ASYMPTOTIC STABILITY OF HARMONIC MAPS FOR THE SCHRÖDINGER MAPS EQUATION IN EQUIVARIANT SYMMETRY

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ABSTRACT. Following [GNT10], we exposit the proof of asymptotic stability for harmonic maps under the Schrödinger maps equation in m-equivariant symmetry for $m \geq 3$.

CONTENTS

1. INTRODUCTION

In [LL35], Landau and Lifshitz proposed a description for the dynamics of the magnetisation vector[1] in an isotropic ferromagnet. The eponymous *Landau-Lifshitz* equation is given by

$$\partial_t u = -\alpha\big(u \times (u \times \Delta u)\big) + \beta\big(u \times \Delta u\big), \tag{LL}$$

eq:LL

where $u : I \times \mathbb{R}^2 \to \mathbb{S}^2$ is the *magnetisation vector*. The parameter $\alpha \geq 0$ is the *Gilbert damping*, representing the strength of dissipation in the model, while the parameter $\beta \in \mathbb{R}$ is the *exchange constant*, representing the strength of dispersion. In the physics literature, the equation is often rescaled so that the parameters are balanced such that $\alpha^2 + \beta^2 = 1$, though mathematically this will be inconsequential for our discussion.

From the perspective of geometric equations, the family of Landau-Lifshitz equations interpolates between the dispersive *Schrödinger maps*, $(\alpha, \beta) = (1, 0)$, and the dissipative *harmonic maps heat flow*, $(\alpha, \beta) = (0, 1)$,

$$\partial_t u = \Delta u + |\nabla u|^2 u, \tag{HMHF}$$

$$\partial_t u = u \times \Delta u. \tag{SM}$$

eq:schrodinge

The equation is naturally associated to the Dirichlet energy,

$$\mathcal{E}[u] := \frac{1}{2}\int_{\mathbb{R}^2} |\partial_1 u|^2 + |\partial_2 u|^2 \, dx,$$

which, for finite-energy (strong) solutions $u \in C^0_t \dot{H}^1_x(I \times \mathbb{R}^2)$ to the Landau-Lifshitz equation (LL), satisfies the energy-balance identity

$$\mathcal{E}[u(t)] + 2\alpha \int_0^t \int_{\mathbb{R}^2} |u \times (u \times \Delta u)|^2 \, dxds = \mathcal{E}[u_0]. \tag{E}$$

eq:energy

Date: October 29, 2024.

[1]This is the direction in which the magnetic moment of a ferromagnet "prefers" to align.

In particular, the energy is non-increasing, and in fact conserved in the dispersive case. Both the equation (LL) and the Dirichlet energy are invariant under the rescaling

$$u(t,x) \mapsto u(t/\lambda^2, x/\lambda),$$

i.e. (LL) is *energy-critical*. It is then natural to study the dynamics of the Landau-Lifshitz equation (LL) in the energy topology $\dot{H}^1(\mathbb{R}^2)$. The energy space further decomposes into infinitely-many connected components $\dot{H}^1_m(\mathbb{R}^2)$ indexed by their topological class $\deg(u) \equiv m$. The degree of a finite-energy map $u \in \dot{H}^1(\mathbb{R}^2)$ is given by

$$\deg(u) := \frac{1}{4\pi} \int_{\mathbb{R}^2} u \cdot (\partial_1 u \times \partial_2 u)\, dx.$$

For continuous maps, the degree captures the number of times the plane wraps around the sphere under $u : \mathbb{R}^2 \to \mathbb{S}^2$ (see Appendix A.2).

The Landau-Lifshitz equation admits stationary solutions in the form of *harmonic maps*. Remarkably, maps from the plane to the sphere $u : \mathbb{R}^2 \to \mathbb{S}^2$ are *self-dual* in the sense that the second-order harmonic maps equation admits a reduction to the first-order *Cauchy-Riemann* equations. One can read off the reduction from the Bogomoln'yi identity (see Appendix A.2),

$$\mathcal{E}[u] = \frac{1}{2} \int_{\mathbb{R}^2} |\partial_1 u - u \times \partial_2 u|^2 + 4\pi \deg(u). \tag{B}$$

eq:B

It follows that solutions to the Cauchy-Riemann equations

$$\partial_1 u = u \times \partial_2 u, \tag{HM}$$

eq:CR

are minimisers of Dirichlet energy within each topological class, i.e. the space of finite energy configurations with fixed degree $\deg(u) \equiv N$. The equations (HM) can be identified with usual formulation of the Cauchy-Riemann equations after stereographic projection $\mathbb{S}^2 \to \mathbb{C}$, under which the solutions to (HM) correspond precisely to rational functions on $\mathbb{C}$. The basic m-equivariant solutions correspond to the maps $z \mapsto z^m$, taking the form

$$Q^m(r,\theta) := e^{m\theta R} Q^m(r), \qquad Q^m(r) := \begin{pmatrix} h_1^m \\ 0 \\ h_3^m \end{pmatrix},$$

where

$$h_1^m(r) := \frac{2r^m}{r^{2m}+1}, \qquad h_2^m(r) := \frac{r^{2m}-1}{r^{2m}+1},$$

and R is the generator for rotation about the z-axis of the sphere $\mathbb{S}^2$.

In view of the energy-balance identity (E), variational heuristics suggest that harmonic maps are stable under the Landau-Lifshitz equation (LL) in the energy topology within each topological class. This is the main subject of this article.

1.1. **Main result.** Following [GNT10], we give an exposition on the *asymptotic stability* of the *harmonic maps* under the Landau-Lifshitz equation (LL) in *equivariant symmetry*.

To set the stage, the m-equivariant maps take the form,

$$u(t,x) = e^{m\theta R} u(t,r),$$

where R is the generator for rotation about the z-axis of the sphere $\mathbb{S}^2$. For such maps, the energy norm reduces to

$$||u||^2_{\dot{H}^1_m} := ||\partial_r u||^2_{L^2_x} + ||\tfrac{u}{r}||^2_{L^2_x}.$$

The Landau-Lifshitz equation (LL) is invariant under rotation, scaling, and translation. Observe that working in equivariant symmetry kills the last symmetry, so the *moduli space of solitons* $\mathcal{Q}_m \subseteq \dot{H}^1_m(\mathbb{R}^2)$ reduces to the two-dimensional family

$$\mathcal{Q}_m := \{Q^m_{\alpha,\lambda} \in \dot{H}^1_m(\mathbb{R}^2) : \alpha \in \mathbb{S}^1 \text{ and } \lambda \in (0,\infty)\},$$

parametrised by the scaling parameter $\lambda \in (0,\infty)$ and rotation parameter $\alpha \in \mathbb{S}^1$ as follows,

$$Q^m_{\alpha,\lambda}(r,\theta) := e^{\alpha R} Q^m(r/\lambda, \theta).$$

The problem of stability for $\mathcal{Q}_m$ under (LL) concerns the dynamics of m-equivariant solutions to (LL) with energy close to the ground state,

$$\mathcal{E}[u] - \mathcal{E}[Q^m] \ll 1. \tag{1.1}$$

eq:close

In [GKT07], Gustafson-Kang-Tsai applied standard variational arguments to show that the family of harmonic maps $\mathcal{Q}_m$ is *orbitally stable* under the equation (LL) in the sense that solutions with energy close to the ground state, i.e. satisfying (1.1), are in fact confined within a small neighborhood of the moduli space of solitons. More precisely,

$$\inf_{\alpha,\lambda} ||Q^m_{\alpha,\lambda} - u||^2_{\dot{H}^1} \lesssim \mathcal{E}[u] - \mathcal{E}[Q^m] \ll 1. \tag{1.2}$$

eq:orbital

Projecting the solution onto $\mathcal{Q}_m$, the solution schematically takes the form

$$u(t) = \underbrace{Q_{\alpha(t),\lambda(t)}}_{\text{projection to } \mathcal{Q}_m} + \underbrace{\varepsilon(t)}_{\text{small error in } \dot{H}^1} \tag{1.3}$$

eq:decomp

The orbital stability result (1.2) does not say much about the precise global dynamics of near solitons since scaling is a non-compact symmetry. Possibilities to consider include

