LOCAL WELL-POSEDNESS FOR QUASILINEAR WAVE EQUATIONS (D'APRÉS SMITH-TATARU)

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ABSTRACT. In this note, we outline the work by Smith-Tataru [ST05] concerning the sharp local well-posedness for generic quasi-linear wave equations. That is, given sufficiently regular Lorentzian metrics $\mathbf{g}_{\mu\nu}(\phi)$ and semi-linear terms $\mathcal{N}(\phi)(\partial\phi, \partial\phi)$, we prove that the initial data problem

$$\Box_{\mathbf{g}(\phi)}\phi = \mathcal{N}(\phi)(\partial\phi, \partial\phi),$$
$$(\phi, \partial_t\phi)_{|t=0} = (\phi_0, \phi_1),$$

is locally well-posed in $H^s_x \times H^{s-1}_x(\mathbb{R}^n)$ for $s > \frac{n}{2} + \frac{1}{2}$ when $n = 3, 4, 5$, and for $s > \frac{n}{2} + \frac{1}{4} + \frac{1}{2}$ when $n = 2$.

CONTENTS

1. INTRODUCTION

In this note, we consider the local well-posedness of *quasilinear wave equations* of the form

$$\begin{aligned} \Box_{\mathbf{g}(\phi)}\phi &= \mathcal{N}(\phi)(\partial\phi, \partial\phi), && \text{on } [0,T] \times \mathbb{R}^n, \\ (\phi, \partial_t\phi) &= (\phi_0, \phi_1), && \text{on } t = 0, \end{aligned} \tag{QNLW}$$

eq:QNLW

where $\mathbf{g}_{\mu\nu}(\phi)$ is a symmetric matrix with signature $(-, +, \cdots +)$, using the convention[1] $\Box_{\mathbf{g}} := \mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$ for its associated wave operator, and $\mathcal{N}(\phi)(\partial\phi, \partial\phi) := \mathcal{N}^{\alpha\beta}(\phi)\partial_\alpha\phi\partial_\beta\phi$ is a bilinear form. Without loss of generality, we can take $t =$ const to be space-like hypersurfaces by reducing to metrics of the form

$$\mathbf{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \mathbf{g}_{ij}dx^i dx^j.$$

We shall also assume sufficient smoothness and boundedness of the metric $\mathbf{g}^{\mu\nu}(\phi)$, its inverse $\mathbf{g}_{\mu\nu}(\phi)$, and of the bilinear form $\mathcal{N}^{\alpha\beta}(\phi)$ as functions of ϕ.

Example. The following can be recast in the form (QNLW),

- the Einstein vacuum equations in wave coordinates,
- the irrotational compressible Euler equations.

For the former, this was observed by Choquet-Bruhat [Fou52], while the later is due to Hughes-Kato-Marsden [HKM77]. The reader may find the lecture notes [Luk] as a more modern reference.

Following the standards set by Hadamard and Kato, we say that the initial data problem for the quasilinear wave equation (QNLW) is *locally well-posed* in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$ if the following hold:

(a) *Existence*: for each initial data $\phi[0] \in (H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$, there exists a time $T > 0$ and a solution $\phi[t] \in C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$ to (QNLW).

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[1] One can equivalently consider the divergence form of the equation, i.e. using $\partial_\mu\mathbf{g}^{\mu\nu}\partial_\nu$ instead of $\mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$ on the left-hand side, as the lower-order terms are encapsulated by the right-hand side.

(b) *(Unconditional) uniqueness*: for each initial data $\phi[0] \in (H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$, the solution $\phi[t]$ to (QNLW) is unique in the space $C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$.

item:cty (c) *Continuity of data-to-solution map*: if $\{\phi_k[0]\}_k$ is a sequence of data converging in the $(H^s_x \times H^{s-1}_x)$-topology to $\phi[0]$, then there exists a common time of existence[2] on which the corresponding sequence of solutions $\{\phi_k[t]\}_k$ to (QNLW) converges to $\phi[t]$ in the $L^\infty_t(H^s_x \times H^{s-1}_x)$-topology,

$$\begin{aligned} & \phi_k[0] \xrightarrow{k\to\infty} \phi[0] \qquad \text{in } H^s_x \times H^{s-1}_x \\ \text{implies} \qquad & \phi_k[t] \xrightarrow{k\to\infty} \phi[t] \qquad \text{in } L^\infty_t(H^s_x \times H^{s-1}_x). \end{aligned}$$

