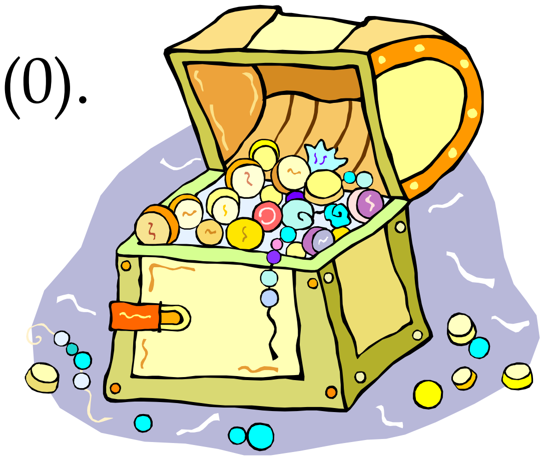
A Majority Game

Three people form a coalition to get a treasure.

- A team of 3 people gets all (1).
- A team of any 2 people gets $\frac{3}{5}$.
- A team of any 1 person gets none (0).



Coalitional Games with Transferable Payoff (TU Games)

There are three *players*.

- $N = \{1,2,3\}$.

The *worth* of the teams is:

- $v(\{1,2,3\}) = 1$.
- $v(\{1,2\}) = \frac{3}{5},\ v(\{2,3\}) = \frac{3}{5},\ v(\{1,3\}) = \frac{3}{5}$.
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0$.

Coalitional Games with Transferable Payoff (TU Games)

DEFINITION. A **coalitional game with transferable payoff** $\langle N, v \rangle$ consists of

- a finite set N (the set of **players**)
- a function v that associates with every nonempty subset S of N (a **coalition**) a real number $v(S)$ (the **worth** of S).

Class Discussion

There are three *players*.

- $N = \{1,2,3\}$.

The *worth* of the teams is:

- $v(\{1,2,3\}) = 1$.
- $v(\{1,2\}) = \frac{3}{5},\ v(\{2,3\}) = \frac{3}{5},\ v(\{1,3\}) = \frac{3}{5}$.
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0$.

Q: How much payoff <u>each player should get</u>?

Class Discussion

Consider <u>another</u> game with 3 players. The *worth* of the teams is:

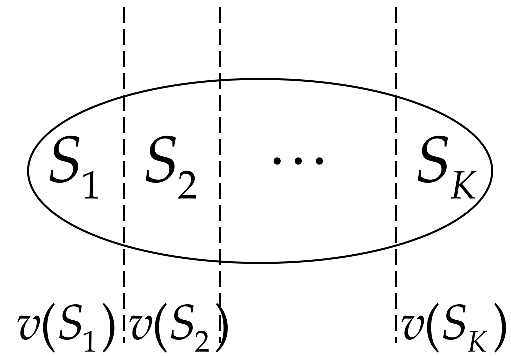
- $v(\{1,2,3\}) = 1$.
- $v(\{1,2\}) = \frac{4}{5}$, $v(\{2,3\}) = \frac{4}{5}$, $v(\{1,3\}) = \frac{4}{5}$.
- $v(\{1\}) = \frac{2}{5}$, $v(\{2\}) = \frac{2}{5}$, $v(\{3\}) = \frac{2}{5}$.

Q: How much payoff <u>each</u> player should get?

Q: Will some set of players break away from the 'grand coalition' {1,2,3}?

Cohesiveness of Coalitional Games with Transferable Payoff

DEFINITION. A coalitional game $\langle N, v \rangle$ with transferable payoff is **cohesive** if



$$v(N) \geq \sum_{k=1}^{K} v(S_k)$$

for every partition $\{S_1, \ldots, S_K\}$ of N (*i.e.*, $i \neq j \rightarrow S_i \cap S_j = \emptyset$ and $S_1 \cup S_2 \cup \ldots \cup S_K = N$).

Class Discussion

Consider a game with 3 players. The *worth* of the teams is:

- $v(\{1{,}2{,}3\}) = 1$.
- $v(\{1{,}2\}) = \frac{3}{5}$, $v(\{2{,}3\}) = \frac{3}{5}$, $v(\{1{,}3\}) = \frac{3}{5}$.
- $v(\{1\}) = 0$, $v(\{2\}) = 0$, $v(\{3\}) = 0$.

Q: Is this game cohesive?

A: First, $v(N) =$ ___. So, it is/is not cohesive.

In this course we assume that all coalitional games with transferable payoff are cohesive.

$$v(N) \geq \sum_{k=1}^{K} v(S_k)$$

Feasible Payoff Profiles

- A *feasible payoff profile* is a profile $x = (x_i)_{i \in N} = (x_1, \dots, x_n)$ of real numbers, such that $v(N) = \sum_{i \in N} x_i$.
- An *S-feasible payoff vector* is a vector $(x_i)_{i \in S}$ of real numbers, such that $x(S) = \sum_{i \in S} x_i = v(S)$.
- A feasible payoff profile is thus an *N*-feasible payoff vector.

