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1 Lorenz Model

The Lorenz Model arises as an approximation to atmospheric flow, starting with the description of the simplest mode of a convection zone and dropping terms that represent higher harmonics. This gives the following system of equations

$$\begin{aligned}\dot{X} &= -\sigma(X - Y) \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= b(XY - Z)\end{aligned}\tag{1}$$

with parameters canonically set at $\sigma = 10, b = 8/3, r = 27 > 1$ (note that σ is 0.7 for an ideal gas, 1–4 for water and > 10 for oils). Note that the equations are dissipative, i.e. volumes in phase space shrink in dynamics. Note also that they are *autonomous* in that there is no explicit time dependence on the right hand side, a convenient property.

It turns out that the solutions to this system are chaotic, or in Lorenz’s original terms “aperiodic.” We do many demonstrations that are no longer accessible, including *strange attractors*, *Poincaré section* and *1D maps*.

The last is going to be covered in a later lecture, but in case I’m interested in simulating the former two, I look them up here (the website has discussions of these with the applets absent):

- Poincaré section examines the intersection of an orbit with a particular plane in phase space, e.g. $Z = 31$ is a common one. So we just simulate and examine every time that $Z = 31$ is crossed, exactly what (X, Y) are.
- Strange attractors are subsets of phase space with fractional dimension (fractals) that orbits at long times lie in.

2 Pendula

2.1 Ideal Pendulum

The ideal pendulum is given by EOM $\frac{d^2\theta}{dt^2} + \frac{g \sin \theta}{l} = 0$. We can plot phase space $(\theta, \omega = \dot{\theta})$ for solution trajectories. There are a few features of interest

- Fixed points — Two types, elliptic (stable but not asymptotically stable) and hyperbolic (unstable)
- Limit Cycles — Solution trajectories.
- Homoclinic orbits — Trajectories that connect the same hyperbolic fixed point. Heteroclinic orbits connect two different hyperbolic fixed points.

We can write down the Hamiltonian for this system $H = \frac{J^2}{2I} + Mgl(1 - \cos \theta)$. The importance of being able to do is that phase space volume is preserved. What do we mean by this? Well, consider Hamilton's canonical equations

$$\begin{aligned}\dot{\theta} &= \frac{\partial H}{\partial J} = \frac{J}{I} \\ \dot{J} &= -\frac{\partial H}{\partial \theta} = -Mgl \sin \theta\end{aligned}\tag{2}$$

and if we define some phase space velocity $\vec{V} = (\dot{\theta}, \dot{J})$ we can verify that $\nabla \cdot \vec{V} = 0$. In other words, no attractors in phase space can exist for a Hamiltonian system.

2.2 Dissipative Pendulum

New EOM is $\frac{d^2\theta}{dt^2} + \eta \frac{d\theta}{dt} + \frac{g}{l} \sin \theta = 0$. This creates a fixed point at $(0, 0)$, asymptotically stable (“linearly stable” is his terminology), and $(\pi, 0)$ is suddenly linearly unstable, in that we will exponentially deviate from the point (before, we still periodically returned).

2.3 Driving + damping

Let's drive the pendulum and rescale variables to $\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \sin \theta = g \cos(\omega_D t)$. Under small angle approximation (small amplitude of solution and driving), we know how to solve this. What about with larger driving amplitudes though?

To handle this, we wish to solve numerically, which is much easier with an autonomous system than with an explicit time dependence. Thus, we introduce $\dot{\theta}_D = \omega_D$ and suddenly we can write

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\gamma\omega - \sin \theta + g \cos(\theta_D) \\ \dot{\theta}_D &= \omega_D\end{aligned}\tag{3}$$

This is a handy trick to get an autonomous system at the expense of an additional equation in the system of ODEs. We might also suspect that the behavior of the system changes when we get an extra dimension of phase space.

