Simultaneous Grasping of Multiple Objects

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***Abstract*—The paper presents novel methods for rearrangement of polygonal object sets to obtain a multi-fingered grasp of this objects with minimal amount of required fingers. That is achieved by exploiting the geometry of the boundaries of the objects. Classification of inter-object contacts is presented, along with simple way of finding complementing fingers locations in order to immobilize each object. Rearrangement algorithm proposed to stack objects together in configurations that will ensure minimal number of required additional contacts. Assessment of obtained objects and grasp configuration along with and correction method are suggested. Proposed algorithms are evaluated in simulations, and the product of the algorithms tested in experimental setup.**

Keywords: ***multi-fingered grasping, form closure, multi-object immobilization,***

I. Introduction

Despite the advancement in robotic grasping and manipulation in last decades, multi-object grasping still haven't been addressed properly. While for humans grasping several items at once is quite ordinary (e.g. wallet and cellphone), robots are not usually programmed for such task and when performing treat each item individually. The subject may be not required at assembly lines but it can be of interest in fields of warehouse stacking, home assisting robots, etc. – in particular where the environment has not been adapted for automatic robotics. In the Amazon Robotics Challenge 2017, teams from across the world were tasked to program robots to transfer items, and classical approach is to move items one by one. Optimization of the process during the preparations for the challenge was the foundation for the research of simultaneous grasping, and this paper presents basic approach for assessment and grasping of multiple objects at once.

Next section summarizes related research and presents the foundations that serve as basis for this work. Followed by section III, which presents the mathematical basis and geometrical considerations for the proposed solution. Section IV presents the algorithms for the objects rearrangement and consecutive finger placement. Assessment of obtained results is described in section V. Finally the results of the algorithms performance is presented in section VI and discussed in section VII.

II. Related work

Object grasping and manipulations rely on basics summarized by [1], who describes motion of point of contact of two contacting objects. Nowadays common approach of configuration space presented by [2], which can be extended to wrench space, force and form closure [3], [4]. Quantitative measures [5], [6] were introduced to optimize grasping [7] in general, and serve as basis for finger placement in this work as well. Planning of grasps is summarized by [8] which served as one of the starting points for this work.

The main approach is to use 4-contact grasps, as shown to be minimum required number to grasp 2D objects [9], overlooking the fact that 3 fingers can successfully grasp polygonal objects [10], [11] or even less when the geometry of the objects give such options as show [12]. Proposed method relies on concepts of convex hulls [13], object-edge interactions [14] and on generalized ideas of graphical methods presented by [15], [16]. Opposite to common approach treating other objects in cluttered environment as obstacles [17], this approach utilizes advantages that other objects can provide. While the work presents a way of obtaining a desired configuration of objects, rearrangement process itself is not part of this work, and other methods like [18] may be utilized to rearrange objects into desired configuration.

III. Problem definition

A. Problem formulation

Given set of fully defined polygonal objects, we wish to find a: 1) A new configuration that will require a grasp with minimal amount of fingers to immobilize all and every object, 2) A set of finger locations for said immobilizing grasp.

B. Inter-object contact description

For the purpose of polygon arrangement evaluation, every contact is assumed to be performed by polygon A acting on the polygon B. The following description of a contact is adapted:

$$C = \{p, \hat{n}, id\,(P_A)\,, id\,(P_B)\} \tag{1}$$

where $id\,(P)$ stands for polygon number/identification entry, p is a contact point location and $\hat{n}$ is the contact normal direction inward the polygon B.

