Miscellaneous Book Notes
So I Don't Have to Keep Reopening Books

Yubo Su

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Chapter 1

Stein & Shakarchi: Princeton Lectures in Analysis

1.1 Book 1: Fourier Analysis

- *Lipschitz continuity* means continuity but also a bounded derivative.
- We define the vector space $\ell^2(\mathbb{Z})$ to be the set of all two-sided infinite sequences of complex numbers satisfying $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$, i.e. the space of Fourier coefficients. This is an infinite-dimensional Hilbert space (inner product space such that the inner product is positive definite and complete, so every Cauchy sequence in the norm converges to a limit in the vector space).
- Note that the partial sums of the Fourier series of a function f are convolututions with the *Dirichlet kernel*, i.e. we have

$$S_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left(\sum_{n=-N}^{N} e^{in(x-y)}\right) \mathrm{d}y, \tag{1.1}$$

$$= (f * D_N)(x), \tag{1.2}$$

where

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}. \tag{1.3}$$

- In general, we can consider a family of kernels $\{K_n\}_{n=1}^{\infty}$. Then families of *good kernels* satisfy:
  - For $n \geq 1$, $\int_{-\pi}^{\pi} K_n(x)\,\mathrm{d}x = 2\pi$.
  - There exists finite M for which $\int_{-\pi}^{\pi} |K_n(x)|\,\mathrm{d}x \leq M$ for all $n \geq 1$.
  - $\int_{\delta\leq|x|\leq\pi} |K_n(x)|\,\mathrm{d}x \to 0$ as $n \to \infty$.

  If f is integrable, and K_n are good kernels, then

$$\lim_{n\to\infty} (f * K_n)(x) = f(x), \tag{1.4}$$

  whenever f is continuous at x. Moreover, if f is continuous everywhere, the above limit is uniform. Sometimes, this is why good kernels are called an *approximation to the identity*.

In particular, the Dirichlet kernel is *not* a good kernel, as the integral of the absolute value diverges $\propto \log N$.

- We know that a Fourier series can fail to converge at individual points, i.e. the limit

$$\lim_{N\to\infty} S_N(f) = f, \tag{1.5}$$

where the S_N are the sums of the first N terms, does not converge. We resolve this with *Cesàro* and *Abel summability*.

Suppose $s_n = \sum\limits_{k=0}^{n} c_k$. Normally, we say s_n converges to s if $\lim_{n\to\infty} s_n = s$, and is the most natural type of "summability". However, if this fails to converge, we can define the Nth Cesàro mean or Cesàro sum by

$$\sigma_N = \frac{1}{N}\sum_{n=0}^{N-1} s_n. \tag{1.6}$$

If σ_N converges to a limit as N tends to infinity, we say that the original series $\sum c_n$ is *Cesàro summable* to σ. The archetypal Cesàro sum is the sum of alternating ± 1, which Cesàro sums to 1/2.

- Earlier, we saw that Dirichlet kernels are not good kernels, but their averages are well behaved. We see this by taking the Nth Cesàro mean of the Fourier series

$$\sigma_N(f)(x) = \frac{1}{N}\sum_{n=0}^{n-1} S_n(f)(x), \tag{1.7}$$

$$= (f * F_n)(x), \tag{1.8}$$

$$F_N(x) = \frac{1}{N}\sum_{n=0}^{n-1} D_n(x), \tag{1.9}$$

$$= \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}. \tag{1.10}$$

This is the *Fejér Kernel*, and is a good kernel. Thus, if f is integrable, then the Fourier series of f is Cesàro summable to f at every point of continuity of f, and is uniformly summable if f is everywhere continous.

- Abel summability is an even more powerful notion of Cesàro summability. Given a series c_k, it is *Abel summable* to s if for every $0 \le r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k \tag{1.11}$$

converges, and

$$\lim_{r\to 1} A(r) = s. \tag{1.12}$$

These $A(r)$ are the *Abel means* of the series. Abel summation shows that $1 - 2 + 3 - 4 \cdots = 1/4$, since

$$A(r) = \sum_{k=0}^{\infty} (-1)^k (k+1) r^k = \frac{1}{(1+r)^2}. \tag{1.13}$$

- Similarly to how Cesàro summation gave the Fejér Kernel, Abel summation gives the *Poisson kernel*:

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}, \tag{1.14}$$

$$= (f * P_r)(\theta), \tag{1.15}$$

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}. \tag{1.16}$$

  Again, the Poisson kernel is a good kernel for $0 \le r < 1$.

- Recall that the Fourier series converges in the mean-square sense:

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 \, d\theta = 0, \tag{1.17}$$

  and moreover the coefficients of the Nth partial sum are the unique best approximation of the first N harmonics.

  Note that the terms of a converging series must tend to 0, so the Fourier coefficients must go to zero as well. This is the *Reimann-Lebesgue Lemma*:

$$\lim_{N\to\infty} \int_0^{2\pi} f(\theta) \sin(N\theta) \, d\theta = 0. \tag{1.18}$$

- Consider f Lipschitz continuous at θ_0 ($|f(\theta) - f(\theta_0)| \le M|\theta - \theta_0|$ for some $M \ge 0$ and all θ) and differentiable. Then the Fourier series converges at θ_0 as $N \to \infty$.

  Construct

$$F(t) = \begin{cases} (f(\theta_0 - t) - f(\theta_0))/t & t \neq 0, \\ -f'(\theta_0) & t = 0. \end{cases} \tag{1.19}$$

  It is easy then to show that

$$S_N(f)(\theta_0) - f(\theta_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta_0 - t) D_n(t) \, dt - f(\theta_0), \tag{1.20}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(\theta_0 - t) - f(\theta_0)) D_n(t) \, dt, \tag{1.21}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) t D_n(t) \, dt, \tag{1.22}$$

$$t D_n(t) = \frac{t}{\sin(t/2)} \sin\left(\left(N + \frac{1}{2}\right) t\right), \tag{1.23}$$

  where $D_n(t)$ is the Dirichlet kernel. Then the Reimann-Lebesgue lemma implies the second to last line vanishes, as the integrand is Reimann-integrable. We should be surprised by this, since this implies pointwise convergence depends only on the behavior of f near θ_0, even though the coefficients are obtained by integrating over all θ.

  Note that above we required the function be differentiable. Otherwise, functions can be very carefully constructed to fail to pointwise converge.