For the working definition, we will need to slightly modify the existence and uniqueness statements, strengthening the former while weakening the latter, and require an additional property of the data-to-solution map:

item:exist (a′) *(Sub-critical) existence*: the time of existence can be taken to depend only on the size of the data

$$T \equiv T(\|\phi[0]\|_{H^s_x \times H^{s-1}_x}).$$

item:unique (b′) *(Conditional) uniqueness*: uniqueness holds only in the smaller Strichartz space,

$$\left\{ \phi[0] \in C^0_t(H^s_x \times H^{s-1}_x) : \partial\phi \in L^2_t L^\infty_x \right\}.$$

item:lipschitz (c+) *Weak Lipschitz continuity of data-to-solution map*: there exists a regularity $s_{\mathrm{Lip}} < s$ such that the data-to-solution map is Lipschitz continuous on bounded sets in $(H^s \times H^{s-1})_x(\mathbb{R}^n)$ with respect to the weaker $(H^{s_{\mathrm{Lip}}} \times H^{s_{\mathrm{Lip}}-1})_x(\mathbb{R}^n)$-topology, i.e. for solutions $\phi[t], \psi[t] \in C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$ to (QNLW) satisfying

$$\|\phi[0]\|_{H^s \times H^{s-1}}, \|\psi[0]\|_{H^s \times H^{s-1}} \leq R,$$

the following stability estimate holds:

$$\|\phi[t] - \psi[t]\|_{L^\infty_t(H^{s_{\mathrm{Lip}}} \times H^{s_{\mathrm{Lip}}-1})_x} \leq C(R) \cdot \|\phi[0] - \psi[0]\|_{(H^{s_{\mathrm{Lip}}} \times H^{s_{\mathrm{Lip}}-1})_x}.$$

In sum, we say that the initial data problem for the quasi-linear wave equation (QNLW) is *locally well-posed* in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$ if (a′), (b′), (c), (c+) hold. This leads us to the following natural question

> *For which values of $s \in \mathbb{R}$ is the initial data problem for the quasi-linear wave equation (QNLW) locally well-posed in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$?*

Scaling

$$\phi(x^\mu) \mapsto \phi(\tfrac{x^\mu}{\lambda})$$

This leads to the scaling critical Sobolev exponent $s_{\mathrm{crit}} := \frac{d}{2}$. The main result: half a derivative above scaling.

$$\begin{aligned} \Box_{\mathbf{g}}\phi &= 0, && \text{on } [0,T] \times \mathbb{R}^n, \\ (\phi, \partial_t\phi) &= (\phi_0, \phi_1), && \text{on } t = 0, \end{aligned} \tag{LW}$$ eq:LW

Theorem 1.1 (Smith-Tataru [ST05]). *The initial data problem for the quasi-linear wave equation (QNLW) is locally well-posed in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$, provided that*

$$\begin{aligned} s &> \frac{n}{2} + \frac{1}{2} + \frac{1}{4} && \text{if } n = 2, \\ s &> \frac{n}{2} + \frac{1}{2} && \text{if } n = 3, 4, 5. \end{aligned}$$

Furthermore, given a solution $\phi[t] \in C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$ to (QNLW), it satisfies the following:

(a) Energy estimate and dispersive estimate: the solution satisfies the estimate

$$\|\partial\phi\|_{L^\infty_t H^{s-1}_x} + \|\partial\phi\|_{L^2_t L^\infty_x} \lesssim \|\phi[0]\|_{H^s_x \times H^{s-1}_x}.$$

[2] To be more precise, one can introduce the notion of the *maximal lifespan* $T \equiv T(\phi[0])$ of a solution, and require it to be lower semi-continuous as a function of initial data $\phi[0] \in (H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$.

