NON-SQUEEZING IN lcs GEOMETRY AND CONFORMAL SYMPLECTIC WEINSTEIN CONJECTURE

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Abstract. We initiate here the study of elements of Gromov-Witten theory of locally conformally symplectic manifolds or lcs manifolds, lcsm's for short, which are a natural generalization of both contact and symplectic manifolds. As a first application we show that in dimension 4 the classical Gromov non-squeezing theorem has a certain C^0 rigidity or persistence with respect to lcs deformations, this is one version of lcs non-squeezing a first result of its kind. In a different direction we study Gromov-Witten theory of the lcsm $C \times S^1$ induced by a contact manifold (C, λ), and show that the Gromov-Witten invariant counting certain elliptic curves in $C \times S^1$ is identified with the classical Fuller index of the Reeb vector field R^λ. Partly inspired by this we conjecture existence of certain 2-dimensional curves we call Reeb curves in some lcsm's, which we call conformal symplectic Weinstein conjecture, and this is a direct extension of the classical Weinstein conjecture. We show that the CSW conjecture holds for a C^0- neighborhood of the Hopf lcs structure. Furthermore, we show that either it holds for any Lichnerowicz exact lcs ω on $S^{2k+1} \times S^1$, with ω exact homotopic to the Hopf lcs structure, or there exist sky catastrophes for families of holomorphic curves in lcs manifold. The latter are analogous to sky catastrophes in dynamical systems discovered by Fuller.

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1. Introduction

The theory of pseudo-holomorphic curves in symplectic manifolds as initiated by Gromov and Floer has revolutionized the study of symplectic and contact manifolds. What the symplectic form gives that is missing for a general almost complex manifold is a priori energy bounds for pseudo-holomorphic curves a fixed class. On the other hand there is a natural structure which directly generalizes both symplectic and contact manifolds, called locally conformally symplectic structure or lcs structure for short. A locally conformally symplectic manifold or lcsm is a smooth $2n$-fold M with an lcs structure: which is a non-degenerate 2-form ω, which is locally diffeomorphic to $f \cdot \omega_{st}$, for some (non-fixed) positive smooth function f, with ω_{st} the standard symplectic form on $\mathbb{R}^{2n}$. It is natural to try to do Gromov-Witten theory for such manifolds. The first problem that occurs is that a priori energy bounds are gone, as since ω is not necessarily closed, the L^2-energy can now be unbounded on the moduli spaces of J-holomorphic curves in such a (M, ω). Strangely a more acute problem is potential presence of holomorphic sky catastrophes - given a smooth family $\{J_t\}$, $t \in [0,1]$, of $\{\omega_t\}$-compatible almost complex structures, we may have a continuous family $\{u_t\}$ of J_t-holomorphic curves s.t. energy$(u_t) \mapsto \infty$ as $t \mapsto a \in (0,1)$ and s.t. there are no holomorphic curves for $t \geq a$. These are analogues of sky catastrophes discovered by Fuller [10].

We are able to tame these problems in certain situations, for example for some 4-d lcsm's, and this is how we arrive at a version of Gromov non-squeezing theorem for such lcsm's. Even when it is impossible to tame these problems we show that there is still a potentially interesting theory which is analogous to the theory of Fuller index in dynamical systems. Inspired by this, we formulate a direct generalization of the Weinstein conjecture for certain lcsm's, we may call this conformal symplectic Weinstein conjecture.

We prove this conjecture for lcs structures C^0 nearby to the Hopf lcs structure on $S^{2k+1} \times S^1$, using a connection with the classical Fuller index. Note that Seifert [23] was likewise initially motivated by a C^0 neighborhood version of the Seifert conjecture. However in our case there is additional evidence, coming from the proof of original Weinstein conjecture. We prove also prove a stronger result that either CSW conjecture holds for any exact lcs structure exactly homotopic to the Hopf lcs structure or holomorphic sky catastrophes exist, which would also be very interesting.

We begin our more detailed discussion with the well known observation:

Theorem 1.1. *[[18], [30]] Let (M, J) be a compact almost complex manifold, and $u : (S^2, j) \to M$ a J-holomorphic map. Given a Riemannian metric g on M, there is an $\hbar = \hbar(g, J) > 0$ s.t. if* $\text{energy}_g(u) < \hbar$ *then u is constant, where* energy_g *is the L^2-energy functional,*

$$e(u) = \text{energy}(u) = \int_{S^2} |du|^2 dvol.$$

Using this we get the following extension of Gromov compactness to this setting. Let

$$\mathcal{M}_{g,n}(J, A) = \mathcal{M}_{g,n}(M, J, A)$$

denote the moduli space of isomorphism classes of class A, J-holomorphic curves in M, with domain a genus g closed Riemann surface, with n marked labeled points. Here an isomorphism between $u_1 : \Sigma_1 \to M$, and $u_2 : \Sigma_2 \to M$ is a biholomorphism of marked Riemann surfaces $\phi : \Sigma_1 \to \Sigma_2$ s.t. $u_2 \circ \phi = u_1$.

Theorem 1.2. *Let (M, J) be an almost complex manifold. Then $\mathcal{M}_{g,n}(J, A)$ has a pre-compactification*

$$\overline{\mathcal{M}}_{g,n}(J, A),$$

by Kontsevich stable maps, with respect to the natural metrizable Gromov topology see for instance [18], for genus 0 case. Moreover given $E > 0$, the subspace $\overline{\mathcal{M}}_{g,n}(J, A)_E \subset \overline{\mathcal{M}}_{g,n}(J, A)$ consisting of elements u with $e(u) \leq E$ is compact. In other words energy *is a proper function.*

Thus the most basic situation where we can talk about Gromov-Witten "invariants" of (M, J) is when the energy function is bounded on $\overline{\mathcal{M}}_{g,n}(J, A)$, and we shall say that J is ***bounded*** (in class A), later on we generalize this a bit in terms of what we call ***finite type***. In this case $\overline{\mathcal{M}}_{g,n}(J, A)$ is compact, and has a virtual moduli cycle as in the original approach of Fukaya-Ono [9], or the more algebraic approach [19]. So we may define functionals:

$$GW_{g,n}(\omega, A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}, \tag{1.3}$$

where $\overline{M}_{g,n}$ denotes the compactified moduli space of Riemann surfaces. Of course symplectic manifolds with any tame almost complex structure is one class of examples, another class of examples comes from some locally conformally symplectic manifolds.

1.1. **Locally conformally symplectic manifolds and Gromov non-squeezing.** Let us give a bit of background on lcsm's. These were originally considered by Lee in [14], arising naturally as part of an abstract study of "a kind of even dimensional Riemannian geometry", and then further studied by a number of authors see for instance, [1] and [28]. This is a fascinating object, an lcsm admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and Murphy show that if a closed almost complex $2n$-fold M has $H^1(M, \mathbb{R}) \neq 0$ then it admits a lcs structure, [5], see also [2].

To see the connection with the first cohomology group, let us point out right away the most basic invariant of a lcs structure ω: the Lee class, $\alpha = \alpha_\omega \in H^1(M, \mathbb{R})$. This has the property that on the associated covering space $\widetilde{M}$, $\widetilde{\omega}$ is globally conformally symplectic. The class α may be defined as the following Cech 1-cocycle. Let $\phi_{a,b}$ be the transition map for lcs charts ϕ_a, ϕ_b of (M, ω). Then $\phi_{a,b}^* \omega_{st} = g_{a,b} \cdot \omega_{st}$ for a positive real constant $g_{a,b}$ and $\{\ln g_{a,b}\}$ gives our 1-cocycle. Thus an lcs form is globally conformally symplectic iff its Lee class vanishes.

As we mentioned lcsm's can also be understood to generalize contact manifolds. This works as follows. First we have a natural class of explicit examples of lcsm's, obtained by starting with a symplectic cobordism (see [5]) of a closed contact manifold C to itself, arranging for the contact forms at the two ends of the cobordism to be proportional (which can always be done) and then gluing together the boundary components. As a particular case of this we get Banyaga's basic example.

Example 1 (Banyaga). Let (C, ξ) be a contact manifold with a contact form λ and take $M = C \times S^1$ with 2-form $\omega = d^\alpha \lambda := d\lambda - \alpha \wedge \lambda$, for α the pull-back of the volume form on S^1 to $C \times S^1$ under the projection.

The operator $d^\alpha : \Omega^k(M) \to \Omega^{k+1}(M)$ is called the Lichnerowicz differential with respect to a closed 1-form α, and satisfies $d^\alpha \circ d^\alpha = 0$ so that we have an associated Lichnerowicz complex.

Using above we may then faithfully embed the category of contact manifolds, and contactomorphism into the category of lcsm's, and certain lcs morphisms as defined below.

Definition 1.4. *A diffeomorphism $\phi : (M_0, \omega_0) \to (M_1, \omega_1)$ is said to be an* lcs **map** *if $\phi^* \omega_1$ is homotopic through lcs forms $\{\omega_t\}$, in the same d^α Lichnerowicz cohomology class, to ω_0, where α is the Lee form of ω_0 as before. In other words, for each $t_0 \in [0,1]$,*

$$d^\alpha\left(\frac{d}{dt}\Big|_{t=t_0} \omega_t\right) = 0.$$

We also define, following Banyaga, ***conformal symplectomorphisms*** $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ to be diffeomorphisms satisfying $\phi^* \omega_2 = f \omega_1$ for a smooth positive function f, see [1] for details on how this embedding works.

Banyaga type lcsm's give immediate examples of almost complex manifolds where the energy function is unbounded on the moduli spaces of fixed class pseudo-holomorphic curves, as well as where null-homologous J-holomorphic curves can be non-constant. We show that it is still possible to extract a variant of Gromov-Witten theory here. The story is closely analogous to that of the Fuller index in dynamical systems, which is concerned with certain rational counts of periodic orbits. In that case sky catastrophes prevent us from obtaining a completely well defined invariant, but Fuller constructs certain partial invariants which give dynamical information. In a very particular situation the relationship with the Fuller index becomes perfect as one of the results of this paper obtains the classical Fuller index for Reeb vector fields on a contact manifold C as a certain genus 1 Gromov-Witten invariant of the lcsm $C \times S^1$. The latter also gives a conceptual interpretation for why the Fuller index is rational, as it is reinterpreted as an (virtual) orbifold Euler number.

1.1.1. *Non-squeezing.* One of the most fascinating early results in symplectic geometry is the so called Gromov non-squeezing theorem appearing in the seminal paper of Gromov [12]. The most well known formulation of this is that there does not exist a symplectic embedding $B_R \to D^2(r) \times \mathbb{R}^{2n-2}$ for $R > r$, with B_R the standard closed radius R ball in $\mathbb{R}^{2n}$ centered at 0. Gromov's non-squeezing is C^0 persistent in the following sense.

