THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH4010 Functional Analysis

Unofficial Solutions[1] to Course Examinations

Zhou Feng

[1]If you find any mistakes or typos, please report them to zfeng@math.cuhk.edu.hk.

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Contents

2022-23 Term 1 Course Examination

1. Let (x_n) be a sequence in the real null sequence space c_0. Show that the sequence (x_n) is weakly convergent in c_0, then $(\|x_n\|_\infty)$ is bounded and $\lim_{n\to\infty} x_n(k)$ is convergent in $\mathbb{R}$ for all $k = 1, 2, \ldots$

   Proof. Let $Q\colon c_0 \to c_0^{**}$ denote the canonical map. Since (x_n) is weakly convergent, then $(Qx_n(x^*)) = (x^*(x_n))$ is convergent for $x^* \in c_0^*$, thus bounded for $x^* \in c_0^*$. Since c_0^* is complete, Uniform Boundedness Theorem implies some $M > 0$ such that

   $$\sup_n \|x_n\|_\infty = \sup_n \|Qx_n\|_\infty \leq M.$$

   This shows that $(\|x_n\|)$ is bounded.

   For $k \in \mathbb{N}$, define

   $$f_k(x) = x(k) \quad \text{for } x = (x(i)) \in c_0.$$

   It is readily checked that $f_k \in c_0^*$. By the weakly convergence of (x_n),

   $$\lim_{n\to\infty} x_n(k) = \lim_{n\to\infty} f_k(x_n)$$

   exists in $\mathbb{R}$. This means that $(x_n(k))$ is convergent for $k \in \mathbb{N}$. $\square$

2. Let X be a normed space. A subset A of X is said to weakly closed if it satisfies the condition: an element $x_0 \in A$ whenever if for any $\varepsilon > 0$ and any $f_1, \ldots, f_n \in X^*$, then there is an element $a \in A$ such that $|f_k(x_0) - f_k(a)| < \varepsilon$ for $k = 1, \ldots, n$.

   (i) Show that if A is a weakly closed subset of X, then it is normed closed. Find an example so that the converse does not hold.

   (ii) If we further assume that A is a vector subspace, show that the converse of Part (i) hold.

   Proof.

   (i) It suffices to prove that the complement A^c is normed open. Let $y \in A^c$. Since A is weakly closed and y is not in A, there exists $\varepsilon > 0$ and $f_1, \ldots, f_n \in X^*$ such that

   $$y \in \bigcap_{k=1}^{n} \{x \in X\colon |f_k(x_0) - f_k(x)| < \varepsilon\} \subset A^c.$$

   Since $f_1, \ldots, f_n$ are continuous, it follows that $\bigcap_{k=1}^{n} \{x \in X\colon |f_k(x_0) - f_k(x)| < \varepsilon\}$ is normed open. This shows that y is an interior point of A^c. Hence A^c is normed open, thus A is normed closed.

   Consider $X = (c_0, \|\cdot\|_\infty)$ and

   $$B := \{x \in c_0\colon \|x\|_\infty = 1\}.$$

   Then B is normed closed since $\|\cdot\|_\infty\colon c_0 \to \mathbb{R}$ is normed continuous. Next we show that B is not weakly closed. Suppose on the contrary that B is weakly closed. Since $c_0^* = \ell^1$, we have $\lim_{n\to\infty} f(e_m) = 0$ for $f \in c_0^*$, where $e_m(i) = \begin{cases} 1 & \text{if } i = m \\ 0 & \text{if } i \neq m \end{cases}$. Then for every $\varepsilon > 0$ and every $f_1, \ldots, f_n \in c_0^*$, there exists $e_m \in B$ with m large enough such that

   $$|f_k(0) - f_k(e_m)| = |f_k(e_m)| < \varepsilon \quad \text{for all } k = 1, \ldots, n.$$

   Then $0 \in B$ by the definition of weakly closedness, which contradicts $0 \notin B$.

(ii) Let A be a normed closed subspace of X. If $A = X$, then A is trivially weakly closed. Next we suppose $A \subsetneq X$. Let $y \in X \setminus A$. Since A is a convex normed closed set, Hyperplane Separation Theorem (or Hahn-Banach Theorem) implies some $f \in X^*$ such that

$$\sup\{f(x)\colon x \in A\} < f(y).$$

Taking $\varepsilon < |f(y) - \sup\{f(x)\colon x \in A\}|$ gives

$$\{x \in X\colon |f(y) - f(x)| < \varepsilon\} \bigcap A = \emptyset.$$

Hence there is no $a \in A$ such that $|f(y) - f(a)| < \varepsilon$. This shows that A is weakly closed since we have checked the contrapositive statement of the definition of weakly closedness.

□

3. Let X be a Hilbert space and let (x_n) be an orthogonal sequence in X. Show that the series $\sum x_n$ is convergent in X if and only if $\sum \langle x_n, y\rangle$ is convergent for all $y \in X$ if and only if $\sum \|x_n\|^2 < \infty$.

Proof.

- Suppose $\sum x_k$ is convergent. Then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq m \geq N$,
$$\left\| \sum_{k=m}^{n} x_k \right\| \leq \varepsilon.$$
Let $y \in X$. By Cauchy-Schwarz inequality,
$$\left| \sum_{k=m}^{n} \langle x_k, y\rangle \right| = \left| \left\langle \sum_{k=m}^{n} x_k, y \right\rangle \right| \leq \left\| \sum_{k=m}^{n} x_k \right\| \|y\| \leq \varepsilon \|y\|.$$
Hence $\sum \langle x_k, y\rangle$ is convergent for $y \in X$ since the scalar field is complete.
- Suppose $\sum \langle x_k, y\rangle$ is convergent for $y \in X$. Then $\sum \langle y, x_k\rangle$ is convergent for $y \in X$ since $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for $x, y \in X$.
For $n \in N$, define
$$f_n(y) := \sum_{k=1}^{n} \langle y, x_k\rangle = \langle y, \sum_{k=1}^{n} x_k\rangle \quad \text{for } y \in X.$$
By Cauchy-Schwarz inequality, $f_n \in X^*$ and $\|f_n\| = \|\sum_{k=1}^{n} x_k\|$. Since $\sum \langle y, x_k\rangle$ is convergent for $y \in X$, we have $(f_n(y))$ is bounded for $y \in X$. Then Uniform Boundedness Theorem implies some $M > 0$ such that
$$\sup_n \|f_n\| \leq M.$$
Thus $\sup_n \|f_n\|^2 \leq M^2$. By the orthogonality of (x_k) and Pythagoras Theorem,
$$\sum_{k=1}^{\infty} \|x_k\|^2 = \sup_n \sum_{k=1}^{n} \|x_k\|^2 = \sup_n \|f_n\|^2 \leq M^2.$$
Hence $\sum \|x_k\|^2 < \infty$.

- Suppose $\sum \|x_k\|^2 < \infty$. Then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq m \geq N$,

$$\sum_{k=m}^{n} \|x_k\|^2 \leq \varepsilon.$$

By the orthogonality of (x_k) and Pythagoras Theorem,

$$\left\| \sum_{k=m}^{n} x_k \right\|^2 = \sum_{k=m}^{n} \|x_k\|^2 \leq \varepsilon.$$

Hence $\sum x_k$ is convergent since X is complete.

□

4. Let X be a Hilbert space and let $L(X)$ be the space of all bounded linear operators. We call an element $S \in L(X)$ *bounded below* if $\inf\{\|S(x)\| : x \in X;\ \|x\| = 1\} > 0$.

   (i) Let $S \in L(X)$. Show that S is invertible in $L(X)$ if and only if S is bounded below and the image $S(X)$ is dense in X.

   (ii) Let $T \in L(X)$. Show that the spectrum $\sigma(T)$ of T is exactly the union

$$\{\mu \in \mathbb{C} : T - \mu \text{ is not bounded below}\} \bigcup \{\mu \in \mathbb{C} : \overline{\mu} \text{ is an eigenvalue of } T^*\}.$$

Proof.

(i) ($\implies$) Since S is invertible in $L(X)$, there exists $T \in L(X)$ such that $ST = TS = I$ where I denotes the identity map. Hence $S(X) = X$ because $X = ST(X) \subset S(X) \subset X$. For $x \in X$ with $\|x\| = 1$,

$$1 = \|x\| = \|TS(x)\| \leq \|T\| \|S(x)\|.$$

Then

$$\inf\{\|S(x)\| : x \in X;\ \|x\| = 1\} \geq \|T\|^{-1} > 0.$$

This shows that S is bounded below.

($\impliedby$) Since S is bounded below, by definition there exists $\delta > 0$ such that

$$\delta \|x\| \leq \|S(x)\| \quad \text{for all } x \in X. \tag{1}$$

Then S is injective because $\|x\| \leq (1/\delta)\|S(x)\| = 0$ if $S(x) = 0$.

Next we show that $S(X)$ is closed. Let $(S(x_n))$ be a Cauchy sequence in $S(X)$. From (1) we see that (x_n) is also a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $x_n \xrightarrow{\|\cdot\|} x$ in X. By the continuity of S,

$$\lim_{n\to\infty} S(x_n) = S(\lim_{n\to\infty} x_n) = S(x) \in S(X).$$

This shows that $S(X)$ is complete, and so closed in X. Then $S(X) = X$ since $S(X)$ is assumed to be dense in X. Hence Open Mapping Theorem (or Bounded Inverse Theorem) implies that S is invertible in $L(X)$.

(ii) By (i),

$$\begin{aligned}\sigma(T) &= \{\mu \in \mathbb{C} \colon T - \mu \text{ is not invertible}\} \\ &= \{\mu \in \mathbb{C} \colon T - \mu \text{ is not bounded below}\} \bigcup \{\mu \in \mathbb{C} \colon (T-\mu)(X) \text{ is not dense}\},\end{aligned}$$

On the other hand, note that

$$\overline{(T-\mu)(X)} = \big(((T-\mu)(X))^{\perp}\big)^{\perp} = (\ker(T-\mu)^*)^{\perp} = (\ker(T^* - \overline{\mu}))^{\perp}.$$

Then

$$\begin{aligned}\{\mu \in \mathbb{C} \colon (T-\mu)(X) \text{ is not dense}\} &= \{\mu \in \mathbb{C} \colon \overline{(T-\mu)(X)} \neq X\} \\ &= \{\mu \in \mathbb{C} \colon (\ker(T^* - \overline{\mu}))^{\perp} \neq X\} \\ &= \{\mu \in \mathbb{C} \colon \ker(T^* - \overline{\mu}) \neq 0\} \\ &= \{\mu \in \mathbb{C} \colon \overline{\mu} \text{ is an eigenvalue of } T^*\}.\end{aligned}$$

Together, we have finished the proof.

□

— THE END —

2021-22 Term 1 Course Examination

1. (a) Let (T_n) be a sequence of bounded linear operators from a Banach space X to a Banach space Y. Let $E := \{x \in X\colon \lim_{n\to\infty} T_n x \text{ exists}\}$. Show that if the sequence (T_n) is uniformly bounded, then E is a closed subspace.

(b) Let $(a_{n,k})_{n,k=0}^{\infty}$ be a sequence of numbers such that $|a_{n,k}| \le 1$ for all $n, k \ge 0$. Suppose that $\lim_{n\to\infty} a_{n,k}$ exists for all k. Let $\phi_n\colon \ell^1 \to \mathbb{C}$ be a sequence of linear functionals defined by $\phi_n(x) := \sum_{k=0}^{\infty} x(k)a_{n,k}$ for all $n = 0, 1, 2, \ldots$ and $x \in \ell^1$. Show that $\lim_{n\to\infty} \phi_n(x)$ exists for all $x \in \ell^1$.

Proof. (a) It is readily checked that E is a subspace. Let $M > 0$ such that $\sup \|T_n\| \le M$. Let x be a closure point of E. For any $\varepsilon > 0$, there exists $\widetilde{x} \in E$ such that $\|x - \widetilde{x}\| \le \frac{\varepsilon}{3M}$. By $\widetilde{x} \in E$, there exists $N \in \mathbb{N}$ such that $\forall\, n, m \ge N$, $\|T_n\widetilde{x} - T_m\widetilde{x}\| \le \frac{\varepsilon}{3}$. Then

$$\begin{aligned}\|T_n x - T_m x\| &\le \|T_n x - T_n\widetilde{x}\| + \|T_n\widetilde{x} - T_m\widetilde{x}\| + \|T_m\widetilde{x} - T_m x\| \\ &\le M \times \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + M \times \frac{\varepsilon}{3M} = \varepsilon.\end{aligned}$$

Hence $(T_n x)$ is a Cauchy sequence, and $\lim_{n\to\infty} T_n x$ exists since X a Banach space. This shows $x \in E$.