Well, let's start by observing that $\vec{\nabla} \cdot \vec{v} = -\gamma$ so phase space volumes contract. This might naively lead us to conclude that any volume of initial conditions must contract to a point, but we mustn't be hasty; the Lorenz model also has contracting phase space volumes. Instead, we know that we might approach a limit cycle in a way such that while the overall phase space contracts, the space might “stretch” along some dimensions and contract in others such that chaotic dynamics are observed! The demonstrations illustrating this are broken, but we will definitely simulate this.

3 Nonlinear Oscillators

3.1 Van der Pol Oscillator

Consider the following model of a harmonic oscillator to include a non-linear damping. Damping is negative at small amplitudes to model local instability, but is positive for larger amplitudes; we thus expect the system to have some limit cycles where the average damping vanishes. The EOM is

$$\ddot{x} - \gamma(1 - x^2)\dot{x} + x = g \cos(\omega_D t) \quad (4)$$

3.1.1 Small γ , secular perturbation

Let's first look at oscillations with no driving, $g = 0$. We return to the second order EOM for this. The natural starting point is to also examine for small γ and try Lindstedt-Poincaré perturbation theory (also called secular perturbation theory) on

$$\ddot{x} + x = \gamma(1 - x^2)\dot{x} \quad (5)$$

We begin with the $\gamma = 0$ solution, which is just $x(t) = ae^{it+\phi}$. Then, by secular perturbation theory¹, when $\gamma \neq 0$, we can substitute $x(t) = A(t) \cos(t) + \gamma x_1 + \dots$ and also expect $A(t)$ to be some slow-varying function such that $\frac{d^n A(t)}{dt^n} \sim \gamma^n$ (this is introducing a second slow time scale, characteristic of secular perturbation theory apparently) and collect $O(\gamma^1)$ terms in the ODE to obtain

$$\begin{aligned} \frac{d^2}{dt^2} [A \cos(t) + \gamma x_1] + A \cos(t) + \gamma x_1 &= \gamma(1 - A^2 \cos^2(t)) \frac{d}{dt} (A \cos(t)) \\ -2 \sin(t) \frac{dA}{dt} + \gamma \frac{d^2 x_1}{dt^2} + \gamma x_1 &= -\gamma A \sin(t) + \gamma A^3 \underbrace{\cos^2(t) \sin(t)}_{\frac{\sin(t)}{4} + \frac{\sin(3t)}{4}} \\ \gamma \left(\frac{d^2 x_1}{dt^2} + x_1 \right) &= -\gamma A \sin(t) + \gamma A^3 \frac{\sin(t)}{4} + 2 \sin(t) \frac{dA}{dt} + O(\sin(3t)) \end{aligned} \quad (6)$$

where we don't care about the $\sin(3t)$ term since it doesn't drive x_1 on resonance. On the other hand, we require the remaining term vanish, which implies that

$$2 \frac{dA}{dt} = \gamma A - \frac{\gamma A^3}{4} = \gamma \frac{A(4 - A^2)}{4} \quad (7)$$

off of which we can read that any initial $A > 0$ tends to grow until $A = 2$, asymptoting at $x(t) = 2 \cos(t)$.

3.1.2 Large γ

For large γ , let's stick to driving $g = 0$ and go to a slightly unconventional choice of phase space variables, namely

$$\dot{y} = \ddot{x} - \gamma(1 - x^2)\dot{x} = -x \quad (8)$$

$$\dot{x} = y + \gamma \left(x - \frac{x^3}{3} \right) \quad (9)$$

We then rescale $y \rightarrow \gamma Y, t \rightarrow \gamma T, x \rightarrow X$ and introduce $\eta = \gamma^{-2}$ and obtain

$$\eta \frac{dX}{dT} = Y + \left(X - \frac{X^3}{3} \right) \quad (10)$$

$$\frac{dY}{dT} = -X \quad (11)$$

and for large γ we now have small η . We first find that $\eta = 0$ is inconsistent; the first equation reduces to $Y = -\left(X - \frac{X^3}{3}\right)$, predicting a minimum of $Y = -\frac{2}{3}$ while the latter suggests $Y \rightarrow 0$ strictly decreasing-ly, contradictory. However, we can still analyze qualitatively what happens for small η , an orbit that comprises two pieces