Theorem 1.5. *Given $R > r$, there is an $\epsilon > 0$ s.t. for any symplectic form ω' on $S^2 \times T^{2n-2}$ C^0-close to a split symplectic form ω and satisfying*

$$\langle \omega, A \rangle = \pi r^2, A = [S^2] \otimes [pt],$$

there is no symplectic embedding $\phi : B_R \hookrightarrow (S^2 \times T^{2n-2}, \omega')$.

On the other hand it is natural to ask:

Question 1. Given $R > r$ and every $\epsilon > 0$ is there a (necessarily non-closed by above) 2-form ω' on $S^2 \times T^{2n-2}$ C^0 or even C^∞ ϵ-close to a split symplectic form ω, satisfying $\langle \omega, A \rangle = \pi r^2$, and s.t. there is an embedding $\phi : B_R \hookrightarrow S^2 \times T^{2n-2}$, with $\phi^* \omega' = \omega_{st}$?

The above theorem follows immediately by Gromov's argument in [12], we shall give a certain extension of this theorem for lcs structures. One may think that recent work of Müller [25] may be related to the question above and our theorem below. But there seems to be no obvious such relation as pull-backs by diffeomorphisms of nearby forms may not be nearby. Hence there is no way to go from nearby embeddings that we work with to ϵ-symplectic embeddings of Müller.

We first give a ridid notion of morphism lcsm's that we later relax.

Definition 1.6. *Given a pair of* lcsm*'s (M_i, ω_i), $i = 0, 1$, we say that $f : M_1 \to M_2$ is a* **symplectomorphism** *if $f^* \omega_2 = \omega_1$. A* **symplectic embedding** *then as usual is an embedding by a symplectomorphism.*

A pair (ω, J) for ω lcs and J compatible will be called a compatible lcs pair, or just a compatible pair, where there is no confusion. Note that the pair of hypersurfaces $\Sigma_1 = S^2 \times S^1 \times \{pt\} \subset S^2 \times T^2$, $\Sigma_2 = S^2 \times \{pt\} \times S^1 \subset S^2 \times T^2$ are naturally foliated by symplectic spheres, we denote by $T^{fol}\Sigma_i$ the sub-bundle of the tangent bundle consisting of vectors tangent to the foliation. The following theorem says that it is impossible to have certain symplectic embeddings into lcs manifolds C^0 nearby to a certain split symplectic manifold, even in the absence of any volume obstruction. So that we have a first basic rigidity phenomenon for lcs structures.

There is a small caveat here, in what follows we take (also natural) C^0 topology on the space of lcs structures that is coarser than the subspace topology of the standard C^0 topology (with respect to a metric) on the space of forms, cf. [1, Section 6].

Theorem 1.7. *Let ω be a split symplectic form on $M = S^2 \times T^2$, and A as above with $\langle \omega, A \rangle = \pi r^2$. Let $R > r$, then there is an $\epsilon > 0$ s.t. if $\{\omega_t\}$ is a continuous* lcs *family on M C^0 ϵ-close to ω, then there is no symplectic embedding*

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1),$$

s.t $\phi_* j$ preserves the bundles $T^{fol}\Sigma_i$, for j the standard almost complex structure. In particular there is no symplectic embedding

$$\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega_1) - (\Sigma_1 \sqcup \Sigma_2),$$

the latter is a full-volume subspace diffeomorphic to $S^2 \times \mathbb{R}^2$.

We note that the image of the embedding ϕ would be of course a symplectic submanifold of (M, ω_1). However it could be highly distorted, so that it might be impossible to complete $\phi_* \omega_{st}$ to a symplectic form on M nearby to ω. We also note that it is certainly possible to have a nearby volume preserving as opposed to lcs embedding which satisfies all other conditions. Take $\omega = \omega_1$, then if the symplectic form on T^2 has enough volume, we can find a volume preserving map $\phi : B_R \to M$ s.t. $\phi_* j$ preserves $T^{fol}\Sigma_i$. This is just the squeeze map, which as a map $\mathbb{C}^2 \to \mathbb{C}^2$ is $(z_1, z_2) \mapsto (\frac{z_1}{a}, a \cdot z_2)$. In fact we can just take any volume preserving map ϕ that doesn't hit Σ_i. Note that ω_1 must necessarily be globally conformally symplectic on $M - \Sigma_1 \sqcup \Sigma_2$, as this is simply connected, however even in this special case the theorem above is not an obvious extension of Gromov non-squeezing.

1.1.2. *Toward direct generalization of contact non-squeezing.* What about non-squeezing for lcs maps as defined above? We can try a direct generalization of contact non-squeezing of Eliashberg-Polterovich [4], and Fraser in [6]. Specifically let $R^{2n} \times S^1$ be the prequantization space of R^{2n}, or in other words the contact manifold with the contact form $d\theta - \lambda$, for $\lambda = \frac{1}{2}(ydx - xdy)$. Let B_R now denote the open radius R ball in $\mathbb{R}^{2n}$.

Question 2. If $R \geq 1$ is there a compactly supported, *lcs* embedding map $\phi : \mathbb{R}^{2n} \times S^1 \times S^1$, so that $\phi(\overline{U}) \subset U$, for $U := B_R \times S^1 \times S^1$ and $\overline{U}$ the topological closure.

We expect the answer is no, but our methods here are not sufficiently developed for this conjecture, as we likely have to extend contact homology rather the Gromov-Witten theory as we do here.

1.2. **Sky catastrophes.** Given a continuous family $\{J_t\}$, $t \in [0, 1]$ we denote by $\overline{\mathcal{M}}_g(\{J_t\}, A)$ the space of pairs (u, t), $u \in \overline{\mathcal{M}}_g(J_t, A)$.

Definition 1.8. *We say that a continuous family $\{J_t\}$ on a compact manifold M has a* **holomorphic sky catastrophe** *in class A if there is an element $u \in \overline{\mathcal{M}}_g(J_i, A)$, $i = 0, 1$ which does not belong to any open compact (equivalently energy bounded) subset of $\overline{\mathcal{M}}_g(\{J_t\}, A)$.*

Let us slightly expand this definition. If the connected components of $\overline{\mathcal{M}}_g(\{J_t\}, A)$ are open, in other words if this space is locally connected, then we have a sky catastrophe in the sense above if and only if there is a $u \in \overline{\mathcal{M}}_g(J_i, A)$ which has a non-compact connected component in $\overline{\mathcal{M}}_g(\{J_t\}, A)$.

Proposition 1.9. *Let M be a closed manifold, and suppose that $\{J_t\}$, $t \in [0, 1]$ has no holomorphic sky catastrophes, then if J_i, $i = 0, 1$ are bounded:*

$$GW_{g,n}(A, J_0) = GW_{g,n}(A, J_1),$$

if $A \neq 0$. If only J_0 is bounded then there is at least one class A J_1-holomorphic curve in M.

The assumption on A is for simplicity in this case. At this point in time there are no known examples of families $\{J_t\}$ with sky catastrophes, cf. [10].

Question 3. Do sky catastrophes exist?

Really what we are interested in is whether they exist generically. The author's opinion is that they do appear even generically. However for locally conformally symplectic deformations $\{(\omega_t, J_t)\}$ as previously defined, it might be possible that holomorphic sky catastrophes cannot exist generically, for example it looks very unlikely that an example can be constructed with Reeb tori (see the following section), cf. [21].

In this direction we have the following result, that will be used in the proof of non-squeezing above.

Theorem 1.10. *Let M be a closed 4-fold and $\{(\omega_t, J_t)\}$, $t \in [0, 1]$, a continuous family of compatible* lcs *pairs on M, and $\{\alpha_t\}$ its associated family of Lee classes. Let $\Sigma_i \subset M$, $i = 0, \ldots, m$ be a collection of hypersurfaces s.t. $PD(\alpha_t) = \sum_i a_{i,t}[\Sigma_i]$ for each t. ($a_{i,t}$ need not be continuous in t.) Let $\{J_t\}$ be such that for each t there is a foliation of Σ_i by J_t-holomorphic class B curves, then $\{J_t\}$ has no sky catastrophes in every class A s.t. $A \cdot B \leq 0$.*

2. Introduction part II conformal symplectic Weinstein conjecture

2.1. **Holomorphic curves in the** lcsm $C \times S^1$**.** Let (C, λ) be a closed contact manifold with a contact form λ. Then $T = S^1$ acts on $C \times S^1$ by rotation in the S^1 coordinate. Let J be an almost complex structure on the contact distribution, compatible with $d\lambda$. There is an induced almost complex structure J^λ on $C \times S^1$, which is T-invariant, coincides with J on the contact distribution

$$\xi \subset TC \oplus \{\theta\} \subset T(C \times S^1),$$

for each θ and which maps the Reeb vector field

$$R^\lambda \in TC \oplus 0 \subset T(C \times S^1)$$

to

$$\frac{d}{d\theta} \in \{0\} \oplus TS^1 \subset T(C \times S^1),$$

for $\theta \in [0, 2\pi]$ the global angular coordinate on S^1. This almost complex structure is compatible with $d^\alpha \lambda$.

We now consider the moduli space of marked holomorphic tori, (elliptic curves) in $C \times S^1$, in a certain class A. Our notation for this is $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$, where A is a class of the maps, (to be explained). The elements are equivalence classes of pairs (u, Σ): u a J^λ-holomorphic map of a stable genus 1, elliptic curve Σ into $C \times S^1$. So Σ is a nodal curve with principal component an elliptic curve, and other components spherical. So the principal component determines an element of $\overline{M}_{1,1}$ the compactified moduli space of elliptic curves, which is understood as an orbifold. The equivalence relation is $(u, \Sigma) \sim (u', \Sigma')$ if there is an isomorphism of marked elliptic curves $\phi : \Sigma \to \Sigma'$ s.t. $u' \circ \phi = u$. When Σ is smooth, we may write $[u, j]$ for an equivalence class where j is understood as a complex structure on the sole principal component of the domain, and u the map. Or we may just write $[u]$, or even just u keeping track of j, and of the fact that we are dealing with equivalence classes, implicitly.

Let us explain what class A means. We need to be careful because it is now possible for non-constant holomorphic curves to be null-homologous. Here is a simple example take $S^3 \times S^1$ with J determined by the Hopf contact form as above, then all the Reeb tori are null-homologous. In many cases we can just work with homology classes $A \neq 0$, and this will remove some headache, but in the above specific situation this is inadequate.

Given $u \in \overline{\mathcal{M}}_{1,1}(J^\lambda, A)$ we may compose $\Sigma \xrightarrow{u} C \times S^1 \xrightarrow{pr} S^1$, for Σ the nodal domain of u.