(b) Since $|a_{n,k}| \le 1$ for all $n, k \in \mathbb{N}$, we have for $x \in \ell^1$,

$$|\phi_n(x)| = |\sum_{k=0}^{\infty} x(k)a_{n,k}| \le= \sum_{k=0}^{\infty} |x(k)||a_{n,k}| \le \|x\|_1.$$

Hence $\|\phi_n\| \le 1$. By (a), it suffices to establish the existence of $\lim_{n\to\infty} \phi_n(x)$ on c_{00} which is dense in ℓ^1 with respect to $\|\cdot\|_1$.
Let $x = (x(k)) \in c_{00}$. Then there exists $n_x \in \mathbb{N}$ such that $x(k) = 0$ for all $k > n_x$. Hence

$$\lim_{n\to\infty} \phi_n = \lim_{n\to\infty} \sum_{k=0}^{n_x} x(k)a_{n,k} = \sum_{k=0}^{n_x} x(k) \lim_{n\to\infty} a_{n,k}$$

exists by the assumption that $\lim_{n\to\infty} a_{n,k}$ exists.

□

2. Let X be a normed space. Let $i\colon X \to X^{**}$ and $j\colon X^* \to X^{***}$ be the canonical mappings, that is $i(x)(x^*) := x^*(x)$ for $x \in X$ and $x^* \in X^*$ and j is defined by the similar way. Put $i^*\colon X^{***} \to X^*$ the dual operator of i.

(a) Show that if we put $Q := j \circ i^*\colon X^{***} \to X^{***}$, then Q is a projection.

(b) Using Part (a), show that the dual space X^* is a complemented closed subspace of X^{***}.

Proof.

$$\begin{tikzcd} X^* \arrow[d, dash, dashed] \arrow[r, bend right, "j"'] & X^{***} \arrow[l, "i^*"'] \arrow[d, dash, dashed] \arrow[loop right, "Q"] \\ X \arrow[r, "i"] & X^{**} \end{tikzcd}$$

(a) Since i^*, j are linear isometries, we have $Q = j \circ i^*$ is a bounded linear map. Since $Q^2 = j \circ i^* \circ j \circ i^*$, to prove $Q^2 = Q$ it suffices to check $i^* \circ j$ is the identity map on X^*. For $x \in X$,

$$\langle x, i^* \circ jx^* \rangle = \langle ix, jx^* \rangle = \langle x^*, ix \rangle = \langle x, x^* \rangle.$$

Hence $i^* \circ j$ is the identity map on X^*.

(b) It follows from (a) that $X^{***} = \operatorname{Im} Q \oplus \operatorname{Ker} Q$. Hence it suffices to check $\operatorname{Im} Q = jX^* \cong X^*$. ($\operatorname{Im} Q \subset jX^*$) For any $x^{***} \in \operatorname{Im} Q$, there is a $y^{***} \in X^{***}$ such that

$$x^{***} = j \circ i^*(y^{***}) = j(i^* y^{***}) \in jX^*.$$

($jX^* \subset \operatorname{Im} Q$) Let $x^{***} \in jX^*$. Then $x^{***} = jx^*$ for some $x^* \in X^*$. Since $j \circ i^*$ is the identity map,

$$(I - Q)x^{***} = jx^* - j \circ i^* \circ jx^* = jx^* - jx^* = 0.$$

Hence $x^{***} \in \operatorname{Ker}(I - Q) = \operatorname{Im} Q$.

□

3. (a) Let H be a Hilbert space. Show that H is separable if and only if the dual space H^* is separable.

Give an example of a Banach space so that the above statement does not hold. Explain your answer.

(b) Given an example of a non-separable Hilbert space. Explain your answer.

Proof. (a) It immediately follows from Riesz Representation Theorem. For example, ℓ^1 is separable but $\ell^\infty = (\ell^1)^*$ is not separable.

(b) For example, $H = L^2([0,1], \#)$ where $\#$ denotes the counting measure. Then H is not separable because H has an uncountable family $\{\delta_x\}_{x \in [0,1]}$ of elements such that $\|\delta_x - \delta_y\| = 2$ for $x \neq y$.

□

4. Let H be a Hilbert space and $T_n \colon H \to H$ be a sequence of bounded linear operators, $n = 1, 2, \ldots$. Suppose that $\lim_{n\to\infty} \langle T_n x, y \rangle$ exists for all $x, y \in H$.

(a) Show that $\sup\{\|T_n\| : n = 1, 2, \ldots\} < \infty$.

(b) Using Part (a) show that there is a bounded linear operator $T \colon H \to H$ such that $(Tx, y) = \lim_{n\to\infty} (T_n x, y)$ for all $x, y \in H$.

Proof. (a) Let $x \in X$. For any $y \in H$, there exists $M_{x,y} > 0$ such that for all $n \in N$,

$$|\langle x, T_n^* y \rangle| = |\langle T_n x, y \rangle| \leq M_{x,y}.$$

Since $\langle \cdot, T_n^* y \rangle \in H^*$, by Uniform Boundedness Theorem, there exists $M_x > 0$ such that

$$\|T_n^* y\| = \|\langle \cdot, T_n^* y \rangle\| \leq M_x \quad \text{for all } n \in \mathbb{N}.$$

Since the above inequality holds for all $x \in H$, by Uniform Boundedness Theorem, there exists $M > 0$ such that for all $n \in N$, we have

$$\|T_n\| = \|T_n^*\| \leq M$$

thus $\sup_n \|T_n\| \leq M$.

(b) Let $y \in H$. Define $f_y(x) \coloneqq \lim_{n\to\infty}\langle T_n x, y\rangle$ for $x \in H$. Then f_x is linear and by Cauchy-Schwarz inequality,

$$|f_y(x)| \leq \lim_{n\to\infty} \|T_n x\| \|y\| \leq M\|x\| \|y\|.$$

Thus $\|f_y\| \leq M\|y\|$, and so $f_y \in H^*$. By Riesz Representation Theorem, there exists an unique $v_y \in H$ such that

$$\langle x, v_y\rangle = f_y(x) = \lim_{n\to\infty}\langle T_n x, y\rangle \quad \text{and} \quad \|v_y\| = \|f_y\| \leq M\|y\|.$$

Define $T\colon H \to H$ by $Ty \coloneqq v_y$. Then for $x, y, \widetilde{y} \in H$ and $\alpha \in \mathbb{C}$,

$$\begin{aligned}
\langle x, T(\alpha y + \widetilde{y})\rangle &= \lim_{n\to\infty}\langle T_n x, \alpha y + \widetilde{y}\rangle \\
&= \overline{\alpha}\lim_{n\to\infty}\langle T_n x, y\rangle + \lim_{n\to\infty}\langle T_n x, \widetilde{y}\rangle \\
&= \overline{\alpha}\langle x, Ty\rangle + \langle x, T\widetilde{y}\rangle \\
&= \langle x, \alpha Ty + T\widetilde{y}\rangle.
\end{aligned}$$

Hence T is linear and $\|T\| \leq M$.

$\square$

— *THE END* —

2018-19 Term 1 Course Examination

1. Let T be a bounded linear operator from a Banach space X to a Banach space Y. Show the following assertions.

(i) If $\widetilde{T}$ is the natural linear isomorphism from $X/\ker T$ onto $T(X)$ induced by T, that is, $\widetilde{T}(x + \ker T) = T(x)$ for $x \in X$, then $\|\widetilde{T}\| = \|T\|$.

(ii) The image $T(X)$ is closed if and only if there is $c > 0$ such that $\|x + \ker T\| \leq c\|Tx\|$ for all $x \in X$. (Hint: use the Open Mapping Theorem.)

Proof. Let $\pi \colon X \to X/\operatorname{Ker} T$ be the natural projection.

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"] & \operatorname{Im} T \subset Y \\ X/\operatorname{Ker} T \arrow[ur, "\widetilde{T}"] & \end{tikzcd}$$

Define $\widetilde{T} \colon X/\operatorname{Ker} T \to \operatorname{Im} T$ by

$$\widetilde{T}(\pi x) := Tx$$

for $\pi x \in X/\operatorname{Ker} T$. Then $\widetilde{T}$ is well defined.

(i) It is readily checked that $\widetilde{T}$ is linear and bijective. Since $\|\pi\| \leq 1$, we have

$$\pi\{x \in X \colon \|x\| < 1\} \subset \{\pi x \in X/\operatorname{Ker} T \colon \|\pi x\| < 1\}.$$

On the other hand, for each $\|\pi x\| \in X/\operatorname{Ker} T$ with $\|\pi x\| < 1$ there exists $h \in \operatorname{Ker} T$ such that $\|x + h\| < 1$ and $\pi(x + h) = \pi(x)$. This implies

$$\pi\{x \in X \colon \|x\| < 1\} = \{\pi x \in X/\operatorname{Ker} T \colon \|\pi x\| < 1\}.$$

Hence

$$\begin{aligned} \|\widetilde{T}\| &= \sup\{\|\widetilde{T}(\pi x)\| \colon \|\pi x\| < 1\} \\ &= \sup\{\|Tx\| \colon \|x\| < 1\} = \|T\|. \end{aligned}$$

(ii) ($\Longrightarrow$) If $\operatorname{Im} T$ is a Banach space as a closed subspace of Banach space Y, then by Open Mapping Theorem $\widetilde{T}$ has a bounded inverse. Hence $\|\pi x\| \leq \|\widetilde{T}^{-1}\|\|Tx\|$.

($\Longleftarrow$) Since $\|\pi x\| \leq c\|Tx\|$, then $\operatorname{Im} T$ is complete since $X/\operatorname{Ker} T$ is complete. Hence $\operatorname{Im} T$ is closed.

$\square$

2. (i) Let X be a Banach space. Suppose that X is the direct sum of some vector subspaces E and F of X, that is, $X = E \oplus F$. Define a linear operator $P \colon X \to X$ by $P(u \oplus v) := u$ for $u \in E$ and $v \in F$.
Show that E and F both are closed subspaces of X if and only if P is bounded.

(ii) Show that if M is a finite dimensional subspace of X, then there is a closed subspace N of X such that X is the direct sum of M and N.

Proof. (i) ($\Longrightarrow$) Let $x_n \xrightarrow{\|\cdot\|} x$ and $Px_n \xrightarrow{\|\cdot\|} u$. Since E is closed, we have $u \in E$. Since $x_n - Px_n \in F$ and F is closed, we have $x_n - Px_n \xrightarrow{\|\cdot\|} x - u \in F$. Thus $x = u + (x - u)$ is the direct sum decomposition of x. Hence $Px = u$. By Closed Graph Theorem, P is bounded.

($\Longleftarrow$) It follows from $P^2 = P$, that is $P(I - P) = (I - P)P = 0$, that $F = \operatorname{Ker} P$ and $E = \operatorname{Ker}(I - P)$. Then E and F are closed since P is closed.

(ii) Let $\{x^i\}_{i=1}^k$ be the coordinate functions of M. Then $\{x^i\}_{i=1}^k$ are linear and continuous on M. By Hahn-Banach Theorem there exist $\{\widetilde{x}^i\}_{i=1}^k$ in X^* extending $\{x^i\}_{i=1}^k$. Define $N := \bigcap_{i=1}^k \operatorname{Ker} \widetilde{x}^i$. Then N is closed and $X = M \oplus N$.

□

3. Let X be a normed space. Let (x_n) be a weakly Cauchy sequence in X, that is $(x^*(x_n))$ is a convergent sequence for all x^* in X^*.

(i) Show that (x_n) is a bounded sequence.

(ii) Show that if we put $\psi(x^*) := \lim_{n\to\infty} x^*(x_n)$ for $x^* \in X^*$, then $\psi \in X^{**}$.

(iii) Show that if X is reflexive, then every weakly Cauchy sequence (x_n) is weakly convergent, that is, there is an element $x \in X$ so that $\lim_{n\to\infty} x^*(x_n) = x^*(x)$ for all $x^* \in X^*$.

Proof. (i) Let $Q\colon X \to X^{**}$ be the natural mapping. Then (Qx_n) is a weak star convergent sequence on X^* since (x_n) is weakly convergent. Hence $(\langle x^*, Qx_n\rangle)_{n=1}^\infty$ is bounded for all $x^* \in X^*$. Since X^* is a Banach space, by Uniform Boundedness Theorem $(Qx_n)_{n=1}^\infty$ is norm bounded. This shows that (x_n) is a bounded sequence since Q is an isometry.

(ii) The linearity of ψ follows from the linearity of Qx_n with respect to x^*. By (i) there exists $M > 0$ such that $\sup_n \|x_n\| \le M$. Hence $|x^*(x_n)| \le \|x^*\|\|x_n\| \le M\|x^*\|$for $x^* \in X^*$. Letting $n \to \infty$ shows that $|\psi(x^*)| \le M\|x^*\|$, thus ψ is bounded.

(iii) By (ii) there is $\psi \in X^{**}$ such that $\psi(x^*) = \lim_{n\to\infty} x^*(x_n)$ for $x^* \in X^*$. Since X is reflexive, there is $x \in X$ such that $\psi(x^*) = x^*(x)$ for $x^* \in X$. Hence $x^*(x) = \lim_{n\to\infty} x^*(x_n)$ for $x^* \in X^*$, and so every weakly Cauchy sequence is weakly convergent.

