A CONFORMAL SYMPLECTIC WEINSTEIN CONJECTURE

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ABSTRACT. We introduce a direct generalization of the Weinstein conjecture to certain locally conformally symplectic manifolds, (for short lcs manifolds). This conjectures existence of certain 1-d immersed closed curves in the manifold, we call Reeb curves. We give partial verifications of this CSW conjecture, based on certain extended ($\mathbb{Q} \sqcup \{\pm\infty\}$ valued) Gromov-Witten, elliptic curve counts in lcs manifolds. In particular, the conjecture is verified for a suitable neighborhood, in the space of lcs structures, of the lcs-fication of any non-degenerate contact 3-manifold (C, λ). As another example, for $(C = S^{2k+1}, \lambda_{standard})$, we show that either this conjecture holds for any exact, rational, first kind lcs structure on $C \times S^1$ (in the same formal class as the lcs-fication of $\lambda_{standard}$) or holomorphic sky catastrophes exist. The latter phenomenon, if it exists, would be analogous to sky catastrophes in dynamical systems discovered by Fuller.

CONTENTS

1. INTRODUCTION

A locally conformally symplectic structure or lcs structure for short, is a natural direct generalization of both symplectic and contact manifolds. The main goal here is to study an lcs variant of a very influential conjecture in contact geometry: the Weinstein conjecture. The latter conjectures existence of closed orbits of the Reeb flow for any contact form on a closed manifold. This is now proved for contact three manifolds by Taubes [24]. We will state and partially verify an analogue of the Weinstein

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conjecture for lcs manifolds, "a conformal symplectic Weinstein conjecture" using suitably extended Gromov-Witten theory, counting certain elliptic curves.

Definition 1.1. *A locally conformally symplectic manifold or just* lcs *manifold, is a smooth $2n$-fold M with an* lcs *structure: which is a non-degenerate 2-form ω, which is locally diffeomorphic to $f \cdot \omega_{st}$, for some (non-fixed) positive smooth function f, with ω_{st} the standard symplectic form on $\mathbb{R}^{2n}$. Explicitly, for every $p \in M$ there is a smooth chart*

$$\phi : V \subset \mathbb{R}^{2n} \to M,$$

so that $\phi(V) \ni p$, and $\phi^* \omega = f_\phi \cdot \omega_{st}$, for some smooth positive f_ϕ.

These kinds of structures were originally considered by Lee in [9], arising naturally as part of an abstract study of "a kind of even dimensional Riemannian geometry", and then further studied by a number of authors see for instance, [2] and [25]. This is a fascinating object, an lcs manifold admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and Murphy show that if a closed almost complex $2n$-fold M has $H^1(M, \mathbb{R}) \neq 0$ then it admits a lcs structure, [4]. As constructed, these lcs structures are Lichnerowitz exact (see (1.2)). Another result of Apostolov, Dloussky [1] is that any complex surface with an odd first Betti number admits an lcs structure, which tames the complex structure. In this case the corresponding lcs structures are generally non-exact.

It is natural to try to do Gromov-Witten theory for lcs manifolds. The first problem that occurs is that a priori energy bounds are gone, as since ω is not necessarily closed, the L^2-energy can now be unbounded on the moduli spaces of J-holomorphic curves in such a (M, ω). Strangely, a more acute problem is potential presence of holomorphic sky catastrophes - given a smooth family $\{J_t\}$, $t \in [0, 1]$, of $\{\omega_t\}$-compatible almost complex structures, we may have a continuous family $\{u_t\}$ of J_t-holomorphic curves s.t. energy$(u_t) \mapsto \infty$ as $t \mapsto a \in (0, 1)$ and s.t. there are no holomorphic curves for $t \geq a$. These are analogues of sky catastrophes discovered by Fuller [8] for closed orbits of dynamical systems.

Even when it is impossible to tame these problems we show that there can still be an extended Gromov-Witten type theory which is analogous to the theory of extended Fuller index in dynamical systems, [22]. In a very particular situation the relationship with the Fuller index becomes perfect as one of the results of this paper obtains the (extended) Fuller index for Reeb vector fields on a contact manifold C as a certain (extended) genus 1 Gromov-Witten invariant of the Banyaga lcs manifold $C \times S^1$, cf. Example 1. The latter also gives a conceptual interpretation for why the Fuller index is rational, as it is reinterpreted as an (virtual) orbifold Euler number.

Partly inspired by this, we conjecture that certain lcs manifolds must poses certain immersed 1-d curves that we call Reeb curves. This is a direct generalization of the Weinstein conjecture, and will be called here the conformal symplectic Weinstein conjecture. For example, we prove this CSW conjecture for lcs structures suitably nearby, in the space of lcs structures, to any non-degenerate contact structure (a contact form) on a three manifold.

This partly uses the above-mentioned connection of Gromov-Witten theory of $C \times S^1$ with the classical Fuller index. In addition to the nearby version of CSW, we also prove a stronger result that relates the CSW conjecture to existence of holomorphic sky catastrophes.

Finally, we should exclaim that the Gromov-Witten theory in this story plays a local (in the space of structures) role, unless addition global geometric control is obtained, as in [21]. This is analogous to what happens with Fuller index in dynamical systems. A global lcs invariant, which takes the form of a homology theory, is under development, but many ingredients for this are already present here. (For example generators, and appropriate almost complex structures.)

1.1. More background.

To see the connection with the first cohomology group $H^1(M, \mathbb{R})$, mentioned above, let us point out right away the most basic invariant of a lcs structure ω, when M has dimension at least 4. This is the Lee class, $\alpha = \alpha_\omega \in H^1(M, \mathbb{R})$. This class has the property that on the associated α-covering space $\widetilde{M}$, the lift $\widetilde{\omega}$ is globally conformally symplectic. The class α may be defined as the following Cech 1-cocycle. Let $\phi_{a,b}$ be the transition map for lcs charts ϕ_a, ϕ_b of (M, ω). Then $\phi_{a,b}^* \omega_{st} = g_{a,b} \cdot \omega_{st}$ for a positive real constant $g_{a,b}$ and $\{\ln g_{a,b}\}$ gives our 1-cocycle. Thus, an lcs form is globally conformally symplectic iff its Lee class vanishes.

Again assuming M has dimension at least 4, the Lee class α has a natural differential form representative, called the Lee form and defined as follows. We take a cover of M by open sets U_a in which $\omega = f_a \cdot \omega_a$ for ω_a symplectic, and f_a a positive smooth function. Then we have 1-forms $d(\ln f_a)$ in each U_a which glue to a well defined closed 1-form on M, as shown by Lee. By slight abuse, we denote this 1-form, its cohomology class and the Cech 1-cocycle from before all by α. It is moreover immediate that for an lcs form ω

$$d\omega = \alpha \wedge \omega,$$

for α the Lee form as defined above.

As we mentioned lcs manifolds can also be understood to generalize contact manifolds. This works as follows. First we have a natural class of explicit examples of lcs manifolds, obtained by starting with a symplectic cobordism (see [4]) of a closed contact manifold C to itself, arranging for the contact forms at the two ends of the cobordism to be proportional (which can always be done) and then appropriately gluing together the boundary components. As a particular case of this we get Banyaga's basic example.

Example 1 (Banyaga). Let (C, λ) be a contact $(2n+1)$-manifold where λ is a contact form, $\forall p \in C :$ $\lambda \wedge \lambda^{2n}(p) \neq 0$, and take $M = C \times S^1$ with 2-form

$$\omega_\lambda = d^\alpha \lambda := d\lambda - \alpha \wedge \lambda,$$

for $\alpha := pr_{S^1}^* d\theta$, $pr_{S^1} : C \times S^1 \to S^1$ the projection, and λ likewise the pull-back of λ by the projection $C \times S^1 \to C$. We call (M, ω_λ) as above the ***lcs-fication*** of (C, λ).

The operator

$$d^\alpha : \Omega^k(M) \to \Omega^{k+1}(M) \tag{1.2}$$

is called the Lichnerowicz differential with respect to a closed 1-form α, and satisfies $d^\alpha \circ d^\alpha = 0$ so that we have an associated Lichnerowicz chain complex.

We assume from now on, unless explicitly stated otherwise, that our lcs manifolds have dimension at least 4.

1.2. **Conformal symplectic Weinstein conjecture.** As previously mentioned lcs manifolds can be understood to generalize both symplectic and contact manifolds. There are two vastly influential conjectures in these two respective areas: the Arnold conjecture and Weinstein conjecture. The statement of Arnold conjecture can be directly generalized to lcs manifolds, but it is very easy to construct counterexamples using Banyaga's example above: there are Hamiltonian lcs maps of the lcs-fication of the standard contact S^3, with no fixed points. (We leave this to the reader as an exercise.) We are instead interested here in extending the Weinstein conjecture.

An ***exact lcs structure*** on M is a pair (λ, α) with α a closed 1-form, s.t. $\omega = d^\alpha \lambda$ is non-degenerate. This determines a generalized distribution $\mathcal{V}_\lambda$:

$$\mathcal{V}_\lambda(p) = \{v \in T_pM | d\lambda(v, \cdot) = 0\},$$

which we call the ***vanishing distribution***.

We also define a generalized distribution ξ_λ that is the ω-orthogonal complement to $\mathcal{V}_\lambda$, which we call ***co-vanishing distribution***. For each $p \in M$, $\mathcal{V}_\lambda(p)$ has dimension at most 2 since $d\lambda - \alpha \wedge \lambda$ is non-degenerate. If M^{2n} is closed $\mathcal{V}_\lambda$ cannot identically vanish since $(d\lambda)^n$ cannot be non-degenerate by Stokes theorem.

Definition 1.3. *If $\mathcal{V}$ is everywhere 2-dimensional and in particular is an actual distribution, then such an* lcs *structure $\omega = d^\alpha \lambda$ is often called to be of the* **first kind**.

A typical example of an lcs structure of the first kind is the mapping torus of a contactomorphism, see Banyaga [2]. In fact Banyaga shows that if (M, ω) is first kind and is compact then it is necessarily a mapping torus as above. So we can understand these first kind compact lcs manifolds as a significant generalization of the class of contact manifolds. Moreover, invariants of lcs manifolds of first kind are then invariants of contactomorphisms and this is one major connection of lcs theory to more classical contact geometry.

Definition 1.4. *Let (M, λ, α) be an exact* lcs *structure. We have a cone structure $C_\lambda \subset \mathcal{V}_\lambda$, with*

$$C_\lambda(p) := \{v \in \mathcal{V}_\lambda(p) | \lambda(v) > 0\}.$$

We propose that C_λ plays the role of the Reeb distribution in this context. And we say that a smooth map $o : S^1 \to M$ is a **Reeb curve** *for (M, λ, α) if it is tangent to C_λ, in other words*

$$\dot{o}(t) \in C_\lambda(o(t))$$

for each t.

