# 1 Laplace Plane Dynamics

## 1.1 Maximum Separatrix Area—Simple

Consider a planet with orbit normal $\hat{\mathbf{l}}_{\rm p}$ that experiences precession driven by stellar oblateness $\hat{\mathbf{l}}_{\rm s}$ and an outer perturber $\hat{\mathbf{l}}_{\rm o}$. We assume that the planet's orbit is circular. The vector form of the precessional dynamics are (e.g. Tremaine+2009, Eq 19):

$$\frac{\mathrm{d}\hat{\mathbf{l}}_{\rm p}}{\mathrm{d}t} = \omega_{\rm sp}\left(\hat{\mathbf{l}}_{\rm p}\cdot\hat{\mathbf{l}}_{\rm s}\right)\left(\hat{\mathbf{l}}_{\rm p}\times\hat{\mathbf{l}}_{\rm s}\right) + \omega_{\rm op}\left(\hat{\mathbf{l}}_{\rm p}\cdot\hat{\mathbf{l}}_{\rm o}\right)\left(\hat{\mathbf{l}}_{\rm p}\times\hat{\mathbf{l}}_{\rm o}\right). \tag{1}$$

We first make an important symmetry argument: in the limits of $\omega_{\rm sp} \ll \omega_{\rm op}$ or $\omega_{\rm sp} \gg \omega_{\rm op}$, the evolution of $\hat{\mathbf{l}}_{\rm p}$ consist of uniform precession about $\hat{\mathbf{l}}_{\rm o}$ and $\hat{\mathbf{l}}_{\rm p}$ respectively, and thus the separatrix area must go to zero in these limits. In fact, the phase portrait must the same under the following transformation: swap the two frequencies $(\omega_{\rm sp}, \omega_{\rm op})$ and the two vectors $(\hat{\mathbf{l}}_{\rm o}, \hat{\mathbf{l}}_{\rm s})$. Swapping the precession frequencies is equivalent to taking $a/r_{\rm M} \mapsto r_{\rm M}/a$ (and rescaling time), since $\omega_{\rm op}/\omega_{\rm sp} = (a/r_{\rm M})^5$. Thus, we arrive at an important conclusion: *the phase portraits are equivalent, up to a rotation of reference frame, for any two $r_{\rm M,1}$ and $r_{\rm M,2}$ satisfying $a/r_{\rm M,1} = r_{\rm M,2}/a$*. This implies that the separatrix area is symmetric about $r_{\rm M} = a$ as well.

It's not clear that the separatrix area must be monotonic between $r_{\rm M} \in [0, a]$, but intuitively **this seems like it should be the case** (?), since there are no special values of $\omega_{\rm sp}/\omega_{\rm op}$ in the equation of motion. If so, then the maximum separatrix area is obtained for $r_{\rm M} = a$. The curve for the separatrix in this case is significantly easier to obtain, though it still seems difficult to integrate explicitly (maybe there's a clever idea?).

To compute the separatrix area for $a = r_{\rm M}$, we note that the low-obliquity Laplace equilibrium P1 is located exactly halfway between $\hat{\mathbf{l}}_{\rm o}$ and $\hat{\mathbf{l}}_{\rm s}$. Thus, we choose the reference frame such that $\hat{\mathbf{z}} \propto \hat{\mathbf{l}}_{\rm o} + \hat{\mathbf{l}}_{\rm s}$, and we choose $\hat{\mathbf{y}}$ to point towards P2 (which is always $\pi/2$ away from P1). Then, defining

$$\cos\epsilon \equiv \hat{\mathbf{l}}_{\rm o}\cdot\hat{\mathbf{l}}_{\rm s}, \tag{2}$$

we can write

$$\hat{\mathbf{l}}_{\rm o} = \cos\frac{\epsilon}{2}\hat{\mathbf{z}} + \sin\frac{\epsilon}{2}\hat{\mathbf{x}}, \tag{3}$$

$$\hat{\mathbf{l}}_{\rm s} = \cos\frac{\epsilon}{2}\hat{\mathbf{z}} - \sin\frac{\epsilon}{2}\hat{\mathbf{x}}. \tag{4}$$