Definition 2.1. *In the setting above we say that u is in class A, if $(pr \circ u)^* d\theta$ can be completed to an integral basis of $H^1(\Sigma, \mathbb{Z})$, and if the homology class of u is A, possibly zero.*

It is easy to see that the above notion of class is preserved under Gromov convergence, and that a class A J-holomorphic map cannot be constant for any A, in particular by Theorem 1.1 a class A map has energy bounded from below by a positive constant, depending on (ω, J). And this holds for any lcs pair (ω, J) on $C \times S^1$.

2.1.1. *Reeb tori.* For the almost complex structure J^λ as above we have one natural class of holomorphic tori in $C \times S^1$ that we call *Reeb tori.* Given a closed orbit o of R^λ, a Reeb torus u_o for o, is the map

$$u_o(\theta_1, \theta_2) = (o(\theta_1), \theta_2),$$

$\theta_1, \theta_2 \in S^1$ A Reeb torus is J^λ-holomorphic for a uniquely determined holomorphic structure j on T^2. If

$$D_t o(t) = c \cdot R^\lambda(o(t)),$$

then

$$j(\frac{\partial}{\partial \theta_1}) = c \frac{\partial}{\partial \theta_2}.$$

Proposition 2.2. *Let (C, λ) be as above. Let A be a class in the sense above, and J^λ be as above. Then the entire moduli space $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$ consists of Reeb tori.*

Note that the formal dimension of $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$ is 0, for A as in the proposition above. It is given by the Fredholm index of the operator (5.2) which is 2, minus the dimension of the reparametrization group (for smooth curves) which is 2. In Theorem 2.13 we relate the count of these curves to the classical Fuller index, which is reviewed in the Appendix A.

What follows is one non-classical application of the above theory.

Theorem 2.3. *Let $(S^{2k+1} \times S^1, d^\alpha \lambda_{st})$ be the* lcsm *associated to a contact manifold (S^{2k+1}, λ_{st}) for λ_{st} the standard contact form. Let A be as in the discussion above. Then for any* lcs *pair (ω, J), homotopic through a path of* lcs *pairs $p = \{(\omega_{t,}, J_t)\}$ to $(d^\alpha \lambda, J^\lambda)$, there exists an elliptic, class A, J-holomorphic curve in $C \times S^1$, provided p has no sky catastrophe in class A.*

While not having sky catastrophes gives us a certain compactness control, the above theorem is not immediate because we can still in principle have total cancellation of the infinitely many components of the moduli space $\mathcal{M}_{1,1}(J^\lambda, A)$. In other words a virtual 0-dimension Kuranishi space $\mathcal{M}_{1,1}(J^\lambda, A)$, with an infinite number of connected components, can certainly be null-cobordant, by a cobordism all of whose components are compact. So for the theorem we need a certain additional geometric control, and a certain algebraic framework, to preclude such total cancellation.

Theorem 2.4. *Let $(S^{2k+1} \times S^1, d^\alpha \lambda_H)$ be the* lcsm *associated to a contact manifold (S^{2k+1}, λ_H) for λ_H the standard contact form. There exists a $\delta > 0$ s.t. for any* lcs *pair (ω, J) C^0 δ-close to $(d^\alpha \lambda_H, J^{\lambda_H})$, there exists an elliptic, class A, J-holomorphic curve in $C \times S^1$. (Where A is as in the discussion above.)*

We shall call $\omega_H := d^\alpha \lambda_H$ the ***Hopf*** lcs ***structure***. Note that Seifert [23] initially found a similar existence phenomenon of orbits on S^{2k+1} for a vector field C^0-nearby to the Hopf vector field. And he asked if the nearby condition can be removed, this was known as the Seifert conjecture. This turned out not to be quite true [13]. Likewise it is natural for us to conjecture that the δ-nearby condition can be removed. This conjecture has some evidence. For if $\omega = d^\alpha \lambda$ for λ the contact form inducing the standard contact structure on S^{2k+1}, or any contact form on a threefold, and $J = J^\lambda$ then we know there are J-holomorphic class A tori, since we know there are λ-Reeb orbits, as the Weinstein conjecture is known to hold in these cases, [29], [27] and hence there are Reeb tori. In order to formally state the conjecture we introduce the following class of almost complex structures.

Definition 2.5. *Let (M, ω) be a Lichnerowicz exact closed lcs manifold, or from now on just* **exact** *for short, which means specifically that $\omega = d^\alpha \lambda = d\lambda - \alpha \wedge \lambda$, for α the Lee form. We say J is ω-**admissible**, if it preserves the vanishing distribution*

$$\mathcal{V}_\omega(p) = \{v \in T_p M | d\lambda(v, \cdot) = 0\},$$

and the distribution $\xi = \xi_\omega$, which is defined to be the ω-orthogonal complement to $\mathcal{V}_\omega$. We call (M, ω, J) as above an **exact lcs triple**.

Lemma 2.6. *The distribution $\mathcal{V}_\omega$ has dimension 2.*

Proof. It has dimension at least 2 since $d\lambda$ cannot be non-degenerate since M is closed, and $\mathcal{V}_\omega$ has dimension at most 2 since $d\lambda - \alpha \wedge \lambda$ is non-degenerate. ☐

The significance of ω-admissible almost complex structure is the following.

Lemma 2.7. *Let (M, ω, J) be an exact lcs triple. Then given a smooth $u : \Sigma \to M$, where Σ is a closed (nodal) Riemann surface, u is J-holomorphic only if $u^* d\lambda = 0$.*

Proof. We have

$$I = \int_\Sigma u^* d\lambda \geq 0$$

since J preserves $\mathcal{V}_\omega$. On the other hand $I > 0$ is impossible by Stokes theorem. So $I = 0$. Since J also preserves ξ_ω, this can happen only if the image of du is contained in the distribution $\mathcal{V}_\omega$. From this our conclusion follows. ☐

Definition 2.8. *Let (M, ω) be an exact* lcs *manifold as above. For Σ a closed (at the moment possibly nodal) Riemann surface, we say that a smooth map $u : \Sigma \to M$ is a* **Reeb curve** *if it a branched cover of a smoothly embedded curve $\Sigma' \to M$, if $u^* d\lambda = 0$, and if*

$$[u^*\alpha] \neq 0 \in H^1_{DR}(\Sigma).$$

This can be understood as a generalization of the condition of being a Reeb torus.

Lemma 2.9. *Let (M, ω, J) be an exact* lcsm *as above. Then every embedded J-holomorphic curve $u : \Sigma \to M$ is Reeb.*

This almost follows by the above discussion except for the final condition that needs to be verified. This is to be proved in Section 5. We then have the following "conformal symplectic Weinstein conjecture".

Conjecture 1. *Let M be closed, and ω be an exact* lcs *structure on M, so that the cohomology class of the Lee form α of ω is rational, then there is a elliptic Reeb curve $u : \Sigma \to M$, meaning that the domain Σ is an elliptic curve* [1].

Theorem 2.10. *The conjecture above holds for a C^0 neighborhood of the Hopf* lcs *structure ω_H.*

This is proved in Section 5.

Conjecture 2. *Suppose we are given an exact* lcs *triple (M, ω, J), with M closed, and so that the cohomology class of the Lee form α of ω is rational. Then there is a non-constant J-holomorphic elliptic curve.*

This immediately implies the CSW conjecture by Proposition 2.2 and by Lemma 2.9. Conjecture 2 is probably too much to hope for in such generality but when M has dimension 4, this looks very close to fundamental results of Taubes [26] on Gromov-Witten theory of symplectic 4-folds.

Conjecture 2 immediately implies the Weinstein conjecture for a closed contact (C, λ). For by the proof of Proposition 2.2, any elliptic curve $u : \Sigma \to M = C \times S^1$, with respect to the Banyaga lcs structure $d^\alpha \lambda$, $\alpha = d\theta$, must cover a Reeb torus. The following is a bit less immediate.

Theorem 2.11. *Conjecture 1 implies the Weinstein conjecture.*

This is also proved in Section 5.

Of course the most promising cases for the above conjecture is when $M = C \times S^1$ with C a 3-fold, or $C = S^{2k+1}$ since the Weinstein conjecture is already proved in these cases as previously mentioned.

The above is not just a curiosity. In contact geometry, rigidity is based on existence phenomena of Reeb orbits, and lcs manifolds should be understood as generalized contact manifolds. To attack rigidity questions in lcs geometry, like Question 2 we need an analogue of Reeb orbits, we propose that this analogue is Reeb curves as above, from which point of view the above conjecture becomes very natural.

We may reformulate Theorem 2.3 as follows.

Theorem 2.12. *Either the conformal symplectic Weinstein conjecture holds for any exact* lcs *structure ω on $M = S^{2k+1} \times S^1$, with ω homotopic through* lcs *structures to the Hopf* lcs *structure, or holomorphic sky catastrophes exist. In fact there is a sky catastrophe for a family $\{J_t\}$, with J_t admissible with respect to an* lcs *form ω_t for each t.*

Proof. Let $\{\omega_t\}$, $t \in [0, 1]$, $\omega_0 = \omega$, and ω_1 the Hopf lcs structure, be a smooth family of exact lcs structures on $S^{2k+1} \times S^1$. Fix a smooth family $\{J_t\}$ of almost complex structures on M, so that J_t is ω_t-admissible for each t. Then by Theorem 2.3 either there is a non-constant elliptic J_0-holomorphic in M, and so a non-constant elliptic Reeb curve, or there exists a holomorphic sky catastrophy for the family $\{(\omega_t, J_t\}$. □

[1] We further conjecture that it must be non-nodal when α is rational.

If holomorphic sky catastrophes are discovered, this will be a tremendous discovery. The original discovery by Fuller [10] of sky catastrophes in dynamical systems is one of the most important in dynamical systems, see also [24] for an overview. On the other hand pseudo-holomorphic curves in lcs manifolds look to be much more rigid objects than periodic orbits of general smooth dynamical systems. So it is possible that holomorphic sky catastrophes do not exist, at least for compatible families $\{(\omega_t, J_t)\}$ as above.

2.2. **Connection with the Fuller index.** If β is a free homotopy class of a loop in C denote by A_β the induced homology class of a Reeb torus in $C \times S^1$. Then we have:

Theorem 2.13.
$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(\widetilde{N}, R^\lambda, \beta),$$
where $N \subset \overline{\mathcal{M}}_1(J^\lambda, A_\beta)$ *is an open compact set,* $\widetilde{N}$ *the corresponding subset of periodic orbits of* R^λ*,* $i(\widetilde{N}, R^\lambda, \beta)$ *is the Fuller index as described in the appendix below, and where the left hand side of the equation is a certain Gromov-Witten invariant, that we discuss in Section 3, just below.*

What about higher genus invariants of $C \times S^1$? Following Proposition 2.2, it is not hard to see that all J^λ-holomorphic curves must be branched covers of Reeb tori. If one can show that these branched covers are regular when the underlying tori are regular, the calculation of invariants would be fairly automatic from this data, see [33], [31] where these kinds of regularity calculation are made.

