THE ENERGY METHOD FOR NON-LINEAR WAVE EQUATIONS

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ABSTRACT. We provide an introduction to the classical local well-posedness theory for non-linear wave equations via the energy. The equations we consider in furthest generality take the form

$$\Box_{\mathbf{g}(\phi,\partial\phi)}\phi = \mathcal{N}(\phi,\partial\phi),$$
$$(\phi,\partial_t\phi)_{|t=0} = (\phi_0,\phi_1),$$

with initial data posed on the scale of L^2_x-based Sobolev spaces $(\phi_0,\phi_1)\in(H^s_x\times H^{s-1}_x)(\mathbb{R}^d)$. Following [Sog95], we present a proof for sufficiently regular data $s\gg 1$ using physical space methods, i.e. integration-by-parts. To reach the classical exponent $s>\frac{d}{2}+1$ due to [FM72, HKM77], we introduce the paradifferential formulation of the equation, drawing from [BCD11, Tay11, IT22].

CONTENTS

1. INTRODUCTION

For any reasonable physical model of *wave propagation*, an individual should be able to predict the evolution of prescribed regular initial conditions on small time scales. In the language of partial differential equations, this is the problem of *well-posedness* of the *initial data problem*. To fix a concrete problem, we consider the evolution of scalar fields $\phi:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ under non-linear wave equations of the form

$$\Box_{g(\phi,\partial\phi)}\phi = \mathcal{N}(\phi,\partial\phi), \tag{NLW}$$ eq:NLW

where the Lorentzian metric $\mathbf{g}(\phi,\partial\phi)$ is a perturbation of the Minkowski metric $\mathbf{m} := \mathrm{diag}(-,+,\dots,+)$ and $\mathcal{N}(\phi,\partial\phi)$ is a smooth non-linearity, posed with initial data in the L^2-based Sobolev spaces

$$(\phi,\partial_t\phi)_{|t=0} = (\phi_0,\phi_1)\in(H^s_x\times H^{s-1}_x)(\mathbb{R}^d).$$

Example.

(a) *Maxwell's equations*: the simplest model for electromagnetic fields $\mathbf{E},\mathbf{B}:\mathbb{R}^{1+3}\to\mathbb{R}^3$, this is a linear system of equations consisting of Ampere's law, Faraday's law, and Gauss's laws,

$$\begin{aligned}\partial_t\mathbf{E} &= \nabla\times\mathbf{B},\\ \partial_t\mathbf{B} &= -\nabla\times\mathbf{E},\\ \nabla\cdot\mathbf{E} &= 0,\\ \nabla\cdot\mathbf{B} &= 0.\end{aligned} \tag{M}$$ eq:maxwell

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While this may not look like the linear wave equation, after differentiating the first and second equations in time, and using the third and fourth equations, we obtain

$$\begin{aligned} \Box \mathbf{E} &= 0, \\ \Box \mathbf{B} &= 0. \end{aligned} \tag{M$*$}$$

eq:maxwell2

The two systems (M) and (M$*$) are equivalent after imposing suitable constraints on the initial data for the latter.

(b) *Wave maps*: a wave map into the sphere, viewed as a Riemannian sub-manifold of Euclidean space $\mathbb{S}^m \hookrightarrow \mathbb{R}^{m+1}$, is a field $\phi : I \times \mathbb{R}^d \to \mathbb{S}^m$ evolving under the semi-linear wave equation

$$\Box\phi = -\phi(\partial^\alpha \phi \cdot \partial_\alpha \phi) \tag{WM}$$

eq:wavemaps

This equation arises in physics as one of the simplest non-trivial models of quantum field theory, often referred to in the literature as a non-linear σ-model.

(c) *Einstein vacuum equations*: in the absence of matter, the propagation of gravitational waves, represented by a Lorentzian manifold $(\mathcal{M}, \mathbf{g})$, is modeled by the equation

$$\mathbf{Ric}_{\mathbf{g}} = 0. \tag{EVE}$$

eq:einstein

After fixing an appropriate choice of coordinates, the Einstein equations reduce to a system of quasi-linear wave equations of the form (NLW) for the metric $\mathbf{g}$.

