Elliptic PDEs

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Lent Term 2018

These notes are produced entirely from the course I took, and my subsequent thoughts. They are not necessarily an accurate representation of what was presented, and may have in places been substantially edited. Please send any corrections to pdtwm2@cam.ac.uk

Recommended books: *Gilbarg & Trudinger,* Elliptic Partial Differential Equations of Second Order

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0. Introduction

In this course we study 2nd order elliptic equations on a domain in $\mathbb{R}^n$. A classical situation in which many elliptic PDEs arise is that of *variational equations*. These lead to nonlinear elliptic PDEs. First we need knowledge of the linear theory to study the non-linear problems.

The general set up is as follows. Consider a function $F : \underbrace{\Omega \times \mathbb{R} \times \mathbb{R}^n}_{\text{coordinates } (x,z,p)} \to \mathbb{R}$, with $\Omega \subset \mathbb{R}^n$ open and bounded. Then consider the functional:

$$\mathscr{F}(u) := \int_\Omega F(x, u(x), Du(x))\, \mathrm{d}x.$$

Assume F is sufficiently regular (i.e. differentiable) so that the following calculations make sense. Let $u \in \mathscr{S}$, for $\mathscr{S}$ a suitable **linear** space of functions (we want a linear space so that we can add elements of $\mathscr{S}$ and not leave $\mathscr{S}$). Here, $u : \Omega \to \mathbb{R}$, and we usually consider cases such as $\mathscr{S} = \mathrm{W}^{1,2}(\Omega)$, or Hölder spaces such as $C^{1,\alpha}(\Omega)$, $C^{2,\alpha}(\Omega)$, etc.

Suppose we have a function $u \in \mathscr{S}$ which minimises $\mathscr{F}$ subject to some further constraint, such as fixed boundary values, i.e. we require $u|_{\partial\Omega} = g$, where $g : \partial\Omega \to \mathbb{R}$ is given. This tends to be needed or else we can get trivial minimisers, such as $u \equiv 0$. Then for any $\varphi \in \mathscr{S}$, we have:

$$\mathscr{F}(u + t\varphi) \geq \mathscr{F}(u) \quad \text{for all } t \in \mathbb{R}.$$

So with enough regularity, this means we need: $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathscr{F}(u + t\varphi) = 0$, i.e. the directional derivative along φ is 0. This means:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_\Omega F(x, u + t\varphi, Du + tD\varphi)\mathrm{d}x = 0.$$

Then from this, after differentiating under the integral sign and integrating by parts, we get a differential equation, which will be elliptic if there are certain bounds on F. So assuming appropriate regularity to exchange the integral and derivative, this becomes:

$$\int_\Omega D_z F(x, u, Du)\varphi + D_{p_j} F(x, u, Du) D_j\varphi \; \mathrm{d}x = 0 \quad \text{(summation convention)}$$

for all $\varphi \in \mathscr{S}$ with $\varphi|_{\partial\Omega} = 0$ (to ensure that $u + t\varphi$ has the correct boundary condition). Then using the divergence theorem, we get

$$\int_\Omega \left[\nabla \cdot D_p F(x, u, D_u) - D_z F(x, u, Du)\right] \varphi \; \mathrm{d}x = 0$$

for all such φ, and so hence:

$$\nabla_x \cdot D_p F(x, u, Du) - D_z F(x, u, Du) = 0 \quad \text{in } \Omega.$$

This is the **Euler-Lagrange Equation** associated with $\mathscr{F}$. This then becomes

$$\sum_{i,j=1}^n D_{p_i p_j} F(x, u, Du) D_{ij} u - D_z F(x, u, Du) = 0$$

a quasi-linear 2nd order PDE. We can write this in the general form:

$$a^{ij}(x, u, Du) D_{ij} u - b(x, u, Du) = 0.$$

This type of equation is **elliptic** if $(a^{ij}(x, u, Du))_{ij}$ is a positive definite matrix in Ω. But then noting that the a^{ij} were the 2nd derivaties of F, this is the same as saying that F is convex in the p-variable.

Example 0.1 (Dirichlet Energy)**.** *Consider the case $F(x,z,p) = |p|^2$. Then repeating the calculation leads to an Euler-Lagrange equation of: $\Delta u = 0$. So the minimisers (or critical points) of the energy functional are harmonic functions.*

Example 0.2 (Minimal Surfaces)**.** *Here we take $F(x,z,p) = \sqrt{1+|p|^2}$, so that*

$$\mathscr{F}(u) = \int_\Omega \sqrt{1+|Du|^2}$$

is the n-dimensional area of the surface determined by u. So the Euler-Lagrange equations here become:

$$\nabla \cdot \left(\frac{Du}{\sqrt{1+|Du|^2}} \right) = 0 \quad \text{in } \Omega,$$

which is the minimal surface equation.

Remark: The existence theory of solutions to Laplace's equation, subject to fixed boundary data, is 'trivial'. However the existence theory of solutions to the minimal surface equation is highly non-trivial. In fact, existence to the Dirichlet problem (i.e. solutions subject to a fixed boundary) fails unless Ω has a suitable convexity property (known as *mean convex*).

Remark: Local properties of solutions to Laplace's equation and the minimal surface equation have much in common. This is intuitively seen from the equations, since locally, $|Du| \sim$ constant, and so the denominator in the minimal surface equation is approximately constant, and so pulling this out gives $\sim C\nabla \cdot Du = C\Delta u$, which has the same form as Laplace's equation.

However for entire solutions, i.e. those defined on all of $\mathbb{R}^n$, the global behaviour is very different. For example, compare the following results.

Theorem 0.1 (Liouville's Theorem for Harmonic Functions)**.** *If $u : \mathbb{R}^n \to \mathbb{R}$ is C^2, bounded and satisfies $\Delta u = 0$, then $u \equiv$ constant.*

Theorem 0.2 (Bernstein's Theorem)**.** *The only entire solutions to the minimal surface equation in $\mathbb{R}^n$ are affine solutions (i.e. planar) $\iff n \leq 7$.*

Note: In the former result, there is an assumption at ∞, namely that u is bounded. In the latter, no assumption at ∞ is needed. So note that the global behaviour is very different (e.g. there is a dependence on dimension in the latter but not the former).

1. Harmonic Functions

1.1. Basic Properties.

Let $\Omega \subset \mathbb{R}^n$ be a domain throughout (i.e. open and connected). As usual we set $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$.

Definition 1.1. *A function $u \in C^2(\Omega)$ is* $\left\{\begin{array}{c}\textbf{\textit{harmonic}}\\ \textbf{\textit{subharmonic}}\\ \textbf{\textit{superharmonic}}\end{array}\right\}$ ***in Ω*** *if* $\left\{\begin{array}{c}\Delta u = 0\\ \Delta u \geq 0\\ \Delta u \leq 0\end{array}\right\}$ *in Ω.*

The first theorem we prove about harmonic functions is the *mean value property* (MVP). It is of critical importance, and simply tells us that harmonic functions are equal to their averages. We write $B_\rho(y) := \{x : |x-y| < \rho\}$.

Theorem 1.1 (Mean Value Property (MVP)). *If $u \in C^2(\Omega)$ is <u>sub</u>harmonic and $\overline{B_r(y)} \subset \Omega$, then:*

$$(\star) \qquad u(y) \leq \frac{1}{\omega_n r^n}\int_{B_r(y)} u$$

and

$$(\dagger) \qquad u(y) \leq \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(y)} u$$

where ω_n = volume of $B_1(0) \subset \mathbb{R}^n$.

Moreover if u is superharmonic, then the inequalities are the same but reversed. If u is harmonic, then both are equalities.

Proof. We have

$$0 \leq \int_{B_\rho(y)} \Delta u = \int_{B_\rho(y)} \nabla \cdot Du \overset{\text{divergence theorem}}{=} \int_{\partial B_\rho(y)} Du(x)\cdot \underbrace{\frac{x-y}{\rho}}_{\text{outward normal}} \mathrm{d}x$$

$$\overset{\rho w = x-y}{=} \rho^{n-1}\int_{S^{n-1}} wDu(y+\rho w)\mathrm{d}w$$

$$\overset{\text{chain rule}}{=} \rho^{n-1}\int_{S^{n-1}} \frac{\partial}{\partial \rho}(u(y+\rho w))\,\mathrm{d}w$$

and this is true for all $\rho \leq r$. So hence we see, for all $\rho \leq r$,

$$0 \leq \int_{S^{n-1}} \frac{\partial}{\partial\rho}(u(y+\rho w))\,\mathrm{d}w = \frac{\partial}{\partial\rho}\int_{S^{n-1}} u(y+\rho w)\mathrm{d}w.$$

Hence $\rho \mapsto \int_{S^{n-1}} u(y+w\rho)$ is increasing with ρ, and so we see that

$$\int_{S^{n-1}} u(y+\rho w)\mathrm{d}w \leq \int_{S^{n-1}} u(y+rw)\mathrm{d}w$$

for all $0 < \rho \leq r$. So letting $\rho \to 0$ in this, we get (by continuity of u)

$$n\omega_n u(y) \leq \frac{1}{r^{n-1}} \int_{\partial B_r(y)} u(x)\mathrm{d}x$$

(after changing variables in the integral on the RHS), thus we get (†).

To see (⋆), note that this gives: $n\omega_n \rho^{n-1} \leq \int_{\partial B_\rho(y)} u$ for all $\rho \leq r$, and so integrating this inequality over $\rho \in (0, r]$ gives:

$$\omega_n r^n u(y) \leq \int_{B_r(y)} u$$

which is (⋆).

This proves the subharmonic case. To see the superharmonic case, just note that then $-u$ is subharmonic and apply that case. For the harmonic case, note that harmonic functions are both superharmonic and subharmonic, and so combine those results to get equalities.

□

Remark: In fact the MVP characterises harmonic functions - see Example Sheet 1.

Theorem 1.2 (The Strong Maximum Principle (SMP)).

Suppose $u \in C^2(\Omega)$ is subharmonic in Ω. Suppose $\exists\ y_0 \in \Omega$ with $u(y_0) = \sup_\Omega u$. Then $u =$ constant in Ω.

[If instead u is superharmonic, then the same result holds if u attains its infimum. If u is instead harmonic, the same conclusion holds if u attains either its supremum or infimum.]

Proof. It suffices to prove the subharmonic case, as the others follow just as before from the subharmonic case.

So let $\Sigma = \{y \in \Omega : u(y) = M\}$ where we have let $M = \sup_\Omega u$ ($< \infty$ by assumption). Then clearly by assumption $M \neq \emptyset$, and is closed (as u is continuous). So by connectedness, it suffices to show that Σ is open.

So pick $y \in \Sigma$. Then by the MVP wehave, for any $\rho > 0$ such that $\overline{B_\rho(y)} \subset \Omega$,

$$M = u(y) \leq \frac{1}{n\omega_n \rho^n} \int_{B_\rho(y)} u \quad \Longrightarrow \quad \int_{B_\rho(y)} (M - u) \leq 0.$$

But then $M - u \geq 0$ and is continuous, and so it follows that we must have $u \equiv M$ on $B_\rho(y)$, i.e. $B_\rho(y) \subset \Sigma$. So hence Σ is open, and so we are done.

□

Here the SMP is easy to prove, given the MVP. However in general PDE's it is not simple to prove, and a weaker statement is proven first. We now deduce it from the SMP for this harmonic function case.

Theorem 1.3 (The Weak Maximum Principle (WMP)).

*Suppose $\Omega \subset \mathbb{R}^n$ is a **bounded** domain, and that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Then, if u is subharmonic in Ω, we have $\sup_\Omega u = \sup_{\partial\Omega} u$.*

[If instead u were superharmonic, the same result holds but with $\inf$'s instead. If instead u is harmonic, both hold.]

Proof. This is immediate from the SMP, as $\sup_\Omega u$ and $\inf_\Omega u$ are attained in $\overline{\Omega}$ (as Ω is bounded, so nothing escapes to infinity), but by the SMP, they are not attained in $\text{Int}(\Omega)$ (unless u is constant, but then this is trivial).

□

The MVP indicates that u is always a average of itself. This suggests that u can never vary too much relative to itself, and thus we should be able to related the sup and inf of u to one another. The Harnack inequality makes this precise. However note that if u ever changed sign, then there would be no hope of bounding $\sup u$ by $\inf u$, as they would be of different signs. So to get around this technicality, we assume u is (wlog) non-negative.

Theorem 1.4 (The Harnack Inequality). *Suppose $u \in C^2(\Omega)$ is **non-negative** and harmonic in Ω. Then if $\Omega' \subset\subset \Omega$ is any bounded subdomain, we have*

$$\sup_{\Omega} u \leq C \inf_{\Omega} u$$

for some $C = C(n, \Omega', \Omega)$.

Proof. First, choose $y \in \Omega$ and $\rho > 0$ such that $\overline{B_{4\rho}(y)} \subset \Omega$. Then pick $x_1, x_2 \in B_\rho(y)$. Then the MVP gives:

$$u(x_1) = \frac{1}{\omega_n \rho^n} \int_{B_\rho(x_1)} u \leq \frac{1}{\omega_n \rho^n} \int_{B_{2\rho}(y)} u \qquad \text{as } B_\rho(x_1) \subset B_{2\rho}(y),$$

$$\text{and} \quad u(x_2) = \frac{1}{\omega_n (3\rho)^n} \int_{B_{3\rho}(x_2)} u \geq \frac{1}{\omega_n (3\rho)^n} \int_{B_{2\rho}(y)} u \qquad \text{as } B_{2\rho}(y) \subset B_{3\rho}(x_2).$$

Hence combining these inequalities we see that $u(x_1) \leq 3^n u(x_2)$ for all $x_1, x_2 \in B_\rho(y)$. So we get such an inequality locally in balls, with constant independent of ρ, y, so long as $\overline{B_{4\rho}(y)} \subset \Omega$.

So now choose $x_1, x_2 \subset \overline{\Omega'} \subset \Omega$ such that $u(x_1) = \sup_{\Omega'} u$ and $u(x_2) = \inf_{\Omega'} u$. Then by (path) connectedness of Ω, $\exists$ a continuous path $\gamma \subset \overline{\Omega'}$ joining x_1 and x_2.

Then as γ is compact in Ω, and Ω is open, we can choose $\rho > 0$ such that $4\rho < \text{dist}(\gamma, \partial\Omega)$. Then choose $N = N(\Omega', \Omega)$[(i)] such that we can cover γ by N balls of radius ρ, so $\gamma \subset \bigcup_{i=1}^N B_\rho(y_i)$ for soem $y_i \in \Omega'$.

Then apply the local result along γ in each ball to get:

$$u(x_1) \leq \underbrace{3^n \cdots 3^n}_{N \text{ times}} u(x_2) = 3^{nN} u(x_2)$$

and so we are done, by the choice of x_1, x_2, with $C = 3^{nN}$.

□

Theorem 1.5 (Derivative Estimates). *Suppose $u \in C^2(\Omega)$ is harmonic in Ω. Then if $B_\rho(y) \subset \Omega$, then*

$$|Du(y)| \leq \frac{C}{\rho} \sup_{\partial B_\rho(y)} |u|$$

where $C = C(n)$.

Proof. $\Delta u = 0 \implies 0 = D_i(\Delta u) = \Delta(D_i u)$ in Ω. So $D_i u$ is harmonic. So by the MVP applied to $D_i u$,

$$D_i u(y) = \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} D_i u = \frac{1}{\omega \rho^n} \int_{B_\rho(y)} \nabla \cdot (0, \ldots, 0, \underbrace{u}_{i\text{'th}}, 0, \ldots, 0)$$

$$= \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} u(x) \nu_i(x) \mathrm{d}x \quad \text{(by the divergence theorem)}$$

where $\nu_i(x) = \frac{x_i - y_i}{\rho}$ is a component of the outward normal. Thus as $|\nu_i(x)| \leq 1$, we have

$$|D_i u(y)| \leq \frac{1}{\omega_n \rho^n} \cdot \sup_{\partial B_\rho(y)} |u| \cdot \int_{\partial B_\rho(y)} \mathrm{d}x = \frac{n}{\rho} \sup_{\partial B_\rho(y)} |u|,$$

and thus

$$|Du(x)| \leq \frac{C}{\rho} \sup_{\partial B_\rho(y)} |u|$$

for some $C = C(n)$.

□

> **Remark:** Applying this result iteratively, we get that for any $\Omega' \subset\subset \Omega$, and for any multi-index α, we have
>
> $$\sup_{\Omega'} |D^\alpha u| \leq C \sup_{\Omega} |u|$$
>
> for some $C = C(n, \alpha, \Omega, \Omega')$.

(i) We need to be careful here: this N depends on γ, which depends on u via its endpoints, so it is not immediately true that N is independent of u. What we need to do is as follows: find a finite cover of $\overline{\Omega'}$ by such balls of radius ρ, where the above inequality holds. Say there are N such balls. Then take a path from x_1 to x_2 going through each of these balls at most once (which we can do as the balls cover, e.g. could take a piecewise linear path). Then we get that this argument is true with $C = 3^{nN}$, independent of u.

It is now very easy to prove the uniqueness of solutions to the Dirichlet problem.

Theorem 1.6 (Uniqueness of Solutions to the Dirichlet Problem). *Suppose Ω is a **bounded** domain, and $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy*

$$\Delta u_1 = \Delta u_2 \quad \text{in } \Omega \quad \text{and} \quad u_1 = u_2 \quad \text{on } \partial\Omega.$$

Then $u_1 = u_2$ on $\overline{\Omega}$.

Proof. Set $w = u_1 - u_2$. Then $\Delta w = 0$ in Ω, and $w = 0$ on $\partial\Omega$. Then by applying the WMP to w we get $w = 0$ in $\overline{\Omega}$.

□

So we know that if $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω, then $\sup_{\Omega'} |D^\alpha u| \leq C \sup_{\Omega'} |u|$ for some $C = C(n, \alpha, \Omega, \Omega')$. Also, in view of the MVP, we have

$$\sup_{\Omega'} |D^\alpha u| \leq C \int_\Omega |u|,$$

where again $C = C(n, \alpha, \Omega, \Omega')$. So in particular we can prove Liouville's theorem from before:

Corollary 1.1 (Liouville's Theorem for Harmonic Functions). *If $u \in C^\infty(\mathbb{R}^n)$ is harmonic in $\mathbb{R}^n$, and if u grows sublinearly at ∞, then $u =$ constant.*

> **Note:** By **growing sublinearly at** ∞, we mean that $|u(x)| \leq C(1 + |x|^\alpha)$, for some $\alpha \in (0,1)$, with C some fixed constant, for all $x \in \mathbb{R}^n$.

Proof. From Theorem 1.5, we know that

$$\sup_{\partial B_{\rho/2}(y)} |Du| \leq \frac{C}{\rho} \sup_{B_\rho(y)} |u|$$

where $C = C(n)$. So fix any $y \in \mathbb{R}^n$. Thus combining this inequality with the growth hypothesis gives

$$|Du(y)| \leq \frac{C}{\rho} \cdot \sup_{B_\rho(y)} |u| \leq \frac{C}{\rho} \cdot (1 + (\rho + |y|)^\alpha).$$

Then letting $\rho \to \infty$ in this to see that $Du(y) = 0$. So hence as y was arbitrary, we see $Du \equiv 0$, and so $u =$ constant.

□

> **Note:** We needed to use this derivative bound, because we needed to know the ρ dependence on the RHS in order to take the limit.

1.2. **Existence Theory for Harmonic Functions.**

A classical problem is to solve the Dirichlet problem for the Laplacian. By this we mean, given $\Omega \subset \mathbb{R}^n$ a bounded domain, and a function $\varphi : \overline{\Omega} \to \mathbb{R}$ continuous, we want to find $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

We will assume for simplicity that $\partial\Omega$ is smooth, and that $\varphi \in C^\infty(\overline{\Omega};\mathbb{R})$. There are at least 3 different methods of solving this problem:

(i) **Hilbert Space Method.** Here the idea is to use the Riesz representation theorem, which depends on the linearity of the equation, to get:
- Existence of a solution in a Sobolev space $\mathrm{W}^{1,2}(\Omega)$,
- Then deal with the regularity separately.

This is the method used in the Analysis of PDE course.

(ii) **Direct Method of the Calculus of Variations.** The idea here is to prove the existence of solutions in $\mathrm{W}^{1,2}(\Omega)$ based on the fact that Laplace's equation is variational, i.e. it is the Euler-Lagrange equation for the Dirichlet energy.

(iii) **Perron's Method.** Here we use the fact that the solvability of the Dirichlet problem in balls implies the solvability of the Dirichlet problem in more general domains. This idea is based on the maximum principles. (We will look at this idea later in more general.)

Note that in all of these methods, we first prove the existence of solutions in some larger space of functions, and then prove the regularity theory afterward to show that it is in the correct space. Let us look at the direct method ((ii) in the above) in more detail.

Define $\mathscr{S} = \{w \in \mathrm{W}^{1,2}(\Omega) : w - \varphi \in W_0^{1,2}(\Omega)\}$, i.e. the functions in $\mathrm{W}^{1,2}(\Omega)$ which agree with φ locally about $\partial\Omega$. Then $\mathscr{S} \neq \emptyset$ since $\varphi \in \mathscr{S}$.

Set $\beta := \inf_{w\in\mathscr{S}} \mathscr{E}(w) \in \mathbb{R}$, where $\mathscr{E}$ is the Dirichlet energy. Note $\beta \geq 0$ clearly. So hence as this is an infimum, we get that $\exists$ a sequence $(w_j)_j \subset \mathscr{S}$ such that $\mathscr{E}(w_i) \to \beta$. We now want to extract a convergent subsequence, and show that the limit is a solution.

So we see that $\int_\Omega |Dw_j|^2 \leq \beta + 1$ for all j sufficiently large. So to hence to bound $\|w_j\|_{W^{1,2}(\Omega)}$, we just need to bound $\|w_j\|_{L^2(\Omega)}$.

But as $w_j - \varphi \in W_0^{1,2}(\Omega)$, we have by the Poincaré inequality,

$$\int_\Omega |w_j - \varphi|^2 \leq C \int_\Omega |D(w_j - \varphi)|^2$$

for some $C = C(n, \Omega)$ which implies that

$$\int_\Omega |w_j|^2 \leq \tilde{C}(C, \beta) < \infty$$

as $\int_\Omega |\varphi|^2 < \infty$. So hence we have control on $\|w_j\|_{W^{1,2}(\Omega)}$ for j large. Hence by the Rellich Compactness Theorem, $\exists$ a subsequence (j') and $w \in \mathrm{W}^{1,2}(\Omega)$ such that

$$w_{j'} \rightharpoonup w \quad \text{in } \mathrm{W}^{1,2}(\Omega)$$

$$w_{j'} \to w \quad \text{in } L^2(\Omega).$$

So in particular,

$$\int_\Omega Dw_{j'} \cdot Dv \to \int_\Omega Dw \cdot Dv \quad \text{for all } v \in \mathrm{W}^{1,2}(\Omega)$$

as the map $w' \mapsto \int_\Omega Dw' \cdot Dv$ is a linear functional on $\mathrm{W}^{1,2}(\Omega)$.

Also we must have $w_{j'} - \varphi \rightharpoonup w - \varphi$ in $\mathrm{W}^{1,2}(\Omega)$. But $w_{j'} - \varphi \in W_0^{1,2}(\Omega)$, which is a norm-closed subspace, and hence is weakly closed (since in a Banach space, a convex subset is norm-closed $\iff$ it is weakly closed).

So hence we must have $w - \varphi \in W_0^{1,2}(\Omega)$, i.e. $w \in \mathscr{S}$. So in particular, $\mathscr{E}(w) \geq \beta$ by definition of β.

Finally, noting that $\mathscr{E}(\cdot)$ is lower semi-continuous(ii) with respect to weak convergence in $\mathrm{W}^{1,2}(\Omega)$, we have

$$\mathscr{E}(w) \leq \liminf_{j' \to \infty} \mathscr{E}(w_{j'}) = \beta$$

and so hence we have $\mathscr{E}(w) = \beta = \inf_{w' \in \mathscr{S}} \mathscr{E}(w')$.

So hence we see that this implies

$$\int_\Omega Dw \cdot Dv = 0 \quad \text{for all } v \in W_0^{1,2}(\Omega),$$

which can be proved via using integration by parts and approximation, by approximating v by smooth compactly supported functions. This is the Euler-Lagrange equation in weak-form. But then this is just the weak form of Laplace's equation, and so $\Delta w = 0$ weakly, and so we have existence.

□

To prove the regularity theory, we will show that weak solutions of Laplace's equation are in fact actual solutions to Laplace's equation (this is Weyl's lemma).

So we have shown the existence of a weak solution to the Dirichlet problem. In the next subsection we will show how to upgrade this weak solution to a classical (in fact smooth) solution.

1.3. **Interior Regularity.**

Next we need to prove the regularity of the weak solution. Indeed, note that we have shown that $\exists\, u \in L^1(\Omega)$ which satisfies

$$\int_\Omega u \cdot \Delta v = 0 \quad \text{for all } v \in C_c^\infty \Omega.$$

(ii)To see this, note that by weak convergence, $\int_\Omega Dw_{j'} \cdot Dv \to \int_\Omega Dw \cdot Dv$ for all $v \in \mathrm{W}^{1,2}(\Omega)$, and so apply this with $v = w$ and use Holder's inequality.

The key result is Weyl's Lemma, which tells us that weakly harmonic functions are in fact smooth harmonic.

Theorem 1.7 (Weyl's Lemma). *Suppose $\Omega \subset \mathbb{R}^n$ is open and $u \in L^1_{\text{loc}}(\Omega)$. Then if*

$$\int_\Omega u \cdot \Delta v = 0 \quad \textit{for all } v \in C_c^\infty \Omega,$$

then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.

Proof. First mollify u: take $\varphi \in C^\infty(\mathbb{R}^n)$ with $\varphi \equiv 0$ in $\mathbb{R}^n \backslash B_1(0)$, $\varphi \geq 0$ in $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} \varphi = 1$. Wlog assume that φ is radially symmetric.

Then for $\sigma > 0$, let

$$\varphi_\sigma(x) = \frac{1}{\sigma^n}\varphi\left(\frac{x}{\sigma}\right).$$

Then $\varphi \in C_c^\infty B_\sigma(0)$ is $\geq$ with $\int_{\mathbb{R}^n} \varphi_\sigma = 1$. The let $u_\sigma = \varphi_\sigma * u$ be this convolution. This is well-defined and finite for $x \in \Omega_\sigma := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \sigma\}$.

