Recall

Hahn-Banach Theorem(s)

Dominated extension. Let Y be a subspace of a vector space X. Let p be a positive homogeneous subadditive function on X. For every linear functional $f \in Y^\sharp$ with $f \le p$ on Y, there exists $F \in X^\sharp$ extending f and $F \le p$ on X.

Continuous extension. Let Y be a subspace of a normed space X. For every $f \in Y^*$, there exists $F \in X^*$ extending f such that $\|F\| = \|f\|$.

Existence of norm-attaining functional. For every x_0 in a normed space X, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x_0) = \|x_0\|$.

Closure point checking. Let Y be a subspace of a normed space X. Then $x \in \overline{Y}$ if and only if for every $f \in X^*$ with $f = 0$ on Y, we have $f(x) = 0$.

Hyperplane separation. Let C be a closed convex subset of a normed space X and $x_0 \in X \setminus C$. Then there exists $f \in X^*$ such that $\sup_{y \in C} f(C) < f(x_0)$.

If the dual space X^* is separable, then X is separable.

Recall that to apply *dominated extension* in the proof of *hyperplane separation*, we have introduced the *Minkowski functional* μ_A defined for a set A. The properties of A determine the behavior of μ_A. The way of defining Minkowski functional is useful to construct natural functions from sets and reveals properties of the space.

Let X, Y be normed spaces and $T \in B(X, Y)$. The adjoint operator $T^*\colon Y^* \to X^*$ is (formally) defined as, for $y^* \in Y^*, x \in X$,

$$T^* y^*(x) := y^*(Tx).$$

Then $T^* \in B(Y^*, X^*)$ and $\|T^*\| = \|T\|$. (In symmetric notation, $\langle x, T^* y^* \rangle := \langle Tx, y^* \rangle := y^*(Tx)$.)

Dual space of $C[a, b]$

Let $[a, b]$ be a closed bounded interval in $\mathbb{R}$. Let $C[a, b]$ be the space of $\mathbb{R}$-valued continuous functions on $[a, b]$ with the sup-norm $\|\cdot\|_\infty$.

Let $\rho\colon [a, b] \to \mathbb{R}$ be a real-valued function and $P\colon \{a = x_0 < \cdots < x_n = b\}$ be a partition of $[a, b]$. Define the variation of ρ with respect to P by

$$V(\rho, P) := \sum_{k=1}^{n} |\rho(x_k) - \rho(x_{k+1})|,$$

and the *total variation* by

$$V(\rho) := \sup_{P \in \mathcal{P}} V(\rho, P)$$

where $\mathcal{P}$ denotes all the paritions of $[a, b]$. A function $\rho\colon [a, b] \to \mathbb{R}$ is called *bounded variation* if $V(\rho) < \infty$. Let $BV[a, b]$ denote the vector space of all the bounded variations.

Let $f \in C[a,b]$ and $\rho \in BV[a,b]$. Let $P : a = x_0 < \cdots < x_n = b$ with tags $t_k \in [x_{k-1}, x_k]$ be a tagged partition. Define the *Riemann-Stieltjes sum* with respect to ρ and P by

$$S(f,\rho,P) = \sum_{k=1}^{n} f(t_k)\Big(\rho(x_k) - \rho(x_{k-1})\Big).$$

Then the *Riemann-Stieltjes integral* is defined by

$$\int_a^b f(x)d\rho(x) \coloneqq \lim_{\|P\|\to 0} S(f,\rho,P).$$

where $\|P\|$ denotes the diameter of a partition. The Riemann-Stieljes integral exists by the uniform continuity of f on $[a,b]$.

Observe that $V(\cdot)$ satisfies non-negativity, scaling property and the triangle inequality. However, $V(\cdot)$ is not non-degenerate since $V(\rho) = 0$ only implies that ρ is constant on $[a,b]$. Hence we restrict to the following subspace (the notation may not be standard)

$$BV_0[a,b] = \{\rho \in BV[a,b] \colon \rho(a) = 0\}.$$

Then it is readily checked that $BV_0[a,b]$ is a Banach space under the norm $V(\cdot)$.

To justify the injectivity in our proof, we further remove the redundancy and modify the space to

$$BV_0^+[a,b] \coloneqq \left\{\rho \in BV_0[a,b] \colon \lim_{y\to x+} \rho(y) = \rho(x),\ \forall\, x \in (a,b)\right\}. \tag{1}$$

It can be checked that $BV_0^+[a,b]$ is closed in $BV_0[0,1]$ since the right continuity is preserved by uniform convergence and $\|\cdot\|_\infty \le V(\cdot)$ on $BV_0[a,b]$.

