Separating out research-related tidbits from non-research ones.

Contents

1 06/29/19—Collisionless Boltzmann Equation in Galaxies: Landau Damping

Inspired by `https://arxiv.org/pdf/1906.08655.pdf`. The problem is basically formulated as thus: consider a kinetic-theoretic description of a fluid using distribution function $f(t,x,p)$ which obeys collisionless Boltzmann equation $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$ (we use p instead of v to work in Hamiltonian coordinates). Introducing a periodic perturbation to this fluid results in a singular dispersion relation, which can be resolved via the usual Landau prescription (consider a perturbation having grown from zero at $t = -\infty$). The dispersion relation describes *Landau damping* (or growth), in which energy from the fluid is exchanged with the perturber.

1.1 Linearized EOM

The point of the paper is instead to analytically compute the impact of the perturber on the distribution function, to quantify the *scarring* of a galaxy upon encounters with a nearby perturber. The equations of motion coupling the distribution function and gravitational potential are given

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{*\}f, \mathscr{H} = 0, \tag{1}$$

where $\mathscr{H} = \frac{p^2}{2} + \Phi$ and $\{*\}\ldots$ denotes the Poisson bracket $\{*\}f, \mathscr{H} = \vec{\nabla}_x f \cdot \vec{\nabla}_p \mathscr{H} - \vec{\nabla}_p f \cdot \vec{\nabla}_x \mathscr{H}$.

If we linearize for perturbation quantities f_1, Φ_1 where $\Phi_1(x)$ does not depend on the momenta, we obtain

$$\begin{aligned}
0 &= \frac{\partial f_1}{\partial t} + \{*\} f_1, \mathscr{H}_0 - \vec{\nabla}_p f \cdot \vec{\nabla}_x \mathscr{H}_0,\\
&= \frac{\partial f_1}{\partial t} + \vec{\nabla}_x f_1 \cdot \vec{p} - \vec{\nabla}_p f_1 \cdot \vec{\nabla}_x \Phi_0 - \vec{\nabla}_p f_0 \cdot \vec{\nabla}_x \Phi_1.
\end{aligned}$$

Oops welp I guess I never solved this.

2 02/16/23—Linear Predictive Coding: Autoregressions and Fourier Transforms

This was a simple enough inquiry initially: given a partial time series that contains sinusoids, how do we extract the frequency? We know one way to do this using the FFT, but there are advantages to other techniques. Courtesy of Jeremy Goodman's pointers.

The trick has to do with autoregressions. Suppose we are looking to extract l frequencies of form $e^{i\omega_m t}$, so that

$$y_n = \sum_m^l C_m e^{i\omega_m n\Delta t}. \tag{2}$$

Thus, if we have at least $2l$ points or so, we should be able to fit for the $2l$ DOF C_m and ω_m. There can be a noise term above if need be, in which case more points will smooth out the noise.

What is the trick? Well, we compute the l-th order autoregression. In other words, for each sequence of l points, we can write down the expression satisfying:

$$y_n = \sum_{m=1}^l a_m y_{n-m}. \tag{3}$$

With l-many such sequences, we have enough equations to solve for the l-many unknowns a_m. These can be written in the matrix form:

$$\begin{bmatrix} y_n \\ y_{n+1} \\ \vdots \\ y_{n+l} \end{bmatrix} = \begin{bmatrix} y_{n-1} & y_{n-2} & \cdots & y_{n-l} \\ y_n & y_{n-1} & \cdots & y_{n-l+1} \\ \vdots & \vdots & \ldots & \vdots \\ y_{n+l-1} & y_{n+l-2} & \cdots & y_{n-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_l \end{bmatrix}. \tag{4}$$

These $\{a_l\}$ form the AR(l) autoregressive model for y_n.

This is great, but how do we get the frequencies, or also maybe the growth rates? Now, we rewrite the above equation as

$$0 = \begin{bmatrix} y_n & y_{n-1} & \cdots & y_{n-l} \\ y_{n+1} & y_n & \cdots & y_{n-l+1} \\ \vdots & \vdots & \ldots & \vdots \\ y_{n+l} & y_{n+l-1} & \cdots & y_{n-1} \end{bmatrix} \begin{bmatrix} -1 \\ a_1 \\ \vdots \\ a_l \end{bmatrix} \equiv \mathbf{B} \cdot \vec{a}. \tag{5}$$

Now, what if the y_n look like λ^n for some complex λ? Then the λ must satisfy

$$1 - a_1\lambda - a_2\lambda^2 - \cdots - a_l\lambda^l = 0. \tag{6}$$

This is the characteristic equation for this AR(l) model. If we solve for the roots of this equation, we get the possible values of λ that satisfy the model. In other words, if the $y_n = \lambda^n$ indeed, then $\mathbf{B}\cdot\vec{a} = 0$ as requested above. Then, if the data are oscillatory, then $\lambda = e^{i\omega_m}$ as requested above.

2.1 Intuitive Understanding

There's something slightly unintuitive here: we began by seeking the frequencies in the y_n, but we eventually obtained this by solving an equation that has *nothing* to do with the y_n, the characteristic equation for the AR(l) model! Why does this make sense?

Well, it should be noted that a particular AR(l) model does not uniquely specify the y_n; this would be impossible, since there are only l DOF in the model and $2l+1$ in Eq. (4). Indeed, this suggests that the amplitudes of the modes C_m, as well as the initial normalization of the autoregressive chain (y_{n-l}) are free parameters. As such, we can imagine that the AR(l) model permits a family of solutions, any with the correct frequencies. In other words, we could also imagine writing:

$$\mathbf{B}\cdot|a\rangle = \sum_{m=1}^{l} C_m\,|b_m\rangle\,\langle b_m|\,|a\rangle\,. \tag{7}$$

Another way to think about the characteristic equation is as exactly a characteristic equation of a matrix. If we consider the matrix that maps the vector

$$\begin{bmatrix} y_N \\ y_{N-1} \\ \vdots \\ y_{N-l} \end{bmatrix} = \mathbf{M}\begin{bmatrix} y_{N-1} \\ y_{N-2} \\ \vdots \\ y_{N-l-1} \end{bmatrix}, \tag{8}$$

then it's clear that $\mathbf{M}$ has the form

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & \dots & a_{l-1} & a_l \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{9}$$

It's then clear $\mathbf{M}$ apparently has exactly the characteristic equation that we prescribe above. This makes sense: the matrix $\mathbf{M}$ tells us whether a vector-valued sequence of y values is growing, decaying, or oscillating.

As such, the final conclusion of this tidbit is this: the autoregression is another way of expressing a Markov chain that allows us to advance the time series. Then, note that any sequence with $y_n = z^n$ where z is complex (allowing periodic or exponential sequences) has z as one of the eigenvalue of its Markov chain matrix, or z is a root of its characteristic polynomial. Turning this on its head, if we compute

the autoregression for a sequence and find a root w of its characteristic polynomial, this implies that the sequence has a geometric component with factor w. Applying this to sequence with a periodic component with frequency ω, we see that $e^{i\omega\Delta t}$ must be a root of the characteristic polynomial of its autoregression.

3 02/21/23—Chaotic vs Diffusive Behavior

This is a short section. Dong (and others) talk about "chaotic tides", where the mode amplitude grows stochastically because the forcing occurs with random amplitude. However, this is not chaos, but should be properly termed "diffusive tides."

How can we argue for this? Well, the defining characteristic of chaos is a positive Lyapunov exponent, i.e. an exponential growth rate of the separation between two trajectories with nearly-idential initial conditions.

$$\delta y(t) = y(t; y_0) - y(t; y_0 - \epsilon), \tag{10}$$

$$\sim \mathcal{O}(\epsilon e^{\lambda t}). \tag{11}$$

What is the growth rate for a random walk, or diffusive growth?

Let's adopt the simplest model for now, a discrete random walk with step size ± 1. Perhaps, for the sake of consistency, we can imagine that the step is determined based on the current value of x, e.g. whether the rounded value of $10^9 x$ is even or odd. Then, consider two initially adjacent x. It is obvious that

$$|\delta x(t)| \leq 2t. \tag{12}$$

So we already see that diffusive behavior is not chaotic.

But now, we have a fun little math problem. Consider two random walks starting at $x = 0$ with step size ± 1. What is the mean and variance of δx? Well, using the usual CLT guidelines, the linearity of expectation gives $\mathrm{E}(X_2 - X_1) = 0$ while linearity of variance gives $\mathrm{Var}(X_2 - X_1) = 2\,\mathrm{Var}(X_1) = 2t$. Thus, the separation between two walkers grows stochastically and $\sim \sqrt{t}$. This is not chaos, where the separation grows deterministically and $\sim \exp(\lambda t)$.

4 02/23/23—Rayleigh Distribution

4.1 2D

The Rayleigh distribution is commonly used for mutual inclinations. Can we briefly give ourselves some intuition for it?

It's known that the Rayleigh distribution is the magnitude of a 2D vector with two normally-distributed components. Thus, consider if X and Y are drawn from $N(0,\sigma)$. Then the CDF of the magnitude $M = \sqrt{X^2 + Y^2}$ of the vector is calculated as

$$F_M(m;\sigma) = \iint\limits_{D(m)} f_U(u;\sigma) f_V(v;\sigma)\,\mathrm{d}u\mathrm{d}v \tag{13}$$

Here, $D(m)$ is the unit disc satisfying $\sqrt{u^2+v^2} \le m$, and $f_U = N(0,\sigma)$ and $f_V = N(0,\sigma)$ are the PDFs of the random variables U, V. Upon changing to polar coordinates and integrating:

$$F_M(m;\sigma) = \int_0^m 2\pi \frac{1}{2\pi\sigma^2} \exp\left[-\frac{u^2+v^2}{2\sigma^2}\right] m' \mathrm{d}m', \tag{14}$$

$$= \frac{1}{\sigma^2} \int_0^m m' \exp\left[-\frac{(m')^2}{2\sigma^2}\right] \mathrm{d}m', \tag{15}$$

$$f_M(m;\sigma) = \frac{m}{\sigma^2} \exp\left[-\frac{m^2}{2\sigma^2}\right]. \tag{16}$$

This is a Rayleigh distribution with width σ.

Now, if we know that two vectors have a magnitude separation that is Rayleigh distributed (like Rayleigh distribution), and we know that they are iid, how can we obtain their individual distributions (under the right assumptions)? Well, let's first assume that the four vector components are all normally distributed with $N(0,\sigma)$. Then their separation vector's components are also normally distributed with:

$$f_{X_1-X_2}(x_1 - x_2;\sigma) = N\left(0, \sigma\sqrt{2}\right), \tag{17}$$

$$f_{Y_1-Y_2}(y_1 - y_2;\sigma) = N\left(0, \sigma\sqrt{2}\right). \tag{18}$$

Thus, the separation vector's magnitude is Rayleigh distributed with width parameter $\sigma\sqrt{2}$. Thus, to generate a pair of vectors whose separation magnitude is Rayleigh distributed with width δ, we can just generate the vector components from $N\left(0, \delta/\sqrt{2}\right)$.

To briefly comment, this obviously doesn't depend on the number of vectors, as long as the measured Rayleigh distribution is for arbitrary pairs in the system.

4.2 3D

What about in 3D? This is just as much for my own practice with manipulating PDFs and CDFs than anything. Consider a vector $\left(v_x, v_y, v_z\right)$ with all three components drawn from $N(0,\sigma)$. What is the distribution of the magnitude? Again:

$$F_M(m;\sigma) = \frac{1}{(2\pi\sigma^2)^{3/2}} \iiint_{D(m)} f_{Vx}(v_x;\sigma) f_{Vy}\left(v_y;\sigma\right) f_{Vz}(v_z;\sigma)\, \mathrm{d}^3 v, \tag{19}$$

$$= \frac{4\pi}{(2\pi\sigma^2)^{3/2}} \int_0^m e^{-(m')^2/(2\sigma^2)} (m')^2 \mathrm{d}m', \tag{20}$$

$$f_M(m;\sigma) = \frac{4\pi m^2}{\left(2\pi\sigma^2\right)^{3/2}} e^{-m^2/(2\sigma^2)}. \tag{21}$$

This of course can easily generalize: it is clear that in N-D:

$$f_M^{(N)}(m;\sigma) = \frac{S_m^{(N)}}{\left(2\pi\sigma^2\right)^{N/2}} e^{-m^2/(2\sigma^2)}, \tag{22}$$

where $S_m^{(N)}$ is the surface area of the N-sphere with radius m.

And what about the separation vector between some $\vec{v}$ and $\vec{w}$ in 3D? Well, each component of $\vec{v}-\vec{w}$ has distribution $N\left(0,\sigma\sqrt{2}\right)$ again, so if $U=|\vec{v}-\vec{w}|$, then its PDF is

$$f_U(u;\sigma)=\frac{u^2}{\sigma^3\sqrt{4\pi}}e^{-m^2/(4\sigma^2)}. \tag{23}$$

5 03/15/23—Mass Loss and Binary Orbit Change

Without kicks, Hills 1983 seems to have the best prescription. Time to revisit this problem now that I've done it wrong literally every time I've tried to do it.

5.1 Brute Force Circular

We start with a circular orbit and in the rest frame of the binary. Call the pre-ML energy E and post-ML energy E'. These are the sums of kinetic and gravitational potential energies, so

$$E=\frac{m_1v_1^2}{2}+\frac{m_2v_2^2}{2}-\frac{Gm_1m_2}{a}, \tag{24}$$

$$E'=E-\frac{m_1v_1^2}{2}(1-f)+\frac{Gm_1m_2}{a}(1-f). \tag{25}$$

Here, $m_1'=fm_1$ is the post-ML mass. Note then that

$$v_1^2=\left(\frac{am_2}{m_{13}}\right)^2\frac{Gm_{12}}{a}=\frac{Gm_2^2}{m_{12}a}, \tag{26}$$

$$K_{\rm cm}=\frac{((1-f)m_1v_1)^2}{2(fm_1+m_2)}. \tag{27}$$

Here, $K_{\rm cm}$ is the kinetic energy associated with the motion of the post-ML binary's center of mass. To undergo unbinding, we need f such that $E'=K_{\rm cm}$, so that in the co-moving frame the binary is unbound. This is a laborious calculation, but we can write ($f'\equiv 1-f$ is the fraction of mass lost from m_1)

$$\begin{aligned}
0 &= E' - K_{\rm cm},\\
&= -\frac{Gm_1m_2}{2a} - \frac{m_1v_1^2}{2}f' + \frac{Gm_1m_2}{a}f' - \frac{\left(f'm_1v_1\right)^2}{2(fm_1+m_2)},\\
&= -\frac{Gm_1m_2}{2a} - \frac{Gm_1m_2^2}{2m_{12}a}f' + \frac{Gm_1m_2}{a}f' - \frac{\left(f'm_1\right)^2}{2(fm_1+m_2)}\frac{Gm_2^2}{m_{12}a},\\
&= -\frac{1}{2} - \frac{m_2}{2m_{12}}f' + f' - \frac{m_1m_2(f')^2}{2(m_{12}-f'm_1)m_{12}},\\
&= -(m_{12}-f'm_1)m_{12} - m_2\left(m_{12}-f'm_1\right)f' + 2(m_{12}-f'm_1)m_{12}f' - m_1m_2(f')^2,\\
&= -m_{12} + f'(m_1 - m_2 + 2m_{12}) - 2m_1(f')^2,\\
&= (f')^2 - \frac{f'}{2}\left(3 + \frac{m_2}{m_1}\right) + \frac{1}{2}\left(1 + \frac{m_2}{m_1}\right),\\
f' &= \frac{(3+q)/2 \pm \sqrt{(3+q)^2/4 - 2(1+q)}}{2},\\
&= \frac{(3+q)/2 \pm ((q-1)/2)}{2},\\
&= \frac{1+q}{2},\\
&= \frac{m_{12}}{2m_1}.
\end{aligned}$$

Whew. This is the canonical result, that we need to lose $f'm_1 = \frac{m_{12}}{2}$ mass to unbind the system.

