1 Mar 14, 2022—Stellar Structure on the Main Sequence

- Have learned that stars are powered by nuclear burning, but this only occurs in the core. What sets the structure outside of the core? Need to understand the energy transfer processes, including: conduction, convection, radiation. Last class, said conduction is unimportant. Today, we will talk about radiation, and possibly a little bit of convection. A way to rephrase our goal today is to answer: *why do stars shine?* Ira's favorite question.

  First, stars must be mostly opaque: when we stare at the Sun, we cannot see the core. So let's understand what happens to radiation being emitted at the core, since it obviously doesn't make it to the surface. Understanding this will let us understand how energy gets transferred via radiation from the core to the surface.

- How does a photon interact with matter? Two ways: scattering, absorption.

  Scattering, Thomson scattering. If photon "hits" electron, then will change direction. But electrons are tiny, roughly point particles; how does it hit? EM scattering,

$$\sigma_{\mathrm{T}} \sim 6.65 \times 10^{-25}\,\mathrm{cm}^2. \tag{1}$$

  Absorption/emission [of photons]: free-free emission/absorption, $e + p \to e + p + \gamma$ or vice versa. Bound-free/bound-bound, change energy levels of atom.

  Often, define an *opacity*, which is just the cross-section/gram. For instance, for Thomson:

$$\begin{aligned}\kappa_{\mathrm{T}} &= \frac{n_{\mathrm{e}}\sigma_{\mathrm{T}}}{\rho}, &(2)\\ &= \frac{\sigma_{\mathrm{T}}}{m_{\mathrm{p}}}\frac{Z}{A}, &(3)\\ &= 0.4\,\mathrm{cm}^2/\mathrm{g}\frac{Z}{A}. &(4)\end{aligned}$$

  Note: independent of density and temperature! And for those curious, free-free and bound-free *do* depend on density and temperature, $\kappa_{\mathrm{ff}} \propto \rho T^{-7/2}$. Thus, for higher temperatures, Thomson scattering is what sets opacity, and for lower temperatures, absorption/emission!

- Okay, so the picture is that photons get jostled as they radiate outwards from the core; how often do they interact? *Mean free path*; we should expect $\ell \ll R_\odot$, else we would see the photons from the core (and the star would not be opaque).

  In general, think of a circle moving through a medium with number density n; how far to move before cylinder contains ~ 1 particle. $\ell\sigma_{\mathrm{T}} = 1/n$, or $\ell = 1/(\rho\kappa)$. Near the solar core, $\ell \sim 0.017\,\mathrm{cm}(Z/A)^2$, dominated by scattering. In the solar atmosphere, $\ell \sim 0.5\,\mathrm{cm}$ (dominated by H^- bound-free, $\kappa_{\mathrm{H}-} \propto \rho^{1/2}T^9$, it turns out, for $T \in [4000, 8000]\,\mathrm{K}$ if H- exists).

- Now, we know that photons travel on average 0.5 cm before encountering something in the solar atmosphere. Can we now show that the Sun is opaque? Photons obey a *random walk* (evolve *diffusively*). Lots of examples and analogies, *drunkard's walk*.

  To get some intuition, let's consider the proporties of this behavior: you flip a coin, and H/T → L/R step by ℓ. Then:

  - What is the average displacement? 0

- However, if you take a million steps, and end up exactly at 0, you would be very surprised, right?
- Calculate mean and *variance* (related to stdev).

$$\left\langle \sum x_i \right\rangle = 0, \tag{5}$$

$$\left\langle \left(\sum x_i\right)^2 \right\rangle = N\ell^2. \tag{6}$$

Thus, central limit theorem, mean 0 but stdev $\ell\sqrt{N}$.

In the Sun's case, we need $\ell\sqrt{N} = R$, setting some scale on the number of times a photon will bounce. Then, the time for a photon to exist the Sun is:

$$t_{\text{diff}} = N\frac{\ell}{c} = \frac{R^2}{\ell c} \sim 10^4\,\text{yr}. \tag{7}$$

This implies that if the Sun stopped fusion right now, it would take 10^4 yr for there to be substantially fewer photons reaching the surface, and the Sun would take $\sim 10^4$ yr to collapse!

Let's also perform a quick estimate: if the core is cooling over timescales of 10^4 yr, then what is the luminosity?

$$L \sim \frac{\left(\sigma T_{\text{c}}^4\right)\left(4\pi R_{\text{c}}^3/3\right)}{R^2/\ell c} \sim 2\times 10^{33}\,\text{erg/s}. \tag{8}$$

In reality, $L_\odot \sim 4\times 10^{33}$ erg/s. This must be balanced by $\dot{E}_{\text{nuc}} = L$.

