Week 10 Lecture Notes
Multivariable Calculus

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Topics : Conservative Vector Field & Independence of Path
Green's Theorem
Surface Integrals

References : Ch. 15.2, 15.4 – 15.5, 16.3 of Adams and Essex Thomas Calculus
Ch. 15.5 – 15.6 of Larson and Edwards CUHK MATH 2020 Lecture Notes

Conservative Vector Fields

Def. (Re-visit) A vector field $\underline{F}$ is conservative if there exists a differentiable function f such that $\underline{F} = \underline{\nabla} f$. f is called the "potential function" of $\underline{F}$.

e.g. A gravitational potential is a scalar function whose gradient field is a gravitational field. An electric potential is a scalar function whose gradient field is an electric field.

Once we can find a potential function f for $\underline{F}$, then all line integrals in the domain of $\underline{F}$ over any path between A and B are evaluated:

$$\int_A^B \underline{F} \cdot d\underline{r} = \int_A^B \underline{\nabla} f \cdot d\underline{r} = f(B) - f(A)$$

$f(B)$: end-pt, $f(A)$: starting pt.

Analogy: $\int_a^b f'(x)\,dx = f(b) - f(a)$ (Fund. thm of Calculus)

i.e. Integral only depends on the end points

Thm (Fund. Thm of Line Integrals)

Let C be a piecewise smooth curve joining the point A to the point B in the plane (for $\mathbb{R}^2$) or in space (for $\mathbb{R}^3$), and C can be parametrized by $\underline{r}(t)$. Let f be a differentiable function with continuous gradient vector $\underline{F} = \underline{\nabla} f$ on a simply connected domain D that contains C. Then $\int_C \underline{F} \cdot d\underline{r} = f(B) - f(A)$.

simply connected : Every loop in D can be contracted to a point in D, without ever leaving D.

e.g. $\mathbb{R}^2$ and $\mathbb{R}^3$ are simply connected domains

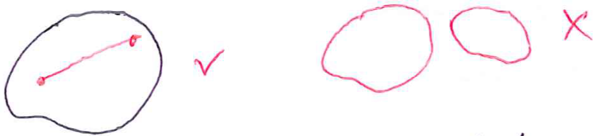
e.g. The plane with a disk removed is a subset of $\mathbb{R}^2$ that is not simply connected.

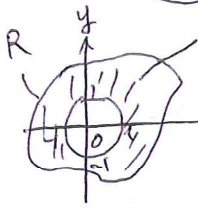
e.g. A loop in the plane that goes around the disk cannot be contracted to a point without going into the "hole" left by the removed disk.

e.g. Removing a line from $\mathbb{R}^3$ is <u>not</u> simply-connected.

<u>Simple</u> – not intersect itself anywhere between end points

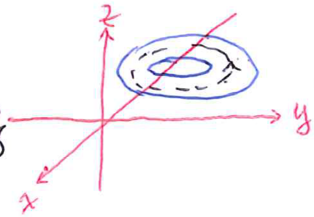
<u>connected</u> For open sets, any two points in a connected domain D can be joined by a smooth curve that lies in the region.



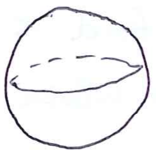


D: <u>not</u> simply-connected

$D = R \setminus \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$



<u>not</u> simply connected



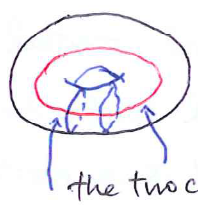
$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$

is simply connected.

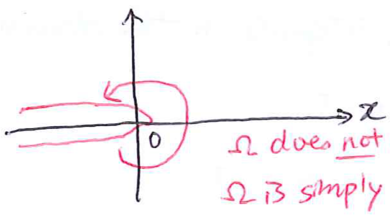
$S^2 \subset \mathbb{R}^3$

Torus $\mathbb{T}^2 \subset \mathbb{R}^3$

<u>not</u> simply connected.



the two closed curves <u>cannot</u> be contracted to a point on the torus.



$\Omega = \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \le 0\}$

Ω does <u>not</u> include the negative x-axis & origin.

Ω is simply connected.

<u>Example</u> (based on Thm.) Suppose $\underline{F} = \underline{\nabla} f$ is the gradient of $f(x,y,z) = -\dfrac{1}{x^2+y^2+z^2}$.

