

1 Maximum Separatrix Area

Consider a planet with orbit normal $\hat{\mathbf{l}}_p$ that experiences precession driven by stellar oblateness $\hat{\mathbf{l}}_s$ and an outer perturber $\hat{\mathbf{l}}_o$. We assume that the planet's orbit is circular. The vector form of the precessional dynamics are (e.g. Tremaine+2009, Eq 19):

$$\frac{d\hat{\mathbf{l}}_p}{dt} = \omega_{sp} (\hat{\mathbf{l}}_p \cdot \hat{\mathbf{l}}_s) (\hat{\mathbf{l}}_p \times \hat{\mathbf{l}}_s) + \omega_{op} (\hat{\mathbf{l}}_p \cdot \hat{\mathbf{l}}_o) (\hat{\mathbf{l}}_p \times \hat{\mathbf{l}}_o). \quad (1)$$

We first make an important symmetry argument: in the limits of $\omega_{sp} \ll \omega_{op}$ or $\omega_{sp} \gg \omega_{op}$, the evolution of $\hat{\mathbf{l}}_p$ consist of uniform precession about $\hat{\mathbf{l}}_o$ and $\hat{\mathbf{l}}_p$ respectively, and thus the separatrix area must go to zero in these limits. In fact, the phase portrait must the same under the following transformation: swap the two frequencies (ω_{sp}, ω_{op}) and the two vectors ($\hat{\mathbf{l}}_o, \hat{\mathbf{l}}_s$). Swapping the precession frequencies is equivalent to taking $a/r_M \mapsto r_M/a$ (and rescaling time), since $\omega_{op}/\omega_{sp} = (a/r_M)^5$. Thus, we arrive at an important conclusion: *the phase portraits are equivalent, up to a rotation of reference frame, for any two $r_{M,1}$ and $r_{M,2}$ satisfying $a/r_{M,1} = r_{M,2}/a$* . This implies that the separatrix area is symmetric about $r_M = a$ as well.

It's not clear that the separatrix area must be monotonic between $r_M \in [0, a]$, but intuitively **this seems like it should be the case** (?), since there are no special values of ω_{sp}/ω_{op} in the equation of motion. If so, then the maximum separatrix area is obtained for $r_M = a$. The curve for the separatrix in this case is significantly easier to obtain, though it still seems difficult to integrate explicitly (maybe there's a clever idea?).

To compute the separatrix area for $a = r_M$, we note that the low-obliquity Laplace equilibrium P1 is located exactly halfway between $\hat{\mathbf{l}}_o$ and $\hat{\mathbf{l}}_s$. Thus, we choose the reference frame such that $\hat{\mathbf{z}} \propto \hat{\mathbf{l}}_o + \hat{\mathbf{l}}_s$, and we choose $\hat{\mathbf{y}}$ to point towards P2 (which is always $\pi/2$ away from P1). Then, defining

$$\cos \epsilon \equiv \hat{\mathbf{l}}_o \cdot \hat{\mathbf{l}}_s, \quad (2)$$

we can write

$$\hat{\mathbf{l}}_o = \cos \frac{\epsilon}{2} \hat{\mathbf{z}} + \sin \frac{\epsilon}{2} \hat{\mathbf{x}}, \quad (3)$$

$$\hat{\mathbf{l}}_s = \cos \frac{\epsilon}{2} \hat{\mathbf{z}} - \sin \frac{\epsilon}{2} \hat{\mathbf{x}}. \quad (4)$$

Finally, upon inspection, $\hat{\mathbf{x}}$ is also an equilibrium point, which must be P3. **In summary, in this reference frame, P1 lies along $\hat{\mathbf{z}}$, P2 lies along $\hat{\mathbf{y}}$, and P3 lies along $\hat{\mathbf{x}}$.**