- Fourier analysis can be used to prove *Weyl's equidistribution theorem*. First, note that, for any real $\gamma \neq 0$, the sequence of numbers $\langle n\gamma \rangle$, where $\langle X \rangle$ denotes the fractional part, is either repeating for rational γ or never repeating for irrational γ. Weyl's equidistribution theorem goes further and says that the $\langle n\gamma \rangle$ are equidistributed (and thus dense) on $0 \le x < 1$.

  The proof is simple. Consider any interval (a, b) on the unit interval, and extend its membership/indicator function over $\mathbb{R}$ by periodicity, defined $\chi_{(a,b)}(x)$. The number of times $\langle n\gamma \rangle$ is in the interval is just the sum of $\chi_{(a,b)}(n\gamma)$, and equidistribution becomes the statement

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) = \int_0^1 \chi_{(a,b)}(x)\, \mathrm{d}x. \tag{1.24}$$

  This can be shown by verifying linearity, then for each of the individual Fourier harmonics. Clever!

  This theorem generalizes: a sequence $\{\xi_n\}$ over the unit interval is equidistributed iff

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} = 0. \tag{1.25}$$

  This is *Weyl's criterion*.

Moving onto Fourier Transforms (continuous):

- The space over which FTs can be taken is $\mathcal{M}(\mathbb{R})$, the set of functions of *moderate decrease*, i.e. that fall off at least as fast as $1/x^2$. This forms a vector space under addition and scalar multiplication. $1/x^2$ is necessary so that the integral over $[-N, N]$ as $N \to \infty$ falls off like $1/N$ and converges.

  However, $\mathcal{M}$ is insufficient to provide guarantees on the integral of the transform. Instead, consider the *Schwartz space*, denoted $\mathcal{S}(\mathbb{R})$, which falls off faster than any power of x (e.g. Gaussian); this implies its derivatives do as well. The Gaussian is a family of good kernels on the real line as their width goes to zero. The FT generally does not require $\mathcal{S}(\mathbb{R})$, and just requires both FT and f to be in $\mathcal{M}$.

- The *Poisson summation formula* gives a way to construct a *periodization* of a function $f \in \mathcal{S}$. It says

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}. \tag{1.26}$$

  This just follows from the definition of the FT, then the periodization is

$$F_1(x) = \sum_{n=-\infty}^{\infty} f(x+n). \tag{1.27}$$

- Some special functions that are related to this are: the *theta function*

$$\vartheta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}. \tag{1.28}$$

  Note that $s^{-1/2}\vartheta(1/s) = \vartheta(s)$ for $s > 0$, which follows from the Poisson summation formula for $\exp(-\pi s x^2)$. This is connected to the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.29}$$

and the Γ function via

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty t^{s/2-1}(\vartheta(t)-1)\,\mathrm{d}t. \tag{1.30}$$

- The *Radon transform* is a useful concept in imaging, and concerns the following: a beam of incident intensity I_0 passes through a medium with variable attenuation coefficient ρ, then the Radon transform is

$$X(\rho)(L) = \int_L \rho(s)\,\mathrm{d}s. \tag{1.31}$$

  We then want to ask whether it is possible to invert $X(\rho)$, which contains the attenuation information for all lines L. Counting dimensions, we see that in 2D, the space of lines is of dimension 2, as is the medium, so the inverse may exist, and it turns out it does, but is tricky to deal with so we discuss 3D.

  In 3D, the space of lines is of dimension 4, while the medium is only of dimension 3. Instead, the more natural generalization is to integrate over *planes*. So the Radon transform maps from a plane (characterized by a tangent vector, so 3D) and a time to a scalar, and is given by

$$\mathscr{R}(f)(t,\gamma) = \int_{P_{t,y}} f\,\mathrm{d}A. \tag{1.32}$$

  It turns out that the result is (and can be given by FTs)

$$\nabla^2\left(\mathscr{R}^*\mathscr{R}(f)\right) = -8\pi^2 f. \tag{1.33}$$

- By applying Fourier analysis to finite spaces, we can prove *Dirichlet's theorem* on primes in arithmetic progression: if q, l are positive integers with no common factor, then $l + kq$ for $k \in \mathbb{Z}$ contains infinitely many prime numbers. We first cover some background in number theory:

  - One key result of number theory is the *fundamental theorem of arithmetic*: every positive integer > 1 can be factored uniquely into a product of primes.
  - A key result on infinite *products* of real numbers is that, if $A_n = 1 + a_n$, and $\sum |a_n|$ converges, then $\prod_n A_n$ also converges, and vanishes iff one of the $a_n = 0$. Furthermore, as long as none of the $a_n \neq 1$, $\prod_n 1/(1-a_n)$ also converges.
  - For $s > 1$, we define the *zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}. \tag{1.34}$$

    Note that ζ is continuous for $s > 1$.
  - Importantly, *Euler's product formula* argues that

$$\zeta(s) = \prod_p \frac{1}{1-1/p^s}, \tag{1.35}$$

    where the sum is only taken over prime p.

    Note that this is just a re-expression of the fundamental theorem of arithmetic. Recall

$$\frac{1}{1-p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots \tag{1.36}$$

But then, by the fundamental theorem of arithmetic, every number n in the zeta funciton sum can be uniquely decomposed as a product of powers of primes. Thus, multiplying all these powers together, we get each n exactly once. I'm not going to work through the proof.

- A next ingredient is another Euler proposition: $\sum_p 1/p$ is divergent, when summed over prime p. To prove this, we take a log of both sides of the Euler formula

$$\log\zeta(s) = -\sum_p \log\left(1-1/p^s\right), \tag{1.37}$$

$$= -\sum_p \left[-1/p^s + \mathscr{O}(1/p^{2s})\right], \tag{1.38}$$

$$= \sum_p 1/p^s + \mathscr{O}(1). \tag{1.39}$$

Here, recall $\sum_p 1/p^{2s} \le \sum_n 1/n^2 = \pi^2/6$. But the zeta function diverges as $s \to 1^+$, as it becomes the harmonic sum.