□

4. For each $x \in \ell^\infty$, define a linear operator M_x from ℓ^2 to itself by $M_x(\xi)(k) := x(k)\xi(k)$ for $\xi \in \ell^2$ and $k = 1, 2, \ldots$.

(i) Show that $\|M_x\| = \|x\|_\infty$ for any $x \in \ell^\infty$.

(ii) Show that M_x is self-adjoint if and only if $x = \overline{x}$, where $\overline{x}(k) := \overline{x(k)}$.

(iii) Show that M_x is a compact operator for all $x \in c_0$.

Proof. (i) Let $x = (x(k)) \in \ell^\infty$. Then for $\xi = (\xi(k)) \in \ell^2$,

$$\|M_x\xi\|^2 = \sum_k |x(k)\xi(k)|^2 \le \|x\|_\infty^2 \sum_k |\xi(k)|^2 = \|x\|_\infty^2 \|\xi\|^2.$$

This shows $\|M_x\| \le \|x\|_\infty$. For $\varepsilon > 0$, by definition there exists $n \in \mathbb{N}$ such that $|x(n)| \ge \|x\|_\infty - \varepsilon$. Let $\{e_n\}_{n=1}^\infty$ be the standard Schauder basis of ℓ^2. Then

$$\|M_x e_n\| = |x(n)| \ge \|x\|_\infty - \varepsilon.$$

Hence $\|M_x\| \ge \|x\|_\infty - \varepsilon$. Letting $\varepsilon \to 0$ gives $\|M_x\| = \|x\|_\infty$.

(ii) ($\Longrightarrow$) Suppose M_x is self-adjoint. Then

$$\langle M_x \xi, \eta \rangle = \langle \xi, M_x \eta \rangle$$

for $\xi, \eta \in \ell^2$. For $n \in \mathbb{N}$, taking $\xi = \eta = e_n$ gives that $x(n) = \overline{x(n)}$. Hence $x = \overline{x}$.

($\Longleftarrow$) Suppose $x = \overline{x}$. Then

$$\langle M_x \xi, \eta \rangle = \sum_k x(k)\xi(k)\overline{\eta(k)} = \sum_k \xi(k)\overline{x(k)\eta(k)} = \langle \xi, M_x \eta \rangle$$

for $\xi = (\xi(k)), \eta = (\eta(k)) \in \ell^2$. This shows that M_x is self-adjoint.

(iii) For $n \in \mathbb{N}$, define

$$x_n(k) := \begin{cases} x(k) & \text{if } k \le n \\ 0 & \text{if } k \ge n+1. \end{cases}$$

Then M_{x_n} is of finite rank, in particular compact. Let $\varepsilon > 0$. Since $x \in c_0$, there exists $N \in \mathbb{N}$ such that $|x(k)| \le \varepsilon$ for all $k \ge N$, thus $\|x - x_N\|_\infty \le \varepsilon$. It follows from (i) that

$$\|M_x - M_{x_N}\| = \|M_{x - x_N}\| = \|x - x_N\|_\infty \le \varepsilon.$$

Hence $\lim_{n\to\infty} \|M_x - M_{x_n}\| = 0$. This implies that M_x is compact as a norm limit of a sequence of compact operators.

$\square$

— *THE END* —

2017-18 Term 1 Course Examination

1. Let X and Y be normed spaces and let $T_n\colon X \to Y$ be a sequence of bounded linear operators.

(i) Show that if X is of finite dimension and $\lim_{n\to\infty} T_n x = 0$ for all $x \in X$, $\|T_n\| \to 0$ as $n \to \infty$.

(ii) If the assumption of $\dim X < \infty$ is removed, does the implication in Part (i) still hold?

Proof. (i) Let $\{e_i\}_{i=1}^k$ be a basis of X. Let $x = \sum_{i=1}^k x^i e_i$ with $\|x\| = 1$. Then there is $M < \infty$ such that $|x_i| \le M$ for $1 \le i \le k$ since $\{x \in X\colon \|x\| = 1\}$ is compact. Hence for x with $\|x\| = 1$,

$$\|T_n x\| = \|\sum_{i=1}^k x^i T_n e_i\| \le M \sum_{i=1}^k \|T_n e_i\|.$$

This implies $\|T_n\| \le M \sum_{i=1}^k \|T_n e_i\|$. By assumption, letting $n \to \infty$ gives $\|T_n\| \to 0$.

(ii) No, the Part (i) will not hold when $\dim X = \infty$. We can show it by example. For $n \in \mathbb{N}$, $T_n(x) := x(n)$ for $x = (x(i)) \in c_0$. Then $T_n \in c_0^*$ and $|T_n x| \to 0$ for all $x \in c_0$. However, $\|T_n\| = 1$ for all $n \in \mathbb{N}$.

□

2. (i) Explain why c_0 is not reflexive.

(ii) Show that every closed subspace of a reflexive space is reflexive.

(iii) Let Y be any Banach space and $X = \ell^\infty \oplus_1 Y$. Suppose that X is equipped with the norm $\|(x, y)\|_1 := \|x\|_\infty + \|y\|_Y$ for $x \in \ell^\infty$ and $y \in Y$. Show that X must not be reflexive.

Proof. (i) We can proving c_0 is not reflexive by showing c_0 does not have one of the properties of a reflexive space. For example, the double dual of a separable reflexive space must separable. However, $c_0^{**} = (\ell^1)^* = \ell^\infty$ is not separable.

(ii) Let M be a closed subspace of a relexive space X. Let $Q\colon X \to X^{**}$ and $\widetilde{Q}\colon M \to M^{**}$ be the canonical mappings and $\iota\colon M \to X$ be the natural inclusion. Then Q is a bijection since X is reflexive.

Let $m^{**} \in M^{**}$. Define $x := Q^{-1}\iota^{**}m^{**}$. Then $x \in X$. For every $x^* \in M^\perp$, then $\iota^* x^* = 0 \in M^*$, and so

$$\langle x, x^*\rangle = \langle Q^{-1}\iota^{**}m^{**}, x^*\rangle = \langle x^*, \iota^{**}m^{**}\rangle = \langle \iota^* x^*, m^{**}\rangle = \langle 0, m^{**}\rangle = 0.$$

Hence $x \in M$ by Hahn-Banach theorem, that is $\iota x = x$. Then for $m^* \in M^*$ with $m^* = \iota^* x^*$ for some $x^* \in X^*$ by Hahn-Banach theorem, since $\iota^{**}\widetilde{Q} = Q\iota$,

$$\begin{aligned}\langle m^*, \widetilde{Q}x\rangle &= \langle \iota^* x^*, \widetilde{Q}x\rangle = \langle x^*, \iota^{**}\widetilde{Q}x\rangle = \langle x^*, Q\iota x\rangle \\ &= \langle x^*, Qx\rangle = \langle x^*, \iota^{**}m^{**}\rangle = \langle \iota^* x^*, m^{**}\rangle = \langle m^*, m^{**}\rangle.\end{aligned}$$

This justifies $\widetilde{Q}x = m^{**}$. Hence M is reflexive.

(iii) Since c_0 is a closed subspace of Banach space ℓ^∞ but not reflexive, we have ℓ^∞ should not be reflexive. Suppose otherwise that X is reflexive. By the natural inclusion $x \mapsto (x, 0)$ and $\|(x,0)\|_1 = \|x\|_\infty$, we have ℓ^∞ should be reflexive as a closed subspace of X, which contradicts that ℓ^∞ is not reflexive.

$\square$

3. (i) Let X be a normed space. Suppose that every 2-dimensional subspace of X is an inner product space. Show that X is an inner product space.

(ii) Let (x_n) and (y_n) be the sequences in a inner product space H. Suppose that the limits $\lim \|x_n\|$, $\lim \|y_n\|$ and $\lim \|\frac{x_n+y_n}{2}\|$ exist and are equal. Show that if (x_n) is convergent, then so is (y_n).

Proof. (i) Let $x, y \in X$. Then $x + y, x - y \in \operatorname{span}\{x, y\}$. If $\dim \operatorname{span}\{x, y\} = 2$, then $\operatorname{span}\{x, y\}$ is a inner product space. Hence Parallelogram law holds, that is

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

If $x \parallel y$, then the above equality holds trivially. Together, we have the Parallelogram law holds for the norm on X. This implies that X is an inner product space.

(ii) By Parallelogram law,

$$\frac{1}{4}\|x_n - y_n\|^2 = \frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|y_n\|^2 - \|\frac{x_n + y_n}{2}\|^2.$$

Hence if $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = \lim_{n\to\infty} \|\frac{x_n+y_n}{2}\|$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$. This shows that if (x_n) convergent then so is (y_n).

$\square$

4. (i) Let $T\colon H \to H$ be a linear operator from a Hilbert space H to itself. Suppose that $\langle Tx, y\rangle = \langle y, Tx\rangle$ for all $x, y \in H$. Show that T is bounded.

(ii) Let $D := \{x \in \ell^2\colon \sum_{n=1}^\infty n^2|x(n)|^2 < \infty\}$. Define a linear operator $T\colon D \to \ell^2$ by $Tx(n) := nx(n)$ for $x \in D$ and $n = 1, 2, \ldots$ Show that the operator T satisfies the condition $\langle Tx, y\rangle = \langle y, Tx\rangle$ for all $x, y \in D$ but T is not bounded.

Proof. (i) Suppose $x_n \xrightarrow{\|\cdot\|} x$ and $Tx_n \xrightarrow{\|\cdot\|} y$. Then for $z \in H$,

$$\langle y, z\rangle = \lim_{n\to\infty} \langle Tx_n, z\rangle = \lim_{n\to\infty} \langle x_n, Tz\rangle = \langle x, Tz\rangle = \langle Tx, z\rangle.$$

Hence $Tx = y$ since z is arbitrary. This implies that T is bounded by Closed Graph Theorem.

(ii) Let $x = (x(n)), y = (y(n)) \in D$. Then

$$\langle Tx, y\rangle = \sum_n nx(n)\overline{y(n)} = \sum_n x(n)\overline{ny(n)} = \langle x, Ty\rangle.$$

Define x_n by $x_n(n) = 1$ and $x_n(i) = 0$ for $i \neq n$. Then $\|x_n\| = 1$ but $\|Tx_n\| = \|nx_n\| = n$. Hence $\|T\| \geq n$. Letting $n \to \infty$ shows that T is unbounded.

$\square$

— *THE END* —

2016-17 Term 2 Course Examination

1. Let X be a normed space. We say that a series $\sum_{n=1}^{\infty} x_n$ in X is absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Show that X is a Banach space if and only if every absolutely convergent series in X is convergent in X.

Proof. ($\Longrightarrow$) Let (x_n) be in a Banach space X such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Then for $\varepsilon > 0$, when k, ℓ large,

$$\| \sum_{n=k}^{\ell} x_n \| \leq \sum_{n=k}^{\ell} \|x_n\| \leq \varepsilon.$$

This implies $(\sum_{n=1}^{k} x_n)_{k=1}^{\infty}$ is a Cauchy sequence. Hence $\sum_{n=1}^{\infty} x_n$ converges the completeness.

($\Longleftarrow$) Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in X. Then there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $\|x_{n_{k+1}} - x_{n_k}\| \leq 1/2^k$. For $k \in \mathbb{N}$, define $y_k = x_{n_{k+1}} - x_{n_k}$. Since $\|y_k\| \leq 1/2^k$, we have $\sum_{k=1}^{\infty} \|y_k\| < \infty$, and so $\sum_{k=1}^{\infty} y_k$ converges. This implies (x_{n_k}) converges. Hence (x_n) converges since any Cauchy sequence with a convergent subsequence will converge. This shows that X is a Banach space. $\square$

2. Let $f\colon X \to Y$ be an isometry, where X and Y are normed spaces. Show that the double dual operator $f^{**}\colon X^{**} \to Y^{**}$ is also an isometry.

Proof. It follows from $\|f^{**}\| = \|f^*\| = \|f\| = 1$ that $\|f^{**}x^{**}\| \leq \|x^{**}\|$ for all $x^{**} \in X^{**}$. Next we show $\|f^{**}x^{**}\| \geq \|x^{**}\|$.

Since f is an isometry, so is the inverse map $f^{-1}\colon fX \to X$. Hence the adjoint operator $(f^{-1})^*\colon X^* \to (fX)^*$ is well defined. Let $x^* \in X^*$. Since fX is a subspace of Y, by Hahn-Banach theorem there exists $y^* \in Y^*$ such that $\|y^*\| = \|(f^{-1})^*x^*\| \leq \|x^*\|$ by $\|(f^{-1})^*)\| = \|f^{-1}\| = 1$ and $y^* = (f^{-1})^*x^*$ on fX. Then

$$\langle x, f^*y^* \rangle = \langle fx, y^* \rangle = \langle fx, (f^{-1})^*x^* \rangle = \langle f^{-1}fx, x^* \rangle = \langle x, x^* \rangle$$

for all $x \in X$. This shows $f^*y^* = x^*$. Hence

$$|\langle x^*, x^{**} \rangle| = |\langle f^*y^*, x^{**} \rangle| = |\langle y^*, f^{**}x^{**} \rangle| \leq \|f^{**}x^{**}\| \|y^*\| \leq \|f^{**}x^{**}\| \|x^*\|.$$

This shows $\|x^{**}\| \leq \|f^{**}x^{**}\|$ since x^* is arbitrary. $\square$

3. Let X be a finite dimensional normed space and (x_n) a sequence in X. Show that (x_n) is convergent in norm if and only if it is weakly convergent.

