Week 9 Lecture Notes

Multivariable Calculus

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Review: In Chapter 11, we discuss vector-valued functions of a single (scalar) variable, e.g. $\underline{r}(t) = x(t)\hat{\underline{i}} + y(t)\hat{\underline{j}} + z(t)\hat{\underline{k}}$.

(we input a scalar quantity, and obtain a curve in $\mathbb{R}^3$ plane)

In Chapter 12, we discuss functions of several variables (real), e.g. $z = f(x_1, x_2, \ldots, x_n)$. Suppose we denote $\underline{r} = \langle x_1, x_2, \ldots, x_n \rangle$, then $z = f(\underline{r})$ (i.e. scalar valued functions of a vector variable $\underline{r}$)

position vector.

i.e. we obtain a scalar field.

Reference: Adam and Essex Chapter 15.1, 15.2, 15.3, 15.4

Larson and Edwards Chapter 15.1 - 15.2

Vector fields

We will study two types of vector-valued functions: functions that assign a vector to ① a point in the plane, or ② a point in the space. Such functions are called "vector fields". They can be used to represent "force fields" and "velocity fields".

Definition ① A vector field over a plane region R is a function $\underline{F}$ that assigns a vector $\underline{F}(\underline{r}) = \underline{F}(x,y)$ to each point in R.

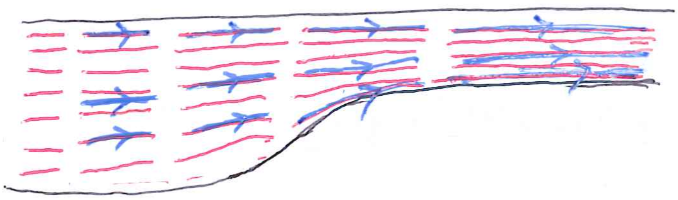
② A vector field over a solid region Q in space is a function $\underline{F}$ that assigns a vector $\underline{F}(\underline{r}) = \underline{F}(x,y,z)$ to each point in Q.

For ①, suppose M and N are two real-valued functions of two variables x and y, defined on a planar region R, the corresponding vector field can be $\underline{F}(x,y) = M(x,y)\hat{\underline{i}} + N(x,y)\hat{\underline{j}}$, where $\underline{r} = \langle x, y \rangle$ → position vector

For ②, suppose M, N and P are three real-valued functions of three variables x, y and z, defined in a solid region R in 3-space, the corresponding vector field can be $\underline{F}(x,y,z) = M(x,y,z)\hat{\underline{i}} + N(x,y,z)\hat{\underline{j}} + P(x,y,z)\hat{\underline{k}}$, where $\underline{r} = \langle x,y,z \rangle$ → position vector.

NOTE: The vector field is continuous if the component functions M, N and P are continuous, it is differentiable if each of the component functions is differentiable.

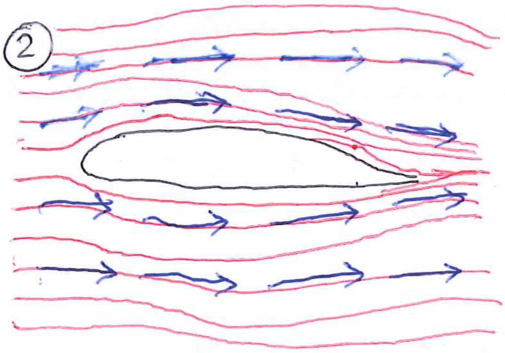
Examples: (1) Streamlines in a contracting channel.



Water speeds up as the channel becomes narrower

i.e. velocity vectors increase in length.

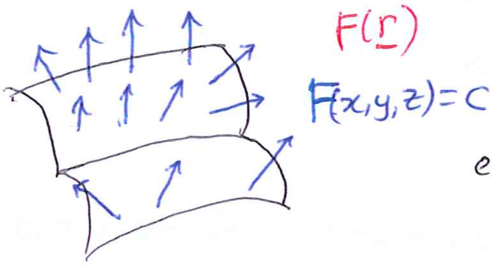
(2)



Airflow past an inclined airfoil

e.g. velocity vectors of a flow around an airfoil in a wind tunnel.

(3)



The field of gradient vectors $\underline{\nabla} f$ on a surface $F(x,y,z)=C$.

e.g. If $f(x,y,z) = x^2+y^2+z^2$

then the level surface is $F(x,y,z) = F(\underline{r})$

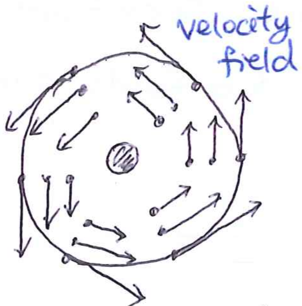
$$= f(x,y,z)-x^2-y^2-z^2$$
$$= C$$

$$\underline{\nabla} F = \left\langle \frac{\partial f}{\partial x}-2x, \frac{\partial f}{\partial y}-2y, \frac{\partial f}{\partial z}-2z \right\rangle$$

$$= \left(\frac{\partial f}{\partial x}-2x\right)\underline{\hat{i}} + \left(\frac{\partial f}{\partial y}-2y\right)\underline{\hat{j}} + \left(\frac{\partial f}{\partial z}-2z\right)\underline{\hat{k}}$$

is a vector field in space.

