Multivariable Calculus

Week 8 NOTES

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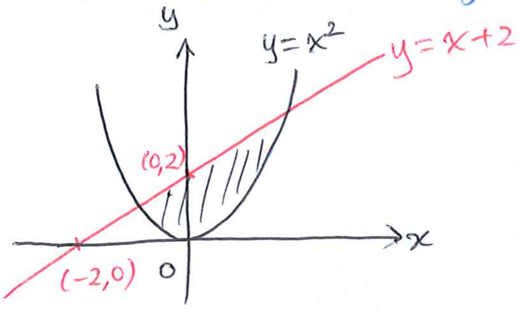
Reference Chapters: Adams and Essex Ch. 14.4, 14.5, 14.6

Larson and Edwards Ch. 14.3, 14.6, 14.7, 14.8

Review of changing limits in integration

Find the area of the region R enclosed by $y=x^2$ and $y=x+2$.



Solving $\begin{cases} y=x^2 \\ y=x+2 \end{cases}$,

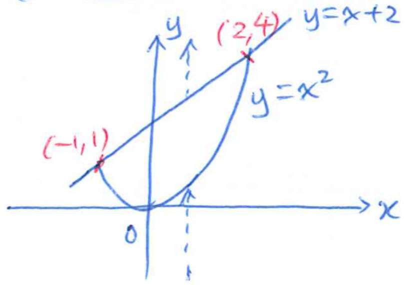
we have $x^2-x-2=0$

$(x-2)(x+1)=0$

$x=2$ or -1

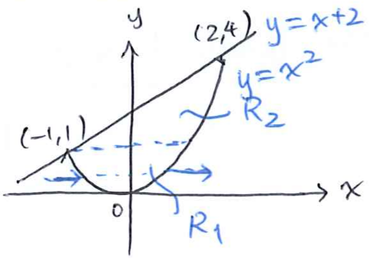
$y=4$ or 1

Method 1: Go in y-direction first, then x-direction



$$\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$$

Method 2: Go in x-direction first, then y-direction



$$\text{Area} = \iint_{R_1} dA + \iint_{R_2} dA$$

$$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy$$

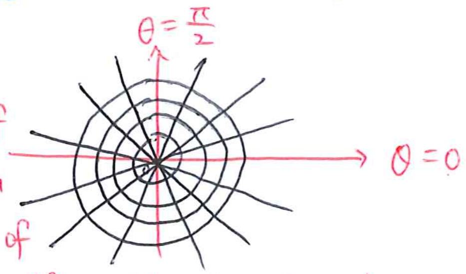
$$+ \int_1^4 \int_{y-2}^{\sqrt{y}} dx\,dy$$

We use Method 1 to carry out the evaluation:

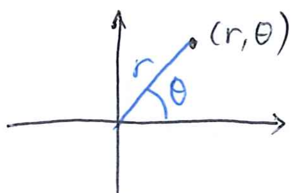
$$\text{Area} = \int_{-1}^{2} [y]_{x^2}^{x+2} dx = \int_{-1}^{2} (x+2-x^2)dx = \left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_{-1}^{2} = \underline{\underline{\frac{9}{2}}}$$

Polar Coordinate System

In polar coordinate system, we adopt a radial and circular net to describe points and curves on the $\mathbb{R}^2$ plane. Points are called "polar points" and curves are called "polar curves". The origin is called the "pole", while the positive x-axis is called the "polar axis".



In this chapter, the radial coor. of P describes the signed distance from origin to P, while the angular coor. of P describes angle whose initial side is the positive x-axis and terminal side lies on the ray that passes through O and P.

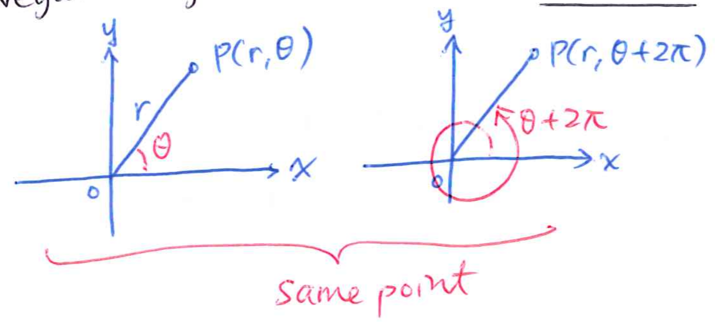


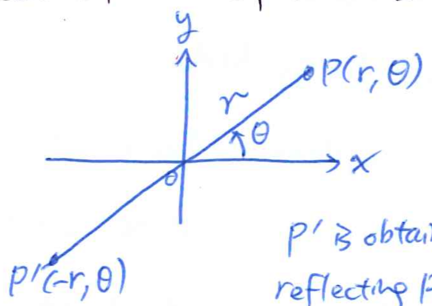
r: radial coordinate

θ: angular coordinate

Positive angles are measured in anti-clockwise direction from the positive x-axis.

Negative angles are measured in clockwise direction from the positive x-axis





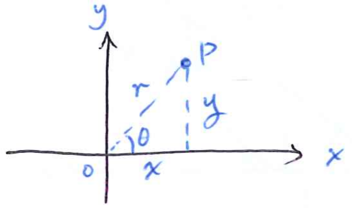
P' is obtained by reflecting P through the origin.

Remarks: In polar coordinate system, each point can have infinitely many representations. Angles are determined up to multiples of 2π rad. r can also be negative.

Conversion of Coordinates

(x, y) : rectangular point

(r, θ) : polar point



Given x and y

Find r and θ by

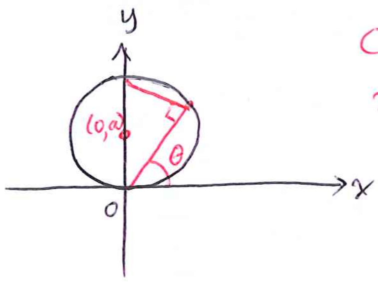
$$\begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

Here we assume $r \geq 0$ for simplicity.

$$\Leftrightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Some examples of curves in Polar coordinate system

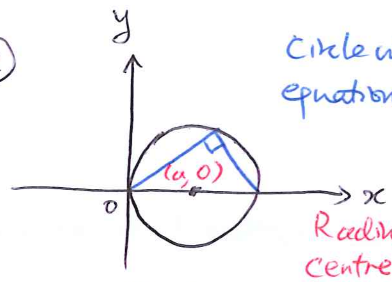
① $r = f(\theta) = 2a\sin\theta$.



Circle with polar equation $r = 2a\sin\theta$, radius a

centre: $(0, a)$

②



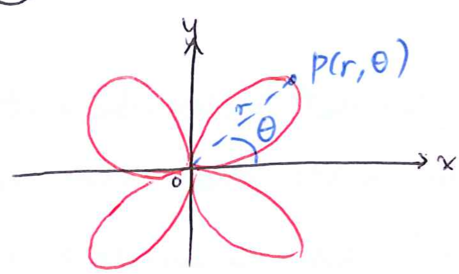
Circle with polar equation $r = 2a\cos\theta$.

