ELLIPTIC CURVES IN LCS MANIFOLDS AND METRIC INVARIANTS

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ABSTRACT. We study invariants defined by count of charged, elliptic J-holomorphic curves in locally conformally symplectic manifolds. We use this to define rational number invariants of certain complete Riemann-Finlser manifolds and their isometries and this is used to find some new phenomena in Riemann-Finlser geometry. In contact geometry this Gromov-Witten theory is used to study fixed Reeb strings of strict contactomorphisms. Along the way, we state an analogue of the Weinstein conjecture in lcs geometry, directly extending the Weinstein conjecture, and discuss various partial verifications. A counterexample for a stronger, also natural form of this conjecture is given.

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1. Introduction

The study of J-holomorphic curves in symplectic manifolds was initiated by Gromov [13]. In that work and since then it have been rational J-holomorphic curves that were central in the subject. We study here certain Gromov-Witten type theory of J-holomorphic elliptic curves in locally conformally symplectic manifolds, for short lcs manifolds. For lcs manifolds it appears that instead elliptic (and possibly higher genus) curves are central. One explanation for this is that rational J-curves in an lcs manifold M have $\widetilde{J}$-holomorphic lifts to the universal cover $\widetilde{M}, \widetilde{J}$, where the form is globally conformally symplectic. Hence, rational Gromov-Witten theory is a priori insensitive to the information carried by the Lee form (Section 2), although it can still be useful [31].

We will present various applications for these elliptic curve counts in contact dynamics and for metric invariants of Riemann-Finlser manifolds. We choose to start the discussion with applications rather than theory, as the latter requires certain buildup. We start with Riemann-Finlser geometry, and this story can be understood as a generalization of the theory in [28], which focuses on more elementary geodesic counts.

In what follows for a manifold X the topology on various functional spaces is usually the topology of C^0 convergence on compact subsets of X, unless specified otherwise. $\pi_1(X)$ will denote the set of free homotopy classes of maps $o : S^1 \to X$.

We recall some definitions from [28].

Definition 1.1. *Let X be a smooth manifold. Fix an exhaustion by nested compact sets $\bigcup_{i\in\mathbb{N}} K_i = X$, $K_i \supset K_{i-1}$ for all $i \geq 1$. We say that a class $\beta \in \pi_1(X)$ is* **boundary compressible** *if β is in the image of*

$$inc_* : \pi_1(X - K_i) \to \pi_1(X)$$

for all i, where $inc : X - K_i \to X$ is the inclusion map. We say that β is **boundary incompressible** *if it is not boundary compressible.*

Let $\pi_1^{inc}(X)$ denote the set of such boundary incompressible classes. When X is compact, we set $\pi_1^{inc}(X) := \pi_1(X) - const$, where *const* denotes the set of homotopy classes of constant loops.

Terminology 1. All our metrics are Riemann-Finsler metrics unless specified otherwise, and usually denoted by just g. Completeness, always means forward completeness. Curvature always means sectional curvature in the Riemannian case and flag curvature in the Finsler case. Thus we will usually just say complete metric g, for a forward complete Riemann-Finsler metric. A reader may certainly choose to interpret all metrics as Riemannian metrics and completeness as standard completeness.

Denote by $L_\beta X$ the class $\beta \in \pi_1^{inc}(X)$ component of the free loop space of X, with its compact open topology. Let g be a complete metric on X, and let $S(g, \beta) \subset L_\beta X$ denote the subspace of all unit speed parametrized, closed g-geodesics in class β.

Definition 1.2. *We say that a metric g on X is β-***taut** *if g is complete and $S(g, \beta)$ is compact. We will say that g is* **taut** *if it is β-taut for each $\beta \in \pi_1^{inc}(X)$.*

As shown in [28], a basic example of a taut metric is a complete metric with non-positive curvature, or more generally a complete metric all of whose boundary incompressible geodesics are minimizing in their homotopy class. Other substantial classes of examples are constructed in [28]. Overall, the class of taut metrics is large and flexible. However, I do not believe it has been much studied, (or even previously, explicitly defined) so there is a wealth of open problems, for example behavior under surgery.

Definition 1.3. *Let $\beta \in \pi_1^{inc}(X)$, and let g_0, g_1 be a pair of β-taut metrics on X. A* **β-taut homotopy or deformation** *between g_0, g_1, is a continuous (in the topology of C^0 convergence on compact sets) family $\{g_t\}$, $t \in [0,1]$ of complete metrics on X, s.t.*

$$S(\{g_t\}, \beta) := \{(o,t) \in L_\beta X \times [0,1] \,|\, o \in S(g_t, \beta)\}$$

is compact. We say that $\{g_t\}$ is a **taut homotopy** *if it is β-taut for each $\beta \in \pi_1^{inc}(X)$.*

As shown in [28], the β-tautness condition is trivially satisfied if g_t have the property that all their class β geodesics are minimal. In particular if g_t have non-positive curvature then $\{g_t\}$ is taut by the Cartan-Hadamard theorem, [3].

Let $\mathcal{E}(X)$ be set of equivalence classes of tuples

$$\{(g,\phi) \,|\, g \text{ is a taut and } \phi \text{ is an isometry of } g\},$$

where (g_0, ϕ_0) is equivalent to (g_1, ϕ_1) whenever there is a homotopy $\{(g_t, \phi_t)\}_{t\in[0,1]}$, s.t. for each t ϕ_t is an isometry of g_t, and $\{g_t\}$ is a taut homotopy. By counting certain charged elliptic curves in a lcs manifold associated to (g, ϕ) we define in Section 4.3 a functional:

Theorem 1.4. *For each manifold X, there is a natural, (generally) non-trivial functional*

$$\mathrm{GWF} : \mathcal{E}(X) \times \pi_1^{inc}(X) \times \mathbb{N} \to \mathbb{Q}.$$

GWF stands for Gromov-Witten-Fuller, as when $\phi = id$ in $\mathrm{GWF}(g, \phi, \beta, n)$ or when $n = 0$ the invariant reduces to a certain geodesic counting invariant studied in [29], and in this case such counts can be defined purely using Fuller's theory [12].

To understand what this functional is counting in general we first define:

Definition 1.5. *Let ϕ be an isometry of X, g. Then a* **charge n fixed geodesic string** *of ϕ is a closed geodesic o whose image is fixed by ϕ^n. That is* $\operatorname{image} o = \operatorname{image} \phi^n \circ o$. *If charge is not specified it is assumed to be one. We say that such a fixed string is in class β if the class of o is β.*

We will see that if $\mathrm{GWF}(g, \phi, \beta, n) \neq 0$ then there is a charge n fixed g-geodesic string of ϕ in class β.

Remark 1.6. Moreover, $\mathrm{GWF}(g, \phi, \beta, n)$ is in fact the "count" of the latter fixed geodesic strings, if by count we mean evaluating the fundamental class of a certain compact virtual dimension zero Kuranishi space with orbifold points. This will be explained once we construct the functional as a Gromov-Witten invariant in Section 4.3.

Here are some basic related phenomena. Let $\beta \in \pi_1^{inc}(X)$ be not a power class (see [28, Definition 1.7]), and suppose that X admits a β-taut metric g, then by Theorem [28, Theorem 1.10] the S^1 equivariant homology $H_*^{S^1}(L_\beta X, \mathbb{Z})$ is finite dimensional. In this case we denote by $\chi^{S^1}(L_\beta X)$ its Euler characteristic.

Theorem 1.7. *Let $\beta \in \pi_1^{inc}(X)$ be not a power class, and suppose that X admits a β-taut metric. Suppose further that $\chi^{S^1}(L_\beta X) \neq 0$. Then for any β-taut g on X, any isometry ϕ of g in the component of the id, has a charge one fixed geodesic string in class β.*

In the next corollary we need that the manifold X admits a metric of negative curvature, and there is a class $\beta \in \pi_1^{inc}(X)$. This condition is automatic if X is compact. It is also readily seen to be satisfied for $X = Y - \{y_0, \ldots, y_k\}$, with Y compact and admitting a metric of negative curvature, and some points $y_i \in Y$. It is of course false for $X = \mathbb{R}^n$.

Corollary 1.8. *Suppose that X admits a metric of negative curvature, and there is a class $\beta \in \pi_1^{inc}(X)$. Then for any other β-taut metric g on X, any isometry ϕ of g in the component of the id, has a charge one fixed geodesic string (possibly not class β). In particular, this holds for any other metric on X with non-positive curvature.*

Proof. Under assumptions we may find a not a power class $\beta' \in \pi_1^{inc}(X)$. Also,

$$\begin{aligned}\chi^{S^1}(L_{\beta'}X) &= \chi(L_{\beta'}X/S^1) \quad \text{as the action is free by the condition that } \beta \text{ is not a power}\\ &= 1,\end{aligned}$$

where the last equality is immediate from the hypothesis that admits a complete metric of negative curvature and the Cartan-Hadamard theorem. Then the result follows by the theorem. □

The following theorem is also a special version of the theorem. For the definition of a β-taut submersion we refer the reader to [28], which also contains a construction of a large class of examples, in particular giving examples for the following theorem.

Theorem 1.9. *Let $Z \hookrightarrow X \to Y$ be a fibration with Y, g_Y a closed hyperbolic Riemann surface, and with Z a manifold admitting a metric with negative curvature. Suppose that $p : X \to Y$ admits the structure of a β-taut submersion, for $\beta \in \pi_1^{inc}(X)$ a fiber class, that is a class in the image of the inclusion $\pi_1^{inc}(Z) \to \pi_1^{inc}(X)$, and s.t. β is not a power. Then for any β-taut metric g on X, any isometry ϕ of g homotopic via isometries to the id, has a fixed geodesic string in class β.*

There is a partially related theory of ϕ-invariant geodesics. The latter are geodesics γ satisfying $\gamma(1) = \phi(\gamma(0))$ for some isometry ϕ of X, g. (These are also analogous to translated points of contactomorphisms mentioned ahead.) A charge 1 fixed geodesic string of ϕ clearly determines a circle family of closed ϕ-invariant geodesics. On the other hand as these ϕ-invariant geodesics are not required to be closed, if we fix a ϕ they can be shown exist under very general conditions using Morse theory, Grove [14].

1.1. **Contact dynamics, fixed Reeb strings and more applications to isometries.**

Let (C^{2n+1}, λ) be a contact manifold with λ a contact form, that is a one form s.t. $\lambda \wedge (d\lambda)^n \neq 0$. Denote by R^λ the Reeb vector field satisfying:

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

We assume throughout that its flow is complete. Recall that a ***closed λ-Reeb orbit*** (or just Reeb orbit when λ is implicit) is a smooth map

$$o : (S^1 = \mathbb{R}/\mathbb{Z}) \to C$$

such that

$$\dot{o}(t) = cR^\lambda(o(t)),$$

with $\dot{o}(t)$ denoting the time derivative, for some $c > 0$ called period. Let $S(R^\lambda, \beta)$ denote the space of all closed Reeb orbits in free homotopy class β, with its compact open topology. And set

$$\mathcal{O}(R^\lambda, \beta) := S(R^\lambda, \beta)/S^1,$$

where S^1 is acting naturally by reparametrization, see Appendix A. We say that the action spectrum is ***discrete*** if the image of the period map $A : S(R^\lambda, \beta) \to \mathbb{R}$, $o \mapsto \int_{S^0} o^*\lambda$ is discrete.

Definition 1.10. *Let $\phi : (C, \lambda) \to (C, \lambda)$ be a strict contactomorphism of a contact manifold. Then a* **fixed Reeb string of** *ϕ is a closed λ-Reeb orbit o whose image is fixed by ϕ. We say that it is in class β if the free homotopy class of o is β.*

Definition 1.11. *Assuming that the class β is non-torsion* [1]*, we say that (C, λ) is* **infinite type** *for class β if the action spectrum of λ is discrete and there is a Reeb perturbation X of the vector field $\mathbb{R}^\lambda$ (in a certain natural sense,* [29, Definition 2.6]*), s.t. all but finitely many class β orbits of X have even Conley-Zehnder index or or all but finitely many orbits of X have odd Conley-Zehnder index.*

A typical example of infinite type is the standard contact form λ_{st} on S^{2k+1}, as shown in [29].

Definition 1.12. *We say that (C, λ) is* **finite type** *for class β if $\mathcal{O}(R^\lambda, \beta)$ is compact. And we say that it is* **finite non-zero type** *if in addition $i(R^\lambda, \beta) \neq 0$, (the Fuller index, see Appendix A).*

We will discuss some examples after the theorem. We say that (C, λ) is ***definite type*** (for class β) if it is either finite non-zero type or infinite type.

Theorem 1.13. *Let (C, λ) be a contact manifold of definite type for class β orbits, then every strict contactomorphism ϕ of (C, λ), homotopic to the id via strict contactomorphisms, has a fixed Reeb string in class β. Furthermore, the same holds for every λ' sufficiently C^1 nearby to λ. In particular, for any contact form λ on S^{2k+1}, sufficiently C^1 nearby to λ_{st}, any strict contactomorphism ϕ of (C, λ) homotopic to the id via strict contactomorphisms has a fixed Reeb string.*

There is a partial connection of the theorem with the theory of translated points.

Definition 1.14 (Sandon [27])**.** *Given a (not necessarily strict) contactomorphism ϕ of (C, λ), a point $p \in C$ is called a* **translated point** *provided that $\phi^* \lambda(p) = \lambda(p)$ and $\phi(p)$ lies on the λ-Reeb flow line passing through p.*

A fixed Reeb string for ϕ in particular determines a special translated point of ϕ (one for each point on the image of the fixed Reeb string). So the above theorem is partly related to the Sandon conjecture [27] on existence of translated points of contactomorphisms. However, also note that the general form of Sandon's conjecture has counterexamples on S^{2k+1} for the standard contact form λ_{st}, see Cant [6]. Partially related to the Sandon conjecture is the Conjecture 1 in Section 3, which is an analogue in lcs geometry of the Weinstein conjecture.

Corollary 1.15. *Let X, g be complete, with a class $\beta \in \pi_1^{inc}(X)$, and such that its unit cotangent bundle is definite type for class $\widetilde{\beta}$, (defined as in Section 4.3). Then every isometry of X, g homotopic through isometries to the id has a class β fixed geodesic string.*

Theorem 1.16. *Suppose that (C, λ) is Morse-Bott and some connected component $N \subset \mathcal{O}(R^\lambda, \beta)$ has non-vanishing Euler characteristic. Then any contact form λ' on C, sufficiently C^1 nearby to λ, any strict contactomorphism ϕ of (C, λ'), homotopic to the id via strict contactomorphisms has class β fixed Reeb string.*

Both of the theorems above are actually special cases of the next theorem proved in Section 7. For more details on the Fuller index see Appendix A. Let λ be a contact form on a closed manifold C, $N \subset \mathcal{O}(R^\lambda, \beta)$ and let $i(N, R^\lambda, \beta) \in \mathbb{Q}$ denote the Fuller index. For example, if λ is Morse-Bott (see [5]) and N is a connected component of $\mathcal{O}(R^\lambda, \beta)$ then by a computation in [29, Section 2.1.1] $i(R^\lambda, N, \beta) \neq 0$ if $\chi(N) \neq 0$ (the Euler characteristic).

Theorem 1.17. *Let (C, λ) be a contact manifold satisfying the condition: $i(N, R^\lambda, \beta) \neq 0$, for some open compact $N \subset \mathcal{O}(R^\lambda, \beta)$. Then any strict contactomorphism $\phi : (C, \lambda) \to (C, \lambda)$, homotopic to the id via strict contactomorphisms has a fixed Reeb string o in class β and moreover $o \in N$.*

[1] In the torsion case the infinite type condition is more complicated see [29].