- Now, to prove Dirichlet's theorem, we just have to show that one further refinement of the Euler proposition diverges:

$$\sum_{p\equiv l \pmod q} 1/p. \tag{1.40}$$

This proof is nontrivial and requires Fourier analysis on the finite group $\mathbb{Z}^*(q)$ (which I skipped), but we proceed:

  - I skipped this, but: $\mathbb{Z}(q)$ is the integers modulo q. Multiplication is unambigious on $\mathbb{Z}(q)$, as we just multiply then take the modulus. An integer $n \in \mathbb{Z}(q)$ is a *unit* if $m \in \mathbb{Z}(q)$ such that $nm \equiv 1 \pmod q$. $\mathbb{Z}^*(q)$ is an abelian group under multiplication modulo q. Alternatively, $\mathbb{Z}^*(q)$ are the elements in $\mathbb{Z}(q)$ that are relatively prime to q. It's somewhat interesting that the multiplicative group $\mathbb{Z}^*(q)$ lies within the additive group $\mathbb{Z}(q)$!
  - Let's specialize to $q = 4, l = 1$ for simplicity. We define the character on $\mathbb{Z}^*(4)$ such that $\chi(1) = 1$ and $\chi(3) = -1$, and zero otherwise. This generalizes to the rest of $\mathbb{Z}$ by just taking modulo 4. This character is multiplicative, so $\chi(nm) = \chi(n)\chi(m)$. Let's define the sum

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.41}$$

Note that $L(1,\chi) = \pi/4 < \infty$. The Euler product can be generalized, since χ is multiplicative, so

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)/p^s}. \tag{1.42}$$

Taking the logarithm of both sides, we find

$$\log L(s,\chi) = \sum_{p\equiv 1} \frac{1}{p^s} - \sum_{p\equiv 3} \frac{1}{p^s} + \mathscr{O}(1). \tag{1.43}$$

Taking $s \to 1^+$, we find the RHS converges. However, since $\sum_p p^{-s}$ diverges when $s \to 1^+$, which is the sum of the two terms on the RHS, the individual terms must diverge as well. This proves that $\sum_{p\equiv 1} 1/p$ diverges, so there must be infinitely many primes of the form $4k+1$.

I omit the general proof, as the last step (summing over the different characters) seems very complicated to do in general.

1.2 Book 2: Complex Analysis

- Recall *holomorphic* functions are differentiable in the complex plane. A *meromorphic* function is holomorphic except over a set of isolated points. A general principle of complex analysis is that analytic functions are effectively determined by their zeros, and meromorphic functions by their zeros and poles.

- For a given power series $\sum\limits_{n=0}^{\infty} a_n z^n$, there exists $0 \le R \le \infty$ such that (i) if $|z| < R$ then the series converges absolutely, and (ii) if $|z| > R$ then the series diverges, and R the radius of convergence is given by *Hadamard's formula*
$$\frac{1}{R} = \limsup |a_n|^{1/n}. \tag{1.44}$$
  Note that a power series defines a holomorphic function in its disc of convergence, the derivative obtained by term-by-term differentiation, which has the same radius of convergence. This implies that a power series is infinitely complex differentiable in its disc of convergence.

  A function f is *analytic* at a point z_0 if it has a local power series expansion with positive radius of convergence. This turns out to be equivalent to being holomorphic.

- Of the three types of singularities (removable, poles, and essential), poles are handled by the Cauchy residue theorem, while removable singularities of $f(z)$ at z_0 require that $f(z)$ be finite in any deleted neighborhood about z_0. Essential singularities are described by the *Casorati-Weierstrass Theorem*: suppose f is holomorphic in the deleted disk $D_r(z_0) - \{z_0\}$, and has an essential singularity at z_0, then the image of $D_r(z_0) - \{z_0\}$ under f is dense in the complex plane.

  This is proven simply by contradiction. If the range of f is not dense, there exists $w \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - w| > \delta$ for all $z \in D_r(z_0) - \{z_0\}$. But if we define $g(z) = 1/(f(z) - w)$ on the deleted disk, it is holomorphic on the deleted neighborhood, and z_0 must be either a removable singularity or pole depending on whether $g(z) \neq 0$ or not, respectively.

  Picard proved a much stronger result, that f about the essential singularity takes on every complex value infinitely many times, with at most one exception! Neat.

- The Cauchy integral formula states (we did this in ACM95, I just am a bit hazy so I rewrite briefly)
$$f(z) = \frac{1}{2\pi i} \int\limits_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta, \tag{1.45}$$
  for any point $z \in D$ if F is holomorphic on D. This has some clear consequences, including the averaging formula ($\zeta - z$ is constant), Liouville's Theorem (entire, bounded functions are constant).

  An complex analog to Weierstrass's theorem in real analysis (continuous functions on a compact interval can be uniformly approximated by polynomials) is Runge's approximation: any function holomorphic in a neighborhood of a compact set K can be uniformly approximated by polynomials if K^c (the complement) is connected.

- Note that the meromorphic functions in the extended complex plane are the rational functions! This is simple: for such a function, its behavior near all poles must be described by *principal parts*, i.e. near pole z_k, $f(z) = f_k(z) + g_k(z)$ where f_k is polynomial in $1/(z - z_k)$ and g_k is holomorphic in the deleted neighborhood about z_k. Then, $H = f - \sum\limits_k f_k$ is nowhere singular in the extended complex plane, so it must be entire and bounded. By Liouville's Theorem, all bounded and entire functions must be constant, so H is in fact constant, and f rational.

- The *argument principle* is a useful application of the Cauchy residue theorem to meromorphic functions, and just requires a bit of thinking if we're willing to be slapdash:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = N - P, \tag{1.46}$$

where N,P are the number of zeros and poles inside of C of f, including multiplicities. The idea is clear: zeros become poles because $f(z)$ is in the denominator, while $f'(z)/f(z)$, if $f(z) = 1/z^p$, becomes $-p/z$, coarsely (this is me reading).

- We learned about deformation of integrals in ACM95. This is rigorously defined by requiring a *homotopy* $\gamma_s, s \in [0,1]$ connecting the two curves γ_0, γ_1.

Harmonic/Fourier stuff:

- Consider power series $f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$. Then

$$a_n = \frac{1}{2\pi r^n}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)e^{-in\theta}\,\mathrm{d}\theta. \tag{1.47}$$

This is interesting, since Fourier analysis and complex functions become related. Continuing.

- A holomorphic function f satisfies that the real and imaginary parts $f : g + ih$ for $g,h : \mathbb{C} \to \mathbb{R}$ are *harmonic*, so $\nabla^2 g = \nabla^2 h = 0$ where $\nabla^2 = \partial_{xx} + \partial_{yy}$.

- Functions for which the Fourier inversion formulae are valid, given by

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi, \tag{1.48}$$

$$\hat{f}(\xi) = -\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x, \tag{1.49}$$

require $|f(x)| \le A/\left(1 + x^2\right)$ and $\left|\hat{f}(\xi)\right| \le A'/(1+\xi^2)$, "moderate decrease".

