LARGE-DATA THEORY FOR WAVE MAPS ON $\mathbb{R}^{1+2}$

JASON ZHAO

Abstract. In these notes we present the large-data theory for the energy-critical wave maps equation a lá the *energy dispersion method* of Sterbenz and Tataru in their series of papers [ST10a, ST10b]. The contents of these notes interpolate in detail between those papers and the summary contained in the Oberwolfach notes of Tataru [KTV14, Geometric Wave Equations].

Contents

1. Introduction 1
1.1. Main results 2
2. Renormalisation 3
3. Energy dispersion method 4
3.1. Induction on energy 4
3.2. Bootstrap argument 6
3.3. Comparing ϕ and $\widetilde{\phi}$ 6
4. Bubbling 10
4.1. Monotonicity formula 10
4.2. Bubbling argument 11
References 11

1. Introduction

Let $(\mathbb{R}^{1+2}, m)$ be the $(1+2)$-dimensional Minkowski space-time, and suppose $(\mathbb{M}, g)$ is a compact Riemannian manifold. By Nash's theorem, we can view the target manifold extrinsically via an isometric embedding $\mathbb{M} \hookrightarrow \mathbb{R}^N$, denoting the second fundamental form by $\mathbf{S} : T\mathbb{M} \times T\mathbb{M} \to T\mathbb{M}^\perp$. We say that $\phi : \mathbb{R}^{1+2} \to \mathbb{M}$ solves the *wave maps equation* if

$$\begin{aligned} \Box \phi^a &= -\mathbf{S}^a_{bc}(\phi)\partial^\alpha \phi^b \partial_\alpha \phi^c, \\ \phi_{|t=0} &= \phi_0, \\ \partial_t \phi_{|t=0} &= \phi_1, \end{aligned} \tag{WM}$$

`eq:wave`

for initial data (ϕ_0, ϕ_1) satisfying the constraints $\phi_0(x) \in \mathbb{M}$ and $\phi_1(x) \in T_{\phi_0(x)}\mathbb{M}$. Formally, wave maps are critical points of the Lagrangian

$$\mathcal{L}[\phi] := \int_{\mathbb{R}^{1+2}} \langle \partial^\alpha \phi, \partial_\alpha \phi \rangle_g \, dtdx$$

of which the wave maps equation (WM) is the Euler-Lagrange equation. There is a corresponding stress-energy tensor

$$T_{\alpha\beta}[\phi] := \langle \partial_\alpha \phi, \partial_\beta \phi \rangle_g - \frac{1}{2} m_{\alpha\beta} \langle \partial^\gamma \phi, \partial_\gamma \phi \rangle_g$$

which is divergence-free

$$\partial^\alpha T_{\alpha\beta}[\phi] = 0.$$

Date: October 18, 2023.

Then, contracting the stress-energy tensor with the vector field ∂_t, applying the divergence-free condition and Stokes' theorem on the space-time slab $I \times \mathbb{R}^2$ furnishes conservation of the *Dirichlet energy* for solutions to the wave maps equation,

$$\mathcal{E}[\phi(t)] := ||\vec{\phi}||^2_{\dot{H}^1 \times L^2}(t) = \int_{\mathbb{R}^2} |\partial_t \phi|^2 + |\nabla_x \phi|^2 \, dx,$$

where we have denoted $\phi[t] = (\phi, \partial_t \phi)(t)$. In view of Noether's theorem, conservation of energy corresponds to the time-translation symmetry of the Lagrangian $\mathcal{L}[\phi]$. Note that the wave maps equation and its Lagrangian are invariant with respect to the scaling

$$\phi(t,x) \mapsto \phi(\lambda t, \lambda x).$$

The Dirichlet energy is also invariant with respect to this scaling in $(1+2)$-dimensions and coincides precisely with the *energy space* $\phi[t] \in \dot{H}^1 \times L^2$, thus, we refer to the wave maps equation (WM) on $\mathbb{R}^{1+2}$ as *energy-critical*.

1.1. **Main results.** We consider the initial data problem for the wave maps equations (WM) on $\mathbb{R}^{1+2}$ with finite energy data $\vec{\phi}_0 \in \dot{H}^1_x \times L^2_x$. Global well-posedness where the target manifold is a sphere $\mathbb{M} = \mathbb{S}^k$ was established in [Tao01] and for general targets in [Tat05]. In this note we turn towards addressing the following questions for large initial data:

- global well-posedness,
- scattering.

Using the symmetries of the equation and the finite speed of propagation, we can reduce the study of the wave map to the forward light cone C. For blow-up, we use time-reversibility of the equation and finite speed of propagation. For scattering, we choose a ball B large so that energy is small outside of this ball, so the small data theory applies. Hence it remains to study the influence cone of B.

Theorem 1 (Bubbling theorem)**.** *Let ϕ be a finite energy solution to the wave maps equation (WM) which either admits time-like energy concentration at the tip of the light cone $(t,x) = (0,0)$ (respectively at infinity $t = \infty$),*

$$\limsup_t \mathcal{E}_{C_\gamma \cap S_t}[\phi] > 0,$$

where we write the limit $t \searrow 0$ (resp. $t \nearrow \infty$). Then there exists a sequence of concentration points $(t_n, x_n) \in C$ such that $(t_n, x_n) \to (0,0)$ (resp. $t_n \nearrow \infty$), and scales $r_n > 0$ with the following properties:

(a) time-like concentration,

$$\limsup_{n\to\infty} \frac{x_n}{t_n} = v,$$

for some velocity $v \in \mathbb{R}^2$ with $|v| < 1$,

(b) below self-similar scale,

$$\limsup_{n\to\infty} \frac{r_n}{t_n} = 0,$$

(c) convergence to a soliton,

$$\lim_{n\to\infty} \phi(t_n + r_n t, x_n + r_n x) = L_v Q(t,x)$$

strongly in $H^1_{\mathrm{loc}}([-\frac{1}{2}, \frac{1}{2}] \times \mathbb{R}^2)$ to a Lorentz transformation with velocity v of a non-trivial harmonic map $Q : \mathbb{R}^2 \to \mathbb{M}$ which contains some of the energy concentration,

$$0 < ||Q||_{\dot{H}^1} \le \lim_t \mathcal{E}_{S_t}[\phi].$$

Remark. The soliton resolution conjecture asks whether the nature of blow-up or non-scattering can be completely characterised by a superposition of solitons. The question of blow-up was partially answered by the thesis of Grinis [Gri16], using the bubble tree argument. In this case, the energy concentration can be completely decomposed into the sum of energies of solitons,

$$\lim_t \mathcal{E}_{S_t}[\phi] = \sum_{Q \text{ bubble}} ||Q||_{\dot{H}^1},$$

sometimes known as the *energy identity*. The remarkable heart of the analysis is showing that no energy is lost between the scales at which the solitons appear.

Theorem 2 (Threshold theorem)**.** *The wave maps equation is globally well-posed for all initial data below the energy threshold and the corresponding solutions scatter in the following sense:*

(a) (regular data) For regular data $\vec{\phi}_0$, then there exists a unique global regular solution which has Lipschitz dependence on the initial data locally in time in the $\dot{H}^1 \times L^2$ topology.
(b) (rough data) The flow map admits an extension to rough data.
(c) (weak Lipschitz dependence) The flow map is globally Lipschitz in the $\dot{H}^\sigma$ topology for $\sigma < 1$ close to 1.
(d) (scattering) The $\mathcal{S}^1$ norm is finite.

Theorem 3 (Dichotomy theorem)**.** *The wave maps equation (WM) is locally well-posed for arbitrary finite energy data. Further, one of the following two properties must hold for the forward maximal solution:*

(a) the solution is global, scatters at infinity,
(b) bubbling off a soliton.

2. Renormalisation

To identify the frequency interactions between the terms of the wave maps equation (WM), we work instead with the paradifferential formulation.

Paradifferential equation

$$\Box\phi_k = -\mathbf{S}(\phi)_{<k}\partial\phi_{<k}\partial\phi_k + \text{perturbative}$$

We can use geometry $\mathbf{S}(\phi)\partial\phi = 0$ so that

$$\Box\phi_k = -\mathbf{A}(\phi)_{<k}\partial\phi_{<k}\partial\phi_k + \text{perturbative}$$

for anti-symmetric $\mathbf{A}$. Gives some conservation structure. One seeks a linear transformation $w_k = U_{<k}\psi_k$ to make the equation

$$\Box w_k = \text{perturbative}$$

$$\begin{aligned} \text{error} &= (\Box U_{<k-m} - U_{<k-m}(\Box + 2\mathbf{A}(\phi)_{<k-m}\partial^\alpha\phi_{<k-m}\partial_\alpha))\,\phi_k \\ &= \Box U_{<k-m}\phi_k + 2(\partial^\alpha U_{<k-m} - U_{<k-m}\mathbf{A}(\phi)_{<k-m}\partial^\alpha\phi_{<k-m})\partial_\alpha\phi_k \end{aligned}$$

$$\begin{aligned} \frac{d}{dk}U_{<k} &= U_{<k}\mathbf{B}_k, \\ \lim_{k\to-\infty} U_{<k} &= I \end{aligned} \tag{1}$$

eq:ODE

One wants

$$\partial U_{<k} = U_{<k}\partial\phi_{<k}$$

Lemma 4 (Diffusion gauge)**.** *Then*

$$\begin{aligned} ||P_k U_k||_{\mathsf{S}} &\lesssim_{\mathcal{F}} 2^{-\delta|k-k'|}2^{-C(k-k')} + c_k, \\ ||P_{k'}\nabla^J_{t,x}U_k||_{L^1_{t,x}} &\lesssim 2^{(|J|-3)k}2^{-C(k'-k)}c_k \end{aligned}$$

and

$$||P_{k'}(U_{<k-20}G)||_{\mathsf{N}} \lesssim_{\mathcal{F}} 2^{-|k'-k|}||G||_{\mathsf{N}}, \tag{2}$$

$$||P_k(\Box U_{k_1}\psi_{k_2})||_{\mathsf{N}} \lesssim_{\mathcal{F}} 2^{-|k-k_2|}2^{-\delta(k_2-k_1)}c_{k_1}||\psi_{k_2}||_{\mathsf{S}} \tag{3}$$

and

$$U^\top_{<k}\nabla_\alpha U_{<k} = \nabla_\alpha\mathbf{B}_{<k} - \int_{-\infty}^k [B_{k'}, U^\top_{<k'}, \nabla_\alpha U_{<k'}]dk' \tag{4}$$

Proposition 5 (Gauge covariant S-estimate)**.** *Let ψ_k be a frequency-localised solution to the linear equation*

$$\Box\psi_k = 2\mathbf{A}(\phi)^\alpha_{<k-m}\partial_\alpha\psi_k + G \tag{5}$$

eq:linwave

where $\mathbf{A}(\phi)^\alpha_{<k-m} : I \times \mathbb{R}^2 \to \mathfrak{so}(\mathbb{R}^2)$ is the anti-symmetric matrix

$$\mathbf{A}(\phi)^\alpha_{<k-m} := \Big(\mathbf{S}(\phi) - \mathbf{S}^\top(\phi)\Big)^\alpha_{<k-m}\partial_\alpha\phi_{<k-m},$$

and ϕ is a smooth wave map on I with bounds

$$||\phi||_{\underline{\mathsf{E}}[I]} + ||\phi||_{\underline{\mathsf{X}}[I]} + ||\phi||_{\mathsf{S}[I]} \leq \mathcal{F}. \tag{6}$$