- Blow-up in either finite-time or infinite time, e.g. concentration into small scales,
$$\lambda(t) \to 0,$$
Finite-time blow-up is possible in the case $m = 1$, see [MRR11, Per14]. Infinite-time blow-up is possible in the case $m = 2$, see [GNT10, Theorem 2].
- Breather-type solutions, e.g. oscillation between disparate scales,
$$\liminf \lambda(t) < \limsup \lambda(t).$$
This is possible in the case $m = 2$, see [GNT10, Theorem 2].
- Global existence and asymptotic stability, e.g. the modulation parameters converge and the error in (1.3) disperses,
$$(\alpha(t), \lambda(t)) \to (\alpha_\infty, \lambda_\infty), \qquad \text{and} \qquad ||\varepsilon||_{\mathcal{S}_{t,x}([0,\infty)\times\mathbb{R}^2)} < \infty.$$
This holds for $m \geq 3$, see [GNT10, Theorem 1].

Remark. Let us also point the interested reader to the results of Bejenaru-Tataru [BT14] on the case $m = 1$, and the recent preprint of Bejenaru-Tataru-Pillai [BPT24] which deals with the most delicate case $m = 2$.

We are primarily interested in the asymptotic stabilty result of [GNT10, Theorem 1] for Schrödinger maps (SM), i.e. the purely dispersive case of the Landau-Lifshitz equation (LL). The dissipative case is an easy modification;

thm:main **Theorem 1.1** (Asymptotic stability of harmonic maps in equivariant symmetry)**.** *Let $m \geq 3$, then given initial data $u_0 \in \dot{H}^1_m(\mathbb{R}^2)$ with energy close to the ground state*

$$\mathcal{E}[u_0] - \mathcal{E}[Q^m] \ll 1,$$

there exists a global solution $u \in C^0_t(\dot{H}^1_m)_x([0,\infty) \times \mathbb{R}^2)$ to the Schrödinger maps equation (SM) converging to a fixed soliton $Q^m_{\alpha_\infty,\lambda_\infty} \in \mathcal{Q}_m$ in the sense that

$$||u(t) - Q_{\alpha_\infty,\lambda_\infty}||_{L^\infty_x} \xrightarrow{t\to\infty} 0. \tag{1.4}$$

eq:convergenc

2. Preliminaries

2.1. **Frame method.** We begin with some geometric generalities; for more details, see the monograph of Tao [Tao06, Chapter 6.2] and the notes of Tataru [KTV14, Geometric Dispersive Equations]. Given a map $u : I \times \mathbb{R}^2 \to \mathbb{S}^2$, we will use $\mathbf{D}_j$ to denote the pullback of the connection on the sphere, and $\mathbf{J}$ the complex structure given by $\frac{\pi}{2}$-rotation. Using this notation, the equation (SM) takes the form

$$\partial_t u = \mathbf{J}(\mathbf{D}_1 \partial_1 u + \mathbf{D}_2 \partial_2 u). \tag{2.1}$$

eq:geoschrod

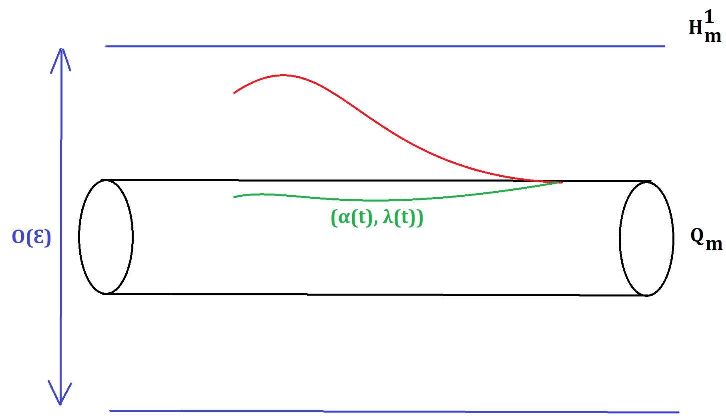

FIGURE 1.1. The near soliton dynamics of (LL) can be decomposed into an ODE for the dynamics on the moduli space and a PDE for the error.

To reveal the linear Schrödinger structure of the equation, it is convenient to fix an orthonormal frame $\{\mathbf{v}, \mathbf{w}\} \subseteq u^* T\mathbb{S}^2$ on the pullback bundle. Using this frame, we can identify each tangent space $T_{u(t,x)}\mathbb{S}^2$ with $\mathbb{C}$ via the isomorphism

$$T_{u(t,x)}\mathbb{S}^2 \longrightarrow \mathbb{C},$$
$$X^1\mathbf{v} + X^2\mathbf{w} \longmapsto X^1 + iX^2.$$

Thus we can identify the field $v : I \times \mathbb{R}^2 \to u^* T\mathbb{S}^2$ with a complex scalar field $\psi : I \times \mathbb{R}^2 \to \mathbb{C}$,

$$\psi := \langle v, \mathbf{v} \rangle \mathbf{v} + i\langle v, \mathbf{w} \rangle \mathbf{w}.$$

In particular, we will denote ψ_α for the differentiated fields $\partial_\alpha u$. In this frame, the action of the covariant derivative on the field v corresponds to the following operation on the complex scalar field ψ,

$$\mathbf{D}_\alpha v \longmapsto (\partial_\alpha + iA_\alpha)\psi,$$

where the frame coefficients $A_\alpha := \langle \mathbf{e}_2, \partial_\alpha \mathbf{e}_1 \rangle$ track the change in the frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ relative to the vector field ∂_α. We abuse notation by denoting the operator on the right by $\mathbf{D}_\alpha\psi$. *A priori*, the connection coefficients satisfy the curl system

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha = \operatorname{Im}(\psi_\alpha \overline{\psi_\beta}), \tag{2.2}$$ eq:curvature

Note that thus far we have not fixed a frame; indeed, any other frame could be obtained from rotating by $e^{i\chi}$. Varying the gauge choice leads to the gauge invariance

$$\psi \mapsto e^{i\chi}\psi, \qquad A_\alpha \mapsto A_\alpha + \partial_\alpha \chi.$$

Fixing a gauge fully determines the connection coefficients in terms of the original field u. We will work with Coulomb gauge,

$$\partial_1 A_1 + \partial_2 A_2 = 0, \tag{2.3}$$ eq:coulomb

which, upon differentiating the curvature relation (2.2) and inverting the Laplacian, we see that the connection coefficients are given precisely by

$$A_\alpha[\psi] = -\frac{1}{2}\Delta^{-1}\partial^\beta \operatorname{Im}(\psi_\alpha \overline{\psi_\beta}). \tag{2.4}$$ eq:coefficien