Finally, upon inspection, $\hat{\mathbf{x}}$ is also an equilibrium point, which must be P3. **In summary, in this reference frame, P1 lies along $\hat{\mathbf{z}}$, P2 lies along $\hat{\mathbf{y}}$, and P3 lies along $\hat{\mathbf{x}}$.**

To get the level curve corresponding to the separatrix, we evaluate the Hamiltonian (factoring out the prefactor $\omega_{\rm sp} = \omega_{\rm op}$) and adopt a spherical coordinate system:

$$H \propto -\frac{1}{2}\left[\left(\hat{\mathbf{l}}_{\rm p}\cdot\hat{\mathbf{l}}_{\rm s}\right)^2 + \left(\hat{\mathbf{l}}_{\rm p}\cdot\hat{\mathbf{l}}_{\rm o}\right)^2\right], \tag{5}$$

$$\tilde{H}\left(\theta, \phi\right) = -\left[\sin^2\frac{\epsilon}{2}\sin^2\theta\cos^2\phi + \cos^2\frac{\epsilon}{2}\cos^2\theta\right], \tag{6}$$

where we have adopted spherical coordinates $(\theta, \phi)$ to describe the orientation of $\hat{\mathbf{l}}_{\rm p}$, and $\theta = \pi/2, \phi = 0$ corresponds to $\hat{\mathbf{x}}$. We first evaluate $H$ (dropping the tilde) at P3:

$$H_3 = -\sin^2\frac{\epsilon}{2}, \tag{7}$$

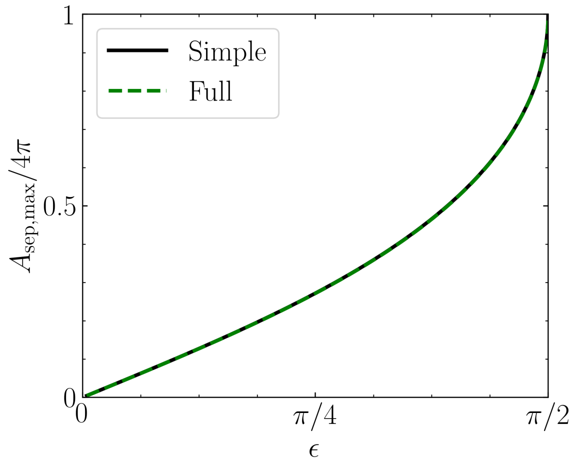


**Figure 1:** Fractional phase space area enclosed by the maximal separatrix as a function of $\epsilon$. Reminder: this is the area surrounding Laplace equilibrium P2 when $r_{\mathrm{M}} = a$, which is also the maximum extent of the separatrix.

then the separatrix is given by

$$H\left(\theta_{\mathrm{sep}}(\phi), \phi\right) = H_3, \tag{8}$$

$$\sin^2\frac{\epsilon}{2}\left(1 - \cos^2\theta_{\mathrm{sep}}\right)\cos^2\phi + \cos^2\frac{\epsilon}{2}\cos^2\theta_{\mathrm{sep}} = \sin^2\frac{\epsilon}{2}, \tag{9}$$

$$\begin{aligned}\cos^2\theta_{\mathrm{sep}} &= \frac{\sin^2\frac{\epsilon}{2}\sin^2\phi}{\cos^2\frac{\epsilon}{2} - \sin^2\frac{\epsilon}{2}\cos^2\phi}\\ &= \frac{\sin^2\phi}{\cot^2\frac{\epsilon}{2} - \cos^2\phi},\end{aligned} \tag{10}$$

$$A_{\mathrm{sep}} = 4\int_0^{\pi} \cos_+\theta_{\mathrm{sep}}\,\mathrm{d}\phi. \tag{11}$$

Here, $\cos_+\theta_{\mathrm{sep}}$ indicates that we take the positive root; one factor of two arises because the vertical extent of the separatrix is $\cos_+\theta_{\mathrm{sep}} - \cos_-\theta_{\mathrm{sep}}$, and a second factor of two arises because we are only integrating $\phi \in [0, \pi]$. Under this convention, the maximum possible phase space area is $4\pi$. We display the value of $A_{\mathrm{sep}}$ in Fig. 1.