Then we know that u_σ is smooth in Ω_σ (via differentiation under the integral sign), and also that $u_\sigma \to u$ in $L^1_{\text{loc}}(\Omega)$. Then we can see that $\Delta u_\sigma = 0$ in Ω_σ: indeed,

$$D_{x_i} u_\sigma(x) = \int_\Omega u(y) \cdot D_{x_i}(\varphi_\sigma(x-y))\,\mathrm{d}y = -\int_\Omega u(y) \cdot D_{y_i}(\varphi_\sigma(x-y))\,\mathrm{d}y$$

and so hence we see:

$$\Delta_x u_\sigma(x) = \int_\Omega u(y) \cdot \Delta_y(\varphi_\sigma(x-y))\,\mathrm{d}y = 0$$

where we have used the definition of weakly harmonic in the last equality. So hence u_σ is harmonic in Ω_σ.

Thus by the a priori derivative estimates we proved for smooth harmonic functions, we see that for any multi-index α,

$$\sup_{\Omega'} |D^\alpha u_\sigma| \leq C \int_{\Omega_{\sigma_1}} |u_\sigma|$$

for some fixed σ_1 small enough, depending on Ω', and where $C = C(\Omega', \sigma, u)$, and this is true for all $\sigma < \sigma_1$. But then as $u_\sigma \to u$ in $L^1_{\text{loc}}(\Omega)$, for small enough σ_1 we can bound the RHS of this by $C\left(\int_{\Omega_{\sigma_1}} |u| + 1\right)$, and thus we get a uniform bound in σ on $\sup_{\Omega'} |D^\alpha u_\sigma|$.

So hence by the Arzela-Ascoli theorem (as bounded derivatives implies equicontinuous), we can find a sequence $(\sigma_j)_{j=1}^\infty$ with $\sigma_j \downarrow \infty$, and $\tilde{u} \in C^\infty(\Omega)$ s.t. $u_{\sigma_j} \to \tilde{u}$ in $C^k(\Omega')$, for any $k \in \mathbb{N}$ and any $\Omega' \subset\subset \Omega$ (this involves a diagonal argument).

Hence we see that $\Delta \tilde{u} = \lim_j \Delta u_{\sigma_j} = 0$ in Ω by continuity of these derivatives. Then since $u_{\sigma_j} \to u$ a.e. in Ω (from above), we must have that $u = \tilde{u}$ a.e. in Ω, and so we are done.

$\square$

The results for boundary regularity we shall omit here - see the Analysis of PDEs Part III course.

Theorem 1.8 (Exist. and Uniq. of Solutions to the DP with Contin. Boundary Data). *Suppose Ω is a bounded domain with sufficiently smooth boundary*[(iii)]*. Then for any $\varphi \in C^0(\partial\Omega)$* ***continuous,*** *$\exists!\ u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ solving the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

> **Warning:** We might not have $\int_\Omega |Du|^2 < \infty$ for this solution u in the theorem!

Proof. Choose a sequence $(\varphi_n)_n \in C^\infty(\mathbb{R}^n)$ s.t. $\varphi_n \to \varphi$ uniformly on $\partial\Omega$. Then by the boundary assumption, we know that we can solve the Dirichlet problem with boundary data φ_n for each n, i.e. $\exists\ u_n \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ such that $\Delta u_n = 0$ in Ω, and $u_n = \varphi_n$ on $\partial\Omega$.

Then for each n, m we have $\Delta(u_n - u_m) = 0$ in Ω and $u_n - u_m = \varphi_n - \varphi_m$ on $\partial\Omega$. By the WMP, we have

$$\sup_{\overline{\Omega}} |u_n - u_m| \leq \sup_{\partial\Omega} |u_n - u_m| = \sup_{\partial\Omega} |\varphi_n - \varphi_m| \to 0$$

as $n, m \to \infty$. Hence $(u_n)_n$ is Cauchy in $C^0(\overline{\Omega})$. So hence as this space is complete, we get that $\exists\ u \in C^0(\overline{\Omega})$ such that $u_n \to u$ uniformly on $\overline{\Omega}$. In particular, $u = \varphi$ on $\partial\Omega$.

Then by the interior derivative estimates for u_n, we see that $(u_n)_n$ also converges in $C^k(\Omega')$ for any $\Omega' \subset\subset \Omega$. Hence we have $u \in C^\infty(\Omega)$, and so by taking limits of second order derivatives, $\Delta u = 0$ in Ω. So done.

□

> **Remarks:**
>
> (i) If Ω is a C^2-domain, then the requirement of the theorem if satisfied. More generally, if Ω satisfies the exterior sphere condition, then it is also satisfied (i.e. $\forall\ z \in \partial\Omega$, $\exists B_\rho(y)$ such that $\overline{B_\rho(y)} \cap \overline{\Omega} = \{y\}$.)
>
> (ii) There are bounded domains for which the conclusion of the theorem does not hold → see "Lebesgue's Example [Dirichlet Energy]" (e.g. if there is a cusp in Ω).

Now we move away from harmonic functions and consider more general 2nd order elliptic operators.

[(iii)] Precisely, $\partial\Omega$ should be smooth enough so that the Dirichlet problem: $\Delta u = 0$ in Ω, with $u = \varphi$ on $\partial\Omega$, has a solution $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ for any given $\varphi \in C^\infty(\overline{\Omega})$, i.e. $\varphi = \tilde{\varphi}|_{\overline{\Omega}}$ for some $\tilde{\varphi} \in C^\infty(\Omega')$, where $\Omega' \supset \overline{\Omega}$ is open.

2. General 2nd Order Elliptic Operators

From now on, we shall write
$$Lu := a^{ij}D_{ij}u + b^i D_i u + cu$$
where we have used summation convention. We work on $\Omega \subset \mathbb{R}^n$ open, and here $u \in C^2(\Omega)$, with $a^{ij}, b^i, c : \Omega \to \mathbb{R}$. We then look at solving the Dirichlet problem (DP),
$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$
for given $f : \Omega \to \mathbb{R}$ and $\varphi : \partial\Omega \to \mathbb{R}$.

If we can write L in **divergence form**, i.e. $Lu = D_i(a^{ij}D_j u) + b^i D_i u + cu$, then we can use Hilbert space methods (see Analysis of PDEs Part III course). However if it cannot, then we need a different method. In this course, we look at **Schauder estimates**: this does not involve Sobolev spaces, and the philosophy is to deform the elliptic operator L into the Laplacian Δ by several rescalings, as we shall see.

Due to $u \in C^2(\Omega)$, by the symmetry of mixed partial derivatives we can wlog assume that $a^{ij} = a^{ji}$ (since if not simply replace a^{ij} with $\tilde{a}^{ij} = \frac{1}{2}(a^{ij} + a^{ji})$. Then as $D_{ij}u = D_{ji}u$, we have $a^{ij}D_{ij}u = \tilde{a}^{ij}D_{ij}u$ (sum over i, j)).

Remark: Summation Convention will always apply here (indices from 1 to n).

Definition 2.1. *We make the following definitions for the L above.*

(i) *L is **elliptic** in Ω if the matrix $(a^{ij}(x))_{ij}$ is positive definite $\forall\ x \in \Omega$. Equivalently,*
$$0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi^i\xi^j \leq \Lambda(x)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n\backslash\{0\}$$
*where λ is the **smallest eigenvalue** of this matrix, and Λ the largest.*

(ii) *L is **strictly elliptic** in Ω if $\exists$ a constant $\lambda_0 > 0$ s.t. $\lambda_0 < \lambda(x)\ \forall\ x \in \Omega$.*

(iii) *L is **uniformly elliptic** in Ω if L is elliptic and Λ/λ is uniformly bounded.*

Note: It is not true in general that uniformly elliptic implies strict ellipticity.

Example 2.1. *Consider the minimal surface equation:*
$$\nabla \cdot \left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$

Then this is:
$$\sum_{i,j=1}^{n}\left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\right)D_{ij}u = 0$$
i.e. $\sum_{i,j} a^{ij}D_{ij}u = 0$, where $a^{ij}(x) = \delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}$. This equation is always elliptic, but not uniformly elliptic, as the smallest eigenvalue $\to 0$ as $|x| \to \infty$.

The goal of this section is to develop an existence theory for solutions $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the DP above, for bounded domains Ω and where f, φ are continuous.

In fact, we will assume, out of necessity, that f, a^{ij}, b^i, c are in $C^{0,\alpha}(\overline{\Omega})$, and prove the existence of $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$. We will also need further assumptions, such as $c \leq 0$, but we give these when they are needed.

Since we will only assume $C^{0,\alpha}$ regularity of the a^{ij}, the operator L cannot be written in divergence form (where energy methods and Euler-Lagrange methods are used). Hence we will not use any Sobolev theory for our approach.

This method will be based on a priori estimates for the $C^{2,\alpha}$ norm of the solutions (interior and boundary estimates). The starting points of these estimates is the **weak maximum principle** (**WMP**).

2.1. Basic Properties.

Theorem 2.1 (Weak Maximum Principle (WMP)). *Suppose that L is elliptic and that $\sup_\Omega |\frac{b^i}{\lambda}| < \infty$. Suppose that Ω is bounded, open, and that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Lu \geq 0$ (i.e. a **subsolution** of $Lu = 0$). Then:*

(i) *If $c = 0$, then $\sup_\Omega u = \sup_{\partial\Omega} u$.*
(ii) *If $c \leq 0$, then $\sup_\Omega u \leq \sup_{\partial\Omega} u^+$, where $u^+ = \max(u, 0)$.*

> **Remark:** in (ii), we cannot drop the assumption that $c \leq 0$ in Ω. For example, consider the Dirichlet eigenfunctions of the Laplacian on bounded domains. Then for example:
>
> - If $n = 1$, consider $u(x) = \sin(x)$ for $x \in (0, \pi)$. Then, $u'' + u = 0$. For this equation, we have $c \equiv 1 > 0$, but $\sup_\Omega u = 1$ whilst $\sup_{\partial\Omega} u^+ = \max u(0), u(\pi) = 0$, a contradiction.
> - If $n = 2$, then consider $u(x) = \sin(x)\sin(y)$ on $\Omega = (0, \pi) \times (0, \pi)$. Then $\Delta u + 2u = 0$ in Ω, but $u \equiv 0$ on $\partial\Omega$, again a contradiction.

Proof. (i): If $Lu > 0$ in Δ, then the conclusion follows easily: in fact, in this case there cannot be a local maximum in $\text{Int}(\Omega)$ (i.e. the strong maximum principle also holds).

Indeed, to see this, note that if $x_0 \in \text{Int}(\Omega)$ is a local maximum, then $Du(x_0) = 0$, and $D^2u(x_0) = [D_{ij}u(x_0)]_{ij}$ is non-positive definite. Thus as $A = [a^{ij}(x_0)]_{ij}$ is positive definite (by ellipticity assumption), we see that

$$a^{ij} D_{ij} u(x_0) = \text{Trace}\left(A \cdot D^2 u(x_0)\right) \leq 0,$$

as then $A \cdot D^2 u(x_0)$ is non-positive (as negative× positive = negative). Hence,

$$0 < Lu(x_0) = a^{ij} D_{ij} u(x_0) + b^i \underbrace{D_i u(x_0)}_{=0} \leq 0,$$

a contradiction (remember that $c = 0$ here). So we cannot have an interior maximum in this case, so done.

In the general case where $Lu \geq 0$, then consider the function $v(x) = e^{\gamma x_1}$, for some $\gamma > 0$ is a constant we will choose. So then we can directly compute:

$$D_1 v = \gamma e^{\gamma x_1}, \quad \text{and} \quad D_i v = 0 \quad \text{if } i > 1$$
$$D_{11} v = \gamma^2 e^{\gamma x_1}, \quad \text{and} \quad D_{ij} v = 0 \quad \text{for } (i,j) \neq (1,1).$$

So then

$$Lv = e^{\gamma x_1}(\gamma^2 a^{11} + b^1\gamma) \overset{\text{by ellipticity}}{\geq} e^{\gamma x_1}(\lambda\gamma^2 + b^1\gamma) = e^{\gamma x_1}\lambda(\gamma^2 + \frac{b^1}{\lambda}\cdot\gamma) > 0$$

in Ω, if we choose γ sufficiently large depending on $\sup_\Omega \left|\frac{b^1}{\lambda}\right| < \infty$.

So hence, $Lu \geq 0 \implies L(u+\varepsilon v) > 0$ for any $\varepsilon > 0$ (where γ is now fixed). So applying the above case gives:

$$u(x) \overset{\text{as } v\geq 0}{\leq} \sup_\Omega (u+\varepsilon v) \overset{\text{by above case}}{\leq} \sup_{\partial\Omega}(u+\varepsilon v) \leq \sup_\Omega u + \varepsilon \underbrace{\sup_\Omega v}_{\text{bounded}}.$$

So hence take $\varepsilon \downarrow 0$ to get that $u(x) \leq \sup_{\partial\Omega} u$ for all $x \in \Omega$, i.e. $\sup_\Omega u \leq \sup_{\partial\Omega} u$. But the other inequality is trivial (by inclusion essentially), and thus we are done.

<u>(ii):</u> Define L_0 by just the 1st and 2nd order terms of L, i.e.

$$L_0 u = a^{ij}D_{ij}u + b^i D_i u.$$

So, in the set $\Omega^+ = \{x \in \Omega : u(x) > 0\}$, we have that, since $cu \leq 0$ here,

$$L_0 u = Lu - cu \geq -cu \geq 0$$

Note that if $\Omega^+ = \emptyset$, then the conclusion is trivially true, as then $u^+ = 0$ and so the RHS of the conclusion is 0.

So we can wlog assume that $\Omega^+ \neq \emptyset$. But then we must have $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$, **and** there must exist $x_0 \in \partial\Omega^+ \cap \partial\Omega$ such that $u(x_0) > 0$ (since if not, then $u \leq 0$ on $\partial\Omega^+$ (since then $\partial\Omega^+ \cap \partial\Omega = \emptyset$, and so $\partial\Omega^+ \subset \text{Int}(\Omega)$, and so $\partial\Omega^+ \subset \Omega\backslash\Omega^+$ (as Ω^+ is open), and so hence $u \leq 0$ here as not in Ω^+). But then this contradicts (i) with $L \to L_0$ and $\Omega \to \Omega^+$).

So hence

$$\sup_\Omega u \overset{\sup_\Omega u>0 \text{ as } \Omega^+\neq\emptyset}{=} \sup_{\Omega^+} u \overset{\text{by (i)}}{=} \sup_{\partial\Omega^+} u \overset{(\star)}{\leq} \sup_{\partial\Omega} u \overset{\text{as } u\leq u^+}{=} \sup_{\partial\Omega} u^+$$

where inequality $(\star)$ is true because $\partial\Omega^+$ is either in $\text{Int}(\Omega)$ (in which case $u = 0$ here, as $\Omega = \Omega^+ \cup \{x \in \Omega : u(x) \leq 0\}$) or on $\partial\Omega$, where we could have $u > 0$. So hence the sup is equal to just that over the $\partial\Omega^+ \cap \partial\Omega$, which is $\leq$ that over $\partial\Omega$. So hence we are done.

□

We now move to proving the Strong maximum principle (SMP), using the WMP. Let $M = \sup_\Omega u$, and consider $\Sigma = \{x \in \Omega : u(x) = M\}$. We first deduce some corollaries of the WMP.

Corollary 2.1. *Let Ω be a bounded, open set, and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Suppose that L is elliptic, with $\sup_\Omega |b|/\lambda < \infty$, where $b = (b_1, \ldots, b_n)$, and $c \leq 0$ in Ω. Then:*

(i) *If $Lu \leq 0$ in Ω, then* $\inf_\Omega u \geq \inf_{\partial\Omega} u^-$ *, where* $u^- = \min(u,0)$.

(ii) *If $Lu = 0$, then* $\sup_\Omega |u| = \sup_{\partial\Omega} |u|$.

Proof. (i): Apply the WMP to $-u$.

(ii): Combine (i) and the usual WMP, as here both $Lu \geq 0$ and $Lu \leq 0$ hold.

□

Corollary 2.2. *Let Ω and L be as above. Suppose that $u, v, w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy in Ω,*

$$Lu \geq 0, \qquad Lv = 0, \qquad Lw \leq 0.$$

Then,

(i) *If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.*

(ii) *If $v \leq w$ on $\partial\Omega$, then $v \leq w$ in Ω.*

Proof. (i): Then $L(u-v) \geq 0$, and so by the WMP, $u - v \leq \sup_{\partial\Omega}(u-v) \leq 0$ on Ω.

(ii): The same (or apply (i) to $-v$, $-w$).

□

> **Remark:** Hence we see the reason for the terminology:
>
> (i) If u satisfies $Lu \leq 0$ then it is called a **subsolution** of the equation $Lu = 0$ (as such functions are below all solutions of the DP, by the above).
>
> (ii) If u satisfies $Lu \leq 0$ then it is called a **supersolution** of the equation $Lu = 0$ (as such functions are above all solutions of the DP, by the above).

Now we work toward the SMP. We need the following lemma, to get the sign of terms.

Theorem 2.2 (The Hopf Boundary Point Lemma). *Let $\Omega \subset \mathbb{R}^n$ be open. Suppose $y \in \partial\Omega$, and suppose that Ω satisfies the* ***interior sphere condition*** *at y (i.e. $\exists R > 0$, $z \in \Omega$ such that $B_R(z) \subset \Omega$ and $y \in \partial B_R(z)$).*

Let $Lu = a^{ij}D_{ij}u + b^iD_iu + cu$ is uniformly elliptic in Ω, and suppose $\sup_\Omega |b|/\lambda + \sup_\Omega |c|/\lambda < \infty$. *Suppose that $u \in C^2(\Omega) \cap C^0(\{y\} \cup \Omega)$ satisifes $u(y) > u(x)$ $\forall x \in \Omega$, and $Lu \geq 0$ in Ω. Suppose also that one of the following holds:*

(i) $c \equiv 0 \in \Omega$,

(ii) *$c \le 0$ in Ω and $u(y) \ge 0$,*

(iii) *$u(y_0) = 0$ (no hypothesis on the sign of c).*

Then, *$\frac{\partial u}{\partial \nu}(y) > \mathbf{0}$ if the derivative exists, where ν is the outward pointing unit normal to $\partial B_R(y)$ to y (and $\frac{\partial u}{\partial \nu}(y) = \lim_{t\uparrow 0} \frac{u(y+t\nu)-u(y)}{t}$).*

Remark: The significance of this theorem is in the **strict** inequality of the conclusion ($\frac{\partial u}{\partial \nu}(y) \ge 0$ is trivially true, since $u(x) < u(y)$ for all $x \in \Omega$).

Proof. Let A be the annulus $B_R(z)\backslash B_r(z)$, for some $o < r < R$ (e.g. $r = R/2$).

Cases (i)+(ii): Consider the function on A defined by

$$v(x) = e^{-\alpha|x-z|^2} - e^{-\alpha R^2}.$$

Then, we clearly have $D_i v(x) = -2\alpha(x_i - z_i)\cdot e^{-\alpha|x-z|^2}$ and

$$D_{ij}v(x) = -2\alpha e^{-\alpha|x-z|^2}\delta_{ij} + 4\alpha^2(x_i-z_i)(x_j-z_j)e^{-\alpha|x-z|^2}.$$

So hence:

$$\begin{aligned}
Lv &= e^{-\alpha|x-z|^2}\left(\underbrace{\sum_{i,j=1}^n a^{ij}(x_i-z_i)(x_j-z_j)}_{\text{bound by ellipticity}} - 2\alpha\underbrace{\sum_{i=1}^n a^{ii}}_{=\text{Trace}(A)=\sum_i \lambda_i \le n\Lambda} - 2\alpha\sum_{i=1}^n b^i(x_i-z_i) + c\right) - \underbrace{ce^{-\alpha R^2}}_{\ge 0 \text{ by condition on } c}\\
&\ge e^{\alpha|x-z|^2}\left(4\alpha^2\lambda(x)\underbrace{|x-z|^2}_{\ge (R/2)^2 \text{ on } A} - 2\alpha n\Lambda(x) - 2\alpha|b||x-z| - |c|\right)\\
&\ge e^{-\alpha|x-z|^2}\lambda(x)\left(\alpha^2R^2 - 2\alpha n\cdot\sup_\Omega\left(\frac{\Lambda}{\lambda}\right) - \alpha R\cdot\sup_\Omega\frac{|b|}{\lambda} - \sup_\Omega\frac{|c|}{\lambda}\right)\\
&> 0 \quad \text{for } \alpha \text{ sufficiently large.}
\end{aligned}$$

Fix such an α so that this is > 0. Then let $w(x) = u(x) - u(y) + \varepsilon v(x)$, where $\varepsilon > 0$ is to be chosen. Then, $Lw = Lu + \varepsilon Lv - cu(y) \ge 0$ in A, by the above. Note also that as $v = 0$ on $\partial B_R(z)$ and $u(x) \le u(y)$ on $\overline{\Omega}$, we have that $w|_{\partial B_R(z)} \le 0$. Also, since $u(x) < u(y)$ on $\partial B_{R/2}(z)$ (which is compact and so $u - u(y)$, v attain their minima, which will be < 0 and > 0 respectively), we can choose $\varepsilon > 0$ small enough such that $w|_{\partial B_{R/2}(z)} < 0$. Fix this $\varepsilon > 0$.

So then $w|_{\partial A} \le 0$. So by applying the WMP to w in A, we get that $u(x) - u(y) + \varepsilon v(x) \le 0$ in A. So hence for $t < 0$ with $|t|$ small, we have (as $v(y) = 0$, since $y \in \partial B_R(z)$),

$$\frac{u(y+t\nu) - u(y)}{t} \ge -\varepsilon\cdot\frac{v(y+t\nu)-v(y)}{t}.$$

(The inequality flips as $t < 0$.) So hence letting $t \uparrow 0$, we get that:

$$\frac{\partial u}{\partial \nu}(y) \ge -\varepsilon\cdot\frac{\partial v}{\partial \nu}(y) = -\varepsilon Dv(y)\cdot\frac{y-x}{R} = 2\varepsilon\alpha Re^{-\alpha R^2} > 0$$

and so done.

Case (iii): Consider the operator $\tilde{L} := L - c^+$, so that

$$\tilde{L}u = a^{ij}D_{ij}u + b^iD_iu + \underbrace{(c-c^+)}_{\leq 0}u.$$

Thus, $\tilde{L}u = \underbrace{Lu}_{\geq 0 \text{ by assumption}} - \overbrace{c^+u}^{\leq c^+u(y)=0 \text{ here}} \geq 0$. Then simply apply case (ii) to $\tilde{L}$ to finish the proof.

□

From the Hopf boundary point lemma, the SMP is now an easy corollary.

Theorem 2.3 (The Strong Maximum Principle (SMP)). *Suppose $\Omega \subset \mathbb{R}^n$ is a domain (not necessarily bounded). Suppose that L is uniformly elliptic, with $\sup_\Omega(|b|/\lambda + |c|/\lambda) < \infty$. Suppose also that $u \in C^2(\Omega)$, $Lu \geq 0$ in Ω, and that $M := \sup_\Omega u < \infty$. Then:*

(i) *If $c = 0$ and $u(y) = M$ for some $y \in \Omega$, then $u \equiv M$ in Ω.*

(ii) *If $c \leq 0$, $M \geq 0$ and $u(y) = M$ for some $y \in \Omega$, then $u \equiv M$ in Ω.*

(iii) *If $M = 0$ any $u(y) = M = 0$ for some $y \in \Omega$, then $u \equiv 0$ in Ω.*

Proof. Let $\Sigma = \{x \in \Omega : u(x) = M\}$. Then by continuity of u, Σ is (relatively) closed in Ω. So by connectedness, it suffices to show that Σ is (relatively) open (as by assumption $\Sigma \neq \emptyset$).

So pick $z \in \Omega\backslash\Sigma$ s.t. $\text{dist}(z, \partial\Omega) > \text{dist}(z, \partial\Sigma)$ (i.e. first pick $z_1 \in \partial\Sigma \cap \Omega$, which we can do as Ω is connected. Then choose $\rho_1 > 0$ such that $B_{\rho_1}(z_1) \subset \Omega$. Then by the triangle inequality, choose our z to be any $z \in B_{\rho_1/2}(z_1)\backslash\Sigma$).

Then let $R = \sup\{\rho : B_\rho(y) \subset \Omega \subset \Sigma\}$ (the balls of increasing radii will intersect Σ first rather than $\partial\Omega$ by how z is chosen). Then $\exists\ y \in \partial B_R(z) \cap \Sigma \cap \Omega$. Then since $Du(y) = 0$ (as it is a maximum), we get a direct contradiction to the Hopf lemma (as this should be > 0). So we have a contradiction, and must have u constant in this ball, equal to M. But then this shows that $\partial\Sigma$ is open, and so Σ is open, and so done.

□

Now we can deduce some corollaries, including uniqueness of the solutions for the Neumann problem.

Corollary 2.3. *Let $L = a^{ij}D_{ij} + b^iD_i + c$ is uniformly elliptic in a domain $\Omega \subset \mathbb{R}^n$. with $\sup_\Omega(|b|/\lambda + |c|/\lambda) < \infty$. Suppose that $u, v \in C^2(\Omega)$ satisfy:*

$$Lu \geq Lv \quad \textit{and} \quad u \leq v \quad \textit{in } \Omega.$$

Then, $u = v$ on Ω, **or** *$u < v$ on all of Ω.*

Proof. Then we have $L(u-v) \geq 0$ in Ω and $u-v \leq 0$ in Ω. Then if $\exists\, x_0 \in \Omega$ with $u(x_0) = v(x_0)$, then by Case (iii) of the SMP, we have $u \equiv v$ in Ω. Otherwise we have $u \neq v$ in Ω, and so by have $u < v$ in Ω (as we have by assumption $u \leq v$).

□

Corollary 2.4 (Uniqueness of Solutions for the Neumann Problem (NP))**.** *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $\partial\Omega$ satisfies the* ***interior sphere condition*** *at each point. Suppose L is uniformly elliptic in Ω, with $\sup_\Omega (|b|/\lambda + |c|/\lambda) < \infty$, and $c \leq 0$. Then if $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy :*

$$\begin{cases} Lu_i = f & \text{in } \Omega \\ \frac{\partial u_i}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

for each i, for some functions $f : \Omega \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$ (i.e. the Neumann problem), then $u_1 = u_2 + c$, for some constant c.

Proof. By considering $u = u_1 - u_2$, it suffices to show that if

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

then $u = c$ is a constant.

So let $M = \sup_{\overline{\Omega}} u$. Assume wlog $M \geq 0$ (or else consider $-u$). Then by the SMP, if $u \not\equiv M$ on $\overline{\Omega}$, then $\exists\, y \in \partial\Omega$ such that $u(y) = M$ **and** $u(x) < u(y)$ for all $x \in \Omega$ (as $c \leq 0$).