Example 2. Let (C, λ) be a closed contact $(2n+1)$-fold with a contact form λ. The ***Reeb vector field*** R^λ on C is a vector field satisfying

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

A ***closed Reeb orbit*** of (C, λ) is a smooth map $o : S^1 \to C$ such that

$$\dot{o}(t) = cR_\lambda(o(t)),$$

for some $c > 0$ called period. If $(C \times S^1, d^\alpha \lambda)$ is the lcs-fication of (C, λ), then identifying $S^1 = \mathbb{R}/\mathbb{Z}$, S^1 acts on $C \times S^1$ by $s \cdot (x, \theta) = (x, \theta + s)$. Let

$$\frac{d}{d\theta} \subset \{0\} \oplus TS^1 \subset C \times S^1$$

denote the vector field generating this S^1 action. Then

$$C_\lambda = \{(v, u) \in R \oplus T \,|\, v \neq 0\},$$

where

$$R = \mathbb{R}_{>0} \cdot (R_\lambda \oplus \{0\})$$

$R^\lambda \oplus \{0\}$ is the section of $T(C \times S^1) \simeq TC \oplus \mathbb{R}$, corresponding to R^λ,

$$T = \{0\} \oplus \mathbb{R} \cdot \frac{\partial}{\partial\theta}.$$

What follows is our basic motivating conjecture, the CSW conjecture.

Conjecture 1. *Let M be closed of dimension at least 4, and (λ, α) an exact* lcs *structure on M with α rational, then there is a Reeb curve for (M, λ, α).*

The dimension 2 case is special, but some version (possibly same version) of the conjecture should hold in this case. As one trivial example, whose verification is left to the reader, given an exact lcs 2-manifold (M, λ, α), with $d\lambda = 0$ and with α rational, the conjecture holds automatically, just take the Reeb curve to parametrize a component of a regular fiber of the map $f : \Sigma \to S^1$ classifying α, that is so that $\alpha = q \cdot f^* d\theta$, for $q \in \mathbb{Q}$.

The CSW conjecture is not just a curiosity. In contact geometry, rigidity is based on existence phenomena of closed Reeb orbits, and lcs manifolds may be understood as generalized contact manifolds. To attack rigidity questions in lcs geometry, it would be of great help to have an analogue of closed Reeb orbits, we propose that this analogue is Reeb curves.

Lemma 1.5. *Conjecture 1 implies the Weinstein conjecture: every closed contact manifold (C, λ) has a closed Reeb orbit.*

Proof. Let λ be a contact form on a closed manifold C. Let $o : S^1 \to C \times S^1$ be a Reeb curve for lcs-fication $(C \times S^1, d^\alpha \lambda)$. Since $o_*(TS^1) \subset \mathcal{V}_\lambda$, $(pr_C)_* \circ o_*(TS^1) \subset \ker d\lambda \subset TC$. Since in addition $o^* \circ u^* \lambda$ is non-vanishing on TS^1, $pr_C \circ o$ is immersed in C and so is the image of a closed Reeb orbit. □

In what follows we use the following C^k metric on the space $\mathcal{L}(M)$ of exact lcs structures on M. For $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in \mathcal{L}(M)$ define:

$$d_k((\lambda_1, \alpha_1), (\lambda_2, \alpha_2)) = d_{C^k}(\lambda_1, \lambda_2) + d_{C^k}(\alpha_1, \alpha_2), \tag{1.6}$$

where d_{C^k} on the right side is the usual C^k metric.

The following theorems are proved in Section 3. We say that an exact lcs structure (λ, α) is ***rational*** if α is rational.

Theorem 1.7. *Let λ be a non-degenerate contact form, (closed Reeb orbits are non-degenerate) on a closed 3-manifold C. Then Conjecture 1 holds for a d_3 neighborhood of the* lcs*-fication (λ, α) in the space of exact, rational* lcs *structures of the first kind on $C \times S^1$.*

Theorem 1.8. *Let C be a closed contact manifold with contact form λ, with $i(R^\lambda, \beta) \neq 0$, for some class β, where the latter is the extended Fuller index, as described in Appendix A. Let (λ, α) be the associated exact* lcs*-structure on $M = C \times S^1$, the* lcs*-fication. Then either the conformal symplectic Weinstein conjecture holds for any exact, rational* lcs *structure of the first kind, homotopic through non-degenerate 2-forms to $\omega_0 = d^\alpha \lambda$, or holomorphic sky catastrophes exist, (the latter are further discussed in Section 1.3).*

Example 3. Take $C = S^{2k+1}$ and $\lambda = \lambda_H$, then $i(R^\lambda, 0) = \pm\infty$, (sign depends on k), [22]. Or take C to be unit cotangent bundle of a hyperbolic manifold (X, g), λ the associated Louiville form, and (λ, α) the associated Banyaga lcs structure, in this case $i(R^\lambda, \beta) = \pm 1$ for every $\beta \neq 0$.

The main tool for the theorems above is a certain theory of elliptic curves in lcs manifolds, and we now discuss this.

1.2.1. *Pseudo-holomorphic curves in exact* lcs *manifolds.* Banyaga type lcs manifolds give immediate examples of almost complex manifolds where the L^2 energy functional is unbounded on the moduli spaces of fixed class J-holomorphic curves, as well as where null-homologous J-holomorphic curves can be non-constant. We are going to see this shortly after developing a more general theory.

Definition 1.9. *Let (M, λ, α) be an exact lcs manifold, so that the* lcs *form is $\omega = d^\alpha \lambda$. We say that an ω-compatible J is* **admissible** *if it preserves the splitting $\mathcal{V}_\lambda \oplus \xi_\lambda$, that is $J(\mathcal{V}_\lambda) \subset \mathcal{V}_\lambda$ and $J(\xi_\lambda) \subset \xi_\lambda$, and if $d\lambda$ tames J on ξ_λ. We call (λ, α, J) as above a* **tamed exact** lcs **structure**, *and (ω, J) is called a tamed exact* lcs *structure if $\omega = d^\alpha \lambda$, for (λ, α, J) an exact* lcs *structure as previously. In this case (M, ω, J), (M, λ, α, J) will be called a* **tamed exact** lcs **manifold**.

Lemma 1.10. *Let (M, λ, α, J) be a tamed exact* lcs *manifold. Then given a smooth $u : \Sigma \to M$, where Σ is a closed (nodal) Riemann surface, u is J-holomorphic only if*

$$\operatorname{image} du(z) \subset \mathcal{V}_\lambda(u(z))$$

for all $z \in \Sigma$, in particular $u^* d\lambda = 0$.

Proof. For u J-holomorphic as above, we have

$$\int_\Sigma u^* d\lambda = 0$$

by Stokes theorem. Let $proj_{\xi_\lambda}(p) : T_p M \to \xi_\lambda(p)$ be the projection induced by the splitting $\mathcal{V}_\lambda \oplus \xi_\lambda$. Then if for some $z \in \Sigma$, $proj_{\xi_\lambda} \circ du(z) \neq 0$, since J is tamed by $d\lambda$ on ξ_λ and since J preserves the splitting $\mathcal{V}_\lambda \oplus \xi_\lambda$, we would have $\int_\Sigma u^* d\lambda > 0$. Thus

$$\forall z \in \Sigma : proj_{\xi_\lambda} \circ du(z) = 0,$$

so

$$\forall z \in \Sigma : \operatorname{image} du(z) \subset \mathcal{V}_\lambda(u(z)).$$

$\square$

1.2.2. *Example, lcs-fication of a contact manifold.* Let (C, λ) be a closed contact $(2n+1)$-fold with a contact form λ. We also denote by λ the pull-back of λ by the projection $C \times S^1 \to C$, and by $\xi \subset T(C \times S^1)$ the distribution $\xi(p) = \ker d\lambda(p)$.

We take J to be an almost complex structure on ξ, which is S^1 invariant, and compatible with $d\lambda$. The latter means that

$$g_J(\cdot, \cdot) := d\lambda|_\xi(\cdot, J\cdot)$$

is a J invariant Riemannian metric on the distribution ξ.

There is an induced almost complex structure J^λ on $C \times S^1$, which is S^1-invariant, coincides with J on ξ and which satisfies:

$$J^\lambda(R^\lambda \oplus \{0\}(p)) = \frac{d}{d\theta}(p),$$

where $R^\lambda \oplus \{0\}$ is the section of $T(C \times S^1) \simeq TC \oplus \mathbb{R}$, as previously, and where

$$\frac{d}{d\theta} \subset \{0\} \oplus TS^1 \subset C \times S^1$$

denotes the vector field generating the action of S^1 on $C \times S^1$ as previously.

In previous terms $(C \times S^1, \lambda, \alpha, J^\lambda)$ is a tamed exact lcs manifold. We now consider a moduli space of holomorphic tori in $C \times S^1$, which have a certain charge, an analogue of this charge condition is also studied Oh-Wang [18], and I am grateful to Yong-Geun Oh for related discussions. Partly the reason for introduction of "charge" is that it is now possible for non-constant holomorphic curves to be null-homologous, so we need additional control. Here is a simple example take $S^3 \times S^1$ with $J = J^\lambda$, for the λ the standard contact form, then all the Reeb holomorphic tori (as defined further below) are null-homologous. In many cases we can just work with homology classes $A \neq 0$, but this is inadequate for our setup for conformal symplectic Weinstein conjecture.

Let Σ be a complex torus with a chosen marked point $z \in \Sigma$, i.e. an elliptic curve over $\mathbb{C}$. An isomorphism $\phi : (\Sigma_1, z_1) \to (\Sigma_2, z_2)$ is a biholomorphism s.t. $\phi(z_1) = z_2$. The set of isomorphism classes forms a smooth orbifold $M_{1,1}$, with a natural compactification, the Deligne-Mumford compactification $\overline{M}_{1,1}$, by adding a point at infinity corresponding to a nodal curve.

Suppose then (M, ω) is an lcs manifold, J ω-compatible almost complex structure, and α the Lee class corresponding to ω. Assuming for simplicity, at the moment, (otherwise take stable maps) that (M, J) does not admit non-constant J-holomorphic maps $(S^2, j) \to (M, J)$, we define:

$$\overline{\mathcal{M}}_{1,1}^{1,0}(J, A)$$

as a set of equivalence classes of tuples (u, S), for $S = (\Sigma, z) \in \overline{M}_{1,1}$, and $u : \Sigma \to M$ a J-holomorphic map satisfying the ***charge (1,0) condition***: there exists a pair of generators ρ, γ for $H_1(\Sigma, \mathbb{Z})$, such that

$$\langle \rho, u_* \alpha \rangle = 1$$
$$\langle \gamma, u_* \alpha \rangle = 0,$$

and with $[u] = A$. The equivalence relation is $(u_1, S_1) \sim (u_2, S_2)$ if there is an isomorphism $\phi : S_1 \to S_2$ s.t. $u_2 \circ \phi = u_1$.

Note that the charge condition directly makes sense for nodal curves. And it is easy to see that the charge condition is preserved under Gromov convergence, and obviously a charge (1,0) J-holomorphic map cannot be constant for any A.

By slight abuse we may just denote such an equivalence class above by u, so we may write $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(J, A)$, with S implicit.