C. Solution existence for combination of internal and external contacts

Given a set of polygonal objects, where some of the objects are in contact one with another. Assuming each object has at least 4 contacts: some of them are fingers and some are inter-object contacts. The equilibrium equation for each object is written in form of total wrench:

$$G\bar{f} = \bar{0} \tag{2}$$

where: $\bar{f} = \begin{bmatrix} f_{f1} & f_{f2} & f_{p12} & f_{p13} & \dots & f_{pMN} \end{bmatrix}^T$ is a set of forces between fingers and objects, and between objects and other objects in contact. Grasp map matrix G can be obtained in a way similar to [19]. The matrix for all objects can be arranged to be block diagonal with some cross-block columns which correspond to body-to-body contacts. Example of such

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matrix for 2 objects with 3 external contacts on each one, and one inter-object contact is shown in (3). Since each block for each object is linearly independent, the whole matrix will be linearly independent and there is a solution (set of solutions) for immobilization of several objects.

$$\begin{bmatrix} f_{Ax} \\ f_{Ay} \\ \tau_A \\ f_{Bx} \\ f_{By} \\ \tau_B \end{bmatrix} = \begin{bmatrix} G_{f[1:3]} & 0 & \left\{ \begin{matrix} -c_{AB} \\ -s_{AB} \\ -\mathbf{x}_{AB} \times \begin{bmatrix} c_{AB} \\ s_{AB} \end{bmatrix} \end{matrix} \right\} \\ 0 & G_{f[4:6]} & \left\{ \begin{matrix} c_{AB} \\ s_{AB} \\ \mathbf{x}_{AB} \times \begin{bmatrix} c_{AB} \\ s_{AB} \end{bmatrix} \end{matrix} \right\} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_{AB} \end{bmatrix} = 0 \tag{3}$$

Caratheodory's theorem shows that a N-dimensional space can be positively spanned by N+1 vectors. For two-dimensional task space, there are 3 degrees of freedom (x, y, θ) and hence both the configuration space and the wrench space are three dimensional ($\mathbb{R}^3$). It has been elaborately shown by [20] and proven by [9] that at least four wrenches are required to immobilize 2D objects. Equilibrium grasp is achieved under condition that the origin of the wrench space lies inside of the convex hull of these wrenches.

Definition 1. A contact between two objects is called ***stable*** if for small perturbations in contact's location (movement of one object in respect to another) and applying same set of forces that kept two objects in equilibrium, the contact will either slip to the previous location or stay in the new location. Given 2 objects A, B located at $[x_A, y_A, x_B, y_B]$, set of forces F acting on the objects and a contact $C = \{p, \hat{n}, A, B\}$ and given a small perturbation ε while applying same set of forces F, new location of the objects will stay in the vicinity of the previous locations :

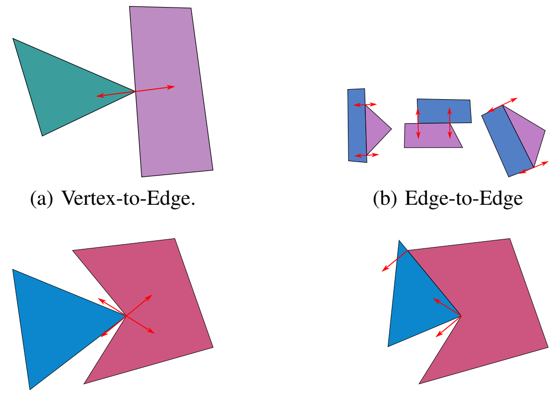
$$\forall \varepsilon \quad C = \{p + \varepsilon, \hat{n}, A, B\} \tag{4}$$
$$\rightarrow |[x_{new}, y_{new}] - [x_{old}, y_{old}]| \leq \varepsilon, \tag{5}$$

Proposition 2. *A set of N polygonal objects in $\mathbb{R}^2$ in stable contact one with another can be immobilized by at most $3N$ fingers assuming first order non-frictional contacts.*

Proof: Given connected set of objects, for each object at least one contact exist. For an object in $\mathbb{R}^2$ to be first order immobilized minimum 4 constraints required. Knowing that at least one constraint already exists because the set is connected, for each object 3 external constrains are needed, yielding $3N$ fingers for the whole set. ■

Proposition 3. *A set of N polygonal objects in $\mathbb{R}^2$ in contact one with another can be immobilized by at least 4 fingers assuming first order contacts and exceptional resulting shape.*

Proof: Assuming optimal configuration of the objects where each of them is immobilized relative to others by form closure conditions and possibly external constraints (EC) the total amount of external constraints for the set immobilization is $\max(4, |EC|)$. Example: egg-like shape cut like jigsaw puzzle which, once assembled, can be immobilized by 4 fingers. ■



(a) Vertex-to-Edge.