On a philosophical note, we argue that it is possible *mathematically* test the physical relevance of the initial data problem for an evolutionary equation. Following Hadamard [Had02], for an equation such as (NLW) to reasonably model physical reality, the initial data problem must satisfy the following three standards for well-posedness:

- *Existence*: If a physical phenomenon is governed by (NLW), then for every choice of initial conditions, the propagation of the conditions should correspond to a solution to the equation.
- *Uniqueness*: In classical physics, physical reality is understood to be deterministic, so each initial data should uniquely determine a solution to (NLW).
- *Continuous dependence on initial data*: Propagation of waves in physical reality is stable under perturbations, so the data to solution map should be continuous.

We say (NLW) is $(H^s_x \times H^{s-1}_x)$-wellposed if there exists a well-defined continuous data-to-solution map,

$$\begin{aligned} H^s_x \times H^{s-1}_x &\longrightarrow C_t H^s_x \cap C^1_t H^{s-1}_x \\ (\phi_0, \phi_1) &\longmapsto \phi. \end{aligned}$$

Theorem 1.1 (Energy estimate)**.** *We prove*

$$||(\phi, \partial_t\phi)||^2_{C^0_t(H^s_x \times H^{s-1}_x)[0,T]} \lesssim \exp\Big(\int_0^T ||\nabla_x\phi||_{L^\infty_x}\, dt\Big) ||(\phi_0,\phi_1)||^2_{H^s_x \times H^{s-1}_x}. \tag{1.1}$$

Theorem 1.2 (Classical local well-posedness)**.** *The quasi-linear wave equation (NLW) is locally well-posed in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^d)$ for $s > \frac{d}{2} + 1$.*

2. Linear wave equations

In the linear setting, energy estimates are essentially equivalent to well-posedness for the initial data problem. To illustrate the argument in a simplified setting, let $\mathsf{L} : X \to Y$ be a linear map between finite-dimensional vector spaces, and denote $\mathsf{L}^* : Y^* \to X^*$ its adjoint map. Then the existence for the original problem,

for every $f \in Y$, there exists a solution $\phi \in X$ to the equation

$$\mathsf{L}\phi = f,$$

is related to uniqueness for the dual problem

for every $f \in X^*$, there is at most one solution $\phi \in Y^*$ to the equation

$$\mathsf{L}^*\phi = f,$$

in that the image of L is equal to the annihilator of the kernel of L^*,

$$\operatorname{Im}\mathsf{L} = (\ker \mathsf{L}^*)^{\perp}.$$

It follows that showing existence for the original problem, i.e. L is surjective, is equivalent to showing uniqueness for the dual problem, i.e. L^* is injective.

$$\mathsf{L} := \mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu + \mathbf{b}^\mu\partial_\nu + \mathbf{a}$$

2.1. **A priori estimates.** Let us begin with the simplest energy estimate, namely the conservation of energy for the linear wave equation on Minkowski space,

$$\Box\phi = f. \tag{W}$$ eq:linear

This equation is invariant under time-translation, so by Noether's theorem, we can produce a conservation law for solutions to the equation by multiplying the equation by $\partial_t\phi$. Differentiating-by-parts appropriately, we obtain the divergence identity

$$\begin{aligned} f\partial_t\phi = \Box\phi\partial_t\phi &= \Big(-\partial_t^2 + \sum_{j=1}^d \partial_j^2\phi\Big)\partial_t\phi \\ &= \partial_t\Big(-\frac{1}{2}|\partial_t\phi|^2\Big) + \sum_{j=1}^d \partial_j(\partial_j\phi\partial_t\phi) - \partial_j\phi\partial_t\partial_j\phi \\ &= \partial_t\Big(-\frac{1}{2}|\partial_t\phi|^2 - \frac{1}{2}\sum_{j=1}^d |\partial_j\phi|^2\Big) + \nabla_x\cdot(\partial_t\phi\nabla_x\phi). \end{aligned}$$

Integrating on the space-time region $[0,T]\times\mathbb{R}^d$ and applying the divergence theorem furnishes