Radius a

centre: $(a, 0)$

$r = f(\theta) = 2a\cos\theta$

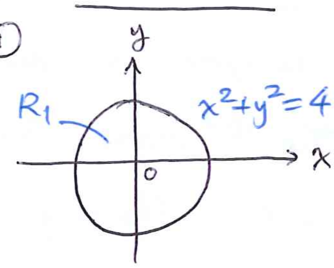
③ $r = f(\theta) = 3\sin 2\theta$



Four-leaf rose

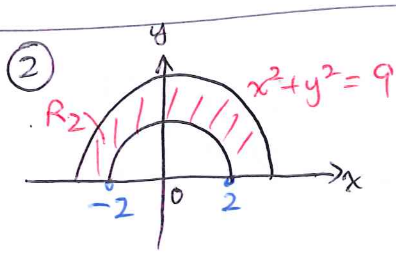
Describing regions in polar coordinates

①



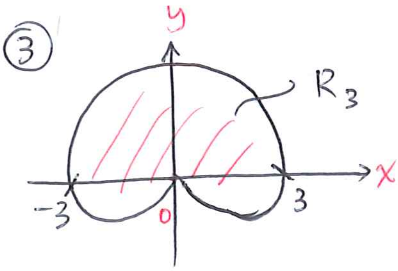
$R_1 = \{(r, \theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$

②



$R_2 = \{(r, \theta) \mid 2 \leq r \leq 3, 0 \leq \theta \leq \pi\}$

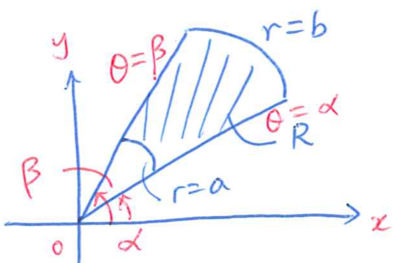
③



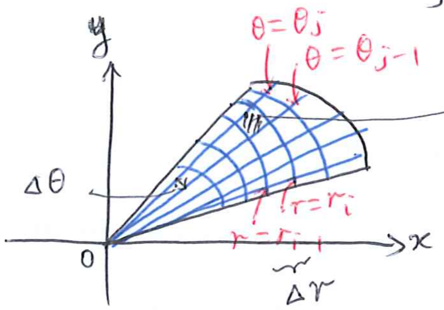
The region R_3 is a cardioid (心臟線) with $r = f(\theta) = 3 + 3\sin\theta$.

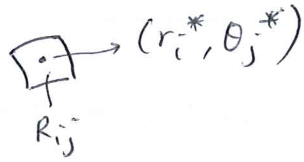
$$R_3 = \{(r,\theta) \mid 0 \le r \le 3+3\sin\theta,\ 0 \le \theta \le 2\pi\}$$

In order to compute the double integral $\iint_R f(x,y)\,dA$, where R is the polar rectangle, we divide the interval $[a,b]$ into m sub-intervals $[r_{i-1}, r_i]$ of equal width $\Delta r = \frac{b-a}{m}$, also we divide the interval $[\alpha,\beta]$ into n sub-intervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = \frac{\beta-\alpha}{n}$.



Then we have the following diagram:





$$R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$$

Division of R into polar subrectangles

$$\boxed{\begin{aligned} r_i^* &= \tfrac{1}{2}(r_{i-1}+r_i) \\ \theta_j^* &= \tfrac{1}{2}(\theta_{j-1}+\theta_j) \end{aligned}}$$

Consider the area of a wedge-shaped sector of a circle having radius r and angle θ

$$= \underbrace{\pi r^2}_{\text{Area of circle}} \cdot \underbrace{\frac{\theta}{2\pi}}_{\text{fraction of the circle's area contained in the wedge}} = \frac{1}{2}r^2\theta$$

$\therefore$ Area of $R_{ij} = \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta$, where $\Delta\theta = \theta_j - \theta_{j-1}$

$$= \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$

$$= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta$$

$$= \underbrace{r_i^*\,\Delta r\,\Delta\theta}_{\text{for discrete sense (in terms of ordinary rectangles)}}$$

Rectangular coordinates of centre of $R_{ij} = (r_i^*\cos\theta_j^*,\ r_i^*\sin\theta_j^*)$, hence a typical Riemann sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*,\ r_i^*\sin\theta_j^*)\,\Delta A_i$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*,\ r_i^*\sin\theta_j^*)\,r_i^*\,\Delta r\,\Delta\theta$$

It can be shown that for continuous functions f, we always obtain the same expression using polar rectangles.

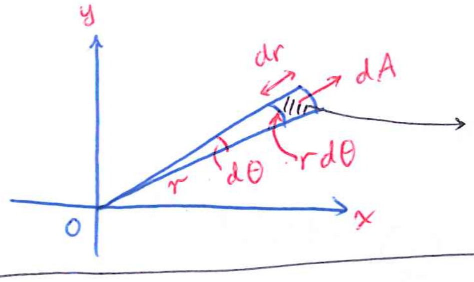
Write $g(r,\theta) = rf(r\cos\theta, r\sin\theta)$, then we have $\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$, which is a Riemann sum for $\int_\alpha^\beta\int_a^b g(r,\theta)\,dr\,d\theta$

$\therefore$ We have $\boxed{\iint_R f(x,y)\,dA} = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*,\ r_i^*\sin\theta_j^*)\Delta A_i$

$$= \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\Delta r\Delta\theta = \int_\alpha^\beta \int_a^b g(r,\theta)\,dr\,d\theta$$

$$= \boxed{\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

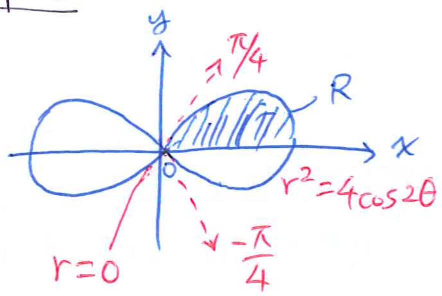
for continuous function f on a polar rectangle R defined by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$.



$\therefore dA = r\,dr\,d\theta$

infinitesimal polar rectangle.