(b) Edge-to-Edge

(c) Vertex-to-Inner-Vertex

(d) Vertex-to-Inner-Vertex with contiguous edge

Figure 1: Contact type examples

D. Classification of inter-object contacts

The examination of given polygon configuration can be subdivided to interaction of each couple of polygons. The polygons can be disconnected - no points belong to 2 polygons simultaneously, or connected - one ore more points belong to 2 or more polygons. The connection between the polygons can be represented by a contact description. A stable contact between 2 polygons can exist in several variants:

1) Vertex to edge: is the discrete contact where a vertex of the first polygon (p_1) is coincident with an edge of the second polygon (p_2). The contact normal direction is defined by the edge the of the second polygon. The set of contacts for this variant consists of a single contact: $C = \{c_1\}$. Contact restrictions can be described as following: velocity of the contact point on the boundary of p_1 relative to the velocity of the contact point on the boundary of p_2 (denoted v_{12}) cannot have component in direction $\hat{n}_1^2$ – polygons cannot penetrate. This can be expressed as follows: $v_{12} \cdot \hat{n}_1^2 \leq 0$. Contact is maintained when the component of the relative velocity in the direction if inward normal of first polygon is 0. Since $\hat{n}_i^2 = -\hat{n}_i^1$, $-v_{12} \cdot \hat{n}_1^2 \leq 0$. Therefore, the contact is maintained when relative velocity of contact points of each polygon are perpendicular to the edge normal: $v_{12} \cdot \hat{n}_1^2 = 0$.

2) Edge to edge : the contact is a continuous contact between two bodies along a continuous segment. Since the edges are finite, there exist at least 2 vertices belonging to either one or two polygons that belong to the contact segment. In a way analogous to distributed load concentrated in a point, the continuous distributed contact can be concentrated in 2 different points. Two different vertices are selected to be such points. Since the vertices lie along the same edge, the contact normal directions are the same in this case. The set of contacts for this variant consists of two contacts: $C = \{c_1, c_2\}$, $c_i = \{r_i, \hat{n}, \dots\}$, where r_i stands for the locations of the contacts and $\hat{n}$ stands for the normal direction of the edge and the polygon names are omitted.

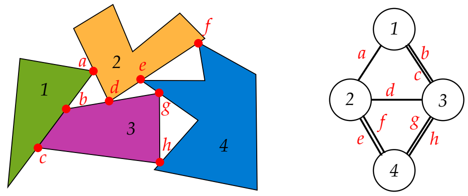


Figure 2: Set of objects and a corresponding graph.

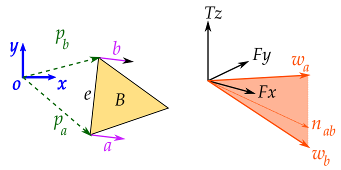


Figure 3: Edge with inner normal (left), Edge Generalized Wrench (right).

3) Vertex-inner-Vertex: is another type of discrete contact with multiple point-normal parameters. Polygon 1 vertex is in contact with 2 edges of p_2. In this case, since the movement of polygon 1 relative to polygon 2 is restricted in 2 directions, 2 normal directions exist at same contact point. The set of contacts for this variant consists of two contacts: $C = \{c_1, c_2\}$, $c_i = \{r, \hat{n}_i, \dots\}$, where$r$ stands for contact location and $\hat{n}$ stands for the normal directions of the edges and the polygon names are omitted.

Other variants of contact descriptions are variations of the mentioned above. Examples: edge and inner vertex, vertex and vertex, edge and edge. Outer vertex to outer vertex contact assumed to be unstable and is discarded in this work. One polygon can contact several others in distinct points, apart from a case where several vertices are coincident.