Interestingly, how do we know that the Sun hasn't already stopped fusion? *Neutrinos* are emitted at the center of the Sun, and the Sun is transparent to these ($a \sim 10^{-44}$ cm^2, or $\ell \sim 10^{20}$ cm), and we can measure them.

- Now, we see that the structure of the stellar atmosphere is set by $\dot{E}_{\text{nuc}}$, but $\dot{E}_{\text{nuc}}$ is set by T_{c} which is set by the stellar structure. Thus, these two are both linked, so we can solve for them to understand the general relation:

$$L \sim \frac{aT_{\text{c}}^4(4\pi R_{\text{c}}^3/3)}{R^2/\ell c} \propto \frac{T_{\text{c}}^4 R_{\text{c}}^3}{R^2\rho\kappa}. \tag{9}$$

Recall though that $kT_{\text{c}} \sim \frac{GMm_{\text{p}}}{R}$ is fixed and set by nuclear burning, so $M \propto R$ and $L \propto M^3/\kappa$. For $\kappa \propto \rho^0$ (Thomson), we obtain $L \propto M^3$. For $\kappa \propto \rho \sim M^{-2}$ (free-free), we obtain $L \propto M^5$. In practice, $L \propto M^{3.5}$ is often found, but we'll use $L \propto M^4$.

We can also calculate the scaling of stellar lifetimes,

$$t_{\text{lifetime}} \sim \frac{0.007 M_{\text{c}} c^2}{L}, \tag{10}$$

$$\propto \frac{1}{M^3}. \tag{11}$$

$$t_\odot = \frac{0.007(0.1M_\odot)c^2}{L_\odot} = 10\,\text{Gyr}. \tag{12}$$

- Thus, we can answer how stars shine: black body radiation at the surface, due to [slow] radiative transfer out of the nuclear-burning core.
- Convection? TODO

- Today, we've covered: photon-matter interactions, how this leads to energy transfer out of the core (random walk), and how this leads to the stellar structure.

  In summary, stellar structure is set by:

  - Hydrostatic equilibrium: force on each each element must vanish, balancing gravity and pressure: $\mathrm{d}P/\mathrm{d}r = -Gm(r)/r^2 \times \rho(r)$.

    In particular, the central pressure $P_\mathrm{c} \sim GM^2/R^4$. Since, for an ideal gas $P_\mathrm{gas} = nkT$, and $P_\mathrm{rad} = aT^4/3$. For stars $\sim M_\odot$, they are pressure-dominated and $kT_\mathrm{c} \sim GMm_\mathrm{p}/R$
  - Energy transfer out from the core: diffusively $t_\mathrm{diff} \sim R^2/\ell c$, and possibly convectively if too large.
  - Energy source is nuclear burning.

  Combining all of these, we get the LMR relation!

  More precisely, we have P, M, L, T dynamical variables, with κ, ρ, Z prescribed variables, and they are all coupled to determine the structure of a star!

2 Mar 16, 2022—Beyond the Main Sequence

Late stellar evolution

- At some point, cores become mostly Helium, what next? Off the *main sequence*. This is necessary: after Big Bang, roughly 75% hydrogen, 25% helium, trace amounts rest. We need heavier elements! Must come from stars, but how?

- As cores run out of H, they lose pressure support, contract, releasing energy and T_c increases. Causes *hydrogen shell burning*.

  Unlike core burning, shell burning is *unstable*. In core burning, any increase in burning rate decreases pressure which cools the core. In shell burning, burning increases core mass which increases pressure which increases burning rate! Red giants, expand by $\sim 100\times$. This terminates when $T_\mathrm{c} \sim 10^8$ K, at which point Helium fusion begins.

- **The Triple Alpha Reaction: 1953, Salpeter & Hoyle**However, how does He fusion work? He-4 is so stable that any Be-8 formed rapidly decays back into He-4! Big puzzle.

  Resolution: at equilibrium, some small amount of Be-8 with short lifetime. However, C-12 is very stable; would be nice if Be-8 + He-4 into C-12, but in typical stellar conditions, such collisions are far too rare (typical atomic cross sections).

  Hoyle suggests: maybe can pump up cross section via some sort of *resonance*. To understand this, recall that photons can be absorbed efficiently when $E_\gamma + E_1 = E_2$, for ground state 1 and excited state 2. Similarly for atoms. However, evaluating the energies for ground-state He-4, Be-8, C-12, we find no such resonance. Hoyle says that there must exist an excited C-12*!

  In particular, He-4 + Be-8 − C-12 = 7.4 MeV, and kinetic energy is about 0.2 MeV, so Hoyle claimed there must be an excited state with about 7.6 MeV above ground. Later experimentally verified.