Examples ① Find the area enclosed by $r^2 = 4\cos 2\theta$



We consider one quarter of the lemniscate.

$$\text{Area of } R = \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\cos 2\theta\,d\theta = \left[\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= 1$$

$\Rightarrow$ Area of the lemniscate $= 4 \times 1 = 4$ (by symmetry)

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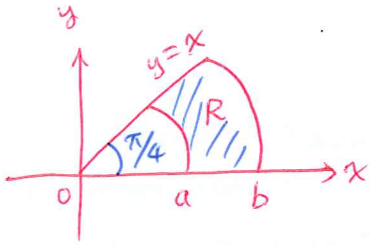
② (P.835 of Adams and Essex) If R is the part of annulus $0 \le a^2 \le x^2+y^2 \le b^2$ lying in the first quadrant and below $y = x$, evaluate $\iint_R \frac{y^2}{x^2}\,dA$



Consider $\frac{y^2}{x^2} = \frac{r^2\sin^2\theta}{r^2\cos^2\theta} = \tan^2\theta$ in polar coordinates.

$$R = \left\{(r,\theta) \mid a \le r \le b,\ 0 \le \theta \le \frac{\pi}{4}\right\}$$

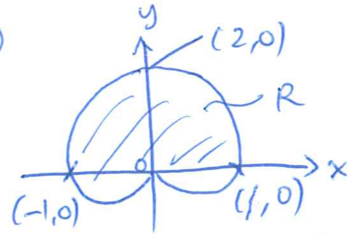
$$\therefore \iint_R \frac{y^2}{x^2}\,dA = \int_0^{\frac{\pi}{4}} \int_a^b \tan^2\theta\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^2\theta\,d\theta \int_a^b r\,dr$$

$$= \frac{1}{2}(b^2-a^2)\int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\,d\theta$$

$$= \frac{1}{2}(b^2-a^2)(\tan\theta - \theta)\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}(b^2-a^2)\left(1-\frac{\pi}{4}\right) \ldots$$

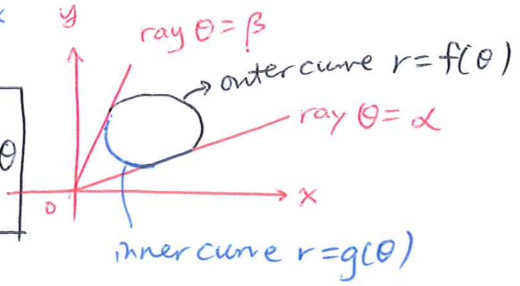
(3)

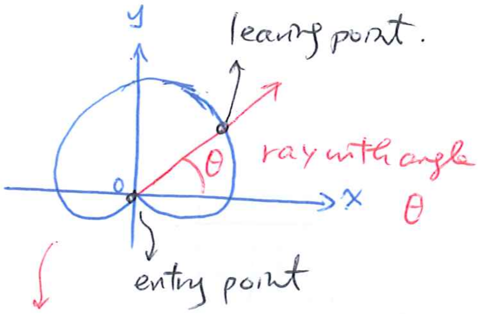


Given $r = f(\theta) = 1 + \sin\theta$.

Find the area of the region enclosed (R).

$$\boxed{\int_{\theta=\alpha}^{\theta=\beta} \int_{r=g(\theta)}^{r=f(\theta)} r\,dr\,d\theta}$$

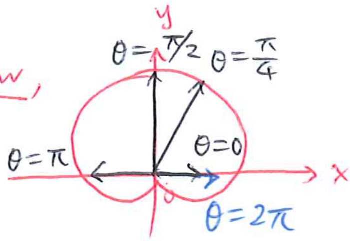




$r_{inner} = g(\theta) = 0$

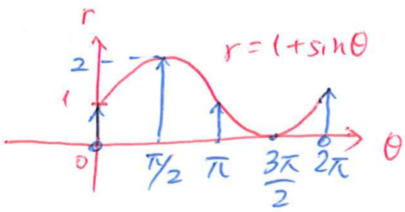
$r_{outer} = f(\theta) = 1 + \sin\theta$

Now,



starting ray : $\theta = 0$, i.e. $\alpha = 0$

ending ray : $\theta = 2\pi$, i.e. $\beta = 2\pi$

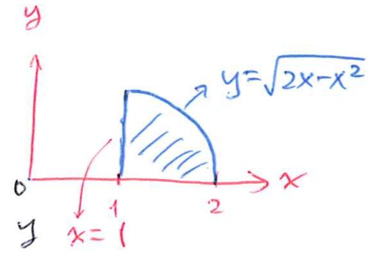


$$\therefore \text{Area} = \int_0^{2\pi} \frac{1}{2}\left(f(\theta)^2 - g(\theta)^2\right) d\theta = \frac{1}{2}\int_0^{2\pi} (1+\sin\theta)^2\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} + \sin\theta + \frac{1-\cos 2\theta}{4}\right] d\theta$$

$$= \left[\frac{\theta}{2} - \cos\theta + \frac{\theta}{4} - \frac{\sin 2\theta}{8}\right]_0^{2\pi} = \underline{\underline{\frac{3\pi}{2}}}$$

(4) Evaluate $\underbrace{\int_1^2 \int_0^{\sqrt{2x-x^2}}}_{R} y\,dy\,dx$ using polar coordinates.



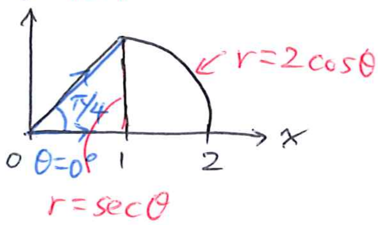
$R = \{(x,y) \mid 1 \le x \le 2 \text{ and } 0 \le y \le \sqrt{2x-x^2}\}$

For $x = 1$, $r\cos\theta = 1 \iff r = \sec\theta$

For $y = \sqrt{2x-x^2}$, $r\sin\theta = \sqrt{2r\cos\theta - r^2\cos^2\theta}$

$$r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$$

$$r = 2\cos\theta$$



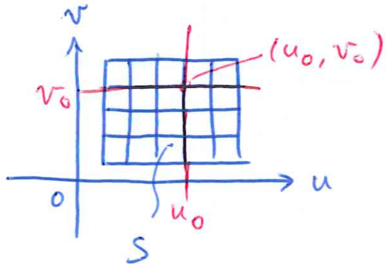
$$\therefore \int_1^2 \int_0^{\sqrt{2x-x^2}} y\,dy\,dx = \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} (r\sin\theta)\, r\,dr\,d\theta$$

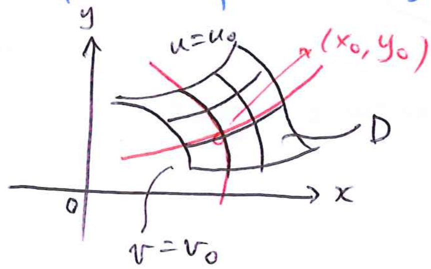
$$= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} r^2 \sin\theta\, dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin\theta\, \frac{1}{3} r^3 \Big|_{\sec\theta}^{2\cos\theta} d\theta \quad \ldots\ldots$$

The transformation of a double integral to polar coordinates is a special case of a general change of variables formula for double integrals.

