Contents

Chapter 1

Flat Spacetime

1.1 Spacetime Physics

- Space tells matter how to move, matter tells space how to curve. Physics is simple when local, and when we eliminate force at a distance life is good.
- *Events* are coordinate-independent points in spacetime that are defined by "what happens there" or more concretely by an intersection of worldlines.
- Spacetime is locally Lorentzian, a.k.a. flat or non-accelerating. Gravitation/curvature is defined as the acceleration of the separation between two nearby geodesics.

  We measure curvature by the following: characterize ξ the separation between two originally parallel geodesics, then propagate each forward by distance s. They are not necessarily parallel anymore (e.g. two great circles on a sphere), and the separation obeys the following EOM

  $$\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} + R\xi = 0 \tag{1.1}$$

  where R is the *Gaussian Curvature* of the surface.

  Generalizing to multiple dimensions, the separation $\boldsymbol{\xi}$ is a vector, and we describe $\frac{\mathrm{D}^2\boldsymbol{\xi}}{\mathrm{d}^2 s}$ with a capital D since the coordinates of the derivative $\frac{\mathrm{d}^2\boldsymbol{\xi}}{\mathrm{d}s^2}$ is subject to the whims of the coordinate lines, which should not affect the separation between these two geodesics that live beyond coordinates. The curvature is instead described by the Reiman curvature tensor, which takes as arguments the 4-velocity $\mathbf{u} = \dfrac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$ and yields

$$\frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}^2\tau^2} + R^\alpha{}_{\beta\gamma\delta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\xi^\gamma\frac{\mathrm{d}x^\delta}{\mathrm{d}\tau} = 0 \tag{1.2}$$

where $\mathbf{R}$ is the 4-component Reimann curvature tensor. We call this the *equation of geodesic deviation*.

These functions are sister functions of the Lorentz force equation in electromagnetism

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} - \frac{e}{m}F^\alpha{}_\beta\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = 0.$$

- $\mathbf{R}$ is a fickle object, subject to perturbation e.g. by gravitational waves. A certain piece of the tensor is generated only by the local mass distribution though, $\mathbf{G}$ the *Einstein curvature tensor*, incidently proportional to the *stress-energy tensor* as $\mathbf{G} = 8\pi\mathbf{T}$. Its interpretation is a local average curvature.

1.1.1 Practice Problems

Exercise 1.1 *Show that the Gaussian curvature of a cylinder $R = 0$.*

Choose cylindrical coordinates (a, θ, z) for the surface of the cylinder of radius a. We assert temporarily that all geodesics can be parameterized as $g(s|\omega, \dot{z}) = (a, \omega s, \dot{z}s)$, i.e. correspond to uniform translation along and rotation about the axis of the cylinder. Then, using the equation of geodesic deviation with Gaussian Curvatures

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} + R\xi = 0$$

for $\xi = g_1 - g_2$ the separation, then since g_1, g_2 are linear functions of s, their second derivatives with respect to s are zero and thus $\dfrac{\mathrm{d}^2\xi}{\mathrm{d}s^2} = 0$. Observe that this happens regardless of the value of ξ, thus $R = 0$.

We now justify our parameterization $g(s|\omega, \dot{z})$. Two points separated infinitisemally (a, θ, z), $(a, \theta + \delta\theta, z + \delta z)$ are separated by distance

$$\sqrt{\delta z^2 + a^2\delta\theta^2} = \mathrm{d}\theta\sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + a^2}.$$

The distance between two points $(\theta_1, z_1), (\theta_2, z_2)$ is then given by

$$D[z(\theta)] = \int_{z_1}^{z_2} \sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + a^2}\,\mathrm{d}\theta$$

where $z(\theta)$ is any trajectory that runs between the desired endpoints. This is then a variational calculus problem, and thus the $z(\theta)$ that minimizes the path must satisfy the Euler-Lagrange Equation

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial I(z', z)}{\partial z'} - \frac{\partial I(z', z)}{\partial z} = 0.$$

We note then that $I(z', z) = \sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + a^2}$ is independent of z and so we instantly find that $I(z', z) = C$ a constant. This furthermore implies that $\frac{\mathrm{d}z}{\mathrm{d}\theta}$ is a constant and so that $z \propto \theta$. Call the constant of proportionality $z \propto \alpha\theta$.

Armed with this, we find that the arclength of the geodesic between the desired endpoints is

$$D[z(\theta)] = \int_{z_1}^{z_2} \sqrt{\alpha^2 + a^2}\,\mathrm{d}\theta \tag{1.3}$$

$$= \sqrt{\alpha^2 + a^2}\Delta z. \tag{1.4}$$

Recalling that s parameterizes the length of our geodesic, we find that s, z, θ must all be proportional. Call the ratios $\frac{\theta}{s} = \omega, \frac{z}{s} = \dot{z}$, justifying our parameterization.

Exercise 1.1b *Alternatively, employ $R = \frac{1}{\rho_1\rho_2}$ where ρ_1, ρ_2 are the principal radii of curvature at the point in question in the enveloping Euclidean 3D space.*

We note that one of the radii in a cylinder is a while the other is infinite, thus $R = 0$.

Exercise 1.3 *Show that given ω, the rotational frequency of a planet about a fixed central mass M, we can not individually determine r the radius of orbit or M. Instead, derive a relation between ω and ρ the* Kepler Density *of the mass, the density if M were spread over the sphere of radius r.*

We know that a central acceleration $\ddot{r} = r\omega^2 = \frac{GM}{c^2r^2}$ under Newton's Law of Gravitation.

Thus, we have

$$\omega^2 = \frac{GM}{c^2 r^3} = \frac{4\pi}{3}\frac{G}{c^2}\rho$$

where since there are no other constraints on the motion we are done.

1.2 Special Relativity and 1-forms

SR/GR aspire to describe all laws as relationships between geometric objects, i.e. events are points, [tangent] vectors connect points, and the metric (defining lengths of vectors) is also geometric. First some definitions

Vector The vector joining A, B as $\frac{\mathrm{d}}{\mathrm{d}\lambda}P(\lambda)$ where $P(\lambda)$ is the straight line joining A, B.

Metric tensor $G(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.

Differential forms Consider the propagation of a plane wave with wavevector $\vec{k}$. There are surfaces of codimension 1 that have equal phase. Define $\tilde{\mathbf{k}}$ such that a vector $\vec{v}$ incurs phase difference $\left\langle \tilde{\mathbf{k}}, \vec{v} \right\rangle$.

This is an example of a 1-form, a differential form that takes a single vector and returns a scalar, often also denoted by $\boldsymbol{\sigma}$ boldface greek letters. We call $\langle \boldsymbol{\sigma}, \vec{v} \rangle$ the contraction of $\boldsymbol{\sigma}$ with $\tilde{\mathbf{v}}$.

Note that there is a one-to-one correspondence $\tilde{\mathbf{p}}$ and $\mathbf{p}$, i.e. $\langle \tilde{\mathbf{p}}, \mathbf{u} \rangle$ is the same as $\mathbf{p} \cdot \mathbf{u}$, and so we sometimes omit the tilde.