1.2.3. *Reeb holomorphic tori in* $(C \times S^1, J^\lambda)$. For the almost complex structure J^λ as above we have one natural class of charge (1,0) holomorphic tori in $C \times S^1$. Let o be a period c closed Reeb orbit o of R^λ, that is a map:

$$o : S^1 \to C,$$
$$\dot{o}(s) = c \cdot R^\lambda(o(s)), c > 0.$$

A ***Reeb torus*** u_o for o, is the map

$$u_o(s,t) = (o(s), t),$$

$s, t \in S^1$. A Reeb torus is J^λ-holomorphic for a uniquely determined holomorphic structure j on T^2 defined by:

$$j(\frac{\partial}{\partial s}) = c \frac{\partial}{\partial t}.$$

Let $\widetilde{S}(\lambda)$ denote the space of general period, closed λ-Reeb orbits. There is an S^1 action on this space, with $\theta \cdot o$ the orbit

$$\theta \cdot o(s) = o(s + \theta).$$

Let $S(\lambda) := \widetilde{S}(\lambda)/S^1$ denote the quotient by this action. We have a map:

$$R : S(\lambda) \to \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A), \quad R(o) = u_o.$$

Proposition 1.11. *The map R is a bijection.* [1]

So in the particular case of J^λ, as above, the domains of elliptic curves in $C \times S^1$ are "rectangular", that is are quotients of the complex plane by a rectangular lattice, however for a more general almost complex structure on $C \times S^1$, tamed by more general lcs forms as we soon consider, the domain almost complex structure on our curves can in principle be arbitrary, in particular we might have nodal degenerations. Also note that the expected dimension of $\overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$ is 0. It is given by the Fredholm index of the operator (3.2) which is 2, minus the dimension of the reparametrization group (for non-nodal curves) which is 2. That is given an elliptic curve $S = (\Sigma, z)$, let $\mathcal{G}(\Sigma)$ be the 2-dimensional group of biholomorphisms ϕ of Σ. Then given a J-holomorphic map $u : \Sigma \to M$, (Σ, z, u) has the equivalence class of $(\Sigma, \phi(z), u \circ \phi)$ in $\overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$, for $\phi \in \mathcal{G}(\Sigma)$.

The following is to be proved in Section 3.

Theorem 1.12. *Let (M, λ, α, J) be a tamed exact rational* lcs *structure, then every non-constant J-holomorphic curve $u : \Sigma \to M$ contains a Reeb curve, meaning that there is a $S_0 \simeq S^1 \subset \Sigma$ s.t. $u|_{S_0}$ is a Reeb curve. If moreover Σ is smooth, connected and immersed then $\Sigma \simeq T^2$.*

1.2.4. *Connection with the extended Fuller index.* One of the main ingredients for the above is a connection of extended Fuller index with certain extended Gromov-Witten invariants. If β is a free homotopy class of a loop in C set

$$A_\beta = [\beta] \times [S^1] \in H_2(C \times S^1).$$

Then we have:

Theorem 1.13. *Suppose that λ is a contact form on a closed manifold C, so that its Reeb flow is definite type, see Appendix A, then*

$$GW_{1,1}(A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(R^\lambda, \beta),$$

where both sides are certain extended rational numbers $\mathbb{Q} \sqcup \{\pm\infty\}$ valued invariants, so that in particular if either side does not vanish then there are λ Reeb orbits in class β.

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[1] It is in fact an equivalence of the corresponding topological action groupoids, but we do not need this explicitly.

What about higher genus invariants of $C \times S^1$? Following the proof of Proposition 1.11, it is not hard to see that all J^λ-holomorphic curves must be branched covers of Reeb tori. If one can show that these branched covers are regular when the underlying tori are regular, the calculation of invariants would be fairly automatic from this data, see [29], [27] where these kinds of regularity calculation are made.

1.3. Sky catastrophes.

This final introductory section will be of a more technical nature. The following is well known.

Theorem 1.14. *[[15, Proposition 4.1.4], [26]] Let (M, J) be a compact almost complex manifold, and $u : (S^2, j) \to M$ a J-holomorphic map. Given a Riemannian metric g on M, there is an $\hbar = \hbar(g, J) > 0$ s.t. if $e_g(u) < \hbar$ then u is constant, where e_g is the L^2-energy functional,*

$$e_g(u) = \text{energy}_g(u) = \int_{S^2} |du|^2 dvol.$$

Using this we get the following (trivial) extension of Gromov compactness. Let

$$\mathcal{M}_{g,n}(J, A) = \mathcal{M}_{g,n}(M, J, A)$$

denote the moduli space of isomorphism classes of class A, J-holomorphic curves in M, with domain a genus g closed Riemann surface, with n marked labeled points. Here an isomorphism between $u_1 : \Sigma_1 \to M$, and $u_2 : \Sigma_2 \to M$ is a biholomorphism of marked Riemann surfaces $\phi : \Sigma_1 \to \Sigma_2$ s.t. $u_2 \circ \phi = u_1$.

The following is proved by the same argument as [14, Theorem 5.6.6]. We claim no originality.

Theorem 1.15. *Let (M, J) be an almost complex manifold. Then $\mathcal{M}_{g,n}(J, A)$ has a pre-compactification*

$$\overline{\mathcal{M}}_{g,n}(J, A),$$

by Kontsevich stable maps, with respect to the natural metrizable Gromov topology see for instance [14, Chapter 5.6], for genus 0 case, [19] for general case. Moreover, given $E > 0$, the subspace $\overline{\mathcal{M}}_{g,n}(J, A)_E \subset \overline{\mathcal{M}}_{g,n}(J, A)$ consisting of elements u with $e(u) \leq E$ is compact, where e is the L^2 energy with respect to an auxiliary metric. In other words e is a proper function.

Thus, the most basic situation where we can talk about Gromov-Witten "invariants" of (M, J) is when the energy function is bounded on $\overline{\mathcal{M}}_{g,n}(J, A)$, and we shall say that J is ***bounded*** (in class A), later on we generalize this in terms of what we call ***finite type***. In this case $\overline{\mathcal{M}}_{g,n}(J, A)$ is compact, and has a virtual moduli cycle as in the original approach of Fukaya-Ono [7], or the more algebraic approach [19]. So we may define functionals:

$$GW_{g,n}(A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}, \tag{1.16}$$

where $\overline{M}_{g,n}$ denotes the compactified moduli space of Riemann surfaces. Of course symplectic manifolds with any tame almost complex structure is one class of examples, another class of examples comes from some locally conformally symplectic manifolds. (We can take for instance the lcs-fication of (C, λ) with the latter the unit cotangent bundle of a hyperbolic manifold, with λ the canonical Louisville form, and J as in Section 1.2.2).

Given a continuous in the C^∞ topology family $\{J_t\}$, $t \in [0, 1]$ we denote by $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ the space of pairs (u, t), $u \in \overline{\mathcal{M}}_{g,n}(J_t, A)$.

Definition 1.17. *We say that a continuous family $\{J_t\}$, $t \in [0, 1]$ on a compact manifold M has a* **holomorphic sky catastrophe** *in class A if there is an element $u \in \overline{\mathcal{M}}_{g,n}(J_i, A)$, $i = 0, 1$ which does not belong to any open compact (equivalently energy bounded) subset of $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$.*

Let us slightly expand this definition. If $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ is locally connected, so that the connected components are open, then we have a sky catastrophe in the sense above if and only if there is a $u \in \overline{\mathcal{M}}_{g,n}(J_i, A)$ which has a non-compact connected component in $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$. At this point in time there are no known examples of families $\{J_t\}$ with sky catastrophes.

Question 1. Do holomorphic sky catastrophes exist?

Really what we are interested in is whether they exist generically. If the answer is no, then Theorem 1.8 tells us that the CSW conjecture holds for a large class of lcs structures on $S^{2k+1} \times S^1$. The author's opinion is that they may appear even generically. However, if we ask that each J_t is tamed by an exact lcs structure, then the question becomes much more subtle, see also [22] for a related discussion on possible obstructions to sky catastrophes.

If holomorphic sky catastrophes are discovered, this would be a very interesting discovery. The original discovery by Fuller [8] of sky catastrophes in dynamical systems is one of the most important in dynamical systems, see also [23] for an overview.

2. Elements of Gromov-Witten theory of an lcs manifold

Suppose (M, J) is a compact almost complex manifold, where the almost complex structures J are assumed throughout the paper to be C^∞, and let $N \subset \overline{\mathcal{M}}_{g,k}(J, A)$ be an open compact subset with energy positive on N. The latter condition is only relevant when $A = 0$. We shall primarily refer in what follows to work of Pardon in [19], only because this is what is more familiar to the author, due to greater comfort with algebraic topology. But we should mention that the latter is a followup to a profound theory that is originally created by Fukaya-Ono [7], and later expanded with Oh-Ohta [6].

The construction in [19] of implicit atlas, on the moduli space $\mathcal{M}$ of curves in a symplectic manifold, only needs a neighborhood of $\mathcal{M}$ in the space of all curves. So more generally if we have an almost complex manifold and an *open* compact component N as above, this will likewise have a natural implicit atlas, or a Kuranishi structure in the setup of [7]. And so such an N will have a virtual fundamental class in the sense of Pardon [19], (or in any other approach to virtual fundamental cycle, particularly the original approach of Fukaya-Oh-Ohta-Ono). This understanding will be used in other parts of the paper, following Pardon for the explicit setup. We may thus define functionals:

$$GW_{g,n}(N, A, J) : H_*(\overline{M}_{g,n}) \otimes H_*(M^n) \to \mathbb{Q}. \tag{2.1}$$

How do these functionals depend on N, J?

Lemma 2.2. *Let $\{J_t\}$, $t \in [0, 1]$ be a Frechet smooth family. Suppose that $\widetilde{N}$ is an open compact subset of the cobordism moduli space $\overline{\mathcal{M}}_{g,n}(\{J_t\}, A)$ and that the energy function is positive on $\widetilde{N}$, (the latter only relevant when $A = 0$). Let*

$$N_i = \widetilde{N} \cap \left(\overline{\mathcal{M}}_{g,n}(J_i, A)\right),$$

then

$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1).$$

In particular if $GW_{g,n}(N_0, A, J_0) \neq 0$, there is a class A J_1-holomorphic stable map in M.

Proof of Lemma 2.2. We may construct exactly as in [19] a natural implicit atlas on $\widetilde{N}$, with boundary $N_0^{op} \sqcup N_1$, (*op* denoting opposite orientation). And so

$$GW_{g,n}(N_0, A, J_0) = GW_{g,n}(N_1, A, J_1),$$

as functionals. □

The most basic lemma in this setting is the following, and we shall use it in the following section.

Definition 2.3. *An* **almost symplectic pair** *on M is a tuple (M, ω, J), where ω is a non-degenerate 2-form on M, and J is ω-compatible, meaning that $\omega(\cdot, J\cdot)$ defines J-invariant Riemannian metric. When ω is lcs we call such a pair an* **lcs pair**.

Definition 2.4. *We say that a pair of almost symplectic pairs (ω_i, J_i) are δ-**close**, if $\{\omega_i\}$ are C^0 δ-close, and $\{J_i\}$ are C^2 δ-close, $i = 0, 1$. Define this similarly for a pair (g_i, J_i) for g a Riemannian metric and J an almost complex structure.*

Definition 2.5. *For an almost symplectic pair (ω, J) on M, and a smooth map $u : \Sigma \to M$ define:*

$$e_\omega(u) = \int_\Sigma u^* \omega.$$

By an elementary calculation this coincides with the L^2 g_J-energy of u, for $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$. That is $e_\omega(u) = e_{g_J}(u)$. In what follows by $f^{-1}(a,b)$, with f a function, we mean the preimage by f of the open set (a,b).