Remark. The elements in (1) are defined explicitly. They can viewed as representatives of classes in a quotient space whose definition shares the same purpose to establish the injectivity. The details are given in the next section.

It follows from Jordan decomposition of $\rho \in BV[a,b]$ that $\rho = \rho_+ - \rho_-$ where ρ_+ and ρ_- are increasing. Hence for $x \in (a,b)$,

$$\rho^* \coloneqq \lim_{y\to x+} \rho = \lim_{y\to x+} \rho_+(x) - \lim_{y\to x+} \rho_-(x) \tag{2}$$

is well defined, i.e., $\rho^* \in BV_0^+[a,b]$. Since ρ_+ and ρ_- are monotone, ρ^* and ρ only differ on the at most countable discountinuities in (a,b). Moreover, for $\rho \in BV_0[a,b]$,

$$V(\rho^*) \le V(\rho) \text{ and } \int_a^b f d\rho = \int_a^b f d\rho^* \text{ for all } f \in C[a,b] \tag{3}$$

and for $\rho_1, \rho_2 \in BV_0^+[a,b]$,

$$\text{if } \int_a^b f d\rho_1 = \int_a^b f d\rho_2 \text{ for all } f \in C[a,b], \text{ then } \rho_1 = \rho_2 \in BV_0^+[a,b]. \tag{4}$$

The proofs of (3) and (4) are given in Appendix. After the above modifications, we are ready to state the main theorem.

Theorem 1. *Under above notation, $C[a,b]^* = BV_0^+[a,b]$.*

Proof. We first introduce some convenient notation. For $f \in C[a,b]$ and $\rho \in BV_0[a,b]$, denote

$$\langle f, \rho \rangle \coloneqq \int_a^b f d\rho.$$

Then $\langle \cdot, \cdot \rangle \colon C[a,b] \times BV_0[a,b] \to \mathbb{R}$ is well defined by the existence of Riemann-Stieltjes integral. It follows from the linearity of summation that for $\alpha \in \mathbb{R}$, $f, \tilde{f} \in C[a,b]$ and $\rho, \tilde{\rho} \in BV_0[a,b]$,

$$\langle \alpha f + \tilde{f}, \rho \rangle = \alpha \langle f, \rho \rangle + \langle \tilde{f}, \rho \rangle \text{ and } \langle f, \alpha\rho + \tilde{\rho} \rangle = \alpha \langle f, \rho \rangle + \langle f, \tilde{\rho} \rangle. \tag{5}$$

And since for any partition P with tags, we have

$$\left| \sum_{k=1}^n f(t_k)\Big(\rho(x_k) - \rho(x_{k-1})\Big) \right| \le \|f\|_\infty \sum_{k=1}^n |\rho(x_k) - \rho(x_{k-1})| = \|f\|_\infty V(\rho, P) \le \|f\|_\infty V(\rho).$$

Then take the limit $\|P\| \to 0$ on the LHS, we have

$$|\langle f, \rho \rangle| \le \|f\|_\infty V(\rho). \tag{6}$$

By (5) and (6), for any fixed $\rho \in BV_0^+[a,b]$, the map $\langle \cdot, \rho \rangle \colon C[a,b] \to \mathbb{R}$ is linear and bounded, i.e., $\langle \cdot, \rho \rangle \in C[a,b]^*$. To complete the proof, we will prove the map

$$\begin{aligned} T \colon BV_0^+[a,b] &\to C[a,b]^* \\ \rho &\mapsto \langle \cdot, \rho \rangle \end{aligned}$$

is an isometric isomorphism.

(i) (linear and injective) By (5), T is linear. By (4), T is injective.

(ii) (surjective) For any $\Lambda \in C[a,b]^*$, we will first find $\rho \in BV_0[a,b]$ such that $\Lambda f = \langle f, \rho \rangle$ for all $f \in C[a,b]$ and then modify ρ to $\rho^* \in BV_0^+[a,b]$.

(Inspired by the 'formal' argument that $\rho(x) - \rho(a) = \int_{[a,x]} d\rho = \langle \chi_{[a,x]}, \rho \rangle \approx \Lambda \chi_{[a,x]}$. But we can NOT apply Λ directly to $\chi_{[a,x]}$ since $\chi_{[a,x]}$ is not continuous, which is where *Hahn-Banach* comes into the stage.) Observing that $C[a,b]$ is a subspace in the normed space $B[a,b]$ of bounded functions, we apply *Hahn-Banach theorem* to extend Λ to $\widetilde{\Lambda} \in B[a,b]^*$ with $\|\widetilde{\Lambda}\| = \|\Lambda\|$. Hence we are able to define $\rho(x) \coloneqq \widetilde{\Lambda}\chi_{[a,x]}$ for $x \in (a,b]$ and $\rho(a) \coloneqq 0$.

