Repeated Games

Consider the game of *Iterated Prisoner's Dilemma*, in which players repeatedly engage in the Prisoner's Dilemma (the '*constituent game*').

Suppose the prisoners encounter for *T* periods.

What are

- **Nash equilibria?**
- **Subgame perfect equilibria?**

(C, C) $(3, 3)$

(D, C) $(5, 0)$

(D, C) $(5, 0)$

$\vdots$

(D, D) $(1, 1)$

Iterated Prisoner's Dilemma

Suppose the prisoners encounter for T periods.

- Are Nash equilibria 'good?'
- Are Subgame perfect equilibria 'good?'
- What should be the best for both?

(C, C) $(3, 3)$

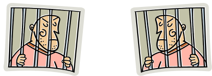
(D, C) $(5, 0)$

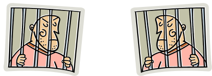
(D, C) $(5, 0)$

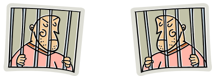
$\vdots$ $\vdots$

(D, D) $(1, 1)$

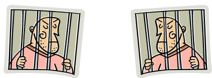
Iterated Prisoner's Dilemma

 (C, C) $(3, 3)$

 (D, C) $(5, 0)$

 (D, C) $(5, 0)$

$\vdots$ $\vdots$

 (D, D) $(1, 1)$

What if the game is infinitely repeated?

Repeated Games

DEFINITION. Let $G = \langle N, (A_i), (\succsim_i) \rangle$ be a strategic game; let $A = \times_{i \in N} A_i$. An **infinitely repeated game** of G is an extensive game with perfect information and simultaneous moves $\langle N, H, P, (\succsim_i^*) \rangle$ in which

- $H = \{\emptyset\} \cup (\cup_{t=1}^{\infty} A^t) \cup A^{\infty}$ (where $\emptyset$ is the initial history and A^{∞} is the set of infinite sequences $(a^t)_{t=1}^{\infty}$ of action profiles in G)

Note:

$$H = \{\emptyset\} \cup (\cup_{t=1}^{\infty} A^t) \cup A^{\infty}$$

A history is a sequence of outcomes (action profiles), of the constituent game.

- $\emptyset$ is the initial history
- A^t is the set of histories of length t. A history $(a^t) = (a^1, a^2, \dots, a^t) \in A^t$ is a sequence of outcomes: a^i is the outcome of period i.
- A^{∞} is the set of infinite sequences ('***terminal history***') $(a^t)_{t=1}^{\infty}$ of action profiles.

- $P(h) = N$ for each nonterminal history $h \in H$.

- $\succsim_i^*$ is a preference relation on A^∞ that extends the preference relation $\succsim_i$ in the constituent game, in the sense that it satisfies the following condition of *weak separability*: if $(a^1, a^2, a^3, \ldots) \in A^\infty$ is a terminal history, $a \in A$ and $a' \in A$ are action profiles of a constituent game, and $a \succsim_i a'$ then

$$(a^1, \ldots, a^{t-1}, a, a^{t+1}, \ldots) \succsim_i^* (a^1, \ldots, a^{t-1}, a', a^{t+1}, \ldots)$$

  for all values of t.

We further assume a payoff function u_i:

$$(a^t) \succsim_i^* (b^t)$$

iff

$$u_i(a^t) \geq u_i(b^t).$$

Preference Relations

If $a \succsim_i a'$ then

$$(a^1, \ldots, a^{t-1}, a, a^{t+1}, \ldots) \succsim_i^* (a^1, \ldots, a^{t-1}, a', a^{t+1}, \ldots).$$

The preference relation $\succsim_i^*$ is too general!

We consider three forms of preference relations:

- Discounting
- Limit of Means
- Overtaking

Discounting Preference Relations

EXAMPLE. Consider two sequences of payoffs of player i:

$(v_i^t) = (v_i^1, v_i^2, v_i^3 \ldots)$ is evaluated by $\sum_{t=1}^{\infty} \delta^{t-1} v_i^t$

$(w_i^t) = (w_i^1, w_i^2, w_i^3 \ldots)$ is evaluated by $\sum_{t=1}^{\infty} \delta^{t-1} w_i^t$

There is some number $\delta \in (0,1)$ (the *discount factor*) such that the sequence (v_i^t) of real numbers is at least as good as the sequence (w_i^t) if and only if $\sum_{t=1}^{\infty} \delta^{t-1}(v_i^t - w_i^t) \geq 0$.

Discounting Preference Relations

EXAMPLE.

$$(1,1,1,1,\underbrace{0,0,0,0,\ldots}_{\text{all } 0})$$

$$(0,0,0,0,\underbrace{2,2,2,2,\ldots}_{\text{all } 2})$$

If $\delta = \frac{1}{2}$, then they are evaluated by

$$1 + 1 \times \frac{1}{2} + 1 \times \frac{1}{4} + 1 \times \frac{1}{8} + 0 \times \frac{1}{16} + 0 \times \frac{1}{32} + \cdots = 1\,\frac{7}{8}$$

$$0 + 0 \times \frac{1}{2} + 0 \times \frac{1}{4} + 0 \times \frac{1}{8} + 2 \times \frac{1}{16} + 2 \times \frac{1}{32} + \cdots = \frac{1}{4}$$

So the first one is ‘better.’