The *Paley-Wiener Theorem* says that if f is continuous and of moderate decrease in $\mathbb{R}$, then f has an extension to the complex plane that is entire with $|f(z)| \le Ae^{2\pi M|z|}$ for some $A > 0$ iff $\hat{f}$ is supported in the interval $[-M, M]$. The intuition is just that the FT of $f(z)$ is bounded:

$$\left|\int_{-M}^{M} \hat{f}(\xi)e^{2\pi i\xi z}\,\mathrm{d}\xi\right| \le \int_{-M}^{M} \left|\hat{f}(\xi)\right| e^{-2\pi\xi(\mathrm{Im}(z))}\,\mathrm{d}\xi \le Ae^{2\pi M|z|}. \tag{1.50}$$

The detailed proof is somewhat more complex, since the real part of z needs to satisfy certain bounds to be integrable as $\mathrm{Re}\, z \to \infty$. A key result is the *Phragmén-Lindelöf theorem*: suppose F is holomorphic on the sector $S = \{z : -\pi/4 < \arg z < \pi/4\}$ and is continuous on the closure of S. Assume $|F(z)| \le 1$ on the boundary of the sector, then there are constants $C, c > 0$ such that $|F(z)| \le Ce^{c|z|}$, then in fact $|F(z)| \le 1$ over S. In other words, if F is bounded on the boundary by 1, and doesn't exhibit unreasonable growth, it is bounded everywhere by 1. A simple counterexample exhibiting unreasonable growth is $f(z)\exp(z^2)$.

The PL theorem can be generalized: if S is a sector whose vertex is at the origin with angle π/β, and F is holomorphic on S that doesn't grow faster than $C\exp(c|z|^\alpha)$ for $0 < \alpha < \beta$, then $|F(z)| \le 1$.

Conformal Mappings

- Conformal mappings answer the question: given two open sets $U, V \subseteq \mathbb{C}$, doees there exist a holomorphic bijection between them? We call U and V *biholomorphic* or *conformally equivalent*.

- Often want to map onto $V = \mathbb{D}$ the unit disk. It turns out that $f : U \to V$ being holomorphic and injective is sufficient for bijectivity. For instance, the upper half plane maps onto the unit disk via injective, holomorphic map $F : \mathbb{H} \to \mathbb{D}$ where $z \mapsto (i - z)/(i + z)$. Indeed, the inverse exists, $G(w) = i(1 - w)/(1 + w)$.

- Why is this useful? Consider the Dirichlet problem (BVP) on an open set Ω, given by $\nabla^2 u = 0$ on Ω and $u = f$ on $\partial\Omega$. The solution where $\Omega = \mathbb{D}$ is well studied and is simple:

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)\tilde{f}\left(e^{i\varphi}\right)\mathrm{d}\varphi, \tag{1.51}$$

  where P_r is the Poisson kernel, and $\tilde{f}$ is the boundary data. We can use conformal maps to generalize this solution to other domains, e.g. in the book we try this for a strip. This works because of the following lemma: if U, V are open sets in $\mathbb{C}$ and $F : V \to U$ holomorphic while $u : U \to \mathbb{C}$, then $u \circ F$ is harmonic on V.

- The Schwarz lemma gives us a bit more understanding of holomorphic functions on $\mathbb{D}$. Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, and $f(0) = 0$. Then over $\mathbb{D}$:

  - $|f(z)| \le |z|$
  - $\left|f'(0)\right| \le 1$.
  - If for any $z_0 \ne 0$, we have $|f(z_0)| = |z_0|$, then f is a rotation.
  - If $\left|f'(0)\right| = 1$, then f is a rotation.

  This defines the group of *automorphisms* (map of an open set to itself) of $\mathbb{D}$, denoted in general $\mathrm{Aut}(\mathbb{D})$, which forms a group under composition.

  The obvious automorphisms are the identity and rotations. An interesting class of automorphisms are $\psi_\alpha(z) = (\alpha - z)/(1 - \bar{\alpha}z)$ where $|\alpha| < 1$, as they are their own inverse. In particular, combining rotations and these automorphisms span $\mathrm{Aut}(\mathbb{D})$.

- Note that, since we have the conformal map $F : \mathbb{H} \to \mathbb{D}$, and $\mathrm{Aut}(\mathbb{D})$, we also can build $\mathrm{Aut}(\mathbb{H})$ as $\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H})$ such that $\varphi \mapsto F^{-1} \circ \varphi \circ F$. So Γ is an isomorphism between these two groups. Carrying this calculation out shows that

$$\mathrm{Aut}(\mathbb{H}) = \left\{f(z) : z \mapsto \frac{az + b}{cz + d} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1\right\}. \tag{1.52}$$

Thus, we see that $\mathrm{Aut}(\mathbb{H})$ is related to $\mathrm{SL}_2(\mathbb{R})$, the group of 2×2 matricies with real entries and determinant 1. However, the same transformation is generated by matrix elements $M = ((a,b),(c,d))$ and $-M$, so the group that is isomorphic with $\mathrm{Aut}(\mathbb{H})$ is the *projective special linear group* $\mathrm{PSL}_2(\mathbb{R})$, for which the sign of M is projected/quotiented out.

- The *Riemann mapping theorem* is the key result. If Ω is proper and simply connected, and $z_0 \in \Omega$, then there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$. Equivalently, any two proper simply connected open subsets in $\mathbb{C}$ are conformally equivalent.

  One special case of the Riemann mapping theorem is when we want to map polygons onto the disk; these maps can be given explicitly by the *Schwarz-Christoffel formula*. First, we define the Schwarz-Christoffel integral

$$S(z) = \int_0^z \prod_{i=1}^N (\zeta - A_i)^{\beta_i}\, \mathrm{d}\zeta, \tag{1.53}$$

  where the A_i are N distinct points on the real axis arranged in increasing order, and β_i satisfy $\beta_k < 1$ and $\sum\limits_k \beta_k > 1$. Up to some branch cut, $\lim_{|z|\to\infty} S(z) = a_\infty$ exists. If we also define $a_k = S(A_k)$, then it turns out that the $\{a_k, a_\infty\}$ will form a polygon of either N or $N+1$ verticies. In these two cases, the formal map of $\mathbb{H}$ to the polygon P is given by $F(z) = c_1 S(z) + c_2$ (in the latter case, the product runs up to N, one fewer than the number of vertices).