Suppose $x = x(u,v)$, $y = y(u,v)$ (i.e. transformation from points (u,v) in a uv-Cartesian plane to points (x,y) in the xy-plane)





Under the transformation, the lines $u=u_0$ and $v=v_0$ in the uv-plane are mapped to the curves $\begin{cases} x = x(u_0, v) \\ y = y(u_0, v) \end{cases}$ and $\begin{cases} x = x(u, v_0) \\ y = y(u, v_0) \end{cases}$ in the xy-plane.

i.e. $(u_0, v_0) \longmapsto (x_0, y_0)$

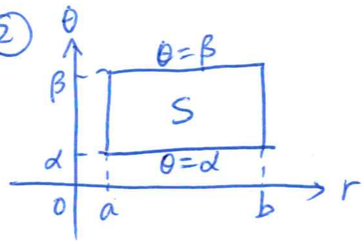
The transformation from the set S in uv-plane to the set D in xy-plane is bijective if
① every point in S gets mapped to a point in D
② every point in D is the image of a point in S
③ different points in S are mapped to different points in D.

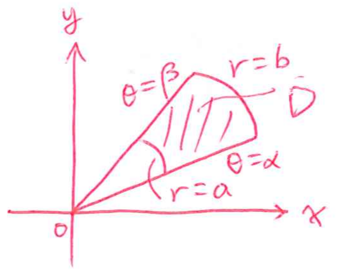
If the transformation is one-to-one, we can write $u = u(x,y)$, $v = v(x,y)$.
Assume the functions $x(u,v)$ and $y(u,v)$ are C^1 and $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$ at (u,v), then the transformation is one-to-one near (u,v) and the inverse transformation $\dfrac{\partial(u,v)}{\partial(x,y)}$ also has continuous first partial derivatives and non-zero Jacobian such that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$ on the set D.

Example

① In particular, $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ has Jacobian $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

Hence, near any point except the origin $(r=0)$, the transformation is one-to-one.

② 



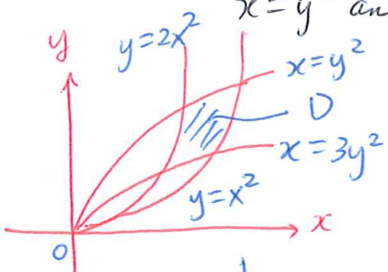
$$\iint_D f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta \quad (r>0)$$
$$= \iint_S f(r\cos\theta, r\sin\theta) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

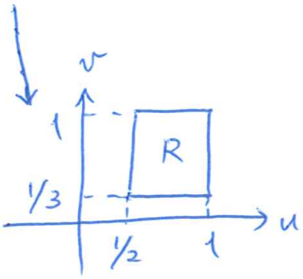
Theorem (Change of variables)

Let $x = x(u,v)$, $y = y(u,v)$ be a one-to-one transformation from a domain S in the uv-plane onto a domain D in the xy-plane.

Suppose x and y and their first partial derivatives with respect to u and v are continuous in S, if $f(x,y)$ is integrable on D, $g(u,v) = f(x(u,v), y(u,v))$, then g is integrable on S and $\iint_D f(x,y)\,dx\,dy = \iint_S g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

Example ① Find the area of the finite plane region bounded by $y = x^2$, $y = 2x^2$, $x = y^2$ and $x = 3y^2$.





Let $u = \frac{x^2}{y}$, $v = \frac{y^2}{x}$, then the region D in xy-plane is mapped to the region R in uv-plane, given by

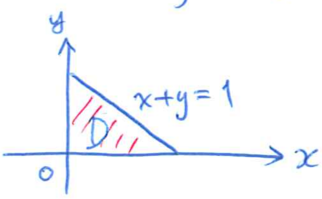
$$\left\{(u,v) \,\middle|\, \tfrac{1}{2} \le u \le 1 \text{ and } \tfrac{1}{3} \le v \le 1\right\}$$

Consider $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & \frac{-x^2}{y^2} \\ \frac{-y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 3$

$$\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3}$$

$$\therefore \text{Area} = \iint_D dx\,dy = \iint_R \frac{1}{3}\,du\,dv = \frac{1}{3}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = \frac{1}{9}$$

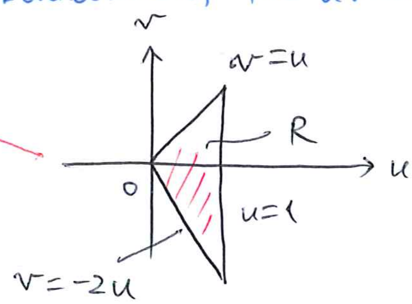
② Evaluate $\underbrace{\int_0^1 \int_0^{1-x}}_{\hookrightarrow \text{ Sketch region } D} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$



→ Integrand suggests we substitute $u = x+y$ and $v = y-2x$.

Solving yields $\begin{cases} x = \frac{u}{3} - \frac{v}{3} \\ y = \frac{2u}{3} + \frac{v}{3} \end{cases}$

Then we can try figure out the boundaries of the uv-region R



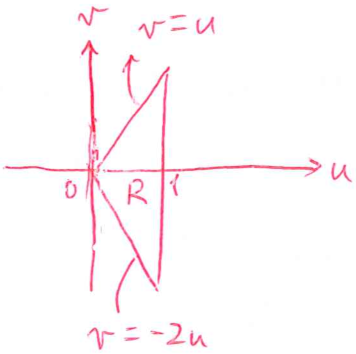
$$\begin{cases} x = \frac{u}{3} - \frac{v}{3} \\ y = \frac{2u}{3} + \frac{v}{3} \end{cases}$$

Note: The equations $\begin{cases} x = \frac{u}{3} - \frac{v}{3} \\ y = \frac{2u}{3} + \frac{v}{3} \end{cases}$ transform R to D.

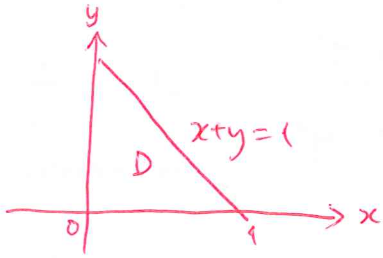
$\begin{cases} u = x+y \\ v = y-2x \end{cases}$ transforms D to R.

We construct a table as follows:

xy-equations for boundary of D	uv-equations for boundary of R	Simplified uv-equations
$x+y=1$	$\left(\frac{u}{3}-\frac{v}{3}\right)+\left(\frac{2u}{3}+\frac{v}{3}\right)=1$	$u=1$
$x=0$	$\frac{u}{3}-\frac{v}{3}=0$	$v=u$
$y=0$	$\frac{2u}{3}+\frac{v}{3}=0$	$v=-2u$



$$\begin{cases} x=\frac{u}{3}-\frac{v}{3} \\ y=\frac{2u}{3}+\frac{v}{3} \end{cases} \longrightarrow$$



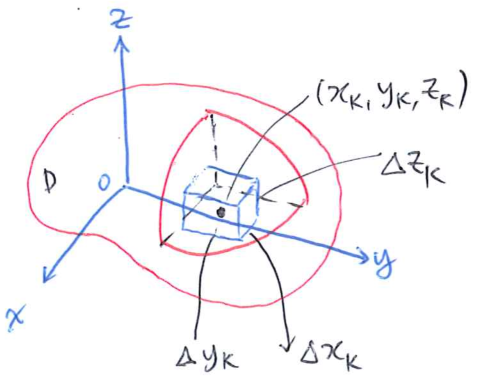
$$\text{Jacobian} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$$

$$\Rightarrow \int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,dy\,dx = \int_0^1\int_{-2u}^{u} u^{\frac{1}{2}}v^2\left(\frac{1}{3}\right)dv\,du$$

$$= \frac{1}{3}\int_0^1 u^{\frac{1}{2}}\left[\frac{1}{3}v^3\right]_{-2u}^{u}du$$

$$= \int_0^1 u^{\frac{7}{2}}du = \frac{2}{9}u^{\frac{9}{2}}\Big|_0^1 = \underline{\underline{\frac{2}{9}}}$$

Triple Integration



$F(x,y,z)$: a function defined on a closed and bounded region D in space, such as the region occupied by a solid ball or a lump of clay.