Limit of Means Preference Relations

EXAMPLE. Consider two sequences of payoffs:

$(v_i^t) = (v_i^1, v_i^2, v_i^3 \dots)$ is evaluated by $\lim\limits_{T\to\infty} \sum_{t=1}^{T} \frac{v_i^t}{T}$

$(w_i^t) = (w_i^1, w_i^2, w_i^3 \dots)$ is evaluated by $\lim\limits_{T\to\infty} \sum_{t=1}^{T} \frac{w_i^t}{T}$

The sequence (v_i^t) of real numbers is preferred to the sequence (w_i^t) iff $\lim\inf \sum_{t=1}^{T} \frac{(v_i^t - w_i^t)}{T} > 0$, (*i.e.*, there exists $\varepsilon > 0$ such that $\sum_{t=1}^{T} \frac{(v_i^t - w_i^t)}{T} > \varepsilon$ for all but a finite number of periods T).

MATHEMATICAL NOTATION

The *infimum* $\inf S$ of a set S of real numbers is a number m such that $m \le x$ for all $x \in S$, but for any positive ε there always exists $x' \in S$ such that $x' < m + \varepsilon$.

EXAMPLES (*http://en.wikipedia.org/wiki/Infimum*)

$$\inf\{x \in \mathbb{R}: 0 < x < 1\} = 0$$

$$\inf\{x \in \mathbb{R}: x^3 > 2\} = 2^{\frac{1}{3}}$$

$$\inf\{(-1)^n + \frac{1}{n}: n = 1,2,3, \dots\} = -1$$

MATHEMATICAL NOTATION

(*http://en.wikipedia.org/wiki/Infimum_limit*)

The *infimum limit* (or *limit inferior*) of a sequence $(a_i)_{i=1}^{\infty}$ of real numbers is

$$\liminf(a_i)_{i=1}^{\infty} = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$$

Note that $\liminf(a_i)_{i=1}^{\infty}$ can be written as $\liminf a_i$.

Q: What is meant by $\liminf a_i > 0$?

A: It means that there exists $\varepsilon > 0$ such that $a_i > \varepsilon$ for all but some (finitely many) i. (We must eventually have $a_i > \varepsilon$ when i is large enough.)

Limit of Means Preference Relations

EXAMPLE. Consider two sequences of payoffs:

$(v_i^t) = (v_i^1, v_i^2, v_i^3 \ldots)$ is evaluated by $\lim_{T\to\infty} \sum_{t=1}^{T} \frac{v_i^t}{T}$

$(w_i^t) = (w_i^1, w_i^2, w_i^3 \ldots)$ is evaluated by $\lim_{T\to\infty} \sum_{t=1}^{T} \frac{w_i^t}{T}$

The sequence (v_i^t) of real numbers is preferred to the sequence (w_i^t) iff $\liminf \sum_{t=1}^{T} \frac{(v_i^t - w_i^t)}{T} > 0$, (*i.e.*, there exists $\varepsilon > 0$ such that $\sum_{t=1}^{T} \frac{(v_i^t - w_i^t)}{T} > \varepsilon$ for all but a finite number of periods T).

EXAMPLE. Consider the following two sequences

$$(\underbrace{0,\ldots,0}_{\text{finite no. of zeros}},\underbrace{2,2,\ldots}_{\text{all 2}})$$

$$(\underbrace{1,1,1,\ldots}_{\text{all 1}})$$

- Which one is more preferred according to discounting preferences?
- Which one is more preferred according to the limit-of-means?

Overtaking Preference Relations

EXAMPLE. Consider two sequences of payoffs:

$(v_i^t) = (v_i^1, v_i^2, v_i^3 \ldots)$ is evaluated by $\lim_{T\to\infty} \sum_{t=1}^{T} v_i^t$

$(w_i^t) = (w_i^1, w_i^2, w_i^3 \ldots)$ is evaluated by $\lim_{T\to\infty} \sum_{t=1}^{T} w_i^t$

> The sequence (v_i^t) is preferred to the sequence (w_i^t) if and only if $\liminf \sum_{t=1}^{T}(v_i^t - w_i^t) > 0$, *i.e.*, there exists $\varepsilon > 0$ such that $\sum_{t=1}^{T}(v_i^t - w_i^t) > \varepsilon$ for all but a finite number of periods T.

EXAMPLES.

- By the discounting criterion, which of $(1, -1, \underbrace{0, 0, \ldots}_{\text{all } 0})$ and $(\underbrace{0, 0, \ldots}_{\text{all } 0})$ is more preferred with $\delta = \_$?
- By the limit of means criterion, which of $(1, -1, \underbrace{0, 0, \ldots}_{\text{all } 0})$ and $(\underbrace{0, 0, \ldots}_{\text{all } 0})$ is more preferred?
- By the overtaking criterion, which of $(1, -1, \underbrace{0, 0, \ldots}_{\text{all } 0})$ and $(\underbrace{0, 0, \ldots}_{\text{all } 0})$ is more preferred?

EXAMPLES.

- By the discounting criterion, which of $(-1,2,\underbrace{0,0,\ldots}_{\text{all } 0})$ and $(\underbrace{0,0,\ldots}_{\text{all } 0})$ is more preferred with $\delta = \_?$
- By the limit of means criterion, which of $(-1,2,\underbrace{0,0,\ldots}_{\text{all } 0})$ and $(\underbrace{0,0,\ldots}_{\text{all } 0})$ is more preferred?
- By the overtaking criterion, which of $(-1,2,\underbrace{0,0,\ldots}_{\text{all } 0})$ and $(\underbrace{0,0,\ldots}_{\text{all } 0})$ is more preferred?

