TS^1 \subset C \times S^1$ denotes the vector field generating the action of $T = \mathbb{R}/\mathbb{Z}$ on $C \times S^1$.

We now consider a certain moduli space of holomorphic tori in $C \times S^1$, which have a certain charge, to be defined. The latter notion is motivated by work of Yong-Geun Oh (private communication). Partly the reason for introduction of "charge" is that it is now possible for non-constant holomorphic curves to be null-homologous. Here is a simple example take $S^3 \times S^1$ with $J = J^\lambda$, for the $\lambda$ the standard contact form, then all the Reeb tori are null-homologous. In many cases we can just work with homology classes $A \neq 0$, but this is inadequate for our setup for conformal symplectic Weinstein conjecture.

Let $\Sigma$ be a complex torus with chosen generators $\gamma, \rho \in H_1(\Sigma, \mathbb{Z})$, so that $\gamma \cdot \rho = 1$, and a chosen marked point $z \in \Sigma$. An isomorphism $\phi : (\Sigma_1, \gamma_1, \rho_1, z_1) \to (\Sigma_2, \gamma_2, \rho_2, z_2)$ is a biholomorphism s.t. $\phi^* \gamma_2 = \gamma_1$, $\phi^* \rho_2 = \rho_1$, and s.t. $\phi(z_1) = z_2$. We will be denote by $\overline{M}^c_{1,1}$ the Deligne-Mumford compactification of the space of isomorphism classes of such structures. Here, the superscript $c$ distinguishes this space from the classical Deligne-Mumford orbifold $\overline{M}_{1,1}$.

Suppose then $(M, \omega)$ is an lcs manifold, $J$ $\omega$-compatible almost complex structure, and $\alpha$ the Lee class corresponding to $\omega$. Assuming for simplicity (otherwise take stable maps) that $(M, J)$ does not admit non-constant $J$-holomorphic maps $(S^2, j) \to (M, J)$, that is rational curves, we define $\overline{\mathcal{M}}_{1,1}^0(J, A)$ as a set of equivalence classes of tuples $(u, S)$, for $S = (\Sigma, \gamma, \rho, z) \in \overline{M}_{1,1}^c$, and $u : \Sigma \to M$ a $J$-holomorphic map satisfying the ***charge (1,0) condition***:

$$\langle \alpha, u_* \rho \rangle = 1$$
$$\langle \alpha, u_* \gamma \rangle = 0,$$

and with $[u] = A$. It is easy to see that the charge condition is preserved under Gromov convergence, and of course a charge (1,0) $J$-holomorphic map cannot be constant for any $A$.

By slight abuse we may just denote such an equivalence class above by $u$, so we may write $u \in \overline{\mathcal{M}}_{1,1}^0(J, A)$, with $S$ implicit.

1.2.2. *Reeb tori.* For the almost complex structure $J^\lambda$ as above we have one natural class of charge (1,0) holomorphic tori in $C \times S^1$ that we call *Reeb tori*. Let $o$ be a period $c$ Reeb orbit $o$ of $R^\lambda$, that is a map:

$$o : S^1 \to C,$$
$$D_s o(s_0) = c \cdot R^\lambda(o(s_0)),$$

for $c > 0$, and $\forall s_0 \in S^1$. A Reeb torus $u_o$ for $o$, is the map

$$u_o(s, t) = (o(s), t),$$

$s, t \in S^1 := \mathbb{R}/\mathbb{Z}$. A Reeb torus is $J^\lambda$-holomorphic for a uniquely determined holomorphic structure $j$ on $T^2$ defined by:

$$j(\frac{\partial}{\partial s}) = c\frac{\partial}{\partial t}.$$

So if $S(\lambda)$ denotes the space of general period $\lambda$-Reeb orbits, we have a map:

$$R : S(\lambda) \to \overline{\mathcal{M}}_{1,1}^0(J^\lambda, A), \quad R(o) = u_o.$$

**Proposition 1.6.** *The map $R$ is a bijection.*

Note that the expected dimension of $\overline{\mathcal{M}}_{1,1}^0(J^\lambda, A)$ is 0. It is given by the Fredholm index of the operator (4.2) which is 2, minus the dimension of the reparametrization group (for non-nodal curves) which is 2. That is given $S = (\Sigma, \gamma, \rho, z) \in \overline{M}_{1,1}^c$, if $\Sigma$ is non-nodal then the group of $\mathcal{G}(S)$ of biholomorphisms $\phi$ of $S$ satisfying the condition $\phi^* \gamma = \gamma$, $\phi^* \rho = \rho$ is isomorphic to $T^2$. And given a $J$-holomorphic map $u : \Sigma \to M$, $(\Sigma, \gamma, \rho, z, u)$ is equivalent to $(\Sigma, \gamma, \rho, \phi(z), u \circ \phi)$ in $\overline{\mathcal{M}}_{1,1}^0(J^\lambda, A)$, for $\phi \in \mathcal{G}(S)$.

In Theorem 4.5 we relate the (extended) count (Gromov-Witten invariant) of these curves to the (extended) Fuller index, which is reviewed in the Appendix A. This will be one ingredient for the following.

**Definition 1.7.** *Given $(\omega, J)$ with $\omega$ an* lcs *form on $M$ and $J$ an almost complex structure compatible with $\omega$, (as previously defined) we call this a* **lcs pair**.

For $\lambda_H$ the standard contact structure on $S^{2k+1}$, so that its Reeb flow is the Hopf flow, we will call $\omega_H := d^\alpha \lambda_H$ the ***Hopf*** lcs ***structure***.

**Theorem 1.8.** *Let $M = S^{2k+1} \times S^1$, $d^\alpha \lambda_H$ the Hopf* lcs *structure. Then there exists a $\delta > 0$ s.t. for any* lcs *pair $(\omega, J)$ on $M$ $C^0$ $\delta$-close to $(d^\alpha \lambda_H, J^{\lambda_H})$, there exists an elliptic, charge (1,0), $J$-holomorphic curve in $S^{2k+1} \times S^1$. Moreover, if $k = 1$, then this curve may be assumed to be embedded.*

**Definition 1.9.** *Let $(M, \lambda, \alpha)$ be an exact lcs structure, $\omega = d^\alpha \lambda$. We say that an $\omega$-compatible $J$ is* **admissible** *if it preserves the generalized distribution $\mathcal{V}_\lambda$:*

$$\mathcal{V}_\lambda(p) = \{v \in T_p M | d\lambda(v, \cdot) = 0\},$$

*and the generalized distribution $\xi$, which is defined to be the $\omega$-orthogonal complement to $\mathcal{V}_\lambda$. We call $(M, \lambda, \alpha, J)$ as above a* **tamed exact** lcs **structure**.

For each $p \in M$ $\mathcal{V}_\lambda(p)$ has dimension at most 2 since $d\lambda - \alpha \wedge \lambda$ is non-degenerate. Moreover $\mathcal{V}_\lambda$ cannot identically vanish, since $M^{2n}$ is closed and $(d\lambda)^n$ cannot be non-degenerate by Stokes theorem.

The significance of an admissible almost complex structure is the following.

**Lemma 1.10.** *Let $(M, \lambda, \alpha, J)$ be a tamed exact* lcs *structure. Then given a smooth $u : \Sigma \to M$, where $\Sigma$ is a closed (nodal) Riemann surface, $u$ is $J$-holomorphic only if $u^* d\lambda = 0$.*

*Proof.* We have

$$I = \int_\Sigma u^* d\lambda \geq 0$$

since $J$ preserves $\mathcal{V}_\lambda$. On the other hand $I > 0$ is impossible by Stokes theorem. So $I = 0$. Since $J$ also preserves $\xi$, this can happen only if

$$\operatorname{image} du(z) \subset \mathcal{V}_\lambda(u(z))$$

for all $z \in \Sigma$. From this our conclusion follows. □

This can be understood as a generalization of the condition of being a Reeb torus.

**Lemma 1.11.** *Let $(M, \lambda, \alpha, J)$ be a tamed exact* lcs *structure, then every immersed $J$-holomorphic curve $u : \Sigma \to M$ is Reeb.*

This almost follows by the above discussion, except for the condition $0 \neq [u^*\alpha] \in H^1_{DR}(\Sigma)$ that needs to be verified. This is to be proved in Section 4. This gives further evidence to the CSW conjecture. For if $\omega = d^\alpha \lambda$ for $\lambda$ the contact form inducing the standard contact structure on $S^{2k+1}$, or any contact form on a threefold, and $J = J^\lambda$ then we know there are immersed $J$-holomorphic tori, since we know there are $\lambda$-Reeb orbits, as the Weinstein conjecture is known to hold in these cases, [29], [27] and hence there are elliptic Reeb curves, by Lemma 1.11.

**Conjecture 2.** *Suppose we are given a tamed exact* lcs *structure $(M, \lambda, \alpha, J)$, with $M$ a closed 4-fold, and so that the cohomology class of the Lee form $\alpha$ of $\omega$ is rational. Then there is an immersed $J$-holomorphic elliptic curve in $M$.*

This immediately implies the CSW conjecture by Proposition 1.6 and by Lemma 1.11. Conjecture 2 is probably too much to hope for in such generality but when $M$ has dimension 4, this looks very close to fundamental results of Taubes [26] on Gromov-Witten theory of symplectic 4-folds.

Conjecture 2 also immediately implies the Weinstein conjecture for a closed contact 3-fold $(C, \lambda)$. For by the proof of Proposition 1.6, any non-constant elliptic curve $u : \Sigma \to M = C \times S^1$, with respect to the Banyaga lcs structure $d^\alpha \lambda$, $\alpha = d\theta$, must cover a Reeb torus.

The above conjectures are not just a curiosity. In contact geometry, rigidity is based on existence phenomena of Reeb orbits, and lcs manifolds should be understood as generalized contact manifolds. To attack rigidity questions in lcs geometry, like Question 2 we need an analogue of Reeb orbits, we propose that this analogue is Reeb curves as above, from which point of view the above conjecture becomes very natural.

1.2.3. *Connection with the extended Fuller index.* One of the main ingredients for the above is a connection of extended Fuller index with certain extended Gromov-Witten invariants. If $\beta$ is a free homotopy class of a loop in $C$ set

$$A_\beta = [\beta] \times [S^1] \in H_2(C \times S^1).$$

Then we have:

**Theorem 1.12.** *Suppose that $\lambda$ is a contact form on a closed manifold $C$, so that its Reeb flow is definite type, see Appendix A, then*

$$GW_{1,1}(A_\beta, J^\lambda)([\overline{M}^c_{1,1}] \otimes [C \times S^1]) = i(R^\lambda, \beta),$$

*where both sides are certain extended rational numbers $\mathbb{Q} \sqcup \{\pm\infty\}$ valued invariants, so that if either side does not vanish then there are $\lambda$ Reeb orbits in class $\beta$.*

What about higher genus invariants of $C \times S^1$? Following the proof of Proposition 1.6, it is not hard to see that all $J^\lambda$-holomorphic curves must be branched covers of Reeb tori. If one can show that these branched covers are regular when the underlying tori are regular, the calculation of invariants would be fairly automatic from this data, see [33], [31] where these kinds of regularity calculation are made.

