

# UNTWISTED GROMOV-WITTEN INVARIANTS OF RIEMANN-FINSLER MANIFOLDS

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**ABSTRACT.** We define a  $\mathbb{Q}$ -valued deformation invariant of certain complete Riemann-Finsler manifolds, in particular of complete Riemannian manifolds with non positive sectional curvature. It is proved that every rational number is the value of this invariant for some compact Riemannian manifold. We use this to find the first and mostly sharp generalizations, to non-compact products and fibrations, of Preissman's theorem on non-existence of negative sectional curvature metrics on compact products. For example,  $\Sigma \times T^n$  admits a metric of negative sectional curvature, where  $\Sigma$  is a non-compact possibly infinite type surface, if and only if  $\Sigma$  has genus zero. We also give novel estimates on counts of closed geodesics with restrictions on multiplicity. Along the way, we also prove that sky catastrophes of smooth dynamical systems are not geodesible by a certain class of forward complete Riemann-Finsler metrics, in particular by complete Riemannian metrics with non-positive sectional curvature. This partially answers a question of Fuller and gives important examples for our theory here.

## 1. INTRODUCTION

We will define certain rational number valued deformation invariants for certain complete Riemann-Finsler manifolds, and in particular for all complete Riemannian manifolds with non-positive sectional curvature. These invariants can be directly interpreted as the untwisted part of certain elliptic Gromov-Witten invariants in an associated lcs manifold, [14]. The twist is coming by way of metric isometries. In the more basic, untwisted setting here, we reduce the invariants to counts of *geodesic strings* (equivalence classes of closed, unit speed geodesics up to reparametrization  $S^1$  action), via the Fuller index [6].

The latter count in modern terms is just an orbifold count of points of virtual dimension 0 Kuranishi spaces, corresponding to spaces of geodesic strings in a fixed free homotopy class. But Fuller's construction makes the latter elementary and geometric.

Indeed, one of the main ideas of this note is that one can make geodesic string counts into a global metric deformation invariant, provided we work with some metric curvature restrictions. Studying connections with metric geometry was suggested by Fuller himself in the 1960's. We now discuss some applications.

Recall that a now classical theorem of Preissman [11] implies that there are no non-trivial compact products with negative sectional curvature. We give the first generalizations of this to the non-compact case. As a basic example:

**Theorem 1.1.** *There is a complete negative sectional curvature metric  $g$  on  $M = \Sigma \times T^n$ , where  $\Sigma$  is a non-compact, possibly infinite type surface, if and only if  $\Sigma$  has genus zero.*

When the genus is zero, existence of such a  $g$  is well known. For example, if  $\Sigma$  is the infinite cylinder  $g$  can be given as the warped product metric.

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If in addition we require that  $g$  be finite volume and that  $M$  be the interior of a compact manifold with boundary, one might try to prove that case of the theorem by the following strategy. Assume such a  $g$  exists. Use “doubling” to get a negatively curved metric on the compact double. Apply Preissman’s theorem to get a contradiction.

*Remark 1.2.* We cannot just double, one needs to do a surgery on ends, glue and deform to a smooth negatively curved metric. For example, doubling of cusped hyperbolic surfaces is already complex.

If  $\Sigma$  is infinite type, for example the standard infinite genus surface, the idea above totally fails. Our argument does not use Preissman’s theorem, and is inherently based on functional analysis and infinite dimensional Morse theory in particular.

The only if part of the above theorem is just a very special case of the following. We denote by  $\pi_1^{inc}(X)$  the set of free homotopy classes of loops incompressible to the ends, in the sense of Definition 2.1, (non-constant classes when  $X$  is closed).

**Theorem 1.3.** *Let  $X = Z \times Y$  where  $Z, Y$  admit complete Riemannian metrics with non-positive sectional curvature,  $Y$  is closed,  $\pi_1^{inc}(Z) \neq 0$ ,  $\chi(Y) \neq \pm 1$ . Then:*

- *$X$  does not admit a complete metric of negative sectional curvature, or a forward complete Finsler metric with negative flag curvature.*
- *Moreover,  $X$  does not even admit a forward complete Finsler metric with a unique and non-degenerate class  $\beta$  geodesic string for any  $\beta \in \pi_1^{inc}(Z)$ .*

This theorem is very close to being sharp, for example the conclusion of the corollary is false if  $Y = S^1$  and  $X = S^1 \times \mathbb{R}$ . As  $Z = T^2 \times \mathbb{R}$  admits the warped product metric  $g_{T^2} \times_{e^t} g_{\mathbb{R}}$  (with respect to the function  $f = e^t$  on  $\mathbb{R}$ ) where  $g_{T^2}, g_{\mathbb{R}}$  are the flat metrics. This warped product has constant negative sectional curvature  $-1$ . So it is essential that not only  $\pi_1(Z) \neq 0$  but also that there is a class incompressible to the ends. Of course,  $\chi(Y) = 1$  is also obviously essential, otherwise we may take  $Y = pt$ . The condition  $\chi(Y) \neq -1$  is however not obviously essential.

We will also give various generalizations of this result to fibrations. For fibrations, the most obvious analogue of Preissman’s theorem fails even assuming compactness. In fact, every closed 3-manifold  $X^3$ , for which there is no injection  $\mathbb{Z}^2 \rightarrow \pi_1(X, x_0)$ , and which fibers over a circle has a hyperbolic structure  $g_h$ , Thurston [15].

A more direct type of application is to geodesic counting with constraints on multiplicity. This is based on the idea that our Gromov-Witten type invariant is a metric deformation invariant provided there is a curvature control. For a non-constant class  $\beta \in \pi_1^{inc}(X)$ , we say that a metric  $g$  on  $X$  is  $\beta$ -**regular** if all of its class  $\beta$  closed geodesics are non-degenerate (meaning that the closed orbits of the associated geodesic flow are non-degenerate, i.e. the associated fixed points are non-degenerate.) For a geodesic string  $o$ ,  $\text{mult}(o)$  will denote its multiplicity, (the order of the corresponding isotropy subgroup of  $S^1$ , where  $S^1$  is acting by reparametrization.) And  $\text{morse}(o)$  will denote the Morse index of  $o$ , (meaning the Morse-Bott index of the associated critical submanifold of the loop space.)

**Definition 1.4.** *We say that a pair  $g_0, g_1$  of forward complete Finsler metrics with non-positive flag curvature are **Hadamard equivalent** if there is a continuous interpolation  $\{g_t\}$ ,  $t \in [0, 1]$ , (with respect to the topology of  $C^0$  convergence on compact sets) s.t. each  $g_t$  has non-positive flag curvature.*

We avoid maximal generality in what follows, as these are just sample applications.

**Theorem 1.5.** *Suppose that  $g_1$  is a forward complete, non-positive flag curvature metric on  $X$ , Hadamard equivalent to a forward complete metric of negative flag curvature. Suppose that  $\beta \in \pi_1^{inc}(X)$*

is a  $k$ -power, (Definition 2.9). Let  $C_\beta = 2 \cdot L_\beta$ , where  $L_\beta$  is the length of a class  $\beta$ ,  $g_1$ -geodesic. For all  $\epsilon > 0$  sufficiently small, whenever  $g'$  is  $C^0$   $\epsilon$  close to  $g_1$ , and is  $\beta$ -regular, we have:

$$\sum_{o \in \mathcal{O}_{C_\beta}(g', \beta)} \frac{(-1)^{\text{morse}(o)}}{\text{mult}(o)} = \frac{1}{k},$$

where  $\mathcal{O}_{C_\beta}(g', \beta)$  is the set of class  $\beta$  geodesic strings with  $g'$ -length less than  $C_\beta$ .

We may use this to estimate the number of closed geodesic strings with constraints on multiplicity and energy. Here is a sample immediate corollary.

**Corollary 1.6.** *Let  $X$ ,  $g_0$  and  $g_1$  be as above. Suppose that  $\beta \in \pi_1^{\text{inc}}(X)$  is a  $p$ -power for  $p$  a prime. Then for all  $\epsilon > 0$  sufficiently small, whenever  $g'$  is  $C^0$   $\epsilon$  close to  $g_1$ , and has a class  $\beta$  geodesic string with  $g'$ -length less than  $C_\beta$ , and with multiplicity  $k \neq p$ , then  $g'$  has at least two such geodesic strings.*

The following theorem in particular says that for regular metrics on  $T^2$  sufficiently nearby to the flat metric, fixed class geodesic strings with “small length” must come at least in pairs, corresponding to multiplicity constraints.