EXAMPLE.

- By the discounting criterion, which of $(\underbrace{0,\dots,0}_{M\text{ zeros}},\underbrace{1,1,\dots}_{\text{all } 1})$ and $(1,\underbrace{0,0,\dots}_{\text{all } 0})$ is more preferred with $\delta = \_$?
- By the limit-of-means criterion, which of $(\underbrace{0,\dots,0}_{M\text{ zeros}},\underbrace{1,1,\dots}_{\text{all } 1})$ and $(1,\underbrace{0,0,\dots}_{\text{all } 0})$ is more preferred?
- By the overtaking criterion, which of $(\underbrace{0,\dots,0}_{M\text{ zeros}},\underbrace{1,1,\dots}_{\text{all } 1})$ and $(1,\underbrace{0,0,\dots}_{\text{all } 0})$ is more preferred?

δ-Discounted Infinitely Repeated Games

Given a strategic game $G = \langle N, (A_i), (u_i) \rangle$, the **$\delta$-Discounted Infinitely Repeated Game** of G is the infinitely repeated game for which the constituent game is G and the preference ordering $\succsim_i^*$ of each player $i \in N$ is derived from u_i using the discounting criterion with a discount factor of δ for each player.

Limit of Means Infinitely Repeated Games

Given a strategic game $G = \langle N, (A_i), (u_i) \rangle$, the **Limit of Means Infinitely Repeated Game** of G is the infinitely repeated game for which the constituent game is G and the preference ordering $\succsim_i^*$ of each player $i \in N$ is derived from u_i using the limit of means criterion for each player.

Overtaking Infinitely Repeated Games

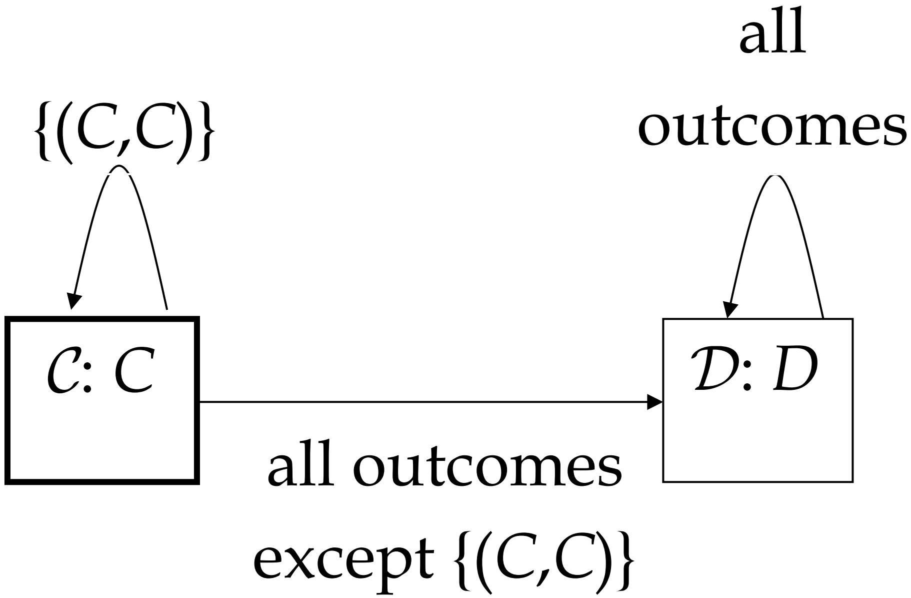
Given a strategic game $G = \langle N, (A_i), (u_i) \rangle$, the **Overtaking Infinitely Repeated Game** of G is the infinitely repeated game for which the constituent game is G and the preference ordering $\succsim_i^*$ of each player $i \in N$ is derived from u_i using the overtaking criterion for each player.

Strategies as Machines

A strategy of player i in an infinitely repeated game is a function that assigns an action in A_i to every finite sequence of outcomes in G.

Strategies as Machines

EXAMPLE. A 'grim' strategy for Iterated Prisoner's Dilemma.