5.2 Easier Circular

The primary difficulty above was that we had this stupid center of mass kinetic energy to carry around. This can be simplified if we just recognize that we only need to compute the contribution of the reduced mass to the energy to understand whether the system remains bound. Recall that the the kinetic energy of of the reduced mass component is just

$$K_{\rm red} = \frac{\mu v_{\rm rel}^2}{2}, \tag{28}$$

$$E_{\rm red} = K_{\rm red} - \frac{Gm_{12}\mu}{a} = -\frac{Gm_{12}\mu}{2a}, \tag{29}$$

$$v_{\rm rel}^2 = \frac{Gm_{12}}{a}, \tag{30}$$

$$E'_{\rm red} = \frac{\mu v_{\rm rel}^2}{2} - \frac{Gm'_{12}\mu'}{a} = 0, \tag{31}$$

$$= \frac{Gm_{12}\mu'}{2a} - \frac{Gm'_{12}\mu'}{a}. \tag{32}$$

Thus, we end up with the result that $m'_{12} = m_{12}/2$ results in a reduced-mass energy = 0 and unbinding. It's important to recognize that $v_{\rm rel}$ is the relative velocity of the two particles, given by $\mathbf{v}_{\rm rel} = \mathbf{v}_2 - \mathbf{v}_1$, which does not change with instantaneous mass loss.

5.3 Eccentric Unbinding

The argument in the previous section is much easier to generalize to a general orbit. Consider that the orbit has semimajor axis a and unbinds when the separation is at r. Then

$$\begin{aligned}
E_{\rm red} &= \frac{1}{2}\mu v_{\rm rel}^2 - \frac{Gm_{12}\mu}{r} = -\frac{Gm_{12}\mu}{2a},\\
\frac{v_{\rm rel}^2}{2} &= Gm_{12}\left(-\frac{1}{2a} + \frac{1}{r}\right),\\
E'_{\rm red} &= \frac{1}{2}\mu' v_{\rm rel}^2 - \frac{Gm'_{12}\mu'}{r},\\
&= \mu'\left[-\frac{Gm_{12}}{2a} + \frac{Gm_{12}}{r}\right] - \frac{Gm'_{12}\mu'}{r}.
\end{aligned}$$

Setting this equal to zero, we find

$$0 = \mu'\left[-\frac{Gm_{12}}{2a} + \frac{Gm_{12}}{r}\right] - \frac{Gm'_{12}\mu'}{r}, \tag{33}$$

$$\frac{m'_{12}}{m_{12}} = -\frac{r}{2a} + 1. \tag{34}$$

If we re-express $m'_{12} \equiv m_{12} - \Delta$, then we can rewrite

$$1 - \frac{\Delta}{m_{12}} = 1 - \frac{r}{2a} = 1 - \frac{1-e^2}{2(1+e\cos f)}. \tag{35}$$

This makes sense: since the mass loss effects a torque on the system, we have to give it the maximum torque to unbind the system, which occurs at pericenter. Thus, qualitatively we need a minimum eccentricity of $(1-e)/2 \sim m'_{12}/m_{12}$ to unbind the system of $m'_{12} > m_{12}/2$, i.e. if the mass loss is too little.

5.4 Bound Orbits: Final Eccentricity

Of course, these exercises can be repeated if we would like for bound orbits, and tracking the angular momentum of the system as well to get eccentricity. I may redo this some day, but for now I will just cite the result from Hills 1983, where the final eccentricity is given by

$$e = \left\{1 - (1-e_0^2)\left[\frac{1-(2a_0/r)(\Delta/m_{12})^2}{1-\Delta/m_{12}}\right]\right\}^{1/2}. \tag{36}$$

6 03/23/2023—Pendulum Periods

It might be helpful to just do the simple pendulum in a few ways to get its period. Specifically, I mean the oscillation of the nondimensionalized EOM

$$\ddot{\theta} = -\sin\theta. \tag{37}$$

In the small angle approximation $\theta \ll 1$, we have that the frequency of the oscillator is just 1, so the period is 2π.

6.1 Lindstedt-Poincaré

I always get this wrong, so let's try again. The zeroth order solution is $\theta = \epsilon\cos(\omega_0 t)$ where $\omega_0 = 1$. Let's next imagine that the frequency has a small ϵ dependence, so that

$$\begin{aligned}
\theta(t) &= \epsilon\cos((1+\epsilon\omega_1)t),\\
\ddot{\theta} &= -\epsilon(1+\epsilon\omega_1)^2\cos((1+\epsilon\omega_1)t),\\
&\approx -\epsilon\cos((1+\epsilon\omega_1)t) + \frac{1}{6}\epsilon^3\cos^3((1+\epsilon\omega_1)t).
\end{aligned}$$

Using the quick identity

$$\begin{aligned}
\cos^3\theta &= \cos\theta - \cos\theta\sin^2\theta\\
&= \cos\theta + \frac{\cos\theta(\cos 2\theta - 1)}{2}\\
&= \frac{\cos\theta}{2} + \frac{\cos(3\theta) + \sin\theta\sin 2\theta}{2}\\
&= \frac{\cos\theta}{2} + \frac{\cos(3\theta)}{2} + \sin^2\theta\cos\theta\\
&= -\cos^3\theta + \frac{3\cos\theta}{2} + \frac{\cos(3\theta)}{2},\\
\cos^3\theta &= \frac{3\cos\theta}{4} + \frac{\cos(3\theta)}{4},
\end{aligned}$$

we obtain

$$-\epsilon(1+\epsilon\omega_1)^2\cos((1+\epsilon\omega_1)t) \approx -\epsilon\cos((1+\epsilon\omega_1)t) + \frac{1}{6}\epsilon^3\cos^3((1+\epsilon\omega_1)t).$$

Matching coefficients of the first frequency term, we find

$$\begin{aligned}
-\epsilon(1+2\epsilon\omega_1) &\approx -\epsilon + \frac{\epsilon^3}{8},\\
\omega_1 &= -\frac{\epsilon}{16},\\
\omega &= 1 - \frac{\epsilon^2}{16}.
\end{aligned}$$

In our Ph106b class notes, we solve the Duffing oscillator with this technique, for which $\ddot{\theta} = -\theta - \epsilon\theta^3$ and we obtain that $\omega = 1 + (3/8)\epsilon A^2$ for oscillation amplitude A. For the simple pendulum, $\epsilon = -1/6$, and so we recover that $\omega = 1 - A^2/16$. We didn't do this strictly correctly, I guess, since our small parameter was the oscillation amplitude A instead of the perturbing term ϵ, but we obtain the right result: **the oscillation period grows with larger amplitude**. We can already anticipate that $\epsilon \to 4$ will produce problems, and indeed $\epsilon = \pi$ corresponds to the upside-down pendulum.

6.2 Explicit Integral

The period of the pendulum can be solved exactly using the method of quadratures. During oscillation, the total energy of the system is conserved,

$$E = -\cos\theta + \frac{\dot{\theta}^2}{2}. \tag{38}$$

Note that in this notation, $\theta = 0$ is the bottom, so $\theta \in [-\pi, \pi]$. We would formally derive this by writing down the Lagrangian and making the Legendre transform, but in the present case if we just identify $p = \dot{\theta}$ the conjugate momentum to θ then we find immediately that $\dot{p} = -\partial E/\partial\theta = -\sin\theta$ and that $\dot{\theta} = \partial E/\partial p = p = \dot{\theta}$. Thus, we can explicitly write:

$$\begin{aligned}\dot{\theta}(\theta) &= \sqrt{2(E + \cos\theta)},\\ &= \sqrt{2(\cos\theta - \cos\theta_0)}.\end{aligned}$$

The period of the pendulum is formally defined such that

$$P = 4\int\limits_0^{\theta_0} \frac{\mathrm{d}t}{\mathrm{d}\theta}\,\mathrm{d}\theta$$

This is the amount of time it takes for the pendulum to go from an initial condition $\pm\theta_0$ to 0, so a quarter-period. For sufficiently small θ_0, the quadrature expression can be expanded

$$\begin{aligned}\dot{\theta} &\approx \sqrt{-\theta^2 + \theta_0^2}\\ P &= 4\int\limits_0^{\theta_0} \frac{1}{\sqrt{\theta_0^2 - \theta^2}}\,\mathrm{d}\theta.\end{aligned}$$

We know this is arcsin, but is it obvious? Actually, yeah:

$$\begin{aligned}\int^{y} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int^{\arcsin(y)} \frac{1}{\sqrt{1-\sin^2 u}}\,\mathrm{d}(\sin u),\\ &= \int^{\arcsin(y)} \mathrm{d}u,\\ &= \arcsin(y).\end{aligned}$$

So then

$$\begin{aligned}P &= 4\left[\arcsin\left(\frac{\theta}{\theta_0}\right)\right]_0^{\theta_0},\\ &= 2\pi.\end{aligned}$$

Great.

What about in the nonlinear limit though? In full generality, we have

$$P = 4\int\limits_0^{\theta_0} \frac{1}{\sqrt{2(\cos\theta - \cos\theta_0)}}\,\mathrm{d}\theta.$$

In the limit where $\theta_0 \to \pm\pi$, we cannot make any approximations since θ will span the full angular interval. However, θ_0 sufficiently close to $\pm\pi$, the unstable points, we recognize that the dominant contribution to P will be near π. Thus, let's instead ask the question: for some fixed $\theta_1 \ll 1$, how long does it take for the trajectory to reach θ_1 as $\theta_0 \to \pi$? We again should be able to make expansions now (let $\phi \equiv \pi - \theta$)

$$P \gtrsim 4\int\limits_{\phi_0}^{\phi_1} \frac{1}{\sqrt{\phi^2 - \phi_0^2}}\,\mathrm{d}\phi.$$

Note here that $\phi > \phi_0$. This one is probably a cosh?

$$\begin{aligned}\int\limits^{y} \frac{1}{\sqrt{x^2 - 1}}\,\mathrm{d}x &= \int\limits^{\cosh^{-1}(y)} \frac{1}{\sqrt{\cosh^2(u) - 1}}\,\mathrm{d}(\cosh(u)),\\ &= \cosh^{-1}(y),\\ P &\gtrsim 4\left[\cosh^{-1}\left(\frac{\phi}{\phi_0}\right)\right]_{\phi_0}^{\phi_1},\\ &\gtrsim 4\ln\left(\phi_1/\phi_0\right) \sim -4\ln\left(\pi_0\right).\end{aligned}$$

Here, we've made use of the fact that $\cosh(x) \approx \exp(x)/2$ for large x. Hence, we recover the logarithmic divergence that is expected.

7 04/15/2023—Distributions of Functions of Random Variables

I can never remember how to do this, so let me just write it down.

If we have a random variable X and a second random variable Y that satisfies $y = y(x)$, then the PDF of Y is simple:

$$\int f_Y(y)\,\mathrm{d}y = \int f_X(x)\,\mathrm{d}x, \tag{39}$$

$$f_Y(y) = f_X(x)\frac{\mathrm{d}x}{\mathrm{d}y}. \tag{40}$$

What if we have a random variable Z that is a function of two random variables X, Y satisfying $z(x, y)$? This is a little trickier, but we need to write down the CDF

$$\int f_Z(z)\,\mathrm{d}z = \iint f_Y(y) f_X(x)\,\mathrm{d}x\mathrm{d}y. \tag{41}$$

This is no longer solvable in general (but we can often do well in statistical cases with moment-generating functions, CLT, and others). But for sufficiently simple dependencies, we can do this. Let's just consider $z = x + y$, for $x, y \in \mathscr{U}_{[0,1]}$. This is then easy to do:

$$\int_0^z f_Z(w)\,\mathrm{d}w = \int_0^{\min(z,1)} \int_0^{\min(z-y,1)} \mathrm{d}x\,\mathrm{d}y, \tag{42}$$

$$= \int_0^{\min(z,1)} \min(z-y, 1)\,\mathrm{d}y, \tag{43}$$

$$= \begin{cases} \int_0^z z - y\,\mathrm{d}y & z < 1 \\ \int_0^{z-1} \mathrm{d}y + \int_{z-1}^1 (z-y)\,\mathrm{d}y & z > 1, \end{cases} \tag{44}$$

$$= \begin{cases} z^2/2 & z < 1 \\ (z-1) + (z-1/2) - z(z-1) + (z-1)^2/2 & z > 1, \end{cases} \tag{45}$$

$$f_Z(z) = \begin{cases} z & z < 1 \\ 2 - z & z > 1. \end{cases} \tag{46}$$

We can do the same for $z = xy$:

8 08/21/2023—Change in Mutual Inclination in Hierarchical Triples due to SNe

We find that when a hierarchical stellar triple has its inner and outer orbits initially isotropically distributed, the final mutual inclination distribution is not isotropic, but is depleted near 90°. We build a simple quantitative model analogous to this phenomenon and show that it has an exact solution.

The essence of this behavior is that: a symmetric SNe in the inner orbit results in an effective kick to the outer orbit in the plane of the inner orbit, denoted $\vec{v}_{\mathrm{k,eff}}$. Only the component of this kick aligned with the outer orbit normal contributes to realignment of the outer AM. Thus,

$$\Delta I \lesssim \Delta I_{\max} \propto v_{\mathrm{k,eff}} \sin I. \tag{47}$$

The actual change in I is approximately symmetrically distributed over the interval $[-\Delta I_{\max}, \Delta I_{\max}]$. Note that, strictly speaking, neither I nor $\cos I$ are uniformly distributed: imagine that the initial AM is along $\hat{\mathbf{l}}_{\mathrm{b},0}$, then the final AMs are distributed in an azimuthally symmetric way about $\hat{\mathbf{l}}_{\mathrm{b},0}$. This is closer to a symmetric distribution in I than $\cos I$ though.

To see what effect this has on the outer inclination, we imagine a diffusion equation for $I \in [0,\pi]$ with diffusion coefficient $D_0 \sin I$. This can be thought of as the cumulative effect of infinitely many, infinitesimally small effective kicks. This has the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial I}\left[D_0 \sin I \frac{\partial f}{\partial I}\right], \tag{48}$$

where $f(t,I)$ is the PDF of I at time t. I really just wanted to solve this PDE lol.

We first consider the steady-state solutions of this PDE. One possible family of solutions requires that

$$D_0 \sin I \frac{\partial f}{\partial I} = C, \tag{49}$$

$$-\sin^2 I \frac{\partial f}{\partial \cos I} = \frac{C}{D_0}, \tag{50}$$

$$\frac{\partial f}{\partial \cos I} = \frac{C}{D_0\left(\cos^2 I - 1\right)}. \tag{51}$$

But since $\operatorname{arctanh}'(x) = (1-x^2)^{-1}$, we find that this is just

$$f(\cos I) = \frac{C}{D_0} \operatorname{arctanh}(\cos I). \tag{52}$$

Then C can be set by the normalization of $f(t,I)$. The second homogeneous solution is simply the family of linear solutions $f(t,I) = A \times I$. If the IC is symmetric, then the symmetry is preserved under evolution, and we can require that $f'(t,0) = 0$ at all times. This thus requires that

$$f(\cos I \in [-1,1]) = \frac{C}{D_0} \left|\operatorname{arctanh}(\cos I) - \cos I\right|. \tag{53}$$

This works since $\operatorname{arctanh}'(0) = 1$, so the derivative is indeed zero at $\cos I = 0$, and the solution is symmetric. C is again set by the normalization, which I'm too lazy to compute. This is thus the steady-state solution, and we see that it vanishes at the origin and is singular at $\cos I = \pm 1$. This is indeed the behavior we were beginning to see!

Of course, with just a single kick, we don't evolve to this steady state f, but only a little bit. Nevertheless, this gives a quantitative model reflecting the depletion near $\cos I = 0$.

9 02/21/24—Jeremy and Rotations Generated by Andoyer Momenta

Jeremy asks a simple question, and as always it's a hard one: the three Andoyer momenta are $p_g = S$, $p_l = S\cos J$, and $p_h = S\cos i$ (not the usual order). The first is the total spin AM, and the second two are projections along the body axis and the orbit axis respectively. The question is: why is $\{p_l, p_h\} = 0$, the PB, when it's clear that rotations about the body and orbit axes do not commute?