**Theorem 1.7.** *Let  $g_0$  be the standard flat metric  $X = T^n$ . For all  $\epsilon > 0$  sufficiently small, whenever  $g'$  is a Finsler metric  $C^0$   $\epsilon$  close to  $g_0$  and is  $\beta$ -regular, the following holds. Suppose  $g'$  has a closed geodesic string  $o$  s.t.*

- (1)  $o$  has class  $\beta$ .
- (2)  $p \mid \text{mult}(o)$ , where  $p$  is prime.
- (3)  $g'$ -length of  $o$  is less than  $C_\beta$ .

*Then  $g'$  has at least one other geodesic string satisfying these conditions. Moreover, any Finsler metric  $g_1$  Hadamard equivalent to  $g_0$  has the same property as  $g_0$ , above.*

We will discuss the statements in more detail in what follows.

## 2. SETUP AND STATEMENTS

**Terminology 1.** From now on, all our metrics are Riemann-Finsler (a.k.a. Finsler) metrics unless specified to be Riemannian, and usually denoted by just  $g$ . Completeness, always means forward completeness, and it is an assumption for all our metrics. Curvature always means sectional curvature in the Riemannian case and flag curvature in the Finsler case. Thus we will usually just say complete metric  $g$ , for a forward complete Riemann-Finsler metric. A reader may certainly choose to interpret all metrics as Riemannian metrics, completeness as standard completeness, and curvature as sectional curvature.

In what follows  $\pi_1(X)$  denotes the set of free homotopy classes of continuous maps  $o : S^1 \rightarrow X$ .

**Definition 2.1.** *Let  $X$  be a smooth manifold. Fix an exhaustion by nested compact sets  $\bigcup_{i \in \mathbb{N}} K_i = X$ ,  $K_i \supset K_{i-1}$  for all  $i \geq 1$ . We say that a class  $\beta \in \pi_1(X)$  is end compressible if  $\beta$  is in the image of*

$$\text{inc}_* : \pi_1(X - K_i) \rightarrow \pi_1(X)$$

*for all  $i$ , where  $\text{inc} : X - K_i \rightarrow X$  is the inclusion map. We say that  $\beta$  is **end incompressible** (or **incompressible to the ends**) if it is not end compressible.*

Let  $\pi_1^{inc}(X)$  denote the set of such end incompressible classes. When  $X$  is compact, we set  $\pi_1^{inc}(X) := \pi_1(X) - \text{const}$ , where  $\text{const}$  denotes the set of homotopy classes of constant loops.

It is easily seen that the above is well defined (independent of the choice of an exhaustion) and moreover any homeomorphism  $X_1 \rightarrow X_2$  of a pair of manifolds induces a set isomorphism  $\pi_1^{inc}(X_1) \rightarrow \pi_1^{inc}(X_2)$ . Denote by  $L_\beta X$  the class  $\beta \in \pi_1^{inc}(X)$  component of the free loop space of  $X$ , with its compact open topology. Let  $g$  be a complete metric on  $X$ , and let  $S(g, \beta) \subset L_\beta X$  denote the subspace of all unit speed parametrized, smooth, closed  $g$ -geodesics in class  $\beta$ .

**Definition 2.2.** We say that a metric  $g$  on  $X$  is  $\beta$ -**taut** if it is complete and  $S(g, \beta)$  is compact. We will say that  $g$  is **taut** if it is  $\beta$ -taut for each  $\beta \in \pi_1^{inc}(X)$ .

**Lemma 2.3.** A complete metric  $g$  with non-positive curvature satisfies:

- All of its closed geodesics are minimizing in their free homotopy class.
- It is taut.

*Proof.* The first part is a standard consequence of the Cartan-Hadamard theorem. The second part follows by the first part and Lemma 6.1.  $\square$

It should be emphasized that taut metrics form a much larger class of metrics than just non-positive curvature metrics. For example any sufficiently  $C^1$  small perturbation of a metric with non-positive curvature will be taut. (Indeed, this is crucial for the construction of our invariant.) Another class of examples comes by way of Lemma 2.18 ahead, these metrics may not be non-positively curved nor nearby to metrics non-positively curved.

**Definition 2.4.** Let  $\beta \in \pi_1^{inc}(X)$ , and let  $g_0, g_1$  be a pair of  $\beta$ -taut metrics on  $X$ . A  $\beta$ -**taut deformation** between  $g_0, g_1$ , is a continuous (in the topology of  $C^0$  convergence on compact sets) family  $\{g_t\}$ ,  $t \in [0, 1]$  of complete metrics on  $X$ , s.t.

$$S(\{g_t\}, \beta) := \{(o, t) \in L_\beta X \times [0, 1] \mid o \in S(g_t, \beta)\}$$

is compact. We say that  $\{g_t\}$  is a **taut deformation** if it is  $\beta$ -taut for each  $\beta \in \pi_1^{inc}(X)$ . The above definitions of tautness are extended naturally to the case of a smooth fibration  $X \hookrightarrow P \rightarrow [0, 1]$ , with a smooth fiber-wise family of metrics.

A useful criterion for  $\beta$ -tautness is the following.

**Theorem 2.5.** Let  $\{g_t\}_{t \in [0, 1]}$  be a continuous family of complete metrics on  $X$ . Suppose that:

$$\sup_t \left| \max_{o \in S(g_t, \beta)} l_{g_t}(o) - \min_{o \in S(g_t, \beta)} l_{g_t}(o) \right| < \infty,$$

where  $l_{g_t}$  is the length functional with respect to  $g_t$ , then  $\{g_t\}$  is  $\beta$ -taut. It follows that sky catastrophes of vector fields on closed manifolds are not geodesible by metrics all of whose geodesics are minimal, Appendix A.1.

For example, the hypothesis is trivially satisfied if  $g_t$  have the property that all their class  $\beta$  closed geodesics are minimal in their homotopy class.

**Corollary 2.6.** If  $g_t$ ,  $t \in [0, 1]$  have non-positive curvature then  $\{g_t\}$  is taut.

*Proof.* This follows by the theorem and by Lemma 2.3.  $\square$

Fuller at the end of [6] has asked for any metric conditions on vector fields to rule out sky catastrophes, see Appendix A.1. By the above, non-positivity of curvature is one such condition. So this is a partial answer to his question.

Note that if sky catastrophes were never geodesible (or at least if geodesible sky catastrophes are necessarily unstable, as was conjectured in [13]) then the geodesible Seifert conjecture would follow, by the main result of [13]. Hence, this is a subtle question. The qualitative structure of such potential geodesible or Reeb sky catastrophes is somewhat understood, [13, Theorem 1.10]. But this does not greatly aid constructing potential examples, which must be topologically very complex, (there are necessarily infinitely many suitably synchronized bifurcation events). No results prior to the theorem above are known to me aside from those mentioned by Fuller himself in [6].

**2.1. The geodesic counting invariant  $F$ .** Let  $\mathcal{G}(X)$  be the set of equivalence classes of taut metrics  $g$ , where  $g_0$  is equivalent to  $g_1$  whenever there is a taut deformation between them. We may denote an equivalence class by its representative  $g$  by a slight abuse of notation.

**Theorem 2.7.** *For each manifold  $X$  there is a natural, non-trivial functional:*

$$F : \mathcal{G}(X) \times \pi_1^{inc}(X) \rightarrow \mathbb{Q}.$$

The value  $F(g, \beta)$  is a certain weighted count of the set of closed  $g$ -geodesic strings in class  $\beta$ . But one must take care of exactly how to count, as in general this set should be understood as an orbifold or rather a Kuranishi space (as introduced by Fukaya-Ono [4]), hence this is why  $F$  is  $\mathbb{Q}$  valued. In the special case when  $g$  is  $\beta$ -regular we have the following formula:

$$F(g, \beta) = \sum_{o \in \mathcal{O}(g, \beta)} \frac{(-1)^{\text{morse}(o)}}{\text{mult}(o)},$$

where  $\text{morse}(o)$  and  $\text{mult}(o)$  are as in the Introduction.

*Question 1.* Do there exist a pair of taut metrics  $g_1, g_2$  on a manifold  $X$  which are not taut homotopic? Or more generally not homotopic via a family of metrics without a sky catastrophe?

