Regarding HW 2 & Selected HW 2 Answers

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14 Feb 2020

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Question 3

(10 %) Find an equation in rectangular coordinates for the following surface, and sketch the corresponding graphs. Label all necessary points (for example intercepts) as well.

(a) $r + 5z = 0$

(b) $\phi = \frac{\pi}{4}$

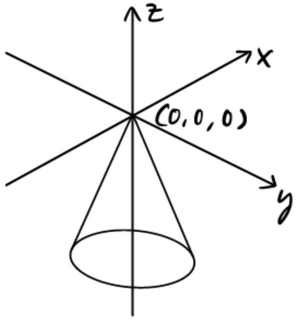
(Students' solution)

(a) $r + 5z = 0$

$\sqrt{x^2+y^2} + 5z = 0$

$\sqrt{x^2+y^2} = -5z$

$z = \dfrac{\sqrt{x^2+y^2}}{-5}$

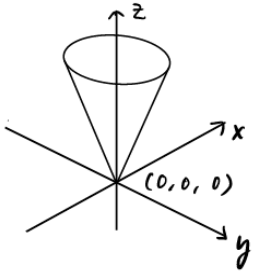


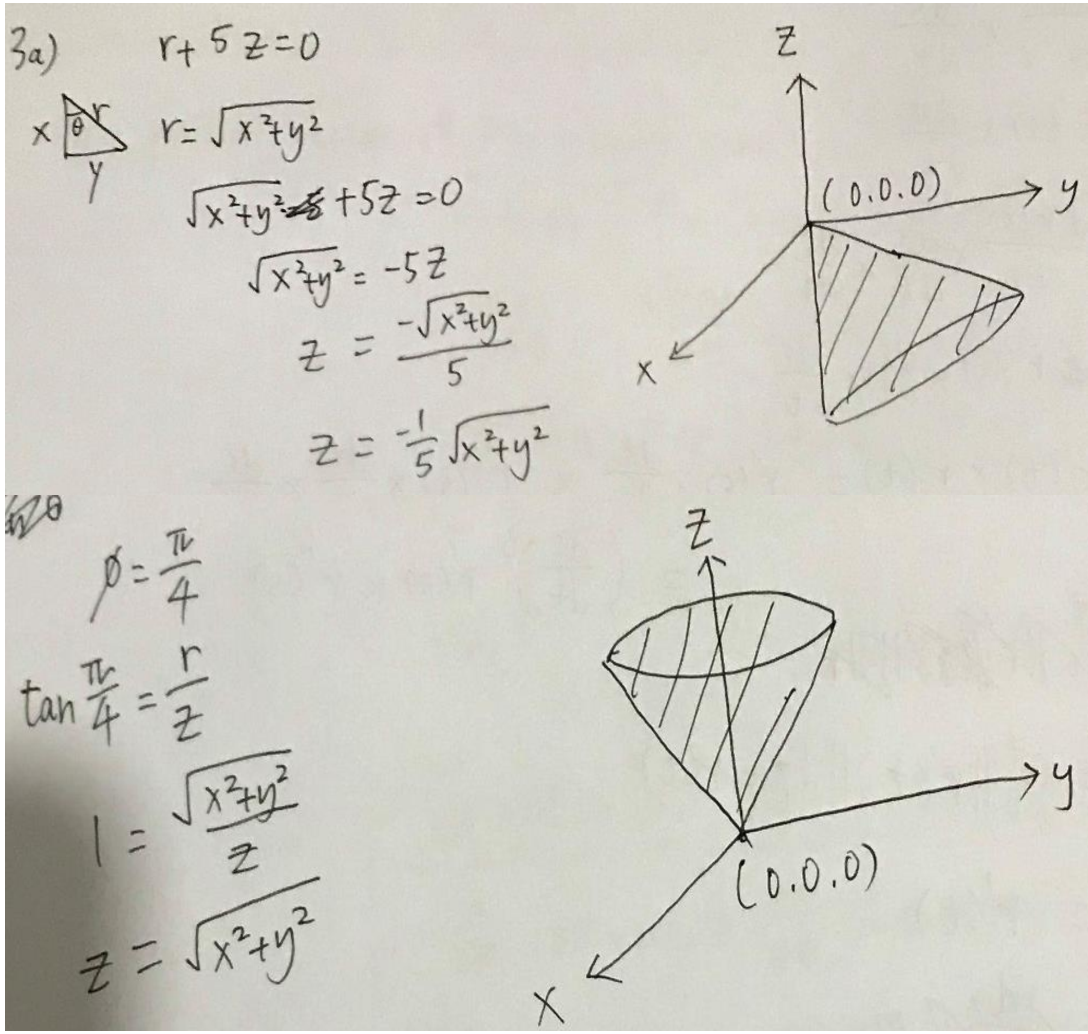
(b) $\phi = \frac{\pi}{4}$

$\tan\phi = \dfrac{r}{z}$

$\tan\frac{\pi}{4} = \dfrac{\sqrt{x^2+y^2}}{z}$

$z = \sqrt{x^2+y^2}$





WHAT A COINCIDENCE!?

Question 3

(10 %) Find an equation in rectangular coordinates for the following surface, and sketch the corresponding graphs. Label all necessary points (for example intercepts) as well.

(a) $r + 5z = 0$ **(Students' solution)**

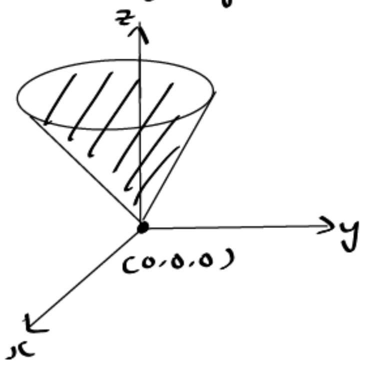
(b) $\phi = \frac{\pi}{4}$

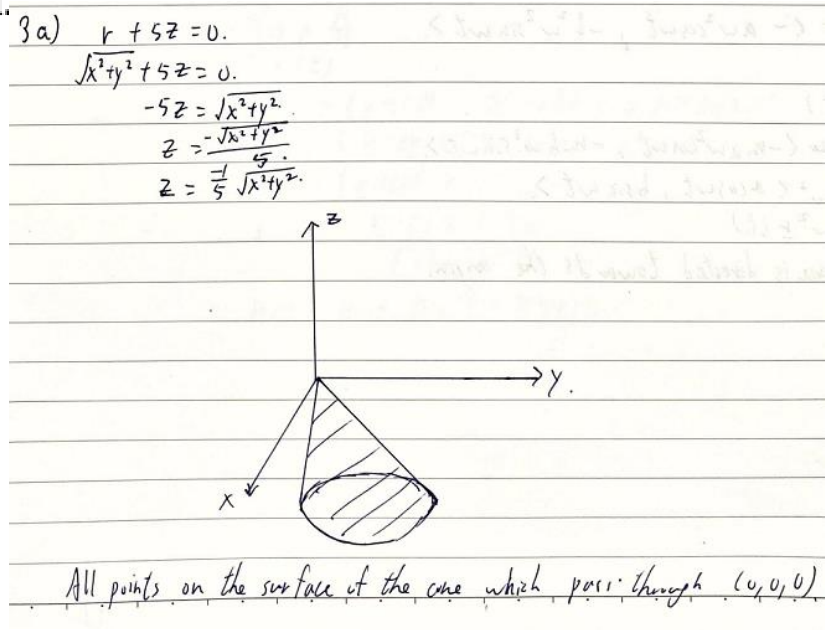
3b $\phi = \frac{\pi}{4}$

$\tan\frac{\pi}{4} = \frac{r}{z}$

$1 = \frac{\sqrt{x^2+y^2}}{z}$

$z = \sqrt{x^2+y^2}$





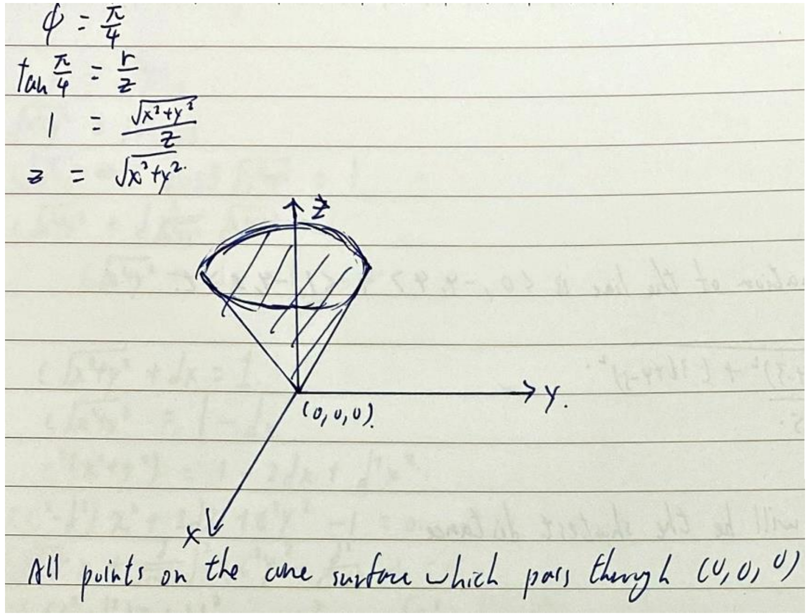
3a $r + 5z = 0$

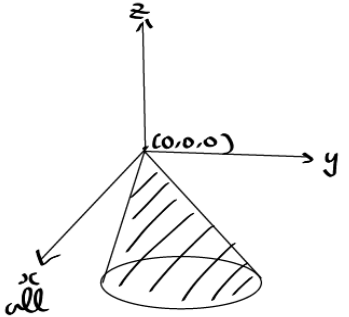
$\sqrt{x^2+y^2} + 5z = 0$

$-5z = \sqrt{x^2+y^2}$

$z = \frac{-\sqrt{x^2+y^2}}{5}$

$z = \frac{-1}{5}\sqrt{x^2+y^2}$





set of all points on a cone surface which pass through (0,0,0)

WHAT A COINCIDENCE!?