- Once C-12 is made, O-16 is easy to make, gives rise to the *cosmic abundances* of H, He, C, O, Ne-20,... all the way through Fe-56. Thus, continued nuclear burning, continued core contraction and shell burning leads to envelope exxpansion and mass ejection. How far a star goes depends on its mass, but once it stops, the star must collapse. Always happens, finite fuel.

- What are their end states? They are:
  - WD: $M \sim 0.7M_\odot$, $R \sim 10^9$ cm $\sim R_\odot/100 \sim R_\oplus \sim 6\times10^8$ cm. For $M_{\rm MS} \lesssim 8M_\odot$ (e.g. the Sun).
  - NS: $M \sim 1.4M_\odot$, $R \sim 10$ km $\sim 10^6$ cm for $M_{\rm MS} \lesssim 40M_\odot$.
  - BH: $M \gtrsim 3M_\odot$, $R = 2GM/c^2 \sim 3\,{\rm km}\,(M/M_\odot)$, for $M_{\rm MS} \gtrsim 40M_\odot$.

  We will start by understanding these objects, then we will understand how MS stars get here.

WDs:

- Sounds like a crazy object: mass of the Sun in the radius of the Earth (density 10^6 g/cm^3, so the weight of a car in the size of a sugar cube)? However, observed! Sirus B, companion to Sirius A ($2M_\odot$). Opik in 1916 analyzed the spectrum, found surface temperature $T_{\rm s} \sim 6000$ K, but $L \lesssim 0.0003L_\odot$ (distance via paralla probably?). This suggests that its volume must be small, and $\bar\rho \sim 2\times10^5\rho_\odot$! Opik concludes "this impossible result indicates that our assumptions are wrong."

- QM was developed in 1920s, *after* Opik. Uncertainty principle, particles cannot both be stationary and unmoving, thus zero-point pressure even at zero temp. WD is supported by this zero-point pressure, *degeneracy pressure*.

- Recall hydrostatic equilibrium $P/R \sim g\rho$, so $P_{\rm c} \sim GM^2/R^4 \sim GM\rho/R$. If $P_{\rm c}$ is provided thermally (i.e. $P = nkT$), then

$$kT_{\rm c} \sim \frac{GMm_{\rm p}}{R} \tag{13}$$

  For $R = R_\odot$, $T_{\rm c} \sim 10^7$ K, but for $R \sim R_\odot/100$, $T_{\rm c} \sim 10^9$ K. This is unlikely: steep temperature gradient to the surface, and would result in nuclear burning (but unstable! And would be explosive due to large ρ).

- Thus, need something else. Let's ignore temperatuer for now, then uncertainty

$$\Delta p \sim \hbar/\Delta x, \tag{14}$$

$$\sim \hbar n_{\rm e}^{1/3} \sim \hbar\left(\frac{ZM}{R^3 m_{\rm p}}\right)^{1/3}, \tag{15}$$

  so the mean kinetic energy per electron is

$$\epsilon_{\rm k} \equiv \frac{p_e^2}{2m_e} \sim \frac{\hbar^2}{m_{\rm e}} n_{\rm e}^{2/3}. \tag{16}$$

  The energy density is then $u = n_e\epsilon_{\rm k}$, which is also similar to the pressure $P \sim 2u/3 \sim u$. Thus, we finally find that the pressure should be equated as

$$\frac{GM^2}{R^4} \sim \frac{\hbar^2}{m_{\rm e}}\left(\frac{ZM}{R^3 A m_{\rm p}}\right)^{5/3}, \tag{17}$$

$$R \sim \frac{\hbar^2}{Gm_{\rm e}}\frac{(Z/A)^{5/3}}{m_{\rm p}^{5/3}}\frac{1}{M^{1/3}} \propto M^{-1/3}. \tag{18}$$

  Often denote $Y = Z/A$. A precise calculation gives a prefactor of 4.8, and we find that

$$R \simeq \frac{R_\odot}{74}\left(\frac{M_\odot}{M}\right)^{1/3}\left(\frac{Z/A}{0.5}\right)^{5/3}. \tag{19}$$

  Note that:

  - Smaller R gives smaller L
  - Larger M gives smaller R, which *increases* density.
  - $v_\mathrm{e} \sim p_\mathrm{e}/m_\mathrm{e} \sim n_\mathrm{e}^{1/3}$. Thus, as density increases, so too does the electron momentum / velocity.

  Of course, $v_\mathrm{e} \lesssim c$ is required, so this sets a maxximum mass. This is the *Chandrasekar mass*, $\sim 1.4 M_\odot$ for $Z/A = 0.5$.

- To get the Chandrasekar mass rigorously, consider *relativistic* electrons.