More generally, a 1-form is a local approximation the same way a vector is a local derivative, i.e. if we want the phase of a point P close to P_0, we have

$$\phi(P) = \phi(P_0) + \left\langle \tilde{\mathbf{k}}, P - P_0 \right\rangle + \dots \tag{1.5}$$

Gradient as a 1-form The gradient is a 1-form, not a vector, since it is the first-order approximation

$$f(P) = f(P_0) + \langle \mathbf{d}f, P - P_0 \rangle + \dots \tag{1.6}$$

Coordinates Choose an orthonormal basis $\mathbf{e}_\alpha$ for vectors, then there is also a basis $\mathbf{w}^\alpha = \mathbf{d}x^\alpha$ for

1-forms, such that

$$\langle \mathbf{w}^\alpha, \mathbf{e}_\beta \rangle = \delta^\alpha_\beta \tag{1.7}$$

$$\langle v^\alpha \mathbf{w}^\alpha, \sigma_\beta \mathbf{e}_\beta \rangle = \langle \boldsymbol{\sigma}, \mathbf{v} \rangle = \sigma_\alpha v^\alpha \tag{1.8}$$

4-velocity To work w/ dynamical variables, we need to start mapping quantities to geometric objects somewhere. Start with the 4-velocity, which in specific Lorentz reference frame has components

$$\begin{aligned} u^0 &= \frac{\mathrm{d}t}{\mathrm{d}\tau} && = \frac{1}{\sqrt{1 - |\vec{v}^2|}} \\ u^j &= \frac{\mathrm{d}x^j}{\mathrm{d}\tau} && = \frac{v^j}{\sqrt{1 - |\vec{v}|^2}} \end{aligned} \tag{1.9}$$

Lorentz Transformations Two types, rotations and boosts, generate a tensor $\Lambda^\alpha{}_\beta$. Rotations look like

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.10}$$

while boosts look like

$$\Lambda = \begin{bmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{bmatrix} \tag{1.11}$$

where velocity $\beta = \tanh\alpha$.

Then both components and basis vectors are simply contracted w/ this tensor. Lorentz transformations obey $\Lambda^T \eta \Lambda = \eta$ transpose.

1.2.1 Exercises

2.2–2.4 *Derive the following formulae:*

$$u_\alpha = \eta_{\alpha\beta} u^\beta$$
$$u^\alpha = \eta^{\alpha\beta} u_\beta$$
$$\mathbf{u} \cdot \mathbf{v} = u^\alpha v^\beta \eta_{\alpha\beta} = u^\alpha v_\alpha = u_\alpha v_\beta \eta^{\alpha\beta}$$

This is trivial,

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= u_\alpha v_\beta \eta^{\alpha\beta} \\
&= \langle \tilde{\mathbf{u}}, \mathbf{v} \rangle = u^\alpha v_\beta \\
&= \langle \mathbf{u}, \tilde{\mathbf{v}} \rangle = u^\alpha v^\beta \\
u_\alpha \eta^{\alpha\beta} &= u^\alpha && (1.12) \\
v_\beta \eta^{\alpha\beta} &= v^\beta && (1.13)
\end{aligned}$$

2.5 *Verify the following coordinate-free relations a particle of mass m and 4-momentum $\mathbf{p}$ measured by an observer with 4-velocity $\mathbf{u}$:*

$$\begin{aligned}
\mathbf{u}^2 &= -1 \\
E &= -\mathbf{p} \cdot \mathbf{u} \\
m^2 &= -\mathbf{p}^2 \\
|\vec{p}| &= \sqrt{(\mathbf{p} \cdot \mathbf{u})^2 + (\mathbf{p} \cdot \mathbf{p})} \\
|\vec{v}| &= \frac{|\vec{p}|}{E} \\
\mathbf{v} &= \frac{\mathbf{p} + (\mathbf{p} \cdot \mathbf{u})\mathbf{u}}{-\mathbf{p} \cdot \mathbf{u}}
\end{aligned}$$

We must go into an arbitrary Lorentz frame for this:

$$\begin{aligned}
\mathbf{u}^2 &= -\frac{1}{1 - v^2} + \sum_i \frac{v_i^2}{1 - v^2} \\
&= -1 && (1.14)
\end{aligned}$$

We know that $E^2 = m^2 + |\vec{p}|^2$, where $\vec{p} = \gamma m \vec{v}$, and so

$$\begin{aligned}
-\mathbf{p} \cdot \mathbf{u} &= -m\mathbf{v} \cdot \mathbf{u} \\
&= m\frac{1}{\sqrt{1-v^2}} \\
m^2 + |\vec{p}|^2 &= m^2\frac{1}{1-v^2} \\
E^2 &= (\mathbf{p} \cdot \mathbf{u})^2 \qquad (1.15)
\end{aligned}$$

and the sign is chosen by defining E as positive.

But then, knowing the definition $\mathbf{p} = m\mathbf{u}$, this yields $m^2 = -\mathbf{p}^2$.

Again, $E^2 = m^2 + |\vec{p}|^2$, so since $E^2 = (\mathbf{p} \cdot \mathbf{u})^2$ and $m^2 = -\mathbf{p} \cdot \mathbf{p}$, we find $|\vec{p}|^2 = (\mathbf{p} \cdot \mathbf{u})^2 + \mathbf{p} \cdot \mathbf{p}$.

Earlier, we used that $\vec{p} = \gamma m \vec{v}$, and moreover we've seen that $E = m\gamma$ so we have $E = \dfrac{|\vec{p}|}{|\vec{v}|} = m\gamma$.

Not sure how to do the last one ☹.

2.6 *Ascribe a function $T(Q)$ to describe the temperature at event Q. To an observer traveling with 4-velocity $\mathbf{u}$, show that he measures temperature change $\dfrac{\mathrm{d}T}{\mathrm{d}\tau} = \langle \mathbf{d}T, \mathbf{u} \rangle$. Why is this reasonable?*

If the observer travels from event P to event $P + \mathbf{u}\Delta\tau$, then the temperature at the new event can be computed as an expansion about the initial point, which to first order is $\Delta T = \langle \mathbf{d}T, \mathbf{u}\Delta\tau \rangle$. Divide both sides by $\Delta\tau$ to obtain the desired result. Note that we want τ the proper time in the above relation because we are expanding in events about the rest frame of the temperature function T.

2.7 *Show the below Lorentz transformation corresponds to motion with velocity $\beta\vec{n}$ and satisfies $\Lambda^T \eta \Lambda = \eta$.*

$$\Lambda^0{}_0 = \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad (1.16)$$

$$\Lambda^0{}_j = \Lambda^j{}_0 = -\beta\gamma n^j \qquad (1.17)$$

$$\Lambda^j{}_k = \Lambda^k{}_j = (\gamma - 1)n^j n^k + \delta^{jk} \qquad (1.18)$$

We know that the above matrix is a Lorentz transformation if it satisfies $\Lambda^T \eta \Lambda = \eta$. Most explicitly, recall that $\Lambda^\mu{}_{\overline{\nu}}$ is the LT from unprimed vectors to primed vectors $v_\mu \Lambda^\mu{}_{\overline{\nu}}$. The transpose is swapping the order of the arguments $\Lambda^\mu{}_{\overline{\nu}} = (\Lambda^T)_{\overline{\nu}}{}^\mu$. Thus, what we aspire to show is

$$(\Lambda^T)_{\bar{\nu}}{}^{\mu}\eta_{\mu\rho}\Lambda^{\rho}{}_{\bar{\sigma}} = \eta_{\bar{\nu}\bar{\sigma}} \tag{1.19}$$
$$\eta_{\mu\rho}\Lambda^{\mu}{}_{\bar{\nu}}\Lambda^{\rho}{}_{\bar{\sigma}} = \eta_{\bar{\nu}\bar{\sigma}} \tag{1.20}$$

Let's examine the following few cases

- $\bar{\nu} = \bar{\sigma} = 0$ —

$$\begin{aligned}\eta_{\mu\rho}\Lambda^{\mu}{}_0\Lambda^{\rho}{}_0 &= (-\gamma^2 + \beta^2\gamma^2 n_j n^j)\\ &= -1\end{aligned}$$

- $\bar{\nu} = \bar{\sigma} \neq 0$ —

$$\begin{aligned}\eta_{\mu\rho}\Lambda^{\mu}{}_i\Lambda^{\rho}{}_i &= -\beta^2\gamma^2(n_i)^2 + \left[(\gamma-1)^2(n_i)^2 n^j n_j + 2(\gamma-1)(n_i)^2 + 1\right]\\ &= (n_i)^2\left[\gamma^2 - 2\gamma + 1 + 2\gamma - 2 - \beta^2\gamma^2 + 1\right] + 1\\ &= 1\end{aligned}$$