We have already mentioned that the index assumption of the theorem holds for Morse-Bott contact forms λ, provided the Euler characteristic of some component of $N \subset \mathcal{O}(R^\lambda)$ is non-vanishing. We may take for instance the standard contact form λ_{st} on S^{2k+1}, the unit contangent bundle of the sphere, or see Bourgeois [5] for more examples. In this Morse-Bott case the theorem may be verified by elementary considerations. To see this suppose we have a connected component $N \subset \mathcal{O}(R^\lambda)$ with $\chi(N) \neq 0$. Then ϕ as above induces a topological endomorphism $\widetilde{\phi}$ of N with non-zero Lefschetz number, so that in this case the result follows by the Lefschetz fixed point theorem.

In general a compact open component $N \subset \mathcal{O}(R^\lambda)$ may not be a finite simplicial complex, or indeed any kind of topological space to which the classical Lefschetz fixed point theorem may apply. Also the relationship of $i(N, R^\lambda, \beta)$ with $\chi(N)$ breaks down in general as $i(N, R^\lambda, \beta)$ is partly sensitive to the dynamics of R^λ.

The following is a variation of Theorem 1.7 in the absence of the condition that β be not a power, and removing all assumptions on the metric except completeness. This is proved in Section 7.

Theorem 1.18. *Let X admit a complete metric with a unique and non-degenerate geodesic in class $\beta \in \pi_1^{inc}(X)$. Then one of the following alternatives holds:*

(1) *Sky catastrophes for families of Reeb vector fields exist, and the sky catastrophe can be essential, see Definition A.3.*

(2) *For any complete metric g on X and every isometry ϕ of X, g homotopic through isometries to the identity, ϕ has a charge 1 fixed geodesic string in class β.*

1.2. **Conformal symplectic Weinstein conjecture.** We introduce in Section 2 certain analogues of Reeb orbits for lcs manifolds. In particular, we define a unifying concept of a Reeb 2-curve on which most of the subsequent theory is based. This leads us to state one analogue in lcs geometry of the classical Weinstein conjecture, and we discuss certain partial verifications. We also state in this section an important counterexample for a stronger, but also natural form of the lcs Weinstein conjecture.

1.3. **Organization.** The main theorems are proved in Section 7. Section 2 presents detailed preliminaries for lcs geometry, which should make this paper self contained and accessible to a general reader. Section 4 defines the Gromov-Witten invariant GWF, which is central to the applications in Riemann-Finlser geometry.

2. Background and preliminaries

Definition 2.1. *A locally conformally symplectic manifold or just an* lcs *manifold, is a smooth $2n$-fold M with an* lcs *structure: which is a non-degenerate 2-form ω, with the property that for every $p \in M$ there is an open $U \ni p$ such that $\omega|_U = f_U \cdot \omega_U$, for some symplectic form ω_U defined on U and some smooth positive function f_U on U.*

These kinds of structures were originally considered by Lee in [15], arising naturally as part of an abstract study of "a kind of even dimensional Riemannian geometry", and then further studied by a number of authors see for instance, [2] and [33]. An lcs manifold admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and Murphy show that if a closed almost complex $2n$-fold M has $H^1(M, \mathbb{R}) \neq 0$ then it admits a lcs structure, [8]. Another result of Apostolov, Dloussky [1] is that any complex surface with an odd first Betti number admits a lcs structure, which tames the complex structure.

To see the connection with the first cohomology group $H^1(M, \mathbb{R})$, mentioned above, let us point out right away the most basic invariant of a lcs structure ω, when M has dimension at least 4. This is the Lee class, $\alpha = \alpha_\omega \in H^1(M, \mathbb{R})$. This class has the property that on the associated α-covering space (see proof of Lemma 6.1) $\widetilde{M}$, the lift $\widetilde{\omega}$ is globally conformally symplectic. Thus, an lcs form is globally conformally symplectic, that is diffeomorphic to $e^f \cdot \omega'$, with ω' symplectic, iff its Lee class vanishes.

Again assuming M has dimension at least 4, the Lee class α has a natural differential form representative, called the Lee form, which is defined as follows. We take a cover of M by open sets U_a in which $\omega = e^{f_a} \cdot \omega_a$ for ω_a symplectic. Then we have 1-forms $d(f_a)$ on each U_a, which glue to a well-defined closed 1-form on M, as shown by Lee. We may denote this 1-form and its cohomology class both by α. It is moreover immediate that for an lcs form ω,

$$d\omega = \alpha \wedge \omega,$$

for α the Lee form as defined above.

As we mentioned lcs manifolds can also be understood to generalize contact manifolds. This works as follows. First we have a class of explicit examples of lcs manifolds, obtained by starting with a symplectic cobordism (see [8]) of a closed contact manifold C to itself, arranging for the contact forms at the two ends of the cobordism to be proportional and then gluing the boundary components, (after a global conformal rescaling of the form on the cobordism, to match the boundary conditions).

Terminology 2. For us a contact manifold is a pair (C, λ) where C is a closed manifold and λ a contact form: $\forall p \in C : \lambda \wedge \lambda^{2n}(p) \neq 0$. This is not a completely common terminology as classically it is the equivalence class of (C, λ) that is called a contact manifold, where $(C, \lambda) \sim (C, \lambda')$ if $\lambda = f\lambda'$ for f a positive function. (Given that the contact structure, in the classical sense, is co-oriented.) A ***contactomorphism*** between (C_1, λ_1), (C_2, λ_2) is a diffeomorphism $\phi : C_1 \to C_2$ s.t. $\phi^* \lambda_2 = f \lambda_1$ for some $f > 0$. It is called ***strict*** if $\phi^* \lambda_2 = \lambda_1$.

A concrete basic example, which can be understood as a special case of the above cobordism construction, is the following.

Example 1 (Banyaga). Let (C, λ) be a contact manifold, $S^1 = \mathbb{R}/\mathbb{Z}$, $d\theta$ the standard non-degenerate 1-form on S^1 satisfying $\int_{S^1} d\theta = 1$. And take $M = C \times S^1$ with the 2-form

$$\omega_\lambda = d_\alpha \lambda := d\lambda - \alpha \wedge \lambda,$$

for $\alpha := pr_{S^1}^* d\theta$, $pr_{S^1} : C \times S^1 \to S^1$ the projection, and λ likewise the pull-back of λ by the projection $C \times S^1 \to C$. We call (M, ω_λ) as above the ***lcs-fication*** of (C, λ). This is also a basic example of a first kind lcs manifold, as in Definition 2.4 ahead.

The operator

$$d_\alpha : \Omega^k(M) \to \Omega^{k+1}(M) \tag{2.2}$$

is called the Lichnerowicz differential with respect to a closed 1-form α, and it satisfies $d_\alpha \circ d_\alpha = 0$ so that we have an associated Lichnerowicz chain complex.

Definition 2.3. *An* **exact lcs form** *on M is an lcs 2-form s.t. there exists a pair of one forms (λ, α) with α a closed 1-form, s.t. $\omega = d_\alpha \lambda$ is non-degenerate. In the case above we also call the pair (λ, α)* **an exact lcs structure**. *The triple (M, λ, α) will be called an* **exact lcs manifold**, *but we may also call (M, ω) an exact lcs manifold when (λ, α) are implicit.*

An exact lcs structure determines a generalized distribution $\mathcal{V}_\lambda$ on M:

$$\mathcal{V}_\lambda(p) = \{v \in T_p M \,|\, d\lambda(v, \cdot) = 0\},$$

which we call the ***vanishing distribution***. We also define a generalized distribution ξ_λ that is the ω-orthogonal complement to $\mathcal{V}_\lambda$, which we call ***co-vanishing distribution***. For each $p \in M$, $\mathcal{V}_\lambda(p)$ has dimension at most 2 since $d\lambda - \alpha \wedge \lambda$ is non-degenerate. If M^{2n} is closed $\mathcal{V}_\lambda$ cannot identically vanish since $(d\lambda)^n$ cannot be non-degenerate by Stokes theorem.

Definition 2.4. *Let (λ, α) be an exact lcs structure on M. We call α integral, rational or irrational if its periods are integral, respectively rational, or respectively irrational. We call the structure (λ, α)* **scale integral**, *if $c\alpha$ is integral for some $0 \neq c \in \mathbb{R}$. Otherwise we call the structure* **scale irrational**. *If $\mathcal{V}$ is non-zero at each point of M, in particular is a smooth 2-distribution, then such a structure is called* **first kind**. *If ω is an exact lcs form then we call ω integral, rational, irrational, first kind if the exists λ, α s.t. $\omega = d_\alpha \lambda$ and (λ, α) is integral, respectively irrational, respectively first kind. Similarly define, scale integral, scale irrational ω.*

Definition 2.5. *A* **conformal symplectomorphism** *of lcs manifolds $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ is a diffeomorphism ϕ s.t. $\phi^* \omega_2 = e^f \omega_1$, for some f. Note that in this case we have an induced relation (when M has dimension at least 4):*

$$\phi^* \alpha_1 = \alpha_0 + df,$$

where α_1 is the Lee form of ω_1 and α_0 is the Lee form of ω_0. If $f = 0$ we call ϕ a **symplectomorphism**. *A (conformal) symplectomorphism of exact lcs structures (λ_1, α_1), (λ_2, α_2) on M_1 respectively M_2 is a (conformal) symplectomorphism of the corresponding lcs 2-forms. If a diffeomorphism $\phi : M_1 \to M_2$ satisfies $\phi^* \lambda_2 = \lambda_1$ and $\phi^* \alpha_2 = \alpha_1$ we call it an* **isomorphism** *of the exact lcs structures. This is analogous to a strict contactomorphism of contact manifolds.*

To summarize, with the above notions we have the following basic points whose proof is left to the reader:

(1) An isomorphism of exact lcs structures (λ_1, α_1), (λ_2, α_2) preserves the first kind condition, and moreover preserves the corresponding vanishing distributions.

(2) A symplectomorphism of lcs forms preserves the first kind condition.

(3) A (conformal) symplectomorphism of exact lcs structures generally does not preserve the first kind condition. (Contrast with 2.)

(4) A (conformal) symplectomorphism of first kind lcs structures generally does not preserve the vanishing distributions. (Similar to a contactomorphism not preserving Reeb distributions.)

(5) A conformal symplectomorphism of lcs forms and exact lcs structures preserves the rationality, integrality, scale integrality conditions.

Remark 2.6. We say that ω_0 is conformally equivalent to ω_1 if $\omega_1 = e^f \omega_0$, i.e. the identity map is a conformal symplectomorphism $id : (M, \omega_0) \to (M, \omega_1)$. It is important to note that for us the form ω is the structure not its conformal equivalence class, as for some authors. In other words conformally equivalent structures on a given manifold determine distinct but isomorphic objects of the category, whose objects are lcs manifolds and morphisms conformal symplectomorphisms.

Example 2. One example of an lcs structure of the first kind is a mapping torus of a strict contactomorphism, see Banyaga [2]. The mapping tori $M_{\phi,c}$ of a strict contactomorphism ϕ of (C, λ) fiber over S^1,

$$\pi : C \hookrightarrow M_{\phi,c} \to S^1,$$

with Lee form of the type $\alpha = c\pi^*(d\theta)$, for some $0 \neq c \in \mathbb{R}$. In particular, these are scale integral first kind lcs structures.

Moreover we have:

Theorem 2.7 (Only reformulating Bazzoni-Marrero [4])**.** *A first kind lcs structure (λ, α) on a closed manifold M is isomorphic to a mapping torus of a strict contactomorphism if and only if it is scale integral.*

The (scaled) integrality condition is of course necessary since the Lee form of a mapping torus of a strict contactomorphism will have this property. Thus we may understand scale irrational first kind lcs structures as first (and rather dramatic) departures from the world of contact manifolds into a brave new lcs world.

Remark 2.8. Note that scale irrational first kind structures certainly exist. A simple example is given by taking λ, α to be closed scale irrational 1-forms on T^2 with transverse kernels. Then $\omega = \lambda \wedge \alpha$ is a scale irrational first kind structure on T^2. In particular (λ, α) cannot be a mapping torus of a strict contactomorphism even up to a conformal symplectomorphism. In general, on a closed manifold we may always perturb a (first kind) scale integral lcs structure to a (first kind) scale irrational one. The examples of the present paper deal with deformations of this sort.

2.1. **Reeb 2-curves.**

Definition 2.9. *Let (M, λ, α) be an exact* lcs *structure and $\omega = d_\alpha \lambda$. Define X_λ by $\omega(X_\lambda, \cdot) = \lambda$ and X_α by $\omega(X_\alpha, \cdot) = \alpha$. Let $\mathcal{D}$ denote the (generalized) distribution spanned by X_α, X_λ, meaning $\mathcal{D}(p) := \operatorname{span}(X_\alpha(p), X_\lambda(p))$. This will be called the* **canonical distribution**.

The (generalized) distribution $\mathcal{D}$ is one analogue for exact lcs manifolds of the Reeb distribution on contact manifolds. A Reeb 2-curve, as defined ahead, will be a certain kind of singular leaf of $\mathcal{D}$, and so is a kind of 2-dimensional analogue of a Reeb orbit.

Example 3. The simplest example of a Reeb 2-curve in an exact lcs (M, λ, α), in the case $\mathcal{D}$ is a true 2-dimensional distribution (for example if (λ, α) is first kind), is a closed immersed surface $u : \Sigma \to M$ tangent to $\mathcal{D}$. However, it will be necessary to consider more generalized curves.

Definition 2.10. *Let Σ be a closed nodal Riemann surface (the set of nodes can be empty). Let $u : \Sigma \to M$ be a smooth map and let $\widetilde{u} : \widetilde{\Sigma} \to M$ be its normalization (see Definition 7.1). We say that u is a* **Reeb 2-curve** *in (M, λ, α), if the following is satisfied:*

(1) *For each $z \in \widetilde{\Sigma}$, $\widetilde{u}_*(T_z\widetilde{\Sigma}) = \mathcal{D}(\widetilde{u}(z))$, whenever $d\widetilde{u}(z) : T_z\Sigma \to T_{\widetilde{u}(z)}M$ is non-zero, and $\dim \mathcal{D}(\widetilde{u}(z)) = 2$.*
(2) *$0 \neq [u^*\alpha] \in H^1(\Sigma, \mathbb{R})$.*

It is tempting to conjecture that every closed exact lcs manifold has a Reeb 2-curve, in analogy to the Weinstein conjecture. However this is false:

Theorem 2.11. *Let T^2, g_{st} be the 2-torus with its standard flat metric. Let $M_{\widetilde{\phi},1}$ be the mapping torus of the unit contangent bundle of T^2, with $\widetilde{\phi}$ corresponding to an isometry $\phi : (T^2, g_{st}) \to (T^2, g_{st})$, which does not fix the image of any closed geodesic (an irrational rotation in both coordinates). Then $M_{\phi,1}$ has no Reeb 2-curves.*

3. Results on Reeb 2-curves and a conformal symplectic Weinstein conjecture

Definition 3.1. *Define* **the set $\mathcal{L}(M)$ of exact** lcs **structures on** M*, to be:*

$$\mathcal{L}(M) = \{(\beta, \gamma) \in \Omega^1(M) \times \Omega^1(M) \,|\, \gamma \text{ is closed, } d_\gamma \beta \text{ is non-degenerate}\}.$$

Define $\mathcal{F}(M) \subset \mathcal{L}(M)$ to be subset of (possibly irrational) first kind lcs structures.

In what follows we use the following C^∞ metric on $\mathcal{L}(M)$. For $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in \mathcal{L}(M)$ define:

$$d_\infty((\lambda_1, \alpha_1), (\lambda_2, \alpha_2)) = d_{C^\infty}(\lambda_1, \lambda_2) + d_{C^\infty}(\alpha_1, \alpha_2), \tag{3.2}$$

where d_{C^∞} on the right side is the usual C^∞ metric.