Question 3

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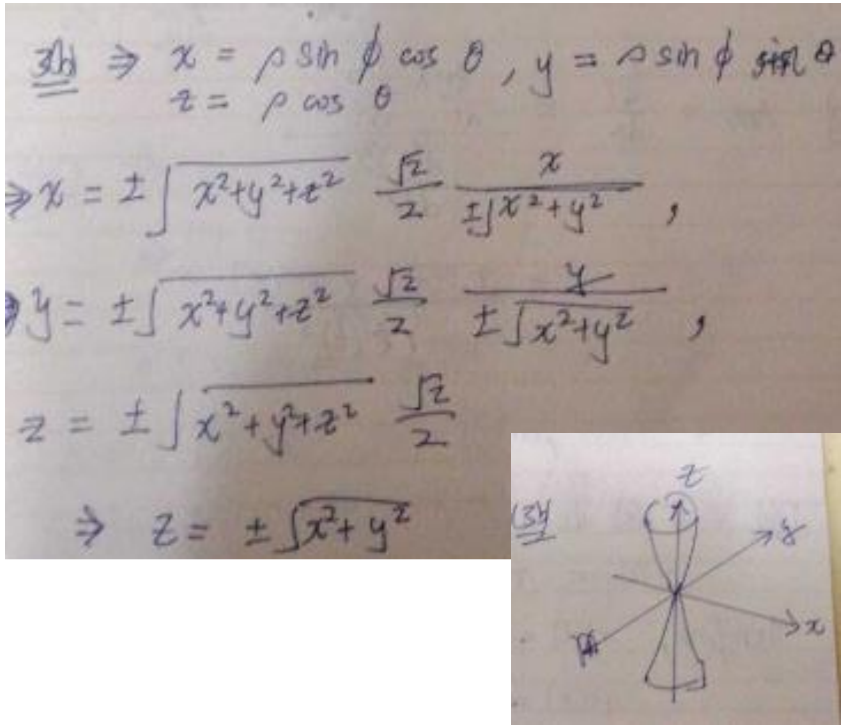
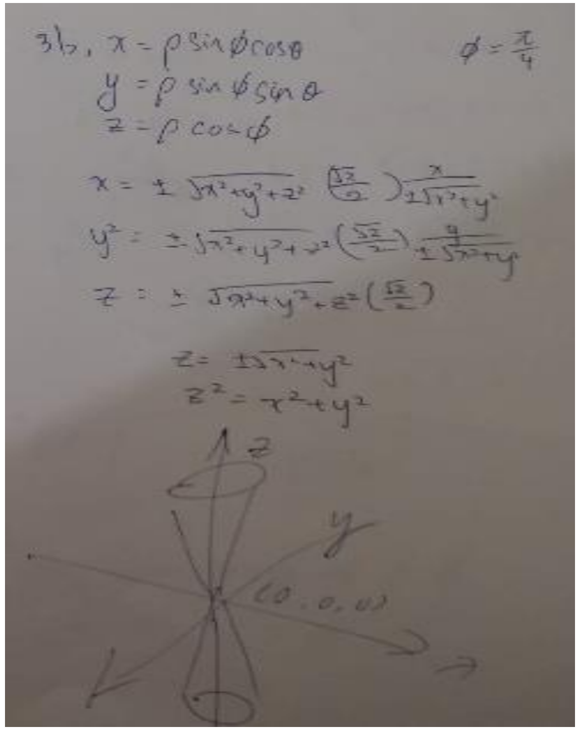
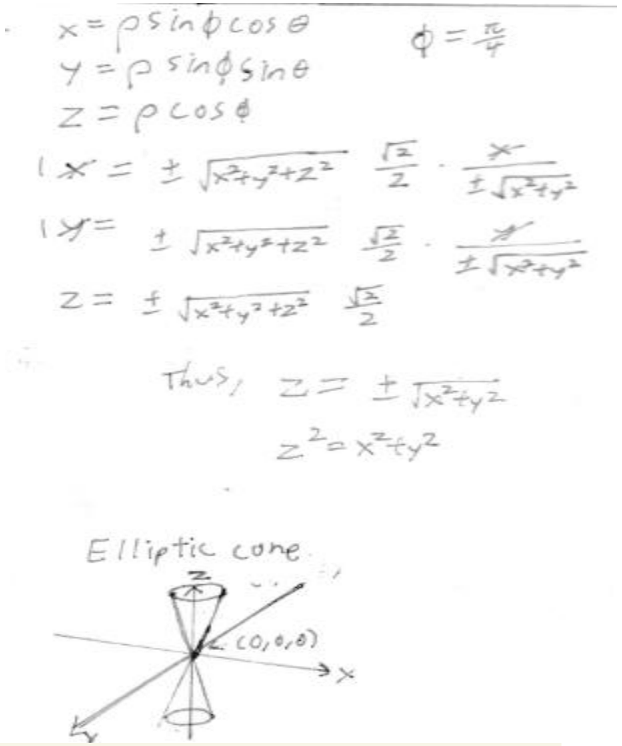
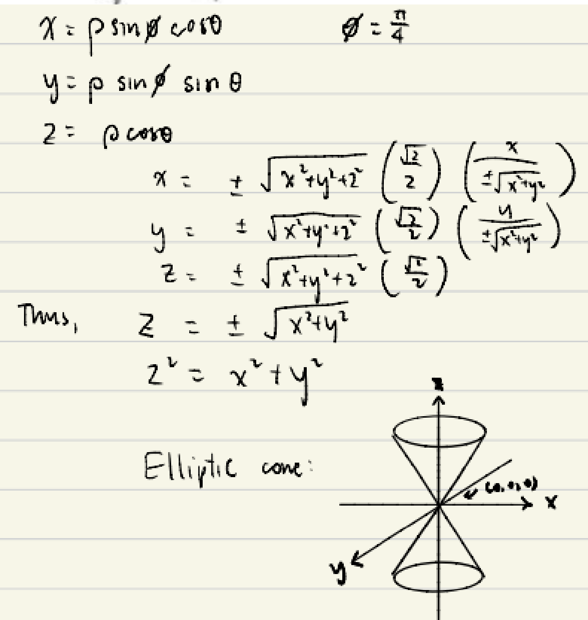
(a) $r + 5z = 0$ **(Students' solution)**

(b) $\phi = \frac{\pi}{4}$









WHAT A COINCIDENCE!?

Question 3

Suggested Approach

(10 %) Find an equation in rectangular coordinates for the following surface, and sketch the corresponding graphs. Label all necessary points (for example intercepts) as well.

(a) $r + 5z = 0$

(b) $\phi = \frac{\pi}{4}$

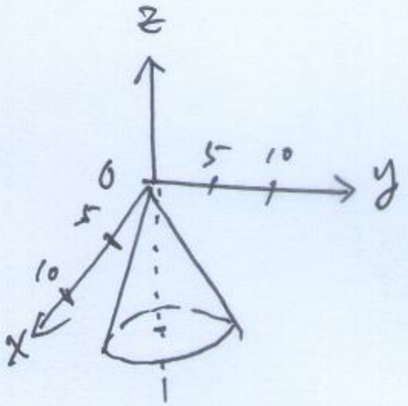
(a) $r + 5z = 0$

Approach 1 Treat r as non-negative

assume $r \geq 0$

$$\sqrt{x^2+y^2} = -5z$$

$$x^2+y^2-25z^2=0$$



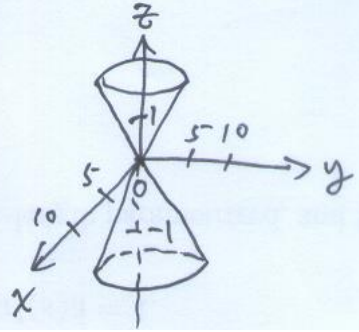
Approach 2 r can be negative (refer to wikipedia definition) – of course r can be ≥ 0 as well

$$r = -5z$$

$$r^2 = 25z^2$$

$$x^2+y^2 = 25z^2$$

$$x^2+y^2-25z^2=0$$



It depends on how you interpret "r" on polar coordinates

When $r < 0$, you move from the pole in the direction opposite to the given positive angle. (dummies.com).

Grading: Both approaches are accepted, and will be awarded full credits

Example

When $z=1$, $r=-5$, we plot the point $(r, \theta, z) = (-5, \theta, 1)$, where $\theta \in [0, 2\pi)$, then we obtain a circle ($\because$ θ can vary from 0 to 2π, revolving a loop) on the horizontal plane $z=1$.