1.3. **Non-squeezing.** One of the most fascinating early results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [12]. The most well known formulation of this is that there does not exist a symplectic embedding $B_R \to D^2(r) \times \mathbb{R}^{2n-2}$ for $R > r$, with $B_R$ the standard closed radius $R$ ball in $\mathbb{R}^{2n}$ centered at 0. Gromov's non-squeezing is $C^0$ persistent in the following sense.

We say that a symplectic form $\omega$ on $M \times N$ is split if $\omega = \omega_1 \oplus \omega_2$ for symplectic forms $\omega_1, \omega_2$ on $M$ respectively $N$.

**Theorem 1.13.** *Given $R > r$, there is an $\epsilon > 0$ s.t. for any symplectic form $\omega'$ on $S^2 \times T^{2n-2}$ $C^0$-close to a split symplectic form $\omega$ and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt],$$

*there is no symplectic embedding $\phi : B_R \hookrightarrow (S^2 \times T^{2n-2}, \omega')$.*

On the other hand it is natural to ask:

*Question* 1. Given $R > r$ and every $\epsilon > 0$ is there a (necessarily non-closed by above) 2-form $\omega'$ on $S^2 \times T^{2n-2}$ $C^0$ or even $C^\infty$ $\epsilon$-close to a split symplectic form $\omega$, satisfying $\langle \omega, A \rangle = \pi r^2$, and s.t. there is an embedding $\phi : B_R \hookrightarrow S^2 \times T^{2n-2}$, with $\phi^* \omega' = \omega_{st}$?

The above theorem follows immediately by Gromov's argument in [12], we shall give a certain extension of this theorem for lcs forms. One may think that recent work of Müller [25] may be related to the question above and our theorem below. But there seems to be no obvious such relation as pull-backs by diffeomorphisms of nearby forms may not be nearby. Hence there is no way to go from nearby embeddings that we work with to $\epsilon$-symplectic embeddings of Müller.

We first give a ridid notion of a morphism of lcsm's.

**Definition 1.14.** *Given a pair of* lcsm*'s $(M_i, \omega_i)$, $i = 0, 1$, we say that $f : M_1 \to M_2$ is a* **symplectomorphism** *if $f^* \omega_2 = \omega_1$. A* **symplectic embedding** *then as usual is an embedding by a symplectomorphism.*

A pair $(\omega, J)$, for $\omega$ lcs and $J$ compatible, will be called a ***compatible*** lcs ***pair***, or just a compatible pair, where there is no confusion.

Let now $M = S^2 \times T^{2n}$, with $\omega$ a split symplectic form on $M$. The following theorem says that it is impossible to have certain symplectic embeddings into $(M, \omega')$ with $\omega'$ $C^0$ nearby to $\omega$, even in the absence of any volume obstruction. So that we have a first basic rigidity phenomenon for lcs structures. Note that in what follows we take a certain natural metric $d$ on the space of general lcs forms, defined in Section 3, whose topology $\mathcal{L}$ is finer than the standard $C^0$ metric topology on the space of forms, cf. [1, Section 6].

We have a real codimension 1 hypersurfaces

$$\Sigma_i = S^2 \times (T^1 \times \ldots \times T^1 \times \{pt\} \times T^1 \times \ldots \times T^1) \subset M,$$

where the singleton $\{pt\} \subset T^1$ replaces the $i$'th factor of $T^{2n} = T^1 \times \ldots \times T^1$. These hypersurfaces are naturally folliated by symplectic submanifolds diffeomorphic to $S^2 \times T^{2n-2}$. We denote by $T^{fol}\Sigma_i \subset TM$, the distribution of all tangent vectors tangent to the leaves of the above mentioned folliation.

**Theorem 1.15.** *Let $\omega$ be a split symplectic form on $M = S^2 \times T^{2n}$, and $A$ as above with $\langle \omega, A \rangle = \pi r^2$. Let $R > r$, then there is an $\epsilon > 0$ s.t. if $\{\omega_t\}$ is a continuous in $\mathcal{L}$ family of* lcs *forms on $M$, with $d(\omega_{t,}, \omega) < \epsilon$ for all $t$, then there is no symplectic embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1) - \cup_i \Sigma_i.$$

*The latter is a full-volume subspace diffeomorphic to* $S^2 \times \mathbb{R}^{2n}$. *More generally there is no symplectic embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1),$$

*s.t* $\phi_* j$ *preserves the bundles* $T^{fol}\Sigma_i$, *for* $j$ *the standard almost complex structure on* $B_R$.

We note that the image of the embedding $\phi$ would be of course a symplectic submanifold of $(M, \omega_1)$. However it could be highly distorted, so that it might be impossible to complete $\phi_* \omega_{st}$ to a symplectic form on $M$ *nearby* to $\omega$, so that it is impossible to deduce the above result directly from symplectic Gromov non-squeezing. We also note that it is certainly possible to have a nearby volume preserving as opposed to lcs embedding which satisfies all other conditions, since as mentioned $(M, \omega_1) - \cup_i \Sigma_i$ is a full $\omega_1$-volume subspace diffeomorphic to $S^2 \times \mathbb{R}^{2n}$.

1.3.1. *Toward direct generalization of contact non-squeezing.* What about non-squeezing for lcs maps as in Definition 1.1? We can try a direct generalization of contact non-squeezing of Eliashberg-Polterovich [4], and Fraser in [6]. Specifically let $R^{2n} \times S^1$ be the prequantization space of $R^{2n}$, or in other words the contact manifold with the contact form $d\theta - \lambda$, for $\lambda = \frac{1}{2}(ydx - xdy)$. Let $B_R$ now denote the open radius $R$ ball in $\mathbb{R}^{2n}$.

*Question* 2. If $R \geq 1$ is there a compactly supported, *lcs* embedding map $\phi : \mathbb{R}^{2n} \times S^1 \times S^1$, so that $\phi(\overline{U}) \subset U$, for $U := B_R \times S^1 \times S^1$ and $\overline{U}$ the topological closure.

We expect the answer is no, but our methods here are not sufficiently developed for this conjecture, as we likely have to extend contact homology rather the Gromov-Witten theory as we do here.

1.4. **Sky catastrophes.** This final introductory section will be of a slightly more technical nature. The following is well known.

**Theorem 1.16.** *[*[18], [30]*] Let* $(M, J)$ *be a compact almost complex manifold, and* $u : (S^2, j) \to M$ *a* $J$*-holomorphic map. Given a Riemannian metric* $g$ *on* $M$, *there is an* $\hbar = \hbar(g, J) > 0$ *s.t. if* $\text{energy}_g(u) < \hbar$ *then* $u$ *is constant, where* $\text{energy}_g$ *is the* $L^2$*-energy functional,*

$$e(u) = \text{energy}(u) = \int_{S^2} |du|^2 dvol.$$

Using this we get the following (trivial) extension of Gromov compactness to this setting. Let

$$\mathcal{M}_{g,n}(J, A) = \mathcal{M}_{g,n}(M, J, A)$$

denote the moduli space of isomorphism classes of class $A$, $J$-holomorphic curves in $M$, with domain a genus $g$ closed Riemann surface, with $n$ marked labeled points. Here an isomorphism between $u_1 : \Sigma_1 \to M$, and $u_2 : \Sigma_2 \to M$ is a biholomorphism of marked Riemann surfaces $\phi : \Sigma_1 \to \Sigma_2$ s.t. $u_2 \circ \phi = u_1$.

**Theorem 1.17.** *Let* $(M, J)$ *be an almost complex manifold. Then* $\mathcal{M}_{g,n}(J, A)$ *has a pre-compactification*

$$\overline{\mathcal{M}}_{g,n}(J, A),$$

*by Kontsevich stable maps, with respect to the natural metrizable Gromov topology see for instance* [18], *for genus 0 case. Moreover given* $E > 0$, *the subspace* $\overline{\mathcal{M}}_{g,n}(J, A)_E \subset \overline{\mathcal{M}}_{g,n}(J, A)$ *consisting of elements* $u$ *with* $e(u) \leq E$ *is compact. In other words* energy *is a proper function.*

Thus the most basic situation where we can talk about Gromov-Witten "invariants" of $(M, J)$ is when the energy function is bounded on $\overline{\mathcal{M}}_{g,n}(J, A)$, and we shall say that $J$ is ***bounded*** (in class $A$), later on we generalize this a bit in terms of what we call ***finite type***. In this case $\overline{\mathcal{M}}_{g,n}(J, A)$ is compact, and has a virtual moduli cycle as in the original approach of Fukaya-Ono [9], or the more algebraic approach [19]. So we may define functionals:

$$GW_{g,n}(\omega, A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}, \tag{1.18}$$

where $\overline{M}_{g,n}$ denotes the compactified moduli space of Riemann surfaces. Of course symplectic manifolds with any tame almost complex structure is one class of examples, another class of examples comes from some locally conformally symplectic manifolds.

Given a continuous in the $C^\infty$ topology family $\{J_t\}$, $t \in [0,1]$ we denote by $\overline{\mathcal{M}}_g(\{J_t\}, A)$ the space of pairs $(u,t)$, $u \in \overline{\mathcal{M}}_g(J_t, A)$.

**Definition 1.19.** *We say that a continuous family $\{J_t\}$ on a compact manifold $M$ has a* **holomorphic sky catastrophe** *in class $A$ if there is an element $u \in \overline{\mathcal{M}}_g(J_i, A)$, $i = 0,1$ which does not belong to any open compact (equivalently energy bounded) subset of $\overline{\mathcal{M}}_g(\{J_t\}, A)$.*

Let us slightly expand this definition. If $\overline{\mathcal{M}}_g(\{J_t\}, A)$ is locally connected, so that the connected components are open, then we have a sky catastrophe in the sense above if and only if there is a $u \in \overline{\mathcal{M}}_g(J_i, A)$ which has a non-compact connected component in $\overline{\mathcal{M}}_g(\{J_t\}, A)$.

At this point in time there are no known examples of families $\{J_t\}$ with sky catastrophes, cf. [10].

*Question* 3. Do sky catastrophes exist?

Really what we are interested in is whether they exist generically. The author's opinion is that they may appear even generically. However, if we further constrain the geometry to exact lcs structures as in Section 1.2, then the question becomes much more subtle, see also [21] for a related discussion on possible obstructions to sky catastrophes.

Related to this we have the following technical result that will be used in the proof of non-squeezing discussed above.

**Theorem 1.20.** *Let $M$ be closed and $\{\omega_t\}$, $t \in [0,1]$, a continuous (with respect to the topology $\mathcal{L}$) family of* lcs *forms on $M$. Let $\{J_t\}$ be a Frechet smooth family of almost complex structures, with $J_t$ compatible with $\omega_t$ for each $t$. Let $A \in H_2(M)$ be fixed, and let $D \subset \widetilde{M}$, with $\pi : \widetilde{M} \to M$ the universal cover of $M$, be a fundamental domain, and $K := \overline{D}$ its topological closure. Suppose that for each $t$, and for every $x \in \partial K$ there is a $\widetilde{J}_t$-holomorphic hyperplane $u_x$ through $x$, with $u_x \subset K$, such that $\pi(u_x) \subset M$ is a closed submanifold and such that $A \cdot \pi_*([u_x]) \leq 0$. Then $\{J_t\}$ has no sky catastrophes in class $A$.*

If holomorphic sky catastrophes are discovered, this would be a very interesting discovery. The original discovery by Fuller [10] of sky catastrophes in dynamical systems is one of the most important in dynamical systems, see also [24] for an overview.