Probably both possibilities are interesting. If the answer is ‘no’ then we can obtain much sharper applications of Theorem 2.7, particularly in the setup of [14]. Moreover, in this case  $F$  becomes a topological invariant, which is a priori unrelated to any classical topological invariant. On the other hand:

**Corollary 2.8.** *Suppose for a pair  $g_1, g_2$  of  $\beta$ -taut metrics on  $X$ :*

$$F(g_1, \beta) \neq F(g_2, \beta),$$

*then any path  $\{g_t\}$ , connecting  $g_0, g_1$ , is not  $\beta$ -taut and in fact has a sky catastrophe. So that if such a pair  $g_1, g_2$  exists the conjecture of [13] would be disproved.*

*Proof.* The fact that any connecting  $\{g_t\}$  is not  $\beta$ -taut is just a direct corollary of the theorem above. The fact that  $\{g_t\}$  has a sky catastrophe follows by [13, Theorem 3.2].  $\square$

If the answer to the question above is ‘yes’, then we should be able to use the above corollary to find non-positively curved metrics which cannot be joined by a continuous family of non-positively curved metrics. (Apparently, existence of such metrics is open.)

**Definition 2.9.** *Let  $\beta \in \pi_1(X)$ . For any based point  $x_0 \in \text{image } \beta \subset X$  (for  $\text{image } \beta$  the image of some representative of  $\beta$ ) there is a naturally determined element  $\beta_{x_0} \in \pi_1(X, x_0)$  well defined up to an inner automorphism, (concatenate a representative of  $\beta$  with a path from  $x_0$  to a point in  $\text{image } \beta$ ).*

- We say that a class  $\beta \in \pi_1(X, x_0)$  is **not a power** if whenever  $\beta = \alpha^k$  for some  $\alpha, k > 0$  then  $k = 1$ .
- We say that a class  $\beta \in \pi_1(X, x_0)$  is a **k-power** if  $\beta = \alpha^k$  for some  $\alpha$  which is not a  $n$ -power for any  $n$ .
- We say that  $\beta$  is **atomic** if it is a  $k$ -power for some  $k$ .<sup>1</sup>
- We say that  $\beta \in \pi_1(X)$  is not a power, respectively is a  $k$ -power, respectively is atomic if for any  $x_0$  as above,  $\beta_{x_0}$  is not a power, respectively is a  $k$ -power, respectively is atomic.

Note that if a class  $\beta \in \pi_1^{inc}(X)$  is not a power then any representative of this class is not multiply covered, but the converse generally does not hold.

*Example 1.* Let  $g$  be a Riemannian metric with negative sectional curvature on a closed manifold  $X$  and  $\beta \in \pi_1(X)$  a class represented by a multiplicity  $n$  closed geodesic, then

$$(2.10) \quad F(g, \beta) = \frac{1}{n}.$$

In particular, if  $\beta$  is not a power then  $F(g, \beta) = 1$ . More generally, (2.10) holds whenever  $g$  has a unique and non-degenerate geodesic string in class  $\beta$ , where non-degenerate is as in the Introduction.

**Theorem 2.11.** *Every rational number has the form  $F(g, \beta)$  for some  $\beta$ -taut Riemannian  $g$  on some compact manifold  $X$  and for some  $\beta \in \pi_1^{inc}(X)$ .*

If  $\beta \in \pi_1^{inc}(X)$  is not a power, then it is easy to see that the reparametrization  $S^1$  action on  $L_\beta X$  is free (see Appendix A), so that  $H_*^{S^1}(L_\beta X, \mathbb{Z}) \simeq H_*(L_\beta X/S^1, \mathbb{Z})$ , where  $H_*^{S^1}(L_\beta X, \mathbb{Z})$  denotes the  $S^1$ -equivariant homology. Moreover, we have:

**Theorem 2.12.** *Suppose that  $\beta \in \pi_1^{inc}(X)$  is not a power, and  $X$  admits a  $\beta$ -taut metric, then  $H_*^{S^1}(L_\beta X, \mathbb{Z})$  is finite dimensional. Denote by  $\chi^{S^1}(L_\beta X)$  the Euler characteristic of this homology. Then for any  $\beta$ -taut metric  $g$  on  $X$ :*

$$F(g, \beta) = \chi^{S^1}(L_\beta X).$$

Explicit examples for the theorem above can be found by the proof of Theorem 2.11. For these types of examples any negative integer may appear as the value of  $F(g, \beta)$ . We leave out the details.

*Remark 2.13.* If  $\beta$  is a power, the idea behind Theorem 2.12 breaks down, as the  $S^1$ -equivariant homology of  $L_\beta X$  may then be infinite dimensional even if  $X$  admits a  $\beta$ -taut  $g$ . As a trivial example this homology is already infinite dimensional when  $g$  is negatively curved, and the class  $\beta$  geodesic is  $k$ -covered, as then this homology is the group homology of  $\mathbb{Z}_k$ . In particular the connection with the Euler characteristic a priori breaks down. It is thus an interesting open problem if the functional  $F$  remains topological, this is related to question 1.

**2.2. Applications to existence of negative curvature metrics.** A celebrated theorem of Preissman [11] says that there are no negative sectional curvature metrics on compact products. Fibration counterexamples to Preissman's product theorem certainly exist as mentioned in the Introduction. We are going to give a certain generalization of Preissman's theorem to fibrations, with possibly non-compact fibers, also replacing the negative sectional curvature condition by a significantly weaker condition.

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<sup>1</sup>As we are working with non-compact manifold, we may in principle have non atomic classes.

**Definition 2.14.** Let  $Z \hookrightarrow X \xrightarrow{p} Y$  be a smooth fiber bundle with  $X$  having a  $\beta$ -taut Riemannian metric  $g$ , for  $\beta \in \pi_1^{inc}(X)$ , and let  $g_Y$  be a metric on  $Y$ . Suppose that:

- (1) The fibers  $Z_y = p^{-1}(y)$  are totally  $g$ -geodesic, for closed geodesics in class  $\beta$ . We denote by  $g_y$  the metric  $g$  restricted to  $Z_y$ .
- (2) The fibers are parallel (the distribution  $T^{vert}X = \ker p_*$  is parallel along any smooth curve in  $X$  with respect to the Levi-Civita connection of  $g$ ).
- (3) For any pair of fibers  $(Z_{y_0}, g_{y_0})$ ,  $(Z_{y_1}, g_{y_1})$ , and a path  $\gamma : [0, 1] \rightarrow Y$  from  $y_0$  to  $y_1$  the fiber family  $\{(Z_{\gamma(t)}, g_{\gamma(t)})\}$  furnishes a taut deformation.
- (4)  $p$  projects  $g$ -geodesics to geodesics of  $Y, g_Y$ .

We then call  $p : X \rightarrow Y$  a  **$\beta$ -taut fibration**, with the metrics  $g, g_Y$  and  $g_Z$  all possibly implicit.

**Definition 2.15.** For  $Z \hookrightarrow X \rightarrow Y$  as above, we say that  $\beta \in \pi_1(X)$  is a **fiber class** if it is in the image of the inclusion  $i_Z : \pi_1(Z) \rightarrow \pi_1(X)$ .

In the above definition of a taut fibration and the following theorem we need the auxiliary metric  $g$  on  $X$  to be Riemannian, and there is no obvious extension of the theorem to the Riemann-Finsler case. However, the conclusions of the theorem are for Riemann-Finsler metrics.

**Theorem 2.16.** Let  $p : (X, g) \rightarrow (Y, g_Y)$  be a  $\beta$ -taut fibration, where  $\beta \in \pi_1^{inc}(X)$  is a fiber class. Suppose further that  $Y$  is connected, closed,  $\chi(Y) \neq \pm 1$  and is such that all smooth closed contractible  $g_Y$ -geodesics in  $Y$  are constant. Then the following holds:

- $X$  does not admit a complete Riemann-Finsler metric with negative curvature.
- Moreover,  $X$  does not admit a complete Riemann-Finsler metric with a unique and non-degenerate class  $\beta$  geodesic string.

Note that  $\chi(Y) \neq 1$  is of course essential, as the trivial fibration  $X \rightarrow \{pt\}$ , with  $X$  admitting a complete negatively curved metric, will satisfy the hypothesis. The condition that there is a fiber class  $\beta \in \pi_1^{inc}(X)$  is also essential, for any vector bundle over a manifold admitting a Riemannian metric of negative curvature admits a metric of negative curvature, Anderson [1].

Theorem 1.3 gives one set of examples. A further basic set of examples for the theorem is obtained by starting with any homomorphism

$$(2.17) \quad \phi : \pi_1(Y, y_0) \rightarrow \text{Isom}(Z, g_Z), \quad (\text{the group of all isometries}).$$

where  $g_Z$  is a taut Riemannian metric, and there is a class  $\beta_Z \in \pi_1^{inc}(Z)$  (for example  $(Z, g_Z)$  is a non-simply connected complete hyperbolic surface). Suppose further:

- (1) The orbit

$$O := \bigcup_{\gamma \in \pi_1(Y, y_0)} \phi_*(\gamma)(\beta_Z)$$

is finite.