The following theorems are proved in Section 7, based on the theory of elliptic pseudo-holomorphic curves in M. We can use C^k metrics for a certain k, instead of C^∞, however we cannot make $k = 0$ (at least not obviously), and the extra complexity of working with C^k metrics, is better left for later developments.

Theorem 3.3. *Let (C, λ) be a closed contact manifold, satisfying one of the following conditions:*

(1) *(C, λ) has at least one non-degenerate Reeb orbit.*

(2) *$i(N, R^\lambda, \beta) \neq 0$ where the latter is the Fuller index of some open compact subset of the orbit space: $N \subset \mathcal{O}(R^\lambda, \beta)$, see Appendix A.*

Then we have the following:

(1) *Then for some d_∞ neighborhood U of the* lcs*-fication (λ, α) of the space $\mathcal{F}(M = C \times S^1)$, every element of U admits a Reeb 2-curve.*

(2) *For any $(\lambda', \alpha') \in U$, the corresponding Reeb 2-curve $u : \Sigma \to M$ can be assumed to be* **elliptic** *meaning that Σ is elliptic (more specifically: a nodal, topological genus 1, closed, connected Riemann surface).*

(3) *u can also be assumed to be α-charge 1 (see Definition 4.3).*

(4) *If M has dimension 4 then u can be assumed to be embedded and normal (the set of nodes is empty). And so in particular, such a u represents a closed, $(\omega = d_\alpha \lambda)$-symplectic torus hypersurface.*

3.1. **Reeb 1-curves.** We have stated some basic new Reeb dynamics phenomena in the introduction. We now discuss an application of a different character.

Definition 3.4. *A smooth map $o : S^1 \to M$ is a* **Reeb 1-curve** *in an exact lcs manifold (M, λ, α), if*

$$\forall t \in S^1 : (\lambda(o'(t)) > 0) \wedge (o'(t) \in \mathcal{D}).$$

The following is proved in Section 7.

Definition 3.5. *We say that an exact lcs manifold (M, λ, α) satisfies the* **Reeb condition** *if:*

$$\lambda(X_\alpha) > 0.$$

Theorem 3.6. *Suppose that (M, λ, α) is an exact lcs manifold satisfying the Reeb condition. If (M, λ, α) has an immersed Reeb 2-curve then it also has a Reeb 1-curve. Furthermore, if it has an immersed elliptic Reeb 2-curve, then this Reeb 2-curve is normal.*

We have an immediate corollary of Theorem 3.3 and Theorem 3.6.

Corollary 3.7. *Let λ be a contact form, on closed 3-manifold C, with at least one non-degenerate Reeb orbit, or more generally satisfying $i(N, R^\lambda, \beta) \neq 0$ for some open compact N as previously. Then there is a d_∞ neighborhood U of the* lcs*-fication (λ, α) in the space $\mathcal{F}(M = C \times S^1)$, s.t. for each $(\lambda', \alpha') \in U$ there is a Reeb 1-curve.*

Corollary 3.8. *Every closed exact lcs surface satisfying the Reeb condition has a Reeb 1-curve.*

Lemma 3.9. *Let (M, λ, α) be an exact lcs manifold with M closed then $0 \neq [\alpha] \in H^1(M, \mathbb{R})$.*

Proof. Suppose by contradiction that α is exact and let g be its primitive. Then computing we get: $d_\alpha \lambda = \frac{1}{f} d(f\lambda)$ with $f = e^g$. Consequently, $d(f\lambda)$ is non-degenerate on M which contradicts Stokes theorem. □

Proof of Corollary 3.8. This follows by Theorem 3.6. As in this case by the lemma above the identity map $X \to X$ is an immersed Reeb 2-curve, by Lemma 3.9. □

Inspired by this we conjecture:

Conjecture 1. *Suppose that (M, λ, α) is a closed exact lcs manifold of dimension 4 satisfying the Reeb condition, then it has a Reeb 1-curve.*

Proposition 3.10. *Then analogue of conjecture 1, in all dimensions, implies the Weinstein conjecture: every closed contact manifold (C, λ) has a closed Reeb orbit.*

Proof of Proposition 3.10. Let $(M = C \times S^1, \lambda, \alpha)$ be the lcs-fication of a closed contact manifold (C, λ). Then it satisfies the Reeb condition. Suppose that $o : S^1 \to M$ is a Reeb 1-curve. Then $\forall t \in [0,1] : \lambda(\dot{o}(t)) > 0$ and o is tangent to $\mathcal{V}_\lambda = \mathcal{D}$. Consequently, $pr_C \circ o$ is tangent to $\ker d\lambda$, and $\forall \tau \in [0,1] : \lambda(\mathrm{D}_t \, pr_C \circ o(\tau)) > 0$. It follows that $pr_C \circ o$ is a Reeb orbit up to parametrization. □

4. J-holomorphic curves in lcs manifolds and the definition of the invariant GWF

Let M, J be an almost complex manifold and Σ, j a Riemann surface. Recall that a map $u : \Sigma \to M$ is said to be *J-holomorphic* if $du \circ j = J \circ du$.

Notation 1. We will often say ***J-curve*** in place of J-holomorphic curve.

First kind lcs manifolds give immediate examples of almost complex manifolds where the L^2 energy functional is unbounded on the moduli spaces of fixed class J-curves, as well as where null-homologous J-curves can be non-constant. We are going to see this shortly after developing a more general theory.

Definition 4.1. *Let (M, λ, α) be an exact lcs manifold, satisfying the* **Reeb condition***: $\omega(X_\lambda, X_\alpha) = \lambda(X_\alpha) > 0$, where $\omega = d_\alpha \lambda$. In this case, $\mathcal{D}$ is a 2-dimensional distribution, and we say that an ω-compatible J is* **(λ, α)-admissible** *or ω-admissible (when λ, α are implicit) if:*

- *J preserves the canonical distribution $\mathcal{D}$ and preserves the ω-orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$. That is $J(V) \subset \mathcal{D}$ and $J(\mathcal{D}^\perp) \subset \mathcal{D}^\perp$.*
- *$d\lambda$ tames J on $\mathcal{D}^\perp$.*

Admissible J exist by classical symplectic geometry, and the space of such J is contractible see [20]. We call (λ, α, J) as above a **tamed exact** lcs **structure***, and (ω, J) is called a tamed exact* lcs *structure if $\omega = d_\alpha \lambda$, for (λ, α, J) a tamed exact* lcs *structure. In this case (M, ω, J), (M, λ, α, J) will be called a* **tamed exact** lcs **manifold**.

Example 4. If (M, λ, α) is first kind then by an elementary computation $\omega(X_\lambda, X_\alpha) = 1$ everywhere. In particular, we may find a J such that (λ, α, J) is a tamed exact lcs structure, and the space of such J is contractible. We will call (M, λ, α, J) a ***tamed first kind*** lcs manifold.

Lemma 4.2. *Let (M, λ, α, J) be a tamed first kind lcs manifold. Then given a smooth $u : \Sigma \to M$, where Σ is a closed (nodal) Riemann surface, u is J-holomorphic only if*

$$\operatorname{image} d\widetilde{u}(z) \subset \mathcal{V}_\lambda(\widetilde{u}(z))$$

for all $z \in \widetilde{\Sigma}$, where $\widetilde{u} : \widetilde{\Sigma} \to M$ is the normalization of u (see Definition 7.1). In particular $\widetilde{u}^* d\lambda = 0$.

Proof. As previously observed, by the first kind condition, $\mathcal{V}_\lambda$ is the span of X_λ, X_α and hence
$$V := \mathcal{V}_\lambda = \mathcal{D}_\lambda.$$
Let u be J-holomorphic, so that $\widetilde{u}$ is J-holomorphic (by definition of a J-holomorphic nodal map). We have
$$\int_\Sigma \widetilde{u}^* d\lambda = 0$$
by Stokes theorem. Let $proj(p) : T_pM \to V^\perp(p)$ be the projection induced by the splitting $TM = V \oplus V^\perp$.

Suppose that for some $z \in \widetilde{\Sigma}$, $proj \circ d\widetilde{u}(z) \neq 0$. By the conditions:

- J is tamed by $d\lambda$ on $V^\perp$.
- $d\lambda$ vanishes on V.
- J preserves the splitting $TM = V \oplus V^\perp$.

we have $\int_{\widetilde{\Sigma}} \widetilde{u}^* d\lambda > 0$, a contradiction. Thus,
$$\forall z \in \widetilde{\Sigma} : proj \circ d\widetilde{u}(z) = 0,$$
so
$$\forall z \in \widetilde{\Sigma} : \operatorname{image} d\widetilde{u}(z) \subset \mathcal{V}_\lambda(\widetilde{u}(z)).$$

□

Example 5. Let $(C \times S^1, \lambda, \alpha)$ be the lcs-fication of a contact manifold (C, λ). In this case
$$X_\alpha = (R^\lambda, 0),$$
where R^λ is the Reeb vector field and
$$X_\lambda = (0, \frac{d}{d\theta})$$
is the vector field generating the natural action of S^1 on $C \times S^1$.

If we denote by $\xi \subset T(C \times S^1)$ the distribution $\xi(p) = \ker \lambda(p)$, then in this case $\xi = V^\perp$ in the notation above. We then take J to be an almost complex structure on ξ, which is S^1 invariant, and compatible with $d\lambda$. The latter means that
$$g_J(\cdot, \cdot) := d\lambda|_\xi(\cdot, J\cdot)$$
is a J invariant Riemannian metric on the distribution ξ.

There is an induced almost complex structure J^λ on $C \times S^1$, which is S^1-invariant, coincides with J on ξ and which satisfies:
$$J^\lambda(X_\alpha) = X_\lambda.$$
Then $(C \times S^1, \lambda, \alpha, J^\lambda)$ is a tamed integral first kind lcs manifold.

4.1. **Charged elliptic curves in an lcs manifold.** We now study moduli spaces of elliptic curves in a lcs manifold, constrained to have a certain charge. [2] In the present context, one reason for the introduction of "charge" is that it is now possible for non-constant holomorphic curves to be null-homologous, so we need additional control. Here is a simple example: take $S^3 \times S^1$ with $J = J^\lambda$, for the λ the standard contact form, then all the Reeb holomorphic tori (as defined further below) are null-homologous.

Let Σ be a complex torus with a chosen marked point $z \in \Sigma$, i.e. an elliptic curve over $\mathbb{C}$. An isomorphism $\phi : (\Sigma_1, z_1) \to (\Sigma_2, z_2)$ is a biholomorphism s.t. $\phi(z_1) = z_2$. The set of isomorphism

[2] The name charge is inspired by the notion of charge in Oh-Wang [24], in the context of contact instantons. However, the respective notions are not obviously related.

classes forms a smooth orbifold $M_{1,1}$. This has a natural compactification - the Deligne-Mumford compactification $\overline{M}_{1,1}$, by adding a point at infinity, corresponding to a nodal genus 1 curve with one node.

The notion of charge can be defined in a general setting.

Definition 4.3. *Let M be a manifold endowed with a closed integral 1-form α. Let $u : T^2 \to M$ be a continuous map. Let $\gamma, \rho : S^1 \to T^2$ represent generators of $H_1(T^2, \mathbb{Z})$, with $\gamma \cdot \rho = 1$, where $\cdot$ is the intersection pairing with respect to the standard complex orientation on T^2. Suppose in addition:*

$$\langle \gamma, u^* \alpha \rangle = 0, \quad \langle \rho, u^* \alpha \rangle \neq 0,$$

where $\langle , \rangle$ is the natural pairing of homology and cohomology. Then we call

$$n = |\langle \rho, u^* \alpha \rangle| \in \mathbb{N}_{>0},$$

*the α-**charge of** u, or just the charge of u when α is implicit. Suppose furthermore that $\langle \rho, u^* \alpha \rangle > 0$, then the class $u_*(\gamma) \in \pi_1(M)$ will be called the π-**class of** u, for $\pi_1(M)$ the set of free homotopy classes of loops as before.*

It is easy to see that charge is always defined and is independent of choices above. We may extend the definition of charge to curves $u : \Sigma \to M$, with Σ a nodal elliptic curve, as follows. If $\rho : S^1 \to \Sigma$ represents the generator of $H_1(\Sigma, \mathbb{Z})$ then define the charge of u to be $|\langle \rho, u^* \alpha \rangle|$. Obviously the charge condition is preserved under Gromov convergence of stable maps. But it is not preserved in homology, so that charge is *not* a functional $H_2(M, \mathbb{Z}) \to \mathbb{N}$. Thus associated to a map $u : \Sigma \to M$ with Σ an elliptic curve, and non-zero α-charge, we have a triple $(A, \beta, n) \in H^2(M, \mathbb{Z}) \times \pi_1(M) \times \mathbb{N}_{>0}$, corresponding to the homology class, the π-class, and the α-charge. This triple will be called the ***charge class*** of u.

Let (M, J) be an almost complex manifold and α a closed integral 1-form on M non vanishing in cohomology, then we call (M, J, α) a ***Lee manifold***. Suppose for the moment that there are no non-constant J-holomorphic maps $(S^2, j) \to (M, J)$ (otherwise we need stable maps), then for $n \geq 1$ we define:

$$\overline{\mathcal{M}}_{1,1}^n(J, A, \beta)$$

as the set of equivalence classes of tuples (u, S), for $S = (\Sigma, z)$ a possibly nodal elliptic curve and $u : \Sigma \to M$ a charge class (A, β, n), J-holomorphic map. The equivalence relation is $(u_1, S_1) \sim (u_2, S_2)$ if there is an isomorphism $\phi : S_1 \to S_2$ s.t. $u_2 \circ \phi = u_1$. It is not hard to see that such an isomorphism of preserves the charge class, so that $\overline{\mathcal{M}}_{1,1}^n(J, A, \beta)$ is well defined.

Also note that the expected dimension of $\overline{\mathcal{M}}_{1,1}^1(J^\lambda, A, \beta)$ is 0. It is given by the Fredholm index of the operator (6.5) which is 2, minus the dimension of the reparametrization group (for non-nodal curves) which is 2. That is given an elliptic curve $S = (\Sigma, z)$, let $\mathcal{G}(\Sigma)$ be the 2-dimensional group of biholomorphisms ϕ of Σ. Then given a J-holomorphic map $u : \Sigma \to M$, (Σ, z, u) is equivalent to $(\Sigma, \phi(z), u \circ \phi)$ in $\overline{\mathcal{M}}_{1,1}^1(J^\lambda, A, \beta)$, for $\phi \in \mathcal{G}(\Sigma)$.

By slight abuse we may just denote such an equivalence class above simply by u, so we may write $u \in \overline{\mathcal{M}}_{1,1}^n(J, A, \beta)$, with S implicit.

4.2. **Reeb holomorphic tori in** $(C \times S^1, J^\lambda)$**.** In this section we discuss an important example. Let (C, λ) be a contact manifold and let α and J^λ be as in Example 5. So that in particular we get a Lee manifold $(C \times S^1, J^\lambda, \alpha)$.

In this case we have one natural type of charge 1 J^λ-holomorphic tori in $M = C \times S^1$. Let o be a period c, closed Reeb orbit o of R^λ, and let β it's class in $\pi_1(C) \subset \pi_1(M)$. A ***Reeb torus*** u_o for o is the map

$$u_o : (S^1 \times S^1 = T^2) \to C \times S^1$$
$$u_o(s,t) = (o(s),t).$$

A Reeb torus is J^λ-holomorphic for a uniquely determined holomorphic structure j on T^2 defined by:

$$j(\frac{\partial}{\partial s}) = c\frac{\partial}{\partial t}.$$

4.3. **Definition of the invariant** GWF**.** Let X be a manifold. For g a taut metric on X, let λ_g be the Liouville 1-form on the unit cotangent bundle C of X. If ϕ is an isometry of g then there is a strict contactomorphism $\widetilde{\phi}$ of (C,λ_g), and this gives the "mapping torus" lcs manifold $(M_{\widetilde{\phi},1},\lambda_{\widetilde{\phi}},\alpha)$ as described in Section 7.1.