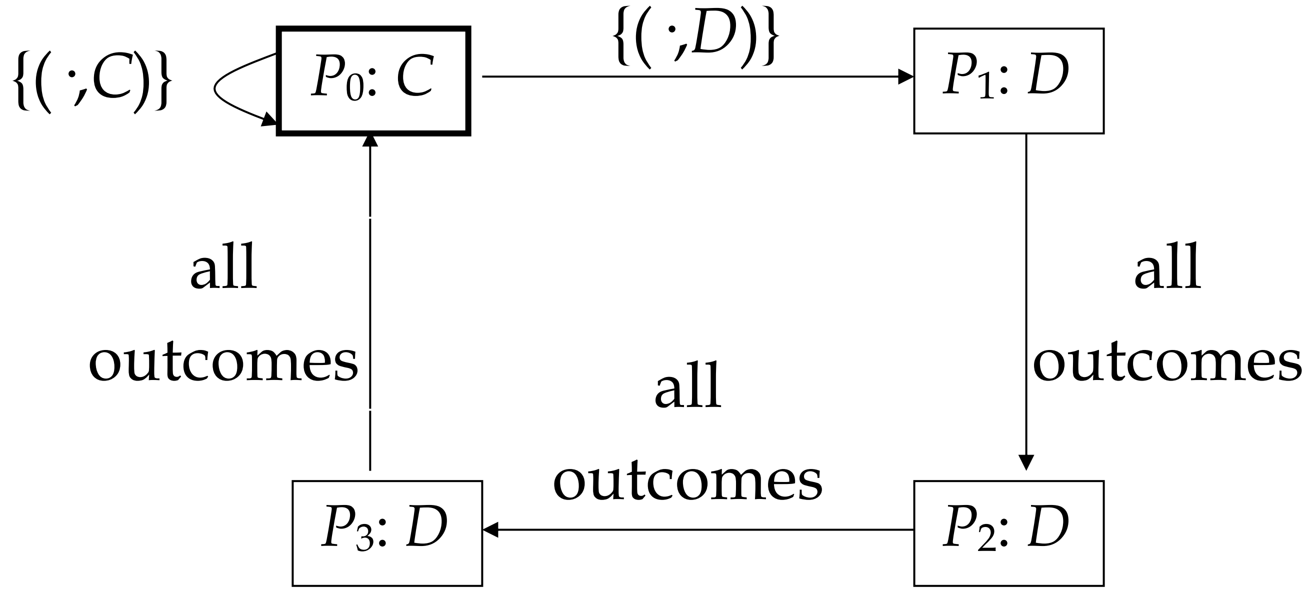


The FA

- $Q_i = \{\mathcal{C}, \mathcal{D}\}$
- $q_i^0 = \mathcal{C}$
- $f_i(\mathcal{C}) = C,\ f_i(\mathcal{D}) = D$
- $\tau_i(\mathcal{C}, (C, C)) = \mathcal{C}$
- $\tau_i(\mathcal{X}, (Y, Z)) = \mathcal{D}$ if $(\mathcal{X}, (Y, Z)) \neq (\mathcal{C}, (C, C))$

Strategies as Machines

EXAMPLE. A 'punishing-forgiving' strategy for Iterated Prisoner's Dilemma.



Strategies in Infinitely Repeated Games as Finite Automata

Not every strategy in an infinitely repeated game can be executed by a machine with a *finite* number of states.

Nash Equilibrium

What are the

Nash equilibria

of Infinitely Repeated Games?

Infinitely Repeated Games

$$\left.\begin{array}{ll} (C, D) & (0, 5) \\ (C, C) & (3, 3) \\ (D, C) & (5, 0) \\ (D, C) & (5, 0) \end{array}\right\} \text{Period 1}$$

$$\left.\begin{array}{ll} (C, D) & (0, 5) \\ (C, C) & (3, 3) \\ (D, C) & (5, 0) \\ (D, C) & (5, 0) \end{array}\right\} \text{Period 2}$$

$$\vdots \qquad\qquad \vdots$$

Infinitely Repeated Games

$$\left.\begin{array}{cc}(C, D) & (0, 5)\\ (C, C) & (3, 3)\\ (D, C) & (5, 0)\\ (D, C) & (5, 0)\end{array}\right\}\text{Period 1}$$

$$\left.\begin{array}{cc}(C, D) & (0, 5)\\ (C, C) & (3, 3)\\ (D, C) & (5, 0)\\ (D, C) & (5, 0)\end{array}\right\}\text{Period 2}$$

$$\begin{array}{cccc}\vdots & \vdots & \vdots & \vdots\end{array}$$

Limit of Means Payoff Profile: $(\frac{13}{4}, 2)$

Infinitely Repeated Games

⋮

?

Period 1

Period 2

Limit of Means Payoff Profile: $(\frac{1}{4}, 4)$

Infinitely Repeated Games

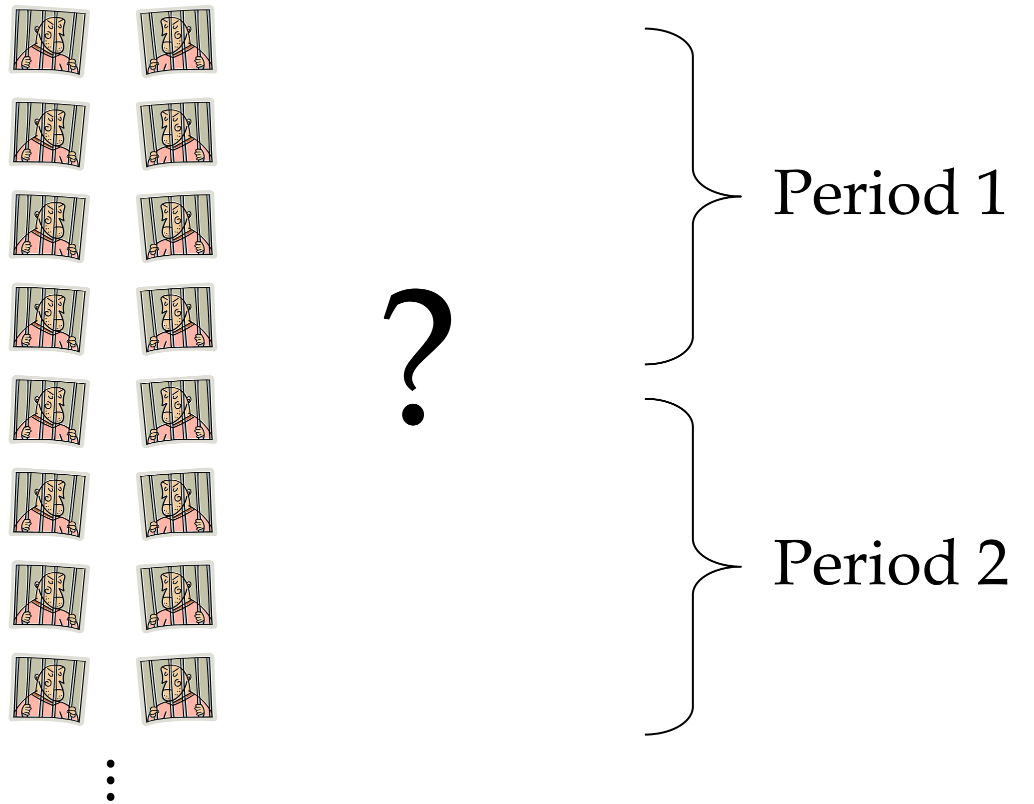
$$\left.\begin{array}{ll}(D, D) & (1, 1)\\ (C, D) & (0, 5)\\ (C, D) & (0, 5)\\ (C, D) & (0, 5)\end{array}\right\}\text{Period 1}$$

