

# GLOBAL FUKAYA CATEGORY II: SINGULAR CONNECTIONS, QUANTUM OBSTRUCTION THEORY AND OTHER APPLICATIONS

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ABSTRACT. We perform computations and give applications of the theory developed in Part I of the homotopy natural classifying map

$$B\mathrm{Ham}(M, \omega) \rightarrow (\mathbb{S}, N\mathrm{Fuk}(M, \omega)),$$

with the right hand side the space ( $\infty$ -groupoid) of infinity categories in the component of the  $A_\infty$  nerve of the Fukaya category of a monotone symplectic manifold  $(M, \omega)$ . For example, we find new curvature phenomena for  $PU(2)$  or  $\mathrm{Ham}(S^2)$  smooth and singular connections on principal bundles over  $S^4$ . Even for  $PU(2)$ , these phenomena are invisible to Chern-Weil theory, and are inaccessible to known Yang-Mills theory techniques, which are the sharpest known “classical” techniques in this setting. So this can be understood as one application of Floer theory and the theory of  $\infty$ -categories in basic differential geometry. We also introduce new, non-trivial integer invariants of smooth manifolds. On the way we also construct what we call quantum Maslov classes, which are higher dimensional versions of the relative Seidel morphism studied by Hu and Lalonde. We compute this in a particular case and discuss an application to Hofer geometry of the space of Lagrangian equators in  $S^2$ .

## 1. INTRODUCTION

Let  $\mathrm{Ham}(M, \omega)$  denote the group of Hamiltonian symplectomorphisms of a symplectic manifold  $(M, \omega)$ , understood as a Frechet Lie group, with its  $C^\infty$  topology. A *Hamiltonian bundle* is a smooth fiber bundle

$$M \hookrightarrow P \rightarrow X,$$

with structure group  $\mathrm{Ham}(M, \omega)$ . Given such a bundle  $P$  with  $M$  monotone, in Part I [19] we have constructed a classifying map

$$f_P : X \rightarrow (\mathbb{S}, N\mathrm{Fuk}(M))$$

where  $\mathbb{S}$  denotes the space of  $\infty$ -categories, concretely quasi-categories, in the component of the  $A_\infty$  nerve  $N\mathrm{Fuk}(M)$  of the Fukaya category of  $M$ , with this nerve having the structure of an  $\infty$ -category. This extends to the universal level, so that there is a universal classifying map:

$$B\mathrm{Ham}(M, \omega) \rightarrow (\mathbb{S}, N\mathrm{Fuk}(M)).$$

From here on we just refer to [19] as Part I.

The construction also induces a kind simplicial fibration over  $X$  called (co)-Cartesian fibration, with fiber modelled on  $N\mathrm{Fuk}(M)$ . We called this the global Fukaya category  $\mathrm{Fuk}_\infty(P)$  of  $P$ . As first computational step, we show that for  $P$  a non-trivial Hamiltonian  $S^2$  fibration over  $S^4$ , the maximal Kan sub-fibration

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of  $Fuk_\infty(P)$ , which is just a combinatorial analogue of a Serre fibration, is non-trivial. In particular  $Fuk_\infty(P)$  is non-trivial as a (co)-Cartesian fibration and so  $f_P$  is homotopically non-trivial. This gives in particular:

**Theorem 1.1.** *The natural homomorphism as constructed in Part I,*

$$\mathbb{Z} = \pi_4(BHam(S^2), id) \xrightarrow{k} \pi_4(\mathbb{S}, NFuk(S^2)),$$

*is injective.*

None of the homotopy groups of  $\mathbb{S}$  are known, so this is in a sense an application of geometry to algebraic topology. Morally, such an application is possible because geometry forces a priori  $A_\infty$ -associativity of certain structures, which may then have formal consequences in algebraic topology.

The calculation is performed by constructing perturbation data in such a way that we are reduced to calculation of a certain higher product in a certain Fukaya type  $A_\infty$  category. Note that this is an actual chain level calculation. To perform this calculation we relate it to the computation of a certain quantum Maslov, which are certain higher dimensional analogues of the relative Seidel element in [9]. The calculation of this quantum Maslov class uses a regularization technique based on “virtual Morse theory” for the Hofer length functional [17].

It is likely that  $k$  is surjective. Surjectivity is in a sense the statement that up to equivalence there are no exotic (co)-Cartesian fibrations over  $S^4$ , with fiber equivalent to  $N(Fuk(S^2))$  - they all come from Hamiltonian  $S^2$  fibrations, via the global Fukaya category.

**1.1. Application in basic Riemannian geometry.** As one less expected application, we can use the computation of Theorem 1.1 to obtain lower bounds for curvature of certain types singular connections.

**Definition 1.2.** *Let  $G \hookrightarrow P \rightarrow X$  be a principal  $G$  bundle, where  $G$  is a Frechet Lie group. A **singular  $G$ -connection** on  $P$  is a closed subset  $C \subset X$ , and a smooth Ehresmann  $G$ -connection  $\mathcal{A}$  on  $P|_{X-C}$ .*

The above definition is basic, as one often puts additional conditions, see for instance [7], [22].

**1.1.1. A non-metric measure of curvature.** Let  $G$  as above be a Frechet Lie group, we denote by  $\text{lie } G$  its Lie algebra and let

$$\mathbf{n} : \text{lie } G \rightarrow \mathbb{R}$$

be an  $Ad$  invariant Finsler norm. For a principal  $G$ -bundle  $P$  over a Riemann surface  $(S, j)$ , and given a  $G$  connection  $\mathcal{A}$  on  $P$  define a 2-form  $\alpha_{\mathcal{A}}$  on  $S$  by:

$$\alpha_{\mathcal{A}}(v, jv) = \mathbf{n}(R_{\mathcal{A}}(v, jv)),$$

where  $R_{\mathcal{A}}$  is the curvature 2-form of  $\mathcal{A}$ , with

$$R_{\mathcal{A}}(v, w) \in \text{lie Aut } P_z,$$

for  $z \in S$ ,  $v, w \in T_z S$ ,  $P_z$  fiber over  $z$ , and  $\text{Aut } P_z \simeq G$  the group of  $G$ -torsor automorphisms of  $P_z$ , where  $\simeq$  means non-canonical group isomorphism.

And define

$$(1.1) \quad \text{area}_{\mathbf{n}}(\mathcal{A}) = \int_S \alpha_{\mathcal{A}}.$$

In the case  $\mathcal{A}$  is singular with singular set  $C$ ,  $\alpha_{\mathcal{A}}$  is defined on  $S - C$  so we define

$$\text{area}_{\mathfrak{n}}(\mathcal{A}) = \int_{S-C} \alpha_{\mathcal{A}},$$

with the right hand side now being an extended integral. This area is a non-metric measurement meaning that no Riemannian metric on  $S$  is needed.

It is possible to extend the functional above to a functional on the space  $\mathcal{C}$  of  $G$ -connections on principal  $G$  bundles  $P \rightarrow \Delta^n$ . It may seem that  $\Delta^n$  has no connection to Riemann surfaces but in fact there is an intriguing connection by way of existence of certain axiomatized systems of maps:

$$u : \bar{\mathcal{S}}_d^{\circ} \rightarrow \Delta^n,$$

where  $\bar{\mathcal{S}}_d$  denotes the universal curve over  $\bar{\mathcal{R}}_d$  - the moduli space of complex structures on the disk with  $d + 1$  punctures on the boundary, and  $\bar{\mathcal{S}}_d^{\circ}$  is  $\bar{\mathcal{S}}_d$  with nodal points of the fibers removed. Such a chosen system is referred to as  $\mathcal{U}$ . This was developed in Part I and plays a principal role there.

There is then a functional:

$$(1.2) \quad \text{area}_{\mathcal{U}} : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0},$$

defined with respect to a choice of  $\mathcal{U}$ . When  $n = 2$  it is just the area functional as previously defined.

**1.1.2. Abstract resolutions of singular connections.** Avoiding generality, suppose that  $\mathcal{A}$  is a singular  $G$ -connection on a principal  $G$ -bundle  $P \rightarrow S^n$ , with a single singularity  $x_0$ . We will show that it is possible to control the curvature of the singular connection  $\mathcal{A}$  if we impose a certain structure on the singularity of  $\mathcal{A}$ . The simplest way to do this is to ask for existence of a certain kind of abstract resolution. First a simplicial  $G$ -connection on  $\mathcal{D}$  on  $P$ , as defined in Section 13.1, is basically a functorial assignment of a smooth  $G$  connection  $\mathcal{D}_{\Sigma}$  on  $\Sigma^*P$  for each smooth

$$\Sigma : \Delta^n \rightarrow S^n.$$

**Definition 1.3.** For  $\mathcal{A}, P$  as above a **simplicial resolution** of  $\mathcal{A}$  is a simplicial  $G$  connection  $\mathcal{A}^{res}$  on  $P$ , with the following properties. Let  $\Sigma_0 : \Delta^n \rightarrow S^n$  represent  $\pi_n(S^n, x_0)$ , then

$$\Sigma_0^*|_{\text{interior } \Delta^n} \mathcal{A} = \mathcal{A}_{\Sigma_0}^{res}|_{\text{interior } \Delta^n}.$$

In the following theorem  $G = PU(2)$ ,  $n = 4$  and the norm  $\mathfrak{n}$  on  $\text{lie } PU(2)$  will be taken to be the operator norm normalized so that the Finsler length of the shortest one parameter subgroup from  $id$  to  $-id$  is  $\frac{1}{2}$ . We will omit  $\mathfrak{n}$  in notation. We also impose an additional constraint on  $\mathcal{A}^{res}$ , so that the curvature at “ $\infty$ ” is bounded by a threshold, which means the following. Let

$$\Sigma_{\infty} : \Delta^4 \rightarrow x_0$$

be the constant map, suppose that:

$$\text{area}_{\mathcal{U}} \mathcal{A}_{\Sigma_{\infty}}^{res} < 1/2,$$

and suppose for simplicity that  $\mathcal{A}_{\Sigma_{\infty}}^{res}$  is trivial along the edges of  $\Delta^4$ , later on this condition is generalized, see Proposition 13.4. We say in this case that  $\mathcal{A}^{res}$  is a **sub-quantum resolution**. The following is proved in Section 13.

**Theorem 1.4.** *Let  $P \rightarrow S^4$  be a non-trivial principal  $PU(2)$  bundle. Let  $\mathcal{A}$  be a singular  $PU(2)$ -connection on  $P$  with a single singularity at  $x_0$ . Then for any sub-quantum resolution  $\mathcal{A}^{res}$  of  $\mathcal{A}$  and for any  $\mathcal{U}$  as above*

$$\text{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_0}^{res}) \geq 1/2.$$

The theorem has certain extensions to Hamiltonian singular connections  $\mathcal{A}$ , understanding  $P$  as a principal  $Ham(S^2)$  bundle, Section 13.

Here is one basic class of examples.

**Example 1.5.** *Let  $P$  be as above, and  $\mathcal{A}'$  be an ordinary smooth  $PU(2)$  connection on  $P$ . Express  $S^4$  as a union of sub-balls  $D_{\pm}^4 \subset S^4$ , intersecting only in the boundary. Suppose that we have the property that  $\text{area}_{\mathcal{U}}(\mathcal{A}')|_{D_{-}^4} < \frac{1}{2}$ . Let  $\mathcal{A}$  be the singular connection on  $P$  obtained as the push-forward of  $\mathcal{A}'$  by the bundle map  $\tilde{q} : P \rightarrow P$  over the singular smooth map  $q : S^4 \rightarrow S^4$  taking  $D_{-}^4$  to a single point  $\infty \in S^4$ , with  $q|_{\text{interior } D_{-}^4}$  an immersion. Then  $\mathcal{A}$  has a sub-quantum resolution  $\mathcal{A}^{res}$  essentially by construction. In this case, the theorem above simply yields that  $\text{area}_{\mathcal{U}}(\mathcal{A}'_{D_{+}^4}) \geq 1/2$ .*

Let us summarize the above example as the following basic differential geometric result. It can be formally understood as a corollary of Theorem 1.4, as partially explained above, but it is more elementary to see it as a corollary of Theorem 13.4, which appears later.

**Corollary 1.6** (Of Theorem 1.4 and of Theorem 13.4). *Let  $P$  be a non-trivial  $PU(2)$  (or  $Ham(S^2)$ ) bundle  $P \rightarrow S^4$ , let  $D_{\pm}^4$  be as above and let  $\mathcal{A}$  be a smooth  $PU(2)$  or  $Ham(S^2)$  connection on  $P$ . Suppose that  $\text{area}_{\mathcal{U}}(\mathcal{A})|_{D_{-}^4} < \frac{1}{2}$ , then  $\text{area}_{\mathcal{U}}(\mathcal{A})|_{D_{+}^4} \geq \frac{1}{2}$ .*

The proof of the above traverses the entirety of the theory here, and it quite non-obvious how it might be simplified, even for  $PU(2)$ . In the case of  $PU(2)$ , one idea might be to replace Floer theory used here by the mathematical Yang-Mills theory over surfaces [2]. If we want to mimic the argument presented in this paper, then we should first extend Yang-Mills theory to work with  $G$ -bundles over surfaces with corners and holonomy constraints over boundary. This might be possible but beyond this things are unclear since we also use certain abstract algebraic topology to glue the data, and it is not clear how this would work for Yang-Mills theory.

We may use the same idea as in the example above to “push forward” simplicial, (not just smooth) connections to singular connections with more complicated singularities, in such a way that we again by construction would have sub-quantum resolutions. In this case Theorem 1.4 no longer has an elementary interpretation as in corollary above.

**Physical Interpretation.** There are possible physical interpretations for singular connections, as appearing in the context here. A  $PU(2) = PSU(2)$  connection  $\mathcal{A}$  on  $P$  in physical terms represents a Yang-Mills field on the space-time  $S^4$ . When the space-time has a black hole singularity, the fields solving the Einstein-Yang-Mills equations (mathematically connections as above) likewise develop singularities. There is a wealth of physics literature on this subject, and I don’t know what has the highest priority, but here is one reference [14]. Note that in the above example we, in a sense, collapse a region in the “space-time”  $S^4$ , also forming a kind of “black-hole”. As quantum gravity is often related to simplicial ideas, it is

not inconceivable that the mathematical sub-quantum resolution condition above also has a physical interpretation.

At this point the reader may be curious why Theorem 1.1 has something to do with Theorem 1.4. We cannot give the full story, but the idea is that the simplicial fibration  $Fuk_\infty(P)$  only sees the principal bundle  $P$  (and its curvature) by the behavior of certain holomorphic curves. When one has the sub-quantum condition on the curvature of  $\mathcal{A}_{\Sigma_\infty}^{res}$ , certain holomorphic curves are ruled out so that from the view point of  $Fuk_\infty(P)$ ,  $\mathcal{A}_{\Sigma_\infty}^{res}$  is the trivial connection, (its curvature is undetectable) but  $Fuk_\infty(P)$  is non-trivial as a fibration so that the aforementioned holomorphic curves and consequently curvature must appear elsewhere.

**1.2. First quantum obstruction and smooth invariants.** It is very tempting to use the theory of the global Fukaya category to find new invariants of smooth manifolds. One such invariant is already discussed in Part I, as the homotopy class of the classifying map  $X \rightarrow \mathbb{S}$  of the projectivized, complexified tangent bundle of a smooth manifold  $X$ . This by itself is not a very practical invariant, but we may try to extract more manageable invariants from this. We present here a construction of an integer invariant which is based on our theory. This is probably just the beginning of the story for invariants of smooth manifolds based on Floer-Fukaya theory.

Let  $P \rightarrow X$  be a Hamiltonian  $M$ -bundle, as previously. Let

$$Fuk_\infty(P) \rightarrow X$$

be the associated (co)-Cartesian fibration, and let

$$K(P) \rightarrow X$$

be its maximal Kan sub-fibration as in Lemma 3.2, which we may understand as just a usual Serre topological fibration. Define

$$\text{q-obs}(P) \in \mathbb{N} \sqcup \{\infty\},$$

to be the degree of the first obstruction to a section of  $K(P)$ . That is  $\text{q-obs}(P)$  is the smallest integer  $n$  such that there is no section of  $K(P)$  over the  $n$  skeleton of  $X$ . over the  $n - 1$  skeleton of  $X$  does not extend to a section over the  $n$ -skeleton. When no such  $n$  exists we set  $\text{q-obs}(P) = \infty$ .

**Theorem 1.7.** *Let  $S^2 \hookrightarrow P \rightarrow S^4$  be a non-trivial Hamiltonian fibration then:*

$$\text{q-obs}(P) = 4.$$

Indeed the proof of Theorem 1.1 can be understood as showing that the associated obstruction class in

$$H^4(S^4, \pi_3(NFuk(S^2)))$$

is non-trivial.

**1.2.1. First quantum obstruction as a manifold invariant.** Let  $X$  be a smooth manifold, and let  $P(X)$  denote the fiber-wise projectivization of  $TX \otimes \mathbb{C}$ . We then define

$$\text{q-obs}(X) := \text{q-obs}(P(X)) \in \mathbb{N} \sqcup \{\infty\},$$

which is then an invariant of the smooth manifold  $X$ . Either this invariant is expressible in terms of classical invariants, which would be fascinating since the construction is in terms pseudo-holomorphic curves or this invariant is new, that is not expressible in classical terms, which would also be interesting. There are

of course gauge theory based invariants of smooth (3,4)-folds, like Donaldson and Seiberg-Witten invariants. I do not see any connections of the above to these invariants at the moment, even in dimension 4. It should be noted that this “first quantum obstruction” invariant is only sensitive to the tangent bundle, whereas for example Donaldson invariants can see finer aspects of the smooth structure. In fact the “quantum Novikov conjecture” of Part I would immediately imply that the first quantum obstruction is only a topological invariant of  $X$ . So that the above mentioned conjecture is in a sense very pessimistic, as its negation means new smooth structure sensitive invariants of manifolds.

**1.3. Hamiltonian rigidity vs flexibility.** By way of the calculation we also obtain an application in Hofer geometry. It can be understood as a relative analogue of a result in [18]. Let  $Lag(M, L_0)$  denote the space of oriented Lagrangian submanifolds of a symplectic manifold  $(M, \omega)$ , Hamiltonian isotopic to  $L_0$ , we may also just write  $Lag(M)$ . Let  $\Omega_{L_0}Lag(M)$  denote the space of based smooth loops in  $Lag(M)$ , constant near end points, and let  $\Omega_{L_0}^{taut}Lag(M) \subset \Omega_{L_0}Lag(M)$  be the subspace of loops taut concordant to the constant loop at  $L_0$ . The notion of taut concordance is defined in more generality in Definition 6.5. In the case above, two loops

$$p_1, p_2 \in \Omega_{L_0}Lag(M)$$

are said to be *taut concordant* if the following holds. We have a Lagrangian subfibration

$$\mathcal{L} \subset Cyl \times M, \quad Cyl = S^1 \times [0, 1],$$

so that  $\mathcal{L}$  over the boundary circles corresponds, in the natural sense, to the pair  $p_1, p_2$ , so that there is a Hamiltonian connection  $\mathcal{A}$  on  $M \times [0, 1]$  preserving  $\mathcal{L}$ , and so that the coupling form  $\Omega_{\mathcal{A}}$  of  $\mathcal{A}$  vanishes on  $\mathcal{L}$ , see Section 6.1 for the definition of coupling forms.

Note that of course  $Lag(S^2)$  is homotopy equivalent to  $Lag^{eq}(S^2) \simeq S^2$  where  $Lag^{eq}(S^2)$  denotes the space of oriented equators in  $S^2$ . Moreover, there is an embedding

$$i : \Omega S^2 \hookrightarrow \Omega_{L_0}^{taut}Lag(S^2),$$

this is because two loops  $p_1, p_2 \in \Omega Lag^{eq}(S^2) \simeq \Omega S^2$  are taut concordant iff they are homotopic in  $Lag^{eq}(S^2)$ , see Lemma 10.4.

