Welcome back to my random tidbits file! When I come up with interesting problems, I will put them here.

Contents

Chapter 1

Probability Distributions and Weight Loss

I was keeping track of my own weight when I realized that my scale was sufficiently inconsistent that my weight loss was dominated by the statistical noise. So then I was curious what the best way of mitigating this is, mean or median of multiple measurements. One would suspect it's the mean, or one would know simply by having taken any real statistics class, but I'm curious.

1.1 Mean-based averaging

This one is easy. Assume we have n iid variables X_i with mean μ and variance σ^2, then the random variable corresponding to their average $\langle X_i \rangle$ has mean μ and variance $\frac{\sigma^2}{n}$, so standard deviation $\frac{\sigma}{\sqrt{n}}$. Thus, we have an unbiased estimator of the true mean and a variance that falls off like $\sim n^{-1/2}$.

1.2 Median-based averaging

This one is a bit more fun. Let's start with $n = 3$, then defining $f(x)$ the probability density function and $F_X(x) = f_X(X \leq x)$ the cumulative distribution function, the probability density of the median $f_\eta(y)$ is given

$$f_\eta(y) = 6 f_X(y) F_X(y)(1 - F_X(y)) \tag{1.1}$$

the probability we choose one value greater than y the median and one less, multiplied by 6 because there 3 ways to choose which element is the median and $\binom{2}{1}$ binomial coefficient for exactly one element on each side. This seems to be a bit difficult to verify to be normalized in the general case, or that

$$\int_{-\infty}^{\infty} f_\eta(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} \left[6 f_X(y) \int_{-\infty}^{y} f_X(\xi)\,\mathrm{d}\xi \int_{y}^{\infty} f_X(\zeta)\,\mathrm{d}\zeta \right] \mathrm{d}y = 1 \tag{1.2}$$

Let's just verify this in the uniform distribution case, and leave the general oase as an exercise to brighter colleagues. We consider the normalized uniform distribution $f_X(x) = 1, x \in [0,1]$, or $F_X(x) = x, x \in [0,1]$. We confirm that the expression for f_η is normalized:

$$\int_0^1 6y(1-y)\,\mathrm{d}y = 1 \tag{1.3}$$

We then wish to examine whether $f_\eta(y)$ is an unbiased estimator of μ. Again, we begin with examining a sub-case, where $f_X(x)$ is symmetric about its mean μ. This yields that $F_X(\mu) = 0.5$ and is odd about μ[1] and so that $F_X(y)(1-F_X(y))$ is also even/symmetric about μ. Finally, this implies that $f_\eta(y)$ as defined in Equation 1.1 is also symmetric about μ and we are done.

However, this analysis breaks down in the asymmetric case. We see that $F_X(y)(1-F_X(y))$ is *always* symmetric about the median η of f_X, since $F_X(\eta) = 0.5$. In general, the mean and median of a probability distribution are not equal, so there is no guarantee that $\langle f_\eta(y)\rangle = \langle f(y)\rangle$, and indeed we can verify for some contrived probability distribution such as

$$f_X(x) = \begin{cases} 2 & 0 \le x \le 0.25 \\ 1 & 0.5 \le x \le 1 \\ 0 & \text{else} \end{cases} \tag{1.4}$$

that $\langle f_X(x)\rangle = 0.4375$ while

$$\left\langle f_\eta(y)\right\rangle = \int_0^{0.25} 24y^2(1-2y)\,\mathrm{d}y + \int_{0.5}^1 6y^2(1-y)\,\mathrm{d}y \tag{1.5}$$

$$\approx 0.4218 \tag{1.6}$$

This should not have surprised us: we're trying to use a median to estimate the mean of a distribution, and the two are equal when the PD is symmetric and unequal otherwise.

The above analyses probably generalizes to median-of-n trials, where with a symmetric PD we have the being a unbiased estimator of the mean and with an asymmetric an biased estimator, for any parity of n, but I'm too lazy to check this out and will take it on faith. For reference, we assert the generalization of Equation 1.1 below for odd $N = 2m+1$ trials below

$$f_{\eta,2m+1}(y) = N\binom{2m}{m} f_X(y)(F_X(y))^m\,(1-F_X(y))^m \tag{1.7}$$

which is simply generalizing to the concept of "m elements on either side of y."

It seems difficult to compare these median-based results (many of which could probably be strengthened) to the mean based results in the case of an arbitrary PDF, so let's specialize to a few tractable cases.

[1] This is a slight abuse of terminology: we mean that $F_X(x-\mu) - 0.5 = -(F_X(-(x-\mu)) - 0.5)$.

1.3 Uniform Distribution

I'm tired of not obtaining usable results, so let's simplify the discussion considerably and assume that we have a uniform probability distribution, or that $X \in [\mu - a, \mu + a]$. In this case the median-of-three also provides for an unbiased estimator as shown above. What is the variance of this estimator then?

1.3.1 Mean-based

Let's first examine the results of a mean-based estimation of μ. Call $\hat{\mu}_N$ the estimator generated by averaging N samplings, then we know that $\langle \hat{\mu}_N \rangle = \mu$ by linearity of expectation and $\sigma^2_{\hat{\mu}_N} = \frac{\sigma^2_X}{N}$ by linearity of variance, so it remains to compute σ^2_X, which is given by

$$\sigma^2_X = \langle X^2 \rangle - \langle X \rangle^2 \tag{1.8}$$

$$= \int_{\mu-a}^{\mu+a} \frac{1}{2a} x^2 \, \mathrm{d}x - \mu^2 \tag{1.9}$$

$$= \frac{6\mu^2 a + 2a^3}{6a} - \mu^2 \tag{1.10}$$

$$= \frac{a^2}{3} \tag{1.11}$$

Thus, $\sigma^2_{\hat{\mu}_N} = \frac{a^2}{3N}$.

1.3.2 Median-based, $N = 3$

Now for the median-based approach. Denote $\tilde{\mu}_N$ the estimator generated by taking the median of N samplings, then we know that $\langle \tilde{\mu}_N \rangle = \mu$ nonetheless because the uniform PD is a symmetric probability distribution. It thus remains to compute $\left\langle \sigma^2_{\tilde{\mu}_N} \right\rangle$. This seems

nontrivial, so let's start with $N = 3$:

$$\left\langle \tilde{\mu}_3^2 \right\rangle = \int_{-\infty}^{\infty} 6 f_X(x) F_X(x)(1 - F_X(x))\, x^2 \mathrm{d}x \tag{1.12}$$

$$= \int_{\mu-a}^{\mu+a} \frac{6}{2a} \frac{x-(\mu-a)}{2a} \frac{(\mu+a)-x}{2a} x^2 \,\mathrm{d}x \tag{1.13}$$

$$= \int_{-a}^{a} \frac{6}{2a} \frac{a+y}{2a} \frac{a-y}{2a} (y+\mu)^2 \,\mathrm{d}y \tag{1.14}$$

$$= \int_{-a}^{a} \left[\frac{6}{8a^3} \left(a^2 y^2 + a^2 2y\mu + a^2\mu^2 - y^4 - 2\mu y^3 - y^2\mu^2 \right) \right] \mathrm{d}y \tag{1.15}$$

$$= \frac{6}{8a^3} \left[\frac{(a^2-\mu^2)y^3}{3} - \frac{y^5}{5} \right]_{-a}^{a} + \frac{3\mu^2}{2} \tag{1.16}$$

$$= \frac{6}{8a^3} \left[\frac{\left(a^2-\mu^2\right)2a^3}{3} - \frac{2a^5}{5} \right] + \frac{3\mu^2}{2} \tag{1.17}$$

$$= \mu^2 + \frac{a^2}{5} \tag{1.18}$$

and so $\sigma^2_{\tilde{\mu}_3} = \frac{a^2}{5}$. Compare this to $\sigma^2_{\hat{\mu}_3} = \frac{a^2}{9}$ and we see that the mean-based estimation has lower uncertainty.

1.3.3 Median-based, arbitrary N

Armed with this, let's also try to compute for arbitrary, odd $N = 2m + 1$, for which we have

$$\left\langle \tilde{\mu}_N^2 \right\rangle = N \binom{2m}{m} \int_{-a}^{a} \frac{1}{2a} \left(\frac{a^2 - y^2}{4a^2} \right)^m (y+\mu)^2 \,\mathrm{d}y \tag{1.19}$$

Now, there's probably a cool combinatorial way to evaluate this, but let's just care about asymptotic behavior. Then

$$\lim_{N\to\infty} \left\langle \tilde{\mu}_N^2 \right\rangle \approx N \frac{2^{2m}\sqrt{2m}}{m} \int_{-a}^{a} \frac{1}{2a} \frac{1}{4^m} \left(1 - \frac{y^2}{a^2} \right)^m (y+\mu)^2 \,\mathrm{d}y \tag{1.20}$$

$$\approx \frac{1}{2a}\sqrt{8m} \int_{-a}^{a} \left(1 - \frac{y^2}{a^2} \right)^m (y+\mu)^2 \,\mathrm{d}y \tag{1.21}$$

where we approximate $N \approx 2m$. Now, we know that $\left(1 - \frac{y^2}{a^2}\right)^m$ is going to fall off sharply to 0

as y increases, so we can approximate (for some normalization factor A)

$$\int_{-a}^{a}\left(1-\frac{y^2}{a^2}\right)^m \mathrm{d}y \sim A\int_{-a/\sqrt{m}}^{a/\sqrt{m}} 1-\frac{my^2}{a^2}\,\mathrm{d}y \tag{1.22}$$

$$\lim_{N\to\infty}\left\langle\tilde{\mu}_N^2\right\rangle \approx \frac{A}{2a}\sqrt{8m}\int_{-a/\sqrt{m}}^{a/\sqrt{m}}\left(1-\frac{my^2}{a^2}\right)(y+\mu)^2\,\mathrm{d}y \tag{1.23}$$

To compute A, we require that the coefficient of μ^2 be 1 so that the difference $\left\langle\tilde{\mu}_N^2\right\rangle-\left\langle\tilde{\mu}_N\right\rangle^2$ does not depend in first order on μ. It's clear that since the integral is symmetric, we need only consider even powers of y, and so our integral becomes

$$\lim_{N\to\infty}\left\langle\tilde{\mu}_N^2\right\rangle = \frac{A\sqrt{8m}}{2a}\int_{-a/\sqrt{m}}^{a/\sqrt{m}}\left(1-\frac{my^2}{a^2}\right)\left(y^2+\mu^2\right)\,\mathrm{d}y \tag{1.24}$$

$$= \frac{A\sqrt{8m}}{2a}\int_{-a/\sqrt{m}}^{a/\sqrt{m}}\mu^2+\left(1-\frac{m\mu^2}{a^2}\right)y^2-\frac{my^4}{a^2}\,\mathrm{d}y \tag{1.25}$$

$$= \frac{A\sqrt{8m}}{2a}\left[\frac{2\mu^2 a}{\sqrt{m}}+\left(1-\frac{m\mu^2}{a^2}\right)\left(\frac{2}{3}\frac{a^3}{m^{3/2}}\right)-\frac{2ma^5}{5a^3m^{5/2}}\right] \tag{1.26}$$

$$= A\frac{\sqrt{32}}{3}\mu^2+A\frac{4\sqrt{2}}{15}\frac{a^2}{m} \tag{1.27}$$

and so we find that $A=\frac{3}{\sqrt{32}}$ and finally

$$\sigma^2_{\tilde{\mu}_N^2} = \frac{a^2}{5m} \tag{1.28}$$

The agreement for $N=3,m=1$ is a bit uncanny, but let's try to verify this computationally before jumping for joy.

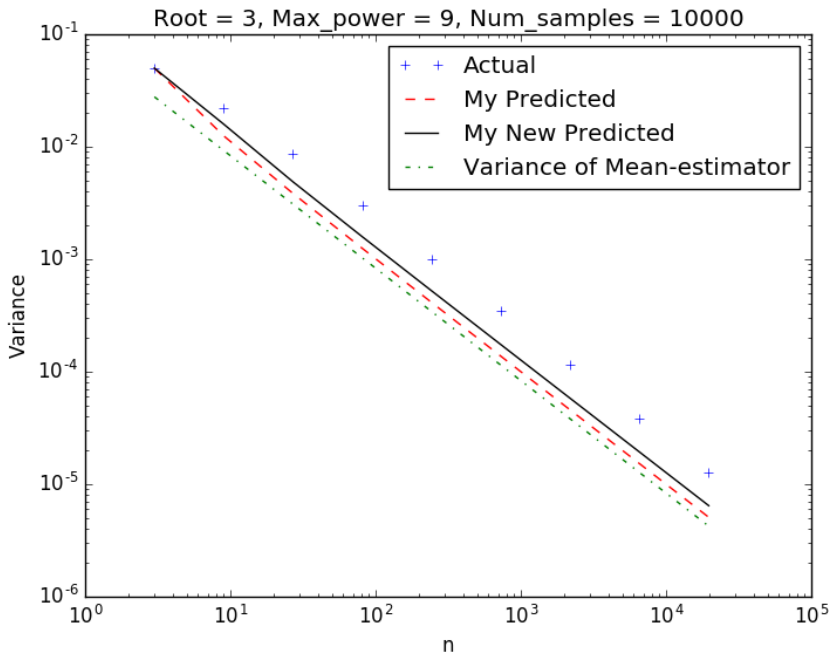
This is a polynomial relationship on m, so we can sample m logarithmically to computationally verify our result. The obtained results are as follows in Figure 1.1.

The histogram is plotted merely out of curiosity, but seems to suggest a normal distribution per the Law of Large Numbers. Nonetheless, Equation 1.28 seems to be slightly off. It perfectly agrees in the $N=3$ case as can be verified by simulation, but eventually grows to be a factor of approximately 2 off.

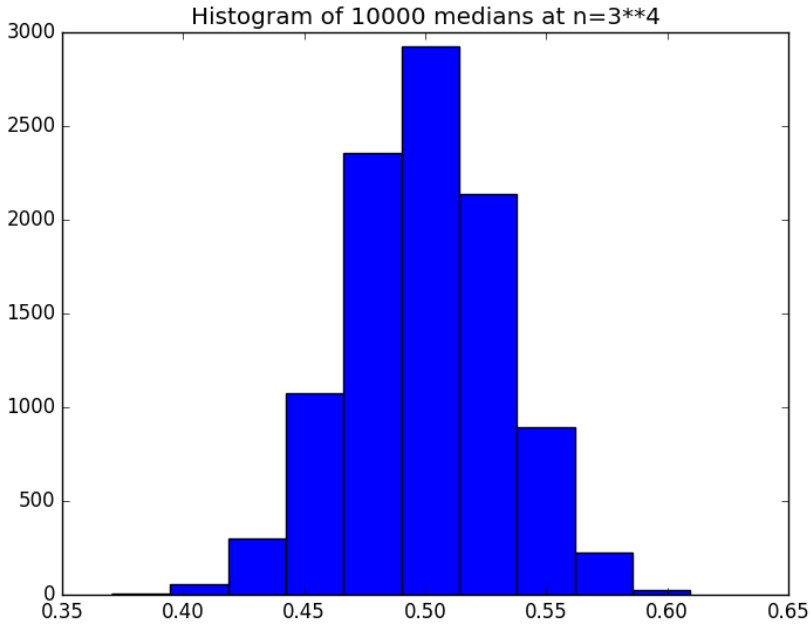
So it turns out our uncanny success for $N=3,m=1$ was a pure stroke of luck, and our expression isn't precisely correct. Nonetheless, we can make a few plots to figure out numerically how well median vs. mean based averaging performs, and the degredation of our estimate over N. These plots are

1.3.4 Median-based, arbitrary N, reworked

Let's try to include the truncated terms in $\left(1-\frac{y^2}{a^2}\right)^m$, since they really are rather non-small compared to the leading term that we kept. Where we had before put $1-\frac{my^2}{a^2}$, we should



(a) Plot of medians as a function of N



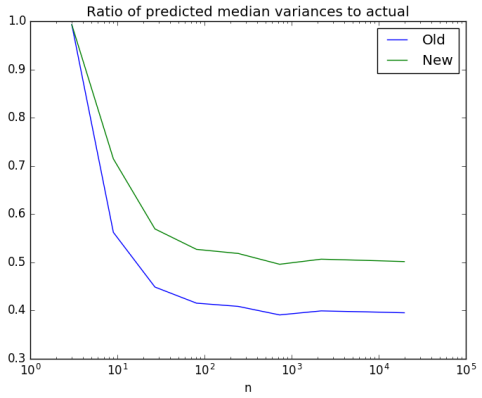
(b) Histogram of 1000 medians at a single value of N

Figure 1.1: Computational results for our medians result. Used $\mu = 0.5, a = 0.5$, or a uniform sampling $[0,1]$. Sampled over $n = 3^{[1,9]}$ with 10000 samples at each value of n.