9.1 Reminder: Standard AM Poisson Brackets

This was a rabbit hole, and I don't think I have the right answer, but I first briefly recall what the PB is. For two functions $f(q_i, p_i)$ and $g(q_i, p_i)$, we have that

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \tag{54}$$

The standard results concern the PBs of the angular momenta, given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, or $L_k = \epsilon_{ijk} r_i p_j$. Then, using the standard Cartesian $\mathbf{r}$ and $\mathbf{p}$ as our canonical coordinates, we can evaluate the PBs for $\mathbf{L}$:

$$\begin{aligned}
\{L_x, L_y\} &= \sum_m \frac{\partial L_x}{\partial r_m}\frac{\partial L_y}{\partial p_m} - \frac{\partial L_x}{\partial p_m}\frac{\partial L_y}{\partial r_m},\\
&= \sum_m \epsilon_{mjx} p_j \epsilon_{myi} r_i - \epsilon_{mxi} r_i \epsilon_{mjy} p_j,\\
&= \sum_m p_j r_i \left(\delta_{jy}\delta_{xi} - \delta_{ji}\delta_{xy}\right) - r_i p_j \left(\delta_{xj}\delta_{iy} - \delta_{ij}\delta_{xy}\right),\\
&= p_y r_x - r_y p_x = L_z.
\end{aligned}$$

In the third line, we've cyclically permuted indicies and used the usual relation to simplify $\epsilon_{ijk}\epsilon_{imn}$. Of course, this can be repeated for the other commutators to obtain that $\{L_i, L_j\} = \epsilon_{ijk} L_k$.

Now, what about for L^2? WLOG

$$\begin{aligned}
\{L^2, L_z\} &= \sum_i L_i \{L_i, L_z\} + \{L_i, L_z\} L_i,\\
&= -L_x L_y + L_y L_x - L_y L_x + L_x L_y = 0.
\end{aligned}$$

And of course, similarly for the other two components. What about for a general power of L? Well, it's not hard to show that

$$\begin{aligned}
\{L^n, L_z\} &= \sum_m \sum_i n L^{n-1} \frac{\mathrm{d}L}{\mathrm{d}L_i}\frac{\partial L_i}{\partial q_i}\frac{\partial L_z}{\partial p_i} - n L^{n-1} \frac{\mathrm{d}L}{\mathrm{d}L_i}\frac{\partial L_i}{\partial p_i}\frac{\partial L_z}{\partial q_i},\\
&= n L^{n-1} \sum_i \frac{L_i}{L} \{L_i, L_z\},\\
&= n L^{n-2} \left(L_x L_y - L_y L_x\right) = 0.
\end{aligned}$$

9.2 What does it mean to generate rotation?

This is the key point that I want to belabor, and hopefully be correct on. When we speak of L_z generating rotations about the z axis, I think that we mean that there exists a canonical set of coordinates such that L_z, ϕ_z are canonically conjugate. However, in isolation, it means nothing to argue that a scalar, which is all that L_z is, *generates* any sort of rotation / action. It is only when we attach it to a Hamiltonian / symplectic manifold that we can promote the coordinate to an operator and assign an action to it.

In this sense, for the standard Delaunay variables, (e, L, L_z) and (M, ω, Ω) (we write $e = \sqrt{GMa}$ for simplicity), L_z generates advancement of Ω at constant M, ω (and of course constant e, L, L_z). However, there are plenty of rotations about $\hat{\mathbf{z}}$ that do not preserve these variables (such as a torque on the orbit), and the one that L_z generates here can only be defined in the context of the other variables.

Put another way, suppose my particle is initially at Cartesian coordinates $(0,-1,0)$. Then L_z will generate rotation about z, which corresponds to motion in the $+\hat{\mathbf{x}}$ direction, but it conserves total AM. For comparison, in the original Cartesian coordinates, p_x generates x velocity at constant (z, y, p_z, p_y), but it does not conserve AM. What is the difference between these two? At the level of the infinitesimal displacement, nothing; we need the Hamiltonian structure and the other canonical coordinates to distinguish between the two.

Thus, for the Andoyer system, consisting of (l, g, h), (p_l, p_g, p_h), it is clear that p_l and p_h generate motion at fixed (g, h) and fixed (l, g) respectively. The angles l and h are indeed Euler angles describing the rotational phase about the orbit and body axes respectively, but the motion generated by p_l and p_h cannot simply correspond to naive rotation about these axes, and is instead a constrained rotation.

After talking to Jeremy, he points out that his question is motivated by "old quantum mechanics", where we only knew how to quantize very limited systems since we didn't have the SE until 1926 (while Planck's quantization was in 1901 or so). When computing the rotational modes of H_2O, there is a third quantum number due to the triaxiality of the molecule, on top of the usual (l, m) due to (L^2, L_z). But we can imagine describing its orientation in Andoyer variables instead; how would the quantization proceed in that case? It would then appear that the canonical momenta are to be promoted to operators, and then quantized; how do the operators commute then? We didn't get this far, but I thought that perhaps one just quantizes the three component rotations; maybe that's what `https://arxiv.org/pdf/2211.11347.pdf` is doing for one of the Euler angles? Since at least the Andoyer angles are Euler angles, though are a mixed set of them. Of course Jeremy's questions are hard to answer...

10 Mass Loss Induced Eccentricity

Consider a particle on a circular orbit with a_0 around a central mass $M_\star$ and initial mass M. It loses δm, ejected behind it with velocity $v_{\rm ej}$; what is the new sma and eccentricity?

First, we note that the new mass of the particle, $M' = M - \delta m$, is now moving at velocity $v' = v_0 + \delta v$ where

$$v_0 = \sqrt{GM_\star/a_0}, \tag{55}$$

$$\delta v = \frac{\delta m}{M'} v_{\rm ej}. \tag{56}$$

Then, computing the new angular momentum of the particle (which now has sma a'), we have (factoring out the particle mass)

$$\sqrt{GM_\star a'(1-e^2)} = a_0 v', \tag{57}$$

$$\frac{GM_\star}{a_0}\frac{a'}{a_0}(1-e^2) = (v')^2, \tag{58}$$

$$1-e^2 = \frac{(v')^2}{v_0^2}\frac{a_0}{a'}. \tag{59}$$

On the other hand, the energy gives us

$$\frac{(v')^2}{2} - \frac{GM_\star}{a_0} = -\frac{GM_\star}{2a'}, \tag{60}$$

$$\frac{(v')^2}{v_0^2} - 2 = -\frac{a_0}{a'}. \tag{61}$$

Combining, we find (define $\Delta \equiv \delta v / v_0$)

$$1-e^2 = \frac{(v')^2}{v_0^2}\left(2 - \frac{(v')^2}{v_0^2}\right), \tag{62}$$

$$= (1+\Delta)^2\left(2-(1+\Delta)^2\right), \tag{63}$$

$$= 2\left(1+2\Delta+\Delta^2\right) - \left(1+4\Delta+6\Delta^2+4\Delta^3+\Delta^4\right), \tag{64}$$

$$= 1-4\Delta^2-4\Delta^3-\Delta^4, \tag{65}$$

$$e \approx 2\Delta = 2\frac{\delta m}{M'}\frac{v_{\rm ej}}{v_0}. \tag{66}$$

11 CS2 Misc

11.1 04/26/2024—Tidal Dissipation into CS2?

Note that the weak friction equations give a nonzero tidal equilibrium obliquity

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{t_{\rm s}}\left(\frac{2n}{\Omega_{\rm s}} - \cos\theta\right). \tag{67}$$

This was used by Valente & Correia 2022, in conjunction with a spin-orbit resonance (non-secular) to excite large obliquities for eccentric orbits, where these asynchronous SORs are common. So maybe this is legit.

If so, what is the condition for CS2 capture? Well, we know that the resonance width is

$$\cos\theta_{\rm sep} \approx \eta\cos I \pm \sqrt{2\eta\sin I\left(1-\cos\phi\right)}, \tag{68}$$

$$\Delta\cos\theta \sim \sqrt{\eta_{\rm sync}\frac{n}{\Omega_{\rm s}}\sin I}. \tag{69}$$

The condition then for guaranteed capture is

$$\frac{2n}{\Omega_s} \lesssim \sqrt{\eta_{\rm sync}\frac{n}{\Omega_s}\sin I}, \tag{70}$$
$$\frac{4n}{\Omega_s} \lesssim \eta_{\rm sync}\sin I, \tag{71}$$
$$\Omega \gtrsim \frac{4n}{\eta_{\rm sync}\sin I}. \tag{72}$$

However, recalling that $\eta_{\rm sync} \lesssim 0.7$ is required for large obliquities—well, $\eta_{\rm sync} \le \eta_c$, where

$$\eta_c = \left(\cos^{2/3} I + \sin^{2/3} I\right)^{-3/2}, \tag{73}$$

then since $\eta_c \in [0.5, 1]$, this is the typical value of η_c as well. Thus, adopting fiducial values $\eta_{\rm sync} \sim \sin I \sim 0.3$, we obtain that

$$\Omega_s \gtrsim 40n. \tag{74}$$

Recall that planets are born near critical rotation, so

$$\Omega_{s,0} \approx \sqrt{\frac{Gm}{R^3}}, \tag{75}$$
$$\frac{\Omega_{s,0}}{n} = \sqrt{\frac{ma^3}{M_\star R^3}}, \tag{76}$$
$$= 6000\left(\frac{(m/M_\star)/(m_\oplus/M_\odot)}{3\times10^{-6}}\right)^{1/2}\left(\frac{(a/R)/(\mathrm{AU}/R_\oplus)}{2.3\times10^4}\right)^{3/2}. \tag{77}$$

Of course, these values are quite optimistic, but it's clear that planets have to spin down substantially before reaching synchronization, i.e. over many t_s, and so there is plenty of time to experience obliquity excitement.

11.2 04/29/2024—Do we believe Obliquity Excitation?

We start with the L12 Eq. (37)

$$S\frac{d\theta}{dt} = -T_x + \frac{S}{L}\left(T_z\sin\theta - T_x\cos\theta\right). \tag{78}$$

Here, θ is the obliquity. In general,

$$\frac{S}{L} = k\left(\frac{R}{a}\right)^2\frac{\Omega_s}{n} \ll 1, \tag{79}$$

so the obliquity evolution is set by T_x. Recall that this is the component of the torque exerted by the companion on the primary that is normal to the rotation axis and is radial (i.e. not azimuthal, does not contribute to precession).

When all the tidal lags are equal to τ, L12 finds that

$$T_x = \frac{6\pi}{5}T_0\sin\theta\left(2\Omega - \Omega_s\cos\theta\right)\tau. \tag{80}$$

Of course, this is negative when $\Omega_{\rm s} > 2\Omega/\cos\theta$, but this is just the standard equilibrium tidal theory and result. Can we be more general?

Let's recall the Gladman96 expression

$$\mathbf{T}_{\rm tide} = \frac{3k_2 G m_{\rm p}^2 R^5}{a^6}\left(\hat{\boldsymbol{\rho}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\boldsymbol{\rho}}\times\hat{\mathbf{r}}\right), \tag{81}$$

where

$$\boldsymbol{\rho} \approx \hat{\mathbf{r}} - \left(n\hat{\mathbf{l}}\times\hat{\mathbf{r}}\right)\tau + \left(\Omega_{\rm s}\hat{\mathbf{k}}\times\hat{\mathbf{r}}\right)\tau, \tag{82}$$

is the time-lagged position (by τ) of the perturber in the rest frame of the primary. It is thus interesting to ask: if in general, a torque scales with $\hat{\boldsymbol{\rho}}\times\hat{\mathbf{r}}$, when does it excite the obliquity? This could have been interesting, but Gladman already showed that the torque, absent its component along $\hat{\mathbf{k}}$, is $\propto -\Omega_{\rm s}\cos\theta + 2n$, as above.

The condition, however, is likely to generalize to: if the dominant, or all, tidal frequencies $m'\Omega - m\Omega_{\rm s}$ are negative, then the primary's obliquity is excited. This is simple to understand: if the perturber is trying to spin the primary down, but it's misaligned from the equator, then it can only exert forces in the plane of the orbit, i.e. along $\hat{\mathbf{l}}$. If $\hat{\mathbf{k}}$ is initially close to $\hat{\mathbf{l}}$, then the only despinning torque along $\hat{\mathbf{l}}$ will tend to drive $\hat{\mathbf{k}}$ *away* from $\hat{\mathbf{l}}$, while spinning it down (draw some arrows).

For a more simple example, call $\hat{\mathbf{l}} = \hat{\mathbf{z}}$, then differentiating $\cos\theta = L_z/L$ gives

$$\frac{\mathrm{d}\cos\theta}{\mathrm{d}t} = \frac{1}{L}T_z - \frac{L_z}{L^2}T, \tag{83}$$

$$= \frac{LT_z - L_zT}{L^2}, \tag{84}$$

$$= \frac{T}{L^2}(1-\cos\theta). \tag{85}$$

Here, T is the component of the torque along $\hat{\mathbf{l}}$; it must be negative to be a despinning torque, so $\cos\theta$ grows.

Maybe we can assert that the dynamical tide generally also acts to increase obliquities as long as the pattern frequency is negative? TODO.

09/02/24 Edit: This is not actually quite true, the torque exerted by the orbit also has an in-plane component, EKE01.

11.3 09/02/24—Tidal Bulge-induced Precession?

As we know, in the apsidal precession problem, there is both a J_2 and a tides-induced precession effect (LML15). At synchronization, the tidal precession is faster, and for rapid rotation, the J_2 precession dominates. What about for spin precession? As in the discussion above, we focus on the effect on the planet's spin, so $\mu = m_{\rm p}$, and the perturber mass is $M_\star$. We want to compare this to the standard spin precession induced by oblateness, given by (e.g. SL22a)

$$\omega_{\rm sl} = \frac{k_2}{2k}\frac{M_\star}{m_{\rm p}}\left(\frac{R}{a}\right)^3\omega\cos\theta. \tag{86}$$

Let's do this two ways to try to ensure correctness. We first follow EKH98, then EKE01 (checked against FT07).

In EKH98, the spin AM $I\mathbf{\Omega}$ and the orbital AM $\mu\mathbf{h} = \mu\mathbf{r}\times\mathbf{v}$ precess about the total AM $\mathbf{H}$, such that

$$\hat{h} = \left(\sin\eta\sin\chi, -\sin\eta\cos\chi, \cos\eta\right), \tag{87}$$

in an inertial frame with $\hat{\mathbf{z}} \parallel \mathbf{H}$. In the limit of no tidal friction (just above Eq. 98), we have that η is constant, and χ advances uniformly with

$$\dot{\chi} = -\frac{Am_2\Omega_0\Omega_\parallel}{2\mu\omega a^5(1-e^2)^2}. \tag{88}$$

We consider circular orbits. EKH uses $A = k_2R^5$, $\mu = m_{\rm p}$, $m_2 = M_\star$, $\Omega_\parallel \approx \mathbf{\Omega}\cdot\hat{\mathbf{h}} = \omega\cos\theta$ where θ is the obliquity and ω is the planet's spin rate, and $\Omega_0 = H/I \approx (L/S)\omega$. Thus, we have instead

$$\begin{aligned}\dot{\chi} &= -\frac{k_2R^5M_\star\omega^2}{2m_{\rm p}\omega a^5}\frac{L}{S}\cos\theta\\ &= -\frac{k_2}{2k}\frac{M_\star}{m_{\rm p}}\left(\frac{R}{a}\right)^3\omega\cos\theta.\end{aligned} \tag{89}$$

The correction of the factor of 2 does not affect this expression (appendix of EKE01). Thus, this is exactly the rotational-induced precession rate. This suggests that EKH98 actually just includes this term somehow.

Let's now follow EKE01 and FT07. We will work out EKE01, and compare to the appendix of FT07. But here, we run into a problem: there is no tidally-induced precession term! Indeed, it's always damping (EKE01). And when we look at FT07, maybe we see the answer: the Hamiltonian for the tidal bulge piece does not depend on the inclination, since it is always pointed at the companion (in the limit of zero dissipation). Thus, it can never contribute a $\dot{\Omega}$, since $\mathscr{F}_{\rm extra}(L,G,H) = \mathscr{F}_{\rm extra}(L,G)$. On the other hand, it *does* contribute apsidal precession, since the bulge amplitude does depend on the eccentricity?

So the EKH98 result must be the same rotational-bulge-induced precession that we have been studying, and indeed, the above two precession frequencies agree. So at the level of approximation considered there, the tidal bulge does *not* affect the spin-orbit precession rate.

Note (09/03/24): Jeremy agrees that since the tidal bulge, described by the apsidal motion constant, cannot induce nodal precession, since it is always a radial distortion. There may be higher-order effects though.