- An example of these conformal maps is the elliptic integral

$$I(z) = \int_0^z \frac{1}{\left[\left(1-\zeta^2\right)\left(1-k^2\zeta^2\right)\right]^{1/2}}\, \mathrm{d}\zeta. \tag{1.54}$$

  This clearly has the same form sa the Schwarz-Christoffel formula above, and indeed we find it is map from the real axis to a rectangle with verticies $\pm K, \pm K + iK'$ for

$$K(k) = \int_0^1 \frac{1}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}\, \mathrm{d}x, \tag{1.55}$$

$$K'(k) = \int_1^{1/k} \frac{1}{\sqrt{\left(x^2-1\right)\left(1-k^2x^2\right)}}\, \mathrm{d}x. \tag{1.56}$$

  Indeed, $I(z)$ maps $\mathbb{H}$ to the interior of this rectangle. If we consider $\mathrm{sn}(z)$ the inverse of $I(z)$, we note that the reflection principle gives $\mathrm{sn}(z) = \overline{\mathrm{sn}(\overline{z})}$. In fact, careful application of such reflections shows that $\mathrm{sn}(z) = \mathrm{sn}(z+4K) = \mathrm{sn}\left(z + 2iK'\right)$, i.e. it is *doubly periodic* in either direction. This is characteristic of *elliptic functions*, which we will study later.

Entire Functions, Theory and Examples

- *Jensen's formula* says that if f is holomorphic in some disk D_R where R is the radius, and z_k are the zeros of f in D_R, then

$$\log|f(0)| = \sum_{k=1}^N \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right|\, \mathrm{d}\theta. \tag{1.57}$$

  If f has no zeros in D_R, $\log|f|$ is a harmonic function, which also has a mean value property, so this is a generalization of that.

- We will want to consider infinite products, so we need some theorems. Note that if the sum $\sum_n |a_n| < \infty$, then the product $\prod_{n=1}^{\infty}(1+a_n)$ converges, and converges to 0 iff any of the $a_n = 1$.

  Given this, if $\{F_n\}$ is a sequence of holomorphic functions on the open set Ω that satisfy $|F_n(z)-1| \le c_n$ and $\sum_n c_n < \infty$, then the infinite product $\prod_{n=1}^{\infty} F_n(z)$ converges uniformly in Ω to a holomorphic function $F(z)$. This just follows from the previous note.

- Before introducing the Weierstrass products, we give an example. Consider the identity

$$\pi\cot(\pi z) = \mathrm{P}\left[\sum_{n=-\infty}^{\infty}\frac{1}{z+n}\right], \tag{1.58}$$

  where we denote the principal value of the sum. It is easy to show that the difference between these two functions is entire, bounded, and odd, implying it is zero. This is then related to the infinite product

$$\frac{\sin(\pi z)}{\pi} = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right), \tag{1.59}$$

  recalling the famous Euler identity.

  The above is a special case of the *Weierstrass infinite product*, which states: if $\{a_n\}$ is a sequence of complex numbers with $|a_n| \to \infty$ (i.e. do not accumulate anywhere finite, and are well-spaced, so no essential singularities), then all entire functions f that vanish at $z = a_n$ are related by some factor $e^{g(z)}$ where $g(z)$ is entire. The second part of this is easy: let f_1 and f_2 be two such f, so f_1/f_2 has removable singularities only at the a_n and vanishes nowhere, so $f_1 = f_2 e^{g(z)}$.

  The first part is a bit more complicated, since we have to construct at least one such f. The naive guess is $\prod_n (1-z/a_n)$, but this does not always converge (see proposition from previous bullet point: the c_n built from this can diverge e.g. if a_n grows like $1/n$). Instead, we regularize these with *canonical factors*

$$E_n(z) = (1-z)\exp\left[\sum_{k=1}^{n} z^k/k\right]. \tag{1.60}$$

  Then the Weierstrass product is

$$f(z) = z^m\prod_{n=1}^{\infty}E_n(z/a_n), \tag{1.61}$$

  where m is the order of zero at $z=0$. Note that the order of the canonical factors grows.

- *Hadamard's factorization Theorem* argues that if the function is of finite order (ρ, where $|f(z)| \le A\exp[B|z|^\rho]$), then the Weierstrass infinite product can be rewritten

$$f(z) = e^{P(z)}z^m\sum_{n=1}^{\infty}E_k(z/a_n), \tag{1.62}$$

  where $k = \lfloor\rho\rfloor$ and P is a polynomial of degree $\le k$.

- Above, we showed that all entire functions are essentially characterized by their zeros. The Gamma function, $\Gamma(s)$, is entire and is such that $1/\Gamma(s)$ has zeros at all non-positive integers, and is defined

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}\,\mathrm{d}t. \tag{1.63}$$

Note that $\Gamma(z)$ as defined is analytic for $\mathrm{Re}(s) > 0$. It can be analytically continued (an analytic function that is equal to $\Gamma(z)$ for $\mathrm{Re}(s) > 0$) by leveraging $\Gamma(s+1) = s\Gamma(s)$. This analytic continuation is unique and has simple poles at $s = 0, -1, \ldots$ with residues $(-1)^n/n!$ at $s = -n$, and is hence meromorphic on $\mathbb{C}$.

An alternative idea is to split the integral for $\Gamma(s)$

$$\Gamma(s) = \int_0^1 e^{-t} t^{s-1}\,\mathrm{d}t + \int_1^\infty e^{-t} t^{s-1}\,\mathrm{d}t, \tag{1.64}$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)} + \int_1^\infty e^{-t} t^{s-1}\,\mathrm{d}t. \tag{1.65}$$

Note that the integral is entire, while the series defines a meromorphic function with poles at the negative integers and the correct residues.

$\Gamma(s)$ has a few more important properties:

- For all $s \in \mathbb{C}$, $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, i.e. Γ has some symmetry about $\mathrm{Re}(s) = 1/2$. Intuitively, $\Gamma(1-s)$ has poles at all positive integers, so the LHS has poles at all integers, as does the RHS.
- $1/\Gamma(s)$ has growth $|1/\Gamma(s)| \le c_1 e^{c_2 |s| \log|s|}$. In fact:

  $$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^\infty \left(1 + \frac{s}{n}\right) e^{-s/n}, \tag{1.66}$$

  where $\gamma = \lim_{N\to\infty} \sum_{n=1}^N \frac{1}{n} - \log N$ is *Euler's Constant*. The proof is a simple application of the Hadamard factorization theorem and requiring $\Gamma(1) = 1$.