There exists then a frequency gap $m \geq m(\mathcal{F}) \geq 20$ of logarithmic growth $m(\mathcal{F}) \sim \log \mathcal{F}$ such that the following energy estimate holds

$$||\psi_k||_{\mathsf{S}[I]} \lesssim_{\mathcal{F}} ||\psi_k[0]||_{\dot{H}^1 \times L^2} + ||G||_{\mathsf{N}}. \tag{7}$$

eq:renormest

Proof. Standard energy estimates

$$||\psi_k||_{\underline{\mathsf{E}}[I]} \lesssim_{\mathcal{F}} ||\psi_k[0]||_{\dot{H}^1 \times L^2} + 2^{\delta m}||G||_{\mathsf{N}[I]} + 2^{-\delta m}||\psi_k||_{\mathsf{S}[I]} \tag{8}$$

□

3. Energy dispersion method

Energy dispersion

$$\mathsf{ED}[\phi] = \sup_{N \in 2^{\mathbb{Z}}} ||P_N \phi||_{L^\infty_{t,x}[I]}$$

Scaling dictates that

$$||P_N \phi||_{L^\infty_x} \lesssim ||\phi||_{\dot{H}^1_x}$$

So while we do not have the endpoint embedding $\dot{H}^1 \subseteq L^\infty$, we do have this embedding into the energy dispersion space.

Theorem 6 (Energy-dispersed regularity theorem)**.** *There exist functions $F(\mathcal{E}) \gg 1$ and $\varepsilon(\mathcal{E}) \ll 1$ of energy such that if $\phi : [t_0, t_1] \times \mathbb{R}^2 \to \mathbb{M}$ is a solution to the wave maps equation (WM) with finite energy $\mathcal{E}[\phi] \equiv \mathcal{E}$ and energy dispersion*

$$||\phi||_{\mathsf{ED}[I]} \leq \varepsilon(\mathcal{E}), \tag{9}$$

eq:ED

then

$$||\phi||_{\mathsf{S}[I]} \leq \mathcal{F}(\mathcal{E}). \tag{10}$$

eq:S

In addition, there exists a polynomial $K(\mathcal{F})$ such that if $\{c_\kappa\}_\kappa$ is any (δ_0, δ_1)-admissible frequency envelope for $\vec{\phi}_0$, we have the bound

$$||\phi||_{\mathsf{S}_c[I]} \leq K(\mathcal{F}(E)).$$

thm:ED *In particular, one may extend ϕ to a finite energy wave-map on the interval $(t_0 - T, t_1 + T)$ for some $T \ll_{\mathcal{E},c,\varepsilon} 1$.*

3.1. **Induction on energy.** To illustrate the induction on energy scheme, we will aim for a qualitative statement, though as we detail the proof we will arrive at the full quantitative energy-dispersion theorem. We say that an energy $\mathcal{E}$ is *regular* if there exists parameters $\varepsilon \ll 1$ sufficiently small and $\mathcal{F} \gg 1$ sufficiently large such that for every wave map $\phi : I \times \mathbb{R}^2 \to \mathbb{M}$ with energy $\mathcal{E}[\phi] = \mathcal{E}$ we have

$$||\phi||_{\mathsf{ED}[I]} \leq \varepsilon \text{ implies } ||\phi||_{\mathcal{S}[I]} \leq \mathcal{F}.$$

We remark that we are free to choose these parameters ε and $\mathcal{F}$, though as we will soon see in the proof we can give a quantitative dependence on $\mathcal{E}[\phi]$. Let us denote the set of regular energies by

$$\mathcal{R} := \{\mathcal{E} \in [0, \infty) : \mathcal{E} \text{ is a regular energy}\}.$$

The global well-posedness theorem for small energy furnishes the base case for our induction on energy, $[0, \mathcal{E}_0] \subseteq \mathcal{R}$ for $\mathcal{E}_0 \ll 1$ sufficiently small. Assume then for induction that energies are regular up to some $\mathcal{E}_0$. Our goal is to construct a positive non-increasing function of energy $e(\mathcal{E}) > 0$ to push the induction forward by showing that $\mathcal{E}_0 + e$ is a regular energy for any $e \leq e(\mathcal{E}_0)$. This induction step allows us to conclude the usual continuous induction argument, as it shows

- $\mathcal{R}$ is open: if $[0, \mathcal{E}_0] \subseteq \mathcal{R}$ then $[0, \mathcal{E}_0 + e(\mathcal{E}_0)] \subseteq \mathcal{R}$,
- $\mathcal{R}$ is closed: if we have a sequence of regular energies $\{\mathcal{E}_n\}_n \subseteq \mathcal{R}$ such that $\mathcal{E}_n \nearrow \mathcal{E}$, then since $e : [0, \infty) \to (0, \infty)$ is positive non-increasing, $e(\mathcal{E}_n) > e(\mathcal{E}) > 0$. Taking n large, we have $\mathcal{E} \leq \mathcal{E}_n + e(\mathcal{E}_n)$ and therefore by the induction step $\mathcal{E}$ is regular.

By connectedness, we conclude $\mathcal{R} = [0, \infty)$, i.e. all energies are regular.

Remark. If instead $e(\mathcal{E}_n) \to 0$ as $\mathcal{E}_n \nearrow \mathcal{E}$, e.g. if we were not precise and allowed e to depend also on ε or $\mathcal{F}$, then we would not be able to prove closedness of the set of regular energies. The Kenig-Merle strategy is to assume towards a contradiction that there exists a *critical element*, i.e. a minimal energy blow-up solution, and then attempt to eliminate this possibility.

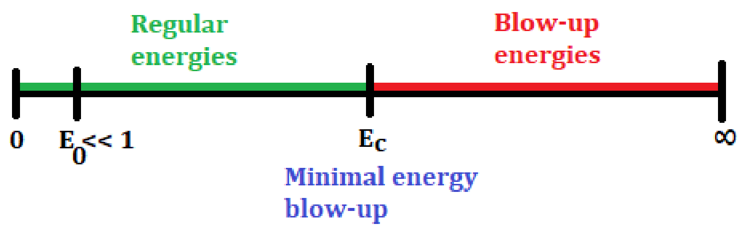

FIGURE 1. If the set of regular energies $\mathcal{R}$ was not closed, i.e. $\mathcal{R} = [0, \mathcal{E}_c)$, then there would exist a minimal energy blow-up wave map $\mathcal{E}[\phi_c] = \mathcal{E}_c$.