- $\bar{\nu} \neq \bar{\sigma}, \bar{\nu} = 0$ —

$$\begin{aligned}\eta_{\mu\rho}\Lambda^{\mu}{}_0\Lambda^{\rho}{}_i &= \beta\gamma^2 n^i - \beta\gamma(\gamma-1)n^i n^j n_j - \beta\gamma n^j\\ &= \beta\gamma(\gamma n^i - (\gamma-1)n^i n^j n_j - n^j)\\ &= 0\end{aligned}$$

- $\bar{\nu} \neq \bar{\sigma}$, none are 0 —

$$\begin{aligned}\eta_{\mu\rho}\Lambda^{\mu}{}_i\Lambda^{\rho}{}_j &= -\beta^2\gamma^2 n^i n^j + (\gamma-1)^2 n^i n^j n^k n_k + 2(\gamma-1)n^i n^j\\ &= n^i n^j\left[-\beta^2\gamma^2 + \gamma^2 - 2\gamma + 1 + 2\gamma - 2\right]\\ &= 0\end{aligned}$$

This verifies our claim that this tensor acts as a Lorentz transformation. To verify that it is a boost of $\beta\hat{n}$, we could simply rotate such that $\hat{n}$ lies along a coordinate axis. This would be very easy if we had a simple form for the rotation matrix, but I don't have one handy, so let's be more creative. Consider starting in a reference frame moving with some 4-velocity $\mathbf{v}$. The components in this comoving frame of the 4-velocity are simply $(-1, \vec{0})$. If we boost with our provided Lorentz transformation, we obtain

$$\mathbf{v} = (-\gamma, -\beta\gamma\hat{n}) \tag{1.21}$$

which, comparing with our earlier provided formula, corresponds to a 3-velocity of $-\beta\hat{n}$. Thus, the boost takes us into a frame that is moving with $+\beta\hat{n}$ relative to the initial reference frame.

1.3 Tensors, Electromagnetic Field and Stress-Energy

Lorentz Force Law The Lorentz force law both defines fields and predicts motions. It is written in geometric form

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = eF^\alpha{}_\beta u^\beta \tag{1.22}$$

where $\mathbf{F}$ is the Faraday tensor satisfying

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = e\mathbf{F}(\mathbf{u}) \tag{1.23}$$

$$F^\alpha{}_\beta = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{1.24}$$

Tensors A tensor with rank $\binom{n}{m}$ takes n 1-forms and m vectors to a scalar. Components of a tensor e.g. $S^{\alpha\beta}{}_\gamma$ can be computed by inserting the basis objects $S^{\alpha\beta}{}_\gamma = S^{\alpha\beta}{}_\gamma \sigma_\alpha \rho_\beta v^\gamma$. These transform with Lorentz transformation too.

We can always raise/lower indicies w/ the metric, so we only discuss the total rank of a tensor usually. In any case, a raised index is contravariant and a lowered index is called covariant.

Tensor Operations In geometric notation, the gradient of a tensor $\mathbf{S}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is

$$\begin{aligned} \boldsymbol{\nabla}\mathbf{S} &= \partial_\xi \left(S_{\alpha\beta\gamma} u^\alpha v^\beta w^\gamma \xi^\delta\right) \\ &= S_{\alpha\beta\gamma,\delta} u^\alpha v^\beta w^\gamma \xi^\delta \end{aligned} \tag{1.25}$$

Contracting a tensor is to sum over one raised and one lowered index, e.g. $M_{\mu\nu} = R_{\alpha\mu}{}^\alpha{}_\nu$.

Divergence (on the first index of $\mathbf{S}$) is defined

$$\boldsymbol{\nabla} \cdot \mathbf{S} = \boldsymbol{\nabla}\mathbf{S}(\mathbf{w}^\alpha, \mathbf{u}, \mathbf{v}, \mathbf{e}_\alpha) S^\alpha{}_{\beta\gamma,\alpha} \tag{1.26}$$

Transpose is interchanging the order of arguments.

Wedge product is antisymmetrized tensor product

$$\mathbf{u} \wedge \mathbf{v} = \mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u} \tag{1.27}$$

Maxwell's Equations These become

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \qquad\qquad F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha \tag{1.28}$$

Stress-Energy Tensor $\mathbf{T}$ is a symmetric, second rank tensor whose components T_{jk} is the j-component of force acting across a unit surface area $\mathbf{e}_k$. Note that $\mathbf{T}^\alpha{}_\beta u^\beta = -\frac{\mathrm{d}p^\alpha}{\mathrm{d}V}$ 4-momentum density, but also $T_{\alpha\beta}u^\beta$ gives the component of the 4-momentum density along the α direction.

It is conserved $\boldsymbol{\nabla} \cdot \mathbf{T} = 0$ (note that thanks to symmetry we can take the divergence on either index).

One example of $\mathbf{T}$ is the following: An observer wants to know at event P_0 how much 4-momentum a volume corresponding to 1-form $\boldsymbol{\Sigma}$ contains. This is expressed $\mathbf{p} = \mathbf{T}(\_, \boldsymbol{\Sigma})$, but the 1-volume of the box is simply $\boldsymbol{\Sigma} = -V\mathbf{u}$, so $\mathbf{p} = V\mathbf{T}(\_, \mathbf{u})$.

Alternatively, in a perfect fluid/ideal gas, we know that the space-space components of the tensor are $T_{ij} = p\delta_{ij}$, i.e. they are equal on the diagonal and vanish off it, so long as we are in the rest frame of the fluid where all velocities are isotropically distributed. The off diagonal terms vanish since we will never measure net x momenta when we are moving only in the y direction if the momenta are isotropic, and the on-diagonal terms must all be equal for isotropy as well. Moreover, the T_{0j} components are the densities of the j components of momentum, while T_{00} is the mass-energy density. Thus, $T_{00} = \rho, T_{ij} = p\delta_{ij}$ and all other terms are zero. Note that ρ is the rest-plus-kinetic energy.

1.3.1 Exercises

3.4 *Formalize the tensor product* $\mathbf{T} = \mathbf{u} \otimes \mathbf{v}$ *e.g. if* $\mathbf{u}, \mathbf{v}$ *are first-rank tensors, then* $T^{\alpha\beta} = u^\alpha v^\beta$.

Let $\mathbf{T} = \bigotimes_i \mathbf{u}_i$ where $\{\mathbf{u}_i\}$ is a collection of tensors of rank n_i. Then $\mathbf{T}$ is a tensor of rank $N = \sum_i n_i$ whose indicies are simply the concatenation of the $\mathbf{u}_i$ indicies. Thus, in the above $\mathbf{T} = \mathbf{u} \otimes \mathbf{v}$, then the first index of $\mathbf{T}$ is the index of $\mathbf{u}$ and the second $\mathbf{v}$.

3.10 *Show that any second rank tensor* $\mathbf{T}$ *can be uniquely decomposed into symmetric and antisymmetric components* $T^{\mu\nu} = A^{\mu\nu} + S^{\mu\nu}$.

For any tensor $T^{\mu\nu}$, consider its transpose $T^{\nu\mu}$. Define

$$S^{\mu\nu} = \frac{T^{\mu\nu} + T^{\nu\mu}}{2} = T^{(\mu\nu)} \qquad\qquad A^{\mu\nu} = \frac{T^{\mu\nu} - T^{\nu\mu}}{2} = T^{[\mu\nu]}$$

and the inverse operation $T^{\mu\nu} = A^{\mu\nu} + S^{\mu\nu}$. Uniqueness is manifest because combining the equations for index $\mu\nu$ and index $\nu\mu$ results in two equations for two unknowns (either $A^{\mu\nu}, S^{\mu\nu}$ or $T^{\mu\nu}, T^{\nu\mu}$ depending on which direction we're going) which exhibits a single unique solution. If each component exhibits a unique solution then the entire system of equations generated by the matrix equation is unique.

