Recall

Reflexive spaces

Let X be a Banach space and $Q\colon X \to X^{**}$ be the *canonical map* (*natural embedding*), i.e.,

$$(Qx)(x^*) \coloneqq x^*(x) \text{ or symmetrically, } \langle x^*, Qx\rangle \coloneqq \langle x, x^*\rangle.$$

If $QX = X^{**}$, then X is called *reflexive*.

Let M be a closed subspace of X.

- X reflexive $\iff$ X^* reflexive.
- X reflexive $\iff$ M & X/M reflexive. The $\impliedby$ direction is called *Three space property* and the proof relies on the isometric isomorphism $\tilde{r}\colon X^*/M^\perp \to M^*$ where $\tilde{r}$ is the split of the restriction map $r\colon X^* \to M^*$ along the natural projection $\pi\colon X^* \to X^*/M^\perp$.

If X is a **separable** Banach space, then:

- (Helley's Thm) bounded sequence in X^* has w^*-convergent subsequence.
- (In X^*, a sequence is w^*-convergent $\implies$ norm convergent.) $\iff$ $\dim X < \infty$.
- X is reflexive $\implies$ bounded sequence in X has weakly convergent subsequence.

Let M be a nonzero proper closed subspace of a Banach space X. In the same notation of Lecture Notes, we may have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M^{**} \arrow[r, "\iota^{**}"] & X^{**} \arrow[r, "\pi^{**}"] & (X/M)^{**} \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[u, "Q"] \arrow[r, "\iota"] \arrow[d, dash, dashed] & X \arrow[u, "Q"] \arrow[r, "\pi"] \arrow[d, dash, dashed] & X/M \arrow[u, "Q"] \arrow[r] \arrow[d, dash, dashed] & 0 \\
0 & M^* \arrow[l] & X^* \arrow[l, "r=\iota^*"'] \arrow[d, "\pi"] & (X/M)^* \arrow[l, "\pi^*"'] & 0 \arrow[l] \\
 & & X^*/M^\perp \arrow[ul, "\widetilde{r}"] & &
\end{tikzcd}$$

where the blue arrows denote the isometries by [LN, Prop. 4.12, Prop. 5.1, & Lem. 5.8]. We denote the dashed lines to show the relationship between a Banach space and its dual. Note that $M^* = X^*/M^\perp$ by $\widetilde{r}$ and $M^\perp = (X/M)^*$ by π^*.

$C[0,1]$ is not reflexive

Example 1. $C[0,1]$ is not reflexive.

In the following, we consider the spaces to be Banach spaces. We prove Example 1 by the necessary conditions or properties of reflexive spaces, i.e., by checking that $C[0,1]$ does not have some property that belongs to a reflexive space. Note that $C[0,1]$ is separable.

Proof by closed subspaces of a reflexive space are reflexive. It suffices to construct a embedding $T\colon c_0 \to C[0,1]$. For $n \in \mathbb{N}$, let $d_n = \frac{1}{n} - \frac{1}{n+1}$ and define a 'triangle' shaped function

$$f_n(t) = \begin{cases} \frac{4[t-(1/(n+1)+d_n/4)]}{d_n} & ,t \in [(1/(n+1)+d_n/4), (1/(n+1)+d_n/2)) \\ -\frac{4[t-(1/(n+1)+d_n/2)]}{d_n} + 1 & ,t \in [(1/(n+1)+d_n/2), (1/(n+1)+3d_n/4)] \\ 0 & \text{otherwise.} \end{cases}$$

Then $\operatorname{supp}(f_n) \subset (1/(n+1), 1/n)$ and $\|f_n\|_\infty = 1$. For $x = (x_n)_{n=1}^\infty \in c_0$, define Tx by

$$Tx(t) = \sum_{n=1}^{\infty} x_n f_n(t) \quad \text{ for } t \in [0,1].$$

Since the supports of $f_n(t)$ are disjoint, for every $t \in [0,1]$, $|Tx(t)| \leq \|x\|_\infty$. Since for each x_n there exists t_n such that $Tx(t_n) = x_n$, we have $\|x\|_\infty = \|Tx\|_\infty$. The injection and linearity is easily checked.

Hence T embeds c_0 into $C[0,1]$. Tc_0 is a closed subspace by the completeness of c_0 and not reflexive since c_0 is not reflexive, thus $C[0,1]$ is not reflexive. □

Proof by the dual of a reflexive separable space is separable. Recall from Tutorial 3 that $(C[0,1])^* = BV_0^+[0,1]$. We will show that $BV_0^+[0,1]$ is not separable. For any $x \in (0,1)$, define

$$f_x(t) = \begin{cases} 0 & ,t \in [0,x) \\ 1 & ,t \in [x,1]. \end{cases}$$

Then $f_x \in BV_0^+[0,1]$ and for any $x \neq y$, $V(f_x - f_y) = 2$. However, the cardinarlity of $\{f_x \colon x \in (0,1)\} \subset BV_0^+[0,1]$ is uncountable. Hence $(C[0,1])^* = BV_0^+[0,1]$ is not separable. □

Proof by the weakly sequentially compactness of closed unit ball in a reflexive separable space. Consider the sequence of functions $f_n(x) = x^n \in C[0,1]$. Then $\|f_n\|_\infty = 1$ and every subsequence of f_n will converge pointwisely to $f = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases} \notin C[0,1]$.

For every $x \in [0,1]$, it follows (from Homework 3) that the evaluation functional $\delta_x(f) = f(x)$ for $f \in C[0,1]$ is bounded, thus $\delta_x \in (C[0,1])^*$. Suppose otherwise that $C[0,1]$ is reflexive. Then by [LN, Coro. 6.12] there exists a subsequence (f_{n_k}) weakly convergent in $C[0,1]$, and hence pointwisely convergent in $C[0,1]$ by $\delta_x \in (C[0,1])^*$, which is a contradiction. □