## 2. Elements of Gromov-Witten theory of an lcs manifold

Suppose $(M,J)$ is a compact almost complex manifold, and let $N \subset \overline{\mathcal{M}}_{g,k}(J,A)$ be an open compact subset with energy positive on $N$. The latter condition is only relevant when $A = 0$. We shall primarily refer in what follows to work of Pardon in [19], only because this is what is more familiar to the author, due to greater comfort with algebraic topology. But we should mention that the latter is a follow up to a profound theory that is originally created by Fukaya-Ono [9], and later expanded with Oh-Ohta [8].

The construction in [19] of implicit atlas, on the moduli space $\mathcal{M}$ of curves in a symplectic manifold, only needs a neighborhood of $\mathcal{M}$ in the space of all curves. So more generally if we have an almost complex manifold and an *open* compact component $N$ as above, this will likewise have a natural implicit atlas, or a Kuranishi structure in the setup of [9]. And so such an $N$ will have a virtual fundamental class in the sense of Pardon [19], (or in any other approach to virtual fundamental cycle, particularly the original approach of Fukaya-Oh-Ohta-Ono). This understanding will be used in other parts of the paper, following Pardon for the explicit setup. We may thus define functionals:

$$GW_{g,n}(N,A,J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}. \tag{2.1}$$

The first question is: how do these functionals depend on $N, J$?

**Lemma 2.2.** *Let $\{J_t\}$, $t \in [0,1]$ be a Frechet smooth family. Suppose that $\widetilde{N}$ is an open compact subset of the cobordism moduli space $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ and that the energy function is positive on $\widetilde{N}$, (the latter only relevant when $A = 0$). Let*

$$N_i = \widetilde{N} \cap \left(\overline{\mathcal{M}}_{g,n}(J_i, A)\right),$$

*then*

$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1).$$

*In particular if $GW_{g,n}(N_0, A, J_0) \neq 0$, there is a class $A$ $J_1$-holomorphic stable map in $M$.*

*Proof of Lemma 2.2.* We may construct exactly as in [19] a natural implicit atlas on $\widetilde{N}$, with boundary $N_0^{op} \sqcup N_1$, (*op* denoting opposite orientation). And so

$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1),$$

as functionals. □

The most basic lemma in this setting is the following, and we shall use it in the following section.

**Definition 2.3.** *An* **almost symplectic pair** *on $M$ is a tuple $(M, \omega, J)$, where $\omega$ is a non-degenerate 2-form on $M$, and $J$ is $\omega$-compatible.*

**Definition 2.4.** *We say that a pair of almost symplectic pairs $(\omega_i, J_i)$ are $\delta$-**close**, if $\{\omega_i\}$, and $\{J_i\}$ are $C^0$ $\delta$-close, $i = 0, 1$.*

**Definition 2.5.** *For an almost symplectic pair $(\omega, J)$ on $M$, and a smooth map $u : \Sigma \to M$ define:*

$$\text{energy}_\omega(u) = \int_\Sigma u^*\omega.$$

By an elementary calculation this coincides with the $L^2$ $g_J$-energy of $u$, for $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$. That is $\text{energy}_\omega(u) = \text{energy}_{g_J}(u)$. In what follows by $f^{-1}(a,b)$, with $f$ a function, we mean the preimage by $f$ of the open set $(a,b)$.

**Lemma 2.6.** *Given a compact $M$ and an almost symplectic tuple $(\omega, J)$ on $M$, suppose that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = \text{energy}_\omega^{-1}(E^0, E^1)\right) \subset \left(V = \text{energy}_\omega^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

*with $\epsilon > 0$, $E^0 > 0$ and with $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$. Suppose also that $GW_{g,n}(N, J, A) \neq 0$. Then there is a $\delta > 0$ s.t. whenever $(\omega', J')$ is a compatible almost symplectic pair $\delta$-close to $(\omega, J)$, there exists $u \in \overline{\mathcal{M}}_{g,n}(J', A) \neq \emptyset$, with*

$$E^0 < \text{energy}_{\omega'}(u) < E^1.$$

*Proof of Lemma 2.6.*

**Lemma 2.7.** *Given a Riemannian manifold $(M, g)$, and $J$ an almost complex structure, suppose that $N \subset \overline{\mathcal{M}}_{d,n}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = \text{energy}_g^{-1}(E^0, E^1)\right) \subset \left(V = \text{energy}_g^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

*with $\epsilon > 0$, $E_0 > 0$, and with $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$. Then there is a $\delta > 0$ s.t. whenever $(g', J')$ is $C^0$ $\delta$-close to $(g, J)$ if $u \in \overline{\mathcal{M}}_{g,n}(J', A)$ and*

$$E^0 - \epsilon < \text{energy}_{g'}(u) < E^1 + \epsilon$$

*then*

$$E^0 < \text{energy}_{g'}(u) < E^1.$$

*Proof of Lemma 2.7.* Suppose otherwise then there is a sequence $\{(g_k, J_k)\}$ $C^0$ converging to $(g, J)$, and a sequence $\{u_k\}$ of $J_k$-holomorphic stable maps satisfying

$$E^0 - \epsilon < \text{energy}_{g_k}(u_k) \leq E^0$$

or

$$E^1 \leq \text{energy}_{g_k}(u_k) < E^1 + \epsilon.$$

By Gromov compactness we may find a Gromov convergent subsequence $\{u_{k_j}\}$ to a $J$-holomorphic stable map $u$, with

$$E^0 - \epsilon \leq \text{energy}_g(u) \leq E^0$$

or

$$E^1 \leq \text{energy}_g(u) \leq E^1 + \epsilon.$$

But by our assumptions such a $u$ does not exist. $\square$

**Lemma 2.8.** *Let $M$ be compact, and let $(M, \omega, J)$ be an almost symplectic triple, so that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is exactly as in the lemma above with respect to some $\epsilon > 0$. Then, there is a $\delta' > 0$ s.t. the following is satisfied. Let $(\omega', J')$ be $\delta'$-close to $(\omega, J)$, then there is a continuous in the $C^\infty$ topology family of almost symplectic pairs $\{(\omega_t, J_t)\}$, $(\omega_0, J_0) = (g, J)$, $(\omega_1, J_1) = (g', J')$ s.t. there is open compact subset*

$$\widetilde{N} \subset \overline{\mathcal{M}}_{g,n}(\{J_t\}, A),$$

*and with*

$$\widetilde{N} \cap \overline{\mathcal{M}}(J, A) = N.$$

*Moreover if $(u, t) \in \widetilde{N}$ then*

$$E^0 < \text{energy}_{g_t}(u) < E^1.$$

*Proof.* For $\epsilon$ as in the hypothesis, let $\delta$ be as in Lemma 2.7.

**Lemma 2.9.** *Given a $\delta > 0$ there is a $\delta' > 0$ s.t. if $(\omega', J')$ is $\delta'$-near $(\omega, J)$ there is an interpolating, continuous in $C^\infty$ topology family $\{(\omega_t, J_t)\}$ with $(\omega_t, J_t)$ $\delta$-close to $(\omega, J)$ for each $t$.*

*Proof.* Let $\{g_t\}$ be the family of metrics on $M$ given by the convex linear combination of $g = g_{\omega_J}, g' = g_{\omega', J'}$. Clearly $g_t$ is $\delta'$-close to $g_0$ for each $t$. Likewise the family of 2 forms $\{\omega_t\}$ given by the convex linear combination of $\omega$, $\omega'$ is non-degenerate for each $t$ if $\delta'$ was chosen to be sufficiently small and is $\delta'$-close to $\omega_0 = \omega_{g,J}$ for each moment.

Let

$$ret : Met(M) \times \Omega(M) \to \mathcal{J}(M)$$

be the "retraction map" (it can be understood as a retraction followed by projection) as defined in [17, Prop 2.50], where $Met(M)$ is space of metrics on $M$, $\Omega(M)$ the space of 2-forms on $M$, and $\mathcal{J}(M)$ the space of almost complex structures. This map has the property that the almost complex structure $ret(g, \omega)$ is compatible with $\omega$, and that $ret(g_J, \omega) = J$ for $g_J = \omega(\cdot, J\cdot)$. Then $\{(\omega_t, ret(g_t, \omega_t)\}$ is a compatible family. As $ret$ is continuous in the $C^0$-topology, $\delta'$ can be chosen so that $\{ret_t(g_t, \omega_t\}$ are $\delta$-nearby. $\square$

Let $\delta'$ be chosen with respect to $\delta$ as in the above lemma and $\{(\omega_t, J_t)\}$ be the corresponding family. Let $\widetilde{N}$ consist of all elements $(u, t) \in \overline{\mathcal{M}}(\{J_t\}, A)$ s.t.

$$E^0 - \epsilon < \text{energy}_{\omega_t}(u) < E^1 + \epsilon.$$

Then by Lemma 2.7 for each $(u, t) \in \widetilde{N}$, we have:

$$E^0 < \text{energy}_{\omega_t}(u) < E^1.$$

In particular $\widetilde{N}$ must be closed, it is also clearly open, and is compact as energy is a proper function, as discussed. $\square$

To finish the proof of the main lemma, let $N$ be as in the hypothesis, $\delta'$ as in Lemma 2.8, and $\widetilde{N}$ as in the conclusion to Lemma 2.8, then by Lemma 2.2

$$GW_{g,n}(N_1, J', A) = GW_{g,n}(N, J, A) \neq 0,$$

where $N_1 = \widetilde{N} \cap \overline{\mathcal{M}}_{g,n}(J_1, A)$. So the conclusion follows. □

While not having sky catastrophes gives us a certain compactness control, the above is not immediate because we can still in principle have total cancellation of the infinitely many components of the moduli space $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$. In other words a virtual 0-dimension Kuranishi space $\overline{\mathcal{M}}^0(J^\lambda, A)$, with an infinite number of connected components, can certainly be null-cobordant, by a cobordism all of whose components are compact. So we need a certain additional algebraic and geometric control to preclude such a total cancellation.

*Proof of Theorem 1.17.* (Outline, as the argument is standard.) Suppose that we have a sequence $u^k$ of $J$-holomorphic maps with $L^2$-energy $\leq E$. By [18, 4.1.1], a sequence $u^k$ of $J$-holomorphic curves has a convergent subsequence if $sup_k ||du^k||_{L^\infty} < \infty$. On the other hand when this condition does not hold rescaling argument tells us that a holomorphic sphere bubbles off. The quantization Theorem 1.16, then tells us that these bubbles have some minimal energy, so if the total energy is capped by $E$, only finitely many bubbles may appear, so that a subsequence of $u^k$ must converge in the Gromov topology to a Kontsevich stable map. □

## 3. Rulling out some sky catastrophes and non-squeezing

The metric topology $\mathcal{L}$ on the set $LCS(M)$ of smooth lcs 2-forms on $M$ is defined with respect to the following metric.