$$\left.\begin{array}{ll}(D, D) & (1, 1)\\ (C, D) & (0, 5)\\ (C, D) & (0, 5)\\ (C, D) & (0, 5)\end{array}\right\}\text{Period 2}$$

$$\vdots \qquad \vdots \qquad \vdots \quad \vdots$$

Limit of Means Payoff Profile: $(\frac{1}{4}, 4)$

Infinitely Repeated Games



Limit of Means Payoff Profile: (0,3)

Infinitely Repeated Games

Period 1

Period 2

⋮

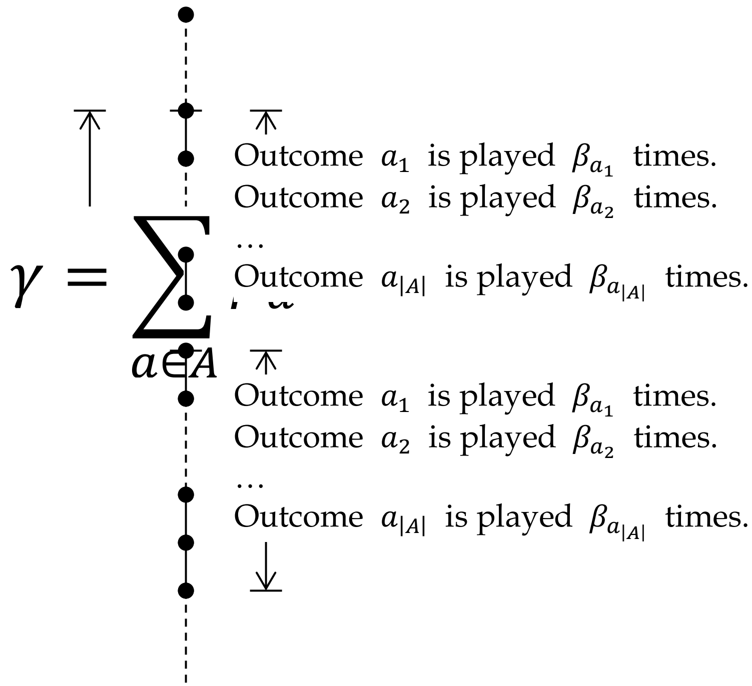
Limit of Means Payoff Profile: (0,3)

Not a feasible Payoff Profile!

Infinitely Repeated Games

Let $G = \langle N, (A_i), (\succsim_i) \rangle$. The set of outcomes is $A = \times_{i \in N} A_i$.

Consider the players play an indefinitely repetition of a cycle of outcomes of length γ, in which each outcome $a \in A$ is played β_a times.

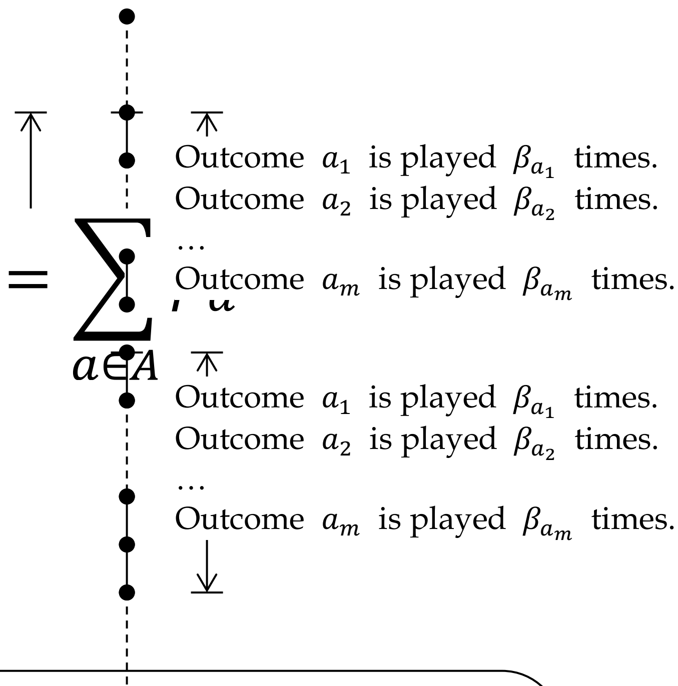


Infinitely Repeated Games

<u>If the limit of means criterion is used</u>, the payoff of player i in each period, **and hence the entire game**, is

$$w_i = \sum_{a \in A} \frac{\beta_a \cdot u_i(a)}{\gamma}.$$

$$\gamma = \sum_{a \in A}$$



Payoff Profile (n users)

$$w = (\underbrace{\sum_{a \in A} \frac{\beta_a \cdot u_1(a)}{\gamma}}_{w_1}, \underbrace{\sum_{a \in A} \frac{\beta_a \cdot u_2(a)}{\gamma}}_{w_2}, \dots, \underbrace{\sum_{a \in A} \frac{\beta_a \cdot u_n(a)}{\gamma}}_{w_n})$$