We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to x, y and z axis.

Consider the kth cell with dimensions Δx_K by Δy_K by Δz_K.

i.e. $\Delta V_K = \Delta x_K \Delta y_K \Delta z_K$.

Select a point (x_K, y_K, z_K) in each cell, let $S_n = \sum_{k=1}^{n} F(x_K, y_K, z_K)\Delta V_K$.

Suppose $\Delta x_K, \Delta y_K, \Delta z_K$ and $\underbrace{\max\{\Delta x_K, \Delta y_K, \Delta z_K\}}_{\substack{\|P\| \\ \text{(norm of partition)}}} \to 0$, i.e. as $\|P\| \to 0$ and the no. of cells $n \to \infty$, the sums S_n approaches a limit. Such limit is the "triple integral of F over D",

$$\lim_{n\to\infty} S_n = \iiint_D F(x,y,z)\,dV$$

or

$$\lim_{\|P\|\to 0} S_n = \iiint_D F(x,y,z)\,dx\,dy\,dz$$

(provided that limit exists)

region over which continuous functions are integrable, i.e. can be closely approximated by small rectangular cells.

Definition The volume of a closed and bounded region D in 3-space is defined as

$$\iiint_D 1\,dV \overset{\text{def}}{=} \iiint_D dV.$$

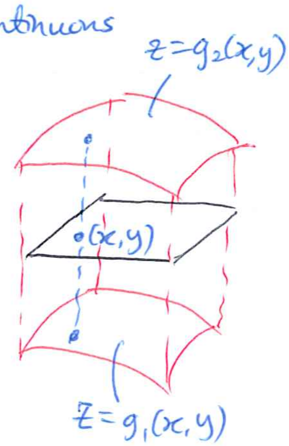
Evaluation by Iterated Integrals

Let f be a continuous function on a solid region D defined by

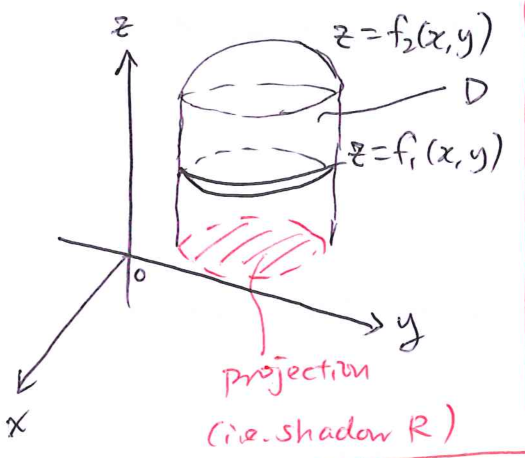
$$a \le x \le b$$
$$h_1(x) \le y \le h_2(x)$$
$$g_1(x,y) \le z \le g_2(x,y)$$

, where h_1, h_2, g_1 and g_2 are continuous functions.

Then $\iiint_D f(x,y,z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dy\,dx$

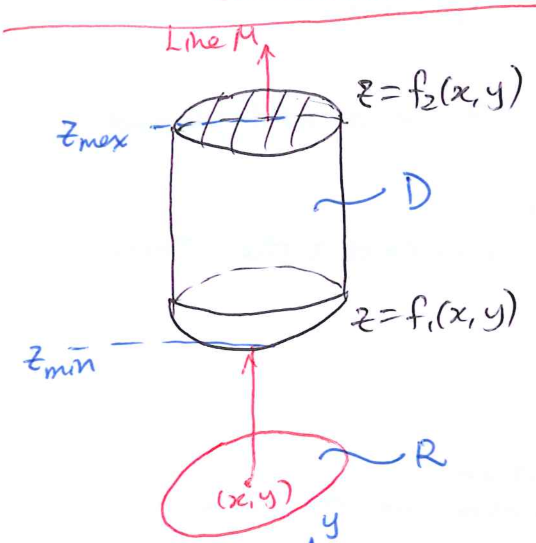


Procedures for finding lower and upper limits of a triple integral over a closed solid region D bounded above by surface $z=f_2(x,y)$, and bounded below by surface $z=f_1(x,y)$



$$\iiint_D F(x,y,z)\,dV = \int_{x=?}^{x=?}\int_{y=?}^{y=?}\underbrace{\int_{z=?}^{z=?} F(x,y,z)\,dz}_{\text{innermost integrals.}}\,dy\,dx$$

① Look at the shadow of the solid region D onto the xy-plane. Fix a point (x,y) within the shadow R. Then at that point, we draw a line M that passes through the solid region D parallel to the z-axis in direction of increasing z.

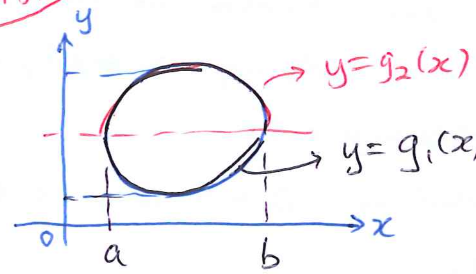


② After evaluating the innermost integral, no more variable z. Let the innermost integral be $G(x,y)$.

$$\iiint_D F(x,y,z)\,dV = \iint_R G(x,y)\,dy\,dx$$

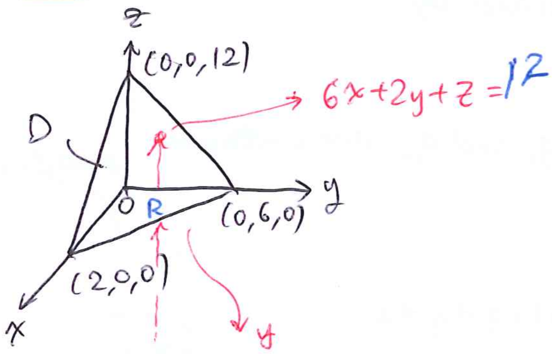
R: vertical shadow of the solid region D in the xy-plane

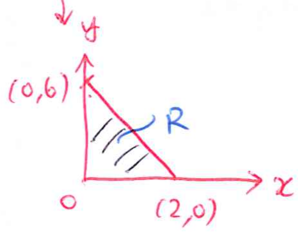
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} G(x,y)\,dy\,dx$$



$$\boxed{\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz\,dy\,dx}$$

Examples ① Find the volume of the following bounded region D in the 1st octant.