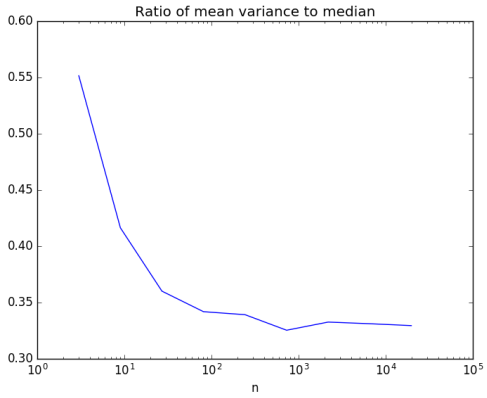
instead put

$$\left(1-\frac{y^2}{a^2}\right)^m = \sum_{k=0}^{m}\binom{m}{k}\left(-\frac{y^2}{a^2}\right)^k \tag{1.29}$$

$$\lim_{N\to\infty}\langle\tilde{\mu}_N^2\rangle \approx \frac{A}{2a}\sqrt{8m}\int_{-a/\sqrt{m}}^{a/\sqrt{m}}\sum_{k=0}^{m}\binom{m}{k}\left(-\frac{y^2}{a^2}\right)^k(y+\mu)^2\,\mathrm{d}y \tag{1.30}$$



(a) Seems asymptotically close to 2/5, 1/2 maybe?



(b) Seems asymptotically close to 1/3?

Figure 1.2: A couple ratios of interest. Same sampling as in Figure 1.1.

We approximate $\binom{m}{k} \approx \frac{m^k}{k!}$ since higher terms in k are attenuated anyways. Using the same parity argument to kill the term odd in y, we rewrite

$$\lim_{N\to\infty} \left\langle \tilde{\mu}_N^2 \right\rangle \approx \frac{A}{2a}\sqrt{8m} \int_{-a/\sqrt{m}}^{a/\sqrt{m}} \sum_{k=0}^{m} \frac{m^k}{k!} \left(-\frac{y^2}{a^2}\right)^k (y^2+\mu^2)\,\mathrm{d}y \tag{1.31}$$

Examine first the μ^2 coefficient

$$1 = \frac{A}{2a}\sqrt{8m} \sum_{k=0}^{m} \frac{m^k}{k!} \int_{-a/\sqrt{m}}^{a/\sqrt{m}} \left(-\frac{y^2}{a^2}\right)^k \mathrm{d}y \tag{1.32}$$

$$= \frac{A}{2a}\sqrt{8m} \sum_{k=0}^{m} \frac{m^k}{k!(2k+1)} 2\left(\frac{a}{m^{k+1/2}}\right)(-1)^k \tag{1.33}$$

$$= A\sqrt{8} \sum_{k=0}^{m} \frac{(-1)^k}{k!(2k+1)} \tag{1.34}$$

and the other term

$$\lim_{N\to\infty} \sigma_{\tilde{\mu}_N}^2 \approx \frac{A}{2a}\sqrt{8m} \int_{-a/\sqrt{m}}^{a/\sqrt{m}} \sum_{k=0}^{m} \frac{m^k}{k!} \left(-\frac{y^2}{a^2}\right)^k y^2\,\mathrm{d}y \tag{1.35}$$

$$= \frac{A}{2a}\sqrt{8m} \sum_{k=0}^{m} \frac{m^k}{k!(2k+3)} 2\left(\frac{a^3}{m^{k+3/2}}\right)(-1)^k \tag{1.36}$$

$$= \frac{A\sqrt{8}a^2}{m} \sum_{k=0}^{m} \frac{(-1)^k}{k!(2k+3)} \tag{1.37}$$

$$= \frac{a^2}{m} \frac{\sum_{k=0}^{m} \frac{(-1)^k}{k!(2k+3)}}{\sum_{k=0}^{m} \frac{(-1)^k}{k!(2k+1)}} \tag{1.38}$$

and we find that we reproduce our previous result for $N = 3$. Crunching the numbers, we get something slightly better, though since factorials fall off so quickly the change is very slight. The results are shown in Figure 1.2.

1.3.5 Further ruminations (TBC)

The approximation where we took the integral over interval $[-a/\sqrt{m}, a/\sqrt{m}]$ seems to be the last point of contention, as it bears noting that if we allow a degree of freedom in the choice of range $\left[-Ba/\sqrt{m}, Ba/\sqrt{m}\right]$ that our choice of B propagates as a factor of B^{2k+3} to the summation in the numerator of Equation 1.38 and B^{2k+1} to the summation in the denominator. Thus, our choice of B has nontrivial implications on the exact prefactor we obtain.

1.4 Open Questions

- Is there any way to find the missing factor on median-based averaging for a uniform-distribution and arbitrary N?
- If we have discretized measurements, what are the statistics of mode-based averaging?
- Did I actually normalize the median-based averaging correctly, for a general probability distribution?

1.5 Delta Function Properties

I can never remember these, so I'll derive some of them below (used in PHYS7653 HW2, FA 2019). These are usually used for marginalizing over delta function PDFs, so I'm not going to include any nontrivial integrand multiplying the delta.

- $\int \delta(*)x - x_0 \,\mathrm{d}x = 1$ axiom.
- $\int \delta(*)cx - y_0 \,\mathrm{d}x = \int \delta(*)y - y_0 \,\mathrm{d}\frac{y}{c} = \frac{1}{c}$.
- $\int \delta(*)f(x) - f_0 \,\mathrm{d}x = \int \delta(*)f(x_0) - f_0 + f'(x_0)(x - x_0) \,\mathrm{d}x$ where $f(x_0) = f_0$, then applying above $= \frac{1}{f'(x_0)}$.
- $\int \delta(*)f(x, y_i) - f_0 \,\mathrm{d}y_i$: this seems a little bit trickier, but I've actually included the generalization in section 6.1, it hinges on

$$\int \delta(*)z - f(x_i) \,\mathrm{d}^N x_i = \int_{\mathscr{S}} \frac{1}{\hat{n} \cdot \vec{\nabla} f(x_i)} \,\mathrm{d}^{N-1} x_i. \tag{1.39}$$

  Note Wikipedia states the denominator as $|*|\vec{\nabla} f(x_i)$, which is equivalent to our definition since $\hat{n}$ is the direction of steepest descent from $f(x_i)$; we will use their definition for clarity. We denote $\mathscr{S}$ to be the $N-1$ surface that satisfies constraint $f(x_i) = z$. This is always a little bit tricky to apply, since the units on the delta function matter,

but using an integrand of $\delta(*)\sqrt{\sum_i x_i^2} - r$ on the left hand side yields exactly 1 for the integrand on the right hand side.

Then the desired form is just

$$\int \delta(*) f(x, y_i) - f_0 \, \mathrm{d}^N y_i = \int\limits_{f(x,y_i)=f_0} \frac{1}{\left|\vec{\nabla} f(*) x, y_i\right|} \, \mathrm{d}^{N-1} y_i. \tag{1.40}$$

Note that the dot product in the denominator of the RHS must run over x as well.

- $\int \delta(*) f(x,y) - f_0 \delta(*) g(x,y) - g_0 \, \mathrm{d}x \, \mathrm{d}y$: didn't end up using this one.

Chapter 2

Feynman-style number theory

In case you have not yet seen `http://www.lbatalha.com/blog/feynman-on-fermats-last-theorem` yet, it's quite a fun read! Would recommend. That sort of thinking inspired this section.

2.1 Asymptotic behavior of primes

Call $\Pi(N)$ the prime number counting function, how many primes are below N. The Prime Number Theorem is a well known result that postulates two approximations to $\Pi(N)$:

$$\Pi(N) \approx \frac{N}{\log N} \approx \int_2^N \frac{1}{\log x}\,\mathrm{d}x \tag{2.1}$$

We will attempt to derive the latter approximation. Consider $P(N)$ the probability density that N is a prime, roughly the statement "if I randomly choose a number near N, what is the probability it is a prime?" The relationship between $P(N)$ and $\Pi(N)$ is then

$$P(N) = \frac{\mathrm{d}\Pi}{\mathrm{d}N} \tag{2.2}$$

To attempt to derive $P(N)$, consider that a number N is prime iff it is not divisible by any primes less than it. Thus, we have that

$$P(N) \approx \prod_{p \in primes}^{N} \left(1 - \frac{1}{p}\right) \tag{2.3}$$

Taking a leap of faith, we recognize that two consecutive contributions to the product above differ roughly by $\frac{1}{P(p)}$, the local inverse probability density that p is prime. Thus, we can rewrite each contribution as $\frac{1}{P(p)}$ contributions of $\left(1 - \frac{1}{p}\right)^{P(p)}$, and then allow p to run over all integers. We thus propose the approximation

$$P(N) \approx \prod_{k=2}^{N} \left(1 - \frac{1}{k}\right)^{P(k)} \tag{2.4}$$

Taking the logarithm of both sides, we obtain

$$\log P(N) = \sum_{k=2}^{N} P(k) \log\left(1 - \frac{1}{k}\right) \tag{2.5}$$

Approximating the right hand side with an integral, we obtain

$$\log P(N) = \int_{2}^{N} P(k) \log\left(1 - \frac{1}{k}\right) \mathrm{d}k \tag{2.6}$$

Differentiating both sides now, we obtain

$$\frac{P'(N)}{P(N)} = P(N) \log\left(1 - \frac{1}{N}\right) \tag{2.7}$$

$$\frac{\mathrm{d}P}{\mathrm{d}N} = P^2 \log\left(1 - \frac{1}{N}\right) \tag{2.8}$$

$$\frac{\mathrm{d}P}{P^2} = \mathrm{d}N \log\left(1 - \frac{1}{N}\right) \tag{2.9}$$

$$-\frac{1}{P} = N \log\left(1 - \frac{1}{N}\right) - \log(N-1) \tag{2.10}$$

$$P(N) = \frac{1}{\log(N-1) + O(1)} \tag{2.11}$$

$$\approx \frac{1}{\log N} \tag{2.12}$$

This recovers the expression $\Pi(N) = \int_{2}^{N} P(N)\,\mathrm{d}N = \int_{2}^{N} \frac{1}{\log N}\,\mathrm{d}N$.

2.2 Scratch work

What follows is me working out loud, which is a lot less interesting.

It's a well-known result (Prime Number Theorem) that the number of primes below N is approximated by $\Pi(N) = N/\log(N)$. Can we try to get a handle on this behavior via application of continuum analysis?

One way of thinking of the problem is to instead look at it from a probabilistic standpoint, that arbitrarily choosing a number n, it has some probability of being prime. Can we estimate this probability and recover the prime number theorem? We should be able to obtain

$$\frac{\mathrm{d}\Pi}{\mathrm{d}N} \approx \frac{\log N - 1}{\log^2(N)} \tag{2.13}$$

2.2.1 First attempt

Let's consider the probability that some large number N is divisible by some divisor d; this is just $\frac{1}{d}$. We might think that the probability that N is prime then just the product of

probabilities it is not divisible by any number smaller than it

$$P(N) = \prod_{k=2}^{N} \left(1 - \frac{1}{k}\right) \tag{2.14}$$

To try to evaluate this product, we take the logarithm of both sides

$$\log P(N) = \sum_{k=2}^{N} \log\left(1 - \frac{1}{k}\right) \tag{2.15}$$

$$\approx \int_{k=2}^{N} \log\left(1 - \frac{1}{k}\right) \mathrm{d}k \tag{2.16}$$

$$\tag{2.17}$$

To compute this antiderivative, it's easiest to separate the integrand

$$\int \log\left(\frac{k-1}{k}\right) \mathrm{d}k = \int \log(k-1)\,\mathrm{d}k - \int \log k\,\mathrm{d}k \tag{2.18}$$

$$= (k-1)\log(k-1) - k - k\log(k) + k + C \tag{2.19}$$

$$= k\log\left(1 - \frac{1}{k}\right) - \log(k-1) + C \tag{2.20}$$

with C some undetermined constant that becomes irrelevant when we consider the definite integral. Thus, we return to our primary expression

$$\log P(N) \sim N\log\left(1 - \frac{1}{N}\right) - \log(N-1) \tag{2.21}$$

where we drop the evaluation of the antiderivative at $k = 2$ since it's a constant in the scaling. Then, we find

$$P(N) \sim \frac{\left(1 - \frac{1}{N}\right)^N}{N-1} = \frac{1/e}{N-1} \tag{2.22}$$

In fact, a quick google search shows that Equation 2.14 evaluates to $\frac{1}{N}$, and so our result is pretty reasonable; we're off by a constant factor since our integral approximation Equation 2.16 misestimates by a constant factor, no surprise there. So where did we go wrong?

2.2.2 Second attempt

The iusse, as some people smarter than me may have noticed, is that our expression Equation 2.14 is faulty: we should only be multiplying *over primes*! While this is correct, primes are not divisible by any primes smaller than them, it's a bit difficult to handle under our present formalism, where we only attach a probability to a number's being prime or not.

Let's think carefully about how to integrate this into our formalism. If a number k is not prime, it should contribute 1 to our product, and if it is prime then it should contribute $\left(1-\frac{1}{k}\right)$. Since we're doing products, the natural way to "average" is via geometric mean, so we modify expression Equation 2.14 to

$$P(N)=\prod_{k=2}^{N}\left(1-\frac{1}{k}\right)^{P(k)} \tag{2.23}$$

where we average each k-th contribution as $\left(1-\frac{1}{k}\right)^{P(k)}(1)^{1-P(k)}$ geometric mean[1]. Doing the usual trick,

$$\log P(N)=\int_{2}^{N}P(k)\log\left(1-\frac{1}{k}\right)\mathrm{d}k \tag{2.24}$$

Differentiating both sides,

$$\frac{P'(N)}{P(N)}=P(N)\log\left(1-\frac{1}{N}\right) \tag{2.25}$$

$$\frac{\mathrm{d}P}{\mathrm{d}N}=P^2(N)\log\left(1-\frac{1}{N}\right) \tag{2.26}$$

$$\frac{\mathrm{d}P}{P^2}=\log\left(1-\frac{1}{N}\right)\mathrm{d}N \tag{2.27}$$

$$-\frac{1}{P}=N\log\left(1-\frac{1}{N}\right)-\log(N-1) \tag{2.28}$$

$$P(N)\approx\frac{1}{\log N} \tag{2.29}$$

Interestingly, this expression is a better approximation to $\Pi(N)$ than the aforementioned $\Pi(N)\approx\frac{N}{\log(N)}$, so it looks like this is a satisfactory conclusion, namely that

$$\Pi(N)\approx\int_{2}^{N}\frac{1}{\log(m)}\,\mathrm{d}m \tag{2.30}$$

However, we pursue one last direction of thought out of curiosity.

2.2.3 Third attempt

In Equation 2.23, maybe we only need to check up until $\sqrt{N}$ in the product. Continuing our thought above, we obtain

$$\frac{P'(N)}{P(N)}=P(\sqrt{N})\log\left(1-\frac{1}{\sqrt{N}}\right) \tag{2.31}$$

$$\approx-\frac{P(\sqrt{N})}{\sqrt{N}} \tag{2.32}$$

[1]Intuitively, this means that we need to multiply $\frac{1}{P(k)}$ of these factors before getting a single one that contributes, i.e. the distance between primes.