Class Discussion

There are three players.

- $N = \{1,2,3\}$.

The worth of the teams is:

- $v(\{1,2,3\}) = 1$.
- $v(\{1,2\}) = \frac{3}{5},\ v(\{2,3\}) = \frac{3}{5},\ v(\{1,3\}) = \frac{3}{5}$.
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0$.

Q: What are the feasible payoff profiles?

Class Discussion

The worth of the teams is:

- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = \frac{3}{5},\ v(\{2,3\}) = \frac{3}{5},\ v(\{1,3\}) = \frac{3}{5}.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

Q: What about the payoff profile $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5})$?

Q: What about the payoff profile $(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$?

Q: What about the payoff profile $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$?

Q: What about the payoff profile $(\frac{3}{10}, \frac{3}{10}, \frac{4}{10})$?

The Core

DEFINITION. The **core of the coalitional game with transferable payoff** $\langle N, v\rangle$ is the set of feasible payoff profiles $(x_i)_{i\in N}$ for which there is no coalition S and S-feasible payoff vector $(y_i)_{i\in S}$ for which $y_i > x_i$ for all $i \in S$.

In other words, the core is the set of feasible payoff profiles $(x_i)_{i\in N}$, *for which* $v(S) \leq x(S)$ *for every coalition* S.

Class Discussion

An expedition of n people discovers treasure. Every two people can carry out one piece.

$$v(S) = \begin{cases} \frac{|S|}{2}, & |S| \text{ is even} \\ \frac{(|S|-1)}{2}, & |S| \text{ is odd} \end{cases}$$

Q: What is the core if $|N| \geq 4$ is even?

Q: What is the core if $|N| \geq 3$ is odd?

Class Discussion

Now consider <u>another</u> game.

Suppose there are n players and n is odd. If a coalition consists of a majority of the players, people in the coalition can have everything.

$$v(S) = \begin{cases} 1, & |S| \geq \frac{n}{2} \\ 0, & \text{otherwise} \end{cases}$$

Q: What is the core?

Class Discussion

In other words, the core is the set of feasible payoff profiles $(x_i)_{i \in N}$, *for which* $v(S) \leq x(S)$ *for every coalition* S.

Q: What is the core?

Some Special Coalitional Games with Transferable Payoff

- A game is **convex** if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all S and T.

- A game is **simple** if $v(S)$ is either 0 or 1 for every S, and $v(N) = 1$.

- A game is **zerosum** (or **constant-sum**) if $v(S) + v(N \backslash S) = v(N)$ for every S.

- A game is **additive** if $v(S) + v(T) = v(S \cup T)$ for all disjoint S and T. Note: $v(S) = \sum_{i \in S} v(i)$.

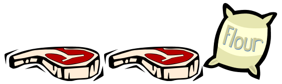
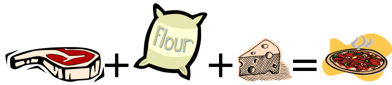
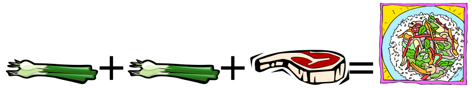
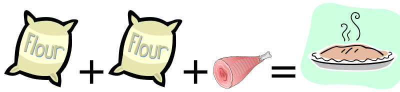
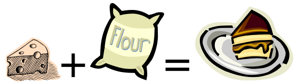
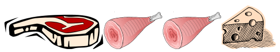
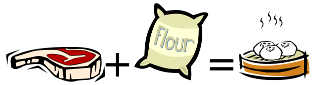
- A game is **superadditive** if $v(S) + v(T) \leq v(S \cup T)$ for all disjoint S and T. A game is **weakly superadditive** if $v(S) + v(\{i\}) \leq v(S \cup \{i\})$ for all disjoint S and $i \notin S$.

- A game is **cohesive** if $v(N) \geq \sum_{k=1}^{K} v(S_k)$ for every partition $\{S_1, \ldots, S_K\}$ of N. This is a special case of the condition of superadditivity.

- A game is **monotonic** if $v(S) \leq v(T)$ for all $S \subseteq T$.

- Let π be a permutation of N, then π is a symmetry of G if $v(\pi(S)) = v(S)$ for all S. A game is **symmetric** if all permutations of N are symmetries of G.

We shall now see an application of the concept of the core: the Market.

