Recall

On a finite dimensional vector space, all the norms are equivalent. All the finite dimensional spaces are isomorphic. For normed spaces, finite dimensionality $\iff$ locally compactness.

Let X, Y be normed spaces and $(T_n)_{n=1}^{\infty}, T\colon X \to Y$ be linear operators.

- $\{ \ T$ continuous $\} \iff \{ \ T$ continuous at $0 \ \} \iff \{ \ T$ bounded $\}$.
- If $\dim X < \infty$, then T must be countinuous. Moreover, $\{ \ T_n x \to Tx$ for all $x \in X \ \} \iff \{ \ T_n \xrightarrow{\|\cdot\|} T \ \}$. However, the direction $\implies$ may not hold when $\dim X = \infty$.
- If $\dim Y < \infty$, then $\{ \ T$ bounded $\} \iff \{ \ \ker T$ closed $\}$. In particular, this holds for linear functionals. However, the direction $\impliedby$ may not hold when $\dim Y = \infty$.
- Equivalent definitions of the operator norm

$$\begin{aligned}\|T\| &= \sup\{\frac{\|Tx\|}{\|x\|} \colon x \in X, \|x\| \neq 0\} \\ &= \sup\{\|Tx\| \colon x \in X, \|x\| = 1\} \\ &= \sup\{\|Tx\| \colon x \in X, \|x\| \leq 1\} \\ &= \inf\{M > 0 \colon \|Tx\| \leq M\|x\|, \ \forall x \in X\}.\end{aligned}$$

  The operator norm depends on both of the norms in the domain X and in the range Y.

Dual space

Example 1 (Dual-space relationship)**.** Let $1 \leq p < \infty$ and $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\ell^p)^* = \ell^q$.

Proof. We begin with some convenient notation. For $x = (x(i))_{i=1}^{\infty} \in \ell^p$ and $y = (y(i))_{i=1}^{\infty} \in \ell^q$, define a pairing

$$\langle x, y \rangle \coloneqq \sum_{i=1}^{\infty} x(i)y(i). \tag{1}$$

By Hölder's inequality,

$$|\langle x, y \rangle| \leq \sum_{i=1}^{\infty} |x(i)y(i)| \leq \|x\|_p \|y\|_q < \infty. \tag{2}$$

Hence $\langle \cdot, \cdot \rangle \colon \ell^p \times \ell^q \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. It is readily checked that for $\alpha \in \mathbb{K}, x, \tilde{x} \in \ell^p$ and $y \in \ell^q$,

$$\langle \alpha x + \tilde{x}, y \rangle = \alpha \langle x, y \rangle + \langle \tilde{x}, y \rangle \text{ and } \langle x, \alpha y + \tilde{y} \rangle = \alpha \langle x, y \rangle + \langle x, \tilde{y} \rangle. \tag{3}$$

By (2) and (3), for any fixed $y \in \ell^q$, the map $\langle \cdot, y \rangle \colon \ell^p \to \mathbb{K}$ is continuous and linear, i.e., $\langle \cdot, y \rangle \in (\ell^p)^*$. To prove $(\ell^p)^* = \ell^q$, we will show that the map

$$\begin{aligned} T \colon \ell^q &\to (\ell^p)^* \\ y &\mapsto \langle \cdot, y \rangle \end{aligned}$$

is an isometric isomorphism.

(i) (linear and injective) By (3), T is linear. If $\langle \cdot, y \rangle$ is identically zero on ℓ^p, then by applying $\langle \cdot, y \rangle$ to e_i, where $e_i(k) = 1$ if $k = 1$ and $e_i(k) = 0$ if $k \neq i$, we get $y = 0 \in \ell^q$, thus T is injective.

(ii) (surjective) Let $\Lambda \in (\ell^p)^*$. We will find $y \in \ell^q$ such that for all $x \in \ell^p$, $\Lambda x = \langle x, y \rangle$. If $\Lambda = 0$, then $y = 0$ satisfying the requirement. Below assume $\Lambda \neq 0$. (Recall basis is like the 'skeleton' of a vector space. To determine the behavior of a linear map Λ on the whole space, it is often enough to determine the how Λ acts on the basis vectors.) For $i \in \mathbb{N}$, let e_i be the sequence taking 1 on i-th term and 0 on all the other terms. Define a sequence $y = (\Lambda e_i)_{i=1}^{\infty}$. We will check y is the desired sequence.

Let $x \in \ell^p$. It is readily proved that $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis in ℓ^p $(1 \leq p < \infty)$. Then $x = \sum_{i=1}^{\infty} x(i) e_i \in \ell^p$. Since Λ is continuous and linear,

$$\Lambda x = \Lambda \left(\sum_{i=1}^{\infty} x(i) e_i \right) = \sum_{i=1}^{\infty} x(i) \Lambda e_i = \langle x, y \rangle. \tag{4}$$

Next we check $y \in \ell^q$.