Then by the Hopf boundary point lemma, we get that $\frac{\partial u}{\partial \nu}(y) \neq 0$, which is a contradiction as we have $\frac{\partial u}{\partial \nu} \equiv 0$ on $\partial\Omega$. So hence we must have $u \equiv M$ is constant in $\overline{\Omega}$.

□

> **Note:** As a consequence of this, we see that the solution to the NP with zero data must be a constant, and so we must have $L(M) = 0$, where M is this constant. Hence as the derivatives of a constant vanish, this says we must have $Mc(x) = 0$ for all $x \in \Omega$. So hence if $c(x) \neq 0$ for any $x \in \Omega$, we must have $M = 0$, and thus the solutions are unique. So it is only when $c \equiv 0$ that the solutions differ by a constant (which makes sense as then constants will always vanish, as then both L and the boundary data depend all on derivatives).

The following result is of critical importance in the theory we develop.

Theorem 2.4 (Maximum Principle a priori Estimate (MPAPE)).

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain, with L elliptic, $c \leq 0$, and $\beta := \sup_\Omega |b|/\lambda < \infty$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $f : \Omega \to \mathbb{R}$. Then:

(i) *If $Lu \geq f$ in Ω, then $\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C \cdot \sup_\Omega \left(\frac{|f|}{\lambda}\right)$.*

(ii) *If $Lu = f$ in Ω, then $\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C \cdot \sup_\Omega \left(\frac{|f|}{\lambda}\right)$*

where in both (i) *and* (ii), *$C = C(\beta, diam(\Omega))$.*

Proof. Let $d = \mathrm{diam}(\Omega)$. Then as Ω is bounded, we have $\Omega \subset \{x : a \leq x_1 \leq a + d\}$ for some $a \in \mathbb{R}$. We can assume wlog that $a = 0$ (or else just consider $\tilde{u}(x) = u(\tilde{x}), \tilde{f}(x) = f(\tilde{x})$, where $\tilde{x} = (x_1 + a_1, x_2, \ldots, x_n)$, for $x \in \tilde{\Omega} := \Omega - (a, 0, \ldots, 0)$).

So $\Omega \subset \{x : 0 \leq x_1 \leq d\}$ wlog. Then let

$$v(x) = \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \cdot \sup_\Omega \frac{|f|}{\lambda}$$

(this expression can be deduced by letting $v(x) = A + Be^{\alpha x_1}$ and working the calculation through to find what constants A and B we need). Then we can just compute:

$$(a^{ij} D_{ij} + bD_i)e^{\alpha x_1} = e^{\alpha x_1}(a^{11}\alpha^2 + b^1\alpha) \overset{a^{11} \geq \lambda \text{ by ellipticity}}{\geq} e^{\alpha x_1}\lambda\left(\alpha^2 + \frac{b^1}{\lambda} \cdot \alpha\right)$$
$$\underset{\text{as } b^1/\lambda \geq -\beta}{\geq} e^{\alpha x_1}\lambda\left(\alpha^2 - \beta\alpha\right) \geq \lambda$$

where the last inequality is true **if** $\alpha = \beta + 1$, since $e^{\alpha x_1} \geq 1$ as $\alpha, x_1 \geq 0$ here.

So then,

$$Lv \leq cv - \lambda \cdot \sup_\Omega \frac{|f|}{\lambda} \leq -\lambda \cdot \sup_\Omega \frac{|f|}{\lambda}$$

since $c \leq 0$, $v \geq 0$.

<u>(i)</u>: Then

$$L(u - v) \geq f + \lambda \cdot \sup_\Omega \frac{|f|}{\lambda} = \lambda\left(\frac{f}{\lambda} + \sup_\Omega \frac{|f|}{\lambda}\right) \geq 0$$

in Ω. Then since $u \leq u^+$, we see that $u(x) \leq v(x)$ for $x \in \partial\Omega$. So by the WMP, $u \leq v$ in Ω. So:

$$\sup_\Omega u \leq \sup_\Omega v \leq \sup_{\partial\Omega} u^+ + C \cdot \sup_\Omega \frac{|f|}{\lambda}$$

where $C = \sup_{x\in\Omega}\left(e^{(\beta+1)d} - e^{(\beta+1)x_1}\right) = C(\beta, d)$. So done with this case.

<u>(ii)</u>: Apply case (i) with $-u$ in place of u and then combine the results.

□

2.2. **Hölder Spaces.**

Let $\Omega \subset \mathbb{R}^n$ be open. Let $\alpha \in (0,1]$. Fix these throughout.

Definition 2.2. *We say that $u : \Omega \to \mathbb{R}$ is* ***uniformly Hölder continuous*** *with* ***exponent α*** *if:*

$$[u]_{\alpha;\Omega} := \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

This quantity is known as the ***Hölder semi-norm****.*

If $\alpha = 1$, then we say that u is ***uniformly Lipschitz****.*

> **Note:** If this were to hold with $\alpha > 1$, then u would be constant (easily seen by considering the derivative of u).

Definition 2.3. *We say that u is* ***locally Hölder continuous with exponent α*** *in Ω, if $u|_K : K \to \mathbb{R}$ is uniformly* Hölder *continuous with exponent α (in K), for every compact $K \subset \Omega$.*

Now let $k \in \mathbb{N} \cup \{\infty\} = \{0,1,\dots\} \cup \{\infty\}$. Then recall that we define:

$$\boldsymbol{C^k(\Omega)} := \left\{ u : \Omega \to R \,\middle|\, D^\beta u \text{ exists and is continuous in } \Omega \text{ for any multi-index } \beta \text{ with } |\beta| = \sum_i \beta_i \le k \right\}.$$

Then we define the **Hölder spaces** by:

$\boldsymbol{C^{k,\alpha}(\Omega)} := \{u \in C^k(\Omega) \mid D^\beta u$ is locally Hölder continuous with exponent α for all β with $|\beta| = k\}$

and

$\boldsymbol{C^{k,\alpha}\left(\overline{\Omega}\right)} := \{u \in C^k(\Omega) \mid D^\beta u$ is uniformly Hölder continuous in Ω with exponent α for all β with $|\beta| = k\}$.

If $\alpha \in (0,1)$, we write $C^\alpha(\Omega)$, $C^\alpha(\overline{\Omega})$ to denote $C^{0,\alpha}(\Omega)$, $C^{0,\alpha}(\overline{\Omega})$ respectively. Also also write $C^{k,0}(\Omega) := C^k(\Omega)$, and $C^{k,0}(\overline{\Omega}) := C^k(\overline{\Omega})$, where here $k \in \mathbb{N} \cup \{\infty\}$.

This then defines $C^{k,\alpha}(\Omega)$, $C^{k,\alpha}(\Omega)$ for any $k \in \mathbb{N} \cup \{\infty\}$, $\alpha \in [0,1]$.

> **Note:** We do not have $C^{k+1}(\Omega) = C^{k,1}(\Omega)$, e.g. $C^1 \neq C^{0,1}$ as Lipschitz functions are not continuously differentiable (e.g. $|x|$).

We then also define:

$$C_0^{k,\alpha}(\Omega) \equiv C_c^{k,\alpha}(\Omega) := \{u \in C^{k,\alpha}(\Omega) : \mathrm{supp}(u) \subset \Omega \text{ is compact (in } \Omega)\},$$

where $\mathrm{supp}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}$ is the support of u.

Now we define the norms and semi-norms we consider on these spaces. For $k \in \mathbb{N}$ and $u \in C^k(\overline{\Omega})$ we define:

$$[u]_{k;\Omega} \equiv |D^k u|_{0;\Omega} := \sup_{|\beta|=k} |D^\beta u|_{0;\Omega} = \sup_{|\beta|=k} \left(\sup_{x\in\Omega} |D^\beta u(x)| \right).$$

For $u \in C^{k,\alpha}(\overline{\Omega})$, we set
$$[u]_{k,\alpha;\Omega} \equiv [D^k u]_{\alpha;\Omega} := \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega}.$$
Note that these are only semi-norms on the respective spaces (as they vanish for constants). Then to get norms on each space, we set:
$$\|u\|_{C^k(\overline{\Omega})} \equiv |u|_{k;\Omega} \equiv |u|_{k,0;\Omega} := \sum_{j=0}^{k} |D^j u|_{0,\Omega}$$
and
$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} \equiv |u|_{k,\alpha;\Omega} := |u|_{k;\Omega} + [D^k u]_{\alpha;\Omega}.$$
One can also check that these spaces, equipped with the relevant norm, are Banach spaces.

We now study the compactness properties of these spaces. With all normed spaces, it is of critical importance to understand the compactness properties of the space, and when we can extract convergent subsequences. Of particular importance here, is the Arzela-Ascoli theorem.

Theorem 2.5 (Arzela-Ascoli). *Let $\Omega \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}$, and $\alpha \in (0,1]$. Then, if $(u_j)_j \subset C^{k,\alpha}(\Omega)$ are such that:*
$$\sup_j \left|u_j\right|_{k,\alpha;\Omega'} < \infty \quad \textit{for all } \Omega' \subset\subset \Omega,$$
then $\exists\ u \in C^{k,\alpha}(\Omega)$ and a subsequence $(u_{j'})_{j'}$ such that $u_{j'} \to u$ in $C^k(\overline{\Omega'})$, for all $\Omega' \subset\subset \Omega$.

Note: There is no conclusion/claim on the convergence in $C^{k,\alpha}(\overline{\Omega'})$!

Proof. Exercise (see example sheet 2 - it is good for the soul). □

Next we prove an interpolation inequality. This is an instance of a result, where is we have Banach spaces $X \subset\subset Y \subset Z$ each with their own norms, we can bounded the norm in Y by that in Z and X, but the norm in X can be made smaller by making the constant in front of the one in Z larger, etc. In this case, we have $C^{k,\alpha}(\overline{\Omega}) \subset\subset C^k(\overline{\Omega}) \subset C^0(\overline{\Omega})$. So we have:

Theorem 2.6 (Interpolation Inequality for Hölder Spaces). *Let $\varepsilon > 0$, $l \in \mathbb{Z}^+$, and $\alpha \in (0,1]$. Then, $\exists\ C = C(n,l,\alpha,\varepsilon) \in (0,\infty)$ such that, if $u \in C^{l,\alpha}(B_R(x_0))$, then:*
$$R^k |D^k u|_{0,B_R(x_0)} \leq \varepsilon R^{l+\alpha} \left[D^l u\right]_{\alpha;B_R(x_0)} + C \cdot |u|_{0;B_R(x_0)},$$
for all $k \in \{0,1,\ldots,l\}$.

Proof. The idea of the proof is as follows - full details can be found on Example Sheet 1.

First, by rescaling and so considering $v(x) = u(x_0 + Rx)$, it suffices to prove this lemma when $R = 1$, $x_0 = 0$, since we have
$$|D^k v|_{0;B_1(0)} = R^k |D^k u|_{0;B_R(x_0)}$$
$$|D^l v|_{\alpha;B_1(0)} = R^{l+\alpha} [D^l u]_{\alpha;B_R(x_0)}.$$
Then one can argue by contradiction using Arzela-Ascoli. □

We will also need the following abstract lemma before starting the Schauder theory.

Theorem 2.7 (Simon's Absorbing Lemma)**.** *Let $\lambda \in [0,\infty)$ and $\theta \in (0,1)$. Then, $\exists\ \delta = \delta(n,\lambda,\theta) \in (0,1)$ such that the following is true:*

Let $B_R(x)$ be a fixed ball in $\mathbb{R}^n$. Let S be a non-negative, subadditive function on the collection of sub-balls of $B_R(x)$, i.e. if $B_\rho(y) \subset \bigcup_{j=1}^N B_{\rho_j}(y_j) \subset B_R(x)$, then

$$S(B_\rho(y)) \leq \sum_{j=1}^N S(B_{\rho_j}(y_j)).$$

Suppose that for all balls $B_\rho(y) \subset B_R(x)$, we have $\rho^\lambda S(B_{\theta\rho}(y)) \leq \delta\rho^\lambda S(B_\rho(y)) + \gamma$, for some fixed γ. Then,

$$R^\lambda S(B_{\theta R}(x)) \leq C\gamma, \quad \textit{where } C = C(n,\theta,\lambda).$$

Note: This is essentially saying if we have some local bound on S, then we can 'absorb' the term dependent on S on the RHS to get a global bound.

Proof. Let

$$Q = \sup_{B_\rho(y)\subset B_R(x)} \rho^\lambda S(B_{\theta\rho}(y)).$$

Note that by sub-additivity of S we have $Q \leq R^\lambda S(B_R(x)) < \infty$, and so Q is finite.

Iterating our bound, with $\theta\rho$ in place of ρ, we have that whenever $B_\rho(y) \subset B_R(x)$,

$$\begin{aligned}(\theta\rho)^\lambda S(B_{\theta^2\rho}(y)) &\leq \delta(\theta\rho)^\lambda S(B_{\theta\rho}(y)) + \gamma \\ &\leq \delta\theta^\lambda Q + \gamma.\end{aligned}$$

So fix any $B_\rho(y) \subset B_R(x)$. Then cover $B_{\theta\rho}(y)$ by balls $\{B_{(1-\theta)\theta^2\rho}(z_j)\}_{j=1}^N$, with $N \leq C(\theta,n)$. To do this, choose a maximal, pairwise disjoint, collection of balls $\{B_{\frac{(1-\theta)\theta^2\rho}{2}}(z_j)\}_{j=1}^N$, subject to the condition that $z_j \in B_{\theta\rho}(y)$. Then these z_j work, since if not, then $\exists\, z \in B_{\theta\rho}(y) \setminus \bigcup_{j=1}^N B_{(1-\theta)\theta^2\rho}(z_j)$. But then we would have z is at least $(1-\theta)\theta^2\rho$ away from each z_j, and so $B_{\frac{(1-\theta)\theta^2\rho}{2}}(z) \cap B_{\frac{(1-\theta)\theta^2\rho}{2}}(z_j) = \emptyset$, which is a contradiction to maximality.

Then to see that bound on N, note that we also have

$$\bigcup_{j=1}^N B_{\frac{(1-\theta)\theta^2\rho}{2}}(z_j) \subset B_{\frac{(1-\theta)\theta^2\rho}{2}+\theta\rho}(y)$$

(just from considering radii), which gives us a volume bound. Indeed, as the balls on the LHS are disjoint of the same volume, we get:

$$\begin{aligned}N\omega_n \cdot \left(\frac{(1-\theta)\theta^2\rho}{2}\right)^n &\leq \omega_n\left(\frac{(1-\theta)\theta^2\rho}{2} + \theta\rho\right)^n \\ \implies N &\leq C(\theta,n).\end{aligned}$$

So hence we get, by sub-additivity,

$$S(B_{\theta\rho}(y)) \leq \sum_{j=1}^{N} S\left(B_{(1-\theta)\theta^2\rho}\left(z_j\right)\right),$$

and so

$$\begin{aligned}
\rho^{\lambda} S(B_{\theta\rho}(y)) &\leq \rho^{\lambda} \sum_{j=1}^{N} S\left(B_{(1-\theta)\theta^2\rho}\left(z_j\right)\right) \\
&\overset{\text{assumption with } \rho \to (1-\theta)\rho}{\leq} ((1-\theta)\theta)^{-\lambda} \cdot \sum_{j=1}^{N} \left(\delta \underbrace{((1-\theta)\theta\rho)^{\lambda} S(B_{(1-\theta)\theta\rho}\left(z_j\right))}_{\leq Q} + \gamma \right) \\
&\leq \delta((1-\theta)\theta)^{-\lambda} N Q + N\gamma((1-\theta)\theta)^{-\lambda},
\end{aligned}$$

where the terms in the sum are $\leq Q$ since $z_j \in B_{\theta\rho}(y)$, and so $B_{(1-\theta)\theta\rho}\left(z_j\right) \subset B_{\rho}(y) \subset B_R(x)$.

So taking the sup over all $B_{\rho}(y) \subset B_R(x)$, we get (as the RHS is independent of ρ, y)

$$Q \leq \delta C Q + C_1 \gamma$$

where C, C_1 only depend on n, θ, λ. So choose δ s.t. $\delta C = 1/2$. Then we get $Q \leq 2C_1\gamma$, and so in particular, we ge the result we are after.

□

3. Schauder Theory

3.1. Interior Schauder Estimates.

We start with proving the interior Schauder estimates. We prove the results in the unit ball, and extend them to more general domains afterward. The point is if the coefficients of L are α-Hölder continuous, then any $C^{2,\alpha}$ solution of $Lu = f$ can have its $C^{2,\alpha}$-norm **on a smaller ball** bounded by its supremum and a Hölder norm of f.

Theorem 3.1 (Unit Scale Interior Schauder Estimate).

Let $\alpha \in (0,1)$, and $\beta > 0$. Suppose $a^{ij}, b^i, c \in C^{0,\alpha}(B_1(0))$, with $\left|a^{ij}\right|_{0,\alpha;B_1(0)} + \left|b^i\right|_{0,\alpha;B_1(0)} + |c|_{0,\alpha;B_1(0)} \leq \beta$. Suppose also that L is strictly elliptic, so $\exists\, \lambda > 0$ fixed such that $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$, for all $x \in B_1(0)$ and $\xi \in \mathbb{R}^n$.

Then if $u \in C^{2,\alpha}(B_1(0)) \cap C^0(\overline{B_1(0)})$ and $f \in C^{0,\alpha}(B_1(0))$ satisfy $Lu = f$ in $B_1(0)$, then:

$$|u|_{2,\alpha;B_{1/2}(0)} \leq C\left(|u|_{0;B_1(0)} + |f|_{0,\alpha;B_1(0)}\right)$$

for some constant $C = C(n,\alpha,\beta,\lambda) < \infty$.

Note:

- We can never take $\alpha = 0$ or 1 in these Schauder estimates - see Example Sheet 1 for details of this.
- The strict ellipticity gives a lower bound on λ, the smallest eignvalue. The bound of $\left|a^{ij}\right|_{0,\alpha;B_1(0)}$ gives us an upper bound on Λ, the largest eigenvalue. Combining these shows that Λ/λ is bounded, and so under these assumptions, strict ellipticity $\Longrightarrow$ uniform ellipticity here.

Remark: This is a remarkable result for many reasons: we get control of u and its derivatives (up to 2nd order) just from $\sup_{B_1(0)}|u|$ and f! In fact:

(i) We will, using this, strengthen this conclusion to:
$$|u|_{2,\alpha;B_\theta(0)} \leq C\left(|u|_{0;B_1(0)} + |f|_{0,\alpha;B_1(0)}\right)$$
for any $\theta \in (0,1)$, provided C can also depend on θ.

(ii) This estimate gives a **compactness property for the space of solutions to $Lu = f$**, for given L, f.

Indeed,if $(u_k)_k \subset C^{2,\alpha}(B_1(0)) \cap C^0(\overline{B_1(0)})$ solve $Lu = f$ in $B_1(0)$, and if $\gamma := \sup_k\left(\sup_{B_1(0)}|u_k|\right) < \infty$, then the estimates give for each $\theta \in (0,1)$,
$$|u_k|_{2,\alpha;B_\theta(0)} \leq C(\gamma, n, \theta, \beta, \lambda, f).$$
So by Arzela-Ascoli, $\exists$ a subsequence (k') and $u \in C^{2,\alpha}(B_1(0))$ such that $u_{k'} \to u$ in $C^2(B_\theta(0))$, for any $\theta \in (0,1)$ (this requires a diagonal argument: we get a subsequence for each θ, and then can take a suitable diagonal subsequence).

Then we can pass to the limit pointwise in $Lu_k = f$ to see that $Lu = f$ as well.

(iii) This also assumes nothing about the C^2 norm of u near the boundary, and makes no "up-to-the-boundary" conclusion.

(iv) The theorem is false if $\alpha = 0$ or $\alpha = 1$, as mentioned above.

Proof. We shall write $B_r := B_r(0)$ throughout for ease of notation.

By working in a slightly smaller ball than B_1, we can assume wlog that $|u|_{2;B_1} < \infty$. The proof will consist of several steps: first a reduction step, then a contradiction step, before lastly finding a PDE satisfied by a certain function, which will be easier to work with. We first claim that:

Step 1: Reduction step.

Claim: It suffices to prove the following: For any given $\delta \in (0,1)$, $\exists$ C such that

$$\left[D^2u\right]_{\alpha;B_{1/2}} \leq \delta\left[D^2u\right]_{\alpha;B_1} + C\left(|u|_{2;B_1} + |f|_{0,\alpha;B_1}\right). \tag{3.1}$$

Proof of Claim. Suppose (3.1) were true. Then by the interpolation inequality, we have

$$\left[D^2u\right]_{\alpha;B_{1/2}} \leq 2\delta\left[D^2u\right]_{\alpha;B_1} + C\left(|u|_{0;B_1} + |f|_{0,\alpha;B_1}\right). \tag{3.2}$$

Then by the absorbing lemma, choosing $\delta = \delta(n,\alpha)$ small enough, we get that the following is true:

"Given any sub-ball $B_\rho(y) \subset B_1(0)$, let $\tilde{u}(x) = u(y+\rho x)$. Then $\tilde{u}$ satisfies in $B_1(0)$ the equation

$$\tilde{a}^{ij}D_{ij}\tilde{u} + \tilde{b}^iD_i\tilde{u} + \tilde{c}\tilde{u} = \tilde{f}$$

where $\tilde{a}^{ij}(x) = a^{ij}(y+\rho x)$, $\tilde{b}^i(x) = \rho b^i(y+\rho x)$, $\tilde{c}(x) = \rho^2 c(y+\rho x)$, and $\tilde{f}(x) = \rho^2 f(y+\rho x)$. Also,

$$\begin{aligned}\left|\tilde{a}^{ij}\right|_{0,\alpha;B_1} + \left|\tilde{b}^i\right|_{0,\alpha;B_1} + |\tilde{c}|_{0,\alpha;B_1} &\leq |a^{ij}|_{0;B_\rho(y)} + \rho^\alpha\left[a^{ij}\right]_{\alpha;B_\rho(y)} + \rho|b^i|_{0;B_\rho(y)} + \rho^{1+\alpha}\left[b^i\right]_{\alpha;B_\rho(y)} + \cdots \\ &\leq \beta."\end{aligned}$$

Also, clearly we have $\tilde{a}^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$, so this PDE is still strictly elliptic.
So we can apply (3.2) with $\tilde{u}$ in place of u and $\tilde{f}$ in place of f to get (after changing variables back to u):

$$\begin{aligned}\rho^{2+\alpha}\left[D^2u\right]_{\alpha;B_{\rho/2}(y)} &\leq 2\delta\rho^{2+\alpha}\left[D^2u\right]_{\alpha;B_\rho(y)} + C\left(|u|_{0;B_\rho(y)} + \rho^2|f|_{0;B_\rho(y)} + \rho^{2+\alpha}[f]_{\alpha;B_\rho(y)}\right) \\ &\leq 2\delta\rho^{2+\alpha}\left[D^2u\right]_{\alpha;B_\rho(y)} + \underbrace{C\left(|u|_{0;B_1} + |f|_{0,\alpha;B_1}\right)}_{=:\gamma,\ \text{independent of } B_\rho(y)\subset B_1(0)}\end{aligned}$$

where for the last inequality we have used that $B_\rho(y) \subset B_1(0)$. So by the absorbing lemma, we get (after choosing δ),

$$\left[D^2u\right]_{\alpha;B_{1/2}} \leq C\underbrace{\left(|u|_{0;B_1} + |f|_{0,\alpha;B_1}\right)}_{=\gamma \text{ from above}}$$

where $C = C(n,\alpha,\lambda,\beta)$, which (after using the interpolation inequality again) is the conclusion of the theorem. So indeed, we have shown that it suffices to prove (3.1). ■

<u>**Step 2:** Argue by contradiction and use Arzela-Ascoli.</u>

If (3.1) does not hold for any $C > 0$, then $\exists\ \delta$ such that $\forall\ k \in \mathbb{N}$, we can find a_k^{ij}, b_k^i, c_k with $\left|a_k^{ij}\right|_{0,\alpha;B_1} + \left|b_k^i\right|_{0,\alpha;B_1} + |c_k|_{0,\alpha;B_1} \leq \beta$ (the same β for each k), with $a_k^{ij}\xi^i\xi^j \geq \lambda|\xi|^2$, and $u_k \in C^{2,\alpha}(B_1) \cap C^0(\overline{B_1})$, solving $L_k u_k = f_k$, for some $f_k \in C^{0,\alpha}(B_1)$, but that

$$\left[D^2u_k\right]_{\alpha;B_{1/2}} > \delta\left[D^2u_k\right]_{\alpha;B_1} + k\left(|u_k|_{2;B_1} + |f_k|_{0,\alpha;B_1}\right). \tag{3.3}$$

Then, by passing to a subsequence of (k), by definition of $\left[D^2u_k\right]_{\alpha;B_{1/2}}$, we can wlog assume $\exists\ x_k, y_k \in B_{1/2}$ and fixed $l, m \in \{1,\ldots,n\}$ (i.e. the same for the entire subsequence) such that

$$\frac{|D_{lm}u_k(x_k) - D_{lm}u_k(y_k)|}{|x_k - y_k|^\alpha} > \frac{1}{2}\left[D^2u_k\right]_{\alpha;B_{1/2}} \quad \forall\ k.$$

[This is simply because for each k, we can find some l, m and x_k, y_k such that this is true. So, as there are only finitely many choices of l, m, we can find some subsequence where the same l, m are used: so restrict to that subsequence.]

Then let $\rho_k = |x_k - y_k|$. Then, by the triangle inequality,

$$\frac{1}{2}\left[D^2u_k\right]_{\alpha;B_{1/2}} < \frac{|D_{lm}u_k(x_k)| + |D_{lm}u_k(y_k)|}{\rho_k^\alpha} \leq \frac{2|u_k|_{2;B_1}}{\rho_k^\alpha} \overset{\text{by (3.3)}}{\leq} \frac{2\left[D^2u_k\right]_{\alpha;B_{1/2}}}{k\rho_k^\alpha}$$

which in particular shows that $\rho_k^\alpha < 4/k$, and so hence $\rho_k \to 0$.

The process from here on is simple: we rescale and subtract the 2-jet from u to get a bounded function which vanishes at 0. These functions will then have some limit, which by all the assumptions will satisfy a Laplace-like equation, which we can work with.

So rescale as follows: set

$$\tilde{u}_k(x) := \frac{u_k(x_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha}[D^2u_k]_{\alpha;B_1}},$$

where $q_k(x) = u_k(x_k) + \rho_k x \cdot Du_k(x_k) + \frac{\rho_k^2}{2}\sum_{i,j} D_{ij}u_k(x_k)x^ix^j$ is the **2-jet** of u_k from the origin (i.e. up to 2nd order Taylor expansion). The $\rho_k^{2+\alpha}$ factor in the definition of $\tilde{u}_k$ is a scaling factor from differentiation $\tilde{u}$. Also, note that here $x = (x^1,\ldots,x^n)$ is a variable, which is completely different to the x_k, which are chosen points.