First we check $\rho \in BV_0[a,b]$. For any partition P, write $\theta_k = \operatorname{Sgn}(\rho(x_k) - \rho(x_{k-1}))$. Then by the linearity and $\|\widetilde{\Lambda}\| = \|\Lambda\|$,

$$\begin{aligned} \sum_{k=1}^n |\rho(x_k) - \rho(x_{k-1})| &= \sum_{k=1}^n \theta_k \left(\rho(x_k) - \rho(x_{k-1})\right) \\ &= \theta_1 \widetilde{\Lambda}\chi_{[a,x_1]} + \sum_{k=2}^n \theta_k \left(\widetilde{\Lambda}\chi_{[a,x_k]} - \widetilde{\Lambda}\chi_{[a,x_{k-1}]} \right) \\ &= \widetilde{\Lambda}\Big(\theta_1 \chi_{[a,x_1]} + \sum_{k=2}^n \theta_k (\chi_{[a,x_k]} - \chi_{[a,x_{k-1}]}) \Big) \\ &\le \|\widetilde{\Lambda}\| \|\theta_1 \chi_{[a,x_1]} + \sum_{k=2}^n \theta_k \chi_{(x_{k-1},x_k]}\|_\infty \\ &= \|\Lambda\|, \end{aligned} \tag{7}$$

where in the last equality holds since the function is of sup-norm 1. Take supremum over partition P on LHS to obtain $V(\rho) \le \|\Lambda\|$. Hence $\rho \in BV_0[a,b]$.

Next we check $\Lambda f = \langle f, \rho \rangle$ for all $f \in C[a,b]$.

(Inspired by the facts that Riemann-Stieljes integral is contuinous w.r.t. $\|\cdot\|_\infty$ and f can by uniformly approximated by step functions.) Let $\varepsilon > 0$. By the uniform continuity of f, there exists δ_1 such that for any partition P with $\|P\| \le \delta_1$, $\sup_{x\in[x_{k-1},x_k]}|f(x) - f(x_k)| \le \varepsilon$ for $1 \le k \le n$. Then define the step function $\tilde{f} = f(x_1)\chi_{[a,x_1]} + \sum_{k=2}^n f(x_k)\chi_{(x_{k-1},x_k]}$

$$\|f - \tilde{f}\|_\infty \le \varepsilon \tag{8}$$

By the definition of Riemann-Stieltjes integral, there exists $\delta_2 > 0$, such that for any partition P with $\|P\| \le \delta_2$ and the tags $t_k = x_k, 1 \le k \le n$, we have

$$|\langle f, \rho \rangle - S(f, \rho, P)| \le \varepsilon. \tag{9}$$

It follows from a similar check in (7) that $S(f, \rho, P) = \widetilde{\Lambda}\tilde{f}$. Hence when $\|P\| < \min\{\delta_1, \delta_2\}$, by (8) and (9),

$$\begin{aligned}
|\langle f, \rho \rangle - \Lambda f| &= \left|\langle f, \rho \rangle - \widetilde{\Lambda} f\right| \\
&\le \left|\langle f, \rho \rangle - \widetilde{\Lambda}\tilde{f}\right| + |\widetilde{\Lambda}\tilde{f} - \widetilde{\Lambda} f| \\
&\le |\langle f, \rho \rangle - S(f, \rho, P)| + \|\widetilde{\Lambda}\|\|f - \tilde{f}\|_\infty \\
&\le (1 + \|\Lambda\|)\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we have $\langle f, \rho \rangle = \Lambda f$.

Replace ρ with ρ^* defined in (2). It follows from (3) that $\langle f, \rho^* \rangle = \langle f, \rho \rangle = \Lambda f$ and $V(\rho^*) \le V(\rho) \le \|\Lambda\|$.

(iii) (isometric) Let $\rho \in BV_0^+[a,b]$. It follows from (6), that $\|T\rho\| \le V(\rho)$. By (ii), there exists $\widetilde{\rho} \in BV_0^+[a,b]$ with $V(\widetilde{\rho}) \le \|T\rho\|$ and $\langle f, \widetilde{\rho} \rangle = \langle f, \rho \rangle$. It follows from (i) that $\rho = \widetilde{\rho}$. Hence $V(\rho) = V(\widetilde{\rho}) \le \|T\rho\|$, thus $V(\rho) = \|T\rho\|$.