**Theorem 1.8.** *Let  $L_0 \subset S^2$  be the equator. And let*

$$f : S^2 \rightarrow \Omega_{L_0}^{taut}Lag(S^2),$$

*represent  $i_*g$ , for  $g$  the generator of*

$$\pi_2(\Omega(S^2)) \simeq \pi_3(S^2) \simeq \mathbb{Z},$$

*and  $i$  as above. Then we have identity for the systole with respect to  $L^+$ :*

$$\min_{f', [f']=[f]} \max_{s \in S^2} L^+(f'(s)) = 1/2 \cdot \text{area}(S^2, \omega),$$

*where  $L^+$  denotes the positive Hofer length functional, as defined in Section 10.1.1. The minimum is attained on a cycle of equators in  $S^2$ .*

Even though everything is now smooth, this is not obvious. For by contrast if we measure a related quantity of the “girth” (infimum of the diameter of a representative) of the generator  $[g]$  of  $\pi_2Lag(S^2)$ , as in [15], then there is an upper bound for this girth which is smaller than the lower bound for girth considered

in the subspace of  $Lag(S^2)$  consisting of equators. So moving from equators to general oriented  $S^1$  Lagrangians in  $S^2$  we may reduce the girth to less than the classically expected girth. By “classical” we mean for the classical objects: great circles. Indeed, it may be that girth of the generator

$$[g] \in \pi_2 Lag(S^2)$$

is actually 0. (This would rather astonishing however.) On the other hand, our theorem says that this kind of non-classical squeezing does not happen at all for the systole we consider. In other words whereas our systole exhibits Hamiltonian rigidity, the girth in [15] while closely related, exhibits flexibility.

Theorem 1.8 is proved in section 12. On the way in Section 9.1 we construct the quantum Maslov classes. We show their non-triviality in Section 10. The Sections 12, 9.1, 10 are mostly logically independent of the  $\infty$ -categorical and even the  $A_\infty$  setup and may be read independently. Theorems 1.1, 1.7 are proved in Section 4.2, they are basic consequences of the main technical lemma.

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### 3. OUTLINE

**Notation 3.1.** We use notation  $\Delta^n$  to denote the standard topological  $n$ -simplex. For the standard representable  $n$ -simplex as a simplicial set we use the notation  $\Delta^n_\bullet$ , in general when  $X$  is a topological space  $X_\bullet$  will mean the singular simplicial set of  $X$ . Likewise if  $p : X \rightarrow Y$  is map of spaces  $p_\bullet : X_\bullet \rightarrow Y_\bullet$  will mean the induced simplicial map. However the notation  $Y_\bullet$  could also mean an abstract simplicial set.

In what follows when we say Part I we will mean [19]. Let us briefly review what we do in Part I. Let  $M \hookrightarrow P \xrightarrow{p} X$  be a Hamiltonian fibration. Denote by  $\Delta(X) := \Delta/X_\bullet$  the smooth simplex category of  $X$ , with objects smooth maps  $\Sigma : \Delta^n \rightarrow X$  and morphisms commutative diagrams:

$$\begin{array}{ccc} \Delta^n & \xrightarrow{mor} & \Delta^m \\ & \searrow \Sigma_0 & \downarrow \Sigma_1 \\ & & X, \end{array}$$

where  $mor : \Delta^n \rightarrow \Delta^m$  is a simplicial map, that is a linear map taking vertices to vertices, preserving the order. The distinction with  $Simp(X)$  is that in  $\Delta(X)$  we allow  $mor$  to be degenerate.

Then given certain auxiliary perturbation data  $\mathcal{D}$ , which in particular involves a choice of a natural system of maps from the universal curves to  $\Delta^n$  and certain choices of Hamiltonian connections, we construct in Part I a functor

$$F : \Delta(X) \rightarrow A_\infty - Cat,$$

where  $A_\infty - Cat$  denotes the category of  $A_\infty$  categories. The properties of this functor are such that we may algebraically, naturally construct from this a functor

$$F^{unit} : \Delta(X) \rightarrow A_\infty - Cat^{unit},$$

with  $A_\infty - Cat^{unit}$  denoting the category of unital  $A_\infty$  categories, by taking unital replacements. In what follows we rename  $F$  by  $F^{raw}$  and  $F^{unit}$  by  $F$ , as this will visually simplify notation.

We then define

$$Fuk_\infty(P) = \operatorname{colim}_{\Delta(X)} NF,$$

which is shown to be an  $\infty$ -category whose equivalence class (under concordance) is independent of all choices. This also has the structure of a co-Cartesian fibration:

$$NFuk(M) \hookrightarrow Fuk_\infty(P) \rightarrow X_\bullet,$$



where  $NFuk(M)$  is the  $A_\infty$  nerve of the Fukaya category of  $M$  and  $X_\bullet$  denotes the simplicial set of singular simplices in  $X$ . The notion of co-Cartesian fibration corresponds to, in categorical language, a relaxation of the notion of Serre/Kan fibrations. Indeed what will do here is extract from the above data a Kan fibration and work with that, since then we can just use standard tools of topology. To this end we have the following elementary lemma.

**Lemma 3.2.** *Suppose we have a (co)-Cartesian fibration  $p : Y \rightarrow X$ , where  $X$  is a Kan complex. Let  $K(Y)$  denote the maximal Kan sub-complex of  $Y$  then  $p : K(Y) \rightarrow X$  is a Kan fibration.*

The proof is given in Appendix A. In particular by the above lemma  $K(P) := K(Fuk_\infty(P))$  is a Kan fibration over  $X_\bullet$ .

**Definition 3.3.** *We say that a Kan fibration or a (co)-Cartesian fibration  $P$  over a Kan complex  $X$  is **non-trivial** if it is not null-concordant. Here  $P$  is **null-concordant** means that there is a Kan respectively (co)-Cartesian fibration*

$$Y \rightarrow X \times \Delta_\bullet^1,$$

whose pull-back by  $i_0 : X \rightarrow X \times \Delta_\bullet^1$  is trivial and by  $i_1 : X \rightarrow X \times \Delta_\bullet^1$  is  $P$ . Here the two maps  $i_0, i_1$  correspond to the two vertex inclusions  $\Delta_\bullet^0 \rightarrow \Delta_\bullet^1$ .

**Theorem 3.4.** *Suppose that  $P \rightarrow S^4$  is a non-trivial Hamiltonian  $S^2$  fibration then  $p_\bullet : K(P) \rightarrow S_\bullet^4$  does not admit a section. In particular  $K(P)$  is a non-trivial Kan fibration over  $S_\bullet^4$  and so  $Fuk_\infty(P)$  is a non-trivial (co)-Cartesian fibration over  $S_\bullet^4$ .*

This is the main technical result of the paper. Although in a sense we just are just deducing existence of a certain holomorphic curve, for this deduction we need a global compatibility condition involving multiple moduli spaces, involved in multiple local datum's of Fukaya categories, so that this computation will not straightforward.

The proof will be aided by constructing suitable perturbation data, and will be split into a number of sections. As previously indicated the arguments are quiet general, however for concreteness and to simplify an already rather complex framework we focus on a special case.

#### 4. QUALITATIVE DESCRIPTION OF THE PERTURBATION DATA

A bit of possibly non-standard terminology: we say that  $A$  is a *model* for  $B$  in some category, with weak equivalences, if there is a morphism  $mod : A \rightarrow B$  which is a weak-equivalence. The map  $mod$  will be called a *modelling map*. In our context the modeling map  $mod$  always turns out to be a monomorphism.

Let  $Fuk(S^2)$  denote the  $\mathbb{Z}_2$ -graded  $A_\infty$  category over  $\mathbb{Q}$ , with objects oriented spin Lagrangian submanifolds Hamiltonian isotopic to the equator. Our particular construction of  $Fuk(M)$  is presented in Part I.

Denote by  $Fuk^{eq}(S^2) \subset Fuk(S^2)$  the full sub-category obtained by restricting our objects to be equators in  $S^2$ . We take our perturbation data  $\mathcal{D}_{pt}$  for construction of  $Fuk(S^2)$  so that the following is satisfied. All the connections  $\mathcal{A}(L, L')$  for  $L, L' \in Fuk^{eq}(S^2)$  are  $PU(2)$ -connections. For  $L$  intersecting  $L'$  transversally the  $PU(2)$  connection  $\mathcal{A}(L, L')$  is the trivial flat connection. For  $L = L'$  the corresponding connection is generated by an autonomous Hamiltonian.

The associated cohomological Donaldson-Fukaya category  $DF(S^2)$  is equivalent as a linear category over  $\mathbb{Q}$  to  $FH(L_0, L_0)$  (considered as a linear category with one object) for  $L_0 \in Fuk(S^2)$ .

It is easily verified that a morphism (1-edge)  $f$  is an isomorphism in the nerve  $NFuk(S^2)$ , see Part I for definitions, if and only if it corresponds, under the nerve construction  $N$ , to a morphism in  $Fuk(S^2)$  that induces an isomorphism in  $DF(S^2)$ . Such a morphism will be called a *c-isomorphism*.

Consequently the maximal Kan subcomplex  $K(S^2)$  of  $NFuk(S^2)$  is characterized as the maximal subcomplex of this nerve with 1-simplices the images by  $N$  of *c-isomorphisms* in  $Fuk(S^2)$ .

**Remark 4.1.** *It would be interesting (and likely elementary) to identify the homotopy type of  $K(S^2)$ .*

**4.1. Extending  $\mathcal{D}_{pt}$  to higher dimensional simplices.** Let us model  $D_\bullet^4$  and  $S_\bullet^3$  as follows. Take the standard representable 3-simplex  $\Delta_\bullet^3$ , and the standard representable 0-simplex  $\Delta_\bullet^0$ . Then collapse all faces of  $\Delta_\bullet^3$  to a point, that is take the colimit of the following diagram:

$$(4.1) \quad \begin{array}{ccccc} & & \Delta_\bullet^0 & & \\ & \nearrow & \uparrow & \nwarrow & \\ \Delta_\bullet^2 & & \Delta_\bullet^2 & & \Delta_\bullet^2 \\ & \searrow i_0 \quad \searrow i_1 \quad \downarrow i_2 \quad \swarrow i_3 & & & \\ & & \Delta_\bullet^3 & & \end{array}$$

Here  $i_j$  are the inclusion maps of the non-degenerate 2-faces. This gives a simplicial set  $S_\bullet^{3,mod}$  modelling the simplicial set  $S_\bullet^3$ , in other words there is a natural a weak-equivalence

$$S_\bullet^{3,mod} \rightarrow S_\bullet^3.$$

Now take the cone on  $S_\bullet^{3,mod}$ , denoted by  $C(S_\bullet^{3,mod})$ , and collapse the one non-degenerate 1-edge. The resulting simplicial set  $D_\bullet^{4,mod}$  is our model for  $D_\bullet^4$ , it may be identified with a subcomplex of  $D_\bullet^4$  so that the inclusion map  $mod : D_\bullet^{4,mod} \rightarrow D_\bullet^4$  induces a weak homotopy equivalence of pairs

$$(4.2) \quad (D_\bullet^{4,mod}, S_\bullet^{3,mod}) \rightarrow (D_\bullet^4, S_\bullet^3).$$

We set  $b_0 \in D_\bullet^4$  to be the vertex which is the image by  $mod$  of the unique 0-vertex in  $D_\bullet^{4,mod}$ .

Suppose we have a commutative diagram:

$$\begin{array}{ccccc} D^4 & \xrightarrow{h_+} & S^4 & \xleftarrow{h_-} & D^4 \\ & \nwarrow i & & \nearrow i & \\ & & S^3 & & \end{array}$$

where  $i : S^3 \rightarrow D^4$  is the natural boundary inclusion, s.t.  $h_\pm : D^4 \rightarrow S^4$  are smooth, with their images covering  $S^4$ , s.t.

$$h_+(D^4) \cap h_-(D^4)$$

is contained in the image  $E$  of

$$h_{\pm} \circ i : S^3 \rightarrow S^4,$$

and so that  $h_{\pm}$  takes  $b_0$  to  $x_0$ . For example we may just let  $h_-$  to represent the generator of  $\pi_4(S^4, x_0)$  and for  $h_+$  to be constant map to  $x_0$ . We call such a pair  $h_{\pm}$  a *complementary pair*.

We set

$$D_{\pm} := h_{\pm}(D_{\bullet}^{4,mod}) \subset S_{\bullet}^4$$

and we set  $\Sigma_{\pm} \in S_{\bullet}^4$  to be the image by  $h_{\pm}$  of the sole non-degenerate 4-simplex of  $D_{\bullet}^{4,mod}$ . We also set

$$\partial D_{\pm} := h_{\pm}(\partial D_{\bullet}^{4,mod}),$$

where  $\partial D_{\bullet}^{4,mod}$  is the image of the natural inclusion  $S_{\bullet}^{3,mod} \rightarrow D_{\bullet}^{4,mod}$ .

Fix a Hamiltonian frame for the fiber  $P_{x_0}$  of  $P$  over  $x_0$ , in other words a diffeomorphism  $S^2 \rightarrow P_{x_0}$  that is in the maximal atlas of trivializations of  $P$  as a  $Ham(S^2)$  structure group bundle. In particular this allows us to identify  $Fuk(S^2)$  with  $F^{raw}(x_0)$ . Denote by  $x_{0,\bullet}$  the image of the map

$$\Delta_{\bullet}^0 \rightarrow S_{\bullet}^{4,mod},$$

induced by the inclusion of the 0-simplex  $x_0$ . Recall from Part I that given perturbation data for a non-degenerate simplex, we assigned extended perturbation data for all degeneracies of this simplex. So our data  $\mathcal{D}_{pt}$  for  $x_0$  induces perturbation data for all degeneracies  $x_0$ , that is for all simplices of  $x_{0,\bullet}$ , this data will again be denoted by  $\mathcal{D}_{pt}$ .

Fix an object  $L_0 \in Fuk^{eq}(S^2) \subset F^{raw}(x_0)$ . Denote by  $\gamma \in \text{hom}_{F^{raw}(x_0)}(L_0, L_0)$  the generator of  $FH_1(L_0, L_0)$ , i.e. the fundamental chain, so that it corresponds to the identity in  $DF(L_0, L_0)$ . This  $\gamma$  is uniquely determined by our conditions and corresponds to a single geometric section. Denote by  $L_0^i$  the image of  $L_0$  by the embedding

$$F^{raw}(x_0) \rightarrow F^{raw}(\Sigma_+)$$

corresponding to the  $i$ 'th vertex inclusion into  $\Delta^4$ ,  $i = 0, \dots, 4$ .

Let  $m_i$  be the edge between  $i-1, i$  vertices and set

$$\overline{m}_i := \Sigma_+ \circ m_i.$$

Let  $\Sigma_i^0$  denote the 0-simplex obtained by restriction of  $\Sigma^4$  to the  $i$ 'th vertex. Note that each  $\overline{m}_i$  is degenerate by construction, so we have a induced morphisms

$$F^{raw}(pr) : F^{raw}(\overline{m}_i) \rightarrow F^{raw}(x_0)$$

corresponding ( $F^{raw}$  is a functor) to the degeneracy morphism in  $\Delta(S^4)$ :

$$pr : \overline{m}_i \rightarrow \Sigma_i^0.$$

Finally, for each  $L_0^{i-1}, L_0^i$  we have a  $c$ -isomorphism

$$\gamma_i : L_0^{i-1} \rightarrow L_0^i$$

in  $F^{raw}(\overline{m}_i) \subset F^{raw}(\Sigma_+)$  which corresponds to  $\gamma$ , meaning that the fully-faithful projection  $F^{raw}(pr)$  takes  $\gamma_i$  to  $\gamma$ . We will denote by  $\gamma_{i,j}$  the analogous morphisms  $L_0^i \rightarrow L_0^j$ .

**Notation 4.2.** Let us abbreviate from now the morphism spaces  $\text{hom}_{F^{raw}(\Sigma_{\pm})}$  by  $\text{hom}_{\Sigma_{\pm}}$ , and  $\mu_{F^{raw}(\Sigma_{\pm})}^d$  by  $\mu_{\Sigma_{\pm}}^d$ .

**Definition 4.3.** We call perturbation data  $\mathcal{D}$  for  $P$  **small** if it extends the data  $\mathcal{D}_{pt}$  as above, and if with respect to  $\mathcal{D}$

$$(4.3) \quad \mu_{\Sigma_+}^d(\gamma^1, \dots, \gamma^d) = 0, \text{ for } 2 < d < 4,$$

where  $(\gamma^1, \dots, \gamma^d)$  is a composable chain, and each  $\gamma^i$  is of the form  $\gamma_{i,j}$  as above.

We will see further on how to construct such small data, assume for now that it is constructed.

Let  $\{f_J\}$ , corresponding to an  $n$ -simplex, be as in the definition of the  $A_\infty$  nerve in Part I, where  $J$  is a subset of  $[n] = \{0, \dots, n\}$ .

**Lemma 4.4.** Let  $\mathcal{D}$  be small as above, then there is a 4-simplex  $\sigma \in NF^{raw}(\Sigma_+)$  with faces determined by the conditions:

- $f_J = 0$ , for  $J$  any subset of  $[4]$  with at least 3 elements.
- $f_{\{i-1, i\}} = \gamma_i$  for  $\gamma_i$  as before.

*Proof.* This follows by (4.3) and by the identity  $\mu_{\Sigma_+}^2(\gamma, \gamma) = \gamma$ .  $\square$

If we take our unital replacements so that  $\gamma$  corresponds to the unit then  $\sigma$  induces a section of  $K(P_+) \rightarrow D_+$ , where  $K(P_\pm)$  will be shorthand for  $K(P)$  restricted over  $D_\pm$ .

Let

$$i : (K(P_+)|_{\partial D_+} := p_\bullet^{-1}(\partial D_+)) \rightarrow K(P_-),$$

be the natural inclusion map. Set

$$sec = i \circ \sigma \circ h_+|_{\partial D_\bullet^{4, mod}}.$$

#### 4.2. The main lemma and immediate consequences.

**Lemma 4.5.** Suppose that  $P$  is a non-trivial Hamiltonian fibration and  $\mathcal{D}$  small data for  $P$  as above, then  $sec$  as above does not extend to a section of  $K(P_-)$ .

The proof of this lemma, which will be broken up in parts, will follow shortly.

*Proof of Theorem 1.7.* Clearly  $\text{q-obs}(P) \geq 4$ , since the 3-skeleton of  $S^4$  is trivial. By Lemma 4.5 above  $K(P)$  does not have a section over the 4-cell representing the generator of  $\pi_4(S^4)$ .  $\square$

**Remark 4.6.** When  $P$  is obtained by clutching with a generator of  $\pi_3(PU(2))$ , and when  $h_\pm$  are embeddings, the class  $[sec]$  in  $\pi_3(K(P_-)) \simeq \pi_3(K(S^2))$  can be thought of as “quantum” analogue of the class of the classical Hopf map.

*Proof of Theorem 3.4.* If we take any small perturbation data  $\mathcal{D}$  for  $P$ , then the first part follows immediately by Lemma 4.5. So  $K(P)$  is non-trivial as a Kan fibration. This then implies that  $Fuk_\infty(P)$  is non-trivial as a (co)-Cartesian fibration, which means specifically that its classifying map

$$f_P : S^4 \rightarrow (\mathcal{S}, NFuk(S^2))$$

is not null-homotopic.

