Contents

Chapter 1

Flat Spacetime

1.1 Spacetime Physics

- Space tells matter how to move, matter tells space how to curve. Physics is simple when local, and when we eliminate force at a distance life is good.

- *Events* are coordinate-independent points in spacetime that are defined by "what happens there" or more concretely by an intersection of worldlines.

- Spacetime is locally Lorentzian, a.k.a. flat or non-accelerating. Gravitation/curvature is defined as the acceleration of the separation between two nearby geodesics.

  We measure curvature by the following: characterize ξ the separation between two originally parallel geodesics, then propagate each forward by distance s. They are not necessarily parallel anymore (e.g. two great circles on a sphere), and the separation obeys the following EOM

  $$\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} + R\xi = 0 \tag{1.1}$$

  where R is the *Gaussian Curvature* of the surface.

  Generalizing to multiple dimensions, the separation ξ is a vector, and we describe $\frac{\mathrm{D}^2\xi}{\mathrm{d}^2 s}$ with a capital D since the coordinates of the derivative $\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2}$ is subject to the whims of the coordinate lines, which should not affect the separation between these two geodesics that live beyond coordinates. The curvature is instead described by the Reiman curvature tensor, which takes as arguments the 4-velocity $\mathbf{u} = \dfrac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$ and yields

$$\frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}^2\tau^2} + R^\alpha{}_{\beta\gamma\delta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\xi^\gamma\frac{\mathrm{d}x^\delta}{\mathrm{d}\tau} = 0 \tag{1.2}$$

where $\mathbf{R}$ is the 4-component Reimann curvature tensor. We call this the *equation of geodesic deviation*.

These functions are sister functions of the Lorentz force equation in electromagnetism

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} - \frac{e}{m}F^\alpha{}_\beta\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = 0.$$

- $\mathbf{R}$ is a fickle object, subject to perturbation e.g. by gravitational waves. A certain piece of the tensor is generated only by the local mass distribution though, $\mathbf{G}$ the *Einstein curvature tensor*, incidently proportional to the *stress-energy tensor* as $\mathbf{G} = 8\pi\mathbf{T}$. Its interpretation is a local average curvature.

1.1.1 Practice Problems

Exercise 1.1 *Show that the Gaussian curvature of a cylinder* $R = 0$.

Choose cylindrical coordinates (a, θ, z) for the surface of the cylinder of radius a. We assert temporarily that all geodesics can be parameterized as $g(s|\omega, \dot{z}) = (a, \omega s, \dot{z}s)$, i.e. correspond to uniform translation along and rotation about the axis of the cylinder. Then, using the equation of geodesic deviation with Gaussian Curvatures

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} + R\xi = 0$$

for $\xi = g_1 - g_2$ the separation, then since g_1, g_2 are linear functions of s, their second derivatives with respect to s are zero and thus $\dfrac{\mathrm{d}^2\xi}{\mathrm{d}s^2} = 0$. Observe that this happens regardless of the value of ξ, thus $R = 0$.

We now justify our parameterization $g(s|\omega, \dot{z})$. Two points separated infinitisemally (a, θ, z), $(a, \theta + \delta\theta, z + \delta z)$ are separated by distance

$$\sqrt{\delta z^2 + a^2\delta\theta^2} = \mathrm{d}\theta\sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + a^2}.$$

The distance between two points $(\theta_1, z_1), (\theta_2, z_2)$ is then given by

$$D[z(\theta)] = \int_{z_1}^{z_2} \sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + a^2}\,\mathrm{d}\theta$$

where $z(\theta)$ is any trajectory that runs between the desired endpoints. This is then a variational calculus problem, and thus the $z(\theta)$ that minimizes the path must satisfy the Euler-Lagrange Equation

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial I(z', z)}{\partial z'} - \frac{\partial I(z', z)}{\partial z} = 0.$$

We note then that $I(z', z) = \sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + a^2}$ is independent of z and so we instantly find that $I(z', z) = C$ a constant. This furthermore implies that $\frac{\mathrm{d}z}{\mathrm{d}\theta}$ is a constant and so that $z \propto \theta$. Call the constant of proportionality $z \propto \alpha\theta$.