(b) *Well-posedness of the linearised equation: for* $1 \le \sigma \le s+1$ *and each* $t_0 \in [0,T]$*, the initial data problem for the linearisation of (QNLW) about the solution* ϕ*,*

$$\begin{aligned} \Box_{\mathbf{g}(\phi)}\psi &= 0, && \text{on } [0,T]\times\mathbb{R}^n, \\ (\psi, \partial_t\psi) &= (\psi_0, \psi_1), && \text{on } t = t_0, \end{aligned} \tag{LW}$$

eq:linearised

is well-posed in $(H^\sigma_x \times H^{\sigma-1}_x)(\mathbb{R}^n)$*, i.e. for each initial data* $\psi[t_0] \in (H^\sigma_x \times H^{\sigma-1}_x)(\mathbb{R}^n)$*, there exists a unique solution* $\psi[t] \in C^0_t(H^\sigma_x \times H^{\sigma-1}_x)([0,T]\times\mathbb{R}^n)$ *to (LW). Furthermore, the solutions to (LW) satisfy the energy estimate,*

$$\|\psi\|_{L^\infty_t H^\sigma_x} + \|\partial_t\psi\|_{L^\infty_t H^{\sigma-1}_x} \lesssim \|\psi[t_0]\|_{H^\sigma_x \times H^{\sigma-1}_x}, \tag{1.1}$$

and the Strichartz estimate,

$$\|\langle\nabla_x\rangle^\rho \psi\|_{L^2_t L^\infty_x} \lesssim \|\psi[t_0]\|_{H^\sigma_x \times H^{\sigma-1}_x}, \tag{1.2}$$

for $\rho < \sigma - \frac{d-1}{2}$.

Remark. The proof breaks down in higher dimensions $n \ge 6$ due to a technical failure in the orthogonality argument for the wave packet decomposition.

	Linear equation	**Strichartz**	**Regularity**
Hughes-Kato-Marsden [HKM77]	generic (LW)	N/A	$s > \frac{n}{2} + 1$
Bahouri-Chemin [BC99]	generic (LW)	$\frac{1}{4}$ loss	$s > \frac{n}{2} + \frac{1}{2} + \frac{1}{4}$
Tataru [Tat01a, Tat01b]	generic (LW)	sharp $\frac{1}{6}$ loss	$s > \frac{n}{2} + \frac{1}{2} + \frac{1}{6}$
Klainerman-Rodnianski [KR03]	linearised (QNLW)	$\frac{2-\sqrt{3}}{2}$ loss	$s > \frac{n}{2} + \frac{1}{2} + \frac{2-\sqrt{3}}{2}$
Smith-Tataru [ST05]	linearised (QNLW)	lossless	$s > \frac{n}{2} + \frac{1}{2}$

table:history

TABLE 1. A non-exhaustive historical overview of the local well-posedness of quasi-linear wave equations for $n \ge 3$, though one can find results concerning $n = 2$ among the references, and the result of Klainerman-Rodnianski [KR03] works only with $n = 3$.

Theorem 1.2 (Energy estimate [HKM77])**.** *Let* $s \ge 0$*, then there exists* $C \gg 1$ *such that any smooth solution* ϕ *to (QNLW) obeys the a priori estimate*

$$\|\partial\phi\|_{L^\infty_t H^{s-1}_x} \lesssim \exp\left(C\int_0^T \|\partial\phi\|_{L^\infty_x}\,dt\right)\|\partial\phi(0)\|_{H^{s-1}_x}. \tag{1.3}$$

In particular, $\phi[t]$ *may be continued as a smooth solution as long as* $\|\partial\phi\|_{L^1_t L^\infty_x} < \infty$.

Remark. The dimension of the $L^p_t L^\infty_x$-norm of $\partial\phi$ under the scaling symmetry reads

$$\|\partial\phi\|_{L^p_t L^\infty_x} \approx [t]^{\frac{1}{p}}[x]^{-1} \approx [\partial]^{1-\frac{1}{p}}.$$

Thus, the continuation criterion $L^1_t L^\infty_x$ is scale-invariant, controlling $L^\infty_{t,x}$ via Sobolev embedding incurs a full derivative difference from scaling $1 - \frac{1}{\infty} = 1$, while control of $L^2_t L^\infty_x$ in $n \ge 3$ via Strichartz leads to half-derivative from scaling $1 - \frac{1}{2} = \frac{1}{2}$, and similarly $L^4_t L^\infty_x$ in $n = 2$ leads to three-quarters $1 - \frac{1}{4} = \frac{1}{2} + \frac{1}{4}$.