To see this, suppose otherwise that we have a null-homotopy  $H$  of  $f_P$ , then this gives a (co)-Cartesian fibration

$$\tilde{P} \rightarrow S_\bullet^4 \times I_\bullet,$$

restricting to  $Fuk_\infty(P)$  over  $S_\bullet^4 \times 0_\bullet$  and to  $NFuk(S^2) \times S_\bullet^4$  over the other end  $S_\bullet^4 \times 1_\bullet$ . Here  $0_\bullet, 1_\bullet$  are notation for simplicial set isomorphic  $\Delta^0$  corresponding to the boundary of  $I_\bullet$ . If we take the maximal Kan sub-fibration of  $\tilde{P}$ , then by Lemma 3.2 we would obtain a trivialization of  $K(P)$  which is a contradiction.  $\square$

*Proof of Theorem 1.1.* Theorem 3.4 implies that the group homomorphism

$$\mathbb{Z} \simeq \pi_4(BHam(S^2, id) \rightarrow \pi_4(\mathcal{S}, NFuk(S^2))),$$

has vanishing kernel, so that the result follows.  $\square$

## 5. TOWARDS THE PROOF OF LEMMA 4.5

We will denote by  $L_{0,\bullet}$  the image of the map  $\Delta_\bullet^0 \rightarrow K(P_-)$ , induced by the inclusion of  $L_0$  into  $K(S^2)$  as a 0-simplex. Suppose that  $sec$  extends to a section of  $K(P_-)$ , so we have map

$$e : D_\bullet^{4,mod} \rightarrow K(P_-)$$

extending  $sec$  over  $\partial D_\bullet^{4,mod}$ . We may assume that  $e$  lies over  $h_-$ , meaning

$$p_\bullet \circ e = h_-.$$

Since it can be homotoped to have this property. To see this, first take a relative homotopy of

$$p_\bullet \circ e : (D_\bullet^{4,mod}, \partial D_\bullet^{4,mod}) \rightarrow (D_-, \partial D_-)$$

to  $h_-$ , using that we have a homotopy equivalence of pairs (4.2), and then lift the homotopy to a relative homotopy upstairs using the defining lifting property of Kan fibrations.

And so we have a 4-simplex

$$T = e(\Sigma^4) \in K(P_-)$$

projecting to  $\Sigma_- \in D_-$  by  $p_\bullet$ . Since  $T$  is in the image of  $e$ , all but one 3-faces of  $T$  are totally degenerate with image in  $L_{0,\bullet}$ , and with this exceptional 3-face being the sole non-degenerate 3-face of  $sec$ , (of  $sec(\partial D_\bullet^{4,mod})$ ).

Then if  $m_{i,j}, \gamma_{i,j}$  are as in the previous section but corresponding now to  $\Sigma_-$  rather than  $\Sigma_+$ , by the boundary condition on  $e$ , the edges of  $T$  correspond under the nerve construction to the  $\gamma_{i,j}$ , since this is the corresponding condition for the edges of  $sec$ .

**Lemma 5.1.** *For  $\mathcal{D}$  small as above, and for the unital replacement  $F$  of  $F^{raw}$  as above, the simplex  $T$  exists if and only if  $\mu_{\Sigma_-}^4(\gamma_1, \dots, \gamma_4)$  is exact.*

*Proof.* The following argument will be over  $\mathbb{F}_2$  as opposed to  $\mathbb{Q}$  as the signs will not matter. Recall that we take the unital replacement so that  $\gamma \in hom_{F^{raw}(P_{x_0})}(L_0, L_0)$  corresponds to the unit in the unital replacement.

Now if  $T \in K(P_-)$  as above exists, then it corresponds under unital replacement to a 4-simplex  $T' \in NF^{raw}(\Sigma_-)$  satisfying the following, by the nerve construction, condition on its 4-face  $f_{[4]} \in hom_{\Sigma_-}(L_0^0, L_0^4)$ :

$$(5.1) \quad \mu_{\Sigma_-}^1 f_{[4]} = \sum_{1 < i < 4} f_{[4]-i} + \sum_s \sum_{(J_1, \dots, J_s) \in decomp_s} \mu_{\Sigma_-}^s(f_{J_1}, \dots, f_{J_s}).$$

By our conditions on the boundary of  $T$ , by the condition on the unital replacement, and by the conditions in Lemma 4.4, we must have  $f_J = 0$ , for every proper subset

$J \subset [4]$ , in some length  $s$  decomposition of  $[4]$ , unless  $J = \{i, j\}$  in which case  $f_{i,j} = \gamma_{i,j}$ . Given this (5.1) holds if and only if  $\mu_{\Sigma-}^4(\gamma_1, \dots, \gamma_4)$  is exact.  $\square$

We are going to show that for a certain small  $\mathcal{D}_0$ ,  $\mu_{\Sigma-}^4(\gamma_1, \dots, \gamma_4)$  does not vanish in homology, which will finish the argument. However the calculation will require significant setup.

## 6. HAMILTONIAN FIBRATIONS AND TAUT STRUCTURES, HOLOMORPHIC SECTIONS AND AREA BOUNDS

We collect here some preliminaries on moduli spaces of holomorphic sections of fibrations with Lagrangian boundary constraints, and the closely related curvature bounds. There is a likely new discussion involving taut Hamiltonian structures, but much of this material has previously appeared elsewhere, perhaps in less generality. We will eventually need all that is presented in this section, but the reader may only skim on the first reading.

**6.1. Coupling forms.** We refer the reader to [11, Chapter 6] for more details on what follows. A Hamiltonian fibration is a smooth fiber bundle

$$M \hookrightarrow P \rightarrow X,$$

with structure group  $\text{Ham}(M, \omega)$  with its  $C^\infty$  Frechet topology. A **Hamiltonian connection** is just an Ehresmann connection for a Hamiltonian fibration.

A *coupling form*, originally appearing in [6], for a Hamiltonian fibration  $M \hookrightarrow P \xrightarrow{p} X$ , is a closed 2-form  $\Omega$  on  $P$  whose restriction to fibers coincides with  $\omega$  and which has the property:

$$\int_M \Omega^{n+1} = 0 \in \Omega^2(X),$$

with integration being integration over the fiber operation. Such a 2-form determines a Hamiltonian connection  $\mathcal{A}_\Omega$ , by declaring horizontal spaces to be  $\Omega$ -orthogonal spaces to the vertical tangent spaces. A coupling form generating a given connection  $\mathcal{A}$  is unique. A Hamiltonian connection  $\mathcal{A}$  in turn determines a coupling form  $\Omega_\mathcal{A}$  as follows. First we ask that  $\Omega_\mathcal{A}$  induces the connection  $\mathcal{A}$  as above. This determines  $\Omega_\mathcal{A}$  up to values on  $\mathcal{A}$ -horizontal lifts  $\tilde{v}, \tilde{w} \in T_p P$  of  $v, w \in T_x X$ . We specify these values by the formula

$$(6.1) \quad \Omega_\mathcal{A}(\tilde{v}, \tilde{w}) = R_\mathcal{A}(v, w)(p),$$

where  $R_\mathcal{A}$  is the Lie algebra valued curvature 2-form of  $\mathcal{A}$ . Specifically, for each  $x$ ,  $R_\mathcal{A}|_x$  is a 2-form valued in  $C_{\text{norm}}^\infty(p^{-1}(x))$  - the space of 0-mean normalized smooth functions on  $p^{-1}(x)$ .

**6.2. Hamiltonian structures on fibrations.** Let  $S$  be a Riemann surface with boundary, with punctures in the boundary, and a fixed structure of strip charts at ends, positive or negative, as in Part I. Let  $M \hookrightarrow \tilde{S} \xrightarrow{pr} S$  be a Hamiltonian fiber bundle, with model fiber monotone symplectic manifold  $(M, \omega)$ , with distinguished Hamiltonian bundle trivializations

$$[0, 1] \times (0, \infty) \times M \rightarrow \tilde{S}$$

at the positive ends, and with distinguished Hamiltonian bundle trivializations

$$[0, 1] \times (-\infty, 0) \times M \rightarrow \tilde{S},$$

at the negative ends. These are collectively called **strip charts**. Given the structure of such bundle trivializations we say that  $\tilde{S}$  has **end structure**.

**Definition 6.1.** *Let*

$$\mathcal{L} \subset (\tilde{S}|_{\partial S} = pr^{-1}(\partial S)) \rightarrow \partial S$$

*be a Lagrangian sub-bundle, with model fiber an object, in the sense of Part I, (in particular a spin oriented Lagrangian submanifold) so that  $\mathcal{L}$  is a constant sub-bundle in the strip chart trivializations above. We say in this case that  $\mathcal{L}$  **respects the end structure**. In the strip chart coordinates at the end  $e_i$ , let  $L_i^j$  be the fibers of  $\mathcal{L}$  over*

$$\{j\} \times \{t\}, j = 0, 1, \text{ by assumption } t \text{ independent.}$$

For  $\mathcal{L}$  as above, we say that a Hamiltonian connection  $\mathcal{A}$  on  $\tilde{S}$  is **compatible** with the connections  $\{\mathcal{A}_i\}$  on  $[0, 1] \times M$  for each end  $e_i$ , if in the strip coordinate chart at the  $e_i$  end,  $\mathcal{A}$  is flat and  $\mathbb{R}$ -translation invariant and has the form  $\bar{\mathcal{A}}_i$  where  $\bar{\mathcal{A}}_i$  denotes its  $\mathbb{R}$ -translation invariant extension  $\mathcal{A}_i$  to  $(0, \pm\infty) \times \mathbb{R} \times M$ , depending on whether the end is positive or negative. We say that a Hamiltonian connection  $\mathcal{A}$  on  $\tilde{S}$  as above, is  **$\mathcal{L}$ -exact** if  $\mathcal{A}$  preserves  $\mathcal{L}$  (this means that the horizontal spaces of  $\mathcal{A}$  are tangent to  $\mathcal{L}$ ).

For  $\mathcal{A}$  compatible with  $\{\mathcal{A}_i\}$  as above, a family  $\{j_z\}$  of fiber wise  $\omega$ -compatible almost complex structures on  $\tilde{S}$  will be said to **respect the end structure** if at each end  $e_i$ , in the strip coordinate chart above, the family  $\{j_z\}$  is  $\mathbb{R}$ -translation invariant and is admissible with respect to  $\mathcal{A}_i$ , in the sense of Part I, Definition 5.3. The data  $\Theta = (\tilde{S}, S, \mathcal{L}, \mathcal{A}, \{j_z\})$ , with  $\mathcal{A}$  compatible with  $\{\mathcal{A}_i\}$ ,  $\{j_z\}$ , respecting the end structure, will be called **Hamiltonian structure**.

We will normally suppress  $\{j_z\}$  in the notation and elsewhere for simplicity, as it will be purely in the background in what follows, (we do not need to manipulate it explicitly).

**Definition 6.2.** *Let  $(\tilde{S}, S, \mathcal{L}, \mathcal{A})$  be a Hamiltonian structure, we say that a smooth section  $\sigma$  of  $\tilde{S} \rightarrow S$  is **asymptotically flat** if at each end  $e_i$  of  $S$ ,  $\sigma$   $C^1$ -converges to an  $\mathcal{A}$ -flat section. Specifically, in the strip coordinates for a positive end, this means that there is a  $\mathcal{A}$ -flat section*

$$\tilde{\sigma} : [0, 1] \times (0, \infty) \rightarrow [0, 1] \times (0, \infty) \times M,$$

*so that for every  $\epsilon > 0$  there is a  $t > 0$  so that*

$$d_{C^1}(\tilde{\sigma}, \sigma|_{[0, 1] \times [t, \infty)}) < \epsilon.$$

*Similarly for a negative end. Given a pair of asymptotically flat sections  $\sigma_1, \sigma_2$ , with boundary in  $\mathcal{L}$ , we say that they have the same **relative class** if they are asymptotic to the same flat sections at each end, and are homologous, relative boundary condition and relative the asymptotic constraints at the ends. The set of relative classes will be denoted by  $H_2^{sec}(\tilde{S}, \mathcal{L})$ .*

### 6.2.1. Families of Hamiltonian structures.

**Definition 6.3.** *A family Hamiltonian structure or henceforth just Hamiltonian structure, consists of the following:*

- (1) *A smooth, connected, compact, oriented manifold  $\mathcal{K}$  with boundary, (or corners).*

- (2) For each  $r \in \mathcal{K}$  a Hamiltonian structure  $(\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$ , so that there are smooth fibrations

$$\tilde{S} \hookrightarrow \tilde{\mathbf{S}} \xrightarrow{p_1} \mathcal{K}, S \hookrightarrow \mathbf{S} \xrightarrow{p} \mathcal{K},$$

and  $\{\tilde{S}_r\}, \{S_r\}$  correspond to the fibers of the first and second fibration, respectively, where the second fibration has fiber a Riemann surface, so that  $\{S_r\} = \{p^{-1}r\}$ . The first fibration is a fibration whose fibers  $p_1^{-1}(r)$  are themselves the total spaces of a smooth Hamiltonian fibration  $\tilde{S}_r \rightarrow S_r$ , so that the structure group of  $\tilde{\mathbf{S}} \xrightarrow{p_1} \mathcal{K}$  can be reduced to smooth Hamiltonian bundle maps of the fiber. To elaborate further, let

$$M \hookrightarrow \tilde{S} \rightarrow S$$

be a Hamiltonian  $M$ -fibration over a Riemann surface  $S$ . Let  $\text{Aut}$  denote the group of Hamiltonian  $M$ -bundle automorphisms of  $\tilde{S}$ . Then  $\tilde{\mathbf{S}} \xrightarrow{p_1} \mathcal{K}$  is the associated bundle  $P \times_{\text{Aut}} \tilde{S}$  for some principal  $\text{Aut}$  bundle  $P$  over  $\mathcal{K}$ .

- (3) The charts

$$e_{i,r} : [0, 1] \times (0, \infty) \times M \rightarrow \tilde{S}_r,$$

for the positive ends, fit into a Hamiltonian  $M$ -bundle diffeomorphism onto the image:

$$(6.2) \quad \tilde{e}_i : [0, 1] \times (0, \infty) \times \mathcal{K} \times M \rightarrow \tilde{\mathbf{S}},$$

similarly for the negative ends.

- (4) We then have an induced smooth  $r$ -family of connections  $\{e_{i,r}^* \mathcal{A}_r\}$  on  $[0, 1] \times (0, \pm\infty) \times M$ , and an induced smooth  $r$ -family of Lagrangian subfibrations  $\{e_{i,r}^{-1} \mathcal{L}_r\}$  over  $\partial[0, 1] \times (0, \pm\infty)$ . We ask that

$$\forall r : \{e_{i,r}^{-1} \mathcal{L}_r\} = \{0\} \times (0, \pm\infty) \times L_i^0 \cup \{1\} \times (0, \pm\infty) \times L_i^1,$$

where  $L_i^j$  are as in Definition 6.1 and we ask that

$$\forall r : \{e_{i,r}^* \mathcal{A}_r\} = \overline{\mathcal{A}}_i$$

for  $\mathcal{A}_i, \overline{\mathcal{A}}_i$  as previously.

- (5) There is a Hamiltonian connection  $\mathcal{A}$  on  $\tilde{\mathbf{S}} \rightarrow \mathbf{S}$  that extends all the connections  $\mathcal{A}_r$  (in the natural sense), and preserving  $\mathbf{L} := \cup_r \mathcal{L}_r$ .

We will write  $\{\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$  for this data,  $\mathcal{K}$  may be omitted from notation when it is implicit.

In the notation above, if there exists a Hamiltonian connection  $\mathcal{A}$  on  $\tilde{\mathbf{S}} \rightarrow \mathbf{S}$  as in the last point, so that  $\Omega_{\mathcal{A}}$  vanishes on  $\mathbf{L}$ , we will say that  $\{\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$  is a **hyper taut Hamiltonian structure**.

6.2.2. *Moduli spaces of sections of Hamiltonian structures.* Let  $\Theta = (\tilde{S}, S, \mathcal{L}, \mathcal{A})$  be a Hamiltonian structure. For a section  $\sigma$  of  $\tilde{S}$  define its vertical  $L^2$  energy by

$$e(\sigma) := \int_S |\pi_{\text{vert}} \circ d\sigma|^2,$$

$$\pi_{\text{vert}} : T\tilde{S} \rightarrow T^{\text{vert}}\tilde{S}$$

is the  $\mathcal{A}$ -projection, for  $T^{\text{vert}}\tilde{S}$  the vertical tangent bundle of  $\tilde{S}$ , that is the kernel of the projection  $T\tilde{S} \rightarrow TS$ . Define  $\overline{\mathcal{M}}(\Theta)$  to be the Gromov-Floer compactification of the space of  $J(\mathcal{A})$ -holomorphic sections  $\sigma$  of  $\tilde{S}$ , with finite vertical  $L^2$  energy, (also



called Floer energy), and with boundary on  $\mathcal{L}$ . Note that for any  $J_{\mathcal{A}}$ -holomorphic  $\sigma$  we have an identity:

$$e(\sigma) = \int_S \sigma^* \Omega_{\mathcal{A}},$$

and  $\Omega_{\mathcal{A}}$  vanishes on  $\mathcal{L}$  by the condition that  $\mathcal{A}$  preserves  $\mathcal{L}$ , so that the standard energy controls apply, to deduce the standard Gromov-Floer compactification structure.

More generally, if  $\{\Theta_r\} = \{\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$  is a Hamiltonian structure, let

$$\overline{\mathcal{M}}(\{\Theta_r\})$$

be the Gromov-Floer compactification of the space of pairs  $(\sigma, r)$ ,  $r \in \mathcal{K}$  with  $\sigma$  a  $J(\mathcal{A}_r)$ -holomorphic, finite vertical  $L^2$  energy section of  $\tilde{\mathcal{S}}_r$ , with boundary on  $\mathcal{L}_r$ .

We also denote by

$$\overline{\mathcal{M}}(\{\Theta_r\}, A) \subset \overline{\mathcal{M}}(\{\Theta_r\})$$

the subset corresponding to relative class  $A$  curves, where the latter is as defined above.

Let  $\{\Theta_r = (\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)\}$  be a Hamiltonian structure, then for each end  $e_i$  of  $S_r$  we have a Floer chain complex

$$CF(L_i^0, L_i^1, \mathcal{A}_i, \{j_z\}),$$

(independent of  $r$  by part 4 of Definition 6.3) generated by  $\mathcal{A}_i$ -flat sections of  $[0, 1] \times M$ , with boundary on  $L_i^0, L_i^1$ . This is defined as in Section 6.1 of Part I. We may also abbreviate  $CF(L_i^0, L_i^1, \mathcal{A}_i, \{j_z\})$  by just  $CF(L_i^0, L_i^1)$ .

**Definition 6.4.** We say that  $\{\Theta_r\}$  is **A-regular** if the pairs  $(\mathcal{A}_i, \{j_z\})$  are regular so that the Floer chain complexes  $CF(\mathcal{A}_i, \{j_z\})$  are defined, and if  $\mathcal{M}(\{\Theta_r\}, A)$  is regular, (transversely cut out). And  $\{\Theta_r\}$  is **regular** if it is A-regular for all  $A$ . We say that it is **A-admissible** if there are no elements

$$(\sigma, r) \in \overline{\mathcal{M}}(\{\Theta_r\}, A),$$

for  $r$  in a neighborhood of the boundary of  $\mathcal{K}$ .

**Definition 6.5.** Given a pair  $\{\Theta_r^i\} = \{\tilde{S}_r^0, S_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}_{\mathcal{K}}$ ,  $i = 1, 2$ , of Hamiltonian structures we say that they are **concordant** if there is a Hamiltonian structure

$$\{\mathcal{T}_r\} = \{\tilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times [0, 1]},$$

with an oriented diffeomorphism (in the natural sense, preserving all structure)

$$\{\tilde{S}_r^0, S_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}_{\mathcal{K}^{op}} \sqcup \{\tilde{S}_r^1, S_r^1, \mathcal{L}_r^1, \mathcal{A}_r^1\}_{\mathcal{K}} \rightarrow \{\tilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times \partial I},$$

where  $op$  denotes the opposite orientation for  $\mathcal{K}$ .

**Definition 6.6.** We say that a Hamiltonian structure  $\{\Theta_r\} = \{\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$  is **taut** if for any pair  $r_1, r_2 \in \mathcal{K}$ ,  $\Theta_{r_1}$  is concordant to  $\Theta_{r_2}$  by a concordance  $\{\tilde{T}_\tau, T_\tau, \mathcal{L}'_\tau, \mathcal{A}'_\tau\}_{[0, 1]}$  which is a hyper taut Hamiltonian structure.

**Definition 6.7.** Given an A-admissible pair  $\{\Theta_r^i\}$ ,  $i = 1, 2$ , of Hamiltonian structures, we say that they are **A-admissibly concordant** if there is an A-admissible Hamiltonian structure

$$\{\tilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times [0, 1]},$$

which furnishes a concordance. If this concordance is in addition a taut Hamiltonian structure, then we say that these pairs are **A-admissibly taut concordant**.