Armed with this, we find that the arclength of the geodesic between the desired endpoints is

$$D[z(\theta)] = \int_{z_1}^{z_2} \sqrt{\alpha^2 + a^2}\,\mathrm{d}\theta \tag{1.3}$$

$$= \sqrt{\alpha^2 + a^2}\Delta z. \tag{1.4}$$

Recalling that s parameterizes the length of our geodesic, we find that s, z, θ must all be proportional. Call the ratios $\frac{\theta}{s} = \omega, \frac{z}{s} = \dot{z}$, justifying our parameterization.

Exercise 1.1b *Alternatively, employ $R = \frac{1}{\rho_1\rho_2}$ where ρ_1, ρ_2 are the principal radii of curvature at the point in question in the enveloping Euclidean 3D space.*

We note that one of the radii in a cylinder is a while the other is infinite, thus $R = 0$.

Exercise 1.3 *Show that given ω, the rotational frequency of a planet about a fixed central mass M, we can not individually determine r the radius of orbit or M. Instead, derive a relation between ω and ρ the* Kepler Density *of the mass, the density if M were spread over the sphere of radius r.*

We know that a central acceleration $\ddot{r} = r\omega^2 = \frac{GM}{c^2r^2}$ under Newton's Law of Gravitation.

Thus, we have

$$\omega^2 = \frac{GM}{c^2 r^3} = \frac{4\pi}{3}\frac{G}{c^2}\rho$$

where since there are no other constraints on the motion we are done.

1.2 Special Relativity and 1-forms

SR/GR aspire to describe all laws as relationships between geometric objects, i.e. events are points, [tangent] vectors connect points, and the metric (defining lengths of vectors) is also geometric. First some definitions

Vector The vector joining A, B as $\frac{\mathrm{d}}{\mathrm{d}\lambda}P(\lambda)$ where $P(\lambda)$ is the straight line joining A, B.

Metric tensor $G(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.

Differential forms Consider the propagation of a plane wave with wavevector $\vec{k}$. There are surfaces of codimension 1 that have equal phase. Define $\tilde{\mathbf{k}}$ such that a vector $\vec{v}$ incurs phase difference $\left\langle \tilde{\mathbf{k}}, \vec{v} \right\rangle$.

This is an example of a 1-form, a differential form that takes a single vector and returns a scalar, often also denoted by σ boldface greek letters. We call $\langle \sigma, \vec{v} \rangle$ the contraction of σ with $\tilde{\mathbf{v}}$.

Note that there is a one-to-one correspondence $\tilde{\mathbf{p}}$ and $\mathbf{p}$, i.e. $\langle \tilde{\mathbf{p}}, \mathbf{u} \rangle$ is the same as $\mathbf{p} \cdot \mathbf{u}$, and so we sometimes omit the tilde.

More generally, a 1-form is a local approximation the same way a vector is a local derivative, i.e. if we want the phase of a point P close to P_0, we have

$$\phi(P) = \phi(P_0) + \left\langle \tilde{\mathbf{k}}, P - P_0 \right\rangle + \dots \tag{1.5}$$

Gradient as a 1-form The gradient is a 1-form, not a vector, since it is the first-order approximation

$$f(P) = f(P_0) + \langle \mathbf{d}f, P - P_0 \rangle + \dots \tag{1.6}$$

Coordinates Choose an orthonormal basis $\mathbf{e}_\alpha$ for vectors, then there is also a basis $\mathbf{w}^\alpha = \mathbf{d}x^\alpha$ for

1-forms, such that

$$\langle \mathbf{w}^\alpha, \mathbf{e}_\beta \rangle = \delta^\alpha_\beta \tag{1.7}$$

$$\langle v^\alpha \mathbf{w}^\alpha, \sigma_\beta \mathbf{e}_\beta \rangle = \langle \sigma, \mathbf{v} \rangle = \sigma_\alpha v^\alpha \tag{1.8}$$

4-velocity To work w/ dynamical variables, we need to start mapping quantities to geometric objects somewhere. Start with the 4-velocity, which in specific Lorentz reference frame has components

$$\begin{aligned} u^0 &= \frac{\mathrm{d}t}{\mathrm{d}\tau} &&= \frac{1}{\sqrt{1 - |\vec{v}^2|}} \\ u^j &= \frac{\mathrm{d}x^j}{\mathrm{d}\tau} &&= \frac{v^j}{\sqrt{1 - |\vec{v}|^2}} \end{aligned} \tag{1.9}$$

Lorentz Transformations Two types, rotations and boosts, generate a tensor $\Lambda^\alpha{}_\beta$. Rotations look like

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.10}$$

while boosts look like

$$\Lambda = \begin{bmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{bmatrix} \tag{1.11}$$

where velocity $\beta = \tanh\alpha$.