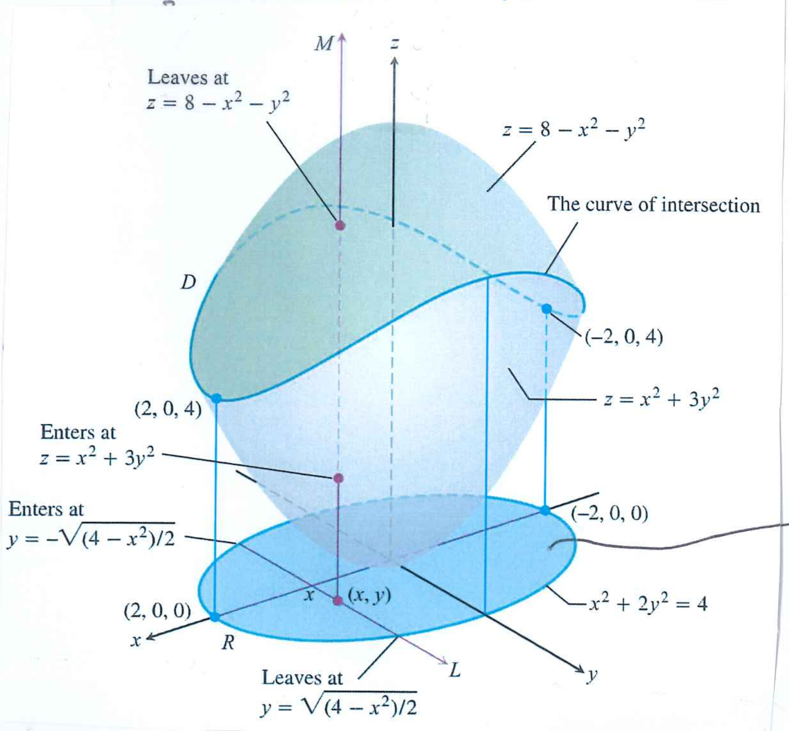
$$\text{Volume} = \iiint_D 1\cdot dV = \iint_R (12-6x-2y)\,dA_R$$

$$= \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} dz\,dy\,dx$$

$$= \int_0^2 \int_0^{6-3x} (12-6x-2y)\,dy\,dx$$

$$= \underline{\underline{24}}$$

② (Thomas Calculus P.1101–1102) Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



Shadow of the solid region D onto xy-plane.

$$\text{Volume of } D = \iiint_D 1\, dV$$

$$= \iint_R \int_{z=x^2+3y^2}^{z=8-x^2-y^2} 1\, dz\, dy\, dx$$

$$= \iint_R (8 - 2x^2 - 4y^2)\, dy\, dx$$

We need to find the equation of the boundary curve of shadow R.

Consider $x^2 + 3y^2 = 8 - x^2 - y^2$ (the place where they intersect together must have the same z-value)

i.e. $x^2 + 2y^2 = 4$ (ellipse)

The line L enters into region R at $y = -\sqrt{\frac{4-x^2}{2}}$ and leaves region R at $y = \sqrt{\frac{4-x^2}{2}}$.

$$\therefore \text{Volume of } D = \int_{-2}^{2} \int_{y=-\sqrt{\frac{4-x^2}{2}}}^{y=\sqrt{\frac{4-x^2}{2}}} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \left(2(8 - 2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}} \right) dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}} \right] dx$$

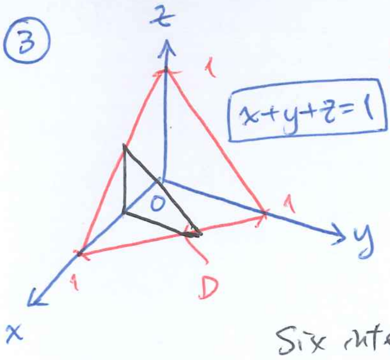
$$= \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{\frac{3}{2}}\, dx$$

Let $x = 2\sin u$

$dx = 2\cos u\, du$

$$= \underline{\underline{8\sqrt{2}\pi}}$$

③



Let D be the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Write out the 6 integrals for $\iiint_D y\,dV$, and evaluate this triple integral via any order.

Six integration orders

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx.$$

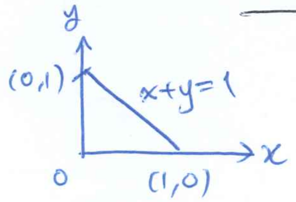
$$\int_0^1\int_0^{1-y}\int_0^{1-x-y} y\,dz\,dx\,dy$$

$$\int_0^1\int_0^{1-x}\int_0^{1-x-z} y\,dy\,dz\,dx$$

$$\int_0^1\int_0^{1-z}\int_0^{1-x-z} y\,dy\,dx\,dz$$

$$\int_0^1\int_0^{1-z}\int_0^{1-y-z} y\,dx\,dy\,dz$$

$$\int_0^1\int_0^{1-y}\int_0^{1-y-z} y\,dx\,dz\,dy.$$



$$= \int_0^1\int_0^{1-x} y(1-x-y)\,dy\,dx$$

$$= \int_0^1 \left[(1-x)\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{1}{6}(1-x)^3\,dx = \underline{\underline{\frac{1}{24}}}$$

Applications

Mass of solid objects in space $M = \iiint_D \delta\,dV$ $\quad (\delta = \delta(x,y,z) = \text{density})$

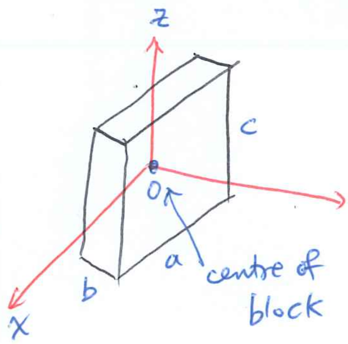
Centre of mass $(\bar{x},\bar{y},\bar{z}) = \left(\dfrac{\iiint_D x\delta\,dV}{M}, \dfrac{\iiint_D y\delta\,dV}{M}, \dfrac{\iiint_D z\delta\,dV}{M}\right)$

first moments about coor. planes

Moment of Inertia about the coordinate axes:

$$I_x = \iiint (y^2+z^2)\,\delta\,dV\ ;\quad I_y = \iiint (x^2+z^2)\,\delta\,dV\ ;\quad I_z = \iiint (x^2+y^2)\,\delta\,dV.$$

Example Find I_x, I_y and I_z for the rectangular solid with constant density δ.



M: mass of the block.

