Miscellaneous Book Notes

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September 15, 2020

Chapter 1

Stein & Shakarchi: Princeton Lectures in Analysis

1.1 Book 1: Fourier Analysis

- *Lipschitz continuity* means continuity but also a bounded derivative.
- We define the vector space $\ell^2(\mathbb{Z})$ to be the set of all two-sided infinite sequences of complex numbers satisfying $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$, i.e. the space of Fourier coefficients. This is an infinite-dimensional Hilbert space (inner product space such that the inner product is positive definite and complete, so every Cauchy sequence in the norm converges to a limit in the vector space).
- Note that the partial sums of the Fourier series of a function f are convolututions with the *Dirichlet kernel*, i.e. we have

$$S_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left(\sum_{n=-N}^{N} e^{in(x-y)}\right) \mathrm{d}y, \tag{1.1}$$

$$= (f * D_N)(x), \tag{1.2}$$

where

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}. \tag{1.3}$$

- In general, we can consider a family of kernels $\{K_n\}_{n=1}^{\infty}$. Then families of *good kernels* satisfy:
  - For $n \geq 1$, $\int_{-\pi}^{\pi} K_n(x)\,\mathrm{d}x = 2\pi$.
  - There exists finite M for which $\int_{-\pi}^{\pi} |K_n(x)|\,\mathrm{d}x \leq M$ for all $n \geq 1$.
  - $\int_{\delta\leq|x|\leq\pi} |K_n(x)|\,\mathrm{d}x \to 0$ as $n \to \infty$.

  If f is integrable, and K_n are good kernels, then

$$\lim_{n\to\infty} (f * K_n)(x) = f(x), \tag{1.4}$$

  whenever f is continuous at x. Moreover, if f is continuous everywhere, the above limit is uniform. Sometimes, this is why good kernels are called an *approximation to the identity*.

In particular, the Dirichlet kernel is *not* a good kernel, as the integral of the absolute value diverges $\propto \log N$.

- We know that a Fourier series can fail to converge at individual points, i.e. the limit

$$\lim_{N\to\infty} S_N(f) = f, \tag{1.5}$$

where the S_N are the sums of the first N terms, does not converge. We resolve this with *Cesàro* and *Abel summability*.

Suppose $s_n = \sum\limits_{k=0}^{n} c_k$. Normally, we say s_n converges to s if $\lim_{n\to\infty} s_n = s$, and is the most natural type of "summability". However, if this fails to converge, we can define the Nth Cesàro mean or Cesàro sum by

$$\sigma_N = \frac{1}{N}\sum_{n=0}^{N-1} s_n. \tag{1.6}$$

If σ_N converges to a limit as N tends to infinity, we say that the original series $\sum c_n$ is *Cesàro summable* to σ. The archetypal Cesàro sum is the sum of alternating ± 1, which Cesàro sums to 1/2.

- Earlier, we saw that Dirichlet kernels are not good kernels, but their averages are well behaved. We see this by taking the Nth Cesàro mean of the Fourier series

$$\begin{aligned}
\sigma_N(f)(x) &= \frac{1}{N}\sum_{n=0}^{n-1} S_n(f)(x), && (1.7)\\
&= (f * F_n)(x), && (1.8)\\
F_N(x) &= \frac{1}{N}\sum_{n=0}^{n-1} D_n(x), && (1.9)\\
&= \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}. && (1.10)
\end{aligned}$$

This is the *Fejér Kernel*, and is a good kernel. Thus, if f is integrable, then the Fourier series of f is Cesàro summable to f at every point of continuity of f, and is uniformly summable if f is everywhere continous.

- Abel summability is an even more powerful notion of Cesàro summability. Given a series c_k, it is *Abel summable* to s if for every $0 \le r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k \tag{1.11}$$

converges, and

$$\lim_{r\to 1} A(r) = s. \tag{1.12}$$

These $A(r)$ are the *Abel means* of the series. Abel summation shows that $1 - 2 + 3 - 4 \cdots = 1/4$, since

$$A(r) = \sum_{k=0}^{\infty} (-1)^k (k+1) r^k = \frac{1}{(1+r)^2}. \tag{1.13}$$

- Similarly to how Cesàro summation gave the Fejér Kernel, Abel summation gives the *Poisson kernel*:

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}, \tag{1.14}$$
$$= (f * P_r)(\theta), \tag{1.15}$$
$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}. \tag{1.16}$$

  Again, the Poisson kernel is a good kernel for $0 \le r < 1$.

- Recall that the Fourier series converges in the mean-square sense:

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 \, \mathrm{d}\theta = 0, \tag{1.17}$$

  and moreover the coefficients of the Nth partial sum are the unique best approximation of the first N harmonics.

  Note that the terms of a converging series must tend to 0, so the Fourier coefficients must go to zero as well. This is the *Reimann-Lebesgue Lemma*:

$$\lim_{N\to\infty} \int_0^{2\pi} f(\theta) \sin(N\theta) \, \mathrm{d}\theta = 0. \tag{1.18}$$

- Consider f Lipschitz continuous at θ_0 ($|f(\theta) - f(\theta_0)| \le M|\theta - \theta_0|$ for some $M \ge 0$ and all θ) and differentiable. Then the Fourier series converges at θ_0 as $N \to \infty$.

  Construct

$$F(t) = \begin{cases} (f(\theta_0 - t) - f(\theta_0))/t & t \ne 0, \\ -f'(\theta_0) & t = 0. \end{cases} \tag{1.19}$$

  It is easy then to show that

$$S_N(f)(\theta_0) - f(\theta_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta_0 - t) D_n(t) \, \mathrm{d}t - f(\theta_0), \tag{1.20}$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(\theta_0 - t) - f(\theta_0)) D_n(t) \, \mathrm{d}t, \tag{1.21}$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) t D_n(t) \, \mathrm{d}t, \tag{1.22}$$
$$tD_n(t) = \frac{t}{\sin(t/2)} \sin\left(\left(N + \frac{1}{2}\right) t\right), \tag{1.23}$$

  where $D_n(t)$ is the Dirichlet kernel. Then the Reimann-Lebesgue lemma implies the second to last line vanishes, as the integrand is Reimann-integrable. We should be surprised by this, since this implies pointwise convergence depends only on the behavior of f near θ_0, even though the coefficients are obtained by integrating over all θ.

- There are a few problems that arise from the traditional notions of integrability/differentiability/continuity. For instance, the Fourier transform maps the space of Reimann-integrable functions $\mathscr{R}$ to the space of Fourier coefficients, denoted $\ell^2(\mathbb{Z})$. However, $\ell^2(\mathbb{Z})$ is *complete*, while $\mathscr{R}$ is not. The question is then: how do we complete $\mathscr{R}$, and how do we integrate these completed functions f?