Ref.:

https://math.stackexchange.com/questions/1390581/how-can-r-be-negative-when-dealing-with-polar-coordinates

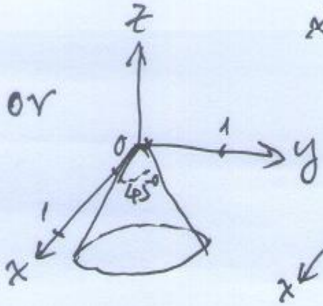
(b) $\phi = \frac{\pi}{4}$ represents a half-cone and can be described as follows

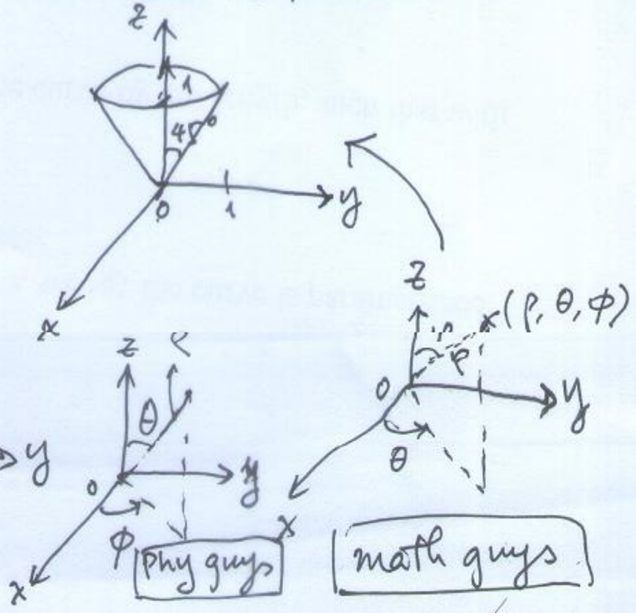
$$\frac{z}{\sqrt{x^2+y^2+z^2}} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$2z^2 = x^2+y^2+z^2$$

$$\boxed{x^2+y^2-z^2=0}$$

or





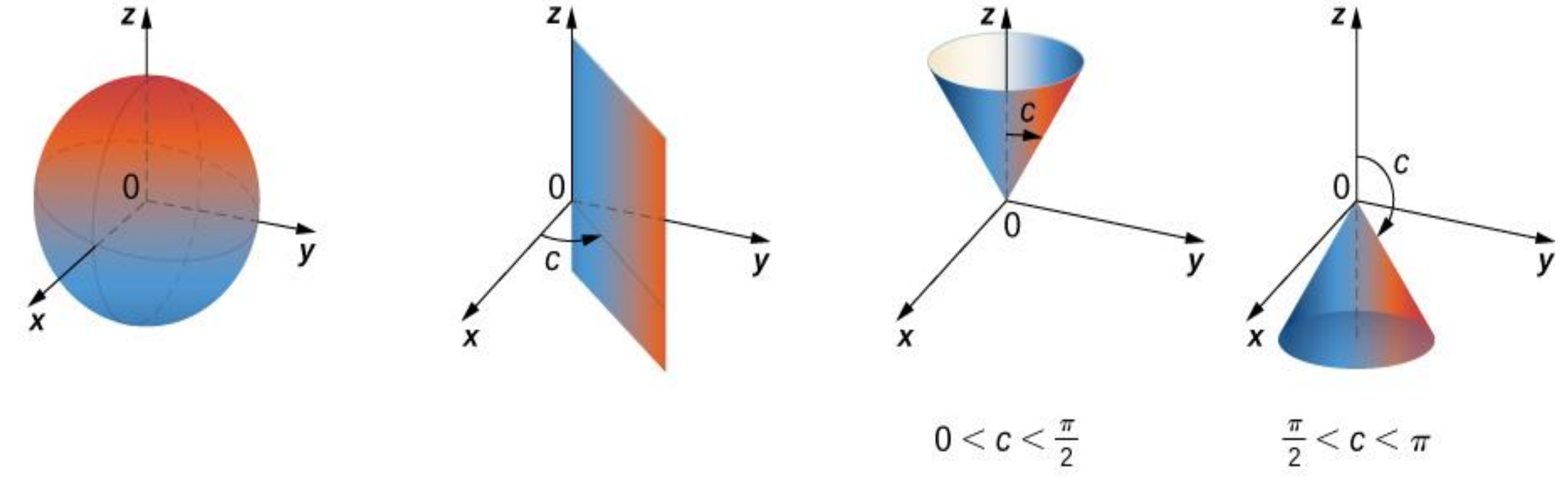
For physics people, they will swap the role of “theta” and “phi”.

Ref.: https://en.wikipedia.org/wiki/Spherical_coordinate_system

Extra Notes for students:

Ref.: https://opentextbc.ca/calculusv3openstax/chapter/triple-integrals-in-cylindrical-and-spherical-coordinates/

In mathematics notation,



Sphere $\rho = c$ (constant) Half plane $\theta = c$ (constant) Half cone $\varphi = c$ (constant)

For grading, we will accept ALL possible answers, provided that your explanations are sufficient. Don't need to worry! We will cater the needs of different students with different thoughts and interpretations regarding the variables in Question 3.

HW 2 Question 5 (Students' solution)

We define the curvature of a path by $\|\mathbf{r}''(s)\|$, where $\mathbf{r}(s)$ is the arc-length parametrization of the path. Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau$.

(a) Show that $\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s)$.

WHAT A COINCIDENCE!?

(a) By chain rule,

$$r'(t) = \frac{dr(s)}{ds} \times \frac{ds}{dt}$$
$$= r'(s) \times \frac{ds}{dt}$$
$$r''(t) = \frac{dr'(s)}{ds} \times \frac{ds}{dt} \times \frac{ds}{dt}$$
$$= r''(s) \times \left(\frac{ds}{dt}\right)^2$$
$$r'(t) \times r''(t) = r'(s) \times \frac{ds}{dt} \times r''(s) \times \left(\frac{ds}{dt}\right)^2$$
$$\therefore r'(t) \times r''(t) = \left(\frac{ds}{dt}\right)^3 r'(s) \times r''(s)$$

$$\underline{r}'(t) = \frac{d\underline{r}(s)}{ds} \times \frac{ds}{dt}.$$
$$= \underline{r}'(s) \times \frac{ds}{dt}.$$
$$r''(t) = \frac{d\underline{r}'(s)}{ds} \times \frac{ds}{dt} \times \frac{ds}{dt}.$$
$$= \underline{r}''(s) \times \left(\frac{ds}{dt}\right)^2$$
$$\therefore \underline{r}'(t) \times \underline{r}''(t)$$
$$= \underline{r}'(s) \times \frac{ds}{dt} \times r''(s) \times \left(\frac{ds}{dt}\right)^2.$$
$$= \left(\frac{ds}{dt}\right)^3 \underline{r}'(s) \times \underline{r}''(s)$$

$$\underline{r}'(t) = \frac{d\underline{r}(s)}{ds} \times \frac{ds}{dt}$$
$$\underline{r}'(t) = \underline{r}'(s) \times \frac{ds}{dt}$$

By using chain rule,

$$\underline{r}''(t) = \frac{dr''(s)}{ds} \times \frac{ds}{dt} \times \frac{ds}{dt}$$
$$= r''(s) \times \frac{ds}{dt} \times \frac{ds}{dt}$$
$$\therefore \underline{r}'(t) \times \underline{r}''(t) = \underline{r}'(s) \times \frac{ds}{dt} \times \underline{r}''(s) \times \frac{ds}{dt} \times \frac{ds}{dt}$$
$$= \left(\frac{ds}{dt}\right)^3 \underline{r}'(s) \times \underline{r}''(s)$$

HW 2 Question 5 (Students' solution)

We define the curvature of a path by $\|\mathbf{r}''(s)\|$, where $\mathbf{r}(s)$ is the arc-length parametrization of the path. Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau$.

(a) Show that $\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s)$.

WHAT A COINCIDENCE!?

$\vec{r}'(s) = \vec{r}'(t)\frac{dt}{ds}$

$\vec{r}'(t) = \vec{r}'(s)\frac{ds}{dt}$

$\vec{r}''(t) = \vec{r}'(s)\frac{d^2s}{dt^2} + \frac{ds}{dt}\vec{r}''(s)\left(\frac{ds}{dt}\right)$

$= \vec{r}'(s)\frac{d^2s}{dt^2} + \vec{r}''(s)\left(\frac{ds}{dt}\right)^2$

$\because s = \int_0^t \|\vec{r}'(\tau)\| d\tau$

$\frac{ds}{dt} = \|\vec{r}'(t)\|$

$=$ constant

$\therefore \frac{d^2s}{dt^2} = 0$

$\therefore \vec{r}''(t) = \vec{r}''(s)\left(\frac{ds}{dt}\right)^2$

$\therefore \vec{r}'(t) \times \vec{r}''(t) = \vec{r}'(s)\frac{ds}{dt} \times \vec{r}''(s)\left(\frac{ds}{dt}\right)^2$

$= \left(\frac{ds}{dt}\right)^3 \vec{r}'(s) \times \vec{r}''(s)$

By chain rule,

$\frac{d\vec{r}(t)}{dt} = \frac{d\vec{r}(t)}{ds}\cdot\frac{ds}{dt}$

$= \vec{r}'(s)\cdot\frac{ds}{dt}$

$\because s = \int_0^t \|\vec{r}'(\tau)\| d\tau$ (given)

$\frac{ds}{dt} = \|\vec{r}'(t)\|$

$\frac{d^2s}{dt^2} = 0$

$\vec{r}'(t) \times \vec{r}''(t) = \vec{r}'(s)\cdot\frac{ds}{dt}\cdot\vec{r}'(s)\cdot\frac{d^2s}{dt^2} + \vec{r}'(s)\cdot\left(\frac{ds}{dt}\right)^3\cdot\vec{r}''(s)$

$= \left(\frac{ds}{dt}\right)^3 \vec{r}'(s) \times \vec{r}''(s)$

$\vec{r}'(t) = \frac{d\vec{r}(t)}{ds}\frac{ds}{dt}. = \vec{r}'(s)\cdot\frac{ds}{dt}.$

$\vec{r}''(t) = \vec{r}'(s)\frac{d^2s}{dt^2} + \frac{ds}{dt}\cdot\vec{r}''(s)\cdot\frac{ds}{dt}$

$= \vec{r}''(s)\left(\frac{ds}{dt}\right)^2 + \vec{r}'(s)\frac{d^2s}{dt^2}.$

$\because s = \int_0^t \|\vec{r}'(\tau)\| d\tau$ ← suppose upper lim

$\frac{ds}{dt} = \|\vec{r}'(t)\|$, which is a constant

$\therefore \frac{d^2s}{dt^2} = 0.$

$\therefore \vec{r}''(t) = \vec{r}'(s)(0) + \vec{r}''(s)\left(\frac{ds}{dt}\right)^2.$

HW 2 Question 5 (Suggested Correct Approach)

We define the curvature of a path by $\|\mathbf{r}''(s)\|$, where $\mathbf{r}(s)$ is the arc-length parametrization of the path. Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau$.