Proof. ($\Longrightarrow$) Let $x_n \xrightarrow{\|\cdot\|} x$ for some $x \in X$. Then for $x^* \in X^*$,

$$|\langle x - x_n, x^* \rangle| \leq \|x - x_n\| \|x^*\| \to 0 \quad \text{as } n \to \infty.$$

($\Longleftarrow$) Let $x_n \xrightarrow{w} x$ for some $x \in X$. Let $\{e_i\}_{i=1}^{k}$ be a basis of X. Write $x_n = \sum_{i=1}^{k} x_n^i e_i$ and $x = \sum_{i=1}^{k} x^i e_i$. For $1 \leq i \leq k$ we have $x_n^i \to x^i$ as $n \to \infty$ since each coordinate function $x \mapsto x^i$ is continuous and linear. Hence

$$\|x_n - x\| = \left\| \sum_{i=1}^{k} (x_n^i - x^i) e_i \right\| \leq (\sum_{i=1}^{k} \|e_i\|) \max_{1 \leq i \leq k} |x_n^i - x^i| \to 0 \quad \text{as } n \to \infty.$$

$\square$

4. Let M be a closed subspace of a Hilbert space H and P be the orthogonal projection from H onto M. Let $T\colon M \to Z$ be a bounded linear operator, where Z is a normed space. Show that $\|T \circ P\| = \|T\|$.

Proof. Since P is an orthogonal projection, we have $\|Px\| \leq \|x\|$ and so $\|P\| \leq 1$. Then $\|TP\| \leq \|T\|\|P\| \leq \|T\|$. On the other hand, since $x = Px$ for each $x \in M$,

$$\begin{aligned} \|T\| &= \sup\{\|Tx\| \colon x \in M, \|x\| \leq 1\} \\ &= \sup\{\|TPx\| \colon x \in M, \|x\| \leq 1\} \\ &\leq \sup\{\|TPx\| \colon x \in H, \|x\| \leq 1\} = \|TP\|. \end{aligned}$$

This finishes the proof. □

5. Let H be a complex Hilbert space.

(i) Let (x_n) be a sequence in H. Show that the sequence (x_n) converges weakly to an element $x \in H$ if and only if $\lim_{n\to\infty} \|x_n\| = \|x\|$.

Errata. This question is wrong since there are counterexamples in both directions.

(ii) Let $T\colon H \to H$ be a linear operator. Show that if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$ then T is bounded.

Proof. (i) Let $\{e_n\}$ denote the standard Schauder basis for ℓ^2. Then we have the following **counterexamples**.

($\implies$) For $x = (x(i)) \in \ell^2$, $\langle e_n, x\rangle = x(n) \to 0$ as $n \to \infty$. This implies $e_n \xrightarrow{w} 0$ but $\|e_n\| = 1$.

($\impliedby$) For $n \in \mathbb{N}$, define $x_n := (-1)^n e_1$. Then $\|x_n\| = 1$. However, it follows from $\langle x_n, e_1\rangle = (-1)^n$ that (x_n) is not weakly convergent.

(ii) Suppose $x_n \xrightarrow{\|\cdot\|} x$ and $Tx_n \xrightarrow{\|\cdot\|} y$. Then for $z \in H$,

$$\langle y, z\rangle = \lim_{n\to\infty} \langle Tx_n, z\rangle = \lim_{n\to\infty} \langle x_n, Tz\rangle = \langle x, Tz\rangle = \langle Tx, z\rangle.$$

Hence $Tx = y$ since z is arbitrary. This implies that T is bounded by Closed Graph Theorem.

□

— *THE END* —

2015-16 Term 1 Course Examination

1. (i) State the Riesz Representation Theorem for a Hilbert space.

(ii) Show that the dual space of a Hilbert space is also a Hilbert space.

(iii) Show that every Hilbert space is reflexive.

Proof. (i) Let H be a Hilbert space. Then for each $f \in H^*$, there is a unique $y \in H$ such that

$$f(x) = \langle x, y \rangle$$

for $x \in H$. Moreover, the map $\phi\colon H^* \to H$ is anti-unitary operator.

(ii) Pull the inner product on H back to H^* by ϕ in (i), that is

$$\langle f, g \rangle := \langle \phi g, \phi f \rangle$$

for $f, g \in H^*$. Note the order of f, g due to the conjugate linearity.

(iii) Let $Q\colon H \to H^{**}$ be the canonical map. Since H^* is a Hilbert space by (ii), it follows from (i) that for there is an anti-unitary operator $\psi\colon H^{**} \to H^*$ such that $x^{**} \in H^{**}$,

$$x^{**}(x^*) = \langle x^*, \psi x^{**} \rangle = \langle \phi\psi x^{**}, \phi x^* \rangle = x^*(\phi\psi x^{**}) = Q(\phi\psi x^{**})(x^*).$$

Hence $x^{**} = Q(\phi\psi x^{**})$. This shows that Q is surjective and so every Hilbert space is reflexive.

$\square$

2. For each $x \in \ell^\infty$, define a linear operator $M_x\colon \ell^2 \to \ell^2$ by

$$M_x(\xi) := x \cdot \xi$$

for $\xi \in \ell^2$, where $(x \cdot \xi)(k) := x(k)\xi(k)$ for $k \in \mathbb{N}$.

(i) Show that $\|M_x\| = \|x\|_\infty$.

(ii) Show that M_x is self-adjoint if and only if $x = \overline{x}$, where $\overline{x}(k) := \overline{x(k)}$.

(iii) Show that if $x \in c_0$, then M_x is a compact operator.

Proof. (i) Let $x = (x(k)) \in \ell^\infty$. Then for $\xi = (\xi(k)) \in \ell^2$,

$$\|M_x\xi\|^2 = \sum_k |x(k)\xi(k)|^2 \le \|x\|_\infty^2 \sum_k |\xi(k)|^2 = \|x\|_\infty^2 \|\xi\|^2.$$

This shows $\|M_x\| \le \|x\|_\infty$. For $\varepsilon > 0$, by definition there exists $n \in \mathbb{N}$ such that $|x(n)| \ge \|x\|_\infty - \varepsilon$. Let $\{e_n\}_{n=1}^\infty$ be the standard Schauder basis of ℓ^2. Then

$$\|M_x e_n\| = |x(n)| \ge \|x\|_\infty - \varepsilon.$$

Hence $\|M_x\| \ge \|x\|_\infty - \varepsilon$. Letting $\varepsilon \to 0$ gives $\|M_x\| = \|x\|_\infty$.

(ii) ($\Longrightarrow$) Suppose M_x is self-adjoint. Then

$$\langle M_x \xi, \eta \rangle = \langle \xi, M_x \eta \rangle$$

for $\xi, \eta \in \ell^2$. For $n \in \mathbb{N}$, taking $\xi = \eta = e_n$ gives that $x(n) = \overline{x(n)}$. Hence $x = \overline{x}$.

($\Longleftarrow$) Suppose $x = \overline{x}$. Then

$$\langle M_x \xi, \eta \rangle = \sum_k x(k)\xi(k)\overline{\eta(k)} = \sum_k \xi(k)\overline{x(k)\eta(k)} = \langle \xi, M_x \eta \rangle$$

for $\xi = (\xi(k)), \eta = (\eta(k)) \in \ell^2$. This shows that M_x is self-adjoint.

(iii) For $n \in \mathbb{N}$, define

$$x_n(k) := \begin{cases} x(k) & \text{if } k \le n \\ 0 & \text{if } k \ge n+1. \end{cases}$$

Then M_{x_n} is of finite rank, in particular compact. Let $\varepsilon > 0$. Since $x \in c_0$, there exists $N \in \mathbb{N}$ such that $|x(k)| \le \varepsilon$ for all $k \ge N$, thus $\|x - x_N\|_\infty \le \varepsilon$. It follows from (i) that

$$\|M_x - M_{x_N}\| = \|M_{x - x_N}\| = \|x - x_N\|_\infty \le \varepsilon.$$

Hence $\lim_{n\to\infty} \|M_x - M_{x_n}\| = 0$. This implies that M_x is compact as a norm limit of a sequence of compact operators.

$\square$

3. Let T be a bounded linear operator from a Banach space X to a normed space Y. Assume that $B_Y \subset \overline{T(B_X)}$, where B_X and B_Y denote the open unit balls of X and Y respectively.

(i) Show that there is $r > 0$ such that $B_Y \subset T(rB_X)$.

(ii) Show that T is an open mapping and is a surjection.

Proof. (i) Since $B_Y \subset \overline{T(B_X)}$ and the scalar product with a nonzero number is a homeomorphism, we have for $n \in \mathbb{N}$,

$$2^{-n} B_Y \subset \overline{T(2^{-n} B_X)}. \tag{2}$$

Let $y_2 \in 2^{-2} B_Y$. Then $y_2 \in \overline{T(2^{-2} B_X)}$. We are going to define two sequences (y_n) in Y and (x_n) in X by induction. Suppose $y_n \in \overline{T(2^{-n} B_X)}$. It follows from the definition of closure point that

$$\left(y_n - 2^{-(n+1)} B_Y\right) \cap T(2^{-n} B_X) \neq \emptyset.$$

Hence by (2) there exists $x_n \in 2^{-n} B_X$ (that is $\|x_n\| < 2^{-n}$) such that

$$y_n - Tx_n \in 2^{-(n+1)} B_Y \subset \overline{T(2^{-(n+1)} B_X)}.$$

Define $y_{n+1} := y_n - Tx_n$.

By the above construction $(x_n)_{n\ge 2}^\infty \in X$ with $\|x_n\| < 2^{-n}$. Then $\sum_{n=2}^\infty \|x_n\| = \frac{1}{2} < \infty$ and so there exists $x = \sum_{n=2}^\infty x_n$ by the completeness of X. Then $\|x\| < 1/2$ and $x \in B_X$. By $y_n \in 2^{-n} B_Y$ we have $y_n \to 0$ as $n \to \infty$. Since T is continuous,

$$T(x) = T(\sum_{n=2}^\infty x_n) = \lim_{m\to\infty} \sum_{n=2}^m Tx_n = \lim_{m\to\infty} \sum_{n=2}^m (y_n - y_{n+1}) = \lim_{m\to\infty} (y_2 - y_{m+1}) = y_2.$$

This shows that $y_2 \in TB_X$ and so $2^{-2} B_Y \subset T(B_X)$, that is $B_Y \subset T(4B_X)$.

(ii) It follows from (i) that T is an open mapping. Then

$$Y = \bigcup_{n=1}^{\infty} nB_Y \subset \bigcup_{n=1} T(nrB_X) \subset TX,$$

which means that T is surjective.

□

4. (i) State the definition of a w^*-convergent sequence in the dual space X^* of a normed space X.

(ii) Prove or disprove the following statement:
"*Every bounded sequence in ℓ^∞ has a w^*-convergent subsequence.*"

Proof. (i) Let (x_n^*) be a w^*-convergent sequence in X^*. This means that for $x \in X$, $\lim_{n\to\infty} x_n^*(x)$ exists in the scalar field.

(ii) Let (x_n) be a bounded sequence in ℓ^∞. Note that ℓ^∞ is the dual space of ℓ^1. Since ℓ^1 is separable, by Helley's Theorem (x_n) has a w^*-convergent subsequence.

□

— *THE END* —

2014-15 Term 2 Course Examination

1. ℓ^∞ denotes the space of all bounded sequences and is endowed with the sup-norm,

(a) State the definition of a Schauder base of a normed space.

(b) Show that ℓ^∞ is non-separable.

(c) Use part (b) to show that the space ℓ^∞ does not have Schauder bases.

(d) Is every infinite dimensional closed subspace of ℓ^∞ is non-separable? (Explain).

Proof. (a) Let X be a normed space. A countable sequence (e_n) in X is a *Schauder base* of X if for each $x \in X$, there is a unique sequence $(x(n))$ of scalars such that

$$x = \sum_{n=1}^{\infty} x(n)e_n.$$

(b) Consider the set of sequence $\{0,1\}^\infty$. Then $\|x-y\|_\infty = 1 > 0$ for $x \neq y \in \{0,1\}^\infty$ and there are uncountably many elements in $\{0,1\}^\infty$. Hence ℓ^∞ is non-separable.

(c) If ℓ^∞ has a Schauder base $\{e_n\}$, the collection of finite linear combinations of $\{e_n\}$ with rational coefficients is a countable dense subset of ℓ^∞. This contradicts that ℓ^∞ is non-separable.