¹We learned as *Lindstedt-Poincaré* in 106, but this seems to differ slightly; instead of setting up this slowly varying timescale as below, we tend to set $s = \omega t$ and expand $\omega = 1 + \epsilon \omega_1 + \dots$

- $Y + \left(X - \frac{X^3}{3}\right)$ is small, so $Y \simeq \frac{X^3}{3} - X$ and Y, X change on timescales $O(1)$.
- $Y + \left(X - \frac{X^3}{3}\right) \gg \eta \frac{dX}{dT}$, then X changes very rapidly over timescale $O(\eta^{-1})$, and since $\frac{dY}{dT} \sim O(1)$ we can treat Y to be constant.

This breaks down to the orbit following $Y = \frac{X^3}{3} - X$ until reaching an extremum, at which point $\frac{dX}{dT} = 0$ and $Y + \left(X - \frac{X^3}{3}\right) \gg \eta \frac{dX}{dT}$, and suddenly we evolve at constant Y until we hit another point on the $Y = \frac{X^3}{3} - X$ curve. The picture is then of following a cubic on $X < -1, X > 1$ and jumping from $\left(-1, \frac{2}{3}\right) \rightarrow \left(2, \frac{2}{3}\right)$ and from $\left(1, -\frac{2}{3}\right) \rightarrow \left(-2, -\frac{2}{3}\right)$ (the constant Y portions of the curve).

This analysis allows us to compute $\dot{x} = v = \gamma \left[Y - \left(-X + \frac{X^3}{3} \right) \right]$ as the difference between $\frac{X^3}{3}$ the cubic and Y which tracks the cubic for $X < -1, X > 1$ and is constant during the jumps. We can simulate via Mathematica, and verify the shape of the trajectory.

This shows that there are free oscillations at any given γ .

3.2 Driven Oscillations: Frequency Locking

In general, when one drives an oscillator, one can either observe oscillations at both the free and driving frequency (in which case the power spectrum would show signatures at both frequencies), or we would see only a single frequency (and maybe harmonics). The latter case is called *frequency locking*, when the driving frequency locks the free oscillation to some fixed frequency. The former case, when the driving and free oscillations are in an irrational ratio, is considered *quasiperiodic*, because the sum of the two frequencies never repeats.

Another way to define frequency locking is that while some parameters of the problem (e.g. drive amplitude, dissipation) are varied, all peaks in the power spectrum must remain fixed for some non-zero parameter space. If the frequency is not locked, at least some peaks should vary continuously as parameters are varied continuously. It turns out that frequency locking occurs for the vDP oscillator for high drive amplitude and small frequency differences, which is generally true.

The basic idea behind analyzing frequency locking (beyond the scope of this class) is to take equation of motion

$$\ddot{x} + x = \gamma(1 - x^2)\dot{x} + g \cos(\omega_D t) \quad (12)$$

and substitute $\omega_D = 1 + \gamma\Delta$ and $g = 2\gamma F$ again in the $\gamma \ll 1$ limit, so weak driving near resonance. Performing the same secular perturbation theory (I concede, it is easier to use $x(t) = A(t)e^{it} + c.c. + \epsilon x_1$ here than cosines) and write driving term $\epsilon F e^{i\Delta\epsilon t} e^{it}$, we obtain that killing the secular term requires

$$\frac{dA}{dt} = \frac{\gamma}{2} \left(A - |A|^2 \right) - \gamma \frac{i}{2} F e^{i\Delta\epsilon t} \quad (13)$$

then we can make the substitution $\tilde{A} = A e^{-i\Delta\epsilon t}$ which yields

$$\frac{d\tilde{A}}{dt} + i\gamma\Delta\tilde{A} = \frac{\gamma}{2} \left(\tilde{A}(1 - |\tilde{A}|^2) - iF \right) \quad (14)$$

upon which whether locking happens boils down to the nature of algebraic solutions to \tilde{A} above. A stable *fixed point* yields a locked solution (harmonics nearby tend to disappear) while stable *limit cycles* produce unlocked solutions (two frequencies), since the Poincaré-Bendixson theorem tells us that no other asymptotic long term dynamics are possible. Unstable solutions are uninteresting; if both stable fixed and limit cycles exist then both locked and unlocked dynamics can manifest. The remainder of this is omitted.