So clearly by definition of q_k (or just by direct calculation) we have

$$\tilde{u}_k(0) = 0, \quad D\tilde{u}_k(0) = 0, \quad D^2\tilde{u}_k(0) = 0,$$

and also by direct calculation, we see that

$$\left[D^2\tilde{u}_k\right]_{\alpha;B_{1/2\rho_k}(0)} \leq 1$$

noting that $\tilde{u}_k$ is defined on $B_{1/\rho_k}\left(-\frac{x_k}{\rho_k}\right) \supset B_{1/2\rho_k}(0)$.

Hence for any $R > 1$, we have

$$|\tilde{u}_k|_{2,\alpha;B_R(0)} \leq CR^{2+\alpha}$$

(since the ρ_k converge and so are bounded). So hence by Arzela-Ascoli, by passing to a subsequence, we get that $\exists\ v \in C^{2,\alpha}(\mathbb{R}^n)$ such that $\tilde{u}_k \to v$ in $C^2(B_R(0))$ for all $R > 0$, and with $[D^2 v]_{\alpha;\mathbb{R}^n} \leq 1$ (by taking a limit in these Hölder norm expressions). [Note that this is the same v for every R].

Step 3: Find the PDE satisfied by v.

Note first that $w_k(x) = u_k(x_k + \rho_k x)$ satisfies

$$\tilde{L}_k w_k = \tilde{f}_k \quad \text{in } B_{1/2\rho_k}(0)$$

where

$$\tilde{L}_k = \tilde{a}_k^{ij} D_{ij} + \tilde{b}_k^i D_i + \tilde{c}_k$$

where the summation is over i, j and not k, where $\tilde{a}_k^{ij}$, etc, are as before. Then since we have

$$w_k(x) = \rho_k^{2+\alpha} \left[D^2 u_k\right]_{\alpha;B_1} \tilde{u}_k(x) + q_k(x),$$

we see that $\tilde{u}_k$ satisfies

$$\tilde{L}_k \tilde{u}_k = g_k$$

where

$$g_k := \frac{\tilde{f}_k - \tilde{L}_k q_k}{\rho_k^{2+\alpha}[D^2 u_k]_{\alpha;B_1}}.$$

Now we shall see that all of these coefficients in $\tilde{L}$ converge uniformly to constants, and thus when we pass to the limit, v will satisfy a PDE with constant coefficients and constant inhomogeneous term.

So note that

$$\left[\tilde{a}_k^{ij}\right]_{\alpha;B_{1/2\rho_k}(0)} \overset{\text{containment + scaling}}{\leq} \rho_k^\alpha \left[a_k^{ij}\right]_{\alpha;B_1} \leq \rho_k^\alpha \beta \to 0$$

as $k \to \infty$, since we know $\rho_k \to 0$. Moreover we need $\alpha > 0$ for this. If $\alpha = 0$, then instead we just have $|\tilde{a}_k^{ij}|_{0;B_{1/2\rho_k}(0)} \leq \beta$, which tells us that $|\tilde{a}_k^{ij}|_{0,\alpha;B_R(0)} \leq C(R)$ for all $R > 0$, for k sufficiently large. Hence we can use Arzela-Ascoli on the $\tilde{a}_k^{ij}$. Hence we get that $\tilde{a}_k^{ij} \to \tilde{a}^{ij}$ as $k \to \infty$ for some $\tilde{a}^{ij}$, and the limit has Hölder semi-norm 0, $\forall \alpha > 0$ (by the above computation), and thus is constant (the convergence is locally uniformly).

Also, $|\tilde{b}_k^i|_{0;B_{1/2\rho_k}(0)} \leq \beta\rho_k \to 0$, and $|\tilde{c}_k|_{0;B_{1/2\rho_k}(0)} \leq \beta\rho_k^2 \to 0$, and so hence we have $\tilde{b}_k^i \to 0$ and $\tilde{c}_k \to 0$ locally uniformly on $\mathbb{R}^n$.

Claim: $g_k \to 0$ locally uniformly.

Proof of Claim: Note that we have (slightly disgusting expression - there might be a nicer way of doing this?)

$$g_k(x) = \frac{\rho_k^2 f_k(x_k') - \left(\rho_k^2 a^{ij}(x_k') D_{ij} u_k(x_k) + \rho_k^2 b_k^i(x_k')(D_i u_k(x_k) + D_{is} u_k(x_k) x^s) + \rho_k^2 c_k(x_k') q_k(x)\right)}{\rho_k^{2+\alpha}[D^2 u_k]_{\alpha;B_1}}$$

where $x_k' = x_k + \rho_k x$, and so summation is over all indices but k. So hence, using the fact that $f_k(x_k) = a_k^{ij}(x_k) D_{ij} u_k(x_k) + b_k^i(x_k) D^i u_k(x_k) + c_k(x_k) u_k(x_k)$, adding and subtracting this in the numerator, using the triangle inequality, and then bounding the terms involving u gives

$$|g_k(x)| \leq \frac{\rho_k^2\left(|f_k(x_k') - f_k(x_k)| + |\left(\left(a^{ij}(x_k') - a^{ij}(x_k)\right)\cdots\right)| \cdot |u_k|_{2;B_1}\right)}{\rho_k^{2+\alpha}[D^2u_k]_{\alpha;B_1}}.$$

Then using the bounds on the Hölder norms of the coefficients, this gives

$$|g_k(x)| \leq \frac{\rho_k^\alpha\left(|f_k|_{\alpha;B_1} + \left(\left[a_k^{ij}\right]_{\alpha;B_1} + \left[b_k^i\right]_{\alpha;B_1} + [c_k]_{\alpha;B_1}\right)|u_k|_{2;B_1}\right)}{\rho_k^\alpha[D^2u_k]_{\alpha;B_1}}$$

$$\leq \frac{|f_k|_{\alpha;B_1} + \beta|u_k|_{2;B_1}}{[D^2u_k]_{\alpha;B_1}} \overset{\text{by (3.3)}}{<} \frac{1+\beta}{k} \to 0.$$

So hence $g_k \to 0$ locally uniformly.

■

With all of this, we can pass to the limit in $\tilde{L}_k\tilde{u}_k = g_k$ to get

$$\tilde{a}^{ij}D_{ij}v = 0 \quad \text{in } \mathbb{R}^n,$$

where the a^{ij} are constants. Also, by passing to the limit in the ellipticity condition (thus it is crucial that this is **strictly** elliptic), we see that $\tilde{a}^{ij}\xi^i\xi^j \geq \lambda|\xi|^2$ as well. So the eigenvalues of this matrix are positive, and so we can diagonalise $A = (\tilde{a}^{ij})_{ij}$. So write

$$PAP^T = Q = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_i \geq \lambda > 0$ for all i. So now let $w(x) = v(P^Tx)$, i.e. $v(x) = w(Px)$. Then since $D^2v(x) = P^TD^2w(Px)P$, our equation becomes:

$$0 = \operatorname{trace}(A \cdot D^2v) = \operatorname{trace}(P^TQPP^T \cdot D^2w(Px)\cdot P) = \operatorname{trace}(Q \cdot D^2w(Px)) = \sum_{i=1}^n \lambda_iD_{ii}w(Px).$$

So as P gives a linear isomorphism $\mathbb{R}^n \to \mathbb{R}^n$, this is the same as

$$\sum_{i=1}^n \lambda_iD_{ii}w(x) = 0 \quad \text{for all } x \in \mathbb{R}^n.$$

So if we rescale via $\tilde{w}(x) = w(\sqrt{\lambda_1}x_1, \ldots, \sqrt{\lambda_n}x_n)$, then we have $\Delta\tilde{w} = 0$ in $\mathbb{R}^n$. We also have $\left[D^2\tilde{w}\right]_{\alpha;\mathbb{R}^n} < \infty$, and so hence we have $\tilde{w} \in C^\infty(\mathbb{R}^n)$. So hence for any $i,j \in \{1,\ldots,n\}$, we have $\Delta(D_{ij}\tilde{w}) = 0$ in $\mathbb{R}^n$.

But by the Hölder continuity condition, we know that

$$|D_{ij}\tilde{w}(x)| \leq |D_{ij}\tilde{w}(0)| + \left[D^2\tilde{w}\right]_{\alpha;\mathbb{R}^n}\cdot|x|^\alpha$$

i.e. $D_{ij}\tilde{w}$ grows sub-linearly, as $\alpha < 1$. So hence Liouville's theorem for harmonic functions implies that $D_{ij}\tilde{w} = \text{constant}$, for all i,j. Hence $D_{ij}v = \text{constant}$ as well, for each i,j. But we know $D^2v(0) = 0$ from taking the limit in $D^2\tilde{u}_k(0) = 0$, and thus we have $D^2v = 0$ identically.

But on the other hand, consider $\zeta = \frac{y_k - x_k}{\rho_k}$. Then $|\zeta_k| = 1$, and $u_k(x_k + \rho_k \zeta_k) = u_k(y_k)$. So hence:

$$|D_{lm}\tilde{u}_k(\zeta_k)| = \left| \frac{\rho_k^2 (D_{lm}u_k(y_k) - D_{lm}u_k(x_k))}{\rho_k^{2+\alpha}[D^2 u_k]_{\alpha;B_1}} \right| \overset{\text{by choice of } x_k, y_k}{\geq} \frac{1}{2} \cdot \frac{\left[D^2 u_k\right]_{\alpha;B_{1/2}}}{[D^2 u_k]_{\alpha;B_1}} \overset{\text{by (3.3)}}{>} \frac{\delta}{2}.$$

So hence as S^{n-1} is compact, and $\zeta_n \in S^{n-1}$ for every k, we can pass to an appropriate subsequence where we have a limit ζ, and then hence taking the limit in this inequality above gives

$$|D_{lm}v(\zeta)| \geq \frac{\delta}{2}$$

but then this directly contradicts $D_{lm}v$ being identically 0, so we have our contradiction, so done.

□

Now we give some corollaries of this result.

Corollary 3.1 (Scale Invariant Interior Schauder Estimate). *Suppose $B_R(x_0) \subset \mathbb{R}^n$, and $a^{ij}, b^i, c \in C^{0,\alpha}(B_R(x_0))$ are such that $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for some fixed $\lambda > 0$, for all $x \in B_R(x_0), \xi \in \mathbb{R}^n$. Suppose also that*

$$|a^{ij}|_{0;B_R(x_0)} + R^\alpha \left[a^{ij}\right]_{\alpha;B_R(x_0)} + R\left(|b^i|_{0;B_R(x_0)} + R^\alpha \left[b^i\right]_{\alpha;B_R(x_0)}\right) + R^2 \left(|c|_{0;B_R(x_0)} + R^\alpha [c]_{\alpha;B_R(x_0)}\right) \leq \beta$$

for some β. Suppose also that $u \in C^{2,\alpha}(B_R(x_0)) \cap C^0(\overline{B_R(x_0)})$ satisfies $Lu = f$ in $B_R(x_0)$ for some $f \in C^{0,\alpha}(B_R(x_0))$.

Then, we have

$$|u|'_{2,\alpha;B_R(x_0)} \leq C\left(|u|_{0;B_R(x_0)} + R^2\left(|f|_{0;B_R(x_0)} + R^\alpha [f]_{\alpha;B_R(x_0)}\right)\right),$$

where $C = C(n, \lambda, \alpha, \beta)$ is independent of u, R, where

$$|u|_{k,\alpha;B_\rho(y)} := \sum_{j=0}^{k} \rho^j |D^j u|_{0;B_\rho(y)} + \rho^{k+\alpha}\left[D^k u\right]_{\alpha;B_\rho(y)}.$$

Proof. Apply Theorem 3.1 (Interior Schauder estimates) with $\tilde{a}^{ij}, \tilde{b}^i, \tilde{c}, \tilde{u}, \tilde{f}$ in place of the relevant quantities, where for a function g we set $\tilde{g}(x) := g(x_0 + Rx)$. □

Now we can further extend these interior estimates to arbitrary domains, instead of just balls.

Corollary 3.2 (Interior Schauder Estimates in General Domains). *Let $\alpha \in (0,1)$, and let $\Omega \subset \mathbb{R}^n$ be an open, bounded set. Suppose that $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$, where*

$$\left|a^{ij}\right|_{0,\alpha;\Omega} + \left|b^i\right|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta,$$

with $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$ for some $\lambda > 0$, for all $x \in \Omega$, $\xi \in \mathbb{R}^n$. Suppose also that $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Lu = f$ in Ω, for some $f \in C^{0,\alpha}(\Omega)$.

Then $\forall\ \tilde{\Omega} \subset\subset \Omega$, we have that

$$|u|_{2,\alpha;\tilde{\Omega}} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right),$$

where $C = C(n,\alpha,\beta,\lambda, dist(\tilde{\Omega},\partial\Omega))$.

Proof. Let $d = \text{dist}(\tilde{\Omega},\partial\Omega) := \sup\{r > 0 : (\tilde{\Omega})_r \subset \Omega\}$, where $(\tilde{\Omega})_r = \bigcup_{x\in\tilde{\Omega}} B_r(x)$ is the r-neighbourhood of $\tilde{\Omega}$.

Then for any $x \in \tilde{\Omega}$, we have $B_d(x) \subset \Omega$, and
$$|a^{ij}|'_{0,\alpha;B_d(x)} + d|b^i|'_{0,\alpha;B_d(x)} + d^2|c|_{0,\alpha,B_d(x)} \leq C(d)\beta$$
for some constant C. Then by Corollary 3.1, we get:
$$|u|_{0;B_{d/2}(x)} + d|Du|_{0;B_{d/2}(x)} + d^2|D^2u|_{0;B_{d/2}(x)} \leq C\left(|u|_{0;B_d(x)} + d^2|f|_{0;B_d(x)} + d^{2+\alpha}[f]_{\alpha;B_d(x)}\right)$$
$$\leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right) \tag{3.4}$$
where $C = C(n,\lambda,\alpha,\beta,d)$. In particular, we have
$$|u(x)| + |Du(x)| + |D^2u(x)| \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right)$$
for all $x \in \tilde{\Omega}$. So, hence this says that we have
$$|u|_{2;\tilde{\Omega}} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right).$$
Then by (3.4), we also have that
$$\sup_{x,y\in\tilde{\Omega}\,:\,x\neq y,\ |x-y|<d/2} \frac{|D_{ij}u(x) - D_{ij}u(y)|}{|x-y|^\alpha} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right).$$

But also, if $x, y \in \tilde{\Omega}$ are such that $|x-y| \geq d/2$, then
$$\frac{|D_{ij}u(x) - D_{ij}u(y)|}{|x-y|^\alpha} \leq \left(\frac{d}{2}\right)^{-\alpha} \cdot \left(|D_{ij}u(x)| + |D_{ij}u(y)|\right) \leq \left(\frac{d}{2}\right)^{-\alpha} |u|_{2;\tilde{\Omega}}$$
$$\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\tilde{\Omega}}),$$
where $C = C(n,\lambda,\alpha,\beta,d)$. So hence combining these we see that
$$\left[D^2u\right]_{\alpha;\tilde{\Omega}} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right).$$
So hence combining we get the result.

□

3.2. **Boundary Schauder Estimates.**

Now we aim to prove boundary Schauder estimates. First we set up some notation.

Notation: We shall write
$$\mathbb{R}^n_+ := \{(x', x^n) : x' \in \mathbb{R}^{n-1}, x^n > 0\}.$$
$$\mathbb{R}^n_- := \{(x', x^n) : x' \in \mathbb{R}^{n-1}, x^n < 0\}.$$
$$B^\pm_R(y) := B_R(y) \cap \mathbb{R}^n_\pm.$$
$$B^\pm_R := B^\pm_R(0).$$
$$S_R(y) = B_R(y) \cap \{x_n = 0\}.$$
$$S_R = S_R(0).$$

Theorem 3.2 (Boundary Schauder Estimate in the Unit Ball). *Suppose* $0 < \alpha < 1$, $a^{ij}, b^i, c \in C^{0,\alpha}\left(B_1^+\right)$, *and* $\left|a^{ij}\right|_{0,\alpha;B_1^+} + \left|b^i\right|_{0,\alpha;B_1^+} + |c|_{0,\alpha;B_1^+} \leq \beta$. *Suppose also that* $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$ *for some fixed* $\lambda > 0$, *for all* $x \in B_1^+$, $\xi \in \mathbb{R}^n$. *Suppose* $u \in C^{2,\alpha}\left(B_1^+\right)$ *satisfies*

$$\begin{cases} Lu = f & \text{in } B_1^+ \\ u = \varphi & \text{on } S_1, \end{cases}$$

for some $f \in C^{0,\alpha}\left(B_1^+\right)$, $\varphi \in C^{2,\alpha}\left(B_1^+\right)$.

Then, we have

$$|u|_{2,\alpha;B_{1/2}^+} \leq C\left(|u|_{0;B_1^+} + |f|_{0,\alpha;B_1^+} + |\varphi|_{2,\alpha;B_1^+}\right)$$

where $C = C(n, \lambda, \alpha, \beta)$.

Proof. By considering $v = u - \varphi$, it suffices to prove the theorem with $\varphi \equiv 0$ (as the coefficients for the equation for v are still of this form, and the bound for v implies it for u).

So first we will prove the following:

Claim: $\forall\ \delta > 0$, $\exists\ C = C(n, \alpha, \lambda, \beta, \delta)$ such that if the hypotheses are as in the theorem, then

$$\left[D^2u\right]_{\alpha;B_{1/2}^+} \leq \delta\left[D^2u\right]_{\alpha;B_1^+} + C\left(|u|_{2;B_1^+} + |f|_{0,\alpha;B_1^+}\right).$$

Proof of Claim: To prove this, we argue by contradiction. Suppose it was not true. Then by passing to a subsequence, as before (in the interior case proof) we get that $\exists\ x_k, y_k \in B_{1/2}^+$ and $u_k \in C^{2,\alpha}(B_1^+)$ such that the hypotheses are satisfied with u_k in place of u, and with some $a_k^{ij}, b_k^i, c_k, f_k \in C^{0,\alpha}\left(B_1^+\right)$ in place of a^{ij}, b^i, c, f, and with, for some fixed $l, m \in \{1, \ldots, n\}$,

$$\frac{|D_{lm}u_k(x_k) - D_{lm}u_k(y_k)|}{|x-y|^\alpha} > \frac{1}{2}\left[D^2u_k\right]_{\alpha;B_{1/2}^+},$$

and,

$$\left[D^2u_k\right]_{\alpha;B_{1/2}^+} > \delta\left[D^2u_k\right]_{\alpha;B_1^+} + k\left(|u_k|_{2;B_1^+} + |f_k|_{0,\alpha;B_1^+}\right).$$

Then as before, we see that $\rho_k := |x_k - y_k| \to 0$ and $k \to \infty$. So we have two cases:

Either: (1) $\limsup_{k\to\infty} \frac{\text{dist}(x_k, S_1)}{\rho_k} = \infty$
or: (2) $\limsup_{k\to\infty} \frac{\text{dist}(x_k, S_1)}{\rho_k} = \mu < \infty$.

We consider each case seperately.

Case (1): For any R and sufficiently large k, we have $\text{dist}(x_k, S_1) \geq R\rho_k$, and so hence (as $x_k \in B_{1/2}^+$) we then have $B_{R\rho_k}(x_k) \subset B_1^+$.

So set

$$\tilde{u}_k(x) = \frac{u_k(x_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha}[D^2u_k]_{\alpha;B_1^+}}$$

where

$$q_k(x) := u_k(x_k) + \rho_k Du_k(x_k)\cdot x + \frac{\rho_k^2}{2}\cdot D_{ij}u_k(x_k)x^ix^j$$

where we are implicitly summing over i,j.

Note that then $\tilde{u}_k$ is defined in $B_R(0)$, and satisfies $|\tilde{u}_k|_{2,\alpha;B_R(0)} \leq C(R)$ for sufficiently large k. Then using the same argument as before, in Theorem 3.1, this case is done.

Case (2): Let $z_k = \mathrm{proj}_{\{x^n=0\}}(x_k)$ be the projection of x_k onto the x^n-axis. So, $z_k = (x_k^1,\dots,x_k^{n-1},0)$. Then subtract off the 2nd order terms as usual (to get the $C^{2,\alpha}$ bounds), so set

$$\tilde{u}_k(x) = \frac{u_k(z_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha}[D^2u_k]_{\alpha;B_1^+}}$$

where $q_k(x) = u_k(z_k) + \rho_k Du_k(z_k)\cdot x + \frac{\rho_k^2}{2}\sum_{i,j=1}^n D_{ij}u_k(z_k)x^ix^j$. So in particular, $\left[D^2\tilde{u}_k\right]_{\alpha;B_R^+(0)} \leq 1$.

Then as before, for any $R > 0$, we have $|\tilde{u}_k|_{2,\alpha;B_R^+(0)} \leq C(R)$ for sufficiently large k, and $\tilde{u}_k|_{S_R} = 0$, since in fact we have

$$q_k(x) = \rho_k D_n u_k(z_k)x^n + \rho_k^2\sum_{j=1}^n D_{nj}u_k(z_k)x^nx^j$$

since $u|_{S_R} = 0$, so the only non-zero derivatives must have a derivative in the x^n-direction.

Also, if we set $\xi_k = \frac{x_k - z_k}{\rho_k}$, $\eta_k = \frac{y_k - z_k}{\rho_k}$, then we have (at least for k sufficiently large) $|\xi_k| \leq 2\mu$ and

$$|\eta_k| \leq \frac{|x_k - y_k| + |x_k - z_k|}{\rho_k} = 1 + \frac{|x_k - z_k|}{\rho_k} \leq 1 + 2\mu.$$

So hence these sequences are bounded, and lie in compact subsets of $\mathbb{R}^n$, and so we can find convergent subsequences for both (which have the same index, wlog). So passing to this subsequence, we can wlog assume that $\xi_k \to \xi$ and $\eta_k \to \eta$. Then we clearly have (just by plugging into the expressions) that

$$D_{lm}\tilde{u}_k(\xi_k) = \frac{D_{lm}u_k(x_k) - D_{lm}u_k(z_k)}{\rho_k^\alpha[D^2u_k]_{\alpha;B_1^+(0)}}$$

and

$$D_{lm}\tilde{u}_k(\eta_k) = \frac{D_{lm}u_k(y_k) - D_{lm}u_k(z_k)}{\rho_k^\alpha[D^2u_k]_{\alpha;B_1^+(0)}}.$$

So hence,

$$|D_{lm}\tilde{u}_k(\xi_k) - D_{lm}\tilde{u}_k(\eta_k)| = \frac{|D_{lm}u_k(x_k) - D_{lm}u_k(y_k)|}{\rho_k^\alpha[D^2u_k]_{\alpha;B_1^+(0)}} \geq \frac{\frac{1}{2}\left[D^2u_k\right]_{\alpha;B_{1/2}(0)^+}}{[D^2u_k]_{\alpha;B_1(0)^+}} \geq \delta/2. \tag{3.5}$$

Then by Arzela-Ascoli, we get that $\exists\ v \in C^{2,\alpha}(\mathbb{R}^n_+ \cup \{x^n = 0\})$ such that, by passing to an appropriate subsequence, we have $\tilde{u}_k \to v$ in C^2 on compact subsets of $\mathbb{R}^n_+ \cup \{x^n = 0\}$. As before, we find that v satisfies an elliptic equation of the form: $\tilde{a}^{ij}D_{ij}v = 0$, where the $\tilde{a}^{ij}$ are constants in $\mathbb{R}^n_+$. Also, $v = 0$ on $\{x^n = 0\}$.

Then again, by rotation and scaling of constants, we get that we have $w \in C^2(\overline{H})$ (H the upper half-plane, $\{x^n > 0\}$) with

$$\begin{cases} \Delta w = 0 & \text{in a half-space } H \\ w|_{\partial H} = 0. \end{cases}$$

So w is harmonic. Then by making an odd reflection on ∂H for w, we can extend w to a harmonic function $\tilde{w}$ on all of $\mathbb{R}^n$ (see below for details of this). Also, $\left[D^2\tilde{w}\right]_{\alpha;\mathbb{R}^n} < \infty$.

But then this implies (as $D^2\tilde{w}$ is the. harmonic in $\mathbb{R}^n$) and sublinear by the above) by Liouville's theorem that $D^2\tilde{w} =$ constant. This then implies that $D^2 v$ is constant. But this then contradicts (3.5), since that shows $D_{lm}v(\xi) \neq D_{lm}v(\eta)$ (after taking the limit to extend it to v instead of $\tilde{u}_k$). So hence we have a contradiction, so hence we are done with proving this claim.

■

So to now finish the proof, we shall use this claim we have just established. By the interpolation inequality and scaling, this claim then gives (just as before in Theorem 3.1), that for any ball $B_\rho(y) \subset B_1(0)$, with $y \in \{x^n = 0\}$, we have

$$\rho^{2+\alpha}\left[D^2u\right]_{\alpha;B_{\rho/2}(y)^+} \leq \delta\rho^{2+\alpha}\left[D^2u\right]_{\alpha;B_\rho(y)^+} + C\left(|u|_{0;B_1^+} + |f|_{0,\alpha;B_1^+}\right).$$

Also, by the interior Schauder estimate, for any ball $\overline{B_\rho(y)} \subset B_1^+$, we have

$$\rho^{2+\alpha}\left[D^2u\right]_{\alpha;B_{\rho/2}(y)} \leq C\left(|u|_{0;B_1^+} + |f|_{0,\alpha;B_1^+}\right).$$

Then the conclusion of the theorem then follows by applying the following variant of the absorbing lemma.

□

Now we quickly state formally the other results we used in this prove, which will be checked on Example Sheet 2.

Proposition 3.1 (Reflection Principle for Harmonic Functions). *Let Ω^+ be an open subset of $\mathbb{R}^n_+$, and let $T = \partial\Omega^+ \cap \{x^n = 0\}$. Let Ω^- be the reflection of Ω^+ in $\{x_n = 0\}$, i.e.*

$$\Omega^- = \{(x^1, \ldots, -x^n : (x^1, \ldots, x^n) \in \Omega^+\}.$$

Let $v \in C^2(\Omega^+) \cap C^0(\Omega^+ \cup T)$. Then let $\overline{v}$ be the odd reflection of v in T, i.e. $\overline{v} : \Omega^+ \cup T \cup \Omega^- \to \mathbb{R}$ is:

$$\overline{v}(x^1, \ldots, x^n) = \begin{cases} v(x^1, \ldots, x^n) & \text{if } (x^1, \ldots, x^n) \in \Omega^+ \cup T \\ -v(x^1, \ldots, -x^n) & \text{if } (x^1, \ldots, x^n) \in \Omega^-. \end{cases}$$

Then if $\Delta v = 0$ in Ω^+ with $v|_T = 0$, then we have $\overline{v} \in C^2(Int(\Omega^+ \cup T \cup \Omega^-))$, and $\Delta\overline{v} = 0$ in $Int(\Omega^+ \cup T \cup \Omega^-)$.