3.11 *Show that* $A_{\mu\nu}S^{\mu\nu} = 0$*, if* A, S *are antisymmetric and symmetric respectively. This in conjunction with the result from the next problem shows that we only need the symmetric/antisymmetric portions of a tensor when multiplying against an antisymmetric/symmetric tensor.*

For each fixed μ, ν, we see that $(A_{\mu\nu})(S_{\mu\nu}) = -(A_{\nu\mu})(S_{\nu\mu})$. Since for each (μ, ν) pair we also sum over (ν, μ) the sum must vanish.

3.12 *Show that for an arbitrary tensor* V*,* $V_{(\mu\nu\dots)}$ *is symmetric,* $V_{[\mu\nu\dots]}$ *is antisymmetric, and that only second rank tensor is fully described by its symmetric/antisymmetric parts.*

Let V be rank N, then we recall that $V_{(\mu\nu\dots)}$ is shorthand for

$$V_{(\mu\nu\dots)} = \frac{1}{N!} \sum_{\alpha\beta\dots\in\text{Permutations}} V_{\alpha\beta\dots}$$

and so when we index V with a one permutation of indicies as opposed to another, we don't change the list of all permutations and thus $V_{(\mu\nu\dots)}$ is invariant under exchange of any indicies, thus rendering it symmetric.

Recall moreover that

$$V_{[\mu\nu\dots]} = \frac{1}{N!} \sum_{\alpha\beta\dots\in\text{Permutations}} V_{\alpha\beta\dots}\varepsilon_{\alpha\beta\dots}$$

We note then that if we interchange two indicies on the left hand side, each of the ε contribute a single sign change (ε is fully antisymmetric in all indicies) and so the left hand side must too be antisymmetric.

As for full decomposition, we note that any N-rank tensor generally has d^N degrees of freedom, assuming there are d allowed values per index. To compute DOF for symmetric tensor, we have N indistinct indicies to deposit into d distinct values, yielding $\binom{d+N-1}{N}$ allowed values. For antisymmetric, we have d allowed values of which we choose N, so $\binom{d}{N}$ allowed values.

The numerator of these expressions is bound by $(2k)^N$, since $N \le d$, while we are dividing by $N!$. Thus, the ratio of the sum of the decomposed DOF to the full DOF is $\frac{2^N}{N!}$ which vanishes. In fact, even for $N = 3$ we have $\binom{d+2}{3} + \binom{d}{3} < d^3$, so we know that for $N \ge 3$ we can't have equal numbers of DOF.

3.14 *Define the* dual *of a tensor J to be obtained by contraction against the Levi-Civita symbol, e.g. $\star J_{\alpha\beta\gamma} = J^{\mu}\varepsilon_{\mu\alpha\beta\gamma}$ in four dimensions. For an arbitrary m-rank tensor, the definition goes $(\star J) = \frac{1}{N!}J\varepsilon$. Show that $\star\star J = (-1)^N J$ for antisymmetric J.*

We leverage the following identity: $\varepsilon^{\{\alpha_i\}}\varepsilon_{\{\beta_i\}} = \det\left(\delta_{\alpha_i\beta_j}\right)$ where the argument of the determinant is the (i,j)th index of a matrix and δ is the Kronecker delta.

Then, $\star\star J$ where J is of rank m in an N dimensional space is simply

$$(\star\star J)^{\alpha_i} = \frac{1}{N!}J^{\beta_i}\varepsilon_{\beta_i\ldots\gamma_j}\varepsilon^{\gamma_j\ldots\alpha_i}$$

We cyclically permute the indicies on the second ε. This incurs N swaps and so a net sign of $(-1)^N$. There are $\frac{N!}{m!}$ choices of these last $N-m$ indicies to sum over.

Of these last indicies, we note that $(\star\star J)^{\alpha_i} = J^{\beta_i}\varepsilon_{\beta_i}\varepsilon^{\alpha_i}$. For each ordering of β_i, they must all be distinct obviously if they are to contribute to the sum, and if they are an even[odd] permutation of the α_i then $J^{\beta_i} = \pm J^{\alpha_i}$, but the ε are same[different] signs too, so the two sides have the same sign regardless of permutation. That's $m!$ orderings, so for each set of indicies α_i, we have $\frac{1}{N!}N!(-1)^N J^{\alpha_i}$ on the right hand side, and the claim is proven.

We leveraged a simple lemma here, the identity initially stated, but it's easy to show that the expression has the desired properties. The two ε symbols must be antisymmetric under exchange of two indicies. If two indicies are exchanged, then either two rows or two columns of the right hand side are exchanged, and by the property of determinants a net sign flip is also incurred. Moreover, in the base case $\{\alpha_i\} \equiv \{\beta_i\}$ element-by-element, then both sides are clearly unity. Thus, the right hand side satisfies the desired properties.

1.4 Accelerated Observers

Overview SR accomodates accelerated observers via local Lorentz frames. It's impossible to tell whether you're accelerating or whether you're in gravity, so it's possible to compute all gravity as if just accelerating. The mathematical result is GR.

Hyperbolic Motion If we require $\mathbf{a} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\tau}$ to satisfy $0 = \mathbf{a} \cdot \mathbf{u}$ (necessitated if we require $\mathbf{u}^2 = -1$ to not instantaneously change), then we can show that this produces a trajectory satisfying $x^2 - t^2 = g^{-2}$, hyperbolic motion (note that $a^\mu a_\mu = g^2$).

Rotations in 4D In 3D, rotation was defined about an angular velocity vector ω_i. In 4D, we instead rotate in a plane $\Omega^{\mu\nu}$, such that

$$\frac{\mathrm{d}v^\mu}{\mathrm{d}\tau} = -\Omega^{\mu\nu} v_\nu \tag{1.29}$$

Compare to $\frac{\mathrm{d}v_i}{\mathrm{d}t} = \epsilon_{ijk}\omega_j v_k$.

Coordinates in Accelerated Frames Choose basis vectors $\mathbf{e}_\alpha$ such that $\mathbf{e}_0 = \mathbf{u}$ the 4-velocity of the frame, and $\mathbf{e}_1$ the acceleration. Thus, $\mathbf{e}_2, \mathbf{e}_3$ must be unaffected by LT in the 1-direction. We lastly require no rotation outside of the above two criteria.

We can verify that the choice $\boldsymbol{\Omega} = \mathbf{a} \wedge \mathbf{u}, \Omega^{\mu\nu} = a^\mu u^\nu - a^\nu u^\mu$ is the rotation that is enforced by our former criteria: any vector orthogonal to $\mathbf{a}, \mathbf{u}$ doesn't get rotated $\Omega^{\mu\nu} w_\nu = 0$.

A vector that undergoes the given Lorentz transformation specified by $\boldsymbol{\Omega}$ (i.e. $\frac{\mathrm{d}v^\mu}{\mathrm{d}\tau} = \Omega^{\mu\nu} v_\nu$) is said to undergo *Fermi-Walker transport* along the wordline of the observer.

These coordinates approximate a Lorentz coordinate system out to g^{-1}.

Gravity cannot exist in flat spacetime We consider three field theories of gravity, a scalar, vector and tensor field. We show that all three are unsatisfactory. I tried the scalar one in the exercise but failed because I am stupid, but one can compute the field generated by a point particle and its effect on a test particle, and in particular that a massless test particle is not deflected by a scalar field.

Gravitational Redshift Photons lose energy climbing out of gravitational wells, otherwise annihilation at different gravitational energies would produce different output energies while paying no penalty in input energy.