If $\beta \in \pi_1^{inc}(X)$, let $\widetilde{\beta} \in \pi_1(C)$ denote the lift of the class, defined by representing β by a unit speed closed geodesic o, taking the canonical lift $\widetilde{o}$ to a closed Reeb orbit, and setting $\widetilde{\beta} = [\widetilde{o}]$. Given $n \geq 1$, suppose that

$$\widetilde{\phi}^n_*(\widetilde{\beta}) = \widetilde{\beta}. \tag{4.4}$$

Then, as explained in Section 7.1, this naturally induces a map $u^n : T^2 \to M$ well defined up to homotopy, whose class in homology is denoted by $A^n_{\widetilde{\beta}} \in H_2(M,\mathbb{Z})$, and s.t. the α-charge of u^n is n.

By the tautness assumption on g the space $\mathcal{O}(R^{\lambda_g},\widetilde{\beta})$ is compact. We then get that $\overline{\mathcal{M}}^n_{1,1}(J^{\lambda_\phi},A^n_{\widetilde{\beta}},\widetilde{\beta})$ is compact and has expected dimension 0 by the Proposition 7.2. We then define

$$\mathrm{GWF}(g,\phi,\beta,n) := \begin{cases} 0, & \text{if (4.4) is not satisfied} \\ GW^n_{1,1}(J^{\lambda_\phi},A^n_{\widetilde{\beta}},\widetilde{\beta})([\overline{M}_{1,1}]\otimes[C\times S^1]), & \text{otherwise} \end{cases},$$

where the Gromov-Witten invariant on the right side is as in (5.2), of the following section. Although we take here a specific almost complex structure J^ϕ, using Proposition 7.2 and Lemma 5.3 we may readily deduce that any (λ_ϕ,α)-admissible almost complex structure gives the same value for the invariant.

Also note that when $\phi = id$

$$\forall n \in \mathbb{N} : GWF(g,id,\beta,n) = F(g,\beta), \tag{4.5}$$

where the latter is the invariant studied in [29]. This readily follows by Theorem 6.8.

5. Elements of Gromov-Witten theory of an almost complex manifold

Suppose that (M,J) is an almost complex manifold (possibly non-compact), where the almost complex structures J are assumed throughout the paper to be C^∞. Let $N \subset \overline{\mathcal{M}}_{g,k}(J,A)$ be an open compact subset with energy positive on N. The latter energy condition is only relevant when $A = 0$. We shall primarily refer in what follows to work of Pardon in [25], being more familiar to the author. But we should mention that the latter is a follow up to a theory that is originally created by Fukaya-Ono [11], and later expanded with Oh-Ohta [10].

The construction in [25] of an implicit atlas, on the moduli space $\mathcal{M}$ of J-curves in a symplectic manifold, only needs a neighborhood of $\mathcal{M}$ in the space of all curves. So for an *open* compact component N as above, we have a well defined natural implicit atlas, (or a Kuranishi structure in the setup of [11]). And so such an N will have a virtual fundamental class in the sense of [25]. This understanding will be used in other parts of the paper, following Pardon for the explicit setup.

We may thus define functionals:

$$GW_{g,n}(N,J,A) : H_*(\overline{M}_{g,n}) \otimes H_*(M) \to \mathbb{Q}. \tag{5.1}$$

In our more specific context we must in addition restrict the charge, which is defined at the moment for genus 1 curves. So supposing (M, J, α) is a Lee manifold we may likewise define functionals:

$$GW^k_{1,1}(N, J, A, \beta) : H_*(\overline{M}_{1,1}) \otimes H_*(M) \to \mathbb{Q}, \tag{5.2}$$

meaning that we restrict the count to charge class (A, β, k) curves, with $N \subset \overline{\mathcal{M}}^k_{1,1}(J, A, \beta)$, an open compact subset. If N is not specified it is understood to be the whole moduli space (if it is known to be compact).

We now study how functionals depend on N, J. To avoid unnecessary generality, we discuss the case of $GW^k_{1,1}(N, J, A, \beta)$. Given a Frechet smooth family $\{J_t\}$, $t \in [0, 1]$, on M, we denote by $\overline{\mathcal{M}}^k_{1,1}(\{J_t\}, A, \beta)$ the space of pairs (u, t), $u \in \overline{\mathcal{M}}^k_{1,1}(J_t, A, \beta)$.

Lemma 5.3. *Let $\{J_t\}$, $t \in [0, 1]$ be a Frechet smooth family of almost complex structures on M. Suppose that $\widetilde{N}$ is an open compact subset of the cobordism moduli space $\overline{\mathcal{M}}^k_{1,1}(\{J_t\}, A, \beta)$, with $k > 0$. Let*

$$N_i = \widetilde{N} \cap \left(\overline{\mathcal{M}}^k_{1,1}(J_i, A, \beta)\right),$$

then

$$GW^k_{1,1}(N_0, J_0, A) = GW^k_{1,1}(N_1, J_1, A, \beta).$$

In particular if $GW^k_{1,1}(N_0, A, J_0, \beta) \neq 0$, there is a J_1-holomorphic, stable, charge class (A, β, k) elliptic curve in M.

Proof of Lemma 5.3. We may construct exactly as in [25] a natural implicit atlas on $\widetilde{N}$, with boundary $N_0^{op} \sqcup N_1$, (op denoting opposite orientation). Note that the condition that $k > 0$ is essential as otherwise we may have boundary components of $\widetilde{N}$ corresponding to degenerations to constant curves. And so we immediately get

$$GW^k_{1,1}(N_0, J_0, A, \beta) = GW^k_{1,1}(N_1, J_1, A, \beta).$$

□

Remark 5.4. In the case the manifold is closed, degenerations to constant curves are impossible when each J_t is tamed by a t-continuous family ω_t of symplectic or lcs forms, [30]. They are also impossible for general closed almost complex manifolds, for rational curves, by energy quantization. But as far as I know, such degenerations might happen for genus 1 curves in general closed almost complex manifolds.

The following generalization of the lemma above will be useful later. First a definition.

Definition 5.5. *Let M be a smooth manifold. Denote by $H^{inc}_2(M)$, the set of boundary incompressible homology classes, defined analogously to Definition 1.1, We say that a Frechet smooth family $\{J_t\}$, $t \in [0, 1]$ on a manifold M has a* **right holomorphic sky catastrophe** *in charge class (A, β, k) for $A \in H^{inc}_2(M)$, if there is an element $u \in \overline{\mathcal{M}}^k_{1,1}(J_0, A, \beta)$, which does not belong to any open compact subset of $\overline{\mathcal{M}}^k_{1,1}(\{J_t\}, A, \beta)$. We say that the sky catastrophe is* **essential** *if the same is true for any smooth family $\{J'_t\}$ satisfying $J'_0 = J_0$ and $J'_1 = J_1$.*

Lemma 5.6. *Let $\{J_t\}$, $t \in [0, 1]$ be a Frechet smooth family of almost complex structures on M, $A \in H^{inc}_2(M)$ and $k > 0$. Suppose that $\overline{\mathcal{M}}^k_{1,1}(J_0, A, \beta)$ is compact, and there is no right holomorphic sky catastrophe for $\{J_t\}$. Then there is a charge class (A, β, k), J_1-holomorphic, stable, elliptic curve in M.*

Proof. By assumption for each $u \in \overline{\mathcal{M}}^k_{1,1}(J_0, A, \beta)$ there is an open compact $u \ni \mathcal{C}_u \subset \overline{\mathcal{M}}^k_{1,1}(\{J_t\}, A, \beta)$. Then $\{\mathcal{C}_u \cap \overline{\mathcal{M}}^k_{1,1}(J_0, A, \beta)\}_u$ is an open cover of $\overline{\mathcal{M}}^k_{1,1}(J_0, A, \beta)$ and so has a finite sub-cover, corresponding to a collection $u_1, \ldots, u_n$.

Set

$$\widetilde{N} = \bigcup_{i \in \{1,\ldots,n\}} \mathcal{C}_i.$$

Then $\widetilde{N}$ is an open-compact subset of $\overline{\mathcal{M}}^k_{1,1}(\{J_t\}, A)$ s.t. $\widetilde{N} \cap \overline{\mathcal{M}}^k_{1,1}(J_0, A, \beta) = N_0 := \overline{\mathcal{M}}^k_{1,1}(J_0, A, \beta)$. Then the result follows by Lemma 5.3.

□

We now state a basic technical lemma, following some standard definitions.

Definition 5.7. *An* **almost symplectic pair** *on M is a tuple (ω, J), where ω is a non-degenerate 2-form on M, and J is ω-compatible, meaning that $\omega(\cdot, J\cdot)$ defines J-invariant Riemannian metric, denoted by g_J (with ω implicit).*

Definition 5.8. *We say that a pair of almost symplectic pairs (ω_i, J_i) are δ-**close**, if ω_0, ω_1 are C^∞ δ-close, and J_0, J_1 are C^∞ δ-close, $i = 0, 1$.*

Let $\mathcal{S}(A)$ denote the space of equivalence classes of all smooth, nodal, stable, charge k, elliptic curves in M in class A, with the standard Gromov topology determined by g_J. That is elements of $\mathcal{S}(A)$ are like elements of $\overline{\mathcal{M}}^k_{1,1}(J, A, \beta)$ but are not required to be J-holomorphic. In particular, we have a continuous function:

$$e = e_{g_J} : \mathcal{S}(A) \to \mathbb{R}_{\geq 0}.$$

Lemma 5.9. *Let (ω, J) be an almost symplectic pair on a compact manifold M and let $N \subset \overline{\mathcal{M}}^k_{1,1}(J, A, \beta)$ be compact and open (as a subset of $\overline{\mathcal{M}}^k_{1,1}(J, A)$). Then there exists an open $U \subset \mathcal{S}(A)$ satisfying:*

1. *e is bounded on $\overline{U}$.*
2. *$U \supset N$.*
3. *$\overline{U} \cap \overline{\mathcal{M}}^k_{1,1}(J, A, \beta) = N$.*

Proof. The Gromov topology on $\mathcal{S}(A)$ has a basis $\mathcal{B}$ satisfying:

1. If $V \in \mathcal{B}$ then e is bounded on $\overline{V}$.
2. If U is open and $u \in U$, then
$$\exists V \in \mathcal{B} : (u \in V) \wedge (\overline{V} \subset U).$$

In the genus 0 case this is contained in the classical text McDuff-Salamon [21, page 140]. The basis $\mathcal{B}$ is defined using a collection of "quasi distance functions" $\{\rho_\epsilon\}_\epsilon$ on the set stable maps. The higher genus case is likewise well known.

Thus, since N is relatively open, using the properties of $\mathcal{B}$ above, we may find a collection $\{V_\alpha\} \subset \mathcal{B}$ s.t.

- $\{V_\alpha\}$ covers N.
- $\overline{V}_\alpha \cap \overline{\mathcal{M}}^k_{1,1}(J, A, \beta) \subset N$.

As N is compact, we have a finite subcover $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$. Set $U := \cup_{i \in \{1,\ldots,n\}} V_{\alpha_i}$. Then U satisfies the conclusion of the lemma. □

Lemma 5.10. *Let (M, ω, J, α) be as above, $N \subset \overline{\mathcal{M}}^k_{1,1}(J, A, \beta)$ an open compact set, and U as in the lemma above. Then there is a $\delta > 0$ s.t. whenever J' is C^2 δ-close to J if $u \in \overline{\mathcal{M}}^k_{1,1}(J', A, \beta)$ and $u \in \overline{U}$ then $u \in U$.*

Proof. Suppose otherwise, then there is a sequence $\{J_k\}$ C^2 converging to J, and a sequence $\{u_k\} \in \overline{U} - U$ of J_k-holomorphic stable maps. Then by property 1 e_{g_J} is bounded on $\{u_k\}$. Hence, by Gromov compactness, specifically theorems [21, B.41, B.42], we may find a Gromov convergent subsequence $\{u_{k_j}\}$ to a J-holomorphic stable map $u \in \overline{U} - U$. But by Properties 3, 2 of the set U,

$$(\overline{U} - U) \cap \overline{\mathcal{M}}^k_{1,1}(J, A, \beta) = \emptyset.$$

So that we obtain a contradiction. □

Lemma 5.11. *Let M, ω, J, α and $N \subset \overline{\mathcal{M}}^k_{1,1}(J, A, \beta)$ be as in the previous lemma. Then there is a $\delta > 0$ s.t. the following is satisfied. Let (ω', J') be δ-close to (ω, J), then there is a continuous in the C^∞ topology family $\{J_t\}$, $J_0 = J$, $J_1 = J'$ s.t. there is an open compact subset*

$$\widetilde{N} \subset \overline{\mathcal{M}}^k_{1,1}(\{J_t\}, A, \beta),$$

satisfying

$$\widetilde{N} \cap \overline{\mathcal{M}}^k_{1,1}(J, A, \beta) = N.$$

Proof. First let δ be as in Lemma 5.10. We then need:

Lemma 5.12. *Given a $\delta > 0$ there is a $\delta' > 0$ s.t. if (ω', J') is δ'-near (ω, J) then there is a continuous in the C^∞ topology family $\{(\omega_t, J_t)\}$ satisfying:*

- *(ω_t, J_t) is δ-close to (ω, J) for each t.*
- *$(\omega_0, J_0) = (\omega, J)$ and $(\omega_1, J_1) = (\omega', J')$.*

Proof. Let $\{g_t\}$ be the family of metrics on M given by the convex linear combination of $g = g_{\omega_J}, g' = g_{\omega', J'}$, $g_t = (1-t)g + tg'$. Clearly g_t is C^∞ δ'-close to g_0 for each t. Likewise, the family of 2 forms $\{\omega_t\}$ given by the convex linear combination of ω, ω' is non-degenerate for each t if δ' was chosen to be sufficiently small. And each ω_t is C^∞ δ'-close to $\omega_0 = \omega_{g,J}$.

Let

$$ret : Met(M) \times \Omega(M) \to \mathcal{J}(M)$$

be the "retraction map" (it can be understood as a retraction followed by projection) as defined in [20, Prop 2.50], where $Met(M)$ is space of metrics on M, $\Omega(M)$ the space of 2-forms on M, and $\mathcal{J}(M)$ the space of almost complex structures. This map has the property that the almost complex structure $ret(g, \omega)$ is compatible with ω, and that $ret(g_J, \omega) = J$ for $g_J = \omega(\cdot, J\cdot)$. Then $\{(\omega_t, ret(g_t, \omega_t))\}$ is a compatible family. As ret is continuous in C^∞-topology, δ' can be chosen such that $\{ret_t(g_t, \omega_t\}$ are C^∞ δ-nearby. □

Returning to the proof of the main lemma. Let $\delta' < \delta$ be chosen as in Lemma 5.12 and let $\{(\omega_t, J_t)\}$ be the corresponding family. Set

$$\widetilde{N} = \overline{\mathcal{M}}^k_{1,1}(\{J_t\}, A, \beta) \cap (U \times [0, 1]),$$

where U is as in Lemma 5.10.

Then $\widetilde{N}$ is an open subset of $\overline{\mathcal{M}}_{1,1}^k(\{J_t\}, A, \beta)$. By Lemma 5.10,

$$\widetilde{N} = \overline{\mathcal{M}}_{1,1}^k(\{J_t\}, A, \beta) \cap (\overline{U} \times [0,1]),$$

so that $\widetilde{N}$ is also closed.