(a) Show that $\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s)$.

By chain rule, $\dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt} = \mathbf{r}'(s)\dfrac{ds}{dt}$ — (1)

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}\left(\mathbf{r}'(s)\frac{ds}{dt}\right) = \frac{d\mathbf{r}'(s)}{dt}\frac{ds}{dt} + \mathbf{r}'(s)\frac{d^2s}{dt^2} \quad \text{— (*)}$$

By chain rule again, $\dfrac{d\mathbf{r}'(s)}{dt} = \dfrac{d\mathbf{r}'(s)}{ds}\dfrac{ds}{dt} = \mathbf{r}''(s)\dfrac{ds}{dt}$ — (**)

Put (**) to (*), $\dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{r}''(s)\left(\dfrac{ds}{dt}\right)^2 + \mathbf{r}'(s)\dfrac{d^2s}{dt^2}$ — (***)

Taking cross product of (1) and (***),

$$\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} = \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s) + \overbrace{\frac{d^2s}{dt^2}\frac{ds}{dt}\,\mathbf{r}'(s) \times \mathbf{r}'(s)}^{=0}$$

$$= \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s) \quad //$$

HW 2 Question 5(b) (Students' solution)

We define the curvature of a path by $\|\mathbf{r}''(s)\|$, where $\mathbf{r}(s)$ is the arc-length parametrization of the path. Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau$. Hence, or otherwise, show that the curvature can be expressed in terms of t. Give the explicit form of the curvature function.

$\|\vec r''(s)\|$ is the curvature

$\because \vec r'(t) // \vec r'(s)$ and $\vec r''(t) // \vec r''(s)$

$\therefore$ angle between $\vec r'(t)$ and $\vec r''(t)$ equals angle between $\vec r'(s)$ and $\vec r''(s)$

From (a), $\vec r'(t) \times \vec r''(t) = \left(\frac{ds}{dt}\right)^3 \vec r'(s) \times \vec r''(s)$

$\|\vec r'(t)\|\|\vec r''(t)\| \cancel{\sin\theta} = \left(\frac{ds}{dt}\right)^3 \|\vec r'(s)\|\|\vec r''(s)\| \cancel{\sin\theta}$

$\|\vec r'(t)\|\|\vec r''(t)\| = \left(\frac{ds}{dt}\right)^3 \left[\|\vec r'(t)\| \left\|\frac{dt}{ds}\right\|\right] \|\vec r''(s)\|$

$\|\vec r'(t)\|\|\vec r''(t)\| = \left(\frac{ds}{dt}\right)^2 \|\vec r'(t)\|\|\vec r''(s)\|$

$\frac{\|\vec r''(t)\|}{\left(\frac{ds}{dt}\right)^2} = \|\vec r''(s)\|$

$\because \frac{ds}{dt} = \|\vec r'(t)\|$

$\therefore \|\vec r''(s)\| = \frac{\|\vec r''(t)\|}{\|\vec r'(t)\|^2}$

Curvature $= \|\vec r''(s)\|$

$\because \vec r'(t) // \vec r'(s)$ and $\vec r''(t) // \vec r''(s)$

$\therefore$ Angle between $\vec r'(t)$ and $\vec r''(t)$ and between $\vec r'(s)$ and $\vec r''(s)$ is the same.

$|\vec r'(t) \times \vec r''(t)| = \left(\frac{ds}{dt}\right)^3 |\vec r'(s) \times \vec r''(s)|$ ($\because \vec r'(t) = \frac{ds}{dt}\vec r'(s)$, $|\vec r'(t)| = \frac{ds}{dt}|\vec r'(s)|$)

$\|\vec r'(t)\|\|\vec r''(t)\| \sin\theta = \left(\frac{ds}{dt}\right)^3 |\vec r'(s)|\,|\vec r''(s)| \sin\theta$.

$\|\vec r'(t)\|\|\vec r''(t)\| = \left(\frac{ds}{dt}\right)^2 |\vec r'(t)|\,\|\vec r''(s)\|$

$\frac{|\vec r''(t)|}{\left(\frac{ds}{dt}\right)^2} = \|\vec r''(s)\|$.

$\|\vec r''(s)\| = \frac{\|\vec r''(t)\|}{|\vec r'(t)|^2}$

$r'(t) = \frac{ds}{dt} \cdot r'(s)$ Curvature $= \|\vec r''(s)\|$

$\|r'(t)\| = \frac{ds}{dt} \cdot \|r'(s)\|$

$\because \vec r'(t) // \vec r'(s)$ and $\vec r''(t) // \vec r''(s)$

$\because \vec r(t)$ and $\vec r(s)$ are refering to the same curve on the graph with different representation

$\therefore \vec r'(t) // \vec r'(s)$ and $\vec r''(t) // \vec r''(s)$

$\therefore$ The angle between $\vec r'(t)$, $\vec r''(t)$ and $\vec r'(s)$, $\vec r''(s)$ is the same

$\vec r'(t) \times \vec r''(t) = \left(\frac{ds}{dt}\right)^3 (\vec r'(s) \times \vec r''(s))$

$\|\vec r'(t)\|\|\vec r''(t)\| \sin\theta = \left(\frac{ds}{dt}\right)^3 \|\vec r'(s)\|\|\vec r''(s)\| \sin\theta$

$\|\vec r'(t)\|\|\vec r''(t)\| = \left(\frac{ds}{dt}\right)^3 \|\vec r'(s)\|\|\vec r''(s)\|$

$\|\vec r'(t)\|\|\vec r''(t)\| = \left(\frac{ds}{dt}\right)^2 \|\vec r'(t)\|\|\vec r''(s)\|$

$\|\vec r''(s)\| = \frac{\|\vec r''(t)\|}{\left(\frac{ds}{dt}\right)^2}$

$\|\vec r''(s)\| = \frac{\|\vec r''(t)\|}{\|\vec r'(t)\|^2}$

WHAT A COINCIDENCE!?

HW 2 Question 5(b) (Students’ solution)

We define the curvature of a path by $\|\mathbf{r}''(s)\|$, where $\mathbf{r}(s)$ is the arc-length parametrization of the path. Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau$. Hence, or otherwise, show that the curvature can be expressed in terms of t. Give the explicit form of the curvature function.

(b) $s = \int_0^t \|r'(\tau)\| d\tau$

$= [r(\tau)]_0^t$

$= r(t)$

$\therefore \frac{ds}{dt} = r'(t)$

$r'(t) \times r''(t) = (\frac{ds}{dt})^3 r'(s) \times r''(s)$

$r'(t) \times r''(t) = (r'(t))^3 r'(s) \times r''(s)$

$r'(t) \times r''(t) = (r'(t))^3 (r'(t) \times \frac{dt}{ds}) \times r''(s)$

$r'(t) \times r''(t) = (r'(t))^3 (r'(t) \times \frac{1}{r'(t)}) \times r''(s)$

$r''(s) = \frac{r'(t) \times r''(t)}{(r'(t))^3}$

$\because$ Curvature $= \|r''(s)\|$

$= \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$

$s = \int_0^t \|r(t)\| dr$

$= r(t).$

$\frac{ds}{dt} = \frac{d}{dt}(r(t))$

$= r'(t).$

$r'(t) \times r''(t) = (\frac{ds}{dt})^3 r'(s) \times r''(s)$

$r'(t) \times r''(t) = (r'(t))^3 r'(s) \times r''(s)$

$r''(t) = r''(s) \times \frac{ds}{dt}$

$r''(s) = \frac{1}{\frac{ds}{dt}} r'(t).$

$\therefore r'(t) \times r''(t) = (r'(t))^3 (\frac{1}{\frac{ds}{dt}} \cdot r'(t)) \times r''(s)$

$= (r'(t))^3 (\frac{1}{r'(t)} \cdot r'(t) \times r''(s)).$

$= (r'(t))^3 \times r''(s)$

$\therefore r''(s) = \frac{r'(t) \times r''(t)}{(r'(t))^3}$

$\|r''(s)\| = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}.$

$s = \int_0^t \|r'(\tau)\| d\tau = r(t)$

$\frac{ds}{dt} = r'(t)$

$r'(t) \times r''(t) = (\frac{ds}{dt})^3 r'(s) \times r''(s)$ (from a.)