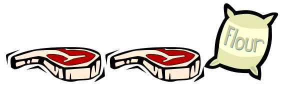
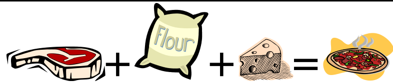
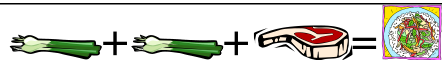
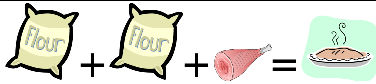
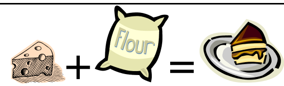
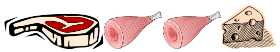
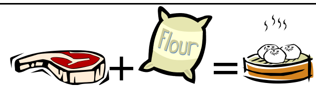
Cooks	*Ingredients*	*Can cook (examples)*
 **cook A**		 ($100)
 **cook B**		 ($80)
 **cook C**		 ($50)
 **cook D**		 ($20)
 **cook E**		 ($50)

Q: How can the cooks form coalition so as to maximise the income?

Markets with Transferable Payoff

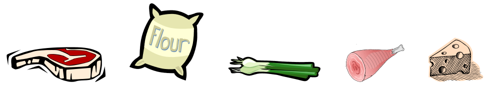
A **market with transferable payoff** consists of

- a finite set N (the set of ***agents***)
- a positive integer ℓ (the number of input goods)
- for each agent $i \in N$ a vector $\omega_i \in \mathbb{R}^{\ell}_{+}$ (the ***endowment*** of agent i)
- for each agent $i \in N$ a continuous, nondecreasing, and concave function $f_i\colon \mathbb{R}^{\ell}_{+} \to \mathbb{R}_{+}$ (the ***production function*** of agent i)

Agents	*Endowments*	*Production Functions*
cook A	 ω_A=(2,1,0,0,0)	 *e.g.*, f_A(1,1,0,0,1)=\$100
cook B	 ω_B=(3,0,1,0,1)	 *e.g.*, f_B(1,0,2,0,0)=\$80
cook C	 ω_C=(0,2,1,1,0)	 *e.g.*, f_C(0,2,0,1,0)=\$50
cook D	 ω_D=(1,0,2,0,1)	 *e.g.*, f_D(0,1,0,0,1)=\$20
cook E	 ω_E=(1,0,0,2,1)	 *e.g.*, f_E(1,1,0,1,0)=\$50

Types of ingredients ($\ell = 5$): , , , , and .

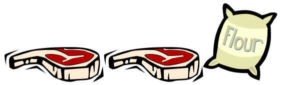
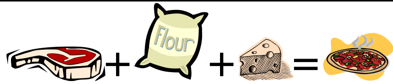
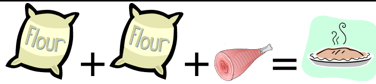
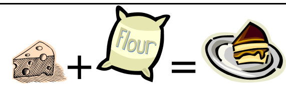
An ***allocation*** is a redistribution of goods.



Agents	*Endowments*	*An example* **allocation**
cook A	$\omega_A=(2,1,0,0,0)$	$z_A=(1,1,1,0,0)$
cook B	$\omega_B=(3,0,1,0,1)$	$z_B=(4,0,0,0,1)$
cook C	$\omega_C=(0,2,1,1,0)$	$z_C=(0,1,1,0,0)$
cook D	$\omega_D=(1,0,2,0,1)$	$z_D=(1,0,2,0,1)$
cook E	$\omega_E=(1,0,0,2,1)$	$z_E=(1,1,0,3,1)$

Formally, a profile $(z_i)_{i \in N}$, where $z_i \in \mathbb{R}^{\ell}_{+}$ for all $i \in N$, for which $\sum_{i \in N} z_i = \sum_{i \in N} \omega_i$, is an ***allocation***.

cook A	ω_A=(2,1,0,0,0)	z_A=(1,1,1,0,0)
cook B	ω_B=(3,0,1,0,1)	z_B=(4,0,0,0,1)
cook C	ω_C=(0,2,1,1,0)	z_C=(0,1,1,0,0)
cook D	ω_D=(1,0,2,0,1)	z_D=(1,0,2,0,1)
cook E	ω_E=(1,0,0,2,1)	z_E=(1,1,0,3,1)

Agents	*Endowments*	*Production Functions*
 **cook A**	 ω_A=(2,1,0,0,0)	 *e.g.*, f_A(1,1,0,0,1)=$100
 **cook C**	 ω_C=(0,2,1,1,0)	 *e.g.*, f_C(0,2,0,1,0)=$50
 **cook D**	 ω_D=(1,0,2,0,1)	 *e.g.*, f_D(0,1,0,0,1)=$20

Q: If cooks A, C and D form a coalition, what is the value of this coalition?

A: $v(S) = \max_{(z_i)_{i\in S}} \left\{ \sum_{i\in S} f_i(z_i) : z_i \in \mathbb{R}_+^{\ell} \text{ and } \sum_{i\in S} z_i = \sum_{i\in S} \omega_i \right\}$.

