THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH4010 Functional Analysis 2022-23 Term 1

Homework 5

Deadline: 2022-10-20 Thursday

Notice:

- All the assignments must be submitted before the deadline.
- Each assignment should include your name and student ID number.

1. We say that that two non-empty subsets A and B of a vector space X may be *separated by a hyperplane* if there is a linear functional $f\colon X \to \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$f(x) < c \text{ for } x \in A \quad \text{and} \quad f(x) > c \text{ for } x \in B.$$

Let c_{00} denote the space of finite real sequences, that is

$$c_{00} = \{(x(i)) \in \mathbb{R}^{\mathbb{N}} \colon \text{there exists } n \in \mathbb{N} \text{ such that } x(i) = 0 \text{ for all } i > n\}.$$

Let $M \subset c_{00}$ be the set of sequences whose leading nonzero term is positive, that is

$$M := \{(x(i)) \in c_{00} \colon \text{there exists } n \in \mathbb{N} \text{ such that } x(n) > 0 \text{ and } x(i) = 0 \text{ for all } i < n\}.$$

Show that the sets M and $-M$ are convex and disjoint, but they cannot be separated by a hyperplane. (Recall $c_{00}^* = \ell_1$.)

2. Let X and Y be Banach spaces and $T\colon X \to Y$ a one-to-one bounded linear operator. Show that $T^{-1}\colon T(X) \to X$ is bounded if and only if $T(X)$ is closed in Y.

— *THE END* —