MEAN CURVATURE VERSUS DIAMETER AND ENERGY QUANTIZATION

YASHA SAVELYEV

Abstract. Using a theorem of Topping we give a simple relation between mean curvature and intrinsic diameter for closed submanifolds of general compact Riemannian manifolds. We use this to prove quantization of energy for immersed pseudo-holomorphic curves in a locally conformally symplectic manifold.

In [9] Topping gave via a concise but sophisticated argument based on ideas of Ricci flows a simple relation between intrinsic diameter and mean curvature for immersed submanifold of $\mathbb{R}^n$. Let us state it here:

Theorem 0.1 ([9]). *For Σ^m a smoothly immersed closed submanifold of $\mathbb{R}^n$ we have:*

$$\operatorname{diam}(\Sigma) \leq Const(m) \int_{\Sigma} |\boldsymbol{H}|^{m-1}\, dvol,$$

for $\boldsymbol{H}$ the mean curvature vector field along Σ, vol the volume measure induced by the standard ambient metric, and diam *the intrinsic diameter:* $\max_{x,y\in\Sigma} dist_{(\Sigma, g_{st})}(x,y)$.

Here we use the above and Nash embedding theorem to give via an otherwise elementary argument the following (mostly) simple relation between diameter and mean curvature of closed submanifolds of general compact Riemannian manifolds. We use this to give an application to energy quantization of holomorphic curves in locally conformally symplectic manifolds. In plain words the main statement is the following: if the volume of a given closed immersed submanifold of a compact Riemannian manifold is "small" but diameter "large" then the mean curvature must be somewhere large. Here is the more precise statement.

Theorem 0.2. *Consider the set $S = S(C)$ of immersed $\Sigma \subset X$, for Σ a closed smooth m-manifold and (X, g) a fixed compact Riemannian manifold, with the mean curvature of Σ bounded from above by $C > 0$. Let $Vol(\Sigma)$ denote the g-volume, and* $\operatorname{diam}(\Sigma)$ *the (intrinsic) diameter in X, g. Then for all $\Sigma \in S$*

$$\operatorname{diam}(\Sigma) \leq F(g, C, m)\, \operatorname{vol}(\Sigma),$$

for some function F.

Proof. Pick an isometric Nash embedding N of (M, g) into $\mathbb{R}^n$, where n is large enough.

Lemma 0.3. *For all $\Sigma \in S(C)$ the magnitude of the mean curvature vector field along $N(\Sigma)$ in $\mathbb{R}^n$ is bounded from above by some C'.*

Proof. In what follows we conflate the notation for Σ and its images $u : \Sigma \to M$, $N \circ f : \Sigma \to \mathbb{R}^n$. In other words we just think in terms of subspaces $\Sigma \subset M \subset \mathbb{R}^n$. Let h be the second fundamental form on T_pM:

$$h(v, w) = \tilde{\tilde{\nabla}}_v w - \nabla_v w,$$

where $\tilde{\tilde{\nabla}}$ is the Levi-Civita connection of $(\mathbb{R}^n, g_{st})$, ∇ is the Levi-Civita connection of (Σ, g) and where we locally extend $v, w \in T_p\Sigma$ to vector fields tangent to Σ. If $\dim \Sigma = m$, the mean curvature vector of Σ in $\mathbb{R}^n$ at p is given by:

$$\mathbf{H}(p) = \frac{1}{m} \sum_i h(e_i, e_i),$$

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where $\{e_i\}$ is an orthonormal basis for $T_p\Sigma$. Likewise $\widetilde{\nabla}$ will denote in what follows the Levi-Civita connection of (M, g). So we have:

$$\begin{aligned} m|\mathbf{H}(p)| = |\sum_i (\widetilde{\widetilde{\nabla}}_{e_i} e_i - \nabla_{e_i} e_i)| &= |\sum_i (\widetilde{\widetilde{\nabla}}_{e_i} e_i - \widetilde{\nabla}_{e_i} e_i + \widetilde{\nabla}_{e_i} e_i - \nabla_{e_i} e_i)| \\ &\leq |\sum_i (\widetilde{\nabla}_{e_i} e_i - \nabla_{e_i} e_i)| + mB \\ &\leq mC + mB, \end{aligned}$$

where $B = \sup_{e \in TM, |e|=1} |\widetilde{\widetilde{\nabla}}_e e - \widetilde{\nabla}_e e|$. □