Lemma 2.6. *Given a Riemannian manifold (M,g), and J an almost complex structure, suppose that $N \subset \overline{\mathcal{M}}_{d,n}(J,A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = e_g^{-1}(E^0, E^1)\right) \subset \left(V = e_g^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

with $\epsilon > 0$, $E_0 > 0$, and with $V \cap \overline{\mathcal{M}}_{g,n}(J,A) = N$. Then there is a $\delta > 0$ s.t. whenever (g', J') is δ-close to (g, J) if $u \in \overline{\mathcal{M}}_{g,n}(J', A)$ and

$$E^0 - \epsilon < e_{g'}(u) < E^1 + \epsilon$$

then

$$E^0 < e_{g'}(u) < E^1.$$

Proof. Suppose otherwise then there is a sequence $\{(g_k, J_k)\}$ converging to (g, J), and a sequence $\{u_k\}$ of J_k-holomorphic stable maps satisfying

$$E^0 - \epsilon < e_{g_k}(u_k) \leq E^0$$

or

$$E^1 \leq e_{g_k}(u_k) < E^1 + \epsilon.$$

By Gromov compactness, specifically theorems [15, B.41, B.42], we may find a Gromov convergent subsequence $\{u_{k_j}\}$ to a J-holomorphic stable map u, with

$$E^0 - \epsilon \leq e_g(u) \leq E^0$$

or

$$E^1 \leq e_g(u) \leq E^1 + \epsilon.$$

But by our assumptions such a u does not exist. □

Lemma 2.7. *Let M be compact, and let (M, ω, J) be an almost symplectic triple, so that $N \subset \overline{\mathcal{M}}_{g,n}(J,A)$ is exactly as in the lemma above with respect to some $\epsilon > 0$. Then, there is a $\delta > 0$ s.t. the following is satisfied. Let (ω', J') be δ-close to (ω, J), then there is a continuous in the C^∞ topology family of almost symplectic pairs $\{(\omega_t, J_t)\}$, $(\omega_0, J_0) = (g, J)$, $(\omega_1, J_1) = (g', J')$ s.t. there is open compact subset*

$$\widetilde{N} \subset \overline{\mathcal{M}}_{g,n}(\{J_t\}, A),$$

and with

$$\widetilde{N} \cap \overline{\mathcal{M}}(J, A) = N.$$

Moreover, if $(u,t) \in \widetilde{N}$ then

$$E^0 < e_{g_t}(u) < E^1.$$

Proof. For ϵ as in the hypothesis, let δ be as in Lemma 2.6.

Lemma 2.8. *Given a $\delta > 0$ there is a $\delta' > 0$ s.t. if (ω', J') is δ'-near (ω, J) there is an interpolating, continuous in C^∞ topology family $\{(\omega_t, J_t)\}$ with (ω_t, J_t) δ-close to (ω, J) for each t.*

Proof. Let $\{g_t\}$ be the family of metrics on M given by the convex linear combination of $g = g_{\omega_J}, g' = g_{\omega', J'}$. Clearly g_t is δ'-close to g_0 for each t. Likewise, the family of 2 forms $\{\omega_t\}$ given by the convex linear combination of ω, ω' is non-degenerate for each t if δ' was chosen to be sufficiently small and is δ'-close to $\omega_0 = \omega_{g,J}$ for each moment.

Let

$$ret : Met(M) \times \Omega(M) \to \mathcal{J}(M)$$

be the "retraction map" (it can be understood as a retraction followed by projection) as defined in [14, Prop 2.50], where $Met(M)$ is space of metrics on M, $\Omega(M)$ the space of 2-forms on M, and $\mathcal{J}(M)$ the space of almost complex structures. This map has the property that the almost complex structure $ret(g, \omega)$ is compatible with ω, and that $ret(g_J, \omega) = J$ for $g_J = \omega(\cdot, J\cdot)$. Then $\{(\omega_t, ret(g_t, \omega_t)\}$ is a compatible family. As ret is continuous in C^2-topology, δ' can be chosen so that $\{ret_t(g_t, \omega_t\}$ are δ-nearby. $\square$

Returning to the proof of the main lemma. Let $\delta' < \delta$ be chosen as in the above lemma and let $\{(\omega_t, J_t)\}$ be the corresponding family. Let $\widetilde{N}$ consist of all elements $(u, t) \in \overline{\mathcal{M}}(\{J_t\}, A)$ s.t.

$$E^0 - \epsilon < e_{\omega_t}(u) < E^1 + \epsilon.$$

Then by Lemma 2.6 for each $(u, t) \in \widetilde{N}$, we have:

$$E^0 < e_{\omega_t}(u) < E^1.$$

In particular $\widetilde{N}$ must be closed, it is also clearly open, and is compact as the energy e is a proper function, as discussed. Renaming $\delta := \delta'$ we are then done. $\square$

Proposition 2.9. *Given a compact M and an almost symplectic pair (ω, J) on M, suppose that $N \subset \overline{\mathcal{M}}_{g,n}(J, A)$ is a compact and open component which is energy isolated meaning that*

$$N \subset \left(U = e_\omega^{-1}(E^0, E^1)\right) \subset \left(V = e_\omega^{-1}(E^0 - \epsilon, E^1 + \epsilon)\right),$$

with $\epsilon > 0$, $E^0 > 0$ and with $V \cap \overline{\mathcal{M}}_{g,n}(J, A) = N$. Suppose also that $GW_{g,n}(N, J, A) \neq 0$. Then there is a $\delta > 0$ s.t. whenever (ω', J') is a compatible almost symplectic pair δ-close to (ω, J), there exists $u \in \overline{\mathcal{M}}_{g,n}(J', A) \neq \emptyset$, with

$$E^0 < e_{\omega'}(u) < E^1.$$

Proof. For N, ϵ as in the hypothesis, let $\delta, \widetilde{N}$ be as in Lemma 2.7, then by Lemma 2.2

$$GW_{g,n}(N_1, J', A) = GW_{g,n}(N, J, A) \neq 0,$$

where $N_1 = \widetilde{N} \cap \overline{\mathcal{M}}_{g,n}(J_1, A)$. $\square$

While not having sky catastrophes gives us a certain compactness control, the above proposition is not immediate because we can still in principle have total cancellation of the infinitely many components of the moduli space $\overline{\mathcal{M}}_{1,1}(J^\lambda, A)$. In other words a virtual 0-dimension Kuranishi space $\overline{\mathcal{M}}^{1,0}(J^\lambda, A)$, with an infinite number of compact connected components, can certainly be null-cobordant, by a cobordism all of whose components are compact. So we need a certain additional algebraic and geometric control to preclude such a total cancellation.

Proof of Theorem 1.15. (Outline, as the argument is standard.) Suppose that we have a sequence u^k of J-holomorphic maps with L^2-energy $\leq E$. By [14, 4.1.1], a sequence u^k of J-holomorphic curves has a convergent subsequence if $sup_k ||du^k||_{L^\infty} < \infty$. On the other hand when this condition does not hold rescaling argument tells us that a holomorphic sphere bubbles off. The quantization Theorem 1.14, then tells us that these bubbles have some minimal energy, so if the total energy is capped by E, only finitely many bubbles may appear, so that a subsequence of u^k must converge in the Gromov topology to a Kontsevich stable map. $\square$

3. Elliptic curves in the lcs manifold $C \times S^1$ and the Fuller index

Proof of Proposition 1.11. Suppose we have a curve without spherical nodal components $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$, represented by $u : \Sigma \to M = C \times S^1$. Since by Lemma 1.10, $u_*(T\Sigma) \subset \mathcal{V}_\lambda$, we get that

$$(pr_C \circ u)_*(T\Sigma) \subset \ker d\lambda \subset TC,$$

where $pr_C : C \times S^1 \to C$ is the projection. Note that this implies in particular that Σ is non-nodal.

By the charge $(1, 0)$ condition $pr_{S^1} \circ u$ is surjective and so by the Sard theorem we have a regular value $\theta_0 \in S^1$, so that $u^{-1} \circ pr_{S^1}^{-1}(\theta_0)$ contains an embedded circle $S_0 \subset \Sigma$, where $pr_{S^1} : C \times S^1 \to S^1$

is the projection. Now $d(pr_{S^1} \circ u)$ is surjective along $T(\Sigma)|_{S_0}$, which means, since u is J^λ-holomorphic, that $pr_C \circ u|_{S_0}$ has non-vanishing differential. From this and the discussion above it follows that image of $pr_C \circ u$ is the image of some Reeb orbit. Consequently, by assumption that u has charge $(1, 0)$, u is equivalent to a Reeb torus for a uniquely determined Reeb orbit o_u.

The statement of the lemma follows when u has no spherical nodal components. On the other hand non-constant J^λ-holomorphic spheres are impossible, which can be seen as follows. Any such a J^λ-holomorphic sphere u lifts to the covering space $\widetilde{M} = C \times \mathbb{R}$ of M, as a $\widetilde{J}$-holomorphic map $\widetilde{u}$, where $\widetilde{J}$ is the lift of J^λ, and is compatible with the lift $\widetilde{\omega}$ of $\omega = d^\alpha \lambda$. On the other had $\widetilde{\omega} = d\lambda - dt \wedge \lambda$ is conformally symplectomorphic to the exact symplectic form $d(e^t \lambda)$, for $t : C \times \mathbb{R} \to \mathbb{R}$ the projection. So that $\widetilde{u}$ is constant by Stokes theorem.

□

Proposition 3.1. *Let (C, ξ) be a general contact manifold. If λ is a non-degenerate contact 1-form for ξ then all the elements of $\overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$ are regular curves. Moreover, if λ is degenerate then for a period c Reeb orbit o the kernel of the associated real linear Cauchy-Riemann operator for the Reeb torus u_o is naturally identified with the 1-eigenspace of $\phi^\lambda_{c,*}$ - the time c linearized return map $\xi(o(0)) \to \xi(o(0))$ induced by the R^λ Reeb flow.*

Proof. We already know by Proposition 1.11 that all $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A)$ are equivalent to Reeb tori. In particular, such curves have representation by a J^λ-holomorphic map

$$u : (T^2, j) \to (Y = C \times S^1, J^\lambda).$$

Since each u is immersed we may naturally get a splitting $u^*T(Y) \simeq N \times T(T^2)$, using the g_J metric, where $N \to T^2$ denotes the pull-back, of the g_J-normal bundle to image u, and which is identified with the pullback of the distribution ξ_λ on Y, (which we also call the co-vanishing distribution).

The full associated real linear Cauchy-Riemann operator takes the form:

$$D^J_u : \Omega^0(N \oplus T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2), N \oplus T(T^2)). \tag{3.2}$$

This is an index 2 Fredholm operator (after standard Sobolev completions), whose restriction to $\Omega^0(N \oplus T(T^2))$ preserves the splitting, that is the restricted operator splits as

$$D \oplus D' : \Omega^0(N) \oplus \Omega^0(T(T^2)) \to \Omega^{0,1}(T(T^2), N) \oplus \Omega^{0,1}(T(T^2), T(T^2)).$$

On the other hand the restricted Fredholm index 2 operator

$$\Omega^0(T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2)),$$

is surjective by classical Teichmuller theory, see also [28, Lemma 3.3] for a precise argument in this setting. It follows that D^J_u will be surjective if the restricted Fredholm index 0 operator

$$D : \Omega^0(N) \to \Omega^{0,1}(N),$$

has no kernel.

The bundle N is symplectic with symplectic form on the fibers given by restriction of $u^*d\lambda$, and together with J^λ this gives a Hermitian structure (g_λ, j_λ) on N. We have a linear symplectic connection $\mathcal{A}$ on N, which over the slices $S^1 \times \{t\} \subset T^2$ is induced by the pullback by u of the linearized R^λ Reeb flow. Specifically the $\mathcal{A}$-transport map from the fiber $N_{(s_0,t)}$ to the fiber $N_{(s_1,t)}$ over the path $[s_0, s_1] \times \{t\} \subset T^2$, is given by

$$(u_*|_{N_{(s_1,t)}})^{-1} \circ (\phi^\lambda_{c(s_1-s_0)})_* \circ u_*|_{N_{(s_0,t)}},$$

where $\phi^\lambda_{c(s_1-s_0)}$ is the time $c \cdot (s_1 - s_0)$ map for the R^λ Reeb flow, where c is the period of the Reeb orbit o_u, and where $u_* : N \to TY$ denotes the natural map, (it is the universal map in the pull-back diagram.)