11.4 09/02/24—Reworking Lai 2012

But Lai 2012 suggests that there should be a dissipation-free component? I guess it's time to follow his calculation and see what's going on. I suspect that the y component will be zero, because it has a $\partial_\theta P_2(\cos\theta)$, which just depends on $P_1(\cos\theta)$, which vanishes when integrating against a $\delta\bar{\rho}$. Nevertheless, it will be instructive to finally understand this paper completely…

We follow his Section 2 through Equation 30. We begin by writing out the tidal potential in the AM frame

$$U(\mathbf{r}_L, t) = -GM' \sum_{m'} \frac{W_{2m'} r^2}{a^3} e^{-im'\Omega t} Y_{2m'}\left(\theta_L, \phi_L\right). \tag{90}$$

Here, $W_{20} = -\sqrt{\pi/5}$ and $W_{\pm 2} = \sqrt{3\pi/10}$. The angles represent the coordinates in the tidal potential frame, as do the m'.

We re-express this in the spin frame using the Wigner $\mathscr{D}$-matrix. Dong has the entries, but it is the matrix that gives

$$Y_{2m'}\left(\theta_L, \phi_L\right) = \sum_m \mathscr{D}_{mm'}(\Theta) Y_{2m}\left(\theta, \phi\right). \tag{91}$$

Here, Θ is the obliquity, or the misalignment angle between the spin and AM frames. Then, the tidal potential in the spin frame (which is where we will work going forwards) is

$$U(\mathbf{r}, t) = -\sum_{mm'} U_{mm'} r^2 Y_{2m}\left(\theta, \phi\right) e^{-im\Omega t}, \tag{92}$$

$$U_{mm'} \equiv \frac{GM'}{a^3} \mathscr{U}_{mm'} \equiv \frac{GM'}{a^3} W_{2m'} \mathscr{D}_{mm'}(\Theta). \tag{93}$$

Now, the planet will in general experience some sort of a tidal response to this perturbing potential. Typically, this is parameterized in terms of the lth Love number, but Dong absorbs all of these into one coefficient, as we shall see. First, for each tidal forcing frequency, associaote to it a phase lag

$$\Delta_{mm'} = \tilde{\omega}_{mm'} t_{mm'}, \tag{94}$$

$$\tilde{\omega}_{mm'} = m'\Omega - m\omega, \tag{95}$$

where ω is the spin frequency of the planet, $\tilde{\omega}_{mm'}$ is the tidal frequency of the mm' component, and $t_{mm'}$ is the lag time. For the purposes of computing the torque, we will need the perturbed structure of the body, expressed in terms of the Lagrangian displacement and density perturbations:

$$\boldsymbol{\xi}_{mm'}(\mathbf{r}, t) = \frac{U_{mm'}}{\omega_0^2} \bar{\boldsymbol{\xi}}_{mm'}(\mathbf{r}) e^{-im'\Omega t + i\Delta_{mm'}}, \tag{96}$$

$$\delta\rho_{mm'}(\mathbf{r}, t) = \frac{U_{mm'}}{\omega_0^2} \delta\bar{\rho}_{mm'}(\mathbf{r}) e^{-im'\Omega t + i\Delta_{mm'}}, \tag{97}$$

$$\delta\bar{\rho}_{mm'} = -\nabla \cdot \left(\rho \bar{\boldsymbol{\xi}}_{mm'}\right). \tag{98}$$

Here, $\bar{\boldsymbol{\xi}}_{mm'}$ and $\delta\bar{\rho}_{mm'}$ describe the shape of the static perturbation to the body; other than a factor of $e^{im\phi}$, Dong claims that they are purely real. Does this mean that I can define them as

$$\bar{\boldsymbol{\xi}}_{mm'}(\mathbf{r}) = \bar{\boldsymbol{\xi}}_{m'}(r, \theta) e^{im\phi}, \tag{99}$$

or not? I'll do that for now.

Now, to evaluate the tidal torque on the star, we need to evaluate

$$\mathbf{T} = \int \mathrm{d}^3\mathbf{r}\, \delta\rho(\mathbf{r}, t) \mathbf{r} \times \left[-\nabla U^*(\mathbf{r}, t)\right]. \tag{100}$$

This is somewhat tedious to write out all at once, so let's start with the gradient of the potential (we drop the $\mathbf{r}$ component, since it will go away with the cross product)

$$
\begin{aligned}
-\nabla_\perp U^*(\mathbf{r},t) &= r^2 \sum_{mm'} U_{mm'} e^{im\Omega t} \nabla_\perp (Y_{2m}^*(\theta,\phi)) \\
&= r^2 \sum_{mm'} U_{mm'} e^{im\Omega t} \left(\frac{1}{r}\frac{\partial}{\partial\theta} Y_{2m}^*(\theta,\phi)\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} Y_{2m}^*(\theta,\phi)\hat{\boldsymbol{\phi}}\right), \qquad (101)\\
\mathbf{r}\times\left[-\nabla_\perp U^*(\mathbf{r},t)\right] &= r^2 \sum_{mm'} U_{mm'} e^{im\Omega t} \left(\frac{\partial}{\partial\theta} Y_{2m}^*(\theta,\phi)\hat{\boldsymbol{\phi}} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} Y_{2m}^*(\theta,\phi)\hat{\boldsymbol{\theta}}\right), \qquad (102)
\end{aligned}
$$

and so

$$
\begin{aligned}
\mathbf{T} &= \sum_{\bar{m}m'} U_{\bar{m}m'} e^{-im'\Omega t} \int \mathrm{d}^3\mathbf{r}\, \delta\rho(\mathbf{r},t) r^2 \left(\frac{\partial}{\partial\theta} Y_{2\bar{m}}^*(\theta,\phi)\hat{\boldsymbol{\phi}} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} Y_{2\bar{m}}^*(\theta,\phi)\hat{\boldsymbol{\theta}}\right)\\
&= \sum_{m\bar{m}m'} \frac{U_{\bar{m}m'}U_{mm'}}{\omega_0^2} e^{i\Delta_{\bar{m}m'}} \int \mathrm{d}^3\mathbf{r}\, \delta\bar{\rho}_{mm'}(\mathbf{r}) r^2 \left(\frac{\partial}{\partial\theta} Y_{2\bar{m}}^*(\theta,\phi)\hat{\boldsymbol{\phi}} + \frac{im}{\sin\theta} Y_{2\bar{m}}^*(\theta,\phi)\hat{\boldsymbol{\theta}}\right). \qquad (103)
\end{aligned}
$$

Then, to compute the $\hat{\mathbf{z}}$ component of this, we note that

$$
\begin{aligned}
\hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}, \qquad (104)\\
\hat{\boldsymbol{\theta}} &= \cos\theta\left(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right) - \sin\theta\hat{\mathbf{z}}. \qquad (105)
\end{aligned}
$$

Thus, the integral of $\delta\bar{\rho}_{mm'}Y_{2\bar{m}}^* \propto \delta_{m\bar{m}}$, and so we have

$$
\begin{aligned}
T_z &= -\sum_{m\bar{m}m'} \frac{U_{\bar{m}m'}U_{mm'}}{\omega_0^2} e^{i\Delta_{\bar{m}m'}} \int \mathrm{d}^3\mathbf{r}\, \delta\bar{\rho}_{mm'}(\mathbf{r}) r^2 (i\bar{m}) Y_{2\bar{m}}^*(\theta,\phi), \qquad (106)\\
\mathrm{Re}(T_z) &= \sum_{mm'} m \frac{U_{mm'}^2}{\omega_0^2} \sin(\Delta_{mm'}) \int \mathrm{d}^3\mathbf{r}\, \delta\bar{\rho}_{mm'}(\mathbf{r}) r^2 Y_{2m}^*(\theta,\phi)\\
&\equiv T_0 \sum_{mm'} m\mathscr{U}_{mm'}^2 \sin(\Delta_{mm'})\kappa_{mm'}, \qquad (107)\\
\kappa_{mm'} &= \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\, \delta\bar{\rho}_{mm'}(\mathbf{r}) r^2 Y_{2m}^*(\theta,\phi). \qquad (108)
\end{aligned}
$$

Note that $U_{mm'}^2/\omega_0^2 = T_0\mathscr{U}_{mm'}/MR^2$.

On the other hand, the xy components are a little less straightforward, since

$$
\begin{aligned}
\sin\phi &= -\frac{e^{i\phi} - e^{-i\phi}}{2i}, \qquad (109)\\
\cos\phi &= \frac{e^{i\phi} + e^{-i\phi}}{2}. \qquad (110)
\end{aligned}
$$

Thus, $m = \bar{m}$ is no longer guaranteed (and in fact, $m = \bar{m} \pm 1$), and we have the full form for

T_x (to compare with Lai 2012):

$$\mathrm{Re}(T_x) = T_0 \sum_{m\bar{m}m'} \mathscr{U}_{mm'}\mathscr{U}_{\bar{m}m'} \frac{i \sin\Delta_{mm'}}{MR^2} \int \mathrm{d}^3\mathbf{r}\, \delta\bar{\rho}_{mm'}(\mathbf{r}) r^2 \left(-\frac{\partial Y_{2\bar{m}}^*}{\partial\theta}\sin\phi + \frac{i\bar{m}}{\tan\theta} Y_{2\bar{m}}^* \cos\phi\right), \tag{111}$$

$$\equiv T_0 \sum_{m\bar{m}m'} \mathscr{U}_{mm'}\mathscr{U}_{\bar{m}m'} \sin\Delta_{mm'} \kappa_{m\bar{m}m'}, \tag{112}$$

$$\kappa_{m\bar{m}m'} \equiv \frac{1}{iMR^2} \int \mathrm{d}^3\mathbf{r}\, \delta\bar{\rho}_{mm'}(\mathbf{r}) r^2 \left(\frac{\partial Y_{2\bar{m}}^*}{\partial\theta}\sin\phi - \frac{i\bar{m}}{\tan\theta} Y_{2\bar{m}}^* \cos\phi\right). \tag{113}$$

Note that $\kappa_{m\bar{m}m'}$ is real, since $\delta\bar{\rho} \propto e^{im\phi}$, and the only terms in the integrand that have forms $e^{i\bar{m}\phi} \pm e^{i\phi}$ have imaginary coefficients.

Finally, to get a torque, we need to relate the $\kappa_{m\bar{m}m'}$ to the $\kappa_{mm'}$. We need the identity

$$\frac{\partial Y_{lm}}{\partial\theta} = \frac{m}{\tan\theta} Y_{lm} + \sqrt{(l-m)(l+m+1)} e^{-i\phi} Y_{l(m+1)}, \tag{114}$$

which then lets us write the geometric piece of the integrand in $\kappa_{m\bar{m}m'}$, denoted $\mathscr{I}$ (simply to have something on the LHS of the equation), as:

$$\begin{aligned}
\mathscr{I} &\equiv \frac{\partial Y_{2\bar{m}}^*}{\partial\theta} \frac{e^{i\phi} - e^{-i\phi}}{2i} + \frac{\bar{m}}{i\tan\theta} Y_{2\bar{m}}^* \frac{e^{i\phi} + e^{-i\phi}}{2} \\
&= \left[\frac{\bar{m}}{\tan\theta} Y_{2\bar{m}}^* + \sqrt{(2-\bar{m})(3+\bar{m})} e^{i\phi} Y_{2(\bar{m}+1)}^*\right] \frac{e^{i\phi} - e^{-i\phi}}{2i} + \frac{\bar{m}}{\tan\theta} Y_{2\bar{m}}^* \frac{e^{i\phi} + e^{-i\phi}}{2i} \\
&= \frac{\bar{m}}{\tan\theta} Y_{2\bar{m}}^* \frac{e^{i\phi}}{i} + \sqrt{(2-\bar{m})(3+\bar{m})} Y_{2(\bar{m}+1)}^* \frac{e^{2i\phi} - 1}{2i}
\end{aligned} \tag{115}$$

Let's evaluate a few cases. First, for $m = 2$, and $\bar{m} = 3, 1$. Of course, $\bar{m} = 3$ is zero, since the $Y_{2\bar{m}}$ are zero. Then

$$\kappa_{21m'} = -\frac{1}{MR^2} \int \mathrm{d}^3\mathbf{r}\, r^2 \delta\bar{\rho}_{2m'}(r,\theta) e^{i2\phi} \left[\frac{Y_{21}^* e^{i\phi}}{\tan\theta} + Y_{22}^* \left(e^{2i\phi} - 1\right)\right], \tag{116}$$

$$= \frac{1}{MR^2} \int \mathrm{d}^3\mathbf{r}\, r^2 \delta\bar{\rho}_{2m'}(r,\theta) e^{i2\phi} Y_{22}^* = \kappa_{2m'}. \tag{117}$$

Nice! Another, for $m = 1$, then there are two cases, and we will take the complex one $\bar{m} = 2$ (and we will take $m = -1$, $\bar{m} = -2$ for comparison):

$$\kappa_{12m'} = -\frac{1}{MR^2} \int \mathrm{d}^3\mathbf{r}\, \frac{r^2}{\tan\theta} \delta\bar{\rho}_{1m'}(r,\theta) e^{i\phi} \left(2Y_{22}^* e^{i\phi}\right), \tag{118}$$

$$Y_{22}^* = -\frac{1}{2} \tan\theta e^{-i\phi} Y_{21}^*, \tag{119}$$

$$\kappa_{12m'} = \kappa_{1m'}, \tag{120}$$

$$\kappa_{(-1)(-2)m'} = \frac{1}{MR^2} \int \mathrm{d}^3\mathbf{r}\, r^2 \delta\bar{\rho}_{(-1)m'}(r,\theta) e^{-i\phi} Y_{2(-1)}^* = \kappa_{(-1)m'}. \tag{121}$$

Oh, this is nifty, we actually have to pull up the list of spherical harmonics to make this work!

On the other hand, for T_y, which was neglected in Dong's treatment:

$$T_y = T_0 \sum_{m\bar{m}m'} \mathscr{U}_{mm'}\mathscr{U}_{\bar{m}m'} \frac{e^{i\Delta_{mm'}}}{MR^2} \int \mathrm{d}^3\mathbf{r}\,\delta\bar{\rho}_{mm'}(\mathbf{r})r^2\left(\frac{\partial Y_{2\bar{m}}^*}{\partial\theta}\cos\phi + \frac{i\bar{m}}{\tan\theta}Y_{2\bar{m}}^*\sin\phi\right), \tag{122}$$

$$= T_0 \sum_{m\bar{m}m'} \mathscr{U}_{mm'}\mathscr{U}_{\bar{m}m'}\cos\Delta_{mm'}\kappa'_{m\bar{m}m'}, \tag{123}$$

$$\kappa'_{m\bar{m}m'} \equiv \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\,\delta\bar{\rho}_{mm'}(\mathbf{r})r^2\left(\frac{\partial Y_{2\bar{m}}^*}{\partial\theta}\cos\phi + \frac{i\bar{m}}{\tan\theta}Y_{2\bar{m}}^*\sin\phi\right). \tag{124}$$

Again, $\kappa'_{m\bar{m}m'}$ is real, since $i\sin\phi$ contributes $e^{\pm i\phi}$ with real coefficients, and so does $\cos\phi$. Here, instead

$$\begin{aligned}
\mathscr{I}' &\equiv \frac{\partial Y_{2\bar{m}}^*}{\partial\theta}\frac{e^{i\phi}+e^{-i\phi}}{2} + \frac{\bar{m}}{\tan\theta}Y_{2\bar{m}}^*\frac{e^{i\phi}-e^{-i\phi}}{2}\\
&= \left[\frac{\bar{m}}{\tan\theta}Y_{2\bar{m}}^* + \sqrt{(2-\bar{m})(3+\bar{m})}e^{i\phi}Y_{2(\bar{m}+1)}^*\right]\frac{e^{i\phi}+e^{-i\phi}}{2} + \frac{\bar{m}}{\tan\theta}Y_{2\bar{m}}^*\frac{e^{i\phi}-e^{-i\phi}}{2}\\
&= \frac{\bar{m}}{\tan\theta}Y_{2\bar{m}}^*e^{i\phi} + \sqrt{(2-\bar{m})(3+\bar{m})}Y_{2(\bar{m}+1)}^*\frac{e^{2i\phi}+1}{2}.
\end{aligned} \tag{125}$$

Now, can we try to directly compute any of these coefficients? Let's try another one

$$\begin{aligned}
\kappa'_{21m'} &= \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\,r^2\delta\bar{\rho}_{2m'}(r,\theta)e^{i2\phi}\left[\frac{Y_{21}^*e^{i\phi}}{\tan\theta} + Y_{22}^*\left(e^{2i\phi}+1\right)\right]\\
&= \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\,r^2\delta\bar{\rho}_{2m'}(r,\theta)e^{i2\phi}Y_{22}^* = \kappa_{2m'}.
\end{aligned} \tag{126}$$

And

$$\kappa'_{12m'} = \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\,\frac{r^2}{\tan\theta}\delta\bar{\rho}_{1m'}(r,\theta)e^{i\phi}2Y_{22}^*e^{i\phi} = -\kappa_{1m'}, \tag{127}$$

$$\begin{aligned}
\kappa'_{(-1)(-2)m'} &= \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\,r^2\delta\bar{\rho}_{(-1)m'}(r,\theta)e^{-i\phi}\left[-\frac{2}{\tan\theta}Y_{2(-2)}e^{i\phi} + Y_{2(-1)}^*\left(e^{2i\phi}+1\right)\right]\\
&= \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\,r^2\delta\bar{\rho}_{(-1)m'}(r,\theta)e^{-i\phi}Y_{2(-1)}^* = \kappa_{(-1)m'}.
\end{aligned} \tag{128}$$

We see that the signs are slightly different: while $\kappa_{12m'}/\kappa_{1m'} = \kappa_{(-1)(-2)m'}/\kappa_{(-1)m'}$ (same sign), we have $\kappa'_{12m'}/\kappa_{1m'} = -\kappa'_{(-1)(-2)m'}/\kappa_{(-1)m'}$. Thus, there should probably be some cancellations.

