THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1
Solution to Homework 1

1. Show that
$$\|x\| = \sum_{k=0}^{n} \sup_{t\in[0,1]} |x^{(k)}(t)| \tag{1}$$
is a norm on $C^n[0,1]$.

Proof. Recall that for any function $x \in C^n[0,1]$, we have for each $1 \leq k \leq n$, $x^{(k)}$ exists and is continuous. It follows from the continuity of $x^{(k)}$, the compactness of $[0,1]$, and the finiteness of the summation that $\|x\| < \infty$.

Next we check $\|\cdot\|$ is indeed a norm.

(i) By definition, $\|\cdot\|$ is non-negative. If $\|x\| = 0$, then $\sup_{t\in[0,1]} |x(t)| \leq \|x\| = 0$, thus $x = 0$ on $[0,1]$.

(ii) For any $\alpha \in \mathbb{K}$ and $x \in C^n[0,1]$,
$$\|\alpha x\| = \sum_{k=0}^{n} \sup_{t\in[0,1]} |\alpha x^{(k)}(t)| = |\alpha| \sum_{k=0}^{n} \sup_{t\in[0,1]} |x^{(k)}(t)| = |\alpha|\|x\|.$$

(iii) For any $x, y \in C^n[0,1]$, by the triangle inequality of $|\cdot|$ in $\mathbb{K}$ and the definition of sup,
$$\begin{aligned}
\|x+y\| &= \sum_{k=0}^{n} \sup_{t\in[0,1]} |x^{(k)}(t) + y^{(k)}(t)| \\
&\leq \sum_{k=0}^{n} \sup_{t\in[0,1]} \left(|x^{(k)}(t)| + |y^{(k)}(t)|\right) \\
&\leq \sum_{k=0}^{n} \sup_{t\in[0,1]} |x^{(k)}(t)| + \sup_{t\in[0,1]} |y^{(k)}(t)| \\
&= \sum_{k=0}^{n} \sup_{t\in[0,1]} |x^{(k)}(t)| + \sum_{k=0}^{n} \sup_{t\in[0,1]} |y^{(k)}(t)| = \|x\| + \|y\|.
\end{aligned}$$

□

2. Let K be a compact topological space. Prove that the spaces $C(K)$ with sup-norm and $C^n[0,1]$ with the norm defined in (1) are Banach spaces.

Proof. Denote the sup-norm on $C(K)$ by $\|\cdot\|_\infty$ and the norm defined in (1) by $\|\cdot\|$. By similar arguments in the previous question $\|\cdot\|_\infty$ and $\|\cdot\|$ are norms. It suffices to check the completeness of the norms.

(a) Let $(x_n)_{n=1}^{\infty}$ be any Cauchy sequence in $C(K)$.
We first find the candidate of the limit. Take any point $t \in K$. By the definition of Cauchy sequence, for any $\varepsilon > 0$, when $m, n \in \mathbb{N}$ large enough, we have
$$|x_m(t) - x_n(t)| \leq \sup_{s\in K} |x_m(s) - x_n(s)| \leq \varepsilon. \tag{2}$$

Hence $(x_n(t))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{K}$. By the completeness of $\mathbb{K}$, there exists $x(t) = \lim_{n\to\infty} x_n(t)$ for any $t \in K$. Define a function $x\colon K \to \mathbb{K}$ by assigning $x(t)$ to each point $t \in K$. Letting $m \to \infty$ in (2), we have $\sup_{t\in K} |x(t) - x(t)| \leq \varepsilon$ when n large enough.

Next we check $x \in C(K)$. Take any $t \in K$. For any $\varepsilon > 0$. Let N be large enough such that $\sup_{s\in K} |x(s) - x_N(s)| \leq \varepsilon/3$. On the other hand, by the continuity of x_N, there exists an neighborhood O of t such that for all $s \in O$, $|x_N(t) - x_N(s)| \leq \varepsilon/3$. Hence for all $s \in O$,

$$|x(t) - x(s)| \leq |x(t) - x_N(t)| + |x_N(t) - x_N(s)| + |x_N(s) - x(s)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence $x \in C(K)$ by the arbitrariness of t. Together we have $x_n \xrightarrow{\|\cdot\|_\infty} x \in C(K)$ as $n \to \infty$

(b) Let $(x_i)_{i=1}^{\infty}$ be any Cauchy sequence in $C^n[0,1]$. Since for any $i, j \in \mathbb{N}$,

$$\|x_i - x_j\|_\infty \leq \|x_i - x_j\| \text{ and } \|x_i^{(1)} - x_j^{(1)}\|_\infty \leq \|x_i - x_j\|,$$

by (a), there exist $x \in C[0,1]$ such that $x_i \xrightarrow{\|\cdot\|_\infty} x$ and $y_1 \in C[0,1]$ such that $x_i^{(1)} \xrightarrow{\|\cdot\|_\infty} y_1$ as $i \to \infty$. By the uniform convergence of $(x_i^{(1)})_{i=1}^{\infty}$ and the convergence of $(x_i)_{i=1}^{\infty}$, we have $x^{(1)} = y_1$ (see e.g. MATH2060). Similarly for $k = 2, \dots, n$, we find $y_k = \lim_{i\to\infty} x_i^{(k)} \in C[0,1]$ in $\|\cdot\|_\infty$. Then sequentially apply the uniform convergence to conclude $x^{(k)} = y_k$. Hence $x \in C^n[0,1]$. Since n is finite, write $y_0 = x$,

$$\lim_{i\to\infty} \|x - x_i\| = \lim_{i\to\infty} \sum_{k=0}^{n} \|y_k - x_i^{(k)}\|_\infty = \sum_{k=0}^{n} \lim_{i\to\infty} \|y_k - x_i^{(k)}\|_\infty = 0.$$

Thus $x_n \xrightarrow{\|\cdot\|} x \in C^n[0,1]$ as $n \to \infty$.

□

— *THE END* —