$r'(t) \times r''(t) = (r'(t))^3 r'(s) \times r''(s)$

$r'(t) = r'(s) \times \frac{ds}{dt}$

$r'(s) = r'(t) \frac{dt}{ds}$

$= r'(t) \frac{1}{\frac{ds}{dt}}$

$\therefore r'(t) \times r''(t) = (r'(t))^3 (\frac{1}{\frac{ds}{dt}} r'(t)) \times r''(s)$

$= (r'(t))^3 (\frac{1}{r'(t)} r'(t)) \times r''(s)$

$= (r'(t))^3 \times r''(s)$

$\therefore r''(s) = \frac{r'(t)(r''(t))}{(r'(t))^3}$

$\|r''(s)\| = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$

WHAT A COINCIDENCE!?

HW 2 Question 5(b) (Students' solution)

We define the curvature of a path by $\|\mathbf{r}''(s)\|$, where $\mathbf{r}(s)$ is the arc-length parametrization of the path. Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau$. Hence, or otherwise, show that the curvature can be expressed in terms of t. Give the explicit form of the curvature function.

$$\vec{r}'(t) \times \vec{r}''(t) = \left(\frac{ds}{dt}\right)^3 \vec{r}'(s) \times \vec{r}''(s)$$
$$= [\vec{r}'(t)]^3 \vec{r}'(s) \times \vec{r}''(s)$$
$$= [\vec{r}'(t)] \frac{1}{\vec{r}'(t)} \vec{r}'(t) \times \vec{r}''(s)$$
$$= [\vec{r}'(t)]^3 \cdot \vec{r}''(s)$$

$\because$ By $\frac{ds}{dt} = \vec{r}'(t)$

By $\vec{r}'(s) = \frac{1}{\frac{ds}{dt}} \vec{r}'(t) = \frac{1}{\vec{r}'(t)} \vec{r}'(t)$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \|\vec{r}'(t)\|^3 \cdot \|\vec{r}''(s)\|$$
$$\|\vec{r}''(s)\| = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \left(\frac{ds}{dt}\right)^3 \vec{r}'(s) \times \vec{r}''(s)$$
$$\vec{r}'(t) \times \vec{r}''(t) = \vec{r}'(t)^3 \left(\vec{r}'(s) \times \vec{r}''(s)\right)$$
$$\therefore r'(t) = \frac{ds}{dt} (\vec{r}'(s))$$
$$\therefore r'(s) = \frac{1}{\frac{ds}{dt}} r'(t) = \frac{1}{r'(t)} r'(t) = 1$$
$$r'(t) \times \vec{r}''(t) = r'(t)^3 \, \vec{r}''(s)$$
$$\vec{r}''(s) = \frac{\vec{r}'(t) \times \vec{r}''(t)}{r'(t)^3}$$
$$\|\vec{r}''(s)\| = \left\| \frac{r'(t) \times \vec{r}''(t)}{r'(t)^3} \right\|$$

$$\vec{r}'(t) \times \vec{r}''(t) = \left(\frac{ds}{dt}\right)^3 \vec{r}'(s) \times \vec{r}''(s)$$
$$\vec{r}'(t) \times \vec{r}''(t) = [\vec{r}'(t)]^3 \vec{r}'(s) \times \vec{r}''(s)$$
$$\vec{r}'(s) = \frac{1}{\frac{ds}{dt}} \cdot \vec{r}'(t)$$
$$\vec{r}'(t) \times \vec{r}''(t) = [\vec{r}'(t)]^3 \frac{1}{\vec{r}'(t)} \cdot \vec{r}'(t) \times \vec{r}''(s)$$
$$= [\vec{r}'(t)]^3 \times \vec{r}''(s)$$
$$\vec{r}'(t) \times \vec{r}''(t) = [\vec{r}'(t)]^3 \cdot \vec{r}''(s)$$
$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \|\vec{r}'(t)\|^3 \cdot \|\vec{r}''(s)\|$$
$$\|\vec{r}''(s)\| = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

WHAT A COINCIDENCE!?

HW 2 Question 5 (Suggested Correct Approach)

We define the curvature of a path by $\|\mathbf{r}''(s)\|$, where $\mathbf{r}(s)$ is the arc-length parametrization of the path. Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau$.

(b) Hence, or otherwise, show that the curvature can be expressed in terms of t. Give the explicit form of the curvature function.

As $\underline{r}(t)$ travels a constant speed, and $\underline{r}'(s)$ and $\underline{r}''(s)$ are two orthogonal vectors, we have

$$\|\underline{r}'(s) \times \underline{r}''(s)\| = \underbrace{\|\underline{r}'(s)\|}_{1} \|\underline{r}''(s)\| \sin\frac{\pi}{2} = \|\underline{r}''(s)\|$$

Taking the magnitude on both sides of the expression in (a),

$$\|\underline{r}'(t) \times \underline{r}''(t)\| = \|\underline{r}''(s)\| \left\|\frac{ds}{dt}\right\|^3$$

$$\|\underline{r}''(s)\| = \frac{\|\underline{r}'(t) \times \underline{r}''(t)\|}{\left\|\frac{ds}{dt}\right\|^3}$$

Now, since $s = \int_0^t \|\underline{r}'(\tau)\| d\tau$

$\Rightarrow \frac{ds}{dt} = \|\underline{r}'(t)\|$,

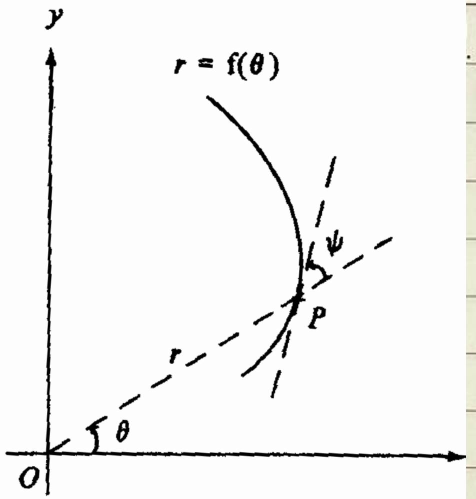
which is in terms of t as well

$$\|\mathbf{r}''(s)\| = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

HW 2 Question 6 (Students' solution)

Given a curve *C* on the *xy*-plane with equation $r = f(\theta)$, where *r* and θ are the polar coordinates used to describe any point lying on *C*, in particular point *P* in the following figure. Let *O* be the origin and ψ be the angle from the line *OP* to the tangent line at point *P*. We assume *f* is continuously differentiable and non-negative.

(a) Express $\tan\psi$ in terms of r and derivative of r with respect to θ.



In particular point P in the figure.
The position vector of P is $\langle r\cos\theta, r\sin\theta\rangle = \langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle$
Suppose that $\underline{r} = \underline{P} = \langle r\cos\theta, r\sin\theta\rangle$ and $r = f(\theta)$.

$$\therefore \frac{dr}{d\theta} = \frac{df(\theta)}{d\theta} = f'(\theta)$$

$$\because \begin{cases} f'(\theta)\cdot P = |f'(\theta)||P|\cos\psi \quad — ① \\ f'(\theta)\times P = |f'(\theta)||P|\sin\psi \quad — ② \end{cases}$$

$$\frac{②}{①} \quad \tan\psi = \frac{f'(\theta)\times P}{f'(\theta)\cdot P}$$

$$= \left| \frac{\langle \frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dr}{d\theta}\sin\theta + r\cos\theta\rangle \times \langle r\cos\theta, r\sin\theta\rangle}{\langle \frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dr}{d\theta}\sin\theta + r\cos\theta\rangle \cdot \langle r\cos\theta, r\sin\theta\rangle} \right|$$

$$= \left| \frac{(\frac{dr}{d\theta}(r)\sin\theta\cos\theta - r^2\sin^2\theta) - (\frac{dr}{d\theta}(r)\sin\theta\cos\theta + r^2\cos^2\theta)}{(\frac{dr}{d\theta}(r)\cos^2\theta - r^2\sin\theta\cos\theta) + (\frac{dr}{d\theta}(r)\sin^2\theta + r^2\sin\theta\cos\theta)} \right|$$

$$= \left| \frac{-r^2}{\frac{dr}{d\theta}(r)} \right|$$

$$= \frac{r}{\frac{dr}{d\theta}} \quad \text{where } \psi\ [0, \tfrac{\pi}{2})$$

6a. In particular point P in the figure
The position vector of $\underline{P}$ is $\langle r\cos\theta, r\sin\theta\rangle = \langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle$
Suppose that $\underline{r} = \underline{P} = \langle r\cos\theta, r\sin\theta\rangle$ and $r = f(\theta)$

$$\therefore \frac{dr}{d\theta} = \frac{df(\theta)}{d\theta} = f'(\theta)$$

$$\because \begin{cases} f'(\theta)\cdot P = |f'(\theta)||\underline{P}|\cos\psi \quad — ① \\ f'(\theta)\times P = |f'(\theta)||\underline{P}|\sin\psi \quad — ② \end{cases}$$

$$\frac{②}{①} : \tan\psi = \frac{f'(\theta)\times P}{f'(\theta)\cdot P}$$

$$= \left| \frac{\langle \frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dr}{d\theta}\sin\theta + r\cos\theta\rangle \times \langle r\cos\theta, r\sin\theta\rangle}{\langle \frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dr}{d\theta}\sin\theta + r\cos\theta\rangle \cdot \langle r\cos\theta, r\sin\theta\rangle} \right|$$

$$= \left| \frac{(\frac{dr}{d\theta}(r)\sin\theta\cos\theta - r^2\sin^2\theta) - (\frac{dr}{d\theta}(r)\sin\theta\cos\theta + r^2\cos^2\theta)}{(\frac{dr}{d\theta}(r)\cos^2\theta - r^2\sin\theta\cos\theta) + (\frac{dr}{d\theta}(r)\sin^2\theta + r^2\sin\theta\cos\theta)} \right|$$

$$= \left| \frac{-r^2}{\frac{dr}{d\theta}(r)} \right|$$

$$= \left| \frac{-r}{\frac{dr}{d\theta}} \right| = \frac{r}{\frac{dr}{d\theta}} \quad \text{where } \psi\ [0, \tfrac{\pi}{2})$$

WHAT A COINCIDENCE!?

