THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1
Solutions to Final Examination

Q1 (15 pts)

(a) (7 pts) It is readily checked that E is a subspace. Next we prove $\overline{E} = E$. Let $x \in \overline{E}$. Let $M > 0$ such that $\sup \|T_n\| \leq M$. For any $\varepsilon > 0$, there exists $\widetilde{x} \in E$ such that $\|x - \widetilde{x}\| \leq \frac{\varepsilon}{3M}$. By $\widetilde{x} \in E$, there exists $N \in \mathbb{N}$ such that $\forall n, m \geq N$, $\|T_n\widetilde{x} - T_m\widetilde{x}\| \leq \frac{\varepsilon}{3}$. Then

$$\begin{aligned}\|T_n x - T_m x\| &\leq \|T_n x - T_n\widetilde{x}\| + \|T_n\widetilde{x} - T_m\widetilde{x}\| + \|T_m\widetilde{x} - T_m x\| \\ &\leq M \times \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + M \times \frac{\varepsilon}{3M} = \varepsilon.\end{aligned}$$

Hence $\lim_{n\to\infty} T_n x$ exists by the completeness of X, thus $\overline{E} = E$.

(b) (8 pts) Since $|a_{n,k}| \leq 1$ for all $n, k \in \mathbb{N}$, we have for $x \in \ell^1$,

$$|\phi_n(x)| = |\sum_{k=0}^{\infty} x(k)a_{n,k}| \leq= \sum_{k=0}^{\infty} |x(k)||a_{n,k}| \leq \|x\|_1.$$

Hence $\|\phi_n\| \leq 1$. By **(a)**, it suffices to establish the existence of $\lim_{n\to\infty}\phi_n(x)$ on c_{00} which is dense in ℓ^1 with respect to $\|\cdot\|_1$.

Let $x = (x(k)) \in c_{00}$. Then there exists $n_x \in \mathbb{N}$ such that $x(k) = 0$ for all $k > n_x$. Hence

$$\lim_{n\to\infty}\phi_n = \lim_{n\to\infty}\sum_{k=0}^{n_x} x(k)a_{n,k} = \sum_{k=0}^{n_x} x(k)\lim_{n\to\infty} a_{n,k}$$

exists by the assumption that $\lim_{n\to\infty} a_{n,k}$ exists.

Q2 (15 pts)

$$\begin{tikzcd} X^* \arrow[r, bend right, "j"'] \arrow[d, dash, dashed] & X^{***} \arrow[l, "i^*"'] \arrow[loop right, red, "Q"] \arrow[d, dash, dashed] \\ X \arrow[r, "i"] & X^{**} \end{tikzcd}$$

(a) (7 pts) Since i^*, j are linear isometries, we have $Q = j\circ c^*$ is a bounded linear map. To prove $Q^2 = Q$, i.e., $j\circ i^*\circ j\circ i^* = j\circ i^*$, it suffices to check $i^*\circ j = id$ on X^*. For any $x \in X$, by i, j being natural,

$$\langle x, i^*\circ jx^*\rangle = \langle ix, jx^*\rangle = \langle x^*, ix\rangle = \langle x, x^*\rangle = \langle x, id\, x^*\rangle.$$

Hence $i^*\circ j = id$ on X^*.

(b) (8 pts) By **(a)**, $X^{***} = \operatorname{Im} Q \oplus \operatorname{Ker} Q$. It suffices to check $\operatorname{Im} Q = jX^* \cong X^*$.

($\operatorname{Im} Q \subset jX^*$) For any $x^{***} \in \operatorname{Im} Q$, there exists y^{***} such that

$$x^{***} = j\circ i^*(y^{***}) = j(i^*y^{***}) \in jX^*.$$

($jX^* \subset \operatorname{Im} Q$) Let $x^{***} \in jX^*$. Then $x^{***} = jx^*$ for some $x^* \in X^*$. Since $j\circ i^* = id$ by **(a)**,

$$(I - Q)x^{***} = jx^* - j\circ i^*\circ jx^* = jx^* - jx^* = 0.$$

Hence $x^{***} \in \operatorname{Ker}(I - Q) = \operatorname{Im} Q$.

Q3 (15 pts)

(a) (pts) Immediately by Riesz-Fréchet Representation. ℓ^1 separable but $\ell^\infty = (\ell^1)^*$ non-separable. ...

(b) (pts) e.g., $L^2([0,1],\#)\ldots$

Q4 (15 pts)

(a) (pts) Let $x \in X$. For any $y \in H$, there exists $M_{x,y} > 0$ such that for all $n \in N$,

$$|\langle T_n x, y\rangle| \leq M_{x,y}.$$

By *Uniform Boundedness Theorem*, there exists $M_y > 0$ such that

$$\|T_n x\| = \|\langle T_n x, \cdot\rangle\| \leq M_x \quad \text{for all } n \in \mathbb{N}.$$

Since the above inequality holds for all $x \in H$, by *Uniform Boundednedd Theorem*, there exists $M > 0$ such that for all $n \in N$, we have

$$\|T_n\| \leq M$$

thus $\sup \|T_n\| \leq M$.

(b) (pts) Fix any $x \in X$. Denote $f_x(y) \coloneqq \lim_{n\to\infty}\langle T_n x, y\rangle$. Then f_x is linear and

$$|f_x(y)| \leq \lim_{n\to\infty} \|T_n x\|\|y\| \leq M|x|\|y\|.$$

Thus $f_x(y) \in H^*$ and $\|f_x\| \leq M\|x\|$. By Riesz-Fréchet, there exists an unique $v_x \in H$ such that

$$\langle v_x, y\rangle = f_x(y) = \lim_{n\to\infty}\langle T_n x, y\rangle \quad \text{and} \quad \|v_x\| = \|f_x\| \leq M\|x\|.$$

Define $T\colon H \to H$ by $Tx \coloneqq v_x$. It is readily check that T is linear and $\|T\| \leq M$, thus $T \in B(H)$.

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