In general, this has unfavourable high $\times$ high $\to$ low interactions, as schematically the right-hand side takes the form $D^{-1}(\psi\overline{\psi})$, though in our setting of equivariant symmetry, the situation is much better. In any case, thinking of the connection coefficients as quadratic terms, we can rewrite the Hasimoto-transformed system in these coordinates as the cubic-type non-linear Schrödinger equation

$$i\partial_t\psi - \Delta\psi = A_t[\psi]\psi + i\partial_{\bar{z}}A_z[\psi]\psi + iA_z[\psi]\partial_{\bar{z}}\psi + iA_{\bar{z}}[\psi]\partial_z\psi - A_z[\psi]A_{\bar{z}}[\psi]\psi_{\bar{z}}. \tag{2.5}$$

$$\begin{tikzcd}
(I \times \mathbb{R}^2) \times \mathbb{C} \arrow[r, "\mathbf{e}"] & u^* T\mathbb{S}^2 \arrow[r, "u"] \arrow[d, "v"] & T\mathbb{S}^2 \arrow[d] \\
I \times \mathbb{R}^2 \arrow[u, "\psi"] \arrow[r] & I \times \mathbb{R}^2 \arrow[r, "u"] & \mathbb{S}^2
\end{tikzcd}$$

FIGURE 2.2. The commutative diagram connecting the trivial bundle $(I \times \mathbb{R}^2) \times \mathbb{C}$, the pullback bundle $u^* T\mathbb{S}^2$, and the tangent bundle $T\mathbb{S}^2$.

2.2. **Notation.** We denote a logarithmic refinement of L^q by

$$||\psi||_{\ell^p L^q} := \Big|\Big| ||\psi||_{L^q_x(R \le r \le 2R)} \Big|\Big|_{\ell^p_R(2^{\mathbb{Z}})}$$

3. OUTLINE OF THE PROOF

Our overview of the argument follows the exposition of [GGKT08], which in turn outlines the stability result for $m \ge 4$ in [GKT07]; we will leave the modifications of the argument in [GNT10] to handle the $m = 3$ case to the details. For our notation, we will instead borrow from [KTV14, Geometric Wave Equations] and [BT14, BPT24].

We decompose the solution into a soliton profile and a dispersive error,

$$u(t,r) = \underbrace{Q_{\alpha(t),\lambda(t)}(r)}_{\text{modulated soliton}} + \underbrace{\varepsilon(t,r)}_{\text{dispersive error}} . \tag{3.1}$$ eq:decomp2

To prove the main theorem, we want construct appropriate modulation parameters (α, λ) and correction term ε such that

(a) the error obeys global-in-time dispersive bounds, $\varepsilon \in \mathsf{S}^1_{t,x}([0,\infty) \times \mathbb{R}^2)$, to obtain the dispersive decay, and
item:goal1
(b) integrability bounds on the derivative of the modulation parameters, $(\dot\alpha, \dot\lambda) \in L^1_t([0,\infty))$, to conclude convergence to a soliton.
item:goal2

Here the $\mathsf{S}^1_{t,x}$-norm is taken to be a scale-invariant Strichartz-type norm at the level of the differentiated field ∂u. For our purposes, it will suffice to take the dispersive norm to consist of the endpoint norms,

$$||\varepsilon||_{\mathsf{S}^1_{t,x}} := ||\varepsilon||_{L^\infty_t \dot H^1_x} + ||r^{-1}\varepsilon||_{L^2_t L^\infty_x}.$$

3.1. **Generalised Hasimoto transformation.** Generally, one would have to study the dynamics of the error coupled with the dynamics of the modulation parameters. However, the Schrödinger maps equation (SM) admits *self-dual* structure, which we first saw from the Bogomoln'yi identity (B). Viewing the Schrödinger maps equation as the Hamiltonian flow of the Dirichlet energy, the Bogomoln'yi identity implies that the equation can be written as

$$\partial_t u = \mathbf{J}\mathbf{D}_z \partial_{\bar z} u, \tag{3.2}$$ eq:selfdual

where

$$\partial_{\bar z} u := \partial_1 u - \mathbf{J}\partial_2 u,$$
$$\mathbf{D}_z v := \mathbf{D}_1 v + \mathbf{J}\mathbf{D}_2 v.$$

Then, applying the covariant Cauchy-Riemann operator to the self-dual formuation (3.2), we obtain the generalised Hasimoto-transformed Schrödinger maps equation, which is an elliptic-dispersive system,

$$\mathbf{D}_t \varepsilon' = \mathbf{J}\mathbf{D}_{\bar z}\mathbf{D}_z \varepsilon', \tag{3.3}$$ eq:hasimoto2

$$\varepsilon' = \partial_{\bar z} u. \tag{3.4}$$ eq:hasimoto3

This is known as the *generalised Hasimoto transformation*[2], derived originally by Chang-Shatah-Uhlenbeck in [CSU00]. This formulation is convenient for two general reasons (see [Tao06, Chapter 6.2] and [KTV14, Chapter 2.5] for more commentary on the *frame method*) and one particular to the self-dual structure:

(a) the differentiated field takes values in a vector bundle $\varepsilon' : I \times \mathbb{R}^2 \to u^*T\mathbb{S}^2$ rather than a manifold like the original field $u : I \times \mathbb{R}^2 \to \mathbb{S}^2$, so one can work in linear function spaces,
(b) we are free to choose an orthonormal frame for the bundle $u^*T\mathbb{S}^2$ to concoct a favourable equation for ε' in coordinates,
(c) in view of the harmonic maps equation (HM), one can think of the map $u \mapsto \varepsilon'$ as a non-linear projection which kills the harmonic maps component, leaving only the error, i.e. the differentiated field is at least linear in the error

$$\varepsilon' = O(\varepsilon) + O(\varepsilon^2).$$

Thus we can view (3.3) as a small L^2_x-data problem for a Schrödinger equation with cubic non-linear interactions. Using standard Strichartz estimates, it is well known that solutions to such equations admit global-in-time dispersive bounds $\varepsilon' \in \mathsf{S}^0_{t,x}([0,\infty) \times \mathbb{R}^2)$.