**Definition 3.1.** *Fix a Riemannian metric $g$ on $M$ and define*

$$d(\omega_1, \omega_2) = d_{mass}(\omega_1, \omega_2) + d_{mass}(\alpha_1, \alpha_2),$$

*for $\alpha_i$ the Lee forms of $\omega_i$ and $d_{mass}$ the metrics induced by the co-mass norms $|\cdot|_{mass}$ with respect to $g$ on differential $k$-forms. That is $|\eta|_{mass} = \sup_v |\eta(v)|$, where the supremum is over all $g$-unit $k$-vectors $v$.*

The following characterization of convergence will be helpful.

**Lemma 3.2.** *Let $M$ be compact and let $\{\omega_k\} \subset LCS(M)$ be a sequence converging to a symplectic form $\omega$. Denote by $\{\widetilde{\omega}_k\}$ the lift sequence on the universal cover $\widetilde{M}$. Then there is a sequence $\{\omega_k^{symp}\}$ of symplectic forms on $\widetilde{M}$, and a sequence $\{f_k\}$ of positive functions pointwise converging to $1$, such that $\widetilde{\omega}_k = f_k \omega_k^{symp}$.*

*Proof.* We may assume that $M$ is connected. Let $\alpha_k$ be the Lee form of $\omega_k$, and $g_k$ functions on $\widetilde{M}$ defined by $g_k([p]) = \int_{[0,1]} p^* \alpha_k$, where the universal cover $\widetilde{M}$ is understood as the set equivalence classes of paths $p$ starting at $x_0 \in M$, with a pair $p_1, p_2$ equivalent if $p_1(1) = p_2(1)$ and $p_2^{-1} \cdot p_1$ is null-homotopic, where $p_2^{-1} \cdot p_1$ is the path concatenation.

Then we get:

$$d\widetilde{\omega}_k = dg_k \wedge \widetilde{\omega}_k,$$

so that if we set $f_k := e^{g_k}$ then

$$d(f_k^{-1} \widetilde{\omega}_k) = 0.$$

Since by assumption $|\alpha_k|_{mass} \to 0$, then pointwise $g_k \to 0$ and pointwise $f_k \to 1$, so that if we set $\widetilde{\omega}_k^{symp} := f_k^{-1} \widetilde{\omega}_k$ then we are done. □

*Proof of Theorem 1.20.* We shall actually prove a stronger statement that there is a universal (for all $t$) energy bound from above for class $A$, $J_t$-holomorphic curves.

**Lemma 3.3.** *Let $M, K$ be as in the statement of the theorem, and $A \in H_2(M)$ fixed. Let $(\omega, J)$ be a compatible* lcs *pair on $M$ such that for every $x \in \partial K$ there is a $\widetilde{J}$-holomorphic (real codimension 2) hyperplane $u_x \subset K \subset \widetilde{M}$ through $x$, such that $\pi(u_x) \subset M$ is a closed submanifold and such that $A \cdot [\pi(u_x)] \leq 0$. Then any genus 0, $J$-holomorphic class $A$ curve $u$ in $M$ has a lift $\widetilde{u}$ with image in $K$.*

*Proof.* For $u$ as in the statement, let $\widetilde{u}$ be a lift intersecting the fundamental domain $D$, (as in the statement of main theorem). Suppose that $\widetilde{u}$ intersects $\partial K$, otherwise we already have image $\widetilde{u} \subset K^\circ$, for $K^\circ$ the interior, since image $\widetilde{u}$ is connected (any by elementary topology). Then $\widetilde{u}$ intersects $u_x$ as in the statement, for some $x$. So $u$ is a $J$-holomorphic map intersecting the closed hyperplane $\pi(u_x)$ with $A \cdot [\pi(u_x)] \leq 0$. By positivity of intersections, [18], image $u \subset \pi(u_x)$, and so image $\widetilde{u} \subset u_x$. And so image $\widetilde{u} \subset \partial K$. □

Suppose otherwise, then there is a sequence $\{u_k\}_{k=1}^{\infty}$, $u_k : \Sigma_k \to M$, of $J_{t_k}$-holomorphic class $A$ curves, with

$$\int_{\Sigma_k} u_k^* \omega_{t_k} \to \infty, \text{as } k \to \infty.$$

We may assume that $t_k$ is convergent to $t' \in [0, 1]$, otherwise take a convergent subsequence.

Now, by the lemma above each $u_t$ has a lift $\widetilde{u}_t$ contained in a compact $K \subset \widetilde{M}$. Then for every $\epsilon > 0$ there is a $N$ so that for $k > N$ we have:

$$\int_{S^2} \widetilde{u}_k^* \omega_{t_k} \leq C_k \langle \widetilde{\omega}_{t_k}^{symp}, A \rangle \leq C \langle \widetilde{\omega}_{t'}^{symp}, A \rangle + \epsilon,$$

where $\widetilde{\omega}_{t_k} = f_k \widetilde{\omega}_k^{symp}$, $\widetilde{\omega}_{t'} = f \widetilde{\omega}^{symp}$ for $\widetilde{\omega}^{symp}, \widetilde{\omega}_k^{symp}$ symplectic on $\widetilde{M}$, $f, f_k : \widetilde{M} \to \mathbb{R}$ positive functions constructed as in the proof of Lemma 3.2, and $C = sup_K f$, $C_k = sup_K f_k$. So we have obtained a contradiction.

□

*Proof of Theorem 1.15.* Fix an $\epsilon' > 0$ s.t. any 2-form $\omega_1$ on $M$, $\epsilon'$-close to $\omega$ with respect to $d_{mass}$, is non-degenerate and is non-degenerate on the leaves of the folliation of each $\Sigma_i$, discussed prior to the formulation of the theorem. Suppose by contradiction that for every $\epsilon > 0$ there is a homotopy $\{\omega_t\}$ of lcs forms, with $\omega_0 = \omega$, such that $\forall t : d(\omega_t, \omega) < \epsilon$ and such that there exists a symplectic embedding

$$\phi : B_R \hookrightarrow (M, \omega_1),$$

satisfying conditions of the statement of the theorem. Take $\epsilon < \epsilon'$, and let $\{\omega_t\}$ be as in the hypothesis above. In particular $\omega_t$ is an lcs form for each $t$, and is non-degenerate on $\Sigma_i$. Extend $\phi_* j$ to an $\omega_1$-compatible almost complex structure $J_1$ on $M$, preserving $T^{fol}\Sigma_i$. We may then extend this to a family $\{J_t\}$ of almost complex structures on $M$, s.t. $J_t$ is $\omega_t$-compatible for each $t$, with $J_0$ is the standard split complex structure on $M$ and such that $J_t$ preserves $T\Sigma_i$ for each $i$. The latter condition can be satisfied since $\Sigma_i$ are $\omega_t$-symplectic for each $t$. (For construction of $\{J_t\}$ use for example the map *ret* from Lemma 2.9). When the image of $\phi$ does not intersect $\cup_i \Sigma_i$ these conditions can be trivially satisfied.

Then the family $\{(\omega_t, J_t)\}$ satisfies the hypothesis of Theorem 1.20, and so has no sky catastrophes in class $A$. In addition if $N = \overline{\mathcal{M}}_{0,1}(J_0, A)$ (which is compact since $J_0$ is tamed by the symplectic form $\omega$) then

$$GW_{0,1}(N, A, J_0)([pt] \otimes [pt]) = 1.$$

Consequently by Lemma 2.2 there is a class $A$ $J_1$-holomorphic curve $u$ passing through $\phi(0)$.

By Lemma 3.3 we may choose a lift $\widetilde{u}$ to $\widetilde{M}$, with homology class $[\widetilde{u}]$ also denoted by $A$, of each $u$ so that the image of $\widetilde{u}$ is contained in a compact set $K \subset \widetilde{M}$, (independent of all choices). Let $\widetilde{\omega}_t^{symp}$ and $f_t$ be as in Lemma 3.2, then by this lemma for every $\delta > 0$ we may find an $\epsilon > 0$ so that if $d(\omega_1, \omega) < \epsilon$ then $d_{mass}(\widetilde{\omega}^{symp}, \widetilde{\omega}_1^{symp}) < \delta$ on $K$.

Since $\langle \widetilde{\omega}^{symp}, A \rangle = \pi r^2$, if $\delta$ above is chosen to be sufficiently small then

$$\left| \int_{S^2} u^* \omega_1 - \pi r^2 \right| \leq \left| \max_K f_1 \langle \widetilde{\omega}_1^{symp}, A \rangle - \pi \cdot r^2 \right| < \pi R^2 - \pi r^2,$$

since

$$\lim_{\epsilon \to 0} |\langle \widetilde{\omega}_1^{symp}, A \rangle - \pi \cdot r^2| = |\langle \widetilde{\omega}^{symp}, A \rangle - \pi \cdot r^2| = 0,$$

and since

$$\lim_{\epsilon \to 0} \max_K f_1 = 1.$$

In particular we get that $\omega_1$-area of $u$ is less then $\pi R^2$.

We may then proceed as in the now classical proof of Gromov [12] of the non-squeezing theorem to get a contradiction and finish the proof. More specifically $\phi^{-1}(\operatorname{image} \phi \cap \operatorname{image} u)$ is a minimal surface in $B_R$, with boundary on the boundary of $B_R$, and passing through $0 \in B_R$. By construction it has area strictly less then $\pi R^2$ which is impossible by the classical monotonicity theorem of differential geometry. □

## 4. Genus 1 curves in the lcsm $C \times S^1$ and the Fuller index

*Proof of Proposition 1.6.* Suppose we a have a curve without spherical nodal components $u \in \overline{\mathcal{M}}_{1,1}^0(J^\lambda, A)$. We claim that $(pr_C \circ u)_*$ everywhere has rank $\leq 1$, for $pr_C : C \times S^1 \to C$ the projection. Suppose otherwise, then it is immediate by construction of $J^\lambda$, that

$$\int_\Sigma (pr_C \circ u)^* d\lambda > 0,$$

for $\Sigma$ domain of $u$, but $d\lambda$ is exact so that that this is impossible. It clearly follows from this that $\Sigma$ must be smooth, (non-nodal).

Next observe that when the rank of $(pr_C \circ u)_*$ is 1, its image is in the Reeb line sub-bundle of $TC$, for otherwise the image has a contact component, but this is $J^\lambda$ invariant and so again we get that $\int_\Sigma (pr_C \circ u)^* d\lambda > 0$. We now show that the image of $pr_C \circ u$ is in fact the image of some Reeb orbit.