**Lemma 6.8.** *Let  $\{\Theta_r\} = \{\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}$  be  $A$ -regular and  $A$ -admissible, with  $S_r$  having one distinguished negative end  $e_0$ , and let  $\gamma_0$  be the asymptotic constraint at  $e_0$  of the class  $A$ . For  $L_0^j$ ,  $j = 0, 1$  as above. Define*

$$ev_A = ev(\{\Theta_r\}, A) := \#\mathcal{M}(\{\Theta_r\}, A)\gamma_0 \in CF(L_0^0, L_0^1),$$

where  $\#\mathcal{M}(\{\Theta_r\}, A, \gamma_0)$  means signed count of elements when the dimension is 0, and is otherwise set to be zero. Suppose that  $CF(L_0^0, L_0^1)$  is defined with respect to the connection  $\mathcal{A}_0$  and that this chain complex is perfect. Then  $ev_A$  is a cycle since  $CF(L_0^0, L_0^1)$  is perfect, and its homology class depends only on the  $A$ -admissible concordance class of  $\{\Theta_r\}$ .

*Proof.* Suppose we are given an  $A$ -admissible concordance (which we may assume to be regular)

$$\mathcal{T} = \{\tilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times [0,1]}$$

between Hamiltonian structures  $\{\Theta_r^0\}$ , and  $\{\Theta_r^1\}$ . Then we get a one dimensional compact moduli space  $\mathcal{M}(\{\tilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}, A)$ . By assumption on the perfection of  $CF(L_0^0, L_0^1)$ , boundary contributions from Floer degenerations cancel out, so that the boundary is:

$$\partial\mathcal{M}(\{\tilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}, A) = \mathcal{M}(\{\Theta_r\}^{op} \sqcup \mathcal{M}(\{\Theta_r\},$$

where  $op$  denotes opposite orientation. From which the result follows.  $\square$

### 6.3. Area of fibrations.

**Definition 6.9.** *For a Hamiltonian connection  $\mathcal{A}$  on a bundle  $M \hookrightarrow \tilde{S} \rightarrow S$ , with  $S$  a Riemann surface, define*

$$(6.3) \quad \text{area}(\mathcal{A}) = \inf_{\alpha} \left\{ \int_S \alpha | \Omega_{\mathcal{A}} + \pi^*(\alpha) \text{ is nearly symplectic} \right\},$$

where  $\Omega_{\mathcal{A}}$  is the coupling form of  $\mathcal{A}$ ,  $\alpha$  is a 2-form on  $S$ , and where  $\Omega_{\mathcal{A}} + \pi^*(\alpha)$  nearly symplectic means that

$$(\Omega_{\mathcal{A}} + \pi^*(\alpha))(\tilde{v}, \tilde{v}) \geq 0,$$

for  $\tilde{v}, \tilde{v}$  horizontal lifts with respect to  $\Omega_{\mathcal{A}}$ , of  $v, v \in T_z S$ , for all  $z \in S$ .

Note that  $\text{area}(\mathcal{A})$  could be infinite if there are no constraints on  $\mathcal{A}$  at the ends. However, when the infimum above is finite it is attained on the 2-form determined by:

$$(6.4) \quad \alpha_{\mathcal{A}}(v, v) := |R_{\mathcal{A}}(v, v)|_+,$$

where  $v \in T_z S$ ,  $R_{\mathcal{A}}(v, w)$  as before identified with zero mean smooth function on the fiber  $\tilde{S}_z$  over  $z$  and  $|\cdot|_+$  is operator:  $|H|_+ = \max_{\tilde{S}_z} H$ .

**Lemma 6.10.** *Let  $(\tilde{S}, S, \mathcal{L}, \mathcal{A})$  be Hamiltonian data. For  $\sigma \in \overline{\mathcal{M}}(\tilde{S}, S, \mathcal{L}, \mathcal{A})$  we have*

$$-\int_S \sigma^* \Omega_{\mathcal{A}} \leq \text{area}(\mathcal{A}).$$

*Proof.* We have

$$\int_S \sigma^*(\Omega_{\mathcal{A}} + \pi^* \alpha) \geq 0,$$

whenever  $\Omega_{\mathcal{A}} + \pi^*(\alpha)$  is nearly symplectic, by the defining properties of  $J_{\mathcal{A}}$  and by  $\sigma$  being  $J_{\mathcal{A}}$ -holomorphic. From which our conclusion follows.  $\square$

**Lemma 6.11.** *Let  $\{(\tilde{S}_t, S_t, \mathcal{L}_t, \mathcal{A}_t)\}_{[0,1]}$  be a taut concordance. Let  $\sigma_j$ ,  $j = 0, 1$  be asymptotically flat sections of  $\tilde{S}_j$  in relative class  $A$ . Then*

$$-\int_{S_1} \sigma_1^* \Omega_{\mathcal{A}_1} = -\int_{S_0} \sigma_0^* \Omega_{\mathcal{A}_0},$$

*whenever both integrals are finite. In particular, for a Hamiltonian structure  $(\tilde{S}, S, \mathcal{L}, \mathcal{A})$   $\int_S \sigma^* \Omega_{\mathcal{A}}$  depends only on the relative class of  $A$ , whenever the integral is finite.*

*Proof.* By the hypothesis, there is a connection  $\mathcal{A}$  on  $\tilde{\mathbf{S}}$ , extending each  $\mathcal{A}_t$  and so that  $\Omega_{\mathcal{A}}$  vanishes on  $\mathbf{L} \subset \tilde{\mathbf{S}}$ . The first part then follows by Stokes theorem. Here are the details. For  $\sigma_j$  as above and for each end  $e_i$ , cut off the part of the section  $\sigma_j$  lying over  $[0, 1] \times (t_{\delta_1, \delta_2}, \infty)$  in the corresponding strip chart at the end. Here  $t_{\delta_1, \delta_2}$  is such that  $\sigma_0|_{[0,1] \times \{t\}}$  is  $C^1$   $\delta_1$ -close to  $\sigma_1|_{[0,1] \times \{t\}}$  for all  $t > t_{\delta_1, \delta_2}$  and for each end, and is such that

$$\int_{[0,1] \times (t_{\delta_1, \delta_2}, \infty)} \sigma_j^*|_{[0,1] \times (t_{\delta_1, \delta_2}, \infty)} \Omega_{\mathcal{A}_j} < \delta_2, \quad j = 1, 2,$$

for each end  $e_i$ . Call the sections with the ends cut off as above by  $\sigma_j^{\delta_1, \delta_2}$ , they are sections over the compact surfaces  $S_j^{cut}$ , with ends correspondingly cut off. Then by Stokes theorem, using that  $\Omega_{\mathcal{A}}$  is closed and using the vanishing of  $\Omega_{\mathcal{A}}$  on  $\mathbf{L}$ : for each  $\epsilon$  there exists  $\delta_1, \delta_2$  such that

$$\int_{S_1^{cut}} (\sigma_1^{\delta_1, \delta_2})^* \Omega_{\mathcal{A}} - \int_{S_0^{cut}} (\sigma_0^{\delta_1, \delta_2})^* \Omega_{\mathcal{A}} < \epsilon,$$

and

$$\int_{S_j^{cut}} (\sigma_j^{\delta_1, \delta_2})^* \Omega_{\mathcal{A}_j} - \int_{S_j} \sigma_j^* \Omega_{\mathcal{A}_j} < \epsilon, \quad j = 1, 2.$$

The last part of the lemma follows, as  $\mathcal{A}$  preserving  $\mathcal{L}$  immediately implies that  $\Omega_{\mathcal{A}}$  vanishes on  $\mathcal{L}$ , so that a constant concordance

$$\{(\tilde{S}_t, S_t, \mathcal{L}_t, \mathcal{A}_t)\}_{[0,1]}$$

is taut. □

**Definition 6.12.** *For  $\sigma$  a relative class  $A$  section of  $\Theta = (\tilde{S}, S, \mathcal{L}, \mathcal{A})$  let us call:*

$$-\int_S \sigma^* \Omega_{\mathcal{A}},$$

*the  $A$ -coupling area of  $\sigma$ , denoted by  $\text{care}(\Theta, \sigma)$ , we may also write  $\text{care}(\Theta, A)$  for the same quantity. By the lemma above this is an invariant of the taut concordance class of  $\Theta$ .*

**Definition 6.13.** *Given Hamiltonian structure  $\Theta = (\tilde{S}, S, \mathcal{L}, \mathcal{A})$  we will say that  $\Theta$  is  $A$ -small if*

$$\text{area}(\Theta) < \text{care}(\Theta, A).$$

*Similarly, given a taut Hamiltonian structure  $\{\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$  we say that it is  $A$ -small near boundary if  $(\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$  is  $A$ -small for  $r$  in a neighborhood of the  $\partial\mathcal{K}$ .*

**Lemma 6.14.** *Suppose that  $\Theta = (\tilde{S}, S, \mathcal{L}, \mathcal{A})$  is  $A$ -small then  $\overline{\mathcal{M}}(\Theta, A)$  is empty. Or as a contrapositive, if  $\overline{\mathcal{M}}(\Theta, A)$  is non-empty then:*

$$\text{care}(\Theta, A) \leq \text{area}(\Theta).$$

*Proof.* This is just a reformulation of Lemma 6.10.  $\square$

**Lemma 6.15.** *Let  $\{\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$  be a taut Hamiltonian structure with  $\mathcal{K}$  connected, so that in particular, for each  $r$ ,  $\Theta_r = (\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$  is taut concordant to a fixed  $\Theta$ . Suppose that  $\{\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$  is  $A$ -small near boundary then  $\{\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$  is  $A$ -admissible for all  $A$  such that  $\text{care}(\Theta, A) > 0$ .*

*Proof.* Follows immediately by the lemma above.  $\square$

**6.4. Gluing Hamiltonian structures.** A Hamiltonian connection  $\mathcal{A}$  on  $[0, 1] \times M$  is determined by a choice of a function  $H : [0, 1] \times M \rightarrow \mathbb{R}$ , normalized to have mean zero at each moment. The holonomy path of  $\mathcal{A}$  is a path  $\phi_{\mathcal{A}} : [0, 1] \rightarrow \text{Ham}(M, \omega)$ , generated by the Hamiltonian  $H$ . Given  $L_0 \in \text{Lag}(M)$  we get a path  $\tilde{\phi}_{\mathcal{A}} : [0, 1] \rightarrow \text{Lag}(M)$  starting at  $L_0$ , defined by  $\tilde{\phi}_{\mathcal{A}}(t) = \phi_{\mathcal{A}}(t)(L_0)$ . We will say that these paths are **generated by  $\mathcal{A}$  or by  $H$** , with the latter as above.

Let  $(\tilde{S}, S, \mathcal{L}, \mathcal{A})$  be a Hamiltonian structure. Where at each end  $e_i$  the corresponding connection  $\mathcal{A}_i$  is determined by  $L^{\pm}$  length  $\kappa_i$ , where  $L^{\pm}$  is as in Section 10.1.1.

Let  $\mathcal{D}$  denote Riemann the surface which is topologically  $D^2 - z_0$ ,  $z_0 \in \partial D^2$ , endowed with a choice of a strip chart at the end (positive or negative depending on context). The complex structure  $j$  here is as induced from  $\mathbb{C}$  under the assumed embedding  $D^2 \subset \mathbb{C}$ .

We may then cap off some of the open ends  $\{e_i\}_{i=0}^n$  of  $S$  by gluing at the ends copies of  $\mathcal{D}$  with oppositely signed end. More explicitly, in the strip coordinate charts at some, say positive, end  $e_i$  of  $S$ , excise  $[0, 1] \times (t, \infty)$  for some  $t > 0$ , call the resulting surface  $S - e_i$ . Likewise excise the negative end of  $\mathcal{D}$ , call this surface  $\mathcal{D} - \text{end}$ . Then glue  $S - e_i$  with  $\mathcal{D} - \text{end}$ , along their new smooth boundary components. Let us denote the capped off surface by  $S'$ . Since  $\tilde{S}$  is naturally trivialized at the ends, we may similarly cap off  $\tilde{S}_r$  over the  $e_i$  end by gluing with a bundle  $\mathcal{D} \times M$  at the end obtaining a Hamiltonian  $M$  bundle  $\tilde{S}'$  over  $S'$ .

More generally we have a certain gluing operation of Hamiltonian structures. In the case of “capping off” as above we glue  $\Theta = (\tilde{S}, S, \mathcal{L}, \mathcal{A})$  with the Hamiltonian structure  $\Theta' = (\mathcal{D} \times M, \mathcal{D}, \mathcal{L}', \mathcal{A}')$  at the  $e_i$  end, provided  $\mathcal{A}'$  is compatible with the connection  $\mathcal{A}_i$ , in the sense of Section 6.2, and provided  $\mathcal{L}$  is compatible with  $\mathcal{L}'$ . The latter means that  $L_i^j = L_i'^j$  where these are Lagrangians corresponding to the strip chart trivialization of  $\mathcal{L}, \mathcal{L}'$  at the corresponding ends, as in Definition 6.1.

Let us name the result of this capping off  $\Theta \#_i \Theta'$ . The following is immediate:

**Lemma 6.16.** *Suppose that  $\{\Theta_r\}_{\mathcal{K}}, \{\Theta'_r\}_{\mathcal{K}}$  with  $\Theta'_r = (\mathcal{D} \times M, \mathcal{D}, \mathcal{L}'_r, \mathcal{A}'_r)$  are taut Hamiltonian structures. Then:*

$$\{\Theta_r \#_i \Theta'_r\}_{\mathcal{K}}$$

*is taut, whenever the gluing operation is well defined, that is whenever we have compatibility of connections and Lagrangian sub-fibrations at the corresponding end.*

**Definition 6.17.** *Let  $\pi : \mathbb{R} \rightarrow [0, 1]$  denote the retraction map, sending  $(-\infty, 0]$  to 0, and sending  $[1, \infty)$  to 1. Fix a parametrization  $\zeta : \mathbb{R} \rightarrow \partial \mathcal{D}$ , which satisfies*

$\zeta(t) \in \{0\} \times (0, \pm\infty)$  for  $t \in (-\infty, 0]$ , and  $\zeta(t) \in \{1\} \times (0, \pm\infty)$  for  $t \in \sqcup[1, \infty)$ , where we are using the coordinates of the strip chart  $e_0 : [0, 1] \times (0, \pm\infty) \rightarrow \mathcal{D}$ . Given a smooth path

$$p : [0, 1] \rightarrow \text{Lag}(M)$$

constant near 0, 1, let  $\mathcal{L}_p \subset \partial\mathcal{D} \times M$  denote the Lagrangian subfibration over  $\partial\mathcal{D}$ , with fiber over  $r \in \partial\mathcal{D}$  given by  $p \circ \pi(r)$ . We say that a Lagrangian subfibration  $\mathcal{L}$  as above is **determined by**  $p$  if  $\mathcal{L} = \mathcal{L}_p$ , after a fixed choice of parametrization of boundary of  $\mathcal{D}$  by  $\mathbb{R}$ .

**Lemma 6.18.** *Let  $p$  and  $\mathcal{L}_p \subset \partial\mathcal{D} \times M$  be as in Definition above with  $L^\pm(\tilde{p}) = \rho$ , where  $\tilde{p}$  is some lift of  $p$  to  $\text{Ham}(M, \omega)$ , that is  $p(t) = \tilde{p}(t)(p(0))$ . Let  $\mathcal{A}_0$  be a Hamiltonian connection on  $[0, 1] \times M$ , generated by a Hamiltonian  $H$  with  $L^\pm$  length  $\kappa$ , constant near the end points. Then there is a Hamiltonian connection  $\tilde{\mathcal{A}}_0^p$  on  $\mathcal{D} \times M$ , preserving  $\mathcal{L}_p$ , and compatible with respect to  $\mathcal{A}_0$ , with  $\text{area}(\tilde{\mathcal{A}}_0) \leq \kappa + \rho$ . The construction is natural in the sense that  $(\tilde{p}, \mathcal{A}_0) \mapsto \tilde{\mathcal{A}}_0$  can be made into a smooth map (of Frechet spaces).*

*Proof.* Let  $q : [0, 1] \rightarrow \text{Ham}(M, \omega)$  be the holonomy path of  $\mathcal{A}_0$ ,  $q(0) = \text{id}$ , generated by  $H$ . Let  $\tilde{p} \cdot q$  be the concatenation in diagrammatic order, and  $H'$  be its generating Hamiltonian.

Define a coupling form  $\Omega'$  on  $D^2 \times M$ :

$$\Omega' = \omega - d(\eta(\text{rad}) \cdot H' d\theta),$$

for  $(\text{rad}, \theta)$  the modified angular coordinates on  $D^2$ ,  $\theta \in [0, 1]$ ,  $0 \leq \text{rad} \leq 1$ , and  $\eta : [0, 1] \rightarrow [0, 1]$  is a smooth function satisfying

$$0 \leq \eta'(\text{rad}),$$

and

$$(6.5) \quad \eta(\text{rad}) = \begin{cases} 1 & \text{if } 1 - \delta \leq \text{rad} \leq 1, \\ \text{rad}^2 & \text{if } \text{rad} \leq 1 - 2\delta, \end{cases}$$

for a small  $\delta > 0$ . By elementary calculation

$$\text{area}(\mathcal{A}') = L^+(p \cdot q) = L^+(p) + L^+(q),$$

where  $\mathcal{A}'$  is the connection induced by  $\Omega'$ . Set  $\text{arc} = \{(1, \theta)\}$ ,  $0 \leq \theta \leq 1/2$ . Let  $\text{arc}^c$  denote the complement of  $\text{arc}$  in  $\partial D^2$ . Fix a smooth embedding  $i : D^2 \hookrightarrow \mathcal{D}$  so that the image of the embedding contains  $\partial\mathcal{D} - \text{end}$  where  $\text{end}$  is the image of the distinguished (say positive) end strip chart

$$[0, 1] \times (0, \infty) \rightarrow \mathcal{D},$$

so that  $i(\text{arc}) \subset \text{end}^c$ , and so that  $i(\text{arc}^c) \subset \text{end}$  as illustrated in the Figure 1.

Next fix a deformation retraction  $\text{ret}$  of  $\mathcal{D}$  onto  $i(D^2)$ , so that in the strip chart above  $\text{ret}$ , for  $r \geq 1$ , is the composition  $i \circ \text{param} \circ \text{pr}$ , for

$$\text{pr} : [0, 1] \times (0, \infty) \rightarrow [0, 1]$$

the projection and for

$$\text{param} : [0, 1] \rightarrow \text{arc}^c \subset D^2$$

a diffeomorphism. Finally set  $\Omega = \text{ret}^* \Omega'$  on  $\mathcal{D} \times S^2$ , and set  $\tilde{\mathcal{A}}_0^p$  to be the induced Hamiltonian connection. As constructed  $\tilde{\mathcal{A}}_0$  will be compatible with  $\mathcal{A}_0$ , when

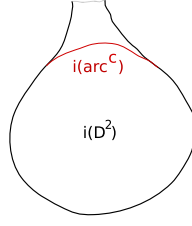


FIGURE 1.

the end of  $\mathcal{D}$  is positive. When the end is negative we take the reverse paths  $p^{-1}, q^{-1}$ .  $\square$

Using the above lemma, we may then put a  $\mathcal{L}'$ -exact Hamiltonian connection,  $\mathcal{A}'$  on  $\tilde{S}'$  (see Definition 6.2), with

$$(6.6) \quad \text{area}(\mathcal{A}') \leq \text{area}(\mathcal{A}) + \kappa_i.$$

This connection is obtained by gluing with the Hamiltonian structure  $(\mathcal{D} \times M, \tilde{\mathcal{A}}_0^{p=\text{const}})$  as obtained in the Lemma 6.18 above.