Markets as Coalitional Games

A **market with transferable payoff** $\langle N, \ell, (\omega_i), (f_i) \rangle$ can be modelled as a coalitional game with transferable payoff $\langle N, v \rangle$, in which for each coalition S

$$v(S) = \max_{(z_i)_{i \in S}} \left\{ \sum_{i \in S} f_i(z_i) : z_i \in \mathbb{R}_+^\ell \text{ and } \sum_{i \in S} z_i = \sum_{i \in S} \omega_i \right\}.$$

($v(S)$ is the maximum total output that the members of S can produce by themselves.)

Nonemptiness of the Cores of Markets as Coalitional Games

The **core** of a **market with transferable payoff** is the core of the associated coalitional game.

PROPOSITION. Every market with transferable payoff has a nonempty core.

Class Discussion

There are seven agents. Two of them have red tickets and five of them have green tickets. If an agent can collect one red ticket and one green ticket, he can get \$1.

RED + GREEN = \$1

Q: Formulate this as a Market with transferable payoff.

- **Agents**: {, , , , , , }
- $\ell = 2$
- **Endowment** of an agent with RED ticket: (1,0).
- **Endowment** of an agent with GREEN ticket: (0,1).
- **Production function** of every agent is the same: $\{(0,0) \mapsto 0, (0,1) \mapsto 0, (1,0) \mapsto 0, (1,1) \mapsto 1\}$.

Q: What is the worth of the different coalitions?

Q: What is the core?

Coalitional Games without Transferable Payoff (NTU Games)

DEFINITION. A **coalitional game** (without transferable payoff)

- a finite set N (the set of **players**)
- a set X (the set of **consequences**)
- a function V that assigns to every nonempty subset S of N a set $V(S) \subseteq X$ (the set of **possible consequences** for **coalition** S).
- for each player $i \in N$ a preference relation $\succsim_i$ on X.

Cores of Coalitional Games without Transferable Payoff (NTU Games)

DEFINITION. The **core of the coalitional game** $\langle N, V, X, (\succsim_i)_{i \in N} \rangle$ is the set of all $x \in V(N)$ for which there is no coalition S and $y \in V(S)$ for which $y \succ_i x$ for all $i \in S$.

Class Discussion

Q: Are coalitional games with transferable payoff special cases of coalitional games without transferable payoff?

- $N = \{1,2,3\}$.
- $v(\{1,2,3\}) = 1$.
- $v(\{1,2\}) = \frac{3}{5},\ v(\{2,3\}) = \frac{3}{5},\ v(\{1,3\}) = \frac{3}{5}$.
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0$.

Q: What are the consequences associated with the coalition $\{1,2\}$?

A: $V(\{1,2\}) = \{(x_1, x_2, 0): x_1 + x_2 = \frac{3}{5}\}$.

Q: What are the consequences associated with the coalition $\{2\}$?

A: $V(\{2\}) = \{(0,0,0)\}$.

Sets of consequences of coalitions:

$$
\begin{aligned}
V(\{1,2,3\}) &= \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1\} \\
V(\{1,2\}) &= \{(x_1, x_2, 0) : x_1 + x_2 = \tfrac{3}{5}\} \\
V(\{2,3\}) &= \{(0, x_2, x_3) : x_2 + x_3 = \tfrac{3}{5}\} \\
V(\{1,3\}) &= \{(x_1, 0, x_3) : x_1 + x_3 = \tfrac{3}{5}\} \\
V(\{1\}) &= \{(0,0,0)\} \\
V(\{2\}) &= \{(0,0,0)\} \\
V(\{3\}) &= \{(0,0,0)\}
\end{aligned}
$$

Players' preferences on the set X of consequences:

Examples.

$$(\frac{2}{5}, 0, \frac{3}{5}) \succ_1 (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$$
$$(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}) \succ_2 (0, \frac{1}{5}, \frac{4}{5})$$
$$\ldots$$

Any coalitional game with transferable payoff $\langle N, v \rangle$ can be associated with a general coalitional game $\langle N, V, X, (\succsim_i)_{i \in N} \rangle$ as follows:

- the set of consequences $X = \mathbb{R}^{|N|}$;
- $V(S) = \{x \in X : \sum_{i \in S} x_i = v(S) \wedge (j \in (N \setminus S) \rightarrow x_j = 0)\}$ for each coalition S;
- $x \succsim_i y$ if and only if $x_i \geq y_i$.

Class Discussion

DEFINITION. The **core of the coalitional game** $\langle N, V, X, (\succsim_i)_{i \in N} \rangle$ is the set of all $x \in V(N)$ for which there is no coalition S and $y \in V(S)$ for which $y \succ_i x$ for all $i \in S$.

Q: Is the consequence $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in the core?

Q: Is the notion of a core in coalitional games with transferable payoff a special case of that in coalitional games without transferable payoff?