Recall

On a finite dimensional vector space, all the norms are equivalent. For normed spaces, finite dimensionality $\iff$ locally compactness.

Let X, Y be normed spaces and $(T_n)_{n=1}^{\infty}, T\colon X \to Y$ be linear operators.

- T continuous $\iff$ T continuous at 0 $\iff$ T bounded.
- If $\dim X < \infty$, then T must be countinous. Moreover, $T_n x \to Tx$ for all $x \in X$ $\iff$ $T_n \xrightarrow{\|\cdot\|} T$. The direction $\implies$ may not hold when $\dim X = \infty$.
- If $\dim Y < \infty$, then T bounded $\iff$ $\ker T$ closed. In particular, this holds for linear functionals. The direction $\impliedby$ may not hold when $\dim Y = \infty$.
- Equivalent definitions of the operator norm

$$\begin{aligned}\|T\| &= \sup\{\frac{\|Tx\|}{\|x\|} : x \in X, \|x\| \neq 0\}\\ &= \sup\{\|Tx\| : x \in X, \|x\| = 1\}\\ &= \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}\\ &= \inf\{M > 0 \colon \|Tx\| \leq M\|x\|,\ \forall x \in X\}.\end{aligned}$$

The operator norm depends on both of the norms in the domain X and in the range Y.

Dual space

Example 1 (Dual-space relationship)**.** Let $1 \leq p < \infty$ and $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\ell^p)^* = \ell^q$.

Proof. We begin with some convenient notations. For $x = (x(i))_{i=1}^{\infty} \in \ell^p$ and $y = (y(i))_{i=1}^{\infty} \in \ell^q$, define a pairing

$$\langle x, y\rangle \coloneqq \sum_{i=1}^{\infty} x(i)y(i). \tag{1}$$

By Hölder's inequality,

$$|\langle x, y\rangle| \leq \sum_{i=1}^{\infty} |x(i)y(i)| \leq \|x\|_p \|y\|_q < \infty. \tag{2}$$

Hence $\langle\cdot,\cdot\rangle\colon \ell^p \times \ell^q \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. It is readily checked that for $\alpha \in \mathbb{K}, x, \tilde{x} \in \ell^p$ and $y \in \ell^q$,

$$\langle \alpha x + \tilde{x}, y\rangle = \alpha\langle x, y\rangle + \langle \tilde{x}, y\rangle \text{ and } \langle x, y\rangle = \langle y, x\rangle. \tag{3}$$

By (2) and (3), for any fixed $y \in \ell^q$, the map $\langle\cdot, y\rangle\colon \ell^p \to \mathbb{K}$ is continuous and linear, i.e., $\langle\cdot, y\rangle \in (\ell^p)^*$. To prove $(\ell^p)^* = \ell^q$, we will show that the map

$$\begin{aligned}T\colon \ell^q &\to (\ell^p)^*\\ y &\mapsto \langle\cdot, y\rangle\end{aligned}$$

is an isometric isomorphism.

(i) (linear and injective) By (3), T is linear. If $\langle \cdot, y\rangle$ is identically zero on ℓ^p, then by applying $\langle \cdot, y\rangle$ to e_i, we get $y = 0 \in \ell^q$, thus T is injective.

(ii) (surjective) Let $\Lambda \in (\ell^p)^*$. We will find $y \in \ell^q$ such that for all $x \in \ell^p$, $\Lambda x = \langle x, y\rangle$. If $\Lambda = 0$, then $y = 0$ satisfying the requirement. Below assume $\Lambda \neq 0$. (Recall basis is like the skeleton of a vector space. To determine the behavior of a linear map Λ on the whole space, it is often enough to determine the how Λ acts on the basis vectors.) For $i \in \mathbb{N}$, let e_i be the sequence taking 1 on i-th term and 0 on all the other terms. Define a sequence $y = (\Lambda e_i)_{i=1}^{\infty}$. We will check y is the desired sequence.

Let $x \in \ell^p$. In Homework 2, we have proved $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis in ℓ^p $(1 \le p < \infty)$. Then $x = \sum_{i=1}^{\infty} x(i) e_i \in \ell^p$. Since Λ is continuous and linear,

$$\Lambda x = \Lambda\left(\sum_{i=1}^{\infty} x(i)e_i\right) = \sum_{i=1}^{\infty} x(i)\Lambda e_i = \langle x, y\rangle. \tag{4}$$

Next we check $y \in \ell^q$.

When $q = \infty$. Suppose on the contrary that $y \notin \ell^\infty$. Then there exist $i_0 \in \mathbb{N}$ such that $|y(i_0)| > 2\|\Lambda\|$. However, $|y(i_0)| = |\Lambda e_{i_0}| \le \|\Lambda\|$, which is a contradiction. Hence $y \in \ell^\infty$.

