THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1

Solution to Homework 2

1. Show that vectors (e_n), where e_n is the sequence whose n-th term is 1 and all other terms are zero,

$$
\begin{aligned}
e_1 &= (1,0,0,\ldots),\\
e_2 &= (0,1,0,\ldots),\\
&\cdots
\end{aligned}
$$

form a Schauder basis in ℓ^p for every $p \in [1,+\infty)$ and in the spaces c_0 and c_{00}.

Proof. Let $x = (x(i))_{i=1}^{\infty}$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$. For every $n \in \mathbb{N}$, define $s_n = \sum_{i=1}^{n} x(i)e_i$. Then $s_n \in c_{00} \subset \ell^{1\le p<\infty} \subset c_0$ for all $n \in \mathbb{N}$. Note that $x - s_n = (0,\ldots,0,x(n+1),\ldots)$.

Convergence of the series

(a) If $x \in \ell^p$, $1 \le p < \infty$, then $(\sum_{i=1}^{\infty} |x(i)|^p)^{1/p} = \|x\|_p < \infty$. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \ge N$, $(\sum_{i=n+1}^{\infty} |x(i)|^p)^{1/p} \le \varepsilon$, thus

$$\|x - s_n\|_p = (\sum_{i=n+1}^{\infty} |x(i)|^p)^{1/p} \le \varepsilon.$$

Hence s_n converges to x in $\|\cdot\|_p$, i.e., $x = \lim_{n\to\infty} s_n = \sum_{i=1}^{\infty} x(i)e_i$ in ℓ^p.

(b) If $x \in c_{00}$ or c_0, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \ge N$, $\sup_{i\ge n+1} |x(i)| \le \varepsilon$, thus

$$\|x - s_n\|_\infty = \sup_{i\ge n+1} |x(i)| \le \varepsilon.$$

Hence s_n converges to x in $\|\cdot\|_\infty$, i.e., $x = \lim_{n\to\infty} s_n = \sum_{i=1}^{\infty} x(i)e_i$ in c_{00} or c_0.

Uniqueness of the expansion

Let $(\alpha(i))_{i=1}^{\infty}$ be a sequence of scalars such that $\sum_{i=1}^{\infty} \alpha(i)e_i = 0 \in \ell^p, 1 \le p < \infty$ or c_{00} or c_0. It suffices to prove that $\alpha(i) = 0$ for all $i \in \mathbb{N}$.

Suppose on the contrary that there exists $n_0 \in \mathbb{N}$ such that $\alpha(n_0) \ne 0$. Let $\|\cdot\|$ denote $\|\cdot\|_\infty$ or $\|\cdot\|_p$. Since $\|\cdot\|_p \ge \|\cdot\|_\infty$, we have $\|\cdot\| \ge \|\cdot\|_\infty$. By the convergence of $\sum_{i=1}^{\infty} \alpha(i)e_i$ in $\|\cdot\|$, there exists $N \ge n_0$ such that $\|\sum_{i=N+1}^{\infty} \alpha(i)e_i\| < |\alpha(n_0)|/2$. It follows from the triangle inequality that

$$
\begin{aligned}
0 = \|\sum_{i=1}^{\infty} \alpha(i)e_i\| &\ge \|\sum_{i=1}^{N} \alpha(i)e_i\| - \|\sum_{i=N+1}^{\infty} \alpha(i)e_i\|\\
&\ge \|\sum_{i=1}^{N} \alpha(i)e_i\|_\infty - \frac{|\alpha(n_0)|}{2}\\
&\ge |\alpha(n_0)| - \frac{|\alpha(n_0)|}{2} > 0,
\end{aligned}
$$

which is a contradiction. Hence $\alpha(n) = 0$ for all $n \in \mathbb{N}$.

□

2. Let $X = \{x \in C[0,1] \colon x(0) = 0\}$ with the sup-norm, and let f be a linear functional on X defined by
$$f(x) = \int_0^1 x(t)dt.$$
Show that $\|f\| = 1$.

Proof. Since $|f(x)| = \left|\int_0^1 x(t)dt\right| \leq \int_0^1 |x(t)|dt \leq \|x\|_\infty$, we have $\|f\| \leq 1$.

For any $\varepsilon > 0$ small, define
$$x_\varepsilon(t) = \begin{cases} \dfrac{t}{\varepsilon} & \text{if } t \in [0, \varepsilon], \\ 1 & \text{if } t \in (\varepsilon, 1]. \end{cases}$$
Then $x_\varepsilon \in X$ with $\|x_\varepsilon\|_\infty = 1$ and $|f(x_\varepsilon)| = |\int_0^1 x_\varepsilon(t)dt| = 1 - \varepsilon/2$. Hence $\|f\| \geq 1 - \varepsilon/2$. Letting $\varepsilon \to 0$, we have $\|f\| \geq 1$, thus $\|f\| = 1$. $\square$

— *THE END* —