Proposition 2.1 (Energy identity)**.** *Let $f\in L^1_tL^2_x([0,T]\times\mathbb{R}^d)$ and suppose $\phi\in C^0_tH^1_x\cap C^1_tL^2_x([0,T]\times\mathbb{R}^d)$ is a solution to the linear wave equation $\Box\phi = f$. Then*

$$\int_{t=T}\frac{1}{2}|\nabla_{t,x}\phi|^2\,dx = \int_{t=0}\frac{1}{2}|\nabla_{t,x}\phi|^2\,dx + \int_0^T\int_{\mathbb{R}^d} f\,\partial_t\phi\,dxdt. \tag{2.1}$$ eq:identity

The solution also satisfies the energy estimates

$$||(\phi,\partial_t\phi)||_{C^0_t(\dot H^1_x\times L^2_x)} \lesssim ||(\phi_0,\phi_1)||_{\dot H^1_x\times L^2_x} + ||f||_{L^1_tL^2_x}, \tag{2.2}$$ eq:linest1

$$||(\phi,\partial_t\phi)||_{C^0_t(H^1_x\times L^2_x)} \lesssim \langle T\rangle\Big(||(\phi_0,\phi_1)||_{H^1_x\times L^2_x} + ||f||_{L^1_tL^2_x}\Big). \tag{2.3}$$ eq:linest2

Proof. To prove (2.2), we simply apply Cauchy-Schwartz and Cauchy's inequality to the right-hand side of the energy identity (2.1) to obtain

$$\begin{aligned} \frac{1}{2}||(\phi,\partial_t\phi)||^2_{C^0_t(\dot H^1_x\times L^2_x)} &\le \frac{1}{2}||(\phi_0,\phi_1)||^2_{\dot H^1_x\times L^2_x} + ||f||_{L^1_tL^2_x}||\partial_t\phi||_{C^0_tL^2_x} \\ &\le \frac{1}{2}||(\phi_0,\phi_1)||^2_{\dot H^1_x\times L^2_x} + \frac{\varepsilon^{-1}}{2}||f||^2_{L^1_tL^2_x} + \frac{\varepsilon}{2}||\partial_t\phi||^2_{C^0_tL^2_x}, \end{aligned}$$

for any choice of $\varepsilon>0$. In particular, choosing $\varepsilon\ll 1$ allows us to absorb the last term in the second line into the left-hand side, completing the proof of (2.2).

To prove (2.3), we apply the fundamental theorem of calculus in time to the estimates on the top-order terms in (2.2) to recover control over the lower-order terms, at the price of linear growth in T. Indeed, it suffices to bound the L^2_x-norm of the solution by the right-hand side. Writing,

$$\phi(T) = \phi_0 + \int_0^T \partial_t\phi(t)\,dt,$$

and applying the L^2_x-norm to both sides, it follows from Minkowski's integral inequality and the triangle inequality that

$$||\phi||_{C^0_tL^2_x} \le ||\phi_0||_{L^2_x} + T||\partial_t\phi||_{C^0_tL^2_x}.$$

Inserting the first linear energy estimate (2.2) into the right-hand side completes the proof of (2.3). □

Remark. The first energy estimate (2.2) states that the top-order terms stay uniformly bounded for all time, while the second energy estimate (2.3) allows the lower-order terms to grow linearly in time.

As a corollary, one arrives at the energy estimate,

Theorem 2.2 (Energy estimate for constant-coefficient wave equation)**.** *Let $f \in L^1_t H^{s-1}_x([0,T] \times \mathbb{R}^d)$ and suppose $\phi \in C^0_t H^s_x \cap C^1_t H^{s-1}_x([0,T] \times \mathbb{R}^d)$ is a solution to the wave equation $\Box\phi = f$. Then*

$$||\phi||_{C^0_t H^s_x} \lesssim \langle T \rangle \Big(||\phi(0)||_{H^s_x} + ||\nabla_{t,x}\phi(0)||_{H^{s-1}_x} + ||f||_{L^1_t H^{s-1}_x} \Big). \tag{2.4}$$

eq:linest4

Proof. The Fourier multiplier $\langle \nabla \rangle^s$ commutes with $\Box$, so the result follows from (2.3). ☐

2.2. Existence-uniqueness duality.

Lemma 2.3 (Existence-uniqueness duality)**.** *Let $\mathsf{L} : X \to Y$ be a linear operator between Banach spaces, and denote $\mathsf{L}^* : Y^* \to X^*$ its adjoint. The following statements hold:*

