

Recall

Reflexive spaces

Let X be a Banach space and $Q: X \rightarrow X^{**}$ be the *canonical map* (*natural embedding*), i.e.,

$$(Qx)(x^*) := x^*(x) \text{ or symmetrically, } \langle x^*, Qx \rangle := \langle x, x^* \rangle.$$

If $QX = X^{**}$, then X is called *reflexive*.

Let M be a closed subspace of a Banach space X . Recall the *annihilator*

$$M^\perp := \{x^* \in X^*: x^*(m) = 0, \forall m \in M\}.$$

By abuse of notation on Q (canonical maps), π (projections for quotient spaces) and ι (inclusions for subspaces), we may have the following (commutative) diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M^{**} & \xrightarrow{\iota^{**}} & X^{**} & \xrightarrow{\pi^{**}} & (X/M)^{**} \longrightarrow 0 \\
 & & \uparrow Q & & \uparrow Q & & \uparrow Q \\
 0 & \longrightarrow & M & \xrightarrow{\iota} & X & \xrightarrow{\pi} & X/M \longrightarrow 0 \\
 & & \vdots & & \vdots & & \vdots \\
 0 & \longleftarrow & M^* & \xleftarrow{r=\iota^*} & X^* & \xleftarrow{\pi^*} & (X/M)^* \longleftarrow 0 \\
 & & & \nwarrow \tilde{r} & \downarrow \pi & & \\
 & & & & X^*/M^\perp & &
 \end{array}$$

where the blue arrows are isometries (which are not necessarily surjective). The dashed lines do not indicate maps. Note that

$$M^* = X^*/M^\perp \text{ by } \tilde{r} \quad \text{and} \quad M^\perp = (X/M)^* \text{ by } \pi^*,$$

and so

$$(X/M)^{**} = (M^\perp)^* = X^{**}/(M^\perp)^\perp \quad \text{also} \quad \iota^{**}M^{**} = (M^\perp)^\perp.$$

$$(i) \quad \{ X \text{ is reflexive} \} \iff \{ X^* \text{ is reflexive} \}.$$

$$(ii) \quad \{ X \text{ is reflexive} \} \iff \{ M \text{ \& } X/M \text{ are reflexive} \}.$$

Remark. In another language, the rows of the above diagram are *short exact sequences*. Then (ii) follows quickly from the *short five lemma* with abstract diagram chasing.

If X is a **separable** Banach space, then:

- (Helly's selection) any bounded sequence in X^* has w^* -convergent subsequence.
- $\{ \text{In } X^*, \text{ a sequence is } w^*\text{-convergent} \implies \text{norm convergent.} \} \iff \{ \dim X < \infty \}.$
- $\{ X \text{ is reflexive} \} \implies \{ \text{any bounded sequence in } X \text{ has weakly convergent subsequence} \}.$

Minkowski functional

Definition 1. Let A be a subset of a normed space (or topological vector space) X . The associated *Minkowski functional* $\mu_A: X \rightarrow [0, \infty]$ is defined by

$$\mu_A(x) := \inf\{t > 0: x \in tA\} \quad (1)$$

for all $x \in X$, with the convention $\inf \emptyset = \infty$.

The property of A affects the behavior of μ_A . Here comes a natural question that when will μ_A become a norm on X .

Proposition 2. Let μ_A be a Minkowski functional defined in (1). Then

- (1) (finiteness) $\{ \mu_A(x) < \infty \text{ for all } x \in X \} \iff \{ 0 \text{ is an interior point of } A \}$.
- (2) (subadditive) $\{ \mu_A(x+y) \leq \mu_A(x) + \mu_A(y) \text{ for } x, y \in X \} \iff \{ A \text{ is convex} \}$.
- (3) (positively homogeneous) Assume $0 \in A$. Then $\{ \mu_A(\alpha x) = \alpha \mu_A(x) \text{ for } \alpha \geq 0 \text{ and } x \in X \}$ always hold.
 - (a) (\mathbb{R} -absolutely homogeneous) $\{ \mu_A(\alpha x) = |\alpha| \mu_A(x) \text{ for } \alpha \in \mathbb{R} \text{ and } x \in X \} \iff \{ A = -A \}$.
 - (b) (\mathbb{C} -absolutely homogeneous) $\{ \mu_A(\alpha x) = |\alpha| \mu_A(x) \text{ for } \alpha \in \mathbb{C} \text{ and } x \in X \} \iff \{ A = e^{i\theta} A \text{ for all } \theta \in \mathbb{R} \}$.
- (4) (positive definiteness) $\{ \text{if } \mu_A(x) = 0, \text{ then } x = 0 \} \iff \{ A \text{ is bounded} \}$.

Proof. Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .

- (1) Let $x \in X$. Let V be an open neighborhood of 0 with $V \subset A$. Since the scalar product $\cdot: \mathbb{K} \times X \rightarrow X$ is continuous and $0 \cdot x = 0$, there exist $\delta > 0$ and an open neighborhood U of x such that

$$B(0, \delta) \cdot U \subset V$$

where $B(0, \delta)$ denotes the open ball of 0 with radius δ in \mathbb{K} . Then taking some $t \in (0, \delta)$, we have $t \cdot x \in V \subset A$, that is $x \in (1/t)A$, and so $\mu_A(x) \leq 1/t < \infty$ by (1).

- (2) Let $x, y \in X$. If $\mu_A(x) = \infty$ or $\mu_A(y) = \infty$, then the subadditivity holds trivially. Below we assume $\mu_A(x), \mu_A(y) < \infty$.