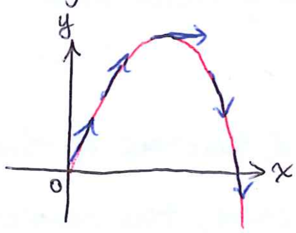
(4)



Rotating wheel

Velocity field determined by a wheel rotating on an axle. The farther a point is from the axle, the greater its velocity.

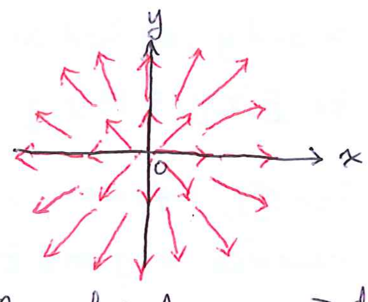
(5) Projectile motion.



velocity vectors $\underline{v}(t)$ of a projectile's motion make a velocity field along the trajectory.

(6) Radial field

$\underline{F} = x\underline{\hat{i}} + y\underline{\hat{j}}$ of position vectors of points in the plane.



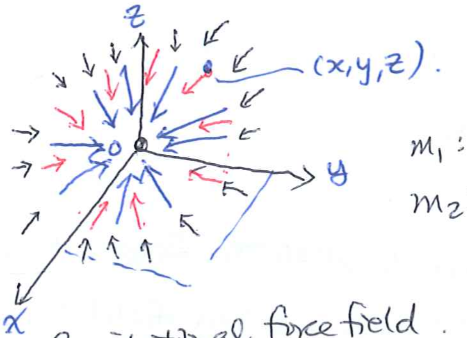
(Remarks: An arrow is drawn with its tail, not the head)

⑦ Gravitational field $\underline{F}(x,y,z)$ due to some object is the force of attraction that the object exerts on a unit mass situated at position (x,y,z).

e.g. $\underline{F}(x,y,z) = \underline{F}(\underline{r}) = \dfrac{-Gm_1m_2}{x^2+y^2+z^2}\,\hat{\underline{u}}$.

G : gravitational constant
$\hat{\underline{u}}$: unit vector in the direction from $(0,0,0)$ to (x,y,z)

(the force of attraction exerted on a particle of mass m_1 located at (x,y,z) by a particle of mass m_2 located at $(0,0,0)$.



Gravitational force field.

m_1 : located at (x,y,z)
m_2 : located at $(0,0,0)$

- $\underline{F}(x,y,z)$ points towards origin
- magnitude of $\underline{F}(x,y,z)$ is the same at all points equidistant from the origin.

Central force field

Suppose the position vector is $\underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}$,

then $\underline{F}(\underline{r}) = \dfrac{-Gm_1m_2}{\|\underline{r}\|^2}\left(\dfrac{\underline{r}}{\|\underline{r}\|}\right) = \underbrace{\dfrac{-Gm_1m_2}{\|\underline{r}\|^2}\,\hat{\underline{u}}}_{\text{Inverse square field}}$

Inverse square field

$\underline{F}(\underline{r}) = \dfrac{k}{\|\underline{r}\|^2}\,\hat{\underline{u}}$ ($k \in \mathbb{R}$, $\hat{\underline{u}} = \dfrac{\underline{r}}{\|\underline{r}\|}$: unit vector in the direction of $\underline{r}$)

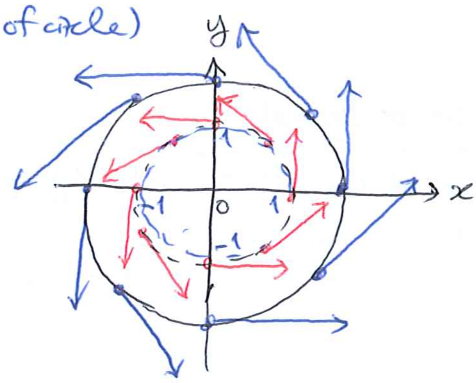
⑧ Sketch some vectors of the vector field $\underline{F}(x,y) = -y\hat{\underline{i}} + x\hat{\underline{j}}$ in $\mathbb{R}^2$.

Idea: To find level curves in scalar fields, i.e. vectors of equal magnitude lie on circles.

$\|\underline{F}\| = c \Rightarrow x^2+y^2=c^2$ (Equation of circle)

e.g.

Point	Vector
$(1,0)$	$\underline{F}(1,0) = \hat{\underline{j}}$
$(0,1)$	$\underline{F}(0,1) = -\hat{\underline{i}}$
$(-1,0)$	$\underline{F}(-1,0) = -\hat{\underline{j}}$
$(0,-1)$	$\underline{F}(0,-1) = \hat{\underline{i}}$



Each arrow is tangent to a circle with centre at origin

Confirmation: Take the dot product of the position vector $\underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}}$ with the vector $\underline{F}(x,y) = -y\hat{\underline{i}} + x\hat{\underline{j}}$.

$\underline{r}\cdot\underline{F}(x,y) = -xy + xy = 0$

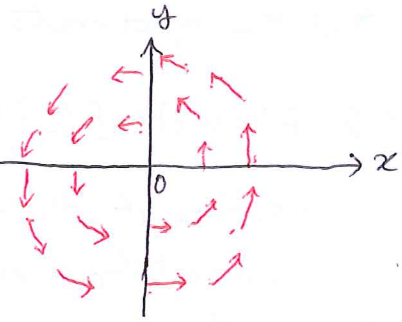
$\therefore \underline{F}(x,y) \perp$ position vector $\underline{r} = \langle x,y\rangle$, i.e. tangent to a circle with centre at $(0,0)$ and radius $\|\underline{r}\| = \sqrt{x^2+y^2}$.