The connection $\mathcal{A}$ is defined to be trivial in the θ_2 direction, where trivial means that the parallel transport maps are the id maps over θ_2 rays. In particular the curvature $R_\mathcal{A}$, understood as a lie algebra valued 2-form, of this connection vanishes. The connection $\mathcal{A}$ determines a real linear CR operator $D_{\mathcal{A}}$ on N in the standard way, take the complex anti-linear part of the vertical differential of a section. Explicitly,

$$D_{\mathcal{A}} : \Omega^0(N) \to \Omega^{0,1}(N),$$

is defined by

$$D_{\mathcal{A}}(\mu)(p) = j_\lambda \circ \pi^{vert}(\mu(p)) \circ d\mu(p) - \pi^{vert}(\mu(p)) \circ d\mu(p) \circ j,$$

where

$$\pi^{vert}(\mu(p)) : T_{\mu(p)}N \to T^{vert}_{\mu(p)}N \simeq N$$

is the $\mathcal{A}$-projection, and where $T^{vert}_{\mu(p)}N$ is the kernel of the projection $T_{\mu(p)}N \to T_p\Sigma$. It is elementary to verify from the definitions that this operator is exactly D. See also [17, Section 10.1] for a computation of this kind in much greater generality.

We have a differential 2-form Ω on the total space of N defined as follows. On the fibers $T^{vert}N$, $\Omega = u_*\omega$, for $\omega = d^\alpha\lambda$, and for $T^{vert}N \subset TN$ denoting the vertical tangent space, or subspace of vectors v with $\pi_* v = 0$, for $\pi : N \to T^2$ the projection. While on the $\mathcal{A}$-horizontal distribution Ω is defined to vanish. The 2-form Ω is closed, which we may check explicitly by using that $R_{\mathcal{A}}$ vanishes to obtain local symplectic trivializations of N in which $\mathcal{A}$ is trivial. Clearly Ω must vanish on the 0-section since it is a $\mathcal{A}$-flat section. But any section is homotopic to the 0-section and so in particular if $\mu \in \ker D$ then Ω vanishes on μ. But then since $\mu \in \ker D$, and so its vertical differential is complex linear, it must follow that the vertical differential vanishes, since $\Omega(v, J^\lambda v) > 0$, for $0 \neq v \in T^{vert}N$ and so otherwise we would have $\int_\mu \Omega > 0$. So μ is $\mathcal{A}$-flat, in particular the restriction of μ over all slices $S^1 \times \{t\}$ is identified with a period c orbit of the linearized at o R^λ Reeb flow, and which does not depend on t as $\mathcal{A}$ is trivial in the t variable. So the kernel of D is identified with the vector space of period c orbits of the linearized at o R^λ Reeb flow, as needed. □

Proposition 3.3. *Let λ be a contact form on a $(2n+1)$-fold C, and o a non-degenerate, period c, R^λ-Reeb orbit, then the orientation of $[u_o]$ induced by the determinant line bundle orientation of $\overline{\mathcal{M}}^{1,0}_{1,1}(J^\lambda, A)$, is $(-1)^{CZ(o)-n}$, which is*

$$\operatorname{sign} \operatorname{Det}(\operatorname{Id}|_{\xi(o(0))} - \phi^\lambda_{c,*}|_{\xi(o(0))}).$$

Proof of Proposition 3.3. Abbreviate u_o by u. Let $N \to T^2$ be associated to u as in the proof of Proposition 3.1. Fix a trivialization ϕ of N induced by any trivialization of the contact distribution ξ along o in the obvious sense: N is the pullback of ξ along the composition

$$T^2 \to S^1 \xrightarrow{o} C.$$

Let the symplectic connection $\mathcal{A}$ on N be defined as before. Then the pullback connection $\mathcal{A}' := \phi^*\mathcal{A}$ on $T^2 \times \mathbb{R}^{2n}$ is a connection whose parallel transport paths $p_t : [0,1] \to \operatorname{Symp}(\mathbb{R}^{2n})$, along the closed loops $S^1 \times \{t\}$, are paths starting at 1, and are t independent. And so the parallel transport path of $\mathcal{A}'$ along $\{s\} \times S^1$ is constant, that is $\mathcal{A}'$ is trivial in the t variable. We shall call such a connection $\mathcal{A}'$ on $T^2 \times \mathbb{R}^{2n}$ *induced by p*.

By non-degeneracy assumption on o, the map $p(1)$ has no 1-eigenvalues. Let $p'' : [0,1] \to \operatorname{Symp}(\mathbb{R}^{2n})$ be a path from $p(1)$ to a unitary map $p''(1)$, with $p''(1)$ having no 1-eigenvalues, and s.t. p'' has only simple crossings with the Maslov cycle. Let p' be the concatenation of p and p''. We then get

$$CZ(p') - \frac{1}{2} \operatorname{sign} \Gamma(p', 0) \equiv CZ(p') - n \equiv 0 \mod 2,$$

since p' is homotopic relative end points to a unitary geodesic path h starting at id, having regular crossings, and since the number of negative, positive eigenvalues is even at each regular crossing of h by unitarity. Here $\operatorname{sign} \Gamma(p', 0)$ is the index of the crossing form of the path p' at time 0, in the notation of [20]. Consequently,

$$CZ(p'') \equiv CZ(p) - n \mod 2, \tag{3.4}$$

by additivity of the Conley-Zehnder index.

Let us then define a free homotopy $\{p_t\}$ of p to p', p_t is the concatenation of p with $p''|_{[0,t]}$, reparametrized to have domain $[0,1]$ at each moment t. This determines a homotopy $\{\mathcal{A}'_t\}$ of connections induced by $\{p_t\}$. By the proof of Proposition 3.1, the CR operator D_t determined by each $\mathcal{A}'_t$ is surjective except at some finite collection of times $t_i \in (0,1)$, $i \in N$ determined by the crossing times of p'' with the Maslov cycle, and the dimension of the kernel of D_{t_i} is the 1-eigenspace of $p''(t_i)$, which is 1 by the assumption that the crossings of p'' are simple.

The operator D_1 is not complex linear. To fix this we concatenate the homotopy $\{D_t\}$ with the homotopy $\{\widetilde{D}_t\}$ defined as follows. Let $\{\widetilde{\mathcal{A}}_t\}$ be a homotopy of $\mathcal{A}'_1$ to a unitary connection $\widetilde{\mathcal{A}}_1$, where the homotopy $\{\widetilde{\mathcal{A}}_t\}$ is through connections induced by paths $\{\widetilde{p}_t\}$, giving a path homotopy of $p' = \widetilde{p}_0$ to h. Then $\{\widetilde{D}_t\}$ is defined to be induced by $\{\widetilde{\mathcal{A}}_t\}$.

Let us denote by $\{D'_t\}$ the concatenation of $\{D_t\}$ with $\{\widetilde{D}_t\}$. By construction in the second half of the homotopy $\{D'_t\}$, D'_t is surjective. And D'_1 is induced by a unitary connection, since it is induced by unitary path $\widetilde{p}_1$. Consequently, D'_1 is complex linear. By the above construction, for the homotopy $\{D'_t\}$, D'_t is surjective except for N times in $(0,1)$, where the kernel has dimension one. In particular the sign of $[u]$ by the definition via the determinant line bundle is exactly

$$-1^N = -1^{CZ(p)-n},$$

by (3.4), which was what to be proved. □

Theorem 3.5.

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(\widetilde{N}, R^\lambda, \beta),$$

where $N \subset \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A_\beta)$ is an open compact set, $\widetilde{N}$ the corresponding under R subset of periodic orbits of R^λ (where R is as in Proposition 1.11), $i(\widetilde{N}, R^\lambda, \beta)$ is the Fuller index as described in the appendix below, and where the left-hand side of the equation is the functional as in (2.1).

Proof. If $N \subset \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A_\beta)$ is open-compact and consists of isolated regular Reeb tori $\{u_i\}$, corresponding to orbits $\{o_i\}$ we have:

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = \sum_i \frac{(-1)^{CZ(o_i)-n}}{mult(o_i)},$$

where the denominator $mult(o_i)$ is there because our moduli space is understood as a non-effective orbifold, see Appendix B.

The expression on the right is exactly the Fuller index $i(\widetilde{N}, R^\lambda, \beta)$. Thus, the theorem follows for N as above. However, in general if N is open and compact then perturbing slightly we obtain a smooth family $\{R^{\lambda_t}\}$, $\lambda_0 = \lambda$, s.t. λ_1 is non-degenerate, that is has non-degenerate orbits. And such that there is an open-compact subset $\widetilde{N}$ of $\overline{\mathcal{M}}_{1,1}^{1,0}(\{J^{\lambda_t}\}, A_\beta)$ with $(\widetilde{N} \cap \overline{\mathcal{M}}_{1,1}^{1,0}(J^\lambda, A_\beta)) = N$, cf. Lemma 2.7. Then by Lemma 2.2 if

$$N_1 = (\widetilde{N} \cap \overline{\mathcal{M}}_{1,1}^{1,0}(J^{\lambda_1}, A_\beta))$$

we get

$$GW_{1,1}(N, A_\beta, J^\lambda)([\overline{M}_{1,1}] \otimes [C \times S^1]) = GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}_{1,1}] \otimes [C \times S^1]).$$

By the previous discussion

$$GW_{1,1}(N_1, A_\beta, J^{\lambda_1})([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(N_1, R^{\lambda_1}, \beta),$$

but by the invariance of Fuller index (see Appendix A),

$$i(N_1, R^{\lambda_1}, \beta) = i(N, R^\lambda, \beta).$$

□

Proof of Theorem 1.12. Let $u : \Sigma \to M$ be a non-constant J-curve. We first show that $[u^*\alpha] \neq 0$. Suppose otherwise. Let $\widetilde{M}$ denote the α-covering space of M, that is the space of equivalence classes of paths p starting at $x_0 \in M$, with a pair p_1, p_2 equivalent if $p_1(1) = p_2(1)$ and

$$\int_{[0,1]} p_1^*\alpha = \int_{[0,1]} p_2^*\alpha.$$

Then the lift of ω to $\widetilde{M}$ is

$$\widetilde{\omega} = \frac{1}{f} d(f\lambda),$$

where $f = e^g$ and where g is a primitive for the lift $\widetilde{\alpha}$ of α to $\widetilde{M}$, that is $\widetilde{\alpha} = dg$. In particular $\widetilde{\omega}$ is conformally symplectomorphic to an exact symplectic form on $\widetilde{M}$. So if $\widetilde{J}$ denotes the lift of J, any closed $\widetilde{J}$-curve is constant by Stokes theorem. Now $[u^*\alpha] = 0$, so u has a lift to a $\widetilde{J}$-holomorphic map $\widetilde{u} : \Sigma \to \widetilde{M}$. Since Σ is closed, it follows by the above that $\widetilde{u}$ is constant, which is a contradiction.