It turns out that the vDP oscillator does not exhibit chaos.

3.3 Duffing Oscillator

The Duffing Oscillator, given by potential

$$V(x) = \pm \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{4} \quad (15)$$

and EOM

$$\ddot{x} + \gamma \dot{x} \pm x + x^3 = g \cos(\omega_D t) \quad (16)$$

does exhibit chaos for both signs apparently!

4 One Dimensional Maps

Let's consider an arbitrary system

$$\dot{U} = f(U|r) \quad (17)$$

with r some control parameters and U a vector of phase space coordinates. We have studied solely such systems that are

autonomous no time appears on RHS

deterministic no stochasticity in EOM

dissipative volumes in phase space shrink, giving attractors

The structure of the phase space is of smooth *vector fields* in \mathbb{R}^N , and the solution $U(t)$ is called a *flow*. Casting in this language gives us strong categorizations of what is/isn't possible such as the *Poincaré-Bendixson Theorem*.

4.1 Maps

In a map, we study an analogous system to (17) but with a discrete timestep $U_{n+1} = F(U_n|r)$. The effect of the evolution on volumes in phase space is then given by the Jacobian

$$J = \left| \frac{\partial^2(F^{(i)})}{\partial(U^{(j)})^2} \right| \quad (18)$$

If $J = 1$ then the map is called *volume preserving*, otherwise *dissipative*. A few ways to map flows to maps:

- Integrate the flow for $n\tau$ with τ some fixed time interval.
- $N - 1$ dimensional Poincaré section.

Some notes about flows:

- Conventionally we scale maps to map $[0, 1]$ onto itself.
- Successive iterations of the map are notated $F^n(x_0)$.
- Defining $x_f : F(x_f) = x_f$ fixed point, we can also ask the whether the fixed point is stable. The usual technique is to linearize the map about $x = x_f + \delta x$ and compute the Taylor expansion $\delta x_{n+1} = F'(x_f)\delta x_n$. We thus observe stability if $F'(x_f) < 1$ and instability for the opposite.

4.1.1 Bifurcations

Suppose we have a map $F(x|a)$ with some parameter a . Then, as a is varied, it is possible that the long time solution of the map changes. These changes are called bifurcations. We depict bifurcations by plotting long time stable $F^k(x|a)$ as a function of a . For instance, a period 2 orbit that is exhibited for some value of a is plotted as two points $(a, x_1), (a, x_2)$ ($F^2(x_1) = F(x_2) = x_1$). Regions of a for which a fixed point orbit exists show up as a single curve, and regions with chaotic dynamics show up as a continuum of points.

To be able to better describe these chaotic regions, we look to a statistical description. The *invariant measure* is the probability density of x values that an orbit visits. We begin by defining the *measure*.

$$\rho(x, x_0) dx = \lim_{N \rightarrow \infty} \frac{1}{N} \times n(x, x_0) \quad (19)$$

where $\rho(x, x_0)$ is the density of points with initial condition x_0 and $n(x, x_0)$ is the number of times the orbit visits $[x-dx, x+dx]$ out of N iterations.