We end this subsection by getting the proof of the induction step started. Suppose for induction $\mathcal{E}_0$ is a regular energy and $e \leq e(\mathcal{E}_0) \ll 1$ a small energy increment to be chosen later. Let $\phi : I \times \mathbb{R}^2 \to \mathbb{M}$ be a wave map with energy $\mathcal{E}[\phi] = \mathcal{E}_0 + e$ and energy dispersion

$$||\phi||_{\mathsf{ED}[I]} \leq \varepsilon.$$

We want to compare with a wave map $\widetilde{\phi} : I \times \mathbb{R}^2 \to \mathbb{M}$ with an energy by hypothesis assumed to be regular $\mathcal{E}[\widetilde{\phi}] = \mathcal{E}_0$. Choose then a *cut frequency* $k_* \in \mathbb{R}$ to truncate the initial data such that its projection Π back onto $T\mathbb{M}$ has regular energy,

$$\mathcal{E}[\Pi P_{\leq k_*} \phi[0]] = \mathcal{E}_0.$$

Such a projection is well-defined since energy-dispersed solutions stay close to the manifold. Local well-posedness guarantees that there exists a solution $\widetilde{\phi} : J \times \mathbb{R}^2 \to \mathbb{M}$ with initial data $\widetilde{\phi}[0] = \Pi P_{\leq k_*} \phi[0]$ on some small time interval $J \subseteq I$. To make use of the fact $\mathcal{E}_0$ is a regular energy, we need to pass the energy dispersion of ϕ to $\widetilde{\phi}$. The heuristic is that $\widetilde{\phi}[0] \approx P_{\leq k_*} \phi[0]$ up to higher-order errors. Indeed,

$$||P_k(P_{\leq k_*} \phi[0] - \widetilde{\phi}[0])||_{\dot{H}^1 \times L^2} \lesssim_{\mathcal{E}_0} \varepsilon^{\frac{1}{4}} 2^{-\frac{1}{2}|k-k_*|}. \tag{11}$$

The proof uses standard Moser-type estimates, i.e. chain rule and Bernstein's inequality, see [ST10b, Section 11]. The gain on the right-hand side allows us to use the Sobolev-Bernstein inequality to estimate the energy-dispersion of $\widetilde{\phi}$ at time $t = 0$ by

$$\begin{aligned} ||P_k \widetilde{\phi}[0]||_{L^\infty_x} &\leq ||P_k(P_{\leq k_*} \phi[0] - \widetilde{\phi}[0])||_{L^\infty_x} + ||P_k(P_{\leq k_*} \phi[0]||_{L^\infty_x} \\ &\lesssim_{\mathcal{E}_0} 2^{\frac{k}{2}} ||P_k(P_{\leq k_*} \phi[0] - \widetilde{\phi}[0])||_{\dot{H}^1 \times L^2} + ||P_k \phi[0]||_{L^\infty_x} \lesssim_{k^*} \varepsilon^{\frac{1}{4}} + \varepsilon \lesssim \varepsilon^{\frac{1}{4}}. \end{aligned}$$

Choosing then $\varepsilon^{\frac{1}{4}} \ll \varepsilon(\mathcal{E}_0)$, the local well-posedness theory guarantees that $\widetilde{\phi}$ is energy-dispersed in that, after possibly choosing a smaller time interval $J_0 \subseteq J \subseteq I$,

$$||\widetilde{\phi}||_{\mathsf{ED}[J_0]} \leq \varepsilon(\mathcal{E}_0).$$

Then the induction hypothesis guarantees that we have the dispersive bound

$$||\widetilde{\phi}||_{\mathsf{S}[J_0]} \leq \mathcal{F}(\mathcal{E}_0).$$

3.2. **Bootstrap argument.** We want to propagate S-control for the truncated solution $\widetilde{\phi}$ to our original solution ϕ. We also want to propagate our estimates from the sub-interval $J \subseteq I$ to the full time interval I. To this end, suppose we have suitable choices of $e, \varepsilon, \mathcal{F}$, and let ϕ be a wave map such that

$$||\phi||_{\mathsf{ED}[I]} \leq \varepsilon. \tag{12}$$ eq:ED1

We make the following bootstrap assumptions on the sub-interval $J \subseteq I$,

$$||\phi||_{\mathsf{S}[J]} \leq 2\mathcal{F}, \tag{13}$$ eq:BS1

$$||\widetilde{\phi}||_{\mathsf{ED}[J]} \leq \widetilde{\varepsilon}, \tag{14}$$ eq:BS2

and aim to improve the bounds,

$$||\phi||_{\mathsf{S}[J]} \leq \mathcal{F}, \tag{15}$$ eq:BS1i

$$||\widetilde{\phi}||_{\mathsf{ED}[J]} \leq \frac{1}{2}\widetilde{\varepsilon}. \tag{16}$$ eq:BS2i

To conclude the proof of Theorem 6 via a continuous induction on time, we need the following:

Lemma 7 (Bootstrapping tool)**.** *Let I be an interval and $\{c_k\}_k$ a frequency envelope. Then for each affinely Schwartz function ϕ in I the following properties hold:*

(a) Seed bound. Let $I_n \subseteq I$ be a decreasing sequence of intervals which converge to $t = 0$. Then

$$\lim_{n\to\infty} ||\phi||_{\mathsf{S}[I_n]} \lesssim ||\phi[0]||_{\dot{H}^1\times L^2},$$
$$\lim_{n\to\infty} ||\phi||_{\mathsf{S}_c[I_n]} \lesssim ||\phi[0]||_{(\dot{H}^1\times L^2)_c}.$$

(b) Continuity properties. For each sub-interval $J \subseteq I$ we have $\phi \in \mathsf{S} \cap \mathsf{S}_c[J]$ and its S-norm, its S_c-norm, and its energy-dispersion norm ED all depend continuously on the endpoints of J.