**Lemma 6.19.** *Let  $L_0 \subset M$  be a monotone Lagrangian submanifold with monotonicity constant  $\text{const} > 0$ :  $\omega(A) = \text{const} \cdot \mu(A)$ ,  $\mu$  the Maslov number. Let*

$$\Theta := \{\Theta_r\} := \{\tilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$$

*be a Hamiltonian structure with  $\mathcal{K}$  connected. And let  $\Theta_r' = (\tilde{S}_r', S_r', \mathcal{L}_r', \mathcal{A}_r')$  be obtained from  $\Theta_r$  by capping off each end  $e_i$ , so that (6.6) is satisfied. Suppose that the Floer chain complex  $CF(L_i^0, L_i^1, \mathcal{A}_i)$  is perfect for each  $i$  and that  $\mathcal{A}_i$  is generated by a time dependent Hamiltonian  $H_i$  with  $L^\pm$  length  $\kappa_i$ .*

*For a given  $A \in H_2^{\text{sec}}(\tilde{S}, \mathcal{L})$ , if*

$$\forall r : \text{area}(\mathcal{A}_r) < \text{carea}(\Theta_r', A') - \sum_i \kappa_i,$$

*where  $A'$  is the capping off of  $A$  as described in the proof, then  $\overline{\mathcal{M}}(\{\Theta_r\}, A)$  is empty. Moreover, if  $\mathcal{L}_r$  is the trivial bundle with fiber  $L_0$  for each  $r$  and  $\mathcal{A}_r$  is likewise trivial over the boundary of  $S_r$  then*

$$\forall r : \text{carea}(\Theta_r', A') = -\text{const} \cdot \text{Maslov}^{\text{vert}}(A'),$$

*where  $\text{Maslov}^{\text{vert}}$  is as in Appendix B.*

*Proof.* Suppose otherwise that we have an element  $(\sigma_0, r_0) \in \overline{\mathcal{M}}(\{\Theta_r\}, A)$ . Suppose for the moment that  $\overline{\mathcal{M}}(\{\Theta_r\}, A)$  is regular.

There is a PSS isomorphism

$$QH(L) \rightarrow FH(L, L),$$

where the right hand side is defined using our construction in terms of flat sections, and the left hand side is interpreted for example as the Pearl complex complex homology, [3]. We won't give the full construction of this map, as it is just reformulation of well known constructions, as presented in [3] for instance. But we can quickly say how to get  $PSS([L])$ . Let

$$\Theta_- = (\mathcal{D} \times M, \mathcal{D}, \mathcal{L}, \mathcal{A}_-),$$

be the Hamiltonian structure with  $e_0$  being a negative end,  $\mathcal{L}$  trivial with fiber  $L$  (which is an object as before), and

$$\mathcal{A}_- := \widetilde{\mathcal{A}}_0^{p=const},$$

with right hand side as in Lemma 6.18, for  $p$  being the constant path. Suppose that  $\Theta_-$  is regular. Define  $PSS([L])$  as the homology class of the chain  $c$  determined by:

$$\langle c, \gamma \rangle = \sum_A \#ev(\Theta_-, A),$$

where the sum is over all classes  $A \in H_2^{sec}(\mathcal{D} \times M, \mathcal{L})$ , which have asymptotic constraint  $\gamma$ , for  $\gamma$  a geometric generator of  $CF(L, L, \mathcal{A}_0)$ . As the PSS map is an isomorphism in our monotone context, as is well known, and since  $CF(L_i^0, L_i^1, \mathcal{A}_i)$  is perfect for each  $i$ , the asymptotic constraint  $\gamma_i$  of  $\sigma_0$  at each  $e_i$  end must be hit by the PSS map, (note that at the negative ends  $\gamma_i$  are interpreted in the dual complex  $CF(L_i^0, L_i^1, \mathcal{A}_i^r)$ , where  $\mathcal{A}_i^r$  is the reverse of  $\mathcal{A}_i$ ).

By gluing with the holomorphic sections of  $M \times \mathcal{D}$  coming from the construction of the PSS maps for each end  $e_i$ , we obtain a  $J(\mathcal{A}'_{r_0})$ -holomorphic, class  $A'$  section  $\sigma'_0$  of  $\Theta'_{r_0}$ .

By Lemma 6.14:

$$carea(\Theta'_{r_0}, A') \leq \text{area}(\mathcal{A}'_{r_0}) \leq \text{area}(\mathcal{A}_{r_0}) + \sum_i \kappa_i,$$

so

$$carea(\Theta'_{r_0}, A') - \sum_i \kappa_i \leq \text{area}(\mathcal{A}_{r_0}),$$

so that we contradict the hypothesis and in the case  $\overline{\mathcal{M}}(\{\Theta_r\}, A)$  is regular we are done with the first part of the lemma. When it is not regular instead of gluing just pre-glue to get a holomorphic building  $\sigma'_0$ , and the conclusion follows by the same argument.

To prove the last part of the lemma, note that in this case each  $\Theta'_r$  is taut concordant to

$$\Theta_0 = (D^2 \times M, D^2, \mathcal{L}, \mathcal{A}^{tr}),$$

with  $\mathcal{L}$  trivial with fiber  $L_0$ , and for  $\mathcal{A}^{tr}$  the trivial connection. And

$$carea(\Theta_0, \cdot) = -const \cdot Maslov^{vert}(\cdot)$$

as functionals on  $H_2^{sec}(D^2 \times M, \mathcal{L})$  with  $const > 0$ . It follows by Lemma 6.11 that

$$carea(\Theta', \sigma'_0) = carea(\Theta_0, \sigma'_0) = -const \cdot Maslov^{vert}(\sigma'_0) = -const \cdot Maslov(A').$$

□

## 7. CONSTRUCTION OF SMALL DATA

To forewarn, we use here notation and notions from Part I, especially from Sections 4, 5 in Part I. Let  $m_i$  denote the edges of  $\Delta^4$  as before.

Let  $\{m^k\}_{k=1}^{k=d}$  be a composable sequence in  $\Pi(\Delta^n)$ , which we recall means that the target of  $m_{i-1}$  is the source of  $m_i$  for each  $i$ . Recall from Part I that the perturbation data  $\mathcal{D}$  in particular specifies for each  $n$  and for each such composable sequence certain maps

$$u(\{m^k\}, n) : \mathcal{E}_d^\circ \rightarrow \Delta^n,$$

where  $\mathcal{E}_d$  is the universal curve over  $\overline{\mathcal{R}}_d$ , and  $\mathcal{E}_d^\circ$  denotes  $\mathcal{E}_d$  with nodal points of the fibers removed. Collective such a chosen collection of maps satisfying certain axioms is denoted  $\mathcal{U}$ . And we have already mentioned this in the introduction. The restriction of  $u(\{m^k\}, n)$  to the fiber  $\mathcal{S}_r$  of  $\mathcal{E}_r^\circ$  over  $r$ , is denoted by  $u(\{m^k\}, n, r)$ , which may also be abbreviated by  $u_r$ .

$\mathcal{D}$  also specifies for each  $\Sigma : \Delta^n \rightarrow X$ , each  $r$ , each composable chain  $(m_1, \dots, m_d)$  in  $\Pi(\Delta^n)$ , and for each chain of objects  $L'_0, \dots, L'_d$  with  $L'_i \subset P|_{\overline{m}_i(1)}$ ,  $i \geq 1$ ,  $L'_0 \subset P|_{\overline{m}_1(0)}$ , a Hamiltonian connection on

$$(7.1) \quad \tilde{\mathcal{S}}_r := (\Sigma \circ u(\{m^k\}, n, r))^* P \rightarrow \mathcal{S}_r.$$

Here as before  $\overline{m}_i = \Sigma \circ m_i$ . When  $n = 4$ ,  $\Sigma = \Sigma_+$ , and  $\forall i : L'_i = L_0$  with the latter as before we write these connections as  $\mathcal{A}_r^+(\{m^k\})$ , further abbreviated by  $\mathcal{A}_r^+$  as  $\{m^k\}$  will be implicit in what follows.

Suppose that  $\mathcal{D}$  extends  $\mathcal{D}_{pt}$  from before. If  $\mathcal{L}_r \subset \tilde{\mathcal{S}}_r|_{\partial \mathcal{S}_r}$  denotes the trivial Lagrangian sub-bundle with fiber  $L_0$ , then we obtain Hamiltonian structure (for each composable  $d$ -chain  $\{m^k\}$ )  $\Theta^+ = \{\Theta_r^+\} = \{\tilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r^+\}_{\overline{\mathcal{R}}_d}$ , which were discussed in the previous section. By the properties of these connections, necessitated by  $\mathcal{D}$ , at each end  $e_i$  of  $\mathcal{S}_r$ ,  $\mathcal{A}_r^+$  is compatible with the connection  $\mathcal{A}_i = \mathcal{A}(L_0, L_0)$ , where  $\mathcal{A}(L_0, L_0)$  is the connection on  $[0, 1] \times S^2$  part of our Floer data  $\mathcal{D}$ . Then  $\Theta^+$  is trivially taut since for each  $r$   $\mathcal{L}_r$  is naturally trivial and  $\mathcal{A}_r$  is likewise trivial over  $\partial \mathcal{S}_r$ , for each  $r$ , by assumed properties of these connections.

Set

$$\hbar := \frac{1}{2} \text{area}(S^2, \omega).$$

Let  $\kappa$  denote the  $L^\pm$  length of the holonomy path in  $\text{Ham}(S^2)$  of  $\mathcal{A}_0 = \mathcal{A}(L_0, L_0)$ . We may suppose that

$$(7.2) \quad \forall r : \text{area}(\mathcal{A}_r^+) < \hbar - 5\kappa,$$

is satisfied after taking  $\kappa$  to be sufficiently small. So that in particular  $\mathcal{A}_r^+$  is small for each  $r$ , with small as defined in the previous section.

Fix a complex structure  $j_0$  on  $M$ , and let  $\{J_r\}$  be the family of complex structures on  $\{\tilde{\mathcal{S}}_r\}$  induced by  $(\{\mathcal{A}_r^+\}, j_0)$ .

**Lemma 7.1.** *As in Part I, let*

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}(\gamma^1, \dots, \gamma^d; \gamma^0, \Sigma_+, \{J_r\}, A),$$

*denote the set of elements of  $\overline{\mathcal{M}}(\Theta^+, A)$  with asymptotic constraints  $\gamma^i$  at each  $e_i$  end. Here each  $\gamma^k$ ,  $k \neq 0$ , is of the form  $\gamma_{i,j}$  where this is as before. Then whenever the class  $A$  is such that  $\mathcal{M}$  has virtual dimension 0, and  $d$  satisfies  $2 < d \leq 4$ ,  $\overline{\mathcal{M}}$  is empty.*

*Proof.* Let

$$\Theta^/ := (\Theta^+)^/,$$

and  $A^/$  be as in Section 6.4. For a fixed  $r$ , by the Riemann-Roch (Appendix B) we get that the expected dimension of  $\mathcal{M}(\Theta^/, A^/)$  is

$$1 + \text{Maslov}^{vert}(A^/).$$

Consequently, when  $\gamma^0 = \gamma$ , the expected dimension of  $\mathcal{M}$  is:

$$(7.3) \quad 1 + \text{Maslov}^{vert}(A^/) - 1 + (\dim \mathcal{R}_d = d - 2).$$



We need the expected dimension of  $\mathcal{M}$  to be 0, and  $d \geq 3$ , so  $Maslov^{vert}(A') \leq -1$ . But  $Maslov^{vert}(A') = -1$  is impossible as the minimal positive Maslov number is 2. Consequently, the result follows by Lemma 6.19 and by the property (7.2).

When  $\gamma^0$  is the Poincare dual to  $\gamma$ , we would get  $Maslov^{vert}(A') \leq -2$  so for the same reason the conclusion follows.  $\square$

So if we choose our data  $\mathcal{D}$  so that the hypothesis of the lemma above are satisfied, then with respect to  $\mathcal{D}$ :

$$(7.4) \quad \mu_{\Sigma_{\pm}^4}^2(\gamma_{i,j}, \gamma_{j,k}) = \gamma_{i,k}$$

$$(7.5) \quad \mu_{\Sigma_{\pm}^4}^3(\gamma^1, \dots, \gamma^3) = 0, \text{ for } \gamma^i \text{ as above}$$

$$(7.6) \quad \mu_{\Sigma_{+}^4}^4(\gamma_1, \dots, \gamma_4) = 0.$$

In particular this  $\mathcal{D}$  is small.

#### 8. THE PRODUCT $\mu_{\Sigma_{-}^4}^4(\gamma_1, \dots, \gamma_4)$ AND THE QUANTUM MASLOV CLASSES

The product

$$\mu_{\Sigma_{-}^4}^4(\gamma_1, \dots, \gamma_4)$$

a priori depends on various choices, like the choices of  $h_{\pm}$ , and then choice of data  $\mathcal{D}_0$ . However by Lemma 6.8, so long as there is a homotopy of the choices, together with a homotopy of associated perturbation data  $\{\mathcal{D}_t\}$ , so that  $\mathcal{D}_t$  is small for all  $t$ , the above product is  $t$ -invariant. Thus, in particular for the purpose of computation we may take  $h_{+}$  to be the constant map to  $x_0$  and

$$h_{-} : (D^4, \partial D^4) \rightarrow (S^4, x_0)$$

the complementary map, that is representing the generator of  $\pi_4(S^4, x_0) \simeq \mathbb{Z}$ , we further suppose that  $h_{-}$  is an embedding in the interior of  $D^4$ .

Let  $\Sigma_{-}$  be the corresponding 4-simplex of  $S_{\bullet}^4$  as before. We need to study the moduli spaces

$$(8.1) \quad \overline{\mathcal{M}}(\gamma_1, \dots, \gamma_4; \gamma^0, \Sigma_{-}, \{\mathcal{A}_r\}, A),$$

where  $\mathcal{A}_r$  now denotes the connections on

$$(8.2) \quad \tilde{\mathcal{S}}_r := (\Sigma_{-} \circ u(m_1, \dots, m_4, 4, r))^* P \rightarrow \mathcal{S}_r,$$

part of some small data  $\mathcal{D}_0$  as above. We abbreviate  $u(m_1, \dots, m_4, 4, r)$  by  $u_r$  in what follows.

By the dimension formula (7.3), since we need the expected dimension of (8.1) to be zero, the class  $A'$  satisfies:

$$Maslov^{vert}(A') = -2,$$

and we must have  $\gamma^0 = \gamma_{0,4}$ , in other words the latter morphism corresponds to the fundamental chain.

**Notation 8.1.** From now on, by slight abuse,  $A_0$  refers to various section classes of various Hamiltonian structures such that the associated class  $A_0'$  satisfies:

$$Maslov^{vert}(A_0') = -2.$$

**8.1. Constructing suitable  $\{\mathcal{A}_r\}$ .** To get a handle on (8.1) we want to construct very special small data  $\mathcal{D}_0$ .

A Hamiltonian  $S^2$  fibration over  $S^4$  is classified by an element

$$[g] \in \pi_3(\text{Ham}(S^2), id) \simeq \pi_3(PU(2), id) \simeq \mathbb{Z}.$$

Such an element determines a fibration  $P_g$  over  $S^4$  via the clutching construction:

$$P_g = D_-^4 \times S^2 \sqcup D_+^4 \times S^2 \sim,$$

with  $D_-^4$ ,  $D_+^4$  being 2 different names for the standard closed 4-ball  $D^4$ , and the equivalence relation  $\sim$  is  $(d, x) \sim \tilde{g}(d, x)$ ,

$$\tilde{g}: \partial D_-^4 \times S^2 \rightarrow \partial D_+^4 \times S^2, \quad \tilde{g}(d, x) = (d, g(d)^{-1}(x)).$$

We suppose that  $x_0 = h_{\pm}(b_0)$  that previously appeared, is a point common to  $D_{\pm}^4 \subset S^4$ .

From now on  $P_g$  will denote such a fibration for a non-trivial class  $[g]$ . Note that the fiber of  $P_g$  over the base point  $x_0 \in S^3 \subset D_{\pm}^4$  (chosen for definition of the homotopy group  $\pi_3(\text{Ham}(S^2), id)$ ) has a distinguished, by the construction, identification with  $S^2$ . Take  $\mathcal{A}$  to be a connection on  $P \simeq P_g$  which is trivial in the distinguished trivialization over  $D_+^4$ . This gives connections

$$\mathcal{A}'_r := (\tilde{u}_r)^* \mathcal{A}$$

on  $\tilde{\mathcal{S}}_r$ ,

$$\tilde{u}_r = \Sigma_- \circ u_r.$$

By the last axiom for the system  $\mathcal{U}$  introduced in Part I, we may choose  $\{u_r\}$  so that the family  $\{\tilde{u}_r(\mathcal{S}_r)\}$  induces a singular foliation of  $S^4$ , that is smooth outside  $x_0$ , with  $x_0$  being the image by  $\tilde{u}_r$  of the ends (images of  $e_i$ ) and the boundary of each  $\mathcal{S}_r$ , and so that each  $\tilde{u}_r$  is an embedding on the complement of  $\tilde{u}_r^{-1}(x_0)$ . Denote by  $E$  the subset  $S^3 \subset S^4$  bounding  $D_{\pm}^4$ . We may in addition suppose that each  $\tilde{u}_r$  intersects  $E$  transversally, again on the complement of  $\tilde{u}_r^{-1}(x_0)$ .

By the above, the preimage by  $\tilde{u}_r$  of  $E$  contains a smoothly embedded curve  $c_r$  as in Figure 2, and  $\tilde{u}_r$  takes  $c_r$  into  $E$ . This  $c_r$  not uniquely determined but we may fix a family  $r \mapsto c_r$ , with parametrizations

$$c_r : \mathbb{R} \rightarrow \mathcal{S}_r$$

so that  $c_r$  maps  $(-\infty, 0)$  diffeomorphically onto the image by  $e_0$  of  $\{0\} \times (-\infty, 0)$ , so that likewise  $c_r$  maps  $(1, \infty)$  diffeomorphically onto the image by  $e_0$  of  $\{1\} \times (-\infty, 0)$ , and so that  $\{c_r\}$  is a  $C^0$  continuous family. We set:

$$\tilde{c}_r := \tilde{u}_r \circ c_r.$$

In Figure 2, the regions  $R_{\pm}$  are the preimages by  $\tilde{u}_r$  of  $D_{\pm}^4 \subset S^4$ , and  $c_r$  bounds  $R_-$ . It follows that  $\{\tilde{c}_r\}$  likewise induces a singular foliation of the equator  $E \simeq S^3$  that is smooth outside  $x_0$ .

So each  $\mathcal{A}'_r$  is flat in the region  $R_+$ , in fact is trivial in the distinguished trivialization of  $\tilde{\mathcal{S}}_r$  over  $R_+$ , corresponding to the distinguished trivialization of  $P$  over  $D_+^4$ . Likewise we have a distinguished trivialization of  $\tilde{\mathcal{S}}_r$  over  $R_-$ , corresponding to the distinguished trivialization of  $P$  over  $D_-^4$ . In this latter trivialization let

$$\phi_r : \mathbb{R} \rightarrow \text{Ham}(S^2, \omega)$$

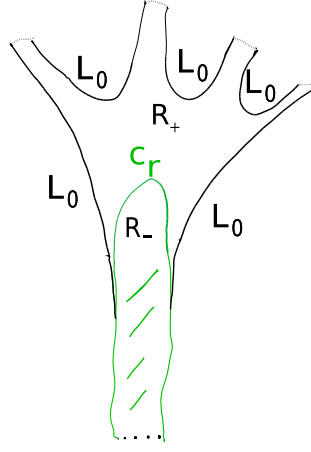


FIGURE 2. The labels  $L_0$  indicate that the Lagrangian subbundle is constant with corresponding fiber  $L_0$ . The curve  $c_r$  bounds  $R_-$ .

be the holonomy path of  $\mathcal{A}'_r$  over  $c_r$ . Then by construction,

$$\phi_r|_{(-\infty, 0] \sqcup [1, \infty)} = id,$$

so that we may define

$$f(r) \in \Omega_{L_0} Lag(M, L_0),$$

by

$$f(r)(t) = \phi_r(t)(L_0), \quad t \in [0, 1]$$

where the right hand side means apply an element of  $Ham(S^2)$  to  $L_0$  to get a new Lagrangian. We will say that  $f(r)$  is **generated by**  $\mathcal{A}'_r$ .