3. Elements of Gromov-Witten theory of an lcs manifold

Suppose (M, J) is a compact almost complex manifold, let $N \subset \overline{\mathcal{M}}_{g,k}(J, A)$ be an open compact subset with energy positive on N. The latter condition is only relevant when $A = 0$. We shall primarily refer in what follows to work of Pardon in [19], only because this is what is more familiar to the author, due to greater comfort with algebraic topology as opposed to analysis. But we should mention that the latter is a follow up to a profound theory that is originally created by Fukaya-Ono [9], and later expanded with Oh-Ohta [8].

The construction in [19] of implicit atlas, on the moduli space $\mathcal{M}$ of curves in a symplectic manifold, only needs a neighborhood of $\mathcal{M}$ in the space of all curves. So more generally if we have an almost complex manifold and an *open* compact component N as above, this will likewise have a natural implicit atlas, or a Kuranishi structure in the setup of [9]. And so such an N will have a virtual fundamental class in the sense of Pardon [19], (or in any other approach to virtual fundamental cycle, particularly the original approach of Fukaya-Oh-Ohta-Ono). This understanding will be used in other parts of the paper, following Pardon for the explicit setup. We may thus define functionals:

$$GW_{g,n}(N, A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}. \tag{3.1}$$

The first question is: how do these functionals depend on N, J?

Lemma 3.2. *Let* $\{J_t\}$*,* $t \in [0, 1]$ *be a continuous in the* C^∞ *topology homotopy. Suppose that* $\widetilde{N}$ *is an open compact subset of the cobordism moduli space* $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ *and that the energy function is positive on* $\widetilde{N}$*, (the latter only relevant when* $A = 0$*). Let*
$$N_i = \widetilde{N} \cap \left(\overline{\mathcal{M}}_{g,n}(J_i, A)\right),$$
then
$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1).$$
In particular if $GW_{g,n}(N_0, A, J_0) \neq 0$*, there is a class* A J_1*-holomorphic stable map in* M*.*

Proof of Lemma 3.2. We may construct exactly as in [19] a natural implicit atlas on $\widetilde{N}$, with boundary $N_0^{op} \sqcup N_1$, (*op* denoting opposite orientation). And so
$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1),$$
as functionals. □

The most basic lemma in this setting is the following, and we shall use it in the following section.

Definition 3.3. *An* **almost symplectic pair** *on M is a tuple (M, ω, J), where ω is a non-degenerate 2-form on M, and J is ω-compatible.*

Definition 3.4. *We say that a pair of almost symplectic pairs (ω_i, J_i) are δ***-close***, if $\{\omega_i\}$, and $\{J_i\}$ are C^0 δ-close, $i = 0, 1$.*

Lemma 3.5. *Given a compact M and an almost symplectic tuple (ω, J) on M, suppose that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = \mathrm{energy}_{\omega}^{-1}(E^0, E^1)\right) \subset \left(V = \mathrm{energy}_{\omega}^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

with $\epsilon > 0$, $E^0 > 0$ and with $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$. Suppose also that $GW_{g,n}(N, J, A) \neq 0$. Then there is a $\delta > 0$ s.t. whenever (ω', J') is a compatible almost symplectic pair δ-close to (ω, J), there exists $u \in \overline{\mathcal{M}}_{g,n}(J', A) \neq \emptyset$, with

$$E^0 < \mathrm{energy}_{\omega'}(u) < E^1.$$

Proof of Lemma 3.5.

Lemma 3.6. *Given a Riemannian manifold (M, g), and J an almost complex structure, suppose that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = \mathrm{energy}_{g}^{-1}(E^0, E^1)\right) \subset \left(V = \mathrm{energy}_{g}^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

with $\epsilon > 0$, $E_0 > 0$, and with $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$. Then there is a $\delta > 0$ s.t. whenever (g', J') is C^0 δ-close to (g, J) if $u \in \overline{\mathcal{M}}_{g,n}(J', A)$ and

$$E^0 - \epsilon < \mathrm{energy}_{g'}(u) < E^1 + \epsilon$$

then

$$E^0 < \mathrm{energy}_{g'}(u) < E^1.$$

Proof of Lemma 3.6. Suppose otherwise then there is a sequence $\{(g_k, J_k)\}$ C^0 converging to (g, J), and a sequence $\{u_k\}$ of J_k-holomorphic stable maps satisfying

$$E^0 - \epsilon < \mathrm{energy}_{g_k}(u_k) \leq E^0$$

or

$$E^1 \leq \mathrm{energy}_{g_k}(u_k) < E^1 + \epsilon.$$

By Gromov compactness we may find a Gromov convergent subsequence $\{u_{k_j}\}$ to a J-holomorphic stable map u, with

$$E^0 - \epsilon < \mathrm{energy}_{g}(u) \leq E^0$$

or

$$E^1 \leq \mathrm{energy}_{g}(u) < E^1 + \epsilon.$$

But by our assumptions such a u does not exist. □

Lemma 3.7. *Given a compact almost symplectic compatible triple (M, ω, J), so that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is exactly as in the lemma above. There is a $\delta' > 0$ s.t. the following is satisfied. Let (ω', J') be δ'-close to (ω, J), then there is a smooth family of almost symplectic pairs $\{(\omega_t, J_t)\}$, $(\omega_0, J_0) = (g, J)$, $(\omega_1, J_1) = (g', J')$ s.t. there is open compact subset*

$$\widetilde{N} \subset \overline{\mathcal{M}}_{g,n}(\{J_t\}, A),$$

and with

$$\widetilde{N} \cap \overline{\mathcal{M}}(J, A) = N.$$

Moreover if $(u, t) \in \widetilde{N}$ then

$$E^0 < \mathrm{energy}_{g_t}(u) < E^1.$$

Proof. Given ϵ as in the hypothesis let δ be as in Lemma 3.6.

Lemma 3.8. *Given a $\delta > 0$ there is a $\delta' > 0$ s.t. if (ω', J') is δ'-near (ω, J) there is an interpolating family $\{(\omega_t, J_t)\}$ with (ω_t, J_t) δ-close to (ω, J) for each t.*

Proof. Let $\{g_t\}$ be the family of metrics on M given by the convex linear combination of $g = g_{\omega_J}, g' = g_{\omega', J'}$. Clearly g_t is δ'-close to g_0 for each t. Likewise the family of 2 forms $\{\omega_t\}$ given by the convex linear combination of ω, ω' is non-degenerate for each t if δ' was chosen to be sufficiently small and is δ'-close to $\omega_0 = \omega_{g,J}$ for each moment.

Let

$$ret : Met(M) \times \Omega(M) \to \mathcal{J}(M)$$

be the "retraction map" (it can be understood as a retraction followed by projection) as defined in [17, Prop 2.50], where $Met(M)$ is space of metrics on M, $\Omega(M)$ the space of 2-forms on M, and $\mathcal{J}(M)$ the space of almost complex structures. This map has the property that the almost complex structure $ret(g, \omega)$ is compatible with ω. Then $\{(\omega_t, ret(g_t, \omega_t)\}$ is a compatible family. As ret is continuous in the C^∞-topology, δ' can be chosen so that $\{ret_t(g_t, \omega_t\}$ are δ-nearby. ◻

Let δ' be chosen with respect to δ as in the above lemma and $\{(\omega_t, J_t)\}$ be the corresponding family. Let $\widetilde{N}$ consist of all elements $(u,t) \in \overline{\mathcal{M}}(\{J_t\}, A)$ s.t.

$$E^0 - \epsilon < \text{energy}_{\omega_t}(u) < E^1 + \epsilon.$$

Then by Lemma 3.6 for each $(u,t) \in \widetilde{N}$, we have:

$$E^0 < \text{energy}_{\omega_t}(u) < E^1.$$

In particular $\widetilde{N}$ must be closed, it is also clearly open, and is compact as energy is a proper function. ◻

To finish the proof of the main lemma, let N be as in the hypothesis, δ' as in Lemma 3.7, and $\widetilde{N}$ as in the conclusion to Lemma 3.7, then by Lemma 3.2

$$GW_{g,n}(N_1, J', A) = GW_{g,n}(N, J, A) \neq 0,$$

where $N_1 = \widetilde{N} \cap \overline{\mathcal{M}}_{g,n}(J_1, A)$. So the conclusion follows. ◻

Proof of Proposition 1.9. For each $u \in \overline{\mathcal{M}}_{g,n}(J_i, A)$, $i = 0, 1$ fix an open-compact subset V_u of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ containing u. We can do this by the hypothesis that there are no sky catastrophes. Since $\overline{\mathcal{M}}_{g,n}(J_i, A)$ are compact we may find a finite subcover

$$\{V_{u_i}\} \cap (\overline{\mathcal{M}}_{g,n}(J_0, A) \cup \overline{\mathcal{M}}_{g,n}(J_1, A))$$

of $\overline{\mathcal{M}}_{g,n}(J_0, A) \cup \overline{\mathcal{M}}_{g,n}(J_1, A)$, considering $\overline{\mathcal{M}}_{g,n}(J_0, A) \cup \overline{\mathcal{M}}_{g,n}(J_1, A)$ as a subset of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ naturally. Then $V = \bigcup_i V_{u_i}$ is an open compact subset of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$, s.t.

$$V \cap \overline{\mathcal{M}}_{g,n}(J_i, A) = \overline{\mathcal{M}}_{g,n}(J_i, A).$$

Now apply Lemma 3.2.