HW 2 Question 6 (Students' solution)

Given a curve *C* on the *xy*-plane with equation $r = f(\theta)$, where *r* and *θ* are the polar coordinates used to describe any point lying on *C*, in particular point *P* in the following figure. Let *O* be the origin and *ψ* be the angle from the line *OP* to the tangent line at point *P*. We assume *f* is continuously differentiable and non-negative.

(a) Express $\tan\psi$ in terms of r and derivative of r with respect to θ.

6a) $r=f(\theta)$

∴ In rectangular coordinates:

$x = r\cos\theta,\ y = r\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(y)}{\frac{d}{d\theta}(x)} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\tan(\theta+\psi) = \frac{\tan\theta+\tan\psi}{1-\tan\theta\tan\psi}$$

$\because \tan(\theta+\psi) = \frac{dy}{dx}$

$$\frac{\tan\theta+\tan\psi}{1-\tan\theta\tan\psi} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{\tan\theta+\tan\psi}{1-\tan\theta\tan\psi} = \frac{\frac{dr}{d\theta}\tan\theta + r}{\frac{dr}{d\theta} - r\tan\theta}$$

$$\frac{dr}{d\theta}\tan\theta - r\tan^2\theta + \frac{dr}{d\theta}\tan\psi - r\tan\theta\tan\psi = \frac{dr}{d\theta}\tan\theta + r - \frac{dr}{d\theta}\tan^2\theta\tan\psi - r\tan\theta\tan\psi$$

$$\frac{dr}{d\theta}\tan\psi - r\tan^2\theta = r - \frac{dr}{d\theta}\tan^2\theta\tan\psi$$

$$\frac{dr}{d\theta}\tan\psi(1+\tan^2\theta) = r(1+\tan^2\theta)$$

$$\frac{dr}{d\theta}\tan\psi = r$$

$$\tan\psi = \frac{d\theta}{dr}\cdot r$$

6a) $\because r = f(\theta)$

$\therefore \vec{OP} = \langle r\cos\theta, r\sin\theta\rangle = \langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right) \div \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right) = \frac{\frac{dr}{d\theta}\tan\theta + r}{\frac{dr}{d\theta} - r\tan\theta}$$

$$\frac{dy}{dx} = \tan(\theta+\psi) = \frac{\tan\theta+\tan\psi}{1-\tan\theta\tan\psi}$$

$$\therefore \frac{\frac{dr}{d\theta}\tan\theta + r}{\frac{dr}{d\theta} - r\tan\theta} = \frac{\tan\theta+\tan\psi}{1-\tan\theta\tan\psi}$$

$$\frac{dr}{d\theta}\tan\theta + r - \frac{dr}{d\theta}\tan^2\theta\tan\psi - r\tan\theta\tan\psi = \frac{dr}{d\theta}\tan\theta + \frac{dr}{d\theta}\tan\psi - r\tan^2\theta - r\tan\theta\tan\psi$$

$$r - \frac{dr}{d\theta}\tan^2\theta\tan\psi = \frac{dr}{d\theta}\tan\psi - r\tan^2\theta$$

$$r(1+\tan^2\theta) = \frac{dr}{d\theta}\tan\psi(1+\tan^2\theta)$$

$$r = \frac{dr}{d\theta}\tan\psi$$

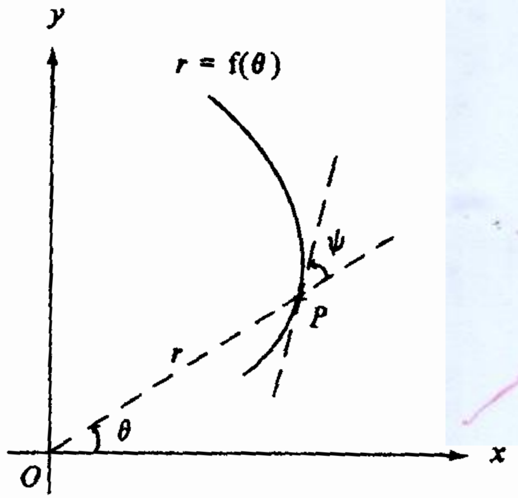
$$\tan\psi = r\cdot\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$$

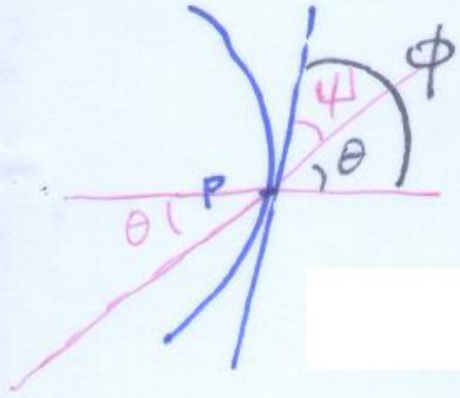
WHAT A COINCIDENCE!?

HW 2 Question 6 (Suggested Approach)

Given a curve *C* on the *xy*-plane with equation $r = f(\theta)$, where *r* and θ are the polar coordinates used to describe any point lying on *C*, in particular point *P* in the following figure. Let *O* be the origin and ψ be the angle from the line *OP* to the tangent line at point *P*. We assume *f* is continuously differentiable and non-negative.

(a) Express $\tan\psi$ in terms of r and derivative of r with respect to θ.





(a) Let (x, y) be the Cartesian coordinates of P, $\tan\phi$ be the gradient of the curve at P.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{aligned} \frac{dx}{d\theta} &= \cos\theta\frac{dr}{d\theta} - r\sin\theta \\ \frac{dy}{d\theta} &= \sin\theta\frac{dr}{d\theta} + r\cos\theta \end{aligned}$$

$$\tan\phi = \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta} = \frac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta}$$

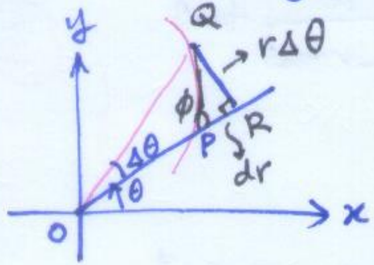
Now $\tan\psi = \tan(\phi - \theta)$

$$= \left(\frac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta} - \tan\theta\right) \times \frac{1}{1 + \frac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta}\tan\theta}$$

$$= \frac{r(1 + \tan^2\theta)}{\frac{dr}{d\theta}(1 + \tan^2\theta)} = \frac{r}{\frac{dr}{d\theta}}$$

Alternatively,

6(a) In the diagram



$PR \approx \Delta r$

$QR \approx r\Delta\theta$

$\therefore \tan\phi \approx \dfrac{r\Delta\theta}{\Delta r}$

important

$\Rightarrow \tan\psi = \lim\limits_{\Delta\theta \to 0} r\dfrac{\Delta\theta}{\Delta r} = \dfrac{r}{dr/d\theta}$

HW 2 Question 6 (Students' solution)

Given a curve C on the xy-plane with equation $r = f(\theta)$, where r and θ are the polar coordinates used to describe any point lying on C, in particular point P in the following figure. Let O be the origin and ψ be the angle from the line OP to the tangent line at point P. We assume f is continuously differentiable and non-negative.

WHAT A COINCIDENCE!?

(b) Given two curves C_1: $r = 2 - 2\cos\theta$ (where $0 \le \theta < 2\pi$) and C_2: $r = 2$.

Find the points of intersection of the two curves.

6bi $C_1: r = 2 - 2\cos\theta \quad C_2: r = 2$

Intersection : $2 = 2 - 2\cos\theta$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

when $\theta = \frac{\pi}{2}$, $P = (r\cos\frac{\pi}{2}, r\sin\frac{\pi}{2}) = (0, 2)$

when $\theta = \frac{3\pi}{2}$, $P = (r\cos\frac{3\pi}{2}, r\sin\frac{3\pi}{2}) = (0, -2)$

$\therefore$ The points of intersection are $(0,2)$ and $(0,-2)$

b)i) $C_1: r = 2 - 2\cos\theta$, $C_2: r = 2$

$2 - 2\cos\theta = 2$

$\cos\theta = 0$

$\theta = \frac{\pi}{2} / \frac{3\pi}{2}$

when $\theta = \frac{\pi}{2}$, $P = (0, 2)$

when $\theta = \frac{3\pi}{2}$, $P = (0, -2)$

$\therefore$ intersection at $(0, 2)$ and $(0, -2)$

$C_1: r = 2 - 2\cos\theta$ and $C_2: r = 2$, $\theta \in [0, 2\pi)$

The intersection:

$2 - 2\cos\theta = 2$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

whe $\theta = \frac{\pi}{2}$, $P = (r\cdot\cos\theta, r\cdot\sin\theta) = (0, 2)$

when $\theta = \frac{3\pi}{2}$, $P = (r\cdot\cos\theta, r\cdot\sin\theta) = (0, -2)$

$\therefore$ The pt. of intersection are $(0, 2)$ and $(0, -2)$.