Proof. See example sheet 2 (just use the mean-value property characterisation of harmonic functions to get that $\overline{v}$ is harmonic here, which implies the result). □

Note: This result is trivial if $T = \emptyset$, since then Ω^+ and Ω^- are disjoint. The important part of this result is that it is C^2 across the boundary T.

Lemma 3.1 (Absorbing Lemma, Boundary Version)**.** *Given $\theta \in (0,1)$, $\mu \in \mathbb{R}$, then $\exists\ \delta = \delta(n,\theta,\mu) \in (0,1)$ and $C = C(n,\theta,\mu) > 0$ such that, if $R > 0$, $\mathscr{B} = \{B_\rho(y) : B_\rho(y) \subset B_R^+(0)\}$ and $\mathscr{B}^+ = \{B_\rho^+(y) : y^n = 0,\ B_\rho^+(y) \subset B_R(0)^+\}$ and $S : \mathscr{B} \cup \mathscr{B}^+ \to \mathbb{R}$ is a sub-additive, non-negative function, satisfying*

$$\rho^\mu S(B^+_{\theta\rho}(y)) \leq \delta\rho^\mu S(B^+_\rho(y)) + \gamma \quad \text{if } B^+_\rho(y) \in \mathscr{B}^+$$

and

$$\rho^\mu S(B_{\theta\rho}(y)) \leq \gamma \quad \text{if } B_\rho(y) \in \mathscr{B}.$$

Then we have $R^\mu S(B^+_{\theta R}(0)) \leq C\gamma$.

Proof. See example sheet 2 (the idea is to first take a maximal pairwise disjoint collection of balls covering strips/boundary, and then double the radii to cover the domain $\mathscr{D}$ and then cover the rest). □

So we have proven the boundary Schauder estimate on balls. So all that is left is to extend this to general domains.

Because of how often we use this hypothesis, we denote the following hypotheses by (H):

> *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain, with $0 < \alpha < 1$. Suppose $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$ are such that*
> $$\left|a^{ij}\right|_{0,\alpha;\Omega} + \left|b^i\right|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta.$$
> *Suppose also that $\exists\ \lambda > 0$ constant such that: $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2\ \forall\ x \in \Omega,\ \xi \in \mathbb{R}^n$. Also let $Lu = a^{ij}D_{ij}u + b^iD_iu + cu$.*

Then we have the following boundary Schauder estimate, as a Corollary of Theorem 3.2.

Corollary 3.3. *Suppose hypotheses (H) hold, and Ω is a $C^{2,\alpha}$ domain. Then, $\exists\ \varepsilon = \varepsilon(\Omega) > 0$ such that, if $u \in C^{2,\alpha}(\overline{\Omega})$, $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$ satisfy*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Then, $\forall\ x \in \partial\Omega$, we have

$$|u|_{2,\alpha;B_\varepsilon(x)\cap\Omega} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right).$$

Remark: Clearly we need Ω to be a $C^{2,\alpha}$ domain for us to have any chance of u being $C^{2,\alpha}$ on $\partial\Omega$.

Proof. For any $z \in \partial\Omega$, by definiton of Ω being a $C^{2,\alpha}$-domain, we know that $\exists\ \psi : B_R(z) \to \mathscr{D} \subset \mathbb{R}^n$ a $C^{2,\alpha}$-diffeomorphism, for some $R > 0$, such that

$$\psi(B_R(z)\cap\Omega) \subset \{y^n > 0\}, \quad \text{and} \quad \psi(B_R(z)\cap\partial\Omega) \subset \{y^n = 0\},$$

i.e. ψ provides a $C^{2,\alpha}$-straightening out of the boundary near z. Let $x = (x^1,\dots,x^n)$ be coordinates for Ω and let $y = (y^1,\dots,y^n)$ be coordinates of $\mathscr{D}$. So let

$$\tilde{u}(y) = u(\psi^{-1}(y)).$$

Then, we have $\tilde{u}|_{\{y^n=0\}\cap\mathscr{D}} = (\varphi\circ\psi^{-1})\big|_{\{y^n=0\}\cap\mathscr{D}} =: \tilde{\varphi}$. So to apply Theorem 3.2, we need to find the PDE satisfied by $\tilde{u}$ and show that it is strictly elliptic, etc.

So note, $u(x) = \tilde{u}(\psi(x))$. So, $D_{x_i}u = D_{y_k}\tilde{u}(\psi(x))\cdot\frac{\partial\psi^k}{\partial x^i}$ (sum over k), and hence

$$D_{x_ix_j}u = \sum_{l,k=1}^n D_{y_ly_k}\tilde{u}(\psi(x))\cdot\frac{\partial\psi^k}{\partial x^i}\cdot\frac{\partial\psi^l}{\partial x_j} + \sum_{k=1}^n D_{y_k}\tilde{u}(\psi(x))\cdot\frac{\partial^2\psi^k}{\partial x^i\partial x^j}.$$

So using this, we can check that $\tilde{u}$ satisfies the equation:

$$\begin{cases} A^{lk}D_{y_ly_k}\tilde{u} + B^lD_{y_l}\tilde{u} + C\tilde{u} = \tilde{f} & \text{on } \mathscr{D}\\ \tilde{u} = \tilde{\varphi} & \text{on } \mathscr{D}\cap\{y^n = 0\}\end{cases}$$

where we can find the coefficients explicitly from the above, e.g. $A^{lk} = \sum_{i,j} a^{ij}\frac{\partial\psi^k}{\partial x_i}\cdot\frac{\partial\psi^l}{\partial x^j}$. Also, $\tilde{f} := f\circ\psi^{-1}$.

Now we want to rescale: choose $\sigma > 0$ such that $B_\sigma(\psi(z)) \subset \mathscr{D}$ (can do as this is open). Then, provided the assumptions hold, apply Theorem 3.1 with $\bar{u}(y) := \tilde{u}(\psi(z) + \sigma y)$ in place of u to see:

$$|\tilde{u}|'_{2,\alpha;B_{\sigma/2}(\psi(z))} \leq C\left(|\tilde{u}|_{0;B_\sigma(\psi(z))} + \sigma^2\left|\tilde{f}\right|_{0,\alpha;B_\sigma(\psi(z))} + |\tilde{\varphi}|'_{2,\alpha;B_\sigma(\psi(z))}\right), \tag{3.6}$$

where $C = C(n,\lambda,\beta,\alpha,\psi)$. So to apply this Theorem 3.1, we need to check that:

$$\left|A^{lk}\right|'_{0,\alpha;B_\sigma(\psi(z))} + \left|B^l\right|'_{0,\alpha;B_\sigma(\psi(z))} + |c|_{0,\alpha;B_\sigma(\psi(z))} \leq \mu\beta$$

for some $\mu = \mu(\psi)$, and we also need to check the ellipticity condition of the $(A^{lk})_{lk}$. For the latter, we know the form of A^{lk} (by the above), and so note

$$\sum_{l,k=1}^n A^{lk}(y)\xi^l\xi^k = \sum_{i,j=1}^n a^{ij}D_i(\xi\cdot\psi)D_j(\xi\cdot\psi) \geq \lambda\left|D(\xi\cdot\psi)|^2\right|_{\psi^{-1}(y)} \geq C(\psi)\lambda|\xi|^2$$

for some $C(\psi) > 0$[iii] (where we have also used the ellipticity condition on the $(a^{ij})_{ij}$). The other condition is checked similarly, and so the application of Theorem 3.2 here is justified.

[iii] Indeed, we have $\xi\cdot y = \xi\cdot\psi(\psi^{-1}(y)) \implies \xi = D(\xi\cdot\psi)|_{\psi^{-1}(y)}\cdot D\psi^{-1}(y)$, which then implies $|\xi| \leq |D(\xi\cdot\psi)||_{\psi^{-1}(y)}\cdot\|D\psi(y)^{-1}\|$. So hence noting that $\|D\psi(y)^{-1}\| = c(\psi)$ is a > 0 constant depending only on ψ, we get this.

So hence transforming back $\tilde{f} \to f$, etc, we get from (3.6) that
$$|\tilde{u}|_{2,\alpha;B_{\sigma/2}(\psi(z))} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right),$$
where $C = C(n, \lambda, \beta, \alpha, \psi, \sigma)$.

Now pick $r > 0$ $r = r(z)$ such that $B_r(z) \subset \psi^{-1}\left(B_{\sigma/2}(\psi(z))\right)$. Then by the above, we get
$$|u|_{2,\alpha;B_r(z)} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right),$$
where $C = C(n, \lambda, \beta, \alpha, \psi, \sigma)$. Do denote the dependence on the boundary point z, we write that this was all done for some $\psi = \psi_z$, $\sigma = \sigma_z$. So here, $C = C_z$ as well.

This is the result we want for a ball. So we finish the proof by a compactness argument. Clearly $\partial\Omega \subset \bigcup_{x\in\partial\Omega} B_{r(z)/2}(z)$. Then by compactness of $\partial\Omega$, we get that there is a finite subcover, and so have $\partial\Omega \subset \bigcup_{j=1}^{N} B_{r(z_j)/2}(z_j)$. Then let
$$\varepsilon = \min_j \left\{\frac{r(z_j)}{2} : j = 1, \ldots, N\right\}$$
$$C = \max_j \left\{C_{z_j} : j = 1, \ldots, N\right\}.$$

Then for any $x \in \partial\Omega$, we have by the above and our choices of ε, C, that
$$|u|_{2,\alpha;B_\varepsilon(x)} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$
and so we are done.

□

3.3. **Global Schauder Estimates.**

Now we finally combine everything we have done to get global Schauder estimates.

Theorem 3.3 (Global Schauder Estimates). *Suppose hypotheses (H) hold. Suppose also that Ω is a $C^{2,\alpha}$-domain. Then, if $u \in C^{2,\alpha}(\Omega)$, $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}(\Omega)$, where*
$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$
then in fact
$$|u|_{2,\alpha;\Omega} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right),$$
where $C = C(n, \lambda, \alpha, \beta, \Omega)$.

Proof. Let $\varepsilon = \varepsilon(\Omega)$ be as in Corollary 3.3. Then let
$$\Omega_\varepsilon := \left\{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \frac{\varepsilon}{4}\right\}.$$
Then by the interior Schauder estimates (Theorem 3.1) we have
$$|u|_{2,\alpha;\Omega_\varepsilon} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right)$$
where $C = C(n, \alpha, \lambda, \beta, \Omega)$.

Then note that $\Omega\backslash\Omega_\varepsilon \subset \bigcup_{x\in\partial\Omega} B_{\varepsilon/4}(x)$. Therefore, for any $x \in \Omega$, we have either:

- If $x \in \Omega_\varepsilon$, then
$$|u(x)| + |Du(x)| + |D^2u(x)| \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right),$$
by the above interior estimate.
- If $x \in \Omega\backslash\Omega_\varepsilon$, then $x \in B_{\varepsilon/4}(y)$ for some $y \in \partial\Omega$, and so by Corollary 3.3, we have
$$|u(x)| + |Du(x)| + |D^2u(x)| \leq |u|_{2,\alpha;B_\varepsilon(y)} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$
where the first inequality is by definition of the norm, and the second is by Corollary 3.3.

Hence combining these, we see that we get
$$|u|_{2;\Omega} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right).$$
So to finish, we just need to bound $\left[D^2u\right]_{\alpha;\Omega}$ (since $|u|_{2,\alpha;\Omega} = |u|_{2;\Omega} + \left[D^2u\right]_{\alpha;\Omega}$). So let $x, y \in \Omega$. Then we have several cases.

- Suppose they are such that $|x-y| < \frac{\varepsilon}{4}$. Then we have two cases.
  - If either of $\text{dist}(x,\partial\Omega) < \frac{\varepsilon}{4}$ or $\text{dist}(y,\partial\Omega) < \frac{\varepsilon}{4}$ hold, then we have $x, y \in B_{\varepsilon/2}(z)$, for some $z \in \partial\Omega$. So then by Corollary 3.3 we get
$$\frac{|D_{ij}u(x) - D_{ij}u(y)|}{|x-y|^\alpha} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$
simply from the bound on the norm. So done in this case (as it incorporates this Hölder coefficient).
  - Otherwise we have both $x, y \in \Omega_\varepsilon$, and so again we are done from the norm bound in the interior estimate.
- Otherwise we have $|x-y| \geq \frac{\varepsilon}{4}$. So in this case we clearly have:
$$\frac{|D_{ij}u(x) - D_{ij}u(y)|}{|x-y|^\alpha} \leq \frac{1}{(\varepsilon/4)^\alpha} \cdot 2 \cdot \overbrace{|u|_{2;\Omega}}^{\text{bounded by above}} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right).$$

So hence in each case we get this bound for some C, and so we are done (by taking the maximum C over the 3 cases).

$\square$

Now with all of our Schauder estimates, we prove some applications. We start with the Dirichlet problem.

4. Solvability of the Dirichlet Problem

We now have a new set of hypotheses which we will assume in this section. We denote the following by (H):

> *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$, and that $\exists\, \lambda > 0$ such that $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n$ (i.e. strict ellipticity). Let $Lu = a^{ij}D_{ij}u + b^iD_iu + cu$.*

With the above set up, we shall prove that if additionally $c \leq 0$ in Ω, then under some mild regularity assumption on $\partial\Omega$ (namely, Ω will need to satisfy the exterior sphere condition), the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

is solvable, for a unique solution $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$, for any given $f \in C^{0,\alpha}(\Omega)$.

> **Recall:** Remember that $\alpha \in (0,1)$ always here: so this does not hold for general continuous coefficient equations.

The first step to doing this is proving the following theorem, which reduces the existence of a solution to this problem to the existence of a solution to an easier Dirichlet problem (namely, Poisson's equation).

Theorem 4.1. *Let hypotheses (H) above hold. Assume further that Ω is a $C^{2,\alpha}$-domain, and $c \leq 0$ in Ω. Then:*

The Dirichlet problem $\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$ *has a solution* $u \in C^{2,\alpha}(\Omega),\ \forall f \in C^{0,\alpha}(\Omega),\ \varphi \in C^{2,\alpha}(\Omega)$

$\iff$ *The Dirichlet problem* $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$ *has a solution* $u \in C^{2,\alpha}(\Omega)$*, for all such* f, φ.

> **Remark:** Since $c \leq 0$, the solution (for either problem), if it exists, is unique.

> **Remark:** Note that there is no assumption on the coefficients of L, as this is an equivalence of existence statements, not a solvability statement.

Proof. By consider $u - \varphi$ it suffices (since $\varphi \in C^{2,\alpha}(\Omega)$) to establish the equivalence in the special case. $\varphi \equiv 0$ (indeed, this may change f, but that does not matter).

So consider $C^{2,\alpha}_0(\overline{\Omega}) := \{u \in C^{2,\alpha}(\Omega) : u \equiv 0 \text{ on } \partial\Omega\}$. This is clearly a Banach space with respect to the usual $C^{2,\alpha}$-norm.

Then consider for $t \in [0,1]$, the 1-parameter family of operators $L_t : C^{2,\alpha}_0(\overline{\Omega}) \to C^{0,\alpha}(\Omega)$, where

$$\begin{aligned} L_t u &:= t \cdot Lu + (1-t)\cdot \Delta u \\ &= a^{ij}_t D_{ij}u + b^i_t D_i u + c_t u \end{aligned}$$

where $a_t^{ij} = ta^{ij} + (1-t)\delta_{ij}$, $b_t^i = tb^i$, $c_t = tc$. Note that clearly $L_0 = \Delta$, and $L_1 = L$. So this is a linear interpolation between these operators.

Clearly we have

$$\left|a_t^{ij}\right|_{0,\alpha;\Omega} + \left|b_t^i\right|_{0,\alpha;\Omega} + |c_t|_{0,\alpha;\Omega} \leq \max(1,\beta)$$

where $\beta = \left|a^{ij}\right|_{0,\alpha;\Omega} + \left|b^i\right|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega}$. Also, clearly we have $a_t^{ij}(x)\xi^i\xi^j \geq \min(1,\lambda)|\xi|^2$ for all $t \in [0,1]$. So hence we can apply the global Schauder estimate to get:

$$|u|_{2,\alpha;\Omega} \leq C_1\left(|u|_{0;\Omega} + |L_t u|_{0,\alpha;\Omega}\right)$$

for all $u \in C_0^{2,\alpha}(\overline{\Omega})$, where $C_1 = C_1(n,\alpha,\lambda,\beta,\Omega)$.

Then by the maximum principle a priori estimate (MPAPE), we always have that (as u in this set is 0 on the boundary) $|u|_{0;\Omega} \leq C_2\,|L_t u|_{0,\alpha;\Omega}$, where $C_2 = C_2(n,\lambda,\Omega)$. So hence, combing these two inequalities we have that

$$|u|_{2,\alpha;\Omega} \leq C\,|L_t u|_{0,\alpha;\Omega} \tag{4.1}$$

is always true, for all $t \in [0,1]$ and all $u \in C_0^{2,\alpha}(\overline{\Omega})$, where $C = C(n,\alpha,\lambda,\beta,\Omega)$. In particular, this tells us that L_t is always injective.

However the solvability of $L_t u = f$ in $C_0^{2,\alpha}(\overline{\Omega})$ (i.e. to ensure correct boundary conditions) for arbitrary $f \in C^{0,\alpha}(\Omega)$ is asking for surjectivity of L_t. (Or equivalently, bijectivity of L_t since we know it is always injective).

So now fix $f \in C^{0,\alpha}(\Omega)$. We will show that if L_t is surjective at some t, then it is surjective at all $t \in [0,1]$ (this is a connectivity argument). So suppose L_s is surjective (and hence bijective) for some $s \in [0,1]$. Fix this s. Then by (4.1),

$$\left|L_s^{-1} g\right|_{2,\alpha;\Omega} \leq C\,|g|_{0,\alpha;\Omega} \tag{4.2}$$

for all $g \in C^{0,\alpha}(\Omega)$. Then, for any $t \in [0,1]$,

$$L_t u = f \iff L_s u + (L_t - L_s)u = f \iff u = L_s^{-1}(f) + L_s^{-1}((L_s - L_t)u)$$

where we have used linearity of L_s^{-1} (which send f to the solution). So hence to find such a solution for L_t, we are looking for a fixed point of a certain map, determined by the above.

So let $T : C_0^{2,\alpha}(\overline{\Omega}) \to C_0^{2,\alpha}(\overline{\Omega})$ be the map above, i.e.

$$T(u) := L_s^{-1}(f) + L_s^{-1}((L_s - L_t)u).$$

So we have seen that solvability of $L_t u = f$ is equivalent to T having a fixed point. We will show that T always has a fixed point by using the contraction mapping theorem (and thus $L_t u = f$ will always have a solution in this case).

Claim: T is a contraction map for t sufficiently close to s.

Proof of Claim: For any $u, v \in C_0^{2,\alpha}(\overline{\Omega})$, we have

$$T(u) - T(v) = L_s^{-1}\left((L_s - L_t)(u-v)\right) = (s-t)L_s^{-1}\left((L-\Delta)(u-v)\right)$$

where we have used the form of L_t. So hence using (4.2), we have

$$|T(u)-T(v)|_{2,\alpha;\Omega} \leq |s-t|\cdot C\,|(L-\Delta)(u-v)|_{0,\alpha;\Omega} \leq |s-t|\cdot \tilde{C}\,|u-v|_{2,\alpha;\Omega}\,,$$

where in the second inequality we have used the fact that both L,Δ are bounded operators here (as they only go up to 2nd order derivatives, which are included in the $C^{2,\alpha}$-norm), where $\tilde{C}=\tilde{C}(n,\alpha,\lambda,\beta,\Omega)$.

So hence if $|s-t|<\frac{1}{2\tilde{C}}$, then T is a contraction map for such t.

■

Hence for t sufficiently close to s, we have that T is a contraction map. Hence by the contraction mapping theorem, T has a fixed point for such t, and so hence $L_t u = f$ is solvable.

Note that the width of t to which we could do this, $1/2\tilde{C}$, did not depend on s. So by breaking up the interval $[0,1]$ into subintervals of length $1/2\tilde{C}$, we can apply this argument to each one (propagating the result onto the next interval) to get that $L_t u = f$ is solvable for all $t\in[0,1]$ if it is solvable for some $s\in[0,1]$ (i.e. apply this argument again to $s\pm\frac{1}{2\tilde{C}}$, etc, until we cover $[0,1]$).

Hence the result follows by assuming we can solve the equation $L_t u = f$ at either $t=0$ or $t=1$.

□

Note: Indeed, we get the equivalence if any $L_t u = f$ is solvable, not just for $L=\Delta$.

Remark: This method is called the **continuity method**: we "continuously deform our operator into a nicer one and relate the two". The above argument is essentially showing that the set $A=\{t\in[0,1]: L_t \text{ is a bijection}\}$ is open, but that around each point, we can find a ball about that point lying in this set which has radius independent of the point. This then clearly implies A must be the whole of $[0,1]$ if it is non-empty.

Recall: $\Omega\subset\mathbb{R}^n$ is a bounded domain, and $\alpha\in(0,1)$. Suppose $a^{ij},b^i,c\in C^{0,\alpha}(\overline{\Omega})$, $\underline{c\leq 0}$ and let L be strictly elliptic in Ω (these were our hypotheses, (H)).

So we now know from Theorem 4.1, we have reduced the existence (in $C^{2,\alpha}(\overline{\Omega})$) of a solution to the Dirichlet problem

$$\begin{cases} Lu=f & \text{in }\Omega\\ u=\varphi & \text{on }\partial\Omega\end{cases}$$

to the existence in $C^{2,\alpha}(\overline{\Omega})$ of a solution to the problem:

$$\begin{cases} \Delta u=f & \text{in }\Omega\\ u=\varphi & \text{on }\partial\Omega,\end{cases}$$

for any given $f\in C^{0,\alpha}(\overline{\Omega})$, $\varphi\in C^{2,\alpha}(\overline{\Omega})$ (this is useful as not only is Δ a simpler operator, but it is also in divergence form).

The idea is to then prove the existence of such solutions to this equations in balls, and then to extend to more general domains using Perron's method.

This then allows us, in the first instance, to prove the existence of a solution to $Lu=f$ in Ω, with $u|_{\partial\Omega}=\varphi$, when the domain is a ball.

Proposition 4.1. *Let $B = B_R(y) \subset \mathbb{R}^n$ be any ball. Then, if $f \in C^\infty(\overline{B})$ and $\varphi \in C^\infty(\overline{B})$, then $\exists!$ $u \in C^\infty(\overline{B})$ such that:*

$$\begin{cases} \Delta u = f & \text{in } B \\ u = \varphi & \text{on } \partial B. \end{cases}$$

Proof. There are several ways of proving this, e.g. using the Riesz representation theorem (taking advantage of the divergence structure of Δ) - See the Analysis of PDE Part III course for details. $\square$

Next we can extend this from balls to more general domains:

Proposition 4.2. *Suppose that B is a ball. Then if $f \in C^{0,\alpha}(\overline{B})$ and $\varphi \in C^0(\overline{B})$, then $\exists!$ $u \in C^{2,\alpha}(B) \cap B^0(\overline{B})$ such that:*

$$\begin{cases} \Delta u = f & \text{in } B \\ u = \varphi & \text{on } \partial B. \end{cases}$$

Moreover, if we have $\varphi \in C^{2,\alpha}(\overline{B})$, then this solution is in fact $u \in C^{2,\alpha}(\overline{B})$.

Proof. Extend f to $\tilde{f} \in C^{0,\alpha}_c(\mathbb{R}^n)$, and φ to $\tilde{\varphi} \in C^0_c(\mathbb{R}^n)$. Then mollify both $\tilde{f}$ and $\tilde{\varphi}$. Then choose $\eta \in C^\infty_c(\mathbb{R}^n)$ such that $\eta|_{\mathbb{R}^n \setminus B_1(0)} = 0$, $\eta \geq 0$, and $\int_{\mathbb{R}^n} \eta = 1$. Let $\eta_\sigma(x) = \sigma^{-n}\eta\left(\frac{x}{\sigma}\right)$.

Then choose a sequence $(\sigma_k)_k$ with $\sigma_k \downarrow 0$, and let

$$f_k(x) := \int_{\mathbb{R}^n} \tilde{f}(y)\eta_{\sigma_k}(x-y)\, \mathrm{d}y \equiv \int_{\mathbb{R}^n} \tilde{f}(x-y)\eta_{\sigma_k}(y)\, \mathrm{d}y$$

and

$$\varphi_k(x) := \int_{\mathbb{R}^n} \tilde{\varphi}(y)\eta_{\sigma_k}(x-y)\, \mathrm{d}y = \int_{\mathbb{R}^n} \tilde{\varphi}(x-y)\eta_{\sigma_k}\, \mathrm{d}y$$

(i.e. convolutions). Then the f_k, φ_k are smooth. So by Proposition 4.1, $\exists$ a smooth u_k such that:

$$\begin{cases} \Delta u_k = f_k & \text{in } B \\ u_k = \varphi_k & \text{on } \partial B. \end{cases}$$

Moreover, we know $f_k \to f$ uniformly on $\overline{B}$, and $\varphi_k \to \varphi$ uniformly on ∂B. We also have $\sup_B |f_k| \leq \sup|\tilde{f}|$ for all k (this is just the usual convolution bound), and so:

$$|f_k(x) - f_k(z)| \leq \int_{\mathbb{R}^n} |\tilde{f}(x-y) - \tilde{f}(z-y)|\eta_{\sigma_k}(y)\, \mathrm{d}y \leq [\tilde{f}]_{0,\alpha} \cdot |x-z|^\alpha$$

$$\implies |f_k|_{0,\alpha} \leq |\tilde{f}|_{0,\alpha}.$$

Also we have (again as this is a convolution) $\sup|\varphi_k| \leq \sup|\tilde{\varphi}|$.

Then as $\Delta(u_k - u_l) = f_k - f_l$, and $(u_k - u_l)|_{\partial B} = \varphi_k - \varphi_l$, by the maximum principle a priori estimate (Theorem 2.4), we have:

$$\sup_{\overline{B}} |u_k - u_l| \le \sup_{\partial B} |\varphi_k - \varphi_l| + C \sup_{\overline{B}} |f_k - f_l|.$$

So hence $(u_k)_k$ is uniformly Cauchy in $\overline{B}$, and so $\exists u \in C^0(\overline{B})$ such that $u_k \to u$ uniformly in $\overline{B}$ (as continuity is preserved under uniform limits). Moreover, by taking limits we see $u = \varphi$ on ∂B.

