Remark. Please be kindly reminded that there is no tutorial this week. We include this note only for the completeness.

Recall

Projection and decomposition in Banach spaces

Let X be a Banach space. A bounded linear operator $P\colon X \to X$ is called a *projection* if $P^2 = P$ (*idempotent*). For each projection P there is a decomposition $X = \operatorname{Im}(P) \oplus \operatorname{Ker}(P)$. A **closed** subspace M is called *complemented* if there exists a **closed** subspace N such that $X = M \oplus N$.

- $\{$ a closed subspace M is complemented $\}$ $\iff$ $\{$ $\exists$ projection P with $\operatorname{Im}(P) = M$ $\}$.
- Any subspace of finite dimension is complemented.
- c_0 is not complemented in ℓ^∞, nor a dual space of any normed space.
- Let $Q\colon X \to X^{**}$, $\widetilde{Q}\colon X^* \to X^{***}$ be the canonical mappings and $Q^*\colon X^{***} \to X^*$ be the adjoint operator of Q, that is

$$\begin{tikzcd} X^* \arrow[r, "\widetilde{Q}"] \arrow[d, dash, dashed] & X^{***} \arrow[l, bend left, "Q^*"] \arrow[d, dash, dashed] \\ X \arrow[r, "Q"] & X^{**} \end{tikzcd}$$

  Then $Q^*\widetilde{Q} = I_{X^*}$, where I_{X^*} denotes the identity map on X^*. Hence $P := \widetilde{Q}Q^*$ is a projection on X^{***}. This implies

$$X^{***} = \operatorname{Im}(P) \oplus \operatorname{Ker}(P) = \widetilde{Q}X^* \oplus (QX)^\perp \cong X^* \oplus X^\perp.$$

  In particular, we have $(\ell^\infty)^* = \ell^1 \oplus c_0^\perp$ by letting $X = c_0$.

- Suppose norms are considered on the direct sum and denote $X = Y \oplus_{\ell_1} Z$ if $X = Y \oplus Z$ and $\|x\| = \|y\| + \|z\|$ for $x = y + z$, $y \in Y, z \in Z$. Then

$$(\ell^\infty)^* = \ell^1 \oplus_{\ell_1} c_0^\perp.$$

Main content

Proposition 1. *Let X, Y be Banach spaces and $T\colon X \to Y$ be a bounded linear operator. Then $\operatorname{Im} T$ is closed in Y if and only if there exists $C < \infty$ such that $d(x, \operatorname{Ker} T) \le C\|Tx\|$ for $x \in X$.*

Proof. Let $\pi\colon X \to X/\operatorname{Ker} T$ be the natural projection, that is,

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"] & \operatorname{Im} T \subset Y \\ X/\operatorname{Ker} T \arrow[ur, "\widetilde{T}"] & \end{tikzcd}$$

Define $\widetilde{T}\colon X/\operatorname{Ker} T \to \operatorname{Im} T$ canonically by $\widetilde{T}(\pi x) := Tx$ for $\pi x \in X/\operatorname{Ker} T$ and some $x \in X$. Then $\widetilde{T}$ is well defined and injective.

($\Longrightarrow$) Since X, Y are Banach spaces and $\operatorname{Im} T$ is closed, then $X/\operatorname{Ker} T$ and $\operatorname{Im} T$ are both Banach spaces. The Open Mapping Theorem implies that $\widetilde{T}^{-1}$ is continuous, thus bounded. Hence $d(x, \operatorname{Ker} T) = \|\pi x\| \leq \|\widetilde{T}^{-1}\|\|Tx\|$.

($\Longleftarrow$) Since $\|\widetilde{T}^{-1}(Tx)\| = \|\pi x\| = d(x, \operatorname{Ker} T) \leq C\|Tx\|$, then $\widetilde{T}$ is continuous. This implies that $\operatorname{Im} T$ is complete since $X/\operatorname{Ker} T$ is complete. Hence $\operatorname{Im} T$ is closed in Y. $\square$

Proposition 2. *Let M be a closed subspace of a normed space X. Then X is complete if and only if M and X/M are both complete.*

Proof. Let $\iota\colon M \to X$ be the natural inclusion and $\pi\colon X \to X/M$ be the natural projection, that is,

$$0 \longrightarrow M \xrightarrow{\ \iota\ } X \xrightarrow{\ \pi\ } X/M \longrightarrow 0.$$

($\Longrightarrow$) The proof of this direction is standard and omitted.

($\Longleftarrow$) Let (x_n) be a Cauchy sequence in X. Then (πx_n) is a Cauchy sequence in X/M since $\|\pi x_n\| \leq \|x_n\|$. By the completeness of X/M, there exists $\pi x \in X/M$ for some $x \in X$ such that $\|\pi(x - x_n)\| = \|\pi x - \pi x_n\| \to 0$ as $n \to \infty$. Hence there exists a sequence (m_n) in M such that

$$\|x_n - x - m_n\| \to 0.$$

This implies that (m_n) is a Cauchy sequence since (x_n) is Cauchy sequence. By the completeness of M, there exists $m \in M$ such that

$$\|m - m_n\| \to 0.$$

Hence

$$\|x_n - (x + m)\| = \|x_n - x - m_n + m_n - m\| \leq \|x_n - x - m_n\| + \|m_n - m\| \to 0$$

as $n \to \infty$, which means $x_n \xrightarrow{\|\cdot\|} x + m$ as $n \to \infty$. $\square$

Remark. A property P is called a *three-space property* if P satisfies a relationship like above. Recall that reflexitivity and separability are three-space properties.

Corollary 3. *Let X, Y be Banach spaces and $T, K \in B(X, Y)$. If $\operatorname{Im} T$ is closed and $\operatorname{Im} K$ is finite dimensional, then $\operatorname{Im}(T + K)$ is closed.*

Proof. Write $Z := \operatorname{Im}(T + K) = \operatorname{Im} T + \operatorname{Im} K$. Then Z is a normed space. Since $\operatorname{Im} T$ is closed in the Banach space Y, we have $\operatorname{Im} T$ is complete, thus closed in Z. It follows from $\dim(Z/\operatorname{Im} T) \leq \dim \operatorname{Im} K < \infty$ that $Z/\operatorname{Im} T$ is complete. Applying Proposition 2 to

$$0 \longrightarrow \operatorname{Im} T \xrightarrow{\ \iota\ } Z \xrightarrow{\ \pi\ } Z/\operatorname{Im} T \longrightarrow 0$$

shows that $Z = \operatorname{Im}(T + K)$ is complete, thus closed in Y. $\square$