1 Laplace Plane Dynamics

1.1 Maximum Separatrix Area—Simple

Consider a planet with orbit normal $\hat{\mathbf{l}}_{\rm p}$ that experiences precession driven by stellar oblateness $\hat{\mathbf{l}}_{\rm s}$ and an outer perturber $\hat{\mathbf{l}}_{\rm o}$. We assume that the planet's orbit is circular. The vector form of the precessional dynamics are (e.g. Tremaine+2009, Eq 19):

$$\frac{\mathrm{d}\hat{\mathbf{l}}_{\rm p}}{\mathrm{d}t} = \omega_{\rm sp}\left(\hat{\mathbf{l}}_{\rm p}\cdot\hat{\mathbf{l}}_{\rm s}\right)\left(\hat{\mathbf{l}}_{\rm p}\times\hat{\mathbf{l}}_{\rm s}\right) + \omega_{\rm op}\left(\hat{\mathbf{l}}_{\rm p}\cdot\hat{\mathbf{l}}_{\rm o}\right)\left(\hat{\mathbf{l}}_{\rm p}\times\hat{\mathbf{l}}_{\rm o}\right). \tag{1}$$

We first make an important symmetry argument: in the limits of $\omega_{\rm sp} \ll \omega_{\rm op}$ or $\omega_{\rm sp} \gg \omega_{\rm op}$, the evolution of $\hat{\mathbf{l}}_{\rm p}$ consist of uniform precession about $\hat{\mathbf{l}}_{\rm o}$ and $\hat{\mathbf{l}}_{\rm p}$ respectively, and thus the separatrix area must go to zero in these limits. In fact, the phase portrait must the same under the following transformation: swap the two frequencies $(\omega_{\rm sp},\omega_{\rm op})$ and the two vectors $(\hat{\mathbf{l}}_{\rm o},\hat{\mathbf{l}}_{\rm s})$. Swapping the precession frequencies is equivalent to taking $a/r_{\rm M} \mapsto r_{\rm M}/a$ (and rescaling time), since $\omega_{\rm op}/\omega_{\rm sp} = (a/r_{\rm M})^5$. Thus, we arrive at an important conclusion: *the phase portraits are equivalent, up to a rotation of reference frame, for any two $r_{\rm M,1}$ and $r_{\rm M,2}$ satisfying $a/r_{\rm M,1} = r_{\rm M,2}/a$*. This implies that the separatrix area is symmetric about $r_{\rm M} = a$ as well.

It's not clear that the separatrix area must be monotonic between $r_{\rm M} \in [0, a]$, but intuitively **this seems like it should be the case** (?), since there are no special values of $\omega_{\rm sp}/\omega_{\rm op}$ in the equation of motion. If so, then the maximum separatrix area is obtained for $r_{\rm M} = a$. The curve for the separatrix in this case is significantly easier to obtain, though it still seems difficult to integrate explicitly (maybe there's a clever idea?).

To compute the separatrix area for $a = r_{\rm M}$, we note that the low-obliquity Laplace equilibrium P1 is located exactly halfway between $\hat{\mathbf{l}}_{\rm o}$ and $\hat{\mathbf{l}}_{\rm s}$. Thus, we choose the reference frame such that $\hat{\mathbf{z}} \propto \hat{\mathbf{l}}_{\rm o}+\hat{\mathbf{l}}_{\rm s}$, and we choose $\hat{\mathbf{y}}$ to point towards P2 (which is always $\pi/2$ away from P1). Then, defining

$$\cos\epsilon \equiv \hat{\mathbf{l}}_{\rm o}\cdot\hat{\mathbf{l}}_{\rm s}, \tag{2}$$

we can write

$$\hat{\mathbf{l}}_{\rm o} = \cos\frac{\epsilon}{2}\hat{\mathbf{z}} + \sin\frac{\epsilon}{2}\hat{\mathbf{x}}, \tag{3}$$

$$\hat{\mathbf{l}}_{\rm s} = \cos\frac{\epsilon}{2}\hat{\mathbf{z}} - \sin\frac{\epsilon}{2}\hat{\mathbf{x}}. \tag{4}$$