Finally, $\sup_{(u,t)\in\widetilde{N}} e_{g_t}(u) < \infty$, by condition 1 of U, and since $\{e_{g_t}\}$, $t \in [0,1]$ is a continuous family. Consequently $\widetilde{N}$ is compact by the Gromov compactness theorem. Resetting $\delta := \delta'$, we are then done with the proof of the main lemma. □

Proposition 5.13. *Given an almost complex manifold M, J suppose that $N \subset \overline{\mathcal{M}}_{1,1}^k(J, A)$ is open and compact. Suppose also that $GW_{1,1}^k(N, J, A, \beta) \neq 0$. Then there is a $\delta > 0$ s.t. whenever J' is C^2 δ-close to J, there exists $u \in \overline{\mathcal{M}}_{1,1}^k(J', A, \beta)$.*

Proof. For N as in the hypothesis, let U, δ and $\widetilde{N}$ be as in Lemma 5.11, then by the conclusion of that lemma and by Lemma 5.3

$$GW_{1,1}^k(N_1, J', A, \beta) = GW_{1,1}^k(N, J, A, \beta) \neq 0,$$

where $N_1 = \widetilde{N} \cap \overline{\mathcal{M}}_{1,1}^k(J_1, A, \beta)$. □

6. Elliptic curves in the lcs-fication of a contact manifold and the Fuller index

The following elementary result is crucial for us.

Lemma 6.1. *Let (M, λ, α, J) be a tamed first kind* lcs *manifold. Then every non-constant (nodal) J-holomorphic curve $u : \Sigma \to M$ is a Reeb 2-curve.*

Proof of Lemma 6.1. Let $u : \Sigma \to M$ be a non-constant, nodal J-curve. By Lemma 4.2 it is enough to show that $[u^*\alpha] \neq 0$. Let $\widetilde{M}$ denote the α-covering space of M, that is the space of equivalence classes of paths p starting at $x_0 \in M$, with a pair p_1, p_2 equivalent if $p_1(1) = p_2(1)$ and

$$\int_{[0,1]} p_1^*\alpha = \int_{[0,1]} p_2^*\alpha.$$

Then the lift of ω to $\widetilde{M}$ is

$$\widetilde{\omega} = \frac{1}{f} d(f\lambda),$$

where $f = e^{-g}$ and where g is a primitive for the lift $\widetilde{\alpha}$ of α to $\widetilde{M}$, that is $\widetilde{\alpha} = dg$. In particular $\widetilde{\omega}$ is conformally symplectomorphic to an exact symplectic form on $\widetilde{M}$. So if $\widetilde{J}$ denotes the lift of J, any closed $\widetilde{J}$-curve is constant by Stokes theorem. Now if $[u^*\alpha] = 0$ then u has a lift to a $\widetilde{J}$-holomorphic map $v : \Sigma \to \widetilde{M}$. Since Σ is closed, it follows by the above that v is constant, so that u is constant, which is impossible. □

6.1. Preliminaries on Reeb tori.

Let $(M = C \times S^1, \lambda, \alpha)$ be the lcs-fication of (C, λ). For $\beta \in \pi_1(C)$ we set $A_\beta^1 = \beta \otimes [S^1] \in H_2(M, \mathbb{Z})$. Let $\mathcal{O}(R^\lambda, \beta)$, be the orbit space as in Section 1.1. Let J^λ on $C \times S^1$ be as in Section 4.2. We have a map:

$$\mathcal{P} : \mathcal{O}(R^\lambda, \beta) \to \overline{\mathcal{M}}_{1,1}^1(J^\lambda, A_\beta^1, \beta), \quad \mathcal{P}(o) = u_o, \tag{6.2}$$

for u_o the Reeb torus as previously. We can say more:

Proposition 6.3. *For any (λ, α)-admissible J there is a natural bijection:* [3]

$$\mathcal{P} : \mathcal{O}(R^\lambda, \beta) \to \overline{\mathcal{M}}^1_{1,1}(J, A^1_\beta, \beta),$$

with $\mathcal{P}$ the map (6.2) in the case $J = J^\lambda$. (Note that there is an analogous bijection $\mathcal{O}(R^\lambda, \beta) \to \overline{\mathcal{M}}^n_{1,1}(J, A^n_\beta, \beta)$, for $n > 1$, where $A^n_\beta = n \cdot \beta \otimes [S^1]$).

In the particular case of J^λ, we see that all elliptic curves in $C \times S^1$ are Reeb tori, and hence the underlying complex structure on the domain is "rectangular". That is, they are quotients of the complex plane by a rectangular lattice. This stops being the case when we consider generalized Reeb tori in Section 7.1 for the mapping torus of some contactomorphism. Moreover, for more general compatible complex structures we might have nodal degenerations.

Proof of Proposition 6.3. We define $\mathcal{P}(o)$ to be the class represented by the unique up to isomorphism J-holomorphic curve $u : T^2 \to M$ determined by the conditions:

- u is charge 1.
- The image of u is the image $\mathcal{T}$ of the map $u_o : T^2 \to M$, $(s,t) \to (o(s), t)$, i.e. the image of the Reeb torus of o.
- The degree of the map $u : T^2 \to \mathcal{T}$ is the multiplicity of o.

We need to show that $\mathcal{P}$ is bijective. Injectivity is automatic. Suppose we have a curve $u \in \overline{\mathcal{M}}^1_{1,1}(J, A, \beta)$, represented by $u : \Sigma \to M$. By Lemma 6.1 u is a Reeb 2-curve. Then u has no spherical components, as such a component corresponds to a J^λ-holomorphic map $u' : \mathbb{CP}^1 \to M$, which by Lemma 6.1 is also a Reeb 2-curve, and this is impossible by second property in the definition.

We first show that u is a finite covering map onto the image of some Reeb torus u_o.

By Lemma 7.7 normalization $\widetilde{u}$ is also a Reeb 2-curve. If u is not normal then $\widetilde{u}$ is a Reeb 2-curve with domain $\mathbb{CP}^1$, which is impossible by the argument above. Hence u is normal.

By the charge 1 condition $pr_{S^1} \circ u$ is surjective, where $pr_{S^1} : C \times S^1 \to S^1$ is the projection. By the Sard theorem we have a regular value $t_0 \in S^1$, so that $u^{-1} \circ pr^{-1}_{S^1}(t_0)$ contains an embedded circle $S_0 \subset \Sigma$. Now $d(pr_{S^1} \circ u)$ is surjective onto $T_{t_0} S^1$ along $T\Sigma|_{S_0}$. And so by first property of u being a Reeb 2-curve, $o = pr_C \circ u|_{S_0}$ has non-vanishing differential $d(o)$. Moreover, again by the first property, o is tangent to $\ker d\lambda$. It follows that o is an unparametrized λ-Reeb orbit.

Also, the image of $d(pr_C \circ u)$ is in $\ker d\lambda$ from which it follows that $\operatorname{image} d(pr_C \circ u) = \operatorname{image} d(o)$. By Sard's theorem and by basic differential topology it follows that the image of u is contained in the image of the Reeb torus u_o, which is an embedded 2-torus $\mathcal{T}$.

By J^λ-holomorphicity of u, since $\Sigma \simeq T^2$, and by basic complex analysis of holomorphic maps $T^2 \to T^2$, u is a holomorphic covering map onto $\mathcal{T}$, of degree $\deg u$.

Let $\widetilde{o}$ be $\deg u$ cover of o. Then $\mathcal{P}(\widetilde{o})$ is also represented by a degree $\deg u$, charge one holomorphic covering map $u' : T^2 \to \mathcal{T}$. By basic covering map theory there is a homeomorphism of covering spaces:

$$\begin{tikzcd} T^2 \arrow[r, "f"] \arrow[d, "u"] & T^2 \arrow[dl, "u'"] \\ \mathcal{T} & \end{tikzcd}.$$

Then f is a biholomorphism, so that u, u' are equivalent. $\square$

[3]It is in fact an equivalence of the corresponding topological action groupoids, but we do not need this explicitly.

Proposition 6.4. *Let (C, ξ) be a general contact manifold. If λ is a non-degenerate contact 1-form for ξ then all the elements of $\overline{\mathcal{M}}^1_{1,1}(J^\lambda, A, \beta)$ are regular curves. Moreover, if λ is degenerate then for a period c Reeb orbit o, the kernel of the associated real linear Cauchy-Riemann operator for the Reeb torus u_o is naturally identified with the 1-eigenspace of $\phi^\lambda_{c,*}$ - the time c linearized return map $\xi(o(0)) \to \xi(o(0))$ induced by the R^λ Reeb flow.*

Proof. We already know by Proposition 6.3 that all $u \in \overline{\mathcal{M}}^1_{1,1}(J^\lambda, A, \beta)$ are equivalent to Reeb tori. In particular, such curves have a representation by a J^λ-holomorphic map

$$u : (T^2, j) \to (Y = C \times S^1, J^\lambda).$$

Since each u is immersed we may naturally get a splitting $u^*T(Y) \simeq N \times T(T^2)$, using the g_J metric, where $N \to T^2$ denotes the pull-back, of the g_J-normal bundle to image u, and which is identified with the pullback of the distribution ξ_λ on Y, (which we also call the co-vanishing distribution).

The full associated real linear Cauchy-Riemann operator takes the form:

$$D^J_u : \Omega^0(N \oplus T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2), N \oplus T(T^2)). \tag{6.5}$$

This is an index 2 Fredholm operator (after standard Sobolev completions), whose restriction to $\Omega^0(N \oplus T(T^2))$ preserves the splitting, that is the restricted operator splits as

$$D \oplus D' : \Omega^0(N) \oplus \Omega^0(T(T^2)) \to \Omega^{0,1}(T(T^2), N) \oplus \Omega^{0,1}(T(T^2), T(T^2)).$$

On the other hand the restricted Fredholm index 2 operator

$$\Omega^0(T(T^2)) \oplus T_j M_{1,1} \to \Omega^{0,1}(T(T^2)),$$

is surjective by classical Teichmuller theory, see also [35, Lemma 3.3] for a precise argument in this setting. It follows that D^J_u will be surjective if the restricted Fredholm index 0 operator

$$D : \Omega^0(N) \to \Omega^{0,1}(N),$$

has no kernel.

The bundle N is symplectic with symplectic form on the fibers given by restriction of $u^*d\lambda$, and together with J^λ this gives a Hermitian structure (g_λ, j_λ) on N. We have a linear symplectic connection $\mathcal{A}$ on N, which over the slices $S^1 \times \{t\} \subset T^2$ is induced by the pullback by u of the linearized R^λ Reeb flow. Specifically the $\mathcal{A}$-transport map from the fiber $N_{(s_0,t)}$ to the fiber $N_{(s_1,t)}$ over the path $[s_0, s_1] \times \{t\} \subset T^2$, is given by

$$(u_*|_{N_{(s_1,t)}})^{-1} \circ (\phi^\lambda_{c(s_1-s_0)})_* \circ u_*|_{N_{(s_0,t)}},$$

where $\phi^\lambda_{c(s_1-s_0)}$ is the time $c \cdot (s_1 - s_0)$ map for the R^λ Reeb flow, where c is the period of the Reeb orbit o_u, and where $u_* : N \to TY$ denotes the natural map, (it is the universal map in the pull-back diagram.)

The connection $\mathcal{A}$ is defined to be trivial in the θ_2 direction, where trivial means that the parallel transport maps are the id maps over θ_2 rays. In particular the curvature $R_\mathcal{A}$, understood as a lie algebra valued 2-form, of this connection vanishes. The connection $\mathcal{A}$ determines a real linear CR operator $D_\mathcal{A}$ on N in the standard way, take the complex anti-linear part of the vertical differential of a section. Explicitly,

$$D_\mathcal{A} : \Omega^0(N) \to \Omega^{0,1}(N),$$

is defined by

$$D_\mathcal{A}(\mu)(p) = j_\lambda \circ \pi^{vert}(\mu(p)) \circ d\mu(p) - \pi^{vert}(\mu(p)) \circ d\mu(p) \circ j,$$

where

$$\pi^{vert}(\mu(p)) : T_{\mu(p)}N \to T^{vert}_{\mu(p)}N \simeq N$$

is the $\mathcal{A}$-projection, and where $T^{vert}_{\mu(p)}N$ is the kernel of the projection $T_{\mu(p)}N \to T_p\Sigma$. It is elementary to verify from the definitions that this operator is exactly D. See also [23, Section 10.1] for a computation of this kind in much greater generality.

We have a differential 2-form Ω on the total space of N defined as follows. On the fibers $T^{vert}N$, $\Omega = u_*\omega$, for $\omega = d_\alpha\lambda$, and for $T^{vert}N \subset TN$ denoting the vertical tangent space, or subspace of vectors v with $\pi_* v = 0$, for $\pi : N \to T^2$ the projection. While on the $\mathcal{A}$-horizontal distribution Ω is defined to vanish. The 2-form Ω is closed, which we may check explicitly by using that $R_{\mathcal{A}}$ vanishes to obtain local symplectic trivializations of N in which $\mathcal{A}$ is trivial. Clearly Ω must vanish on the 0-section since it is a $\mathcal{A}$-flat section. But any section is homotopic to the 0-section and so in particular if $\mu \in \ker D$ then Ω vanishes on μ.

Since $\mu \in \ker D$, and so its vertical differential is complex linear, it follows that the vertical differential vanishes. To see this note that $\Omega(v, J^\lambda v) > 0$, for $0 \neq v \in T^{vert}N$ and so if the vertical differential did not vanish we would have $\int_\mu \Omega > 0$. So μ is $\mathcal{A}$-flat, in particular the restriction of μ over all slices $S^1 \times \{t\}$ is identified with a period c orbit of the linearized at o R^λ Reeb flow, and which does not depend on t as $\mathcal{A}$ is trivial in the t variable. So the kernel of D is identified with the vector space of period c orbits of the linearized at o R^λ Reeb flow, as needed. □

Proposition 6.6. *Let λ be a contact form on a $(2n+1)$-fold C, and o a non-degenerate, period c, λ-Reeb orbit, then the orientation of $[u_o]$ induced by the determinant line bundle orientation of $\overline{\mathcal{M}}^1_{1,1}(J^\lambda, A)$, is $(-1)^{CZ(o)-n}$, which is*

$$\operatorname{sign}\operatorname{Det}(\operatorname{Id}|_{\xi(o(0))} - \phi^\lambda_{c,*}|_{\xi(o(0))}).$$

Proof of Proposition 6.6. Abbreviate u_o by u. Let $N \to T^2$ be the vector bundle associated to u as in the proof of Proposition 6.4. Fix a trivialization ϕ of N induced by any trivialization of the contact distribution ξ along o in the obvious sense: N is the pullback of ξ along the composition

$$T^2 \to S^1 \xrightarrow{o} C.$$

Let the symplectic connection $\mathcal{A}$ on N be defined as before. Then the pullback connection $\mathcal{A}' := \phi^*\mathcal{A}$ on $T^2 \times \mathbb{R}^{2n}$ is a connection whose parallel transport paths $p_t : [0,1] \to \operatorname{Symp}(\mathbb{R}^{2n})$, along the closed loops $S^1 \times \{t\}$, are paths starting at 1, and are t independent. And so the parallel transport path of $\mathcal{A}'$ along $\{s\} \times S^1$ is constant, that is $\mathcal{A}'$ is trivial in the t variable. We shall call such a connection $\mathcal{A}'$ on $T^2 \times \mathbb{R}^{2n}$ *induced by* p.