Then both components and basis vectors are simply contracted w/ this tensor. Lorentz transformations obey $\Lambda^T \eta \Lambda = \eta$ transpose.

1.2.1 Exercises

2.2–2.4 *Derive the following formulae:*

$$\begin{aligned} u_\alpha &= \eta_{\alpha\beta} u^\beta \\ u^\alpha &= \eta^{\alpha\beta} u_\beta \\ \mathbf{u} \cdot \mathbf{v} &= u^\alpha v^\beta \eta_{\alpha\beta} = u^\alpha v_\alpha = u_\alpha v_\beta \eta^{\alpha\beta} \end{aligned}$$

This is trivial,

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_\alpha v_\beta \eta^{\alpha\beta} \\ &= \langle \tilde{\mathbf{u}}, \mathbf{v} \rangle = u^\alpha v_\beta \\ &= \langle \mathbf{u}, \tilde{\mathbf{v}} \rangle = u^\alpha v^\beta \\ u_\alpha \eta^{\alpha\beta} &= u^\alpha \qquad (1.12) \\ v_\beta \eta^{\alpha\beta} &= v^\beta \qquad (1.13) \end{aligned}$$

2.5 *Verify the following coordinate-free relations a particle of mass m and 4-momentum $\mathbf{p}$ measured by an observer with 4-velocity $\mathbf{u}$:*

$$\begin{aligned} \mathbf{u}^2 &= -1 \\ E &= -\mathbf{p} \cdot \mathbf{u} \\ m^2 &= -\mathbf{p}^2 \\ |\vec{p}| &= \sqrt{(\mathbf{p} \cdot \mathbf{u})^2 + (\mathbf{p} \cdot \mathbf{p})} \\ |\vec{v}| &= \frac{|\vec{p}|}{E} \\ \mathbf{v} &= \frac{\mathbf{p} + (\mathbf{p} \cdot \mathbf{u})\mathbf{u}}{-\mathbf{p} \cdot \mathbf{u}} \end{aligned}$$

We must go into an arbitrary Lorentz frame for this:

$$\begin{aligned} \mathbf{u}^2 &= -\frac{1}{1 - v^2} + \sum_i \frac{v_i^2}{1 - v^2} \\ &= -1 \qquad (1.14) \end{aligned}$$

We know that $E^2 = m^2 + |\vec{p}|^2$, where $\vec{p} = \gamma m\vec{v}$, and so

$$\begin{aligned}
-\mathbf{p}\cdot\mathbf{u} &= -m\mathbf{v}\cdot\mathbf{u} \\
&= m\frac{1}{\sqrt{1-v^2}} \\
m^2 + |\vec{p}|^2 &= m^2\frac{1}{1-v^2} \\
E^2 &= (\mathbf{p}\cdot\mathbf{u})^2 \qquad (1.15)
\end{aligned}$$

and the sign is chosen by defining E as positive.

But then, knowing the definition $\mathbf{p} = m\mathbf{u}$, this yields $m^2 = -\mathbf{p}^2$.

Again, $E^2 = m^2 + |\vec{p}|^2$, so since $E^2 = (\mathbf{p}\cdot\mathbf{u})^2$ and $m^2 = -\mathbf{p}\cdot\mathbf{p}$, we find $|\vec{p}|^2 = (\mathbf{p}\cdot\mathbf{u})^2 + \mathbf{p}\cdot\mathbf{p}$.

Earlier, we used that $\vec{p} = \gamma m\vec{v}$, and moreover we've seen that $E = m\gamma$ so we have $E = \frac{|\vec{p}|}{|\vec{v}|} = m\gamma$.