At this point, our expression doesn't seem particularly amenable to solution, but we can at least check how well $P(N) \sim \frac{1}{\log N}$ works:

$$\frac{-\frac{1}{N\log^2 N}}{\frac{1}{\log N}} = -\frac{2}{\sqrt{N}\log N} \tag{2.33}$$

$$-\frac{1}{N\log N} = \frac{2}{\sqrt{N}\log N} \tag{2.34}$$

which doesn't seem to work too well. How about the original estimate $P(N) \sim \frac{\log N - 1}{\log^2 N}$?

$$\frac{\frac{2-\log N}{N\log^3 N}}{\frac{\log N - 1}{\log^2 N}} = -\frac{\frac{\log\sqrt{N}-1}{\log^2\sqrt{N}}}{\sqrt{N}} \tag{2.35}$$

$$\frac{2-\log N}{N\log N(\log N - 1)} = \frac{2(2-\log N)}{\sqrt{N}\log^2 N} \tag{2.36}$$

which is even worse. The obvious problem is that the $\sqrt{N}$ has nowhere tho go since the probability density P depends only on the logarithm of N. So interesting, considering the further optimization of only going up to $\sqrt{N}$ ruins the accuracy of our prediction!

Chapter 3

Ellipsoidal surface areas

We all know that ellipses do not have a closed form for their arclength, but their enclosed area is well defined, namely $A = \pi ab$. This can be seen by defining an ellipse as a projection of a circle by unevenly scaling the axes, and noting that an area element $\mathrm{d}x\mathrm{d}y$ scales linearly with the projection factors.

One series approximation to the arclength can be computed by noting the following: if S is the arclength of an ellipse, then $S\mathrm{d}n$ for some small $\mathrm{d}n$ estimates the change in area by enlarging the ellipse.

Systematically, exhibit an ellipse with axis lengths a, b, such that its area is πab. Then, say that we extend both axes by some $\mathrm{d}\epsilon$, then its area becomes $\pi ab + \pi(a+b)\mathrm{d}\epsilon + \mathcal{O}(\mathrm{d}\epsilon^2)$, and the change in area is $\pi(a+b)\mathrm{d}\epsilon + \mathcal{O}(\mathrm{d}\epsilon^2)$. This implies that the arclength of an ellipse to first order is $\pi(a+b)$, which seems to make sense for $a = b$.

This isn't particularly radical, and neither is this entire section, but we can verify it to be reasonable for three dimensions as well:

$$V = \frac{4}{3}\pi abc + \frac{4}{3}\pi(ac + bc + ab)\mathrm{d}\epsilon + \mathcal{O}(\mathrm{d}\epsilon^2) \tag{3.1}$$

$$S = \frac{4}{3}\pi(ac + bc + ab) + \mathcal{O}(\mathrm{d}\epsilon) \tag{3.2}$$

which again agrees with intuition for $a = b = c$

Chapter 4

12/04/16—Musings on Hamiltonian Chaos

We learned in our chaos readings that given an integrable Hamiltonian (can be written in terms of action-angle variables, has N constants of motion for $2N$ dimensional phase space), a small perturbation generally breaks the toroidal phase space trajectory into chaotic motion. Let's see how much of this we can actually understand.

4.1 Action-Angle variables

I don't have my 106 notes handy, so let's rederive some action-angle stuff. The archetypal Hamiltonian to use is the SHO $H = (p^2 + q^2)/2$. While we may have suspicions for the choice of action-angle, we look up that

$$I = \frac{1}{2\pi} \oint \mathrm{d}(pq) \tag{4.1}$$

the integral over one period. For us, let's note that $E = H = \frac{p^2+q^2}{2}$ is a constant of motion, thus we can write

$$p = \sqrt{2E - q^2} \tag{4.2}$$

$$I = \frac{1}{2\pi} \left[2 \int_{-\sqrt{2E}}^{\sqrt{2E}} \sqrt{2E - q^2}\, \mathrm{d}q \right] \tag{4.3}$$

$$= E \tag{4.4}$$

where we recognize the integral to just be the integral of the circle. This makes sense, as we recognize that the action integral I is just the area of phase space enclosed within a full period, which for us is just $2\pi E$ since we enclose a circle in phase space with radius $r^2 = 2E$.

The angle θ must be such that the above expression also holds, i.e.

$$\oint \mathrm{d}(pq) = \oint \mathrm{d}(I\theta) \tag{4.5}$$

so that the phase space volume enclosed in one rotation is the same for both variables. We can then differentiate both sides by I to obtain

$$\theta = \frac{\partial}{\partial I} \oint \mathrm{d}pq. \tag{4.6}$$

Since the $\oint$ depends only on the bounds of integration, we see that θ is simply the limit on the integral, which further algebra shows to be $\arctan \frac{q}{p}$. We can verify that this is canonical by computing the PB

$$\frac{\partial I}{\partial p}\frac{\partial \theta}{\partial q} - \frac{\partial I}{\partial q}\frac{\partial \theta}{\partial p} = p\frac{1}{p}\frac{1}{1+\left(\frac{q}{p}\right)^2} - q\left(-\frac{q}{p^2}\right)\frac{1}{1+\left(\frac{q}{p}\right)^2} \tag{4.7}$$

$$= 1 \tag{4.8}$$

so we're in good shape.

4.2 Multi-dimensional SHOs

In more generality, if we have a multidimensional SHO, we see that the Hamiltonian is just their sum, and so H in terms of action angle variables is still the sum of the actions, while their angles evolve separately.

What is the rate at which the angle evolves? For our above single-dimensional oscillator, it's easy to simply solve the EOM and find that $\frac{q}{p} = \tan t$, and so that the angle evolves with unit frequency. More generally, if the Hamiltonian is of form $H = p^2 + Cq^2$, it is easy to associate $C = \omega^2$ thanks to Hamilton's canonical equations $\dot{p} = H_q, \dot{q} = -H_p$ or something like that up to a sign. Thus, in general our Hamiltonian takes on form

$$H = \frac{\sum_j p_j^2 + \omega_j^2 q_j^2}{2} = \sum_j \omega_j I_j \tag{4.9}$$

with each of the I_j having a corresponding angle θ_j that evolves at ω_j.

4.3 With perturbation (incorrect result and faulty method)

How can we handle the perturbation? I have no idea, but I'll give it a shot. Let's adopt a phase space where each (q_j, p_j) are components of a complex number. Then the I_j are the magnitudes of each component, the θ_j the phases, and the Hamiltonian acts simply to rotate each component. It's easy to write down a system of equations that reproduces this behavior, but can we express it in terms of the Hamiltonian? Put another way, is there a way we can cast the $2N$-dimensional real Hamiltonian system above into an N-dimensional complex system with similar rules?

The defining property of a Hamiltonian system is Hamilton's canonical equations $\dot{p} = \frac{\partial H}{\partial q}, \dot{q} = -\frac{\partial H}{\partial p}$. If each dynamical variable is instead $z_j = p_j + iq_j$, we instead want a property that looks something like $\dot{z}_j = i\frac{\partial H}{\partial z_j}$[1].

[1] We have made a choice of convention in putting the i with the partial derivative rather than in the Hamiltonian, motivated by keeping the Hamiltonian clean and most in analog with the real-variable Hamiltonian

Specializing to the SHO, given an ω in the SHO, we should make the correspondence $z_j = p_j + i\omega_j q_j$. Then we can write $H = \sum_j \omega_j z_j^2/2$ which gives us results something like $\dot{p}_j = -\omega_j q_j, \omega_j \dot{q}_j = p_j$ which is in accordance with what we expect. Thus, for an SHO we have

$$\dot{z_j} = i\frac{\partial H}{\partial z_j} = i\omega_j z_j \tag{4.10}$$

In other words, the evolution of the system is fully diagonal with eigenvalues ω_j. This is awfully reminescent of quantum mechanics! However, in QM, first order perturbation theory always gives us a result for a new orthogonal basis; why would any tori in classical mechanics break down? What does it even mean for a torus to break down? Why am I so stupid? These are not rhetorical questions but rather live musings as I type this up.

Consider if we then perturb the above EOM to something like

$$\dot{z}_j = i\omega_j z_j + i\epsilon\frac{\partial \delta H(\vec{z})}{\partial z_j} \tag{4.11}$$

Let's do the sensible thing and linearize δH so that $\delta H(\vec{z}) = \delta\mathbf{H}\vec{z}$. We seek a new set of z_j such that the EOM is again diagonal. If we treat the z_j coordinates as vectors in a vector space, we can do this via perturbation theory analogous to QM. Let's write down ansatz for new eigenbasis

$$\vec{z}_j = \vec{z}_j + \sum_k A_{jk}\vec{z}_k \tag{4.12}$$

We then seek that $\frac{\partial(\mathbf{H}+\delta\mathbf{H})(\vec{z}_j')}{\partial z_j'} = \omega_j'\vec{z}_j'$, and so (goodness the algebra below is so so wrong, but suck it up)

$$\frac{\partial \mathbf{H}\vec{z}_j}{\partial z_j} + \frac{\partial \delta\mathbf{H}\vec{z}_j}{\partial z_j} + \frac{\partial}{\partial z_j}\mathbf{H}\sum_k A_{jk}\vec{z}_k = \omega_j z_j\hat{\jmath} + \delta\omega_j\vec{z}_j + \omega_j\sum_k A_{jk}\vec{z}_k \tag{4.13}$$

$$\sum_k (\delta H)_{kj}\hat{k} + \sum_k A_{jk}\omega_k\vec{z}_k = \delta\omega_j z_j\hat{\jmath} + \omega_j\sum_k A_{jk}\vec{z}_k \tag{4.14}$$

$$A_{jk} = \frac{(\delta H)_{jk}}{\omega_j - \omega_k} \tag{4.15}$$

which is in line with the result from QM. I don't trust the algebra but I think the result is reasonable.

Not really sure where to go from here, but I found old lecture notes on the topic so I'll just consult them. Oops. It's been fun!

The correct approach (which we may or may not work through) attacks this from the generating function perspective, computing the Hamilton-Jacobi generating function for the canonical transformation in terms of the small perturbation, then showing that for purely rational ω_j, the perturbation theory fails to converge. It would appear then that the transformation of coordinates we propose above in general is not canonical for a rational winding number or something like that, if H_1 has a sufficiently high frequency component.

The moral of the story is that in classical mechanics, to change coordinates we must approach from a generating function to show that the transformation is canonical. Duly noted.

I continue this discussion in a separate section in my chaos notes, and conclude this discussion to pursue more interesting musings about the classical-quantum correspondence we began to uncover above.

Chapter 5

12/07/16—Musings on Quantum-Classical correspondence

I'll repeat a small amount of the earlier discussion for my own benefit as I write this up.

The definiting characteristic of the Hamiltonian formalism are Hamilton's canonical equations

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad\qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i \tag{5.1}$$

Above, we tried to map $z_i = q_i + ip_i$. However, part of what we did incorrectly in the above section was to assume that $\frac{\partial H}{\partial z_i}$ exists, i.e. H is analytic in the z_i. This is generally not true, obviously, so we need to do some work here.

I still want to try and make some sort of z_i transformation, so let's consider the following: let's try to construct some $\mathscr{H} = H + iH'$ such that $\mathscr{H}$ is an analytic function of the $z_i = q_i + ip_i$. Recall the Cauchy-Riemann equations for differentiability of a function $f(z) = u(z) + iv(z), z = x + iy$:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \qquad\qquad i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} \tag{5.2}$$

and so for us

$$\frac{\partial H'}{\partial p_i} = \frac{\partial H}{\partial x_i} \qquad\qquad \frac{\partial H'}{\partial q_i} = -\frac{\partial H}{\partial p_i} \tag{5.3}$$

We then wish to compute $\frac{\partial \mathscr{H}}{\partial z_i}$. Since $\mathscr{H}$ is analytic, it doesn't matter along which direction in the complex plane we approach z_i, so we choose to fix p_i, and we obtain

$$\left.\frac{\partial \mathscr{H}}{\partial z_i}\right|_{p_i} = \frac{\partial H(q_i, p_i)}{\partial q_i} + i\frac{\partial H'(q_i, p_i)}{\partial q_i} \tag{5.4}$$

$$= \frac{\partial H}{\partial q_i} - i\frac{\mathrm{d}H}{\mathrm{d}p_i} \tag{5.5}$$

Hey, that's super nice, since now we can write $\dot{z}_i = -i\frac{\partial \mathscr{H}}{\partial z_i}$, or in slightly more familiar form

$$i\frac{\partial z_i}{\partial t} = \frac{\partial \mathscr{H}}{\partial z_i} \tag{5.6}$$

But wait! We can do even better. Since for any N-dimensional system, we have N conserved quantities, we can always make canonical transformation for $H(p_i, q_i) \to H(I_i)$ the action-angle transformation. I won't bother checking canonical-ness here, but the new Hamiltonian $H(I_i) = \omega_i I_i$.

Furthermore, from here we can always make a canonical transformation $H(I_i) \to H(a_i, b_i)$ where $a_i = \sqrt{2I_i}\cos\theta, b_i = \sqrt{2I_i}\sin\theta$ with θ the angle variable. Let's verify that this is canonical

$$\frac{\partial a_i}{\partial I_i}\frac{\partial b_i}{\partial \theta_i} - \frac{\partial b_i}{\partial I_i}\frac{\partial a_i}{\partial \theta_i} = \left(\frac{\cos\theta}{\sqrt{2I_i}}\right)\left(\sqrt{2I_i}(-\cos\theta)\right) - \left(\frac{\sin\theta}{\sqrt{2I_i}}\right)\left(\sqrt{2I_i}\sin\theta\right) \tag{5.7}$$

$$= -1 \tag{5.8}$$

which means the Hamiltonian becomes

$$H = \sum_i \frac{\omega_i}{2}(a_i^2 + b_i^2) \tag{5.9}$$

which is basically an SHO in a_i, b_i. But what's nice about this formulation in particular is that

$$\frac{\partial \mathcal{H}}{\partial z_i} = \frac{\partial H}{\partial a_i} - i\frac{\partial H}{\partial b_i} \tag{5.10}$$

$$= \omega_i(a_i + ib_i) = \omega_i z_i \tag{5.11}$$

and ergo we obtain

$$i\frac{\partial z_i}{\partial t} = \omega_i z_i \tag{5.12}$$

which is exactly the Schödinger equation in the energy eigenbasis! In other words, write the matrix $H_{QM} = \text{diag}(\hbar\omega_i)$, then we have

$$i\hbar\frac{\partial \vec{z}}{\partial t} = H_{QM}\vec{z} \tag{5.13}$$

I'm off by a sign, which is simply a convention of the definition of z_i, or alternatively a definiton on i. But we can start to see the emergence of the QM interpretation: there are certain dynamical orbits that are invariant under evolution by the Hamiltonian, and these correspond to the energy levels in QM that have energy $\hbar\omega$ associated with them.

It should be noted in hindsight that our jubilance is slightly overeager: we have showed that the Hamiltonian in classical mechanics can be interpreted to dictate each component of $\vec{z}$ in a way that corresponds to ω_i the rate of change of the angle variable, but we have in no way derived any energy levels or the sort. This should not surprise us, as the quantization of energy levels is a purely quantum mechanical phenomenon, but the resemblance in Equation 5.13 is still slightly superficial.

5.1 Significance of $\frac{\partial \mathscr{H}}{\partial z_i} \Rightarrow H_{QM}\vec{z}$

Note that to arrive at (5.13), we made what amounted to the correspondence $\frac{\partial \mathscr{H}}{\partial z_i} \Rightarrow \left(H_{QM}\vec{z}\right)_i$ up to $\hbar$. This result makes perfect sense when H is a quadratic sum of the q_i, p_i, but how the hell do the other operators enter the picture?