- The Reimann zeta function is initially defined for real $s > 1$ by

  $$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}. \tag{1.67}$$

  This can be analytically continued into $\mathbb{C}$, but it is slightly tricky. It is easy to continue to $\mathrm{Re}(s) > 1$ by just taking s complex. To proceed further, we must introduce the theta function as well (discussed above),

  $$\vartheta(t) = \sum_{n=-\infty}^\infty e^{-\pi n^2 t} = t^{-1/2}\vartheta(1/t). \tag{1.68}$$

  If we then recall the identity

  $$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty u^{(s/2)-1}\left[\vartheta(u) - 1\right]\mathrm{d}u, \tag{1.69}$$

  then a natural modification of the ζ function is the ξ function, which is slightly more symmetric:

  $$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s). \tag{1.70}$$

  Note that ξ can be analytically continued to be meromorphic with simple poles at $s = 0, 1$, and $\xi(s) = \xi(1-s)$, by simply expanding the identity above. Then, dividing the above by $\Gamma(s/2)$, we see that $\zeta(s)$ only has a simple pole at $s=1$, since the simple pole at $s=0$ multiplies by the simple zero of $\Gamma^{-1}(s/2)$.

This is a lot of machinery, and doesn't give a lot of insight into the actual behavior of ζ. Instead, there are a few other analytic continuations for ζ that prove useful:

- Given the family of functions $\{\delta_n(s)\}_{n=1}^{\infty}$ by

$$\delta_n(s) = \int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s}\,\mathrm{d}x, \tag{1.71}$$

  then

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \delta_n(s), \tag{1.72}$$

  where $\mathrm{Re}(s) > 0$ (breaking down at $s=0$ because of the $\delta_n(s)$) and the summation is holomorphic. This can be analyzed to show that the growth of ζ near $\mathrm{Re}(s)=1$ is generally modest.

- Note that $\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}\,\mathrm{d}x$, and separating the integral into $[0,1]$ and $(1,\infty)$ shows that the first term has a simple pole at $s=0$ and nowhere else.

- Lastly,

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty}\frac{\{x\}}{x^{s+1}}\,\mathrm{d}x, \tag{1.73}$$

  where $\{x\}$ is the fractional part of x. This can be analytically continued by introducing the periodic function $Q(x) = \{x\} - 1/2$ and taking k successive integrals of Q to push the zero leftwards to $s=-k$.

- The ζ function is of importance for the distribution of primes. We recall that $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ for prime p, and furthermore it does not vanish if $\mathrm{Re}(s) > 1$, as none of its product terms are zero. Furthermore, recall

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s). \tag{1.74}$$

If $\mathrm{Re}(s) < 0$, then $\zeta(1-s)$ has no zeros, but $1/(\Gamma(s/2))$ has zeros at $s=-2,-4,-6,\dots$ (not zero due to domain restriction). Thus, outside of the strip $\mathrm{Re}(s)\in[0,1]$, the only zeros are at the negative even numbers. These are called the *trivial zeros*.

Inside the *critical strip*, $\mathrm{Re}(s)\in[0,1]$, analysis is a bit trickier. The *Riemann hypothesis* claims that the zeros of $\zeta(s)$ lie on the line $\mathrm{Re}(s)=1/2$. We won't see a proof of the RH in this book, but we can show that ζ has no zeros on the line $\mathrm{Re}(s)=1$, which by the functional equation above forces no zeros on $\mathrm{Re}(s)=0$. In particular for every $\epsilon>0$, we can find $1/|\zeta(s)| \le c_\epsilon |t|^\epsilon$ when $s=\sigma+it$, where $\sigma\ge 1$ and $t\ge 1$.

Our ultimate goal is the function $\pi(x)$, the number of primes $\le x$ (we guesstimate this in tidbits, and is proven by Tchebychev, $\pi(x)\approx x/\log x$). A key intermediate step is the Tchebychev ψ-

function and its integral ψ_1

$$\psi(x) = \sum_{p,m|p^m \le x} \log p, \tag{1.75}$$

$$= \sum_{p \le x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p, \tag{1.76}$$

$$\psi_1(x) = \int_1^x \psi(u)\,\mathrm{d}u. \tag{1.77}$$

The goal is to show $\psi_1(x) \sim x^2/2$ as $x \to \infty$, which gives $\psi(x) \sim x$ and therefore that $\pi(x) \sim x/\log x$, all as $x \to \infty$. To be precise, this requires

$$1 \le \liminf_{x\to\infty} \pi(x) \frac{\log x}{x} \qquad \limsup_{x\to\infty} \pi(x) \frac{\log x}{x} \le 1. \tag{1.78}$$

The key trick is that , if we define the function $\Lambda(n) = \log p$ if $n = p^m$ for some prime p, we can obtain

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m}, \tag{1.79}$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \tag{1.80}$$

and then that, for all $c > 1, \mathrm{Re}(s) = c$

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \mathrm{d}s. \tag{1.81}$$

This can be verified with direct computation. But then, to compute the integral on the RHS, we deform the contour to be an intended contour with $\mathrm{Re}(s) = 1$ except for a small neighborhood about $s = 1$. The residue is $x^2/2$, and the rest is mostly bookkeeping.

Elliptic Functions

- The elliptic functions were introduced by Jacobi, and his sn/cn/dn functions, but that is messy. Instead, consider the set of *doubly periodic* functions f such that $f(z + w_1) = f(z + w_2) = f(z)$ for linearly independent w_1 and w_2, and consider f meromorphic. For convenience, rescale these to 1 and τ with $\mathrm{Im}(\tau) > 0$. We then consider the lattice

  $$\Lambda = \{n + m\tau : n, m \in \mathbb{Z}\}. \tag{1.82}$$

  Thus, f is constant up to translation by elements in Λ. The *fundamental parallelogram* of Λ is the set of points $P_0 = \{z = a + b\tau \mid 0 \le a, b < 1\}$. f is completely determined by its values on P_0, and in fact any translation of P (called a *period parallelogram*).

  Liouville's theorem says that any entire doubly periodic function is constant.

- A function is *elliptic* if it is doubly periodic, meromorphic, and non-constant. Moreover, it must have at least two poles (they can be degenerate): integrate around ∂P_0, periodicity ensures this integral is zero, so by the residue theorem the sum of enclosed residues is zero as well. This cannot be the case with just one simple pole, and we know there must be at least one pole, else the function is entire.