Note: the integral for $A_{\rm sep}$ is analytic:

$$\begin{aligned}
A_{\rm sep} &= 4\int_0^{\pi} \frac{\sin\phi}{\sqrt{\cot^2\frac{\epsilon}{2} - \cos^2\phi}}\,\mathrm{d}\phi \\
&= 4\int_{-1}^{1} \frac{1}{\sqrt{\cot^2\frac{\epsilon}{2} - \cos^2\phi}}\,\mathrm{d}\cos\phi \\
&= 4\left[\tan^{-1}\left(\frac{u}{\sqrt{\cot^2\frac{\epsilon}{2} - u^2}}\right)\right]_{u=-1}^{u=1} \\
&= 8\left[\tan^{-1}\sqrt{\frac{\sin^2(\epsilon/2)}{\cos\epsilon}}\right]. \qquad (12)
\end{aligned}$$

## 1.2 Maximum Separatrix Area—Melaine

Melaine says that the separatrix is given by the solutions to the equation ($I_Q$ is the satellite inclination to the planet equator, and $\delta Q$ is the corresponding phase angle; we call these $\theta, \phi$ above)

$$\tan I_{\rm Q,\pm} = \frac{\cos\delta Q\sin(2\epsilon) \pm \sin\delta Q\sin\epsilon\sqrt{2\left(u - 1 + \sqrt{1 + u^2 + 2u\cos(2\epsilon)}\right)}}{u + 1 - 2\cos^2\delta Q\sin^2\epsilon - \sqrt{1 + u^2 + 2u\cos(2\epsilon)}}. \qquad (13)$$

Here $u = r_{\rm M}^5/a^5$. However, this expression is singular for $\cos\delta_Q = w_\pm$, where

$$w_\pm = \pm\sqrt{\frac{1 + u - \sqrt{1 + u^2 + 2u\cos(2\epsilon)}}{2\sin^2\epsilon}}. \qquad (14)$$

This is where the denominator vanishes. Note that this must be a removable/coordinate singularity: physically, there is no special value of $\delta_Q$. The separatrix area is then just given by integrating $\cos I_Q$ but taking the correctly-signed roots, which Melaine works out to be

$$\frac{A}{2} = \int_0^{\pi} \cos I_{\rm Q,+} - \cos I_{\rm Q,-}\,\mathrm{d}\delta_Q, \qquad (15)$$

$$\begin{aligned}
&= \int_0^{\arccos w_+} \left(\frac{-1}{\sqrt{1 + x_+^2}} - \frac{-1}{1 + x_-^2}\right)\mathrm{d}\delta_Q \\
&\quad + \int_{\arccos w_+}^{\arccos w_-} \left(\frac{1}{\sqrt{1 + x_+^2}} - \frac{-1}{1 + x_-^2}\right)\mathrm{d}\delta_Q \\
&\quad + \int_{\arccos w_-}^{\pi} \left(\frac{1}{\sqrt{1 + x_+^2}} - \frac{1}{1 + x_-^2}\right)\mathrm{d}\delta_Q. \qquad (16)
\end{aligned}$$

We find that the two expressions agree, see Fig. 1. Is it obvious that they should? Setting $u = 1$, we find that

$$\begin{aligned}\tan I_{\mathrm{Q},\pm} &= \frac{\cos\delta Q\sin(2\epsilon)\pm\sin\delta Q\sin\epsilon\sqrt{4\cos\epsilon}}{2-2\cos^2\delta Q\sin^2\epsilon-2\cos\epsilon}\\ &= \frac{\cos\delta Q\sin(2\epsilon)\pm\sin\delta Q\sin\epsilon\sqrt{4\cos\epsilon}}{2-2\cos^2\delta Q\sin^2\epsilon-2\cos\epsilon}.\end{aligned} \tag{17}$$

Not obvious.