It would seem that the key identification we must make here is how the complex $\vec{z}$ vector relates to dynamical measurements in classical mechanics, since the correspondence between $\vec{z}$ and state vector $|\psi\rangle$ is clear. In order to measure, say, the q_j position of the system given $\vec{z}$, we need simply to compute the change-of-basis operation from the $z_i = a_i + ib_i$ to the q_i, p_i basis, then project out the jth index.

Another way to extract q_j that perhaps seems more promising is to construct the generating function S (which exists for all canonical transformations). Let's choose the generating function $S(a,b)$ such that $q = S_a, p = S_b$. It seems clear that we can use the same complexification trick above to obtain a $\mathscr{S}$ such that

$$q + ip = \frac{\partial \mathscr{S}(a+ib)}{\partial(a+ib)} \tag{5.14}$$

Let's call $w = q + ip$, then we find that the q_j component can be computed by computing

$$q_j = \mathrm{Re}\left(-i\hat{w}_j \cdot \frac{\partial \mathscr{S}}{\partial \vec{z}}\right) \tag{5.15}$$

where I use the shorthand $\left[\frac{\partial \mathscr{S}}{\partial \vec{z}}\right]_j = \frac{\partial \mathscr{S}}{\partial z_j}$. Compare this to the QM result, which would be $\langle q_j | \psi \rangle$, where in order to evaluate the dot product we must compute the components of ψ in the $\{q_j\}$ basis. We now interpret $\frac{\partial \mathscr{S}}{\partial \vec{z}}$ to be the $\vec{z}$ under the basis tranformation dictated by $\mathscr{S}$. The quantum mechanical equivalent of this operator would be $\sum_k |q_k\rangle\langle q_k|$ the resolution of identity but also the change of basis operator.

There seems to be at first glance a trend here: quantum mechanical operators tend to correspond to classical functions that act on a variable by *differentiating* about that variable. In other words, the classical correspondence of some operator Ω that acts on a wavefunction $|\psi\rangle$ is $\Omega = \frac{\partial \Omega}{\partial [\,]}$ to be such that $\Omega\vec{v} = \frac{\partial \Omega}{\partial \vec{v}}$ and where Ω is some complexified analytic function of its real/Hamiltonian counterpart. We have seen this to be true for the resolution of identity and for the Hamiltonian itself.

Let's note that, heuristically, this explains the mapping of Poisson Brackets to commutators. Recall that a set of coordinates is classically canonical if its PB with q, p is unity, and a set of variables is conjugate if their commutator is $i\hbar$. These are both phase space-preserving constraints, so it's clear that the latter is really the stipulation that the commutator of conjugate variables must equal $[X, P] = i\hbar$, just as the PB is just that the change of coordinates has Jacobian with unit determinant. Let's try to flesh this out a bit more.

5.2 Classical Observables and Quantum Mechanical Operators

Let's formalize the above a little bit. When we speak of a classical dynamical variable a, it means that given a full specification of a state $\{x_i, p_i\}$, we can measure a for the given state.

Similarly, when we speak of a quantum mechanical Hermitian operator (an observable) A, we mean that given a $|\psi\rangle$ we can determine the probability of measuring A given $|\psi\rangle$ and observing any eigenvalue λ of A, the probability $|\psi\rangle$ is eigenstate $|\lambda\rangle$.

Under our above formalism, we mean the following: when we speak of a classical dynamical variable a, we mean that the value of a in a classical state described by the complexified $\vec{z}_i$ can be computed by

$$a = \mathrm{Re}\left(-i\hat{a}\cdot\frac{\partial\mathscr{S}}{\partial\vec{z}}\right) \tag{5.16}$$

where $\mathscr{S} = S + iS'$, S being the generating function describing a canonical transformation from $\{x_i, p_i\}$ to some set of canonical variables involving a and S' such that $\mathscr{S}$ is analytic.

For instance, if we want to compute x_j, we note that the generating function has form $S = x_i p_i$, and so

$$\left[\frac{\partial\mathscr{S}}{\partial\vec{z}}\right]_j = \frac{\partial\mathscr{S}}{\partial z_j} \tag{5.17}$$

$$= p_j + ix_j \tag{5.18}$$

$$x_j = \mathrm{Re}(-i*(p_i + ix_j)) = x_j \tag{5.19}$$

The insertion of the $-i$ is a bit inconvenient, and clearly would have been different were we trying to measure p_j. This seems a bit inconvenient, until we remember that we can probably absorb these factors into $\mathscr{S}$ by defining different operators; indeed, X, P are different operators in QM, and there seems to be little reason for us to use the same $\mathscr{S}$ for them!

Thus, under our formalism, classical dynamical variables a map to second-order tensors (matricies), which is again right for our correspondence.

As one last parting shot of the merit of our formalism, let's observe that if we define a canonical transformation $\{x_i, p_i\} \to \{z_i, z_i^*\}$ a classic trick, where $z_i = \frac{x_i + p_i}{\sqrt{2}}$, we obtain that the PB is equal to i. The significance of this eludes me, but certainly the agreement with QM up to an omnipresent $\hbar$ ought to be encouraging?

5.3 Normalizing the state vector

An astute reader (all zero of you) would have objected a while ago that our Hamiltonian only superficially resembled a quantum mechancical one, since as I mentioned, the frequencies above correspond to the rate of change of the action variable, which is a constant of the Hamiltonian and the *system*, not the state of the system. Contrast this with quantum mechanics, where the $\hbar\omega$ eigenvalues of the Hamiltonian depend on the *state* of the system equally well as the Hamiltonian/configuration of the system itself.

We can see why we err: the ω above does not change when the amplitude of motion changes, but the energy of the system clearly increases. To better understand where this comes from, let's focus on a single dimension for now, so two dimensions of phase space. If we recall how we wrote down the formula to measure x_j above, it relied on the fact that the x_j component of some vector was equal to the x_j position of the taste. But $x_j \in (-\infty, \infty)$, so

this won't do if we try to normalize the state vector. The only way we can get around this is by turning the single x_j component into an infinity of components, one for each value of x_j. Then we can ask whether the system's x_j coordinate is some x_0 value by examining whether the x_0 component is nonzero! (Now we see why we prefer to discuss only a single dimension for now...)

Consider our system to be in a state $z_0 = x_0 + ip_0$, and map it to a continuous function $\psi(z = x + ip) = \delta(x - x_0)\delta(p - p_0)$. It is easy to verify that ψ is normalized $\int |\psi^2| = 1$, and we may measure the position of ψ by integrating it against an operator

$$x_0 = \int x |\psi|^2 \, \mathrm{d}x\mathrm{d}p \tag{5.20}$$

Lest I be accused of cherry-picking my normalization criterion for ψ, I remind you this is the standard normalization procedure for vectors. My argument above holds equally well whether using $\int \psi$ or $\int |\psi|^2$, since we're only dealing with δ functions. So I guess it's a tossup.

We can find similar expressions to measure the energy etc. of the system, and we see that we have moved from a $\mathbb{C}^n$-component vector representation of the state of the system to a representation that has unit norm albeit is a continuous function on $\mathbb{C}^n$. One can easily contest that this is an obfuscation of the physics, but is a natural starting ground for quantum mechanical states, where the phase space is discretized and we can have linear superpositions of states.

5.4 Afterword

We are officially at an impasse: without foresight of the quantization that quantum mechanics brings, without supposing that systems have finite precision in conjugate dynamical variables, it is impossible to proceed. These assumptions have no place in a classical worldview, and it seems that we have largely exhausted the range of speculations we can accomplish. But what a rich theory it is already! We have shown that it is possible to

- Map a $2N$ real phase space to a normalized state vector without losing any expressiveness.
- Proposed a second-order tensor form for all classical dynamical variables that act similarly to quantum observables.
- Accidentally come across the canonical commutation relation and an eerily remeniscent version of the Schrödinger Equation.

A symplectic manifold is a system endowed with an antisymmetric two-form, called the symplectic form. I think that this association of the antisymmetric two-form with a complexified function is generalizable and perhaps broadly applicable as a useful way of thinking about these two-forms. Of course, n-forms are probably just generalizations of the complex numbers, on second thought, but alas that is far beyond the scope of my intellect.

With this I close my analysis. Thanks for listening!

Chapter 6

12/14/16—Matricies

We saw earlier in quantum chaos that the way that the continuous energy spectrum breaks down into discrete eigenvalues (which governs the power spectrum of the trajectory) is described by random matrix theory, specifically the distribution of gaps between eigenvalues of random matricies follows the same distribution. Let's see how much ground we can make on these results.

6.1 Prelim: Distributions of combinations of random variables

Suppose we want the difference of two random variables X, Y. For simplicity, let's say that they're normally distributed. Of course, if we're just talking expectation and variance then this is a trivial problem; how about the shape of the distribution? Differences of normal variables are also normally distributed, but let's see one way we can describe this. Call $Z = X - Y$, with both X, Y chosen with zero mean and standard deviation σ, then we should obtain

$$P_Z(z) = \iint P_X(x) P_y(y) \delta(x - y - z)\, \mathrm{d}x \mathrm{d}y \tag{6.1}$$

$$= \int_{-\infty}^{\infty} P_X(x) P_y(x - z)\, \mathrm{d}x \tag{6.2}$$

and if we specialize $P_X = P_Y = N(0, \sigma)$ then (dropping normalizations)

$$P_Z(z) \propto \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-\frac{(x-z)^2}{2\sigma^2}}\, \mathrm{d}x \tag{6.3}$$

$$\propto \int_{-\infty}^{\infty} \exp\left[\frac{-(2x^2 - 2xz + z^2)}{2\sigma^2}\right] \mathrm{d}x \tag{6.4}$$

$$\propto \int_{-\infty}^{\infty} e^{-\frac{2(x - z/2)^2}{2\sigma^2}} e^{-\frac{z^2/2}{2\sigma^2}}\, \mathrm{d}x \tag{6.5}$$

and since the former exponent integrates to a constant and the latter a Gaussian of width $\sigma\sqrt{2}$, we haeve our answer. It is worth noting the former exponent integrates to a constant with an extra factor of $\sqrt{2}$ that provides exactly the correct normalization for the latter.

In fact, this is trivially generalizable. If we want the distribution of a random variable $Z = f(\{X_i\})$ with each X_i obeying its own probability distribution $P_i(x_i)$, the distribution is simply

$$P_Z(z) = \int \delta(z - f(\{x_i\})) \prod_i P_i(x_i)\, \mathbf{d}^N \vec{x} \tag{6.6}$$

$$= \int \frac{1}{\hat{n} \cdot \nabla f(x_i)} \prod_i P_i(x_i)\, \mathrm{d}^{N-1} \vec{x} \tag{6.7}$$

Thus, for any old $Z = f(X_i)$ we theoretically know how to compute the distribution $P_Z(z)$. What if the probabilities are not independent though, or similarly what if $f(X_i)$ is not differentiable? One such tricky example is $Z = \max_i X_i$. Note that this problem is very similar to our eventual problem of finding the distribution of *differences* in eigenvalues, though perhaps even more non-differentiable.

One interesting way to tackle this is to solve for the $U(0,1)$ uniform distribution on the unit interval case, then apply to an arbitrary PDF by mapping the two CDFs. This only works for a small number of distribution functions at an analytic level however.

Instead, let's attack from this perspective: let's consider $P_Z(z)$ subject to the assumption that $X_1 > X_2$. This is the probability $P_Z(z|X_1 \geq X_2)$, which is simply $P_X(x|X \geq X_2) = \frac{P_X(x)}{P(x_2 \geq x)}$. If we then define $C_X(x_0) = P_X(x \geq x_0)$ the CDF, the form of the solution is simply

$$P_X(x|X > X_2) = \frac{P_X(x)}{C_X(x)} \tag{6.8}$$

More generally, if we have N iid variables distributed as $P_X(x)$, and we want to compute their successive differences, then to compute the kth largest value X_k where the X are ordered, we have

$$P_k(x_k) = \frac{P_X(x)}{C_X^{k-1}(x)(1 - C_X(x))^{N-k+1}} \tag{6.9}$$

6.2 Back to matricies

We've now laid out how to compute the distribution of a difference of two random variables as well as the how to leverage the CDF to compute the kth largest in a set of random variables. This is more than sufficient to generate the distribution of differences of random variables; we really only need to compute the distribution of the differences between two eigenvalues! Oops, did a bit of extra work, though it was pretty fun.

We now must turn to exactly what the distribution of eigenvalues looks like.

UNFINISHED.

Chapter 7

12/16/16—Infinite Grid of Resistors

It is a well known problem, `https://xkcd.com/356/`. Let's first attack this problem under the continuous limit, to get some intuition for how to solve the problem, then attack the discrete problem.

7.1 Continuous Problem

The problem is as follows: given an infinite plane with resistance $R = \rho L$, what is the resistance between two points separated by distance d?

It would appear that the majority of the tools we have at our disposal to attack such a problem allow us to compute V the voltage everywhere given the voltage at the two points separated by d. The current that the system draws is directly related to the divergence of V at the points of contact. The ratio between these gives the effective resistance. Ok.

We know there are only two current sources, which means we know $\vec{\nabla}\cdot\vec{J} = \sigma\vec{\nabla}\cdot\vec{E} = \sigma\vec{\nabla}\cdot(\nabla V) = \sigma\nabla^2 V$ is just two delta functions, at the source and sink. Moreover, we can look up the Green's function for a 2D infinite plane for the Poisson Equation $\nabla^2 V = \delta(\vec{r}-\vec{r}_0)$. This alone means we are given sufficient information to solve for $V(\vec{r})$ and with it the effective resistance. Great!

I'll not do this exercise here since computing the GF is hard (ok, non-easy, not really that bad) and using it is uninteresting yet grungy.

7.2 Discrete Problem

This is a much harder problem since we don't have the GF! Nor for that matter, do we even have an equation of state like the Poisson Equation. Time to translate that formalism.

Label each point (i,j) discrete indicies, and call V_{ij} the voltage at i,j. We know that the voltage drop between two points is simply proportional to the current, and we know that there are exactly one source and one sink in the entire grid, so we have something like

$$V_{i+1,j} - 2V_{i,j} + V_{i-1,j} + V_{i,j+1} - 2V_{i,j} + V_{i,j-1} = R\left(\mathscr{J}_{ij}\right) \tag{7.1}$$

where $\mathscr{J}_{ij}$ we define to be the "current source" at (i,j) (i.e. zero everywhere except $\pm$ unity at source/sink). This is just a discrete version of the Poisson Equation.

Now we need the equivalent of the Green's Function. ACM95 style. We need to develop the homogeneous equations first, then solve for the BVP at the boundary induced by the delta function. We separate variables

$$V_{i+1,j} - 2V_{i,j} + V_{i-1,j} = C_{ij} \tag{7.2}$$

where the only stipulation with C_{ij} is that its negative goes into the j coordinate of the separation of variables.

But wait, this is just the coupled SHO equation! So we will have an SHO along one direction. Thus, C_{ij} is simply λV_{ij} where λ is selected by the normal mode frequencies. What frequencies are allowed? Any wavelength longer than $2a$ with a the separation between nodes is allowed, above the Nyquist frequency.

Seems like can solve now. **UNFINISHED**.

Chapter 8

04/28/17—Bridges, Chains and others

8.1 Bridges and Dem Chainz

Inspired by a visit to the Golden Gate bridge. I will first rehash a problem we solved in a previous tidbits doc, then work on a new one.

8.1.1 Shape

Easy problem. What is the shape of a light chain holding up a horizontal bridge, e.g. a suspension bridge? How about the shape of a chain freely suspended?

Consider first the former problem first, a light chain holding up a horizontal bridge, as depicted magnificently in Figure 8.1.

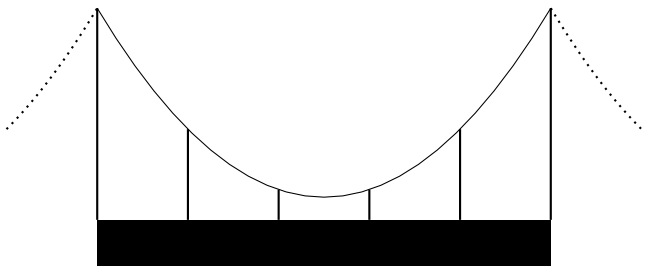


Figure 8.1: Picture of light chain holding up bridge.