- (2)  $Y$  is closed and connected.
- (3) All contractible smooth closed  $g_Y$  geodesics in  $Y$  are constant.

We have the obvious induced diagonal action

$$\begin{aligned} \pi_1(Y, y_0) &\rightarrow \text{Diff}(Z \times \tilde{Y}), \quad (\text{the group of all diffeomorphisms}), \\ \gamma &\mapsto ((z, y) \mapsto (\phi(\gamma)(z), \gamma \cdot y)), \end{aligned}$$



for  $\tilde{Y}$  the universal cover of  $Y$ . Taking the quotient of  $Z \times \tilde{Y}$  by this action, we get an associated “flat” bundle  $Z \hookrightarrow X_\phi \xrightarrow{p} Y$ , with a metric  $g_\phi$  induced from the product metric  $\tilde{g} = g_Z \oplus g_Y$ , on the covering space  $q : Z \times \tilde{Y} \rightarrow Z \times Y$ .

**Lemma 2.18.** *Let  $p : (X_\phi, g_\phi) \rightarrow (Y, g_Y)$  be as above, then this is a  $\beta$ -taut fibration, where  $\beta = i_*(\beta_Z)$ , for  $i_* : \pi_1^{\text{inc}}(Z) \rightarrow \pi_1^{\text{inc}}(X_\phi)$  induced by inclusion.*

By the lemma above,  $p : (X_\phi, g_\phi) \rightarrow (Y, g_Y)$  satisfies the hypothesis of the theorem above. Yet more concretely:

*Example 2.* Suppose we have  $\beta_Z \in \pi_1^{\text{inc}}(Z)$ , and let  $\phi : Z \rightarrow Z$  be an isometry of a taut metric  $g_Z$ . Then by the construction above, the mapping torus

$$(Z, g_Z) \hookrightarrow (X_\phi, g_\phi) \xrightarrow{\pi} S^1$$

has the structure of a  $\beta$ -taut fibration, satisfying the hypothesis of the theorem, for  $\beta = i_*(\beta_Z)$  as above.

The next corollary of Theorem 2.16 is immediate.

**Corollary 2.19.** *Let*

$$(Z_{g,Z}) \hookrightarrow (X_\phi, g_\phi) \rightarrow (Y, g_Y)$$

*be as in the construction above for  $Z, g_Z$  having non-positive curvature, and let  $\beta_Z \in \pi_1^{\text{inc}}(Z)$ . Then if  $\chi(Y) \neq \pm 1$ :*

- (1)  $X_\phi$  does not admit a complete Riemann-Finsler metric with negative curvature.
- (2) Moreover,  $X_\phi$  does not admit a Riemann-Finsler metric with a unique and non-degenerate class  $\beta$  geodesic string, for  $\beta = i_*(\beta_Z)$  as above.

*As a special case, this applies to the mapping tori  $X_\phi$ , for  $\phi : Z \rightarrow Z$  an isometry of a complete Riemannian non-positively curved metric on  $Z$ , satisfying the finiteness condition 1. (The non-positive curvature hypothesis is for concreteness we may of course replace this condition by tautness.)*

In the special case when  $Z$  is compact, the first part of the above corollary can be deduced, with some work, from Preissman’s theorem (specifically, because of the condition 1), see also [3, Theorem 9.3.4] for a generalization that fits our Finsler setting. The second part is new even when  $Z$  is compact. In the non-compact case, as far as I know, the above corollary and the more basic Theorem 1.3, are the only presently known extensions of Preissman’s theorem.

### 3. PROOF OF THEOREM 2.5

The first part of the theorem clearly follows by the second part. So let  $\{g_t\}$ ,  $t \in [0, 1]$  be as in the hypothesis, with

$$(3.1) \quad \sup_t \left| \max_{o \in S(g_t, \beta)} l_{g_t}(o) - \min_{o \in S(g_t, \beta)} l_{g_t}(o) \right| < C,$$

and suppose that

$$\sup_{(o,t) \in \mathcal{O}(\{g_t\}, \beta)} l_{g_t}(o) = \infty.$$

Then we have a sequence  $\{o_k\}$ ,  $k \in \mathbb{N}$ , of closed class  $\beta$   $g_{t_k}$ -geodesics in  $X$ , satisfying:

- (1)  $\lim_{k \rightarrow \infty} t_k = t_\infty \in [0, 1]$ .
- (2)  $\lim_{k \rightarrow \infty} l_{g_{t_k}}(o_k) = \infty$ , where  $l_{g_{t_k}}(o_{t_k})$  is the length with respect to  $g_{t_k}$ .



Let  $o_\infty$  be a minimal, class  $\beta$ ,  $g_\infty = g_{t_\infty}$  geodesic in  $X$ . And let  $L$  denote its length  $g_\infty$  length. Let  $g_{aux}$  be a fixed auxiliary metric on  $X$ , and let  $L_{aux}$  be the  $g_{aux}$  length of  $o_\infty$ .

Define a pseudo-metric  $d_{C^0}$  on the space of metrics on  $X$  as follows. Set  $K = \text{image } o_\infty$ . And set

$$V \subset TX = \{v \in TX \mid \pi(v) \in K \text{ for } \pi : TX \rightarrow X \text{ the canonical projection, and } |v|_{aux} = 1\},$$

where  $|v|_{aux}$  is the norm taken with respect to  $g_{aux}$ .

Then define:

$$d_{C^0}(g_1, g_2) = \sup_{v \in V} ||v|_{g_1} - |v|_{g_2}|.$$

By Properties 1 and 2 we may find a  $k > 0$  such that:

$$(3.2) \quad d_{C^0}(g_{t_k}, g_{t_\infty}) < \epsilon$$

and

$$(3.3) \quad l_{g_{t_k}}(o_k) > C + L + L_{aux} \cdot \epsilon.$$

By (3.2), we have:

$$l_{g_{t_k}}(o_\infty) < l_{g_{t_\infty}}(o_\infty) + L_{aux} \cdot \epsilon = L + L_{aux} \cdot \epsilon.$$

Combining with (3.3) we get:

$$l_{g_{t_k}}(o_k) > l_{g_{t_k}}(o_\infty) + C.$$

Since we may find a closed  $g_{t_k}$ -geodesic  $o'$  satisfying  $l_{g_{t_k}}(o') \leq l_{g_{t_k}}(o_\infty)$ , we get that

$$|\max_{o \in S(g_{t_k}, \beta)} l_{g_{t_k}}(o) - \min_{o \in S(g_{t_k}, \beta)} l_{g_{t_k}}(o)| > C,$$

and so we are in contradiction.

Thus,

$$\sup_{(o, t) \in \mathcal{O}(\{g_t\}, \beta)} l_{g_t}(o) < \infty.$$

It follows, by an analogue of Lemma 6.2, that the images of all elements  $o \in S(\{g_t\}, \beta)$  are contained in a fixed compact  $T \subset X$ . Compactness of  $S(\{g_t\}, \beta)$  then readily follows by the Arzella-Ascoli theorem.  $\square$

#### 4. PROOF OF LEMMA 2.18

Let  $\phi_* : \pi_1(Y, y_0) \rightarrow \text{Aut}(\pi_1^{inc}(Z))$  be the natural induced action, where  $\text{Aut}(\pi_1^{inc}(Z))$  denotes the group of set isomorphisms of  $\pi_1^{inc}(Z)$ . And such that the orbit

$$O := \bigcup_{\gamma \in \pi_1(Y, y_0)} \phi_*(\gamma)(\beta_Z)$$

is finite.