Now we shall upgrade the regularity of u via the interior Schauder estimates.

By the interior Schauder estimate, for any sub-ball $\tilde{B} \subset\subset B$, we have:

$$|u_k|_{2,\alpha,\tilde{B}} \le C\left(\sup_B |u_k| + |f_k|_{0,\alpha,B}\right) \le C\left(\sup_{\partial B} |\varphi_k| + |\tilde{f}|_{0,\alpha,\mathbb{R}^n}\right) \le C\left(\sup_{\partial B} |\varphi| + |\tilde{f}|_{0,\alpha,\mathbb{R}^n}\right)$$

where $C = C(\tilde{B}, B)$.

So this is a bound independent of k, and so by Arzela-Ascoli, (a subsequence of) $(u_k)_k$ converges in $C^2_{\text{loc}}(B)$ to some limit in $C^2(B)$, which must be u (by uniqueness of limits).

So hence we get that $u \in C^2(B)$ and (by now passing to the limit in $\Delta u_k = f_k$), we get $\Delta u = f$ in B (we get the Hölder condition on u by taking limits).

For the "Moreover" part, if $\varphi \in C^{2,\alpha}(\overline{B})$, then use the same argument but this time we can use the global Schauder estimates in place of the interior Schauder estimates to get the result.

$\square$

> **Remark:** By combining Theorem 4.1 and the argument of Proposition 4.2, we get that if hypothesis (H) hold in a ball B, then the Dirichlet problem $Lu = f$ in B, $u = \varphi$ on ∂B, has a unique solition $u \in C^{2,\alpha} \cap C^0(\overline{B})$, for <u>any</u> given $f \in C^{0,\alpha}(\overline{B})$, $\varphi \in C^0(\partial B)$.

Now we extend the solvability of the Dirichlet problem associated with L from balls to more general bounded domains.

4.1. **Perron's Method.**

The idea of Perron's method is that solvability in balls implies solvability in general domains. It uses the notion of subsolutions, extended to <u>continuous</u> functions. It is based on the maximum principle.

We first make two observations.

<u>Observation 1:</u> Fix $f \in C^{0,\alpha}(\overline{\Omega})$. Let $u \in C^2(\Omega)$. Then, we have that: u is a subsolution to $Lu = f$ (i.e. $Lu \ge f$ in Ω) $\iff$ for every ball $B \subset\subset \Omega$, we have $u \le u_B$ in B, where $u_B \in C^2(B) \cap C^0(\overline{B})$ is the (unique) solution to the Dirichlet problem:

$$\begin{cases} Lu_B = f & \text{in } B \\ u_B = u & \text{on } \partial B. \end{cases}$$

[**Exercise** to check this equivalence.]

Note that such a u_B exists by the previous discussion. So note that this equivalent notion of subsolution can be extended to include continuous functions (i.e. we can use it to define when u is a subsolution, even if u is just continuous, whereas we cannot always substitute a continuous function into L).

Observation 2: Let $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$. Then set:

$$S_\varphi := \{v \in C^2(\Omega) \cap C^0(\overline{\Omega}) : Lv \geq f \text{ in } \Omega \text{ and } v \leq \varphi \text{ on } \partial\Omega\}.$$

Then if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solves:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

then in fact we have $u(x) = \sup_{v \in S_\varphi} v(x)$ [**Exercise** to check].

(i.e. subsolutions give us a representation of the solution in this way.)

Now modify our assumptions (H) to include $c \leq 0$ as an assumption.

Definition 4.1. *Let $f \in C^{0,\alpha}(\overline{\Omega})$. Then a function $u \in C^0(\Omega)$ is said to be a **sub (super) solution** to $Lu = f$ in Ω if, for all open balls B with $\overline{B} \subset \Omega$, we have that $u \leq u_B$ ($u \geq u_B$) in B, where u_B is the unique function in $C^{2,\alpha}(B) \cap C^0(\overline{B})$ satisfying:*

$$\begin{cases} Lu_B = f & \text{in } B \\ u_B = u & \text{on } \partial B. \end{cases}$$

We tend to check that things are subsolutions, etc, by using the maximum principle and it's corollaries.

Note: The point is that as sub/super solutions need only be continuous, if we take the maximum/minimum of sub/super solutions, we will get another sub/super solution. This would not be true for C^2 functions, as taking max/min might destroy the regularity.

Lemma 4.1. *We have the following.*

(i) *Let $u, v \in C^0(\overline{\Omega})$ be sub and super solutions respectively to $L\tilde{u} = f$ in Ω. Then if $u \leq v$ on $\partial\Omega$, then we have $u \leq v$ on $\overline{\Omega}$.*

(ii) *If $u_1, u_2 \in C^0(\Omega)$ are both subsolutions to $Lu = f$ in Ω, then $v := \max\{u_1, u_2\}$ is again a subsolution.*

(iii) *Let $u \in C^0(\Omega)$ be a subsolution to $Lu = f$ in Ω. Let B be a ball with $\overline{B} \subset \Omega$, and let $u_B \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ be the unique solution to:*

$$\begin{cases} Lu_B = f & \text{in } B \\ u_B = u & \text{on } \partial B. \end{cases}$$

Then define:

$$U := \begin{cases} u_B & \text{on } B \\ u & \text{on } \Omega \backslash B \end{cases}$$

*which is called the **L-lift of u**. Then, $U \in C^0(\Omega)$, and U is a subsolution to $Lu = f$ in Ω.*

Proof. See Example Sheet 3. □

> **Note:** The latter two properties are the usefulness in the generalisation of this notion of subsolution/supersolution.

Now fix $\varphi \in C^0(\partial\Omega)$. Then, let

$$\mathscr{S}_\varphi := \{v \in C^0(\overline{\Omega}) : v \text{ is a subsolution to } Lu = f \text{ in } \Omega \text{ such that } v \leq \varphi \text{ on } \partial\Omega\}.$$

Theorem 4.2. *Let hypothesis (H) holds, with $c \leq 0$. Suppose that $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$, and define $\mathscr{S}_\varphi$ as above. Then set:*

$$u = \sup_{v \in \mathscr{S}_\varphi} v.$$

Then, $u \in C^{2,\alpha}(\Omega)$, and $Lu = f$ in Ω.

Proof. We first show that $\mathscr{S}_\varphi$ is non-empty, so that this is well-defined.

Claim: $\mathscr{S}_\varphi \neq \emptyset$.

Proof of Claim. In fact, we see that if $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is such that $\Omega \subset \{x \in \mathbb{R}^n : x_1 \geq y_1\}$, and if $\varphi(y) := \sup_{x \in \Omega} |x - y|$, then,

$$s(x) := -\sup_{\partial\Omega} |\varphi| - (e^{\gamma d} - e^{\gamma(x_1 - y_1)}) \sup_\Omega |f|$$

for γ sufficiently large, is a subsolution to $Lu = f$ (and of course $s \leq \varphi$ on $\partial\Omega$).

Given this explicit expression, the fact that $s \in \mathscr{S}_\varphi$ is easily checked by direction calculation of $L(S)$.

■.

Also, the function $s_1 = -s$ satisfies $Ls_1 \leq f$ in Ω, and $s_1 \geq \varphi$ on $\partial\Omega$.

So by Lemma 4.1, we have that $v \leq s_1$ for all $v \in \mathscr{S}_\varphi$, and so in particular, $u \leq s_1$ in Ω. So thus u is well-defined.

Now fix $z \in \Omega$, and fix a ball $B_R(z)$ with $\overline{B}_R(z) \subset \Omega$. Then by definition of u, $\exists$ a sequence $v_j \subset \Omega$ such that $v_j(z) \to u(z)$. Then by replacing v_j with $\max\{v_j, s\}$, we may assume wlog that $\sup_\Omega |v_j| \leq \mu$, for some $\mu \in \mathbb{R}$, for all j.

Then let V_j be the L-lift of v_j in $B = B_R(z)$. Then, we know that $V_j \in C^{2,\alpha}(B) \cap C^0(\overline{B})$, with $LV_j = f$ in B, $V_j = v_j$ on ∂B, and $V_j = v_j$ in $\Omega \backslash B$.

So hence by Lemma 4.1 (iii), $V_j \in \mathscr{S}_\varphi$.

So, $V_j \leq u$ in Ω. Then let $w_j = V_j|_{B_R(z)}$. Then by the interior Schauder estimate,

$$\begin{aligned}|w_j|_{2,\alpha;B_{R/2}(z)} &\leq C\left(|w_j|_{0;B_R(z)} + |f|_{0,\alpha;B_R(z)}\right)\\ &\leq C(|v_j|_{0;\partial B_R(z)} + C_1|f|_{0;\Omega} + |f|_{0,\alpha;\Omega})\\ &\leq C(\mu + C_1|f|_{0,\alpha;\Omega})\end{aligned}$$

where ion the second line we have used that $w_j = v_j$ on $\partial B_R(z)$ and the maximum principle a priori estimate.

So hence by Arzela-Ascoli, by passing to a subsequence, $\exists V | in C^{2,\alpha}(\overline{B}_{R/2}(z))$ such that $V_j \to V$ in $C^2(\overline{B}_{R/2}(z))$, and $LV = f$ in $B_{R/2}(z)$.

Then we know that $V \leq u$ in $B_{R/2}(z)$ (as each $V_j \leq u$). But then as the v_j are subsolution and V_j solves the problem, we have that $v_j(x) \leq V_j(x)$ in $B_R(z)$. So hence taking $j \to \infty$, we get that $u(z) = V(z)$.

Claim: $u = V$ in $B_{R/16}(z)$.

(Note that since $z \in \Omega$ was arbitrary, and V is $C^{2,\alpha}$ here, with $LV = f$, it then follows (by covering Ω with balls) that $u \in C^{2,\alpha}(\Omega)$ and $Lu = f$ in Ω.)

Proof of Claim: Suppose not. Then, $\exists\ z_1 \in B_{R/16}(z)$ such that $V(z_1) < u(z_1)$ (as we know in general that $V \leq u$ in $B_{R/2}(z)$).

So then as u is defined as a supremum, we get that $\exists \tilde{w} \in \mathscr{S}_\varphi$ such that $V(z_1) < \tilde{w}(z_1) \leq u(z_1)$. So then let $\tilde{v}_j = \max\{\tilde{w}, V_j\}$. Then note that $\tilde{v}_j \in \mathscr{S}_\varphi$, and let $\tilde{V}_j$ be the L-lift of $\tilde{v}_j$ in $B_{R/4}(z_1)$.

Then as before, by the interior Schauder estimates, $\exists \tilde{V} \in C^{2,\alpha}(B_{R/8}(z_1))$ such that $\tilde{V}_j \to \tilde{V}$ in $C^2(B_{R/4}(z_1))$ (by passing to a subsequence).

So hence as: $V(z_1) < \tilde{w}(z_1) \leq \tilde{v}_j(z_1) \leq \tilde{V}_j(z_1)$, we have $V(z_1) < \tilde{V}(z_1)$. But then as $V_j \leq \tilde{V}_j$ in $B_{R/4}(z_1) \Rightarrow V \leq \tilde{V}$ in $B_{R/8}(z)$. But then as $V(z) = u(z)$, we know $V(z) = \tilde{V}(z)$.

So since $LV = f$, $L\tilde{V} = f$ in $B_{R/8}(z_1)$, we have $L(V - \tilde{V}) = 0$ in $B_{R/8}(z_1)$.

So hence by the SMP (as they agree at a point), we get that $V \equiv \tilde{V}$, which contradicts $V(z_1) < \tilde{V}(z_1)$. So hence we must have $u = V$ on $B_{R/16}(z)$.

■

So hence as V solves the equation, so does u. But then as this ball was arbitrary, we get that u satisfies the equation on all of Ω, and that $u \in C^{2,\alpha}(\Omega)$ as required.

□

Now we want to look at the boundary behaviour of the Perron solution u, since the above only gives us that u satisfies the equation in Ω with no information on its boundary values. We will see that under mild assumptions on $\partial\Omega$ (i.e. if $\partial\Omega$ satisfies the exterior sphere condition at every point of $\partial\Omega$), then the Perron solution about satisfies $u \in C^0(\overline{\Omega})$, with $u = \varphi$ on $\partial\Omega$.

To do this, we will need to important concept of **barriers**.

Definition 4.2. *Assume the standard hypotheses hold. Let $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$, and fix $x_0 \in \partial\Omega$. Then:*

(i) *A sequence of functions $w_i^+ : \overline{\Omega} \to \mathbb{R}$ is said to be an **upper barrier at x_0** to L, f, φ if:*
- *$w_i^+ \in C^0(\overline{\Omega})$ and w_i^+ is a supersolution to $Lu = f$ in Ω, with $w_i^+ \geq \varphi$ on $\partial\Omega$.*
- *$w_i^+(x_0) \to \varphi(x_0)$ as $i \to \infty$.*

(ii) *Similarly, a sequence $w_i^- : \overline{\Omega} \to \mathbb{R}$ is a **lower barrier at x_0** relative to L, f, φ if:*
- *$w_i^- \in C^0(\overline{\Omega})$ and w_i^- is a subsolution to $Lu = f$ in Ω, and $w_i^- \leq \varphi$ on $\partial\Omega$.*
- *$w_i^-(x_0) \to \varphi(x_0)$ as $i \to \infty$.*

Proposition 4.3. *Suppose hypothesis (H) hold, and that $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^0(\partial\Omega)$ and $x_0 \in \partial\Omega$. Suppose $\exists$ upper and lower barriers at x_0 relative to L, f, φ. Then, the Perron solution (given by Theorem 4.2) satisfies $u(x) \to \varphi(x_0)$ as $x \to x_0$ with $x \in \Omega$*

i..e if the boundary obeys this barrier condition, the Perron solution agrees on the boundary.

Proof. From the defining expression for the Perron solution u, we clearly have $u(x) \geq w_i^-(x)$ for all $x \in \Omega$ and $\forall i$, where the w_i^- are the lower barriers.

For the other inequality, note that for any $v \in S_\varphi$, we have $w_i^+(x) \geq v(x)$ for all $x \in \Omega$ (as supersolutions lie above subsolutions if they are so at the boundary). So hence we get

$$u(x) := \sup_{v \in S_\varphi} v(x) \leq w_i^+(x) \quad \text{for all } i, x.$$

So hence $w_i^-(x) \leq u(x) \leq w_i^+(x)$ for all $x \in \Omega$ and i.

So now given $\varepsilon > 0$, we can choose i such that $0 \leq w_i^+(x_0) - \varphi(x_0) \leq \varepsilon/2$ and $0 \leq \varphi(x_0) - w_i^-(x) \leq \varepsilon/2$ (from the convergence in the definition of barriers).

Then we can choose by continuity, $\delta > 0$ such that $|w_i^\pm(x) - w_i^\pm(x_0)| < \varepsilon/2$ for all $x \in B_\delta(x_0) \cap \overline{\Omega}$. Then it is easy to see by the triangle inequality that $|u(x) - \varphi(x)| \leq \varepsilon$ if $x \in B_\delta(x_0) \cap \Omega$, and so hence we are done.

□

Proposition 4.4. *Suppose hypotheses (H) hold. Let $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$. Let $x_0 \in \partial\Omega$. Then, if $\exists w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that:*

(i) $Lw \leq -1$ *in* Ω

(ii) $w(x_0) = 0$ *and* $w(x) > 0$ *for all* $x \in \partial\Omega \backslash \{x_0\}$

then $\exists$ *lower and upper barriers at x_0 relative to L, f, φ.*

In fact, for any sequence $\varepsilon \downarrow 0$, $\exists$ constants $k_i > 0$ such that:

$$w_i^{\pm} = \varphi(x_0) \pm \varepsilon_i \pm k_i w(x)$$

are upper and lower barriers at x_0.

Proof. For any $\varepsilon > 0$, we can choose $\delta > 0$ such that $\varphi(x) \leq \varphi(x_0) + \varepsilon$ for all $x \in \partial\Omega \cap B_\delta(x_0)$, by continuity of φ. So as φ is bounded on $\partial\Omega$, and w is positive and continuous on the compact set $\partial\Omega \backslash B_\delta(x_0)$ (so attains a positive minima), we can choose $ł_\varepsilon > 0$ constant such that $l_\varepsilon w(x) \geq \varphi(x) - \varphi(x_0) - \varepsilon$ for all $x \in \partial\Omega \backslash B_\delta(x_0)$.

So hence, $\varphi(x_0) + \varepsilon + l_\varepsilon w(x) \geq \varphi(x)$ for all $x \in \partial\Omega$ (since on $B_\delta(x_0)$ this is true by choice of δ/ε).

So let $k_\varepsilon = \max\{l_\varepsilon, \sup_{x\in\partial\Omega} |f(x) - c(x)\varphi(x_0)|\}$ (i.e. make the constant l_ε larger if necessary).

Then we have $Lw_\varepsilon \leq f$ in Ω, where $w_\varepsilon = \varphi(x_0) + \varepsilon + k_\varepsilon w$. So hence taking a sequence $\varepsilon_i \downarrow 0$, we ge tthat $w_{\varepsilon_i}^+ := w_{\varepsilon_i}$ is an upper barrier. Similarly, $w_i^- = \varphi(x_0) - \varepsilon_i - k_{\varepsilon_i} w(x)$ is a lower barrier, so done.

□

So the question now becomes, when does such a w exist? Well, it turns out that it suffices for $\partial\Omega$ to satisfy the exterior sphere condition.

Proposition 4.5. *If Ω satisfies the exterior sphere condition at $x_0 \in \partial\Omega$, then upper and lower barriers exist at x_0 relative to L, f, φ.*

In particular, if $\partial\Omega$ is C^2, then the Perron solution agrees with φ on $\partial\Omega$ (and so is a solution).

Proof. We know by assumption that $\exists R > 0$ such that $\overline{B_R(y)} \cap \overline{\Omega} = \{x_0\}$. So for some constants $\mu, \sigma > 0$, set:

$$w(x) := \mu\left(R^{-\sigma} - |x-y|^{-\sigma}\right)$$

for $x \in \Omega$. Then by direct calculation, we can check that for $\mu, \sigma > 0$ sufficiently larger (depending on diam(Ω)) we have $Lw \leq -1$ on Ω.

Then clearly $w(x_0) = 0$, and $w(x) > 0$ for all $x \in \partial\Omega \backslash \{x_0\}$. So hence the claim follows from Proposition 4.4.

□

Now putting together all of Propositions 4.3-4.5 with Theorem 4.2, we get our existence theorem:

Theorem 4.3. *Let hypotheses (H) hold. Let $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$. Then, $\exists!\ u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ solving:*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

provided that Ω satisfies the exterior sphere condition at every point on $\partial\Omega$ (e.g. if Ω is a C^2 domain).

Proof. Just combine Theorem 4.2 with Propositions 4.3-4.4. □

Finally, we prove that if φ is $C^{2,\alpha}$ as well, then u is $C^{2,\alpha}$ up to and including the boundary as well.

Theorem 4.4. *Suppose hypothesis (H) hold, and also suppose Ω is a bounded $C^{2,\alpha}$ domain. Then, for any $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$, $\exists!\ u \in C^{2,\alpha}(\overline{\Omega})$ such that*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Proof. Let $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ be the unique solution to this problem, given by Theorem 4.3. Note that we know this is the unique such solution, so we need to show that it is more regular on $\partial\Omega$ in this case.

By definition of Ω being $C^{2,\alpha}$, we know $\exists \psi_1 : B_R(x_0) \to \tilde{U}$ which is a $C^{2,\alpha}$-map, where $\tilde{U} \subset \mathbb{R}^n$, and $\psi_1(B_R(x_0) \cap \Omega) = U \cap \{x_n > 0\}$.

Then further compose ψ_1 with a map ψ_2 such that ψ_2 pushes $\{x_n = 0\}$ to touch a ball B. Let ψ be the composition of these.

Hence we get that $\exists \psi : B_R(x_0) \to U$ a $C^{2,\alpha}$ diffeomorphism such that $\exists$ a ball B with $\overline{B} \subset \psi(\overline{\Omega} \cap B_R(x_0))$, such that: $T := \psi(\partial\Omega \cap B_\rho(x_0)) \subset \partial B$, for some $\rho > 0$.

Letting $y := \psi(x)$ be the coordinates on U, the equation $Lu(x) = f(x)$ transforms to a strictly elliptic equation $\tilde{L}\tilde{u}(y) = \tilde{f}(y)$ in U, with $\tilde{u}(y) = \tilde{\varphi}(y)$ on T.

So now solve the Dirichlet problem:

$$\begin{cases} \tilde{L}v = \tilde{f} & \text{in } B \\ v = \tilde{u} & \text{on } \partial B \end{cases}$$

(which we know we can do as this is a ball). Then we know that $v \in C^{2,\alpha}(B) \cap C^0(\overline{B})$. Moreover (**key**) we know $v \in C^{2,\alpha}(B \cup T)$ (since $\tilde{u} = \tilde{\varphi}$ is $C^{2,\alpha}$ on T) (this is by the same mollification and compactness argument used before in a previous theorem).

Now we will show via the maximum principle that $v = \tilde{u}$, and so hence $\tilde{u}$ is $C^{2,\alpha}$ on T, and so u is $C^{2,\alpha}$ on some neighbourhood of x on $\partial\Omega$, which completes the proof.

On the other hand, since $\tilde{L}(v-\tilde{u}) = 0$ in B and $v - \tilde{u} = 0$ on ∂B, and $v - \tilde{u} \in C^0(\overline{B})$, it follows from the maximum principle that $v = \tilde{u}$ in $\overline{B}$. So hence $\tilde{u} \in C^{2,\alpha}(B \cup T)$. So transforming back, we have $u \in C^{2,\alpha}(\overline{B} \cap B_\rho(x_0))$. So since x_0 was arbitrary, we get $u \in C^{2,\alpha}(\overline{\Omega})$.

□

4.2. **Fredholm Alternative for Elliptic PDEs.**

Let V be a normed space, and let $T : V \to V$ be a bounded, linear compact map (note that T compact means that if $(x_j)_j$ is a bounded sequence in V, then $(T(x_j))_j$ has a convergent subsequence). Then the general Fredholm alternative says that, either:

(i) The equation $x + T(x) = 0$ has a non-zero solution, or

(ii) For any given $y \in V$, the equation $x + T(x) = y$ has a unique solution $x \in V$.

[See Gilbarg + Trudinger, Chapter 5, for a proof.]

We can apply this to get a Fredholm alternative for our setting:

Theorem 4.5 (Fredholm Alternative). *Let Ω be a bounded, $C^{2,\alpha}$ domain in $\mathbb{R}^n$ and $\alpha \in (0,1)$. Let $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$, and L be a strictly elliptic operator in Ω. Then either:*

(i) *The homogeneous problem $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$, has a non-zero solution $u \in C^{2,\alpha}(\overline{\Omega})$*

(ii) *<u>or</u> for any given $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$, the Dirichlet problem $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$, has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.*

i.e. so what we really need to uniquely solve the Dirichlet problem is the fact that the homogeneous problem has only the zero solution, <u>not</u> that $c \leq 0$.

[The way to think about this Fredholm alternative is that if uniqueness holds, then existence also holds, i.e. if both u_1, u_2 satisfy $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$, then we know $w = u_1 - u_2$ solves the homogeneous problem, and so we want $w = 0$, etc.]

Proof. It suffices to establish the dichotomy in the special case $\varphi = 0$, since:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \text{has a unique solution} \iff \begin{cases} Lv = f - L\varphi & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \text{has a unique solution.}$$

So let σ be a constant such that $\sigma \geq \sup_{\overline{\Omega}} c$. Then consider the map $L_\sigma : C_0^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ given by $L_\sigma u := Lu - \sigma u$ (note we work in $C_0^{2,\alpha}$ as our boundary condition is 0).

Then by Theorem 4.4, we know L_σ is a bijection. Then by the global Schauder estimates and the MPAPE, we have:

$$|u|_{2,\alpha;\overline{\Omega}} \leq C|L_\sigma u|_{0,\alpha;\Omega}$$

for all $u \in C_0^{2,\alpha}(\overline{\Omega})$, and some fixed constant C. Equivalently, this says:

$$|L_\sigma^{-1}(f)|_{2,\alpha;\overline{\Omega}} \leq C|f|_{0,\alpha;\Omega} \quad \forall f \in C^{0,\alpha}(\overline{\Omega})$$

i.e. L_σ^{-1} is a bounded linear map. The inclusion $I : C^{2,\alpha}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ is a compact map (by Arzela-Ascoli). So hence $T_\sigma = I \circ L_\sigma^{-1} : C^{0,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ is a compact linear map, and so hence so is σT_σ. So applying the abstract Fredholm alternative to σT_σ, we see that either:

(i) $u + \sigma T_\sigma(u) = 0$ has a non-zero solution $u \in C^{0,\alpha}(\overline{\Omega})$
(ii) <u>or</u> for any $f \in C^{0,\alpha}(\overline{\Omega})$, $\exists!\ u \in C^{0,\alpha}(\overline{\Omega})$ such tha $u + \sigma T_\sigma(u) = L_\sigma^{-1}(f)$.

Note that in either case, we know that since $u, f \in C^{0,\alpha}(\overline{\Omega})$, that $L_\sigma^{-1}(u), L_\sigma^{-1}(f) \in C_0^{2,\alpha}(\overline{\Omega})$ (by definition of L_σ). But then in each case, u is expressible in terms of these, and so hence we have that $u \in C_0^{2,\alpha}(\overline{\Omega})$.

So then we can apply L_σ to both sides (which we can do now as we know u is in the correct space), and then we get that either (as $L = L_\sigma + \sigma$):

(i) $Lu = 0$ has a non-zero solution $u \in C_0^{2,\alpha}(\overline{\Omega})$
(ii) <u>or</u> for any $f \in C^{0,\alpha}(\overline{\Omega})$, $\exists!\ u \in C_0^{2,\alpha}(\overline{\Omega})$ such that $Lu = f$ in Ω.

So hence as we have 0 boundary conditions, this implies the result.