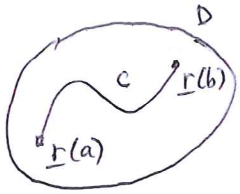
Find the work done by $\underline{F}$ in moving an object along a smooth curve C joining $(2,0,0)$ to $(0,0,4)$ that does <u>NOT</u> pass through the origin.

$$\int_C \underline{F} \cdot d\underline{r} = f(0,0,4) - f(2,0,0) = \frac{-1}{16} - \left(\frac{-1}{4}\right) = \underline{\underline{\frac{3}{16}}}$$

<u>Proof of Thm.</u> Suppose A and B are two points in simply connected domain D, C is $\underline{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$, $a \le t \le b$ is a smooth curve in D joining A to B,

then $\int_C \underline{F} \cdot d\underline{r} = \int_A^B \underline{\nabla} f \cdot d\underline{r} = \int_a^b \underline{\nabla} f(\underline{r}(t)) \cdot \underline{r}'(t)\,dt = \int_a^b \frac{d}{dt} f(\underline{r}(t))\,dt$

$= f(\underline{r}(b)) - f(\underline{r}(a)) = f(B) - f(A)$.



$$\left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt = \frac{df}{dt}\,dt$$

Thm: (Path independence of Conservative Fields).

If $\underline{F}$ is continuous on an open connected region, then the line integral

$\int_C \underline{F}\cdot d\underline{r}$ is path-independent if and only if $\underline{F}$ is conservative.

Proof: If $\underline{F}$ is conservative, by Fund. Thm. of Line Integrals, the line integral is path independent.

Now, let's prove the converse for a plane region R.

Let $\underline{F} = M\hat{i} + N\hat{j}$, (x_0, y_0) be a fixed point in R. For any (x,y) in R, choose a piecewise smooth curve C running from (x_0,y_0) to (x,y), define f as follows:

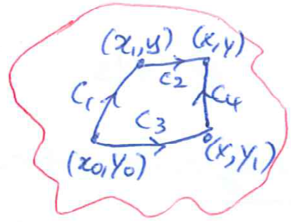
$$f(x,y) = \int_C \underline{F}\cdot d\underline{r} = \int_C M\,dx + N\,dy$$

(which exists since R is connected)

Consider two different paths between (x_0,y_0) and (x,y).

Path 1: Choose $(x_1, y) \in R$, where $x \neq x_1$ ((x_1,y) exists because R is open)

Then choose C_1, C_2 as shown.



$$f(x,y) = \int_C M\,dx + N\,dy$$

$$= \underbrace{\int_{C_1} M\,dx + N\,dy}_{\text{not depend on } x} + \underbrace{\int_{C_2} M\,dx + N\,dy}_{dy=0}$$

$$\therefore f(x,y) = g(y) + \int_{C_2} M\,dx$$

$$\Rightarrow f_x(x,y) = M.$$

Path 2: Choose $(x, y_1) \in R$, where $y \neq y_1$. Repeat above arguments, $f_y(x,y) = N$.

$$\Rightarrow \nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j} = M\hat{i} + N\hat{j} = \underline{F}(x,y)$$

i.e. $\underline{F}$ is conservative.

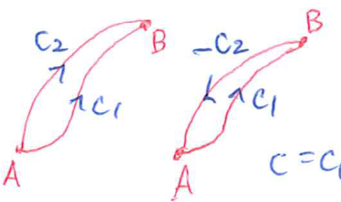
Thm: (Loop Property of Conservative Fields).

The following statements are equivalent

① $\oint_C \underline{F}\cdot d\underline{r} = 0$ around every loop (closed curve C) in D.

② $\underline{F}$ is conservative on D.

① ⇒ ②) We wish to show that for any two points A and B in D, the integral of $\underline{F}\cdot d\underline{r}$ has the same value over any two paths C_1 and C_2 from A to B.