When $q < \infty$. Define $y_n = \begin{cases} |y(i)|^{q-1}\exp(-\theta_i) & , i \le n; \\ 0 & , i > n. \end{cases}$ Then $y_n \in \ell^p$. By (4) and the boundedness of Λ,

$$\sum_{i=1}^{n} |y(i)|^q = |\langle y_n, y\rangle| = |\Lambda y_n| \le \|\Lambda\| \|y_n\|_p = \|\Lambda\| (\sum_{i=1}^{n} |y(i)|^q)^{1/p}.$$

Dividing both sides by $(\sum_{i=1}^{n} |y(i)|^q)^{1/p}$ (that is nonzero when n large enough),

$$(\sum_{i=1}^{n} |y(i)|^q)^{1/q} \le \|\Lambda\|.$$

Letting $n \to \infty$, we have $y \in \ell^q$.

(iii) (isometric) Let $y \in \ell^q$. By (2), $\|Ty\| \le \|y\|_q$. If $y = 0$, then $Ty = 0$. Below assume $y \neq 0$.

When $q = \infty$. For any $\varepsilon > 0$, there exists $i \in \mathbb{N}$ such that $|y(i)| \ge \|y\|_\infty - \varepsilon$. Hence

$$|\langle e_i, y\rangle| = |y(i)| \ge \|y\|_\infty - \varepsilon.$$

Letting $\varepsilon \to 0$ and since $\|e_i\|_1 = 1$ for all $i \in \mathbb{N}$, we have $\|Ty\| \ge \|y\|_\infty$.

When $q < \infty$. Write $y(i) = |y(i)|\exp(\theta_i)$ for $i \in \mathbb{N}$. Define the 'conjugate function'

$$y^* = \|y\|_q^{1-q} \left(|y(i)|^{q-1}\exp(-\theta_i)\right)_{i=1}^{\infty}. \tag{5}$$

Then $\|y^*\|_p = 1$ and $\langle y^*, y\rangle = \|y\|_q$. Hence $\|Ty\| \ge \|y\|_q$.

$\square$

Remark. Example 1 can be generalized to $L^p(\mu)$ with μ being σ-finite. In the general proof, we can use $\int fg d\mu$ as the pairing in (1), integral-version Hölder inequality in (2), apply Radon-Nikodym to find the candidate 'y' in (ii). The proof idea of the other parts is similar. The construction in (5) is an explicit example of [LN, Prop. 4.5].

For vector spaces A and B, denote $A \hookrightarrow B$ if $A \subset B$. For Banach spaces X and Y, denote $X \xrightarrow{*} Y$ if $Y = X^*$. Recall $(c_0)^* = \ell^1$. Let $1 < p < 2$ and $2 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we can summarize

$$\begin{tikzcd}
c_{00} \arrow[r, hook] & \ell^1 \arrow[r, hook] \arrow[rrrrr, bend left, "*"] & \ell^p \arrow[r, hook] \arrow[rr, bend left, "*"] & \ell^2 \arrow[r, hook] \arrow[loop above, "*"] & \ell^q \arrow[r, hook] \arrow[ll, bend left, "*"] & c_0 \arrow[r, hook] \arrow[llll, bend left, "*"] & \ell^\infty
\end{tikzcd}$$

Remark. For a locally compact Hausdorff space X (Here $X = \mathbb{N}$ for c_0), Riesz representation characterizes that $C_0(X)^* = M(X)$, where $M(X)$ denotes the space of regular Borel measures.

(Take a σ-finite measure μ on X and define the $L^1(\mu)$. There is a natural way to conclude $L^1(\mu) \subset M(X)$ where the inclusion is usually strict. However, in our case where $X = \mathbb{N}$ and $\mu = \#$ (the counting measure), we have $C_0(\mathbb{N}) = c_0$ and $L^1(\#) = \ell^1$. The counting measure $\#$ has a special property that every measure $\eta \in M(\mathbb{N})$ is absolutely continuous with respect to $\#$. By Radon-Nikodym, we can identify $M(\mathbb{N}) = L^1(\#) = \ell^1$. Hence $M(\mathbb{N}) = c_0^* = \ell^1$ becomes reasonable.)

Below is an application of the representation of $(\ell^2)^*$.

Example 2. For $x = (x(i))_{i=1}^\infty \in \ell^2$, define $\Lambda x = \sum_{i=1}^\infty \dfrac{x(2i)}{i}$. Show that $\Lambda \in (\ell^2)^*$ and compute $\|\Lambda\|$.

Proof. For $i \in \mathbb{N}$, define

$$y(i) = \Lambda(e_i) = \begin{cases} \dfrac{1}{k} & , i = 2k, \\ 0 & , i = 2k-1, \end{cases}$$

where $\{e_i\}_{i=1}^\infty$ is the standard Schauder basis of ℓ^2. Let $y = (y(i))_{i=1}^\infty$. Then $\Lambda \cdot = \langle \cdot, y \rangle$.

Since $\|y\|_2 = (\sum_{k=1}^\infty \frac{1}{k^2})^{1/2} = \pi/\sqrt{6}$, it follows from Example 1 that $\Lambda \in (\ell^2)^*$ and $\|\Lambda\| = \|y\|_2 = \pi/\sqrt{6}$. □