Identify the domain $\Sigma$ of $u$, via an isomorphism as defined in Section 1.2.1, with a marked Riemann surface $(T^2, j)$, $T^2$ the standard torus. We use coordinates $(s,t)$ on $T^2$ $s,t \in S^1$. Then

$$t \mapsto pr_{S^1} \circ u(\{s_0\} \times \{t\}),$$

is a degree 1 curve, where $pr_{S^1} : C \times S^1 \to S^1$ is the projection. And so by the Sard theorem we have a regular value $\theta_0$, so that $u^{-1} \circ pr_{S^1}^{-1}(\theta_0)$ contains an embedded circle $S_0 \subset T^2$. Now $d(pr_{S^1} \circ u)$ is surjective along $T(T^2)|_{S_0}$, which means, since $u$ is $J^\lambda$-holomorphic, that $pr_C \circ u|_{S_0}$ has non-vanishing differential. From this and the discussion above it follows that image of $pr_C \circ u$ is the image of some Reeb orbit. Consequently, by assumption that $u$ has charge $(1,0)$, $u$ is a Reeb torus map for a uniquely determined Reeb orbit $o_u$.

The statement of the lemma follows when $u$ has no spherical nodal components. On the other hand non-constant holomorphic spheres are impossible also by the previous argument. So there are no nodal elements in $\overline{\mathcal{M}}_{1,1}^0(J^\lambda, A)$ which completes the argument. □

**Proposition 4.1.** *Let $(C,\xi)$ be a general contact manifold. If $\lambda$ is a non-degenerate contact 1-form for $\xi$ then all the elements of $\overline{\mathcal{M}}_{1,1}^0(J^\lambda, A)$ are regular curves. Moreover, if $\lambda$ is degenerate then for a period $c$ Reeb orbit $o$ the kernel of the associated real linear Cauchy-Riemann operator for the Reeb torus $u_o$ is naturally identified with the 1-eigenspace of $\phi^\lambda_{c,*}$ - the time $c$ linearized return map $\xi(o(0)) \to \xi(o(0))$ induced by the $R^\lambda$ Reeb flow.*

*Proof.* We already known that all $u \in \overline{\mathcal{M}}_{1,1}^0(J^\lambda, A)$, are Reeb tori. Since each $u$ is immersed we may naturally get a splitting $u^*T(C \times S^1) \simeq N \times T(T^2)$, using the $g_J$ metric, where $N \to T^2$ denotes the pull-back, of the $g_J$-normal bundle to $\operatorname{image} u$, and which is identified with the pullback of the distribution $\xi$ on $C \times S^1$.

The full associated real linear Cauchy-Riemann operator takes the form:

$$D_u^J : \Omega^0(N \oplus T(T^2)) \oplus T_j M_{1,1}^0 \to \Omega^{0,1}(T(T^2), N \oplus T(T^2)). \tag{4.2}$$

This is an index 2 Fredholm operator (after standard Sobolev completions), whose restriction to $\Omega^0(N \oplus T(T^2))$ preserves the splitting, that is the restricted operator splits as

$$D \oplus D' : \Omega^0(N) \oplus \Omega^0(T(T^2)) \to \Omega^{0,1}(T(T^2), N) \oplus \Omega^{0,1}(T(T^2), T(T^2)).$$

On the other hand the restricted Fredholm index 2 operator

$$\Omega^0(T(T^2)) \oplus T_j M^0_{1,1} \to \Omega^{0,1}(T(T^2)),$$

is surjective by classical Teichmuller theory, see also [32, Lemma 3.3] for a precise argument in this setting. It follows that $D^J_u$ will be surjective if the restricted Fredholm index 0 operator

$$D : \Omega^0(N) \to \Omega^{0,1}(N),$$

has no kernel.

The bundle $N$ is symplectic with symplectic form on the fibers given by restriction of $u^*d\lambda$, and together with $J^\lambda$ this gives a Hermitian structure on $N$. We have a linear symplectic connection $\mathcal{A}$ on $N$, which over the slices $S^1 \times \{t\} \subset T^2$ is induced by the pullback by $u$ of the linearized $R^\lambda$ Reeb flow. Specifically the $\mathcal{A}$-transport map from the fiber $N_{(s_0,t)}$ to the fiber $N_{(s_1,t)}$ over the path $[s_0, s_1] \times \{t\} \subset T^2$, is given by

$$(u_*|_{N_{(s_1,t)}})^{-1} \circ \phi^\lambda_{c(s_1-s_0)} \circ u_*|_{N_{(s_0,t)}},$$

where $\phi^\lambda_{c(s_1-s_0)}$ is the time $c \cdot (s_1 - s_0)$ map for the $R^\lambda$ Reeb flow, where $c$ is the period of the Reeb orbit $o_u$, and where $u_* : N \to T(C \times S^1)$ denotes the natural map, (it is the universal map in the pull-back diagram.)

The connection $\mathcal{A}$ is defined to be trivial in the $\theta_2$ direction, where trivial means that the parallel transport maps are the $id$ maps over $\theta_2$ rays. In particular the curvature $R_\mathcal{A}$, understood as a lie algebra valued 2-form, of this connection vanishes. The connection $\mathcal{A}$ determines a real linear CR operator on $N$ in the standard way (take the complex anti-linear part of the vertical differential of a section). It is elementary to verify from the definitions that this operator is exactly $D$.

We have a differential 2-form $\Omega$ on the total space of $N$ defined as follows. On the fibers $T^{vert}N$, $\Omega = u_*\omega$, for $\omega = d^\alpha\lambda$, and for $T^{vert}N \subset TN$ denoting the vertical tangent space, or subspace of vectors $v$ with $\pi_* v = 0$, for $\pi : N \to T^2$ the projection. While on the $\mathcal{A}$-horizontal distribution $\Omega$ is defined to vanish. The 2-form $\Omega$ is closed, which we may check explicitly by using that $R_\mathcal{A}$ vanishes to obtain local symplectic trivializations of $N$ in which $\mathcal{A}$ is trivial. Clearly $\Omega$ must vanish on the 0-section since it is a $\mathcal{A}$-flat section. But any section is homotopic to the 0-section and so in particular if $\mu \in \ker D$ then $\Omega$ vanishes on $\mu$. But then since $\mu \in \ker D$, and so its vertical differential is complex linear, it must follow that the vertical differential vanishes, since $\Omega(v, J^\lambda v) > 0$, for $0 \neq v \in T^{vert}N$ and so otherwise we would have $\int_\mu \Omega > 0$. So $\mu$ is $\mathcal{A}$-flat, in particular the restriction of $\mu$ over all slices $S^1 \times \{t\}$ is identified with a period $c$ orbit of the linearized at $o$ $R^\lambda$ Reeb flow, and which does not depend on $t$ as $\mathcal{A}$ is trivial in the $t$ variable. So the kernel of $D$ is identified with the vector space of period $c$ orbits of the linearized at $o$ $R^\lambda$ Reeb flow, as needed. □

**Proposition 4.3.** *Let $\lambda$ be a contact form on a $(2n+1)$-fold $C$, and $o$ a non-degenerate, period $c$, $R^\lambda$-Reeb orbit, then the orientation of $[u_o]$ induced by the determinant line bundle orientation of $\overline{\mathcal{M}}^0_{1,1}(J^\lambda, A)$, is $(-1)^{CZ(o)-n}$, which is*

$$\operatorname{sign} \operatorname{Det}(\operatorname{Id}|_{\xi(o(0))} - \phi^\lambda_{c,*}|_{\xi(o(0))}).$$

*Proof of Proposition 4.3.* Abbreviate $u_o$ by $u$. Let $N \to T^2$ be associated to $u$ as in the proof of Proposition 4.1. Fix a trivialization $\phi$ of $N$ induced by any trivialization of the contact distribution $\xi$ along $o$ in the obvious sense: $N$ is the pullback of $\xi$ along the composition

$$T^2 \to S^1 \xrightarrow{o} C.$$

Let the symplectic connection $\mathcal{A}$ on $N$ be defined as before. Then the pullback connection $\mathcal{A}' := \phi^*\mathcal{A}$ on $T^2 \times \mathbb{R}^{2n}$ is a connection whose parallel transport paths $p_t : [0, 1] \to \operatorname{Symp}(\mathbb{R}^{2n})$, along the closed loops $S^1 \times \{t\}$, are paths starting at 1, and are $t$ independent. And so the parallel transport path of

$\mathcal{A}'$ along $\{s\} \times S^1$ is constant, that is $\mathcal{A}'$ is trivial in the $t$ variable. We shall call such a connection $\mathcal{A}'$ on $T^2 \times \mathbb{R}^{2n}$ *induced by* $p$.

By non-degeneracy assumption on $o$, the map $p(1)$ has no 1-eigenvalues. Let $p'' : [0,1] \to \mathrm{Symp}(\mathbb{R}^{2n})$ be a path from $p(1)$ to a unitary map $p''(1)$, with $p''(1)$ having no 1-eigenvalues, and s.t. $p''$ has only simple crossings with the Maslov cycle. Let $p'$ be the concatenation of $p$ and $p''$. We then get

$$CZ(p') - \frac{1}{2}\operatorname{sign}\Gamma(p',0) \equiv CZ(p') - n \equiv 0 \mod 2,$$

since $p'$ is homotopic relative end points to a unitary geodesic path $h$ starting at $id$, having regular crossings, and since the number of negative, positive eigenvalues is even at each regular crossing of $h$ by unitarity. Here $\operatorname{sign}\Gamma(p',0)$ is the index of the crossing form of the path $p'$ at time 0, in the notation of [20]. Consequently

$$CZ(p'') \equiv CZ(p) - n \mod 2, \tag{4.4}$$

by additivity of the Conley-Zehnder index.

Let us then define a free homotopy $\{p_t\}$ of $p$ to $p'$, $p_t$ is the concatenation of $p$ with $p''|_{[0,t]}$, reparametrized to have domain $[0,1]$ at each moment $t$. This determines a homotopy $\{\mathcal{A}'_t\}$ of connections induced by $\{p_t\}$. By the proof of Proposition 4.1, the CR operator $D_t$ determined by each $\mathcal{A}'_t$ is surjective except at some finite collection of times $t_i \in (0,1)$, $i \in N$ determined by the crossing times of $p''$ with the Maslov cycle, and the dimension of the kernel of $D_{t_i}$ is the 1-eigenspace of $p''(t_i)$, which is 1 by the assumption that the crossings of $p''$ are simple.

The operator $D_1$ is not complex linear. To fix this we concatenate the homotopy $\{D_t\}$ with the homotopy $\{\widetilde{D}_t\}$ defined as follows. Let $\{\widetilde{\mathcal{A}}_t\}$ be a homotopy of $\mathcal{A}'_1$ to a unitary connection $\widetilde{\mathcal{A}}_1$, where the homotopy $\{\widetilde{\mathcal{A}}_t\}$ is through connections induced by paths $\{\widetilde{p}_t\}$, giving a path homotopy of $p' = \widetilde{p}_0$ to $h$. Then $\{\widetilde{D}_t\}$ is defined to be induced by $\{\widetilde{\mathcal{A}}_t\}$.