Since α is rational we may construct a smooth $p : M \to S^1$, so that $\alpha = c \cdot p^*d\theta$ for $c \in \mathbb{Q}$. Let $u : \Sigma \to M$ be a non-constant J-curve. Let $s_0 \in S^1$ be a regular value of $p \circ u$, and let $S_0 \subset \Sigma$, $S_0 \simeq S^1$ be a component of $(p \circ u)^{-1}(s_0)$. Since the critical points of u are isolated we may suppose that u is non-critical along S_0. In particular $u^*\omega$ is non-vanishing everywhere on $T\Sigma|_{S_0}$, which together with Lemma 1.10 implies that $u^*\lambda \wedge u^*\alpha$ is non-vanishing everywhere on $T\Sigma|_{S_0}$. So if $o : S^1 \to S_0$ is any parametrization, $u \circ o$ is a Reeb curve.

Now if u is an immersion then $u^*\omega$ is symplectic and by Lemma 1.10 $u^*d\lambda = 0$, so that $\omega_0 = u^*\alpha \wedge u^*\lambda$ is non-degenerate on Σ. Let $\widetilde{\Sigma}$ be the $u^*\alpha$-covering space of Σ so that $\omega_0 = dH \wedge u^*\lambda$ for some proper $H : \widetilde{\Sigma} \to \mathbb{R}$. Since ω_0 is non-degenerate, H has no critical points so that $\widetilde{\Sigma} \simeq S^1 \times \mathbb{R}$ by basic Morse theory. It follows that $\Sigma \simeq T^2$. $\square$

Lemma 3.6. *Let (M, λ, α, J) be a tamed exact* lcs *structure. Suppose that α is rational, then every non-constant J-curve $u : \Sigma \to M$, with Σ a closed possibly nodal Riemann surface, is smooth, that is Σ is a smooth Riemann surface.*

Proof. Since α is rational we may construct a smooth $p : M \to S^1$, so that $\alpha = c \cdot p^*d\theta$ for $c \in \mathbb{Q}$. Let $u : \Sigma \to M$ be a non-constant J-curve. Let $s_0 \in S^1$ be a regular value of $p \circ u$, and let $S_0 \subset \Sigma$, $S_0 \simeq S^1$ be a component of $(p \circ u)^{-1}(s_0)$. Since the critical points of u are isolated we may suppose that u is non-critical along S_0. Suppose by contradiction that Σ is nodal. We may then find an embedded disk $i : D^2 \to \Sigma$ with $\partial i(D^2) = S$.

Since $u^*d\lambda = 0$ by Lemma 1.10, $\int_{S^1} i^*u^*\lambda = 0$ by Stokes theorem, and so $u^*\lambda(v) = 0$ for some $v \in TS_0(z) \subset T_z\Sigma$, $z \in S_0$. And let $w \in T_z\Sigma$ be such that v, w form a basis for $T_z\Sigma$. Now $u^*\omega$ is symplectic along S_0 so that $u^*\omega(v, w) \neq 0$ which implies that $u^*\alpha \wedge u^*\lambda(v, w) \neq 0$ since $u^*d\lambda(v, w) = 0$, but $u^*\alpha(v) = 0$ and $u^*\lambda(v) = 0$, so that we have a contradiction. $\square$

Theorem 3.7. *Let (C, λ) be a closed contact 3-manifold with λ non-degenerate. Then there exists a $\epsilon > 0$ s.t. for any almost symplectic pair (ω, J) on $M = C \times S^1$, ϵ-close to $(d^\alpha\lambda, J^\lambda)$ (as in Definition 2.4), there exists an elliptic, charge $(1, 0)$, J-holomorphic curve in M. Moreover, this curve may be assumed to be non-nodal and embedded.*

Proof. For λ as in the hypothesis by Taubes theorem [24] there is a non-degenerate Reeb orbit o. Let $N \subset \overline{\mathcal{M}}_{1,1}^{1,0}(A, J^\lambda)$, be the subspace consisting of a single point, corresponding to o (under the correspondence of Proposition 1.11). Clearly N is open and compact by the non-degeneracy assumption on λ. By Theorem 3.5 $GW_{1,1}(N, J^\lambda, A) \neq 0$, since the Fuller index of o is non-zero by non-degeneracy. N is also clearly energy isolated, where this is as in Proposition 2.9. The first part of the theorem then follows by Proposition 2.9.

We now verify the second part. Let (ω, J) be an almost symplectic pair. Suppose that $u \in \overline{\mathcal{M}}_{1,1}^{1,0}(A, J)$. Let $\underline{u}$ be a simple J-holomorphic curve covered by u, which is non-nodal by Lemma 3.6. For convenience, we recall the adjunction inequality.

Theorem 3.8 (McDuff-Micallef-White [16], [11])**.** *Let (M, J) be an almost complex 4-manifold and $A \in H_2(M)$ be a homology class that is represented by a simple J-holomorphic curve u. Then*

$$2\delta(u) - \chi(\Sigma) \leq A \cdot A - c_1(A),$$

with equality if and only if u is an immersion with only transverse self-intersections.

In our case $A = 0$, $\chi(\Sigma) = 0$, so that $\delta(\underline{u}) = 0$, and so $\underline{u}$ is an embedding. ∎

Proof of Theorem 1.7. Let

$$U \ni (\omega_0 := d^\alpha \lambda, J_0 := J^\lambda)$$

be a set of pairs (ω, J), where $\omega \in \mathcal{L}(M)$, is an exact, first kind, rational lcs structure and satisfying the following.

- For each $(\omega, J) \in U$, J is ω-compatible and admissible.
- Let ϵ be chosen as in the first part of Theorem 3.7. Each $(\omega, J) \in U$, is ϵ-close to (ω_0, J_0), (as in Definition 2.4).

To prove the theorem we need to construct a map $E : V \to \mathcal{J}(M)$, where V is some d_3 neighborhood of ω_0 in the space of exact, first kind rational lcs structures, and where

$$\forall \omega \in V : (\omega, E(\omega)) \in \mathcal{U}.$$

As then Theorem 3.7 tells us that for each $\omega \in V$, there is a class A, $E(\omega)$-holomorphic, elliptic curve u in M. Since $E(\omega)$ is ω-admissible, using Theorem 1.12 we would conclude that there is a Reeb curve for ω, and so we will be done.

Define a metric ρ_0 measuring the distance between subspaces W_1, W_2, of same dimension, of an inner product space (T, g) as follows.

$$\rho_0(W_1, W_2) := |P_{W_1} - P_{W_2}|,$$

for $|\cdot|$ the g-operator norm, and P_{W_i} g-projection operators onto W_i. We may of course generalize this to a C^2 metric ρ_2 again in terms of these projection operators.

Let $\delta > 0$ be given. Suppose that $\omega = d^{\alpha'} \lambda'$ is a first kind lcs structure δ-close to ω_0 for the C^3 metric d_3 as in the statement of the theorem. Then $\mathcal{V}_{\lambda'}, \xi_{\lambda'}$ are smooth distributions by the assumption that (α', λ') is a lcs structure of the first kind. Moreover, for each $p \in M$,

$$\rho_2(\mathcal{V}_{\lambda'}(p), \mathcal{V}_\lambda(p)) < \epsilon_\delta$$

and

$$\rho_2(\xi_{\lambda'}(p), \xi_\lambda(p)) < \epsilon_\delta$$

where $\epsilon_\delta \to 0$ as $\delta \to 0$, and where ρ_2 is the metric as defined above for subspaces of the inner product space $(T_p M, g)$.

Then choosing δ to be suitably be small, for each $p \in M$ we have an isomorphism

$$\phi(p) : T_p M \to T_p M,$$

$\phi_p := P_1 \oplus P_2$, for $P_1 : \mathcal{V}_{\lambda_0}(p) \to \mathcal{V}_{\lambda'}(p)$, $P_2 : \xi_{\lambda_0}(p) \to \xi_{\lambda'}(p)$ the g-projection operators. Define $E(\omega)(p) := \phi(p)_* J_0$. Then clearly, if δ was chosen to be sufficiently small, E defined on the d_3 δ-neighborhood V has the needed property. ∎

Proof of Theorem 1.8. Let $\{\omega_t\}$, $t \in [0, 1]$, be a continuous in the usual C^∞ topology homotopy of non-degenerate 2-forms on $M = C \times S^1$, with $\omega_0 = d^\alpha \lambda$ as in the hypothesis and with ω_1 an exact, rational, first kind lcs structure. As $\omega_1 = (\alpha', \lambda')$ is first kind, we may fix an almost complex structure J_1 on M admissible with respect to (α', λ'). And let J_0 be the almost complex structure J^λ, as in Section 1.2.2. Extend J_0, J_1 to a Frechet smooth family $\{J_t\}$ of almost complex structures on M, so that J_t is ω_t-compatible for each t. Then in the absence of holomorphic sky catastrophes, by Theorem 4.11, there is a non-constant elliptic J_1-holomorphic curve in M, so that the result follows by Theorem 1.12.

∎

4. Extended Gromov-Witten invariants and the extended Fuller index

In what follows M is a closed oriented $2n$-fold, $n \geq 2$, and J an almost complex structure on M. Much of the following discussion extends to general moduli spaces $\mathcal{M}_{g,n}(J, A, a_1, \ldots, a_n)$ with $a_1, \ldots, a_n$ homological constraints in M. We shall however restrict for simplicity to the case (ω, J) is a compatible lcs pair on M, $g = 1, n = 1$, the homological constraint is $[M]$, as this is the main interest in this paper. Moreover, we restrict our moduli space to consist of non-zero charge pair (for example $(1, 0)$) curves, with charge defined with respect to the Lee form α of ω as in Section 1.2.1, and this will be implicit, so that we no longer specify this in notation.

In what follows, for an almost symplectic pair (ω, J), and $u : \Sigma \to M$ a smooth map, $e(u) := e_{g_J}(u)$ the L^2 energy with respect to g_J, where the latter is as previously.

Definition 4.1. *Let $h = \{(\omega_t, J_t)\}$ be a homotopy of almost symplectic pairs on M, so that $\{J_t\}$ is Frechet smooth, and $\{\omega_t\}$ C^0 continuous. We say that it is* **partially admissible for** A *if every element of*

$$\overline{\mathcal{M}}_{1,1}(M, J_0, A)$$

is contained in a compact open subset of $\overline{\mathcal{M}}_{1,1}(M, \{J_t\}, A)$. We say that h is **admissible for** A *if every element of*

$$\overline{\mathcal{M}}_{1,1}(M, J_i, A),$$

$i = 0, 1$ is contained in a compact open subset of $\overline{\mathcal{M}}_{1,1}(M, \{J_t\}, A)$.

Thus, in the above definition, a homotopy is partially admissible if there are no sky catastrophes going one way, and admissible if there are no sky catastrophes going either way. Partly to simplify notation, we denote by a capital X a general almost symplectic pair (ω, J) on a smooth manifold M. Then we introduce the following simplified notation.

$$\begin{aligned} S(X, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(X, A)\} \\ S(X, a, A) &= \{u \in S(X, A) \mid e(u) \leq a\} \\ S(h, A) &= \{u \in \overline{\mathcal{M}}_{1,1}(h, A)\}, \text{ for } h = \{(\omega_t, J_t)\} \text{ a homotopy as above} \\ S(h, a, A) &= \{u \in S(h, A) \mid e(u) \leq a\} \end{aligned} \tag{4.2}$$

Definition 4.3. *For an isolated element u of $S(X, A)$, which means that $\{u\}$ is open as a subset, we set $gw(u) \in \mathbb{Q}$ to be the local Gromov-Witten invariant of u. This is defined as:*

$$gw(u) = GW_{1,1}(\{u\}, A, J)([\overline{M}_{1,1}] \otimes [M]),$$

with the right-hand side as in (2.1).