Feasible Payoff Profiles

$$
\begin{aligned}
w &= (w_1, w_2, \dots, w_n) \\
&= (\underbrace{\textstyle\sum_{a \in A} \frac{\beta_a \cdot u_1(a)}{\gamma}}_{w_1}, \underbrace{\textstyle\sum_{a \in A} \frac{\beta_a \cdot u_2(a)}{\gamma}}_{w_2}, \dots, \underbrace{\textstyle\sum_{a \in A} \frac{\beta_a \cdot u_n(a)}{\gamma}}_{w_n})
\end{aligned}
$$

A vector $w = \sum_{a \in A} \frac{\beta_a}{\gamma} u(a)$, where $\gamma = \sum_{a \in A} \beta_a$ and $\beta_a \geq 0$ for all a, is called a **feasible payoff profile** of $G = \langle N, (A_i), (u_i) \rangle$. Note that a feasible payoff profile is not necessarily a payoff profile of $G = \langle N, (A_i), (u_i) \rangle$.

Feasible Payoff Profiles

(C, D)	$(0, 5)$	(D, D)	$(1, 1)$	(D, D)	$(1, 1)$
(C, C)	$(3, 3)$	(C, C)	$(3, 3)$	(C, D)	$(0, 5)$
(D, C)	$(5, 0)$	(D, D)	$(1, 1)$	(C, D)	$(0, 5)$
(D, C)	$(5, 0)$	(C, D)	$(0, 5)$	(C, D)	$(0, 5)$
(C, D)	$(0, 5)$	(D, D)	$(1, 1)$	(D, D)	$(1, 1)$
(C, C)	$(3, 3)$	(C, C)	$(3, 3)$	(C, D)	$(0, 5)$
(D, C)	$(5, 0)$	(D, D)	$(1, 1)$	(C, D)	$(0, 5)$
(D, C)	$(5, 0)$	(C, D)	$(0, 5)$	(C, D)	$(0, 5)$
$\vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots \quad \vdots$	$\vdots$	$\vdots \quad \vdots$
	$\left(\frac{13}{4}, 2\right)$		$\left(\frac{5}{4}, \frac{10}{4}\right)$		$\left(\frac{1}{4}, 4\right)$

Not a feasible Payoff Profile!

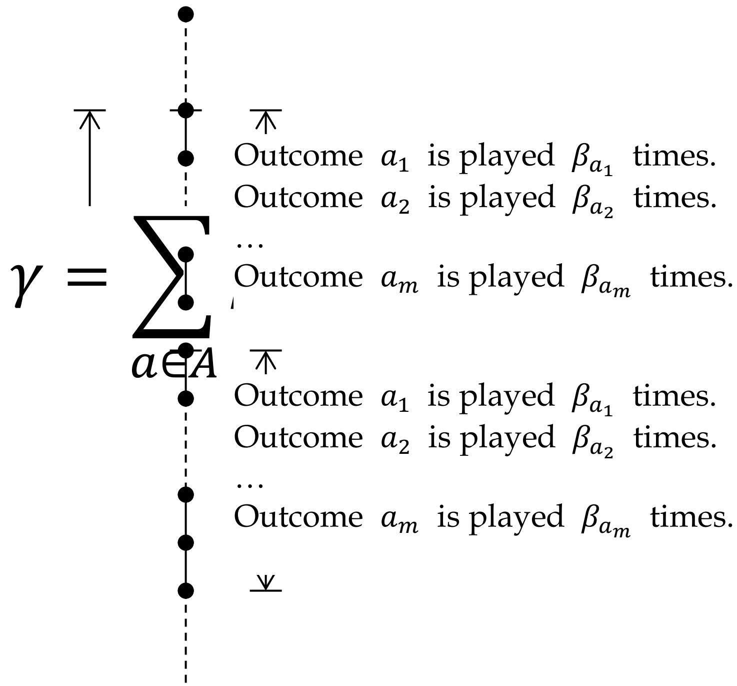
$(0,3)$

Feasible Payoff Profiles

$$w = (w_i)_{i \in N} = \left(\sum_{a \in A} \frac{\beta_a \cdot u_i(a)}{\gamma} \right)_{i \in N} = \sum_{a \in A} \frac{\beta_a \cdot u(a)}{\gamma}$$

But how can we enforce this feasible payoff profile?

The feasible payoff profile is enforceable if no player can increase his payoff by deviating!