Let us denote by $\{D'_t\}$ the concatenation of $\{D_t\}$ with $\{\widetilde{D}_t\}$. By construction in the second half of the homotopy $\{D'_t\}$, $D'_t$ is surjective. And $D'_1$ is induced by a unitary connection, since it is induced by unitary path $\widetilde{p}_1$. Consequently $D'_1$ is complex linear. By the above construction, for the homotopy $\{D'_t\}$, $D'_t$ is surjective except for $N$ times in $(0,1)$, where the kernel has dimension one. In particular the sign of $[u]$ by the definition via the determinant line bundle is exactly

$$-1^N = -1^{CZ(p)-n},$$

by (4.4), which was what to be proved. □

**Theorem 4.5.**

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}^c_{1,1}] \otimes [C \times S^1]) = i(\widetilde{N}, R^\lambda, \beta),$$

*where $N \subset \overline{\mathcal{M}}^0_{1,1}(J^\lambda, A_\beta)$ is an open compact set, $\widetilde{N}$ the corresponding subset of periodic orbits of $R^\lambda$, $i(\widetilde{N}, R^\lambda, \beta)$ is the Fuller index as described in the appendix below, and where the left hand side of the equation is a certain Gromov-Witten invariant, that we discuss in Section 2.*

*Proof.* If $N \subset \overline{\mathcal{M}}^0_{1,1}(J^\lambda, A_\beta)$ is open-compact and consists of isolated regular Reeb tori $\{u_i\}$, corresponding to orbits $\{o_i\}$ we have:

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}^c_{1,1}] \otimes [C \times S^1]) = \sum_i \frac{(-1)^{CZ(o_i)-n}}{mult(o_i)},$$

where the denominator $mult(o_i)$ is there because our moduli space is understood as a non-effective orbifold, see Appendix B.

The expression on the right is exactly the Fuller index $i(\widetilde{N}, R^\lambda, \beta)$. Thus the theorem follows for $N$ as above. However in general if $N$ is open and compact then perturbing slightly we obtain a smooth family $\{R^{\lambda_t}\}$, $\lambda_0 = \lambda$, s.t. $\lambda_1$ is non-degenerate, that is has non-degenerate orbits. And such that there is an open-compact subset $\widetilde{N}$ of $\overline{\mathcal{M}}^0_{1,1}(\{J^{\lambda_t}\}, A_\beta)$ with $(\widetilde{N} \cap \overline{\mathcal{M}}^0_{1,1}(J^\lambda, A_\beta)) = N$, cf. Lemma 2.8. Then by Lemma 2.2 if

$$N_1 = (\widetilde{N} \cap \overline{\mathcal{M}}^0_{1,1}(J^{\lambda_1}, A_\beta))$$

we get

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}^c_{1,1}] \otimes [C \times S^1]) = GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}^c_{1,1}] \otimes [C \times S^1]).$$

By the previous discussion

$$GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}^c_{1,1}] \otimes [C \times S^1]) = i(N_1, R^{\lambda_1}, \beta),$$

but by the invariance of Fuller index (see Appendix A),

$$i(N_1, R^{\lambda_1}, \beta) = i(N, R^\lambda, \beta).$$

□

*Proof of Theorem 1.8.* Let $N \subset \overline{\mathcal{M}}^0_{1,1}(A, J^\lambda)$, be the subspace corresponding, (under the Reeb tori, Reeb orbit correspondence) to the subspace $\widetilde{N}$ of all period $2\pi$ $R^\lambda$-orbits. It is easy to compute, see for instance [11],

$$i(\widetilde{N}, R^\lambda) = \pm\chi(\mathbb{CP}^k) \neq 0.$$

By Theorem 4.5 $GW_{1,1}(N, J^\lambda, A) \neq 0$. The first part of the theorem then follows by Lemma 2.6. We now verify the second part.

Let $U$ be a $\delta$-neighborhood of $(d^\alpha\lambda_H, J^{\lambda_H})$ guaranteed by the first part of the theorem. Let $(\omega, J) \in U$ and $u \in \overline{\mathcal{M}}^0_{1,1}(A, J)$ guaranteed by the first part of the theorem. Let $\underline{u}$ be a simple $J$-holomorphic curve covered by $u$. Let us recall for convenience the adjunction inequality of McDuff-Micallef-White.

**Theorem 4.6.** *Let $(M, J)$ be an almost complex 4-manifold and $A \in H_2(M)$ be a homology class that is represented by a simple $J$-holomorphic curve $u$. Then*

$$2\delta(u) - \chi(\Sigma) \leq A \cdot A - c_1(A),$$

*with equality if and only if $u$ is an immersion with only transverse self-intersections.*

In our case $A = 0$, $\chi(\Sigma) = 0$, so that $\delta(\underline{u}) = 0$, and so $\underline{u}$ is an embedding. □

*Proof of Lemma 1.11.* Let $(M, \omega, J)$ be an exact triple with $\omega = d^\alpha\lambda$. Suppose that $u : \Sigma \to M$ is an immersed $J$-holomorphic curve. By Lemma 1.10 we have that $u^*d\lambda = 0$. We only need to check that $[u^*\alpha] \neq 0$. Suppose otherwise. Let $\widetilde{M}$ denote the universal covering space of $M$, then the lift of $\omega$ to $\widetilde{M}$ is $\widetilde{\omega} = \frac{1}{f}d(f\lambda)$, where $f = e^g$ and where $g$ is the primitive for the lift $\widetilde{\alpha}$ of $\alpha$ to $\widetilde{M}$, that is $\widetilde{\alpha} = dg$. In particular $\widetilde{\omega}$ is conformally exact on $\widetilde{M}$. Now $[u^*\alpha] = 0$, so $u$ has a lift to a $\widetilde{J}$-holomorphic map $\widetilde{u} : \Sigma \to \widetilde{M}$, where $\widetilde{J}$ is the lift of $J$, which is compatible with $\widetilde{\omega}$. Since $\Sigma$ is closed, it follows that $\widetilde{u}$ is constant, which contradicts the fact that $u$ is immersed. □

*Proof of Theorem 1.4.* Define a pseudo-metric $d_g$ measuring distance between subspaces $W_1, W_2$ of an inner product space $(T, g)$ as follows. If $\dim W_1 = \dim W_2$ then

$$d_g(W_1, W_2) := |P_{W_1} - P_{W_2}|,$$

for $|\cdot|$ the $g$-operator norm, and $P_{W_i}$ $g$-projection operators onto $W_i$. If $W_1 = T$ or $W_2 = T$ define $d_g(W_1, W_2) := 0$, in all other cases set $d_g(W_1, W_2) := 1$.

Let $U$ be a $C^0$ metric $\epsilon$-ball neighborhood of $(\omega_H, J_H := J^{\lambda_H})$ as in Theorem 1.8. To prove the theorem we need to construct an admissible pair $(\omega, J) \in U$, as Theorem 1.8 then tells us that there is a class $A$, $J$-holomorphic elliptic curve $u$ in $M$, and since $J$ is admissible, by Lemma 1.11 there is an elliptic Reeb curve for $(M, \omega)$.

Suppose that $\omega = d^{\alpha'}\lambda'$ is $\delta$-close to $\omega_H$ for the metric $d$ as in the statement of the theorem. Then for each $p \in M$, $d_g(\mathcal{V}_\omega(p), \mathcal{V}_{\omega_H}(p)) < \epsilon_\delta$ and $d_g(\xi_\omega(p), \xi_{\omega_H}(p)) < \epsilon_\delta$ where $\epsilon_\delta \to 0$ as $\delta \to 0$, and where $d_g$ is the pseudo-metric as defined above for subspaces of the inner product space $(T_pM, g)$.

Then choosing $\delta$ to be suitably be small, for each $p \in V := \mathcal{V}(M, \lambda')$ we have an isomorphism $\phi(p) : T_pM \to T_pM$, $\phi_p := P_1 \oplus P_2$, for $P_1 : \mathcal{V}_{\lambda_H}(p) \to \mathcal{V}_{\lambda'}(p)$, $P_2 : \xi_{\omega_H}(p) \to \xi_\omega(p)$ the $g$-projection operators. Define $J(p) := \phi(p)_* J_H$, and this defines $J$ in the sub-bundle $\pi_{TM}^{-1} V$, for $\pi_{TM} : TM \to M$ the bundle projection. In addition, if $\delta$ was chosen to be sufficiently small $(\omega, J)$ is a compatible pair in $\pi_{TM}^{-1} V$, and is $\epsilon$-close to $(\omega_H, J_H)$ in $\pi_{TM}^{-1} V$.

Now take any extension of $J$ to $TM$ so that $(\omega, J)$ is a compatible pair $\epsilon$-close to $(\omega_H, J_H)$ in $\pi_{TM}^{-1} V$. This can be obtained by using a partition of unity. Explicitly, $J$ defined in $\pi_{TM}^{-1} V$, and $\omega$ give a Riemannian metric $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$ in $\pi_{TM}^{-1} V$. Use a partition of unity to extend this metric to $TM$, and then use the map:

$$ret : Met(M) \times \Omega(M) \to \mathcal{J}(M),$$

as in Lemma 2.9. $\square$

*Proof of Theorem 1.3.* Let $u : \Sigma \to C \times S^1$ be an immersed elliptic Reeb curve for the Banyaga lcs structure $d^\alpha \lambda$ on $M = C \times S^1$,

$$\alpha = d\theta := pr_1^* d\theta,$$

where $pr_1 : C \times S^1 \to S^1$ is the projection. As in the proof of Proposition 1.6 we obtain that $(pr_C \circ u)_*$ everywhere has rank $\leq 1$, for $pr_C : C \times S^1 \to C$ the projection, and when it has rank 1 the image is contained in $\ker d\lambda$.

By assumptions $[u^* \alpha] = [u^* \circ pr_1^* d\theta] \neq 0$, then applying Sard's theorem as in the proof of Proposition 1.6, we find an embedded circle $S_0 \subset \Sigma$, $S_0 \subset (pr_1 \circ u)^{-1}(\theta_0)$ for some regular value $\theta_0 \subset S^1$ for the map $pr_1 \circ u$. Then $pr_C \circ u|_{S_0}$ must be a smooth immersion, for otherwise, since $S_0 \subset (pr_1 \circ u)^{-1}(\theta_0)$, $u|_{S_0}$ would not be a smooth immersion. Then by the first paragraph, the image of $u|_{S_0}$ must be the image of a Reeb orbit. $\square$

*Proof of Theorem 1.5.* Let $\{\omega_t\}$, $t \in [0, 1]$, be a continuous (in $C^0$ topology) homotopy of lcs forms on $M = C \times S^1$, as in the hypothesis. Fix an almost complex structure $J_1$ on $M$ admissible with respect to $(\alpha', \lambda')$. Extend to a Frechet smooth family $\{J_t\}$ of almost complex structures on $M$, so that $J_t$ is $\omega_t$-compatible for each $t$. Then in the absence of holomorphic sky catastrophes, by Theorem 5.11, there is a non-constant elliptic $J_1$-holomorphic curve in $M$. Then since $\chi(\Sigma) = 0$, $A_\beta \cdot A_\beta = 0$, and $c_1(A_\beta) = 0$ (as $A_\beta$ is the class of a Reeb torus), as in the second part of the proof of Theorem 1.8, we obtain an embedded $J_1$-holomorphic elliptic curve in $M$. $\square$

## 5. Extended Gromov-Witten invariants and the extended Fuller index

In what follows $M$ is a closed oriented $2n$-fold. Much of the following discussion extends to general moduli spaces $\mathcal{M}_{g,n}(J, A, a_1, \ldots, a_n)$ with $a_1, \ldots, a_n$ homological constraints in $M$. We shall however restrict for simplicity to the case $g = 1, n = 1$ with trivial constraint $[M]$, as this is the main interest in this paper. Moreover, we restrict our moduli space to consist of charge $(1, 0)$ curves, with respect to a closed 1-form $\alpha$ on $M$, as in Section 1.2.1, and this will be implicit.