To get the level curve corresponding to the separatrix, we evaluate the Hamiltonian (factoring out the prefactor $\omega_{sp} = \omega_{op}$) and adopt a spherical coordinate system:

$$H \propto -\frac{1}{2} \left[(\hat{\mathbf{l}}_p \cdot \hat{\mathbf{l}}_s)^2 + (\hat{\mathbf{l}}_p \cdot \hat{\mathbf{l}}_o)^2 \right], \quad (5)$$

$$\tilde{H}(\theta, \phi) = - \left[\sin^2 \frac{\epsilon}{2} \sin^2 \theta \cos^2 \phi + \cos^2 \frac{\epsilon}{2} \cos^2 \theta \right], \quad (6)$$

where we have adopted spherical coordinates (θ, ϕ) to describe the orientation of $\hat{\mathbf{l}}_p$, and $\theta = \pi/2, \phi = 0$ corresponds to $\hat{\mathbf{x}}$. We first evaluate H (dropping the tilde) at P3:

$$H_3 = -\sin^2 \frac{\epsilon}{2}, \quad (7)$$

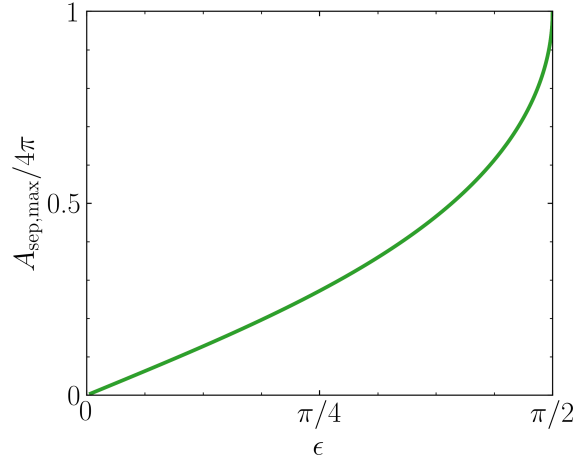


Figure 1: Fractional phase space area enclosed by the maximal separatrix as a function of ϵ . Reminder: this is the area surrounding Laplace equilibrium P2 when $r_M = a$, which is also the maximum extent of the separatrix.

then the separatrix is given by

$$H(\theta_{\text{sep}}(\phi), \phi) = H_3, \quad (8)$$

$$\sin^2 \frac{\epsilon}{2} (1 - \cos^2 \theta_{\text{sep}}) \cos^2 \phi + \cos^2 \frac{\epsilon}{2} \cos^2 \theta_{\text{sep}} = \sin^2 \frac{\epsilon}{2}, \quad (9)$$

$$\begin{aligned} \cos^2 \theta_{\text{sep}} &= \frac{\sin^2 \frac{\epsilon}{2} \sin^2 \phi}{\cos^2 \frac{\epsilon}{2} - \sin^2 \frac{\epsilon}{2} \cos^2 \phi} \\ &= \frac{\sin^2 \phi}{\cot^2 \frac{\epsilon}{2} - \cos^2 \phi}, \end{aligned} \quad (10)$$

$$A_{\text{sep}} = 4 \int_0^\pi \cos_+ \theta_{\text{sep}} d\phi. \quad (11)$$

Here, $\cos_+ \theta_{\text{sep}}$ indicates that we take the positive root; one factor of two arises because the vertical extent of the separatrix is $\cos_+ \theta_{\text{sep}} - \cos_- \theta_{\text{sep}}$, and a second factor of two arises because we are only integrating $\phi \in [0, \pi]$. Under this convention, the maximum possible phase space area is 4π . We display the value of A_{sep} in Fig. 1.

Note: the integral for A_{sep} is analytic:

$$\begin{aligned}
A_{\text{sep}} &= 4 \int_0^\pi \frac{\sin \phi}{\sqrt{\cot^2 \frac{\epsilon}{2} - \cos^2 \phi}} d\phi \\
&= 4 \int_{-1}^1 \frac{1}{\sqrt{\cot^2 \frac{\epsilon}{2} - \cos^2 \phi}} d\cos \phi \\
&= 4 \left[\tan^{-1} \left(\frac{u}{\sqrt{\cot^2 \frac{\epsilon}{2} - u^2}} \right) \right]_{u=-1}^{u=1} \\
&= 8 \left[\tan^{-1} \sqrt{\frac{\sin^2(\epsilon/2)}{\cos \epsilon}} \right].
\end{aligned} \tag{12}$$