Likewise if only J_0 is bounded, for each $u \in \overline{\mathcal{M}}_{g,n}(J_0, A)$, fix an open-compact subset V_u of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ containing u. Since $\overline{\mathcal{M}}_{g,n}(J_0, A)$ is compact we may find a finite subcover

$$\{V_{u_i}\} \cap (\overline{\mathcal{M}}_{g,n}(J_0, A)$$

of $\overline{\mathcal{M}}_{g,n}(J_0, A)$. Then $V = \bigcup_i V_{u_i}$ is an open compact subset of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$, s.t.

$$V \cap \overline{\mathcal{M}}_{g,n}(J_i, A) = \overline{\mathcal{M}}_{g,n}(J_i, A).$$

Again apply Lemma 3.2. ◻

Proof of Theorem 1.2. (Outline, as the argument is standard.) Suppose that we have a sequence u^k of J-holomorphic maps with L^2-energy $\leq E$. By [18, 4.1.1], a sequence u^k of J-holomorphic curves has a convergent subsequence if $sup_k ||du^k||_{L^\infty} < \infty$. On the other hand when this condition does not hold rescaling argument tells us that a holomorphic sphere bubbles off. The quantization Theorem 1.1, then tells us that these bubbles have some minimal energy, so if the total energy is capped by E, only finitely many bubbles may appear, so that a subsequence of u^k must converge in the Gromov topology to a Kontsevich stable map. ◻

4. Rulling out sky catastrophes and non-squeezing

Proof of Theorem. 1.10 We shall actually prove a stronger statement that there is a universal energy bound from above for class A, J_t-holomorphic curves. Suppose otherwise, then there is a sequence $\{u_k\}$ of J_{t_k}-holomorphic class A curves, with $\text{energy}_{\omega_{t_k}} u_k \mapsto \infty$ as $k \mapsto \infty$. We may assume that t_k is convergent to $t_0 \in [0,1]$. Let $\{\widetilde{u}_k\}$ be a lift of the curves to the covering space $\widetilde{M} \xrightarrow{\pi} M$ determined by the class α as described prior to the formulation of the theorem. If the image of $\{\widetilde{u}_k\}$ is contained (for a specific choice of lifts) in a compact $K \subset \widetilde{M}$ then we have:

$$\text{energy}_{t_k}(\widetilde{u}_k) \simeq \text{energy}_{t_0}(\widetilde{u}_k) \leq C\langle \widetilde{\omega}^{symp}_{t_0}, A\rangle,$$

where $\widetilde{\omega}_{t_0} = f\widetilde{\omega}^{symp}$ for $\widetilde{\omega}^{symp}$ symplectic on $\widetilde{M}$, $f > 0$ and $C = sup_K f$. Hence energy would be universally bounded for all $\{u_k\}$.

Suppose there is no such K. Let $\{u_k\}$ be the corresponding sequence. We may suppose that image u_k does not intersect Σ_i for all k and i, since otherwise u_k must be a branched covering map of a leaf of the J_{t_k}-holomorphic foliation of Σ_i by the positivity of intersections, and consequently all such u_k have lifts contained in a specific compact subset of $\widetilde{M}$.

The Lee class α has a natural differential form representative, called the Lee form and defined as follows. We take a cover of M by open sets U_a in which $\omega = f_a \cdot \omega_a$ for ω_a symplectic, and f_a a positive smooth function. Then we have 1-forms $d(\ln f_a)$ in each U_a which glue to a well defined closed 1-form on M. We conflate the notation for this 1-form with its cohomology class, and the Cech 1-cocycle α defined as before.

By our hypothesis that $PD(\alpha) = \sum_i a_i[\Sigma_i]$ we have that $\pi^{-1}(M - \bigcup_i \Sigma_i)$ is a disjoint union $\sqcup_i K_i$ of bounded subsets, with respect to the proper function on $\widetilde{M}$ determined by the Lee 1-form α. Then for some k' sufficiently large, the image of some lift $\widetilde{u}_{k'}$ intersects more then one of the K_i, and so $u_{k'}$ intersects some Σ_i, a contradiction.

□

Proof of Theorem 1.7. We need to say what is our C^0 topology on the space of lcs forms. This is defined via the following metric.

Definition 4.1. *The metric d on on the space $\mathcal{L}(M)$ of* lcs *2-forms on M, is defined with respect to a fixed Riemannian metric g on M, and is given by*

$$d(\omega_1, \omega_2) = d_{mass}(\omega_1, \omega_2) + d_{mass}(\alpha_1, \alpha_2),$$

for α_i the Lee forms as above and d_{mass} the metrics induced by the co-mass norms $||\cdot||_{mass}$ with respect to g on differential k-forms. That is $||\eta||_{mass}$ is the supremum over unit k-vectors v of $|\eta(v)|$.

Explicitly this means the following: a sequence $\{\omega_k\} \subset \mathcal{L}(M)$ converges to a symplectic form ω, iff $\{\omega_k\}$, converges to ω, and iff given the lift sequence $\widetilde{\omega}_k = f_k\widetilde{\omega}^{symp}_k$, on the universal cover $\widetilde{M}$, with ω^{symp}_k symplectic, the sequence $\{f_k\}$ is a sequence of positive functions converging to 1 on compact sets.

Fix an $\epsilon' > 0$ s.t. any 2-form ω_1 on M, C^0 ϵ'-close to ω, is non-degenerate, and is non-degenerate on the leaves of Σ_i. Suppose by contradiction that for every $\epsilon > 0$ there exists an lcs embedding

$$\phi: B_R \hookrightarrow (M, \omega_1),$$

satisfying the conditions. Assume that $\epsilon < \epsilon'$, and let $\{\omega_t\}$ be as in the hypothesis. By assumptions ω_t is an lcs form for each t, and is non-degenerate on the leaves of Σ_i. Extend $\phi_* j$ to an almost complex structure J_1 on M, preserving $T^{fol}\Sigma_i$. We may then extend this to a family $\{J_t\}$ of almost complex structures M, s.t. J_t is ω_t-compatible for each t, and such that J_t preserves $T^{fol}\Sigma$ for each i, since the foliation of Σ_i is ω_t-symplectic for each t. (For construction of $\{J_t\}$ use for example the map *ret* from Lemma 3.8). When the image of ϕ does not intersect Σ_i this is vacuous. Then the family $\{(\omega_t, J_t)\}$ satisfies the hypothesis of Theorem 1.10, and so has no sky catastrophes in class A. Consequently by Lemma 3.2 there is a class A J_1-holomorphic curve u passing through $\phi(0)$. By the proof of Theorem

1.10 we may choose a lift to $\widetilde{M}$ for each such curve u so that it is contained in a compact set $K \subset \widetilde{M}$, (independent of ϵ and all other choices). Now by definition of our d on $\mathcal{L}(M)$, for every δ we may find an ϵ so that if ω_1 is ϵ-close to ω then $\widetilde{\omega}^{symp}$ is δ-close to $\widetilde{\omega}_1^{symp}$ on K. Since $\langle \widetilde{\omega}^{symp}, [\widetilde{u}] \rangle = \pi r^2$, if δ above is chosen to be sufficiently small then

$$|\max_K f_1 \langle \widetilde{\omega}_1^{symp}, [\widetilde{u}] \rangle - \pi \cdot r^2| < \pi R^2 - \pi r^2,$$

since

$$|\langle \widetilde{\omega}_1^{symp}, [\widetilde{u}] \rangle - \pi \cdot r^2| \simeq |\langle \widetilde{\omega}^{symp}, [\widetilde{u}] \rangle - \pi \cdot r^2| = 0,$$

for δ small enough, and $\max_K f_1 \simeq 1$ for δ small enough, where $\simeq$ denotes approximate equality. In particular we get that ω_1-area of u is less then πR^2.

We may then proceed as in the classical proof Gromov [12] of the non-squeezing theorem to get a contradiction and finish the proof. More specifically $\phi^{-1}(\operatorname{image} \phi \cap \operatorname{image} u)$ is a (nodal) minimal surface in B_R, with boundary on the boundary of B_R, and passing through $0 \in B_R$. By construction it has area strictly less then πR^2 which is impossible by a classical result of differential geometry, (the monotonicity theorem.) □

5. Connections of GW-theory of an lcsm $C \times S^1$ with the Fuller index

Proof of Proposition 2.2. Suppose we a have a curve without spherical nodal components $u \in \overline{\mathcal{M}}_{1,1}(J^\lambda, A)$. We claim that $(pr_C \circ u)_*$ everywhere has rank ≤ 1, for $pr_C : C \times S^1 \to C$ the projection. Suppose otherwise, then it is immediate by construction of J^λ, that

$$\int_\Sigma (pr_C \circ u)^* d\lambda > 0,$$

for Σ domain of u, but $d\lambda$ is exact so that that this is impossible. It clearly follows from this that Σ must be smooth, (non-nodal).

Next observe that when the rank of $(pr_C \circ u)_*$ is 1, its image is in the Reeb line sub-bundle of TC, for otherwise the image has a contact component, but this is J^λ invariant and so again we get that $\int_\Sigma (pr_C \circ u)^* d\lambda > 0$. We now show that the image of $pr_C \circ u$ is in fact the image of some Reeb orbit.

Pick an identification of the domain Σ of u with a marked Riemann surface (T^2, j), T^2 the standard torus. We shall use throughout coordinates (θ_1, θ_2) on T^2 $\theta_1, \theta_2 \in S^1$, with S^1 unit complex numbers. Then by assumption on the class A (and WLOG)

$$\theta \mapsto pr_{S^1} \circ u(\{\theta_0^1\} \times \{\theta\}),$$

is a degree 1 curve, where $pr_{S^1} : C \times S^1 \to S^1$ is the projection. And so by the Sard theorem we have a regular value θ_0, so that $u^{-1} \circ pr_{S^1}^{-1}(\theta_0)$ contains an embedded circle $S_0 \subset T^2$. Now $d(pr_{S^1} \circ u)$ is surjective along $T(T^2)|_{S_0}$, which means, since u is J^λ-holomorphic, that $pr_C \circ u|_{S_0}$ has non-vanishing differential. From this and the discussion above it follows that image of $pr_C \circ u$ is the image of some embedded Reeb orbit o_u. Consequently the image of u is contained in the image of the Reeb torus of o_u, and so (again by the assumption on A) u is a itself a Reeb torus map for some o covering o_u.

The statement of the lemma follows when u has no spherical nodal components. On the other hand non-constant holomorphic spheres are impossible also by the previous argument. So there are no nodal elements in $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$ which completes the argument. □

Proposition 5.1. *Let (C, ξ) be a general contact manifold. If λ is non-degenerate contact 1-form for ξ, then all the elements of $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$ are regular curves. Moreover if λ is degenerate then for a period P Reeb orbit o the kernel of the associated real linear Cauchy-Riemann operator for the Reeb torus of o is naturally identified with the 1-eigenspace of $\phi^\lambda_{P,*}$ - the time P linearized return map $\xi(o(0)) \to \xi(o(0))$ induced by the R^λ Reeb flow.*

Proof. We have previously shown that all $[u, j] \in \overline{\mathcal{M}}_{1,1}(J^\lambda, A)$, are represented by smooth immersed curves, (covering maps of Reeb tori). Since each u is immersed we may naturally get a splitting $u^*T(C \times S^1) \simeq N_u \times T(T^2)$, using g_J metric, where N_u denotes the pull-back normal bundle, which is identified with the pullback along the projection of $C \times S^1 \to C$ of the distribution ξ.

The full associated real linear Cauchy-Riemann operator takes the form:

$$D_u^J : \Omega^0(N_u \oplus T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2), N_u \oplus T(T^2)). \tag{5.2}$$

This is an index 2 Fredholm operator (after standard Sobolev completions), whose restriction to $\Omega^0(N_u \oplus T(T^2))$ preserves the splitting, that is the restricted operator splits as

$$D \oplus D' : \Omega^0(N_u) \oplus \Omega^0(T(T^2)) \to \Omega^{0,1}(T(T^2), N_u) \oplus \Omega^{0,1}(T(T^2), T(T^2)).$$

On the other hand the restricted Fredholm index 2 operator

$$\Omega^0(T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2)),$$

is surjective by classical Teichmuller theory, see also [32, Lemma 3.3] for a precise argument in this setting. It follows that D_u^J will be surjective if the restricted Fredholm index 0 operator

$$D : \Omega^0(N_u) \to \Omega^{0,1}(N_u),$$

has no kernel.