Note that by construction

$$(8.3) \quad \phi_r(t) = g(\tilde{c}_r(t)), \quad t \in [0, 1],$$

if we identify  $\tilde{c}_r(t)$  with an element of  $S^3$ .

Let  $D_0^2 \subset \overline{\mathcal{R}}_4$  be an embedded closed disk  $D^2$ , not intersecting the boundary  $\partial \overline{\mathcal{R}}_4$ , so that  $\partial D_0^2$  is in the normal gluing neighborhood  $N$  of  $\partial \overline{\mathcal{R}}_4$ , where  $N$  is as in Part I.

So we have a continuous map

$$f : D_0^2 \rightarrow \Omega_{L_0} Lag(S^2).$$

And  $f(\partial D_0^2) = L_0$ , with the right hand side denoting the constant loop at  $L_0$ . Then by construction and (8.3) in particular,  $f \simeq lag$ , where  $\simeq$  is a homotopy equivalence, and where

$$(8.4) \quad lag : S^2 \rightarrow \Omega_{L_0} Lag(S^2)$$

is the composition

$$S^2 \xrightarrow{g'} \Omega_{id} PU(2) \rightarrow \Omega_{L_0} Lag^{eq}(S^2),$$

for  $g'$  naturally induced by  $g$ , and for the second map naturally induced by the map

$$PU(2) \rightarrow Lag^{eq}(S^2), \quad \phi \mapsto \phi(L_0).$$

We then deform each  $\mathcal{A}'_r$  to a connection  $\mathcal{A}_r$ , which is as follows. In the region  $R_+$   $\mathcal{A}_r$  is still flat, but at each end  $e_i$ ,  $\mathcal{A}_r$  is compatible with  $\mathcal{A}(L_0, L_0)$ , where this is as in Section 6.2, and so that  $\mathcal{A}_r$  is still trivial over the boundary of  $\mathcal{S}_r$ .

Since  $\tilde{\mathcal{S}}_r$  and  $\mathcal{A}'_r$  are trivial for  $r \in \overline{\mathcal{R}}_d - D_0^2$ , with trivialization induced by the trivialization of  $P_+$ , and since the condition (7.2) holds, we may insure that

$$(8.5) \quad \text{area}(\mathcal{A}_r) < \hbar - 5\kappa,$$

for  $r$  in the complement of  $D_0^2$ . In other words  $\{\mathcal{A}_r\}$  extends to a system of connections corresponding to small data  $\mathcal{D}_0$  for  $P$ , as intended.

**8.2. Restructuring the data  $\{\mathcal{A}_r\}$ .** Applying Lemma 6.19 we see that the resulting Hamiltonian structure  $\mathcal{H} := \{\tilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r\}$  is  $A_0$ -admissible. We now further mold this data for the purposes of computation.

First cap off the ends  $e_i$ ,  $i \neq 0$ , of  $\mathcal{S}_r$  as in Section 6.4. This gives a Hamiltonian structure

$$\mathcal{H}^\wedge := \{\tilde{\mathcal{S}}_r^\wedge, \mathcal{S}_r^\wedge, \mathcal{L}_r^\wedge, \mathcal{A}_r^\wedge\}_{\mathcal{K}=D_0^2},$$

satisfying

$$\text{area}(\mathcal{A}_r^\wedge) + \kappa < \hbar,$$

for each  $r$ . Again by Lemma 6.19  $\mathcal{H}^\wedge$  is  $A_0$ -admissible.

By the now classical gluing of holomorphic curves it follows that

$$(8.6) \quad [\mu_{\Sigma^4}^4(\gamma_1, \dots, \gamma_4)] = [ev(\mathcal{H}^\wedge, A_0)].$$

It remains to compute the right hand side, to this end we further restructure.

Let  $p_1 : [0, 1] \rightarrow \text{Lag}(S^2, L_0)$  be the path generated by  $\mathcal{A}(L_0, L_0)$ , with  $p_1$  starting at  $L_0$ , and where generated has the same meaning as in the previous section. Suppose we have defined  $p_{i-1}$ , set  $L_{i-1} := p_{i-1}(1)$  and define  $p_i$  to be the path in  $\text{Lag}(S^2, L_0)$  starting at  $L_{i-1}$ , generated by  $\mathcal{A}(L_0, L_0)$ . Set  $p_0 := p_1 \cdots p_d$ , where  $\cdot$  is path concatenation in diagrammatic order. We may assume that  $L_0$  is transverse to  $L_4 = p_0(1)$  by adjusting the connection  $\mathcal{A}(L_0, L_0)$  if necessary. Then deform  $\mathcal{L}_r^\wedge$  in the Hamiltonian data  $\{\tilde{\mathcal{S}}_r^\wedge, \mathcal{S}_r^\wedge, \mathcal{L}_r^\wedge, \mathcal{A}_r^\wedge\}$  as in Figure 3. The resulting Lagrangian subbundle over  $\partial\mathcal{S}_r$  will be denoted  $\mathcal{L}_r^n$ ,  $n$  here stands for ‘new’.

We simultaneously deform  $\mathcal{A}_r^\wedge$  to an  $\mathcal{L}_r^n$  exact Hamiltonian connection  $\mathcal{A}_r^n$  which satisfies the following conditions.  $\mathcal{A}_r^n$  is flat in the entire region  $R_+$  (which includes the red shaded finger regions). Along the dotted line (which is contained in the strip chart at the  $e_0$  end)  $\mathcal{A}_r^n$  is the trivial connection in the distinguished trivialization at the end, and such that at the  $e_0$  end, which is down in the Figure 3, the connection is unchanged over  $[0, 1] \times (t, \infty)$ , for  $t$  large. In order to get such a deformation, we introduce curvature in the blue stripped region of Figure 3. We name this new Hamiltonian data by

$$\mathcal{H}^n := \{\tilde{\mathcal{S}}_r^\wedge, \mathcal{S}_r^\wedge, \mathcal{L}_r^n, \mathcal{A}_r^n\}.$$

As  $L^\pm(p_i)$ , for each  $p_i$ , can be arranged to be arbitrarily small, it is clear that we may choose the deformation from  $\mathcal{H}^\wedge$  to  $\mathcal{H}^n$  to be small near boundary (Definition 6.13) and hence be an  $A_0$ -admissible concordance. More specifically, we may choose a concordance from  $\mathcal{H}^\wedge$  to  $\mathcal{H}^n$  so that for the associated family of connections  $\{\mathcal{A}_{r,t}\}$ ,

$$\mathcal{A}_{r,0} = \mathcal{A}_r^\wedge, \mathcal{A}_{r,1} = \mathcal{A}_r^n,$$

so that for each  $r$ ,

$$\frac{d}{dt}|R_{\mathcal{A}_{r,t}}(v, jv)|_+ < 0, \quad v \in T_z\mathcal{S}_r$$

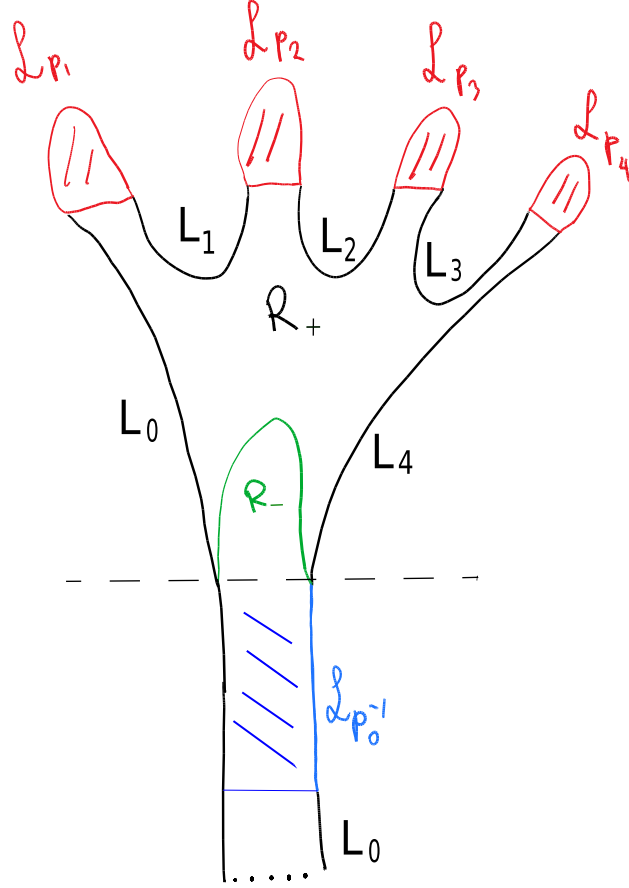


FIGURE 3. Over the boundary components with black labels  $L_i$  the Lagrangian subbundle  $\mathcal{L}_r^n$  is constant with corresponding fiber  $L_i$ . Over the  $i$ 'th red boundary component the Lagrangian subbundle corresponds to the path of Lagrangians  $p_i$ , analogously to Definition 6.17 further below. Likewise over the blue boundary component the Lagrangian subbundle corresponds to the path of Lagrangians  $p_0^{-1}$ . In the red striped regions we have removed the curvature of the connection, the blue striped regions we have added it.

for each  $z \in \mathcal{S}_r$ , except for  $z$  in the region which is blue striped in Figure 3. However, the area increase in this blue region is bounded from above by  $L^+(p_0^{-1})$ , so that

$$\forall t : |\text{area}(\mathcal{A}_{r,t}) - \text{area}(\mathcal{A}_r^\wedge)| \leq L^+(p_0^{-1}).$$

In fact we can arrange that

$$\frac{d}{dt} \text{area}(\mathcal{A}_{r,t}) = 0,$$

since the gain of area in the blue striped region is exactly equal to the loss of area in the red striped regions, but this extra precision is not necessary.

Of course:

$$[ev(\mathcal{H}^\wedge, A_0)] = [ev(\mathcal{H}^n, A_0)]$$

since the corresponding Hamiltonian data are  $A_0$ -admissibly concordant. This finishes our restructuring.

**8.3. Computing  $[ev(\mathcal{H}^n, A_0)]$ .** If we stretch the neck along the dashed line in Figure 3, the upper half of the resulting building gives us new Hamiltonian data

$$\mathcal{H}^0 = \{\tilde{S}_r^0, S_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}.$$

By the classical theory of continuation maps in Floer homology we clearly have that

$$[ev_n] := [ev(\mathcal{H}^n, A_0)] \in FH(L_0, L_0)$$

is non-zero iff

$$[ev_0] := [ev(\mathcal{H}^0, A_0)] \in FH(L_0, L_4)$$

is non-zero.

Let  $\mathcal{P}_{L_0, L_4} Lag(M)$  denote the space of smooth paths in  $Lag(M)$  from  $L_0$  to  $L_4$ . Let

$$f' : D_0^2 \rightarrow \mathcal{P}_{L_0, L_4} Lag(S^2),$$

be like  $f$  but defined with respect to  $\{\mathcal{A}_r^0\}$ , so that

$$(8.7) \quad f'(t) = g(\tilde{c}_r(t))(p_0(t)),$$

if we suppose that the holonomy path of  $\mathcal{A}_r^0$  over  $c_r$  in the trivialization over  $R_+$  generates  $p_0$  (which can be insured by adjusting  $\{\mathcal{A}_r^0\}$  or the parametrizations  $\{c_r\}$ ). Here the right hand side of (8.7) means as before: apply an element of  $Ham(S^2)$  to a Lagrangian to get a new Lagrangian. In this case  $f'$  takes  $\partial D_0^2$  to  $p_0 \in \mathcal{P}_{L_0, L_4} Lag(M)$ . In particular,  $f'$  represents a class  $a \in \pi_2(\mathcal{P}_{L_0, L_4} Lag(M), p_0)$ .

In what follows we omit specifying the parameter space  $D_0^2$  for  $r$ , since it will be the same everywhere. Let  $\mathcal{L}_p$  be as in the Definition 6.17.

**Lemma 8.2.** *The  $A_0$ -admissible Hamiltonian structure  $\mathcal{H}_0 = \{\tilde{S}_r^0, S_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}$  is  $A_0$ -admissibly concordant to*

$$\Theta' = \{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f'}(r), \mathcal{B}_r\},$$

for certain Hamiltonian connections  $\{\mathcal{B}_r\}$  (which are not explicitly relevant yet).

*Proof.* Let  $R_\pm \subset \mathcal{S}_r^\wedge$  be as before. Fix a deformation retraction

$$ret_r : \mathcal{S}_r^0 \times I \rightarrow \mathcal{S}_r^0,$$

of  $\mathcal{S}_r^0$  onto  $R^-$ , smoothly in  $r$ . Since  $\mathcal{A}_r^0$  is flat over  $R^+$ , the pull-back by  $ret_r$  of the data  $\mathcal{H}_0$  then induces an  $A_0$ -admissible concordance between  $\mathcal{H}_0$  and

$$\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f'}(r), \mathcal{B}_r = ret_r^* \mathcal{A}_r^0\},$$

once we use smooth Riemann mapping theorem to identify each  $R^- \subset \mathcal{S}_r^0$  with its induced complex structure  $j_r$  with  $(\mathcal{D}, j_{st})$ , smoothly in  $r$ .  $\square$

## 9. QUANTUM MASLOV CLASSES

The class of  $ev(\Theta', A_0)$  is related to what we christen as quantum Maslov classes. These are relative analogues of the quantum characteristic classes [16]. We chose the name quantum Maslov class, because the classical Maslov numbers are relative analogues of Chern numbers, while quantum characteristic classes are directly related (via semi-classical approximation) to Chern classes, [18].

We will not give extensive detail here since we don't need the full theory, we present it because it gives extra perspective. The ordinary relative Seidel morphism appears in Seidel's [21] in the exact case and further developed in [9] in the monotone case. Let  $Lag(M)$  denote the space whose components are objects of  $Fuk(M)$  in the previous sense, so in particular oriented, spin, Hamiltonian isotopic Lagrangian submanifolds of  $M$ . We may also denote the component of  $L$  by  $Lag(M, L)$ . Then the relative Seidel morphism is a functor

$$S : \Pi Lag(M) \rightarrow DF(M),$$

where  $\Pi Lag(M)$  is the fundamental groupoid of  $Lag(M)$  and  $DF(M)$  is the Donaldson-Fukaya category of  $M$ , see also [5], [4] which can be understood as an extension.

We sketch how this works. To a path  $p$  in  $Lag(M)$  from  $L_0$  to  $L_1$  we have an associated Lagrangian subbundle  $\mathcal{L}_p$  of  $\mathcal{D} \times M$  over the boundary, as in Definition 6.17. Extend this to a Hamiltonian structure

$$\Theta_p = (\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_p, \mathcal{A}_0^p)$$

where  $\mathcal{A}_0^p$  is as in Lemma 6.18. Assuming  $\Theta_p$  is regular, we define  $S([p]) \in DF(L_0, L_1)$  by

$$S([p]) = \sum_A [ev(\Theta_p, A)],$$

where by monotonicity only finitely many  $A$  can have non-zero contribution.

**9.1. Definition of the quantum Maslov classes.** Let  $M$  be as before, and let  $\mathcal{P}(L_0, L_1)$  denote the space of smooth paths in  $Lag(M)$  from  $L_0$  to  $L_1$ , constant in  $[0, \epsilon] \cup [1 - \epsilon, 1]$  for some  $0 < \epsilon < 1$ . There is then an additive group homomorphism:

$$(9.1) \quad \Psi : H_*(\mathcal{P}(L_0, L_1), \mathbb{Q}) \rightarrow FH(L_0, L_1)$$

defined analogously to above and to [16] in non-relative context. Although formally we will only need the restriction of  $\Psi$  to spherical classes.

This works as follows. To a smooth cycle

$$f : B \rightarrow \mathcal{P}(L_0, L_1)$$

for  $B$  a smooth closed oriented manifold, we may associate a Hamiltonian structure

$$\{\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_b\}_B,$$

$\mathcal{L}_b := \mathcal{L}_{f(b)}$  a Lagrangian subbundle of  $M \times \mathcal{D}$  over  $\partial\mathcal{D}$  determined by  $f(b)$  as before. The end of  $\mathcal{D}$  here is negative. Now let  $\mathcal{A}_0$  be a Hamiltonian connection on  $[0, 1] \times M$ , so that  $\mathcal{A}_0(L_0)$  is transverse to  $L_1$  where  $\mathcal{A}_0(L_0) \subset \{1\} \times M$  denotes the  $\mathcal{A}_0$ -transport over  $[0, 1]$  of  $L_0 \subset \{0\} \times M$ .

For each  $b$  the space of Hamiltonian connections  $\mathcal{L}_b$ -exact with respect to  $\mathcal{A}_0$ , (as in Section 6.2) is contractible, c.f. [1]. So we get an induced Hamiltonian structure:

$$\Theta_f = \{\mathcal{D} \times M, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}$$

well defined up to concordance.

We may then define  $\Psi([f])$  by:

$$\Psi([f]) = \sum_A [ev(\Theta_f, A)],$$

where again by monotonicity only finitely many  $A$  can give non-zero contribution. It is immediate that  $\Psi$  is an additive group homomorphism.

**Remark 9.1.** *We should mention that the morphism  $\Psi$  extends to a certain functor to  $DF(M)$ , see [4] for a related discussion.*

Given the definition above,

$$[ev(\Theta', A_0)] = \Psi(a)$$

clearly holds, as  $A_0$  is the only class that can contribute to  $\Psi(a)$ , since by the dimension formula (B.1) only a class  $A$  with  $Maslov^{vert}(A') = -2$  can contribute.

## 10. COMPUTATION OF THE QUANTUM MASLOV CLASS $\Psi(a)$

**10.1. Morse theory for the Hofer length functional.** Under certain conditions the spaces of perturbation data for certain problems in Gromov-Witten theory admit a Hofer like functional. Although these spaces of perturbations are usually contractible, there may be a gauge group in the background that we have to respect, so that working equivariantly there is topology. The reader may think of the analogous situation in Yang-Mills theory [2].

Without elaborating too much, the basic idea of the computation that we will perform consists of cooling the perturbation data as much as possible (in the sense of the functional) to obtain a mini-max (for the functional) data, using which we may write down our moduli spaces explicitly. This idea was first used in [17].

**10.1.1. Hofer length.** For  $p : [0, 1] \rightarrow Ham(M, \omega)$  a smooth path, define

$$\begin{aligned} L^+(p) &:= \int_0^1 \max_M H_t^p dt, \\ L^-(p) &:= \int_0^1 \max_M (-H_t^p) dt, \\ L^\pm(p) &:= \max\{L^+(p), L^-(p)\}, \end{aligned}$$

where  $H^p : M \times [0, 1] \rightarrow \mathbb{R}$  generates  $p$  normalized by the condition that for each  $t$ ,  $H_t^p := H^p|_{M \times \{t\}}$  has mean 0, that is  $\int_M H_t^p dvol_\omega = 0$ . Also define

$$L_{lag}^+ : PLag(M) \rightarrow \mathbb{R},$$

$$L_{lag}^+(p) := \int_0^1 \max_{p(t)} H_t^p dt,$$

$p(0) = L$  and where  $H^p : M \times [0, 1] \rightarrow \mathbb{R}$  is normalized as above and generates a lift  $\tilde{p}$  of  $p$  to  $Ham(M)$  starting at  $id$ . By lift we mean that  $p(t) = \tilde{p}(t)(p(0))$ . (That is  $H^p$  generates a path in  $Ham(M)$ , which moves  $L_0$  along  $p$ .) Some theory of this latter functional is developed in [10]. We may however omit the subscript  $lag$  from notation, as usually there can be no confusion which functional we mean.



Note that  $Lag^{eq}(S^2)$  is naturally diffeomorphic to  $S^2$  and moreover it is easy to see that the functional  $L^+|_{Lag^{eq}(S^2)}$  is proportional to the Riemannian length functional  $L_{met}$  on the path space of  $S^2$ , with its standard round metric  $met$ .