Not sure how to do the last one ☹.

2.6 *Ascribe a function $T(Q)$ to describe the temperature at event Q. To an observer traveling with 4-velocity $\mathbf{u}$, show that he measures temperature change $\frac{\mathrm{d}T}{\mathrm{d}\tau} = \langle \mathbf{d}T, \mathbf{u}\rangle$. Why is this reasonable?*

If the observer travels from event P to event $P + \mathbf{u}\Delta\tau$, then the temperature at the new event can be computed as an expansion about the initial point, which to first order is $\Delta T = \langle \mathbf{d}T, \mathbf{u}\Delta\tau\rangle$. Divide both sides by $\Delta\tau$ to obtain the desired result. Note that we want τ the proper time in the above relation because we are expanding in events about the rest frame of the temperature function T.

2.7 *Show the below Lorentz transformation corresponds to motion with velocity $\beta\vec{n}$ and satisfies $\Lambda^T\eta\Lambda = \eta$.*

$$\Lambda^0{}_0 = \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad (1.16)$$

$$\Lambda^0{}_j = \Lambda^j{}_0 = -\beta\gamma n^j \qquad (1.17)$$

$$\Lambda^j{}_k = \Lambda^k{}_j = (\gamma - 1)n^j n^k + \delta^{jk} \qquad (1.18)$$

We know that the above matrix is a Lorentz transformation if it satisfies $\Lambda^T\eta\Lambda = \eta$. Most explicitly, recall that $\Lambda^\mu{}_{\overline{\nu}}$ is the LT from unprimed vectors to primed vectors $v_\mu\Lambda^\mu{}_{\overline{\nu}}$. The transpose is swapping the order of the arguments $\Lambda^\mu{}_{\overline{\nu}} = (\Lambda^T)_{\overline{\nu}}{}^\mu$. Thus, what we aspire to show is

$$(\Lambda^T)_{\bar{\nu}}{}^{\mu}\eta_{\mu\rho}\Lambda^{\rho}{}_{\bar{\sigma}} = \eta_{\bar{\nu}\bar{\sigma}} \tag{1.19}$$

$$\eta_{\mu\rho}\Lambda^{\mu}{}_{\bar{\nu}}\Lambda^{\rho}{}_{\bar{\sigma}} = \eta_{\bar{\nu}\bar{\sigma}} \tag{1.20}$$

Let's examine the following few cases

- $\bar{\nu} = \bar{\sigma} = 0$ —

$$\begin{aligned}\eta_{\mu\rho}\Lambda^{\mu}{}_0\Lambda^{\rho}{}_0 &= (-\gamma^2 + \beta^2\gamma^2 n_j n^j)\\ &= -1\end{aligned}$$

- $\bar{\nu} = \bar{\sigma} \neq 0$ —

$$\begin{aligned}\eta_{\mu\rho}\Lambda^{\mu}{}_i\Lambda^{\rho}{}_i &= -\beta^2\gamma^2(n_i)^2 + \left[(\gamma-1)^2(n_i)^2 n^j n_j + 2(\gamma-1)(n_i)^2 + 1\right]\\ &= (n_i)^2\left[\gamma^2 - 2\gamma + 1 + 2\gamma - 2 - \beta^2\gamma^2 + 1\right] + 1\\ &= 1\end{aligned}$$

- $\bar{\nu} \neq \bar{\sigma}, \bar{\nu} = 0$ —

$$\begin{aligned}\eta_{\mu\rho}\Lambda^{\mu}{}_0\Lambda^{\rho}{}_i &= \beta\gamma^2 n^i - \beta\gamma(\gamma-1)n^i n^j n_j - \beta\gamma n^j\\ &= \beta\gamma(\gamma n^i - (\gamma-1)n^i n^j n_j - n^j)\\ &= 0\end{aligned}$$

- $\bar{\nu} \neq \bar{\sigma}$, none are 0 —

$$\begin{aligned}\eta_{\mu\rho}\Lambda^{\mu}{}_i\Lambda^{\rho}{}_j &= -\beta^2\gamma^2 n^i n^j + (\gamma-1)^2 n^i n^j n^k n_k + 2(\gamma-1)n^i n^j\\ &= n^i n^j\left[-\beta^2\gamma^2 + \gamma^2 - 2\gamma + 1 + 2\gamma - 2\right]\\ &= 0\end{aligned}$$

