THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1
Solution to Homework 5

1. Prove that in a complex (resp. real) inner product space, $x \perp y$ if and only if

$$\|x + \lambda y\| = \|x - \lambda y\| \tag{1}$$

for all scalars $\lambda \in \mathbb{C}$ (resp. $\mathbb{R}$).

Proof. Let X denote an inner product space with scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.
$(\implies)$ Let $x, y \in X$. If $x \perp y$, then $x \perp \pm\lambda y$ for all $\lambda \in \mathbb{K}$. By Pythagorean theorem,

$$\|x + \lambda y\|^2 = \|x\|^2 + \|\lambda y\|^2 = \|x - \lambda y\|^2.$$

$(\impliedby)$ It follows from Polarization identities (see e.g., [Tutorial 6, Theorem 2]) that for $\forall\, x, y \in X$, if $\mathbb{K} = \mathbb{R}$, then

$$\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2), \tag{2}$$

and if $\mathbb{K} = \mathbb{C}$, then

$$\langle x, y\rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\right). \tag{3}$$

Hence if $\mathbb{K} = \mathbb{R}$, then it follows from (2) that $\langle x, y\rangle = 0$ by taking $\lambda = 1$ in (1). If $\mathbb{K} = \mathbb{C}$, then it follows from (3) that $\langle x, y\rangle = 0$ by taking $\lambda = 1$ and i in (1). $\square$

— *THE END* —