(c) Closure and extension property. Let I_n be an increasing sequence of intervals and $\bigcup_n I_n = I$. Let ϕ be a classical wave map in I which satisfies the uniform bounds

$$||\phi||_{\mathsf{S}[I_n]} \leq \mathcal{F},$$
$$||\phi||_{\mathsf{ED}[I_n]} \leq \varepsilon,$$

with $\varepsilon \leq \varepsilon(\mathcal{F})$. Then $\phi \in \mathsf{S}[I]$. Furthermore, it can be extended to a classical wave map in a larger interval.

Proof.

(a) Follows from the energy estimate.
(b) Scale invariance along with convergence in Schwartz space is stronger than convergence in S-norm.
(c) Frequency envelope bound allows us to extend.

□

3.3. **Comparing ϕ and $\widetilde{\phi}$.** It remains to improve our bootstrap assumptions (13), (14) to the estimates (15), (16). We divide the analysis between comparing low frequencies and comparing high frequencies.

3.3.1. *Low frequencies.* We use the *a priori* space-time control (13) for our solution ϕ to pass good energy dispersion estimates (12) for ϕ back down to the truncated solution $\widetilde{\phi}$, improving (14) to (16). Towards this end it suffices to compare $\widetilde{\phi}$ against the frequencies of ϕ below the cut frequency k_*. We claim that there exists a non-decreasing positive function $K_1(\mathcal{F}) > 0$ of polynomial growth such that

$$||\widetilde{\phi} - P_{\leq k_*}\phi||_{\mathsf{S}[J]} \leq K_1(\mathcal{F})\varepsilon^{\delta_0}.$$

Given this estimate, choosing $\varepsilon \ll \widetilde{\varepsilon} \ll 1$ such that $K_1(\mathcal{F})\varepsilon^{\delta_0} \ll \widetilde{\varepsilon}$, it follows that

$$||\widetilde{\phi}||_{\mathsf{ED}[J]} \lesssim ||\widetilde{\phi} - P_{\leq k_*}\phi||_{\mathsf{S}[J]} + ||\phi||_{\mathsf{ED}[J]} \leq K_1(\mathcal{F})\varepsilon^{\delta_0} + \varepsilon \leq \frac{1}{2}\widetilde{\varepsilon},$$

improving our energy dispersion bound (14) to (16), as desired.

Proposition 8 (Low frequency evolution)**.** *Let ϕ be a wave map with energy $\mathcal{E}[\phi] = \mathcal{E} + e$, and denote $\widetilde{\phi}$ the wave map with initial data $\widetilde{\phi}[0] = \Pi P_{\leq k_*}\phi[0]$ and energy $\mathcal{E}[\widetilde{\phi}] = \mathcal{E}$. If ϕ and $\widetilde{\phi}$ are defined on the time interval J with bounds*

$$||\phi||_{\mathsf{ED}[J]} \leq \varepsilon, \tag{17}$$
$$||\phi||_{\mathsf{S}[J]} \leq \mathcal{F}, \tag{18}$$

and

$$||\widetilde{\phi}||_{\mathsf{ED}[J]} \leq \widetilde{\varepsilon}, \tag{19}$$
$$||\widetilde{\phi}||_{\mathsf{S}[J]} \leq \widetilde{\mathcal{F}}, \tag{20}$$

for appropriate choices of $\varepsilon, \widetilde{\varepsilon}, \mathcal{F}, \widetilde{\mathcal{F}}$, then there exists a non-decreasing positive function $K_1(\mathcal{F}) > 0$ of polynomial growth such that

$$||P_k(P_{\leq k_*}\phi - \widetilde{\phi})||_{\mathsf{S}[J]} \leq K_1(\mathcal{F}) 2^{-\delta_0|k-k_*|}\varepsilon^{\delta_0}. \tag{21}$$ eq:lowfreq

prop:lowfreq

Proof. Our strategy will be to linearise the wave maps equation (WM) around the solution $\widetilde{\phi}$. The difference $\psi := \widetilde{\phi} - P_{\leq k_*}\phi$ satisfies the equation

$$\Box\psi = -\mathfrak{D}(\widetilde{\phi}, \psi) + \mathfrak{C}(\phi) \tag{22}$$ eq:wavelow

where the difference $\mathfrak{D}$ and the generalised commutator $\mathfrak{C}$ are defined as

$$\mathfrak{D}(\widetilde{\phi}, \psi) = \mathbf{S}(\widetilde{\phi})\partial^\alpha\widetilde{\phi}\partial_\alpha\widetilde{\phi} - \mathbf{S}(\widetilde{\phi}+\psi)\partial^\alpha(\widetilde{\phi}+\psi)\partial_\alpha(\widetilde{\phi}+\psi)$$
$$\mathfrak{C}(\phi) = P_{<k_*}\left(\mathbf{S}(\phi)\partial^\alpha\phi\partial_\alpha\phi\right) - \mathbf{S}(P_{<k_*}\phi)\partial^\alpha P_{<k_*}\phi\partial_\alpha P_{<k_*}\phi.$$