E. Graphs depicting the interaction between objects

The interaction between several objects can be qualitatively described by a graph where nodes describe objects, and edges describe contact existence between the objects. For the purposes of multi-object grasp evaluation graphs allow search of connected sub-sets and their correspondent contacts. Fig. 2 shows an example of 4 polygonal objects, with contact points shown in red and the qualitatively descriptive graph.

F. Edge Generalized Wrench

EGW is a graphical way of describing possible contact wrenches for a given linear edge of some 2D object.

Definition 4. Given a polygon's edge e and a reference point o, the set of all possible wrenches w_e can can be defined by a two-dimensional convex cone in a vertical plane of a wrench space.

Denote the edge's normal by $\hat{n}$ and edge's ends by $\vec{e}_1, \vec{e}_2$ relative to o, the edge is parameterized by parameter $s \in [0,1]$ the wrench of a contact along the edge can vary between:

$$(1-s)\begin{bmatrix}\hat{n}\\ \vec{e}_1\times\hat{n}\end{bmatrix}+s\begin{bmatrix}\hat{n}\\ \vec{e}_2\times\hat{n}\end{bmatrix},\qquad s\in[0,1] \tag{6}$$

which is exactly a linear combination of two marginal wrenches. In a geometrical representation, it is a planar angle

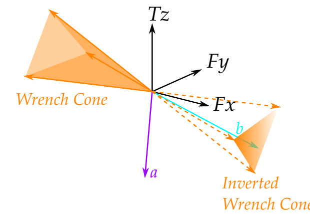


Figure 4: Inverted Cone, with vector a inside and vector b outside.

contained in a vertical plane (plane's normal direction has no torsional component) in the wrench space. Illustration shows an example of a generalized edge wrench. An edge of polygonal object B has inward normal direction $\hat{n}_{ab}$. Two marginal contacts a, b are shown in violet, with locations defined by vectors p_a, p_b respectively. Wrenches induced by these contacts are calculated by:

$$w_a=\begin{bmatrix}\hat{n}_a\\ \vec{p}_a\times\hat{n}_a\end{bmatrix},\ w_b=\begin{bmatrix}\hat{n}_b\\ \vec{p}_b\times\hat{n}_b\end{bmatrix}. \tag{7}$$

As can be seen from the fig. 3, contact a induces positive torque about the selected reference point, while contact b induces negative torque. Hence, the 3-rd components of a wrench have opposite signs and are located on different sides of the xy plane. The expression for the edge generalized wrench can be rewritten to a form of:

$$w_e(s)=(1-s)\begin{bmatrix}\hat{n}\\ \vec{e}_1\times\hat{n}\end{bmatrix}+s\begin{bmatrix}\hat{n}\\ \vec{e}_2\times\hat{n}\end{bmatrix}=\dots,\qquad s\in[0,1]$$
$$=\begin{bmatrix}n_x\\ n_y\\ (x_1\cdot n_y-y_1\cdot n_x)+s\left(n_y(x_2-x_1)+n_x(y_1-y_2)\right)\end{bmatrix} \tag{8}$$

We can note that only the torque component of the wrench vector changes along the edge, and hence the edge generalized wrench is strictly "vertical" in the wrench space, and that it is linear along the parameter$s$.

G. Inverted cone test

As stated above, a convex hull of the contact wrenches should contain the origin of a wrench space. Given several contacts, and corresponding wrenches in the wrench space that do not contain the origin, a simple test can show whether additional vector will make total convex hull to contain the origin or not.