□

Remark. A similar proof shows Theorem 1 also holds when the scalar field is $\mathbb{C}$. These results are the special cases of Riesz representation of $C_0(X)^*$ via Borel regular measures when X is locally compact Hausdorff.

It's good to stop here.

Remark. Actually the explicit candidate in (1) is found in a 'cheated' way. We can reason as following, for $\Lambda \in C[a,b]^*$, by the general Riesz representation, there exists a unique Borel regular measure $\mu \in M[a,b]$ such that $\Lambda f = \int_{[a,b]} f d\mu$ (the integral is defined in Lebesgue way). Then the cumulative distribution function $F_\mu(t) \coloneqq \mu[a,t]$ is right countinuous (by the continuity of measure) and of bounded variation. Moreover, μ is the measure extension of the premeasure induced by F_μ on semiring $\{a, \emptyset, (c,d], a \le c < d \le b\}$. However, notice that for the Dirac measure δ_a (representing the evaluation $\Lambda f = f(a)$), the Riemann-Stieljes integral w.r.t. $F_{\delta_a} = \chi_{[a,b]}$

identically vanish on $C[a,b]$! Then we realize the Riemann-Stieltjies integral will 'forget' the jump at a if the ρ is right-continuous at a. Hence we made the following modification for $\mu \in M[a,b]$,

$$\Lambda f = \int_{[a,b]} f d\mu = \mu\{a\} f(a) + \int_a^b f dF_\mu = \int_a^b f d(F_\mu + G_{\mu\{a\}})$$

where $G_{\mu\{a\}}(a) \coloneqq -\mu\{a\}\chi_{\{a\}}$. Hence $\widetilde{F_\mu} \coloneqq F_\mu + G_{\mu\{a\}} \in BV_0^+[a,b]$.

A quotient space perspective

Instead of explicitly finding the representatives like (1), another natural way to achieve the injectivity is to define a quotient space. First, we define a subspace of $BV_0[a,b]$ as

$$H \coloneqq \{\rho \in BV_0[a,b] \colon \langle f, \rho \rangle = 0, \ \forall f \in C[a,b]\}. \tag{10}$$

By (6), H is closed as the intersection of the kernels of continuous function $\langle f, \cdot \rangle$. Explicitly, H is exactly the subspace of $BV_0[a,b]$ consisting of the functions differing from 0 only on at most countable points on (a,b). Hence $BV_0[a,b]/H$ is well-defined. Let π be the natural projection. Define $T \colon BV_0[a,b]/H \to C[a,b]^*$ by $T(\pi(\rho)) = \langle \cdot, \rho \rangle$. Recall that $\|\pi(\rho)\| \leq \|\rho\|$ for $\pi(\rho) \in BV_0[a,b]/H$ and $|\langle f, \rho \rangle| = |\langle f, \rho + h \rangle| \leq \|f\|_\infty \|\rho + h\|$ for all $h \in H$. Then taking infimum with respect to $h \in H$ leads to $\|T\pi(\rho)\| \leq \|\pi(\rho)\|$. The linear and injectivity follows as we expected. The surjectivity is obtained in the same way in the proof of Theorem 1. Thus

$$C[a,b]^* = BV_0[a,b]/H.$$

Appendix

In this Appendix, we will establish the intuition that for $f \in C[a,b]$, the Riemann-Stieltjes integral $\langle f, \rho \rangle$ is invariant under a change of values at the countable **interior** discountinuites of $\rho \in BV_0[a,b]$.

Lemma 2. *Let $c \in (a,b)$ and $\alpha \in \mathbb{K}$. The Riemann-Stieljes integral $\int_a^b f d(\alpha \chi_{\{c\}}) = 0$ for all $f \in C[a,b]$.*

Proof. Let $f \in C[a,b]$ and P be any tagged partition of $[a,b]$. If c is in the interior of some $[x_k, x_{k-1}]$, then $S(f, \chi_{\{c\}}, P) = 0$. If $c = x_k$ for some $x_k \in (a,b)$, then by choosing the tag $c = x_k$ at both $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$, we have $S(f, \chi_{\{c\}}, P) = \alpha f(c) - \alpha f(c) = 0$. Hence $\int_a^b f d(\alpha \chi_{\{c\}}) = 0$. □