2. PARADIFFERENTIAL DECOMPOSITION

Proposition 2.1.

$$\|\psi\|_{L^2_t L^\infty_x} \lesssim \varepsilon_0^{-\frac{1}{2}}\lambda^{\sigma-1}\|\psi[-2]\|_{H^1_x \times L^2_x}, \tag{2.1}$$

3. Wave packet parametrix

$$\mathfrak{w} := (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}}\lambda^{-\frac{1}{2}+1}\mathsf{T}_{<\lambda}\left(w\,\mathrm{dS}_{\Sigma_{\theta,u}}\right)$$

where

$$w = w_0\big((\varepsilon_0\lambda)^{\frac{1}{2}}(x'_\theta - \gamma'_\theta(t))\big)$$

Proposition 3.1 (Existence of wave packet parametrix)**.** *Let $n = 2,3,4,5$ then there exists a superposition of wave packets*

$$\phi := \sum_{\theta,j} a_{\theta,j}\,\mathfrak{w}^{\theta,j}$$

which is an approximate solution to the paralinearised equation in the sense that

(a) matches the initial data with

$$\Big(\sum_{\theta,j}|a_{\theta,j}|^2\Big)^{\frac{1}{2}} \lesssim \|\phi[0]\|_{(H^1\times L^2)_x}. \tag{3.1}$$

(b) the energy estimate

$$\|\partial\phi_\lambda\|_{L^\infty_t L^2_x} \lesssim \Big(\sum_{\theta,j}|a_{\theta,j}|^2\Big)^{\frac{1}{2}} \tag{3.2}$$

(c) error estimate

$$\|\Box_{\mathbf{g}_{<\lambda}}\phi_\lambda\|_{L^1_t L^2_x} \lesssim \varepsilon_0\Big(\sum_{\theta,j}|a_{\theta,j}|^2\Big)^{\frac{1}{2}}. \tag{3.3}$$

3.1. Wave packets as approximate solutions.

Lemma 3.2 (Energy estimate for $\mathfrak{w}$)**.** *Wave packets have $O(1)$-energy,*

$$\|\partial\mathsf{P}_\lambda\mathfrak{w}\|_{L^\infty_t L^2_x} \lesssim 1. \tag{3.4}$$ eq:WPenergy

Proof. Obvious. □

Lemma 3.3 (Error estimate for $\mathfrak{w}$)**.** *Each wave packet has small error,*

$$\|\Box_{\mathbf{g}_{<\lambda}}\mathsf{P}_\lambda\mathfrak{w}\|_{L^1_t L^2_x} \lesssim \varepsilon_0, \tag{3.5}$$ eq:L1error

$$\|\Box_{\mathbf{g}_{<\lambda}}\mathsf{P}_\lambda\mathfrak{w}\|_{L^2_{t,x}} \lesssim \varepsilon_0. \tag{3.6}$$ eq:L2error

Obviously (3.6) is stronger than (3.5), so we focus on proving an $L^2_{t,x}$-error estimate. We write

$$\begin{aligned}
\Box_{\mathbf{g}_{<\lambda}}\mathsf{P}_\lambda\mathfrak{w} &= (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}}\lambda^{-\frac{1}{2}+1}\Big([\Box_{\mathbf{g}_{<\lambda}},\mathsf{P}_\lambda\mathsf{T}_{<\lambda}] + \mathsf{P}_\lambda\mathsf{T}_{<\lambda}\Box_{\mathbf{g}_{<\lambda}}\Big)w\,\mathrm{dS}_\Sigma\\
&= (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}}\lambda^{-\frac{1}{2}+1}[\Box_{\mathbf{g}_{<\lambda}},\mathsf{P}_\lambda\mathsf{T}_{<\lambda}]w\,\mathrm{dS}_{\Sigma_{\theta,u}}\\
&\quad + (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}}\lambda^{-\frac{1}{2}+1}\mathsf{P}_\lambda\mathsf{T}_{<\lambda}\left(\Box_{\mathbf{g}_{<\lambda}}w\cdot\mathrm{dS}_\Sigma + 2\mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha w\cdot\partial_\beta\mathrm{dS}_\Sigma + w\cdot\Box_{\mathbf{g}_{<\lambda}}\mathrm{dS}_\Sigma\right)\\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},
\end{aligned}$$

commuting $\Box$ with the frequency projections and then applying the product rule.