The bundle N_u is symplectic with symplectic form on the fibers given by restriction of $u^* d\lambda$, and together with J^λ this gives a Hermitian structure on N_u. We have a linear symplectic connection A on N_u, which over the slices $S^1 \times \{\theta_2'\} \subset T^2$ is induced by the pullback by u of the linearized R^λ Reeb flow. Specifically the A-transport map from $N|_{(\theta_1', \theta_2')}$ to $N|_{(\theta_1'', \theta_2')}$ over $[\theta_1', \theta_2''] \times \{\theta_2'\} \subset T^2$, $0 \leq \theta_1' \leq \theta_2'' \leq 2\pi$ is given by

$$(u_*|_{N|_{(\theta_1'', \theta_2')}})^{-1} \circ \phi^\lambda_{mult \cdot (\theta_1'' - \theta_1')} \circ u_*|_{N|_{(\theta_1', \theta_2')}},$$

where $mult$ is the multiplicity of o and where $\phi^\lambda_{mult\cdot(\theta_1''-\theta_1')}$ is the time $mult \cdot (\theta_1'' - \theta_1')$ map for the R^λ Reeb flow.

The connection A is defined to be trivial in the θ_2 direction, where trivial means that the parallel transport maps are the id maps over θ_2 rays. In particular the curvature R_A of this connection vanishes. The connection A determines a real linear CR operator on N_u in the standard way (take the complex anti-linear part of the vertical differential of a section). It is elementary to verify from the definitions that this operator is exactly D.

We have a differential 2-form Ω on the N_u which in the fibers of N_u is just the fiber symplectic form and which is defined to vanish on the horizontal distribution. The 2-form Ω is closed, which we may check explicitly by using that R_A vanishes to obtain local symplectic trivializations of N_u in which A is trivial. Clearly Ω must vanish on the 0-section since it is a A-flat section. But any section is homotopic to the 0-section and so in particular if $\mu \in \ker D$ then Ω vanishes on μ. But then since $\mu \in \ker D$, and so its vertical differential is complex linear, it must follow that the vertical differential vanishes, since $\Omega(v, J^\lambda v) > 0$, for $0 \neq v \in T^{vert} N_u$ and so otherwise we would have $\int_\mu \Omega > 0$. So μ is A-flat, in particular the restriction of μ over all slices $S^1 \times \{\theta_2'\}$ is identified with a period P orbit of the linearized at o R^λ Reeb flow, which does not depend on θ_2' as A is trivial in the θ_2 variable. So the kernel of D is identified with the vector space of period P orbits of the linearized at o R^λ Reeb flow, as needed. □

Proposition 5.3. *Let λ be a contact form on a $2n+1$-fold C, and o a non-degenerate, period P, R^λ-Reeb orbit, then the orientation of $[u_o]$ induced by the determinant line bundle orientation of $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$, is $(-1)^{CZ(o)-n}$, which is*

$$\operatorname{sign} \operatorname{Det}(\operatorname{Id}|_{\xi(o(0))} - \phi^\lambda_{P,*}|_{\xi(o(0))}).$$

Proof of Proposition 5.3. Abbreviate u_o by u. Fix a trivialization ϕ of N_u induced by a trivialization of the contact distribution ξ along o in the obvious sense: N_u is the pullback of ξ along the composition

$$T^2 \to S^1 \xrightarrow{o} C.$$

Then the pullback A' of A (as above) to $T^2 \times \mathbb{R}^{2n}$ is a connection whose parallel transport path along $S^1 \times \{\theta_2\}$, $p : [0,1] \to \operatorname{Symp}(\mathbb{R}^{2n})$, starting at 1, is θ_2 independent and so that the parallel transport path of A' along $\{\theta_1\} \times S^1$ is constant, that is A' is trivial in the θ_2 variable. We shall call such a connection A' on $T^2 \times \mathbb{R}^{2n}$ *induced by* p. By non-degeneracy assumption on o, the map $p(1)$ has no 1-eigenvalues. Let $p'' : [0,1] \to \mathrm{Symp}(\mathbb{R}^{2n})$ be a path from $p(1)$ to a unitary map $p''(1)$, with $p''(1)$ having no 1-eigenvalues, s.t. p'' has only simple crossings with the Maslov cycle. Let p' be the concatenation of p and p''. We then get

$$CZ(p') - \frac{1}{2}\operatorname{sign}\Gamma(p', 0) \equiv CZ(p') - n \equiv 0 \mod 2,$$

since p' is homotopic relative end points to a unitary geodesic path h starting at id, having regular crossings, and since the number of negative, positive eigenvalues is even at each regular crossing of h by unitarity. Here $\operatorname{sign}\Gamma(p', 0)$ is the index of the crossing form of the path p' at time 0, in the notation of [20]. Consequently

$$CZ(p'') \equiv CZ(p) - n \mod 2, \tag{5.4}$$

by additivity of the Conley-Zehnder index.

Let us then define a free homotopy $\{p_t\}$ of p to p', p_t is the concatenation of p with $p''|_{[0,t]}$, reparametrized to have domain $[0,1]$ at each moment t. This determines a homotopy $\{A'_t\}$ of connections induced by $\{p_t\}$. By the proof of Proposition 5.1, the CR operator D_t determined by each A'_t is surjective except at some finite collection of times $t_i \in (0,1)$, $i \in N$ determined by the crossing times of p'' with the Maslov cycle, and the dimension of the kernel of D_{t_i} is the 1-eigenspace of $p''(t_i)$, which is 1 by the assumption that the crossings of p'' are simple.

The operator D_1 is not complex linear so we concatenate the homotopy $\{D_t\}$ with the homotopy $\{\widetilde{D}_t\}$ induced by the homotopy $\{\widetilde{A}_t\}$ of A'_1 to a unitary connection $\widetilde{A}_1$, where the homotopy $\{\widetilde{A}_t\}$, is through connections induced by paths $\{\widetilde{p}_t\}$, giving a homotopy relative end points of $p' = \widetilde{p}_0$ to a unitary path $\widetilde{p}_1$ (for example h above). Let us denote by $\{D'_t\}$ the concatenation of $\{D_t\}$ with $\{\widetilde{D}_t\}$. By construction in the second half of the homotopy $\{D'_t\}$, D'_t is surjective. And D'_1 is induced by a unitary connection, since it is induced by unitary path $\widetilde{p}_1$. Consequently D'_1 is complex linear. By the above construction, for the homotopy $\{D'_t\}$, D'_t is surjective except for N times in $(0,1)$, where the kernel has dimension one. In particular the sign of $[u]$ by the definition via the determinant line bundle is exactly

$$-1^N = -1^{CZ(p)-n},$$

by (5.4), which was what to be proved. □

Thus if $N \subset \overline{\mathcal{M}}_{1,1}(J^\lambda, A_\beta)$ is open-compact and consists of isolated regular Reeb tori $\{u_i\}$, corresponding to orbits $\{o_i\}$ we have:

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = \sum_i \frac{(-1)^{CZ(o_i)-n}}{mult(o_i)},$$

where the denominator $mult(o_i)$ is there because our moduli space is understood as a non-effective orbifold, see Appendix B.

The expression on the right is exactly the Fuller index $i(\widetilde{N}, R^\lambda, \beta)$. Thus the theorem follows for N as above. However in general if N is open and compact then perturbing slightly we obtain a smooth family $\{R^{\lambda_t}\}$, $\lambda_0 = \lambda$, s.t. λ_1 is non-degenerate, that is has non-degenerate orbits. And such that there is an open-compact subset $\widetilde{N}$ of $\overline{\mathcal{M}}_{1,1}(\{J^{\lambda_t}\}, A_\beta)$ with $(\widetilde{N} \cap \overline{\mathcal{M}}_{1,1}(J^\lambda, A_\beta)) = N$, cf. Lemma 3.7. Then by Lemma 3.2 if

$$N_1 = (\widetilde{N} \cap \overline{\mathcal{M}}_{1,1}(J^{\lambda_1}, A_\beta))$$

we get

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}_{1,1}] \otimes [C \times S^1]).$$

By previous discussion

$$GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(N_1, R^{\lambda_1}, \beta),$$

but by the invariance of Fuller index (see Appendix A),

$$i(N_1, R^{\lambda_1}, \beta) = i(N, R^\lambda, \beta).$$

This finishes the proof of Theorem 2.13 □

Proof of Theorem 2.4. Let $N \subset \overline{\mathcal{M}}_{1,1}(A, J^\lambda)$, be the subspace corresponding, (under the Reeb tori, Reeb orbit correspondence) to the subspace $\widetilde{N}$ of all period 2π R^λ-orbits. It is easy to compute see for instance [11]

$$i(\widetilde{N}, R^\lambda) = \pm\chi(\mathbb{CP}^k) \neq 0.$$

By Theorem 2.13 $GW_{1,1}(N, J^\lambda, A) \neq 0$. The theorem then follows by Lemma 3.5. □

Proof of Lemma 2.9. Let (M, ω, J) be an exact triple with $\omega = d^\alpha\lambda$. Suppose that $u : \Sigma \to M$ is an embedded J-holomorphic curve. By Lemma 2.7 we have that $u^*d\lambda = 0$. We only need to check that $[u^*\alpha] \neq 0$. Suppose otherwise. Let $\widetilde{M}$ denote the α-covering space of M, then the lift of ω to $\widetilde{M}$ is $\widetilde{\omega} = \frac{1}{f}d(f\lambda)$, where $f = e^g$ and where g is the primitive for the lift $\widetilde{\alpha}$ of α to $\widetilde{M}$, that is $\widetilde{\alpha} = dg$. In particular $\widetilde{\omega}$ is conformally exact on $\widetilde{M}$. Now $[u^*\alpha] = 0$, so u has a lift to a $\widetilde{J}$-holomorphic map $\widetilde{u} : \Sigma \to \widetilde{M}$, where $\widetilde{J}$ is the lift of J, which is compatible with $\widetilde{\omega}$. Since Σ is closed, it follows that $\widetilde{u}$ is constant, which contradicts our assumptions. □

Proof of Theorem 2.10. Let U be a C^0 metric ϵ-ball neighborhood of $(\omega_H, J_H := J^{\lambda_H})$ as in Theorem 2.4. If ω is C^0 δ-close to ω_H, then by our Definition 4.1, the pairs of distributions $(\mathcal{V}_\omega, \xi_\omega)$ and $(\mathcal{V}_{\omega_H}, \xi_{\omega_H})$ are C^0 δ'-close for δ' arbitrarily small if δ is taken to be sufficiently small. Here, the C^0 metric distance on pairs of distributions is defined naturally by fixing an auxiliary Riemannian metric g on M, and then defining one k-distribution $\mathcal{V}_1$ to be δ-close to a distribution $\mathcal{V}_2$ if for each $p \in M$ the g-unit spheres in $\mathcal{V}_1(p)$, $\mathcal{V}_2(p)$ are Hausdorff δ-close with respect to g on T_pM.