$C_1 = C_2$

$2 - 2\cos\theta = 2$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

polar coordinates: $(2, \frac{\pi}{2})$; $(2, \frac{3\pi}{2})$

rectangular coordinates: $(0, 2)$; $(0, -2)$

The curves intersect at $(0, 2)$ and $(0, -2)$

$2 = 2 - 2\cos\theta$

$1 = 1 - \cos\theta$

$\cos\theta = 0 \qquad 0 \le \theta \le 2\pi$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

In polar plane: $(2, \frac{\pi}{2})$, $(2, \frac{3\pi}{2})$

In rectangular plane: $(0, 2)$, $(0, -2)$

6b(i) The curve cuts at $(0,2)$ & $(0,-2)$

$\Rightarrow \begin{cases} 2 = 2 - 2\cos\theta \\ 1 = 1 - \cos\theta \end{cases} \Rightarrow 0 = \cos\theta$

$\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ for $0 \le \theta \le 2\pi$

Polar: $(2, \frac{\pi}{2})$, $(2, \frac{3\pi}{2})$

Rectangular: $(0,2)$, $(0,-2)$

HW 2 Question 6 (Suggested Approach)

(b) Given two curves C_1: $r = 2 - 2\cos\theta$ (where $0 \le \theta < 2\pi$) and C_2: $r = 2$.

(i) Find all points of intersection of these two curves.

(ii) Find the angle between the tangent lines at each point of intersection you obtained in (i).

Here we also accept C_2

(b)(i) Sub $r=2$ in C_1: $r = 2 - 2\cos\theta$, we get

$$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

The points of intersection are $(2, \frac{\pi}{2})$ and $(2, \frac{3\pi}{2})$ //

(b)(ii)

At $\theta = \frac{\pi}{2}$, the tangent to C_2 is parallel to the x-axis.

For C_1, $\frac{dr}{d\theta} = 2\sin\theta = 2 \Rightarrow \tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{2}{2} = 1$

$\therefore$ The angle between C_1 and C_2 is $\frac{\pi}{2} - \tan^{-1} 1 = \underline{\frac{\pi}{4}}$. (2)

By symmetry, the angle between C_1 and C_2 at $\theta = \frac{3\pi}{2}$ is also $\underline{\underline{\frac{\pi}{4}}}$ (1)

If students write down $\frac{3\pi}{4}$ or $-\frac{\pi}{4}$, OK! I understand how you obtain these 2 answers.

$\frac{\pi}{2}$: just deduct 1 mark

HW 2 Question 6 (Students' solution)

(c) Given two curves C_1: $r = 2 - 2\cos\theta$ (where $0 \le \theta < 2\pi$) and C_2: $r = 2$.

Some part of C_1 is inside C_2, find the arc length of such part.

6c) when C_1 is inside C_2,

$r_2 > r_1$

$2 > 2 - 2\cos\theta$

$\cos\theta > 0$

$\theta < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \theta < 2\pi$ $\because \theta \in [0, 2\pi)$

$\vec{r}(\theta) = \langle r\cos\theta, r\sin\theta \rangle$

$\vec{r}'(\theta) = \langle \frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dr}{d\theta}\sin\theta + r\cos\theta \rangle$

For C_1, $r_1 = 2 - 2\cos\theta$

$\frac{dr_1}{d\theta} = 2\sin\theta$

$\therefore \vec{r}_1'(\theta) = \langle 2\sin\theta\cos\theta - 2\sin\theta + 2\sin\theta\cos\theta, 2\sin^2\theta + 2\cos\theta - 2\cos^2\theta \rangle$

$= \langle 2\sin 2\theta - 2\sin\theta, 2\cos\theta - 2\cos 2\theta \rangle$

6C $\because$ Some part of C_1 is inside C_2 $\therefore C_2 > C_1$, $2 > 2 - 2\cos\theta$, $\cos\theta > 0$

$\theta < \frac{\pi}{2}$ or $\theta > \frac{3}{2}\pi$ where $0 \le \theta \le 2\pi$

$\because \frac{dr}{d\theta} = \langle \frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dr}{d\theta}\sin\theta + r\cos\theta \rangle$

$\because r = 2 - 2\cos\theta$

$\therefore r' = 2\sin\theta$

$\frac{dr}{d\theta} = \langle 2\sin\theta\cos\theta - (2 - 2\cos\theta)\sin\theta, 2\sin\theta\sin\theta + (2 - 2\cos\theta)\cos\theta \rangle$

$= \langle 2\sin\theta\cos\theta - 2\sin\theta + 2\sin\theta\cos\theta, 2\sin^2\theta + 2\cos\theta - 2\cos^2\theta \rangle$

$= \langle 2\sin 2\theta - 2\sin\theta, 2\cos\theta - 2\cos 2\theta \rangle$

WHAT A COINCIDENCE!?

If $C_2 > C_1$

$2 > 2 - 2\cos\theta$, $\cos\theta > 0$, $\theta < \frac{\pi}{2}$ / $\frac{3\pi}{2}$

$r'(\theta) = \langle r'\cos\theta - r\sin\theta, r'\sin\theta + r\cos\theta \rangle$

$= \langle 2\sin\theta\cos\theta - 2\sin\theta + 2\sin\theta\cos\theta, 2\sin^2\theta + 2\cos\theta - 2\cos^2\theta \rangle$

$\therefore \langle 2\sin 2\theta - 2\sin\theta, 2\cos\theta - 2\cos 2\theta \rangle$

$C_1 < C_2$

$2 - 2\cos\theta < 2$

$-2\cos\theta < 0$

$\cos\theta > 0$

$\theta < \frac{\pi}{2}$ or $\theta > \frac{3}{2}\pi$ where $0 \le \theta \le 2\pi$

For $r = 2 - 2\cos\theta$

$\frac{dr}{d\theta} = 2\sin\theta$

$\vec{r}'(\theta) = \langle \frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dr}{d\theta}\sin\theta + r\cos\theta \rangle$

$= \langle 2\sin\theta\cos\theta - (2 - 2\cos\theta)\sin\theta, 2\sin^2\theta + (2 - 2\cos\theta)\cos\theta \rangle$

$= \langle 2\sin\theta\cos\theta - 2\sin\theta + 2\cos\theta\sin\theta, 2\sin^2\theta + 2\cos\theta - 2\cos^2\theta \rangle$

$= \langle 2\sin 2\theta - 2\sin\theta, 2\cos\theta - 2\cos 2\theta \rangle$

HW 2 Question 6 (Students' solution)

(c) Given two curves C_1: $r = 2 - 2\cos\theta$ (where $0 \le \theta < 2\pi$) and C_2: $r = 2$.

Some part of C_1 is inside C_2, find the arc length of such part.

$\left|\frac{dr}{d\theta}\right| = \sqrt{(2\sin 2\theta - 2\sin\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2}$

$= \sqrt{4\sin^2 2\theta - 8\sin 2\theta \sin\theta + 4\sin^2\theta + 4\cos^2\theta - 8\cos\theta\cos 2\theta + 4\cos^2 2\theta}$

$= \sqrt{4 + 4 - 8(\sin 2\theta \sin\theta + \cos\theta\cos 2\theta)}$

$= \sqrt{8 - 8(\sin 2\theta\sin\theta + \cos\theta\cos 2\theta)}$

$= \sqrt{8 - 8\cos\theta}$

$= \sqrt{8}\,\sqrt{1-\cos\theta}$

$= \sqrt{8}\,\sqrt{2\cdot(\sin\frac{\theta}{2})^2}$

$= 2\sqrt{2}\sqrt{2}\cdot\sin\frac{\theta}{2}$

$= 4\sin\frac{\theta}{2}$

The arc length of such part

$= \int_0^{\frac{\pi}{2}} |r'(\theta)|\,d\theta + \int_{\frac{3}{2}\pi}^{2\pi} |r'(\theta)|\,d\theta = \int_0^{\frac{\pi}{2}} 4\sin\frac{\theta}{2}\,d\theta + \int_{\frac{3}{2}\pi}^{2\pi} 4\sin\frac{\theta}{2}\,d\theta$

$= \left[8\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} + \left[8\cos\frac{\theta}{2}\right]_{\frac{3\pi}{2}}^{2\pi} = 8(2-\sqrt{2}) = 16 - 8\sqrt{2}$ units

$\|\vec{r}_1'(\theta)\| = \sqrt{(2\sin 2\theta - 2\sin\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2}$

$= \sqrt{4\sin^2 2\theta - 8\sin 2\theta\sin\theta + 4\sin^2\theta + 4\cos^2\theta - 8\cos 2\theta\cos\theta + 4\cos^2 2\theta}$

$= \sqrt{4 + 4 - 8(\sin 2\theta\sin\theta + \cos 2\theta\cos\theta)}$

$= \sqrt{8 - 8\left(\frac{1}{2}\cos\theta - \frac{1}{2}\cos 3\theta + \frac{1}{2}\cos\theta + \frac{1}{2}\cos 3\theta\right)}$

$= \sqrt{8 - 8\cos\theta}$

$= 2\sqrt{2}\cdot\sqrt{(1-\cos\theta)}$

$= 2\sqrt{2}\cdot\sqrt{2\left(\frac{1-\cos\theta}{2}\right)}$

$= 2\sqrt{2}\cdot\sqrt{2\left(\sin^2\frac{\theta}{2}\right)}$

$= 4\sin\frac{\theta}{2}$

$= \int_0^{\frac{\pi}{2}} \|\vec{r}_1'(\theta)\|\,d\theta + \int_{\frac{3\pi}{2}}^{2\pi} \|\vec{r}_1'(\theta)\|\,d\theta$