□

> **Remark:** Clearly $K = \{u \in C_0^{2,\alpha}(\overline{\Omega} : Lu = 0 \text{ in } \Omega\}$, which is a linear subspace, of $C_0^{2,\alpha}(\overline{\Omega})$, is always a finite dimensional subspace of $C^{2,\alpha}(\overline{\Omega})$ (with the $C^{0,\alpha}(\overline{\Omega})$ norm). This is because $Lu = 0 \Leftrightarrow u = -\sigma L_\sigma^{-1}(u)$, and by the estimate in the proof, we have $|u|_{2,\alpha;\overline{\Omega}} = |\sigma| \cdot |L_\sigma^{-1}(u)| \leq C|u|_{0,\alpha;\overline{\Omega}}$.
>
> i.e. if we take the unit ball in the $C^{0,\alpha}$-norm, this gives rise to a uniform bound on the $|u|_{2,\alpha;\overline{\Omega}}$ norms, and so by Arzela-Ascoli we get a convergent subsequence, which shows that the closed unit ball in the space $(K, \|\cdot\|_{0,\alpha;\overline{\Omega}})$ is compact, and so hence K is finite dimensional (as compactness of the unit ball characterises finite-dimensionality).

4.3. **Higher Regularity of Solutions.**

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and $\alpha \in (0,1)$. Let $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$. Suppose that L is strictly elliptic in Ω. Then suppose $f \in C^{0,\alpha}(\Omega)$ and $u \in C^2(\Omega)$ solves $Lu = f$ in Ω. Then in fact $u \in C^{2,\alpha}(\Omega)$, and:*

$$|u|_{2,\alpha;\tilde{\Omega}} \leq C\left(|u|_{0,\Omega'} + |f|_{0,\alpha;\Omega'}\right)$$

for all $\tilde{\Omega} \subset\subset \Omega' \subset\subset \Omega$, where $C = C(n, \alpha, \lambda, \text{dist}(\tilde{\Omega}, \partial\Omega'))$.

> **Note:** The estimate just comes from that of the interior Schauder theory. The interesting part is the fact that we automatically have $C^{2,\alpha}$ regularity.

Proof. Define $L_1 u := a^{ij}D_{ij}u + b^i D_i u = f_1$, where $f_1 = f - cu \in C^{0,\alpha}(\Omega)$.

So fix $B = B_\rho(y) \subset\subset \Omega$. Then by Theorem 4.3, $\exists v \in C^{2,\alpha}(B) \cap B^0(\overline{B})$ such that $L_1 v = f_1$ in B, and $v = u$ on ∂B.

But then $L_1(v-u) = 0$ in B, and $v-u = 0$ on ∂B, and $v-u \in C^0(\overline{B})$. So hence by the weak maximum principle, we get that $v = u$ in B, and so $u \in C^{2,\alpha}(\Omega)$.

□

With this we can now prove higher regularity results.

Theorem 4.6 (Higher Interior Regularity). *Suppose $\Omega \subset \mathbb{R}^n$ is open and $\alpha \in (0,1)$. Let $a^{ij}, b^i, c \in C^{k,\alpha}(\Omega)$ for some $k \geq 0$. Suppose L is strictly elliptic in Ω, and $f \in C^{k,\alpha}(\Omega)$. Suppose that $u \in C^2(\Omega)$ solves $Lu = f$ in Ω.*

Then, $u \in C^{k+2,\alpha}(\Omega)$, and:
$$|u|_{k+2,\alpha,\Omega'} \leq C\left(|u|_{0;\Omega_1} + |f|_{k,\alpha;\Omega_1}\right)$$
for all $\Omega' \subset\subset \Omega_1 \subset\subset \Omega$, where $C = C(n,k,\alpha,\lambda,L,\mathrm{dist}(\Omega',\partial\Omega_1))$.

Proof. We prove this by induction on k (we have already done the $k=0$ case, as this is just Lemma 4.2 above).

Step 1: Establish the $k=1$ case, i.e. if $a^{ij}, b^i, c, f \in C^{1,\alpha}(\Omega)$, and $u \in C^{2,\alpha}(\Omega)$, then $u \in C^{3,\alpha}(\Omega)$ and
$$|u|_{3,\alpha;\Omega'} \leq C\left(|u|_{0;\Omega_1} + |f|_{1,\alpha;\Omega_1}\right).$$

[Note that we can wlog assume that $u \in C^{2,\alpha}$ by the $k=0$ case.]

To see this, let $\Omega'' \subset\subset \Omega'$ and fix a direction $l \in \{1,\dots,n\}$. Then for $0 < |h| < \mathrm{dist}(\Omega'',\Omega_1)$, let $\delta_{l,h}$ be the difference quotient operator, i.e.
$$\delta_{l,h}g(x) := \frac{g(x+he_l) - g(x)}{h}$$

for $g : \Omega_1 \to \mathbb{R}$. Then, we have

$$\begin{aligned}
L(\delta_{l,h}u)(x) &= \frac{1}{h}\left[L(u(x+he_l)) - \overbrace{Lu(x)}^{=f(x)}\right]\\
&= \frac{1}{h}\left[a^{ij}(x)D_{ij}u(x+he_l) + b^i(x)D_iu(x+he_l) + c(x)u(x+he_l) - f(x)\right]\\
&= \frac{1}{h}\left[(a^{ij}(x) - a^{ij}(x+he_l))D_{ij}u)x+he_l) + (b^i(x) - b^i(x+he_l))D_iu(x+he_l)\right.\\
&\qquad \left. +(c(x)-c(x+he_l))u(x+he_l) + f(x+he_l) - f(x)\right]\\
&= -(\delta_{l,h}a^{ij})(x)D_{ij}u(x+he_l) - (\delta_{l,h}b^i)(x)D_iu(x+he_l) - (\delta_{l,h}c)(x)u(x+he_l) + \delta_{h,l}f(x)\\
&=: g_{l,h}(x)
\end{aligned}$$

where in the long expression we have added and subtracted $f(x+he_l) \equiv Lu(x+he_l)$.

Now, as f is differentiable, we have

$$(\delta_{l,h}f)(x) = \frac{1}{h}(f(x+he_l) - f(x)) = \frac{1}{h}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}f(x+the_l)\,\mathrm{d}t = \int_0^1 (D_lf)(x+the_l)\,\mathrm{d}t.$$

So, we have $\sup_{\Omega''}|\delta_{l,h}f| \leq \sup_{\Omega'}|Df|$. Moreover, for $x \neq y$, we have

$$(\delta_{l,h}f)(x) - (\delta_{l,h}f)(y) = \int_0^1 (D_lf)(x+the_l) - (D_lf)(y+the_l)\,\mathrm{d}t$$

which implies $[\delta_{l,h}f]_{\alpha;\Omega''} \leq [D_lf]_{\alpha;\Omega'}$. Similarly, we get that same for the other coefficients, e.g. $[\delta_{l,h}a^{ij}]_{\alpha;\Omega''} \leq [D_la^{ij}]_{\alpha;\Omega'}$.

Hence by the interior Schauder estimates, we get:

$$|\delta_{l,h}u|_{2,\alpha;\Omega''} \leq C\left(|g_{l,h}|_{0,\alpha;\Omega_1} + |\delta_{l,h}u|_{0;\Omega_1}\right) \leq C(|u|_{2,\alpha;\Omega_1} + |f|_{1,\alpha;\Omega_1}) \tag{4.3}$$

where the second inequality comes from our bounds above. Hence we have a uniform bound on this, independent on h. Hence by Arzela-Ascoli, we can find $(h_j)_j \downarrow 0$ such that $\delta_{l,h_j} \to w$, with the convergence in $C^2(\overline{\Omega''})$, where $w \in C^{2,\alpha}(\overline{\Omega''})$. But we know that $\delta_{l,h_j}u \to D_lu$, and so hence we have $D_lu = w \in C^{2,\alpha}(\Omega'')$, i.e. $u \in C^{3,\alpha}(\Omega'')$. So hence by (4.3) and the previous Lemma, this proves the $k=1$ case.

Step 2: Prove the $k \geq 2$ case.

Now we want to prove that if $a^{ij}, b^i, c, f \in C^{k,\alpha}(\Omega)$ and $u \in C^{k+1,\alpha}(\Omega)$, then in fact $u \in C^{k+2,\alpha}(\Omega)$, with the required estimate.

[Note by the induction hypothesis, we can assume that $u \in C^{k+1,\alpha}$.]

To see this, let γ be any multi-index with $|\gamma| = k-1$. Then we know from then product rule of derivatives,

$$L(D^\gamma u) = D^\gamma f - \sum_{\beta<\gamma}\frac{\gamma!}{\beta!(\gamma-\beta)!}\left(D^{\gamma-\beta}a^{ij}\cdot D^\beta D_{ij}u + D^{\gamma-\beta}b^i \cdot D^\beta D_iu + D^{\gamma-\beta}c\cdot D^\beta u\right)$$

where here $\beta < \gamma$ means that $\beta_i \leq \gamma_i$ for all i and $\beta_i < \gamma_i$ for some i.

Then one can check to see that the regularity assumptions on this hold (i.e. the RHS is in $C^{1,\alpha}(\Omega)$), and so we can apply Step 1/the $k=1$ case to get that $D^\gamma u \in C^{3,\alpha}(\Omega)$, and so hence $u \in C^{k+2,\alpha}(\Omega)$, and the estimate holds.

Step 3: Prove the theorem with the estimate.

This just follows by induction on k, from Step 2. So done.

□

There is also a higher global regularity result as well (i.e. up to the boundary - see Gilbarg and Trudinger, Chapter 6).

> **Note:** We can apply Theorem 4.6 to certain non-linear problems, e.g. the minimal surface equation.

Example 4.1. *Suppose $u \in C^2(\Omega)$ solves the minimal surface equation,*

$$D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) = 0$$

in Ω. Then, in fact we have $u \in C^\infty(\Omega)$.

Proof. The equation in non-divergence form is:

$$\underbrace{\left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\right)}_{=a^{ij}} D_{ij}u = 0.$$

So hence if $u \in C^2(\Omega)$, then the lemma gives $u \in C^{2,\alpha}(\Omega)$. Then we know $D_i u \in C^{1,\alpha}(\Omega)$ for all i, and so hence in fact $a^{ij} \in C^{1,\alpha}(\Omega)$. So hence by Theorem 4.6, we get that $u \in C^{3,\alpha}(\Omega)$. But then in the same way, we now have $a^{ij} \in C^{2,\alpha}$, and so by induction we get $u \in C^{k,\alpha}(\Omega)$ for all k, i.e. $u \in C^\infty(\Omega)$.

□

5. Quasilinear Elliptic Theory and the De Giorgi-Nash-Moser Theory

Let $\alpha \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ be a $C^{2,\alpha}$ bounded domain. Let $a^{ij}, b \in C^{0,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ for all $i,j \in \{1,\dots,n\}$, with coordinates $(x,z,p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$, and suppose that the matrix $(a^{ij}(x,z,p))_{ij}$ is positive definite in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.

Now consider the quasilinear operator:

$$Qu := a^{ij}(x,u,Du)D_{ij}u + b(x,u,Du).$$

The Dirichlet problem is then, given $\varphi \in C^{2,\alpha}(\overline{\Omega})$, find a solution $u \in C^{2,\alpha}(\overline{\Omega})$ such that:

$$\text{(DP)} \qquad \begin{cases} Qu = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega. \end{cases}$$

We will use the following abstract fixed point theorem, which reduces the solvability of (DP) in $C^{2,\alpha}(\overline{\Omega})$ to establishing a certain a priori estimate on solutions to related equations (which is what we have been doing for the rest of the course, which was the main reason we were interested in the Schauder estimates).

Theorem 5.1 (Leray-Schauder Fixed Point Theorem)**.** *Let X be a Banach space and $T : X \to X$ a continuous compact map*[iv]*. Suppose that $\exists$ a constant $M > 0$ such that, whenever $x \in X$ satisfies $x = \sigma T(x)$ for some $\sigma \in [0,1]$, then we have $\|x\| < M$ (with M independent of σ).*

Then $\exists x_0 \in X$ with $x_0 = T(x_0)$.

Proof. See Gilbarg & Trudinger, §11. □

Let us see how we can apply this fixed point theorem to establish solvability of (DP) to establishing an a priori estimate. First set $X = C^{1,\beta}(\overline{\Omega})$ for some $\beta > 0$ (which will be determined later). Then define $T : X \to X$ by setting, for a given $v \in X$, $T(v) := u$, where u solves the linear Dirichlet problem:

$$(\star) \quad \begin{cases} a^{ij}(x,v,Dv)D_{ij}u + b(x,v,Dv) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

i.e. use this given v to determine the coefficients, and then with these coefficients we have a linear strictly elliptic PDE with just second order and 0'th order parts which we know from before we can solve (i.e. we are essentially solving $a^{ij}D_{ij}u = F$). Note that $x \mapsto a^{ij}(x,v(x),Dv(x))$ and $x \mapsto b(x,v(x),Dv(x))$ are in $C^{0,\alpha\beta}(\overline{\Omega})$, and so by our linear theory, $\exists!\ u \in C^{2,\alpha\beta}(\overline{\Omega}) \subset C^{1,\beta}(\overline{\Omega})$ satisfying $(\star)$, and thus T is well-defined. Clearly solvability of our original (DP) is equivalent to finding a fixed point of T.

We can then check ([**Exercise**] using the Schauder estimates) that T is a continuous and compact map. Also, if $v \in C^{1,\beta}(\overline{\Omega})$ has $v = \sigma T(v)$ for some $\sigma \in [0,1]$, this means (for $\sigma \neq 0$) that $u := v/\sigma$ solves $(\star)$, which by multiplying everything in $(\star)$ by σ is the same as solving that v solves:

$$\text{(DP)}_* \qquad \begin{cases} a^{ij}(x,v,Dv)D_{ij}v + \sigma b(x,v,Dv) = 0 & \text{in } \Omega \\ v = \sigma\varphi & \text{on } \partial\Omega. \end{cases}$$

[iv] Note that we do not assume T is linear.

Note this also implies that $v \in C^{2,\alpha}(\overline{\Omega})$: indeed, above we argued that if $v \in C^{1,\beta}(\overline{\Omega})$ then in fact $T(v) \in C^{2,\alpha\beta}(\overline{\Omega})$, hence $v = \sigma T(v) \in C^{2,\alpha\beta}(\overline{\Omega})$, and so in fact the coefficients of $(\star)$ are $C^{1,\alpha^2\beta}(\overline{\Omega})$, and thus are in fact $C^{0,\alpha}(\overline{\Omega})$, and so Schauder theory then implies the solution is in fact $C^{2,\alpha}(\overline{\Omega})$.

Hence by the Leray-Schauder fixed point theorem, given $\varphi \in C^{2,\alpha}(\overline{\Omega})$, solvability of (DP) will follow if we can show that the following holds:

"$\exists$ *a constant* $M = M(\varphi, a^{ij}, b) > 0$ *and* $\beta > 0$ *such that whenever* $v \in C^{1,\beta}(\overline{\Omega})$ *solves* (DP)$_\sigma$ *for some* $\sigma \in [0,1]$, *we have* $|v|_{1,\beta;\overline{\Omega}} < M$."

Note: We can in fact assume by the above Schauder theory arguments that $v \in C^{2,\alpha}(\overline{\Omega})$.

This theory was all developed in the mid 20th century with the minimal surface equation in mind. The problem was solved independently by De Giorgi and Nash, with then Moser later giving different techniques to solve the same problem. So for now we will look at Moser's techniques, and only look at equations of variational type, such as the minimal surface equation, which the De Giorgi-Nash-Moser (DGNM) theory we present here was used for.

So consider operators Q which arise as Euler-Lagrange operators of functionals of the form:

$$\mathscr{F}(u) := \int_\Omega F(x,u,Du)\, \mathrm{d}x.$$

For simplicity we will only focus on integrands $F = F(x,z,p)$ that depend on the p-variable, i.e. $F(x,z,p) \equiv F(p)$, for $p \in \mathbb{R}^n$. So the Euler-Lagrange equation in its weak form becomes (i.e. consider $F(u+t\varphi) - F(u)$ and take the term linear in t):

$$\int_\Omega F_{p_i}(Du) D_i\varphi = 0 \quad \varphi \in C^1_c(\Omega).$$

In divergence form this is $D_i(F_{p_i}(Du)) = 0$, and in non-divergence form this is $F_{p_ip_j}(Du)D_{ij}u = 0$.

To get a bound on $|u|_{1,\beta}$, we need to first get a bound on Du so that we may bound the coefficients $F_{p_ip_j}(Du)$. To do this (remembering first that by assumption in fact $u \in C^{2,\alpha}$) we want to find an equation for Du. Thus if $u \in C^{2,\alpha}(\overline{\Omega})$ is a critical point of $\mathscr{F}(u) = \int_\Omega F(Du)$, the idea is to "differentiate" the Euler-Lagrange equation to derive equations for all partial derivatives of u. So since we know

$$\int_\Omega F_{p_i}(Du) D_i\varphi = 0 \quad \forall \varphi \in C^1_c(\Omega)$$

for each $k \in \{1,\ldots,n\}$ we can replace φ by $D_k\varphi$ for some $\varphi \in C^2_c(\overline{\Omega})$ to get:

$$\int_\Omega F_{p_i}(Du) D_i(D_k\varphi) = 0$$

and then integrating by parts with respect to x_k we get:

$$\int_\Omega F_{p_ip_j}(Du) D_j(D_ku) D_i\varphi = 0$$

and so if we set $w := D_ku \in C^{1,\alpha}(\overline{\Omega})$, we have a divergence form equation for w, namely

$$\int_\Omega F_{p_ip_j}(Du) D_jw \cdot D_i\varphi = 0$$

or equivalently,

$$D_i(F_{p_ip_j}(Du)D_jw) = 0 \quad \text{weakly in } \Omega.$$

Now if F is for instance smooth and convex, then this implies that this equation is elliptic, with coefficients (in this divergence form) in $C^{1,\alpha}$. But we must be careful: note that the coefficients still depend on u, and so the usual Schauder estimates would give estimates which depend on u. This is why we need to develop a deeper a priori estimate theory than the Schauder theory for quasilinear equations.

> **Remark:** Our current aim to establish the estimate we want is to find a uniform $C^{0,\beta}(\overline{\Omega})$ estimate for w, independent of u, so that we may bound the coefficients above. This will come from the DGNM theory, for which to bound the coefficients of the equation, we first need to find an a priori estimate for Du, i.e. an estimate of the form $\sup_{\overline{\Omega}}|Du| \leq K$, for some $K = K(\varphi)$. We won't discuss this, but it is instructive to look at the example of the minimal surface equation to see how one might get this. This will then give us the coefficient bound, and the DGNM theory will then tell us that if we have a divergence form equation with bounded coefficients, then in fact the solution is $C^{0,\beta}$, with an estimate.
>
> Hence we are heading towards a result of the form: if $(a^{ij})_{ij}$ is strictly elliptic in Ω with $A^{ij}\xi^i\xi^j \geq \lambda|\xi|^2$, <u>and</u> $a^{ij} \in L^\infty_{\text{loc}}(\Omega)$, and if $w \in W^{1,2}_{\text{loc}}(\Omega)$ solves the divergence form equation $D_i(a^{ij}D_jw) = 0$ weakly in Ω, <u>then</u> DGMN tells us that $\exists \beta = \beta(\lambda, \Omega', K) \in (0,1)$ (where $K = \max_{i,j}\|a^{ij}\|_{L^\infty(\Omega')}$) such that $w \in C^{0,\beta}(\Omega')$, with an estimate on $|w|_{C^{1,\beta}(\Omega')}$. The β found here will then be the β we use for $X = C^{1,\beta}(\overline{\Omega})$ in the application of the Leray-Schauder fixed point theorem.

5.1. **De Giorgi-Nash-Moser Theory.**

We have the following set up. Consider a linear divergence form operator of the form:

$$Lu = D_i(a^{ij}D_ju)$$

on an open domain $\Omega \subset \mathbb{R}^n$.[(v)] We call the following hypothesis (H):

$$\text{(H)} \quad \begin{cases} a^{ij} \in L^\infty(\Omega) \\ a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega \text{ and some } \lambda > 0 \\ \sum_{i,j}\|a^{ij}\|^2_{L^\infty(\Omega)} \leq \Lambda \quad \text{for some } \Lambda > 0. \end{cases}$$

Definition 5.1. *We say that $u \in W^{1,2}(\Omega)$ is a* ***weak sub (super)solution to $Lu = 0$*** *in Ω if*

$$\int_\Omega a^{ij}D_iuD_jv \leq 0 \ \ (\geq 0) \qquad \forall v \in W^{1,2}_0(\Omega) \ \ \underline{\text{with } v \geq 0}.$$

> **Remark:** Clearly $u \in W^{1,2}(\Omega)$ is a weak solution to $Lu = 0$ if and only if u is a weak subsolution <u>and</u> a weak supersolution.

We now state the main results which we will prove.

[(v)]The theory does hold when L has lower order terms as well. Introducing a zeroth order term is a relatively easy extension of the theory we present here for such L, but adding a first order b^iD_iu term requires non-trivial technical changes. So to illustrate the main ideas, we just consider the above case.

Theorem 5.2 (Local Boundedness of subsolutions)**.** *Let (H) hold. Then if $u \in W^{1,2}(\Omega)$ is a weak subsolution to $Lu = 0$ in Ω, then for each $B_R(y) \subset\subset \Omega$ and $p > 1$ we have that:*

$$\sup_{B_{R/2}(y)} u \leq CR^{-\frac{n}{p}} \|u^+\|_{L^p(B_R(y))}$$

where $C = C(n, \lambda, \Lambda, p)$, and $u^+ = \max\{u, 0\}$ is the positive part of u.

Remark: The power of R in the above statement is the correct power to guarantee scale invariance of C, i.e. that C is independent of R.

Theorem 5.3 (Weak Harnack Inequality for non-negative supersolutions)**.** *Let (H) hold. Then if u is a non-negative weak supersolution to $Lu = 0$ in Ω, then for any $B_R(y) \subset\subset \Omega$ and any $p \in \left[1, \frac{n}{n-2}\right)$ we have:*

$$R^{-\frac{n}{p}} \|u\|_{L^p(B_R(y))} \leq C \inf_{B_{R/2}(y)} u$$

where $C = C(n, \lambda, \Lambda, p)$.

Theorem 5.3 will in fact be a consequence of Theorem 5.2.

Note: We are aiming for a Harnack inequality for non-negative weak solutions of $Lu = 0$. Theorem 5.3 tells us that if we can prove Theorem 5.2 for any $p > 1$, then applying both Theorem 5.2 and Theorem 5.3 we get a Harnack inequality for weak solutions. So the key tricky part is getting Theorem 5.2 for all $p > 1$. It isn't too hard to prove it for $p = 2$ and then for all $p \geq 2$ from the $p = 2$ case and Hölder's inequality, but it is more tricky to get that it holds for $p \in (1, 2)$.

So the straightforward corollary of these two results is the Harnack inequality for weak solutions:

Corollary 5.1 (Harnack Inequality for non-negative weak solutions)**.** *Let (H) hold. Suppose that $u \in W^{1,2}(\Omega)$ is a non-negative weak solution to $Lu = 0$ in Ω, for $\Omega \subset \mathbb{R}^n$ a domain. Then for any subdomain $\tilde{\Omega} \subset\subset \Omega$ we have:*

$$\sup_{\tilde{\Omega}} u \leq C \inf_{\tilde{\Omega}} u$$

where $C = C(n, \lambda, \Lambda, \tilde{\Omega}, \Omega)$.

Proof. By Theorem 5.2 and Theorem 5.3, if $B_{2R}(y) \subset\subset \Omega$, then choosing any $p \in \left(1, \frac{n}{n-2}\right)$ we have:

$$\sup_{B_R(y)} u \leq CR^{-\frac{n}{p}} \|u^+\|_{L^p(B_{2R}(y))} = CR^{-\frac{n}{p}} \|u\|_{L^p(B_{2R}(y))} \leq C \inf_{B_R(y)} u$$

where $C = C(u, \lambda, \Lambda)$. Given this inequality, we can then complete the proof in exactly the same way as in the Harnack inequality for harmonic functions, so we are done.

□

With these theorems we can prove the following regularity result (note that there is now no sign assumption on u - we will need to apply Theorem 5.3, for which we needed Theorem 5.2):

Theorem 5.4 (Hölder Continuity of Weak Solutions). *Let (H) hold. Then if $u \in W^{1,2}(\Omega)$ is a weak solution to $Lu = 0$ in Ω, then for any $B_R(x_0) \subset\subset \Omega$:*

(i) *For all $R \leq R_0$ we have*
$$\mathrm{osc}_{B_R(x_0)} u \leq C\left(\frac{R}{R_0}\right)^{\mu} \mathrm{osc}_{B_{R_0}(x_0)} u$$
for some $\mu = \mu(n,\lambda,\Lambda) \in (0,1]$ and $C = C(n,\lambda,\Lambda) > 0$.

(ii) *$u \in C^{0,\mu}(\Omega)$, with the estimate*
$$R_0^{\mu}[u]_{0,\mu;B_{R_0/4}(x_0)} \leq C \sup_{B_{R_0}(x_0)} |u|.$$

Recall: The **oscillation** of u is defined by:
$$\mathrm{osc}_{B_R(x)} u := \sup_{B_R(x)} u - \inf_{B_R(x)} u$$
i.e. just measures the biggest variation in the values of u on some set. Clearly this is related to Hölder continuity of u.

Proof of Theorem 5.4. (ii): This follows from (i). Indeed, for any $x, y \in B_{R_0/4}(x_0)$, set $d := \frac{5}{4}|x-y| \leq \frac{5}{4}\cdot 2(R_0/4) = \frac{5}{8}R_0$. Thus $y \in B_d(x) \subset B_{R_0}(x_0)$ and so:
$$\begin{aligned}\frac{|u(x)-u(y)|}{|x-y|^{\mu}} \leq \frac{\mathrm{osc}_{B_d(x)} u}{\left(\frac{4}{5}d\right)^{\mu}} &\leq \frac{C\cdot\left(\frac{d}{\frac{5}{8}R_0}\right)^{\mu}\cdot \mathrm{osc}_{B_{\frac{5R_0}{8}}(x)} u}{\left(\frac{4}{5}d\right)^{\mu}} \qquad \text{by (i) applied to } B_d(x),\ B_{\frac{5R_0}{8}}(x)\\ &= CR_0^{-\mu}\mathrm{osc}_{B_{\frac{5R_0}{8}}(x)} u\\ &\leq CR_0^{-\mu}\mathrm{osc}_{B_{R_0}(x_0)} u \qquad \text{as } B_{\frac{5R_0}{8}}(x) \subset B_{R_0}(x_0).\end{aligned}$$
Thus taking the supremum over all $x \neq y$ in $B_{R_0/4}(x_0)$ we get that
$$[u]_{0,\mu;B_{R_0/4}(x_0)} \leq CR_0^{-\mu}\mathrm{osc}_{B_{R_0}(x_0)} u$$
which then gives the estimate, since $\mathrm{osc}_{\Omega} u \leq 2\sup_{\Omega}|u|$.