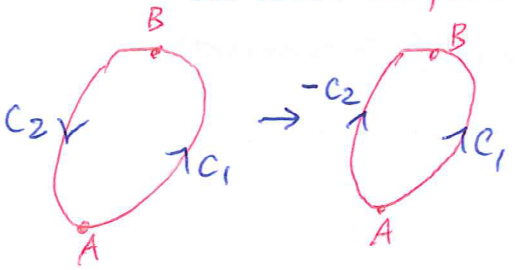


$$\int_{C_1} \underline{F}\cdot d\underline{r} - \int_{C_2} \underline{F}\cdot d\underline{r} = \int_{C_1} \underline{F}\cdot d\underline{r} + \int_{-C_2} \underline{F}\cdot d\underline{r} = \int_C \underline{F}\cdot d\underline{r} = 0$$

$$\therefore \underbrace{\int_{C_1} \underline{F}\cdot d\underline{r} = \int_{C_2} \underline{F}\cdot d\underline{r}}_{\text{Path independence.}}$$ (true for all curves C_1 and C_2)

② ⇒ ①) We wish to show that $\oint_C \underline{F}\cdot d\underline{r} = 0$ over any closed loop C.

We select two points (distinct) A and B, and break C into 2 pieces: C_1 and C_2 respectively.



$$\oint_C \underline{F}\cdot d\underline{r} = \int_{C_1}\underline{F}\cdot d\underline{r} + \int_{C_2}\underline{F}\cdot d\underline{r} = \int_A^B \underline{F}\cdot d\underline{r} - \int_A^B \underline{F}\cdot d\underline{r} = 0$$

Summary: A continuous vector field $\underline{F}$ defined in a simply connected domain D is conservative if and only if, it has one of the following properties:

① $\underline{F}$ is the gradient of a scalar function, $\underline{F}(\underline{r}) = \underline{\nabla} f(\underline{r})$.

② Its line integral along any regular curve extending from a point A to a point B is independent of the path.

$$\int_{C_1}\underline{F}\cdot d\underline{r} = \int_{C_2}\underline{F}\cdot d\underline{r} = \int_{C_3}\underline{F}\cdot d\underline{r} = \cdots\cdots$$

③ Its line integral around any regular closed loop is zero, i.e. $\oint_C \underline{F}\cdot d\underline{r} = 0$.

Necessary condition for existence of a potential function f is $\underline{\nabla}\times\underline{F} = \underline{0}$.

Even if $\underline{\nabla}\times\underline{F} = \underline{0}$ at every point in D, there may still be no potential function f.

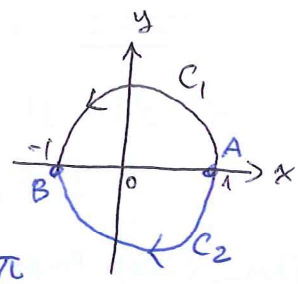
We need an additional restriction on D: D must be open and simply-connected.

Example: Let $\underline{F}(x,y) = \dfrac{-y\hat{\underline{i}} + x\hat{\underline{j}}}{x^2+y^2} = f(x,y)\hat{\underline{i}} + g(x,y)\hat{\underline{j}}$.

$\dfrac{\partial f}{\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2} = \dfrac{\partial g}{\partial x}$, i.e. curl test is satisfied.

However, let C_1: $x=\cos\theta$, $y=\sin\theta$, $\theta\in[0,\pi]$

C_2: $x=\cos\theta$, $y=\sin\theta$, θ runs from 2π to π



$$\int_{C_1}\underline{F}\cdot d\underline{r} = \int_0^\pi \langle -\sin\theta, \cos\theta\rangle\cdot\langle -\sin\theta, \cos\theta\rangle d\theta = \int_0^\pi d\theta = \pi$$

But $\int_{C_2}\underline{F}\cdot d\underline{r} = \int_{2\pi}^{\pi} d\theta = -\pi \neq \int_{C_1}\underline{F}\cdot d\underline{r}$

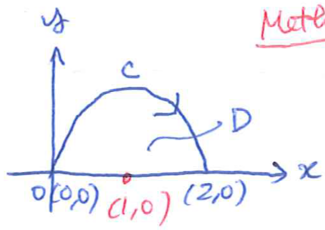
$\therefore \int_C \underline{F}\cdot d\underline{r}$ is dependent of path.

Remarks: If $\tilde{D} = \{(x,y)\in\mathbb{R}^2 \mid x>0\}$, then $\tilde{D}$ is an open simply-connected region (a subset of $\mathbb{R}^2$), and the necessary condition $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$ is satisfied on $\tilde{D}$.

Then $\exists\, f(x,y)$ such that $\underline{F} = \underline{\nabla} f$.

Take Home Exercise: Find such f in $\tilde{D}$.