(d) No. c_0 is an infinite dimensional closed subspace of ℓ^∞ but c_0 is separable with a countable dense subset c_{00}.

□

2. (a) State te Uniform Boundedness Theorem.

(b) State the Closed Graph Theorem.

(c) Let X be a Banach space and $T\colon X \to X$ be a linear operator. Let A be a subset of X^* which separates the points in X, that is if $0 \neq x \in X$, then there is an element $f \in A$ such that $f(x) \neq 0$.
Show that if $f \circ T \in X^*$ for all $f \in A$, then T is bounded.

Proof. (a) Omit.

(b) Omit.

(c) Let $x_n \xrightarrow{\|\cdot\|} x$ and $Tx_n \xrightarrow{\|\cdot\|} y$ in X. By $fT \in X^*$ for $f \in A \subset X^*$,

$$f \circ T(x) = f \circ T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f \circ T(x_n) = f(\lim_{n\to\infty} T_{x_n}) = f(y).$$

Hence $f(Tx-y) = 0$ for all $f \in A$. Since A separates the points in X, we have $Tx - y = 0$ because otherwise we can find a $f \in A$ such that $f(Tx-y) = 0$. This shows $Tx = y$. Then T is bounded by Closed Graph Theorem.

□

3. Let H be a Hilbert space and T be a bounded linear operator from H into itself. Put I the identity operator on H. Assume that $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

(a) Show that $\|x\| \leq \|(I+T)x\|$ for all $x \in H$.

(b) Show that $I+T$ is injective.

(c) Show that the image of $I+T$ is closed.

(d) Show that $I+T$ is surjective.

Proof. (a) Let $x \in H$. By $\langle Tx, x\rangle \geq 0$,

$$\langle (I+T)x, (I+T)x\rangle = \langle x, x\rangle + 2\langle Tx, x\rangle + \langle Tx, Tx\rangle \geq \langle x, x\rangle.$$

Hence $\|x\| \leq \|(I+T)x\|$.

(b) Suppose $(I+T)x = 0$. Then $\|x\| \leq \|(I+T)x\| = 0$ by (a). Hence $x = 0$. This shows that $I+T$ is injective.

(c) Let $((I+T)x_n)$ be a Cauchy sequence in the image of $I+T$. Then (x_n) is a Cauchy sequence in H by (a). Since H is complete, there exists $x \in H$ such that $x = \lim_{n\to\infty} x_n$. By the continuity of $I+T$,

$$\lim_{n\to\infty}(I+T)(x_n) = (I+T)(\lim_{n\to\infty} x_n) = (I+T)(x).$$

This finishes the proof.

(d) By $\langle Tx, x\rangle \geq 0$ for all $x \in H$,

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle T^*x, x\rangle.$$

Hence $T = T^*$, and so $(I+T)^* = (I+T)$. Since $\mathcal{R}(I+T)$ is closed by (c) and $\operatorname{Ker}(I+T) = \{0\}$ by (b),

$$\mathcal{R}(I+T) = (\mathcal{R}(I+T)^\perp)^\perp = (\operatorname{Ker}(I+T)^*)^\perp = (\operatorname{Ker}(I+T))^\perp = \{0\}^\perp = H$$

where the second equality is by $\mathcal{R}(S)^\perp = \operatorname{Ker} S^*$ for every bounded linear operator S.

□

4. Let H be a Hilbert space and let (P_n) be a sequence of orthogonal projections on H. Suppose that $P_nP_m = 0$ for $n \neq m$.

(a) If we define $S_nx := \sum_{k=1}^n P_kx$ for $x \in H$, then show that $\lim_{n\to\infty} S_nx$ exists for all $x \in H$.

(b) By using part (a) or otherwise show that there is an orthogonal projection P on H such that $Px = \sum_k P_kx$ for all $x \in H$.

(c) With the notation as above, do we have $\|P - S_n\| \to 0$ in general?

Proof. (a) Let $x \in H$. Since for all $P_iP_j = 0$ for $i \neq j$ and P_i are orthogonal projections,

$$\langle P_ix, P_jx\rangle = \langle x, P_iP_jx\rangle = \langle x, 0\rangle = 0$$

and for $n \in \mathbb{N}$,

$$S_n^2 = (\sum_{i=1}^n P_i)^2 = \sum_i^n P_i^2 = \sum_i^n P_i = S_n.$$

Hence P_ix, P_jx are orthogonal for $i \neq j \in \mathbb{N}$ and S_n are orthogonal projections for $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, by Pythagoras Theorem,

$$\sum_{k=1}^{n} \|P_k x\|^2 = \left\| \sum_{k=1}^{n} P_k x \right\|^2 = \|S_n x\|^2 \le \|x\|^2. \tag{3}$$

Letting $n \to \infty$ gives

$$\sum_{i=1}^{\infty} \|P_k x\|^2 \le \|x\|^2.$$

Let $\varepsilon > 0$. Then for $m < n$ sufficiently large, by Pythagoras Theorem,

$$\|S_n x - S_m x\|^2 = \left\| \sum_{k=m+1}^{n} P_k x \right\|^2 = \sum_{k=m+1}^{n} \|P_k x\|^2 \le \varepsilon.$$

This shows that $(S_n x)$ is a Cauchy sequence in H. Hence $\lim_{n\to\infty} S_n x$ exists by the completeness of H.

(b) By (a), we can define

$$Px = \sum_{k=1}^{\infty} P_k x = \lim_{n\to\infty} S_n x \quad \text{for } x \in H.$$

It is readily checked that P is linear. By Uniform Boundedness Theorem, there exists $M > 0$ such that $\sup_n \|S_n\| \le M$. Then

$$\|Px\| = \left\| \sum_{k=1}^{\infty} P_k x \right\| = \liminf_{n\to\infty} \|S_n x\| \le \liminf_{n\to\infty} \|S_n\| \|x\| \le M\|x\|.$$

Hence P is a bounded linear operator.

Next we check the P is an orthogonal projection. Let $x \in H$. Since (S_n) are orthogonal projections by the proof of (a) and the inner product is norm continuous,

$$\langle Px, x\rangle = \lim_{n\to\infty} \langle S_n x, x\rangle = \lim_{n\to\infty} \|S_n x\|^2 = \|Px\|^2.$$

This shows that P is self-adjoint since

$$\langle Px, x\rangle = \overline{\langle Px, x\rangle} = \langle x, Px\rangle = \langle P^* x, x\rangle.$$

Then

$$\langle P^2 x, x\rangle = \langle Px, Px\rangle = \lim_{n\to\infty} \|Px\|^2 = \lim_n \|S_n x\|^2 = \lim_n \langle S_n x, x\rangle = \langle Px, x\rangle.$$

Hence P is a projection. This finishes the proof.

(c) No. Since $(P - S_n)x = \sum_{k=n+1}^{\infty} P_k x$, the same proof of (b) shows that $P - S_n$ is also an orthogonal projection. Hence $\|P - S_n\| = 1$ if (P_k) are non-trivial.

For example, consider $H = \ell^2$. For $k \in \mathbb{N}$, define

$$P_k(x) = (0, \ldots, x(k), 0, \ldots) \quad \text{for } x = (x(n)) \in \ell^2.$$

Then P is the identity operator of ℓ^2. Moreover,

$$(P - S_n)(x) = (0, \ldots, x(n), x(n+1), \ldots) \quad \text{for } x = (x(n)) \in \ell^2.$$

This shows that $\|P - S_n\| = 1$ for $n \in \mathbb{N}$.

$\square$

— *THE END* —

2013-14 Term 1 Course Examination

1. Let $C[0,1]$ be the space of continuous functions with the supremum norm. Define $T\colon C[0,1] \to C[0,1]$ by

$$y(t) = (Tx)(t) = \int_0^t x(s)\,ds.$$

(a) Show that T is a bounded operator, and find $\|T\|$.

(b) Let $\mathcal{R}(T)$ denote the range of T, consider $T^{-1}\colon \mathcal{R}(T) \to C[0,1]$. Specify $\mathcal{R}(T)$ and T^{-1}.

(c) Is T^{-1} bounded? Justify your answer.

Proof. (a) $\|T\| = 1$.

(b) $\mathcal{R}(T) = C^1[0,1]$ is the set of continuously differentiable functions. The inverse operator T^{-1} is the differentiation operator by the Fundamental theorem of Calculus.

(c) Consider (y_n) where $y_n = \sin(nx)$.

$\square$

2. Let X be a normed linear space, and let Y be a finite dimensional subspace of X. Show that there exists $z \in X$ such that $\|z\| = 1$ and $d(z, Y) = 1$.

Proof. Since Y is a finite dimensional proper subspace of X, we can find a finite dimensional subspace Z of X such that $Y \subsetneq Z$. Then by Riesz Lemma, we can find a sequence (z_n) in S_Z such that $\lim_{n\to\infty} d(z_n, Y) = 1$. Since Z is finite dimensional, then S_Z is norm sequentially compact, there exits a subsequence (z_{n_k}) such that $\lim_{k\to\infty} z_{n_k} = z$ such that $\|z\| = 1$, and $d(z, Y) = \lim_{k\to\infty} d(z_{n_k}, Y) = 1$. $\square$

3. Evaluate the Fourier series for $f(x) = |x|$, $x \in [-\pi, \pi]$.

Proof. The simplified form of the Fourier series is

$$f(x) = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)^2} \cos((2k+1)x).$$

$\square$

4. Let $L^2[0,1]$ be the Banach space of square integrable functions on $[0,1]$ with norm $\|x\| = (\int_0^1 |x(t)|^2\,dt)^{1/2}$. Let $K(t,s)$ be a continuous function on $[0,1]\times[0,1]$. Define

$$(Tx)(t) = \int_0^1 K(t,s)x(s)\,ds.$$

(a) Show that $T\colon L^2[0,1] \to L^2[0,1]$ is a bounded linear operator.

(b) Find the adjoint operator T^* of T.

Proof. (a) By Minkowski's inequality (or Cauchy-Schwarz inequality),

$$\|T\| \le \left(\int_0^1 \int_0^1 |K(t,s)|^2 dtds\right)^{1/2} \le \sup_{t,s} |K(t,s)| < \infty.$$

(b) By definition,

$$(T^*x)(t) = \int_0^1 \overline{K(s,t)}x(t)\,ds.$$

$\square$

5. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H. Show that for any $x, y \in H$,

$$\left|\sum_{n=1}^{\infty} \langle x, e_n\rangle\langle y, e_n\rangle\right| \le \|x\|\,\|y\|.$$

Proof. For $N \in \mathbb{N}$, by the orthonormality of $\{e_n\}$ and Cauchy-Schwarz inequality,

$$\begin{aligned}\left|\sum_{n=1}^{N} \langle x, e_n\rangle\langle y, e_n\rangle\right| &= \left|\left\langle \sum_{n=1}^{N} \langle x, e_n\rangle e_n, \sum_{n=1}^{N} \langle y, e_n\rangle e_n \right\rangle\right| \\ &\le \left\|\sum_{n=1}^{N} \langle x, e_n\rangle e_n\right\| \left\|\sum_{n=1}^{N} \langle y, e_n\rangle e_n\right\| \\ &\le \|x\|\,\|y\|\end{aligned}$$

where the last inequality is by $x - \sum_{n=1}^{N}\langle x, e_n\rangle e_n$ is orthogonal to $\sum_{n=1}^{N}\langle x, e_n\rangle e_n$. Letting $N \to \infty$ finishes the proof. $\square$

6. Let X be a normed linear space. Use the Hahn-Banach theorem to prove

(a) For $x, y \in X$, if $f(x) = f(y)$ for all $f \in X^*$, then $x = y$.

(b) Let Y be a closed proper subspace in X, then there exists $f \in X^*$ such that $f(Y) = 0$ and $\|f\| = 1$.

Proof. (a) If $x \neq y$, then there exists $f \in X^*$ with $\|f\| = 1$ and $f(x - y) = \|x - y\| \neq 0$.

(b) Let $x_0 \in X \setminus Y$. Since Y is a closed subspace of X, we have $\delta = \inf_{y \in Y} \|x_0 + y\| > 0$. Write $Z = \operatorname{span}\{x_0\} + Y$. For $z = \alpha x_0 + y \in Z$ for some $y \in Y$, define

$$f(x) = f(\alpha x_0 + y) = \alpha.$$

Then f is a linear functional on Z and $f(Y) = 0$. Moreover, for $\alpha \neq 0$,

$$\|\alpha x_0 + y\| = |\alpha| \left\|x_0 + \frac{y}{\alpha}\right\| \ge \alpha\delta = |f(\alpha x_0 + y)|\delta.$$

Together with the trivial case $\alpha = 0$, this shows

$$|f(z)| \le \frac{1}{\delta}\|z\|$$

for $z \in Z$. Hence $f \in Z^*$. By Hahn-Banach Theorem, there exists $\widetilde{f} \in X^*$ such that $\widetilde{f}|_Z = f$ and $\|\widetilde{f}\| = \|f\| < \infty$. In particular, $\widetilde{f}(Y) = 0$. Finally, the normalized functional $\widetilde{f}/\|\widetilde{f}\|$ finishes the proof.