## 2 Constraints on Perturber

For the mechanism to work, we need a host star with mass $m_\star$ and a distant perturber with mass $m_2$. We will denote the planet's properties with $p$ subscripts.

In order for the mechanism to work, we need to know the Laplace radius of the star as it spins down. This is easy to do by generalizing the expression in my paper with Melaine. The Laplace radius should be:

$$r_{\rm M}^5 = \frac{2k_{2,\star}}{3}\frac{m_\star + m_2}{m_2}\left(\frac{\omega_\star}{n_b}\right)^2 R_\star^5, \tag{18}$$

$$r_{\rm M} = 0.41\,\text{au}\left(\frac{k_2}{0.02}\right)^{1/5}\left(\frac{\omega_\star}{2\pi/(3\,\text{day})}\right)^{2/5}\left(\frac{m_2}{M_\odot}\right)^{-1/5}\left(\frac{a_{\rm b}}{300\,\text{AU}}\right)^{3/5}\left(\frac{R_\star}{R_\odot}\right). \tag{19}$$

Here, b subscripts denotes properties of the binary, and we've taken $m_\star = m_2 = M_\odot$. Thus, if a star spins down by $10\times$, its Laplace radius migrates by $10^{2/5} \approx 2.5$, or $r_{\rm M} \sim 0.16$ au. For reference, a planet on a 10 day orbit (typical of polar Neptunes) is located at $\sim 0.1$ au.

In order for this mechanism to work, we really need to be looking at stellar companions a factor of $\sim 10\times$ closer, since $10^{3/5} \sim 4$, so that it sweeps through the 0.1 au range. Moreover, considering that high-e migration typically circularizes the planet at $\sim 3$ day (a few times the tidal radius $R_{\rm p}/(M_\star/m_{\rm p})^{1/3} \sim 0.01$ au), the Laplace plane transition will generally struggle to do much for post-high-e-migration.

However, interestingly, if we consider instead CJ companions:

$$r_{\rm M} = 0.16\,\text{au}\left(\frac{k_2}{0.02}\right)^{1/5}\left(\frac{\omega_\star}{2\pi/(3\,\text{day})}\right)^{2/5}\left(\frac{m_2}{M_{\rm J}}\right)^{-1/5}\left(\frac{a_{\rm b}}{5\,\text{AU}}\right)^{3/5}\left(\frac{R_\star}{R_\odot}\right). \tag{20}$$

Thus, a CJ misaligned from an inner Neptune may introduce nontrivial Laplace plane transitions. Such a mechanism naturally arises from Petrovich+20, the current (only?) favored mechanism for polar Neptunes. However, if there is just primordial disk misalignment, then as the star spins down, the SE+CJ systems may experience this transition. Of course, there is no preferennce for 90° once the star spins down, and since there is no dissipation, we should roughly have $\theta_{\star-{\rm p,i}} \sim \theta_{\rm p-CJ,f}$. But the obliquity will broadly oscillate!

### 2.1 ZLK and Laplace Plane Transition Parameter Space

If we want the planet to undergo high-eccentricity migration via the ZLK + tides mechanism, then it will circularize no closer[1] than $\simeq 2a_{\rm peri,min}$, where $a_{\rm peri,min}$ is set by the combination of ZLK and short range forces. This is because orbital AM ($\propto \sqrt{a(1-e^2)}$) is conserved by the eccentricity tide (driven by the pseudosynchronously-spinning planet).

