THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH4010 Functional Analysis 2021-22 Term 1

Solutions to Final Examination

Q1 (15 pts)

(a) (7 pts) It is readily checked that E is a subspace. Next we prove $\overline{E} = E$. Let $x \in \overline{E}$. Let M > 0 such that $\sup \|T_n\| \le M$. For any $\varepsilon > 0$, there exists $\widetilde{x} \in E$ such that $\|x - \widetilde{x}\| \le \frac{\varepsilon}{3M}$. By $\widetilde{x} \in E$, there exists $N \in \mathbb{N}$ such that $\forall n, m \ge N$, $\|T_n\widetilde{x} - T_m\widetilde{x}\| \le \frac{\varepsilon}{3}$. Then

$$||T_n x - T_m x|| \le ||T_n x - T_n \widetilde{x}|| + ||T_n \widetilde{x} - T_m \widetilde{x}|| + ||T_m \widetilde{x} - T_m x||$$

$$\le M \times \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + M \times \frac{\varepsilon}{3M} = \varepsilon.$$

Hence $\lim_{n\to\infty} T_n x$ exists by the completeness of X, thus $\overline{E} = E$.

(b) (8 pts) Since $|a_{n,k}| \leq 1$ for all $n, k \in \mathbb{N}$, we have for $x \in \ell^1$,

$$|\phi_n(x)| = |\sum_{k=0}^{\infty} x(k)a_{n,k}| \le \sum_{k=0}^{\infty} |x(k)||a_{n,k}| \le ||x||_1.$$

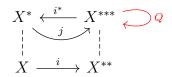
Hence $\|\phi_n\| \leq 1$. By (a), it suffices to establish the existence of $\lim_{n\to\infty} \phi_n(x)$ on c_{00} which is dense in ℓ^1 with respect to $\|\cdot\|_1$.

Let $x = (x(k)) \in c_{00}$. Then there exists $n_x \in \mathbb{N}$ such that x(k) = 0 for all $k > n_x$. Hence

$$\lim_{n \to \infty} \phi_n = \lim_{n \to \infty} \sum_{k=0}^{n_x} x(k) a_{n,k} = \sum_{k=0}^{n_x} x(k) \lim_{n \to \infty} a_{n,k}$$

exists by the assumption that $\lim_{n\to\infty} a_{n,k}$ exists.

Q2 (15 pts)



(a) (7 pts) Since i^* , j are linear isometries, we have $Q = j \circ c^*$ is a bounded linear map. To prove $Q^2 = Q$, i.e., $j \circ i^* \circ j \circ i^* = j \circ i^*$, it suffices to check $i^* \circ j = id$ on X^* . For any $x \in X$, by i, j being natural,

$$\langle x, i^* \circ jx^* \rangle = \langle ix, jx^* \rangle = \langle x^*, ix \rangle = \langle x, x^* \rangle = \langle x, id \, x^* \rangle.$$

Hence $i^* \circ j = id$ on X^* .

(b) (8 pts) By (a), $X^{***} = \operatorname{Im} Q \oplus \operatorname{Ker} Q$. It suffices to check $\operatorname{Im} Q = jX^* \cong X^*$.

 $(\operatorname{Im} Q \subset jX^*)$ For any $x^{***} \in \operatorname{Im} Q$, there exists y^{***} such that

$$x^{***} = j \circ i^*(y^{***}) = j(i^*y^{***}) \in jX^*.$$

 $(jX^* \subset \operatorname{Im} Q)$ Let $x^{***} \in jX^*$. Then $x^{***} = jx^*$ for some $x^* \in X^*$. Since $j \circ i^* = id$ by (a),

$$(I-Q)x^{***} = jx^* - j \circ i^* \circ jx^* = jx^* - jx^* = 0.$$

Hence $x^{***} \in \text{Ker}(I - Q) = \text{Im } Q$.

Q3 (15 pts)

- (a) (pts) Immediately by Riesz-Fréchet Representation. ℓ^1 separable but $\ell^{\infty} = (\ell^1)^*$ non-separable. . . .
- **(b)** (pts) e.g., $L^2([0,1], \#)...$

Q4 (15 pts)

(a) (pts) Let $x \in X$. For any $y \in H$, there exists $M_{x,y} > 0$ such that for all $n \in N$,

$$|\langle T_n x, y \rangle| \le M_{x,y}.$$

By Uniform Boundedness Theorem, there exists $M_y > 0$ such that

$$||T_n x|| = ||\langle T_n x, \cdot \rangle|| \le M_x$$
 for all $n \in \mathbb{N}$.

Since the above inequality holds for all $x \in H$, by Uniform Boundednedd Theorem, there exists M > 0 such that for all $n \in N$, we have

$$||T_n|| \leq M$$

thus $\sup ||T_n|| \leq M$.

(b) (pts) Fix any $x \in X$. Denote $f_x(y) := \lim_{n \to \infty} \langle T_n x, y \rangle$. Then f_x is linear and

$$|f_x(y)| \le \lim_{n \to \infty} ||T_n x|| ||y|| \le M|x|||y||.$$

Thus $f_x(y) \in H^*$ and $||f_x|| \leq M||x||$. By Riesz-Fréchet, there exists an unique $v_x \in H$ such that

$$\langle v_x, y \rangle = f_x(y) = \lim_{n \to \infty} \langle T_n x, y \rangle$$
 and $||v_x|| = ||f_x|| \le M||x||$.

Define $T: H \to H$ by $Tx := v_x$. It is readily check that T is linear and $||T|| \leq M$, thus $T \in B(H)$.

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