Since Σ is closed we get by Topping's theorem:

$$\operatorname{diam}(N\Sigma) \leq Const(m) \int_\Sigma |\mathbf{H}_{\mathbb{R}^n}|^{m-1} \, dVol.$$

By the lemma above the function $|\mathbf{H}_{\mathbb{R}^n}|$ on Σ is universally (independently of u) bounded from above by some C'. So we get:

$$\operatorname{diam}(\Sigma) = \operatorname{diam}(N\Sigma) \leq Const(m) \cdot (C')^{m-1} \cdot \operatorname{vol}(\Sigma),$$

and so we get the required inequality. □

1. Application to energy quantization

A locally conformally symplectic manifold or l. c. s. manifold, is a smooth $2n$-fold M, with a non-degenerate 2-form ω, which is locally diffeomorphic to $e^f \omega_0$ for some functions f, and ω_0 the standard symplectic form on $\mathbb{R}^{2n}$. There has been some interest recently in developing Gromov-Witten theory for l. c. s. manifolds, [7], [1]. This is partly impeded by our lack of understanding of how badly can holomorphic curves behave in such a manifold. For instance, in the context of holomorphic curves in symplectic manifolds, we have the following energy quantization phenomenon:

Theorem 1.1. *Given a closed symplectic (M, ω), J an ω-compatible almost complex structure there exists a constant $\hbar = \hbar(\omega, J)$, s.t. for any $u : (\Sigma, j) \to M$ a non-constant J-holomorphic map of a closed Riemann surface (Σ, j), the energy of the map satisfies:*

$$e(u) = \int_\Sigma |du|^2 dvol = \int_\Sigma u^* \omega \geq \hbar,$$

for vol the measure induced by the ambient (ω, J) metric.

For $\Sigma = S^2$ this holds via a generalized mean value inequality, [5], [10] and the symplectic condition on M can be loosened to just almost complex. For more general Σ, but with M symplectic we can give a simple argument via geometric measure theory. We give this proof here for completeness, as I am not aware if this previously appeared and the proof is relatively elementary, given state of the art.

Proof. Suppose otherwise, then we have a sequence $\{u_i\}$ of J-holomorphic curves with $e(u_i) \to 0$ as $i \to \infty$. In particular, $|u_i| \to 0$, where $|\cdot|$ is the mass norm, and u_i are understood as integral 2-currents, Federer [2]. By the main compactness theorem for currents, $\{u_i\}$ has a convergent subsequence $\{u_{i_k}\}$ to an integral 2-current with mass necessarily 0, and hence 0 in the vector space of closed integral 2-currents $\mathcal{I}_2(M)$, since mass norm is a norm. Next it is proved in [2] that the space $\mathcal{H}_k(M)$ of closed integral k-currents modulo exact integral k-currents, is discreet with respect to the topology induced by the mass norm, and isomorphic to singular integral k-homology. Moreover the natural map

$$q_k : \mathcal{I}_k(M) \to \mathcal{H}_k(M)$$

is continuous. Thus $\{q_2(u_{i_k})\}$ is eventually constant, which means that u_{i_k} are eventually in class 0, which is a contradiction, since all u_i have positive symplectic area. □

We point out here that a partial extension of the above can be made to l. c. s. manifolds.