By non-degeneracy assumption on o, the map $p(1)$ has no 1-eigenvalues. Let $p'' : [0,1] \to \operatorname{Symp}(\mathbb{R}^{2n})$ be a path from $p(1)$ to a unitary map $p''(1)$, with $p''(1)$ having no 1-eigenvalues, and s.t. p'' has only simple crossings with the Maslov cycle. Let p' be the concatenation of p and p''. We then get

$$CZ(p') - \frac{1}{2}\operatorname{sign}\Gamma(p', 0) \equiv CZ(p') - n \equiv 0 \mod 2,$$

since p' is homotopic relative end points to a unitary geodesic path h starting at id, having regular crossings, and since the number of negative, positive eigenvalues is even at each regular crossing of h by unitarity. Here $\operatorname{sign}\Gamma(p', 0)$ is the index of the crossing form of the path p' at time 0, in the notation of [26]. Consequently,

$$CZ(p'') \equiv CZ(p) - n \mod 2, \tag{6.7}$$

by additivity of the Conley-Zehnder index.

Let us then define a free homotopy $\{p_t\}$ of p to p', p_t is the concatenation of p with $p''|_{[0,t]}$, reparametrized to have domain $[0,1]$ at each moment t. This determines a homotopy $\{\mathcal{A}'_t\}$ of connections induced by $\{p_t\}$. By the proof of Proposition 6.4, the CR operator D_t determined by each $\mathcal{A}'_t$ is surjective except at some finite collection of times $t_i \in (0,1)$, $i \in N$ determined by the crossing times of p'' with the

Maslov cycle, and the dimension of the kernel of D_{t_i} is the 1-eigenspace of $p''(t_i)$, which is 1 by the assumption that the crossings of p'' are simple.

The operator D_1 is not complex linear. To fix this we concatenate the homotopy $\{D_t\}$ with the homotopy $\{\widetilde{D}_t\}$ defined as follows. Let $\{\widetilde{\mathcal{A}}_t\}$ be a homotopy of $\mathcal{A}'_1$ to a unitary connection $\widetilde{\mathcal{A}}_1$, where the homotopy $\{\widetilde{\mathcal{A}}_t\}$ is through connections induced by paths $\{\widetilde{p}_t\}$, giving a path homotopy of $p' = \widetilde{p}_0$ to h. Then $\{\widetilde{D}_t\}$ is defined to be induced by $\{\widetilde{\mathcal{A}}_t\}$.

Let us denote by $\{D'_t\}$ the concatenation of $\{D_t\}$ with $\{\widetilde{D}_t\}$. By construction, in the second half of the homotopy $\{D'_t\}$, D'_t is surjective. And D'_1 is induced by a unitary connection, since it is induced by unitary path $\widetilde{p}_1$. Consequently, D'_1 is complex linear. By the above construction, for the homotopy $\{D'_t\}$, D'_t is surjective except for N times in $(0,1)$, where the kernel has dimension one. In particular the sign of $[u]$ by the definition via the determinant line bundle is exactly

$$-1^N = -1^{CZ(p)-n},$$

by (6.7), which was what to be proved. □

Theorem 6.8.
$$GW^1_{1,1}(N, J^\lambda, A_\beta, \beta)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(\mathcal{P}^{-1}(N), R^\lambda, \beta),$$

where $N \subset \overline{\mathcal{M}}^1_{1,1}(J^\lambda, A_\beta, \beta)$ is an open compact set (where $\mathcal{P}$ is as in Proposition 6.3), $i(\mathcal{P}^{-1}(N), R^\lambda, \beta)$ is the Fuller index as described in the Appendix below, and where the left-hand side of the equation is the functional as in (5.2).

Proof. Suppose that $N \subset \overline{\mathcal{M}}^1_{1,1}(J^\lambda, A_\beta, \beta)$ is open-compact and consists of isolated regular Reeb tori $\{u_i\}$, corresponding to orbits $\{o_i\}$. Denote by $mult(o_i)$ the multiplicity of the orbits as in Appendix A. Then we have:

$$GW^1_{1,1}(N, J^\lambda, A_\beta, \beta)([\overline{M}_{1,1}] \otimes [C \times S^1]) = \sum_i \frac{(-1)^{CZ(o_i)-n}}{mult(o_i)},$$

where n half the dimension of M, the numerator is as in (A.2), and $mult(o_i)$ is the order of the corresponding isotropy group, see Appendix B.

The expression on the right is exactly the Fuller index $i(\mathcal{P}^{-1}(N), R^\lambda, \beta)$. Thus, the theorem follows for N as above. However, in general if N is open and compact then perturbing slightly we obtain a smooth family $\{R^{\lambda_t}\}$, $\lambda_0 = \lambda$, s.t. λ_1 is non-degenerate, that is has non-degenerate orbits. And such that there is an open-compact subset $\widetilde{N}$ of $\overline{\mathcal{M}}^1_{1,1}(\{J^{\lambda_t}\}, A_\beta, \beta)$ with $(\widetilde{N} \cap \overline{\mathcal{M}}^1_{1,1}(J^\lambda, A_\beta, \beta) = N$, see Lemma 5.11. Then by Lemma 5.3 if

$$N_1 = (\widetilde{N} \cap \overline{\mathcal{M}}^1_{1,1}(J^{\lambda_1}, A_\beta, \beta))$$

we get

$$GW^1_{1,1}(N, J^\lambda, A_\beta, \beta)([\overline{M}_{1,1}] \otimes [C \times S^1]) = GW^1_{1,1}(N_1, J^{\lambda_1}, A_\beta, \beta)([\overline{M}_{1,1}] \otimes [C \times S^1]).$$

By the previous discussion

$$GW^1_{1,1}(N_1, J^{\lambda_1}, A_\beta, \beta)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(N_1, R^{\lambda_1}, \beta),$$

but by the invariance of Fuller index (see Appendix A),

$$i(N_1, R^{\lambda_1}, \beta) = i(N, R^\lambda, \beta).$$

□

What about higher genus invariants of $C \times S^1$? Following the proof of Proposition 6.3, it is not hard to see that all J^λ-holomorphic curves must be branched covers of Reeb tori. If one can show that these branched covers are regular when the underlying tori are regular, the calculation of invariants would be fairly automatic from this data. See [36], [34] where these kinds of regularity calculation are made.

7. Proofs of main theorems

To set notation and terminology we review the basic definition of a nodal curve.

Definition 7.1. *A* **nodal Riemann surface** *(without boundary) is a pair $\Sigma = (\widetilde{\Sigma}, \mathcal{N})$ where $\widetilde{\Sigma}$ is a Riemann surface, and $\mathcal{N}$ a set of pairs of points of $\widetilde{\Sigma}$: $\mathcal{N} = \{(z_0^0, z_0^1), \ldots, (z_n^0, z_n^1)\}$, $n_i^j \neq n_k^l$ for $i \neq k$ and all j, l. By slight abuse, we may also denote by Σ the quotient space $\widetilde{\Sigma}/\sim$, where the equivalence relation is generated by $n_i^0 \sim n_i^1$. Let $q_\Sigma : \widetilde{\Sigma} \to (\widetilde{\Sigma}/\sim)$ denote the quotient map. The elements $q_\Sigma(\{z_i^0, z_i^1\}) \in \widetilde{\Sigma}/\sim$, are called* **nodes**. *Let M be a smooth manifold. By a map $u : \Sigma \to M$ of a nodal Riemann surface Σ, we mean a set map $u : (\widetilde{\Sigma}/\sim) \to M$. u is called smooth or immersion or J-holomorphic (when M is almost complex) if the map $\widetilde{u} = u \circ q_\Sigma$ is smooth or respectively immersion or respectively J-holomorphic. We call $\widetilde{u}$* **normalization of** *u. u is called an embedding if u is a topological embedding and its normalization is an immersion. The cohomology groups of Σ are defined as $H^\bullet(\Sigma) := H^\bullet(\widetilde{\Sigma}/\sim)$, likewise with homology. The genus of Σ is the topological genus of $\widetilde{\Sigma}/\sim$.*

We shall say that $(\widetilde{\Sigma}, \mathcal{N})$ is normal if $\mathcal{N} = \emptyset$. Similarly, $u : \Sigma \to M$, $\Sigma = (\widetilde{\Sigma}, \mathcal{N})$ is called ***normal*** if $\mathcal{N} = \emptyset$. The normalization of u is the map of the nodal Riemann surface $\widetilde{u} : \widetilde{\Sigma} \to M$, $\widetilde{\Sigma} = (\widetilde{\Sigma}, \emptyset)$. Note that if u is a Reeb 2-curve, its normalization $\widetilde{u}$ may not be a Reeb 2-curve (the second condition may fail).

7.1. Mapping tori and Reeb 2-curves.

Let (C, λ) be a contact manifold and ϕ a strict contactomorphism and let $M = (M_{\phi,1}, \lambda_\phi, \alpha)$ denote the mapping torus of ϕ, as also appearing in Theorem 2.7. More specifically, $M = C \times \mathbb{R}/\sim$, where the equivalence $\sim$ is generated by $(x, \theta) \sim (\phi(x), \theta + 1)$, for more details on the corresponding lcs structure see for instance [4]. Then $(M, \lambda_\phi, \alpha)$ is an integral first kind lcs manifold.

In this case $\mathcal{V}_\lambda = \mathcal{D}$, and in mapping torus coordinates at a point (x, θ), it is spanned by $X_\lambda = (0, \frac{\partial}{\partial \theta}), X_\alpha = (R^{\lambda_\theta}, 0)$ for R^{λ_θ} the λ_θ-Reeb vector field, where $\lambda_\theta = \lambda_{C_\theta}$ the fiber over θ of the projection $M \to S^1$. Analogously to the Example 5 there is an S^1-invariant almost complex structure on M, which we call J^{λ_ϕ}.

We now show that all Reeb 2-curves in M must be of a certain type. Let $o : S^1 \to C$ be a λ-Reeb and suppose that image $\phi^n(o) =$ image o, for some $n > 0$, so that

$$\forall t \in [0, 1] : \phi^n(o)(t) = o(t + \theta_0)$$

for some uniquely determined $\theta_0 \in [0, 1)$. Let $\widetilde{o} : S^1 \times [0, n] \to C \times \mathbb{R}$ be the map

$$\widetilde{o}(t, \tau) = (o(t + \theta_0 \cdot \frac{\tau}{n}), \tau).$$

Then $\widetilde{o}$ is well defined on the quotient $T^2 \simeq S^1 \times ([0, n]/0 \sim n)$, and we denote the quotient map by u_o^n, called the *charge n generalized Reeb torus of o*. If the class $[o] \in \pi_1(C) = \beta$ we denote by A_β^n the class of u_o^n in $H_2(M, \mathbb{Z})$. The class A_β^n can be defined more generally whenever $\phi_*^n(\beta) = \beta$, it is the class of a torus map $T^2 \to M$ defined analogously to the map u_o^n, but no longer having the Reeb 2-curve property. We may abbreviate A_β^1 by A_β.

By construction, u_o^n is a charge n Reeb 2-curve and its image is an embedded J^{λ_ϕ}-holomorphic torus $\mathcal{T}$. Moreover, u_o^n is J^{λ_ϕ}-holomorphic with respect to a uniquely determined complex structure on T^2, similarly to the case of Reeb tori of Section 4.2. However, unlike the case of Reeb tori, this complex structure is not "rectangular" unless $\phi^n \circ o = o$.

Proposition 7.2. *Let $M = (M_\phi, \lambda_\phi, \alpha)$ be the mapping torus of a strict contactomorphism ϕ as above. Then:*

(1) *Every charge n Reeb 2-curve u in M has a factorization:*

$$u = u^n_o \circ \rho, \tag{7.3}$$

for $\rho : \Sigma \to T^2$ some degree one map, and for some orbit string o uniquely determined by u.

(2) *Every element $u \in \overline{\mathcal{M}}^n_{1,1}(J^{\lambda_\phi}, A^n_\beta, \beta)$ is represented (as an equivalence class in this moduli space) by u^n_o, where the latter is as above, for some o uniquely determined.*

(3) *The Fredholm index of the corresponding real linear CR operator is 2, so that the expected dimension of $\overline{\mathcal{M}}^n_{1,1}(J^{\lambda_\phi}, A^n_\beta, \beta)$ is 0.*

(4) *Let J be a (λ_ϕ, α)-admissible almost complex structure on M. There is a natural proper topological embedding:*

$$emb : \overline{\mathcal{M}}^n_{1,1}(J, A^n_\beta, \beta) \to \mathcal{O}(R^\lambda, \beta),$$

defined by $u \mapsto o$, where o is uniquely determined by the condition (7.3).

Proof. The proof of part one is completely analogous to the proof of Proposition 6.3. To prove the second part, first note that by the first part u has image $\mathcal{T} = \operatorname{image} u^n_o$ for some n, o, and $\mathcal{T}$ is an embedded J^{λ_ϕ}-holomorphic torus. By basic theory of mappings of complex tori u must be a covering map $\Sigma \to \mathcal{T}$. Since we know the charge n, as in final part of the proof of Proposition 6.3, we may conclude that $u \simeq u^n_o$ for some uniquely determined o, where $\simeq$ is an isomorphism.

We prove Part 3. Note that $c_1(A^n_\beta) = 0$, as by construction the complex tangent bundle along u^n_o admits a flat connection, induced by the natural $\mathcal{G}$-connection on $M_\phi \to S^1$, for $\mathcal{G}$ the group of strict contactomorphisms of (C, λ), cf. Proof of Proposition 6.4. The needed fact then follows by the index/Riemann-Roch theorem.

The last part of the proposition readily follows from the first part. □

Proof of Theorem 2.11. Let $u : \Sigma \to (M = M_{\widetilde{\phi},1})$ be a Reeb 2-curve in the mapping torus as in the statement. By part one of Proposition 7.2, there must be a generalized charge n Reeb torus in M. By definitions this means that $\widetilde{\phi}$ has a charge n fixed Reeb string, so that ϕ has a charge n fixed geodesic string, which is impossible by assumptions. □

Proposition 7.4. *Let (C, λ) be a contact manifold with λ satisfying one of the following conditions:*

(1) *There is a non-degenerate λ-Reeb orbit.*

(2) *$i(N, R^\lambda, \beta) \neq 0$ for some open compact $N \subset \mathcal{O}(R^\lambda, \beta)$, and some β.*

Then:

(1) *Let (λ, α) be the lcs-fication of (C, λ). There exists an $\epsilon > 0$ s.t. for any tamed exact lcs structure (λ', α', J) on $M = C \times S^1$, with $(d_{\alpha'}\lambda', J)$ ϵ-close to $(d_\alpha \lambda, J^\lambda)$ (as in Definition 5.8), there exists an elliptic, J-holomorphic α-charge 1 curve u in M.*

(2) *In addition, if (M, λ', α') is first kind and has dimension 4 then u may be assumed to be normal and embedded.*

Proof. If we have a closed non-degenerate λ-Reeb orbit o then we also have an open compact subset $N = \{o\} \subset S_\lambda$. Thus suppose that the condition 2 holds.

Set

$$(\widetilde{N} := \mathcal{P}(N)) \subset \overline{\mathcal{M}}^1_{1,1}(J^\lambda, A_\beta, \beta),$$

which is an open compact set. By Theorem 6.8, and by the assumption that $i(N, R_\lambda, \beta) \neq 0$

$$GW^1_{1,1}(N, J^\lambda, A_\beta, \beta) \neq 0.$$

The first part of the conclusion then follows by Proposition 5.13.

We now verify the second part. Suppose that M has dimension 4. Let U be an ϵ-neighborhood of $(\lambda, \alpha, J^\lambda)$, for ϵ as given in the first part, and let $(\lambda', \alpha', J) \in U$. Suppose that $u \in \overline{\mathcal{M}}^1_{1,1}(J, \beta)$. Let $\underline{u}$ be a simple J-holomorphic curve covered by u, (see for instance [21, Section 2.5].

For convenience, we now recall the adjunction inequality.