$\|\underline{F}(x,y)\| = \sqrt{x^2+y^2} = \|\underline{r}\|$ (magnitude of vector $\underline{F}(x,y)$ = radius of circle).

⑨ Velocity field of a solid rotating about the z-axis with angular velocity

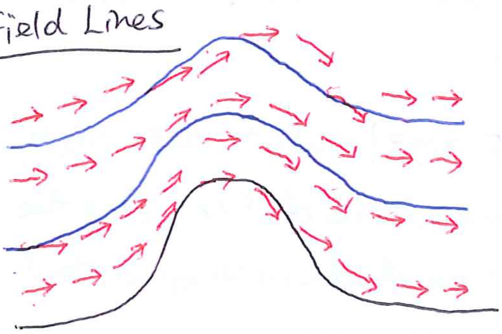
$\underline{\Omega} = \Omega \hat{k}$ is $\underline{v}(x,y,z) = \underline{\Omega} \times \underline{r} = -\Omega y \hat{\underline{i}} + \Omega x \hat{\underline{j}}$.

plane vector field



Velocity field of a rigid body rotating about the z-axis.

Field Lines



Velocity field and streamlines of wind blowing over a hill

vector field.

field lines

All previous graphical representations of vector fields suggest a pattern of motion through space or in the plane.

- The path will be a curve to which the field is tangent at every point. Such curves are called field lines, or trajectories for the given vector field.

The field lines of $\underline{F}$ only depend on the direction of the field, but NOT depend on the magnitude of $\underline{F}$.

Suppose the field line through some point has parametric equation $\underline{r} = \underline{r}(t)$, then its tangent vector $\frac{d\underline{r}}{dt}$ is parallel to $\underline{F}(\underline{r}(t))$ for all t, i.e. $\frac{d\underline{r}}{dt} = \lambda(t)\, \underline{F}(\underline{r}(t))$.

Let $\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$, $\underline{F} = \langle F_1, F_2, F_3 \rangle$, then
$$\boxed{\frac{dx}{F_1(x,y,z)} = \frac{dy}{F_2(x,y,z)} = \frac{dz}{F_3(x,y,z)}}$$

Differential equation for the field lines

e.g. Re-visit of Example ⑨

Find the field lines of the velocity field $\underline{v} = \Omega(-y\hat{\underline{i}} + x\hat{\underline{j}})$, assume $\Omega, y, x \neq 0$.

sol^n
$$\frac{dx}{-\Omega y} = \frac{dy}{\Omega x} \iff \frac{dx}{-y} = \frac{dy}{x}$$
$$\iff x\,dx = -y\,dy$$
$$\iff \frac{x^2}{2} = \frac{-y^2}{2} + C$$
$$\iff \boxed{x^2 + y^2 = C_0}$$

field lines are circles centered at the origin in the xy-plane

Regard $\underline{v}$ as a vector field in 3-space, the field lines are horizontal circles centered on the z-axis:
$$\boxed{x^2 + y^2 = C_0, \quad z = C_1}$$

example Find the field lines of the velocity field $\underline{F}(x,y,z) = xz\,\hat{\underline{i}} + yz\,\hat{\underline{j}} + x\,\hat{\underline{k}}$, where $x, y, z \neq 0$

Soln: $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{x}$

Consider $\frac{dx}{x} = \frac{dy}{y} \iff y = C_1 x$ (why?)

and $\frac{dx}{z} = dz \iff x = \frac{z^2}{2} + C_2$

Hence the field lines have parametric equations $\begin{cases} x = \frac{t^2}{2} + C_2 & \rightarrow \text{parabola} \\ y = C_1\left(\frac{t^2}{2} + C_2\right) = C_1\frac{t^2}{2} + C_3, & \rightarrow \text{parabola} \\ z = t \end{cases}$ $t \in \mathbb{R}$.

From previous plots, it seems that the vectors appear to be normal to the level curve from which they emanate. Then, we have a natural question: whether or not the vector field $\underline{F}(x,y)$ or $\underline{F}(x,y,z)$ is the gradient of some differentiable function f?

Ans: not always possible.

Conservative vector field

Def. A vector field $\underline{F}$ is conservative when there exists a differentiable function f such that $\underline{F} = \underline{\nabla} f$. We call f as the "potential function" of $\underline{F}$.

Remarks: potentials are not determined uniquely like anti-derivatives (can have constants).

$\underline{F}$ is conservative in a domain D if and only if $\underline{F} = \underline{\nabla} f$ at every point of D; the potential f cannot have any singular points in D.

e.g. ① Every inverse square field is conservative.

let $\underline{F}(x,y,z) = \frac{k}{\|\underline{r}\|^2}\hat{\underline{u}}$ and $f(x,y,z) = \frac{-k}{\sqrt{x^2+y^2+z^2}}$, where $\hat{\underline{u}} = \frac{\underline{r}}{\|\underline{r}\|}$

Consider $\underline{\nabla} f = \frac{kx}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{\underline{i}} + \frac{ky}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{\underline{j}} + \frac{kz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{\underline{k}}$

$= \frac{k}{x^2+y^2+z^2}\left(\frac{x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}}{\sqrt{x^2+y^2+z^2}}\right) = \frac{k}{\|\underline{r}\|^2}\frac{\underline{r}}{\|\underline{r}\|} = \frac{k}{\|\underline{r}\|^2}\hat{\underline{u}} = \underline{F}$

$\therefore \underline{F}$ is conservative.