Example: Given a vector field $\underline{F}(x,y) = \langle y^3+1, 3xy^2+1 \rangle$. Evaluate $\int_C \underline{F}\cdot d\underline{r}$, where C is the upper semi-circular path, traversed in clockwise direction, with centre at $(1,0)$ and radius 1, starting from $(0,0)$ and ending at $(2,0)$.



Method 1

Step 1: Check whether the vector field $\underline{F}$ is conservative.

$$\nabla \times \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3+1 & 3xy^2+1 & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy^2+1 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y^3+1 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^3+1 & 3xy^2+1 \end{vmatrix} \hat{k}$$

$$= 0\hat{i} + 0\hat{j} + (3y^2 - 3y^2)\hat{k}$$

$$= \underline{0}$$

Since the domain D (enclosed by C) is simply connected,

$\underline{F}$ is conservative, i.e. $\underline{F}(x,y)$ has a potential function over the entire 2-space.

Step 2: Seeking potential function f such that $\underline{F} = \nabla f$.

$$\frac{\partial f}{\partial x} = y^3+1, \quad \frac{\partial f}{\partial y} = 3xy^2+1$$

$$f(x,y) = (y^3+1)x + g(y)$$

Diff. $f(x,y)$ w.r.t. y: $f_y = 3xy^2 + \frac{dg}{dy} = 3xy^2+1 \Rightarrow \frac{dg}{dy} = 1 \Rightarrow g = y + C$

$$\therefore f(x,y) = xy^3 + x + y + C$$

Step 3: By Fund. Thm of Line Integral, $\int_C \underline{F}\cdot d\underline{r} = f(2,0) - f(0,0) = \underline{\underline{2}}$.

Method 2 As the vector field $\underline{F}$ is conservative (by Step 1 of Method 1),

$$\int_C \underline{F}\cdot d\underline{r} = \int_{C'} \underline{F}\cdot d\underline{r}, \text{ where } C' \text{ is any smooth curve from } (0,0) \text{ to } (2,0)$$

Hence, we may replace the semicircular path by a line segment from $(0,0)$ to $(2,0)$. For such line segment, $\underline{r}(t) = t\hat{i} + 0\hat{j}$, where $0 \le t \le 2$.

$$\therefore \int_C \underline{F}\cdot d\underline{r} = \int_{C'} \underline{F}\cdot d\underline{r} = \int_0^2 \underline{F}(t,0)\cdot \frac{d\underline{r}}{dt}\,dt = \int_0^2 \langle 0^3+1, 3t(0)^2+1\rangle \cdot \langle 1,0\rangle\,dt$$

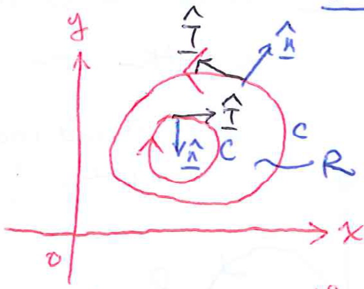
$$= \int_0^2 \langle 1,1\rangle\cdot\langle 1,0\rangle\,dt = \int_0^2 1\,dt = \underline{\underline{2}}$$

Green's Theorem

Green's Theorem is a 2D version of the Fund. Thm. of Calculus that expresses the double integral of a certain kind of derivative of a 2D vector field $\underline{F}(x,y)$ – the k-component of $\nabla \times \underline{F}$, over a region R in the xy-plane as a line integral of the tangential component of $\underline{F}$ around the curve C (the oriented boundary of R).

i.e. $\iint_R (\nabla \times \underline{F}) \cdot \hat{\underline{k}}\, dA = \oint_C \underline{F} \cdot d\underline{r}$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \oint_C F_1(x,y)\,dx + F_2(x,y)\,dy .$$



Planar region with positively oriented boundary.

holes of R: oriented clockwise boundary.

C : oriented boundary of R ⇝ surface with orientation provided by $\hat{\underline{n}} = \hat{\underline{k}}$.