Incompatibility with SR Now consider sending a wave of frequency f to a point with different gravitational potential energy for n cycles. Thanks to gravitational redshift, they will receive a wave of different frequency. The four events, the start/end of the signal at both points, form a parallelogram in Minkowski spacetime but have different durations at either point, violating SR!

GR GR is really just meshing all these Lorentz frames together, a mathematical way of curving spacetime.

1.4.1 Exercises

Self *Derive hyperbolic motion.*

The assumptions are: $\mathbf{a}^\mu \mathbf{a}_\mu = g^2$ constant magnitude, $\mathbf{u}^\mu \mathbf{u}_\mu = -1$ unaccelerated, and $\mathbf{a} \cdot \mathbf{u} = 0$ at all times. Go to the instaneous rest frame of the motion at any time, then since $\mathbf{u} = (1, 0, 0, 0)$, we note $a_0 = 0$. Choose $\mathbf{a} \propto \mathbf{e}_1$ for simplicity. Writing out these equations in time and 3-vector coordinates gives

$$\begin{aligned} a_0 u_0 &= a_1 u_1 \\ u_0^2 &= 1 + u_1^2 \\ a_1^2 - a_0^2 &= g^2 \end{aligned}$$

Solving (with much difficulty wow I'm rusty) we obtain

$$\begin{aligned} a_1^2 - a_1^2 \left(\frac{u_1}{u_0}\right)^2 &= g^2 & \qquad a_0^2 \left(\frac{u_0}{u_1}\right)^2 - a_0^2 &= g^2 \\ a_1^2 \left(\frac{1}{u_0^2}\right) &= & a_1^2 \left(\frac{1}{u_1^2}\right) &= \\ a_1 &= g u_0 & a_0 &= g u_1 \\ \frac{\mathrm{d}u_1}{\mathrm{d}\tau} &= g u_0 & \frac{\mathrm{d}u_0}{\mathrm{d}\tau} &= g u_1 \end{aligned}$$

which exhibits solutions $u_1(\tau) = \sinh(g\tau), u_0(\tau) = \cosh(g\tau)$, which can exhibit the desired property $x_0^2 - x_1^2 = g^{-2}$ with appropriate choice of initial conditions.

6.6 *Show that in an LLT coordinate system, the proper time measured at a coordinate* (ξ^1, ξ^2, ξ^3) *differs from its coordinate time* $\frac{\mathrm{d}\tau}{\mathrm{d}\xi^0} = 1 + g\xi^1$.

Thinking of the proper time "at" a coordinate to be the proper time measured by a co-moving observer at that coordinate, we see that τ plays the role of ξ^0 in the frame of reference of the observer co-moving with the reference frame of the origin, and ξ^0 the coordinate time is simply τ, the time of the LLT which is non-accelerating.. Moreover, $\xi^1(\tau = 0) = 0$ which explains the $+1$, and we trivially note from our earlier solved trajectories that

$$\frac{\mathrm{d}\xi^0}{\mathrm{d}\tau} = \cosh(g\tau) = 1 + g\frac{\cosh g\tau - 1}{g}$$

6.8 *Consider an observer that allows his LLT to rotate, not Fermi-Walker transport. Specifically, we rotate as*

$$\frac{\mathrm{d}\mathbf{e}_\alpha}{\mathrm{d}\tau} = -\mathbf{\Omega} \cdot \mathbf{e}_\alpha \tag{1.30}$$

where $\mathbf{\Omega} = \mathbf{a} \wedge \mathbf{u} + u_\alpha \omega_\beta \varepsilon^{\alpha\beta\mu\nu}$, *the second component being the spatial rotation, where* $\boldsymbol{\omega}$ *is perpendicular to 4-velocity* $\mathbf{u}$. *Show that*

- $\mathbf{e}_0 = \mathbf{u}$ is permitted by the above.

  By permitted, we mean that $\frac{\mathrm{d}\mathbf{e}_0}{\mathrm{d}\tau} = \mathbf{a}$, i.e. the time coordinate is not rotated any more than by the 4-acceleration $\mathbf{a}$. We note that $(\mathbf{a} \wedge \mathbf{u}) \cdot \mathbf{u} = \mathbf{u}(\mathbf{a} \cdot \mathbf{u}) - \mathbf{a}(\mathbf{u} \cdot \mathbf{u}) = \mathbf{a}$. Moreover, if $\mathbf{e}_0 = \mathbf{u}$ then $\mathbf{u}$ only has nonzero component in a single index $\alpha = 0$, and so contracting against the Levi-Civita symbol automatically implies either we're summing for a set of indicies where $u^\mu = 0$ or the two u share the same index $\mu = 0$, both cases which result in the second term having zero contribution. We thus obtain

  $$\frac{\mathrm{d}\mathbf{e}_0}{\mathrm{d}\tau} = \mathbf{a}$$

  as is required.

- The second term, call it $\mathbf{\Omega}_{SR}$, obeys

  $$\mathbf{\Omega}_{SR} \cdot \mathbf{u} = 0 \qquad\qquad \mathbf{\Omega}_{SR} \cdot \boldsymbol{\omega} = 0$$

  We showed the first of the two relations, and the second is equally obvious since contracting the same vector twice against the Levi-Civita symbol always yields zero, since for every contribution $u^\alpha u^\beta$ we also have a term $-u^\beta u^\alpha$ thanks to full antisymmetry.

This implies $\mathbf{\Omega}_{SR}$ is orthogonal to both $\mathbf{u}, \boldsymbol{\omega}$.

- Compare the basis vectors transported via the above rules with those obtained via Fermi-Walker transport. Show that the space-like components are rotated by $\boldsymbol{\omega}$. Hint: pick a moment where the tetrads coincide and show that the change in their difference is rotating like $\vec{\omega} \cdot \vec{e}_j$.

  We've chosen $\mathbf{e}_0 = \mathbf{u}$, and so $\boldsymbol{\omega}$ must have only space-like components. Then, since $\mathbf{\Omega}_{SR} = u_\alpha \omega_\beta \boldsymbol{\varepsilon}^{\alpha\beta\mu\nu}$, and $\mathbf{u}$ only has nonzero $\alpha = 0$ component, we find that $\mathbf{\Omega}_{SR} = \omega_i \epsilon^{ijk}$ where $i, j, k \in [1, 3]$. Thus, it is clear that the effect of $\mathbf{\Omega}_{SR}$ is to rotate by $\vec{\omega}$ in space-like coordinates.

7.1 *Determine the consequences of assuming a scalar gravitational field satisfying Newton's equations of motion $x_i'' = -\dfrac{\partial \Phi}{\partial x_i}$ and $\nabla^2 \Phi = 4\pi G \rho$.*

- Compute EOM for variational principle with action for a particle of mass m following path $z^\alpha(\lambda)$ in field $\Phi(\mathbf{z})$

$$I = -m \int e^{\Phi(\mathbf{z})} \left(-\eta_{\alpha\beta} \frac{\mathrm{d}z^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}z^\beta}{\mathrm{d}\lambda} \right)^{1/2} \mathrm{d}\lambda$$

  We want $\delta I = 0$, so $\dfrac{\partial I}{\partial z^\alpha} - \dfrac{\mathrm{d}}{\mathrm{d}\lambda} \dfrac{\partial I}{\partial {z^\alpha}_{,\lambda}} = 0$. We obtain $\forall \nu$:

$$\frac{\partial \Phi(\mathbf{z})}{\partial z^\nu} e^{\Phi(\mathbf{z})} \left(-\eta_{\alpha\beta} \frac{\mathrm{d}z^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}z^\beta}{\mathrm{d}\lambda} \right)^{1/2} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{m e^\Phi \eta_{\nu\nu} \frac{\mathrm{d}z^\nu}{\mathrm{d}\lambda}}{\left(-\eta_{\alpha\nu} \frac{\mathrm{d}z^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}z^\nu}{\mathrm{d}\lambda} \right)^{1/2}} = 0$$

  Expressing in terms of the proper time

$$\mathrm{d}\tau = \left(-\eta_{\alpha\beta} \frac{\mathrm{d}z^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}z^\beta}{\mathrm{d}\lambda} \right)^{1/2} \mathrm{d}\lambda$$

  we obtain

$$\frac{\partial \Phi(\mathbf{z})}{\partial z^\nu} = \eta_{\nu\nu} \frac{\mathrm{d}^2 z^\nu}{\mathrm{d}\tau^2}$$

- Actually I changed my mind, I do not want to do the rest of this ☺. Too hard ☹.