② Show that the velocity field $\underline{v} = -\Omega y\hat{\underline{i}} + \Omega x\hat{\underline{j}}$ of rigid body rotation about the z-axis is not always conservative.

We try to seek for a potential function $f(x,y)$ for the vector field $\underline{v}(x,y)$.

i.e. $\frac{\partial f}{\partial x} = -\Omega y$ and $\frac{\partial f}{\partial y} = \Omega x \longrightarrow f(x,y) = \Omega xy + C_2(x)$

$f(x,y) = -\Omega xy + C_1(y)$

$\therefore -\Omega xy + C_1(y) = \Omega xy + C_2(x)$

$2\Omega xy = C_1(y) - C_2(x)$ for all $(x,y) \in \mathbb{R}^2$.

This is not possible for any choice of the single-variable functions C_1 & C_2 unless $\Omega = 0$.

The following theorem gives a necessary and sufficient condition for a vector field in the plane to be conservative.

Theorem Let M and N be functions with continuous first partial derivatives on an open disk R. The vector field $\underline{F}(x,y) = M\hat{\underline{i}} + N\hat{\underline{j}}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Proof: Let us prove the necessary condition at this moment.

Suppose there exists a potential function f such that $\underline{F}(x,y) = \underline{\nabla} f(x,y)$
$= M\hat{\underline{i}} + N\hat{\underline{j}}$

Then $f_x(x,y) = M$ and $f_y(x,y) = N \Rightarrow f_{xy}(x,y) = \frac{\partial M}{\partial y}$ and $f_{yx}(x,y) = \frac{\partial N}{\partial x}$

Since $f_{xy} = f_{yx}$ (by Clairaut's Theorem), then $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ $\forall (x,y)$ in R.

Example ① Find a potential function for $\underline{F}(x,y) = \underbrace{2xy}_{F_1(x,y)}\hat{\underline{i}} + \underbrace{(x^2-y)}_{F_2(x,y)}\hat{\underline{j}}$.

Sol^n From the theorem, $\underline{F}$ is conservative because

$$\frac{\partial F_1}{\partial y} = 2x \quad \text{and} \quad \frac{\partial F_2}{\partial x}(x^2-y) = 2x$$

If f is a function whose gradient is equal to $\underline{F}(x,y)$, then

$$\underline{\nabla} f(x,y) = \underbrace{2xy}_{f_x(x,y)}\hat{\underline{i}} + \underbrace{(x^2-y)}_{f_y(x,y)}\hat{\underline{j}}$$

Consider $f(x,y) = \int f_x(x,y)\,dx = \int 2xy\,dx = x^2y + g(y)$.

$f_y(x,y) = x^2 + g'(y) \Rightarrow$ i.e. $g'(y) = -y$

$\Rightarrow g(y) = -\frac{y^2}{2} + C$.

$\therefore f(x,y) = x^2y - \frac{y^2}{2} + C$ gives potential functions of $\underline{F}(x,y)$.

e.g. $f(x,y) = x^2y - \frac{y^2}{2} + 2020$ is a potential function.

Curl and Divergence

① Curl of a vector field $\underline{F}$ is a vector field that gives us at each point, an indication of how the field swirls in the vicinity of that point.

curl of $\underline{F}(x,y,z) = M\hat{\underline{i}} + N\hat{\underline{j}} + P\hat{\underline{k}}$ is $\underline{\nabla} \times \underline{F}(x,y,z)$

$$= \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{\underline{i}} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\hat{\underline{j}} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{\underline{k}}$$

- If curl $\underline{F} = \underline{0}$, then $\underline{F}$ is "irrotational"

Theorem (Test for Conservative Vector Field in Space).

Suppose that M, N and P are functions with continuous first partial derivatives in an open domain Q in space. The vector field $F(x,y,z) = M\hat{i} + N\hat{j} + P\hat{k}$ is conservative if and only if $\nabla \times \underline{F} = \underline{0}$, i.e. $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$, $\frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$ and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$.

Example Determine whether $\underline{F}(x,y,z) = (xy - \sin z)\hat{\underline{i}} + \left(\frac{1}{2}x^2 - \frac{e^y}{z}\right)\hat{\underline{j}} + \left(\frac{e^y}{z^2} - x\cos z\right)\hat{\underline{k}}$ is conservative in $D = \{(x,y,z) \in \mathbb{R}^3 \mid z \neq 0\}$.

Find a potential if it is.