Proposition 5. *Given a set of 3 contacts, whose wrenches do not span the wrench space (the origin of the wrench space is not inside the convex hull of these wrenches) while not being in the same plane and additional contact, the contact can complete the force closure grasp of the object if the contact's wrench lies within the inverted cone of the wrenches of given contacts .*

Proof: Each couple of wrenches along with the origin point form a plane, Equilibrium condition for a grasp is that the total wrench for all contacts equals to zero:

$$\sum_{i=1}^{4} G_i f_i = 0 \tag{9}$$

Where G_i is $\begin{bmatrix} n_i \\ p_{c_i} \times n_i \end{bmatrix}$ for each contact. Rewriting the (9):

$$G_4 f_4 = -\sum_{i=1}^{3} G_i f_i \tag{10}$$

Given the fact that G columns are wrench space vectors, and f are scalar values, this allows the representation of a 4th wrench space vector as a linear combination of other wrench vectors, which is exactly the cone formed by these wrenches. If the 4th vector lies outside the cone then it cannot be represented as a linear combination of other wrenches and hence will not allow equilibrium grasp. ■

H. Maximizing the Inscribed Sphere

One of the proposed quality measures is a volume/radius of the origin centered sphere inscribed in the convex hull of grasp map basis vectors. Maximizing the sphere radius can be done by increasing the minimal of the distances between the origin end the faces of the convex hull. Since the hull is convex, some of the faces will be tangent to the sphere, and in no case a edge will be tangent. In cases where several contacts act on the object, it is always possible to construct a *convex* hull and thus eliminate redundant wrench vectors.

IV. Algorithms

Relying on the classification, an preferable order of desired inter-object contact types can be defined:

- Vertex-to-Inner-Vertex with 2 contiguous edges
- Vertex-to-inner-vertex with 1 contiguous edge and point contact
- Vertex-to-inner-vertex with 1 contiguous edge
- Edge to edge / Vertex-to-inner-vertex

A. Algorithm for objects rearrangement

First step for the object stacking is to define a root object, which all others will lean on. The algorithm searches among the objects for either one with biggest concavity, or for a couple of objects to form a concavity together. When root object is selected, further stacking can be done by iteratively selecting objects and stacking them in concavities including newly formed concavities from previously stacked objects. Algorithms 1 and 2 present pseudo-code for the objects rearrangement.

B. Algorithm for complementing fingers placement

When rearrangement of desired objects is proposed, a search for fingers need to complete the immobilize the objects can be performed. The search is done by iterating over each object and testing for immobilization. If the object is not immobilized, required number of fingers is found by examining the convex hull of existing wrenches and testing EGWs for all edges until suitable grasp is found.

Algorithm 1: Defining the root object

Data: Set of polygons $P = \{p_i\}$ $i \in [1, K]$
Result: Root object and possibly one child.

if $\exists$ *1+ concave vertices* **then**
  Root object = p_i with widest concave vertex;
else
  if $\exists$ *2 vertices with inner angles* $\geq 90°$ **then**
    Root = p with greatest angle;
    Root.Child(k) = p with second greatest angle;
    Align vertices to conctact, child leans on longest adjacent edge of the parent;
  else
    Root = p with longest edge;
    Root.Child(k) = p with second longest edge;
    Align longest edges to be collinear, vertex on child's longest edge coincident with midpoint of parent's longest edge.;
  end
end

Algorithm 2: Objects stacking

Data: Set of polygons $P = \{p_i\}$, Stacked Objects $T = \{p_i\}$
Result: Complete configuration of the set of of objects.

C = list of concavities;
initialize C with the root concavity;
L = $\{v\}$ sorted list of polygon vertices;
foreach $v \in L$ **do**
  if $\angle(v) \leq \angle(c_i \in C)$ **then**
    Stack the child object in the c_i, *vertex to innter vertex*, edge aligned preferrably to a edge with non equal length;
    move $L.v$ to $T.v$ and save child location and orientation;
    $C.c_i$ = reduced concavity;
    C = new formed concavities;
  else
    Find smallest concavity that allows *contiguous edge with vertex contact* interaction, stack object there. C = new formed concavities;
  end
end

V. Multi object grasp evaluation and grasp securing

Once every given object is immobilized by other objects or/and fingers, the constellation still might be not fully immobilized. Total immobilization can be achieved by ensuring that every possible subset of objects is immobilized. This requires extraction of all possible connected subsets from the given set object and evaluation whether they are or are not immobilized. Each constellation ought to have at least 4 external contacts. If there are less - additional contacts required to immobilize it. If the amount of external contacts is higher than 4, assessment of intersection of contacts external to the subset can show