- *uniqueness furnishes existence for the dual problem, i.e. the energy estimate for L*
$$||u||_X \lesssim ||\mathsf{L}u||_Y$$
implies the adjoint operator is surjective, $\operatorname{Im} \mathsf{L}^* = X^*$,
- *existence furnishes uniqueness for the dual problem, i.e. if L is surjective, $\operatorname{Im} \mathsf{L} = Y$, then the adjoint satisfies the energy estimate,*
$$||v||_{Y^*} \lesssim ||\mathsf{L}^*||_{X^*}.$$

In particular, if X is reflexive, then the energy estimate furnishes existence and uniqueness for the problem $\mathsf{L}u = f$.

Lemma 2.4 (Hahn-Banach theorem)**.** *Let X be a normed vector space and suppose $Y \hookrightarrow X$ is a linear subspace. If $f \in Y^*$ is a bounded linear functional on the subspace Y, then there exists an extension $\widetilde{f} \in X^*$ to a bounded linear functional on the entire space X such that*

$$||\widetilde{f}||_{X^*} = ||f||_{Y^*}.$$

$$\mathsf{L} := g^{\mu\nu}\partial_\mu\partial_\nu + b^\mu\partial_\mu + a$$

$$\begin{aligned} \mathsf{L}\phi &= f, \\ (\phi, \partial_t\phi)_{|t=0} &= (\phi_0, \phi_1). \end{aligned} \tag{VW}$$

eq:varlinear

Theorem 2.5 (Existence and uniqueness)**.** *Let $s \in \mathbb{R}$, then for every forcing term $f \in L^1_t H^{s-1}([0,T] \times \mathbb{R}^d)$, there exists a unique solution $\phi \in (C^0_t H^s_x \cap C^1_t H^{s-1}_x)([0,T] \times \mathbb{R}^d)$ to the initial data problem*

$$\begin{aligned} \mathsf{L}\phi &= f, \\ (\phi, \partial_t\phi)_{|t=0} &= (0, 0). \end{aligned}$$

3. Energy methods

With our discussion of linear wave equations at hand, we are ready to begin our study of non-linear wave equations.

$$\Box_{\mathbf{g}(\phi)} := \mathbf{g}^{\mu\nu}(\phi)\partial_\mu\partial_\nu$$

$$|\mathbf{g}^{\mu\nu} - \mathbf{m}^{\mu\nu}| \ll 1. \tag{P}$$

eq:perturb

$$\begin{aligned} \Box_{\mathbf{g}(\phi)}\phi &= \mathcal{N}(\phi, \partial\phi), \\ (\phi, \partial_t\phi)_{|t=0} &= (\phi_0, \phi_1), \end{aligned} \tag{QLW}$$

eq:QLW

The simplest model to consider is the semi-linear wave equation with power-type non-linearity,

$$\Box\phi = \phi^3$$

4. Paradifferential calculus

References

houriEtAl2011 [BCD11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*, volume 343 of *Grundlehren Der Mathematischen Wissenschaften*. Springer, Berlin, Heidelberg, 2011.

erMarsden1972 [FM72] Arthur E. Fischer and Jerrold E. Marsden. The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system, I. *Communications in Mathematical Physics*, 28(1):1–38, October 1972.

hadamard [Had02] J. Hadamard. Sur les problèmes aux dérivés partielles et leur signification physique. *Princeton University Bulletin*, 13:49–52, 1902.

ughesEtAl1977 [HKM77] Thomas J. R. Hughes, Tosio Kato, and Jerrold E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Archive for Rational Mechanics and Analysis*, 63(3):273–294, September 1977.

imTataru2022b [IT22] Mihaela Ifrim and Daniel Tataru. Local well-posedness for quasi-linear problems: A primer. *Bulletin of the American Mathematical Society*, 60(2):167–194, July 2022.

Sogge1995 [Sog95] Christopher Donald Sogge. *Lectures on Nonlinear Wave Equations*. International Press, 1995.

Taylor2011b [Tay11] Michael E. Taylor. *Partial Differential Equations III: Nonlinear Equations*, volume 117 of *Applied Mathematical Sciences*. Springer, New York, NY, 2011.