- As an example of an elliptic function is to place a double pole at every lattice point in Λ. To ensure convergence as $|w| \to \infty$, we consider the following series

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda - \{0\}} \left[\frac{1}{(z+w)^2} - \frac{1}{w^2} \right]. \tag{1.83}$$

  The sum converges for all z. It isn't immediately obviously periodic (though can be shown by rearranging terms in the absolutely convergent sum), but $\wp'(z)$ is very obviously periodic.

  Note that $\wp'(z)$ vanishes at the three points $z = 1/2$, $z = \tau/2$, and $z = (1+\tau)/2$; these are the *half-periods* of $\wp$. It is useful to note further that

$$(\wp')^2 = 4\left(\wp - e_1\right)\left(\wp - e_2\right)\left(\wp - e_3\right) \tag{1.84}$$

  for e_i these three half-periods. The importance of these two together is that: every elliptic function f with periods $1, \tau$ is a rational function of $\wp$ (even) and $\wp'$ (odd).

- Note that in general, if the periods $(1, \tau)$ generate Λ, so too do $(1, 1+\tau)$. Additionally, switching the two periods w_1, w_2 gives $\tau \to -1/\tau$. Indeed, it turns out that $\wp(z;\tau) = \wp(z;\tau+1) = \wp(z; -1/\tau)/\tau^2$. These two operations generate the *modular group*, under which all functions related to $\wp$ should obey similar transformation laws.

  The *Eisenstein series* of order k is related to $\wp$, and is written

$$E_k(\tau) = \sum_{w \in \Lambda - \{0\}} \frac{1}{w^k}, \tag{1.85}$$

  for $k \geq 3$ (recall $k = 2$, the $\wp$ case, is not convergent). In general, $E_k(\tau+1) = E_k(\tau) = \tau^{-k} E_k(-1/\tau)$. Note also that if k is odd, the Eisenstein series is identically zero. The Eisenstein series are important because $\wp = 1/z^2 + 3E_4 z^2 + 5E_6 z^4 + \dots$.

- The Eisenstein series are actually related to the divisor functions. First, by the Poisson summation identity,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\tau)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{l=1}^{\infty} l^{k-1} e^{22\pi i \tau l}. \tag{1.86}$$

  Then, if $\sigma_l(r) \equiv \sum_{d|r} d^l$, then

$$\begin{aligned} E_k(\tau) &= \sum_{n \neq 0} \frac{1}{n^k} + \sum_{m \neq 0} \sum_{n=-\infty}^{\infty} \frac{1}{(n + m\tau)^k}, && (1.87) \\ &= 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!} \sum_{m>0} \sum_{l=1}^{\infty} l^{k-1} e^{2\pi i \tau m l}, && (1.88) \\ &= 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} \sigma_{k-1}(r) e^{2\pi i \tau r}. && (1.89) \end{aligned}$$

  The divisor function comes in to count the number of times each ml is encountered when summing over m, l.

- The general Θ function is defined

$$\Theta(z;\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}. \tag{1.90}$$

Note that earlier in this book, we used $\vartheta(t) = \Theta(0; it)$. In particular, $\Theta(z+1;\tau) = \Theta(z;\tau)$ while $\Theta(z+\tau;\tau) = \Theta(z;\tau)e^{-\pi i\tau}e^{-2\pi iz}$, so it is quasi-elliptical. Surprisingly, the function

$$\Pi(z;\tau) = \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+q^{2n-1}e^{2\pi iz}\right)\left(1+q^{2n-1}e^{-2\pi iz}\right), \tag{1.91}$$

where $q = e^{\pi i\tau}$ is traditional, has almost the same properties as $\Theta(z;\tau)$, and in fact for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ we have $\Theta(z;\tau) = \Pi(z;\tau)$! This is the *product formula* for Θ.

- The *partition function* studies how many ways to partition a number n. For instance, if $p(n)$ is the number of ways n can be written as a sum of positive integers,

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}\frac{1}{1-x^k}. \tag{1.92}$$

This just gives us the generating function, and the proof qualitatively is clear: $(1-x^k)^{-1} = \sum\limits_m x^{km}$, and we multiply a bunch of these together.

Interestingly, a much deeper result is in store. Call $p_e(n)$ and $p_o(n)$ the number of partitions of n into an even and odd number of unequal parts respectively. It turns out that $p_e(n) - p_o(n) = (-1)^k$ when $n = k(3k+1)/2$ otherwise 0. These numbers are called the *pentagonal numbers*, the number of pieces when the numbers are arranged in a pentagon (versus triangular numbers $k(k-1)/2$ and squares k^2). This profound result comes by observing from generating functions

$$\sum_{n=1}^{\infty}\left[p_e(n) - p_o(n)\right]x^n = \prod_{m=1}^{\infty}\left(1-x^m\right) = \sum_{k=-\infty}^{\infty}(-1)^k x^{x(3k+1)/2}. \tag{1.93}$$

The first equality comes by noting that the number of negative signs gives the number of ways to combine a bunch of x^m into an x^n. The second equality comes from the Θ function: take $x = e^{2\pi iu}$, $q = e^{3\pi iu}$ and $z = 1/2 + u/2$, then apply the product formula.

- The Θ function also comes into play when discussing the sum of squares problems. I won't copy this down, but it's good to know. Call $r_q(n)$ the number of ways to write n as a sum of q squares of integers (including repetitions and signs), then we have:

  - If $n \geq 1$, then $r_2(n) = 4(d_1(n) - d_3(n))$, where $d_j(n)$ is the number of divisors of n of the form $4k + j$. The proof relies on generating functions, or the identity

$$\theta(t)^2 = \sum_{n=0}^{\infty} r_2(n)q^n, \tag{1.94}$$

    when $q = e^{\pi it}$.

  - Every number is a sum of four squares, and it turns out that $r_4(n) = 8\sigma_1^*(n)$ where $\sigma_1^*(n)$ is the sum of divisors of n that are not divisible by 4. In fact, in general,

$$\theta(t)^4 = \sum_{n=0}^{\infty} r_4(n)q^n. \tag{1.95}$$

These proofs look somewhat involved. But they are a testament to the power of the product formula for Θ and generating functions (more in exercises).

1.2.1 Asymptotic Integrals

The guiding principle is the study of integrals

$$I(s) = \int_a^b e^{-s\Phi(x)}\,\mathrm{d}x. \tag{1.96}$$

If Φ is complex, for large s it is difficult to evaluate $I(s)$ since the integrand is highly oscillatory. We will study a few characteristic cases of evaluating such integrals.