But let's just do it all the way, we have 3/7 nontrivial terms already...

$$\kappa'_{(-2)(-1)m'} = \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\, r^2 \delta\bar{\rho}_{(-2)m'}(r,\theta)e^{-i2\phi}\left[-\frac{Y_{2(-1)}^* e^{i\phi}}{\tan\theta} + \sqrt{\frac{3}{2}}Y_{20}^*\left(e^{2i\phi}+1\right)\right]$$

$$-\frac{Y_{2(-1)}^* e^{i\phi}}{\tan\theta} + \sqrt{\frac{3}{2}}Y_{20}^* e^{2i\phi} = -Y_{22}^*$$

$$\kappa'_{(-2)(-1)m'} = -\kappa_{(-2)m'}, \tag{129}$$

$$\kappa'_{10m'} = \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\, r^2 \delta\bar{\rho}_{1m'}(r,\theta)e^{i\phi}\sqrt{\frac{3}{2}}Y_{21}^* = \sqrt{\frac{3}{2}}\kappa_{1m'}, \tag{130}$$

$$\kappa'_{(-1)0m'} = \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\, r^2 \delta\bar{\rho}_{(-1)m'}(r,\theta)e^{-i\phi}\sqrt{\frac{3}{2}}Y_{21}^* e^{2i\phi} = -\sqrt{\frac{3}{2}}\kappa_{(-1)m'}, \tag{131}$$

$$\begin{aligned}\kappa'_{01m'} &= \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\, r^2 \delta\bar{\rho}_{0m'}(r,\theta)\left[\frac{Y_{21}^* e^{i\phi}}{\tan\theta} + Y_{22}^*\left(e^{2i\phi}+1\right)\right]\\ &= -\frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\, r^2 \delta\bar{\rho}_{0m'}(r,\theta)Y_{20} = -\sqrt{\frac{3}{2}}\kappa_{0m'},\end{aligned} \tag{132}$$

$$\begin{aligned}\kappa'_{0(-1)m'} &= \frac{1}{MR^2}\int \mathrm{d}^3\mathbf{r}\, r^2 \delta\bar{\rho}_{0m'}(r,\theta)\left[-\frac{Y_{2(-1)}^* e^{i\phi}}{\tan\theta} + \sqrt{\frac{3}{2}}Y_{20}^*\left(e^{2i\phi}+1\right)\right]\\ &= \sqrt{\frac{3}{2}}\kappa_{0m'}.\end{aligned} \tag{133}$$

In summary, our direct computation yields the following results:

$$\begin{aligned}
\kappa_{21m'} &= \kappa^-_{2m'} = \kappa_{2m'}, & \kappa_{(-2)(-1)m'} &= \kappa^+_{(-2)m'} = -\kappa_{(-2)m'},\\
\kappa_{12m'} &= \kappa^+_{1m'} = -\kappa_{1m'}, & \kappa_{10m'} &= \kappa^-_{1m'} = \sqrt{3/2}\kappa_{1m'},\\
\kappa_{0(\pm1)m'} &= \kappa^\pm_{0m'} = \mp\sqrt{3/2}\kappa_{0m'}, & &\\
\kappa_{(-1)0m'} &= \kappa^+_{(-1)m'} = -\sqrt{3/2}\kappa_{(-1)m'}, & \kappa_{(-1)(-2)m'} &= \kappa^-_{(-1)m'} = \kappa_{(-1)m'},
\end{aligned} \tag{134}$$

$$\kappa_{m\bar{m}m'} = -\kappa_{(-m)(-\bar{m})m'}. \tag{135}$$

Looking back at the torque, we need to sum an expression of the form

$$T_y \propto \sum_{m\bar{m}m'} \mathscr{U}_{mm'}\mathscr{U}_{\bar{m}m'}\kappa'_{m\bar{m}m'}, \tag{136}$$

$$\mathscr{U}_{mm'} = W_{2m'}\mathscr{D}_{mm'}(\Theta), \tag{137}$$

$$(-1)^{m-m'}\mathscr{D}_{mm'} = \mathscr{D}_{(-m)(-m')}, \tag{138}$$

$$\mathscr{D}_{mm'}\mathscr{D}_{\bar{m}m'} = (-1)^{m+\bar{m}-2m'}\mathscr{D}_{(-m)(-m')}\mathscr{D}_{(-\bar{m})(-m')}. \tag{139}$$

There seems to be no natural symmetry to get this torque to vanish... What a mystery.

Update, next day: Let's resume this for fun. It must vanish. The algebra is a little tedious, but fundamentally, when all the $\Delta_{mm'} = 0$, we're talking about a $\delta\bar{\rho}$, symmetric about the $\hat{\mathbf{l}}$ axis, integrated against a torque that is antisymmetric; the complications just arise because the Y_{lm} are not suited for evaluating the torque, whose symmetries are in the L frame. Nevertheless, we're almost there...

Recall that our goal is to evaluate, for $\Delta_{mm'} = 0$:

$$T_y = T_0 \sum_{m\bar{m}m'} \mathscr{U}_{mm'}\mathscr{U}_{\bar{m}m'}\kappa'_{m\bar{m}m'}, \tag{140}$$

with

$$\kappa'_{m(m\pm1)m'} \equiv (\kappa'_{mm'})^{\pm} = \mp\kappa_{mm'}, \tag{141}$$

$$\mathscr{U}_{mm'} = W_{2m'}\mathscr{D}_{mm'}(\Theta). \tag{142}$$

Again, $W_{20} = -\sqrt{\pi/5}$ and $W_{2(\pm2)} = \sqrt{3\pi/10}$. Here, the Wigner $\mathscr{D}$ matrix elements, more explicitly, are (using L12 6–11)

$$\begin{aligned}
&\mathscr{D}_{22} = \frac{(1+\cos\Theta)^2}{4}, && \mathscr{D}_{12} = \frac{\sin\Theta(1+\cos\Theta)}{2}, && \mathscr{D}_{02} = \frac{\sqrt{6}\sin^2\Theta}{4}, && \mathscr{D}_{(-1)2} = \frac{\sin\Theta(1-\cos\Theta)}{2}, && \mathscr{D}_{(-2)2} = \frac{(1-\cos\Theta)^2}{4},\\
&\mathscr{D}_{20} = \frac{\sqrt{6}\sin^2\Theta}{4}, && \mathscr{D}_{10} = -\frac{\sqrt{6}\sin\Theta\cos\Theta}{2}, && \mathscr{D}_{00} = \frac{3\cos^2\Theta-1}{2}, && \mathscr{D}_{(-1)0} = \frac{\sqrt{6}\sin\Theta\cos\Theta}{2}, && \mathscr{D}_{(-2)0} = \frac{\sqrt{6}\sin^2\Theta}{4},\\
&\mathscr{D}_{2(-2)} = \frac{(1-\cos\Theta)^2}{4}, && \mathscr{D}_{1(-2)} = -\frac{\sin\Theta(1-\cos\Theta)}{2}, && \mathscr{D}_{0(-2)} = \frac{\sqrt{6}\sin^2\Theta}{4}, && \mathscr{D}_{(-1)(-2)} = -\frac{\sin\Theta(1+\cos\Theta)}{2}, && \mathscr{D}_{(-2)(-2)} = \frac{(1+\cos\Theta)^2}{4}.
\end{aligned} \tag{143}$$

We will eventually compare this to the equilibrium tidal scenario, so we eventually take all of the $\kappa_{mm'}$ to be equal, to some multiple of k_2, the usual Love number (we'll keep the indexing for bookkeeping). Then:

$$\begin{aligned}
\hat{T}_y ={}& \sum_{m\bar{m}m'} \mathscr{U}_{mm'}\mathscr{U}_{\bar{m}m'}\kappa'_{m\bar{m}m'}\\
={}& \mathscr{U}_{22}\mathscr{U}_{12}\kappa_{22} - \mathscr{U}_{12}\mathscr{U}_{22}\kappa_{12} + \mathscr{U}_{12}\mathscr{U}_{02}\kappa_{12} - \mathscr{U}_{02}\mathscr{U}_{12}\kappa_{02} + \mathscr{U}_{02}\mathscr{U}_{(-1)2}\kappa_{02}\\
&+ -\mathscr{U}_{(-1)2}\mathscr{U}_{02}\kappa_{(-1)2} + \mathscr{U}_{(-1)2}\mathscr{U}_{(-2)2}\kappa_{(-1)2} + \mathscr{U}_{(-2)2}\mathscr{U}_{(-1)2}\kappa_{(-2)2}\\
&+ \mathscr{U}_{20}\mathscr{U}_{10}\kappa_{20} - \mathscr{U}_{10}\mathscr{U}_{20}\kappa_{10} + \mathscr{U}_{10}\mathscr{U}_{00}\kappa_{10} - \mathscr{U}_{00}\mathscr{U}_{10}\kappa_{00} + \mathscr{U}_{00}\mathscr{U}_{(-1)0}\kappa_{00}\\
&+ -\mathscr{U}_{(-1)0}\mathscr{U}_{00}\kappa_{(-1)0} + \mathscr{U}_{(-1)0}\mathscr{U}_{(-2)0}\kappa_{(-1)0} + \mathscr{U}_{(-2)0}\mathscr{U}_{(-1)0}\kappa_{(-2)0}\\
&+ \mathscr{U}_{2(-2)}\mathscr{U}_{1(-2)}\kappa_{2(-2)} - \mathscr{U}_{1(-2)}\mathscr{U}_{2(-2)}\kappa_{1(-2)}\\
&\quad + \mathscr{U}_{1(-2)}\mathscr{U}_{0(-2)}\kappa_{1(-2)} - \mathscr{U}_{0(-2)}\mathscr{U}_{1(-2)}\kappa_{0(-2)} + \mathscr{U}_{0(-2)}\mathscr{U}_{(-1)(-2)}\kappa_{0(-2)}\\
&+ -\mathscr{U}_{(-1)(-2)}\mathscr{U}_{0(-2)}\kappa_{(-1)(-2)} + \mathscr{U}_{(-1)(-2)}\mathscr{U}_{(-2)(-2)}\kappa_{(-1)(-2)} - \mathscr{U}_{(-2)(-2)}\mathscr{U}_{(-1)(-2)}\kappa_{(-2)(-2)}.
\end{aligned} \tag{144}$$

I guess if all the κ's are equal, this vanishes by exchange of $m\bar{m}$ to $\bar{m}m$, but this seems wrong, since it is the

$$\tau_{mm'} = \frac{\Delta_{mm'}}{\tilde{\omega}_{mm'}}\kappa_{mm'}, \tag{145}$$

that are set equal in L12. In fact, maybe we did something wrong: Dong says $\kappa_{mm'} = \kappa_{(-m)(-m')}$, but this would seem to set a bunch of cancellations in the above expression for the T_x components. Edit: probably not, since $\Delta_{mm'} = -\Delta_{(-m)(-m')}$, so that saves the T_x expression. But nevertheless, the T_y expression still doesn't obviously vanish.

12 WASP-12b

12.1 Quick Summary of Spin-Orbit Resonance: WASP-12b Case

The spin $\hat{\mathbf{s}}$ of WASP-12b precesses about its orbit normal $\hat{\mathbf{l}}$, and its orbit precesses about the invariable plane of the planetary system $\hat{\boldsymbol{j}}$ following (Su & Lai 2022)

$$\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t} = \omega_{\mathrm{sl}}\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}\right) \equiv \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}\right), \tag{146}$$

$$\frac{\mathrm{d}\hat{\mathbf{l}}}{\mathrm{d}t} = \omega_{\mathrm{lp}}\left(\hat{\mathbf{l}}\times\hat{\boldsymbol{j}}\right)\cos I, \tag{147}$$

where

$$\omega_{\mathrm{sl}} \equiv \alpha = \frac{3GJ_2 mR^2 M_\star}{2a^3 \mathscr{I}\Omega_{\mathrm{s}}} = \frac{k_2}{2k}\frac{M_\star}{m_b}\left(\frac{R_b}{a_b}\right)^3 \Omega_{\mathrm{b}}. \tag{148}$$

Ω_{b} is the spin rate of WASP-12b. The value of ω_{lp} comes from Laplace-Lagrange secular theory, and is given in general by (Millholland & Laughlin 2018)

$$\omega_{\mathrm{lp}} = \frac{b_{3/2}^{(1)}(\alpha)}{4}\alpha\left(n_b\frac{m_c}{M_\star + m_b}\alpha + n_c\frac{m_b}{M_\star + m_c}\right), \tag{149}$$

where the subscripts denote the values for planets b and c, and $\alpha = a_b/a_c$. In our system, we anticipate that $m_b \gg m_c$, in which case this simplifies to

$$\omega_{\mathrm{lp}} \approx \frac{\tilde{b}}{4}\alpha^2\left(n_b\frac{m_b}{M_\star}\right)\alpha^{3/2}, \tag{150}$$

where $\tilde{b} \equiv b_{3/2}^{(1)}(\alpha)/\alpha \sim 1$, around 1.25 for $\alpha = 1.29$ (Su & Lai 2022).

In intuitive terms, a spin-orbit resonance occurs when $\alpha\cos\theta = \omega_{\mathrm{lp}}$, where $\theta \equiv \arccos\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}$ is the obliquity. In less-intuitive terms, the second Cassini State (CS2; Su & Lai 2022, Fig 1) is at substantial obliquity when the ratio $\eta_{\mathrm{sync}} \gtrsim 1$, where

$$\eta_{\mathrm{sync}} \equiv \frac{\omega_{\mathrm{lp}}}{\alpha_{\Omega_{\mathrm{s}}=n_b}} = \frac{\tilde{b}k}{2k_2}\left(\frac{m_b}{M_\star}\right)^2\left(\frac{a_b}{a_c}\right)^{7/2}\left(\frac{a_b}{R_b}\right)^3, \tag{151}$$

$$= 0.0176\left(\frac{a_{\mathrm{b}}}{0.029\,\mathrm{AU}}\right)^3\left(\frac{a_{\mathrm{c}}}{1.29a_{\mathrm{b}}}\right)^{-7/2}. \tag{152}$$

We've taken $k_2 \approx 1$ (Love number), $k \approx 1/3$ (normalized moment of inertia), and $\tilde{b} \approx 1.25$, and I've dropped the scalings $m_b = 1.41M_J$, $M_\star = 1.36M_\odot$, and $R_b = 1.89R_J$ since they are well-constrained.

Here, I've focused on $\alpha = 1.29$ because Hill stability (Gladman 1993) requires

$$\Delta > 2\sqrt{3}\left(\frac{2+\Delta}{2}\right)\left(\frac{m_b + m_c}{3M_\star}\right)^{1/3}, \tag{153}$$

$$a_c \gtrsim \frac{9}{7}a_b \approx 1.28a_b. \tag{154}$$

I've used a slightly more accurate value quoted from my paper.

12.2 Probabilistic Capture

One way of capturing WASP-12b into an obliquity tide scenario is to use the Su & Lai 2022 mechanism. Naively, this scenario can be understood simply: an obliquity of $\sim 90^\circ$ is the CS2 spin-orbit resonance. If the planet forms with an isotropic spin orientation, then half the time its obliquity will damp to zero without crossing the resonance, and half the time it will. Thus, the "probabilistic" capture is when marginalizing over the initial spin orientation. Of course, a giant planet is much less likely to form with a large obliquity than a small planet, which undergoes a phase of giant impacts due to late-time dynamical instabilities (probably).