$\hat{\underline{n}} = \hat{\underline{T}} \times \hat{\underline{k}}$ | $\hat{\underline{T}}$: unit tangent vector

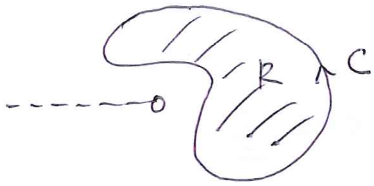
$\hat{\underline{n}}$: unit exterior normal (pointing out of R) on C

Green's Theorem

Let R be a simply connected region in the xy-plane whose boundary C, consists of one or more piecewise smooth, simple closed curves that are positively oriented w.r.t. R. If $\underline{F} = F_1(x,y)\hat{\underline{i}} + F_2(x,y)\hat{\underline{j}}$ is a smooth vector field on R, then

$$\oint_C F_1(x,y)\,dx + F_2(x,y)\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

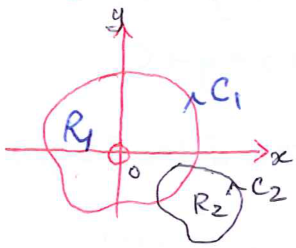
NOTE $\Omega = \mathbb{R}^2 \setminus \{x \le 0\}$



Green's Theorem will apply, since

$R \subset \Omega$

$\Omega' = \mathbb{R}^2 \setminus \{(0,0)\}$



$(0,0) \in R_1$

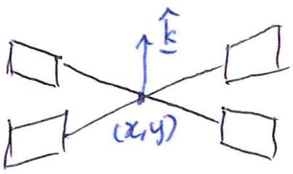
$(0,0) \notin R_2$

Green's Theorem applies for R_2, since $R_2 \subset \Omega'$, but it does not apply for R_1, since $R_1 \not\subset \Omega'$.

Physical interpretation of $(\nabla \times \underline{F}) \cdot \hat{\underline{k}}$.

In the flow of an incompressible fluid over planar region, $(\nabla \times \underline{F}) \cdot \hat{\underline{k}}$ measures the rate of the fluid's rotation at (x,y).

The k-component of $\nabla \times \underline{F}$ at (x,y) measures how fast and in what direction a small paddle wheel spins if it is put into water at (x,y) with an axis ⊥ plane, parallel to $\hat{\underline{k}}$.



- If $(\nabla \times \underline{F}) \cdot \hat{\underline{k}} > 0$, then paddle rotates in counterclockwise direction.
- If $(\nabla \times \underline{F}) \cdot \hat{\underline{k}} < 0$, then paddle rotates in clockwise direction.

$$\underbrace{\oint_C \underline{F} \cdot \hat{\underline{T}}\, ds}_{\text{circulation integral of } \underline{F} \text{ along } C} = \iint_R (\nabla \times \underline{F}) \cdot \hat{\underline{k}}\, dA$$

<u>Examples</u> ① Evaluate $\oint_C (x-y^3)dx + (y^3+x^3)dy$, where C is the positively oriented boundary of the quarter-disk R: $0 \le x^2+y^2 \le a^2$, $x \ge 0$, $y \ge 0$.

$$\text{Integral} = \iint_R \left(\frac{\partial}{\partial x}(y^3+x^3) - \frac{\partial}{\partial y}(x-y^3)\right) dA = 3\iint_R (x^2+y^2)\,dA$$

simply-connected region.

$$= 3\int_0^{\frac{\pi}{2}}\int_0^a r^3\,dr\,d\theta = \underline{\underline{\frac{3}{8}\pi a^4}}$$

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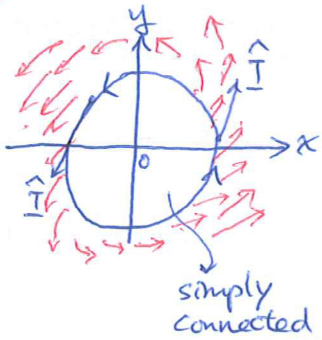
② Verify Green's Theorem for the vector field $\underline{F}(x,y) = \underbrace{(x-y)}_{F_1(x,y)}\hat{i} + \underbrace{x}_{F_2(x,y)}\hat{j}$, with region R bounded by the unit circle C.

<u>Sol^n</u> Let $C: \underline{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, where $t \in [0, 2\pi]$.



Hence $F_1 = \cos t - \sin t$, $dx = d(\cos t) = -\sin t\,dt$

$F_2 = \cos t$, $dy = d(\sin t) = \cos t\,dt$

$$\frac{\partial F_1}{\partial x} = 1, \quad \frac{\partial F_1}{\partial y} = -1, \quad \frac{\partial F_2}{\partial x} = 1, \quad \frac{\partial F_2}{\partial y} = 0$$

Consider $\oint_C F_1\,dx + F_2\,dy = \int_0^{2\pi} (\cos t - \sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt)$

$$= \int_0^{2\pi} (-\sin t\cos t + 1)\,dt = 2\pi$$

and $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R (1-(-1))\,dx\,dy = 2(\text{area of unit-circle})$

$$= 2\pi.$$

$\therefore \oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$ is verified for $\underline{F}(x,y)$.