$\square$

7. (a) Let $f\colon C[0,1] \to \mathbb{R}$ be defined by $f(x) = \int_0^1 x(t)\,dw(t)$ where w is a function on $[0,1]$ with bounded variation. Show that $f \in C[0,1]^*$. What is $\|f\|$?

(b) Evaluate $\|f\|$ for the case $w(t) = \begin{cases} t & \text{if } 0 \le t < \frac{1}{2}, \\ \frac{1}{2} - t & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$

Proof. (a) $\|f\| \le V(\omega) + \omega(0)$.

(b) Since ω is continuous and $\omega(0) = 0$, we have $\|f\| = V(\omega)$. Moreover, by the piecewise monotoneness, we have

$$\|f\| = V(\omega) = \left|\frac{1}{2} - 0\right| + \left|0 - \frac{1}{2}\right| = 1.$$

□

8. Let X, Y be Banach spaces. Suppose there exists a continuous bijection $\tau\colon X \to Y$, then X, Y have equivalent norms.

Proof. This is a direct corollary of Open Mapping Theorem. □

— *THE END* —

2012-13 Term 1 Course Examination

1. Find the dual space of ℓ^1 and justify your answer.

Proof. $(\ell^1)^* = \ell^\infty$. See the corresponding Lecture Notes or Tutorial Notes for details. □

2. (a) Let $C[a,b]$ be the space of continuous functions on $[a,b]$ with the supremum norm $\|\cdot\|_\infty$. Is the subspace $Y = \{f \in C[a,b] \colon \int_a^b f(t)\,dt = 0\}$ closed? Justify your answer.

(b) Let $C^1[a,b]$ be the space of continuously differentiable functions on $[a,b]$ with norm

$$\|f\| := \|f\|_\infty + \|f'\|_\infty.$$

Show that $C^1[a,b]$ is a Banach space.

Proof. (a) Yes. Since $C[a,b]$ is a metric space, the for every $f \in \overline{Y}$, there exists (f_n) in Y such that $f_n \xrightarrow{\|\cdot\|} f$. Then

$$\left|\int_a^b f(t)\,dt\right| = \left|\int_a^b f(t) - f_n(t)\,dt\right| \le [b-a]\|f - f_n\|_\infty \to 0$$

as $n \to \infty$. Hence $\int_a^b f(t)\,dt = 0$. This shows $f \in Y$, thus Y is closed.

(b) Let (f_n) be a Cauchy sequence in $C^1[a,b]$. Then $(f_n) \xrightarrow{\|\cdot\|_\infty} g$ and $(f_n') \xrightarrow{\|\cdot\|_\infty} h$ for some functions $g, h \in C[a,b]$. By the limit process changing results about the uniform convergence, we have $g' = h$. Hence $f_n \xrightarrow{\|\cdot\|} g \in C^1[a,b]$ as $n \to \infty$.

□

3. Let H be a Hilbert space, and let $C(\neq \emptyset)$ be a closed convex subset in H. Use the parallelogram law to show that there is a unique $x \in C$ such that $\|x\| = \inf\{\|y\| : y \in C\}$.

Proof. (Existence) Let (y_n) be a sequence in C such that

$$\lim_{n\to\infty} \|y_n\| = \inf_{y \in C} \|y\|.$$

By Parallelogram Law,

$$\|y_n - y_m\|^2 + \|y_n + y_m\|^2 = 2\|y_n\|^2 + 2\|y_m\|^2.$$

Since $(y_n + y_m)/2 \in C$ by the convexity of C,

$$\begin{aligned}
\frac{1}{4}\|y_n - y_m\|^2 &= \frac{1}{2}\|y_n\|^2 + \frac{1}{2}\|y_m\|^2 - \|\frac{y_n + y_m}{2}\|^2 \\
&\le \frac{1}{2}\|y_n\|^2 + \frac{1}{2}\|y_m\|^2 - (\inf_{y \in C}\|y\|)^2 \to 0
\end{aligned}$$

as $n, m \to \infty$, it follows that (y_n) is a Cauchy sequence. Since C is closed in the Hilbert space H, there exists some $y_0 \in C$ such that $y_n \xrightarrow{\|\cdot\|} y_0$ as $n \to \infty$. Hence

$$\|y_0\| = \lim_{n\to\infty} \|y_n\| = \inf_{y \in C} \|y\|.$$

(Uniqueness) Let $y_1, y_2 \in C$ such that

$$\|y_1\| = \|y_2\| = \inf_{y \in C} \|y\|.$$

Then by Parallelogram Law,

$$\begin{aligned}\frac{1}{4}\|y_1 - y_2\|^2 &= \frac{1}{2}\|y_1\|^2 + \frac{1}{2}\|y_m\|^2 - \|\frac{y_n + y_m}{2}\|^2 \\ &\le \frac{1}{2}\|y_1\|^2 + \frac{1}{2}\|y_1\|^2 - \|y_1\|^2 = 0.\end{aligned}$$

Hence $y_1 = y_2$. □

4. Evaluate the Fourier series of $f(x) = x$, $x \in [-\pi, \pi]$.

Proof. For $n \in \mathbb{Z}$,

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx = \begin{cases} \frac{i(-1)^n}{n} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0. \end{cases}$$

Hence the Fourier series is

$$g(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n)e^{inx} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

□

5. Let T be a bounded linear operator on a Hilbert space H, and let T^* be the adjoint of T. Show that

(a) $\|T\| = \sup\{|\langle Tx, y\rangle| \colon \|x\| \le 1, \|y\| \le 1\}$.

(b) $\|T\| = \|T^*\|$.

(c) $\|TT^*\| = \|T^*T\| = \|T\|^2$.

Proof. (a) By Hahn-Banach theorem and Riesz representation theorem, for $Tx \in H$, there exists $y \in H$ with $\|y\| = 1$ such that $\langle Tx, y\rangle = \|Tx\|$. Hence

$$\begin{aligned}\|T\| &:= \sup\{\|Tx\| \colon \|x\| \le 1\} \\ &= \sup\{|\langle Tx, y\rangle| \colon \|x\| \le 1, \|y\| \le 1\}.\end{aligned}$$

(b) By (a),

$$\begin{aligned}\|T^*\| &= \sup\{|\langle T^*y, x\rangle| \colon \|x\| \le 1, \|y\| \le 1\} \\ &= \sup\{|\langle x, T^*y\rangle| \colon \|x\| \le 1, \|y\| \le 1\} \\ &= \sup\{|\langle Tx, y\rangle| \colon \|x\| \le 1, \|y\| \le 1\} = \|T\|.\end{aligned}$$

(c) Since $T^{**} = T$, we have $\|TT^*\| = \|T^*T\| \le \|T\|\|T^*\| = \|T\|^2$. On the other hand, for x with $\|x\| = 1$,

$$\|Tx\|^2 = |\langle Tx, Tx\rangle| = |\langle x, T^*Tx\rangle| \le \|T^*T\|\|x\|^2 = \|T^*T\|.$$

Hence $\|T\|^2 \le \|T^*T\|$. This finishes the proof.

□

6. (a) State the Hahn-Banach theorem of a normed linear space X.

(b) Use the theorem to prove that for x_1, x_2 with $x_1 \neq x_2$, there exists an $f \in X^*$ such that $f(x_1) \neq f(x_2)$.

Proof. (a) (Dominated extension) Let p be a real-valued subadditive positive homogeneous function on X. Let Y be a subspace of X and f is a real-valued linear functional on Y. Suppose

$$f \leq p \quad \text{on } Y.$$

Then there exists a linear functional $\widetilde{f}$ on X such that $\widetilde{f} \leq p$.

(Norm attaining) For any $x \in X$, there exists a continuous functional $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

(b) Since $\|x_1 - x_2\| > 0$ by $x_1 \neq x_2$, it follows from Hahn-Banach theorem that there exists $f \in X^*$ such that $f(x_1 - x_2) = \|x_1 - x_2\| > 0$, thus $f(x_1) > f(x_2)$.

□

7. Let $w \in BV[a, b]$, the space of functions with bounded variation. Let

$$f(x) = \int_a^b x(t)\, dw(t), \quad x \in C[a, b].$$

Show that f is a bounded linear functional on $C[a, b]$, and find $\|f\|$.

Proof. Let $P = \{a = t_0 < \cdots t_n = b\}$ be any partition of $[a, b]$ with left endpoint tags. Let $x \in C[a, b]$ Then the Riemann sum

$$R(x, \rho, P) \leq \left| \sum_{k=1}^{n} x(t_{k-1})(w(t_k) - w(t_{k-1})) \right| \leq \|x\|_\infty V(w, P) \leq \|x\|_\infty V(w).$$

where $V(w)$ denotes the total variation of w. Hence letting the diameter of P go to zero gives that $|f(x)| \leq \|x\|_\infty V(w)$. This implies $\|f\| \leq V(w)$.

As for the other inequality, it may not hold for all bounded variation functions and we need some modifications. Define

$$\widetilde{w}(t) := \begin{cases} 0 & \text{if } t = a \\ \lim_{\tau \to t+} w(\tau) & \text{if } t \in (a, b) \\ w(b) & \text{if } t = b. \end{cases}$$

Hence $\widetilde{w} \in BV_0^+[a, b]$ and so $\|f\| = V(\widetilde{w})$. Please refer to the corresponding Tutorial Notes for the details. □

8. Use the Baire Category theorem to prove the following: Let X be a Banach space. Suppose $\{x_n\}_{n=1}^\infty \subset X$ is such that $\{f(x_n)\}_{n=1}^\infty$ is a bounded sequence for each $f \in X^*$, then $\{x_n\}_{n=1}^\infty$ is a bounded sequence.

Proof. For $k \in \mathbb{N}$, define

$$\Lambda_k := \{f \in X^* \colon |f(x_n)| \le k \text{ for all } n \in \mathbb{N}\}.$$

Then it follows from the assumption that

$$X^* = \bigcup_{k \in \mathbb{N}} \Lambda_k.$$

Since X^* is a complete metric space, by Baire Category Theorem there exists $K \in \mathbb{N}$ such that

$$B(x_0, r) \subset \Lambda_K$$

for some $f_0 \in \Lambda_K$ and $r > 0$. Since for every $f \in B_{X^*}$, we have $f_0 + rf \in B(f_0, r)$. Then for all $n \in \mathbb{N}$ and $f \in B_{X^*}$

$$r|f(x_n)| \le |(f_0 + rf)(x_n)| + |f_0(x_n)| \le K + K = 2K.$$

Then Hahn-Banach theorem implies that $\|x_n\| \le \frac{2K}{r}$ for all $n \in \mathbb{N}$. $\square$

— *THE END* —

2011-12 Term 1 Course Examination

Part I

1. Let $b = (b(1), b(2), \ldots)$ be a sequence in $\mathbb{R}$ satisfying, for every $a \in \ell^2$, $\sum_j a(j)b(j)$ is finite. Show that $b \in \ell^2$.

Proof. For $n \in \mathbb{N}$, define

$$\Lambda_n(a) = \sum_{j=1}^{n} a(j)b(j)$$

for $a = (a(j)) \in \ell^2$. Then $\|\Lambda_n\| = (\sum_{j=1}^{n} |b(j)|^2)^{1/2}$ by $(\ell^2)^* = \ell^2$. By assumption, (Λ_n) is pointwise bounded on ℓ^2. Then Uniform Boundedness Theorem implies that $\sup_n \|\Lambda_n\| \leq M$ for some $M < \infty$. This implies $(\sum_{j=1}^{\infty} |b(j)|^2)^{1/2} \leq M$, thus $b \in \ell^2$. $\square$

2. Let (x_k) be a sequence in the Hilbert space H and $z \in H$ which satisfy, for every $\Lambda \in H^*$, $\Lambda x_k \to \Lambda z$ and $\|x_k\| \to \|z\|$ as $k \to \infty$. Show that $\|x_k - z\| \to 0$ in H.

Proof. Note that

$$\|x_k - z\|^2 = \|x_k\|^2 - 2\Re\langle x_k, z\rangle + \|z\|^2$$

where $\Re z$ denotes the real part of a complex number z. Since $\langle x_k, z\rangle \to \langle z, z\rangle$ and $\|x_k\| \to \|z\|$ as $k \to \infty$ and $\Re\colon \mathbb{C} \to \mathbb{R}$ is continuous, we have

$$\|x_k - z\|^2 \to \|z\|^2 - 2\Re\langle z, z\rangle + \|z\|^2 = 0$$

as $k \to \infty$. $\square$

3. Let X_1 and X_2 be two closed subspaces of the Banach space X which form a direct sum of X. That is, every $x \in X$ can be expressed uniquely as $x_1 + x_2, x_i \in X_i, i = 1, 2$. Show that the projection map $P\colon x \mapsto x_1$ is bounded. You may assume that it is linear.