We can express $a_{\rm peri,min}$ in terms of $j_{\min} \equiv \sqrt{1-e_{\max}^2}$

$$a_{\rm peri,min} = a j_{\min}, \tag{21}$$

where $j_{\min}$ satisfies (Liu, Munoz & Lai 2015, with $m_\star + m_{\rm p} \approx m_\star$ taken)

$$\frac{9}{8}e_{\max}^2 = \frac{\epsilon_{\rm tide}}{15}\left(\frac{1+3e_{\max}^2+\frac{3e_{\max}^4}{8}}{j_{\min}^9}-1\right)+\frac{\epsilon_{\rm rot}}{3}\left(\frac{1}{j_{\min}^3}-1\right)+\epsilon_{\rm GR}\left(\frac{1}{j_{\min}}-1\right), \tag{22}$$

$$\epsilon_{\rm GR} = \frac{3Gm_\star^2 a_{\rm out}^3}{a^4c^2m_2}, \tag{23}$$

$$\epsilon_{\rm tide} = \frac{15m_{\rm p}a_{\rm out}^3k_{2,\star}R_\star^5}{a^8m_2}, \tag{24}$$

$$\begin{aligned}\epsilon_{\rm rot} &= \frac{a_{\rm out}^3k_{2,\star}R_\star^5}{2Ga^5m_2}\omega_\star^2\\ &= \frac{(m_\star+m_2)k_{2,\star}R_\star^5}{2a^5m_2}\left(\frac{\omega_\star}{n_{\rm out}}\right)^2,\end{aligned} \tag{25}$$

where the host star uses $\star$ subscripts, the planet $p$, and the perturbing/binary companion 2; $a$ is the sma of the inner binary, and $a_{\rm out}$ refers to that of the outer binary (which we take to be circular for convenience).

For young stellar hosts, which are typically rotating rapidly, $\epsilon_{\rm rot}$ dominates the short range force contributions. Taking $e_{\max} \approx 1$, we obtain

$$\frac{9}{8} \approx \frac{\epsilon_{\rm rot}}{3j_{\min}^3}, \tag{26}$$

$$j_{\min} \approx \frac{2}{3}\epsilon_{\rm rot}^{1/3}, \tag{27}$$

$$\frac{a_{\rm peri,min}}{a} \approx \frac{2}{3}\epsilon_{\rm rot}^{1/3}, \tag{28}$$

$$\frac{a_{\rm peri,min}}{R_\star} = \frac{2}{3}\left(\frac{(m_\star+m_2)k_{2,\star}}{2m_2}\right)^{1/3}\left(\frac{R_\star}{a_0}\right)^{2/3}\left(\frac{\omega_{\star,0}}{n_{\rm out}}\right)^{2/3}, \tag{29}$$

where we have denoted $\omega_{\star,0}$ to point out that $a_{\rm peri,min}$ is set by the star's early spin rate, and $a_0$ to remind us that this is the *starting* sma of the planet.

As such, we need $r_{\rm M}$, the Laplace radius, to cross $2a_{\rm peri,min}$, where:

$$\frac{r_{\rm M,[0,f]}}{R_\star} = \left(\frac{2k_{2,\star}}{3}\frac{m_\star+m_2}{m_2}\right)^{1/5}\left(\frac{\omega_{\star,[0,f]}}{n_{\rm out}}\right)^{2/5}, \tag{30}$$

[1] It can circularize wider, if the eccentricity is so extreme that the eccentricity tide circularize the orbit before $a_{\rm peri,min}$ is reached, and it may not circularize at all, if the pericenter distance is too large to give efficient tidal dissipation.