Our strategy then will be to prove dispersive estimates for the differentiated variable ε' using the non-linear Schrödinger equation (3.3). Using elliptic estimates for the inhomogeneous Cauchy-Riemann equation (3.4), we can pass these dispersive decay estimates onto the original error ε.

$$||\varepsilon'||_{\mathsf{S}_{t,x}} := ||\varepsilon'||_{L^\infty_t L^2_x} + ||\varepsilon'||_{L^2_t L^\infty_x}$$

3.2. **Orthogonality and modulation.** To identify the main enemy to proving decay estimates for the error, let us consider the linearisations of the Schrödinger maps equation (3.2) and the Cauchy-Riemann equation (3.4) about a fixed soliton profile Q. A solution to the linear flow is a field $u_{\text{lin}} : I \times \mathbb{R}^2 \to Q^*T\mathbb{S}^2$, which, upon choosing appropriate coordinates, e.g. Coulomb gauge $\{\mathbf{v}_Q, \mathbf{w}_Q\} \subseteq Q^*T\mathbb{S}^2$, corresponds to a complex scalar field $\phi_{\text{lin}} : I \times \mathbb{R}^2 \to \mathbb{C}$ satisfying

$$(i\partial_t - \mathsf{H}_Q)\phi_{\text{lin}} = N, \tag{3.5}$$

eq:linearised

$$\mathsf{L}_Q\phi_{\text{lin}} = F, \tag{3.6}$$

eq:linearised

where H_Q is the *linearised operator* and L_Q is the *linearised Cauchy-Riemann operator*, given by

$$\mathsf{H}_Q := \mathsf{L}_Q^*\mathsf{L}_Q, \qquad \mathsf{L}_Q := h_1\partial_r {h_1}^{-1}.$$

The factorisation of the linearised operator a lá the linearised Cauchy-Riemann operator can be read off from the self-dual structure of the equation. We see that the main enemy to decay of the error $\varepsilon \approx \phi_{\text{lin}}$, either via dispersive estimates from (3.5) or elliptic estimates from (3.6), would be the kernel of the linearised operators. From the perspective of the dispersive equation, the kernel elements lead to non-decaying, constant-in-time solutions, while from the perspective of the elliptic equation, the non-trivial kernel leads to non-uniqueness when attempting to invert the operator.

The kernel is given by

$$\ker \mathsf{H}_Q = \ker \mathsf{L}_Q = \operatorname{span}_{\mathbb{C}} h_1.$$

One can either read this off from the conjugation $\mathsf{L}_Q = h_1\partial_r h_1^{-1}$ or invariance of the non-linear equations under scaling and rotation, which tells us that differentiating the modulated soliton $Q_{\alpha,\lambda}$ in these parameters generates elements of the kernel,

$$\begin{aligned} \frac{\partial Q_{\alpha,\lambda}}{\partial \alpha}\Big|_{(\alpha,\lambda)=(0,1)} &= h_1\mathbf{v}_Q, \\ \frac{\partial Q_{\alpha,\lambda}}{\partial \lambda}\Big|_{(\alpha,\lambda)=(0,1)} &= h_1\mathbf{w}_Q. \end{aligned} \tag{3.7}$$

eq:generators

To kill these enemies, it will suffice to impose an orthogonality condition on the error, see Figure 3.2.

[2]The original Hasimoto transformation was derived in the context of fluids mechanics to transform the vortex filament equation into the one-dimensional cubic non-linear Schrödinger equation. The one-dimensional Schrödinger maps arises as the equation for the unit tangent field to the vortex filament. We point the interested reader to the classical book of Majda-Bertozzi [MB02, Chapter 7.1] for details.

fig:orth

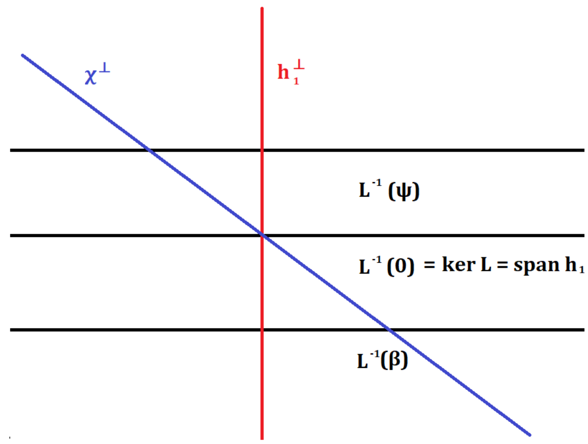

FIGURE 3.3. The level sets of the linear operator L_Q are affine subspaces of the domain which are parallel to the kernel

$$\mathsf{L}_Q^{-1}(\psi) = \phi + \ker \mathsf{L}_Q.$$

To identify a unique member ϕ in the level set, one must restrict the operator L_Q to a subspace which is transverse to the kernel.

Fix $\chi : (0, \infty) \to (0, \infty)$ a radial, non-negative function which does not lie in the kernel,

$$\langle \varepsilon, \chi^\lambda \rangle_{L^2_x} = 0. \tag{3.8}$$

eq:orthogonal

To enforce the orthogonality condition for all time, it suffices to solve the ODE furnished by differentiating the condition in time and imposing the condition holds initially at $t = 0$. This fixes the decomposition (3.1), as substituting it into the resulting ODE and using the equation (SM) furnishes the *modulation equation* for the parameters (α, λ).

The linearised equations motivate two goals of the decomposition (3.1),

(a) respects the linear flow (3.5),
(b) good elliptic estimates for the linearised Cauchy-Riemann operator (3.6).

Our first take would be to impose the condition that the error "$\varepsilon \approx \phi$" is orthogonal to the kernel of the linearised operator H_Q,

$$\phi_{\text{lin}} \perp \ker \mathsf{H}_Q \qquad \text{"i.e."} \qquad \int_0^\infty \phi_{\text{lin}} \overline{h_1} \, r dr = 0. \tag{3.9}$$

eq:orthogonal

This is a decomposition[3] into invariant subspaces of the linearised flow,

$$L^2_r([0, \infty)) = \ker \mathsf{H}_Q \oplus (\ker \mathsf{H}_Q)^\perp.$$

In view of the orthogonality of the error (3.9) to the generators of the kernel (3.7), the forcing terms in the modulation equations which are *linear* in the error ε are killed, so the derivatives of the parameters $(\dot\alpha, \dot\lambda)$ only see *quadratic* forcing terms and higher,

$$|\dot\alpha| + |\dot\lambda| = O(|\varepsilon|^2). \tag{3.10}$$

eq:quadratic

Thus, ignoring any technical problems with spatial asymptotics of ε, an L^2_t-dispersive estimate for ε would imply an L^1_t-estimate for the parameters. This is favourable from the perspective of dispersive estimates, since our earlier discussion gives us access to $L^2_t L^\infty_x$-bounds on the differentiated error ε'. At the level of the linearised Cauchy-Riemann operator, we would like to invert the operator and prove the fixed-time estimate

$$||r^{-1}\phi||_{L^\infty_x} \lesssim ||\mathsf{L}_Q \phi||_{L^\infty_x}.$$

[3]Since the error is in the energy space, $\varepsilon \in \dot H^1_m(\mathbb{R}^2)$, the orthogonality condition on the right is well-defined provided that $rh_1 \in L^2_r([0, \infty))$ by Cauchy-Schwartz. The spatial asymptotics of the kernel elements are precisely $h_1(r) = O(r^{-m})$, so one must restrict to the high equivariance case $m \geq 3$ to make sense of the orthogonality condition.

By duality, one necessarily needs to place the function we are projecting away from in the dual space, i.e. we need $r\chi \in L^1_x(\mathbb{R}^2)$. In view of the asymptotics $h_1 = O(r^{-m})$, one necessarly needs $m \geq 4$. To reach $m = 3$, we need to replace our orthogonality condition with a generic test function $\chi \in C^\infty_c(0,\infty)$; this price one pays is that the linearised flow (3.5) no longer preserves the decomposition, so $(\dot\alpha, \dot\lambda)$ are forced by linear terms in ε in the modulation equations,

$$|\dot\alpha| + |\dot\lambda| = O(\varepsilon) + O(\varepsilon^2).$$

To handle the linear terms, we make a *normal form transformation*.