Denote by $\mathcal{S}(M, A)$ the set of equivalence classes of all smooth stable maps $\Sigma \to M$, in class A, for Σ an (non-fixed) elliptic curve, and where equivalence has the same meaning as in Section 1.2.1.

Definition 4.4. *Suppose that $S(X, A)$ has open connected components. And suppose that we have a collection of almost symplectic pairs*

$$\{X^a = (M, \omega^a, J^a)\},\ a \in \mathbb{R}_+$$

satisfying the following:

- *$S(X^a, a, A)$ consists of isolated curves for each a.*
- $$S(X^a, a, A) = S(X^b, a, A),$$
  (equality of subsets of $\mathcal{S}(M, A)$) if $b > a$,
- *For $b > a$, and for each $u \in S(X^a, a, A) = S(X^b, a, A)$:*
  $$GW_{1,1}(\{u\}, A, J^a) = GW_{1,1}(\{u\}, A, J^b),$$
  thus we may just write $gw(u)$ for the common number.

- *There is a prescribed homotopy $h^a = \{X^a_t\}$ of each X^a to X, called* **structure homotopy**, *with the following property. For every*
$$y \in S(X^a_0, A)$$
there is an open compact subset $\mathcal{C}_y \subset S(h^a, A)$, $y \in C_y$, which is **non-branching**, *where the latter means that*
$$\mathcal{C}_y \cap S(X^a_i, A),$$
$i = 0, 1$ are connected.
- $$S(h^a, a, A) = S(h^b, a, A),$$
(similarly equality of subsets) if $b > a$ is sufficiently large.

We will then say that
$$\mathcal{P}(A) = \{(X^a, h^a)\}$$
is a **perturbation system** *for X in the class A.*

We shall see shortly that, given a contact (C, λ), the associated Banyaga lcs structure on $C \times S^1$ always admits a perturbation system for the moduli spaces of charge (1,0) curves in any class, if λ is Morse-Bott.

Definition 4.5. *Suppose that X admits a perturbation system $\mathcal{P}(A)$ so that there exists an $E = E(\mathcal{P}(A))$ with the property that*
$$S(X^a, a, A) = S(X^E, a, A)$$
for all $a > E$, where this as before is equality of subsets, and the local Gromov-Witten invariants of the identified elements are also identified. Then we say that X is **finite type** *and set:*
$$GW(X, A) = \sum_{u \in S(X^E, A)} gw(u).$$

Definition 4.6. *Suppose that X admits a perturbation system $\mathcal{P}(A)$ and there is an $E = E(\mathcal{P}(A)) > 0$ so that $gw(u) > 0$ for all*
$$\{u \in S(X^a, A) \,|\, E \leq e(u) \leq a\}$$
respectively $gw(u) < 0$ for all
$$\{u \in S(X^a, A) \,|\, E \leq e(u) \leq a\},$$
and every $a > E$. Suppose in addition that
$$\lim_{a \mapsto \infty} \sum_{u \in S(X, a, A)} gw(u) = \infty, \text{ respectively } \lim_{a \mapsto \infty} \sum_{u \in S(X, a, \beta)} gw(u) = -\infty.$$
Then we say that X is **positive infinite type**, *respectively* **negative infinite type** *and set*
$$GW(X, A) = \infty,$$
respectively $GW(X, A) = -\infty$. These are meant to be interpreted as extended Gromov-Witten invariants, counting elliptic curves in class A. We say that X is **infinite type** *if it is one or the other.*

Definition 4.7. *We say that X is* **definite** *type if it admits a perturbation system and is infinite type or finite type.*

With the above definitions
$$GW(X, A) \in \mathbb{Q} \sqcup \infty \sqcup -\infty,$$
when it is defined.

Proof of Theorem 1.13. Given the definitions above, and the definition of the extended Fuller index in [22], this follows by the same argument as the proof of Theorem 3.5. ☐

4.0.1. *Perturbation systems for Morse-Bott Reeb vector fields.*

Definition 4.8. *A contact form λ on M, and its associated flow R^λ are called* Morse-Bott *if the λ action spectrum $\sigma(\lambda)$ - that is the space of critical values of $o \mapsto \int_{S^1} o^*\lambda$, $o : S^1 \to M$, is discreet and if for every $a \in \sigma(\lambda)$, the space*

$$N_a := \{x \in M |\, F_a(x) = x\},$$

F_a the time a flow map for R^λ - is a closed smooth manifold such that rank $d\lambda|_{N_a}$ is locally constant and $T_x N_a = \ker(dF_a - I)_x$.

Proposition 4.9. *Let λ be a contact form of Morse-Bott type, on a closed contact manifold C. Then the corresponding* lcs *pair $X_\lambda = (C \times S^1, d^\alpha\lambda, J^\lambda)$ admits a perturbation system $\mathcal{P}(A)$, for moduli spaces of charge (1,0) curves for every class A.*

Proof. This follows immediately by [22, Proposition 2.12], and by Proposition 1.11. □

Lemma 4.10. *The Hopf* lcs *pair $(S^{2k+1} \times S^1, d^\alpha\lambda_H, J^{\lambda_H})$, for λ_H the standard contact structure on S^{2k+1} is infinite type.*

Proof. This follows immediately by [22, Lemma 2.13], and by Proposition 1.11. □

Theorem 4.11. *Let (C, λ) be a closed contact manifold so that R^λ has definite type, and suppose that $i(R^\lambda, \beta) \neq 0$. Let $\omega_0 = d^\alpha\lambda$ be the Banyaga structure, and suppose we have a partially admissible homotopy $h = \{(\omega_{t,}, J_t)\}$, for class A_β, then there in an element $u \in \overline{\mathcal{M}}^{1,0}_{1,1}(J_1, A_\beta)$.*

The proof of this will follow.

4.1. Preliminaries on admissible homotopies.

Definition 4.12. *Let $h = \{X_t\}$ be a smooth homotopy of almost symplectic pairs. For $b > a > 0$ we say that h is* **partially** a, b-**admissible***, respectively* a, b-**admissible** *(in class A) if for each*

$$y \in S(X_0, a, A)$$

there is a compact open subset $\mathcal{C}_y \subset S(h, A)$, $y \in \mathcal{C}_y$ with $e(u) < b$, for all $u \in \mathcal{C}_y$. Respectively, if for each

$$y \in S(X_i, a, A),$$

$i = 0, 1$ there is a compact open subset $\mathcal{C}_y \ni y$ of $S(h, A)$ with $e(u) < b$, for all $u \in \mathcal{C}_y$.

Lemma 4.13. *Suppose that X_0 has a perturbation system $\mathcal{P}(A)$, and $\{X_t\}$ is partially admissible, then for every a there is a $b > a$ so that $\{\widetilde{X}^b_t\} = \{X_t\} \cdot \{X^b_t\}$ is partially a, b-admissible, where $\{X_t\} \cdot \{X^b_t\}$ is the (reparametrized to have t domain $[0, 1]$) concatenation of the homotopies $\{X_t\}, \{X^b_t\}$, and where $\{X^b_t\}$ is the structure homotopy from X^b to X_0.*

Proof. This is a matter of pure topology, and the proof is completely analogous to the proof of [22, Lemma 3.8]. □

The analogue of Lemma 4.13 in the admissible case is the following:

Lemma 4.14. *Suppose that X_0, X_1 and $\{X_t\}$ are admissible, then for every a there is a $b > a$ so that*

$$\{\widetilde{X}^b_t\} = \{X^b_{1,t}\}^{-1} \cdot \{X_t\} \cdot \{X^b_{0,t}\} \tag{4.15}$$

is a, b-admissible, where $\{X^b_{i,t}\}$ are the structure homotopies from X^b_i to X_i.

4.2. **Invariance.**

Theorem 4.16. *Suppose X_0 is definite type, with $GW(X_0, A) \neq 0$, and suppose it is joined to X_1 by a partially admissible homotopy $\{X_t\}$, then X_1 has non-constant elliptic class A curves.*

Proof of Theorem 4.11. This follows by Theorem 4.16 and by Theorem 1.13. □

We also have a more precise result.

Theorem 4.17. *If X_0, X_1 are definite type pairs and $\{X_t\}$ is admissible then $GW(X_0, A) = GW(X_1, A)$.*

Proof of Theorem 4.16. Suppose that X_0 is definite type with $GW(X_0, A) \neq 0$, $\{X_t\}$ is partially admissible and $\overline{\mathcal{M}}_{1,1}(X_1, A) = \emptyset$. Let a be given and b determined so that $\widetilde{h}^b = \{\widetilde{X}_t^b\}$ is a partially (a, b)-admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\widetilde{h}^b, A),$$

for $y \in S(X_0^b, a, A)$. Here we use a natural identification of $S(X^b, a, A) = S(\widetilde{X}_0^b, a, A)$ as a subset of $S(\widetilde{h}^b, A)$ by its construction. Then S_a is an open-compact subset of $S(h, A)$ and so admits an implicit atlas (Kuranishi structure) with boundary, (with virtual dimension 1) s.t:

$$\partial S_a = S(X^b, a, A) + Q_a,$$

where Q_a as a set is some subset (possibly empty), of elements $u \in S(X^b, b, A)$ with $e(u) \geq a$. So we have for all a:

$$\sum_{u \in Q_a} gw(u) + \sum_{u \in S(X^b, a, A)} gw(u) = 0. \tag{4.18}$$

4.3. **Case I, X_0 is finite type.** Let $E = E(\mathcal{P})$ be the corresponding cutoff value in the definition of finite type, and take any $a > E$. Then $Q_a = \emptyset$ and by definition of E we have that the left side is

$$\sum_{u \in S(X^b, E, A)} gw(u) \neq 0.$$

Clearly this gives a contradiction to (4.18).

4.4. **Case II, X_0 is infinite type.** We may assume that $GW(X_0, A) = \infty$, and take $a > E$, where $E = E(\mathcal{P}(A))$ is the corresponding cutoff value in the definition of infinite type. Then

$$\sum_{u \in Q_a} gw(u) \geq 0,$$

as $a > E(\mathcal{P}(A))$. On the other hand,

$$\lim_{a \mapsto \infty} \sum_{u \in S(X^b, a, A)} gw(u) = \infty,$$

as $GW(X_0, A) = \infty$. This also contradicts (4.18). □

Proof of Theorem 4.17. This is somewhat analogous to the proof of Theorem 4.16. Suppose that $X_i, \{X_t\}$ are definite type as in the hypothesis. Let a be given and b determined so that $\widetilde{h}^b = \{\widetilde{X}_t^b\}$, see (4.15) is an (a, b)-admissible homotopy. We set

$$S_a = \bigcup_y \mathcal{C}_y \subset S(\widetilde{h}^b, A)$$

for $y \in S(X_i^b, a, A)$. Then S_a is an open-compact subset of $S(h, A)$ and so has admits an implicit atlas (Kuranishi structure) with boundary, (with virtual dimension 1) and satisfies the following.

$$\partial S_a = (S(X_0^b, a, A) + Q_{a,0})^{op} + S(X_1^b, a, A) + Q_{a,1},$$

with *op* denoting the opposite orientation, where $Q_{a,i}$ as sets are some subsets (possibly empty), of elements $u \in S(X_i^b, b, A)$ with $e(u) \geq a$. So we have for all a:

$$\sum_{u \in Q_{a,0}} gw(u) + \sum_{u \in S(X_0^b, a, A)} gw(u) = \sum_{u \in Q_{a,1}} gw(u) + \sum_{u \in S(X_1^b, a, A)} gw(u). \tag{4.19}$$

4.5. **Case I, X_0 is finite type and X_1 is infinite type.** Suppose in addition $GW(X_1, A) = \infty$ and let $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$, for $\mathcal{P}_i$, the perturbation systems of X_i. Take any $a > E$. Then $Q_{a,0} = \emptyset$ and the left-hand side of (4.19) is

$$\sum_{u \in S(X_0^b, E, A)} gw(u) < \infty.$$

The right-hand side tends to ∞ as a tends to infinity since,

$$\sum_{u \in Q_{a,1}} gw(u) \geq 0,$$

as $a > E(\mathcal{P}_1(A))$, and since

$$\lim_{a \mapsto \infty} \sum_{u \in S(X_1^b, a, A)} gw(u) = \infty.$$

Clearly this gives a contradiction to (4.19).