In what follows $e(u)$ denotes the energy of a map $u : \Sigma \to M$, with respect to implicit metric induced by an lcs pair $(\omega, J)$.

**Definition 5.1.** *Let $h = \{(\omega_t, J_t)\}$ be a homotopy of* lcs *pairs on $M$, so that $\{J_t\}$ is Frechet smooth, and $\{\omega_t\}$, $C^0$ continuous. We say that it is* **partially admissible for** *$A$ if every element of*

$$\overline{\mathcal{M}}_{1,1}(M, J_0, A)$$

*is contained in a compact open subset of $\overline{\mathcal{M}}_{1,1}(M, \{J_t\}, A)$. We say that $h$ is* **admissible for** *$A$ if every element of*

$$\overline{\mathcal{M}}_{1,1}(M, J_i, A),$$

*$i = 0, 1$ is contained in a compact open subset of $\overline{\mathcal{M}}_{1,1}(M, \{J_t\}, A)$.*

Thus in the above definition, a homotopy is partially admissible if there are sky catastrophes going one way, and admissible if there are no sky catastrophes going either way.

Partly to simplify notation we denote by a capital $X$ a compatible general lcs triple $(M, \omega, J)$, then we introduce the following simplified notation.

$$\begin{aligned} S(X, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(X, A)\} \\ S(X, a, A) &= \{u \in S(X, A) \mid e(u) \le a\} \\ S(h, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(h, A)\}, \text{ for } h = \{(\omega_t, J_t)\} \text{ a homotopy as above} \\ S(h, a, A) &= \{u \in S(h, A) \mid e(u) \le a\} \end{aligned} \tag{5.2}$$

**Definition 5.3.** *For an isolated element $u$ of $S(X, A)$, which means that $\{u\}$ is open as a subset, we set $gw(u, p) \in \mathbb{Q}$ to be the local Gromov-Witten invariant of $u$. This is defined as:*

$$gw(u, p) = GW_{1,1}(\{u\}, A, J)([\overline{M}^c_{1,1}] \otimes [M]),$$

*with the right hand side as in (2.1).*

**Definition 5.4.** *Suppose that $S(X, A)$ has open connected components. And suppose that we have a collection of* lcs *pairs*

$$\bigcup_{a>0} (X^a = (M, \omega^a, J^a)),$$

*satisfying the following:*

- *$S(X^a, a, A)$ consists of isolated curves for each $a$.*
- $$S(X^a, a, A) = S(X^b, a, A),$$
  *(equality of subsets of $\overline{\mathcal{M}}_{1,1}(X, A) \times \mathbb{R}_+$) if $b > a$, and the local Gromov-Witten invariants corresponding to the identified elements of these sets coincide.*
- *There is a prescribed homotopy $h^a = \{X^a_t\}$ of each $X^a$ to $X$, called* **structure homotopy***, with the property that for every*
  $$y \in S(X^a_0, A)$$
  *there is an open compact subset $\mathcal{C}_y \subset S(h^a, A)$, $y \in C_y$, which is* **non-branching** *which means that*
  $$\mathcal{C}_y \cap S(X^a_i, A),$$
  *$i = 0, 1$ are connected.*
- $$S(h^a, a, A) = S(h^b, a, A),$$
  *(equality of subsets) if $b > a$ is sufficiently large.*

*We will then say that*

$$\mathcal{P}(A) = (\{X^a\}_a, h^a)$$

*is a* **perturbation system** *for $X$ in the class $A$.*

We shall see shortly that, given a contact $(C, \lambda)$, the associated lcs structure on $C \times S^1$ always admits a perturbation system for the moduli spaces of charge (1,0) curves in any class, if $\lambda$ is Morse-Bott.

**Definition 5.5.** *Suppose that $X$ admits a perturbation system $\mathcal{P}(A)$ so that there exists an $E = E(\mathcal{P}(A))$ with the property that*

$$S(X^a, a, A) = S(X^E, a, A)$$

*for all $a > E$, where this as before is equality of subsets, and the local Gromov-Witten invariants of the identified elements are also identified. Then we say that $X$ is* **finite type** *and set:*

$$GW(X, A) = \sum_{u \in S(X^E, A)} gw(u).$$

**Definition 5.6.** *Suppose that $X$ admits a perturbation system $\mathcal{P}(A)$ and there is an $E = E(\mathcal{P}(A)) > 0$ so that $gw(u) > 0$ for all*

$$\{u \in S(X^a, A) \,|\, E \leq e(u) \leq a\}$$

*respectively $gw(u) < 0$ for all*

$$\{u \in S(X^a, A) \,|\, E \leq e(u) \leq a\},$$

*and every $a > E$. Suppose in addition that*

$$\lim_{a \mapsto \infty} \sum_{u \in S(X,a,A)} gw(u) = \infty, \text{ respectively } \lim_{a \mapsto \infty} \sum_{u \in S(X,a,\beta)} gw(u) = -\infty.$$

*Then we say that $X$ is* **positive infinite type**, *respectively* **negative infinite type** *and set*

$$GW(X, A) = \infty,$$

*respectively $GW(X, A) = -\infty$. These are meant to be interpreted as extended Gromov-Witten invariants, counting elliptic curves in class $A$. We say that $X$ is* **infinite type** *if it is one or the other.*

**Definition 5.7.** *We say that $X$ is* **definite** *type if it admits a perturbation system and is infinite type or finite type.*

With the above definitions

$$GW(X, A) \in \mathbb{Q} \sqcup \infty \sqcup -\infty,$$

when it is defined.

*Proof of Theorem 1.12.* Given the definitions above, and the definition of the extended Fuller index in [21], this follows by the same argument as the proof of Theorem 4.5. □

5.0.1. *Perturbation systems for Morse-Bott Reeb vector fields.*

**Definition 5.8.** *A contact form $\lambda$ on $M$, and its associated flow $R^\lambda$ are called* Morse-Bott *if the $\lambda$ action spectrum $\sigma(\lambda)$ - that is the space of critical values of $o \mapsto \int_{S^1} o^* \lambda$, is discreet and if for every $a \in \sigma(\lambda)$, the space*

$$N_a := \{x \in M |\, F_a(x) = x\},$$

*$F_a$ the time $a$ flow map for $R^\lambda$ - is a closed smooth manifold such that rank $d\lambda|_{N_a}$ is locally constant and $T_x N_a = \ker(dF_a - I)_x$.*

**Proposition 5.9.** *Let $\lambda$ be a contact form of Morse-Bott type, on a closed contact manifold $C$. Then the corresponding* lcs *pair $X_\lambda = (C \times S^1, d^\alpha \lambda, J^\lambda)$ admits a perturbation system $\mathcal{P}(A)$, for moduli spaces of charge (1,0) curves for every class $A$.*

*Proof.* This follows immediately by [22, Proposition 2.12 ], and by Proposition 1.6. □

**Lemma 5.10.** *The Hopf* lcs *pair $(S^{2k+1} \times S^1, d^\alpha \lambda_H, J^{\lambda_H})$, for $\lambda_H$ the standard contact structure on $S^{2k+1}$ is infinite type.*

*Proof.* This follows immediately by [21, Lemma 2. 13], and by Proposition 1.6. □

The following is the most basic technical result that we need for our applications.

**Theorem 5.11.** *Let $(C, \lambda)$ be a closed contact manifold so that $R^\lambda$ has definite type. And suppose that $i(R^\lambda, \beta) \neq 0$. Let $\omega_0 = d^\alpha \lambda$ be the Banyaga structure, and suppose we have a partially admissible homotopy $p = \{(\omega_{t,}, J_t)\}$, for class $A_\beta$, then there in an element $u \in \overline{\mathcal{M}}^0_{1,1}(J_1, A_\beta)$.*

## 5.1. Preliminaries on admissible homotopies.

**Definition 5.12.** *Let $h = \{X_t\}$ be a smooth homotopy of* lcs *pairs. For $b > a > 0$ we say that $h$ is* **partially** $a, b$-**admissible**, *respectively* $a, b$-**admissible** *(in class $A$) if for each*

$$y \in S(X_0, a, A)$$

*there is a compact open subset $\mathcal{C}_y \subset S(h, A)$, $y \in \mathcal{C}_y$ with $e(u) < b$, for all $u \in \mathcal{C}_y$. Respectively, if for each*

$$y \in S(X_i, a, A),$$

*$i = 0, 1$ there is a compact open subset $\mathcal{C}_y \ni y$ of $S(h, A)$ with $e(u) < b$, for all $u \in \mathcal{C}_y$.*

**Lemma 5.13.** *Suppose that $X_0$ has a perturbation system $\mathcal{P}(A)$, and $\{X_t\}$ is partially admissible, then for every $a$ there is a $b > a$ so that $\{\widetilde{X}^b_t\} = \{X_t\} \cdot \{X^b_t\}$ is partially $a, b$-admissible, where $\{X_t\} \cdot \{X^b_t\}$ is the (reparametrized to have $t$ domain $[0, 1]$) concatenation of the homotopies $\{X_t\}, \{X^b_t\}$, and where $\{X^b_t\}$ is the structure homotopy from $X^b$ to $X_0$.*

*Proof.* This is a matter of pure topology, and the proof is completely analogous to the proof of [21, Lemma 3.8]. □

The analogue of Lemma 5.13 in the admissible case is the following:

**Lemma 5.14.** *Suppose that $X_0, X_1$ and $\{X_t\}$ are admissible, then for every $a$ there is a $b > a$ so that*

$$\{\widetilde{X}^b_t\} = \{X^b_{1,t}\}^{-1} \cdot \{X_t\} \cdot \{X^b_{0,t}\} \tag{5.15}$$

*is $a, b$-admissible, where $\{X^b_{i,t}\}$ are the structure homotopies from $X^b_i$ to $X_i$.*

## 5.2. Invariance.

**Theorem 5.16.** *Suppose we have a definite type* lcs *pair $X_0$, with $GW(X_0, A) \neq 0$, which is joined to $X_1$ by a partially admissible homotopy $\{X_t\}$, then $X_1$ has non-constant elliptic class $A$ curves.*

*Proof of Theorem 5.11.* This follows by Theorem 5.16 and by Theorem 1.12. □

We also have a more a more precise result.