The alternative to probabilistic capture is Sarah Millholland's scenarios, where orbital migration is invoked. The idea is that when $\eta_{\rm sync} > 1$, CS2 is at low obliquity, so if the planets form at $\eta_{\rm sync} > 1$, then orbital migration drives $\eta_{\rm sync} < 1$, the planet follows CS2 from low to high obliquity.

What conditions are necessary to set Eq. (152) to be > 1? Well, $a_{\rm b}$ would have been larger at earlier times. How much larger? Well, to borrow Eq. (62) of SL22

$$\frac{1}{a_b}\frac{\mathrm{d}a_b}{\mathrm{d}t} = -\frac{1}{\tau_{\rm tide}}\left(\frac{a_b}{0.038\,\mathrm{AU}}\right)^{-13/2}\left(1 - \frac{\Omega_b}{n_b}\cos\theta\right), \tag{155}$$

where $\tau_{\rm tide} \simeq 1\,\mathrm{Gyr}$ for $Q = 10^6$ is the tidal decay timescale at 0.038 AU for a WASP-12b like planet. However, even at early times, we still must satisfy $1.29a_{\rm b,init} < a_{\rm c}$. Thus, with tides alone, the initial sma of WASP-12b cannot have been much larger than 0.038 AU, which only pushes $\eta_{\rm sync} = 0.04$ (Eq. 152).

We must invoke Sarah's other idea with orbital migration. However, this too keeps the bc period ratio roughly constant, assuming that the orbital migration rate doesn't depend much on sma. Thus, we actually need

$$\eta_{\rm sync,init} = \left(\frac{a_{\rm b,init}}{0.12\,\mathrm{AU}}\right)^3\left(\frac{a_{\rm c}}{1.29a_{\rm b}}\right)^{-7/2} > 1 \tag{156}$$

in order for the HJ to have formed in CS2 (this assumes that the HJ had a puffy radius even at such wide separations, also somewhat unlikely). This requires a planet c at 0.1548 AU, or a 19 day orbit.

For such a planet, does our approximation of neglecting the first term in the general $\omega_{\rm lp}$ work?

$$\frac{n_b m_c \alpha}{n_c m_b} = \frac{m_c}{m_b}\alpha^{-1/2} = \frac{L_c}{L_b} = 0.58\frac{m_c}{80M_\oplus}. \tag{157}$$

Not great, but it should be fine. Also, as Sarah points out, the AM constraint makes it even harder to tilt the HJ during migration (since the AM must come out of the outer planet's orbit, instead of the inner planet's orbit as would occur with tides). Thus, the tidal dissipation scenario is probably easier to make work.

13 Tides with Fluid Envelope

Question: for a planet w/ thin fluid envelope (by mass), it affects the dissipation of the rocky core, since the tidal bulge of the envelope will shield the perturbing potential somewhat.

Can we estimate this effect? Moreover, how much does this effect depend on the radial extent of the envelope; can we check the effect of tidal inflation on the dissipation of a SN that is a rocky core + gaseous envelope?

TODO look into Storch & Lai 2014.

14 Adiabaticity Calculation

We are curious about adiabiticity and resonance advection. Consider the simplest normal form for this, the 1D complex oscillator:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = i\omega(z - t), \tag{158}$$

where $z(0) = 0$ (or $z(0) = 1$), and we've normalized the time and distance scales to the motion of the equilibrium: the time scale is the total time passed in the evolution, and the distance scale is the total distance the equilibrium moves.

The explicit solution to this equation can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[ze^{-i\omega t}\right] = -i\omega t e^{-i\omega t}. \tag{159}$$

The distance of the solution from the equilibrium at the end of the interval is easiest to express when correcting for the oscillatory phase of $z(t)$ (which is conveniently just the integrating factor). Thus:

$$ze^{-i\omega t} - t = \int\limits_0^1 -\omega t \sin\omega t - 1\,\mathrm{d}t. \tag{160}$$

This clearly shows that when $\omega = 0$ that the distance is -1 (correct), and when $\omega = \infty$ that the rapid oscillations should cancel, and the distance is small.

How small? Well, this comes down to the expression (define $x = \omega t$)

$$\int\limits_0^1 \omega t \sin\omega t\,\mathrm{d}t = \frac{1}{\omega}\int\limits_0^\omega x\sin x\,\mathrm{d}x. \tag{161}$$

To estimate the value of $x\sin x$, we define the integral

$$I_n \equiv \int\limits_{n\pi}^{(n+1)\pi} y\sin y\,\mathrm{d}y, \tag{162}$$

$$\approx \left(n + \frac{1}{2}\right)\pi \int\limits_{n\pi}^{(n+1)\pi} \sin y\,\mathrm{d}y, \tag{163}$$

$$= (2n+1)\pi(-1)^n. \tag{164}$$

Then, defining $N = \lfloor \omega/\pi \rfloor$, we find

$$\int_0^\omega x \sin x \,\mathrm{d}x. = \sum_{n=0}^{N-1} I_n + \int_N^\omega x \sin x \,\mathrm{d}x, \tag{165}$$

$$= 1 - 3 + 5 - 7 \cdots + E, \tag{166}$$

$$\left| \sum_{n=0}^{N-1} I_n \right| < N - 1, \tag{167}$$

$$|E| < \omega(\omega - N). \tag{168}$$

Note that the summation is oscillating, and that the sign of the error term and magnitude terms are opposite. But this simply means that

$$\int_0^1 \omega t \sin \omega t \,\mathrm{d}t \in [-1, 1]. \tag{169}$$

This really doesn't seem like adiabaticity... I wonder what went wrong

15 Distribution of a Cut Sphere

We've come across this problem in two contexts, but I finally solved it haha.

Consider the surface of the unit sphere $S = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$. It's well-known that each of x, y, z are uniformly distributed on $[-1, 1]$ when uniformly sampling over S. However, now, consider if y is constrained such that $|y| < y_0$. What is the new distribution in z?

This question came up once already when I was playing with the Madigan scenario of a self-gravitating scattered disk. However, more recently, it came up as follows: suppose that an inner transiting planet is known to form via ZLK. We thus know that the mutual misalignment between the transiting planet and the outer companion must be in the range $[40, 140]^\circ$ (if we want octupole, then it can be much smaller). The question then is: what is the distribution of $\sin i$ of the outer companion, its misalignment from the line of sight? If we choose the LOS to be along $\hat{\mathbf{z}}$, and the transiting planet orbit normal to be along $\hat{\mathbf{y}}$, then it is clear that the orbit normal of the companion cannot be within 40° of the y axis, i.e. it must be sampled on the restricted unit sphere with $|y| < \cos 40^\circ$.

With this insight, let's rearrange slightly and call the bound on y to be $y_0 = \sin i_0$.

Now, the key insight is to consider the amount of surface area on the restricted sphere within some dz of z_0 (proportional to $P(z_0)$, the distribution of z). For each z_0, it corresponds to some polar angle $\cos\theta = z_0$, and the intersection with the unit sphere is a circle and thus has length $2\pi \sin\theta$. Now, the key step: if $\theta > i_0$, then part of this arclength is restricted away by y cut. In particular, we are just calculating the arclength of a circle satisfying $x^2 + y^2 = \sin^2\theta$ but restricting $|y| < \sin i_0$. Doing some basic geometry, it can be shown that the angle of the arc subtended by $x > 0, 0 < y < \sin i_0$ (i.e. the contribution from the first quadrant) must be $\arcsin(\sin i_0 / \sin\theta)$. Thus, the total amount of arclength at z_0 on the

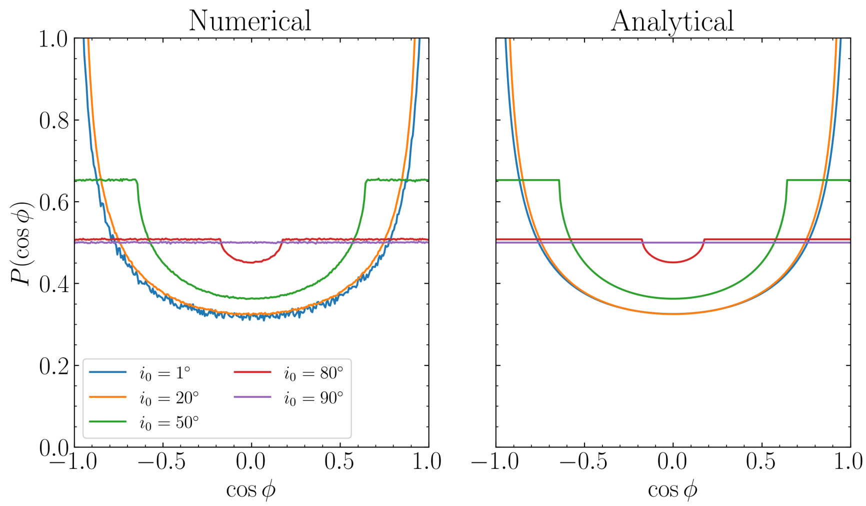

Figure 1: Numerical and analytical agreement of surface area of truncated unit sphere. Numerical results are from 10^8 draws from the unit sphere (runs in ~ 4 seconds).

restricted unit sphere is

$$L = 4\sin\theta \arcsin\left(\frac{\sin i_0}{\sin\theta}\right). \tag{170}$$

Of course, we now want the area contained within a range dz of this. The area element for a sphere scales like $1/\sin\theta$, and so we find that the amount of surface area δA within δz of $z = \cos\theta$ is

$$\frac{\delta A}{\delta z} = 4\arcsin\left(\frac{\sin i_0}{\sin\theta}\right). \tag{171}$$

Of course, when $\sin\theta < \sin i_0$, we just need the total arclength of the circle instead, $2\pi\sin\theta$, and dividing through by the mass element, we obtain the full expression for the PDF of z:

$$f(z) \propto \begin{cases} 2\pi & \sin\theta < \sin i_0, \\ 4\arcsin\left(\frac{\sin i_0}{\sin\theta}\right) & \sin\theta > \sin i_0. \end{cases} \tag{172}$$

This works well, see Fig. 1.

16 11/11/24—Love Number and Tidal Inflation

I talked to Jeremy about this, better write it down before I forget. Inspired by discussions with Tiger. Tldr I think his model is quite optimistic (but may be demanded by nature), and I think that more accurate prescriptions should be easily doable.

Consider a mini Neptune with a rocky core and envelope mass fraction $f_{\rm env} \lesssim 10\%$ and radius R. Now, inject some heat into the planet; this likely occurs at the CEB. Suppose this puffs the planet up to R'. What is the effect on the dynamics of the planet, most notably its k_2?

Recall that k_2 defines the induced exterior potential of the planet in response to an external perturbation:

$$\Phi_{\rm resp}(r > R) = k_2 \Phi_{\rm ext}\left(\frac{R}{r}\right)^3 \tag{173}$$

Recall that $k_2 = 1.5$ for a homogeneous (uniform density) fluid, and is something like 1 for an $n = 1$ polytrope (it can be done analytically, and I think it's in Jeremy's notes somewhere).

Let's assume for now that k_2 is dominated by the planet's core. This probably occurs as long as the core is not perfectly rigid (otherwise, it would not deform, and $\Phi_{\rm resp} = 0$) and dominates the mass of the planet.

If this is the case, then inflating the radius will not appreciably change the $\Phi_{\rm resp}$. However, the radius of the planet increased. Thus,

$$\Phi'_{\rm resp}\left(r > R'\right) = k_2'\Phi_{\rm ext}\left(\frac{R'}{r}\right)^3 = \Phi_{\rm resp}(r > R'). \tag{174}$$

Equating these two, we find that the quantity k_2R^3 should be conserved.

17 11/13/24—When can Torques Overcome Accretion?

Maybe this will go into a future paper, but I need to write it down. This is relevant for both planetary and stellar obliquities: if it accretes from the PPD, how can I check whether my obliquity excitation mechanisms can overcome the accretion?

The key quantity to compare is the accreted angular momentum to the torques (which also effect a change to the angular momentum). While gas is accreted once it enters the Bondi radius, it cannot be added to the star unless it loses enough angular momentum to reach the stellar surface. Thus, I think that the specific AM of a parcel of gas that is successfully accreted onto the host star cannot exceed the Keplerian AM at the stellar surface, or

$$\mathrm{d}L \lesssim \mathrm{d}m\sqrt{GMR} = R^2\Omega_{\rm dyn}\,\mathrm{d}m, \tag{175}$$

$$\dot{J}_{\rm acc} \lesssim \dot{M}R^2\Omega_{\rm dyn}. \tag{176}$$

On the other hand, precessional torques have characteristic strength (e.g. from a disk on equator of the star) [Lai 2014].

$$\mathbf{T} \sim \frac{3GM_{\rm d}}{4r_{\rm in}^2 r_{\rm out}}M_\star R_\star^2 J_2. \tag{177}$$

Comparing these two, we find that

$$\frac{\dot{J}_{\rm acc}}{\mathbf{T}} \sim \frac{4\dot{M}\sqrt{GM_\star R_\star}r_{\rm in}^2 r_{\rm out}}{3GM_{\rm d}M_\star R_\star^2 J_2}, \tag{178}$$

$$= \frac{4\dot{M}}{3M_{\rm d}\Omega_{\rm dyn}}\frac{r_{\rm in}^2 r_{\rm out}}{R_\star^3}\frac{1}{J_2}. \tag{179}$$

Note that $J_2 = k_2\epsilon^2/3$, where $\epsilon = \Omega_\star/\Omega_{\rm dyn}$ should be $\sim 10^{-2}$ for a young star, where $\epsilon \simeq 0.1$. Evaluating, we find

$$\frac{\dot{J}_{\rm acc}}{\mathbf{T}} = \frac{10^3\mathrm{yr}}{M_{\rm d}/\dot{M}}\left(\frac{r_{\rm in}/R_\star}{4}\right)^2\left(\frac{r_{\rm out}}{50\,\mathrm{AU}}\right)\left(\frac{R_\star}{2R_\odot}\right)^{1/2}\left(\frac{M_\star}{M_\odot}\right)^{-1/2}\left(\frac{J_2}{10^{-2}}\right)^{-1}. \tag{180}$$

For typical values (Fig. 3 of Manara review), $\dot{M} \lesssim 10^{-8} M_\odot$ /yr. As such, this expression evaluates to $\sim 10^{-4}$. Thus, we can have a much larger inner disk truncation radius without breaking this inequality. In conclusion, accretion is significantly subdominant to gravitational torques in changing the angular momentum of a young protostar.

18 11/18/24—ZLK Fun Stuff

18.1 Maximum Eccentricity due to SRFs

We start with LML15's equation evaluated at $i_0 = 90°$, the maximal value

$$\epsilon_{\rm GR}\left(\frac{1}{j_{1,\rm min}} - 1\right) + \frac{\epsilon_{\rm tide}}{15}\left(\frac{1 + 3e_{1,\rm max}^2 + 3e_{1,\rm max}^4/8}{j_{1,\rm min}^9} - 1\right) + \frac{\epsilon_{\rm rot}}{3}\left(\frac{1}{j_{1,\rm min}^3} - 1\right) = \frac{9}{8}\left(1 - j_{1,\rm min}^2\right). \quad (181)$$

What are the maximum eccentricities obtained when only one of the SRFs is relevant?

Let's first do the GR case. This is simple:

$$\epsilon_{\rm GR}\left(\frac{1 - j_{1,\rm min}}{j_{1,\rm min}}\right)u = \frac{9}{8}\left(1 - j_{1,\rm min}^2\right), \quad (182)$$

$$\frac{8}{9}\epsilon_{\rm GR} = j_{1,\rm min}\left(1 + j_{1,\rm min}\right). \quad (183)$$

Since $j_{1,\rm min} \le 1$, it is clear that the RHS maximizes to be 2, and therefore no eccentricity excitation is possible if $\epsilon_{\rm GR} \ge 9/4$. This is in agreement with Figure 6 of LML15.

What about the rotation case? This is not much harder:

$$\frac{\epsilon_{\rm rot}}{3}\left(\frac{1 - j_{1,\rm min}^3}{j_{1,\rm min}^3}\right) = \frac{9}{8}\left(1 - j_{1,\rm min}^2\right), \quad (184)$$

$$\frac{8\epsilon_{\rm rot}}{27} = \frac{j_{1,\rm min}^3\left(1 + j_{1,\rm min}\right)}{1 + j_{1,\rm min} + j_{1,\rm min}^2}. \quad (185)$$

The RHS maximizes when $j_{1,\rm min} = 1$, where it is equal to 2/3, and so again no ZLK when $\epsilon_{\rm rot} \ge 9/4$.