(i): For R proportionally large compared to R_0, this is straightforward: for instance, if $R \geq R_0/4$ (any $R \geq \gamma R_0$ would work for $\gamma \in (0,1)$) we have
$$\begin{aligned}\mathrm{osc}_{B_R(x_0)} u &\leq \mathrm{osc}_{B_{R_0}(x_0)} u \qquad \text{as } R \leq R_0\\ &\leq 4^{\mu}\cdot\left(\frac{R}{R_0}\right)^{\mu}\cdot \mathrm{osc}_{B_{R_0}(x_0)} u \qquad \text{for any } \mu > 0 \text{ since } \frac{4R}{R_0} \geq 1\end{aligned}$$
which is what we wanted with $C = 4^{\mu}$.

So now assume that $R \leq \frac{R_0}{4}$. Then set:
$$M_1 := \sup_{B_R(x_0)} u, \quad m_1 := \inf_{B_R(x_0)} u, \quad M_4 := \sup_{B_{4R}(x_0)} u, \quad m_4 := \inf_{B_{4R}(x_0)} u.$$

Then both $M_4 - u$ and $u - m_4$ are non-negative in $B_{4R}(x_0)$ and satisfy (as L has no 0'th order term - if it did this is where we would need to modify the argument slightly):

$$L(M_4 - u) = 0 \quad \text{and} \quad L(u - m_4) = 0 \qquad \text{in } B_{4R}(x_0).$$

So by Theorem 5.3 with $p = 1$ we have:

$$R^{-n} \int_{B_{2R}(x_0)} (M_4 - u) \leq C \inf_{B_R(x_0)} (M_4 - u) = C(M_4 - M_1)$$

and

$$R^{-n} \int_{B_{2R}(x_0)} (u - m_4) \leq C \inf_{B_R(x_0)} (u - m_4) = C(m_1 - m_4).$$

Adding these expressions we get:

$$M_4 - m_4 \leq C(M_4 - m_4 - (M_1 - m_1))$$

which upon rearranging gives

$$(M_1 - m_1) \leq \gamma(M_4 - m_4) \qquad \text{where } \gamma := \frac{C-1}{C}$$

(note that since C comes from an upper bound, we can increase C such that the upper bound is still true - hence we can assume here that $C > 1$, so that $\gamma \in (0,1)$). This then is exactly saying:

$$\text{osc}_{B_R(x_0)} u \leq \gamma \cdot \text{osc}_{B_{4R}(x_0)} u.$$

Take $R = R_0/4$. Iterating the above inequality, we get for any $m \geq 1$:

$$\text{osc}_{B_{R_0/4^{m+1}}(x_0)} u \leq \gamma \cdot \text{osc}_{B_{R_0/4^m}(x_0)} u \leq \cdots \leq \gamma^m \text{osc}_{B_{R_0/4}(x_0)} u.$$

This has effectively proved the result for any R of the form $R = \frac{R_0}{4^m}$. To get the result for any $R \leq \frac{R_0}{4}$, we need to interpolate between these $R_0/4^m$ powers.

So for an arbitrary $R < \frac{R_0}{4}$, since the intervals $\left(\frac{R_0}{4^2}, \frac{R_0}{4}\right], \left(\frac{R_0}{4^3}, \frac{R_0}{4^2}\right], \ldots$ partition $\left(0, \frac{R_0}{4}\right]$, we can find $m \geq 1$ with

$$\frac{R_0}{4^{m+1}} < R \leq \frac{R_0}{4^m}.$$

In particular, this gives $m + 1 \geq \frac{\log(R_0/R)}{\log(4)}$. So in particular, if we set $\mu := \frac{\log(\gamma)}{\log(1/4)} > 0$ (and if C was again sufficiently large, i.e. $C \geq \frac{4}{3}$, then we have $\mu \leq 1$) then we have

$$\gamma^{m+1} \leq \left(\frac{R}{R_0}\right)^{\mu}$$

(since $\gamma \in (0,1)$) and thus since $B_R(x_0) \subset B_{R_0/4^m}(x_0)$ we have:

$$\begin{aligned} \text{osc}_{B_R(x_0)} u &\leq \text{osc}_{B_{R_0/4^m}(x_0)} u \qquad \text{as } B_R(x_0) \subset B_{R_0/4^m}(x_0) \\ &\leq \gamma^m \text{osc}_{B_{R_0}(x_0)} u \qquad \text{by the above} \\ &\leq \frac{1}{\gamma} \cdot \left(\frac{R}{R_0}\right)^{\mu} \cdot \text{osc}_{B_{R_0}(x_0)} u \qquad \text{by choice of } \mu \end{aligned}$$

and this is in the form of the inequality we want, and this holds for all $R < R_0/4$. Hence combining this with the $R \geq R_0/4$ case (take the same μ, and then take the largest constant) we get (i).

□

Note: The boundedness of the supremum's here will come from an application of Theorem 5.2.

Remark: For applications to the quasilinear equations (e.g. as in the Leray-Schauder method) we need a global version of Theorem 5.4, which would follow from a global version of Theorem 5.3. But here we omit that completely and only discuss local results as stated in Theorem's 5.2, 5.3, 5.4.

Proof of Theorem 5.2. First note that u^+ is also a weak subsolution [**Exercise** - see Example Sheet 4]. So we can assume wlog that $u \geq 0$ a.e., since $\sup_{B_{R/2}(y)} u \leq \sup_{B_{R/2}(y)} u^+$, and thus if we can prove the result for u^+ (i.e. the result when we assume $u \geq 0$) we get the result for any u. (We will need $u \geq 0$ so that we can take a test function involving u in the definition of subsolution).

Assume further that $u > \varepsilon$ a.e. in Ω, for small $\varepsilon > 0$. We can do this since once we know the result is true under this assumption, for general $u \geq 0$ we can apply this result to $u + \varepsilon$, and then let $\varepsilon \downarrow 0$ to get the result for general $u \geq 0$.

So we have $u > \varepsilon$ a.e. in Ω. As usual by translating and scaling we can assume wlog that $y = 0$ and $R = 1$. The idea is to try and get a uniform bound on the L^p norms of u for all p sufficiently large, and then take $p \to \infty$ for which the L^p norms then converge to the supremum. The Moser iteration trick we use ensures that we any sets we are working in don't collapse down to a point.

So to get bounds on the L^p norms of u for p large, we would like to take $v = u^\beta \eta$ in the definition of a subsolution, for some (large) $\beta > 0$ and where $\eta \in C_c^1(B)$, $B = B_1(0)$. But we will need to be more careful. So to stop this from getting too large, we cut it off via defining for $k > 0$,

$$v_k := \min\{u^\beta, ku\}.$$

Note that for $v_k \to u^\beta$ pointwise as $k \to \infty$, since $u > \varepsilon$ and so $u \neq 0$ anywhere.

Claim: $v_k \in W^{1,2}(B)$ with

$$Dv_k = \begin{cases} \beta u^{\beta-1} Du & \text{if } x \in \Omega_k \\ kDu & \text{if } x \in B \backslash \Omega_k \end{cases}$$

where $\Omega_k := \{x \in B : u^\beta(x) \leq ku(x)\}$ (i.e. $v_k = u^\beta$ on Ω_k).

To see this, note $v_k = u \min\{u^{\beta-1}, k\} = u w_k^{\beta-1} = ug(w_k)$ where $w_k = \min\{u, k^{\frac{1}{\beta-1}}\}$ and $g(t) := t^{\beta-1}$ for $t \in \left[\varepsilon, k^{1/(\beta-1)}\right]$. Then since g and g' are bounded in $[\varepsilon, k^{1/(\beta-1)}]$, it follows from standard facts about Sobolev spaces that $g(w_k) \in W^{1,2}(B)$ and thus $ug(w_k) \in W^{1,2}(B)$.

Now let $\eta \in C_c^1(B)$, $\eta \geq 0$, and take $v = v_k \eta^2$ in the definition of a subsolution (we can do this as $u \geq 0$). Then we have

$$\int_B a^{ij} D_i u D_j(v_k \eta^2) \leq 0$$

which gives after expanding:

$$\beta \int_{\Omega_k} a^{ij} D_i u \cdot u^{\beta-1} D_j u \cdot \eta^2 + k \int_{B \backslash \Omega_k} a^{ij} D_i u D_j u \cdot \eta^2 \leq -2 \int_B a^{ij} D_i u \cdot v_k \cdot \eta D_j \eta$$

So using ellipticity in the terms on the LHS, the bound on the a^{ij} on the RHS, and the definition of v_k, we get

$$\lambda\beta\int_{\Omega_k}|Du|^2u^{\beta-1}\eta^2+\lambda k\int_{B\backslash\Omega_k}|Du|^2\eta^2\leq 2\Lambda\int_{\Omega_k}|Du||D\eta|\eta u^\beta+2\Lambda k\int_{B\backslash\Omega_k}|Du||D\eta|\cdot\eta u.$$

Now we do the usual thing, applying $ab\leq\varepsilon a^2+\frac{1}{4\varepsilon}b^2$ for suitable $\varepsilon>0$ twice on the RHS to absorb the Du terms onto the LHS. So dividing the above line by λ and applying such an inequality (matching up terms with those on the LHS) we get:

$$\begin{aligned}\beta\int_{\Omega_k}|Du|^2u^{\beta-1}\eta^2+k\int_{B\backslash\Omega_k}|Du|^2\eta^2&\leq\frac{2\Lambda}{\lambda}\left(\int_{\Omega_k}|Du|\eta u^{\frac{\beta-1}{2}}\cdot u^{\frac{\beta+1}{2}}|D\eta|+k\int_{B\backslash\Omega_k}|Du|\eta\cdot|D\eta|u\right)\\&\leq\frac{\beta}{2}\int_{\Omega_k}|Du|^2\eta^2u^{\beta-1}+\frac{C}{\beta}\int_{\Omega_k}u^{\beta+1}|D\eta|^2\\&\qquad+\frac{k}{2}\int_{B\backslash\Omega_k}|Du|^2\eta^2+Ck\int_{B\backslash\Omega_k}|D\eta|^2u^2\end{aligned}$$

and then we can absorb the 1st and 3rd term on the RHS onto the LHS (this moves all derivatives of u onto the LHS). So we end up with:

$$\frac{\beta}{2}\int_{\Omega_k}|Du|^2u^{\beta-1}\eta^2+\frac{k}{2}\int_{B\backslash\Omega_k}|Du|^2\eta^2\leq\frac{C}{\beta}\int_{\Omega_k}u^{\beta+1}|D\eta|^2+Ck\int_{B\backslash\Omega_k}|D\eta|^2u^2.$$

Thus in particular, as the 2nd term on the LHS is positive, this tells us that

$$\frac{\beta}{2}\int_{\Omega_k}|Du|^2u^{\beta-1}\eta^2\leq\frac{C}{\beta}\int_{\Omega_k}u^{\beta+1}|D\eta|^2+Ck\int_{B\backslash\Omega_k}|D\eta|^2u^2.$$

Now we want to let $k\uparrow\infty$. Note for the 2nd term on the RHS we know on $B\backslash\Omega_k$ that $ku\leq u^\beta$, and so we get

$$\frac{\beta}{2}\int_{\Omega_k}|Du|^2u^{\beta-1}\eta^2\leq\frac{C}{\beta}\int_{\Omega_k}u^{\beta+1}|D\eta|^2+C\int_{B\backslash\Omega_k}|D\eta|^2u^{\beta+1}.$$

Now since Ω_k is increasing in k and $\bigcup_{k>0}\Omega_k=B$, so $\mathbb{1}_{\Omega_k}\to\mathbb{1}_B$ pointwise as $k\to\infty$, by Fatou's lemma, for the LHS we have

$$\int_B|Du|^2u^{\beta-1}\eta^2=\int_B\liminf_j\left(\mathbb{1}_{\Omega_j}|Du|^2u^{\beta-1}\eta^2\right)\leq\liminf_j\int_B\mathbb{1}_{\Omega_j}|Du|^2u^{\beta-1}\eta^2=\liminf_j\int_{\Omega_j}|Du|^2u^{\beta-1}\eta^2$$

and by dominated convergence on the RHS we get that, <u>provided</u> $\int_B u^{\beta+1}|D\eta|^2<\infty$ (so that we can use $u^{\beta+1}|D\eta|^2$ as a dominating function for each $\mathbb{1}_{\Omega_j}u^{\beta+1}|D\eta|^2$), that

$$\int_{\Omega_j}u^{\beta+1}|D\eta|^2\to\int_B u^{\beta+1}|D\eta|^2\quad\text{and}\quad\int_{B\backslash\Omega_j}u^{\beta+1}|D\eta|^2\to 0$$

since $\mathbb{1}_{B\backslash\Omega_j}\to 0$ pointwise. Thus combining we see:

$$\int_B|Du|^2u^{\beta-1}\eta^2\leq\frac{C}{\beta^2}\int_B u^{\beta+1}|D\eta|^2$$

where $C=C(\Lambda/\lambda)$. The finiteness assumption on the integral will be important for the iterating scheme, as we are going to see shortly.

Write $\beta + 1 =: \alpha$. So we have:

$$\int_B |Du|^2 u^{\alpha-2}\eta^2 \leq \frac{C}{(\alpha-1)^2}\int_B u^\alpha |D\eta|^2, \quad \underline{\text{provided}} \quad \int_B u^\alpha |D\eta|^2 < \infty.$$

Now note, since $(a+b)^2 \leq 2a^2 + 2b^2$:

$$\begin{aligned}\int_B \left|D\left(u^{\alpha/2}\eta\right)\right|^2 &\leq \frac{\alpha^2}{2}\int_B \left(u^{\alpha-2}|Du|^2\eta^2 + u^\alpha|D\eta|^2\right)\\ &\leq \frac{C\alpha^2}{(\alpha-1)^2}\int_B u^\alpha|D\eta|^2 \qquad \text{by the above line.}\end{aligned}$$

Now let $\sigma := \frac{n}{n-2}$ if $n \geq 3$ (and $\sigma > 1$ is arbitrary if $n = 2$). Then from the Sobolev inequalities we know (see Analysis of PDEs notes, the GNS inequality), we have

$$\|u^{\alpha/2}\eta\|_{L^{2\sigma}(\mathbb{R}^n)} \leq C\|D(u^{\alpha/2}\eta)\|_{L^2(\mathbb{R}^n)}$$

and then since η has support in B, this gives using the above bound on the L^2 norm on the RHS

$$\left(\int_B \left(u^{\alpha/2}\eta\right)^{2\sigma}\right)^{\frac{1}{\sigma}} \leq \frac{C\alpha^2}{(\alpha-1)^2}\int_B u^\alpha|D\eta|^2$$

i.e.

$$\left(\int_B u^{\alpha\sigma}\eta^{2\sigma}\right)^{\frac{1}{\sigma\alpha}} \leq \left(\frac{C\alpha^2}{(\alpha-1)^2}\right)^{1/\alpha}\left(\int_B u^\alpha|D\eta|^2\right)^{1/\alpha}.$$

Now since $\sigma > 1$, $\alpha\sigma > \alpha$, so we have increased the power of u on the LHS. Note that the above held whenever the integral on the RHS is finite. We want to iterate this, but we need to be careful that due to the test function on the LHS, the LHS integral only sees u in some smaller subset than B. But equally to iterate it we need to ensure that the integral on the RHS is finite, so we might need to shrink the ball at each stage to iterate (to guarantee this finiteness). But also when iterating, we need to be careful that the test functions don't cause the region to decay to a point. We will do this with a clever choice of test function.

For any $r' < r < 1$, choose η such that $\eta|_{B_{r'}} \equiv 1$, $\eta|_{B\setminus B_r} \equiv 0$, and $|D\eta| \leq \frac{2}{r-r'}$. Then we get from the above:

$$\left(\int_{B_{r'}} u^{\alpha\sigma}\right)^{\frac{1}{\sigma\alpha}} \leq \left(\frac{C\alpha^2}{(\alpha-1)^2}\right)^{1/\alpha}\frac{2^{2/\alpha}}{(r-r')^{2/\alpha}}\left(\int_{B_r} u^\alpha\right)^{1/\alpha}, \quad \underline{\text{provided}} \quad \int_{B_r} u^\alpha < \infty.$$

(Thus we see why we need to be careful about the domain not collapsing to a point: we increase the power on the RHS at any given stage, but then we need to shrink the radius to iterate, as to iterate we need to ensure that the integral to a larger power is finite. We want to increase the exponents say that we can hit any exponent p we want. We also need to be careful that the constants on the RHS don't explode. We choose nice radii to ensure this.)

So let $r_j = \frac{1}{2} + \frac{1}{2^{j+1}}$ for $j = 0, 1, 2, \ldots$. Use the above with $r' = r_j$, $r = r_{j-1}$. Since $\beta > 0$ was arbitrary, $\alpha = \beta + 1 > 1$ is arbitrary. So take $\alpha = p\sigma^{j-1}$, $p > 1$. Then we get:

$$\left(\int_{B_{r_j}} u^{p\sigma^j}\right)^{\frac{1}{p\sigma^j}} \leq C(p)^{\frac{1}{p\sigma^{j-1}}}\cdot 2^{\frac{2(j+2)}{p\sigma^{j-1}}}\left(\int_{B_{r_{j-1}}} u^{p\sigma^{j-1}}\right)^{\frac{1}{p\sigma^{j-1}}}$$

where we have a new constant $C = C(p)$ on the RHS. This comes from the fact that α increases with j, and thus $\frac{\alpha}{\alpha-1} = 1 + \frac{1}{\alpha-1}$ decreases with j, and thus is bounded by its value when $j = 1$, i.e. $\frac{p}{p-1}$.

Now iterate this inequality, as it was true for any $j \geq 1$. Then we get (since the domain on the LHS integral always contains $B_{1/2}$ and so provides an upper bound on it, and on the RHS it is always contained in B):

$$\left(\int_{B_{1/2}} u^{p\sigma^j}\right)^{\frac{1}{p\sigma^j}} \leq C(p)^{\sum_{l=1}^{\infty} \frac{1}{p\sigma^{l-1}}} \cdot 2^{\frac{2}{p}\sum_{l=1}^{\infty} \frac{l+2}{\sigma^{l-1}}} \cdot \left(\int_B u^p\right)^{1/p}.$$

The constants on the RHS are finite as the infinite sums we say are finite. Now let $j \to \infty$, since in general $\lim_{q\to\infty} \|w\|_{L^q} \to \|w\|_{L^\infty}$, the LHS $\to \sup_{B_{1/2}} |u|$ (as $p\sigma^j \to \infty$), which is what we wanted for Theorem 5.2.

□

Remark: The about technique of choosing the test functions and their supports in this way is known as **Moser's iteration technique**. It is clever because normally we would take the test functions to half the ball radius at any point, but this would not work in the above as then the domains would collapse to a point. So by choosing the radii in this different way, we enable that the domain actually collapses onto $B_{1/2}$.

Remark: The power of Theorem 5.2 and the other theorems is that we get these regularity results without any regularity assumptions on the coefficients a^{ij}.

Now let us look at Theorem 5.3. To prove this, we need:

Lemma 5.1 (John-Nirenberg). *Suppose $u \in W^{1,1}(B)$, where $B = B_1(0)$, and that $\exists M$ such that*

$$\frac{1}{\rho^{n-1}}\int_{B_\rho(y)} |Du| \leq M \qquad \text{for all } B_\rho(y) \subset B.$$

Then we have:

$$\int_B e^{\frac{p_0}{M}|u-u_B|} \leq C$$

where $u_B = \frac{1}{|B|}\int_B u$ and $p_0 = p_0(n)$, $C = C(n)$.

Proof. None given. □

Sketch Proof of Theorem 5.3. Wlog translate and rescale so that $y = 0$ and $R = 2$. So $u \in W^{1,2}(B_2)$, $u \geq 0$, and u is a supersolution. Again as in the proof of Theorem 5.2, assume that $u \geq \varepsilon$, or else we can apply the theorem to $u + \varepsilon$ and then let $\varepsilon \downarrow 0$.

Let $v = \frac{1}{u}$. Then by replacing the test function φ in the definition of u being a supersolution by $v^2\varphi$, we get by direct calculation that v satisfies for any $\varphi \geq 0$ of compact support in B_2 (from the supersolution condition of u):

$$\int_{B_2} a^{ij} D_i v D_i \varphi \leq -2\int_{B_2} \frac{a^{ij} D_i v D_j v}{v} \cdot \varphi \leq -2\lambda \int_{B_2} \frac{|Dv|^2}{v}\varphi \leq 0$$

and so v is a subsolution.

Then by Theorem 5.2 we have:

$$\sup_{B_{3/4}} v \leq C\left(\fint_{B_{3/2}} v^p\right)^{1/p}$$

where $C = C(n,p,\Lambda/\lambda)$, and thus as $v = 1/u$ this gives:

$$\begin{aligned} C\inf_{B_{3/4}} u &\geq \left(\fint_{B_{3/2}} u^{-p}\right)^{-\frac{1}{p}} \\ &= \left(\fint_{B_{3/2}} u^p\right)^{1/p} \times \left[\left(\fint_{B_{3/2}} u^p\right)\cdot\left(\fint_{B_{3/2}} u^{-p}\right)\right]^{-\frac{1}{p}} \end{aligned}$$

So we are naturally left with looking at the awkward term on the RHS. We want to then prove the following, which would complete the result:

Claim: $\exists p_0 > 0$ such that for all $p \in (0, p_0]$,

$$\left(\fint_{B_{3/2}} u^{-p}\right)\left(\fint_{B_{3/2}} u^p\right) \leq C(n,\Lambda/\lambda).$$

Indeed to see this, let $w = \log(u) - \frac{1}{|B_{3/2}|}\int_{B_{3/2}} \log(u)$ (we will be able to do all of this calculation because $u > \varepsilon$). Then by direct calculation, using the fact that u is a supersolution so

$$\int_{B_2} a^{ij} D_i u D_j \varphi \geq 0$$

and using this with $u^{-1}\varphi$ instead of φ, we get

$$\int_{B_2} a^{ij} u D_i w\left(u^{-1} D_j\varphi - \frac{\varphi}{u^2} D_j u\right) \geq 0$$

which implies

$$\int_{B_2} a^{ij} D_i w D_j \varphi \geq \int_{B_2} a^{ij} D_i w D_j w \cdot \varphi \geq \lambda \int_{B_2} |Dw|^2 \varphi.$$

Now replace φ by φ^2 to get

$$2\int_{B_2} a^{ij} D_i w D_j \varphi \cdot \varphi \geq \lambda \int_{B_2} |Dw|^2 \varphi^2.$$

Using the usual inequality bounds on $2ab$ on the LHS we get

$$\int_{B_2} |Dw|^2\varphi^2 \leq \frac{\Lambda}{\lambda}\int_{B_2} |Dw|\cdot|D\varphi|\cdot|\varphi| \leq \frac{1}{2}\int_{B_2} |Dw|^2\varphi^2 + C\cdot\frac{\Lambda}{\lambda}\int_{B_2} |D\varphi|^2$$

and so we get:

$$\int_{B_2} |Dw|^2\varphi^2 \leq C\int_{B_2} |D\varphi|^2.$$

So since φ is so far any arbitrary positive test function, given any $B_\rho(y) \subset B_{3/2}(0)$ we can choose $\varphi|_{B_\rho(y)} \equiv 1$, $\varphi|_{B_2\setminus B_{5\rho/4}(y)} \equiv 0$ with $|D\varphi| \leq \frac{C}{\rho}$. Then we get

$$\int_{B_\rho(y)} |Dw|^2 \leq C\rho^{n-2}$$

where $C = C(n)$. Thus we get from Hölder's inequality:

$$\int_{B_\rho(y)} |Dw| \leq \left(\int_{B_\rho(y)} 1^2\right)^{1/2} \cdot \left(\int_{B_\rho(y)} |Dw|^2\right)^{1/2} \leq C\,(\rho^n)^{1/2} \cdot C\left(\rho^{n-2}\right)^{1/2} = C\rho^{n-1}.$$

Thus we have the criterion of John-Nirenberg, and so we get

$$\int_{B_{3/2}} e^{p_0|w|} \leq C$$

for some $p_0 = p_0(n) > 0$ and $C = C(n)$ (note $w_{B_{3/2}} = 0$ by definition of w). Thus we get

$$\left(\int_{B_{3/2}} u^{-p_0}\right) \cdot \left(\int_{B_{3/2}} u^{p_0}\right) = \left(\int_{B_{3/2}} e^{-p_0(w+b)}\right) \cdot \left(\int_{B_{3/2}} e^{p_0(w+b)}\right) \leq \left(\int_{B_{3/2}} e^{p_0|w|}\right)^2 \leq C$$

since $e^{-p_0 w}, e^{p_0 w} \leq e^{p_0|w|}$ and the $e^{\pm p_0 b}$ terms are constant which cancel out. This gives the result for $p = p_0$. To get this for all $p \leq p_0$, we just apply Hölder's inequality to reduce the p case to the p_0 case, i.e.

$$\int_{B_{3/2}} u^{\pm p} = \|u^{\pm p}\|_{L^1(B_{3/2})} \leq \|u^{\pm p}\|_{L^{p_0/p}(B_{3/2})} \cdot \|1\|_{L^{(p_0/p)^*}(B_{3/2})} = C\|u^{\pm p_0}\|^{p/p_0}_{L^1(B_{3/2})}.$$

Combined with what we previously had, this proves the theorem for $p \leq p_0$. To get it for any $p \in \left(0, \frac{n}{n-2}\right)$ we apply the Moser iteration trick again to increase the power. So use the supersolution condition

$$\int_{B_2} a^{ij} D_i u D_j \varphi \geq 0$$

with $\varphi = u^{-\beta}\eta^2$ for $\beta \in (0,1)$, and proceed exactly as in the proof of Theorem 5.2. What we end up with is:

$$\left(\int_{B_{r'}} u^{\sigma\gamma}\right)^{\frac{1}{\sigma\gamma}} \leq \frac{1}{(r-r')^{\frac{2}{\gamma}}} \cdot C^{\frac{1}{\gamma}} \cdot \left(\int_{B_r} u^{\gamma}\right)^{\frac{1}{\gamma}}$$

where again $\sigma = \frac{n}{n-2}$ if $n \geq 3$ and $\sigma > 1$ is arbitrary if $n = 2$. Here, $\gamma = 1 - \beta \in (0,1)$, which is why we need $\beta < 1$. Then choosing $\gamma = \sigma^{j-1}\min\{1, p_0\}$, $r' = 1 + \frac{1}{2^{j+1}}$ and $r = 1 + \frac{1}{2^j}$, iterate this a finite number of times N (enough times to ensure that the power $\sigma\gamma = \sigma^j \min\{1, p_0\}$ is $\geq p_1$, for a given $p_1 \in \left(0, \frac{n}{n-2}\right)$) and then applying Hölder's inequality, we get the result for any such p and so we are done.

□

End of lecture course.