4. Elliptic estimates

To pass the dispersive bounds on the differentiated field ε' back onto the error ε, we need to prove appropriate non-linear elliptic estimates for non-linear Cauchy-Riemann operator relating the two fields via (3.4), i.e. we study the equation

$$\varepsilon' = \partial_{\bar z} u, \tag{4.1}$$

eq:CR3

where the Cauchy-Riemann operator acts on m-equivariant fields by

$$\partial_{\bar z} u := \partial_r u - \frac{m}{r} u \times Ru.$$

Recall the linearised operator takes the form

$$\mathsf{L}_u \phi := u_1 \partial_r {u_1}^{-1}.$$

4.1. **Linearised equation.** To identify the main terms, we linearise the equation by decomposing the error into a component perpindicular to the soliton profile (i.e. in the tangent space $T_Q\mathbb{S}^2$) and a component parallel to the soliton profile,

$$\begin{aligned} u(t,r) &= \underbrace{Q(r)}_{\text{soliton profile}} + \underbrace{\varepsilon(t,r)}_{\text{perturbation}} \\ &= \underbrace{Q(r)}_{\text{soliton profile}} + \underbrace{u_{\mathrm{lin}}(t,r)}_{\text{perpindicular perturbation}} + \underbrace{\gamma(t,r)Q(r)}_{\text{parallel perturbation}} \end{aligned} \tag{4.2}$$

eq:perpparall

For the more visually-inclined reader, see Figure 4.4 below,

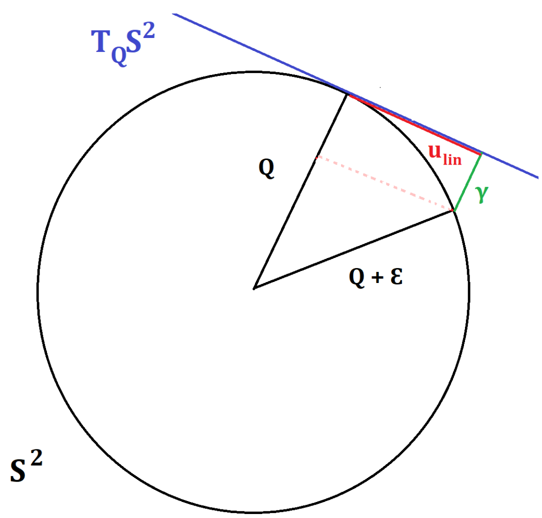

Figure 4.4. Decomposition of the error

$$\varepsilon = u_{\mathrm{lin}} + \gamma Q,$$

into a component parallel to the soliton $\gamma Q \in \operatorname{span} Q$ and a component in the tangent space $u_{\mathrm{lin}} \in T_Q\mathbb{S}^2$. For small error, the main term is the latter.

fig:decomp

For small perturbation $|\varepsilon| \ll 1$, the main term in the decomposition (4.2) is the perpindicular component $\varepsilon \approx u_{\mathrm{lin}}$ in view of the constraint $|u| \equiv |Q| \equiv 1$; indeed one can compute that the parallel part is of quadratic order in ε,

$$\gamma = \sqrt{1 - |u_{\mathrm{lin}}|^2} - 1 = O(|u_{\mathrm{lin}}|^2).$$

Analogous to the study of the dispersive equation (3.3), we would like to fix a coordinate system which reveals the elliptic structure of the equations (3.4) and furthermore is well-adapted to the decomposition (4.2). In this case, we choose the Coulomb frame $\{\mathbf{v}_Q, \mathbf{w}_Q\} \subseteq Q^*T\mathbb{S}^2$ adapted to the soliton profile Q, which takes the explicit form

$$\mathbf{v}_Q(r) = \begin{pmatrix} h_3(r) \\ 0 \\ -h_1(r) \end{pmatrix}, \qquad \mathbf{w}_Q(r) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.3}$$

eq:coulombQfr

In these coordinates, the fields $u_{\mathrm{lin}} : I \times \mathbb{R}^2 \to Q^*T\mathbb{S}^2$ and differentiated field $\varepsilon' : I \times \mathbb{R}^2 \to u^*T\mathbb{S}^2$ correspond respectively to the complex scalar fields $\phi : I \times \mathbb{R}^2 \to \mathbb{C}$ and $\widetilde{\psi} : I \times \mathbb{R}^2 \to \mathbb{C}$, defined by

$$\phi := \langle u_{\mathrm{lin}}, \mathbf{v}_Q \rangle + i \langle u_{\mathrm{lin}}, \mathbf{w}_Q \rangle,$$
$$\widetilde{\psi} := \langle \varepsilon', \mathbf{v}_Q \rangle + i \langle \varepsilon', \mathbf{w}_Q \rangle.$$

Using the decomposition (4.2), we can write the inhomogeneous Cauchy-Riemann equation (4.1) in the Coulomb frame as

$$\underbrace{\mathsf{L}_Q \phi}_{\text{linearised operator}} = \underbrace{\widetilde{\psi}}_{\text{dispersive bound}} - \underbrace{\frac{m}{r}\varepsilon_3 \phi + \frac{m}{r} h_1 \gamma}_{\text{perturbative non-linearity}}. \tag{4.4}$$

eq:CRcoulomb

Thus, we see that, at least on the linear level, passing bounds on the differentiated field $\widetilde{\psi}$ to the main error ϕ amounts to inverting the linearised operator L_Q.

4.2. **Elliptic estimates.** The right-inverse is given by

$$\mathsf{R}_\chi g(r) := 2\pi h_1(r) \int_0^\infty \left(\int_{r'}^r h_1(r'') g(r'') dr'' \right) \overline{\chi(r')} h_1(r')\, r' dr'.$$

inearelliptic **Lemma 4.1** (Linear elliptic estimate)**.** *For each $\lambda > 0$ and radial function $\chi : (0,\infty) \to (0,\infty)$, there exists a linear operator $\mathsf{R}_{\lambda,\chi}$ which serves as a right-inverse for L^λ and also a left-inverse for L_λ modulo the kernel,*

$$\mathsf{L}^\lambda \mathsf{R}^\lambda_\chi g = g, \tag{4.5}$$
$$\mathsf{R}^\lambda_\chi \mathsf{L}^\lambda g = g - h_1(r/\lambda) \int_0^\infty g(r') \overline{\tfrac{1}{\lambda^2} \chi(\tfrac{r}{\lambda})} r' dr'. \tag{4.6}$$

Furthermore, for $1 \le p \le \infty$ and $|\theta| < m$, if $\chi \in r^{-1}(\ell^{p'} L^1)_x(0,\infty)$, then the right-inverse satisfies the bound

$$||r^{-\theta} \mathsf{R}^\lambda_\chi g||_{(\ell^p L^\infty)_x} \lesssim ||r^\theta \chi||_{(\ell^{p'} L^1)_x} ||r^{-\theta-1} g||_{(\ell^p L^1)_x} \tag{4.7}$$

Proof. Note that

$$g(r) \mapsto g\left(\tfrac{r}{\lambda}\right), \qquad \chi(r) \mapsto \tfrac{1}{\lambda^2} \chi\left(\tfrac{r}{\lambda}\right)$$

are adjoint operators, so it suffices to prove the result for $\lambda = 1$. For the details, we refer the reader to [GNT10, Section 10.1]. □