Class Discussion.

$$\left.\begin{array}{cc}(D, D) & (1, 1)\\ (C, D) & (0, 5)\\ (C, D) & (0, 5)\\ (C, D) & (0, 5)\end{array}\right\}\text{Period 1}$$

$$\left.\begin{array}{cc}(D, D) & (1, 1)\\ (C, D) & (0, 5)\\ (C, D) & (0, 5)\\ (C, D) & (0, 5)\end{array}\right\}\text{Period 2}$$

$$\vdots \qquad \vdots \qquad \vdots \quad \vdots$$

Is this payoff profile enforceable? $(\frac{1}{4}, 4)$

Class Discussion.

$$\left.\begin{array}{ll} (C, C) & (3, 3) \\ (C, C) & (3, 3) \\ (C, D) & (0, 5) \\ (C, D) & (0, 5) \end{array}\right\} \text{Period 1}$$

$$\left.\begin{array}{ll} (C, C) & (3, 3) \\ (C, C) & (3, 3) \\ (C, D) & (0, 5) \\ (C, D) & (0, 5) \end{array}\right\} \text{Period 2}$$

$$\vdots \qquad \vdots$$

Is this payoff profile enforceable? $(\frac{3}{2}, 4)$

Enforceable Payoffs

In iterated prisoner's dilemma with limit of means preferences, payoff 1 is the worst possible payoff any player can get. Hence,

- the payoff profile $(\frac{1}{4}, 4)$ *is not* **enforceable** because player 1 gets a payoff of less than 1 — player 1 can always deviates and gets 1.
- the payoff profile $(\frac{3}{2}, 4)$ *is* **enforceable** because every player gets a payoff of more than 1. (*Reason follows soon…*)

Enforceability of Payoffs and Outcomes of Constituent Game

Consider the constituent game G.

Player i's **minmax payoff**:

$$v_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_{-i}, a_i)$$

The collection of actions

$$p_{-i} = \arg \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_{-i}, a_i)$$

is the most severe 'punishment' that other players can inflict upon player i in the constituent game G.

Enforceability of Payoffs and Outcomes of Constituent Game

Player i's **minmax payoff**:

$$v_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_{-i}, a_i).$$

- A payoff profile w is **enforceable** if and only if $w_i \geq v_i$ for all $i \in N$; it is **strictly enforceable** if and only if $w_i > v_i$ for all $i \in N$.
- An outcome $a \in A$ is a **(strictly) enforceable outcome** of G, if $u(a)$ is (strictly) enforceable in G.

Class Discussion

EXAMPLE. Consider the Iterated Prisoner's Dilemma with the limit of means criterion.

- What is the player 1's minmax payoff v_1?
- What is the player 2's minmax payoff v_2?
- Is the payoff profile $w = (\frac{1}{4}, 4)$ enforceable? Is it strictly enforceable?
- Is the payoff profile $w = (1,1)$ enforceable? Is it strictly enforceable?

Class Discussion

EXAMPLE. Consider an infinitely repeated game of the game shown on the right.

	A	D
A	2,3	1,5
D	0,1	0,1

- What is a player's minmax payoff v_i?
- What is the corresponding 'most severe punishment' p_{-i}?
- Name some feasible enforceable payoff profiles.
- Do you think player 1 will punish player 2?

Nash Equilibrium of Limit of Means Infinitely Repeated Games

We consider a payoff profile w and $w_i < v_i$, then a player can change his strategy to increase his payoff to at least v_i in any period. Hence w cannot be a Nash equilibrium payoff profile.

Therefore, a Nash equilibrium payoff profile of a limit of means infinitely repeated game must be an enforceable payoff profile.

Nash Equilibrium of Limit of Means Infinitely Repeated Games

$(v_i^t) = (v_i^1, v_i^2, v_i^3 \dots)$ is evaluated by $\lim_{T\to\infty} \sum_{t=1}^{T} \frac{v_i^t}{T}$

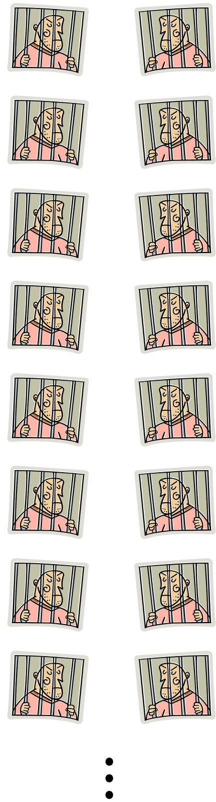
PROPOSITION. Every Nash equilibrium payoff profile of the limit of means infinitely repeated game of $G = \langle N, (A_i), (u_i) \rangle$ is an enforceable payoff profile of G.

Trigger Strategies

Consider the following strategy:

- Always play according to the agreement, <u>unless</u>

- in a previous period, the other player deviated from the agreement. In this case, play D forever.

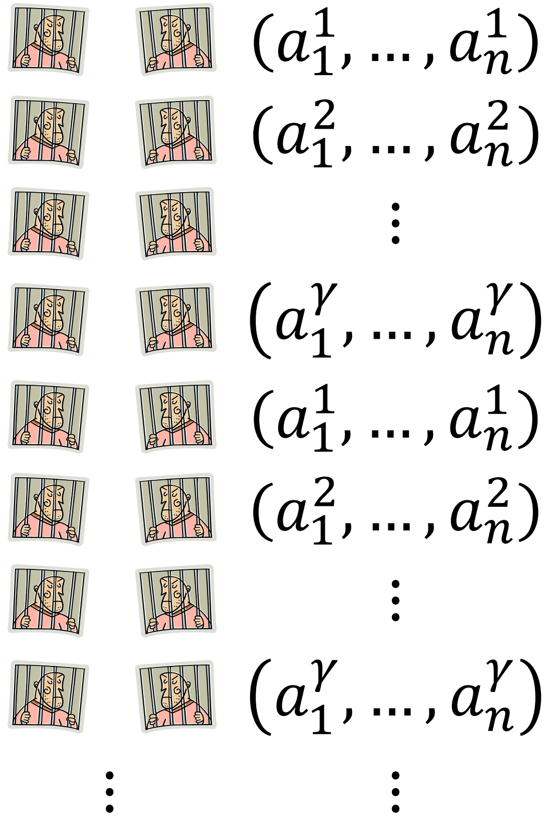


(C, C)	$(3, 3)$
(C, C)	$(3, 3)$
(C, D)	$(0, 5)$
(C, D)	$(0, 5)$
(C, C)	$(3, 3)$
(C, C)	$(3, 3)$
(C, D)	$(0, 5)$
(C, D)	$(0, 5)$
$\vdots$	$\vdots \quad \vdots$
	$(\frac{3}{2}, 4)$