Call the linear density of the bridge ρ. Let's compute $\vec{T}(x)$ the tension in the rope as a function of x, where $x \in [-L/2, L/2]$, L the distance between the supports. Because the setup is vertically symmetric, it must satisfy $T_x(x) = -T_x(-x)$ i.e. it is odd, and $T_y(x) = T_y(-x)$ is even.

At any point x, we know that $T_y(x + \mathrm{d}x) - T_y(x) = \rho \operatorname{sgn}(x)(\mathrm{d}x)g$ the difference in y components must support the differential weight (where $\operatorname{sgn}(x)$ is the sign of x), or

$$\frac{\mathrm{d}T_y}{\mathrm{d}x} = \operatorname{sgn}(x)\rho g$$
$$T_y(x) = \rho g\,|x|$$

But then consider that $\vec{T}$ must lie along the chain, so $\frac{T_y}{T_x} = \frac{\mathrm{d}y}{\mathrm{d}x}$. We note that T_x must be constant almost everywhere, else the chain would accelerate, but we note that it must point left on the left side and right on the right side! We note that $T_x \propto \operatorname{sgn}(x)$ is thus what we are looking for, and we obtain simply

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\rho g x}{|T_x|} \\ y(x) &= \frac{\rho g}{2|T_x|} x^2 \end{aligned} \tag{8.1}$$

where we've chosen the constant of integration such that $y(0) = 0$. What exactly is T_x then? We see that it's the shape parameter; the looser the chain is horizontally, the lower it drops, so given one we can compute the other, but indeed a family of solutions exists for any ρ, L determined by their shape and tension.

How about for a freely hanging chain, as in Figure 8.2?

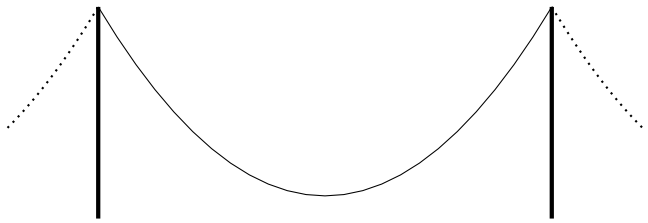


Figure 8.2: Picture of free hanging chain.

We have mostly the same math as before, except

$$\frac{\mathrm{d}T_y}{\mathrm{d}x} = \operatorname{sgn}(x)\rho g\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \qquad\qquad \frac{T_y}{T_x} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

where the extra term comes in since $\mathrm{d}m = \rho(\mathrm{d}s) = \rho\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$ with $\mathrm{d}s$ the length of chain which is also the *arclength* now. We observe that

$$\begin{aligned} \frac{\mathrm{d}T_y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(T_x \frac{\mathrm{d}y}{\mathrm{d}x}\right) \\ &= |T_x| \operatorname{sgn}(x) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \end{aligned}$$

and combining this with the above we obtain

$$\begin{aligned} \operatorname{sgn}(x)\rho g\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} &= |T_x| \operatorname{sgn}(x) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\rho g}{|T_x|}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \\ y(x) &= \frac{|T_x|}{\rho g}\cosh\left(\frac{\rho g x}{|T_x|}\right) + y_0 \\ y(x) &= \frac{|T_x|}{\rho g}\left(\cosh\left(\frac{\rho g x}{|T_x|}\right) - 1\right) \end{aligned} \tag{8.2}$$

and we see again $|T_x|$ plays the role of a shape parameter and we've chosen the solution such that $y(0) = 0$.

Note: We were a bit haphazard in asserting that $T_x(x) = |T_x|\operatorname{sgn}(x)$. It of course made sense, since at each point the x tension should be equal on both sides, meaning $\frac{\mathrm{d}T_x(x)}{\mathrm{d}x} = 0$, but we also knew that $T_x(x)$ is odd. This is furthermore justifiable since it satisfies $\vec{T}(0) = 0$, intuitive since the tension pulls in $\pm\hat{x}$ on either side of $x = 0$ which cancel, so any tension "at" $x = 0$ would produce an acceleration. So it seems a justifiable conclusion.

8.1.2 Force to Pull

Let's hang these bridges and chains between two towers separated by a distance L such that the distance from the bottom of the chain to the top of the towers is h. What is the tension in the chain? From this, we can compute what the minimum force is to change h.

This is a simple exercise. h is directly related to T_x, and from the shape of the parabola we can get the ratio of $\frac{T_y}{T_x}$ and from this get $\left|\vec{T}\right|(h)$. So in the parabolic solution

$$\begin{aligned}
h = y\left(\pm\frac{L}{2}\right) &= \frac{\rho g L^2}{8|T_x|} \\
|T_x| &= \frac{\rho g L^2}{8h} \\
\frac{T_y}{T_x} &= \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=L/2} = \frac{\rho g L}{2|T_x|} = \frac{4h}{L} \\
\left|\vec{T}\right| &= \sqrt{T_x^2 + T_y^2} \\
&= |T_x|\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \\
&= \frac{\rho g L^2}{8h}\sqrt{1 + \frac{16h^2}{L^2}}
\end{aligned} \tag{8.3}$$

and in the hyperbolic

$$\begin{aligned}
h = y\left(\pm\frac{L}{2}\right) &= \frac{|T_x|}{\rho g}\left(\cosh\left(\frac{\rho g L}{2|T_x|}\right) - 1\right) \\
\frac{T_y}{T_x} &= \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=L/2} = \sinh\left(\frac{\rho g L}{2|T_x|}\right) \\
\left|\vec{T}\right| &= \sqrt{T_x^2 + T_y^2} \\
&= |T_x|\cosh\left(\frac{\rho g L}{2|T_x|}\right) \\
&= \rho g h + |T_x|
\end{aligned} \tag{8.4}$$

and since we can't analytically solve for a relation between $|T_x|, h$ we are probably stuck here. We can both in a couple limits though:

- High h—For large h in the parabolic case, the square root is roughly $\frac{4h}{L}$ and we obtain the trivial $\left|\vec{T}\right| = \frac{\rho g L}{2}$ which is totally as expected: $|T_x|$ vanishes as we grow higher, so we just need to support the weight of the bridge.

  For the hyperbolic case, large h equates equates to large argument to the cosh which is for small $|T_x|$ as in the parabolic case. This is again a dead end to evaluate analytically: the cosh goes to an exponential but then we're stuck. Instead, we see that $\left|\vec{T}\right| \approx \rho g h$, which for large h is roughly half the weight of the chain (since $h \gg L$, the length of the chain is approximately $2h$).

- Small h—For small h in the parabolic solution, we have $T \propto h^{-1}$ so as we further shrink h, T grows with its inverse.

  To get small h for the hyperbolic case, we expect we need large $|T_x|$, and indeed that lets us approximate

$$\begin{aligned} h &= \frac{|T_x|}{\rho g}\left(\cosh\left(\frac{\rho g L}{2|T_x|}\right) - 1\right) \\ &\approx \frac{|T_x|}{\rho g}\frac{\rho^2 g^2 L^2}{8|T_x|^2} = \frac{\rho g L^2}{8|T_x|} \end{aligned}$$

  and we see that we recover the same limit as the parabolic solution, since at that point the chain's length is roughly L as well and we have the same mass being suspended in both configurations.

To summarize, in the low h case, both exhibit $\left|\vec{T}\right| \propto h^{-1}$ behavior, but in the high h case, the suspension bridge yields constant tension while the chain yields $\left|\vec{T}\right| \propto h$.

8.2 Misc

8.2.1 Sum of Digits vs Base

For large numbers, determine the dependence of the sum of the digits of the number in some base b as a function of b. When does the approximation break down?

Let's just throw the simplest approximation we can at the problem first. For a sufficiently large number, the distribution of one of its non-leading digits is approximately uniform. Thus, in base b, the average digit is $\frac{0+1+\cdots+(b-1)}{b} = \frac{b-1}{2}$. For any number N, the number of digits is $\log_b N$, so we conclude that the sum of digits of N in base b is approximately $S_b(N) = \frac{b-1}{2}\log_b N$.

This seems to make sense: to build $S_b(N)$, we start with $S_b(0) = 0$, then we can build $S_b(N)$ as

$$S_b(b^i \le N < b^{i+1}) = 1 + S_b(0 \le N < b^i)$$

Thus, $S_b(N)$ lies in the range $[1,(b-1)k]$ when N is in the range $[b^k, b^{k+1}-1]$ where k is the number of digits in N expressed in base b, otherwise written $\lfloor \log_b N \rfloor$. What is the

distribtion of $S_b(N)$ in this range? That's simply the number of ways we can pick k numbers from $[0, b-1]$ and sum them to $S_b(N)$.

We know that a single uniform distribution over $[0, b-1]$ has mean $\frac{b-1}{2}$ and variance $\frac{b^2-1}{12}$[1], so the sum of k such variables has variance $\frac{k(b^2-1)}{12}$ since variance is a linear operator. Thus, we obtain that the sum of digits of a number N in base b has distribution

$$S_b(N) = \frac{b-1}{2}\log_b N \pm \sqrt{\frac{\log_b N(b^2-1)}{12}} \tag{8.5}$$

$$\approx \frac{b-1}{2}\log_b N\left(1 \pm \sqrt{\frac{1}{3\log_b N}}\right) \tag{8.6}$$

Intuitively, this approximation fails when $\log_b N \sim b$, i.e. the number of digits is of the same order as the number of possible digits, at which point we can't assume each digit is uniformly distributed over $[0, b-1]$ and instead have to account for a greater concentration towards lower digits e.g. Benford's law.

8.2.2 Approximations to Birthday Problem

See what we can get for a back-of-the-envelope usable solution to the birthday problem (draw with replacement from a set of size N, how many do you draw before you expect a duplicate?).

If we draw k elements with replacement from a set of size N, the probability that we do not get any duplicates is $\frac{N!}{N^k(N-k)!}$. Throwing Stirling's approximation at the two factorials, we obtain

$$\begin{aligned} P &\approx \sqrt{\frac{N}{N-k}}\left(\frac{N}{e}\right)^N \left(\frac{e}{N-k}\right)^{N-k} \frac{1}{N^k} \\ &= \left(\frac{N}{N-k}\right)^{N-k+1/2} \frac{1}{e^k} \\ &= \left(1+\frac{k}{N-k}\right)^{N-k+1/2} \frac{1}{e^k} \end{aligned}$$

We recall that $\left(1+\frac{k}{n}\right)^n$ as $n \to \infty$ is e^k, but that would imply that $P > 1$. Instead, we have to recall that we canot take the $N \to \infty$ limit since we're leaving out terms that themselves depend on N.

[1] https://en.wikipedia.org/wiki/Discrete_uniform_distribution

Instead, let's find the first order correction to the log formula:

$$\begin{aligned}\log\left(1+\frac{k}{n}\right)^n &= n\log\left(1+\frac{k}{n}\right)\\ &\approx n\left(\frac{k}{n}-\frac{k^2}{2n^2}+\mathcal{O}\left(\left(\frac{k}{n}\right)^3\right)\right)\\ &= k-\frac{k^2}{2n}\\ \left(1+\frac{k}{n}\right)^n &\approx \exp\left[k-\frac{k^2}{2n}\right]\end{aligned} \tag{8.7}$$

and so

$$P = e^{-\frac{k^2}{2(N-k)}}\sqrt{1+\frac{k}{N-k}} \tag{8.8}$$

How well does this do? We can compare the shape this curve produces with the original $P = \frac{N!}{N^k(N-k)!}$, and we obtain

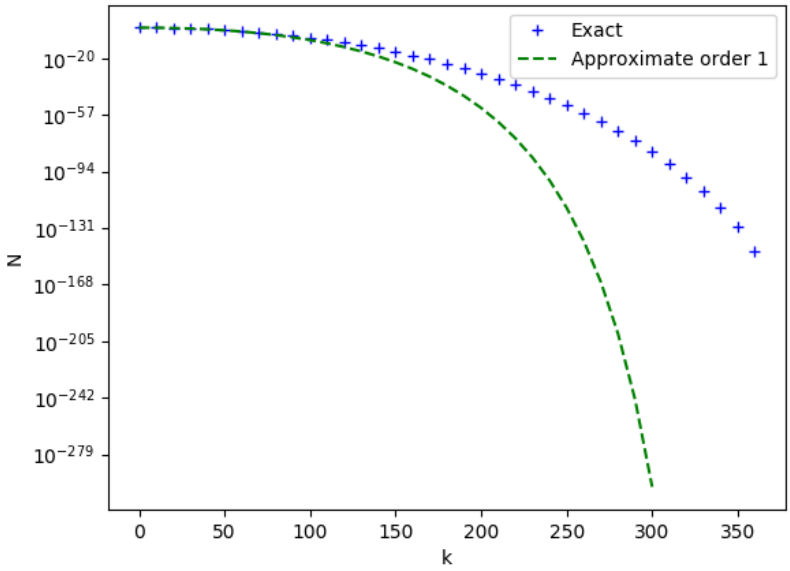


Figure 8.3: Plot of Exact and Approximate solutions for 365-day Birthday Problem

So we see that at least in the regime $k \ll N$, we do reasonably well. We recall that in (8.7), we only grabbed the first order term in $\frac{k}{N}$; we can grab a few higher order terms and see what happens in Figure 8.4.

Not so great! But this is because we note that our expansion is in order $\frac{k}{N-k}$, so it's expected that as this approaches unity our expansion falls apart, which happens when $k = N/2$. A quick exercise replacing the Taylor expansion with the full exponent recovers the exactly correct curve.

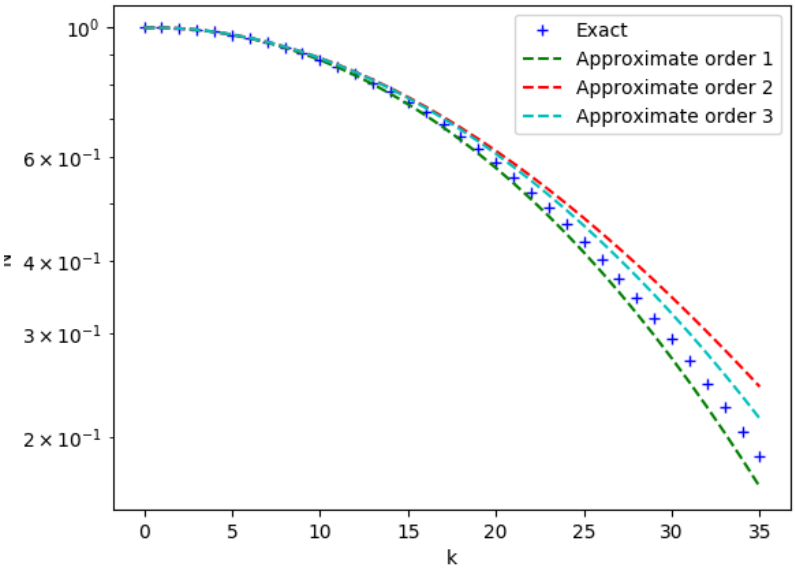


Figure 8.4: Plot of Exact and Approximate Solutions with higher order terms in logarithm expansion

Chapter 9

05/02/17—L'Hôpital's Rule and Complex Analysis

This ought to be a short chapter. Had a sudden shower thought to prove L'Hôpital's rule using complex analysis, let's just carry it out.

Consider if some function $f(x) = \frac{g(x)}{h(x)}$ such that $g(a) = h(a) = 0$. Then the well-known solution to evaluating $f(a)$ is to differentiate both numerator and denominator until at least one is non-vanishing, then evaluate, this is *L'Hôpital's Rule*.