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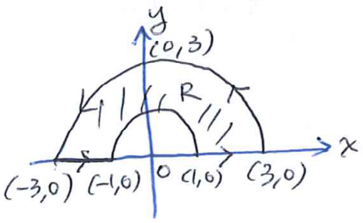
③ Evaluate $\int_C (\arctan x + y^2)dx + (e^y - x^2)dy$, where C is the path that encloses the annular region.



Note that C is piecewise smooth, R is <u>simply-connected</u>

In polar coordinates, $R = \{(r,\theta) \mid 1 \le r \le 3,\ 0 \le \theta \le \pi\}$.

Let $F_1(x,y) = \arctan x + y^2$, $F_2(x,y) = e^y - x^2$.

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -2x - 2y = -2(r\cos\theta + r\sin\theta)$$

By <u>Green's Theorem</u>, $\int_C (\arctan x + y^2)dx + (e^y - x^2)dy$

$$= \iint_R -2(x+y)\,dA = \int_0^{\pi}\int_1^3 -2r(\cos\theta + \sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi} -\frac{52}{3}(\cos\theta + \sin\theta)\,d\theta = \frac{-52}{3}\Big[\sin\theta - \cos\theta\Big]_0^{\pi}$$

$$= \frac{-104}{3}.$$

<u>Remarks</u>: Green's Theorem are normally used to evaluate line integrals as double integrals. We can also evaluate double integrals as line integrals, e.g. when $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$. Then $\int_C F_1\,dx + F_2\,dy = \text{area of region } R$.

Extension: Alternative Forms of Green's Theorem

① For a vector field $\underline{F}$ in the plane, we can write $\underline{F}(x,y,z) = F_1\hat{\imath} + F_2\hat{\jmath} + 0\hat{k}$.

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = -\frac{\partial F_2}{\partial z}\hat{\imath} + \frac{\partial F_1}{\partial z}\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}.$$

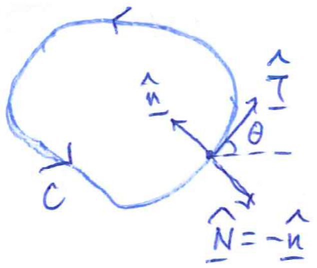
$$(\underline{\nabla} \times \underline{F}) \cdot \hat{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

With appropriate conditions and assumptions on $\underline{F}$, C and R, we may write Green's Theorem in vector form: $\int_C \underline{F} \cdot d\underline{r} = \iint_R (\underline{\nabla} \times \underline{F}) \cdot \hat{k}\, dA$

Extension of vector form of Green's Theorem to surfaces in spaces produces Stokes's Thm.

② Assume same conditions for $\underline{F}$, C, R, using the arc length parameter s for C,

$\underline{r}(s) = x(s)\hat{\imath} + y(s)\hat{\jmath}$.



$\underline{r}'(s) = \hat{T} = x'(s)\hat{\imath} + y'(s)\hat{\jmath}$.

↳ unit tangent vector $\hat{T}$ to curve C

Outward normal vector $\hat{N} = y'(s)\hat{\imath} - x'(s)\hat{\jmath}$.

$\hat{T} = \langle \cos\theta, \sin\theta \rangle$

$\hat{n} = \langle \cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2}) \rangle$

$= \langle -\sin\theta, \cos\theta \rangle$

$\hat{N} = \langle \sin\theta, -\cos\theta \rangle$

For $\underline{F}(x,y) = F_1\hat{\imath} + F_2\hat{\jmath}$, $\int_C \underline{F} \cdot \hat{N}\, ds = \int_a^b \langle F_1, F_2 \rangle \cdot \langle y'(s), -x'(s) \rangle\, ds$

$$= \int_a^b \left(F_1 \frac{dy}{ds} - F_2 \frac{dx}{ds}\right) ds$$

$$= \int_C F_1\, dy - F_2\, dx$$

$$= \iint_R \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dA = \iint_R \underline{\nabla} \cdot \underline{F}\, dA.$$

$\therefore \boxed{\int_C \underline{F} \cdot \hat{N}\, ds = \iint_R \underline{\nabla} \cdot \underline{F}\, dA}$ Divergence Theorem in the plane.