When $q = \infty$. Suppose on the contrary that $y \notin \ell^\infty$. Then there exist $i_0 \in \mathbb{N}$ such that $|y(i_0)| > 2\|\Lambda\|$. However, $|y(i_0)| = |\Lambda e_{i_0}| \leq \|\Lambda\|$, which is a contradiction. Hence $y \in \ell^\infty$.

When $q < \infty$. Define $x_n = \begin{cases} |y(i)|^{q-1} \exp(-\theta_i) & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}$, where $y(i) = |y(i)| e^{i\theta_i}$. Then $x_n \in \ell^p$. By (4) and the boundedness of Λ,

$$\sum_{i=1}^{n} |y(i)|^q = |\langle x_n, y \rangle| = |\Lambda x_n| \leq \|\Lambda\| \|x_n\|_p = \|\Lambda\| (\sum_{i=1}^{n} |y(i)|^q)^{1/p}.$$

Dividing both sides by $(\sum_{i=1}^{n} |y(i)|^q)^{1/p}$ (which is nonzero when n large enough),

$$(\sum_{i=1}^{n} |y(i)|^q)^{1/q} \leq \|\Lambda\|.$$

Letting $n \to \infty$ gives $\|y\|_q \leq \|\Lambda\|$, thus $y \in \ell^q$.

(iii) (isometric) Let $y \in \ell^q$. Note that (2) implies $\|Ty\| \leq \|y\|_q$. If $y = 0$, then $Ty = 0$. Below assume $y \neq 0$.

When $q = \infty$. For any $\varepsilon > 0$, there exists $i \in \mathbb{N}$ such that $|y(i)| \geq \|y\|_\infty - \varepsilon$. Hence

$$|\langle e_i, y \rangle| = |y(i)| \geq \|y\|_\infty - \varepsilon.$$

Letting $\varepsilon \to 0$ and since $\|e_i\|_1 = 1$ for all $i \in \mathbb{N}$, we have $\|Ty\| \geq \|y\|_\infty$.

When $q < \infty$. Write $y(i) = |y(i)| \exp(\theta_i)$ for $i \in \mathbb{N}$. Define the 'conjugate function'

$$y^* = \|y\|_q^{1-q} \left(|y(i)|^{q-1} \exp(-\theta_i) \right)_{i=1}^{\infty}. \tag{5}$$

Then $\|y^*\|_p = 1$ and $\langle y^*, y \rangle = \|y\|_q$. Hence $\|Ty\| \geq \|y\|_q$.

$\square$

We can summarize the dual-space relationships of sequence spaces. Recall c_0 is the space of sequences converging to zero and c is the space of convergent sequences, both of which are equipped with sup-norm while considering as normed spaces. For vector spaces A and B, denote $A \hookrightarrow B$ if $A \subset B$. For Banach spaces X and Y, denote $X \xrightarrow{*} Y$ if $Y = X^*$. Recall $(c_0)^* = c^* = \ell^1$. Let $1 < p < 2$ and $2 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{tikzcd}
c_{00} \arrow[r, hook] & \ell^1 \arrow[r, hook] \arrow[rrrrrr, bend left, "*"] & \ell^p \arrow[r, hook] \arrow[rr, bend left, "*"] & \ell^2 \arrow[r, hook] \arrow[loop above, "*"] & \ell^q \arrow[r, hook] \arrow[ll, bend left, "*"] & c_0 \arrow[r, hook] \arrow[llll, bend left, "*"] & c \arrow[r, hook] \arrow[lllll, bend left, "*"] & \ell^\infty
\end{tikzcd}$$

Remark. So far, ℓ^∞ is the only sequence space whose dual space is not yet characterized. One reason why our justification in Example 1 doesn't work is that ℓ^∞ is not separable, thus lack of a Schauder basis. However, in the future lectures we may see that we can decompose $(\ell^\infty)^*$ into a direct sum of ℓ^1 and another space.

Below is a simple application of the representation of $(\ell^2)^*$.

Example 2. For $x = (x(i))_{i=1}^\infty \in \ell^2$, define $\Lambda x = \sum_{i=1}^\infty \dfrac{x(2i)}{i}$. Show that $\Lambda \in (\ell^2)^*$ and compute $\|\Lambda\|$.

Proof. For $i \in \mathbb{N}$, define

$$y(i) = \Lambda(e_i) = \begin{cases} \dfrac{1}{k} & , i = 2k, \\ 0 & , i = 2k-1, \end{cases}$$

where $\{e_i\}_{i=1}^\infty$ is the standard Schauder basis of ℓ^2. Let $y = (y(i))_{i=1}^\infty$. Then $\Lambda(\cdot) = \langle \cdot, y \rangle$.

Since $\|y\|_2 = (\sum_{k=1}^\infty \frac{1}{k^2})^{1/2} = \pi/\sqrt{6}$, it follows from Example 1 that $\Lambda \in (\ell^2)^*$ and $\|\Lambda\| = \|y\|_2 = \pi/\sqrt{6}$. $\square$