Theorem 1.2. *Given a closed* l. c. s. (M, ω), J *an* ω*-compatible almost complex structure there exists a constant* $\hbar = \hbar(\omega, J)$, *s.t. for any* $u : (\Sigma, j) \to M$ *a* J*-holomorphic immersion of a closed Riemann surface* (Σ, j), *the energy of the map satisfies:*

$$e(u) = \int_\Sigma |du|^2 dvol = \int_\Sigma u^* \omega \geq \hbar,$$

for vol *the measure induced by the ambient* (ω, J) *metric.*

Remark 1.3. Note that Theorem 1.1 is not particularly relevant in Gromov-Witten theory for $\Sigma \neq S^2$, because one tends to work relative to fixed homology classes, and so have automatic energy bounds, given by the symplectic form. This changes dramatically in l. c. s. geometry, because in this case there are no a priori bounds on energy, and we may need to work with trivial homology classes, how do we know that upon deforming J our holomorphic curves don't disapear into the energy floor, or into the energy sky, cf. [8], [3], [6]? Thus the above theorem is much more relevant in l. c. s. geometry, from the Gromov-Witten theory angle.

Proof. First we need the following.

Theorem 1.4. *For* $u : \Sigma \to M$ *a* J*-holomorphic immersion, as above, the mean curvature* $|\boldsymbol{H}|$ *of the image of* u *is bounded from above by a universal constant* $C(\omega, J) > 0$.

Proof. We may fix a finite cover of M by charts $\phi_i : U_i \subset \mathbb{R}^{2n} \to M$, $\phi_i^* \omega = e^{f_i} \omega_0$ with ω_0 the standard symplectic form, and with U_i contractible. Then $\phi_i^{-1} \circ u$ is a $\phi_i^* J$-holomorphic map defined on $u^{-1}(\phi_i(U_i))$ into $\mathbb{R}^{2n}$, and $\phi_i^* J$ is compatible with $f_i \omega_0$ and hence with ω_0. Fix a cover of Σ by disk domains $\{V_{i_j}\}$, with each $V_{i_j} \subset u^{-1}(\phi_i(U_i))$ for some i. Then $\phi_i^{-1} \circ u|_{V_{i_j}}$ is an immersed $\phi_i^* J$-holomorphic curve in $\mathbb{R}^{2n}$. So $(\omega_0, \phi_i^* J)$ is an almost Kahler manifold, and the image D_{i_j} of $\phi_i^{-1} \circ u|_{V_{i_j}}$ is hence a minimal surface, with respect to the metric $(\omega_0, \phi_i^* J)$, since ω is then a calibration, Harvey-Lawson [4]. Thus D_{i_j} has mean curvature 0. Since the cover $\{U_i\}$ is finite, the C^∞ norm of the functions f_i is universally bounded: $A < |f_i| < B$, for some some A, B and all i. It follows that the magnitude of the mean curvature of the surface D_{i_j} with respect to the metric induced by $(e^{f_i} \omega_0, \phi_i^* J)$ is bounded from above by some universal constant $C(\omega, J)$. Since the distortion of the mean curvature corresponding to the conformal distortion e^{f_i} can be readily bounded in terms of the functions f_i and hence A, B. Consequently the mean curvature of image $u|_{V_{i_j}}$ with respect to (ω, J), is likewise bounded by $C(\omega, J)$, from which the result follows. □

Now let ϵ be the Lebesgue covering number of $\{U_i\}$ with respect to the metric g. Combining Lemma 1.4 and Theorem 0.2 we get that for u non-constant as in the hypothesis if area$(u) < \hbar$ then diam$(u) < \epsilon$, for some $\hbar$ independent of u. Consequently the image of u is contained in some U_i, and so $\phi_i^{-1} \circ u$ is a $\phi_i^* J$-holomorphic map of a sphere into the almost Kahler contractible manifold $(U_i, \omega_0, \phi_i^* J)$ and so must be constant, which is a contradiction. □

Question 1. Can the condition of u being an immersion in the statement of Theorem 1.2 be replaced by u being non-constant?

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E-mail address: yasha.savelyev@gmail.com

University of Colima, CUICBAS