Chapter 2

Curving Spacetime

2.1 Mathematics of GR

Below follows a brief overview of the formalism:

Tensor Algebra with curved spacetime, the metric Instead of $\eta_{\alpha\beta}$ purely diagonal, we have $g_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$, satisfying $g_{\alpha\beta}g^{\beta\gamma} = \delta_\beta{}^\gamma$.

Moreover, instead of $\Lambda^{\alpha'}{}_\beta$ Lorentz transformation matrix, we have an arbitrary $L^{\alpha'}{}_\beta$.

Finally, the usual Levi-Civita symbol is scaled by a factor of $(-g)^{1/2}$ where $g \equiv \det g_{\alpha\beta}$.

Vectors in Curved Geometry Vectors are directional derivatives $\mathbf{u} \equiv \partial_\mathbf{u}$. Recall that these satisfy $\partial_\mathbf{u} f = \langle \mathbf{d}f, \mathbf{u}\rangle$. Makes more sense in the context of "tangent spaces", where we look at the flat tangent space instead of the full curved spacetime, where "distance between points" is ill-defined.

Basis Vector Commutation Coefficients Consider $\mathbf{u}, \mathbf{v}$ vectors/directional derivatives as operators. Define their commutator $c_{\beta\gamma}{}^\alpha$ as $[\mathbf{u}, \mathbf{v}]$. Demonstrably, this is also a vector.

Specialize to basis vectors, where this is called the *commutation coefficient* $c_{\beta\gamma}{}^\alpha$

$$[\mathbf{e}_\beta, \mathbf{e}_\gamma] = c_{\beta\gamma}{}^\alpha \mathbf{e}_\alpha \tag{2.1}$$

Parallel Transport Intuitively, moving a vector in on LLF to the same one in a neighboring LLF, same meaning same components.

Covariant Derivative Define $\boldsymbol{\nabla}_\mathbf{u}\mathbf{T}$ to be the *covariant derivative* of $\mathbf{T}$ along a curve $P(\lambda)$ whose tangent is $\mathbf{u}$. More precisely, parallel transport $\mathbf{T}$ along $P(\lambda)$ for infinitisemally small increments to compute the derivative. Define $\boldsymbol{\nabla}_\mathbf{u} \equiv \boldsymbol{\nabla}(\_, \mathbf{u})$. the gradient of $\mathbf{T}$.

Connection Coefficients In an LLF, the components of $\boldsymbol{\nabla}\mathbf{T}$ are just directional derivatives of $\mathbf{T}$ i.e. $T^{\beta}{}_{\alpha,\gamma}$. Generally though, if the basis vectors vary continuously from point to point, then the gradient will contain contributions from $\boldsymbol{\nabla}\mathbf{e}_\beta$, $\boldsymbol{\nabla}\mathbf{w}^\alpha$ as well as from $T^{\beta}{}_{\alpha,\gamma}$.

To quantify this, we define *connection coefficients*

$$\Gamma^{\alpha}{}_{\beta\gamma} = \langle \mathbf{w}^\alpha, \boldsymbol{\nabla}_\gamma \mathbf{e}_\beta \rangle \tag{2.2}$$

and likewise $-\Gamma^{\alpha}{}_{\beta\gamma} = \langle \boldsymbol{\nabla}_\gamma \mathbf{w}^\alpha, \mathbf{e}_\beta \rangle$. Then the components of the gradient are

$$T^{\beta}{}_{\alpha;\gamma} = T^{\beta}{}_{\alpha,\gamma} + \Gamma^{\beta}{}_{\mu\gamma} T^{\mu}{}_{\alpha} - \Gamma^{\mu}{}_{\alpha\gamma} T^{\beta}{}_{\mu} \tag{2.3}$$

$$\boldsymbol{\nabla}_{\boldsymbol{\mu}} \mathbf{T} = \left(T^{\beta}{}_{\alpha;\gamma} u^\gamma\right) \mathbf{e}_\beta \otimes \mathbf{w}^\alpha \tag{2.4}$$

Note that these vanish at the origin of any LLF.

Define also $\dfrac{\mathrm{D}T^{\beta}{}_{\alpha}}{\mathrm{d}\lambda} = T^{\beta}{}_{\alpha;\gamma} u^\gamma$ shorthand.

Computing these in a given basis follows the **#totallySimple** procedure:

- Evaluate the metric $g_{\beta\gamma}$ in a given basis.
- Differentiate along the basis directions $g_{\beta\gamma,\mu}$.
- Using the commutation coefficients for the basis,

$$\Gamma_{\mu\beta\gamma} = \frac{1}{2}\left[g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu} + c_{\mu\beta\gamma} + c_{\mu\gamma\beta} - c_{\beta\gamma\mu}\right] \tag{2.5}$$

- Raise an index $\Gamma^{\alpha}{}_{\beta\gamma} = g^{\alpha\mu}\Gamma_{\mu\beta\gamma}$.

Geodesic A geodesic is such that a particle's trajectory parameterized in terms of some λ (called the affine parameter since it uniformly ticks along as the path moves) is a multiple of the particle's proper time. More succinctly, a geodesic exhibits parallel transports its tangent vector along itself $\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{u} = 0$.

In component form, this takes on

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^{\alpha}{}_{\mu\gamma} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} = 0 \tag{2.6}$$

Note that a geodesic is usually intuited as a "non-accelerating trajectory", and indeed the correspondence $\mathbf{a} = \boldsymbol{\nabla}_{\mathbf{u}}$ holds.

Geodesic Deviation and the Reimann Curvature Tensor Consider a family of geodesics $P(\lambda, n)$ parameterized by some continuous n. Note that $\mathbf{n} \equiv \dfrac{\partial P}{\partial n}$ measures the separation between points on neighboring geodesics at the same λ. While $\mathbf{n}$, $\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{n}$ for any two geodesics are set by initial conditions, $\boldsymbol{\nabla}_{\mathbf{u}}\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{n}$ is set by the curvature, and concisely

$$\boldsymbol{\nabla}_{\mathbf{u}}\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{n} + \mathbf{R}(\_, \mathbf{u}, \mathbf{n}, \mathbf{u}) = 0 \tag{2.7}$$

$$\frac{\mathrm{D}^2 n^\alpha}{\mathrm{d}\lambda^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta n^\gamma u^\delta = 0 \tag{2.8}$$

We can compute in coordinate basis then

$$R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha{}_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha{}_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} \tag{2.9}$$

Other curvature tensors We can form the Ricci, scalar and Einstein curvature tensors respectively:

$$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu} \tag{2.10}$$

$$R \equiv R^\mu{}_\mu \tag{2.11}$$

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R \tag{2.12}$$

The Einstein curvature tensor is the only part of the Reimann curvature tensor that does not vanish in contributing to the Bianchi Identities

$$R^\alpha{}_{\beta[\lambda\mu;\nu]} = G^{\mu\nu}{}_{;\nu} = 0$$

2.1.1 Exercises

8.1 *Note: I am only picking one of the problems. Prove that the corresponding one form to a directional derivative $\partial_{\mathbf{e}^q}$ or vector $\mathbf{e}^q$ for $\mathbf{e}^q$ basis vector along the q coordinate is $\mathbf{d}q$.*

$\partial_{\mathbf{u}}$ encodes "take the gradient and project along $\mathbf{u}$". To show that these two are duals, it suffices to show $\partial_{\mathbf{e}^q} f = \langle \mathbf{d}f, \mathbf{e}^q\rangle$ is equal to $\mathbf{d}f \cdot \mathbf{d}q$. But $\mathbf{d}q$ is clearly 1 along the q componment and 0 everywhere else, so it's clear that both expressions simply pick out the q component of $\mathbf{d}f$.