$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \underbrace{(y^2+z^2)\,\delta}_{\text{even function of } x,y,z}\,dx\,dy\,dz$$

totally 8 symmetric pieces, one in each octant.

$$= 8\int_0^{c/2}\int_0^{b/2}\int_0^{a/2} (y^2+z^2)\,\delta\,dx\,dy\,dz$$

$$= 4a\delta\int_0^{c/2}\int_0^{b/2} (y^2+z^2)\,dy\,dz$$

$$= 4a\delta\int_0^{c/2}\left[\frac{y^3}{3}+z^2y\right]_0^{b/2} dz$$

$$= 4a\delta\int_0^{\frac{c}{2}}\left(\frac{b^3}{24}+\frac{z^2b}{2}\right)dz = \frac{abc\delta}{12}(b^2+c^2) = \underline{\underline{\frac{M}{12}(b^2+c^2)}}$$

By symmetry,

$$I_y = \frac{M}{12}(a^2+c^2)$$

$$I_z = \frac{M}{12}(a^2+b^2)$$

Triple Integrals in Cylindrical Coordinates and Spherical Coordinates

The change of variables formula in double integral can be extended to triple (or higher order) integrals.

Consider $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$, where x, y and z have continuous first partial derivatives with respect to u, v, w. Then near any point where $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$,

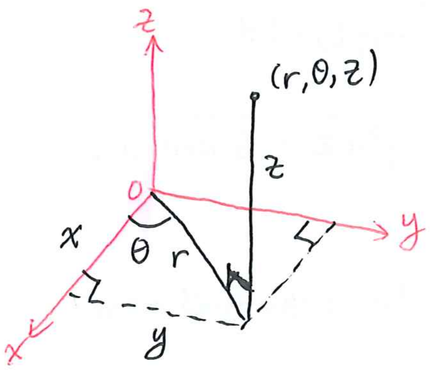
$$dV = dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

If the transformation is one-to-one from a domain S in uvw-space onto a domain D in xyz-space, if $g(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w))$, then

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_S g(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

Integration in Cylindrical Coordinates

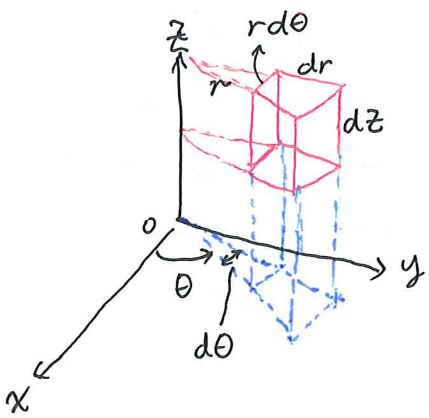
Def: Cylindrical coordinates represent a point P in space defined by ordered triples (r, θ, z), where we assume $r \geq 0$ in our following context.



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

Volume element in cylindrical coordinates: $\boxed{dV = r\,dr\,d\theta\,dz}$

We examine the infinitesimal 'box' bounded by the coordinate surfaces corresponding to values $r, r+dr, \theta, \theta+d\theta, z$ and $z+dz$.



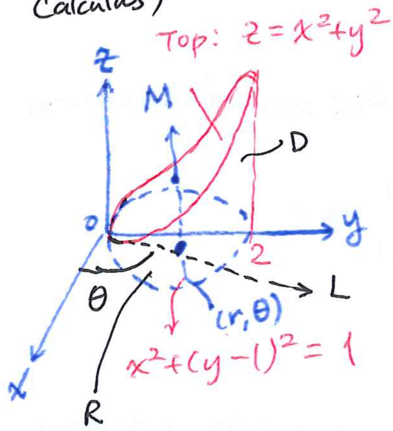
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\therefore \iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_D f(r,z,\theta)\,r\,dr\,d\theta\,dz$$

$\theta: \alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$

Examples ① Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over the region D bounded by $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$, and above by paraboloid $z=x^2+y^2$.

(due to Thomas Calculus)



Project solid D onto xy-plane, the projection is denoted as R. The boundary of R is the circle $x^2+(y-1)^2=1$.

Consider its polar coor. equation: $r^2-2r\sin\theta=0$

$r=2\sin\theta$.

z-limits of integration

Consider a line M through a typical point (r,θ) in R parallel to the z-axis: It enters D at $z=0$ and leaves at $z=x^2+y^2$.

r-limits of integration : A ray L through (r,θ) from origin enters R at $r=0$ and leaves R at $r=2\sin\theta$.

θ-limits of integration : As L sweeps across R, the angle θ it makes with the positive x-axis run from $\theta=0$ to $\theta=\pi$.

$$\iiint_D f(r,\theta,z)\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)\, r\,dz\,dr\,d\theta$$

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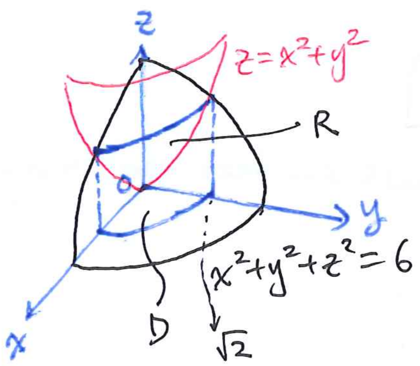
② Find the volume of the solid region inside the sphere $x^2+y^2+z^2=6$ and above the paraboloid $z=x^2+y^2$.



One quarter of the required volume lies in the first octant. Denote one quarter of the solid region as R, and its projection onto xy-plane: one quarter of the disk be D.

The two surfaces intersect on the vertical cylinder:

$$6-x^2-y^2=z^2=(x^2+y^2)^2, \text{ i.e. } 6-r^2=r^4$$

$$(r^2+3)(r^2-2)=0$$

The only relevant soln is $r=\sqrt{2}$, hence the required volume lies above the disk D centered at origin in the xy-plane.

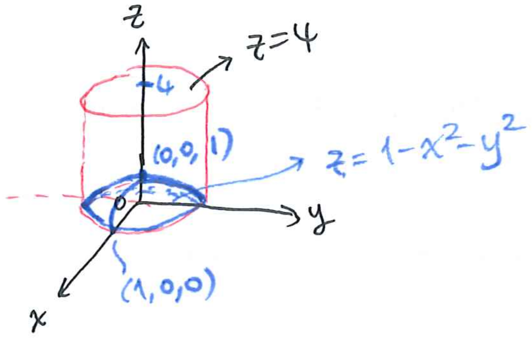
$$\therefore\ V=\iiint_R dV=\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} r\,dz\,dr\,d\theta = 2\pi\int_0^{\sqrt{2}}\left(r\sqrt{6-r^2}-r^3\right)dr$$

$$= 2\pi\left[\frac{-1}{3}(6-r^2)^{\frac{3}{2}}-\frac{r^4}{4}\right]_0^{\sqrt{2}} = \frac{2\pi}{3}(6\sqrt{6}-11)$$

③ (mass of an object) A solid G lies within the cylinder $x^2+y^2=1$, bounded below by $z=4$ and bounded above by the paraboloid $z=1-x^2-y^2$.

The density at any point is proportional to its distance from the axis of the cylinder.

