## Recall

## Reflexive spaces

Let X be a Banach space and  $Q: X \to X^{**}$  be the canonical map (natural embedding), i.e.,

$$(Qx)(x^*) := x^*(x)$$
 or symmetrically,  $\langle x^*, Qx \rangle := \langle x, x^* \rangle$ .

If  $QX = X^{**}$ , then X is called *reflexive*.

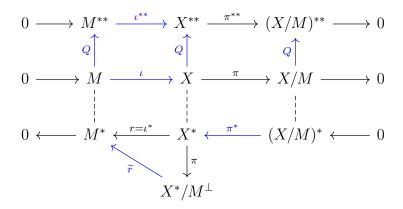
Let M be a closed subspace of X.

- X reflexive  $\iff X^*$  reflexive.
- X reflexive  $\iff M \& X/M$  reflexive. The  $\iff$  direction is called Three space property and the proof relies on the isometric isomorphism  $\tilde{r} \colon X^*/M^{\perp} \to M^*$  where  $\tilde{r}$  is the split of the restriction map  $r \colon X^* \to M^*$  along the natural projection  $\pi \colon X^* \to X^*/M^{\perp}$ .

If X is a **separable** Banach space, then:

- (Helley's Thm) bounded sequence in  $X^*$  has  $w^*$ -convergent subsequence.
- (In  $X^*$ , a sequence is  $w^*$ -convergent  $\implies$  norm convergent.)  $\iff$  dim  $X < \infty$ .
- X is reflexive  $\implies$  bounded sequence in X has weakly convergent subsequence.

Let M be a nonzero proper closed subspace of a Banach space X. In the same notation of Lecture Notes, we may have the following diagram



where the blue arrows denote the isometries by [LN, Prop. 4.12, Prop. 5.1, & Lem. 5.8]. We denote the dashed lines to show the relationship between a Banach space and its dual. Note that  $M^* = X^*/M^{\perp}$  by  $\widetilde{r}$  and  $M^{\perp} = (X/M)^*$  by  $\pi^*$ .

## C[0,1] is not reflexive

**Example 1.** C[0,1] is not reflexive.

In the following, we consider the spaces to be Banach spaces. We prove Example 1 by the necessary conditions or properties of reflexive spaces, i.e., by checking that C[0,1] does not have some property that belongs to a reflexive space. Note that C[0,1] is separable.

Proof by closed subspaces of a reflexive space are reflexive. It suffices to construct a embedding  $T: c_0 \to C[0,1]$ . For  $n \in \mathbb{N}$ , let  $d_n = \frac{1}{n} - \frac{1}{n+1}$  and define a 'triangle' shaped function

$$f_n(t) = \begin{cases} \frac{4[t - (1/(n+1) + d_n/4)]}{d_n} &, t \in [(1/(n+1) + d_n/4), (1/(n+1) + d_n/2)) \\ -\frac{4[t - (1/(n+1) + d_n/2)]}{d_n} + 1 &, t \in [(1/(n+1) + d_n/2), (1/(n+1) + 3d_n/4)] \\ 0 & \text{otherwise.} \end{cases}$$

Then supp $(f_n) \subset (1/(n+1), 1/n)$  and  $||f_n||_{\infty} = 1$ . For  $x = (x_n)_{n=1}^{\infty} \in c_0$ , define Tx by

$$Tx(t) = \sum_{n=1}^{\infty} x_n f_n(t)$$
 for  $t \in [0, 1]$ .

Since the supports of  $f_n(t)$  are disjoint, for every  $t \in [0,1]$ ,  $|Tx(t)| \leq ||x||_{\infty}$ . Since for each  $x_n$  there exists  $t_n$  such that  $Tx(t_n) = x_n$ , we have  $||x||_{\infty} = ||Tx||_{\infty}$ . The injection and linearity is easily checked.

Hence T embeds  $c_0$  into C[0,1].  $Tc_0$  is a closed subspace by the completeness of  $c_0$  and not reflexive since  $c_0$  is not reflexive, thus C[0,1] is not reflexive.

Proof by the dual of a reflexive separable space is separable. Recall from Tutorial 3 that  $(C[0,1])^* = BV_0^+[0,1]$ . We will show that  $BV_0^+[0,1]$  is not separable. For any  $x \in (0,1)$ , define

$$f_x(t) = \begin{cases} 0 & , t \in [0, x) \\ 1 & , t \in [x, 1]. \end{cases}$$

Then  $f_x \in BV_0^+[0,1]$  and for any  $x \neq y$ ,  $V(f_x - f_y) = 2$ . However, the cardinarlity of  $\{f_x \colon x \in (0,1)\} \subset BV_0^+[0,1]$  is uncountable. Hence  $(C[0,1])^* = BV_0^+[0,1]$  is not separable.

Proof by the weakly sequentially compactness of closed unit ball in a reflexive separable space. Consider the sequence of functions  $f_n(x) = x^n \in C[0,1]$ . Then  $||f_n||_{\infty} = 1$  and every subsequence of  $f_n$  will converge pointwisely to  $f = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases} \notin C[0,1]$ .

For every  $x \in [0, 1]$ , it follows (from Homework 3) that the evaluation functional  $\delta_x(f) = f(x)$  for  $f \in C[0, 1]$  is bounded, thus  $\delta_x \in (C[0, 1])^*$ . Suppose otherwise that C[0, 1] is reflexive. Then by [LN, Coro. 6.12] there exists a subsequence  $(f_{n_k})$  weakly convergent in C[0, 1], and hence pointwisely convergent in C[0, 1] by  $\delta_x \in (C[0, 1])^*$ , which is a contradiction.