Might as well do tides while we're here lmao. Again, we use the same idea: $e_{1,\rm max} \approx 0$ when ZLK is fully suppressed, so we should just solve

$$\frac{\epsilon_{\rm tide}}{15}\left(\frac{1}{j_{1,\rm min}^9} - 1\right) = \frac{9}{8}\left(1 - j_{1,\rm min}^2\right), \quad (186)$$

$$\frac{8\epsilon_{\rm tide}}{135} = \frac{j_{1,\rm min}^9\left(1 + j_{1,\rm min}\right)}{1 + j_{1,\rm min} + \cdots + j_{1,\rm min}^8}. \quad (187)$$

Again, the LHS maximizes at 1/4, so no eccentricity excitation is obtained when $\epsilon_{\rm tide} \ge 135/32$.

18.2 Nonzero e_0 Effects

Two questions to answer here. (i) What is the range of $e_{\max}$ for a given e_0, i_0 and ω_0, and (ii) is there such a thing as an "inclination window" for finite e_0?

For the first one, it's clear that we just need to evaluate using the conserved quantities. The two are:

$$K = \left(1 - e^2\right)\cos^2 i \equiv x\cos^2 i, \tag{188}$$

$$C_K = e^2\left(5\sin^2 i \sin^2\omega - 2\right) = (1 - x)\left(5\left(\frac{x - K}{x}\right)\sin^2\omega - 2\right). \tag{189}$$

For a given e_0, the maximum $e_{\max}$ is attained when $\omega_0 = 0$ (a circulating orbit), and the minimum $e_{\max}$ is attained when $\omega_0 = \pi$ (which can even be equal to e_0 if e_0 is below the center of the ZLK resonance, i.e. the fixed point) We can solve these two cases separately:

- For the circulating case ($C_K < 0$), i.e. the maximum $e_{\max}$, we simply need to equate:

$$\begin{aligned}(1 - x_{\min})\left[5\left(\frac{x_{\min} - K}{x_{\min}}\right) - 2\right] &= -2(1 - x_0)\\ \frac{5K}{2} - \frac{3x_{\min}}{2} &= x_{\min}\frac{1 - x_0}{1 - x_{\min}}.\end{aligned} \tag{190}$$

  Of course, when $x_0 = 1$, this reduces to the standard ZLK result. Note that $x_{\min} > 0$, but the LHS must be positive as well, so $0 < x_{\min} < 5K/3$. Since the RHS vanishes for $x_{\min} = 0$, the LHS vanishes for $x_{\min} = 5K/3$, and both sides are positive, it's clear that by the intermediate value theorem that there will be an intersection that determines $x_{\min}$. Furthermore, if we simply rearrange

$$\frac{5K}{2} - x_{\min}\frac{1 - x_0}{1 - x_{\min}} = \frac{3x_{\min}}{2}, \tag{191}$$

  it is clear that as x_0 decreases from 1 (i.e. nonzero initial eccentricity) that the LHS decreases, thus decreasing $x_{\min}$. A perturbative calculation can probably give us the leading-order dependence if we must.

- For the librating case ($C_K > 0$), we instead must solve for the two roots of (I've included the second line to show that it's manifestly quadratic)

$$\begin{aligned}C_K &= (1 - x)\left[5\left(\frac{x - K}{x}\right) - 2\right]\\ C_K x &= (1 - x)[5(x - K) - 2x].\end{aligned} \tag{192}$$

  I've plotted this function on Wolframalpha, but we can see the broad features quite easily: when $x = 0$, this function is negative (not a valid solution to our "librating" assumption), becomes positive at just above $x = K$, and asymptotes to 0 again as $x \to 1$.

  Thus, for all values of $C_K > 0$ (as long as it's not too large...), there must be at least two roots, one just above $x = K$ and one out at $x \to 1$. We can hammer out the details when we need this expression, but this is the core result.

Now, the second point: what does it mean to have a ZLK window? Even at low mutual inclinations, there will still be eccentricity and inclination oscillations. I guess what we want to describe is the window within which a libration region appears. But if libration is set by $C_K > 0$, we see that it is actually *eccentricity independent*, since it just depends on whether the expression $(5\sin^2 i \sin^2\omega - 2)$ exceeds zero anywhere. Since this is maximized at $\omega = \pi/2$ of course, we recover the standard result, that $\cos^2 i < 3/5$ for $C_K > 0$! Since when $C_K < 0$, there is only one root to Eq. (192), it is clear that $C_K > 0$ is indeed the condition for the ZLK resonance feature to appear.

So then, why do eccentricity oscillations become larger in amplitude, even at small mutual inclinations, and when we don't rely on different resonances (coplanar ZLK, Li+14, Petrovich15)? This must be a consequence of the circulating regime calculation. So the question is: when x_0 and K are comparable, but are small, what does $x_{\rm min}$ look like? We first note that

$$\frac{5K}{2} - \frac{5x_{\rm min}}{2} = x_{\rm min}\frac{x_{\rm min} - x_0}{1 - x_{\rm min}}, \tag{193}$$

$$\frac{5}{2}\sin^2 i_{x_{\rm min}} = \frac{x_0 - x_{\rm min}}{1 - x_{\rm min}}. \tag{194}$$

Note that $i_{x_{\rm min}}$ occurs at eccentricity maximum, which is also the inclination minimum, and at $\omega = \pi/2$. Taking $x_{\rm min} \ll 1$, we can finally arrive at

$$\frac{5}{2}\sin^2 i_{\pi/2} \approx x_{\pi/2} - x_0. \tag{195}$$

Thus, when the LHS is of order the RHS, the eccentricity cycles can still result in substantial changes to the x values. Thus, we arrive at the condition that the inclination can still cycle due to non-resonant secular dynamics as long as

$$1 - e_0^2 \lesssim \frac{5}{2}\sin^2 i_{\pi/2}. \tag{196}$$

19 12/07/24—Viscosity and Convective Front Propagation

As Daniel [Lecoanet] rightfully points out, in Masa's simulations, the convective front propagates due to microturbulent viscosity alone, so this complicates the interpretation of our results somewhat. What is the shape and rate of this propagation? We just need to recall that

$$N^2 = N^2_{\rm structure} - R^{-1}_{\rm rho}\frac{\partial \mu}{\partial z}, \tag{197}$$

$$\frac{\partial \bar{\mu}}{\partial t} = \frac{\tau_0}{\rm Pe}\nabla^2\bar{\mu} + \ldots. \tag{198}$$

Considering the IC as $\bar{\mu}$ being a wedge function

$$\bar{\mu} = \begin{cases} 1 & z < 1, \\ (2 - z) & 1 < z < 2. \end{cases} \tag{199}$$

we can pose the question: what is the rate of advance of the Ledoux-stabilized region due to the diffusivity of the mean $\bar{\mu}$ instead? We've neglected the $z \in [2,3]$ region from our paper.

This can be solved analytically. Consider the 1D diffusion equation

$$\frac{\partial y}{\partial t} = D\frac{\partial^2 y}{\partial x^2}, \tag{200}$$

where $D = \tau_0/\mathrm{Pe}$ and $y = \bar{\mu}$. Take the initial condition above. We want to calculate where N^2 increases above some critical value, which is equivalently where $\bar{\mu}$ decreases below some critical value $1-\epsilon$.

Let's take the second of y_0, which also obeys a diffusion equation

$$\frac{\partial y''}{\partial t} = D\frac{\partial^2 y''}{\partial x^2}. \tag{201}$$

y_0'' is a delta function $y_0'' = \delta(x-1)$. It has analytic solution to the diffusion equation (I looked this up, too lazy to do)

$$y''(t,x) = \sqrt{\frac{D}{4\pi t}}\exp\left[-\frac{Dx^2}{4t}\right]. \tag{202}$$

Thus, the analytic solution to $y'(t)$ must then be

$$\begin{aligned} y'(t,x) &= \sqrt{\frac{D}{4\pi t}}\int\limits_{-\infty}^{x} \mathrm{d}\xi \exp\left[-\frac{D\xi^2}{4t}\right] \\ &= \sqrt{\frac{1}{\pi}}\int\limits_{-\infty}^{s} \mathrm{d}\sigma\, e^{-\sigma^2} \\ &= \sqrt{\frac{1}{\pi}}\,\mathrm{erf}\left[x\sqrt{\frac{D}{4t}}\right]. \end{aligned} \tag{203}$$

Not sure if I have the right definition of erf, but yeah it should be like this. Thus, the x at which $y'(t,x)$ crosses some critical $y'_{\rm crit}$ at a given t should be related to

$$x_{\rm crit} = \sqrt{\frac{4t}{D}}\,\mathrm{erf}^{-1}\left(y'_{\rm crit}\sqrt{\pi}\right) \propto \sqrt{t}. \tag{204}$$

I have trouble seeing how this can be avoided.

20 12/09/24—Ivection Resonance and Broken Disks?

Let's check some frequencies first. Consider a host star, binary companion, inner and outer disk. Then

$$\omega_{\rm io} = \frac{3}{16}\frac{m_{\rm o}}{M_\star}\left(\frac{r_{\rm i,out}^3}{r_{\rm o,in}^2 r_{\rm o,out}}\right) n_{\rm i,out}, \tag{205}$$

$$\frac{\omega_{\rm io}}{n_{\rm b}} = \frac{3}{16}\frac{m_{\rm o}}{M_\star}\left(\frac{r_{\rm i,out}^3}{r_{\rm o,in}^2 r_{\rm o,out}}\right)\left(\frac{m_\star a_{\rm b}^3}{(m_\star + m_{\rm b}) r_{\rm i,out}^3}\right)^{1/2}, \tag{206}$$

$$\approx \frac{3}{16}\frac{m_{\rm o}}{M_\star}\left(\frac{r_{\rm i,out}^{3/2} a_{\rm b}^{3/2}}{r_{\rm o,in}^2 r_{\rm o,out}}\right), \tag{207}$$

$$\approx 0.1\left(\frac{m_{\rm o}/M_\star}{10^{-2}}\right)\left(\frac{r_{\rm i,out}}{15\,{\rm AU}}\right)^{3/2}\left(\frac{a_{\rm b}}{10^3\,{\rm AU}}\right)^{3/2}\left(\frac{r_{\rm o,in}}{20\,{\rm AU}}\right)^{-2}\left(\frac{r_{\rm o,out}}{80\,{\rm AU}}\right)^{-1}. \tag{208}$$

This is if the disk obeys $\Sigma \propto r^{-1}$. But if it's a $r^{-3/2}$ disk, then... well, we should do this more carefully.

But also, for a $p = 1$ disk, the mass is uniformly distributed dr. For a $p = 3/2$ disk, the local AM $2\pi r \Sigma \sqrt{r}\,{\rm d}r$ is uniformly distributed. Thus, the outer disk doesn't have much more AM than the inner disk. Does this help us at all? TBD.

21 12/17/24—IW Dissipation in Envelope

We often hear this $(r_c/R)^5$ scaling of the IW Q, but can we do better in a thin envelope? Just writing this down after chatting it through with Eliot [Quataert].

From Ogilvie 2013 (Eq. B3) and Barker 2020 (Eq. 39), the full tidal Q for IW should be

$$\begin{aligned}\frac{1}{Q'} = \frac{3k_2}{2Q} = \epsilon^2 \frac{200\pi}{189}\left(\frac{\alpha^5}{1-\alpha^5}\right)(1-\gamma)^2(1-\alpha)^4 \\ \times\left(1 + 2\alpha + 3\alpha^2 + \frac{3\alpha^3}{2}\right)^2\left[1 + \left(\frac{1-\gamma}{\gamma}\right)\alpha^3\right] \\ \times\left(1 + \frac{3\gamma}{2} + \frac{5}{2\gamma}\left(1 + \frac{\gamma}{2} - \frac{3\gamma^2}{2}\right)\alpha^3 - \frac{9(1-\gamma)\alpha^5}{4}\right)^{-2}.\end{aligned} \tag{209}$$

Note that this is an exact result, for the simplified two-zone model, despite its appearance as an expansion in α. Note that

$$\epsilon = \frac{\Omega_\star}{\sqrt{GM_\star/R_\star^3}}, \qquad \alpha = \frac{r_{\rm c}}{R}, \qquad \gamma = \frac{\alpha(1-\beta)}{\beta(1-\alpha^3)}, \qquad \beta = \frac{m_{\rm c}}{M}. \tag{210}$$

It's common to see this expression written as $\propto \epsilon^2\alpha^5$. However, this only holds when $\alpha \ll 1$. In the other limit, that we're working on with Luke, we want $\alpha \to 1$ and $\beta \to 1$, but where β approaches much faster. Thus, we first take the limit for $\beta \to 1$, or $\gamma \to 0$ (if $\alpha \to 1$ slowly, certainly $\alpha^3 \to 1$ even more slowly):

$$\begin{aligned}\lim_{\gamma\to 0}\frac{1}{Q'} \approx{}& \epsilon^2\frac{200\pi}{189}\left(\frac{\alpha^5}{1-\alpha^5}\right)(1-\alpha)^4\\ &\times\left(1+2\alpha+3\alpha^2+\frac{3\alpha^3}{2}\right)^2\left[\frac{\alpha^3}{\gamma}\right]\left(\frac{5\alpha^3}{2\gamma}\right)^{-2} \qquad (211)\\ \lim_{\gamma\to 0}\frac{1}{\gamma Q'} \approx{}& \epsilon^2\frac{200\pi}{189}\left(\frac{\alpha^5}{1-\alpha^5}\right)(1-\alpha)^4\\ &\times\left(1+2\alpha+3\alpha^2+\frac{3\alpha^3}{2}\right)^2\frac{4}{25\alpha^3}. \qquad (212)\end{aligned}$$

Now, we correspondingly take $\alpha \to 1$, and obtain

$$\begin{aligned}\lim_{\alpha\to 1}\frac{1}{\gamma Q'} ={}& \epsilon^2\frac{40\pi}{21}(1-\alpha)^3, \qquad (213)\\ \lim_{\alpha\to 1}\lim_{\gamma\to 0}\frac{1}{Q'} ={}& \epsilon^2\frac{40\pi}{21}\gamma(1-\alpha)^3\\ ={}& \epsilon^2\frac{40\pi}{21}\frac{\alpha(1-\beta)}{\beta\left(1+\alpha+\alpha^2\right)}(1-\alpha)^2\\ \lim_{r_c\to R}\lim_{m_c\to M}\frac{1}{Q'} \approx{}& \epsilon^2\frac{40\pi}{63}(1-\beta)(1-\alpha)^2. \qquad (214)\end{aligned}$$

22 02/18/25—Pseudosynchronization at High Obliquity

Josh asked me once why pseudosynchronization is so slow at high obliquity. To analyze this, let's look to the tidal torque along the spin axis, with all time lags set to be equal (Eq. 27 of Lai2012):

$$\frac{T_z}{T_0\tau} = \frac{3\pi}{20}\left[(1+c)^4(\Omega-\omega)+s^2(1+c)^2(2\Omega-\omega)-s^2(1-c)^2(2\Omega+\omega)-(1-c)^4(\Omega+\omega)\right]-\frac{3\pi}{5}\omega\left(s^4+s^2c^2\right). \qquad (215)$$

Here, $s = \sin\theta$ and $c = \cos\theta$. When θ is close to 90°, we can set $s = 1$ and expand $c \ll 1$ to obtain

$$\begin{aligned}\frac{T_z}{T_0\tau} \approx{}& \frac{3\pi}{20}\left[(1+4c)(\Omega-\omega)+(1+2c)(2\Omega-\omega)-(1-2c)(2\Omega+\omega)-(1-4c)(\Omega+\omega)\right]-\frac{3\pi}{5}\omega\left(1-c^4\right), \qquad (216)\\ ={}& \frac{3\pi}{20}\left[16c\Omega-4\omega\right]-\frac{3\pi}{5}\omega\left(1-c^4\right). \qquad (217)\end{aligned}$$

Setting this equal to zero, we get the usual result

$$2\cos\theta = \frac{\omega}{\Omega}+\mathcal{O}\left(\cos^2\theta\right). \qquad (218)$$

Expanding to higher order gives better precision. But the key point is that, instead of the dominant torque being the (22) component ($(1+c)^4(\Omega-\omega)$, the only one that doesn't vanish when $s = 0$ and $c = 1$), the only component overlapping with the orbital motion Ω scales with $\cos\theta$ (for a highly misaligned $m' = 2$ component, it gets broken up into nearly-equal $m = \pm 2$ components, with only the $\cos\theta$ splitting the degeneracy). Put another way, the highly misaligned potential is much more ring-like than orbital like, and ring-like will tend to favor zero-spin.

