Recall

Projection and decomposition in Banach spaces

Let X be a Banach space. A bounded linear operator $P \in B(X, X)$ is a *projection* if $P^2 = P$ (*idempotent*). For any projection P, we have $X = \operatorname{Im}(P) \oplus \operatorname{Ker}(P)$. A closed subspace $M \subset X$ is *complemented* if there exists a closed subspace $N \subset X$ such that $X = M \oplus N$.

- closed subspace $M \subset X$ complemented $\iff$ $\exists$ projection P with $\operatorname{Im}(P) = M$.
- Any finite dimensional subspace in normed space is complemented.
- c_0 is not complemented in ℓ^∞ and $c_0 \neq X^*$ for any normed space X.
- (Dixmier) Let $i\colon X \to X^{**}$ and $j\colon X^* \to X^{***}$ be the natural embeddings.

$$\begin{tikzcd} X^* \arrow[d, dashed, no head] \arrow[r, bend right, "j"'] & X^{***} \arrow[l, "i^*"'] \arrow[d, dashed, no head] \\ X \arrow[r, "i"] & X^{**} \end{tikzcd}$$

By viewing X as a subspace of X^{**}, the projection $D \coloneqq j \circ i^*$ implies

$$X^{***} = \operatorname{Im}(D) \oplus \operatorname{Ker}(D) = X^* \oplus X^\perp,$$

where the annihilator $X^\perp \coloneqq \{y \in X^{***} \colon y(x) = 0\,, \forall\, x \in X\}$. In particular, letting $X = c_0$, we have $(\ell^\infty)^* = \ell^1 \oplus c_0^\perp$.

- By further considering norms on direct sum, we denote $X = M \oplus_{\ell_1} N$ if $X = M \oplus N$ and $\|x\| = \|y\| + \|z\|$ for every $x = y + z$ with $y \in M, z \in N$. Then

$$(\ell^\infty)^* = \ell^1 \oplus_{\ell_1} c_0^\perp.$$

About norm and inner product

Example 1. $C[0,1]$ with sup-norm $\|\cdot\|_\infty$ is not an inner product space.

Proof. We prove by showing $\|\cdot\|_\infty$ does not satisfy Parallelogram Law.

Consider functions $x(t) = 1$ and $y(t) = t$ for $t \in [0,1]$. Then

$$\|x\|_\infty = 1 \quad \text{and} \quad \|y\|_\infty = 1$$

while

$$\|x+y\|_\infty = \sup_{t\in[0,1]} (1+t) = 2 \quad \text{and} \quad \|x-y\|_\infty = \sup_{t\in[0,1]} (1-t) = 1.$$

Hence

$$\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 5 \neq 4 = 2(\|x\|_\infty^2 + \|y\|_\infty^2).$$

$\square$

Theorem 2 (Polarization identities)**.** *If X is a real inner product space, then*

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2), \quad \forall x, y \in X. \tag{1}$$

For a complex inner product space X, we have

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right), \quad \forall x, y \in X. \tag{2}$$

Proof. (i) (Real case) By the bilinearity of real inner product,

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \tag{3}$$

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2. \tag{4}$$

Then (1) is obtained by simplifying (3) $-$ (4).

(ii) (Complex case) By the sequilinearity of complex inner product,

$$\|x+y\|^2 = \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2 = \|x\|^2 + 2\Re\langle x, y\rangle + \|y\|^2 \tag{5}$$

$$\|x-y\|^2 = \|x\|^2 - 2\Re\langle x, y\rangle + \|y\|^2 \tag{6}$$

$$\|x+iy\|^2 = \|x\|^2 - i\langle x, y\rangle + i\overline{\langle x, y\rangle} + \|y\|^2 = \|x\|^2 + 2\Im\langle x, y\rangle + \|y\|^2 \tag{7}$$

$$\|x-iy\|^2 = \|x\|^2 - 2\Im\langle x, y\rangle + \|y\|^2. \tag{8}$$

Then $\Re\langle x, y\rangle$ follows from (5) $-$ (6) and $\Im\langle x, y\rangle$ follows from (7) $-$ (8), thus (2).

$\square$

Example 3. Let X be a normed space with norm $\|\cdot\|$. Then

$\|\cdot\|$ is induced by an inner product $\iff$ $\|\cdot\|$ satisfies the Parallelogram Law.

Proof. $\Longrightarrow$ is the property of inner product.

Since it is fun and meaningful to check $\Longleftarrow$ by ourselves, we omit the details but leave a possible sketch: Define $\langle\cdot,\cdot\rangle\colon X \times X \to \mathbb{C}$ as (2). Then $\langle x, x\rangle \geq 0$ and $\langle y, x\rangle = \overline{\langle x, y\rangle}$ directly follows from (2). As for the linearity on the first argument, firstly we can prove the additivity $\langle x + \tilde{x}, y\rangle = \langle x, y\rangle + \langle \tilde{x}, y\rangle$ via Parallelogram Law. To achieve $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$, we may check in the following order

$$\text{additivity} \to \alpha \in \mathbb{N} \xrightarrow{\text{“}-\text{”}} \alpha \in \mathbb{Z} \xrightarrow{\text{“}n/m\text{”}} \alpha \in \mathbb{Q} \xrightarrow{\text{continuity}} \alpha \in \mathbb{R} \xrightarrow{\text{“}i\text{”}} \alpha \in \mathbb{C}.$$

$\square$