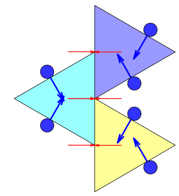
Algorithm 3: Search of complementing fingers

Data: Polygon: P, Contacts set: $C = \{c_i\}$ $i = 1, ..N3$
Result: $CF = \{f_i, GQM\}$
$AntiCone = -1 \cdot Cone\left(\{w_{C_i}\}\right)$;
for e_i = *edge of* P **do**
  $GW = GeneralizedWrench(e_i)$;
  $CS = CommonSegment(GW, AntiCone)$;
  if $CS \neq \emptyset$ **then**
    $w_{optimal} = \underset{w}{\operatorname{argmax}}\ GQM$;
    $CF \twoheadleftarrow \{w_{optimal}\}$;
  end
end
if $CF \neq \emptyset$ **then**
  for e_i, e_j = *edge pairs of* P **do**
    $W_{optimal} = \underset{w1,w2}{\operatorname{argmax}}\ GQM$;
    $CF \twoheadleftarrow \{W_{optimal}\}$;
  end
end

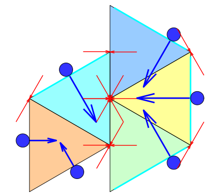
whether the subset is immobilized or not. If the contacts intersect at one point, contact positions of added fingers can be altered in a allowed regions that were found for that contact.

Algorithm 4: Grasp evaluation

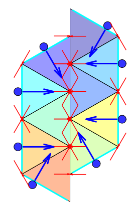
Data: Polygons: $P = \{p_i\}$, Finger contacts set: $F = \{f_i\}$ $i = 1, ..N3$
Result: $CF = \{f_i, GQM\}$
for *subset* $P' \in P$ **do**
  if *contacts intersect in one point* ***and*** *more than 4 contacts* **then**
    if *some contacts can be moved* **then**
      move them to obtain a valid grasp
    else
      add more contacts
    end
end



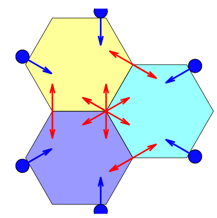
(a) 3 triangles



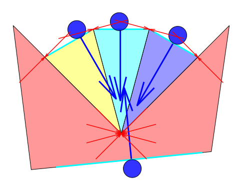
(b) 5 triangles



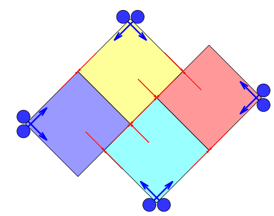
(c) 10 triangles



(d) 3 hexagons

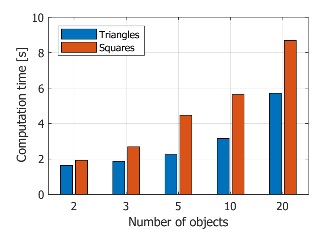


(e) 3 triangles and a concave object

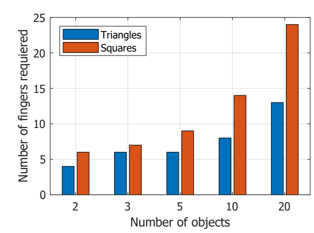


(f) 4 squares

Figure 5: Generated object configurations and finger locations found



(a) Computation time



(b) Number of fingers

Figure 6: Comparison of algorithm results and performance for different sets.

VI. Results

A. Simulations

The algorithms were tested in MATLAB environment for consistence, and performance. The calculations are performed offline, and the execution is tested on Intel i7-6700K with 32gB RAM in MATLAB r2018a.