Let now  $L_0, L_1 \in Lag^{eq}(S^2)$  be any transverse pair, and

$$f' : S^2 \rightarrow \mathcal{P}(L_0, L_1) := \mathcal{P}_{L_0, L_1} Lag^{eq}(S^2),$$

be the generator of the group  $H_2(\mathcal{P}(L_0, L_1), \mathbb{Z})$ . The idea of the computation is then this: perturb  $f'$  to be transverse to the (infinite dimensional) stable manifolds for the Riemannian length functional on

$$\mathcal{P}(L_0, L_1) := \mathcal{P}_{L_0, L_1} Lag^{eq}(S^2),$$

push the cycle down by the “infinite time” negative gradient flow for this functional, and use the resulting representative to compute  $\Psi(a = [f'])$ . Although, we will not actually need infinite dimensional topology.

10.1.2. *The “energy” minimizing perturbation data.* Classical Morse theory [13] tells us that the energy functional

$$E(p) = \int_{[0,1]} \langle \dot{p}(t), \dot{p}(t) \rangle_{met} dt$$

on  $\mathcal{P}(L_0, L_1)$  is Morse non-degenerate with a single critical point in each degree. Consequently  $a$  (as a homology class) has a representative in the 2-skeleton of  $\mathcal{P}(L_0, L_1)$ , for the Morse cell decomposition induced by  $E$ . This follows by Whitehead’s compression lemma which is as follows.

**Lemma 10.1** (Whitehead, see [8]). *Let  $(X, A)$  be a CW pair and let  $(Y, B)$  be any pair with  $B \neq \emptyset$ . For each  $n$  such that  $X - A$  has cells of dimension  $n$ , assume that  $\pi_n(Y, B, y_0) = 0$  for all  $y_0 \in B$ . Then every map  $f : (X, A) \rightarrow (Y, B)$  is homotopic relative to  $A$  to a map  $X \rightarrow B$ .*

Suppose that  $a$  has a representative  $f' : S^2 \rightarrow \mathcal{P}_{L_0, L_1}(S^2)$  mapping into the  $n$ -skeleton  $B^n$  for the Morse cell decomposition for  $E$ ,  $n > 2$ . Apply the lemma above with  $(X, A) = (S^2, pt)$ ,  $Y = B^n$  and  $B = B^{n-1}$  as above. Then the quotient  $B^n/B^{n-1}$  is a wedge of  $n$ -spheres and since  $\pi_2(S^n) = 0$  for  $n > 2$ ,  $f$  can be homotoped into  $B^{n-1}$  by the Whitehead lemma. Descend this way until we get a representative mapping into  $B^2$ .

Furthermore since  $\pi_2(S^1) = 0$  such a representative cannot entirely lie in the 1-skeleton. It follows, since we have a single Morse 2-cell that there is a representative  $f : S^2 \rightarrow \mathcal{P}_{L_0, L_1}(S^2)$ , for  $a$ , s.t. the function  $f^*E$  is Morse with a maximizer  $max$ , of index 2, and s.t.  $\gamma_0 = f(max)$  is the index 2 geodesic. We call such a representative **minimizing**.

**Remark 10.2.** *In principle there maybe more than one such maximizer  $max$ , but recall that we assumed that  $a$  is the generator, so by further deformation we may insure that there is only one maximizer. The relevant representative  $f$ , with a single maximizer  $max$  as above, can also be constructed by hand.*

It follows that  $max$  is likewise the unique index 2 maximizer of the function  $f^*L_{met}$  by the classical relation between the energy functional and length functional. And so  $max$  is the index 2 maximizer of  $f^*L^+$ .

10.1.3. *The corresponding minimizing data.*

**Lemma 10.3.** *There is a minimizing representative  $f_0$  for the class  $a$  and a taut Hamiltonian structure*

$$\Theta_{f_0} = \{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f_0(b)}, (\mathcal{A}_b = \mathcal{A}_0^{f_0(b)})\},$$

where  $\mathcal{A}_0^{f_0(b)}$  is again as in Lemma 6.18, satisfying:

$$(10.1) \quad \forall b : \text{area}(\mathcal{A}_b) = L^+(f_0(b)).$$

*Proof.* Note that a geodesic segment  $p : [0, 1] \rightarrow S^2$  for the round metric  $met$  on  $S^2$  has a unique lift

$$\tilde{p} : [0, 1] \rightarrow PU(2) \simeq SO(3),$$

$\tilde{p}(0) = id$  with  $\tilde{p}$  a segment of a one parameter subgroup, and in this case

$$L_{lag}^+(p) = L^+(\tilde{p}).$$

It then follows that for a piecewise geodesic path  $p$  in  $S^2$ , there is likewise a unique lift  $\tilde{p}$  satisfying

$$L_{lag}^+(p) = L^+(\tilde{p}).$$

Now if  $f$  is a minimizing representative of  $a$ , we may homotop it to a likewise minimizing representative  $f_0$  so that for all  $b$   $f_0(b)$  is piecewise geodesic, this follows by the piecewise geodesic approximation theorem Milnor [13, Theorem 16.2] of the loop space.

Let  $\mathcal{A}_0$  be the trivial Hamiltonian connection on  $[0, 1] \times M$ . Use the construction of Lemma 6.18, to get a family of Hamiltonian connections  $\{\mathcal{A}_0^{f_0(b)}\}$ . In this case since  $\mathcal{A}_0$  is trivial

$$\text{area}(\mathcal{A}_0^{f_0(b)}) = L^+(f_0(b)).$$

It remains to verify that  $\Theta_{f_0} = \{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_{f_0(b)}, \mathcal{A}_b\}$ , is taut. This follows by the following more general lemma.

**Lemma 10.4.** *Let  $Lag^{eq}(\mathbb{CP}^n)$  denote the space of oriented Lagrangian submanifolds Hamiltonian isotopic to  $\mathbb{RP}^n$ , then two loops  $p_1, p_2 : S^1 \rightarrow Lag^{eq}(\mathbb{CP}^n)$  are taut concordant as defined in Section 1.3 iff they are homotopic.*

*Proof.* Let  $\mathcal{L}$  be a sub-fibration of  $Cyl \times M$  as in the definition of taut concordance of loops. Let  $\mathcal{A}$  be any  $PU(n)$  connection on  $Cyl \times \mathbb{CP}^n$  which preserves  $\mathcal{L}$  (there are no obstructions to constructing this). Then  $R_{\mathcal{A}}$  is a lie  $PU(n)$  valued 2-form, such that for all  $v, w \in T_z Cyl$  the vector field  $X = R_{\mathcal{A}}(z)(v, w)$  is tangent to  $\mathcal{L}_z$ . In particular if  $H_X$  is the Hamiltonian generating  $X$ , then since  $X$  is an infinitesimal unitary isometry preserving  $\mathcal{L}_z$ ,  $H_X$  vanishes on  $\mathcal{L}_z$ . It follows by the definition of  $\Omega_{\mathcal{A}}$ , that it vanishes on  $\mathcal{L}$  and so we are done.  $\square$

$\square$

So given  $\{\mathcal{A}_b\}$  as in the lemma above, since

$$\forall b : \text{area}(\mathcal{A}_b) = L^+(f_0(b)),$$

we immediately deduce:

**Lemma 10.5.** *The function  $\text{area} : b \mapsto \text{area}(\mathcal{A}_b)$  has a unique maximizer, coinciding with the maximizer  $\max$  of  $f_0^* L_{met}$  and  $\text{area}$  is Morse at  $\max$  with index 2.*

10.1.4. *Finding class  $A_0$  holomorphic sections for the data.* Let us now rename  $f_0$  by  $f$ ,  $\mathcal{L}_{f_0(b)}$  by  $\mathcal{L}_b$ , and  $\Theta_{f_0}$  by  $\Theta = \{\Theta_b\}$ .

As  $f(\max)$  is a geodesic for  $met$ , its lift  $\tilde{f}(\max)$  to  $SO(3)$  is a rotation around an axis intersecting  $L_0 = f(\max)(0)$  in a pair of points, in particular there is a unique point

$$x_{\max} \in \bigcap_t (L_t = f(\max)(t))$$

maximizing  $H_t^{\max}$  for each  $t$ . In our case this follows by elementary geometry but there is a more general phenomenon of this form c.f. [10].

Define

$$\sigma_{\max} : \mathcal{D} \rightarrow \mathcal{D} \times S^2$$

to be the constant section  $z \mapsto x_{\max}$ . Then  $\sigma_{\max}$  is a  $\mathcal{A}_{\max}$ -flat section with boundary on  $\mathcal{L}_{\max}$ , and is consequently  $J(\mathcal{A}_{\max})$ -holomorphic.

**Lemma 10.6.**

$$[\sigma_{\max}] = A_0.$$

*Proof.* Set

$$T_z^{vert} \mathcal{L}_{\max} := \{v \in T\mathcal{L} \subset T_z(\mathcal{D} \times S^2) \mid pr_* v = 0\}$$

where  $pr : \mathcal{D} \times S^2 \rightarrow \mathcal{D}$  is the projection. Denote by

$$Lag(T_{x_{\max}} S^2 \simeq Lag(\mathbb{R}^2) \simeq S^1$$

the space of oriented linear Lagrangian subspaces of  $T_{x_{\max}} S^2$ . Let  $\rho$  be the path in  $Lag(T_{x_{\max}} S^2)$  defined by

$$\rho(t) = T_{(\zeta(t), x_{\max})}^{vert} \mathcal{L}_{\max}, \quad t \in [0, 1]$$

where  $\zeta : \mathbb{R} \rightarrow \partial\mathcal{D}$  is a fixed parametrization as in Definition 6.17.

By our conventions for the Hamiltonian vector field:

$$\omega(X_H, \cdot) = -dH(\cdot),$$

$\rho$  is a clockwise oriented path from

$$T_{x_{\max}} L_0 := T_{(\zeta(0), x_{\max})}^{vert} \mathcal{L}_{\max}$$

to

$$T_{x_{\max}} L_1 := T_{(\zeta(1), x_{\max})}^{vert} \mathcal{L}_{\max}$$

for the orientation induced by the complex orientation on  $T_{x_{\max}} S^2$ .

By the Morse index theorem in Riemannian geometry [13] and by the condition that  $f(\max)$  has Morse index 2,  $\rho$  visits initial point  $\rho(0)$  exactly twice if we count the start, as this corresponds to the geodesic  $f(\max)$  passing through two conjugate points in  $S^2$ . So the concatenation of  $\rho$  with the minimal counter-clockwise path from  $T_{x_{\max}} L_1$  back to  $T_{x_{\max}} L_0$  is a degree  $-1$  loop, if  $S^1 \simeq Lag(\mathbb{R}^2)$  is given the counter-clockwise orientation. Consequently

$$Maslov^{vert}(\sigma_{\max}^{\prime}) = -2,$$

cf. Appendix B, in other words  $[\sigma_{\max}] = A_0$ . □

**Proposition 10.7.**  $(\sigma_{\max}, \max)$  is the sole element of  $\overline{\mathcal{M}}(\Theta, A_0)$ .

*Proof.* By Stokes theorem, since  $\omega$  vanishes on  $\sigma_{\max}$ , it is immediate:

$$(10.2) \quad \text{carea}(\Theta_{\max}, A_0) = - \int_{\mathcal{D}} \sigma_{\max}^* \tilde{\Omega}_{\max} = L^+(f(\max)).$$

Moreover, since  $\Theta = \{\Theta_b\}$  is taut  $\text{carea}(\Theta_b, A_0) = L^+(f(\max))$ . So by (10.1) and by Lemmas 6.10, 6.11 we have:

$$L^+(f(\max)) \leq \text{area}(\mathcal{A}_b) = L^+(f(b)),$$

whenever there is an element

$$(\sigma, b) \in \overline{\mathcal{M}}(\{\Theta_b\}, A_0).$$

But clearly this is impossible unless  $b = \max$ , since  $L^+(f(b)) < L^+(f(\max))$  for  $b \neq \max$ . So to finish the proof of the proposition we just need:

**Lemma 10.8.** *There are no elements  $\sigma$  other than  $\sigma_{\max}$  of the moduli space*

$$\overline{\mathcal{M}}(\Theta_{\max}, A_0).$$

*Proof.* We have by (10.2), and by (10.1)

$$0 = \langle [\tilde{\Omega}_{\max} + \alpha_{\tilde{\Omega}_{\max}}], [\sigma_{\max}] \rangle,$$

and so given another element  $\sigma$  we have:

$$0 = \langle [\tilde{\Omega}_{\max} + \alpha_{\tilde{\Omega}_{\max}}], [\sigma] \rangle.$$

It follows that  $\sigma$  is necessarily  $\tilde{\Omega}_{\max}$ -horizontal, since

$$(\tilde{\Omega}_{\max} + \alpha_{\tilde{\Omega}_{\max}})(v, J_{\tilde{\Omega}_{\max}} v) \geq 0.$$

Since  $J_{\tilde{\Omega}_{\max}}$  by assumptions preserves the vertical and  $\mathcal{A}_{\max}$ -horizontal subspaces of  $T(\mathcal{D} \times S^2)$ , and since the inequality is strict for  $v$  in the vertical tangent bundle of

$$S^2 \hookrightarrow \mathcal{D} \times S^2 \rightarrow \mathcal{D},$$

the above inequality is strict whenever  $v$  is not horizontal. So  $\sigma$  must be  $\mathcal{A}_{\max}$ -horizontal. But then  $\sigma = \sigma_{\max}$  since  $\sigma_{\max}$  is the only flat section asymptotic to  $\gamma_0$ .  $\square$

$\square$

10.1.5. *Regularity.* It will follow that

$$\Psi(a) = \pm[\gamma_0]$$

if we knew that  $(\sigma_{\max}, \max)$  be a regular element of

$$\overline{\mathcal{M}}(\{\Theta_b\}, A_0).$$

We won't answer directly if  $(\sigma_{\max}, \max)$  is regular, although it likely is. But it is regular after a suitably small Hamiltonian perturbation of the family  $\{\mathcal{A}_r\}$  vanishing at  $\mathcal{A}_{\max}$ . We call this essentially automatic regularity.

**Lemma 10.9.** *There is a family  $\{\mathcal{A}_b^{\text{reg}}\}$  arbitrarily  $C^\infty$ -close to  $\{\mathcal{A}_b\}$  with  $\mathcal{A}_{\max}^{\text{reg}} = \mathcal{A}_{\max}$  and such that*

$$(10.3) \quad \overline{\mathcal{M}}(\{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b^{\text{reg}}\}, A_0),$$

*is regular, with  $(\sigma_{\max}, \max)$  its sole element. In particular*

$$\Psi(a) = \pm[\gamma_0].$$

*Proof.* The associated real linear Cauchy-Riemann operator

$$D_{\sigma_{\max}} : \Omega^0(\sigma_{\max}^* T^{\text{vert}} \mathcal{D} \times S_{\max}^2) \rightarrow \Omega^{0,1}(\sigma_{\max}^* T^{\text{vert}} \mathcal{D} \times S_{\max}^2),$$

has no kernel, by Riemann-Roch [12, Appendix C], as the vertical Maslov number of  $[\sigma_{\max}]$  is  $-2$ . And the Fredholm index of  $(\sigma_{\max}, \max)$  which is  $-2$ , is  $-1$  times the Morse index of the function  $\text{area}$  at  $\max$ , by Lemma 10.5. Given this, our lemma follows by a direct translation of [20, Theorem 1.20], itself elaborating on the argument in [17].  $\square$

To summarize:

**Theorem 10.10.** *For  $0 \neq a \in H_2(\mathcal{P}_{L_0, L_1} \text{Lag}(S^2), \mathbb{Z})$ ,*

$$0 \neq \Psi(a) \in HF(L_0, L_1).$$

*Proof.* We have shown that  $0 \neq \Psi(a) \in HF(L_0, L_1)$ , for  $a$  the generator of the group  $H_2(\mathcal{P}_{L_0, L_1} \text{Lag}(S^2), \mathbb{Z})$ . Since  $\Psi$  is an additive group homomorphism the conclusion follows.  $\square$

## 11. FINISHING UP THE PROOF OF LEMMA 4.5

Starting with (8.6) we showed that  $[\mu_{\Sigma^4}^4(\gamma_1, \dots, \gamma_4)]$  is non-vanishing in Floer homology iff

$$[ev(\mathcal{H}_0, A_0)] \in HF(L_0, L_4),$$

is non-vanishing. We then use Lemma 8.2 to identify  $[ev(\mathcal{H}_0, A_0)]$  with  $[ev(\Theta', A_0)]$ , which is also identified with  $\Psi(a)$ , for a certain spherical 2-class  $a$ . Finally, in Section 10 we compute  $\Psi(a)$  and show that it is non-zero. This together with Lemma 5.1 imply Lemma 4.5.  $\square$

## 12. PROOF OF THEOREM 1.8

Suppose otherwise, so that

$$\min_{f, [f]=a'} \max_{b \in S^2} L^+(f(b)) = U < \hbar,$$

for  $a' = i_* g$  as in the statement of the theorem. Fix  $L_1 \in \text{Lag}^{eq}(S^2)$  so that  $L_0$  intersects  $L_1$  transversally, and so that there is a geodesic path  $p_0 \in \mathcal{P}\text{Lag}^{eq}(L_0, L_1)$  with

$$\kappa := L^\pm(\tilde{p}_0) < \epsilon = (\hbar - U)/2.$$

Here  $\tilde{p}_0$  is the geodesic lift to  $PU(2)$  starting at  $id$ . Then concatenating  $f$  with  $p_0$  we obtain a smooth family of paths

$$\begin{aligned} g : S^2 &\rightarrow \mathcal{P}(L_0, L_1) \\ g(0) &= p_0, \end{aligned}$$

and  $g$  represents the previously appearing class  $a$ , that is the generator of the corresponding  $\pi_2$  group. Let

$$\{\Theta_b\} = \{\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_b, \mathcal{A}_b\}_{\mathcal{K}=S^2},$$

be the corresponding Hamiltonian structure, where  $\mathcal{A}_b = \mathcal{A}_0^{g(b)}$  is as in Lemma 10.3, for  $\mathcal{A}_0$  the trivial connection, and where  $\mathcal{L}_b := \mathcal{L}_{g(b)}$ . In particular  $\{\Theta_b\}$  is taut and satisfies:

$$(12.1) \quad \forall b \in S^2 : \text{area}(\mathcal{A}_b) = L^+(g(b)) < \hbar - \kappa.$$

By assumption that each  $f(b)$  is taut concordant to the constant loop at  $L_0$ , each  $\Theta_b$  is taut concordant to

$$\mathcal{H} = (\mathcal{D} \times S^2, \mathcal{D}, \mathcal{L}_0, \mathcal{A}_0),$$

where  $\mathcal{L}_0 = \mathcal{L}_{p_0}$ ,  $\mathcal{A}_0 = \mathcal{A}_0^{p_0}$ , where the latter is again as in Lemma 6.18. And so for each  $b$   $\Theta_b'$  is taut concordant to  $\mathcal{H}'$  by Lemma 6.16.