Finally, upon inspection, $\hat{\mathbf{x}}$ is also an equilibrium point, which must be P3. **In summary, in this reference frame, P1 lies along $\hat{\mathbf{z}}$, P2 lies along $\hat{\mathbf{y}}$, and P3 lies along $\hat{\mathbf{x}}$.**

To get the level curve corresponding to the separatrix, we evaluate the Hamiltonian (factoring out the prefactor $\omega_{\rm sp} = \omega_{\rm op}$) and adopt a spherical coordinate system:

$$H \propto -\frac{1}{2}\left[\left(\hat{\mathbf{l}}_{\rm p}\cdot\hat{\mathbf{l}}_{\rm s}\right)^2 + \left(\hat{\mathbf{l}}_{\rm p}\cdot\hat{\mathbf{l}}_{\rm o}\right)^2\right], \tag{5}$$

$$\tilde{H}(\theta,\phi) = -\left[\sin^2\frac{\epsilon}{2}\sin^2\theta\cos^2\phi + \cos^2\frac{\epsilon}{2}\cos^2\theta\right], \tag{6}$$

where we have adopted spherical coordinates (θ,ϕ) to describe the orientation of $\hat{\mathbf{l}}_{\rm p}$, and $\theta = \pi/2, \phi = 0$ corresponds to $\hat{\mathbf{x}}$. We first evaluate H (dropping the tilde) at P3:

$$H_3 = -\sin^2\frac{\epsilon}{2}, \tag{7}$$

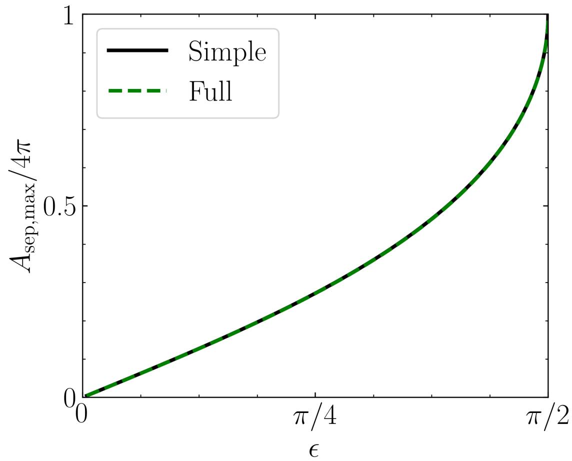

Figure 1: Fractional phase space area enclosed by the maximal separatrix as a function of ϵ. Reminder: this is the area surrounding Laplace equilibrium P2 when $r_{\rm M} = a$, which is also the maximum extent of the separatrix.

then the separatrix is given by

$$H\left(\theta_{\rm sep}(\phi), \phi\right) = H_3, \tag{8}$$

$$\sin^2\frac{\epsilon}{2}\left(1-\cos^2\theta_{\rm sep}\right)\cos^2\phi + \cos^2\frac{\epsilon}{2}\cos^2\theta_{\rm sep} = \sin^2\frac{\epsilon}{2}, \tag{9}$$

$$\begin{aligned}\cos^2\theta_{\rm sep} &= \frac{\sin^2\frac{\epsilon}{2}\sin^2\phi}{\cos^2\frac{\epsilon}{2} - \sin^2\frac{\epsilon}{2}\cos^2\phi}\\ &= \frac{\sin^2\phi}{\cot^2\frac{\epsilon}{2} - \cos^2\phi},\end{aligned} \tag{10}$$

$$A_{\rm sep} = 4\int_0^\pi \cos_+\theta_{\rm sep}\,\mathrm{d}\phi. \tag{11}$$

Here, $\cos_+\theta_{\rm sep}$ indicates that we take the positive root; one factor of two arises because the vertical extent of the separatrix is $\cos_+\theta_{\rm sep} - \cos_-\theta_{\rm sep}$, and a second factor of two arises because we are only integrating $\phi \in [0, \pi]$. Under this convention, the maximum possible phase space area is 4π. We display the value of $A_{\rm sep}$ in Fig. 1.

