1 4a

Our objective is to compute the following integral

$$\varepsilon = \int \frac{\Theta(\vec{k}) - \Theta(\vec{k} + \vec{q})}{E(\vec{k} + \vec{q}) - E(\vec{k})}\, \mathrm{d}\vec{k} \tag{1}$$

for Θ the zero temperature Fermi-Dirac distribution and $E(\vec{v}) = \dfrac{\hbar^2 v^2}{2m}$.

First, we separate the integral into two parts. Define $\vec{k}' = \vec{k} + \vec{q}$, then

$$\varepsilon = \int \frac{\Theta(\vec{k})}{E\left(\vec{k} + \vec{q}\right) - E\left(\vec{k}\right)}\, \mathrm{d}\vec{k} - \int \frac{\Theta(\vec{k}')}{E\left(\vec{k}'\right) - E\left(\vec{k}' - \vec{q}\right)}\, \mathrm{d}\vec{k}' \tag{2}$$

where both $\vec{k}, \vec{k}'$ are integrated in k_F-spheres about their respective origins. Thus, we can drop the prime and write

$$\varepsilon = \int \frac{\Theta(\vec{k})}{E\left(\vec{k} + \vec{q}\right) - E\left(\vec{k}\right)}\, \mathrm{d}\vec{k} - \int \frac{\Theta(\vec{k})}{E\left(\vec{k}\right) - E\left(\vec{k} - \vec{q}\right)}\, \mathrm{d}\vec{k} \tag{3}$$

Let's focus on the first integral, since the results for the second integral follow. Recall that $\left|\vec{a}+\vec{b}\right|^2 = a^2 + b^2 + 2\vec{a}\cdot\vec{b}$. This allows us to write

$$\begin{aligned}
\frac{\hbar^2}{2m} I_1 &= \int \frac{1}{2\vec{q}\cdot\vec{k} + q^2}\,\mathrm{d}\vec{k} && (4)\\
&= 2\pi \int_0^{k_F}\int_0^{\pi} \frac{1}{2qk\cos\theta + q^2}\,k^2\sin\theta\mathrm{d}\theta\mathrm{d}k && (5)\\
&= 2\pi \int_0^{k_F}\int_0^{\pi} \frac{(k/2q)\sin\theta}{\cos\theta + q/2k}\,\mathrm{d}\theta\mathrm{d}k && (6)\\
&= 2\pi \int_0^{k_F} \frac{k}{2q}\left[-\log\left|\cos\theta + \frac{q}{2k}\right|\right]_0^{\pi}\mathrm{d}\theta\mathrm{d}k && (7)\\
&= 2\pi \int_0^{k_F} \frac{k}{2q}\log\left|\frac{1+\frac{q}{2k}}{-1+\frac{q}{2k}}\right|\mathrm{d}\theta\mathrm{d}k && (8)\\
&= 2\pi \int_0^{k_F} \frac{k}{2q}\log\left|\frac{q/2+k}{q/2-k}\right|\mathrm{d}\theta\mathrm{d}k && (9)
\end{aligned}$$

Now, we can look up the following integral (derived later):

$$\int k\log\frac{C+k}{C-k}\,\mathrm{d}k = \frac{1}{2}\left(\left(k^2 - C^2\right)\log\frac{C+k}{C-k}\right) + kC \tag{10}$$

For us, $C = q/2$, and so we instantly obtain

$$I_1 = \frac{\pi m}{\hbar^2 q}\left(\frac{\left(k_F^2 - \frac{q^2}{4}\right)}{2}\log\left|\frac{q/2+k_F}{q/2-k_F}\right| + \frac{k_F q}{2}\right) \tag{11}$$

where we leverage the fact that $k = 0$ forces both the kC and the logarithm to vanish in (10).

For I_2, we note that in (4), the q^2 term changes sign. Propagating this all the way through, we see that in (9), the $q/2$ changes sign. Pulling out a negative sign from the top and bottom, we find that I_2 has the reciprocal of the log found in I_1 (i.e. if we were to pursue the same algebra as above, in (9) we would obtain $\frac{q/2-k}{q/2+k}$ as the argument of the logarithm). Thus, $I_2 = -I_1$[1] , and so we find that

$$\varepsilon = \frac{2\pi m}{\hbar^2 q}\left(\frac{\left(k_F^2 - \frac{q^2}{4}\right)}{2}\log\left|\frac{q/2+k_F}{q/2-k_F}\right| + \frac{k_F q}{2}\right) \tag{12}$$

$$= \frac{2\pi m k_F}{\hbar^2}\left(\frac{\left(k_F^2 - \frac{q^2}{4}\right)}{2k_F q}\log\left|\frac{q/2+k_F}{q/2-k_F}\right| + \frac{1}{2}\right) \tag{13}$$

$$= \frac{2\pi m k_F}{\hbar^2}\left(\frac{(4k_F^2 - q^2)}{8k_F q}\log\left|\frac{q+2k_F}{q-2k_F}\right| + \frac{1}{2}\right) \tag{14}$$

which agrees with the desired answer up to a sign in the absolute value.

Lemma: We argue for the integral formula used above. We simply compute for one sign

$$\begin{aligned}\int k\log(C+k)\,\mathrm{d}k &= k\left[(C+k)\log(C+k) - k\right] \\ &\qquad - \int (C+k)\log(C+k) - k\,\mathrm{d}k \qquad (15)\\ &= k(C+k)\log(C+k) - k^2 - C\left[(C+k)\log(C+k) - k\right] \\ &\qquad + \frac{k^2}{2} - \int k\log(C+k)\,\mathrm{d}k \qquad (16)\\ 2\int k\log(C+k)\,\mathrm{d}k &= (k-C)(C+k)\log(C+k) - \frac{k^2}{2} - Ck \qquad (17)\\ \int k\log(C+k)\,\mathrm{d}k &= \frac{1}{2}\left[(k^2 - C^2)\log(C+k) - \frac{k^2}{2} - Ck\right] \qquad (18)\end{aligned}$$

The second sign is obtained by flipping the sign of k, and the desired integral is the

[1] This should have been obvious in hindsight. Since we are integrating a spherical region of $\vec{k}$ about the origin, for every $\vec{k}$ we also include the contribution of $-\vec{k}$, and so the integrand of I_2 is equivalently $\left(E(\vec{k}) - E(\vec{k}+\vec{q})\right)^{-1}$ which even more obviously yields $-I_1$.

difference between the two. The log terms in (18) then combine, the $\frac{k^2}{2}$ term cancels since it's the same in both signs, and Ck doubles since it incurs a sign flip. This reproduces (10).