It turns out that this measure $\rho(x, x_0)$ doesn't actually depend on x_0 for almost all choices of x_0 , so we define $\rho(x)$ the invariant measure of our attractor. Sometimes the invariant measure can be constructed directly from its definition

$$\rho_n(y) = \rho_{n+1}(y) = \int dx \delta[y - F(x)] \rho_n(x) \quad (20)$$

4.1.2 Lyapunov Exponents

We can better quantify “sensitive dependence on initial conditions” by Lyapunov Exponents. We demand that for some pair of initial conditions $x_0, x_0 + \epsilon$ for the distance to grow as $|\delta x_n| = \epsilon e^{n\lambda(x_0)}$. $\lambda(x_0)$ is the Lyapunov Exponent

$$\lambda(x_0) = \lim_{n \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \log \left| \frac{F^n(x_0 + \epsilon) - F^n(x_0)}{\epsilon} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \frac{dF^n(x_0)}{dx_0} \right| \quad (21)$$

For systems with ergodic invariant measure, λ is independent of x_0 , so we will call λ *the Lyapunov Exponent of the map*. Note that the derivative is of the n -th iteration, and it turns out that

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \frac{dF^n(x_0)}{dx_0} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} \log |F'(x_{n-1})F'(x_{n-2}) \dots F'(x_1)F'(x_0)| = \langle \log |F'| \rangle \quad (22)$$

A positive λ means that closely spaced initial conditions diverge exponentially, which is a signature of chaos and often used as a definition thereof.

5 Two Dimensional Maps

One-dimensional maps are a restricted view of most dynamical systems, which cannot simply be iterated in reverse to find the initial condition (many non-invertible systems). On the other hand, integrating most dynamical equations in reverse can yield the preimage of a trajectory, which leads us to *two*-dimensional maps as a faithful representation of a smooth flow in 3D phase space. Here, we will investigate four examples of 2D maps.

5.1 Henon Map

Consider the Henon map, which iterates

$$\begin{aligned} x_{n+1} &= y_n + 1 - ax_n^2 \\ y_{n+1} &= bx_n \end{aligned} \tag{23}$$

The Jacobian is $J = \begin{vmatrix} -2ax_n & 1 \\ b & 0 \end{vmatrix} = |b|$, so for $b = 1$ the map is area preserving and for $b < 1$ the map is dissipative, where areas in phase space contract. This is where the chaos lives. To get some intuition about what the strongly dissipative limit looks like, we can substitute $x_{n+1} = bx_{n-1} + 1 - ax_n^2$ which for $b \rightarrow 0$ is simply $x_{n+1} \approx 1 - ax_n^2$ the quadratic map we studied in the previous chapter (that I ignored).

Canonically chosen values are $a = 1.4, b = 0.3$.

5.2 Baker's Map

The Baker's map is given by the mapping

$$x_{n+1} = \begin{cases} \lambda_a x_n & y_n < \alpha \\ (1 - \lambda_b) + \lambda_b x_n & y_n > \alpha \end{cases} \quad y_{n+1} = \begin{cases} y_n / \alpha & y_n < \alpha \\ (y_n - \alpha) / \beta & y_n > \alpha \end{cases} \tag{24}$$

with $\beta = 1 - \alpha, \lambda_a + \lambda_b \leq 1$, so points in 2D phase space on either side of the $y = \alpha$ boundary are mapped to $x < \lambda_a, x > 1 - \lambda_b$ respectively. Then when $\lambda_a + \lambda_b < 1$, the total phase space area decreases and the map is *dissipative*. Qualitatively, we see that the number of vertical striations increases with increasing iterations, specifically with the n th iteration we have $\binom{n}{m}$ stripes of width $\lambda_a^m \lambda_b^{n-m}$.

Note that for the Bakers' map, the attractor is *hyperbolic* in that, crudely, at each point phase space both an expanding and contracting direction can be defined such that the directions vary continuously across the map and are bounded to be nonzero. In the case of the Bakers' map, the map is expanding in the y direction and contracting in the x across the entire map and so is hyperbolic. Not many maps/flows are hyperbolic, but many useful properties can be proven in such cases.

5.3 Other 2D maps

Other interesting maps are the Duffing and the Kaplan-Yorke, also simulated in our tests. We are unable to simulate the Baker's Map because its chaos relies on how regions of phase space map and our simulation module is not presently powerful enough to handle this.