Note: the integral for A_{sep} is analytic:

$$\begin{aligned}
A_{\mathrm{sep}} &= 4\int_0^{\pi} \frac{\sin\phi}{\sqrt{\cot^2\frac{\epsilon}{2} - \cos^2\phi}}\,\mathrm{d}\phi \\
&= 4\int_{-1}^{1} \frac{1}{\sqrt{\cot^2\frac{\epsilon}{2} - \cos^2\phi}}\,\mathrm{d}\cos\phi \\
&= 4\left[\tan^{-1}\left(\frac{u}{\sqrt{\cot^2\frac{\epsilon}{2} - u^2}}\right)\right]_{u=-1}^{u=1} \\
&= 8\left[\tan^{-1}\sqrt{\frac{\sin^2(\epsilon/2)}{\cos\epsilon}}\right]. \qquad (12)
\end{aligned}$$

1.2 Maximum Separatrix Area—Melaine

Melaine says that the separatrix is given by the solutions to the equation (I_Q is the satellite inclination to the planet equator, and δQ is the corresponding phase angle; we call these θ, ϕ above)

$$\tan I_{\mathrm{Q},\pm} = \frac{\cos\delta Q\sin(2\epsilon) \pm \sin\delta Q\sin\epsilon\sqrt{2\left(u - 1 + \sqrt{1 + u^2 + 2u\cos(2\epsilon)}\right)}}{u + 1 - 2\cos^2\delta Q\sin^2\epsilon - \sqrt{1 + u^2 + 2u\cos(2\epsilon)}}. \qquad (13)$$

Here $u = r_{\mathrm{M}}^5/a^5$. However, this expression is singular for $\cos\delta_Q = w_\pm$, where

$$w_\pm = \pm\sqrt{\frac{1 + u - \sqrt{1 + u^2 + 2u\cos(2\epsilon)}}{2\sin^2\epsilon}}. \qquad (14)$$

This is where the denominator vanishes. Note that this must be a removable/coordinate singularity: physically, there is no special value of δ_Q. The separatrix area is then just given by integrating $\cos I_Q$ but taking the correctly-signed roots, which Melaine works out to be

$$\frac{A}{2} = \int_0^{\pi} \cos I_{\mathrm{Q},+} - \cos I_{\mathrm{Q},-}\,\mathrm{d}\delta_Q, \qquad (15)$$

$$\begin{aligned}
&= \int_0^{\arccos w_+} \left(\frac{-1}{\sqrt{1 + x_+^2}} - \frac{-1}{1 + x_-^2}\right)\mathrm{d}\delta_Q \\
&+ \int_{\arccos w_+}^{\arccos w_-} \left(\frac{1}{\sqrt{1 + x_+^2}} - \frac{-1}{1 + x_-^2}\right)\mathrm{d}\delta_Q \\
&+ \int_{\arccos w_-}^{\pi} \left(\frac{1}{\sqrt{1 + x_+^2}} - \frac{1}{1 + x_-^2}\right)\mathrm{d}\delta_Q. \qquad (16)
\end{aligned}$$

We find that the two expressions agree, see Fig. 1. Is it obvious that they should? Setting $u = 1$, we find that

$$\begin{aligned}\tan I_{\mathrm{Q},\pm} &= \frac{\cos\delta Q\,\sin(2\epsilon) \pm \sin\delta Q\,\sin\epsilon\sqrt{4\cos\epsilon}}{2 - 2\cos^2\delta Q\,\sin^2\epsilon - 2\cos\epsilon} \\ &= \frac{\cos\delta Q\,\sin(2\epsilon) \pm \sin\delta Q\,\sin\epsilon\sqrt{4\cos\epsilon}}{2 - 2\cos^2\delta Q\,\sin^2\epsilon - 2\cos\epsilon}.\end{aligned} \tag{17}$$

Not obvious.