Projecting to each frequency $|\xi| \sim 2^k$, we obtain the following paradifferential form for the equation

$$\Box\psi_k + 2\widetilde{\mathbf{A}}^\alpha(\widetilde{\phi})_{<k-m}\partial_\alpha\psi_k = \mathfrak{D}^m_k(\widetilde{\phi}, \psi) + \mathfrak{L}^m_k(\widetilde{\phi}, \psi) + \mathfrak{C}^m_k(\phi). \tag{23}$$ eq:wavelowpar

where

$$\widetilde{\mathbf{A}}^\alpha(\widetilde{\phi})_{<k-m} := \left(\mathbf{S}(\widetilde{\phi})_{<k-m} - \mathbf{S}^\top(\widetilde{\phi})_{<k-m}\right)\partial^\alpha\widetilde{\phi}_{<k-m}.$$

and

$$\mathfrak{L}^m_k := 2\left(\mathbf{A}^\alpha_{<k-m}(\widetilde{\phi}) - 2\mathbf{A}^\alpha_{<k-m}(\widetilde{\phi}+\psi)\right)\partial_\alpha(\widetilde{\phi}_k + \psi_k)$$

Then

$$||\psi_k||_{\mathsf{S}} \tag{24}$$

ADD ALGEBRAIC DECOMPOSITION AND ESTIMATES FOR PERTURBATIVE TERMS $\square$

3.3.2. *High frequencies.* We now turn towards comparing the solution with truncated data $\widetilde{\phi}$ with the original solution ϕ to improve the space-time control (13) to (15). We claim that there exists a non-decreasing function $K_2(\mathcal{F}) > 0$ of polynomial growth such that

$$||\widetilde{\phi} - \phi||_{\mathsf{S}[J]} \leq K_2(\mathcal{F}).$$

In view of the previous section, in which we showed that $\widetilde{\phi} - P_{\leq k_*}\phi$ is negligible, one can view this as an estimate on the evolution of the high frequencies of ϕ. Given this estimate, choosing $\mathcal{F} \gg \widetilde{\mathcal{F}} \gg 1$ such that $K_2(\widetilde{\mathcal{F}}) \ll \mathcal{F}$, it follows from the triangle inequality that

$$||\phi||_{\mathsf{S}[J]} \leq ||\widetilde{\phi}||_{\mathsf{S}[J]} + ||\phi - \widetilde{\phi}||_{\mathsf{S}[J]} \leq \widetilde{\mathcal{F}} + K_2(\widetilde{\mathcal{F}}) \ll \mathcal{F},$$

improving our space-time control (13) to (15), completing the bootstrap argument.

The difference $\psi := \widetilde{\phi} - \phi$ satisfies the equation

$$\Box\psi = -\mathbf{S}(\widetilde{\phi})\partial^\alpha\widetilde{\phi}\partial_\alpha\widetilde{\phi} + \mathbf{S}(\widetilde{\phi}+\psi)\partial^\alpha(\widetilde{\phi}+\psi)\partial_\alpha(\widetilde{\phi}+\psi).$$

As in the previous section, our strategy will be to reduce the problem to a perturbation of the gauge covariant equation (**??**). We want to apply the paradifferential argument as in the preceding proof for low frequencies, however the coefficients for this equation $\widetilde{\phi}$ are not small. To remedy this lack of smallness, we need three intermediate steps as outlined in the following diagram:

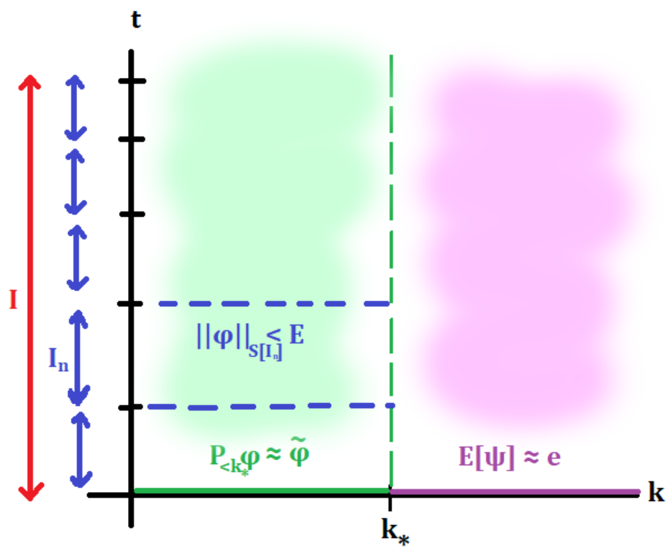

FIGURE 2. Proposition 8 shows that the low frequencies are close. The high frequencies are controlled by the energy step via conservation of energy. The size of $\widetilde{\phi}$ in the S-norm is large, so we divide into $O_{\widetilde{\mathcal{F}}}(1)$-many sub-intervals on which the norm is comparable to the energy. We conclude the argument using perturbation theory on each sub-interval.

We first establish uniform energy bounds for ψ. As ϕ and $\widetilde{\phi}$ are wave maps, they have conserved energy $\mathcal{E}[\phi] = \mathcal{E}$ and $\mathcal{E}[\widetilde{\phi}] \equiv \mathcal{E} + e$ respectively. Thus, using Proposition 8, one can conclude their difference has energy on the order $\mathcal{E}[\psi] \lesssim e$.

Lemma 9 (Almost energy conservation)**.** *Let ϕ be a wave map with energy $\mathcal{E}[\phi] = \mathcal{E} + e$, and denote $\widetilde{\phi}$ the wave map with initial data $\widetilde{\phi}[0] = \Pi P_{\le k_*}\phi[0]$ and energy $\mathcal{E}[\widetilde{\phi}] = \mathcal{E}$. If ϕ and $\widetilde{\phi}$ are defined on the time interval J, their difference $\psi := \phi - \widetilde{\phi}$ satisfies the almost conservation of energy*

$$\mathcal{E}[\psi(t)] \lesssim e \tag{25}$$

eq:psiconserv

for all $t \in J$.