Then if δ was chosen to be suitably be small we may clearly find an ω-admissible J, so that (ω, J) is ϵ-close to (ω_H, J_H). The details for the construction of such J are in Lemma 3.8 using the map:

$$ret : Met(M) \times \Omega(M) \to \mathcal{J}(M).$$

Then the result immediately follows. □

Proof of Theorem 2.11. Let $u : \Sigma \to C \times S^1$ be an embedded elliptic Reeb curve for the Banyaga lcs structure $d^\alpha\lambda$ on $M = C \times S^1$,

$$\alpha = d\theta := pr_1^* d\theta,$$

where $pr_1 : C \times S^1 \to S^1$ is the projection. As in the proof of Proposition 2.2 we obtain that $(pr_C \circ u)_*$ everywhere has rank ≤ 1, for $pr_C : C \times S^1 \to C$ the projection, and when it has rank 1 the image is contained in $\ker d\lambda$.

By assumptions $[u^*\alpha] = [u^* \circ pr_1^* d\theta] \neq 0$, then applying Sard's theorem as in the proof of Proposition 2.2, we find an embedded circle $S_0 \subset \Sigma$, $S_0 \subset (pr_1 \circ u)^{-1}(\theta_0)$ for some regular value $\theta_0 \subset S^1$ for the map $pr_1 \circ u$. Then $pr_C \circ u|_{S_0}$ must be an embedding, for otherwise, since $S_0 \subset (pr_1 \circ u)^{-1}(\theta_0)$, $u|_{S_0}$ would not be an embedding. By the first paragraph the image of $u|_{S_0}$ must be the image of a Reeb orbit. □

6. Preliminaries for the Proof of Theorem 2.3

Much of the following discussion holds verbatim for general moduli spaces $\mathcal{M}_{g,n}(J, A, a_1, \ldots, a_n)$ with $a_1, \ldots, a_n$ homological constraints. We shall however restrict for simplicity to the case $g = 1, n = 1$ with trivial constraint $[M]$, as this is the interest of Theorem 2.4.

Definition 6.1. *For a smooth homotopy $h = \{(\omega_t, J_t)\}$ of* lcs *pairs on M, we say that it is* **partially admissible for** *A, if every element of*

$$\overline{\mathcal{M}}_{1,1}(M, J_0, A)$$

is contained in a compact open subset of $\overline{\mathcal{M}}_{1,1}(M, \{J_t\}, A)$*. We say that* h *is* **admissible for** A *if every element of*

$$\overline{\mathcal{M}}_{1,1}(M, J_i, A),$$

$i = 0, 1$ *is contained in a compact open subset of* $\overline{\mathcal{M}}_{1,1}(M, \{J_t\}, A)$*.*

Thus in the above definition, a homotopy is partially admissible if there are sky catastrophes going one way, and admissible if there are no sky catastrophes going either way.

Partly to simplify notation we denote by a capital X a general lcs pair (M, ω, J), then we introduce the following simplified notation.

$$\begin{aligned} S(X, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(X, A)\} \\ S(X, a, A) &= \{u \in S(X, A) \mid e(u) \leq a\}. \\ S(h, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(h, A)\} \\ S(h, a, A) &= \{u \in S(h, A) \mid e(u) \leq a\} \end{aligned} \tag{6.2}$$

Definition 6.3. *For an isolated element* u *of* $S(X, A)$*, which means that* $\{u\}$ *is open as a subset, we set* $gw(u, p) \in \mathbb{Q}$ *to be the local Gromov-Witten invariant of* u*. This is defined as:*

$$gw(u, p) = GW_{1,1}(\{u\}, A, J)([\overline{M}_{1,1}] \otimes [M]),$$

with the right hand side as in (3.1)*.*

Definition 6.4. *Suppose that* $S(X, A)$ *has open connected components. And suppose that we have a collection of* lcs *pairs*

$$\bigcup_{a>0} (X^a = (M, \omega^a, J^a)),$$

satisfying the following:

- $S(X^a, a, A)$ *consists of isolated curves for each* a*.*
- $$S(X^a, a, A) = S(X^b, a, A),$$
  (equality of subsets of $\overline{\mathcal{M}}_{1,1}(X, A) \times \mathbb{R}_+$*) if* $b > a$*, and the local Gromov-Witten invariants corresponding to the identified elements of these sets coincide.*
- *There is a prescribed homotopy* $h^a = \{X^a_t\}$ *of each* X^a *to* X*, called* **structure homotopy***, with the property that for every*
  $$y \in S(X^a_0, A)$$
  there is an open compact subset $\mathcal{C}_y \ni y$ *of* $S(h^a, A)$ *which is* **non-branching** *which means that*
  $$\mathcal{C}_y \cap S(X^a_i, A),$$
  $i = 0, 1$ *are connected.*
- $$S(h^a, a, A) = S(h^b, a, A),$$
  (equality of subsets) if $b > a$ *is sufficiently large.*

We will then say that

$$\mathcal{P}(A) = (\{X^a\}_a, h^a)$$

is a **perturbation system** *for* X *in the class* A*.*

We shall see shortly that given a contact (C, λ), the associated lcs structure on $C \times S^1$ always admits a perturbation system as above, if λ is Morse-Bott.

Definition 6.5. *Suppose that* X *admits a perturbation system* $\mathcal{P}(A)$ *so that there exists an* $E = E(\mathcal{P}(A))$ *with the property that*

$$S(X^a, a, A) = S(X^E, a, A)$$

for all $a > E$, where this as before is equality of subsets, and the local Gromov-Witten invariants of the identified elements are also identified. Then we say that X is **finite type** *and set:*

$$GW(X, A) = \sum_{u \in S(X^E, A)} gw(u).$$

Definition 6.6. *Suppose that X admits a perturbation system $\mathcal{P}(A)$ and there is an $E = E(\mathcal{P}(A)) > 0$ so that $gw(u) > 0$ for all*

$$\{u \in S(X^a, A) \,|\, E \leq e(u) \leq a\}$$

respectively $gw(u) < 0$ for all

$$\{u \in S(X^a, A) \,|\, E \leq e(u) \leq a\},$$

and every $a > E$. Suppose in addition that

$$\lim_{a \mapsto \infty} \sum_{u \in S(X,a,A)} gw(u) = \infty, \text{ respectively } \lim_{a \mapsto \infty} \sum_{u \in S(X,a,\beta)} gw(u) = -\infty.$$

Then we say that X is **positive infinite type**, *respectively* **negative infinite type** *and set*

$$GW(X, A) = \infty,$$

respectively $GW(X, A) = -\infty$. These are meant to be interpreted as extended Gromov-Witten invariants, counting elliptic curves in class A. We say that X is **infinite type** *if it is one or the other.*

Definition 6.7. *We say that X is* **definite** *type if it is infinite type or finite type.*

With the above definitions

$$GW(X, A) \in \mathbb{Q} \sqcup \infty \sqcup -\infty,$$

when it is defined.

Definition 6.8. *An* lcs *pair X is* **admissible** *if it admits a perturbation system, and if it is definite type.*

6.0.1. *Perturbation systems for Morse-Bott Reeb vector fields.*

Definition 6.9. *A contact form λ on M, and its associated flow R^λ are called* Morse-Bott *if the λ action spectrum $\sigma(\lambda)$ - that is the space of critical values of $o \mapsto \int_{S^1} o^* \lambda$, is discreet and if for every $a \in \sigma(\lambda)$, the space*

$$N_a := \{x \in M |\, F_a(x) = x\},$$

F_a the time a flow map for R^λ - is a closed smooth manifold such that rank $d\lambda|_{N_a}$ is locally constant and $T_x N_a = \ker(dF_a - I)_x$.

Proposition 6.10. *Let λ be a contact form of Morse-Bott type, on a closed contact manifold C. Then the corresponding* lcs *pair $X_\lambda = (C \times S^1, d^\alpha \lambda, J^\lambda)$ admits a perturbation system $\mathcal{P}(A)$, for every class A.*

Proof. This follows immediately by [22, Proposition 2.12], and by Proposition 2.2. ☐

Lemma 6.11. *The Hopf* lcs *pair $(S^{2k+1} \times S^1, d^\alpha \lambda_H, J^{\lambda_H})$, for λ_H the standard contact structure on S^{2k+1} is infinite type.*

Proof. This follows immediately by [21, Lemma 2. 13], and by Proposition 2.2. ☐

6.1. Preliminaries on admissible homotopies.

Definition 6.12. *Let $h = \{X_t\}$ be a smooth homotopy of* lcs *pairs. For $b > a > 0$ we say that h is* **partially** a, b**-admissible**, *respectively* a, b-**admissible** *(in class A) if for each*

$$y \in S(X_0, a, A)$$

there is a compact open subset $\mathcal{C}_y \ni y$ of $S(h, A)$, with $e(u) < b$, for all $u \in \mathcal{C}_y$. Respectively, if for each

$$y \in S(X_i, a, A),$$

$i = 0, 1$ there is a compact open subset $\mathcal{C}_y \ni y$ of $S(h, A)$ with $e(u) < b$, for all $u \in \mathcal{C}_y$.

Lemma 6.13. *Suppose that X_0 has a perturbation system $\mathcal{P}(A)$, and $\{X_t\}$ is partially admissible, then for every a there is a $b > a$ so that $\{\widetilde{X}^b_t\} = \{X_t\} \cdot \{X^b_t\}$ is partially a, b-admissible, where $\{X_t\} \cdot \{X^b_t\}$ is the (reparametrized to have t domain $[0, 1]$) concatenation of the homotopies $\{X_t\}, \{X^b_t\}$, and where $\{X^b_t\}$ is the structure homotopy from X^b to X_0.*

Proof. This is a matter of pure topology, and the proof is completely analogous to the proof of [21, Lemma 3.8]. □

The analogue of Lemma 6.13 in the admissible case is the following:

Lemma 6.14. *Suppose that X_0, X_1 and $\{X_t\}$ are admissible, then for every a there is a $b > a$ so that*

$$\{\widetilde{X}^b_t\} = \{X^b_{1,t}\}^{-1} \cdot \{X_t\} \cdot \{X^b_{0,t}\} \tag{6.15}$$

is a, b-admissible, where $\{X^b_{i,t}\}$ are the structure homotopies from X^b_i to X_i.

The proof of this is completely analogous to the proof of Lemma 6.13.