**Theorem 5.17.** *If $X_0, X_1$ are definite type* lcs *pairs and $\{X_t\}$ is admissible then $GW(X_0, A) = GW(X_1, A)$.*

*Proof of Theorem 5.16.* Suppose that $X_0$ is definite type with $GW(X_0, A) \neq 0$, $\{X_t\}$ is partially admissible and $\overline{\mathcal{M}}_{1,1}(X_1, A) = \emptyset$. Let $a$ be given and $b$ determined so that $\widetilde{h}^b = \{\widetilde{X}^b_t\}$ is a partially $(a, b)$-admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\widetilde{h}^b, A),$$

for $y \in S(X^b_0, a, A)$. Here we use a natural identification of $S(X^b, a, A) = S(\widetilde{X}^b_0, a, A)$ as a subset of $S(\widetilde{h}^b, A)$ by its construction. Then $S_a$ is an open-compact subset of $S(h, A)$ and so has admits an implicit atlas (Kuranishi structure) with boundary, (with virtual dimension 1) s.t:

$$\partial S_a = S(X^b, a, A) + Q_a,$$

where $Q_a$ as a set is some subset (possibly empty), of elements $u \in S(X^b, b, A)$ with $e(u) \geq a$. So we have for all $a$:

$$\sum_{u \in Q_a} gw(u) + \sum_{u \in S(X^b, a, A)} gw(u) = 0. \tag{5.18}$$

5.3. **Case I, $X_0$ is finite type.** Let $E = E(\mathcal{P})$ be the corresponding cutoff value in the definition of finite type, and take any $a > E$. Then $Q_a = \emptyset$ and by definition of $E$ we have that the left side is

$$\sum_{u \in S(X^b, E, A)} gw(u) \neq 0.$$

Clearly this gives a contradiction to (5.18).

5.4. **Case II, $X_0$ is infinite type.** We may assume that $GW(X_0, A) = \infty$, and take $a > E$, where $E = E(\mathcal{P}(A))$ is the corresponding cutoff value in the definition of infinite type. Then

$$\sum_{u \in Q_a} gw(u) \geq 0,$$

as $a > E(\mathcal{P}(A))$. While

$$\lim_{a \mapsto \infty} \sum_{u \in S(X^b, a, A)} gw(u) = \infty,$$

as $GW(X_0, A) = \infty$. This also contradicts (5.18). □

*Proof of Theorem 5.17.* This is somewhat analogous to the proof of Theorem 5.16. Suppose that $X_i, \{X_t\}$ are definite type as in the hypothesis. Let $a$ be given and $b$ determined so that $\widetilde{h}^b = \{\widetilde{X}^b_t\}$, see (5.15) is an $(a, b)$-admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\widetilde{h}^b, A)$$

for $y \in S(X^b_i, a, A)$. Then $S_a$ is an open-compact subset of $S(h, A)$ and so has admits an implicit atlas (Kuranishi structure) with boundary, (with virtual dimension 1) s.t:

$$\partial S_a = (S(X^b_0, a, A) + Q_{a,0})^{op} + S(X^b_1, a, A) + Q_{a,1},$$

with *op* denoting opositite orientation and where $Q_{a,i}$ as sets are some subsets (possibly empty), of elements $u \in S(X^b_i, b, A)$ with $e(u) \geq a$. So we have for all $a$:

$$\sum_{u \in Q_{a,0}} gw(u) + \sum_{u \in S(X^b_0, a, A)} gw(u) = \sum_{u \in Q_{a,1}} gw(u) + \sum_{u \in S(X^b_1, a, A)} gw(u) \tag{5.19}$$

5.5. **Case I, $X_0$ is finite type and $X_1$ is infinite type.** Suppose in addition $GW(X_1, A) = \infty$ and let $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$, for $\mathcal{P}_i$, the perturbation systems of $X_i$. Take any $a > E$. Then $Q_{a,0} = \emptyset$ and the left hand side of (5.19) is

$$\sum_{u \in S(X^b_0, E, A)} gw(u).$$

While the right hand side tends to $\infty$ as $a$ tends to infinity since,

$$\sum_{u \in Q_{a,1}} gw(u) \geq 0,$$

as $a > E(\mathcal{P}_1(A))$, and

$$\lim_{a \mapsto \infty} \sum_{u \in S(X^b_1, a, A)} gw(u) = \infty,$$

Clearly this gives a contradiction to (5.19).

5.6. **Case II, $X_i$ are infinite type.** Suppose in addition $GW(X_0, A) = -\infty$, $GW(X_1, A) = \infty$ and let $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$, for $\mathcal{P}_i$, the perturbation systems of $X_i$. Take any $a > E$. Then $\sum_{u \in Q_{a,0}} gw(u) \leq 0$, and $\sum_{u \in Q_{a,1}} gw(u) \geq 0$. So by definition of $GW(X_i, A)$ the left hand side of (5.18) tends to $-\infty$ as $a$ tends to $\infty$, and the right hand side tends to $\infty$. Clearly this gives a contradiction to (5.19).

5.7. **Case III, $X_i$ are finite type.** The argument is analogous. □

## A. Fuller index

Let $X$ be a vector field on $M$. Set

$$S(X) = S(X,\beta) = \{(o,p) \in L_\beta M \times (0,\infty) \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\},$$

where $L_\beta M$ denotes the free homotopy class $\beta$ component of the free loop space. Elements of $S(X)$ will be called orbits. There is a natural $S^1$ reparametrization action on $S(X)$, and elements of $S(X)/S^1$ will be called *unparametrized orbits*, or just orbits. Slightly abusing notation we write $(o,p)$ for the equivalence class of $(o,p)$. The multiplicity $m(o,p)$ of a periodic orbit is the ratio $p/l$ for $l > 0$ the least period of $o$. We want a kind of fixed point index which counts orbits $(o,p)$ with certain weights - however in general to get invariance we must have period bounds. This is due to potential existence of sky catastrophes as described in the introduction.

Let $N \subset S(X)$ be a compact open set. Assume for simplicity that elements $(o,p) \in N$ are isolated. (Otherwise we need to perturb.) Then to such an $(N,X,\beta)$ Fuller associates an index:

$$i(N,X,\beta) = \sum_{(o,p)\in N/S^1} \frac{1}{m(o,p)} i(o,p),$$

where $i(o,p)$ is the fixed point index of the time $p$ return map of the flow of $X$ with respect to a local surface of section in $M$ transverse to the image of $o$. Fuller then shows that $i(N,X,\beta)$ has the following invariance property. Given a continuous homotopy $\{X_t\}$, $t \in [0,1]$ let

$$S(\{X_t\},\beta) = \{(o,p,t) \in L_\beta M \times (0,\infty) \times [0,1] \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX_t\}.$$

Given a continuous homotopy $\{X_t\}$, $X_0 = X$, $t \in [0,1]$, suppose that $\widetilde{N}$ is an open compact subset of $S(\{X_t\})$, such that

$$\widetilde{N} \cap (LM \times \mathbb{R}_+ \times \{0\}) = N.$$

Then if

$$N_1 = \widetilde{N} \cap (LM \times \mathbb{R}_+ \times \{1\})$$

we have

$$i(N,X,\beta) = i(N_1,X_1,\beta).$$

In the case where $X$ is the $R^\lambda$-Reeb vector field on a contact manifold $(C^{2n+1},\xi)$, and if $(o,p)$ is non-degenerate, we have:

$$i(o,p) = \operatorname{sign} \operatorname{Det}(\operatorname{Id}|_{\xi(x)} - F^\lambda_{p,*}|_{\xi(x)}) = (-1)^{CZ(o)-n}, \tag{A.1}$$

where $F^\lambda_{p,*}$ is the differential at $x$ of the time $p$ flow map of $R^\lambda$, and where $CZ(o)$ is the Conley-Zehnder index, (which is a special kind of Maslov index) see [20].

There is also an extended Fuller index $i(X,\beta) \in \mathbb{Q} \sqcup \{\pm\infty\}$, for certain $X$ having definite type. This is constructed in [21], and is completely analogous to the extended Gromov-Witten invariant constructed in this paper.

## B. Virtual fundamental class

This is a small note on how one deals with curves having non-trivial isotropy groups, in the virtual fundamental class technology. We primarily need this for the proof of Theorem 4.5. Given a closed oriented orbifold $X$, with an orbibundle $E$ over $X$ Fukaya-Ono [9] show how to construct using multi-sections its rational homology Euler class, which when $X$ represents the moduli space of some stable curves, is the virtual moduli cycle $[X]^{vir}$. (Note that the story of the Euler class is older than the work of Fukaya-Ono, and there is possibly prior work in this direction.) When this is in degree 0, the corresponding Gromov-Witten invariant is $\int_{[X]^{vir}} 1$. However they assume that their orbifolds are effective. This assumption is not really necessary for the purpose of construction of the Euler class but is convenient for other technical reasons. A different approach to the virtual fundamental class which emphasizes branched manifolds is used by McDuff-Wehrheim, see for example McDuff [15], which does not have the effectivity assumption, a similar use of branched manifolds appears in [3]. In the case of a non-effective orbibundle $E \to X$ McDuff [16], constructs a homological Euler class $e(E)$ using multi-sections, which extends the construction [9]. McDuff shows that this class $e(E)$ is Poincare dual to the completely formally natural cohomological Euler class of $E$, constructed by other authors. In other words there is a natural notion of a homological Euler class of a possibly non-effective orbibundle. We shall assume the following black box property of the virtual fundamental class technology.

**Axiom B.1.** *Suppose that the moduli space of stable maps is cleanly cut out, which means that it is represented by a (non-effective) orbifold $X$ with an orbifold obstruction bundle $E$, that is the bundle over $X$ of cokernel spaces of the linearized CR operators. Then the virtual fundamental class $[X]^{vir}$ coincides with $e(E)$.*

Given this axiom it does not matter to us which virtual moduli cycle technique we use. It is satisfied automatically by the construction of McDuff-Wehrheim, (at the moment in genus 0, but surely extending). It can be shown to be satisfied in the approach of John Pardon [19]. And it is satisfied by the construction of Fukaya-Oh-Ono-Ohta [7], although not quiet immediately. This is also communicated to me by Kaoru Ono. When $X$ is 0-dimensional this does follow immediately by the construction in [9], taking any effective Kuranishi neighborhood at the isolated points of $X$, (this actually suffices for our paper.)

As a special case most relevant to us here, suppose we have a moduli space of elliptic curves in $X$, which is regular with expected dimension 0. Then its underlying space is a collection of oriented points. However as some curves are multiply covered, and so have isotropy groups, we must treat this is a non-effective 0 dimensional oriented orbifold. The contribution of each curve $[u]$ to the Gromov-Witten invariant $\int_{[X]^{vir}} 1$ is $\frac{\pm 1}{[\Gamma([u])]}$, where $[\Gamma([u])]$ is the order of the isotropy group $\Gamma([u])$ of $[u]$, in the McDuff-Wehrheim setup this is explained in [15, Section 5]. In the setup of Fukaya-Ono [9] we may readily calculate to get the same thing taking any effective Kuranishi neighborhood at the isolated points of $X$.

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