Sol^n. Note that $\underline{F}$ is not defined when $z = 0$. Consider

$$M(x,y,z) = xy - \sin z$$
$$N(x,y,z) = \frac{1}{2}x^2 - \frac{e^y}{z}$$
$$P(x,y,z) = \frac{e^y}{z^2} - x\cos z$$

$$\frac{\partial P}{\partial y} = \frac{e^y}{z^2} = \frac{\partial N}{\partial z}, \quad \frac{\partial P}{\partial x} = -\cos z = \frac{\partial M}{\partial z} \quad \text{and} \quad \frac{\partial N}{\partial x} = x = \frac{\partial M}{\partial y}.$$

i.e. $\underline{F}$ is conservative in domains that do not intersect the xy-plane.

Consider
$$\begin{cases} \dfrac{\partial f}{\partial x} = xy - \sin z & -(1) \\ \dfrac{\partial f}{\partial y} = \dfrac{1}{2}x^2 - \dfrac{e^y}{z} & -(2) \\ \dfrac{\partial f}{\partial z} = \dfrac{e^y}{z^2} - x\cos z & -(3) \end{cases}$$

From (1), $f(x,y,z) = \frac{1}{2}x^2 y - x\sin z + C_1(y,z)$.

Diff. f w.r.t. y, we have
$$f_y(x,y,z) = \frac{1}{2}x^2 + C_{1y}(y,z)$$

Comparing with (2), $C_{1y}(y,z) = \frac{-e^y}{z}$
$$C_1(y,z) = \frac{-e^y}{z} + C_2(z).$$

$$\therefore f(x,y,z) = \frac{1}{2}x^2 y - x\sin z - \frac{e^y}{z} + C_2(z)$$

Diff. f w.r.t. z, we have
$$f_z(x,y,z) = -x\cos z + \frac{e^y}{z^2} + C_2'(z)$$

By comparing with (3), $C_2'(z) = 0$
$$\Rightarrow C_2(z) = C$$

$\therefore$ Potential: $f(x,y,z) = \frac{1}{2}x^2 y - x\sin z - \frac{e^y}{z} + C$

(C may have diff. values in two regions $z > 0$ and $z < 0$ whose union constitutes D).

(2) Divergence of a vector field is a scalar field that tells us, at each point, the extent to which the field "diverges" or "spreads away" from that point. Let $\underline{F}(x,y) = M\hat{\underline{i}} + N\hat{\underline{j}}$

$$\operatorname{div} \underline{F}(x,y) = \underline{\nabla} \cdot \underline{F}(x,y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$
(in plane)

Let $\underline{F}(x,y,z) = M\hat{\underline{i}} + N\hat{\underline{j}} + P\hat{\underline{k}}$,

then $\operatorname{div} \underline{F}(x,y,z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$
$= \underline{\nabla} \cdot \underline{F}(x,y,z)$
(in space).

If $\operatorname{div} \underline{F} = 0$, then $\underline{F}$ is "divergence free".

Example: $\underline{F}(\underline{r}) = y\hat{\underline{i}} - x\hat{\underline{j}}$

$\underline{\nabla} \cdot \underline{F} = 0 \rightarrow$ Amount flowing into $\underline{F}$ = amount flowing out of $\underline{F}$

$\underline{\nabla} \times \underline{F} = -2\hat{\underline{k}}$

NOTE: For vector fields that represent velocities of moving particles, divergence measures the rate of particle flow per unit volume at a point.

In hydrodynamics (study of fluid motion), a divergence free velocity field is called "an incompressible field".

In electricity and magnetism, a divergence free vector field is called "a solenoidal field".

Hint: Let $\underline{F}(x,y,z) = M\hat{i} + N\hat{j} + P\hat{k}$, M, N, P have continuous 2nd partial derivatives

Try to show: ① $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{F}) = 0$ (div of curl of a vector field $= 0$)

② $\underline{\nabla} \times (\underline{\nabla} f) = \underline{0}$ (curl of gradient of a scalar field $= \underline{0}$)

Def. Laplacian Operator (scalar differential operator)
$$\nabla^2 = \underline{\nabla} \cdot \underline{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

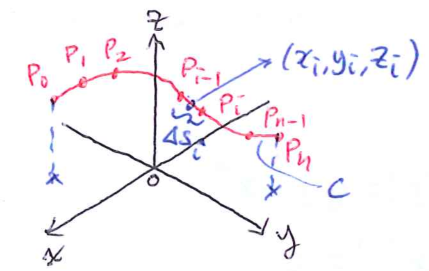
Line Integrals

Recall: $\int_a^b f(x)\,dx$: total amount of a quantity distributed along the x-axis between a and b in terms of line density, $f(x)$ at point x.

$\iint_R f(x,y)\,dA$ and $\iiint_D f(x,y,z)\,dV$: total amounts of quantities distributed over regions R in the plane and D in space in terms of areal or volume densities of these quantities.

It may happen that a quantity is distributed with specified line density along a curve in the plane or in 3-space, for which the integration is conducted over a piecewise smooth curve C.

Consider the mass of a wire of finite length, given by a curve C in space. The density (mass per unit length) of the wire at (x,y,z) is given by $f(x,y,z)$.



Total Mass of wire $\approx \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i$

Partition the curve C by $P_0, P_1, P_2, \ldots, P_n$.

Length of the ith subarc $= \Delta s_i$

Let $\|\Delta\|$ = length of the longest subarc, and let $\|\Delta\| \to 0$, limit of this sum approaches the mass of the wire (line integral of f along curve C)

$$\int_C f(x,y)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta s_i \quad \text{(plane)}$$

$$\int_C f(x,y,z)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i \quad \text{(space)}$$

provided that limit exists.