Let's try instead to compute the residue of $\frac{f(x)}{x-a}$ at $x = a$. Now, if $f(x)$ is not singular, then this function has a simple pole at $x = a$ and the residue is simply $f(a)$. Let this be our motivating factor.

Let there be some $h'(x)$ such that $h(x) = (x-a)^n h'(x)$ and $h'(x)$ is analytic at a. Then we know that the residue of $f(x)$ is given

$$\begin{aligned} \operatorname{Res}(f,a) &= \lim_{x\to a} \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x-a)^{n+1}f(x) \\ &= \lim_{x\to a} \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{g(x)}{h'(x)} \end{aligned} \tag{9.1}$$

There are then two cases when $f(a)$ is analytic:

- $g^{(n)} \neq 0$, but all lower-order derivatives vanish. Some trivial reapplication of the quotient rule to compute the derivatives shows that then

$$\begin{aligned} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{g(x)}{h'(x)} &= \frac{g^{(n)}(x)(h'(x))^{2^n-1}}{(h'(x))^{2^n}} \\ &= \frac{g^{(n)}(x)}{h'(x)} \end{aligned}$$

  but then $h^{(n)}(x=a) = n!h'(x=a)$, and so we find the above result agrees with L'Hôpital's Rule.

- $g^{(n)} = 0$ and all lower-order derivatives also vanish. Since we only differentiate $g(x)$ a maximum of n times, all terms vanish and $\operatorname{Res}(f,a) = f(a) = 0$ as also expected.

What about when $f(a)$ is singular? Then this means that for some $k < n$, $f'(x) = f(x)(x-a)^k$ is analytic at $x = a$, and $f'(x)$ must then also be equal to

$$f'(x) = \frac{(x-a)^k g(x)}{(x-a)^n h'(x)}$$

and so if we apply L'Hôpital's Rule, we will obtain a diverging result after only k iterations.

The above proof could be tightened but I'm not going to bother, it seems the equivalence is pretty clear (at least when $f(x)$ is analytic in a neighborhood of a and the usual).

How about for $f(x) = \frac{g(x)}{h(x)}$ where $g(x), h(x)$ are both divergent at $x = a$? This should be a simple exercise if we just take $f(x) = \frac{1/h(x)}{1/g(x)}$ and perform the above proof on this.

Chapter 10

05/28/17—Parabolas

We're hiking the Grand Canyon, and halfway down we rediscover signal that we did not have at the top. I joke "clearly it is because the Grand Canyon approximates a parabolic dish reflector, and since we are now closer to the focus of the parabola we will have better signal than up top." To follow up on this, we first demonstrate that the parabola has the unique property that all plane waves of incidence perpendicular to the axis of symmetry of the parabola reflect off the surface to a particular point, called the *focus*.

We then consider the more interesting problem: if we project the reflections of incoming rays onto the axis of symmetry, they form a perfect delta function at the focus if the surface is a perfect parabola. If we want the distribution to be normally distributed and centered on the focus of the parabola, what uncertainty tolerances do we have on height measurements of the parabola? More generally, how do various sources of error (error in parabola parameter, error in height measurements) propagate to uncertainties in focusing ability?

10.1 Parabola Properties

We wish to find a 1D surface in $\mathbb{R}^2$ that has the following property: for any ray traveling in the $-\hat{y}$ direction, if the ray reflects according to the Law of Reflection (symmetric about the normal to the surface at the point of incidence) then it must go through the *focus*.

Call the *focus* $(0,a)$. Consider also a surface $y(x)$. Then the ray from the focus to the surface has slope $b(x) = \frac{y-a}{x}$. Thus, we wish to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ such that the normal $-\frac{\mathrm{d}x}{\mathrm{d}y}$ *bisects* the angle formed by $b(x)$ and $\hat{y}$. This implies that the tangent must bisect $\hat{x}$ and the slope formed by $-\frac{1}{b(x)}$, the normal to $b(x)$. Calling the angle formed by $\frac{x}{a-y}$ and $\hat{x}$ 2θ, then this implies that

$$\tan 2\theta = \frac{x}{a-y} = \frac{2\frac{x}{2a}}{1-y/a}$$

Recall double angle identity $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$. Making an inspired guess, assume $\left(\frac{x}{2a}\right)^2 = \frac{y}{a}$, in which case $\tan\theta = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2a}$, while additionally $y'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{4a}\right) = \frac{x}{2a}$. Wow!

10.2 Uncertainties

It is clear that if the uncertainties are in the parameter $a = a(x)$, then the distribution of rays along the axis of symmetry will simply be the PDF of a over all x, or more quantitatively, calling $L(y)$ the luminosity (density of rays) incident at $(0, y)$, we have

$$L(y) = \int_{-\infty}^{\infty} \delta(y - a(x))\,\mathrm{d}x$$

and so if there are random variations in a, we can obtain $L(y)$ as well.

What about if the uncertainty is in $y(x)$ though? It seems perhaps sensible to propagate the uncertainty to $\frac{\mathrm{d}y}{\mathrm{d}x}$ and translate that into an uncertainty on $a(x)$. For example, if $y(x) = y_0(x) + N_{0,\sigma}$ where N is a random variable with zero mean and σ variance, then we know that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y_0}{\mathrm{d}x} + N_{0,\sigma\sqrt{2}} \\ &= \frac{x}{2a} + N_{0,\sigma\sqrt{2}} \\ &= \frac{x}{2a} \pm \sigma\sqrt{2}\end{aligned}$$

We aspire instead to express this uncertainty as one in a, so we instead write $a = a_0 + N_{0,\sigma_a}$. We can rewrite

$$\frac{x}{2(a_0 + N_{0,\sigma_a})} = \frac{x}{2a_0} \pm \frac{x\sigma_a}{2a_0^2}$$

and matching terms we find $\sigma = \frac{x\sigma_a}{2a_0^2\sqrt{2}}$.

However, we must consider one more thing: this corresponds to a global uncertainty on a by σ_a, and it assumes that the point at some x value has position given by $y(x) = \frac{x^2}{4(a_0+\sigma_a)}$ rather than position $\frac{x^2}{4a_0}$. This can be remedied: for any given parameter $a = a_0 \pm N_{0,\sigma_a}$, the position of the focus $(0, y_a)$ is given by

$$\begin{aligned}y_a &= a + \left(\frac{x^2}{4a_0} - \frac{x^2}{4a}\right) \\ &= a_0 + N_{0,\sigma_a} + \frac{x^2}{4a_0^2} N_{0,\sigma_a}\end{aligned}$$

Chapter 11

07/16/17—Principal Component Analysis

My friend and I were watching a Youtube video called "18 types of Asian girls" when we both had the same reaction: why 18? As von Neumann famously said, four parameters fits an elephant, five wiggles its trunk. So instead, we both arrived at Principal Component Analysis (PCA) as a viable way to determine just how many categories are needed. And this made me think again about how this technique works, so I looked up a few definitions.

SVD *Singular Value Decomposition* is a fancy name for change of basis, diagonalizing a matrix. In the particular case of a rectangular matrix, diagonalizing the matrix means the rectangle becomes the concatenation of a diagonal square matrix and a zeroes matrix.

PCA *Principal Component Analysis* is looking for linear combinations of feature vector components that are strongly correlated in a dataset. These linear combinations are the eigenvectors that the SVD yields.

In our particular case, consider if we have N Asian girls, each with a k-dimensional feature vector. Then our data M form an $N \times k$ matrix, and so the SVD $\mathbf{U\Sigma V}$ consists of three matricies:

- $\mathbf{\Sigma}$ is a $k \times k$ diagonal matrix, with $(N-k) \times k$ zero entries concatenated.
- $\mathbf{U}$ is $N \times N$, but obviously in each row only the first k values matter, since the remainder are being multiplied by the zero entries of Σ
- $\mathbf{V}$ is a $k \times k$ matrix, the k eigenvectors that go into the PCA. Generally we expect only a few of these to correspond to larger eigenvalues, and these are the principal components we expect.

As a demonstration, let's try!

Conclusion from my demonstration: Forgot that PCA and clustering are different. Oops. Probably not gonna resume this one.

Chapter 12

08/15/17—Sum until > 1

Sum random numbers uniformly distributed $\in [0,1]$ until you exceed 1. What's the expected value of the number you end up with?

12.1 Simulation

We can first simulate, as in Figure 12.1. The simulation concludes that the mean is pretty close to $\frac{e}{2}$. The Python code used is:

```
#!/usr/bin/env python
''' runs math problem '''
import matplotlib.pyplot as plt
import numpy as np

TRIALS_POW = 6
BIN_SIZE = 100
if __name__ == '__main__':
    nums = np.zeros(10 ** TRIALS_POW)
    while not all(nums > 1):
        rand_incs = np.random.random(10 ** TRIALS_POW)
        mask = 1 - (nums > 1) # 0 if nums is greater than 1
        nums += rand_incs * mask

    n, bins, patches = plt.hist(nums, BIN_SIZE)

    bin_centers = (bins[:-1] + bins[1:]) / 2
    unnormalized_fit = np.e - np.exp(bin_centers - 1)
    fit = (unnormalized_fit / sum(unnormalized_fit)) * (10 ** TRIALS_POW)

    plt.title('Histogram of Ending Values with 10E{} trials'.format(TRIALS_POW))
    plt.xlabel('Ending Value. Mean is {}'
               .format(round(sum(nums) / 10 ** TRIALS_POW, 4))
              )
    plt.ylabel('Occurrence Count')
    plt.plot(bin_centers, fit, 'r-', label='Fit')
    plt.legend()
    plt.savefig('trials10e{}.png'.format(int(TRIALS_POW)))
```

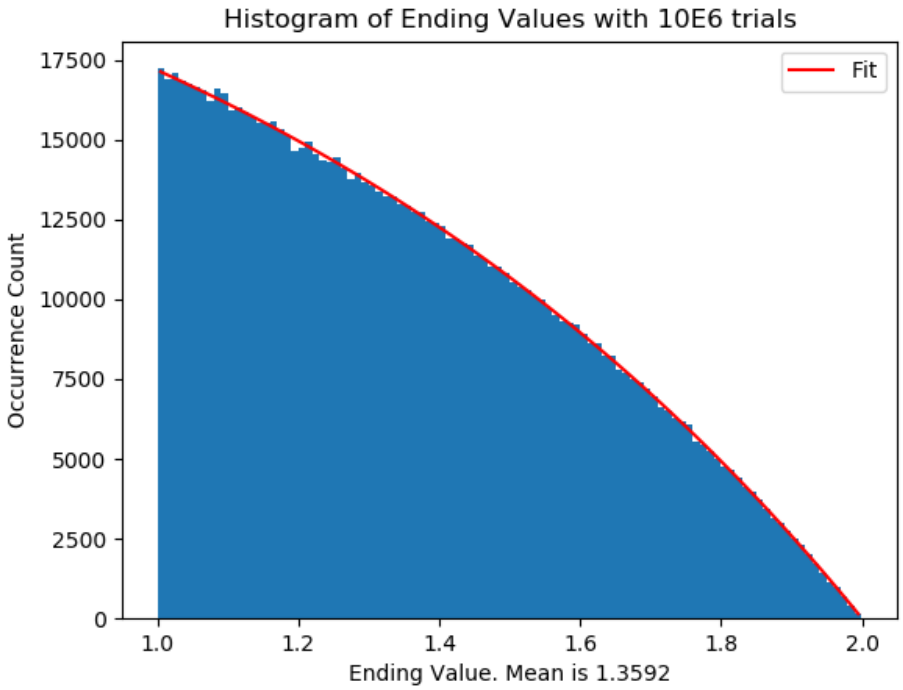


Figure 12.1: Simulation with 10^6 data points.

12.2 Solution

So what's the theory behind this?

The probability that you to take $> k$ additions to exceed 1 is the probability that k U_0^1 numbers sum to ≤ 1. This is the volume of the volume formed by the origin and each of the unit vectors in k dimensions, also $\frac{1}{k!}$ (lemma coming).

Given this, the probability that we exceed exactly on the kth addition is the probability that we have not yet succeeded by the $k-1$th addition $(1/(k-1)!)$ but subsequently fail by the kth addition. The probability that we *do not* fail on the kth addition is $1/k!$, but everything that does not fail on the kth addition also does not on the $k-1$th addition, so the probability that we do not fail on the kth addition conditional on not failing on the $k-1$th addition is just $1/k$, from which we easily deduce that *failing* on the kth addition is $(k-1)/k!$.

Then, the expected number of additions is just $\sum_k kP(k)$ where $P(k)$ is the probability of terminating exactly on the kth addition as computed above. The sum starts at $k = 2$ since this is the first point at which we can first exceed 1, and so we have $\mathrm{E} = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} = e$. Then, since each addition contributes on average 1/2, we have that the expected value is $e/2$.

12.2.1 Lemma

Claim: In D dimensions, the volume of points satisfying $q_k \geq 0$ and $\sum_{k=1}^{D} q_k \leq a$ has volume $\frac{a^D}{D!}$. Geometrically, it is the volume bound by the origin and the points distance a along each of the D dimensions (i.e. $(a,0,0,\ldots),(0,a,0,\ldots)$).

Proof: We prove via induction. The base case is simply $D = 1$, in which case the line segment of length a has length $a = \frac{a^1}{1!}$.

Inductively, let $V_D(a)$ be the volume in D dimensions summing to $\leq a$. Then we notice that $V_D(a)$ is just the sum of all $V_{D-1}(a')$ where $0 \leq a' \leq a$ multiplied by some thickness $\mathrm{d}a'$. But since $\int \frac{a^{D-1}}{D!}\mathrm{d}a = \frac{a^D}{D!}$ we are done.

This proves the result we used where the volume is $\frac{1}{D!}$.

To be fair, this is a fairly trivial result if we use the result that $|\bigwedge_k a\hat{e}_k|$ is the above described volume, since the norm of the wedge product is the $1/D!$ times the determinant of the matrix containing the basis vectors, and the matrix is just $a\mathbb{I}_D$ where $\mathbb{I}_D$ is the identity matrix in D dimensions.

12.2.2 Distribution

A much more interesting problem is deriving the distribution that we found in the simulation!

Consider $P_k(x)$ the probability of being on value x after exactly k additions. The probability of ending on some value x_f is then

$$P_k(1 \leq x_f \leq 2) = \int_{x_f-1}^{1} P_{k-1}(x_i)\,\mathrm{d}x_i \tag{12.1}$$

But we know from above that $P_{k-1}(x_i) = \frac{(x_i)^{k-2}}{(k-2)!}$ since this is the PDF, we must differentiate one time from the CDF. Then, summing over all k, we have that

$$P(1 \leq x_f \leq 2) = \int_{x_f-1}^{1} P(x_i)\,\mathrm{d}x_i \tag{12.2}$$

$$\propto \int_{x_f-1}^{1} e^{x_i}\,\mathrm{d}x_i \tag{12.3}$$

$$\propto e - e^{x_f-1} \tag{12.4}$$

Normalization is a bit tricky to get from first principles since there's some "leakage" when summing over $P(x_i)$, but we can always just normalize $P(x_f)$ directly instead. The fit is shown in Figure 12.1.

Chapter 13

8/18/2017—Random Processes

Motivated by stock prices. Consider if for every price P_1 there is a time-independent probability $\Delta(P_2, P_1)$ of transitioning to P_2 from P_1. Compute the PSD of the random trajectories this produces.

The analysis is most easily carried out by looking at mixed states of prices. Consider a state $|\psi\rangle = \sum_{k=1}^{N} a_k P_k$ that is an eigenvector of Δ, such that $\Delta|\psi\rangle = |\psi\rangle$ (let's assume N the number of states is finite for now). The significance of such a state is that if we choose $M \gg N$ prices, distributed with $\langle P_k|\psi\rangle$ probability distribution, then the distribution after the action of Δ is still $|\psi\rangle$.

The way that we capture the "transition with probability" is as follows: suppose you start at some value P_0, then apply Δ a few, q, times. Then the probability of measuring each of the P_k prices is simply $P[P_k] = \langle P_k|\Delta^q|P_0\rangle$, just like in quantum mechanics.