On the other hand by Lemma 10.4  $\mathcal{H}'$  is taut concordant to the trivial Hamiltonian structure  $(D^2 \times S^2, D^2, \mathcal{L}_{tr}, \mathcal{A}_{tr})$ ,  $\mathcal{L}_{tr}$  the trivial bundle with fiber  $L_0$  and  $\mathcal{A}_{tr}$  the trivial Hamiltonian connection. So for each  $b$ :

$$(12.2) \quad \text{carea}(\Theta_b', A_0) = \text{carea}(\mathcal{H}', A_0') = \hbar.$$

Now by Theorem 10.10

$$ev(\{\Theta_b\}, A_0) = \Psi(a) \neq 0.$$

And so:

$$\overline{\mathcal{M}}(\{\Theta_b\}, A_0) \neq \emptyset,$$

but this contradicts the conjunction of (12.1), (12.2), and Lemma 6.19.  $\square$

### 13. SINGULAR AND SIMPLICIAL CONNECTIONS AND CURVATURE BOUNDS

Let  $\mathcal{A}$  be a  $G$  connection on a principal  $G$  bundle  $P \rightarrow \Delta^n$ , and the Finsler norm  $\mathbf{n}$  on  $\text{lie } G$  be as in Section 1.1.1 of the introduction. A given system  $\mathcal{U}$  in particular specifies maps:

$$u(m_1, \dots, m_n, r, n) : \mathcal{S}_r \rightarrow \Delta^n,$$

where  $r \in \overline{\mathcal{R}}_n$ ,  $\mathcal{S}_r$  is the fiber of  $\overline{\mathcal{S}}_n^\circ$  over  $r$ , and where  $(m_1, \dots, m_n)$  is the composable chain of morphisms in  $\Pi(\Delta^n)$ ,  $m_i$  being the edge morphism from the vertex  $i-1$  to  $i$ . Then define

$$(13.1) \quad \text{area}_{\mathcal{U}}(\mathcal{A}) = \sup_r \text{area}_{\mathbf{n}}(u(m_1, \dots, m_n, r, n)^* \mathcal{A}),$$

where  $\text{area}_{\mathbf{n}}$  on the right hand side is as defined in equation (1.1). In the case  $G = \text{Ham}(M, \omega)$  we take

$$\mathbf{n} : \text{lie } \text{Ham}(M, \omega) \rightarrow \mathbb{R}$$

to be

$$\mathbf{n}(H) = |H|_+ = \max_M H.$$

Let  $\omega$  be the area 1 Fubini-Study symplectic 2-form on  $M = \mathbb{CP}^1$ . Then the pull-back by the natural map

$$\text{lie } h : \text{lie } PU(2) \rightarrow \text{lie } \text{Ham}(\mathbb{CP}^1, \omega) \simeq C_0^\infty(\mathbb{CP}^1)$$

of the semi-norm:  $|H|_+ = \max_M H$  is the operator norm on  $PU(2)$ , up to normalization. This will be used to get the specific form of Theorem 1.4, from the more general form here.

**13.1. Simplicial connections.** We now introduce a certain abstraction of singular connections, which can partly be understood as simplicial resolutions of singular connections. Let  $G \hookrightarrow P \rightarrow X$  be a principal  $G$  bundle, where  $G$  is a Frechet Lie group. Denote by  $X_\bullet$  the simplicial set whose set of  $n$ -simplices,  $X_\bullet(n)$ , consists smooth maps  $\Sigma : \Delta^n \rightarrow X$ , with  $\Delta^n$  standard topological  $n$ -simplex with vertices ordered  $0, \dots, n$ . And denote by  $\text{Simp}(X_\bullet)$  the category with objects  $\cup_n X_\bullet(n)$  and with  $\text{hom}(\Sigma_0, \Sigma_1)$  commutative diagrams:

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\text{mor}} & \Delta^m \\ & \searrow \Sigma_0 & \downarrow \Sigma_1 \\ & & X, \end{array}$$

for  $\text{mor}$  a simplicial face map, that is an injective linear map preserving order of the vertices.

**Definition 13.1.** Define a **singular simplicial  $G$ -connection**  $\mathcal{A}$  or hereby just **simplicial  $G$ -connection** on  $P$  to be the following data:

- For each  $\Sigma : \Delta^n \rightarrow X$  in  $X_\bullet(n)$  a smooth  $G$ -connection  $\mathcal{A}_\Sigma$  on  $\Sigma^* P \rightarrow \Delta^n$ , (a usual Ehresmann  $G$ -connection.)
- For a morphism  $\text{mor} : \Sigma_0 \rightarrow \Sigma_1$  in  $\text{Simp}(X_\bullet)$ , we ask that  $\text{mor}^* \mathcal{A}_{\Sigma_1} = \mathcal{A}_{\Sigma_0}$ .

**Example 13.2.** If  $\mathcal{A}$  is a smooth  $G$ -connection on  $P$  define a simplicial connection by  $\mathcal{A}_\Sigma = \Sigma^* \mathcal{A}$  for every simplex  $\Sigma \in X_\bullet$ . We call such a simplicial connection **induced**.

If we try to “push forward” a simplicial connection to get a “classical” connection on  $P$  over  $X$ , then we get a kind of multi-valued singular connection. Multi-valued because each  $x \in X$  may be in the image of a number of  $\Sigma : \Delta^n \rightarrow X$  and  $\Sigma$  itself may not be injective, and singular because each  $\Sigma$  is in general singular so that the naive push-forward may have blow up singularities. We will call the above the naive pushforward of a simplicial connection.

*Proof of Theorem 1.4 and Corollary 1.6.* We will prove this by way of a stronger result. Let  $P$  be a Hamiltonian fibration  $S^2 \hookrightarrow P \rightarrow S^4$ . And let  $\mathcal{A}$  a simplicial  $\text{Ham}(S^2)$  connection on  $P$ . Let  $\sigma_0^1 \in S_\bullet^4$  be the degenerate 1-simplex at  $x_0$ , in other words the constant map:  $\sigma_0^1 : [0, 1] \rightarrow x_0$ . Let  $\kappa$  be the  $L^\pm$  length of the holonomy path of  $\mathcal{A}_{\sigma_0^1}$  over  $[0, 1]$ .

Let  $\Sigma_\pm \in X_\bullet(4)$  be a complementary pair as in Section 4.1. The connection  $\mathcal{A}$  gives us a simplicial connection as in Example 13.2. By inductive procedure as in Part I, Lemma 5.6, we may find perturbation data  $\mathcal{D}$  for  $P$  so that for this data

$$(13.2) \quad \forall r : pr_1 \mathcal{F}(L_0^0, \dots, L_0^n, \Sigma_\pm, r) \simeq_\delta u(m_1, \dots, m_s, r, n)^* \mathcal{A}_{\Sigma_\pm},$$

$$(13.3) \quad \mathcal{A}(L_0, L_0) \simeq_\delta \mathcal{A}_{\sigma_0^1},$$

where  $L_0^i$  are the objects as before, where  $\simeq_\delta$  means  $\delta$ -close in the metrized  $C^\infty$  topology, and  $\delta$  is as small as we like. Here we are using notation of Part I as before. Set

$$\tilde{u}_r := \Sigma_- \circ u(m_1, \dots, m_s, r, 4),$$

so  $\tilde{u}_r : \mathcal{S}_r \rightarrow S^4$ . Set  $\tilde{\mathcal{S}}_r := \tilde{u}_r^* P$ , set  $\mathcal{A}'_r := pr_1 \mathcal{F}(L_0^0, \dots, L_0^n, \Sigma_-, r)$  and set

$$\{\Theta_r\} := \{\tilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}'_r\}.$$

**Definition 13.3.** We say that  $\mathcal{A}$  is **perfect**, if for every  $\delta$ , as above, arbitrarily small,  $\mathcal{A}(L_0, L_0)$  as above can be chosen so that the corresponding Floer chain complex  $CF(L_0, L_1, \mathcal{A}(L_0, L_0))$  is perfect.

**Theorem 13.4.** Let  $\mathcal{A}$  be a perfect simplicial Hamiltonian connections on  $P$ . If  $P$  is non-trivial as a Hamiltonian bundle then

$$(\text{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_+}) \geq \hbar - 5\kappa) \vee (\text{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_-}) \geq \hbar - 5\kappa),$$

*Proof.* Suppose

$$(13.4) \quad \text{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_+}) < \hbar - 5\kappa,$$

then by (13.2), (13.2) and by Lemma 6.19  $\mathcal{D}$  as defined above can be assumed to be small provided  $\delta$  is chosen to be sufficiently small. Take the unital replacement as in Lemma 5.1. Since we know that  $K(P)$  does not admit a section by Theorem 3.4, the simplex  $T$  of the Lemma 5.1 does not exist. Hence again by this lemma

$$ev(\{\Theta_r\}, A_0) = [\mu_{\Sigma_-}^4(\gamma_1, \dots, \gamma_4)] \neq 0.$$

In particular

$$\overline{\mathcal{M}}(\{\Theta_r\}, A_0) \neq \emptyset.$$

So by Lemma 6.19 there exists an  $r_0$  so that

$$(13.5) \quad \text{area}(\mathcal{A}'_{r_0}) \geq \hbar - 5\kappa'.$$

where  $\kappa'$  denotes the  $L^\pm$  length of the holonomy path in  $Ham(S^2)$  of  $\mathcal{A}(L_0, L_0)$ . By (13.3)  $\kappa' \rightarrow \kappa$  as  $\delta \rightarrow 0$ . By (13.2), (13.5) passing to the limit as  $\delta \rightarrow 0$  we get:

$$\text{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_-}) \geq \hbar - 5\kappa.$$

□

**Corollary 13.5.** Let  $\mathcal{A}$  be a  $PU(2)$  connection connections on a non-trivial principal  $PU(2)$  bundle  $P \rightarrow S^4$ . Then

$$(\text{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_+}) \geq \hbar - 5\kappa) \vee (\text{area}_{\mathcal{U}}(\mathcal{A}_{\Sigma_-}) \geq \hbar - 5\kappa).$$

*Proof.* A simplicial  $PU(2)$  connection  $\mathcal{A}$  on a principal  $PU(2)$  bundle  $PU(2) \hookrightarrow P' \rightarrow S^4$  is automatically perfect, when understood as a Hamiltonian connection on the associated bundle  $S^2 \hookrightarrow P \rightarrow S^4$ . So that this is an immediate consequence of the theorem above. □

To prove Corollary 1.6, we just note that for an induced simplicial Hamiltonian connection  $\mathcal{A}$ , as defined in Example 13.2,  $\mathcal{A}_{\sigma_0^1}$  is trivial. And hence  $\mathcal{A}$  is automatically perfect. So that this corollary follows by Theorem 13.4. □

## APPENDIX A. HOMOTOPY GROUPS OF KAN COMPLEXES

For convenience let us quickly review Kan complexes just to set notation. This notation is also used in Part I. Let

$$\Delta_\bullet^n(k) := \text{hom}_\Delta([k], [n]),$$

be the standard representable  $n$ -simplex, where  $\Delta$  is the simplicial category with objects ordered finite sets  $[n] = \{1, \dots, n\}$  and morphisms order preserving set maps.



Let  $\Lambda_k^n \subset \Delta_\bullet^n$  denote the sub-simplicial set corresponding to the “boundary” of  $\Delta_\bullet^n$  with the  $k$ 'th face removed,  $0 \leq k \leq n$ . By  $k$ 'th face we mean the face opposite to the  $k$ 'th vertex. A simplicial map

$$h : \Lambda_k^n \subset \Delta_\bullet^n \rightarrow X_\bullet$$

will be called a **horn**. A simplicial set  $S_\bullet$  is said to be a **Kan complex** if for all  $n, k \in \mathbb{N}$  given a diagram with solid arrows

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{h} & S_\bullet \\ \downarrow i & \nearrow \tilde{h} & \\ \Delta_\bullet^n & & \end{array},$$

there is a dotted arrow making the diagram commute. The map  $\tilde{h}$  will be called **the Kan filling** of the horn  $h$ . The  $k$ 'th face of  $\tilde{h}$  will be called **Kan filled face along  $h$** .

Given a pointed Kan complex  $(X_\bullet, x)$  and  $n \geq 1$  the  $n$ 'th *simplicial homotopy group* of  $(X_\bullet, x)$ :  $\pi_n(X_\bullet, x)$  is defined to be the set of equivalence classes of maps

$$\Sigma : \Delta_\bullet^n \rightarrow X_\bullet,$$

such that  $\Sigma$  takes  $\partial\Delta_\bullet^n$  to  $x_\bullet$ , with the latter denoting the image of  $\Delta_\bullet^0 \rightarrow X$ , induced by the vertex inclusion  $x \rightarrow X_\bullet$ .

More precisely, we have a commutative diagram:

$$\begin{array}{ccc} \Delta_\bullet^n & \longrightarrow & \Delta_\bullet^0 \\ & \searrow \Sigma & \downarrow x \\ & & X_\bullet \end{array}$$

**Example A.1.** When  $X_\bullet$  is the singular set associated to a topological space  $X$ , the maps above are in complete correspondence with maps:

$$\Sigma : \Delta^n \rightarrow X,$$

taking the topological boundary of  $\Delta^n$  to  $x$ .

For  $X_\bullet$  general simplicial set, a pair of maps  $\Sigma_1 : \Delta_\bullet^n \rightarrow X_\bullet, \Sigma_2 : \Delta_\bullet^n \rightarrow X_\bullet$ , are equivalent if there is a diagram, called simplicial homotopy:

$$\begin{array}{ccc} \Delta_\bullet^n & & \\ \downarrow i_0 & \searrow \Sigma_1 & \\ \Delta_\bullet^n \times I_\bullet & \longrightarrow & X_\bullet \\ \uparrow i_1 & \nearrow \Sigma_2 & \\ \Delta_\bullet^n & & \end{array}$$

such that  $\partial\Delta_\bullet^n \times I_\bullet$  is taken by  $H$  to  $x_\bullet$ . The simplicial homotopy groups of a Kan complex  $(X_\bullet, x)$  coincide with the classical homotopy groups of the geometric realization  $(|X_\bullet|, x)$ . But the power of the above definition is that if we know our Kan complex well, (like in the example of the present paper) the simplicial

homotopy groups are very computable since they are completely combinatorial in nature.

*Proof of Lemma 3.2.* We refer the reader to Part I, Appendix A.2, for more details on the notions here. We prove a stronger claim.

**Lemma A.2.** *Let  $p : Y \rightarrow X$  be an inner fibration of quasi-categories  $Y, X$ , with  $X$  a Kan complex. And let  $K(Y) \subset Y$  denote the maximal Kan subcomplex. Then  $p : K(Y) \rightarrow X$  is a Kan fibration.*

The above is probably well known, but it is simple to just provide the proof for convenience.

*Proof.* By definition of an inner fibration, whenever we are given a commutative diagram with solid arrows and with  $0 < k < n$ :

$$(A.1) \quad \begin{array}{ccccc} \Lambda_k^n & \xrightarrow{\sigma} & K(Y) & \hookrightarrow & Y \\ \downarrow & & \searrow \Sigma & & \nearrow p \\ \Delta^n & \longrightarrow & X, & & \end{array}$$

there exists a dashed arrow  $\Sigma$  as indicated, making the whole diagram commutative. When  $n > 2$  the edges of  $\Sigma$  are all automatically invertible in  $Y$ , as  $\Sigma$  extends  $\sigma$ , and all edges of  $\sigma$  are invertible by definition. For  $n = 2$  the edges of  $\Sigma$  are either edges of  $\sigma$ , or are compositions of edges of  $\sigma$  in the quasi-category  $Y$ , and hence again always invertible.

It follows that  $\Sigma$  maps into  $K(Y) \subset Y$ . Since the starting diagram was arbitrary, we just proved that  $p : K(Y) \rightarrow X$  is an inner fibration. In particular the pre-images  $p^{-1}(\Sigma(\Delta^n)) \subset K(Y)$  are quasi-categories, for all  $n$ , where  $\Sigma : \Delta^n \rightarrow X$  is any  $n$ -simplex, see Part I, Appendix A.2. But  $K(Y)$  is a Kan complex, so that also the above pre-images  $p^{-1}(\Sigma(\Delta^n))$  are Kan complexes. It readily follows from this that  $p : K(Y) \rightarrow X$  is a Kan fibration.  $\square$

The main lemma then follows, since if  $p : Y \rightarrow X$  is a (co)-Cartesian fibration, it is in particular an inner fibration.  $\square$

## APPENDIX B. ON THE MASLOV NUMBER

Let  $S$  be obtained from a compact connected Riemann surface  $S'$  with boundary, by removing a finite number of points  $\{e_i\}$  removed from the boundary of  $S'$ .

Let  $\mathcal{V} \rightarrow S$  be a rank  $r$  complex vector bundle, trivialized at the open ends  $\{e_i\}$ , that is so we have distinguished bundle charts  $\mathbb{C}^r \times [0, 1] \times [0, \infty) \rightarrow \mathcal{V}$  at the ends.

Let

$$\Xi \rightarrow \partial S \subset S$$

be a totally real rank  $r$  subbundle of  $\mathcal{V}$ , which is constant in the coordinates

$$\mathbb{C}^r \times [0, 1] \times [0, \infty),$$

at the ends. For each end  $e_i$  and its distinguished chart  $e_i : [0, 1] \times [0, \infty) \rightarrow S$  let  $b_i^j : [0, \infty) \rightarrow \partial S$ ,  $j = 0, 1$  be the restrictions of  $e_i$  to  $\{i\} \times [0, \infty)$ .

We then have a pair of real vector spaces

$$\Xi_i^j = \lim_{\tau \rightarrow \infty} \Xi|_{b_i^j(\tau)}.$$

There is a Maslov number  $Maslov(\mathcal{V}, \Xi, \{\Xi_i^j\})$  associated to this data coinciding with the boundary Maslov index in the sense of [12, Appendix C3], in the case  $\Xi_i^0 = \Xi_i^1$ , for the modified pair  $(\mathcal{V}', \Xi')$  obtained from  $(\mathcal{V}, \Xi, \{\Xi_i^j\})$  by naturally closing off each  $e_i$  end of  $\mathcal{V} \rightarrow S$ .

When  $\Xi_i^0$  is transverse to  $\Xi_i^1$  for each  $i$ ,  $Maslov(\mathcal{V}, \Xi, \{\Xi_i^j\})$  is obtained as the Maslov index for the modified pair  $(\mathcal{V}', \Xi')$  by again closing off the ends  $e_i$  via gluing (at each end  $e_i$ ) with

$$(\mathbb{C}^r \times \mathcal{D}, \tilde{\Xi}, \{\tilde{\Xi}_0^j\}),$$

where  $\mathcal{D}$  as before is diffeomorphic to  $D^2$  with a point  $e_0$  on the boundary removed. Here  $\tilde{\Xi}_0^0 = \Xi_0^1$  and  $\tilde{\Xi}_0^1 = \Xi_0^0$ , while  $\tilde{\Xi}$  over the boundary of  $\mathcal{D}$  is determined by the “shortest path” from  $\tilde{\Xi}_0^0$  to  $\tilde{\Xi}_0^1$ , meaning that since these are a pair of transverse totally real subspaces up to a complex isomorphism of  $\mathbb{C}^r$  (whose choice will not matter) we may identify them with the subspaces  $\mathbb{R}^r$ , and  $i\mathbb{R}^r$  after this identification our shortest path is just  $e^{i\theta}\mathbb{R}^r$ ,  $\theta \in [0, \pi_2]$ .

For a real linear Cauchy-Riemann operator on  $\mathcal{V}$ , with suitable asymptotics, the Fredholm index is given by:

$$r \cdot \chi(S) + Maslov(\mathcal{V}, \Xi, \{\Xi_i\}).$$

The proof of this is analogous to [12, Appendix C], we can also reduce it to that statement via a gluing argument. (This kind of argument appears for instance in [21])

**B.1. Dimension formula for moduli space of sections.** Suppose we are given a Hamiltonian fiber bundle  $M^{2r} \hookrightarrow \tilde{S} \rightarrow S$ , with end structure and  $S$  as above. Let  $\mathcal{L}$  be a Lagrangian sub-bundle of  $\tilde{S}$  over the boundary of  $S$ , compatible with the end structure, and such that the Lagrangian submanifolds

$$L_i^j = \lim_{\tau \rightarrow \infty} \mathcal{L}|_{b_i^j(\tau)},$$

intersect transversally (identifying the corresponding fibers) or coincide.

Given an  $\mathcal{L}$ -exact Hamiltonian connection  $\mathcal{A}$ , on  $\tilde{S}$ , (see Definition 6.2) which is additionally assumed to be trivial in the strip coordinate charts at the ends, and a choice of a family  $\{j_z\}$  of compatible almost complex structures on the fibers of  $\tilde{S}$ , set  $\mathcal{M}(A)$  to be the moduli space of (relative) class  $A$  finite vertical  $L^2$  energy holomorphic sections of  $\tilde{S} \rightarrow S$  with boundary in  $\mathcal{L}$ . Define

$$Maslov^{vert}(A)$$

to be the Maslov number of the triple  $(\mathcal{V}, \Xi, \{\Xi_i\})$  determined by the pullback by  $\sigma \in \mathcal{M}(A)$  of the vertical tangent bundle of  $\tilde{S}$ ,  $\mathcal{L}$ . Then the expected dimension of  $\mathcal{M}(A)$  is:

$$(B.1) \quad r \cdot \chi(S) + Maslov^{vert}(A).$$

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