Lemma 3. *Let $\rho \in BV_0[a,b]$. Denote $(c_n)_{n=1}^\infty$ the discountinuous points of ρ in (a,b) and $(\alpha_n)_{n=1}^\infty$ the oscillations of ρ, more precisely, $\alpha_n = |\lim_{y \to c_n -} \rho(y) - \lim_{y \to c_n +} \rho(y)|$. For any sequence $(\beta_n)_{n=1}^\infty$ such that $|\beta_n| \leq \alpha_n$ for all $n \in \mathbb{N}$, define $\eta = \sum_{n=1}^\infty \beta_n \chi_{\{c_n\}}$. Then $\eta \in BV_0[a,b]$ and $\int_a^b f d\eta = 0$ for all $f \in C[a,b]$.*

If ρ has only finitely many discontinuities, Lemma 2 finishes the proof.

Proof. Since $\rho \in BV_0[a,b]$, $\sum_{n=1}^{\infty}|\beta_n| \leq \sum_{n=1}^{\infty} \alpha_n \leq V(\rho) < \infty$. Hence for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty}|\beta_n| \leq \varepsilon/2$. Then

$$V(\sum_{n=N+1}^{\infty} \beta_n \chi_{\{c_n\}}) \leq 2 \sum_{n=N+1}^{\infty} |\beta_n| \leq \varepsilon.$$

Let $f \in C[a,b]$. By Lemma 2 and (6),

$$|\langle f, \eta \rangle| = \left| \langle f, \sum_{n=1}^{N} \beta_n \chi_{\{c_n\}} \rangle + \langle f, \sum_{n=N+1}^{\infty} \beta_n \chi_{\{c_n\}} \rangle \right| \leq 0 + \|f\|_\infty \varepsilon.$$

Letting $\varepsilon \to 0$, we have $\langle f, \eta \rangle = 0$. ☐

Proof of (3). Let $\rho \in BV_0[a,b]$. Define $\rho_n := \begin{cases} \rho(x+1/n) & \text{if } x \in [a, b-1/n] \\ \rho(b) & \text{if } x \in (b-1/n, b]. \end{cases}$ Then it is readily checked that $V(\rho_n) \leq V(\rho)$ and $\rho^* = \lim_{n\to\infty} \rho_n$. By the lower semi-continuity of $V(\cdot)$ (see e.g. [Royden-Fitzpatrick Real Analysis, Sec 6.3 Problem 33]), $V(\rho^*) \leq V(\rho)$.

By the definition of ρ^*, we have $\rho^* - \rho = \sum_{n=1}^{\infty} \beta_n \chi_{\{c_n\}}$ for some sequence $(\beta_n)_{n=1}^{\infty}$ satisfying the condition in Lemma 3. Hence $\langle f, \rho^* \rangle = \langle f, \rho \rangle$ for all $f \in C[a,b]$. ☐

Proof of (4). It suffices to prove that if $\rho \in BV_0^+[a,b]$ and $\langle f, \rho \rangle = \int_a^b f\, d\rho = 0$ for all $f \in C[a,b]$, then $\rho = 0$. Define a right continuous function ρ^* by $\rho^*(a) := \lim_{x\to a+} \rho(0)$ and $\rho^*(x) = \rho(x)$ for $x \in (a,b]$. Then we can generate a measure μ on $[a,b]$ by setting $\mu[a,x] = \rho^*(x)$.

For $n \in \mathbb{N}$, let $f_n(x) := \begin{cases} 1 & [a,c] \\ \text{linear} & (c, c+1/n] \\ 0 & (c+1/n, b] \end{cases}$. The definition of Riemann-Stieljes integral implies that $\int_a^b f_n d(\rho^*(a)\chi_{\{a\}}) = -\rho^*(a) f_n(a)$ and recall $\rho^* = \rho + \rho^*(a)\chi_{\{a\}}$. Then for any $c \in (a,b]$,

$$\begin{aligned} \rho^*(c) = \mu[a,c] = \lim_{n\to\infty} \int f_n d\mu &= \lim_{n\to\infty} \left(\rho^*(a) f_n(a) + \int_a^b f_n d\rho^* \right) \\ &= \lim_{n\to\infty} \left(\rho^*(a) f_n(a) + \int_a^b f_n d\rho + \int_a^b f_n d(\rho^*(a)\chi_{\{a\}}) \right) = 0 \end{aligned}$$

where the second equality is by Lebesgue dominated convergence theorem. Hence $\rho^* = 0$ on $(a,b]$, and so $\rho = 0$ on $[a,b]$. ☐