Proof. Suppose $x_n \xrightarrow{\|\cdot\|} x$ and $Px_n \xrightarrow{\|\cdot\|} y$. Then $y \in X_1$ since X_1 is closed, and so $Py = y$. Since $x_n - Px_n \in X_2$ and X_2 is closed, we have $x - y \in X_2$. Then $P(x - y) = 0$, thus $Px = Py = y$. Hence P is bounded by Closed Graph Theorem. $\square$

4. Consider the linear operator $S_R x = (x(2), x(3), x(4), \ldots)$ for $x = (x(i)) \in \ell^2$ over $\mathbb{C}$. Determine the spectrum of S_R.

Proof. Let $\lambda \in \mathbb{C}$ and $x = (x(i)) \in \ell^2$. Write $y = (\lambda I - S_R)x$. Then

$$y(i) = \lambda x(i) - x(i+1) \qquad \text{for } i \in \mathbb{N}.$$

Suppose $y = 0 \in \ell^2$. Then $x(i+1) = \lambda x(i)$. By induction $x = (\lambda^{i-1}x(1))$. Then $x \in \ell^2$ if and only if $|\lambda| < 1$. Hence $\ker(\lambda I - S_R) \neq 0$ if $|\lambda| < 1$, thus

$$\sigma(S_R) \supset \{\lambda \in \mathbb{C}\colon |\lambda| < 1\}.$$

By $\|S_R\| = 1$,

$$\sigma(S_R) \subset \{\lambda \in \mathbb{C} \colon |\lambda| \le 1\}.$$

Since $\sigma(S_R)$ is a closed set, we have

$$\sigma(S_R) = \{\lambda \in \mathbb{C} \colon |\lambda| \le 1\}.$$

$\square$

5. Define

$$Lf(x) = \int_0^1 a(x-y)f(y)\,dy$$

for $f \in L^2((0,1))$ over $\mathbb{C}$. Find conditions on the function a so that L defines a bounded, self-adjoint linear operator on $L^2((0,1))$.

Proof. The linearity is readily justified if L is well defined.

- Suppose $a \in L^2[-1,1]$. Then by Cauchy-Schwarz inequality,

$$\begin{aligned}
\|Lf\| &= (\int_0^1 \left|\int_0^1 a(x-y)f(y)\,dy\right|^2 dx)^{1/2} \\
&\le \|f\| \left(\int_0^1 \int_0^1 |a(x-y)|^2\,dydx\right)^{1/2} \\
&= \|f\|\|a\|.
\end{aligned}$$

  Hence $\|L\| \le \|a\|$, which shows that L is a bounded linear operator.

- Suppose $a(x) = \overline{a(-x)}$ for $x \in [0,1]$. Then $a(x-y) = \overline{a(y-x)}$ for $x, y \in (0,1)$. For $f, g \in L^2((0,1))$,

$$\begin{aligned}
\langle Lf, g\rangle &= \iint a(x-y)f(y)\overline{g(x)}\,dydx \\
&= \iint \overline{a(y-x)}f(y)\overline{g(x)}\,dydx \\
&= \iint f(y)\overline{a(y-x)g(x)}\,dydx \\
&= \int f(y)\left(\int \overline{a(y-x)g(x)}\,dx\right) dy \\
&= \langle f, Lg\rangle.
\end{aligned}$$

  Hence $L = L^*$, which shows that L is a self-adjoint operator.

$\square$

Part II

6. Describe how to construct the dual space of $C[0,1]$. You do not have to include proofs, but all notations should be clearly explained. In particular, point out how the Hahn-Banach theorem is used in a crucial step.

Proof. For each $\Lambda \in C[0,1]^*$, there exists some bounded variation function ρ such that

$$\Lambda(f) = \int f \, d\rho$$

for all $f \in C[0,1]$ and the above integral is Riemann-Stieljes integral.

The Hahn-Banach theorem is used to extend Λ to $\widetilde{\Lambda}$ defined on bounded functions, so that we can define the value for $x \in (0,1]$,

$$\rho(x) := \widetilde{\Lambda}(\chi_{(0,x]})$$

which χ_A denotes the indicator function for a set A. Note that $\chi_{(0,x]}$ is not continuous.

For more details, please refer to the corresponding Tutorial Notes. $\square$

7. Let $A \in B(X)$ where X is a Banach space. Propose a definition for $\cos A$ so that it belongs to $B(X)$. Then show that the function $\phi(t) = (\cos tA)x_0$, $t \geq 0$, solves the initial value problem

$$x'' + A^2 x = 0, \quad x(0) = x_0, \quad x'(0) = 0.$$

Errata: Replace A with A^2.

Proof. Similar to the definition of e^A, we can define

$$\cos A := \sum_{k=0}^{\infty} (-1)^k \frac{A^{2k}}{(2k)!}.$$

Since $A \in B(X)$ and X is a Banach space, then $\cos A$ is well defined and $\cos A \in B(X)$ with $\|\cos A\| \leq \cos(\|A\|)$.

Similarly, define

$$\sin A := \sum_{k=0}^{\infty} (-1)^k \frac{A^{2k+1}}{(2k+1)!}.$$

Fix any $t \in$. Then $|t|, \|A\| \leq M$ for some $M > 0$. For every $h \in \mathbb{R}$ with $|h|$ small, we have

$$\begin{aligned}
&\|\phi(t+h) - \phi(t) + A(\sin tA)x_0\| \\
&\leq \|\cos(t+h)A - \cos tA + A\sin tA\| \, \|x_0\| \\
&\leq \left\| \sum_{k=0}^{\infty} (-1)^k \left(\frac{(t+h)^{2k}A^{2k}}{(2k)!} - \frac{t^{2k}A^{2k}}{(2k)!} \right) + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}A^{2k+1}}{(2k+1)!} \right\| \|x_0\| \\
&\leq \left\| \sum_{k=0}^{\infty} \left((-1)^{k+1} \frac{t^{2k+1}A^{2k+2}}{(2k+1)!} + (-1)^k \frac{t^{2k+1}A^{2k+1}}{(2k+1)!} \right) + \sum_{k=1}^{\infty} \frac{C_k h^2}{(2k)!} \right\| \|x_0\| \\
&= \left\| \sum_{k=1}^{\infty} \frac{C_k}{(2k)!} \right\| \|x_0\| |h|^2
\end{aligned}$$

where $\|C_k\| \leq (2M^2)^{2k}$. Then $\left\| \sum_{k=1}^{\infty} \frac{C_k}{(2k)!} \right\| < \infty$, and so

$$\|\phi(t+h) - \phi(t) + A(\sin tA)x_0\| = o(|h|).$$

This implies

$$\phi'(t) = -A(\sin tA)x_0.$$

A similar argument gives

$$\phi''(t) = -A^2(\cos tA)x_0.$$

Next we justify that $\phi(t)$ is indeed the solution.

- $\phi''(t) + A^2\phi(t) = -A^2(\cos tA)x_0 + A^2(\cos tA)x_0 = 0.$
- $\phi'(0) = -A(\sin(0))x_0 = 0.$
- $\phi(0) = \cos(0)x_0 = x_0.$

This finishes the proof. □

8. Let B be a nonempty set and $\ell^2(B)$ be the real vector space consisting of all functions B to $\mathbb{R}$ such that (i) $x(b) \neq 0$ for at most countably many b and (ii) $\Sigma_b x(b)^2 < \infty$. Establish the followings:

(a) $\ell^2(B)$ forms a Hilbert space under the inner product

$$\langle x, y\rangle = \sum_b x(b)y(b).$$

(b) There exists a complete orthonormal set of $\ell^2(B)$ whose cardinality is equal to the cardinality of B.

Proof. (a) Let $x, y \in \ell^2(B)$ with $x(b) \neq 0, y(b) \neq 0$ for at most countably many b. Then $x(b)y(b) \neq 0$ for at most countably many b. By Cauchy-Schwarz inequality,

$$|\langle x, y\rangle| \leq \sum_b |x(b)y(b)| \leq (\sum_b |x(b)|^2)^{1/2}(\sum_b |y(b)|^2)^{1/2} = \|x\|\,\|y\|.$$

Then $\langle x, y\rangle$ is well defined.

- $\langle x, x\rangle = \sum_b (x(b))^2 \geq 0$. If $\langle x, x\rangle = 0$, then $x(b) = 0$ for all $b \in B$, thus $x = 0$.
- $\langle x, y\rangle = \sum_b x(b)y(b) = \sum_b y(b)x(b) = \langle y, x\rangle$.
- $\langle \alpha x + \widetilde{x}, y\rangle = \sum_b (\alpha x(b) + \widetilde{x}(b))y(b) = \alpha \sum_b x(b)y(b) + \sum_b \widetilde{x}(b)y(b) = \alpha\langle x, y\rangle + \langle \widetilde{x}, y\rangle$.

Hence $\langle \cdot, \cdot\rangle$ is an inner product. Next we justify the completeness.
Let (x_n) be a Cauchy sequence in $\ell^2(B)$. Since for each $b \in B$,

$$|x_n(b) - x_m(b)| \leq \|x_n - x_m\|,$$

we have $(x_n(b))$ is a Cauchy sequence for each $b \in B$. Hence $\lim_{n\to\infty} x_n(b) = x(b)$ for some $x(b) \in \mathbb{R}$ by the completeness of $\mathbb{R}$. Let $A = \{a_i\}_{i=1}^{\infty}$ be the union of the nonzero points of $\{x_n\}$. Then $x(b) \neq 0$ only if $b \in A$.
Let $\varepsilon > 0$ and $K \in \mathbb{N}$. Then when n, m are large,

$$\left(\sum_{i=1}^{K} |x_n(a_i) - x_m(a_i)|^2\right)^{1/2} \leq \|x_n - x_m\| \leq \varepsilon.$$

Letting $m \to \infty$ gives

$$\left(\sum_{i=1}^{K} |x_n(a_i) - x(a_i)|^2\right)^{1/2} \leq \|x_n - x_m\| \leq \varepsilon.$$

Letting $K \to \infty$ gives

$$\left(\sum_{a\in A} |x_n(a) - x(a)|^2\right)^{1/2} = \left(\sum_{i=1}^{\infty} |x_n(a_i) - x(a_i)|^2\right)^{1/2} \leq \|x_n - x_m\| \leq \varepsilon.$$

Hence

$$\|x_n - x\| = \left(\sum_{b\in B} |x_n(a) - x(a)|^2\right)^{1/2} = \left(\sum_{a\in A} |x_n(a) - x(a)|^2\right)^{1/2} \leq \varepsilon.$$

This implies $x_n \xrightarrow{\|\cdot\|} x$, and so $x \in \ell^2(B)$.

(b) For each $b \in B$, define δ_b by $\delta_b(b) = 1$ and $\delta_b(a) = 0$ for $a \neq b$. Then $\|b\| = 1$ and $\delta_b \in \ell^2(B)$. For $a \neq b$, we have $\langle a, b\rangle = 0$. Moreover, for each $x \in \ell^2$,

$$x = \sum_b x(b)\delta_b.$$

If $\langle x, \delta_b\rangle = 0$ for all b, then $x(b) = 0$ for all $b \in B$, thus $x = 0$. Hence $\{\delta_b\}_{b\in B}$ is a complete orthonormal basis. Since $b \mapsto \delta_b$ is a bijection, the cardinality of $\{\delta_b\}$ is the same as that of B.

□

9. Let T be a compact, self-adjoint linear operator on the Hilbert space X and let

$$R(x) = \frac{\langle Tx, x\rangle}{\|x\|^2}, x \neq 0$$

be its Rayleigh quotient. Let λ_n, $n \geq 1$, be the positive eigenvalues of T in decreasing order.

(a) Show that for $n \geq 0$,

$$\lambda_{n+1} = \min_{E_n} \max\{R(x) \colon x \neq 0, x \perp E_n\},$$

where E_n is any n-dimensional subspace of X. This is called *Courant's principle* for eigenvalues.

(b) Let S be another compact, self-adjoint linear operator with positive eigenvalues μ_n ordered in decreasing order. Suppose that $\langle (T - S)x, x\rangle \geq 0, x \in X$. Prove $\mu_n \leq \lambda_n$, $\forall\, n$.

Proof. This is standard result on Hilbert-Schmidt operators, (see e.g., Sec 16.6 of Royden's).

□

— *THE END* —

2010-11 Term 1 Course Examination

Part I

1. Describe the dual space of $C[a,b]$.

Proof. For each $\Lambda \in C[a,b]^*$, there exists some bounded variation function ρ such that

$$\Lambda(f) = \int f\, d\rho$$

for all $f \in C[a,b]$ and the above integral is Riemann-Stieljes integral. For a complete characterization of $C[a,b]^*$, we can modify the space of bounded variation function $BV[a,b]$ to obtain

$$BV_0^+[a,b] := \{\rho \in BV[a,b] \colon \rho(a) = 0 \text{ and } \rho \text{ is right continuous}\}.$$

Then $C[a,b]^* \cong BV_0^+[a,b]$. For the details, please refer to the related Tutorial Note. □

2. Determine the spectrum of the linear operator ℓ^2 given by

$$Tx = (x(1), -x(2), x(3), -x(4), \ldots)$$

for $x = (x(i)) \in \ell^2$.