This verifies our claim that this tensor acts as a Lorentz transformation. To verify that it is a boost of $\beta\hat{n}$, we could simply rotate such that $\hat{n}$ lies along a coordinate axis. This would be very easy if we had a simple form for the rotation matrix, but I don't have one handy, so let's be more creative. Consider starting in a reference frame moving with some 4-velocity $\mathbf{v}$. The components in this comoving frame of the 4-velocity are simply $(-1, \vec{0})$. If we boost with our provided Lorentz transformation, we obtain

$$\mathbf{v} = (-\gamma, -\beta\gamma\hat{n}) \tag{1.21}$$

which, comparing with our earlier provided formula, corresponds to a 3-velocity of $-\beta\hat{n}$. Thus, the boost takes us into a frame that is moving with $+\beta\hat{n}$ relative to the initial reference frame.

1.3 Electromagnetic Field and Tensors

Lorentz Force Law The Lorentz force law both defines fields and predicts motions. It is written in geometric form

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = eF^\alpha{}_\beta u^\beta \tag{1.22}$$

where $\mathbf{F}$ is the Faraday tensor satisfying

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = e\mathbf{F}(\mathbf{u}) \tag{1.23}$$

$$F^\alpha{}_\beta = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{1.24}$$

Tensors A tensor with rank $\binom{n}{m}$ takes n 1-forms and m vectors to a scalar. Components of a tensor e.g. $S^{\alpha\beta}{}_\gamma$ can be computed by inserting the basis objects $S^{\alpha\beta}{}_\gamma = S^{\alpha\beta}{}_\gamma \sigma_\alpha \rho_\beta v^\gamma$. These transform with Lorentz transformation too.

We can always raise/lower indicies w/ the metric, so we only discuss the total rank of a tensor usually. In any case, a raised index is contravariant and a lowered index is called covariant.

Tensor Operations In geometric notation, the gradient of a tensor $\mathbf{S}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is

$$\begin{aligned} \nabla\mathbf{S} &= \partial_\xi \left(S_{\alpha\beta\gamma} u^\alpha v^\beta w^\gamma \xi^\delta \right) \\ &= S_{\alpha\beta\gamma,\delta} u^\alpha v^\beta w^\gamma \xi^\delta \end{aligned} \tag{1.25}$$

Contracting a tensor is to sum over one raised and one lowered index, e.g. $M_{\mu\nu} = R_{\alpha\mu}{}^\alpha{}_\nu$.

Divergence (on the first index of $\mathbf{S}$) is defined

$$\nabla \cdot \mathbf{S} = \nabla \mathbf{S}(\mathbf{w}^\alpha, \mathbf{u}, \mathbf{v}, \mathbf{e}_\alpha) S^\alpha{}_{\beta\gamma,\alpha} \tag{1.26}$$

Transpose is interchanging the order of arguments.

Wedge product is antisymmetrized tensor product

$$\mathbf{u} \wedge \mathbf{v} = \mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u} \tag{1.27}$$

Maxwell's Equations These become

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \qquad F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha \tag{1.28}$$

1.3.1 Exercises

3.4 *Formalize the tensor product* $\mathbf{T} = \mathbf{u} \otimes \mathbf{v}$ *e.g. if* $\mathbf{u}, \mathbf{v}$ *are first-rank tensors, then* $T^{\alpha\beta} = u^\alpha v^\beta$.

3.11 *Show that* $A_{\mu\nu} S^{\mu\nu} = 0$, *if* A, S *are antisymmetric and symmetric respectively. This in conjunction with the result from the next problem shows that we only need the symmetric/antisymmetric portions of a tensor when multiplying against an antisymmetric/symmetric tensor.*

3.12 *Show that for an arbitrary tensor* V, $V_{(\mu\nu\dots)}$ *is symmetric,* $V_{[\mu\nu\dots]}$ *is antisymmetric, and that only second rank tensor is fully described by its symmetric/antisymmetric parts.*