As  $g_Z$  is taut,  $S(g_Z, \phi_*(\gamma)(\beta_Z))$  is compact for each  $\gamma$ , where  $S(g_Z, \phi_*(\gamma)(\beta_Z))$  is the space of geodesics as in Definition 2.2. By the condition on contractible geodesics of  $g_Y$ , we get:

$$\begin{aligned} S(g_\phi, \beta) &= q_*(S(g_Z \oplus g_Y, \beta)) \\ &= \bigcup_{\beta \in O} q_*(S(g_Z, \beta) \times \tilde{Y}), \end{aligned}$$

for  $q_* : L(Z \times \tilde{Y}) \rightarrow L(Z \times Y)$  induced by the quotient map  $q : Z \times \tilde{Y} \rightarrow Z \times Y$ , (as in the preamble to the statement of the lemma) and where  $S(g_Z, \gamma) \times \tilde{Y}$  is understood as a subset

$$S(g_Z, \gamma) \times \tilde{Y} \subset L(Z) \times \tilde{Y} \subset L(Z \times \tilde{Y}).$$

Given that  $O$  is finite, this then readily implies our claim.  $\square$

## 5. PRELIMINARIES ON REEB FLOW

Let  $(C^{2n+1}, \lambda)$  be a contact manifold with  $\lambda$  a contact form, that is a one form s.t.  $\lambda \wedge (d\lambda)^n \neq 0$ . Denote by  $R^\lambda$  the Reeb vector field satisfying:

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

Recall that a **closed  $\lambda$ -Reeb orbit** (or just Reeb orbit when  $\lambda$  is implicit) is a smooth map

$$o : (S^1 = \mathbb{R}/\mathbb{Z}) \rightarrow C$$

such that

$$\dot{o}(t) = cR^\lambda(o(t)),$$

with  $\dot{o}(t)$  denoting the time derivative, for some  $c > 0$  called period. Let  $S(R^\lambda, \beta)$  denote the space of all closed  $\lambda$ -Reeb orbits in free homotopy class  $\beta$ , with its compact open topology. And set

$$\mathcal{O}(R^\lambda, \beta) = S(R^\lambda, \beta)/S^1,$$

where  $S^1 = \mathbb{R}/\mathbb{Z}$  acts by reparametrization  $t \cdot o(\tau) = o(t + \tau)$ .

## 6. DEFINITION OF THE FUNCTIONAL F AND PROOFS OF AUXILIARY RESULTS

Let  $X$  be a manifold with a taut metric  $g$ . Let  $C$  be the unit cotangent bundle of  $X$ , with its Liouville contact 1-form  $\lambda_g$ . If  $o : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow X$  is a unit speed closed geodesic, it has a canonical lift  $\tilde{o} : S^1 \rightarrow C$ . If  $\beta \in \pi_1^{inc}(X)$ , let  $\tilde{\beta} \in \pi_1(C)$  denote class  $[\tilde{o}] \in \pi_1(C)$ , where  $o$  is a unit speed closed geodesic representing  $\beta$ .

Let  $S(R^{\lambda_g}, \tilde{\beta})$  be the orbit space as in Section 5, for the Reeb flow of the contact form  $\lambda_g$ . And set

$$\mathcal{O}_{g,\beta} = \mathcal{O}(R^{\lambda_g}, \tilde{\beta}) := S(R^{\lambda_g}, \tilde{\beta})/S^1,$$

i.e. this can be identified with the space of class  $\beta$   $g$ -geodesic strings. By the tautness assumptions  $\mathcal{O}_{g,\beta}$  is compact.

We then define

$$F(g, \beta) = i(\mathcal{O}_{g,\beta}, R^{\lambda_g}, \tilde{\beta}) \in \mathbb{Q}$$

where the right hand side is the Fuller index as outlined in the Appendix A. As a basic example we have:

**Lemma 6.1.** *Suppose that  $g$  is a complete metric on  $X$ , all of whose class  $\beta \in \pi_1^{inc}(X)$  geodesics are minimal, then  $g$  is  $\beta$ -taut.*

*Proof.* First we state a more basic lemma.

**Lemma 6.2.** *Suppose that  $g$  is a complete metric on  $X$ ,  $\beta \in \pi_1^{inc}(X)$  and let  $S \subset L_\beta X$  be a subset on which the  $g$ -length functional is bounded from above. Then the images in  $X$  of elements of  $S$  are contained in a fixed compact subset of  $X$ .*

*Proof.* Suppose otherwise. Fix an exhaustion by nested compact sets

$$\bigcup_{i \in \mathbb{N}} K_i = X, \quad K_i \supset K_{i-1}.$$

Then either there is sequence  $\{o_i\}_{i \in \mathbb{N}}$ ,  $o_i \in S$  s.t.  $o_i \in K_i^c$ , for  $K_i^c$  the complement of  $K_i$ , which contradicts the fact that  $\beta$  is end incompressible. Or there is a sequence  $\{o_k\}_{k \in \mathbb{N}}$ ,  $o_k \in S$  s.t.:

- (1) Each  $o_k$  intersects  $K_{i_0}$  for some  $i_0$  fixed.
- (2) For each  $i \in \mathbb{N}$  there is a  $k_i > i$  s.t.  $o_{k_i}$  is not contained in  $K_i$ .

Now if  $\text{diam}(o_k)$  is bounded in  $k$ , then condition 1 implies that  $o_k$  are contained in a set of bounded diameter. (Here  $\text{diam}(o_k)$  denotes the diameter of image  $o_k$ .) Consequently, by Hopf-Rinow theorem [2],  $o_k$  are contained in a compact set. But this contradicts condition 2, and the fact that  $K_i$  form an exhaustion of  $X$ .

Thus, we conclude that  $\text{diam}(o_k)$  is unbounded, but this contradicts the hypothesis.  $\square$

Returning to the proof of the main lemma. By assumption, closed, class  $\beta \in \pi_1^{\text{inc}}(X)$  geodesics are  $g$ -minimizing in their homotopy class and in particular have fixed length. By the lemma above there is a fixed  $K \subset X$  s.t. every class  $\beta$  closed geodesic has image contained in  $K$ . Then compactness of  $S(g, \beta)$  follows by Arzella-Ascoli theorem.  $\square$

*Proof Theorem 2.7.* Let  $\beta \in \pi_1^{\text{inc}}(X)$ , be given and let  $g$  be  $\beta$ -taut. We just need to prove that  $F(g, \beta)$  is invariant under a  $\beta$ -taut deformation of  $g$ . So let  $\{g_t\}$ ,  $t \in [0, 1]$  be a  $\beta$ -taut deformation of metrics on a compact manifold  $X$ . Let  $R^{\lambda_{g_t}}$  be the geodesic flow on the  $g_t$  unit cotangent bundle  $C_t$ . Trivializing the family  $\{C_t\}$  we get a family  $\{R_t\}$  of flows on  $C \simeq C_t$ , with  $R_t$  conjugate to  $R^{\lambda_{g_t}}$ .

Let  $\mathcal{O}(\{R_t\}, \tilde{\beta})$  be the cobordism as in (A.2), where  $\tilde{\beta} \in \pi_1(C)$  is as above. Then  $\mathcal{O}(\{R_t\}, \tilde{\beta})$  is compact as  $S(\{g_t\}, \beta)$  is compact by assumption.

Basic invariance of the Fuller index, that is (A.3), immediately yields:  $F(g_0, \beta) = F(g_1, \beta)$ .  $\square$

## 7. PROOF OF THEOREM 1.5

We already know by Example 1 that  $F(g_0, \beta) = \frac{1}{k}$  and hence by Theorem 2.7  $F(g_1, \beta) = \frac{1}{k}$ . Let  $U$  denote the open subset of  $L_\beta X$  consisting of loops with  $g_1$ -length less than  $C_\beta$ . By [13, Lemma 4.1], for all  $\epsilon > 0$  sufficiently small, for any  $g'$ ,  $C^0$   $\epsilon$  close to  $g$  the following holds. Set  $g'_t = (t-1) \cdot g_1 + t \cdot g'$ , for  $t \in [0, 1]$ , then

$$\tilde{N} = \mathcal{O}(\{g'_t\}, \beta) \cap (U \times [0, 1])$$

is an open and compact subset of  $\mathcal{O}(\{g'_t\}, \beta)$ .

Now set

$$N_1 = \tilde{N} \cap (L_\beta X \times \{1\}),$$

and  $N_0 = \mathcal{O}(g_1, \beta)$ . By the invariance property (A.3) of the Fuller index, we then have that

$$\frac{1}{k} = i(N_0, R^{\lambda_{g_1}}) = i(N_1, R^{\lambda_{g'}}).$$

On the other hand, by construction and by index computations as in [13, Section 2]), we get:

$$i(N_1, R^{\lambda_{g'}}) = \sum_{o \in \mathcal{O}(g', \beta) \cap U} \frac{(-1)^{\text{morse}(o)}}{\text{mult}(o)}.$$

If  $\epsilon$  is chosen to be sufficiently small then  $\mathcal{O}(g', \beta) \cap U = \mathcal{O}_{C_\beta}(g', \beta)$ . So that we are done.  $\square$

## 8. PROOF OF THEOREM 1.7

Let  $\beta$  be a non-zero class, as in the statement. Let  $Y \subset T^n$  be a totally geodesic submanifold diffeomorphic to  $T^{n-1}$ , containing image  $\beta$ . Let  $\alpha \in H^1(T^n, \mathbb{Z})$  be the Poincare dual of  $Y$  and let  $p : T^n \rightarrow S^1$  be the classifying map of  $\alpha$ . Then clearly  $p$  is a  $\beta$ -taut fibration.