Proposition 4.2 (Non-linear elliptic estimates)**.** *For $m \ge 2$, consider the inhomogeneous non-linear Cauchy-Riemann equation (4.4). If the error satisfies the orthogonality condition*

$$\int_0^\infty \phi(r) \overline{\widetilde{h}_1(r/\lambda)}\, r dr = 0, \tag{4.8}$$

eq:ellipticor

where $\widetilde{h}_1 \in C_c^\infty(0,\infty)$ is a smooth compactly-supported radial function satisfying $\langle h_1, \widetilde{h}_1 \rangle_{L^2_r} = 1$, then

(a) L^2_x-type bound

$$||\phi||_{\dot{H}^1_m} \lesssim ||\psi||_{L^2_r} + ||\phi||_{L^\infty_r} ||\phi||_{\dot{H}^1_m}, \tag{4.9}$$

eq:ellipticer

(b) *L^∞_x-type bound*

$$||r^{-1}\phi||_{(\ell^p L^\infty)_r} \lesssim ||\psi||_{L^\infty_r} + ||\phi||_{L^\infty_r} ||\phi||_{(\ell^p L^\infty)_r}. \tag{4.10}$$

eq:ellipticur

Proof. By the orthogonality condition (4.8), we have

$$\phi = \mathsf{R}_{\lambda,\widetilde{h}_1} \mathsf{L}^\lambda \phi.$$

The equation (4.4) schematically takes the form

$$\mathsf{L}_Q \phi = \widetilde{\psi} + O(|\phi|^2).$$

Using the estimates from Lemma 4.1 and the triangle inequality, placing one factor of ϕ in the non-linearity in L^∞_r-norm and the other in either $\dot{H}^1_m$-norm or $(\ell^p L^\infty)_r$-norm, furnishes the L^2_x-bound and L^∞_x-bound respectively. □

Proposition 4.3 (Hardy-Sobolev inequality)**.** *Let $m \geq 1$ and suppose $\phi \in \dot{H}^1_m(0,\infty)$ is m-equivariant, then*

$$||\phi||_{L^\infty_r} \lesssim ||\phi||_{\dot{H}^1_m}. \tag{4.11}$$

eq:ellipticha

Proof. Fundamental theorem of calculus and Cauchy-Schwartz. □

5. Dispersive estimates

To prove dispersive bounds for the differentiated field ε', we fix a gauge to write (3.3) as a cubic non-linear Schrödinger equation with potential. Using standard arguments, one can prove Strichartz estimates for the linearised equation, and, in view of the small data assumption, the non-linearity is perturbative. Again, the equation we consider is

$$\mathbf{D}_t \varepsilon' = \mathbf{D}_z \mathbf{D}_{\overline{z}} \varepsilon'. \tag{5.1}$$

eq:hasimotodi

5.1. **Linearised equation.** To reveal the Schrödinger structure of the equation, we work in coordinates, fixing an m-equivariant frame $\{\mathbf{v}, \mathbf{w}\} \subseteq u^* T\mathbb{S}^2$. Using this frame, we can identify the differentiated field $\varepsilon' : I \times \mathbb{R}^2 \to u^* T\mathbb{S}^2$ with the complex scalar field $\psi : I \times \mathbb{R}^2 \to \mathbb{C}$ via

$$\psi := \langle \varepsilon', \mathbf{v} \rangle + i \langle \varepsilon', \mathbf{w} \rangle.$$

The connection coefficients are given by

$$\begin{aligned} A_t &= \langle \partial_t \mathbf{v}, \mathbf{w} \rangle, \\ A_r &= \langle \partial_r \mathbf{v}, \mathbf{w} \rangle, \\ A_r &= \langle \partial_\theta \mathbf{v}, \mathbf{w} \rangle. \end{aligned}$$

Proposition 5.1 (Hasimoto transform in Coulomb gauge)**.** *Let $u : I \times \mathbb{R}^2 \to \mathbb{S}^2$ be a solution to Schrödinger maps (SM), and denote $\{\mathbf{v}, \mathbf{w}\} \subseteq u^* T\mathbb{S}^2$ the frame satisfying the Coulomb gauge condition*

$$\partial_1 A_1 + \partial_2 A_2 = 0. \tag{5.2}$$

Then the connection coefficients are given by

$$A_r[u] = 0, \tag{5.3}$$

$$A_\theta[u] = m u_3, \tag{5.4}$$

$$A_t[u] = A_t[u] = \left(\frac{1}{2}|\varepsilon'|^2 + \frac{m}{r}\varepsilon'_3\right) + \int_r^\infty 2\left(\frac{1}{2}|\varepsilon'|^2 + \frac{m}{r'}\varepsilon'_3\right)\frac{dr'}{r'}. \tag{5.5}$$

Proof.

(a) In polar coordinates, the Coulomb gauge condition reads

$$\partial_r A_r + \frac{1}{r^2}\partial_\theta A_\theta = 0.$$

Since the connection coefficients are radial and satisfy appropriate boundary conditions at infinity, we immediately conclude $A_r = 0$.

(b) Since $\mathbf{v}$ and $\mathbf{w}$ are equivariant and $\{u, \mathbf{v}, \mathbf{w}\} \subseteq \mathbb{R}^3$ are orthonormal vectors,

$$\begin{aligned} A_\theta &= \partial_\theta \mathbf{v} \cdot \mathbf{w} \\ &= (m\mathbf{k} \times \mathbf{v}) \cdot \mathbf{w} \\ &= m\mathbf{k} \cdot (\mathbf{v} \times \mathbf{w}) = m\mathbf{k} \cdot \mathbf{u} = mu_3. \end{aligned}$$

(c) Exercise.

□

The equation takes the form

$$(\partial_t + iA_t[u])\psi = -i\mathsf{L}_u\mathsf{L}_u^*\psi. \tag{5.6}$$

Expanding, and regarding the modulation as slow and thus the differences in some terms in the potential as perturbative, we can write

$$(i\partial_t - \widetilde{\mathsf{H}}_Q)\psi = \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3, \tag{5.7}$$

eq:hasimotoS

where the main linear part is given by

$$\begin{aligned} \widetilde{\mathsf{H}}_Q &:= -\Delta + \widetilde{V}_Q, \\ \widetilde{V}_Q(r) &:= \frac{2m(1-h_3(r))}{r^2} = \frac{4m}{r^2(r^2+1)} \end{aligned}$$

and the pertubrative part is given by

$$\begin{aligned} \mathcal{N}_1 &:= iA_t[u]\psi, \\ \mathcal{N}_2 &:= -2im\frac{h_3 - h_3^\lambda}{r^2}\psi \\ \mathcal{N}_3 &:= -2im\frac{\varepsilon_3}{r^2}\psi - im\frac{\varepsilon_3'}{r}\psi. \end{aligned}$$

We can estimate the perturbative terms in the dual Strichartz space

$$||\mathcal{N}_1||_{L_t^2(\ell^2 L^1)_x} \lesssim ||\psi||^2_{L_t^2(\ell^2 L^\infty)_x}, \tag{5.8}$$

$$||\mathcal{N}_2||_{L_t^2(\ell^2 L^1)_x} \lesssim \left|\left|h_3\left(\tfrac{\lambda(t)}{\lambda(0)}\right)\right|\right|_{L_t^\infty} ||\psi||_{L_t^2(\ell^2 L^\infty)_x} \tag{5.9}$$

$$||\mathcal{N}_3||_{L_t^2(\ell^2 L^1)_x} \lesssim ||\psi||^2_{L_t^2(\ell^2 L^\infty)_x}. \tag{5.10}$$

In the case of the linear term $\mathcal{N}_2$ we assume λ is slowly varying to regard this term as perturbative; this is the main goal of the modulation theory.