23 03/21/25—Mixed Modes Maybe?

We already know that the "mixed modes" Dong and I found in our Paper III are due to high-order SORs. These must also be identifiable in the vector formalism too though, but we can't seem to find it. The vector formulation is important to characterize since, when $\hat{\mathbf{l}}$ experiences crazy evolution, no good coordinate system can be written down.

We will approximate a bit. Consider the inertial frame equations

$$\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t} = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}\right), \tag{219}$$

$$\hat{\mathbf{l}} \approx \hat{\mathbf{z}} + \mathrm{Re}(\mathscr{I})\hat{\mathbf{x}} + \mathrm{Im}(\mathscr{I})\hat{\mathbf{y}}, \tag{220}$$

$$\mathscr{I} = I_1 e^{ig_1 t} + I_2 e^{ig_2 t}. \tag{221}$$

23.1 Total AM approach

Let's instead adopt the celestial mechanics approach, where we keep the total AM $\hat{\mathbf{l}}_{\mathrm{p}}$ along the z-axis.

Start again from the inertial frame equations, Eq. (221). Consider a rotation with rate $\bar{g}$.

$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)' = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}\right) + \bar{g}\left(\hat{\mathbf{s}}\times\hat{\mathbf{z}}\right), \tag{222}$$

$$\mathscr{I} = I_1 e^{i(g_1-\bar{g})t} + I_2 e^{i(g_2-\bar{g})t}. \tag{223}$$

This problem is somewhat difficult because $\hat{\mathbf{l}}$ contains two parameters: $I_2 \ll I_1 \ll 1$, the last expressing the point that $\hat{\mathbf{l}} \approx \hat{\mathbf{z}}$. Denote $\hat{\mathbf{s}}_{ij}$ to be ith order in I_1 and jth order in I_2

Let's first evaluate the ODE at the lowest order, $\hat{\mathbf{l}} \approx \hat{\mathbf{z}}$:

$$\frac{\mathrm{d}\hat{\mathbf{s}}_{00}}{\mathrm{d}t} = \alpha(\hat{\mathbf{s}}_{00}\cdot\hat{\mathbf{z}})(\hat{\mathbf{s}}_{00}\times\hat{\mathbf{z}}) + \bar{g}(\hat{\mathbf{s}}_{00}\times\hat{\mathbf{z}}), \tag{224}$$

$$\alpha(\hat{\mathbf{s}}_{00}\cdot\hat{\mathbf{z}}) = -\bar{g}. \tag{225}$$

This is the simple result.

Now, what about at leading order in I_1, assuming $I_2 = 0$? Note that the next-order correction to $\hat{\mathbf{s}}$, denoted $\hat{\mathbf{s}}_{10}$, is not a unit vector.

$$\begin{aligned}\frac{\mathrm{d}\mathbf{s}_{10}}{\mathrm{d}t} ={}& \alpha(\mathbf{s}_{10}\cdot\hat{\mathbf{z}})(\hat{\mathbf{s}}_{00}\times\hat{\mathbf{z}}) + \cancel{\alpha(\hat{\mathbf{s}}_{00}\cdot\hat{\mathbf{z}})(\mathbf{s}_{10}\times\hat{\mathbf{z}}) + \bar{g}(\mathbf{s}_{10}\times\hat{\mathbf{z}})}\\ &+ \alpha(\hat{\mathbf{s}}_{00}\cdot\mathbf{l}_\perp)(\hat{\mathbf{s}}_{00}\times\hat{\mathbf{z}}) + \cancelto{-\bar{g}}{\alpha(\hat{\mathbf{s}}_{00}\cdot\hat{\mathbf{z}})}(\hat{\mathbf{s}}_{00}\times\mathbf{l}_\perp).\end{aligned} \tag{226}$$

Note now that $\mathbf{s}_{10}$ is not constant. We expect it to look like a uniformly rotating matrix though, so let's demand that

$$\frac{\mathrm{d}\mathbf{s}_{10}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\mathbf{s}}_{10}, \tag{227}$$

where $\boldsymbol{\omega} \perp \mathbf{s}_{10}$.

There is one solution. Let's guess that $\boldsymbol{\omega} \propto \hat{\mathbf{s}}_{00}$, and $\mathbf{s}_{10}$ oscillates in phase with $\mathbf{l}_\perp$, i.e.

$$\mathbf{s}_{10} \propto \mathbf{l}_\perp - (\hat{\mathbf{s}}_{00} \cdot \mathbf{l}_\perp)\hat{\mathbf{s}}_{00}, \tag{228}$$

$$\boldsymbol{\omega} = -\bar{g}\hat{\mathbf{s}}_{00}, \tag{229}$$

The amplitude of $\mathbf{s}_{10}$ is fixed by demanding that the first two terms in Eq. (226) cancel:

$$0 = \mathbf{s}_{10} \cdot \hat{\mathbf{z}} + \hat{\mathbf{s}}_{00} \cdot \mathbf{l}_\perp, \tag{230}$$

$$= C(\hat{\mathbf{s}}_{00} \cdot \mathbf{l}_\perp)\frac{\bar{g}}{\alpha} + \hat{\mathbf{s}}_{00} \cdot \mathbf{l}_\perp, \tag{231}$$

$$C = -\frac{\alpha}{\bar{g}}, \tag{232}$$

$$\mathbf{s}_{10} = -\frac{\alpha}{\bar{g}}\left[\mathbf{l}_\perp - (\hat{\mathbf{s}}_{00} \cdot \mathbf{l}_\perp)\hat{\mathbf{s}}_{00}\right]. \tag{233}$$

Thus, indeed such a solution for $\mathbf{s}_{10}$ exists.

Now, note that $\mathbf{s}_{10}$ has time-varying components along $\mathbf{l}_\perp$ (implying oscillation frequencies $(g_1 - \bar{g})$ and $(g_2 - \bar{g})$), while its rotation has frequency $\bar{g}$ (Eq. 229). Consistency among all of these frequencies demands $\bar{g} = (g_1 + g_2)/2$ (excepting the trivial solutions $\bar{g} = g_1$ and g_2).

Looking back at my paper, it looks like we can have $\mathbf{s}_{10}$ oscillate in the xy plane, just with no z component. But this makes sense: the z component oscillation is just because the true z axis should be $\hat{\mathbf{l}}$, not $\hat{\mathbf{l}}_\mathrm{p}$. So our $\mathbf{s}_{10}$ should be such that

$$(\hat{\mathbf{s}}_{00} + \mathbf{s}_{10}) \cdot (\hat{\mathbf{z}} + \mathbf{l}_\perp) = \hat{\mathbf{s}}_{00} \cdot \hat{\mathbf{z}} + \frac{\alpha}{\bar{g}}(\hat{\mathbf{s}}_{00} \cdot \mathbf{l}_\perp)\hat{\mathbf{s}}_{00} \cdot \hat{\mathbf{z}} + \hat{\mathbf{s}}_{00} \cdot \mathbf{l}_\perp, \tag{234}$$

$$= \hat{\mathbf{s}}_{00} \cdot \hat{\mathbf{z}}. \tag{235}$$

Yay!

Well, I guess at this point, we've successfully shown that $\mathbf{s}_{10}$ must have an azimuthal component $\sim \mathcal{O}(I_1)$, and the polar component must be $\lesssim \mathcal{O}(I_2)$ (which it is), and the frequency of oscillation still does demand that $\bar{g} = g_1 - \bar{g} = -g_2 + \bar{g}$. Is this enough? Maybe we should expand to one more order, $\mathbf{s}_{01}$, to show that we've truly found the correct state...

23.2 Co-precessing, co-nutating approach

For posterity, I include the erroneous approach I started with...

We will go to the co-precessing, co-nutating frame, and point $\hat{\mathbf{z}}$ along the vertical axis. This means that $\hat{\mathbf{l}} \to \hat{\mathbf{z}}$ while $\hat{\mathbf{l}}_\mathrm{p} = \hat{\mathbf{z}} \to -\hat{\mathbf{l}}$. So

$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\mathrm{co-p}} = \alpha(\hat{\mathbf{s}} \cdot \hat{\mathbf{z}})(\hat{\mathbf{s}} \times \hat{\mathbf{z}}) - \dot{\Omega}\left(\hat{\mathbf{l}}_\mathrm{p} \times \hat{\mathbf{s}}\right) - \dot{I}\left[\left(\hat{\mathbf{l}}_\mathrm{p} \times \hat{\mathbf{z}}\right) \times \hat{\mathbf{s}}\right], \tag{236}$$

$$\hat{\mathbf{l}}_\mathrm{p} \approx \hat{\mathbf{z}} - |\mathscr{I}|\hat{\mathbf{x}} \tag{237}$$

Note that $\dot{\Omega}$ has contributions from both modes, but $\dot{I} \sim \mathcal{O}(I_2/I_1)$ is small in amplitude. While there are a lot more terms (and this looks ugly), it will be easier to do expansions since there are no crazy cross terms with $s^2 l^2$. Of course, in the limit $I_2 = 0$, we have $\dot{I} = 0$ and we recover the standard CS EOM.

Note that in the limit $I_2 \ll I_1$, we should have that

$$\begin{aligned}
\Omega &= \arctan\left(\frac{I_1 \sin g_1 t + I_2 \sin g_2 t}{I_1 \cos g_1 t + I_2 \cos g_2 t}\right), && (238)\\
&\approx \arctan\left(\tan g_1 t + \frac{I_2 \cos g_2 t}{I_1 \cos g_1 t}(\tan g_2 t - \tan g_1 t)\right), && (239)\\
&\approx g_1 t + \frac{I_2 \cos g_1 t \cos g_2 t}{I_1}(\tan g_2 t - \tan g_1 t)\\
&= g_1 t + \frac{I_2}{I_1}\sin(g_2 - g_1)t, && (240)\\
\dot{\Omega} &= g_1 + \Delta g \frac{I_2}{I_1}\sin \Delta g t, && (241)
\end{aligned}$$

while

$$\begin{aligned}
I(t) &= \left(I_1^2 + I_2^2 \cos(g_2 - g_1)t\right)^{1/2}, && (242)\\
&\approx I_1 + \frac{I_2^2}{2I_1^2}\cos(g_2 - g_1)t, && (243)\\
\dot{I} &= \Delta g \frac{I_2^2}{2I_1^2}\cos \Delta g t. && (244)
\end{aligned}$$

Now, let's consider further rotating into a new reference frame. Consider a rotation by some $-\bar{g}$ (which we will later set to be $-(g_1 + g_2)/2$):

$$\begin{aligned}
\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\text{co-p}} &= \alpha(\hat{\mathbf{s}} \cdot \hat{\mathbf{z}})(\hat{\mathbf{s}} \times \hat{\mathbf{z}}) - \dot{\Omega}\left(\hat{\mathbf{l}}_{\mathrm{p}} \times \hat{\mathbf{s}}\right) - \dot{I}\left[\left(\hat{\mathbf{l}}_{\mathrm{p}} \times \hat{\mathbf{z}}\right) \times \hat{\mathbf{s}}\right] + \bar{g}(\hat{\mathbf{z}} \times \hat{\mathbf{s}}), && (245)\\
\hat{\mathbf{l}}_{\mathrm{p}} &\approx \hat{\mathbf{z}} - \mathrm{Re}(\mathscr{I})\hat{\mathbf{x}} - \mathrm{Im}(\mathscr{I})\hat{\mathbf{y}}, && (246)\\
\mathscr{I} &= I e^{i\bar{g}t}. && (247)
\end{aligned}$$

At leading order, $\hat{\mathbf{l}}_{\mathrm{p}} \approx \hat{\mathbf{z}}$, while $\dot{\Omega} \approx g_1$ (this may not be the dominant term by magnitude, but in a time-averaged sense the $g_2 I_2$ contribution vanishes), and $\dot{I} \approx 0$, and we obtain

$$\frac{\mathrm{d}\hat{\mathbf{s}}_0}{\mathrm{d}t} = \alpha(\hat{\mathbf{s}}_0 \cdot \hat{\mathbf{z}})(\hat{\mathbf{s}}_0 \times \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}} \times \hat{\mathbf{s}}) + \bar{g}(\hat{\mathbf{z}} \times \hat{\mathbf{s}}). \qquad (248)$$

This admits constant solutions if $\hat{\mathbf{s}}_0 \cdot \hat{\mathbf{z}} = (g_1 - \bar{g})/\alpha$. Easy enough. In order for there to be librating solutions, let's further demand that $\hat{\mathbf{s}}_0$ be in the $\hat{\mathbf{x}}$–$\hat{\mathbf{z}}$ plane.

What about the next order terms? (there should be seven, because in Eq. 245, the first term has two terms that depend on $\hat{\mathbf{s}}$, the second term has three terms that have leading order behavior, the third term is small so only contributes one term at leading (small) order, and the fourth only needs to be expanded in $\hat{\mathbf{s}}$)

$$\frac{\mathrm{d}\hat{\mathbf{s}}_1}{\mathrm{d}t} = \alpha(\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{z}})(\hat{\mathbf{s}}_0 \times \hat{\mathbf{z}}) + \cancel{\alpha(\hat{\mathbf{s}}_0 \cdot \hat{\mathbf{z}})(\hat{\mathbf{s}}_1 \times \hat{\mathbf{z}})} - \frac{\Delta g I_2}{I_1} \sin \Delta g t (\hat{\mathbf{z}} \times \hat{\mathbf{s}}_0) - \cancel{g_1(\hat{\mathbf{z}} \times \hat{\mathbf{s}}_1)}$$
$$- g_1\left(\hat{\mathbf{l}}_{\mathrm{p},\perp} \times \hat{\mathbf{s}}_0\right) - \Delta g \frac{I_2^2}{2I_1^2} \cos \Delta g t \left[\left(\hat{\mathbf{l}}_{\mathrm{p}} \times \hat{\mathbf{z}}\right) \times \hat{\mathbf{s}}_0\right] + \cancel{\bar{g}(\hat{\mathbf{z}} \times \hat{\mathbf{s}}_1)}. \tag{249}$$

The cancelled terms vanish from the zeroth order term, so

$$\frac{\mathrm{d}\hat{\mathbf{s}}_1}{\mathrm{d}t} = \alpha(\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{z}})(\hat{\mathbf{s}}_0 \times \hat{\mathbf{z}}) - \frac{\Delta g I_2}{I_1} \sin \Delta g t (\hat{\mathbf{z}} \times \hat{\mathbf{s}}_0)$$
$$- g_1\left(\hat{\mathbf{l}}_{\mathrm{p},\perp} \times \hat{\mathbf{s}}_0\right) - \Delta g \frac{I_2^2}{2I_1^2} \cos \Delta g t \left[\left(\hat{\mathbf{l}}_{\mathrm{p},\perp} \times \hat{\mathbf{z}}\right) \times \hat{\mathbf{s}}_0\right]. \tag{250}$$

Note that $(\hat{\mathbf{l}}_{\mathrm{p},\perp} \times \hat{\mathbf{z}}) \times \hat{\mathbf{s}}_0 = \hat{\mathbf{z}}\left(\hat{\mathbf{l}}_{\mathrm{p},\perp} \cdot \hat{\mathbf{s}}_0\right) - \hat{\mathbf{l}}_{\mathrm{p},\perp}(\hat{\mathbf{s}} \cdot \hat{\mathbf{z}})$.

The first two terms point along the $\hat{\mathbf{y}}$ axis, while the last two point along all three. The third term is much larger in magnitude than the fourth term though, so its non-$\hat{\mathbf{y}}$ contributions must be cancelled by corrections to the zeroth-order solution. From this, we see that $\hat{\mathbf{s}}_1 \times \hat{\mathbf{z}}$ must have a $\bar{g}$ time dependence. But the first two terms immediately show that $\hat{\mathbf{s}}_1$ must have a Δg time dependence. Thus, we recover that $\bar{g} = \pm\Delta g$. We've just shown that the other first-order solution are the mode-2 Cassini States! That won't do.

The reason is obvious: the spin-orbit resonance we were looking for must exist at higher order, and we've only taken our expansion up to first order. To expand this to second order would be quite a lengthy task.