By Theorem 2.12,  $F(g_0, \beta) = 0$ . By the latter proof, for all  $\epsilon > 0$  sufficiently small, for any  $g'$  as in the statement of our theorem we have:

$$\sum_{o \in \mathcal{O}_{C_\beta}(g', \beta)} \frac{(-1)^{\text{morse}(o)}}{\text{mult}(o)} = 0.$$

The conclusion then readily follows by basic arithmetic.  $\square$

### 9. PROOF OF THEOREM 2.12

This is an application of Morse theory. As  $g$  is  $\beta$ -taut,  $S(g, \beta)$  is compact. Let

$$L = \sup_{o \in S(g, \beta)} \text{energy}_g(o),$$

where

$$\text{energy}_g : L_\beta X \rightarrow \mathbb{R},$$

is the function:

$$(9.1) \quad \text{energy}_g(o) = \int_{S^1} \langle \dot{o}(t), \dot{o}(t) \rangle_g dt.$$

Choose  $C > L$  and let  $U$  denote the subspace of  $L_\beta X$  consisting of loops with  $g$ -energy less than  $C$ . Now  $U$  has the homotopy type of  $L_\beta X$ . This can be proved without infinite dimensional Morse theory. We may use the finite dimensional broken geodesic approximation as in Milnor [9], (passing to the limit) and the fact that there are no geodesics in the complement of  $U$ .

If  $g'$  is sufficiently  $C^0$  nearby to  $g$  and is  $\beta$ -regular then

$$F(g, \beta) = \sum_{o \in \mathcal{O}(g', \beta) \cap U} (-1)^{\text{morse}(o)}.$$

The latter assertion is shown similarly to the proof of Theorem 1.5, except now there is no multiplicity weight since our geodesics are forced have multiplicity one, by the condition that  $\beta$  is not a power. To finish the proof we just need to show that  $\sum_{o \in \mathcal{O}(g', \beta) \cap U} (-1)^{\text{morse}(o)}$  is the Euler characteristic of  $U/S^1$ , since the latter is the Euler characteristic of  $L_\beta X/S^1$ .

Let us now denote by  $\mathcal{L}_\beta X$  the Hilbert manifold of  $H^1$  loops, in class  $\beta$ , as used for example in the classical work of Gromoll-Meyer [7]. We also denote by  $\mathcal{U}$  the  $C$ -sublevel set analogous to  $U$ . The Hilbert manifold  $\mathcal{L}_\beta X$  is well known to be homotopy equivalent to  $L_\beta X$  with its previously used compact open topology.

The energy function  $\text{energy}_{g'} : \mathcal{L}_\beta X \rightarrow \mathbb{R}$ , defined as above, is smooth,  $S^1$  invariant and satisfies the Palais-Smale condition. The flow for its negative gradient vector field  $V$  is complete, and we can do Morse theory mostly as usual. This is understood starting with the work of Klingenberg [8], with the framework of Palais and Smale [10]. Note that all this also applies to  $\mathcal{U}$ . In our case,  $\text{energy}_{g'}$  is moreover a Morse-Bott function with critical manifolds  $C_o$  corresponding to  $S^1$  families of geodesics, for each geodesic string  $o$ .

There is an induced Morse-Bott cell decomposition on  $\mathcal{U}$ , meaning a stratification formed by  $V$  unstable manifolds of the above mentioned critical manifolds  $C_o$ . This is Bott's extension of the fundamental Morse decomposition theorem. Now the  $S^1$  action on  $\mathcal{L}_\beta X$  is free by the condition that  $\beta$  is not a power. This action is not smooth, but it is continuous. So taking the topological  $S^1$  quotient, we get a CW cell decomposition of  $\mathcal{U}/S^1$ , with one  $k$ -cell for each closed  $g'$ -geodesic string  $o$  in  $\mathcal{U}$ , with Morse index  $\text{morse}(o) = k$ . (Here the Morse index is the Morse-Bott index of the critical manifold  $C_o$ .) All of the above is well understood, see for instance [7].

From the above cell decomposition, we readily get that the homology

$$H_*(\mathcal{U}/S^1, \mathbb{Z}) = H_*(U/S^1, \mathbb{Z}) = H_*(L_\beta X/S^1, \mathbb{Z}) = H_*^{S^1}(L_\beta X, \mathbb{Z})$$

is finite dimensional. And we get that:

$$\chi(U/S^1) = \sum_{o \in \mathcal{O}(g', \beta) \cap U} (-1)^{\text{morse}(o)} \quad (\text{immediate from the cell decomposition}).$$

$\square$

## 10. PROOF OF THEOREM 2.16 AND ITS COROLLARIES

We first prove:

**Theorem 10.1.** *Let  $p : X \rightarrow Y$  be a  $\beta$ -taut fibration as in the statement of Theorem 2.16 and  $\beta \in \pi_1^{inc}(X)$  a fiber class. Then*

$$(10.2) \quad F(g, \beta) = \text{card} \cdot \chi(Y) \cdot F(g_Z, \beta_Z),$$

where  $\text{card} \in \mathbb{N} - \{0\}$  is the cardinality of a certain orbit of the holonomy group (as explained in the proof), and where  $\beta_Z$  is as in Lemma 2.18.

*Proof.* We have a natural subset of  $\mathcal{O}' \subset \mathcal{O}g, \beta$ , consisting of all vertical geodesics, that is  $g$ -geodesics contained in fibers  $p^{-1}(y) = Z_y$ . In fact,

$$(10.3) \quad \mathcal{O}' = \mathcal{O}g, \beta,$$

for if  $o$  is any class  $\beta$  closed geodesic, the projection  $p(o)$  is a contractible closed  $g_Y$ -geodesic, and by assumptions is constant.

In particular, there a natural continuous projection

$$\tilde{p} : \mathcal{O}g, \beta \rightarrow Y, \quad \tilde{p}(o) = y$$

where  $y$  is determined by the condition that

$$Z_y \supset \text{image } o.$$

We will use this to construct a suitable (in a sense abstract i.e. not Reeb) perturbation of the vector field  $R^{\lambda_g}$ , using which we can calculate the invariant  $F(g, \beta)$ .

Fix a Morse function on  $f$  on  $Y$ , let  $C = S^*X$  denote the  $g$ -unit cotangent bundle of  $X$ . For  $v \in T_x X$  let  $\langle v |$  denote the functional

$$T_x X \rightarrow \mathbb{R}, \quad w \mapsto \langle v, w \rangle_g.$$

Define  $\tilde{f} : C \rightarrow \mathbb{R}$  by

$$\tilde{f}(\langle v |) := f(p(v)),$$

also define

$$P : C \rightarrow \mathbb{R}$$

by

$$P(\langle v |) := |P^{vert}(v)|_g^2,$$

where  $P^{vert}(v)$  denotes the  $g$ -orthogonal projection of  $v$  onto the  $T_x^{vert} X \subset T_x X$ , for  $T^{vert} X$  the vertical tangent bundle of  $X$ , i.e. the kernel of the bundle map  $p_* : TX \rightarrow TY$ .

Next define  $F : C \rightarrow \mathbb{R}$  by:

$$F(\langle v |) := P(\langle v |) + \tilde{f}(\langle v |).$$

Set

$$V_t = R^{\lambda_g} - t \text{grad}_{g_S} F,$$

where the gradient is taken with respect to the Sasaki metric  $g_S$  on  $C$  [12] induced by  $g$ . The latter Sasaki metric is the natural metric for which we have an orthogonal splitting  $TC = T^{vert}C \oplus T^{hor}C$ , where  $T^{vert}C$  is the kernel of  $pr_* : TC \rightarrow TX$ , induced by the natural projection  $pr : C \rightarrow X$ , and where  $T^{hor}C$  is the  $g$  Levi-Civita horizontal sub-bundle.

Set  $\mathcal{O}_t = \mathcal{O}(V_t, \tilde{\beta})$ , where  $\tilde{\beta}$  is as in Section 6.

**Lemma 10.4.** *We have:*

- (1) For all  $t \in [0, 1]$ ,  $N_t := \mathcal{O}_t \cap \mathcal{O}_{g, \beta}$  is open and closed in  $\mathcal{O}_t$ .