Theorem 7.5 (McDuff-Micallef-White [22], [17])**.** *Let (M, J) be an almost complex 4-manifold and let $A \in H_2(M)$ be a homology class that is represented by a simple J-holomorphic curve $u : \Sigma \to M$. Let $\delta(u)$ denote the number of self-intersections of u, then*

$$2\delta(u) - \chi(\Sigma) \leq A \cdot A - c_1(A),$$

with equality if and only if u is an immersion with only transverse self-intersections.

In our case $A = A_\beta$ so that $c_1(A) = 0$ and $A \cdot A = 0$. If u is not normal its normalization is of the form $\widetilde{u} : \mathbb{CP}^1 \to M$ with at least one self intersection and with $0 = [\widetilde{u}] \in H_2(M)$, but this contradicts positivity of intersections. So u and hence $\underline{u}$ are normal. Moreover, the domain Σ' of $\underline{u}$ satisfies: $\chi(\Sigma') = \chi(T^2) = 0$, so that $\delta(\underline{u}) = 0$, and the above inequality is an equality. In particular $\underline{u}$ is an embedding, which of course implies our claim.

□

Proof of Theorem 3.3. Let

$$U \ni (\omega_0 := d_\alpha \lambda, J_0 := J^\lambda)$$

be a set of pairs (ω, J) satisfying the following:

- ω is a first kind lcs structure.
- For each $(\omega, J) \in U$, J is ω-compatible and admissible.
- Let ϵ be chosen as in the first part of Proposition 7.4. Then each $(\omega, J) \in U$ is ϵ-close to (ω_0, J_0), (as in Definition 5.8).

To prove the theorem we need to construct a map $E : V \to \mathcal{J}(M)$, where V is some neighborhood of ω_0 in the space $(\mathcal{F}(M), d_\infty)$ (see Definition 3.1) and where

$$\forall \omega \in V : (\omega, E(\omega)) \in U.$$

As then Proposition 7.4 tells us that for each $\omega \in V$, there is a class A, $E(\omega)$-holomorphic, elliptic curve u in M. Using Lemma 6.1 we would then conclude that there is an elliptic Reeb 2-curve u in (M, ω). If M has dimension 4 then in addition u may be assumed to be normal and embedded. If ω is integral, by Proposition 7.4, u may be assumed to be charge 1. And so we will be done.

Define a metric ρ_0 measuring the distance between subspaces W_1, W_2, of same dimension, of an inner product space (T, g) as follows.

$$\rho_0(W_1, W_2) := |P_{W_1} - P_{W_2}|,$$

for $|\cdot|$ the g-operator norm, and P_{W_i} g-projection operators onto W_i.

Let $\delta > 0$ be given. Suppose that $\omega = d^{\alpha'} \lambda'$ is a first kind lcs structure δ-close to ω_0 for the metric d_∞. Then $\mathcal{V}_{\lambda'}, \xi_{\lambda'}$ are smooth distributions by the assumption that (α', λ') is a lcs structure of the first kind and $TM = \mathcal{V}_{\lambda'} \oplus \xi_{\lambda'}$. Moreover,

$$\rho_\infty(\mathcal{V}_{\lambda'}, \mathcal{V}_\lambda) < \epsilon_\delta$$

and

$$\rho_\infty(\xi_{\lambda'}, \xi_\lambda) < \epsilon_\delta$$

where $\epsilon_\delta \to 0$ as $\delta \to 0$, and where ρ_∞ is the C^∞ analogue of the metric ρ_0, for the family of subspaces of the family of inner product spaces (T_pM, g).

Then choosing δ to be suitably be small, for each $p \in M$ we have an isomorphism

$$\phi(p) : T_pM \to T_pM,$$

$\phi_p := P_1 \oplus P_2$, for $P_1 : \mathcal{V}_{\lambda_0}(p) \to \mathcal{V}_{\lambda'}(p)$, $P_2 : \xi_{\lambda_0}(p) \to \xi_{\lambda'}(p)$ the g-projection operators. Define $E(\omega)(p) := \phi(p)_* J_0$. Then clearly, if δ was chosen to be sufficiently small, if we take V to be the δ-ball in $(\mathcal{F}(M), d_\infty)$ centered at ω_0, then it has the needed property.

□

Definition 7.6. *Let α be a scale integral closed 1-form on a closed smooth manifold M. Let $0 \neq c \in \mathbb{R}$ be such that $c\alpha$ is integral. A* **classifying map** *$p : M \to S^1$ of α is a smooth map s.t. $c\alpha = p^* d\theta$. A map p with these properties is of course not unique.*

Lemma 7.7. *Let $u : \Sigma \to M$ be a Reeb 2-curve in a closed, scale integral, first kind lcs manifold (M, λ, α), then its normalization $\widetilde{u} : \widetilde{\Sigma} \to M$ is a Reeb 2-curve.*

Proof. By Lemma 3.9 we have a surjective classifying map $p : M \to S^1$ of α. Note that the fibers M_t of p, for all $t \in S^1$, are contact with contact form $\lambda_t = \lambda|_{C_t}$, as $0 \neq \omega^n = \alpha \wedge \lambda \wedge d\lambda^{n-1}$ and $c \cdot \alpha = 0$ on M_t, where c is as in the definition of p.

Let $\widetilde{u} : \widetilde{\Sigma} \to M$ be the normalization of u. Suppose it is not a Reeb 2-curve, which in this case, by definitions, just means that $0 = [\widetilde{u}^*\alpha] \in H^1(\widetilde{\Sigma}, \mathbb{R})$. Since $0 \neq [u^*\alpha] \in H^1(\Sigma, \mathbb{R})$, some node z_0 of Σ lies on closed loop $o : S^1 \to \Sigma$ with $\langle [o], [u^*\alpha] \rangle \neq 0$.

Let $q_\Sigma : \widetilde{\Sigma} \to \Sigma$ be the quotient map as previously appearing. In this case, we may find a smooth embedding $\eta : D^2 \to \widetilde{\Sigma}$, s.t. $q_\Sigma \circ \eta(D^2)|_{\partial D^2}$ is a component of a regular fiber C_t, of the classifying map $p' : \Sigma \to S^1$ of $u^*\alpha$. See Figure 1, $\eta(D^2)$ is a certain disk in $\widetilde{\Sigma}$, whose interior contains an element of $\phi^{-1}(z_0)$.

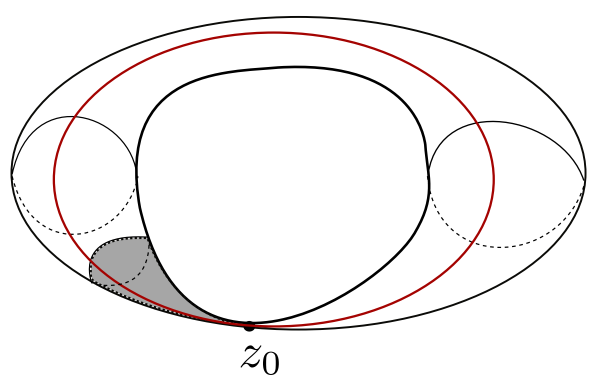

FIGURE 1. The figure for Σ. The gray shaded area is the image $q_\Sigma \circ \eta(D^2)$. The red shaded curve is the image of the closed loop o as above.

Then analogously to the proof of Proposition 6.3 $\widetilde{u} \circ \eta|_{\partial D^2}$ is a (unparametrized) λ_t-Reeb orbit in M_t. (The classifying maps can be arranged, such that $u(C_t) \subset M_t$.) And in particular $\int_{\partial D} \widetilde{u}^*\lambda \neq 0$. Now u (not $\widetilde{u}$) is a Reeb 2-curve, and the first condition of this implies that $\int_D d\widetilde{u}^*\lambda = 0$, since $\ker d\lambda$ on

M is spanned by X_λ, X_α. So we have a contradiction to Stokes theorem. Thus, $\widetilde{u}$ must be a Reeb 2-curve.

□

Proof of Theorem 1.17. Let (C, λ) and ϕ be as in the hypothesis. Let $\{\phi_t\}$, $t \in [0,1]$ be a smooth family of strict contactomorphisms $\phi_0 = id$, $\phi_1 = \phi$. This gives a smooth fibration $\widetilde{M} \to [0,1]$, with fiber over $t \in [0,1]$: $M_{\phi_t,1}$, which is moreover endowed with the first kind lcs structure $(\lambda_{\phi_t}, \alpha)$, where this is the "mapping torus structure" as above. Let $tr : \widetilde{M} \to (C \times S^1) \times [0,1]$ be a smooth trivialization, restricting to the identity $C \times S^1 \to C \times S^1$ over 0. Pushing forward by the bundle map tr, the above mentioned family of lcs structures, we get a smooth family $\{(\lambda_t, \alpha)\}$, $t \in [0,1]$, of first kind integral lcs structures on $C \times S^1$, with $(\lambda_0, \alpha) = (\lambda, \alpha)$ the standard lcs-fication of λ.

Fix a family $\{J^{\lambda_t}\}$ of almost complex structures on $C \times S^1$ with each J^{λ_t} admissible with respect (λ_t, α). Let $N \subset \mathcal{O}(R^\lambda, \beta)$ be an open compact set satisfying $i(N, R^\lambda, \beta) \neq 0$. The embedding emb from part 4 of Proposition 7.2 induces a proper embedding

$$\widetilde{emb} : \widetilde{\mathcal{M}} = \overline{\mathcal{M}}^1_{1,1}(\{J^{\lambda_t}\}, A_\beta, \beta) \to \mathcal{O}(R^\lambda, \beta) \times [0,1],$$

defined by $\widetilde{emb}(u,t) = (emb(u), t)$.

Set $N_0 = \mathcal{P}(N) \subset \overline{\mathcal{M}}^1_{1,1}(J^{\lambda_0}, A_\beta, \beta) \subset \widetilde{\mathcal{M}}$. So that by construction $\widetilde{emb}(N_0) = N \times \{0\}$. Set

$$\widetilde{N} = \widetilde{emb}^{-1}(\widetilde{emb}(\widetilde{\mathcal{M}}) \cap (N \times [0,1])),$$

then this is an open and compact subset of $\widetilde{\mathcal{M}}$. And by construction $\widetilde{N} \cap \overline{\mathcal{M}}^1_{1,1}(J^{\lambda_0}, A_\beta, \beta) = N_0$. Set $N_1 = \widetilde{N} \cap \overline{\mathcal{M}}^1_{1,1}(J^{\lambda_1}, A_\beta, \beta)$.

It follows by Theorem 6.8 that

$$GW^1_{1,1}(N_0, J^{\lambda_0}, A_\beta, \beta)([\overline{M}_{1,1}] \otimes [C \times S^1]) = i(N, R^\lambda, \beta) \neq 0.$$

Then applying Lemma 5.3 we get that

$$GW^1_{1,1}(N_1, J^{\lambda_1}, A_\beta, \beta)([\overline{M}_{1,1}] \otimes [C \times S^1]) \neq 0.$$

By part two of Proposition 7.2 there is a charge 1 generalized Reeb torus u_o in M. In particular, image $\phi(o) =$ image(o). Also by construction, $o \in N$ and so we are done. □

Proof of Theorem 1.13. If R^λ is finite type and $i(R^\lambda, \beta) \neq 0$ then the theorem follows immediately by Theorem 1.17.

We leave the full definition of infinite type vector fields to the reference [29, Definition 2.5]. We have that R^λ is infinite type in class β. WLOG assume that it is positive infinite type. In particular, we may find a perturbation X^a of R^λ together with a homotopy X_t, $t \in [0,1]$, s.t.:

(1) $X_0 = X^a$, $X_1 = R^\lambda$.

(2) $\mathcal{O}(X^a, a, \beta) = \{o \in \mathcal{O}(X^a, \beta) \,|\, A(o) \leq a\}$ is discrete, where A is the period map as in the introduction.

(3) Each $o \in \mathcal{O}(X^a, a, \beta)$ is contained in a non-branching open compact subset $K_o \subset \mathcal{O}(\{X^a_t\}, \beta)$. Where the latter means that:

   (a) $K_o \cap \mathcal{O}(X_1, \beta)$ is connected.

   (b) For $o, o' \in \mathcal{O}(X^a, a, \beta)$ $K_o = K_{o'}$ or $K_o \cap K_{o'} = \emptyset$.

   (c) $\mathcal{O}(X_1, \beta) = \cup_o (K_o \cap \mathcal{O}(X_1, \beta))$.

   (d) $(K_o \cap \mathcal{O}(X^a, \beta)) \subset \mathcal{O}(X^a, a, \beta)$.

(4) $\sum_{o \in \mathcal{O}(X^a, a, \beta)} i(o) > 0$.

Set $N := \mathcal{O}(X^a, a, \beta)$, then by the condition 3 and by the non-branching property,

$$N = \sqcup_{i \in \{1,\ldots,n\}} \mathcal{O}(X^a, \beta) \cap K_{o_i},$$

disjoint union for some $o_1, \ldots o_n \in \mathcal{O}(X^a, a, \beta)$. Set $\widetilde{N} := \cup_{i \in \{1,\ldots,n\}} K_{o_i}$. Then $\widetilde{N} \cap \mathcal{O}(X^a, \beta) = N$. Set $N_1 := \widetilde{N} \cap \mathcal{O}(R^\lambda, \beta)$, then this is an open compact subset of $\mathcal{O}(X_1, \beta)$.

Finally, using invariance of the Fuller index we get that $i(N_1, R^\lambda, \beta) \neq 0$. Then the result follows by Theorem 1.17. □

Proof of Theorem 1.16. Suppose that λ is Morse-Bott and we have an open compact component $N \subset \mathcal{O}(R^\lambda, \beta)$, with $\chi(N) \neq 0$ so that $i(N, R^\lambda, \beta) \neq 0$, [29, Section 2.1.1]. If λ' is sufficiently C^1 nearby to λ then we may find open compact $N' \in \mathcal{O}(R^{\lambda'})$ s.t.

$$i(N', R^{\lambda'}, \beta) = i(N, R^\lambda, \beta) \neq 0.$$

See [29, Lemma 1.6]. Then the result follows by Theorem 1.17. □

Proof of Theorem 1.7. Under the assumptions on the Euler characteristic by [28, Theorem 1.10] for any β-taut g on X:

$$\mathrm{GWF}(g, id, \beta, 1) = F(g, \beta) = \chi^{S^1}(L_\beta X) \neq 0.$$

Then for ϕ as in the hypothesis, $\mathrm{GWF}(g, \phi, \beta, 1) \neq 0$ and the conclusion follows. □

Proof of Theorem 1.9. For $(X, g') \xrightarrow{p} (Y, g_y)$ a β-taut fibration as in the hypothesis, by the proof of [29, Theorem 1.7]

$$\mathrm{GWF}(g', id, \beta, 1) = F(g', \beta) \neq 0.$$

By [28, Theorem 1.10], $\chi^{S^1}(L_\beta X) \neq 0$. Then the result follows by Theorem 1.7. □

Proof of Corollary 1.15. Let g and $\beta \in \pi_1^{inc}(X)$ be as in the hypothesis. By assumption the corresponding unit cotangent bundle (C, λ) is definite type in class $\widetilde{\beta}$. If ϕ is some isometry of g homotopic through isometries to the identity, then the induced strict contactomorphism $\widetilde{\phi}$ is homotopic through strict contactomorphisms to the id, and so by Theorem 1.13 $\widetilde{\phi}$ has a class $\widetilde{\beta}$ fixed Reeb string. So that ϕ has a class β fixed geodesic string. □

Proof of Theorem 1.18. Let g_0 be a complete metric on X with a unique class $\beta \in \pi_1^{inc}(X)$ geodesic. Let $\lambda_0 = \lambda_{g_0}$ be the Liouville 1-form on the g_0-unit contangent bundle C of X. Let g be as in the hypothesis and let λ_1 be Liouville 1-form on C corresponding to g.