5.2. **Strichartz estimates.** We want to prove a Strichartz estimate for a Schrödinger equation with potential. More generally, one can regard this is a variable-coefficient Schrödinger equation. Let us begin with some generalities; consider a self-adjoint Schrödinger operator H, and its corresponding linear flow

$$\begin{aligned} (i\partial_t - \mathsf{H})\psi &= f, \\ \psi_{|t=0} &= \psi_0. \end{aligned} \tag{5.11}$$

eq:generic

We say that H satisfies the *(double) endpoint Strichartz estimate* if

$$||\psi||_{L_t^2(\ell^2 L^\infty)_x} \lesssim ||\psi_0||_{L_r^2} + ||f||_{L_t^2(\ell^2 L^1)_x}. \tag{S}$$

eq:strichartz

In view of the bounds on the non-linearity in the dual space and assuming the scaling parameter is slowly moving $|\log(\lambda/\lambda_0)| \ll 1$, the right-hand side can be regarded as perturbative and we conclude the dispersive estimate

$$||\psi||_{L_t^\infty L_x^2 \cap L_t^2 (L_2^\infty)_x} \ll 1. \tag{5.12}$$

eq:dispersive

The standard proof of the Strichartz estimate for the Laplacian $\mathsf{H} = -\Delta$ relies on the explicit kernel for the linear propagator $e^{it\Delta}$ and the method of stationary phase. This method is unfortunately not very robust, as classical Fourier analysis is ill-suited for variable-coefficient operators. Instead, we will rely on a weaker form of dispersion as a stepping stone for proving (S); we say H satisfies *integrated local energy decay* (also known as the *local smoothing estimate*) if

$$||r^{-1}\psi||_{L^2_{t,x}} \lesssim ||\psi_0||_{L^2_x} + ||rf||_{L^2_{t,x}}. \tag{ILED}$$

eq:ILED

The heuristic is as follows; given a wave packet ψ localised to frequency $|\xi| \approx N$, the packet travels under a homogeneous Schrödinger-type flow with group velocity $|v| \approx N$. Thus for a compact region $K \subseteq \mathbb{R}^2$ of radius R, the bulk of the mass only remains in the region for time scale $T \approx R/N$, so integrating-in-time the mass in the fixed region gives

$$\int_{\mathbb{R}} \int_K |\psi|^2 \, dxdt \lesssim \frac{R}{N} \int_{\mathbb{R}^2} |\psi|^2 \, dx.$$

This represents a *gain* of $\frac{1}{2}$-derivatives on the left-hand side after localising-in-space and averaging-in-time. The proof is more robust, using little more than integration-by-parts (in the form of the *positive commutator method*). Furthermore, in the time-independent case $\mathsf{H}(t) \equiv \mathsf{H}$, the problem can be further reduced to studying the spectral properties of the operator.

Our strategy can then be summarised as follows:

$$(S) \text{ for } \Delta + (\text{ILED}) \text{ for } \mathsf{H}_Q \implies (S) \text{ for } \mathsf{H}_Q$$

Lemma 5.2 (Endpoint Strichartz for m-equivariant Laplacian)**.** *The endpoint Strichartz estimate (S) holds for $\mathsf{H} = -\Delta$ upon restricting to the class of m-equivariant functions for $m \geq 1$.*

Proof. C.f. [GNT10, Theorem 10.1]. □

Remark. In the case of radial functions $m = 0$, the homogeneous Strichartz estimate holds for the Laplacian, however, the inhomogeneous estimate fails, c.f. the classical paper of Tao [Tao00].

Lemma 5.3 (ILED implies Strichartz)**.** *Let H_0 be a self-adjoint operator for which the endpoint Strichartz estimate (S) holds, and consider the Schrödinger operator with potential $\mathsf{H} := \mathsf{H}_0 + V$. If $V(x)$ is a real-valued potential satisfying the growth condition*

$$\sup_{x \in \mathbb{R}^2} |x|^2 |V(x)| < \infty, \tag{5.13}$$

eq:growth

and integrated local energy decay (ILED) holds for H, then the endpoint Strichartz estimate (S) also holds for H.

Proof. We can rewrite the equation for the operator with potential as

$$\begin{aligned} (i\partial_t - \mathsf{H}_0)\psi &= f + V\psi, \\ \psi_{|t=0} &= \psi_0. \end{aligned}$$

Applying Strichartz for H_0, the growth condition (5.13), and integrated local energy decay (ILED), and the embedding

$$\begin{aligned} ||\psi||_{L^2_t(\ell^2 L^\infty)_x} &\lesssim ||\psi_0||_{L^2_x} + ||f||_{L^2_t(\ell^2 L^1)_x} + ||V\psi||_{L^2_t(\ell^2 L^1)_x} \\ &\lesssim ||\psi_0||_{L^2_x} + ||f||_{L^2_t(\ell^2 L^1)_x} + ||r^2 V||_{L^\infty_x} ||r^{-1}\psi||_{L^2_{t,x}} \\ &\lesssim ||\psi_0||_{L^2_x} + ||rf||_{L^2_{t,x}}. \end{aligned}$$

By duality,

$$||r^{-1}\psi||_{L^2_{t,x}} \lesssim ||\psi_0||_{L^2_x} + ||f||_{L^2_t(\ell^2 L^1)_x}.$$

Feeding this into the second line of the previous inequality finishes the proof. □

Proposition 5.4. *Let $V \in C^1_r(0,\infty)$ be a radial real-valued potential satisfying the growth condition (5.13) along with the conditions*

$$\inf_{r>0} r^2 V(r) > 0, \tag{5.14}$$

$$\inf_{r>0} -r^2 \partial_r (rV(r)) > 0. \tag{5.15}$$

Then the Laplacian with potential $\mathsf{H} := -\Delta + V$ satisfies both integrated local energy decay (ILED) and the endpoint Strichartz estimate (S) in the class of m-equivariant functions.

Proof. See Burq-Planchon-Stalker-Tahvildar-Zadeh [BPST04]. The proof boils down to proving the resolvent satisfies the uniform bounds

$$\sup_{\lambda\neq 0} ||(\mathsf{H}-\lambda)^{-1}f||_{\mathsf{LE}_x} \lesssim ||f||_{\mathsf{LE}_x^*}. \tag{5.16}$$

To see how this is sufficient, we point the interested reader to our notes on the wave case. □