7. Proof of Theorem 2.3

Let us state a more general claim.

Theorem 7.1. *Suppose we have an admissible* lcs *pair X_0, with $GW(X_0, A) \neq 0$ which is joined to X_1 by a partially admissible homotopy $\{X_t\}$, then X_1 has non-constant elliptic class A curves.*

Theorem 2.3 clearly follows by the above and by Lemma 6.11. We also have a more a more precise result.

Theorem 7.2. *If X_0, X_1 are admissible* lcs *pairs and $\{X_t\}$ is admissible then $GW(X_0, A) = GW(X_1, A)$.*

Proof of Theorem 7.1. Suppose that X_0 is admissible with $GW(X_0, A) \neq 0$, $\{X_t\}$ is partially admissible and $\mathcal{M}_{1,1}(X_1, A) = \emptyset$. Let a be given and b determined so that $\widetilde{h}^b = \{\widetilde{X}^b_t\}$ is a partially (a, b)-admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\widetilde{h}^b, A),$$

for $y \in S(X^b_0, a, A)$. Here we use a natural identification of $S(X^b, a, A) = S(\widetilde{X}^b_0, a, A)$ as a subset of $S(\widetilde{h}^b, A)$ by its construction. Then S_a is an open-compact subset of $S(h, A)$ and so has admits an implicit atlas (Kuranishi structure) with boundary, (with virtual dimension 1) s.t:

$$\partial S_a = S(X^b, a, A) + Q_a,$$

where Q_a as a set is some subset (possibly empty), of elements $u \in S(X^b, b, A)$ with $e(u) \geq a$. So we have for all a:

$$\sum_{u \in Q_a} gw(u) + \sum_{u \in S(X^b, a, A)} gw(u) = 0. \tag{7.3}$$

7.1. **Case I, X_0 is finite type.** Let $E = E(\mathcal{P})$ be the corresponding cutoff value in the definition of finite type, and take any $a > E$. Then $Q_a = \emptyset$ and by definition of E we have that the left side is

$$\sum_{u \in S(X^b, E, A)} gw(u) \neq 0.$$

Clearly this gives a contradiction to (7.4).

7.2. **Case II, X_0 is infinite type.** We may assume that $GW(X_0, A) = \infty$, and take $a > E$, where $E = E(\mathcal{P}(A))$ is the corresponding cutoff value in the definition of infinite type. Then

$$\sum_{u \in Q_a} gw(u) \geq 0,$$

as $a > E(\mathcal{P}(A))$. While

$$\lim_{a \mapsto \infty} \sum_{u \in S(X^b, a, A)} gw(u) = \infty,$$

as $GW(X_0, A) = \infty$. This also contradicts (7.4). □

Proof of Theorem 7.2. This is somewhat analogous to the proof of Theorem 7.1. Suppose that $X_i, \{X_t\}$ are admissible as in the hypothesis. Let a be given and b determined so that $\widetilde{h}^b = \{\widetilde{X}^b_t\}$, see (6.15) is an (a, b)-admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\widetilde{h}^b, A)$$

for $y \in S(X^b_i, a, A)$. Then S_a is an open-compact subset of $S(h, A)$ and so has admits an implicit atlas (Kuranishi structure) with boundary, (with virtual dimension 1) s.t:

$$\partial S_a = (S(X^b_0, a, A) + Q_{a,0})^{op} + S(X^b_1, a, A) + Q_{a,1},$$

with *op* denoting opositite orientation and where $Q_{a,i}$ as sets are some subsets (possibly empty), of elements $u \in S(X^b_i, b, A)$ with $e(u) \geq a$. So we have for all a:

$$\sum_{u \in Q_{a,0}} gw(u) + \sum_{u \in S(X^b_0, a, A)} gw(u) = \sum_{u \in Q_{a,1}} gw(u) + \sum_{u \in S(X^b_1, a, A)} gw(u) \tag{7.4}$$

7.3. **Case I, X_0 is finite type and X_1 is infinite type.** Suppose in addition $GW(X_1, A) = \infty$ and let $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$, for $\mathcal{P}_i$, the perturbation systems of X_i. Take any $a > E$. Then $Q_{a,0} = \emptyset$ and the left hand side of (7.4) is

$$\sum_{u \in S(X^b_0, E, A)} gw(u).$$

While the right hand side tends to ∞ as a tends to infinity since,

$$\sum_{u \in Q_{a,1}} gw(u) \geq 0,$$

as $a > E(\mathcal{P}_1(A))$, and

$$\lim_{a \mapsto \infty} \sum_{u \in S(X^b_1, a, A)} gw(u) = \infty,$$

Clearly this gives a contradiction to (7.4).

7.4. **Case II, X_i are infinite type.** Suppose in addition $GW(X_0, A) = -\infty$, $GW(X_1, A) = \infty$ and let $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$, for $\mathcal{P}_i$, the perturbation systems of X_i. Take any $a > E$. Then $\sum_{u \in Q_{a,0}} gw(u) \leq 0$, and $\sum_{u \in Q_{a,1}} gw(u) \geq 0$. So by definition of $GW(X_i, A)$ the left hand side of (7.4) tends to $-\infty$ as a tends to ∞, and the right hand side tends to ∞. Clearly this gives a contradiction to (7.4).

7.5. **Case III, X_i are finite type.** The argument is analogous. □

A. Fuller index

Let X be a vector field on M. Set

$$S(X) = S(X, \beta) = \{(o, p) \in L_\beta M \times (0, \infty) \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\},$$

where $L_\beta M$ denotes the free homotopy class β component of the free loop space. Elements of $S(X)$ will be called orbits. There is a natural S^1 reparametrization action on $S(X)$, and elements of $S(X)/S^1$ will be called *unparametrized orbits*, or just orbits. Slightly abusing notation we write (o, p) for the equivalence class of (o, p). The multiplicity $m(o, p)$ of a periodic orbit is the ratio p/l for $l > 0$ the least period of o. We want a kind of fixed point index which counts orbits (o, p) with certain weights - however in general to get invariance we must have period bounds. This is due to potential existence of sky catastrophes as described in the introduction.

Let $N \subset S(X)$ be a compact open set. Assume for simplicity that elements $(o, p) \in N$ are isolated. (Otherwise we need to perturb.) Then to such an (N, X, β) Fuller associates an index:

$$i(N, X, \beta) = \sum_{(o,p) \in N/S^1} \frac{1}{m(o, p)} i(o, p),$$

where $i(o, p)$ is the fixed point index of the time p return map of the flow of X with respect to a local surface of section in M transverse to the image of o. Fuller then shows that $i(N, X, \beta)$ has the following invariance property. Given a continuous homotopy $\{X_t\}$, $t \in [0, 1]$ let

$$S(\{X_t\}, \beta) = \{(o, p, t) \in L_\beta M \times (0, \infty) \times [0, 1] \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX_t\}.$$

Given a continuous homotopy $\{X_t\}$, $X_0 = X$, $t \in [0, 1]$, suppose that $\widetilde{N}$ is an open compact subset of $S(\{X_t\})$, such that

$$\widetilde{N} \cap (LM \times \mathbb{R}_+ \times \{0\}) = N.$$

Then if

$$N_1 = \widetilde{N} \cap (LM \times \mathbb{R}_+ \times \{1\})$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta).$$

In the case where X is the R^λ-Reeb vector field on a contact manifold (C^{2n+1}, ξ), and if (o, p) is non-degenerate, we have:

$$i(o, p) = \operatorname{sign} \operatorname{Det}(\operatorname{Id}|_{\xi(x)} - F^\lambda_{p,*}|_{\xi(x)}) = (-1)^{CZ(o)-n}, \tag{A.1}$$

where $F^\lambda_{p,*}$ is the differential at x of the time p flow map of R^λ, and where $CZ(o)$ is the Conley-Zehnder index, (which is a special kind of Maslov index) see [20].

B. Virtual fundamental class

This is a small note on how one deals with curves having non-trivial isotropy groups, in the virtual fundamental class technology. We primarily need this for the proof of Theorem 2.13. Given a closed oriented orbifold X, with an orbibundle E over X Fukaya-Ono [9] show how to construct using multi-sections its rational homology Euler class, which when X represents the moduli space of some stable curves, is the virtual moduli cycle $[X]^{vir}$. (Note that the story of the Euler class is older than the work of Fukaya-Ono, and there is possibly prior work in this direction.) When this is in degree 0, the corresponding Gromov-Witten invariant is $\int_{[X]^{vir}} 1$. However they assume that their orbifolds are effective. This assumption is not really necessary for the purpose of construction of the Euler class but is convenient for other technical reasons. A different approach to the virtual fundamental class which emphasizes branched manifolds is used by McDuff-Wehrheim, see for example McDuff [15], which does not have the effectivity assumption, a similar use of branched manifolds appears in [3]. In the case of a non-effective orbibundle $E \to X$ McDuff [16], constructs a homological Euler class $e(E)$ using multi-sections, which extends the construction [9]. McDuff shows that this class $e(E)$ is Poincare dual to the completely formally natural cohomological Euler class of E, constructed by other authors. In other words there is a natural notion of a homological Euler class of a possibly non-effective orbibundle. We shall assume the following black box property of the virtual fundamental class technology.

Axiom B.1. *Suppose that the moduli space of stable maps is cleanly cut out, which means that it is represented by a (non-effective) orbifold X with an orbifold obstruction bundle E, that is the bundle over X of cokernel spaces of the linearized CR operators. Then the virtual fundamental class $[X]^{vir}$ coincides with $e(E)$.*

Given this axiom it does not matter to us which virtual moduli cycle technique we use. It is satisfied automatically by the construction of McDuff-Wehrheim, (at the moment in genus 0, but surely extending). It can be shown to be satisfied in the approach of John Pardon [19]. And it is satisfied by the construction of Fukaya-Oh-Ono-Ohta [7], although not quiet immediately. This is also communicated to me by Kaoru Ono. When X is 0-dimensional this does follow immediately by the construction in [9], taking any effective Kuranishi neighborhood at the isolated points of X, (this actually suffices for our paper.)

As a special case most relevant to us here, suppose we have a moduli space of elliptic curves in X, which is regular with expected dimension 0. Then its underlying space is a collection of oriented points. However as some curves are multiply covered, and so have isotropy groups, we must treat this is a non-effective 0 dimensional oriented orbifold. The contribution of each curve $[u]$ to the Gromov-Witten invariant $\int_{[X]^{vir}} 1$ is $\frac{\pm 1}{[\Gamma([u])]}$, where $[\Gamma([u])]$ is the order of the isotropy group $\Gamma([u])$ of $[u]$, in the McDuff-Wehrheim setup this is explained in [15, Section 5]. In the setup of Fukaya-Ono [9] we may readily calculate to get the same thing taking any effective Kuranishi neighborhood at the isolated points of X.

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