Let $\varepsilon > 0$. It follows from (1) that there exist $0 < \alpha \leq \mu_A(x) + \varepsilon$ and $0 < \beta \leq \mu_A(y) + \varepsilon$ such that $x \in \alpha A$ and $y \in \beta A$, that is, $x/\alpha, y/\beta \in A$. By the convexity of A ,

$$\frac{x+y}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta} \cdot \frac{x}{\alpha} + \frac{\beta}{\alpha+\beta} \cdot \frac{y}{\beta} \in A.$$

This shows that $x+y \in (\alpha+\beta)A$, then $\mu_A(x+y) \leq \alpha+\beta \leq \mu_A(x) + \mu_A(y) + 2\varepsilon$. The proof is completed by letting $\varepsilon \rightarrow 0$.

- (3) By the assumption that $0 \in A$, we have $\mu_A(0 \cdot x) = \mu_A(0) = 0 \cdot \mu_A(x) = 0$ (with the convention $0 \cdot \infty = 0$). Let $\alpha > 0$. Then it is directly checked that

$$\{t > 0: \alpha x \in tA\} = \alpha\{t > 0: x \in tA\}.$$

Thus $\mu_A(\alpha x) = \alpha\mu_A(x)$ by taking infimum.

- (a) It follows from $A = -A$ that

$$\{t > 0: x \in tA\} = \{t > 0: x \in t(-A)\} = \{t > 0: -x \in tA\}.$$

Thus $\mu_A(x) = \mu_A(-x)$ by taking infimum. By (3) it suffices to check for $\alpha < 0$. In that case

$$\mu_A(\alpha x) = \mu_A((- \alpha)(-x)) = (-\alpha)\mu_A(-x) = |\alpha|\mu_A(x).$$

- (b) Let $\theta \in \mathbb{R}$. It follows from $A = e^{i\theta}A$ that

$$\{t > 0: x \in tA\} = \{t > 0: x \in t(e^{-i\theta}A)\} = \{t > 0: e^{i\theta}x \in tA\}.$$

Thus $\mu_A(x) = \mu_A(e^{i\theta}x)$ by taking infimum. Let $\alpha = |\alpha|e^{i\theta} \in \mathbb{C}$. By (3),

$$\mu_A(\alpha x) = \mu_A(|\alpha|e^{i\theta}x) = |\alpha|\mu_A(e^{i\theta}x) = |\alpha|\mu_A(x).$$

- (4) Let $x \in X \setminus \{0\}$. Then (by the separation of vector space topology) there exists an open neighborhood V of 0 such that $x \notin V$. Since A is bounded, there exists $s > 0$ such that $A \subset tV$ for all $t > s$. This implies $x \notin (1/t)A$ for all $t > s$ since $x \notin V$. Then

$$\{\tau > 0: x \in \tau A\} \subset [1/s, +\infty).$$

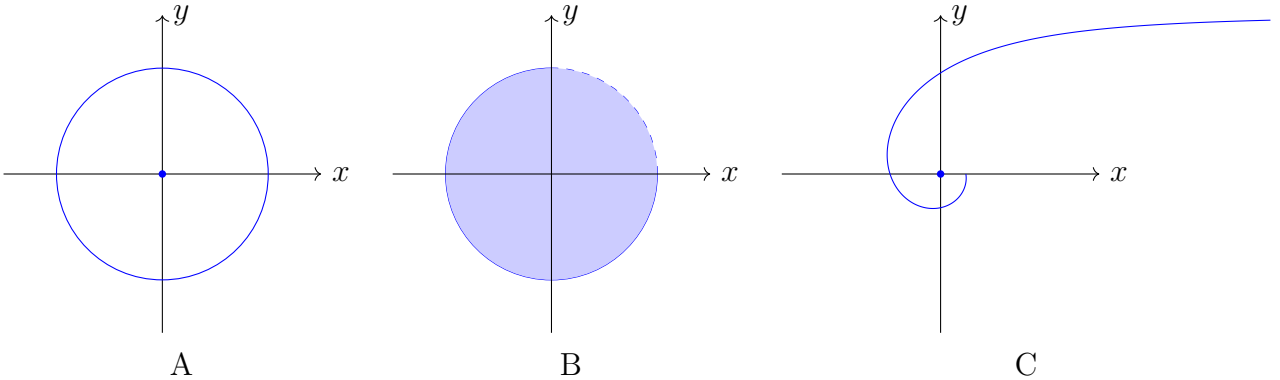
Hence $\mu_A(x) \geq 1/s > 0$ by (1).

□

It turns out that the above conditions are sufficient but not necessary. Counterexamples can be found in the plane to show that all the inverse directions “ \implies ” are false.

Example 3. Consider $X = \mathbb{C}$ and denote the absolute value of $x \in \mathbb{C}$ by $|x|$.

- Let $A = \{x \in \mathbb{C}: |x| = 1\} \cup \{0\}$. Then $\mu_A(x) = |x|$.
- Let $B = \{x \in \mathbb{C}: |x| \leq 1\} \setminus \{e^{i\theta}: \theta \in (0, \pi/2)\}$. Then $\mu_B(x) = |x|$.
- Let $C = \{e^{i\theta}/\theta: \theta \in (0, 2\pi]\} \cup \{0\}$. Then $\mu_C(x) = \theta(x)|x|$ if $x = |x|e^{i\theta(x)}$ and $\theta(x) \in (0, 2\pi]$.



Set A is for (1) and (2); Set B is for (a) and (b); Set C is for (4).