(2) For all  $t \in (0, 1]$ ,  $N_t = \cup_{y \in \text{crit}(f)} \tilde{p}^{-1}(y)$ , where  $\text{crit}(f)$  is the set of critical points of  $f$ .

*Proof.* It is easy to see that  $V_t$  is complete and without zeros. Suppose that  $t > 0$ . Let  $\langle v_\tau |$ ,  $\tau \in \mathbb{R}$  be the flow line of  $V_t$ , through  $\langle v_0 |$ , i.e.  $\langle v_\tau | = \phi_\tau(\langle v_0 |)$ , for  $\phi_\tau$  the time  $\tau$  flow map of  $V_t$ . By the fact that the fibers of  $p$  are assumed to be parallel, we have that

$$R^{\lambda_g}(P) = 0, \quad \text{using the derivation notation.}$$

Also,

$$\text{grad}_{g_S} \tilde{f}(P) = 0,$$

which readily follows by the conjunction of  $g_S$  being Sasaki and the fibers of  $p$  being parallel. Consequently, the function

$$\tau \mapsto P(\langle v_\tau |) = |P^{\text{vert}}(v_\tau)|_g^2$$

is monotonically decreasing unless either:

- (1)  $v_0$  is tangent to  $T^{\text{vert}}X$ , in which case for all  $\tau$ ,  $v_\tau$  are tangent to  $T^{\text{vert}}X$  and  $|P^{\text{vert}}(v_\tau)|_g^2 = 1$ .
- (2) For all  $\tau$ ,  $|P^{\text{vert}}(v_\tau)|_g^2 = 0$ .

In particular, the closed orbits of  $V_t$  split into two types.

- (1) Closed orbits  $o(\tau) = \langle v_\tau |$  with  $v_\tau$  always tangent to  $T^{\text{vert}}X$ . In this case we may immediately conclude that  $o$  is a lift to  $C$  of a closed  $g$ -geodesic contained in the fiber over a critical point of  $f$ .
- (2) Closed orbits  $o(\tau) = \langle v_\tau |$  for which  $v_\tau$  is always  $g$ -orthogonal to  $T^{\text{vert}}X$ .

Clearly, the conclusion follows.  $\square$

*Remark 10.5.* It would be very fruitful to remove the condition on the fibers of  $p$  being parallel. But our argument would need to substantially change.

We return to the proof of the theorem. Set

$$\tilde{N} = \{(o, t) \in L_{\tilde{\beta}}C \times [0, \epsilon] \mid o \in N_t\},$$

where  $L_{\tilde{\beta}}C$  denotes the  $\tilde{\beta}$  component of the free loop space as previously. By part I of Lemma 10.4, this is an open compact subset of  $\mathcal{O}(\{V_t\}, \tilde{\beta})$  s.t.

$$\tilde{N} \cap (L_{\tilde{\beta}}C \times \{0\}) = \mathcal{O}(R^{\lambda_g}, \tilde{\beta}),$$

(equalities throughout are up to natural set theoretic identifications.)

By definitions:

$$N_t = \tilde{N} \cap (L_{\tilde{\beta}}C \times \{t\}).$$

Now the basic invariance of the Fuller index, (A.3) gives:

$$i(N_0, R^{\lambda_g}, \tilde{\beta}) = i(N_1, V_1, \tilde{\beta}).$$

We proceed to compute the right hand side. Fix any smooth Ehresmann connection  $\mathcal{A}$  on the fiber bundle  $p : X \rightarrow Y$ . This induces a holonomy homomorphism:

$$\text{hol}_y : \pi_1(Y, y) \rightarrow \text{Aut } \pi_1(Z_y) \text{ (the right-hand side is the group of set automorphisms),}$$

with image denoted  $\mathcal{H}_y \subset \text{Aut } \pi_1(Z_y)$ .

Let  $\beta_Z$  denote a class in  $\pi_1(Z_y)$  s.t.  $(i_{Z_y})_*(\beta_Z) = \beta$ , for  $i_{Z_y} : Z_y \rightarrow X$  the inclusion map. Set

$$S_y := \bigcup_{g \in \mathcal{H}_y} g(\beta_Z) \subset \pi_1(Z_y).$$

Then for another  $y' \in Y$ ,

$$(10.6) \quad h_* : S_{y'} \rightarrow S_y,$$

is an isomorphism, where  $h : Z_y \rightarrow Z_{y'}$  is a smooth map given by the  $\mathcal{A}$ -holonomy map determined by some path from  $y$  to  $y'$ , and where  $h_*$  is the naturally induced map.

Denoting by  $g_y$  the restriction of  $g$  to the fiber  $Z_y$ , let  $R^y$  denote the  $\lambda_{g_y}$  Reeb vector field on the  $g_{Z_y}$ -unit cotangent bundle  $C_y$  of  $Z_y$ . The cardinality  $\text{card}$  of  $S_y$  is finite, as otherwise we get a contradiction to the compactness of  $S(g, \beta)$ . Now

$$\tilde{p}^{-1}(y) = \bigcup_{\alpha \in S_y} \mathcal{O}(R^{\lambda_y}, \alpha).$$

From part 2 of Lemma 10.4 and by index computations as in [13, Section 2]), we get:

$$i(N_1, V_1, \tilde{\beta}) = \sum_{y \in \text{crit}(f)} (-1)^{\text{morse}(y)} \cdot i(\tilde{p}^{-1}(y), R^{\lambda_y}, \tilde{\beta}),$$

where  $\text{morse}(y)$  denotes the Morse index of  $y$ . Now

$$\begin{aligned} i(\tilde{p}^{-1}(y), R^{\lambda_y}, \tilde{\beta}) &= \sum_{\alpha \in S_y} i(\mathcal{O}(R^{\lambda_y}), R^{\lambda_y}, \tilde{\alpha}) \\ &= \sum_{\alpha \in S_y} F(g_y, \alpha). \\ &= \text{card} \cdot F(g_Z, \beta_Z), \end{aligned}$$

where the last equality follows by (10.6), and by the condition 3 in the Definition 2.2. And so the result follows.  $\square$

We return to the proof of the main theorem. The first part immediately follows from the second, as any class  $\beta \in \pi_1^{\text{inc}}(X)$  geodesic strings of a complete negatively curved Riemannian manifold  $X$  are unique. We prove the second part first. Suppose first that  $\beta$  is an  $n$ -power:  $\beta = \alpha^n$ , for some  $n \geq 1$  where  $\alpha$  is not a power. By the assumption that all contractible  $g_Y$  geodesics are constant, classical Morse theory Milnor [9] tells us that  $Y$  has vanishing higher homotopy groups  $\pi_k(Y, y_0)$ ,  $k \geq 2$ . And in particular  $i_{Z,*} : \pi_1(Z, p_0) \rightarrow \pi_1(X, p_0)$  is a group injection, by the long exact sequence of a fibration. It follows that  $\alpha \in \pi_1^{\text{inc}}(X)$  is also a fiber class.

Now, any  $\alpha$ -class  $g$ -geodesic string must be contained in a fiber of  $p$ . For otherwise we may find a  $\beta$  class  $g$ -geodesic string, which is not contained in a fiber of  $p$ , which would contradict (10.3). It readily follows that  $p : X \rightarrow Y$  is also  $\alpha$ -taut.

Now, if  $\chi(Y) \neq \pm 1$  then by (10.2)  $F(g, \alpha) \neq 1$ , since  $F(g_Z, \alpha_Z)$  is an integer by Theorem 2.12. By Theorem 2.12

$$F(g, \alpha) = \chi^{S^1}(L_\alpha X).$$

So if  $X$  admits a complete metric with a unique and non-degenerate class  $\alpha$   $g$ -geodesic string then we have:

$$F(g, \alpha) = \chi^{S^1}(L_\alpha X) = 1$$

(see Proof of Theorem 2.12), which is impossible. It follows that  $X$  does not admit a metric with a unique and non-degenerate class  $\beta$   $g$ -geodesic string. For if  $o, o'$  are distinct, non-degenerate, class  $\alpha$   $g$ -geodesics strings, then the  $n$ -fold covers  $o^n, (o')^n$  are class  $\beta$ , distinct non-degenerate  $g$ -geodesic strings.

We now prove the general case. Suppose by contradiction that  $X$  admits a metric with a unique and non-degenerate class  $\beta$   $g$ -geodesic string  $o$ . By the above,  $\beta$  is not atomic. Now  $o$  covers a multiplicity one geodesic string  $\tilde{o}$  in some class  $\tilde{\beta} \in \pi_1^{\text{inc}}(X)$ . Moreover,  $\tilde{o}$  is the unique geodesic string in its



class, otherwise  $o$  would not be unique in its class. We prove that  $\tilde{\beta}$  is not a power, which will be a contradiction to  $\beta$  not being atomic and will complete the proof.