- Asymptotic behavior of Bessel functions and contour deformation: one formula for the Bessel functions is

$$J_\nu(s) = \frac{(s/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)}\int_{-1}^{1} e^{isx}\left(1-x^2\right)^{\nu-1/2}\,\mathrm{d}x. \tag{1.97}$$

  This is hard to evaluate when $s \to \infty$. We are basically interested in the function

$$I(s) = \int_{-1}^{1} e^{isx}\left(1-x^2\right)^{\nu-1/2}\,\mathrm{d}x. \tag{1.98}$$

  If we raise the integrand to the complex plane $x \to z$ and consider the contour integral defined by $1 \to 1+i\infty \to -1+i\infty \to -1 \to 1$, we see that $I(s)$ is just the difference of the integral along the imaginary legs. But along these legs, $x = \pm 1 + iy$ where $0 \le y < \infty$, so the integral rapidly converges, and we can obtain the approximation $J_\nu(s) \approx \sqrt{2/\pi s}\cos(s - \pi\nu/2 - \pi/4)$.

- Another example of this is Laplace's Method / the stationary phase approximation (when s is large but real/imaginary, respectively, and Φ attains a minimum on a, b).

- The *Airy* function has slow-to-converge integral

$$\mathrm{Ai}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(x^3/3+sx\right)}\,\mathrm{d}x. \tag{1.99}$$

  It satisfies $\mathrm{Ai}''(s) = s\mathrm{Ai}(s)$. Again, we want to understand the behavior for $s \to \pm\infty$. Making the transformation $x \mapsto \sqrt{u}x$, we find that in the two limits

$$\mathrm{Ai}(-u) = \frac{u^{1/2}}{2\pi}\int_{-\infty}^{\infty} e^{it\left(x^3/3-x\right)}\,\mathrm{d}x, \tag{1.100}$$

$$\mathrm{Ai}(+u) = \frac{u^{1/2}}{2\pi}\int_{-\infty}^{\infty} e^{-t\left(-i\left(x^3/3+x\right)\right)}\,\mathrm{d}x, \tag{1.101}$$

  where $t = |u|^{3/2}$. In the former case, the stationary phase approximation gives the answer. In general, calling $F(z) = -i(z^3/3+x)$, we seek to deform the integration onto a contour Γ such that $\mathrm{Im}(F) = 0$ on Γ and $\mathrm{Re}(F)$ has a minimum at some z_0 on Γ; this is the *method of steepest descent*.

- A much more complex example, which I will note briefly, is the derivation of the Hardy-Ramanujan asymptotic formula for the partition function. The basic idea is as follows: suppose $\{F_n\}$ is some sequence such that the generating function $F(w) = \sum\limits_{n=0}^{\infty} F_n w^n$ converges on the unit disc but has a pole of order r at $w = 1$. Then there is a polynomial P of degree $r-1$ such that $F_n \approx P(n)$ for large n and $\sum\limits_n P(n)w^n$ carries the principal part of $F(w)$. This is the principle of the argument.

  Recall the partition function $p(n)$ satisfies

$$\sum_{n=0}^{\infty} p(n)w^n = \prod_{n=1}^{\infty} \frac{1}{1-w^n}. \tag{1.102}$$

  This must be holomorphic on the unit disk, so we can map to $\mathbb{H}$ by $w = e^{2\pi i z}$, since it is easier to deform contours, and

$$\sum_{n=0}^{\infty} p(n)e^{2\pi i n z} = f(z), \tag{1.103}$$

  where

$$f(z) = \prod_{n=1}^{\infty} \frac{1}{1-e^{2\pi i n z}}, \tag{1.104}$$

$$p(n) = \int\limits_{\gamma} f(z)e^{-2\pi i n z}\,\mathrm{d}z. \tag{1.105}$$

  Note that γ is the segment joining $-1/2 + i\delta$ to $1/2 + i\delta$ with $\delta > 0$; this corresponds to a small circle about $w = 0$ in the original coordinates. We will fix δ in terms of n.

  It turns out that $f(z) = e^{i\pi z/12}/\eta(z)$ where $\eta(z)$ is the Dedekind eta function satisfying $\eta(-1/z) = \sqrt{z/i}\eta(z)$. Since $f(-1/z)$ rapidly approaches 1, we estimate that

$$p(n) \approx \int_{\gamma} \sqrt{z/i}e^{i\pi/12z}e^{i\pi z/12}e^{-2\pi i n z}\,\mathrm{d}z. \tag{1.106}$$

  The idea now is to rewrite the integral into form

$$p(n) = \mu^{3/2}\int\limits_{\Gamma} e^{-sf(z)}\sqrt{z/i}\,\mathrm{d}z, \tag{1.107}$$

  where $F(z) = i(z - 1/z)$ and $s = \pi\sqrt{(n-1/24)/6}$, where the endpoints of Γ are now from $-a_n + i\delta'$ to $a_n + i\delta'$, where $a_n = 1/2\mu$ and $\delta' = \delta/\mu$, where $\mu = 1/2\sqrt{6(n-1/24)}$.

  Now, we again want to deform Γ such that $F(z)$ is real on Γ and contains a critical point. The only two choices are the imaginary axis and the unit circle, and we choose the latter. At this point, we really want the endpoints on the real line, such that the integration over the unit circle spans exactly $\theta \in [0,\pi]$, so we will add little "tails" to Γ that contribute negligibly to the integral but move the endpoints onto the real number line. Finally, this deformed Γ has three parts, the unit semicircle and two componetns on the real number line, and the latter two are small, so we are finally left with just the integral on the unit semicircle. This last is done via Laplace's method, and we obtain

$$p(n) \approx \frac{1}{4n\sqrt{3}}e^{k\sqrt{n}}. \tag{1.108}$$

  There is a much more precise result, the main improvement in accuracy comes from handling the integral on the semicircle much more carefully (instead of Laplace's method).

NB: we touch on asymptotic series above. One characteristic about these series is that they often diverge for fixed x and increasingly many terms, but their accuracy improves at fixed terms and increasingly large x. This is because they are expansions about the point at infinity.

1.3 Book 3: Measure Theory, Integration & Hilbert Spaces

- There are a few problems that arise from the traditional notions of integrability/differentiability/continuity. For instance, the Fourier transform maps the space of Reimann-integrable functions $\mathscr{R}$ to the space of Fourier coefficients, denoted $\ell^2(\mathbb{Z})$. However, $\ell^2(\mathbb{Z})$ is *complete*, while $\mathscr{R}$ is not. The question is then: how do we complete $\mathscr{R}$, and how do we integrate these completed functions f?