8.2 *Compute the components of the commutator $[\partial_{\mathbf{u}}, \partial_{\mathbf{v}}]$ in a coordinate basis.*

Let's try acting on a single arbitrary coordinate q^α such that $\mathbf{d}q^\alpha \propto \mathbf{e}^\alpha$ (e.g. one dimension):

$$\begin{aligned}
\partial_{\mathbf{u}}\partial_{\mathbf{v}}q^\alpha - \partial_{\mathbf{v}}\partial_{\mathbf{u}}q^\alpha &= \partial_{\mathbf{u}}\left\langle \mathbf{d}q^\alpha, \mathbf{v}\right\rangle - \partial_{\mathbf{v}}\left\langle \mathbf{d}q^\alpha, \mathbf{u}\right\rangle\\
&= \partial_{\mathbf{u}}v^\alpha - \partial_{\mathbf{v}}u^\alpha\\
&= \left\langle \mathbf{d}v^\alpha, \mathbf{u}\right\rangle - \left\langle \mathbf{d}u^\alpha, \mathbf{v}\right\rangle\\
&= u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta}
\end{aligned}$$

where we have used the coordinate basis assumption to expand the gradient. Then, if we consider arbitrary dimensions, we simply must expand $\mathbf{d} = \sum\limits_\alpha \mathbf{e}_\alpha$. Substituting this in above gives us the desired

$$[\partial_{\mathbf{u}}, \partial_{\mathbf{v}}] = \partial_{\mathbf{u}}\partial_{\mathbf{v}} - \partial_{\mathbf{v}}\partial_{\mathbf{u}} = \left(u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta}\right)\mathbf{e}_\alpha \tag{2.13}$$

Recall that the commutation coefficients for any basis are defined $[\mathbf{e}_\beta, \mathbf{e}_\gamma] = c_{\beta\gamma}{}^\alpha \mathbf{e}_\alpha$. For any coordinate basis, the basis vectors are orthonormal so $u^\alpha{}_{,\beta} = \delta^\alpha{}_\beta$ and (2.13) vanishes.

8.3 *Compute the Levi-Civita symbol in nonorthonormal bases.*

We follow the hints. First, we enforce Lorentz invariance in an LLF in the hatted coordinate system.

$$\epsilon_{\alpha\beta\gamma\delta} = L^{\hat\mu}{}_\alpha L^{\hat\nu}{}_\beta L^{\hat\lambda}{}_\gamma L^{\hat\rho}{}_\delta \epsilon_{\hat\mu\hat\nu\hat\lambda\hat\rho}$$

We leverage the matrix identity $\det \mathbf{A} = \frac{1}{n!}\varepsilon_{\{\dots i_n\}}\varepsilon_{\{\dots j_n\}}\left\{\dots a_{i_n,j_n}\right\}$ by first multiplying both sides by $\varepsilon^{\alpha\beta\gamma\delta}$, giving us

$$\begin{aligned}
\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} &= L^{\hat\mu}{}_\alpha L^{\hat\nu}{}_\beta L^{\hat\lambda}{}_\gamma L^{\hat\rho}{}_\delta \varepsilon^{\alpha\beta\gamma\delta}\epsilon_{\hat\mu\hat\nu\hat\lambda\hat\rho}\\
&= n! \det L^{\hat\nu}{}_\gamma
\end{aligned}$$

There are $n!$ permutations of $\alpha\beta\gamma\delta$, so we see that $\varepsilon_{\alpha\beta\gamma\delta} = \det L^{\hat\nu}{}_\gamma \varepsilon_{\hat\mu\hat\nu\hat\lambda\hat\rho}$ satisfies the above relations.

But then, $g_{\alpha\beta} = \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, while $\mathbf{e}_{\hat\mu} = \mathbf{e}_\alpha L^\alpha{}_{\hat\mu}$, and so let's consider

$$\det g = \frac{1}{n!}\varepsilon^{\{i_n\}}\varepsilon^{\{j_n\}}\left\{\dots g_{i_n j_n}\right\}$$

We know that all non-zero contributions to the sum must have distinct i_n, j_n, so we see that for each component, there will be two factors of $L^{\alpha}{}_{\hat{\nu}}$. We see from the previous exercise that each factor imposes a global $\det L^{\alpha}{}_{\hat{\nu}}$ factor, so we find that

$$\begin{aligned}\det g &= \left(\det L^{\alpha}{}_{\hat{\nu}}\right)^2 \det \hat{g}\\ &= -\left(\det L^{\alpha}{}_{\hat{\nu}}\right)^2\end{aligned}$$

To recover the expression in the book, recall that $L^{\alpha}{}_{\hat{\nu}} = \left(L^{\hat{\nu}}{}_{\alpha}\right)^{-1}$, which yields

$$\det g\left(\det L^{\hat{\nu}}{}_{\alpha}\right)^2 = -1$$

Combining these two results, we derive

$$\begin{aligned}\varepsilon_{\alpha\beta\gamma\delta} &= \det L^{\hat{\nu}}{}_{\gamma}\varepsilon_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}}\\ &= \sqrt{-g}\varepsilon_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}}\end{aligned}$$

16.3 *Demonstrate that covariant derivatives do not commute by demonstrating the below:*

$$\begin{aligned}B^{\mu}{}_{;\alpha\beta} &= B^{\mu}{}_{;\beta\alpha} + R^{\mu}{}_{\nu\beta\alpha}B^{\nu}\\ S^{\mu\nu}{}_{;\alpha\beta} &= S^{\mu\nu}{}_{;\beta\alpha} + R^{\mu}{}_{\rho\beta\alpha}S^{\rho\nu} + R^{\nu}{}_{\rho\beta\alpha}S^{\mu\rho}\\ B^{\mu;\alpha}{}_{\mu} &= B^{\mu}{}_{;\mu}{}^{\alpha} + R^{\alpha}{}_{\mu}B^{\mu}\end{aligned}$$

Recall

$$T^{\beta}{}_{\alpha;\gamma} = T^{\beta}{}_{\alpha,\gamma} + \Gamma^{\beta}{}_{\mu\gamma}T^{\mu}{}_{\alpha} - \Gamma^{\mu}{}_{\alpha\gamma}T^{\beta}{}_{\mu}$$

Let's simply evaluate the second expression, since it is a strict generalization of the second. Then

$$\begin{aligned}S^{\mu\nu}{}_{;\alpha\beta} =& \left(S^{\mu\nu}{}_{,\alpha} + \Gamma^{\mu}{}_{\sigma\alpha}S^{\sigma\nu} + \Gamma^{\nu}{}_{\rho\alpha}S^{\mu\rho}\right)_{;\beta}\\ =& S^{\mu\nu}{}_{,\alpha\beta} + \Gamma^{\mu}{}_{\sigma\alpha}\left(S^{\sigma\nu}{}_{,\beta} + \Gamma^{\sigma}{}_{\gamma\beta}S^{\gamma\nu} + \Gamma^{\nu}{}_{\delta\beta}S^{\sigma\delta} + \right) + \Gamma^{\nu}{}_{\rho\alpha}\left(S^{\mu\rho}{}_{,\beta} + \Gamma^{\mu}{}_{\gamma\beta}S^{\gamma\rho} + \Gamma^{\rho}{}_{\delta\beta}S^{\mu\delta}\right)\\ &+ \Gamma^{\mu}{}_{\sigma\alpha;\beta}S^{\sigma\nu} + \Gamma^{\nu}{}_{\rho\alpha;\beta}S^{\mu\rho}\end{aligned}$$

