Contents

Chapter 1

Flat Spacetime

1.1 Spacetime Physics

1.1.1 Notes

- Space tells matter how to move, matter tells space how to curve. Physics is simple when local, and when we eliminate force at a distance life is good.

- *Events* are coordinate-independent points in spacetime that are defined by "what happens there" or more concretely by an intersection of worldlines.

- Spacetime is locally Lorentzian, a.k.a. flat or non-accelerating. Gravitation/curvature is defined as the acceleration of the separation between two nearby geodesics.

  We measure curvature by the following: characterize ξ the separation between two originally parallel geodesics, then propagate each forward by distance s. They are not necessarily parallel anymore (e.g. two great circles on a sphere), and the separation obeys the following EOM

  $$\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} + R\xi = 0 \tag{1.1}$$

  where R is the *Gaussian Curvature* of the surface.

Generalizing to multiple dimensions, the separation ξ is a vector, and we describe $\frac{\mathrm{D}^2\xi}{\mathrm{d}^2 s}$ with a capital D since the coordinates of the derivative $\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2}$ is subject to the whims of the coordinate lines, which should not affect the separation between these two geodesics that live beyond coordinates. The curvature is instead described by the Reiman curvature tensor, which takes as arguments the 4-velocity $\mathbf{u} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$ and yields

$$\frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}^2\tau^2} + R^\alpha_{\beta\gamma\delta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\xi^\gamma\frac{\mathrm{d}x^\delta}{\mathrm{d}\tau} = 0 \tag{1.2}$$

where $\mathbf{R}$ is the 4-component Reimann curvature tensor. We call this the *equation of geodesic deviation*.

These functions are sister functions of the Lorentz force equation in electromagnetism

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} - \frac{e}{m}F^\alpha_\beta\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = 0.$$

- $\mathbf{R}$ is a fickle object, subject to perturbation e.g. by gravitational waves. A certain piece of the tensor is generated only by the local mass distribution though, $\mathbf{G}$ the *Einstein curvature tensor*, incidently proportional to the *stress-energy tensor* as $\mathbf{G} = 8\pi\mathbf{T}$. Its interpretation is a local average curvature.

1.1.2 Practice Problems

Exercise 1.1 *Show that the Gaussian curvature of a cylinder* $R = 0$.

Choose cylindrical coordinates (a, θ, z) for the surface of the cylinder of radius a. We assert temporarily that all geodesics can be parameterized as $g(s|\omega, \dot{z}) = (a, \omega s, \dot{z}s)$, i.e. correspond to uniform translation along and rotation about the axis of the cylinder. Then, using the equation of geodesic deviation with Gaussian Curvatures

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} + R\xi = 0$$

for $\xi = g_1 - g_2$ the separation, then since g_1, g_2 are linear functions of s, their second derivatives with respect to s are zero and thus $\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} = 0$. Observe that this happens regardless of the value of ξ, thus $R = 0$.

We now justify our parameterization $g(s|\omega, \dot{z})$. Two points separated infinitisemally $(a, \theta, z), (a, \theta + \delta\theta, z + \delta z)$ are separated by distance

$$\sqrt{\delta z^2 + a^2\delta\theta^2} = \mathrm{d}\theta\sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + a^2}.$$

The distance between two points $(\theta_1, z_1), (\theta_2, z_2)$ is then given by

$$D[z(\theta)] = \int\limits_{z_1}^{z_2} \sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + a^2}\,\mathrm{d}\theta$$

where $z(\theta)$ is any trajectory that runs between the desired endpoints. This is then a variational calculus problem, and thus the $z(\theta)$ that minimizes the path must satisfy the Euler-Lagrange Equation

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial I(z', z)}{\partial z'} - \frac{\partial I(z', z)}{\partial z} = 0.$$

We note then that $I(z', z) = \sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + a^2}$ is independent of z and so we instantly find that $I(z', z) = C$ a constant. This furthermore implies that $\frac{\mathrm{d}z}{\mathrm{d}\theta}$ is a constant and so that $z \propto \theta$. Call the constant of proportionality $z \propto \alpha\theta$.

Armed with this, we find that the arclength of the geodesic between the desired endpoints is

$$D[z(\theta)] = \int_{z_1}^{z_2} \sqrt{\alpha^2 + a^2}\,\mathrm{d}\theta \tag{1.3}$$

$$= \sqrt{\alpha^2 + a^2}\Delta z. \tag{1.4}$$

Recalling that s parameterizes the length of our geodesic, we find that s, z, θ must all be proportional. Call the ratios $\frac{\theta}{s} = \omega, \frac{z}{s} = \dot{z}$, justifying our parameterization.

Exercise 1.1b *Alternatively, employ $R = \frac{1}{\rho_1\rho_2}$ where ρ_1, ρ_2 are the principal radii of curvature at the point in question in the enveloping Euclidean 3D space.*

We note that one of the radii in a cylinder is a while the other is infinite, thus $R = 0$.

Exercise 1.3 *Show that given ω, the rotational frequency of a planet about a fixed central mass M, we can not individually determine r the radius of orbit or M. Instead, derive a relation between ω and ρ the* Kepler Density *of the mass, the density if M were spread over the sphere of radius r.*

We know that a central acceleration $\ddot{r} = r\omega^2 = \frac{GM}{c^2r^2}$ under Newton's Law of Gravitation. Thus, we have

$$\omega^2 = \frac{GM}{c^2r^3} = \frac{4\pi}{3}\frac{G}{c^2}\rho$$

where since there are no other constraints on the motion we are done.