Find the mass of G.



In cylindrical coordinates, the cylinder: $r=1$

the paraboloid: $z=1-r^2$

$\therefore G=\{(r,\theta,z)\,|\,0\le\theta\le 2\pi,\ 0\le r\le 1,\ 1-r^2\le z\le 4\}$

Density function $f(x,y,z)=k\underbrace{\sqrt{x^2+y^2}}_{\text{distance from } z\text{-axis}}=kr$, where k is a constant.

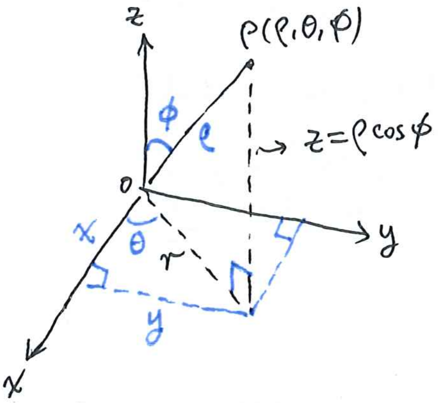
$$\therefore \text{Mass}=\iiint_G k\sqrt{x^2+y^2}\,dV=\int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (kr)r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 kr^2(3+r^2)\,dr\,d\theta$$

$$=2\pi k\left[r^3+\frac{r^5}{5}\right]_0^1=\frac{12\pi k}{5}$$

Integration in Spherical Coordinates

Def.: Spherical coordinates represent a point P in space defined by ordered triples (ρ,θ,ϕ) such that
① ρ is the distance from P to the origin ($\rho\ge 0$)
② θ: angle from cylindrical coordinates
③ ϕ: angle $\overrightarrow{OP}$ makes with the positive z-axis ($0\le\phi\le\pi$)

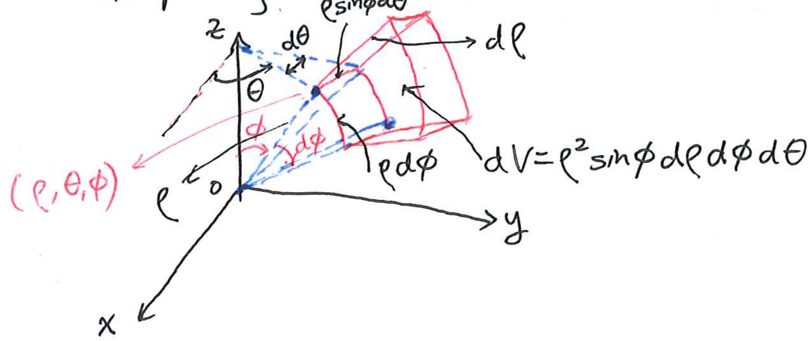


$$\begin{cases} r=\rho\sin\phi \\ z=\rho\cos\phi\end{cases} \longrightarrow \begin{aligned} x&=r\cos\theta=\rho\sin\phi\cos\theta \\ y&=r\sin\theta=\rho\sin\phi\sin\theta\end{aligned}$$

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{r^2+z^2}$$

Volume element in spherical coordinates: $\boxed{dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$

We examine the infinitesimal coordinate box bounded by the coordinate surfaces corresponding to values ρ, $\rho+d\rho$, ϕ, $\phi+d\phi$, θ, $\theta+d\theta$.



$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}=\begin{vmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{vmatrix}$$

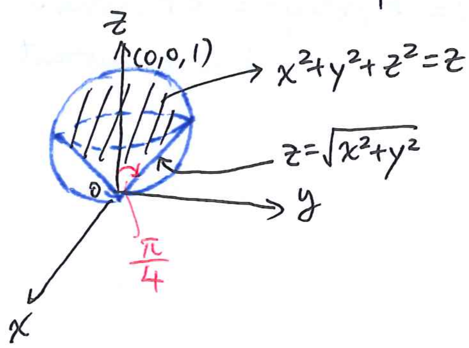
$$=\rho^2\sin\phi$$

Remarks: spherical coordinates are suitably designed for problems involving spherical symmetry, e.g., regions bounded by spheres centered at origin, circular cones with axes along the z-axis, and vertical planes containing the z-axis.

$$\iiint_D f(\rho,\phi,\theta)\,dV = \iiint_D f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\alpha \le \theta \le \beta,\quad 0 \le \beta-\alpha \le 2\pi$$

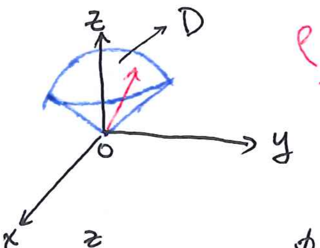
Examples ① Find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.



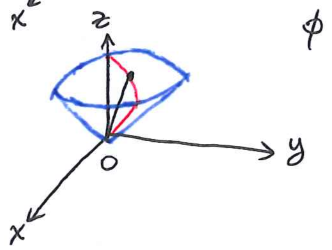
The sphere passes through the origin and has centre $(0,0,\frac{1}{2})$.

Spherical equation of sphere: $\rho^2=\rho\cos\phi$
$\rho=\cos\phi$

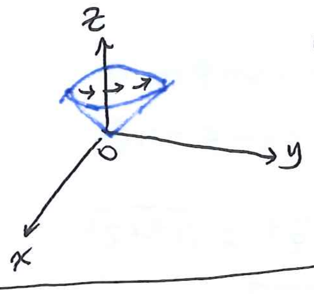
Spherical equation of cone: $\rho\cos\phi=\sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}$
$=\rho\sin\phi$
$\phi=\frac{\pi}{4}$.



ρ varies from 0 to $\cos\phi$



ϕ varies from 0 to $\frac{\pi}{4}$



θ varies from 0 to 2π

$$\therefore \text{Volume}=\iiint_D dV=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=2\pi\int_0^{\frac{\pi}{4}}\sin\phi\left[\frac{\rho^3}{3}\right]_0^{\cos\phi}d\phi$$

$$=\frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\sin\phi\cos^3\phi\,d\phi$$

$$=\frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\frac{\pi}{4}}=\underline{\underline{\frac{\pi}{8}}}$$

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② A solid half-ball B of radius a has density δ depending on distance ρ from the centre of the base disk. Given that $\delta=k(2a-\rho)$, with k being a constant. Find the mass of the half-ball B.

We select the coordinates with origin at the centre of the base (projection), then the half-ball B lies above the xy-plane.

$\boxed{\text{half-ball}}$

$$\text{Mass}=\iiint_B k(2a-\rho)\,dV=\iiint_B k(2a-\rho)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=k\cdot 2\pi\int_0^{\frac{\pi}{2}}\sin\phi\,d\phi\int_0^a(2a-\rho)\rho^2\,d\rho$$

$$=2k\pi\left[\frac{2a}{3}\rho^3-\frac{\rho^4}{4}\right]_0^a=\underline{\underline{\frac{5}{6}\pi k\,a^4}}$$