Several cases of planned grasps for various number polygonal objects are presented below in fig. 5. Polygonal objects are shown in colors; inter-object contacts are shown in red; selected finger combination with normal direction lines are shown by dark blue circles with arrows; allowed regions if exist marked with cyan. Fig. 6 shows some of the results for different sets of primitive shapes. It is easily seen that the execution time does grow with the increase of the number of objects, and with increase of number of edges of polygons.

B. Experiments

Experiments were conducted to physically test the output of the algorithms. Two model sets were tested to move along with the custom made gripper. The griper consists of several static fingers which constitute a form closure grasp. For both models insignificant relative motion of objects in the set was captured, the motion results from imperfections of the 3D printed parts, and the set remains caged. Fig. 8 shows 4 frames of a footage of manipulation of 6 equilateral triangles. The images show bottom view, where red gripper grasps 6 triangles color in white, gray and dark blue. Translation with simultaneous rotation are applied by the robotic manipulator which results on translation and rotation of the set of objects

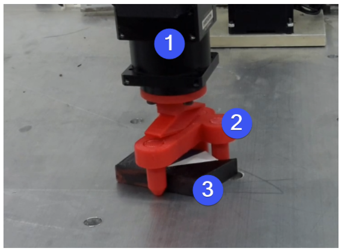
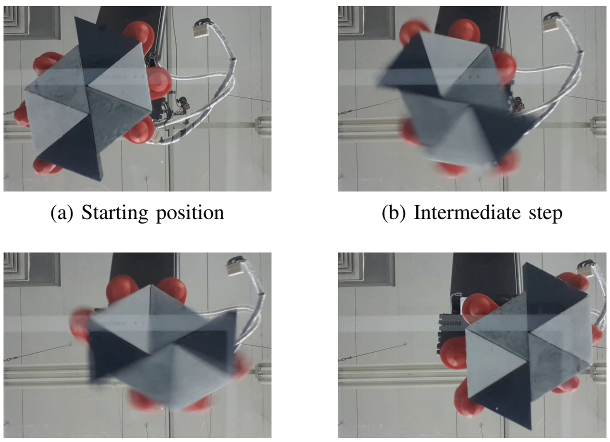


Figure 7: Experimental setup: part of robotic manipulator (1), gripper (2), set of objects (3).



(a) Starting position

(b) Intermediate step

(c) Intermediate step

(d) Final position

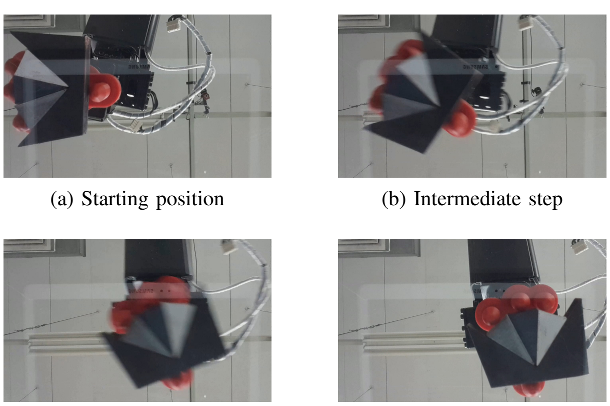
Figure 8: Translation and counter-clockwise rotation by 180° of 6 equilateral triangles.

altogether. Fig. 9 presents a setup of 3 isosceles triangles and one concave object which were shown in fig. 5e.

VII. Conclusions

In this work we introduced a novel method for object set rearrangement and grasp planning while exploiting the geometrical properties of polygonal objects. The descriptions of inter-object contacts presented and classified as to the advantages they give for multi-object configurations. Algorithms for object stacking and finger placement proposed and validated in simulations ans experiments.

In future we plan to extend the methods for objects with curvilinear boundaries, incorporating methods for exploiting such boundaries to lower the amount of fingers needed for the grasp as [12] proposed.



(a) Starting position

(b) Intermediate step

(c) Intermediate step

(d) Final position

Figure 9: Translation and clockwise rotation by 90° of 3 isosceles triangles and one concave object.

Appendix

Accompanying code with MATLAB simulations is presented at https://github.com/yossioo/SGMO.

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