THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1
Solution to Homework 5

1. Let (x_n) be a sequence in an inner product space. Show that the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ imply $x_n \to x$.

   Proof. Note that $\|x - x_n\|^2 = \langle x - x_n, x - x_n\rangle = \|x\|^2 - 2\Re\langle x_n, x\rangle + \|x_n\|^2$.

   It follows from $\langle x_n, x\rangle \to \langle x, x\rangle$ that $\Re\langle x_n, x\rangle \to \Re\langle x, x\rangle = \langle x, x\rangle$ since $\Re\colon \mathbb{C} \to \mathbb{R}$ is continuous and $\langle x, x\rangle \geq 0$. Combining with with $\|x_n\| \to \|x\|$, we have
   $$\|x - x_n\|^2 \to \|x\|^2 - 2\langle x, x\rangle + \|x\|^2 = 0, \quad \text{as } n \to \infty.$$
   Thus $x_n \to x$ in $\|\cdot\|$. □

2. Show that
   $$X = \left\{x = (x_n) \in \ell^2 \colon \sum_{n=1}^{\infty} \frac{x_n}{n} = 0\right\}$$
   is a closed subspace of ℓ^2.

   Proof. For $x = (x_n) \in \ell^2$, define
   $$f(x) := \sum_{n=1}^{\infty} \frac{x_n}{n}.$$
   Write $y = (1/n)_{n=1}^{\infty}$. Note $\|y\|^2 = \sum_{n=1}^{\infty} 1/n^2 < \infty$. For $x \in \ell^2$, by Cauchy-Schwarz inequality,
   $$|f(x)| = |\langle x, y\rangle| \leq \|x\|^2\|y\|^2 < \infty.$$
   Hence f is well-defined and continuous since f is readily checked to be linear. Thus X is a closed subspace as the kernel of a linear continuous functional. □

3. (a) Prove that for every two subspaces X_1 and X_2 of a Hilbert space,
   $$(X_1 + X_2)^\perp = X_1^\perp \cap X_2^\perp.$$
   (b) Prove that for every two closed subspaces X_1 and X_2 of a Hilbert space,
   $$(X_1 \cap X_2)^\perp = \overline{X_1^\perp + X_2^\perp}.$$

   Proof. (a) By $X_1, X_2 \subset (X_1 + X_2)$, we have $(X_1 + X_2)^\perp \subset (X_1)^\perp, (X_2)^\perp$, thus $(X_1 + X_2)^\perp \subset (X_1)^\perp \cap (X_2)^\perp$.

   Let $x^* \in (X_1)^\perp \cap (X_2)^\perp$. Let $y \in X_1 + X_2$ and write $y = x_1 + x_2$ for some $x_1 \in X_1, x_2 \in X_2$. Then $\langle y, x^*\rangle = \langle x_1 + x_2, x^*\rangle = \langle x_1, x^*\rangle + \langle x_2, x^*\rangle = 0$. Hence $x^* \in (X_1 + X_2)^\perp$, thus $(X_1)^\perp \cap (X_2)^\perp \subset (X_1 + X_2)^\perp$. Together we have $(X_1 + X_2)^\perp = (X_1)^\perp \cap (X_2)^\perp$.

   (b) Since X_1, X_2 are closed, we have $(X_i^\perp)^\perp = X_i$, $i = 1, 2$. Then by applying (a) to $X_1^\perp, X_2^\perp$, we have
   $$(X_1^\perp + X_2^\perp)^\perp = (X_1^\perp)^\perp \cap (X_2^\perp)^\perp = X_1 \cap X_2.$$
   Hence
   $$\overline{X_1^\perp + X_2^\perp} = \left((X_1^\perp + X_2^\perp)^\perp\right)^\perp = (X_1 \cap X_2)^\perp.$$
   □

— *THE END* —