Thus, decompose any initial state $|\varphi_0\rangle$ (which may be a pure state) into some sum of eigenvectors $a_j|\psi_j\rangle$. Then at some future time t the probability of finding $|\varphi_t\rangle$ in state P_k is given simply

$$\begin{aligned} P[\varphi_t = P_k] &= \sum_j \langle P_k|\Delta^t a_j|\psi_j\rangle \\ &= \sum_j a_j \langle P_k|\psi_j\rangle \end{aligned} \tag{13.1}$$

This is not very helpful to us though, since to compute any spatial correlations we have to discount spurious transitions (e.g. in a random walk, one cannot go from $t = 1, x = -1$ to $t = 2, x = 2$, but by simply integrating the probability distribution such contributions are included). We'll have to find something more clever.

To compute the PSD, we look to the autocorrelation. Recall that the PSD of a continuous signal $f(t)$ is given by the FT of the autocorrelation, defined to be

$$R_{ff}(\tau) \propto \int_{-\infty}^{\infty} f(u+\tau) f(u)\,\mathrm{d}u. \tag{13.2}$$

Chapter 14

10/16/17—Hopping Bunny

A bunny in two dimensions takes three hops of unit distance in random directions. What's the probability the bunny ends up within unit distance of its starting position?

14.1 Simple Solution

Label the points that the bunny hops to be labeled by P_i and the circles of unit radius about each of the P_i be labeled C_i. Let P_0 be the origin, then P_1 must lie on the unit circle C_0. WLOG choose $P_1 = (0,1)$.

Now, we know that P_3 must lie on C_2, and the probability that P_3 lies inside the unit circle is the arclength of C_2 inside C_0. We can draw diagram Figure 14.1.

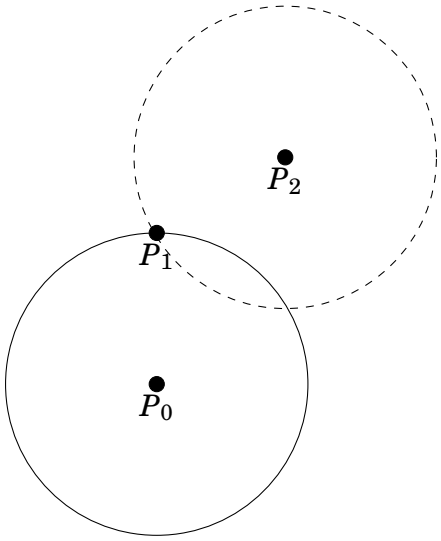


Figure 14.1: Schematic drawing of the argument for section 14.1.

We note that if we define θ to be the angle formed by P_0–P_1–P_2 then the intersection of C_2, C_0 has arclength $|\pi - \theta|$. Thus, the probability given θ of ending up inside C_0 is $\frac{|\pi-\theta|}{2\pi}$. Then since θ ranges uniformly over $[0, 2\pi]$, we simply average and find probability P of

ending up inside C_0 to be

$$P = \frac{1}{2\pi}\int_0^{2\pi} \frac{|\pi-\theta|}{2\pi}\,\mathrm{d}\theta = \frac{1}{4}. \tag{14.1}$$

14.2 Generalizable Solution

Let's develop a more powerful solution capable of explaining the probability trends in two dimensions:

```
2 jumps: 0.331158
3 jumps: 0.250073
4 jumps: 0.199248
5 jumps: 0.166664
6 jumps: 0.143223
7 jumps: 0.124646
8 jumps: 0.110964
9 jumps: 0.100208
10 jumps: 0.090960
```

and three dimensions:

```
2 jumps: 0.247005
3 jumps: 0.166980
4 jumps: 0.118910
5 jumps: 0.091725
6 jumps: 0.072513
7 jumps: 0.059951
8 jumps: 0.050485
9 jumps: 0.043220
10 jumps: 0.037371
```

14.2.1 Computing $P_n(r)$

We wish to compute a general formula for $P_n(r)$, the probability distribution of r after n jumps. To compute this, WLOG suppose we are at a point $(a,0)$ and we wish to compute the probability that we jump to a distance r away. We note that r,a are related by

$$r^2 = a^2 + 1 - 2a\cos\theta, \tag{14.2}$$

$$\mathrm{d}r = \frac{r}{2a\sin\theta}\mathrm{d}\theta, \tag{14.3}$$

$$\begin{aligned} P(r|a) &= P(\theta)\frac{\mathrm{d}r}{\mathrm{d}\theta}, \\ &= \frac{1}{2\pi}\frac{r}{2a\sin\theta}, \\ &= 2\frac{1}{2\pi}\frac{r}{2a\sqrt{1-\left(\frac{r^2-a^2-1}{2a}\right)^2}}, \end{aligned} \tag{14.4}$$

where a factor of 2 arises since there are two values of θ that can satisfy the given square root.

But this means that we can just integrate over all a given an existing $P_{n-1}(a)$ to obtain $P_n(r)$, namely

$$P_n(r) = \int_0^{n-1} P_{n-1}(a) 2\frac{1}{2\pi}\frac{r}{2a\sqrt{1-\left(\frac{r^2-a^2-1}{2a}\right)^2}}\,\mathrm{d}a. \tag{14.5}$$

We can verify this expression by computing explicitly $P_2(a)$, since $P_1(a) = \delta(1)$. We first compute it from scratch then take the limit of Equation 14.4 to verify that the two expressions agree. First

$$r^2 = 2 - 2\cos\theta \qquad r = 2\sin\frac{\theta}{2}, \tag{14.6}$$

$$\mathrm{d}r = \cos\frac{\theta}{2}\mathrm{d}\theta, \tag{14.7}$$

$$\begin{aligned} P_2(r) &= \frac{1}{2\pi}\frac{1}{\cos\frac{\theta}{2}}, \\ &= 2\frac{1}{2\pi}\sqrt{\frac{1}{1-\left(\frac{r}{2}\right)^2}}. \end{aligned} \tag{14.8}$$

It is clear that $\int_0^1 P_2(r)\,\mathrm{d}r$ must be $\frac{1}{3}$, since C_1 must have arclength $\frac{\pi}{3}$ inside C_0, and indeed performing the integral we obtain this result.

We then need to set $a = 1$ in Equation 14.4. This gives us

$$\begin{aligned} P(r|a=1) &= 2\frac{1}{2\pi}\frac{r}{2\sqrt{1-\left(\frac{r^2-2}{2}\right)^2}}, \\ &= \frac{r}{2\pi}\frac{1}{\sin\theta} = \frac{1}{2\pi}\frac{1}{\cos\frac{\theta}{2}}. \end{aligned} \tag{14.9}$$

The results thankfully agree!

14.2.2 Computing P_n

Define $P_n = P_n(r < 1)$. Now, we do some seriously hacky algebra. We can express P_n in terms of Equation 14.5 as follows

$$P_n = \int_0^1\int_0^2 P_{n-1}(a)2\frac{1}{2\pi}\frac{r}{\sqrt{1-\left(\frac{r^2-a^2-1}{2a}\right)^2}}\,\mathrm{d}a\mathrm{d}r. \tag{14.10}$$

We know that we only need to integrate out to $a = 2$ since any larger cannot produce $r < 1$.

We exchange the order of integration and perform the r integral first. We recognize the integrand to be of form $-\frac{f'(x)}{\sqrt{1-f(x)}}$ which integrates to a $\sin^{-1}$, or more precisely

$$P_n = -\int_0^2 P_{n-1}(a)\frac{1}{2\pi}\arcsin\left[\frac{r^2-a^2-1}{2a}\right]_0^1\mathrm{d}a. \tag{14.11}$$

Now we drop the $r = 0$ evaluation since it's just evaluating the CDF at the lower boundary (I agree it's scary that the integrand doesn't seem to vanish stop giving me a hard time),

then the integrand simplifies to

$$P_n = \int_0^2 P_{n-1}(a) \frac{1}{2\pi} \arcsin \frac{a}{2} \, \mathrm{d}a, \tag{14.12}$$

Now, just to verify this expression, let's evaluate for $n = 3$ using our $P_2(a)$ that we have from before. This integral is then

$$\begin{aligned} P_3 &= \int_0^2 \frac{1}{2\pi^2} \sqrt{\frac{1}{1 - \left(\frac{a}{2}\right)^2}} \arcsin \frac{a}{2} \, \mathrm{d}a, \\ &= \int_0^1 \frac{1}{\pi^2} \sqrt{\frac{1}{1 - \left(\frac{a}{2}\right)^2}} \arcsin \frac{a}{2} \, \mathrm{d}\frac{a}{2}, \\ &= \frac{1}{\pi^2} \arcsin^2 x \Big|_{x=0}^{x=1}, \\ &= \frac{1}{4}. \end{aligned} \tag{14.13}$$

So technically this is prescriptive. And we can feasibly see that the successive antiderivatives might go something like $\arcsin^{n-1}(x)$ which somehow fortuitously yields a $\frac{1}{n+1}$ relationship? At least there is something linearly increasing. But this is far beyond our algebraic abilities sadly.

Chapter 15

10/22/17—Other Leaping Bunny

A bunny starts at $x = 0$ on the one-dimensional real number line. Each time step, it jumps with constant integer v (positive or negative) to a new integer position. You can only examine one x value at each time. Find the bunny in finite time.

15.1 Rigorously Defining "In Finite Time"

What does it mean to find the bunny in finite time? This means that given an initial bunny velocity $v < V$, we are guaranteed to find the bunny in some time $t < T(V)$ where $T(V)$ is finite for all finite V.

15.2 Positive Velocity Starting at Origin

Consider if we are given integer $v \geq 0$. Then for any given t, we examine what position the bunny would be at if $v = t$. This means at time $t = 0$, we examine $x = 0$ because $x = vt = 0 \times 0 = 0$. At $t = 1$, we examine $x = 4$ because $x = vt = 1 \times 1 = 1$, and so on.

15.3 Integer Velocity Starting at Origin

This is a bit trickier; we obviously cannot examine all positive v then all negative v since then if v is negative we will not find it in finite time! Instead, we search as follows: for odd $t = 2\tau - 1$, we search as if $v = -\tau$, and for even $t = 2\tau'$ we search as if $v = \tau'$.

We prove that the bunny velocity can be found in finite time: for a bunny velocity $v < V$, we are guaranteed to find the bunny in $t < 2|V|$ time, and since $T(V) = 2|V|$ we guarantee that for any finite bunny velocity we find the bunny in finite time. After the bunny is found, the position of the bunny one-to-one maps to the v of the bunny, so the bunny velocity can be computed in constant time as well.

This follows the same train of thought as Cantor cardinality counting arguments: two sets are of the same cardinality if for any finite member of one we can determine to which member of another it corresponds in finite time, very intuitively and roughly.

15.4 Integer Velocity, Unknown Integer Initial Position

The initial conditions for the bunny live in some countable configuration space (x_0, v), i.e. given x_0, v of the bunny we can determine for any time where the bunny is located. We need to map a linear search t to this configuration space. The intuitive algorithm suffices; within time $t \in [k^2, k+1^2]$ time for some k, we search over rectangular perimeter $(k/2, -k/2)$–$(k/2, k/2)$–$(-k/2, k/2)$–$(-k/2, -k/2)$–$(k/2, -k/2)$. This traces out a spiral in (x_0, v) space. Then of course at each (x_0, v), we examine the position the bunny would be found at.

However, this problem is harder than the previous problem; once we have located the bunny, we cannot invert to find the initial (x_0, v) because the map $(x_0, v) \to x$ is not injective and hence not invertible! More precisely, the map is many-to-one, since multiple (x_0, v)s can end up at a given position x where we find the bunny, even for a constant time.

However, once we have found the bunny once, we can let this position x be the new origin, then this problem reduces to the previously solved problem, bunny starting at the origin with integer velocity, and we can apply our previous algorithm to compute v and disambiguate between the degenerate (x_0, v) incurred by the previous step.

We prove that this finds the bunny in finite time: for any bunny in configuration space bound by $|x_0| < X, |v| < V$, we are guaranteed to find the bunny in time $t < 4\max(X, V)^2$ which is finite for all finite values of X, V (this is the natural extension of the earlier "finite time" definition now that configuration space is larger), thus for any finite initial position and velocity the bunny can be found in finite time. Then, once the bunny is found, the second step to determine v has also been proven to take finite time for finite v, therefore we are guaranteed to determine the bunny's velocity v in finite time.

This can obviously be generalized to any countable configuration space, giving some more intuition to Cantor's notion of *countable sets* having equivalent cardinality.

Chapter 16

02/04/18—Conformal Mapping

Conformal maps locally preserve angles.

16.1 Reviewing Analytic Functions as Conformal Maps

Consider an analytic function $f : \mathbb{C} \to \mathbb{C}$, and write $w = f(z)$. We wish to see what happens to angles in the z plane under the action of f. Angles are a local property, so we expand $f(z)$ about some local point z_0 in a Taylor series

$$f(z_0 + \Delta z) \approx f(z_0) + \sum_{k=1}^{\infty} \frac{f^{(k)}(z_0)}{k!}(*)\Delta z^k, \tag{16.1}$$

$$\Delta w = \sum \frac{f^{(k)}(z_0)}{k!}(*)\Delta z^k, \tag{16.2}$$

where $\Delta w = f(z_0 + \Delta z) - f(z_0)$. For small Δz, it's clear that Δw is set by the leading order in the summation above.

Notably, if $f'(z_0) \neq 0$ then we have $\Delta w = f'(z_0)\Delta z$. In this case, $|\Delta w| = \left|f'(z_0)\right||\Delta z|$ while $\operatorname{Arg}\Delta w = \operatorname{Arg} f'(z_0) + \operatorname{Arg}\Delta z$. It is of worth to note that it preserves relative lengths (two Δz of the same length produce two Δw of the same length) though not area, and that it preserves angles (two Δz with different arguments will correspond to two Δw that differ by the same argument).

If $f'(z_0) = 0$, then the relation between Δw and Δz is more complex; namely $\Delta w = \frac{f^{(k)}(z_0)}{k!}(\Delta z)^k$ for a k-th order zero $f(z_0)$. We still have shape-preserving properties, but angles are *enlarged* in w space, $\operatorname{Arg}\Delta w = \operatorname{Arg} f^{(k)}(z_0) + k \operatorname{Arg}\Delta z$.

16.2 Complex Analysis and Conformal Mapping

It was mentioned in the book I'm reading out of, Churchill *Complex Variables and Applications*, that conformal mapping is only used in 2D problems. One naturally wonders why conformal mapping is restricted to 2D; one can easily imagine e.g. why should quaternions not be able to extend conformal mapping to 4D domains?

This turns out to be *Liouville's Theorem*, for which the proof is sufficiently complicated it is not outlined on Wikipedia ☹. But I think we can get some understanding of the factors at play.

Consider two vector spaces V, W and a function $f : V \to W$. By choosing vector spaces for our domain and range, we guarantee that members of V, W are closed under addition and scalar multiplication. Furthermore, in the context of conformal mapping, endow both V, W with an inner product satisfying linearity, which we will denote by $\langle u, v\rangle, u, v \in V$ such that $\langle u, u\rangle \neq 0$ unless $u = 0$.

With this structure, we are able to introduce *components*. Call two vectors $\langle u, v\rangle = 0$ *orthogonal*. For vector space V, consider the maximal set of vectors that are pairwise orthogonal and have norm 1; call these $\langle e_i e_j\rangle = \delta_{ij}$ (we will take the set to be finite for now, but natural extensions follow for countably infinite and continuously infinite sets I imagine). Then, by linearity, vectors $v \in V$ can be written $\sum_i v_i e_i$; this is obviously possible by decomposing v into two other vectors v_1, v_2 such that $\langle v_1 e_i\rangle = 0, \langle v_2 e_j\rangle = 0 \forall j \neq i$, doable since the $\{*\}e_i$ are maximal. It is then obvious that $\langle u, v\rangle = \sum_i u_i v_i$, and we have our usual dot product.