Proof. Let $\sigma(T)$ denote the spectrum of T. Recall

$$\sigma(T) := \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ does not have bounded inverse}\}.$$

It follows from Open Mapping Theorem that $\lambda \in \sigma(T)$ if and only if $\lambda I - T$ is not bijective.

(Not injective) Let $x = (x(i)) \in \ell^2$. If $(\lambda I - T)x = 0$, then

$$(\lambda - 1)x(2k-1) = 0 \quad \text{and} \quad (\lambda + 1)x(2k) = 0$$

for all $k \in \mathbb{N}$. Hence $\lambda I - T$ is not injective if and only if $\lambda = \pm 1$. Hence $\{-1, 1\} \subset \sigma(T)$.

(Not surjective) Let $\lambda \in \mathbb{C} \setminus \{-1, 1\}$. Let $y = (y(i)) \in \ell^2$. We can define x by

$$x(i) = \begin{cases} \frac{y(i)}{\lambda - 1} & \text{if } i \text{ is odd} \\ \frac{y(i)}{\lambda + 1} & \text{if } i \text{ is even} \end{cases}.$$

Then $x \in \ell^2$ and $(\lambda I - T)x = y$. This implies that $\lambda I - T$ is surjective.

Together we have $\sigma(T) = \{-1, 1\}$. □

3. Find the orthogonal projection of the function $\sin(x)$ to the subspace spanned by $\cos(x)$ and $1 - x$ in $L^2(-\pi, \pi)$.

Proof. Write $f(x) = \sin x$, $g_1(x) = \cos x$ and $g_2(x) = 1 - x$. Note that

$$\langle g_1, g_2 \rangle = \int_{-\pi}^{\pi} (1-x) \cos x \, dx = 0.$$

First we transfer $\{g_1, g_2\}$ into an orthonormal basis by Hilbert-Schmidt process. Define

$$e_1 := \frac{g_1}{\|g_1\|_2} = \frac{\cos x}{\sqrt{\pi}} \tag{4}$$

$$e_2 := \frac{g_2}{\|g_2\|_2} = \frac{1-x}{\sqrt{2\pi}}. \tag{5}$$

Hence the orthogonal projection is

$$Px = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2$$

for $x \in L^2(-\pi, \pi)$. In particular, since $\langle f, e_1 \rangle = 0$ and

$$\langle f, e_2 \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (1-x) \sin x \, dx = -\sqrt{2\pi},$$

we have $Pf(x) = -\sqrt{2\pi} \times \frac{1-x}{\sqrt{2\pi}} = x - 1$. □

4. Find the adjoint operator of the linear operator $S \colon \ell^2 \to \ell^2$ given by

$$Sx = (0, 0, x(1), x(2), x(3), \ldots),$$

for $x = (x(i)) \in \ell^2$.

Proof. Let $x = (x(i)), y = (y(i)) \in \ell^2$. Then

$$\langle x, S^* y \rangle = \langle Sx, y \rangle = \sum_{i=1}^{\infty} x(i) \overline{y(i+2)} = \langle x, \widetilde{y} \rangle$$

where $\widetilde{y} = (y(i+2))_{i=1}^{\infty}$. Hence $S^* y = (y(i+2))$ for $y = (y(i)) \in \ell^2$. □

5. Consider the linear operator $T \colon L^2(0,1) \to L^2(0,1)$ given by

$$(Tf)(x) = \int_0^1 K(x,y) f(y) \, dy$$

where K and $\partial K / \partial x$ belong to $C([0,1] \times [0,1])$. Show that T is compact.

Proof. For $h \in C([0,1] \times [0,1])$, denote $\|h\|_\infty = \sup_{x,y \in [0,1]} |x, y|$. Then $\|h\|_\infty < \infty$. Denote the norm on $L^2(0,1)$ by $\| \cdot \|$.

Since

$$\begin{aligned} \|Tf\|^2 &= \int_0^1 \left| \int_0^1 K(x,y) f(y) \, dy \right|^2 dx \\ &\le \int_0^1 \left(\int_0^1 |K(x,y)||f(y)| \, dy \right)^2 dx \\ &\le \|K\|_\infty^2 \|f\|^2, \end{aligned}$$

we have $\|T\| \leq \|K\|_\infty$. This implies T is bounded.

Let (f_n) be a bounded sequence in L^2, that is, $\sup_n \|f_n\| \leq M$ for some $M > 0$. To prove T is compact, it suffices to show that (Tf_n) has a convergent subsequence. Notice that for $x \in [0,1]$,

$$|(Tf_n)(x)| \leq \int_0^1 |K(x,y)||f_n(y)|\,dy \leq \|K\|_\infty \int_0^1 |f_n(y)|\,dy \leq \|K\|_\infty \|f_n\| \leq \|K\|_\infty M.$$

Hence (Tf_n) is pointwise bounded. On the other hand, for any $x, \widetilde{x} \in [0,1]$, it follows from Finite-increment Theorem (Mean Value Inequality) that

$$\begin{aligned}
|(Tf_n)(x) - (Tf_n)(\widetilde{x})| &\leq \int_0^1 |K(x,y) - K(\widetilde{x},y)|\,|f_n(y)|\,dy \\
&\leq \int_0^1 \|\partial K/\partial x\|_\infty |x - \widetilde{x}||f_n(y)|\,dy \\
&\leq \|\partial K/\partial x\|_\infty M|x - \widetilde{x}|.
\end{aligned}$$

This implies that (Tf_n) is equicontinuous. Since $[0,1]$ is compact, by Arzelà–Ascoli theorem there exists a subsequence (Tf_{n_k}) uniformly converging to a function g. Then

$$\|g - Tf_{n_k}\| \leq \|g - Tf_{n_k}\|_\infty \to 0 \text{ as } k \to \infty.$$

Hence T is compact. □

Part II

6. (a) State Baire Theorem.

(b) Show that any (Hamel) basis of an infinite dimensional Banach space is uncountable.

Proof. (a) In a complete metric space (or locally compact Hausdorff space), any countable intersection of open dense sets is dense, or equivalently any countable union of closed sets with empty interior has no interior points.

(b) Suppose otherwise there exists a countable Hamel basis $\{x_n\}_{n=1}^\infty$ of an infinite dimensions Banach space X. For $n \in \mathbb{N}$, define

$$V_n = \operatorname{span}\{x_1, \ldots, x_n\}.$$

Then V_n is complete as a finite dimensional space, thus closed in X. The interior of V_n is empty. It follows from the definition of Hamel basis that

$$X = \bigcup_{n \in \mathbb{N}} V_n.$$

Since X is a Banach space, by Baire Category Theorem there exists $N \in \mathbb{N}$ such that V_N has non-empty interior. Hence after a translation there is an open set U such that $0 \in U \subset V_N$. This implies $X \subset V_N$ because $X = \bigcup_{k \in \mathbb{N}} kU \subset V_N$, which contradicts the assumption that X is infinite dimensional.

□

7. For $g \in C[a,b]$, define

$$\Lambda f = \int_a^b f(x)g(x)\,dx.$$

(a) Show that $\Lambda \in C[a,b]^*$ and

$$\|\Lambda\| = \int_a^b |g(x)|\,dx.$$

You may assume g has finitely many zeros in $[a,b]$.

(b) Describe how Λ can be viewed as a bounded linear functional on $L^2(a,b)$ and determine its operator norm.

Proof. (a) By $g \in C[a,b]$ we have $\|g\|_1 \leq (b-a)\|g\|_\infty < \infty$. Hence

$$|\Lambda f| \leq \int_a^b |f(x)||g(x)|\,dx \leq \|f\|_\infty \|g\|_1.$$

Then $\|\Lambda\| \leq \|g\|_1$. On the other hand, let $\{x_s\}_{s=1}^n$ be the zeros of g. Then for k large enough such that $1/k < 1/2 \min_{s\neq t}|x_s - x_t|$, define

$$f_k(x) = \begin{cases} \overline{g(x)}/|g(x)| & \text{if } x \notin \bigcup_{s=1}^n [x_s - 1/k, x_s + 1/k] \\ \text{linearly connect the argument of end points} & \text{if } x \in (x_s - 1/k, x_s + 1/k). \end{cases}$$

Then f_k is continuous and $\|f_k\|_\infty = 1$. Moreover,

$$\begin{aligned} |\|g\|_1 - \Lambda f_k| &= \left| \int_a^b |g(x)|\,dx - \int_a^b f_k(x)g(x)\,dx \right| \\ &\leq \sum_{s=1}^n \int_{x_s-1/k}^{x_s+1/k} |1 - f_k(x)|\,dx \leq \frac{2n}{k}. \end{aligned}$$

Hence $\|\Lambda\| \geq \|g\|_1 - \frac{2n}{k}$. Letting $k \to \infty$ gives $\|\Lambda\| = \|g\|_1$.

(b) By Cauchy-Schwarz inequality, for $f \in L^2$,

$$|\Lambda f| \leq \|g\|_2 \|f\|_2 < \infty.$$

Hence $\|\Lambda\| \leq \|g\|_2$. Without loss of generality we can assume $\|g\|_2 > 0$. Define $f(x) = \overline{g(x)}/\|g\|_2$. Then

$$|\Lambda f| = \frac{1}{\|g\|_2} \int \overline{g(x)} g(x)\,dx = \|g\|_2.$$

Hence $\|\Lambda\| = \|g\|_2$.

□

8. Let $T \in L(X,Y)$ where X and Y are normed spaces. The operator T is called weakly continuous if $Tx_n \to Tx$ weakly in Y if $x_n \to x$ weakly in X.

(a) T is weakly continuous if T is continuous.

(b) T is continuous if T is weakly continuous provided X and Y are complete.

Proof. (a) Since T is continuous, then the adjoint $T^*\colon Y^* \to X^*$ is well defined. Let $x_n \to x$ weakly in X. Then for $y^* \in Y^*$,

$$\langle Tx_n, y^* \rangle = \langle x_n, T^* y^* \rangle \to \langle x, T^* y^* \rangle = \langle Tx, y^* \rangle.$$

Hence $Tx_n \to Tx$ weakly in Y since y^* is arbitrary.

(b) Suppose $x_n \xrightarrow{\|\cdot\|} x$ in X and $Tx_n \xrightarrow{\|\cdot\|} y$ in Y. Then $x_n \xrightarrow{w} x$ and $Tx_n \xrightarrow{w} y$ since the weak topology is smaller than the norm topology. On the other hand, we have $Tx_n \xrightarrow{w} Tx$ since T is weakly continuous. Since the weak topology is Hausdorff by Hahn-Banach theorem, the weak limit is unique, thus $Tx = y$. Hence T is continuous by Closed Graph Theorem.

□

9. Let K be a closed, convex set in the Hilbert space H. For any x_0 in H, prove that there exists a unique y_0 in K satisfying

$$\|y_0 - x_0\| \le \|y - x_0\|, \quad \forall\, y \in K.$$

Proof. By considering $K - x_0$ instead, without loss of generality we can assume $x_0 = 0$.

(Existence) Let (y_n) be a sequence in K such that

$$\lim_{n\to\infty} \|y_n\| = \inf_{y\in K} \|y\|.$$

By Parallelogram Law,

$$\|y_n - y_m\|^2 + \|y_n + y_m\|^2 = 2\|y_n\|^2 + 2\|y_m\|^2.$$

Since $(y_n + y_m)/2 \in K$ by the convexity of K,

$$\begin{aligned}\frac{1}{4}\|y_n - y_m\|^2 &= \frac{1}{2}\|y_n\|^2 + \frac{1}{2}\|y_m\|^2 - \|\frac{y_n + y_m}{2}\|^2 \\ &\le \frac{1}{2}\|y_n\|^2 + \frac{1}{2}\|y_m\|^2 - (\inf_{y\in K}\|y\|)^2 \to 0\end{aligned}$$

as $n, m \to \infty$, it follows that (y_n) is a Cauchy sequence. By the closedness of K in the Hilbert space H, we have $y_n \xrightarrow{\|\cdot\|} y_0$ for some $y_0 \in K$. Hence

$$\|y_0\| = \lim_{n\to\infty}\|y_n\| = \inf_{y\in K}\|y\|.$$

(Uniqueness) Let $y_1, y_2 \in K$ such that

$$\|y_1\| = \|y_2\| = \inf_{y\in K}\|y\|.$$

Then by Parallelogram Law,

$$\begin{aligned}\frac{1}{4}\|y_1 - y_2\|^2 &= \frac{1}{2}\|y_1\|^2 + \frac{1}{2}\|y_m\|^2 - \|\frac{y_n + y_m}{2}\|^2 \\ &\le \frac{1}{2}\|y_1\|^2 + \frac{1}{2}\|y_1\|^2 - \|y_1\|^2 = 0.\end{aligned}$$

Hence $y_1 = y_2$. □

— *THE END* —