where the subscripts $0, f$ denote the dependence of the initial/final Laplace radii on the initial/final stellar spin frequency. The hierarchy demanded by our scenario is, to be explicit:

$$\left(\frac{2k_{2,\star}}{3}\frac{m_\star+m_2}{m_2}\right)^{1/5}\left(\frac{\omega_{\star,\mathrm{f}}}{n_{\mathrm{out}}}\right)^{2/5} < \frac{4}{3}\left(\frac{(m_\star+m_2)k_{2,\star}}{2m_2}\right)^{1/3}\left(\frac{R_\star}{a_0}\right)^{2/3}\left(\frac{\omega_{\star,0}}{n_{\mathrm{out}}}\right)^{2/3} < \left(\frac{2k_{2,\star}}{3}\frac{m_\star+m_2}{m_2}\right)^{1/5}\left(\frac{\omega_{\star,0}}{n_{\mathrm{out}}}\right)^{2/5}, \quad (31)$$

$$\left(\frac{\omega_{\star,\mathrm{f}}}{\omega_{\star,0}}\right)^{2/5} < \frac{4}{3}\left(\frac{3}{2}\right)^{1/5}\left(\frac{1}{2}\right)^{1/3}\left(\frac{R_\star}{a_0}\right)^{2/3}\left(\frac{k_{2,\star}(m_\star+m_2)}{m_2}\right)^{2/15}\left(\frac{\omega_{\star,0}}{n_{\mathrm{out}}}\right)^{4/15} < 1. \quad (32)$$

The key quantity to evaluate then is the middle quantity:

$$\frac{2a_{\mathrm{peri,min}}}{r_{\mathrm{M},0}} \approx 0.5\left(\frac{R_\star}{R_\odot}\right)^{2/3}\left(\frac{a_0}{\mathrm{au}}\right)^{-2/3}\left(\frac{\omega_{\star,0}}{2\pi/(3\,\mathrm{day})}\right)^{4/15}\left(\frac{(m_\star+m_2)/m_2}{2}\right)^{2/15}\left(\frac{a_{\mathrm{out}}}{100\,\mathrm{au}}\right)^{2/5}. \quad (33)$$

We've also taken $m_\star = m_2 = M_\odot$ and $k_{2,\star} = 0.04$, typical for Sun-like stars.

Recalling that spindown can take us from 3 day to 30 day spin periods, roughly, the quantity on the left hand side of Eq. (32) evaluates to

$$\left(\frac{\omega_{\star,\mathrm{f}}}{\omega_{\star,0}}\right)^{2/5} \simeq 0.4. \quad (34)$$

So for the fiducial parameters above, the required hierarchy of scales works: $r_{\mathrm{M}}$ will indeed sweep through the location of the planet's present-day semi-major axis ($\simeq 2a_{\mathrm{peri,min}}$).

Of course, this is all conditional on the requirement that $a_{\mathrm{peri,min}}$ be close-in enough to induce merger. Using the standard planetary equilibrium tide, (e.g. Leconte+2010) and taking $e \to 1$, we obtain

$$\frac{\mathrm{d}\ln a}{\mathrm{d}t} \sim \frac{3k_{2\mathrm{p}}}{Q}\frac{M_\star}{M_{\mathrm{p}}}\left(\frac{R_{\mathrm{p}}}{a}\right)^5 n\frac{231}{480(1-e)^6}, \quad (35)$$

$$\sim \frac{1}{8\,\mathrm{Gyr}}\left(\frac{a_{\mathrm{peri}}}{0.06\,\mathrm{au}}\right)^{-6}\left(\frac{a}{\mathrm{au}}\right)^{-1/2}\left(\frac{M_\star/M_{\mathrm{p}}}{M_\odot/M_{\mathrm{Nep}}}\right)\left(\frac{k_{2\mathrm{p}}/Q}{0.3/10^3}\right)\left(\frac{R_{\mathrm{p}}}{R_{\mathrm{Nep}}}\right)^5. \quad (36)$$

This can be helped somewhat by using that $R_{\mathrm{p}}$ likely larger for young planets, and may also be inflated by tidal heating.