What this shows is that *the linear norm for a vector space is necessarily an $\mathscr{L}^2$ norm*. In fact, any bilinear form on the vector space is necessarily quadratic in the components of an arbitrarily vector in a spanning basis of the bilinear form. If we only definitions of angles that only use two vectors from V in a bilinear form (e.g. $\cos\theta = \langle u, v\rangle$), then any angle-preserving transformation must preserve the $\mathscr{L}^2$ norm of elements of V. This is in agreement with Liouville's Theorem which basically restricts conformal mappings in Euclidean space to mappings that are conformal in $2D$. Of course, my result assumes one only uses angles related to a bilinear form, but that seems reasonable from a practical perspective.

It lastly bears noting that our requirement that V, W have an inner product satisfying linearity is natural. Only a vector space with a linear inner product exhibits notions of proximity, and we want to study "close" to some vector v_0 to employ linearization such as the Taylor Series.

Thus, we see that angle-preserving transformations can only operate in a 2D slice of a higher dimensionality system and no generalizations to higher dimension systems can capture additional complexity. This is obviously not meant to be a proof, but it seems reasonably clear that performing an analysis like the one we do above can only be done in 2D topologies.

16.3 Harmonic Functions and Conformal Mappings

Recall a harmonic function $u(x, y)$ satisfies Laplace equation $\nabla^2 u = 0$. We can show that there exists another harmonic function $v(x, y)$ such that $u + iv$ is analytic in $z = x + iy$; this is just inverting the Cauchy-Reimann conditions. Thus, a function being harmonic over a domain is equivalent to the sum of it and its harmonic conjugate being analytic over the same domain.

Harmonic functions are important because they crop up everywhere, with Neumann or Dirichlet BCs. It's also important that every harmonic function of $z = x + iy$ is also harmonic under change of variables $z = f(u + iv)$ if f is analytic; the proof follows because for the

harmonic function, there is a harmonic conjugate that makes it an analytic function of z and ergo of $u+iv$, which forces the original harmonic function te be harmonic in $u+iv$ as well.

Finally, to finish applying conformal mappings to harmonic functions, we must study how boundary conditions map. It turns out that Dirichlet and Neumann BCs for a function $H(x,y)$ on a *level curve* (curve where $H(x_c,y_c)=C$) are unchanged under conformal mapping; H will be constant on the image of the level curve.

Consider a boundary parameterized by $p(s)$, whose image boundary is $f(p(s))$. Then at some $p(s_0)$ and neighboring point $p(s_0+\Delta s)$, the harmonic function H takes on value $H(s_0),H(s_0+\Delta s)$, their difference being approximately $H'(s)\Delta s$. Under conformal mapping $w=f(z)$, path $p(s)$ maps to $p'(\sigma)$ where σ the new path length is related to the old by $\sigma=\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|s$. Thus the new path length difference is $\frac{\mathrm{d}H}{\mathrm{d}\sigma}=\frac{\mathrm{d}H}{\mathrm{d}s}\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|\neq\frac{\mathrm{d}H}{\mathrm{d}s}$ unless $\frac{\mathrm{d}H}{\mathrm{d}s}=0$, exactly a level curve.

16.4 Conformal Mapping and Laplace's Equation, Examples

Let's consider trying to compute steady-state $T(x,y)$, when $T(x=\pm\pi/2)=0$, $T(y=0)=1$ and $T(x,y)<\infty$ everywhere (equivalently, $T(y\to\infty)\to 0$), in the semi-infinite slab. With no heat sources, T must satisfy $\nabla^2 T=0$ in the steady state limit. Of course, we could solve using separation of varibables and obtain a series solution, but there is a slicker solution.

Let $z=x+iy$. We consider a two-step transformation: $z'=\sin z$ maps the semi-infinite slab in z to the half plane $y'>0$ onto . Then $w=\mathrm{Log}\frac{z'-1}{z'+1}$ maps the half plane $y'>0$ onto an infinite horizontal slab $w=u+iv, v\in[0,\pi]$.

Let's now keep careful track of where the boundaries of the original z domain map to. In z', it is clear that the real axis of the z domain maps to the $x'\in[-1,1]$ region in z' as well. The other two boundaries, $z=\pm 1+iy, y>0$ map to $z'=x, |x|>1$. Thus, $T(z')=1$ on the real axis between $[-1,1]$ and is zero along the rest of the real axis.

Then, when making the w mapping, the two boundaries $w=u, w=u+i\pi$, we note that $\mathrm{Re}(z')\in[-1,1]$ maps to $w=u+i\pi$ and the remainder of the axis maps to $w=u$. Thus, $T(u)=0, T(u+i\pi)=1$.

Now, a simple harmonic function that is bounded in the strip and satisfies the boundary conditions is just $T(w)=\frac{v}{\pi}=\frac{\mathrm{Im}(w)}{\pi}$. We note that it is level on the boundary (to satisfy the BCs) and is bounded in the domain. Inverting the transformations gives us

$$T(w)=\frac{\mathrm{Im}(w)}{\pi}, \tag{16.3}$$

$$T(z')=\frac{1}{\pi}\arctan\frac{2y'}{x'^2+y'^2-1}, \tag{16.4}$$

$$T(z)=\frac{2}{\pi}\arctan\frac{\cos x}{\sinh y}. \tag{16.5}$$

It is worth noting that if we do not require T remain finite at infinity, we could have added any homogeneous solution to $T(w)$; one class of such functions are $T(u,v)=\sin nv e^{\pm nu}$,

obtained by a simple separation of variables ansatz. Inverting this to find its effect on $T(z)$ is a bit too much work for the aging author, but the book's example of $T(z)[+A\sin z]$ suggests this will map to some class of functions $A\sin nz$ or the ilk.

Chapter 17

10/05/18—Plane Boarding

Anybody that has boarded a plane knows that one always has to wait for the person in front to put up their bag before getting a seat. It's so infuriating! So, on a flight right now, let's give some thought to how we might treat this mathematically.

17.1 Counting Inversions

Let's think about the simplest problem: given a sequence x_i of N numbers, consider an *inversion* to be a subsequence $x_i < x_{i+1}$, a discrepancy from being sorted in descending order. This is of course motivated by the fact that, if everybody in the back of the plane boarded first, there would be no waiting for people in front of you (assuming each row has only one seat). What is the distribution of inversions?

Well, it should be obvious first that you expect $\frac{N-1}{2}$ inversions: each subsequence of length 2 is either ordered or not ordered, with probability 1/2, and so by linearity of expectation we expect $\frac{N-1}{2}$ inversions. By another argument, if we flipped the definition of inversion, there is exactly one other sequence (reversing the numbers) that contributes to the original inversion definition in the exact same way. Thus, the definition of an inversion should not depend on the sign, and so the average number of inversions must be exactly half so that they sum to be $N-1$ the total number of subsequences.

Additionally, since each X_i seems to be just a Bernoulli process, the variance is $p(1-p) = \frac{1}{4}$, and so the standard deviation of the distribution should be $\frac{\sqrt{N-1}}{2}$. Instead, the standard deviation seems to be $\sqrt{(N-1)/12}$! See Figure 17.1.

I'm not sure what the discrepancy is for the time being.

17.2 Number of people to sit

This is a slightly more involved problem and a bit more realistic: if everybody in front of you in line has a seat number further to the back, you can sit! Thus, if we have a sequence x_i of N numbers, a number x_i can sit *if it is smaller than every number before it*.

The problem may seem more intimidating, but a closed formula exists. Consider the first person on line, this person always can sit. Of the remaining people, everybody who is larger than the first person is immediately ineligible; only those who are smaller can continue to

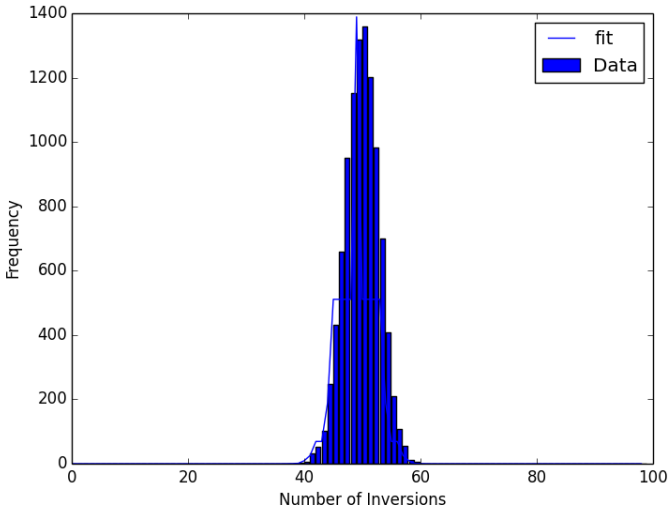


Figure 17.1: Histogram of inversions, fitted with $\mu = \frac{N-1}{2}, \sigma^2 = \frac{N-1}{12}$.

be eligible for a seat. The first person is equally distributed among the N possible values, and so letting $f(n)$ be the expected number of people that can sit of a total of n, we may write down recursive formula

$$f(n) = \frac{1}{n}\sum_{m=1}^{n} f(n-m) + 1. \tag{17.1}$$

The base case of the recursive formula is just $f(0) = 0$, with no people in line, nobody can be seated. We may examine the agreement of this recursive formula in Figure 17.2.

Of course, of interest is evaluation of Equation 17.1 in closed form. Perhaps easiest is to observe that $(n-1)f(n-1) - (n-1) = \sum_{m=1}^{n-1} f(n-m)$, and so $f(n) - 1 = \frac{1}{n}[*]f(n-1) - \frac{n-1}{n}$, or $f(n) = f(n-1) + \frac{1}{n}$! This is easily $f(n) = \sum_{m=1}^{n} \frac{1}{m}$. This agreement can be seen in

In hindsight, this is another instance of the power of linearity of expectation: suppose you add people to the line randomly in decreasing order, then for each subsequent person, there is a $1/m$ probability they're placed at the front of the line, if they're the mth addition.

This has a simple relation to wait times: unlike a geometric series, e.g. where n/k of the people at each iteration get to be seated for some number k, we can only seat $\ln n/n$. This will be a reasonably large number for small lines, but as the line grows the number of people we can seat *decreases* rather than increasing as in exponential decay. So longer lines are actually increasingly inefficient in this model.

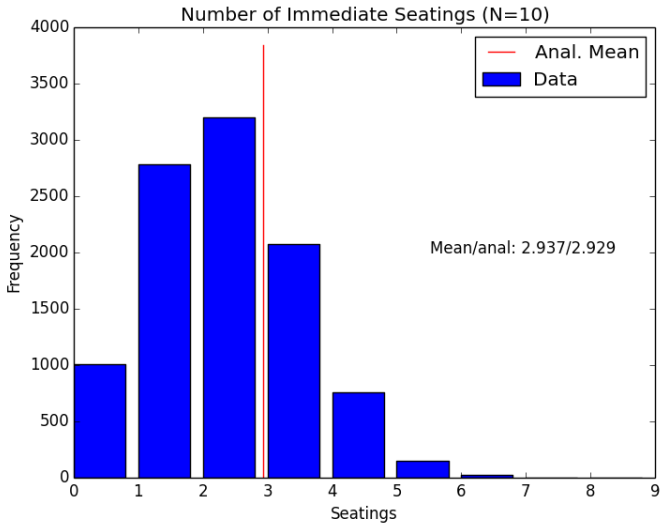


Figure 17.2: Agreement of Equation 17.1 with simulation.

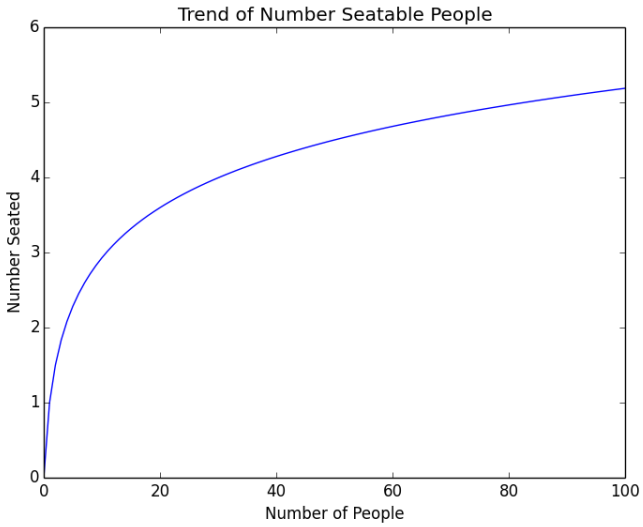


Figure 17.3: $f(n)$ evaluated using the recursive formula. The harmonic sum is evident in the logarithmic increase.

Chapter 18

06/29/19—Research Learning

18.1 Collisionless Boltzmann Equation in Galaxies: Landau Damping

Inspired by `https://arxiv.org/pdf/1906.08655.pdf`. The problem is basically formulated as thus: consider a kinetic-theoretic description of a fluid using distribution function $f(t,x,p)$ which obeys collisionless Boltzmann equation $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$ (we use p instead of v to work in Hamiltonian coordinates). Introducing a periodic perturbation to this fluid results in a singular dispersion relation, which can be resolved via the usual Landau prescription (consider a perturbation having grown from zero at $t = -\infty$). The dispersion relation describes *Landau damping* (or growth), in which energy from the fluid is exchanged with the perturber.

18.1.1 Linearized EOM

The point of the paper is instead to analytically compute the impact of the perturber on the distribution function, to quantify the *scarring* of a galaxy upon encounters with a nearby perturber. The equations of motion coupling the distribution function and gravitational potential are given

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{*\}f, \mathscr{H} = 0, \tag{18.1}$$

where $\mathscr{H} = \frac{p^2}{2} + \Phi$ and $\{*\}\ldots$ denotes the Poisson bracket $\{*\}f, \mathscr{H} = \vec{\nabla}_x f \cdot \vec{\nabla}_p \mathscr{H} - \vec{\nabla}_p f \cdot \vec{\nabla}_x \mathscr{H}$.

If we linearize for perturbation quantities f_1, Φ_1 where $\Phi_1(x)$ does not depend on the momenta, we obtain

$$\begin{aligned} 0 &= \frac{\partial f_1}{\partial t} + \{*\}f_1, \mathscr{H}_0 - \vec{\nabla}_p f \cdot \vec{\nabla}_x \mathscr{H}_0, \\ &= \frac{\partial f_1}{\partial t} + \vec{\nabla}_x f_1 \cdot \vec{p} - \vec{\nabla}_p f_1 \cdot \vec{\nabla}_x \Phi_0 - \vec{\nabla}_p f_0 \cdot \vec{\nabla}_x \Phi_1. \end{aligned}$$

Chapter 19

03/07/21—Random Youtube Problem

I thought the name of this was something like the "Pythagorean Problem", but I can't find the youtube video I got it from. The premise is simple: start with 1, and repeatedly take away k/N of the remainder. Which time do you take away the most?

Call a_k the remainder after taking away k/N, where $a_0 = 1$, then

$$\begin{aligned}
a_k &= \left(1 - \frac{k}{N}\right) a_{k-1}, \\
&= \prod_{j=1}^{k} \left(1 - \frac{j}{N}\right), \\
\ln a_k &= \sum_{j=1}^{k} \ln\left(1 - \frac{j}{N}\right), \\
&\approx \sum_{j=1}^{k} -\frac{j}{N}, \\
&\approx -\frac{k(k+1)}{2N}, \\
a_k &\approx -e^{-\frac{k(k+1)}{2N}}, \\
&\approx -e^{\frac{1}{8N}} e^{-\frac{(k+1/2)^2}{2N}}.
\end{aligned}$$

The approximation is valid assuming that we will only consider $k \ll N$ (which is validated in hindsight), and we numerically checked that this expression is reasonably accurate. The objective is then to find the time that you take away the most, i.e. the maximum of $a_k - a_{k-1}$. In the continuum limit, this is just the zero of $\mathrm{d}a_k/\mathrm{d}k$, which occurs where $k = \sqrt{N}$.