4.6. **Case II, X_i are infinite type.** Suppose in addition $GW(X_0, A) = -\infty$, $GW(X_1, A) = \infty$ and let $E = \max(E(\mathcal{P}_0(A)), E(\mathcal{P}_1(A)))$, for $\mathcal{P}_i$, the perturbation systems of X_i. Take any $a > E$. Then $\sum_{u \in Q_{a,0}} gw(u) \leq 0$, and $\sum_{u \in Q_{a,1}} gw(u) \geq 0$. So by definition of $GW(X_i, A)$ the left hand side of (4.18) tends to $-\infty$ as a tends to ∞, and the right hand side tends to ∞. Clearly this gives a contradiction to (4.19).

4.7. **Case III, X_i are finite type.** The argument is analogous. $\square$

A. Fuller index

Let X be a vector field on M. Set

$$S(X) = S(X, \beta) = \{(o, p) \in L_\beta M \times (0, \infty) \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\},$$

where $L_\beta M$ denotes the free homotopy class β component of the free loop space. Elements of $S(X)$ will be called orbits. There is a natural S^1 reparametrization action on $S(X)$, and elements of $S(X)/S^1$ will be called *unparametrized orbits*, or just orbits. Slightly abusing notation we write (o, p) for the equivalence class of (o, p). The multiplicity $m(o, p)$ of a periodic orbit is the ratio p/l for $l > 0$ the least period of o. We want a kind of fixed point index which counts orbits (o, p) with certain weights - however in general to get invariance we must have period bounds. This is due to potential existence of sky catastrophes as described in the introduction.

Let $N \subset S(X)$ be a compact open set. Assume for simplicity that elements $(o, p) \in N$ are isolated. (Otherwise we need to perturb.) Then to such an (N, X, β) Fuller associates an index:

$$i(N, X, \beta) = \sum_{(o,p) \in N/S^1} \frac{1}{m(o, p)} i(o, p),$$

where $i(o, p)$ is the fixed point index of the time p return map of the flow of X with respect to a local surface of section in M transverse to the image of o. Fuller then shows that $i(N, X, \beta)$ has the following invariance property. Given a continuous homotopy $\{X_t\}$, $t \in [0, 1]$ let

$$S(\{X_t\}, \beta) = \{(o, p, t) \in L_\beta M \times (0, \infty) \times [0, 1] \mid o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX_t\}.$$

Given a continuous homotopy $\{X_t\}$, $X_0 = X$, $t \in [0, 1]$, suppose that $\widetilde{N}$ is an open compact subset of $S(\{X_t\})$, such that

$$\widetilde{N} \cap (LM \times \mathbb{R}_+ \times \{0\}) = N.$$

Then if

$$N_1 = \widetilde{N} \cap (LM \times \mathbb{R}_+ \times \{1\})$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta).$$

In the case where X is the R^λ-Reeb vector field on a contact manifold (C^{2n+1}, ξ), and if (o, p) is non-degenerate, we have:

$$i(o,p) = \operatorname{sign} \operatorname{Det}(\operatorname{Id}|_{\xi(x)} - F^\lambda_{p,*}|_{\xi(x)}) = (-1)^{CZ(o)-n}, \tag{A.1}$$

where $F^\lambda_{p,*}$ is the differential at x of the time p flow map of R^λ, and where $CZ(o)$ is the Conley-Zehnder index, (which is a special kind of Maslov index) see [20].

There is also an extended Fuller index $i(X, \beta) \in \mathbb{Q} \sqcup \{\pm\infty\}$, for certain X having definite type. This is constructed in [22], and is conceptually completely analogous to the extended Gromov-Witten invariant constructed in this paper.

B. Remark on multiplicity

This is a small note on how one deals with curves having non-trivial isotropy groups, in the virtual fundamental class technology. We primarily need this for the proof of Theorem 3.5. Given a closed oriented orbifold X, with an orbibundle E over X Fukaya-Ono [7] show how to construct using multi-sections its rational homology Euler class, which when X represents the moduli space of some stable curves, is the virtual moduli cycle $[X]^{vir}$. When this is in degree 0, the corresponding Gromov-Witten invariant is $\int_{[X]^{vir}} 1$. However, they assume that their orbifolds are effective. This assumption is not really necessary for the purpose of construction of the Euler class but is convenient for other technical reasons. A different approach to the virtual fundamental class which emphasizes branched manifolds is used by McDuff-Wehrheim, see for example McDuff [10], [13] which does not have the effectivity assumption, a similar use of branched manifolds appears in [3]. In the case of a non-effective orbibundle $E \to X$ McDuff [12], constructs a homological Euler class $e(E)$ using multi-sections, which extends the construction [7]. McDuff shows that this class $e(E)$ is Poincare dual to the completely formally natural cohomological Euler class of E, constructed by other authors. In other words there is a natural notion of a homological Euler class of a possibly non-effective orbibundle. We shall assume the following black box property of the virtual fundamental class technology.

Axiom B.1. *Suppose that the moduli space of stable maps is cleanly cut out, which means that it is represented by a (non-effective) orbifold X with an orbifold obstruction bundle E, that is the bundle over X of cokernel spaces of the linearized CR operators. Then the virtual fundamental class $[X]^{vir}$ coincides with $e(E)$.*

Given this axiom it does not matter to us which virtual moduli cycle technique we use. It is satisfied automatically by the construction of McDuff-Wehrheim, (at the moment in genus 0, but surely extending). It can be shown to be satisfied in the approach of John Pardon [19]. And it is satisfied by the construction of Fukaya-Oh-Ono-Ohta [5], the latter is communicated to me by Kaoru Ono. When X is 0-dimensional this does follow immediately by the construction in [7], taking any effective Kuranishi neighborhood at the isolated points of X, (this actually suffices for our paper.)

As a special case most relevant to us here, suppose we have a moduli space of elliptic curves in X, which is regular with expected dimension 0. Then its underlying space is a collection of oriented points. However, as some curves are multiply covered, and so have isotropy groups, we must treat this is a non-effective 0 dimensional oriented orbifold. The contribution of each curve $[u]$ to the Gromov-Witten invariant $\int_{[X]^{vir}} 1$ is $\frac{\pm 1}{[\Gamma([u])]}$, where $[\Gamma([u])]$ is the order of the isotropy group $\Gamma([u])$ of $[u]$, in the McDuff-Wehrheim setup this is explained in [10, Section 5]. In the setup of Fukaya-Ono [7] we may readily calculate to get the same thing taking any effective Kuranishi neighborhood at the isolated points of X.

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References

[1] V. Apostolov and G. Dloussky, *Locally conformally symplectic structures on compact non-Kähler complex surfaces*, Int. Math. Res. Not., 2016 (2016), pp. 2717–2747.
[2] A. Banyaga, *Some properties of locally conformal symplectic structures.*, Comment. Math. Helv., 77 (2002), pp. 383–398.
[3] K. Cieliebak, I. Mundet i Riera, and D. A. Salamon, *Equivariant moduli problems, branched manifolds, and the Euler class.*, Topology, 42 (2003), pp. 641–700.
[4] Y. Eliashberg and E. Murphy, *Making cobordisms symplectic*, arXiv.
[5] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Technical details on Kuranishi structure and virtual fundamental chain*, arXiv.
[6] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Intersection Floer theory, Anomaly and Obstruction I and II*, AMS/IP, Studies in Advanced Mathematics, 2000.
[7] K. Fukaya and K. Ono, *Arnold Conjecture and Gromov–Witten invariant*, Topology, 38 (1999), pp. 933 – 1048.
[8] F. Fuller, *Note on trajectories in a solid torus.*, Ann. Math. (2), 56 (1952), pp. 438–439.
[9] H.-C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus.*, Am. J. Math., 65 (1943), pp. 433–438.
[10] D. McDuff, *Notes on Kuranishi Atlases*, arXiv.
[11] D. McDuff, *The local behaviour of holomorphic curves in almost complex 4-manifolds*, J. Differ. Geom., 34 (1991), pp. 143–164.
[12] ———, *Groupoids, branched manifolds and multisections.*, J. Symplectic Geom., 4 (2006), pp. 259–315.
[13] ———, *Constructing the virtual fundamental class of a Kuranishi atlas*, Algebr. Geom. Topol., 19 (2019), pp. 151–238.
[14] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford Math. Monographs, The Clarendon Oxford University Press, New York, second ed., 1998.
[15] ———, *J–holomorphic curves and symplectic topology*, no. 52 in American Math. Society Colloquium Publ., Amer. Math. Soc., 2004.
[16] M. J. Micallef and B. White, *The structure of branch points in minimal surfaces and in pseudoholomorphic curves*, Ann. Math. (2), 141 (1995), pp. 35–85.
[17] Y.-G. Oh and Y. Savelyev, *Pseudoholomorphic curves on the lcs-fication of contact manifolds*, https://arxiv.org/pdf/2107.03551.pdf, Accepted by Advances in geometry.
[18] Y.-G. Oh and R. Wang, *Analysis of contact Cauchy-Riemann maps. I: A priori C^k estimates and asymptotic convergence*, Osaka J. Math., 55 (2018), pp. 647–679.
[19] J. Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of J-holomorphic curves*, Geometry and Topology.
[20] J. Robbin and D. Salamon, *The Maslov index for paths.*, Topology, 32 (1993), pp. 827–844.
[21] Y. Savelyev, *Locally conformally symplectic deformation of Gromov non-squeezing*, submitted.
[22] ———, *Extended Fuller index, sky catastrophes and the Seifert conjecture*, International Journal of mathematics, 29 (2018).
[23] A. Shilnikov, L. Shilnikov, and D. Turaev, *Blue-sky catastrophe in singularly perturbed systems.*, Mosc. Math. J., 5 (2005), pp. 269–282.
[24] C. H. Taubes, *The Seiberg-Witten equations and the Weinstein conjecture.*, Geom. Topol., 11 (2007), pp. 2117–2202.
[25] I. Vaisman, *Locally conformal symplectic manifolds.*, Int. J. Math. Math. Sci., 8 (1985), pp. 521–536.
[26] K. Wehrheim, *Energy quantization and mean value inequalities for nonlinear boundary value problems.*, J. Eur. Math. Soc. (JEMS), 7 (2005), pp. 305–318.
[27] C. Wendl and C. Gerig, *Generic transversality for unbranched covers of closed pseudoholomorphic curves*, arXiv:1407.0678, (2014).
[28] C. Wendle, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four.*, Comment. Math. Helv., 85 (2010), pp. 347–407.
[29] ———, *Transversality and super-rigidity for multiply covered holomorphic curves*, arXiv:1609.09867, (2016).

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