$= \int_0^{\frac{\pi}{2}} 4\sin\frac{\theta}{2}\,d\theta + \int_{\frac{3\pi}{2}}^{2\pi} 4\sin\frac{\theta}{2}\,d\theta$

$= 8\left[-\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} + 8\left[-\cos\frac{\theta}{2}\right]_{\frac{3\pi}{2}}^{2\pi}$

$= 8\left(-\cos\frac{\pi}{4} + \cos 0 - \cos\pi + \cos\frac{3\pi}{4}\right)$

$= 8\left(2 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$

$= 16 - 8\sqrt{2}$

$\|\vec{r}'(\theta)\| = \sqrt{(2\sin 2\theta - 2\sin\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2}$

$= \sqrt{4\sin^2 2\theta - 8\sin\theta\sin 2\theta + 4\sin^2\theta + 4\cos^2\theta - 8\cos\theta\cos 2\theta + 4\cos^2 2\theta}$

$= \sqrt{4 + 4 - 8\sin\theta\sin 2\theta - 8\cos\theta\cos 2\theta}$

$= \sqrt{8 - 8\cos\theta}$

$= \sqrt{8}\sqrt{1-\cos\theta}$

$= \sqrt{8}\sqrt{2\sin^2\frac{\theta}{2}}$

$= 4\sin\frac{\theta}{2}$

arc length

$= \int_0^{\frac{\pi}{2}} 4\sin\frac{\theta}{2}\,d\theta + \int_{\frac{3}{2}\pi}^{2\pi} 4\sin\frac{\theta}{2}\,d\theta$

$= \int_0^{\frac{\pi}{4}} 8\sin u\,du + \int_{\frac{3}{4}\pi}^{\pi} 8\sin u\,du$

$= \left[-8\cos u\right]_0^{\frac{\pi}{4}} + \left[-8\cos u\right]_{\frac{3}{4}\pi}^{\pi}$

$= -8\left(\frac{1}{\sqrt{2}} - 1\right) - 8\left(-1 + \frac{1}{\sqrt{2}}\right)$

$= 8\left(1 - \frac{\sqrt{2}}{2}\right) + 8\left(1 - \frac{\sqrt{2}}{2}\right)$

$= 16 - 8\sqrt{2}$

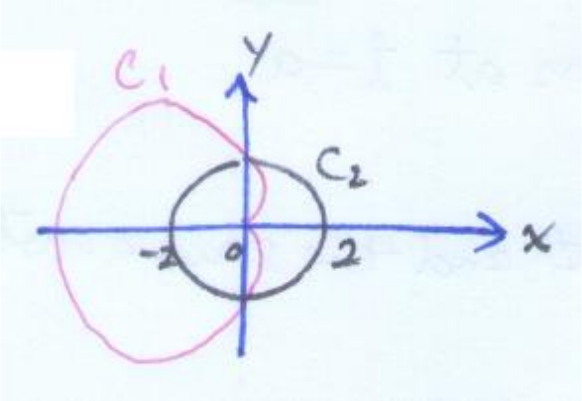
WHAT A COINCIDENCE!?

HW 2 Question 6 (Suggested Approach)

Given a curve *C* on the *xy*-plane with equation $r = f(\theta)$, where *r* and θ are the polar coordinates used to describe any point lying on *C*, in particular point *P* in the following figure. Let *O* be the origin and ψ be the angle from the line *OP* to the tangent line at point *P*. We assume *f* is continuously differentiable and non-negative.

(c) Given two curves C_1: $r = 2 - 2\cos\theta$ (where $0 \le \theta < 2\pi$) and C_2: $r = 2$.

Some part of C_1 is inside C_2, find the arc length of such part.



For any point (r, θ) of C_1 lying inside C_2

$r = 2(1-\cos\theta)$, $r < 2$

$\therefore 2(1-\cos\theta) < 2 \Rightarrow 0 < \theta < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \theta < 2\pi$

$\therefore$ Length of C_1 inside C_2

$$= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta + \int_{3\pi/2}^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \quad \text{(by symmetry)}$$

$$= 2\int_0^{\pi/2} \sqrt{\left(\cos\theta \frac{dr}{d\theta} - r\sin\theta\right)^2 + \left(\sin\theta \frac{dr}{d\theta} + r\cos\theta\right)^2}\, d\theta$$

$$= 2\int_0^{\pi/2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{4\sin^2\theta + 4(1-\cos\theta)^2}\, d\theta$$

$$= 4\sqrt{2}\int_0^{\pi/2} \sqrt{1-\cos\theta}\, d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \sin\frac{\theta}{2}\, d\theta = 16\left[-\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} = \underline{\underline{8(2-\sqrt{2})}}$$

Alternatively,

$$ds^2 = (r\,d\theta)^2 + (dr)^2$$

Length of C_1 inside $C_2 = \int ds = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \;\ldots\ldots = 8(2-\sqrt{2})$

Question 7(a) (Students' solution)

(a) $\lim_{t\to 0}\left(|t|^t\mathbf{i} + \cos^{\frac{1}{t}}(t)\mathbf{j} + (1-t)^t\mathbf{k}\right)$

(a) $\lim_{t\to 0^-}\left(|t|^t\hat{\underline{i}} + \cos^{\frac{1}{t}}(t)\hat{\underline{j}} + (1-t)^t\hat{\underline{k}}\right)$

$= \hat{\underline{i}} + \hat{\underline{j}} + \hat{\underline{k}}$

$\lim_{t\to 0^+}\left(|t|^t\hat{\underline{i}} + \cos^{\frac{1}{t}}(t)\hat{\underline{j}} + (1-t)^t\hat{\underline{k}}\right)$

$= \hat{\underline{i}} + \hat{\underline{j}} + \hat{\underline{k}}$

$\because \lim_{t\to 0^-} = \lim_{t\to 0^+}$

$\therefore$ the limit exist and it is equal to $\hat{\underline{i}} + \hat{\underline{j}} + \hat{\underline{k}}$

7a. $\lim_{t\to 0^-}\left(|t|^t\underline{i} + \cos^{\frac{1}{t}}(t)\underline{j} + (1-t)^t\underline{k}\right)$

$= \underline{i} + \underline{j} + \underline{k}$

$\lim_{t\to 0^+}\left(|t|^t\underline{i} + \cos^{\frac{1}{t}}(t)\underline{j} + (1-t)^t\underline{k}\right)$

$= \underline{i} + \underline{j} + \underline{k}$

$\because$ Left-hand limit is equal to the right hand limit

$\therefore$ The limit exists and $\lim_{t\to 0}\left(|t|^t\underline{i} + \cos^{\frac{1}{t}}(t)\underline{j} + (1-t)^t\underline{k}\right) = \underline{i} + \underline{j} + \underline{k}$ ※

$\lim_{t\to 0^-}\left(|t|^t\hat{i} + \cos^{\frac{1}{t}}(t)\hat{j} + (1-t)^t\hat{k}\right)$

$= 1\hat{i} + 1\hat{j} + 1\hat{k}$

$\lim_{t\to 0^+}\left(|t|^t\hat{i} + \cos^{\frac{1}{t}}(t)\hat{j} + (1-t)^t\hat{k}\right)$

$= 1\hat{i} + 1\hat{j} + 1\hat{k}$

$\because$ LH limit = RH limit

$\therefore$ The limit exists and it is $\hat{i} + \hat{j} + \hat{k}$

$\lim_{t\to 0}\left(|t|^t\hat{i} + \cos^{\frac{1}{t}}(t)\hat{j} + (1-t)^t\hat{k}\right)$

$= \hat{i} + \hat{j} + \hat{k}$

$\lim_{t\to 0^-} = \hat{i} + \hat{j} + \hat{k}$ L.H.S = R.H.S

$\therefore \lim_{t\to 0}$ exists $= \hat{i} + \hat{j} + \hat{k}$

WHAT A COINCIDENCE!?

Question 7(a) (Suggested Approach)

(a) $\lim_{t \to 0} \left(|t|^t \mathbf{i} + \cos^{\frac{1}{t}}(t) \mathbf{j} + (1-t)^t \mathbf{k} \right)$

7 (a) $\lim_{t\to 0} |t|^t = \lim_{t\to 0} e^{t\ln(|t|)} = \exp\left(\lim_{t\to 0} \frac{\ln(|t|)}{t^{-1}}\right)$

$= \exp\left(\lim_{t\to 0} \frac{t^{-1}}{-t^{-2}}\right) = 1$

2 need steps

$\lim_{t\to 0} \cos^{\frac{1}{t}}(t) = \lim_{t\to 0} e^{\frac{\ln(\cos t)}{t}} = \exp\left(\lim_{t\to 0} \frac{\ln(\cos(t))}{t}\right)$

$= \exp\left(\lim_{t\to 0} \frac{-\tan t}{1}\right) = 1$

2 need steps!

$\lim_{t\to 0} (1-t)^t = 1$

2 for ans.

$\therefore$ limit $= \hat{\underline{i}} + \hat{\underline{j}} + \hat{\underline{k}}$ //

What can you conclude?

HOPE YOU WILL GET SOME INSIGHTS AFTER READING THIS PPT!

Justice must be done and upheld!