Suppose otherwise, so that  $\tilde{\beta}_{x_0} = \alpha^k$  for  $k > 1$  and  $\alpha \in \pi_1(X, x_0)$ , ( $\beta_{x_0}$  is as in Definition 2.9). Let  $u$  be a class  $\alpha$ , closed  $g$ -geodesic string (where  $\alpha$  also denotes the class in  $\pi_1^{inc}(X)$  corresponding to the based class  $\alpha$ .) It is immediate that the  $k$  cover of  $u$ ,  $u^k$  represents  $\tilde{\beta}$  and is a  $g$ -geodesic string. By the uniqueness,  $\tilde{o} = u^k$ . But this contradicts simplicity of  $\tilde{o}$ . So  $\tilde{\beta}$  is not a power.

□

*Proof of Theorem 1.3.* Let  $g_Z, g_Y$  be complete Riemannian metrics on  $Z$  respectively  $Y$  with non-positive curvature. Take the product metric  $g = g_Z \times g_Y$  on  $X = Z \times Y$ . For a class  $\beta \in \pi_1(Z)$  in the image of the inclusion  $\pi_1(Z) \rightarrow \pi_1(X)$ , the natural projection  $X \rightarrow Y$  is automatically a  $\beta$ -taut fibration. Then the conclusion readily follows from Theorem 2.16.

□

*Proof of Theorem 2.11.* By Theorem 10.1 0 is certainly a value of the invariant  $F$ . We first prove that every negative rational number is the value of the invariant. Let  $p, q$  be positive integers. Let  $Y$  be a closed surface of genus  $(p+1) > 1$  with a hyperbolic metric  $g_Y$ , let  $Z$  be the genus 2 closed surface with a hyperbolic metric  $g_Z$  and let  $\beta_Z \in \pi_1^{inc}(Z)$  be the class represented by a  $2 \cdot q$ -fold covering of a simple closed loop representing a generator of the fundamental group of  $Z$ .

Let  $X = Y \times Z$  with the product metric  $g = g_Y \times g_Z$  and  $p : X \rightarrow Y$  the canonical projection. By Theorem 10.1

$$F(g, \beta) = \chi(Y) \cdot F(g_Z, \beta_Z) = (-2p) \cdot \frac{1}{2q} = -\frac{p}{q},$$

where  $\beta$  is as in Lemma 2.18. So we proved our first claim.

Let again  $p, q$  be positive integers. Let  $Y$  be closed surface of genus 2, with a hyperbolic metric  $g_Y$ . And let  $Z$  be a manifold satisfying  $F(g_Z, \beta_Z) = -\frac{p}{2q}$  for some  $\beta_Z$ -taut metric  $g_Z$  on  $Z$  and for some class  $\beta_Z \in \pi_1^{inc}(Z)$ . This exist by the discussion above. Let  $g = g_Y \times g_Z$  be the product metric on  $Y \times Z$ , and  $\beta$  as above. Analogously to the discussion above we get:

$$F(g, \beta) = \chi(Y) \cdot F(g_Z, \beta_Z) = (-2) \cdot \frac{-p}{2q} = \frac{p}{q}.$$

□

#### A. FULLER INDEX AND SKY CATASTROPHES

Let  $X$  be a complete vector field without zeros on a manifold  $M$ . Set

$$(A.1) \quad S(X, \beta) = \{o \in L_\beta M \mid \exists p \in (0, \infty), \ o : \mathbb{R}/\mathbb{Z} \rightarrow M \text{ is a periodic orbit of } pX\}.$$

The above  $p$  is uniquely determined and we denote it by  $p(o)$  called the period of  $o$ .

There is a natural  $S^1$  reparametrization action on  $S(X, \beta)$ :  $t \cdot o$  is the loop  $t \cdot o(\tau) = o(t + \tau)$ . The elements of  $\mathcal{O}(X, \beta) := S(X, \beta)/S^1$  will be called **orbit strings**. Slightly abusing notation we just write  $o$  for the equivalence class of  $o$ .

The multiplicity  $m(o)$  of an orbit string is the ratio  $p(o)/l$  for  $l > 0$  the period of a simple orbit string covered by  $o$ .

We want a kind of fixed point index which counts orbit strings  $o$  with certain weights. Assume for simplicity that  $N \subset \mathcal{O}(X, \beta)$  is finite. (Otherwise, for a general open compact  $N \subset \mathcal{O}(X, \beta)$ , we need

to perturb.) Then to such an  $(N, X, \beta)$  Fuller associates an index:

$$i(N, X, \beta) = \sum_{o \in N} \frac{1}{m(o)} i(o),$$

where  $i(o)$  is the fixed point index of the time  $p(o)$  return map of the flow of  $X$  with respect to a local surface of section in  $M$  transverse to the image of  $o$ .

Fuller then shows that  $i(N, X, \beta)$  has the following invariance property. For a continuous homotopy  $\{X_t\}$ ,  $t \in [0, 1]$  set

$$S(\{X_t\}, \beta) = \{(o, t) \in L_\beta M \times [0, 1] \mid o \in S(X_t)\}.$$

And given a continuous homotopy  $\{X_t\}$ ,  $X_0 = X$ ,  $t \in [0, 1]$ , suppose that  $\tilde{N}$  is an open compact subset of

$$(A.2) \quad \mathcal{O}(\{X_t\}, \beta) := S(\{X_t\}, \beta) / S^1,$$

such that

$$\tilde{N} \cap (L_\beta M \times \{0\}) / S^1 = N.$$

Then if

$$N_1 = \tilde{N} \cap (L_\beta M \times \{1\}) / S^1$$

we have

$$(A.3) \quad i(N, X, \beta) = i(N_1, X_1, \beta).$$

We call this **basic invariance**. In the case  $\mathcal{O}(X_0, \beta)$  is compact,  $\mathcal{O}(X_1, \beta)$  is compact for any sufficiently  $C^0$  nearby  $X_1$ , and in this case basic invariance implies (see for instance [13, Proof of Lemma 1.6]):

$$(A.4) \quad i(\mathcal{O}(X_0, \beta), X, \beta) = i(\mathcal{O}(X_1, \beta), X_1, \beta).$$

### A.1. Blue sky catastrophes.

**Definition A.5** (Preliminary). A **sky catastrophe** for a smooth family  $\{X_t\}$ ,  $t \in [0, 1]$ , of non-vanishing vector fields on a closed manifold  $M$  is a continuous family of closed orbit strings  $\tau \mapsto o_{t_\tau}$ ,  $o_{t_\tau}$  is an orbit string of  $X_{t_\tau}$ ,  $\tau \in [0, \infty)$ , such that the period of  $o_{t_\tau}$  is unbounded from above.

A sky catastrophe as above was initially constructed by Fuller [5]. Or rather his construction essentially contained this phenomenon. A more general definition appears in [13], we slightly extend it here to the case of non-compact manifolds. All these definitions become equivalent given certain regularity conditions on the family  $\{X_t\}$  and assuming  $M$  is compact.

**Definition A.6.** Let  $\{X_t\}$ ,  $t \in [0, 1]$  be a continuous family of non-zero, complete smooth vector fields on a manifold  $M$  and  $\beta \in \pi_1^{inc}(X)$ .

We say that  $\{X_t\}$  has a **catastrophe in class  $\beta$** , if there is an element

$$y \in \mathcal{O}(X_0, \beta) \sqcup \mathcal{O}(X_1, \beta) \subset \mathcal{O}(\{X_t\}, \beta)$$

such that there is no open compact subset of  $\mathcal{O}(\{X_t\}, \beta)$  containing  $y$ .

A vector field  $X$  on  $M$  is **geodesible** if there exists a metric  $g$  on  $M$  s.t. every flow line of  $X$  is a unit speed  $g$ -geodesic. A family  $\{X_t\}$  is **geodesible** if there is a continuous family  $\{g_t\}$  of metrics, with  $X_t$  geodesible with respect to  $g_t$  for each  $t$ . A family  $\{X_t\}$  is **geodesible** if there is a continuous family  $\{g_t\}$  of metrics with  $X_t$  geodesible with respect to  $g_t$  for each  $t$ . A **geodesible sky catastrophe** is a geodesible family  $\{X_t\}$  with a sky catastrophe. A **Reeb sky catastrophe** is a family of Reeb vector fields  $\{X_t\}$  with a sky catastrophe.

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