To evaluate a line integral over a plane curve C given by $\underline{r}(t) = x(t)\hat{\underline{i}} + y(t)\hat{\underline{j}}$,

$ds = \|\underline{r}'(t)\| dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

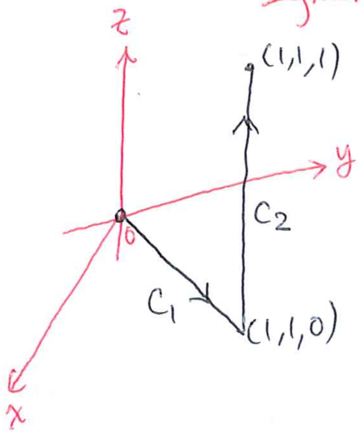
Hence, we have the following theorem

(a) Let $z = f(x,y)$ be continuous in a region containing a smooth curve C. If the plane curve C is traced out by a vector function $\underline{r}(t) = x(t)\hat{\underline{i}} + y(t)\hat{\underline{j}}$, where $a \le t \le b$, then $\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t)) \left\| \frac{d\underline{r}}{dt} \right\| dt = \int_a^b f(x(t),y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

(b) Let $w = f(x,y,z)$ be continuous in a region containing a smooth curve C. If the space curve C is traced out by a vector function $\underline{r}(t) = x(t)\hat{\underline{i}} + y(t)\hat{\underline{j}} + z(t)\hat{\underline{k}}$, where $a \le t \le b$, then $\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$.

Examples ① Consider the following path from origin to $(1,1,1)$, the union of line segments C_1 and C_2. Integrate $f(x,y,z) = x - 3y^2 + z$ over $C_1 \cup C_2$.

(from Thomas Calculus)



Sol^n We choose the simplest parametrizations for C_1 and C_2.

C_1: $\underline{r}_1(t) = \langle t, t, 0 \rangle$, where $0 \le t \le 1$

$$\left\| \frac{d\underline{r}_1}{dt} \right\| = \sqrt{\left(\frac{dt}{dt}\right)^2 + \left(\frac{dt}{dt}\right)^2 + \left(\frac{d}{dt}0\right)^2} = \sqrt{2}$$

C_2: $\underline{r}_2(t) = \langle 1, 1, t \rangle$, where $0 \le t \le 1$

$$\left\| \frac{d\underline{r}_2}{dt} \right\| = \sqrt{\left(\frac{d}{dt}1\right)^2 + \left(\frac{d}{dt}1\right)^2 + \left(\frac{d}{dt}t\right)^2} = 1$$

Then $\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds = \int_0^1 f(t,t,0)\cdot\sqrt{2}\,dt + \int_0^1 f(1,1,t)\cdot 1\,dt$

$$= \int_0^1 (t - 3t^2)\sqrt{2}\,dt + \int_0^1 (t-2)\,dt$$

$$= \sqrt{2}\left[\frac{t^2}{2} - t^3\right]_0^1 + \left[\frac{t^2}{2} - 2t\right]_0^1 = \frac{-\sqrt{2}}{2} - \frac{3}{2}$$

---

② Find the centroid of the circular helix given by $\underline{r}(t) = a\cos t\,\hat{\underline{i}} + a\sin t\,\hat{\underline{j}} + bt\,\hat{\underline{k}}$, where $0 \le t \le 2\pi$, a, b are constants.

Sol^n

Consider $ds = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}\,dt = \sqrt{a^2+b^2}\,dt$.

Length of circular helix $L = \int_0^{2\pi} \sqrt{a^2+b^2}\,dt = 2\pi\sqrt{a^2+b^2}$

Moment about $z=0$ is $M_{z=0} = \int_C z\,ds = \int_0^{2\pi} bt\sqrt{a^2+b^2}\,dt = 2\pi^2 b\sqrt{a^2+b^2}$

$\Rightarrow$ z-component of its centroid $= \frac{M_{z=0}}{L} = b\pi$.

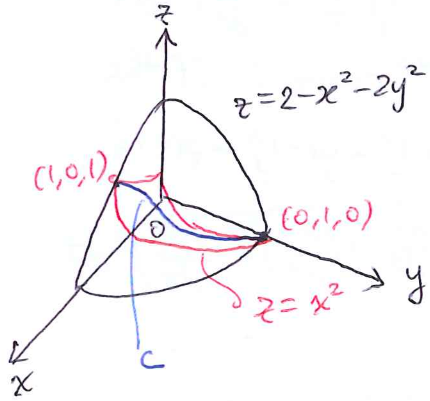
Moment about $x=0$ and $y=0$ are $M_{x=0} = \int_C x\,ds = a\sqrt{a^2+b^2}\int_0^{2\pi} \cos t\,dt = 0$ and.

$$M_{y=0} = \int_C y\,ds = a\sqrt{a^2+b^2}\int_0^{2\pi}\sin t\,dt = 0$$

$\therefore$ The centroid is $\underline{\underline{(0,0,b\pi)}}$

---

(3) (Adams and Essex § (15.3)) Find the mass of a wire lying along the first octant part of the curve of intersection of $z=2-x^2-2y^2$ and $z=x^2$ between $(0,1,0)$ and $(1,0,1)$, given that the density of the wire at position (x,y,z) is $\delta(x,y,z)=xy$.