Trigger Strategies

Player i's strategy s_i:

- In round t, player i always chooses a_i^t, unless
- in round t' in a previous period, player $j \neq i$ deviated and did not play $a_j^{t'}$. In this case, player i plays $(p_{-j})_i$, the most severe 'punishment' that can be inflicted upon player j by all players other than j.



Trigger Strategies

The strategy s_i is a trigger strategy.

The 'normal' payoff profile should be an enforceable payoff profile $w = \sum_{a \in A} \frac{\beta_a}{\gamma} u(a)$ (which is feasible). If player j deviates, he receives his minmax payoff $v_j \leq w_j$ in every subsequent period (**Harsh!**).

What action should player j play when he is being punished?

Trigger Strategies

Therefore, the strategy profile s, in which all users are using a trigger strategy, is in Nash equilibrium.

Class Discussion

EXAMPLE. Consider the Iterated Prisoner's Dilemma with the limit of means criterion.

$$v = (\underbrace{1}_{v_1}, \underbrace{1}_{v_2})$$

- What is the corresponding 'most severe punishment' p_{-i}?
- Name some feasible enforceable payoff profiles.
- Name some trigger strategies.

Nash Equilibrium of Limit of Means Infinitely Repeated Games

PROPOSITION. *(Nash folk theorem for the limit of means criterion)* Every feasible enforceable payoff profile of $G = \langle N, (A_i), (u_i) \rangle$ is a Nash equilibrium payoff profile of the limit of means infinitely repeated game of G.

Summary for
Limit of Means Infinitely Repeated Games:

PROPOSITION. Every Nash equilibrium payoff profile of the limit of means infinitely repeated game of $G = \langle N, (A_i), (u_i) \rangle$ is an enforceable payoff profile of G.

PROPOSITION. *(Nash folk theorem for the limit of means criterion)* Every feasible enforceable payoff profile of $G = \langle N, (A_i), (u_i) \rangle$ is a Nash equilibrium payoff profile of the limit of means infinitely repeated game of G.

Do these results hold for δ-Discounted Infinitely Repeated Games?

Nash Equilibrium of δ-Discounted Infinitely Repeated Games

We consider a payoff profile w and $w_i < v_i$, then a player can change his strategy to increase his payoff to at least v_i in any period. Hence w cannot be a Nash equilibrium payoff profile.

Therefore, a Nash equilibrium payoff profile of a δ-discounted infinitely repeated game must be an enforceable payoff profile.

Nash Equilibrium of δ-Discounted Infinitely Repeated Games

$$(v_i^t) = (v_i^1, v_i^2, v_i^3 \ldots) \text{ is evaluated by } \sum_{t=1}^{\infty} \delta^{t-1} v_i^t$$

PROPOSITION. Every Nash equilibrium payoff profile of the <u>δ-discounted</u> infinitely repeated game of $G = \langle N, (A_i), (u_i) \rangle$ is an enforceable payoff profile of G, for any $\delta \in (0,1)$.

Are Trigger Strategies Effective in δ-Discounted Infinitely Repeated Games?

Player i's strategy s_i:

- Player i always chooses a_i^t, unless
- in a previous period t', player $j \neq i$ deviated and did not play $a_j^{t'}$. In this case, player i plays $(p_{-j})_i$ the most severe 'punishment' that can be inflicted upon player j by all players other than j.

Player i's strategy s_i:

- Player i always chooses a_i^t, unless
- in a previous period t', player $j \neq i$ deviated and did not play $a_j^{t'}$. In this case, player i plays $(p_{-j})_i$ the most severe 'punishment' that can be inflicted upon player j by all players other than j.

If δ is too small, then the deviant j cannot be 'punished!'

Nash Equilibrium of δ-Discounted Infinitely Repeated Games

PROPOSITION. *(Nash folk theorem for the discounting criterion)* Let w be a strictly enforceable feasible payoff profile of $G = \langle N, (A_i), (u_i) \rangle$. For all $\varepsilon > 0$ there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ then the δ-discounted infinitely repeated game of G has a Nash equilibrium whose payoff profile w' satisfies $|w' - w| < \varepsilon$.

Summary for δ-Discounted Infinitely Repeated Games:

PROPOSITION. Every Nash equilibrium payoff profile of the δ-discounted infinitely repeated game of $G = \langle N, (A_i), (u_i) \rangle$ is an enforceable payoff profile of G.

PROPOSITION. *(Nash folk theorem for the discounting criterion)* Let w be a strictly enforceable feasible payoff profile of $G = \langle N, (A_i), (u_i) \rangle$. For all $\varepsilon > 0$ there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ then the δ-discounted infinitely repeated game of G has a Nash equilibrium whose payoff profile w' satisfies $|w' - w| < \varepsilon$.