Estimating the term I. Since the metric is cut-off to frequencies much lower than λ, the commutator clearly projects to frequencies $|\xi| \sim \lambda$. Thus, one can harmlessly insert fattened projections $\widetilde{\mathsf{P}}_\lambda\widetilde{\mathsf{T}}_{<\lambda}$ in front of the commutator. Furthermore, while two derivatives fall on the wave packet, standard commutator arguments[3] allow us to move one derivative onto the metric. In total, we can rewrite

$$\mathrm{I} = [\mathbf{g}^{\alpha\beta}_{<\lambda},\mathsf{P}_\lambda\mathsf{T}_{<\lambda}]\partial_\alpha\partial_\beta\widetilde{\mathfrak{w}} = \mathcal{L}(\partial\mathbf{g},\partial\widetilde{\mathfrak{w}})$$

[3] In a word, the principal symbol of the commutator is given by the Poisson bracket, so one can, to leading order, write $[\mathbf{g}(x),\chi(\nabla/\lambda)] \approx \{\mathbf{g}(x),\chi(\xi/\lambda)\} \approx \partial_x\mathbf{g}\cdot\partial_\xi\chi(\xi/\lambda) \approx \frac{1}{\lambda}\partial_x\mathbf{g}$.

for another wave packet $\widetilde{\mathfrak{w}}$ and some translation-invariant bilinear operator $\mathcal{L}(-,-)$ with finite-measure kernel. To estimate in $L^2_{t,x}$, we place $\partial \mathbf{g}$ in $L^2_t L^\infty_x$, gaining smallness from our bootstrap assumption, and $\partial\widetilde{\mathfrak{w}}$ in $L^\infty_t L^2_x$, in which it is unit size by construction (3.4), yielding

$$\|\mathrm{I}\|_{L^2_{t,x}} \lesssim \|\partial \mathbf{g}\|_{L^2_t L^\infty_x} \|\partial\widetilde{\mathfrak{w}}\|_{L^\infty_t L^2_x} \lesssim \varepsilon_2.$$

Taking $\varepsilon_2 \leq \varepsilon_0$ is an acceptable contribution towards (3.6). □

Estimating the term II. We are left to compute two derivatives of the bump function on $\mathbb{R}^{n-1}$ localised to the null geodesic γ,

$$\partial_\alpha \partial_\beta w = \begin{cases} O(\varepsilon_0 \lambda) & \text{if two spatial derivatives,} \\ O((\varepsilon_0\lambda)^{\frac{1}{2}}\ddot{\gamma}) & \text{if two time derivatives,} \\ O(\varepsilon_0 \lambda \dot{\gamma}) & \end{cases}$$

Since $\|\ddot{\gamma}\|_{L^2_t} \lesssim \varepsilon_1$ this is acceptable. □

Estimating the term III. □

Estimating the term IV. Surface measure is an approximate solution. □

3.2. **Almost orthogonality of wave packets.**

3.3. **Matching the initial data.**

4. Strichartz estimates

The analysis in the previous sections tell us that the geometry of slabs is approximately that of Minkowski space. Thus, one can expect that the same harmonic analysis counting arguments used to prove Strichartz estimate (or, alternatively, Fourier restriction estimates) hold in this setting.

Proposition 4.1. *Let*

$$\phi := \sum_{T \in \mathcal{T}} a_T \mathbb{1}_T$$

then

$$\|\phi\|_{L^2_t L^\infty_x} \lesssim (\varepsilon_0 \lambda)^{-\frac{1}{2}\frac{n-1}{2}} \lambda^{\frac{1}{2}} \tag{4.1}$$

Proposition 4.2 (Dispersive decay, I). *For all pairs of points P_1, P_2 in space-time, the number of slabs at scale λ containing both is*

$$\#_\lambda(P_1, P_2) \lesssim \begin{cases} \varepsilon_0 & \text{if } m \in I_1, \\ \varepsilon_0 & \text{if } m \in I_2, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

where

$$I_1 :=$$
$$I_2 :=$$

Corollary 4.3 (Dispersive decay, II). *For all pairs of points P_1, P_2 in space-time, the number of slabs at scale λ containing both is*

$$\#_\lambda(P_1, P_2) \lesssim \varepsilon_0^{-\frac{n-1}{2}} \lambda^{\frac{n-3}{2}} |t_1 - t_2|^{-1} \tag{4.3}$$

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