Let $\{\lambda_t\}$, $t \in [0,1]$, be a smooth homotopy between λ_0, λ_1. We may in addition assume that this homotopy is constant near the end points. We then get a family $\{(\lambda_t, \alpha)\}$ of first kind integral lcs structures on $C \times S^1$.

Now let $\{\widetilde{\phi}_t\}$, $t \in [0,1]$ be a smooth homotopy of strict contactomorphisms of (C, λ_1) with $\widetilde{\phi}_0 = id$, corresponding to a homotopy $\{\phi_t\}$ of isometries of X, g, with $\phi_0 = id$. We may suppose that $\{\widetilde{\phi}_t\}$ is constant near end points. As in the Proof of Theorem 1.17, this gives a smooth fibration over $\widetilde{M} \to [0,1]$. And as before we get a family $\{(\lambda''_t, \alpha)\}$, $t \in [0,1]$, of first kind integral lcs structures on $C \times S^1$, s.t.:

- (λ''_0, α) is the lcs-fication of λ_0.
- For each t (λ''_t, α) is isomorphic to the mapping torus structure $(\lambda_{\widetilde{\phi}_t}, \alpha)$.

Let $\{(\lambda'_t, \alpha)\}$ be the concatenation of the families $\{(\lambda_t, \alpha)\}$, $\{(\lambda''_t, \alpha)\}$, i.e.:

$$\lambda'_t = \begin{cases} \lambda_{2t}, & \text{if } t \in [0, \frac{1}{2}] \\ \lambda''_{2t-1}, & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Let $\{J'_t\}$, $t \in [0, 1]$ be a family of almost complex structures on $M = C \times S^1$ s.t. J'_t is (λ'_t, α)-admissible for each t. Let $\widetilde{\beta}$ be the lift of $\beta \in \pi_1^{inc}(X)$, as in Section 4.3. Now, by construction and by Theorem 6.8

$$GW^1_{1,1}(J'_0, A_{\widetilde{\beta}}, \beta) = i(R^{\lambda_{g_0}}, \widetilde{\beta}) = 1,$$

with the last equality due to the assumption that there is a unique non-degenerate g_0-geodesic in class β. Then by Lemma 5.6, we get that one of the following holds:

- $(\lambda_{\widetilde{\phi}}, \alpha)$ has an elliptic charge 1, class $A_{\widetilde{\beta}}$ Reeb 2-curve u, and hence by part one of the Proposition 7.2, $\widetilde{\phi}$ has a charge 1 fixed Reeb string in class $\widetilde{\beta}$, and so ϕ has a charge 1 fixed geodesic in class β.
- The family $\{J'_t\}$, $t \in [0, 1]$ has an essential right holomorphic sky catastrophy, of charge 1, class $A_{\widetilde{\beta}}$ curves.

Suppose that the latter holds. By the admissibility condition and by Lemma 6.1, if

$$(u, t) \in \overline{\mathcal{M}}^1_{1,1}(\{J'_t\}, A_{\widetilde{\beta}}, \beta)$$

then u is a charge 1 elliptic Reeb 2-curve. Let $\{X_t\}$, $t \in [0, 1]$, be the smooth family of vector fields satisfying $X_t = R^{\lambda_{2t}}$ for $t \in [0, \frac{1}{2}]$, $X_t = R^{\lambda_1}$ for $t \in [\frac{1}{2}, 1]$. Analogously to part 4 of Proposition 7.2, we have a proper topological embedding

$$emb : \overline{\mathcal{M}}^1_{1,1}(\{J'_t\}, A_{\widetilde{\beta}}, \beta) \to \mathcal{O}(\{X_t\}, \widetilde{\beta}),$$

It follows, by the previous hypothesis, that the family $\{X_t\}_{t \in [0,1]}$ has a sky catastrophe in class $\beta = 0$. In addition, this sky catastrophe must be essential, as otherwise the original holomorphic sky catastrophe would not be essential.

□

Proof of Theorem 1.4. We have to prove invariance of the counts. Let $\{(g_t, \phi_t)\}_{t \in [0,1]}$, be a smooth family furnishing an equivalence between (g_0, ϕ_0), (g_1, ϕ_1) in $\mathcal{E}(X)$. Let $(C, \lambda_t = \lambda_{g_t})$, be the g_t-unit cotangent bundle of X, and λ_{g_t} the Liouville 1-form. (Fixing an implicit identification of the unit cotangent bundles with a fixed manifold C.) Let $\widetilde{\beta} \in \pi_1(C)$, be the lift of a class $\beta \in \pi_1^{inc}(X)$ as in Section 4.3. Denote by $(M_t, \lambda_t, \alpha_t)$ the mapping torus of $\widetilde{\phi}_t$ action on (C, λ_t), where $\widetilde{\phi}_t$ is the strict contactomorphism induced by the isometry ϕ_t. Finally, let $\{J_t\}_{t \in [0,1]}$, be a smooth family with J_t (λ_t, α_t)-admissible almost complex structures on M_t.

By the tautness assumption the family $\{R^{\lambda_t}\}$ has no sky catastrophe in the class $\widetilde{\beta}$. Then by the third part of Proposition 7.2, $\{J_t\}$ has no sky catastrophe in class in $A_{\widetilde{\beta}}$, and so by Theorem 5.6 we have

$$\mathrm{GWF}(g_0, \phi_0, \beta, n) = \mathrm{GWF}(g_1, \phi_1, \beta, n)$$

and we are done. □

Proof of Theorem 3.6. Suppose that $u : \Sigma \to M$ is an immersed Reeb 2-curve, we then show that M also has a Reeb 1-curve. Let $\widetilde{u} : \widetilde{\Sigma} \to M$ be the normalization of u, so that $\widetilde{u}$ is an immersion. We have a pair of transverse 1-distributions $D_1 = \widetilde{u}^* \mathbb{R}\langle X_\alpha \rangle$, $D_2 = \widetilde{u}^* \mathbb{R}\langle X_\lambda \rangle$ on $\widetilde{\Sigma}$. We may then find an embedded path $\gamma : [0, 1] \to \widetilde{\Sigma}$, tangent to D_1 s.t. $\lambda(\gamma'(t)) > 0$, $\forall t \in [0, 1]$, and s.t. $\gamma(0)$ and $\gamma(1)$ are on a leaf of D_2. It is then simple to obtain from this a Reeb 1-curve o, by joining the end points of γ by an embedded path tangent to D_2, and perturbing, see Figure 2. This proves the first part of the

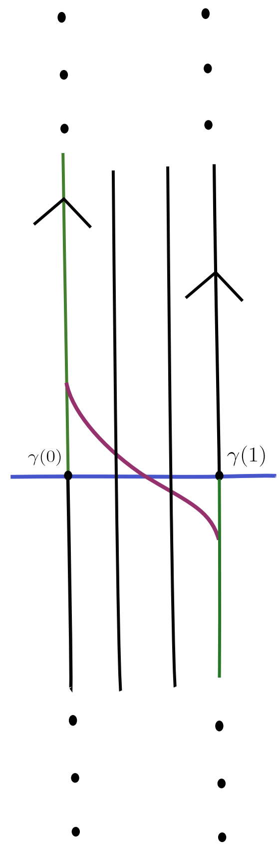

Figure 2. The green shaded path is γ, the indicated orientation is given by $u^*\lambda$, the D_1 folliation is shaded in black, the D_2 folliation is shaded in blue. The purple segment is part of the loop $o: S^1 \to \Sigma$, which is is smooth and satisfies $\lambda(o'(t)) > 0$ for all t.

theorem.

To prove the second part, suppose that $u : \Sigma \to M$ is an immersed elliptic Reeb 2-curve. Suppose that u is not normal. Let $\widetilde{u} : \widetilde{\Sigma} \to M$ be its normalization. Then $\widetilde{\Sigma}$ has a genus 0 component $\mathcal{S}$. So that $\widetilde{u} : \mathcal{S} \simeq \mathbb{CP}^1 \to M$ is immersed. The distribution $D_1 = \widetilde{u}^* \mathbb{R}\langle X_\alpha \rangle$, as appearing above, is then a $\widetilde{u}^*\lambda$-oriented 1-dimensional distribution on $\mathbb{CP}^1$ which is impossible. $\square$

A. Fuller index

Let X be a complete vector field without zeros on a smooth manifold M. Set

$$S(X, \beta) = \{o \in L_\beta M \mid \exists p \in (0, \infty),\ \ o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\}, \tag{A.1}$$

where $L_\beta M$ denotes the free homotopy class β component of the free loop space

$$LM = \{o : S^1 \to M \mid o \text{ is smooth}\}.$$

And where recall that $S^1 = \mathbb{R}/\mathbb{Z}$. The above p is uniquely determined and we denote it by $p(o)$ called the period of o.

There is a natural S^1 reparametrization action on $S(X, \beta)$: $t \cdot o$ is the loop $t \cdot o(\tau) = o(t + \tau)$. The elements of $\mathcal{O}(X, \beta) := S(X, \beta)/S^1$ will be called *orbit strings*. Slightly abusing notation we just write o for the equivalence class of o.

The multiplicity $m(o)$ of an orbit string is the ratio $p(o)/l$ for $l > 0$ the period of a simple orbit covered by o.

We want a kind of fixed point index of an open compact subset $N \subset \mathcal{O}(X, \beta)$, which counts orbit strings o with certain weights. Assume for simplicity that $N \subset \mathcal{O}(X)$ is finite. (Otherwise, for a general open compact $N \subset \mathcal{O}(X, \beta)$, we need to perturb.) Then to such an (N, X, β) Fuller associates an index:

$$i(N, X, \beta) = \sum_{o \in N} \frac{1}{m(o)} i(o),$$

where $i(o)$ is the fixed point index of the time $p(o)$ return map of the flow of X with respect to a local surface of section in M transverse to the image of o.

Fuller then shows that $i(N, X, \beta)$ has the following invariance property. For a continuous homotopy $\{X_t\}$, $t \in [0, 1]$ set

$$S(\{X_t\}, \beta) = \{(o, t) \in L_\beta M \times [0, 1] \,|\, o \in S(X_t)\}.$$

And given a continuous homotopy $\{X_t\}$, $X_0 = X$, $t \in [0, 1]$, suppose that $\widetilde{N}$ is an open compact subset of $S(\{X_t\}, \beta)/S^1$, such that

$$\widetilde{N} \cap (L_\beta M \times \{0\}) / S^1 = N.$$

Then if

$$N_1 = \widetilde{N} \cap (L_\beta M \times \{1\}) / S^1$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta).$$

In the case where X is the R^λ-Reeb vector field on a contact manifold (C^{2n+1}, λ), and if o is non-degenerate, we have:

$$i(o) = \operatorname{sign} \operatorname{Det}(\operatorname{Id}|_{\xi(x)} - F^\lambda_{p(o),*}|_{\xi(x)}) = (-1)^{CZ(o)-n}, \tag{A.2}$$

where $F^\lambda_{p(o),*}$ is the differential at x of the time $p(o)$ flow map of R^λ, and where $CZ(o)$ is the Conley-Zehnder index, see [26].

There is also an extended Fuller index $i(X, \beta) \in \mathbb{Q} \sqcup \{\pm\infty\}$, for certain X having definite type. This is constructed in [29], and is conceptually analogous to the extended Gromov-Witten invariant described in this paper.

The following is a version of the definition of sky catastrophes first appearing in Savelyev [29], generalizing a notion commonly called a "blue sky catastrophe", see Shilnikov-Turaev [32].

Definition A.3. *Let $\{X_t\}$, $t \in [0, 1]$ be a continuous family of non-zero, complete smooth vector fields on a closed manifold M, and let $\beta \in \pi_1^{inc}(X)$. And let $S(\{X_t\})$ be as above. We say that $\{X_t\}$ has a* **right sky catastrophe in class** *β, if there is an element*

$$y \in S(X_0, \beta) \subset S(\{X_t\}, \beta)$$

so that there is no open compact subset of $S(\{X_t\}, \beta)$ containing y. We say that $\{X_t\}$ has a **left sky catastrophe in class** *β, if there is an element*

$$y \in S(X_1, \beta) \subset S(\{X_t\}, \beta)$$

so that there is no open compact subset of $S(\{X_t\}, \beta)$ containing y. We say that $\{X_t\}$ has a **sky catastrophe in class** *β, if it has either left or right sky catastrophe in class β.*

Definition A.4. *In the case that $X_t = R^{\lambda_t}$ for $\{\lambda_t\}$, $t \in [0, 1]$ smoothly varying, we say that a sky catastrophe of Reeb vector fields $\{X_t\}$ is* **essential** *if the condition of the definition above holds for any family $\{X'_t = R^{\lambda'_t}\}$ satisfying $X'_0 = X_0$ and $X'_1 = X_1$, and such that $\{\lambda'_t\}$ is smooth.*

B. Remark on multiplicity

This is a small note on how one deals with curves having non-trivial isotropy groups, in the virtual fundamental class technology. We primarily need this for the proof of Theorem 6.8.

Given a closed oriented orbifold X, with an orbibundle E over X Fukaya-Ono [11] show how to construct using multi-sections its rational homology Euler class, which when X represents the moduli space of some stable curves, is the virtual moduli cycle $[X]^{vir}$. When this is in degree 0, the corresponding Gromov-Witten invariant is $\int_{[X]^{vir}} 1$. However, they assume that their orbifolds are effective. This assumption is not really necessary for the purpose of construction of the Euler class but is convenient for other technical reasons. A different approach to the virtual fundamental class which emphasizes branched manifolds is used by McDuff-Wehrheim, see for example McDuff [16], [19] which does not have the effectivity assumption, a similar use of branched manifolds appears in [7]. In the case of a non-effective orbibundle $E \to X$ McDuff [18], constructs a homological Euler class $e(E)$ using multi-sections, which extends the construction [11]. McDuff shows that this class $e(E)$ is Poincare dual to the completely formally natural cohomological Euler class of E, constructed by other authors. In other words there is a natural notion of a homological Euler class of a possibly non-effective orbibundle. We shall assume the following black box property of the virtual fundamental class technology.

Axiom B.1. *Suppose that the moduli space of stable maps is cleanly cut out, which means that it is represented by a (non-effective) orbifold X with an orbifold obstruction bundle E, that is the bundle over X of cokernel spaces of the linearized CR operators. Then the virtual fundamental class $[X]^{vir}$ coincides with $e(E)$.*

Given this axiom it does not matter to us which virtual moduli cycle technique we use. It is satisfied automatically by the construction of McDuff-Wehrheim, (at the moment in genus 0, but surely extending). It can be shown to be satisfied in the approach of John Pardon [25]. And it is satisfied by the construction of Fukaya-Oh-Ono-Ohta [9], the latter is communicated to me by Kaoru Ono. When X is 0-dimensional this does follow immediately by the construction in [11], taking any effective Kuranishi neighborhood at the isolated points of X, (this actually suffices for our paper.)

As a special case most relevant to us here, suppose we have a moduli space of elliptic curves in X, which is regular with expected dimension 0. Then its underlying space is a collection of oriented points. However, as some curves are multiply covered, and so have isotropy groups, we must treat this is a non-effective 0 dimensional oriented orbifold. The contribution of each curve $[u]$ to the Gromov-Witten invariant $\int_{[X]^{vir}} 1$ is $\frac{\pm 1}{[\Gamma([u])]}$, where $[\Gamma([u])]$ is the order of the isotropy group $\Gamma([u])$ of $[u]$, in the McDuff-Wehrheim setup this is explained in [16, Section 5]. In the setup of Fukaya-Ono [11] we may readily calculate to get the same thing taking any effective Kuranishi neighborhood at the isolated points of X.

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