5.3. **Dispersive decay.** Use the Fraunhofer formula, dispersive bounds (5.12) and the elliptic bounds (4.9)-(4.10), we can prove

$$||\phi||_{L_x^\infty} \overset{t\to\infty}{\longrightarrow} 0. \tag{5.17}$$

6. Modulation theory

It remains to show that the modulation parameters converge $(\lambda(t),\alpha(t)) \to (\lambda_\infty,\alpha_\infty)$. Our strategy will be to show suitable L_t^1-bounds for the perturbative part of the equation for the derivatives $(\dot\lambda,\dot\alpha)$, and rewrite the non-perturbative part as a total derivative. To derive the modulation equations, recall that the Schrodinger maps equation can be written using the decomposition as

$$\partial_t\varepsilon = \partial_t u - \partial_t Q_{\alpha(t),\lambda(t)} = -\mathbf{D}_z\varepsilon' - h_1^\lambda(\mathbf{v}_Q+\mathbf{w}_Q)\dot\mu. \tag{6.1}$$

eq:schromod

Then, writing in coordinates, and then integrating against $\widetilde{h}_1$, using the orthogonality condition (3.8) to kill the left-hand side, we obtain the modulation equation

$$\dot\mu = -i\langle \mathsf{L}_u^*\psi, \tfrac{1}{\lambda^2}\widetilde{h}_1^{\lambda}\rangle_{L_x^2} + \text{higher order terms}. \tag{6.2}$$

eq:modulate1

If we had h_1 instead of $\widetilde{h}_1$, then the linear term would vanish up to terms which are higher order. Unfortunately, we must contend with this term if we want to pass good L_x^∞ estimates from ε' onto ε. Let us disregard the higher order terms, one has

$$||\text{higher order}||_{L_t^1} \lesssim ||\psi||^2_{L_t^2L_x^\infty}.$$

Since h_1 is in the kernel of L_Q, we can test the equation (6.1) against h_1, and compare against what we obtained for the modulation equation (6.2).

$$\dot\mu = -i\langle \mathsf{L}_u^*\psi, \tfrac{1}{\lambda^2}(\widetilde{h}_1^{\lambda} - ch_1^\lambda)\rangle_{L_x^2} + \text{higher order terms}$$

The difference between testing against $\widetilde{h}_1$ and h_1 is as follows; let $c = ||h_1||_{L^2}^{-2}$, then

$$\langle \widetilde{h}_1 - ch_1, h_1\rangle = 1 - c||h_1||_{L^2}^2 = 0.$$

This tells us that $\widetilde{h}_1 - ch_1 \perp \ker \mathsf{L}_Q$, so it follows that one can write

$$\widetilde{h}_1 - ch_1 = L_Q^* R_{\widetilde{h}_1}^*(\widetilde{h}_1 - ch_1) =: L_Q^* g.$$

Thus we can rewrite the linear term, using the equation (5.1),

$$\langle \mathsf{L}_u^*\psi, \tfrac{1}{\lambda^2}(\widetilde{h}_1^{\lambda} - ch_1^\lambda)\rangle_{L_x^2} = \langle \mathsf{L}_u^*\psi, \tfrac{1}{\lambda}L_Q^* g\rangle_{L_x^2} = \langle \partial_t\psi, \tfrac{1}{\lambda}g\rangle_{L_x^2} + \text{higher order}.$$

Now we can differentiate by parts in time to recover a full derivative, plus some terms which are higher-order. For the remainder of the proof, see [GNT10, Section 7, 8].

Appendix A.

ndix:hasimoto

A.1. **Generalised Hasimoto transform.** It is convenient to write the Schrödinger maps equation (SM) in geometric formulation,

$$\partial_t u = \mathbf{J}(\mathbf{D}_1\partial_1 u + \mathbf{D}_2\partial_2 u).$$

Then

$$\begin{aligned}\mathbf{J}\mathbf{D}_z\partial_{\bar z}u &= \mathbf{J}(\mathbf{D}_1+\mathbf{J}\mathbf{D}_2)(\partial_1 u - \mathbf{J}\partial_2 u)\\ &= \mathbf{J}(\mathbf{D}_1\partial_1 u - \mathbf{J}\mathbf{J}\mathbf{D}_2\partial_2 u) + \mathbf{J}\mathbf{J}(\mathbf{D}_1\partial_2 u - \mathbf{D}_2\partial_1 u)\\ &= \mathbf{J}(\mathbf{D}_1\partial_1 u + \mathbf{D}_2\partial_2 u).\end{aligned}$$

appendix:bogo

A.2. **Bogomoln'yi identity.** We compute

$$\begin{aligned}|\partial_1 u - u \times \partial_2 u|^2 &= (\partial_1 u - u \times \partial_2 u) \cdot (\partial_1 u - u \times \partial_2 u)\\ &= |\partial_1 u|^2 + |u \times \partial_2 u|^2 - 2(\partial_1 u) \cdot (u \times \partial_2 u).\end{aligned}$$

In the last line, the first two terms are exactly the Dirichlet energy density, while the last term is the pull-back of the volume form on $\mathbb{S}^2$ to $\mathbb{R}^2$ under u. Indeed, since the almost complex structure acts isometrically on the tangent space,

$$|\partial_1 u|^2 + |u \times \partial_2 u|^2 = |\partial_1 u|^2 + |\partial_2 u|^2.$$

To see the pull-back of the volume form, recall that

$$d\,\mathrm{Vol}_{\mathbb{S}^2} := u^1 du^2 \wedge du^3 - u^2 du^1 \wedge du^3 + u^3 du^1 \wedge du^2.$$

Then, pulling back by $u : \mathbb{R}^2 \to \mathbb{S}^2$, we obtain

$$\begin{aligned}u^* d\,\mathrm{Vol}_{\mathbb{S}^2} &= u^1(\partial_1 u^2 dx^1 + \partial_2 u^2 dx^2) \wedge (\partial_1 u^3 dx^1 + \partial_2 u^3 dx^2)\\ &\quad - u^2(\partial_1 u^1 dx^1 + \partial_2 u^1 dx^2) \wedge (\partial_1 u^3 dx^1 + \partial_2 u^3 dx^2)\\ &\quad + u^3(\partial_1 u^1 dx^1 + \partial_2 u^1 dx^2) \wedge (\partial_1 u^2 dx^1 + \partial_2 u^2 dx^2)\\ &= u \cdot (\partial_1 u \times \partial_2 u)\, dx^1 \wedge dx^2\\ &= (\partial_1 u) \cdot (u \times \partial_2 u)\, dx^1 \wedge dx^2,\end{aligned}$$

where the last line we have used the scalar triple product identity. By the degree theorem,

$$\int_{\mathbb{R}^2} (\partial_1 u) \cdot (u \times \partial_2 u)\, dx = \int_{\mathbb{R}^2} u^* d\,\mathrm{Vol} = \deg(u) \int_{\mathbb{S}^2} d\,\mathrm{Vol} = 4\pi \deg(u)$$

This completes the proof of the Bogomoln'yi identity (B).

A.3. **Harmonic maps from (LL).** Here we show that

$$\partial_t u = \alpha(\Delta u + |\nabla u|^2 u) + \beta(u \times \Delta u)$$

is equivalent to

$$\partial_t u = -\alpha(u \times (u \times \Delta u)) + \beta(u \times \Delta u).$$

Consider the vector cross product formula

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c.$$

Then

$$u \times (u \times \Delta u) = (u \cdot \Delta u)u - |u|^2 \Delta u.$$

Observe that

$$u \cdot \nabla u = 0.$$

Thus

$$u \cdot \Delta u = -|\nabla u|^2.$$

A.4. **Linearised Cauchy-Riemann.** We write

$$\begin{aligned}\varepsilon' &= \partial_{\bar z} u\\ &= \partial_r u - \frac{m}{r} u \times Ru\\ &= \left(\partial_r u_{\mathrm{lin}} - \frac{m}{r} Q \times Ru_{\mathrm{lin}}\right) - \frac{m}{r} u_{\mathrm{lin}} \times Ru\\ &= \mathsf{L}_Q u_{\mathrm{lin}} + \frac{m}{r} h_1 u.\end{aligned}$$

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