Proof. Decomposing into high and low frequencies $\phi = P_{>k_*}\phi + P_{\le k_*}\phi$ and viewing the energy as arising from the inner product on $\dot{H}^1 \times L^2$, we can write

$$\mathcal{E}[\phi] = \mathcal{E}[P_{>k_*}\phi] + \mathcal{E}[P_{\le k_*}\phi] + 2\langle P_{>k_*}\phi, P_{\le k_*}\phi\rangle_{\dot{H}^1 \times L^2}.$$

The Littlewood-Paley projections are non-negative operators, so the inner product on the right is non-negative. On the other hand, the estimate (21) from Proposition 8 states that the low frequency terms $\widetilde{\phi}$ and $P_{\le k_*}\phi$ are close, so, in particular, the reverse triangle inequality implies

$$\left|\mathcal{E}[P_{\le k_*}\phi] - \mathcal{E}[\widetilde{\phi}]\right| + \left|\mathcal{E}[\psi] - \mathcal{E}[P_{>k_*}\phi]\right| \le 100 K_1(\mathcal{F})\varepsilon^{\delta_0}.$$

Collecting our results, we obtain

$$\begin{aligned}
\mathcal{E}[\psi] &\le \mathcal{E}[P_{>k_*}\phi] + 100\varepsilon^{\delta_0}K(\mathcal{F}) \\
&\le \mathcal{E}[\phi] - \mathcal{E}[P_{\le k_*}\phi] + 100\varepsilon^{\delta_0}K(\mathcal{F}) \\
&\le \mathcal{E}[\phi] - \mathcal{E}[\widetilde{\phi}] + 200\varepsilon^{\delta_0}K(\mathcal{F}) \\
&\le e + 200\varepsilon^{\delta_0}K(\mathcal{F})
\end{aligned}$$

Making appropriate choices of ε and $\mathcal{F}$ completes the proof. □

Next, we prove partial divisibility for the S-norm. For functions with finite L^p_t-norm for $1 \le p < \infty$, we can divide the interval into sub-intervals on which the L^p_t-norms are small. However, since the S-norm contains L^∞_t-norms, one cannot hope for divisibility into arbitrarily small norms, but one can divide into norms on the order of the energy.

Lemma 10 (Partial divisibility). *Let $\widetilde{\phi}$ be a wave map on the interval J with energy $\mathcal{E}[\widetilde{\phi}] = \mathcal{E}$ and space-time control $||\phi||_{\mathsf{S}[J]} = \widetilde{\mathcal{F}}$. Then there exists a non-decreasing positive function $K_2(\widetilde{\mathcal{F}}) > 0$ of polynomial growth such that we can partition the time interval into $K_2(\widetilde{\mathcal{F}})$-many sub-intervals $J = \bigsqcup J_k$ such that*

$$||\widetilde{\phi}||_{\mathsf{S}[J_k]} \lesssim \mathcal{E}. \tag{26}$$
`eq:divis`

Proof. See [ST10b, Section 10.2] □

Finally, we can use the perturbation theory as in the previous section to obtain good estimates for the S-norm of ψ on each sub-interval. The coefficients $\widetilde{\phi}$ have size on the order of the energy, we can choose the energy step $e \ll 1$ sufficiently small, depending *only* on energy, to close the continuity argument.

Lemma 11 (Perturbation theory for ψ). *Let $\widetilde{\phi}$ be a wave map and ψ a solution to the equation for the evolution of high frequencies*

$$\Box\psi = -\mathbf{S}(\widetilde{\phi})\partial^\alpha\widetilde{\phi}\partial_\alpha\widetilde{\phi} + \mathbf{S}(\widetilde{\phi}+\psi)\partial^\alpha(\widetilde{\phi}+\psi)\partial_\alpha(\widetilde{\phi}+\psi). \tag{27}$$
`eq:highwave`

Suppose $\widetilde{\phi}$ has space-time norm controlled by the energy $\mathcal{E} > 0$ on an interval J as in (26), and ψ has energy smaller than $e(\mathcal{E}) \ll 1$ uniformly in time as in (25), that is

$$||\widetilde{\phi}||_{\mathsf{S}[J]} \lesssim \mathcal{E}, \tag{28}$$
`eq:lowphicont`

$$\mathcal{E}[\psi(t)] \lesssim e(\mathcal{E}), \qquad \text{for } t \in J \tag{29}$$
`eq:psicontrol`

then ψ also admits the space-time control over the time interval J,

$$||\psi||_{\mathsf{S}[J]} \le 1. \tag{30}$$
`eq:pert`

Proof. Without loss of generality, suppose J is centered at $t = 0$. Using the estimate (29) for $e(\mathcal{E}) \ll 1$ and the energy estimate, there exists a small time interval $J_0 \subseteq J$ on which the conclusion of the lemma holds. We propagate this base case by making the bootstrap assumption

$$||\psi||_{\mathsf{S}[I]} \le 2 \tag{31}$$
`eq:pertBA`

for some sub-interval $I \subseteq J$, and aim to improve to (30).

$$||\psi + \widetilde{\phi}||_{\mathsf{S}[I]} + ||\widetilde{\phi}||_{\mathsf{S}[I]} \lesssim_{\mathcal{E}} 1. \tag{32}$$

Claim

$$||\psi||_{\mathsf{S}_{\lambda c}[I]} \lesssim_{\mathcal{E}} 1 \tag{33}$$

for $\lambda = e + \widetilde{\varepsilon}^{\delta_0\delta_1^2}$. The paradifferential version

$$\Box\psi_k + 2\mathbf{A}^\alpha(\phi)_{<k-m}\partial_\alpha\psi_k = \text{perturbative} \tag{34}$$

ADD ALGEBRAIC DECOMPOSITION AND ESTIMATES FOR PERTURBATIVE TERMS □

Proposition 12 (High frequency evolution). *Let ϕ and $\widetilde{\phi}$ be defined as in Lemma 8. There exists a function of energy $e(\mathcal{E})$ and a non-decreasing positive function $K_2(\widetilde{\mathcal{F}}) > 0$ of polynomial growth such that if $e \le e(\mathcal{E})$ then*

$$||\phi - \widetilde{\phi}||_{\mathsf{S}[J]} \le K_2(\widetilde{\mathcal{F}}). \tag{35}$$