Going to the LLF, all connection coefficients vanish (but not their derivatives) and so we are left with only

$$S^{\mu\nu}{}_{;\alpha\beta} = S^{\mu\nu}{}_{,\alpha\beta} + \Gamma^{\mu}{}_{\sigma\alpha,\beta}S^{\sigma\nu} + \Gamma^{\nu}{}_{\rho\alpha,\beta}S^{\mu\rho}$$

Similarly, we could have arrived at

$$S^{\mu\nu}{}_{;\beta\alpha} = S^{\mu\nu}{}_{,\alpha\beta} + \Gamma^{\mu}{}_{\sigma\beta,\alpha}S^{\sigma\nu} + \Gamma^{\nu}{}_{\rho\beta,\alpha}S^{\mu\rho}$$

Recall Reimann coordinate components (2.9), of which we only have the derivative components

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta}$$

and it is obvious that

$$S^{\mu\nu}{}_{;\alpha\beta} - S^{\mu\nu}{}_{;\beta\alpha} = R^{\mu}{}_{\sigma\beta\alpha}S^{\sigma\nu} + R^{\nu}{}_{\rho\beta\alpha}S^{\mu\rho}$$

For the third expression, we note that we are allowed to lower even covariant derivatives with $g_{\alpha\beta}$ since the partial derivative is just a regular component and the "ΓT" terms we just lower the T. Thus, we contract

$$\begin{aligned} B^{\mu}{}_{;\alpha\mu} &= B^{\mu}{}_{;\mu\alpha} + R^{\mu}{}_{\nu\mu\alpha}B^{\nu} \\ &= B^{\mu}{}_{;\mu\alpha} + R_{\nu\alpha}B^{\nu} \\ B^{\mu;\alpha}{}_{\mu} &= B^{\mu}{}_{;\mu}{}^{\alpha} + R^{\alpha}{}_{\nu}B^{\nu} \end{aligned}$$

where we lastly use that $R_{\nu\alpha} = R_{\alpha\nu}$.

2.2 General Relativity

Equivalence Principle Dynamics in all LLFs is equivalent to SR. In practice, this usually entails replacing commas (partial derivative) with semicolons (covariant derivatives), with some subtlety in non-commuting covariant derivatives (see ex. 16.3 previous).

Ideal Measurements Define ideal rods and clocks to measure $\mathrm{d}s = (g_{\alpha\beta}\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta})^{1/2}$ and $\mathrm{d}\tau = (-g_{\alpha\beta}\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta})^{1/2}$ respectively. These can be constructed from geodesic worldlines.

Stress-Energy to Einstein Curvature We postulate mass-energy sources gravity. Recall that an observer with 4-velocity $\mathbf{u}$ measures mass-energy density $\rho = \mathbf{u} \cdot \mathbf{T} \cdot \mathbf{u}$ with $\mathbf{T}$ the stress-energy tensor a symmetric tensor satisfying $\boldsymbol{\nabla} \cdot \mathbf{T} = 0$. Thus, $\mathbf{T}$ must be the frame-independent object that sources gravity.

There must then be some tensor $\mathbf{G} \propto \mathbf{T}$ that describes spatial curvature. We know that it must be constructed from no more than the Reimann curvature tensor and the metric, must $\boldsymbol{\nabla} \cdot \mathbf{G} = 0$. This must be proportional to the Einstein curvature tensor (proved in an exercise), and requiring agreement with $R_{00} = R^{\alpha}{}_{0\alpha0} = 4\pi\rho$ yields proportionality constant $\mathbf{G} = 8\pi\mathbf{T}$.

2.2.1 Exercises

17.1 *Prove uniqueness of the Einstein tensor following steps in the book.*

First, we aim to compute the most general second-rank symmetric tensor constructable from the Reimann curvature tensor and the metric. We know that the metric is symmetric, the Ricci curvature tensor is symmetric, so we already know of three possible terms

$$\mathbf{G} = aR_{\alpha\beta} + bRg_{\alpha\beta} + \Lambda g_{\alpha\beta}$$

Since the only procedure to lower rank in a tensor is contraction, we can either contract $R^{\alpha}{}_{\beta\gamma\delta}$ once or twice, producing either the first or second term above. Thus, no other second-rank tensors can be constructed.

Taking the divergence of the above, we first note that in any LLF that $g_{\alpha\beta,\gamma} = 0$ by definition, and so the divergence of $g_{\alpha\beta}$ must itself vanish as well. As for the rest of the equation, we have

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{G} &= aR_{\alpha\beta}{}^{,\alpha} + bR^{,\alpha}g_{\alpha\beta} \\ &= aR_{\alpha\beta}{}^{,\alpha} + bR_{\beta\beta}{}^{,\alpha} \end{aligned} \tag{2.14}$$

But by the Bianchi identity for the Reimann curvature tensor, we know that $R^{\alpha}{}_{\beta\gamma\delta;\mu} + R^{\alpha}{}_{\beta\delta\mu;\gamma} + R^{\alpha}{}_{\beta\mu\gamma;\delta} = 0$. Contracting α and γ and going to an LLF, we obtain

$$R_{\beta\delta,\mu} + R^{\alpha}{}_{\beta\delta\mu,\alpha} - R_{\beta\mu,\delta} = 0$$

where for the last term we recall $R^{\alpha}{}_{\beta\mu\gamma} = -R^{\alpha}{}_{\beta\gamma\mu}$. But then we can swap the first two indicies of the second term above and contract $\beta\delta$ as follows

$$\begin{aligned} R_{\beta\delta,\mu} - R^{\beta}{}_{\alpha\delta\mu,\alpha} - R_{\beta\mu,\delta} &= 0 \\ R_{\beta\beta,\mu} - R_{\alpha\mu,\alpha} - R_{\beta\mu,\beta} &= \\ R_{\beta\beta,\mu} - 2R_{\alpha\mu,\alpha} &= \end{aligned}$$

Comparing to (2.14), we find that $b = -a/2$ is necessary and sufficient to make the equation vanish.

We also require $\mathbf{G}$ vanish in flat spacetime. Since $R^{\alpha}{}_{\beta\gamma\delta} = 0$ in flat spacetime, and $g_{\alpha\beta} = \eta_{\alpha\beta}$, the only way we can have vanishing $\mathbf{G}$ is for $\Lambda = 0$, thus producing

$$\mathbf{G} = a\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right)$$

where the constant of proportionality is determined by requiring agreement with Newton's law of gravitation.

Finally, to determine this constant of proportionality, we recall the equation of geodesic deviation

$$\frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}^2\tau^2} + R^{\alpha}{}_{\beta\gamma\delta}u^\beta\xi^\gamma u^\delta = 0$$

in which for the Newtonian regime we approximate $\mathbf{u} \approx \mathbf{e}^0$. Thus, we arrive at

$$\frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}^2\tau^2} + R^{\alpha}{}_{0\gamma0}\xi^\gamma = 0$$

Moreover, if we consider the EOM inside a ball of constant density ρ, we might recall that the EOM are of form $\ddot{q} + \frac{4\pi\rho}{3}q = 0$. Comparing with our above expression, we find that $R^{\alpha}{}_{0\gamma0} = \frac{4\pi\rho}{3}\delta^{\alpha}{}_{\gamma}, \alpha, \gamma \neq 0$, and so the Ricci curvature tensor has component $R_{00} = 4\pi\rho$. At the same time, tracing over $\mathbf{G}$ we obtain

$$\operatorname{Tr}\mathbf{G} = a$$