6 Power

One interesting potential criterion for chaos is the power spectrum of a signal. Periodic/quasiperiodic signals all give peaks at countably many frequencies, while chaotic dynamics tend to yield broad bands in the power spectrum. Nonetheless, due to observation time constraints and their impact on observed power spectra, we must characterize the impacts of these phenomena first before using this as a solid criterion.

If we had an infinite time series $y(t)$, then the power spectrum of the signal is given simply by $P(\omega) \propto |y(\omega)|^2$, and our above analysis would make sense. However, realistically we have a signal y_i measured in discrete intervals over a finite interval $t \in [0, T]$, so the power spectrum is estimated by the Fourier series

$$\tilde{y}_k = \sum_{j=0}^{N-1} y_j \exp\left(\frac{2\pi i j k}{N}\right) \quad (25)$$

$$= \sum_j y(t_j) \exp(i\omega_k t_j) \quad (26)$$

and the power spectrum is given

$$P(\omega) \simeq \frac{1}{N^2} \begin{cases} |\tilde{y}_0|^2 & \omega = 0 \\ |\tilde{y}_{N/2}|^2 & \omega = \frac{\pi N}{T} \\ (|\tilde{y}_k|^2 + |\tilde{y}_{N-k}|^2) & \end{cases} \quad (27)$$

To figure out how well this power spectrum estimates the real power spectrum of some signal $y(t)$ that we sample with y_i , we can realize that our \tilde{y}_k can be written in terms of $y(t)$ like

$$\tilde{y}(t) = [(y(t) \times H(t, T)) * S(t, T)] \times S(t, \Delta) \quad (28)$$

with $S(t, \tau)$ a Dirac comb with period τ and H the tophat function. The convolution represents the periodic assumption, and the product by the Dirac comb samples in intervals of $\Delta = \frac{T}{N}$. But we can compute the FT of the above function to be

$$\tilde{y}(\omega) \propto \left[\left(\tilde{y}(\omega) * \tilde{H}\left(\omega, \frac{2\pi}{T}\right) \right) \times S\left(\omega, \frac{2\pi}{T}\right) \right] * S\left(\omega, \frac{2\pi}{\Delta}\right) \quad (29)$$

where we know that the FT of H is just a sinc function with width $\sim 1/T$ and falls off like ω^{-1} . The prescription is thus “Take the ideal FT, broaden with a sinc, sample every $2\pi/T$, then convolve to periodically repeat.” This last convolution means that all higher frequencies will just be aliased back into the Nyquist range $\pm\pi/\Delta$.

It is also possible to convolve by a less sharp windowing function than H , which means that the FT falls off faster and the ideal power spectrum is less broadened.

7 Lyapunov Exponents

Lyapunov exponents describe how quickly nearby trajectories diverge. In one dimension, it was just $\left\langle \log \left| \frac{df}{dx} \right| \right\rangle$. Let's generalize this.

7.1 The Jacobian

Consider the map $U_{n+1} = F(U_n)$. We want to know how different the trajectory is starting from U_0 and from some small change. This is described by the Jacobian matrix

$$K_{ij}(U_n) = \left. \frac{\partial F_i}{\partial U^{(j)}} \right|_{U=U_n} \quad (30)$$

and so if two trajectories are at $U_n, U_n + \epsilon_n$ then $\epsilon_{n+1} = \mathbf{K}(U_n)\epsilon_n$.

For continuous systems $\dot{U} = f(U)$, then we have instead $\frac{d\epsilon}{dt} = \mathbf{K}(\mathbf{U})\epsilon$ for $K^{(ij)} = \frac{\partial f_i}{\partial U^{(j)}}$. But if we introduce a new matrix \mathbf{M} satisfying $\frac{d\mathbf{M}}{dt} = \mathbf{K}(U(t))\mathbf{M}$, then we have

$$\frac{\partial U^{(i)}(t)}{\partial U^{(j)}(t_0)} = M_{ij}(t, t_0) \quad (31)$$