Proof. Triangle inequality

$$||\phi||_{\mathsf{S}[J]} \le \sum_{k=1}^{K_2(\widetilde{\mathcal{F}})} ||\phi||_{\mathsf{S}[J_k]} \le K_2(\widetilde{\mathcal{F}}).$$

□

4. Bubbling

4.1. **Monotonicity formula.** If we instead integrated the stress-energy tensor over the light cone, we can obtain a monotonicity formula for the energy when restricted to slices of the light cone in time. Before we state the formula, we will need to introduce some notation. We denote the forward light cone by

$$C := \{(t,x) \in \mathbb{R}^{1+2} : r \leq t\}$$

and its restrictions to some time interval $I \subseteq [0,\infty)$ as well as time-slices by

$$C_I := C \cap (I \times \mathbb{R}^2),$$
$$S_t := C \cap (\{t\} \times \mathbb{R}^2).$$

The *null boundary* ∂C_I denotes the boundary of the time-slab C_I modulo the top and bottom time-slices. Due to singularities on the null boundary, we will also consider the shifted light cone

$$C^\delta := (\delta, 0) + C.$$

Accordingly, we have

$$C_I^\delta := C_I \cap C^\delta,$$
$$S_t^\delta := S_t \cap C^\delta,$$

In view of the null boundary, we define the null frame $\{L, \underline{L}, \not\partial\}$ to be the vector fields given by

$$L := \partial_t + \partial_r,$$
$$\underline{L} := \partial_t - \partial_r,$$
$$\not\partial := \frac{1}{r}\partial_\theta.$$

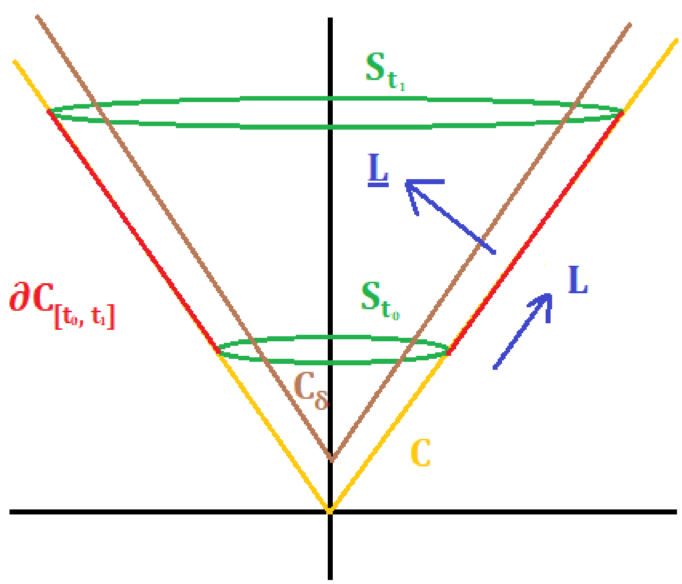

FIGURE 3. The light cone C, its vertical shift C^δ, the time-slices S_t, the null boundary $\partial C_{[t_0,t_1]}$, and the null frame $\{L, \underline{L}, \not\partial\}$.

Contracting the stress-energy tensor $T_{\alpha\beta}$ with the vector field ∂_t and then integrating then over the slab of the light cone $C_{[t_0,t_1]}$, we obtain in view of the divergence-free property and Stokes' theorem the *monotonicity formula*

$$\mathcal{E}_{S_{t_1}}[\phi] = \mathcal{F}_{[t_0,t_1]}[\phi] + \mathcal{E}_{S_{t_0}}[\phi], \tag{↑}$$ eq:mono

where $\mathcal{E}_{S_{t_0}}[\phi]$ denotes the energy on the time-slice S_{t_0},

$$\mathcal{E}_{S_{t_0}}[\phi] := \int_{S_{t_0}} |\partial_t \phi|^2 + |\nabla_x \phi|^2 \, dx,$$

and $\mathcal{F}_{[t_0,t_1]}[\phi]$ denotes the *flux* of the wave map on the null-boundary of the light cone,

$$\mathcal{F}_{[t_0,t_1]}[\phi] := \int_{\partial C_{[t_0,t_1]}} \left(\frac{1}{4}|L\phi|^2 + \frac{1}{2}|r^{-1}\partial_\theta \phi|^2 \right) dA.$$

The key observation is that the flux is non-negative, hence (↑) is indeed a monotonicity formula for the energy on time-slices of the light cone $\mathcal{E}_{S_t}[\phi] \nearrow$.

4.2. **Bubbling argument.** SKETCH BUBBLING ARGUMENT

References

Grinis2016 [Gri16] Roland Grinis. *On the Soliton Resolution Conjecture for Wave Maps*. PhD thesis, November 2016.

KochEtAl2014 [KTV14] Herbert Koch, Daniel Tataru, and Monica Vişan. *Dispersive Equations and Nonlinear Waves: Generalized Korteweg–de Vries, Nonlinear Schrödinger, Wave and Schrödinger Maps*, volume 45 of *Oberwolfach Seminars*. Springer, Basel, 2014.

nzTataru2010a [ST10a] Jacob Sterbenz and Daniel Tataru. Energy Dispersed Large Data Wave Maps in 2 + 1 Dimensions. *Communications in Mathematical Physics*, 298(1):139–230, August 2010.

enzTataru2010 [ST10b] Jacob Sterbenz and Daniel Tataru. Regularity of Wave-Maps in dimension 2+1. *Communications in Mathematical Physics*, 298(1):231–264, August 2010.

Tao2001 [Tao01] Terence Tao. Global regularity of wave maps. II. Small energy in two dimensions. *Comm. Math. Phys.*, 224(2):443–544, 2001.

Tataru2005 [Tat05] Daniel Tataru. Rough solutions for the wave maps equation. *Amer. J. Math.*, 127(2):293–377, 2005.