Denote C as the first octant part of the curve of intersection.

The curve C lies on the cylinder $z=x^2$, where $x\in[0,1]$.

$\Rightarrow$ let $x=t$, $z=t^2$.

$\therefore\ 2y^2 = 2-x^2-z = 2-t^2-t^2 \Rightarrow y^2 = 1-t^2$

$y=\sqrt{1-t^2}$ ($\because$ C lies in the first octant).

$\therefore\ x(t)=t,\ y(t)=\sqrt{1-t^2},\ z(t)=t^2$, where $t\in[0,1]$

$$\therefore\ ds = \sqrt{1+\frac{t^2}{1-t^2}+4t^2}\,dt = \frac{\sqrt{1+4t^2-4t^4}}{\sqrt{1-t^2}}\,dt$$

Hence, mass of wire $= \int_C xy\,ds = \int_0^1 t\sqrt{1-t^2}\,\frac{\sqrt{1+4t^2-4t^4}}{\sqrt{1-t^2}}\,dt$

$$= \int_0^1 t\sqrt{1+4t^2-4t^4}\,dt$$

We let $u=t^2$, $du=2t\,dt$ $= \int_0^1 \frac{1}{2}\sqrt{1+4u-4u^2}\,du$

$$= \frac{1}{2}\int_0^1\sqrt{2-(2u-1)^2}\,du$$

$\boxed{\text{sub } v=\sqrt{2}\sin w}$

let $v=2u-1$, $dv=2du$ $= \frac{1}{4}\int_{-1}^1\sqrt{2-v^2}\,dv = \frac{1}{2}\int_0^1\sqrt{2-v^2}\,dv$ (even)

$$= \frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right) = \underline{\underline{\frac{\pi+2}{8}}}$$

(check!)

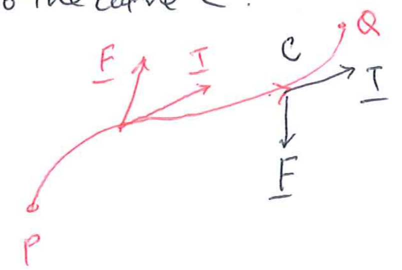
Line integrals of vector fields

Recall that work done by a constant force of magnitude f in moving an object from a to b along the x-axis $= W = f(b-a)$

If f is variable, then $W=\int_a^b f(x)\,dx$

Now, imagine we have a force field $F(x,y,z) = \underline{F}(\underline{r}) = M(x,y,z)\hat{\underline{i}} + N(x,y,z)\hat{\underline{j}} + P(x,y,z)\hat{\underline{k}}$, the work done by this force field in moving an object along a smooth curve C (in space) given by $\underline{r}(t) = \langle x(t), y(t), z(t)\rangle$, where $t\in[a,b]$ is defined as follows:

$\int_C \underline{F}(x,y,z)\cdot \underline{T}(x,y,z)\,ds$, where $\underline{T}(x,y,z)=\dfrac{\underline{r}'(t)}{\|\underline{r}'(t)\|}$ is the unit tangent vector to the curve C.



$W=\int_C \underline{F}\cdot d\underline{r}=\int_a^b \underline{F}\cdot \underline{r}'(t)\,dt$ (scalar).

This line integral changes sign if the orientation of C is reversed, and is independent of parametrization of C.

We have 6 possible ways to represent the work done integrals.

① $\int_C \underline{F}(x,y,z)\cdot\underline{T}(x,y,z)\,ds$ ② $\int_a^b \underline{F}(x(t),y(t),z(t))\cdot\dfrac{\underline{r}'(t)}{\|\underline{r}'(t)\|}\|\underline{r}'(t)\|\,dt$

③ $\int_a^b \underline{F}(x(t),y(t),z(t))\cdot\underline{r}'(t)\,dt$ ④ $\int_C \underline{F}(x,y,z)\cdot d\underline{r}$

⑤ $\int_a^b \left[M(x(t),y(t),z(t))\dfrac{dx}{dt}+N(x(t),y(t),z(t))\dfrac{dy}{dt}+P(x(t),y(t),z(t))\dfrac{dz}{dt}\right]dt$

⑥ $\int_C M\,dx+N\,dy+P\,dz$

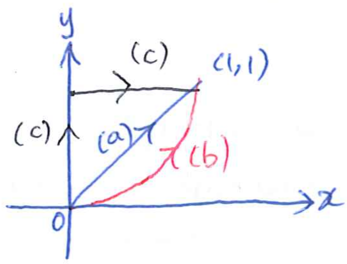
Remarks: If C is a closed curve, $\oint_C \underline{F}\cdot d\underline{r}$ denotes the circulation of $\underline{F}$ around the closed curve C.

Examples ① Let $\underline{F}(x,y)=y^2\hat{\underline{i}}+2xy\hat{\underline{j}}$.

Evaluate $\int_C \underline{F}\cdot d\underline{r}$ from $(0,0)$ to $(1,1)$ along (a) the straight line $y=x$

(b) the curve $y=x^2$

(c) the piecewise smooth path that consists of the line segments from $(0,0)$ to $(0,1)$, then from $(0,1)$ to $(1,1)$.



(a) Parametrize the straight path (a) as $\underline{r}(t)=t\hat{\underline{i}}+t\hat{\underline{j}}$ $(0\le t\le 1)$

$\therefore \underline{F}\cdot d\underline{r}=(t^2\hat{\underline{i}}+2t^2\hat{\underline{j}})\cdot(\hat{\underline{i}}+\hat{\underline{j}})\,dt=3t^2\,dt$

$\int_C \underline{F}\cdot d\underline{r}=\int_0^1 3t^2\,dt=t^3\Big|_0^1=\underline{\underline{1}}$

(b) Parametrize the parabola as $\underline{r}(t)=t\hat{\underline{i}}+t^2\hat{\underline{j}}$, where $0\le t\le 1$.

$\therefore \underline{F}\cdot d\underline{r}=(t^4\hat{\underline{i}}+2t^3\hat{\underline{j}})\cdot(\hat{\underline{i}}+2t\hat{\underline{j}})\,dt=5t^4\,dt$

$\int_C \underline{F}\cdot d\underline{r}=\int_0^1 5t^4\,dt=t^5\Big|_0^1=\underline{\underline{1}}$

(c) For the vertical segment, use y as the parameter $(x=0, dx=0)$

For the horizontal segment, use x as the parameter $(y=1, dy=0)$

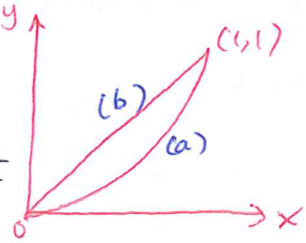
$\int_C \underline{F}\cdot d\underline{r}=\int_C y^2\,dx+2xy\,dy=\int_0^1 0\,dy+\int_0^1 1\,dx=\underline{\underline{1}}$

same value along all 3 paths ($\because$ $\underline{F}(x,y)$ is conservative) on a simply connected domain

② Find $\int_C \underline{F} \cdot d\underline{r}$, where $\underline{F}(x,y) = e^{x-1}\hat{\underline{i}} + xy\hat{\underline{j}}$, and C is given by

(a) $\underline{r}(t) = t^2\hat{\underline{i}} + t^3\hat{\underline{j}}$, where $0 \le t \le 1$

(b) $\underline{r}(t) = t\hat{\underline{i}} + t\hat{\underline{j}}$, where $0 \le t \le 1$



Soln (a) $\int_C \underline{F} \cdot d\underline{r} = \int_0^1 (e^{t^2-1}\hat{\underline{i}} + t^5\hat{\underline{j}}) \cdot (2t\hat{\underline{i}} + 3t^2\hat{\underline{j}})\,dt$

$= \int_0^1 (2te^{t^2-1} + 3t^7)\,dt$

$= \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - \frac{1}{e}$

(b) $\int_C \underline{F} \cdot d\underline{r} = \int_0^1 (e^{t-1}\hat{\underline{i}} + t^2\hat{\underline{j}}) \cdot (\hat{\underline{i}} + \hat{\underline{j}})\,dt = \int_0^1 (e^{t-1} + t^2)\,dt$

$= \left[e^{t-1} + \frac{1}{3}t^3\right]_0^1 = \frac{4}{3} - \frac{1}{e}$

The line integral depends on the path from $(0,0)$ to $(1,1)$ along which the integral is evaluated. ($\because$ The vector field is NOT conservative)

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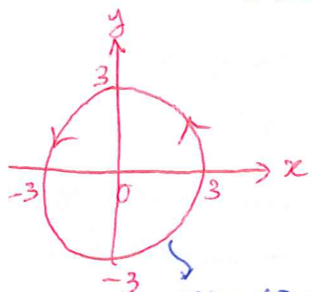
③ Let C be the circle of radius 3 given by $\underline{r}(t) = 3\cos t\,\hat{\underline{i}} + 3\sin t\,\hat{\underline{j}}$, where $0 \le t \le 2\pi$.
Evaluate $\int_C y^3\,dx + (x^3 + 3xy^2)\,dy$.



$\underline{r}(t) = \langle 3\cos t, 3\sin t \rangle$

$x(t) = 3\cos t$, $y(t) = 3\sin t$

$dx = -3\sin t\,dt$, $dy = 3\cos t\,dt$

We use the formula

$\int_C M\,dx + N\,dy$, where $M(x,y) = y^3$ and $N(x,y) = x^3 + 3xy^2$

$= \int_C (27\sin^3 t)(-3\sin t)\,dt + (27\cos^3 t + 81(\cos t \sin^2 t))(3\cos t)\,dt$

$= 81\int_0^{2\pi} (\cos^4 t - \sin^4 t + 3\sin^2 t\cos^2 t)\,dt$

$= 81\int_0^{2\pi} (\cos^2 t - \sin^2 t + \frac{3}{4}\sin^2 2t)\,dt$

$= 81\int_0^{2\pi} \left[\cos 2t + \frac{3}{4}\left(\frac{1-\cos 4t}{2}\right)\right]dt$

$= 81\left[\frac{\sin 2t}{2} + \frac{3}{8}t - \frac{3}{32}\sin 4t\right]_0^{2\pi} = \frac{243}{4}\pi.$

Remarks: The orientation of C affects the value of the line integral.
Denote $-C$ as the curve with opposite orientation as C, then

$\int_{-C} M\,dx + N\,dy = -\int_C M\,dx + N\,dy.$