LARGE-DATA THEORY FOR WAVE MAPS ON $\mathbb{R}^{1+2}$

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ABSTRACT. In these notes we present the large-data theory for the energy-critical wave maps equation a lá the *energy dispersion method* of Sterbenz and Tataru in their series of papers [ST10a, ST10b]. The contents of these notes interpolate in detail between those papers and the summary contained in the Oberwolfach notes of Tataru [KTV14, Geometric Wave Equations].

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1. INTRODUCTION

Let $(\mathbb{R}^{1+2}, m)$ be the $(1+2)$-dimensional Minkowski space-time, and suppose $(\mathbb{M}, g)$ is a compact Riemannian manifold. By Nash's theorem, we can view the target manifold extrinsically via an isometric embedding $\mathbb{M} \hookrightarrow \mathbb{R}^N$, denoting the second fundamental form by $\mathbf{S} : T\mathbb{M} \times T\mathbb{M} \to T\mathbb{M}^\perp$. We say that $\phi : \mathbb{R}^{1+2} \to \mathbb{M}$ solves the *wave maps equation* if

$$\begin{aligned} \Box \phi^a &= -\mathbf{S}^a_{bc}(\phi)\partial^\alpha \phi^b \partial_\alpha \phi^c, \\ \phi_{|t=0} &= \phi_0, \\ \partial_t \phi_{|t=0} &= \phi_1, \end{aligned} \tag{WM}$$

for initial data (ϕ_0, ϕ_1) satisfying the constraints $\phi_0(x) \in \mathbb{M}$ and $\phi_1(x) \in T_{\phi_0(x)}\mathbb{M}$. Formally, wave maps are critical points of the Lagrangian

$$\mathcal{L}[\phi] := \int_{\mathbb{R}^{1+2}} \langle \partial^\alpha \phi, \partial_\alpha \phi \rangle_g \, dtdx$$

of which the wave maps equation (WM) is the Euler-Lagrange equation. There is a corresponding stress-energy tensor

$$T_{\alpha\beta}[\phi] := \langle \partial_\alpha \phi, \partial_\beta \phi \rangle_g - \frac{1}{2} m_{\alpha\beta} \langle \partial^\gamma \phi, \partial_\gamma \phi \rangle_g$$

which is divergence-free

$$\partial^\alpha T_{\alpha\beta}[\phi] = 0.$$

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Then, contracting the stress-energy tensor with the vector field ∂_t, applying the divergence-free condition and Stokes' theorem on the space-time slab $I \times \mathbb{R}^2$ furnishes conservation of the *energy* for solutions to the wave maps equation,

$$\mathcal{E}[\phi(t)] := ||\vec{\phi}||^2_{\dot{H}^1 \times L^2}(t) = \int_{\mathbb{R}^2} |\partial_t \phi|^2 + |\nabla_x \phi|^2 \, dx,$$

where we have denoted $\phi[t] = (\phi, \partial_t \phi)(t)$. In view of Noether's theorem, conservation of energy corresponds to the time-translation symmetry of the Lagrangian $\mathcal{L}[\phi]$. Note that the wave maps equation and its Lagrangian are invariant with respect to the scaling

$$\phi(t, x) \mapsto \phi(\lambda t, \lambda x).$$

The Dirichlet energy is also invariant with respect to this scaling in $(1+2)$-dimensions and coincides precisely with the *energy space* $\phi[t] \in \dot{H}^1 \times L^2$, thus, we refer to the wave maps equation (WM) on $\mathbb{R}^{1+2}$ as *energy-critical*.

1.1. **Main results.** We consider the initial data problem for the wave maps equations (WM) on $\mathbb{R}^{1+2}$ with finite energy data $\vec{\phi}_0 \in \dot{H}^1_x \times L^2_x$. Global well-posedness where the target manifold is a sphere $\mathbb{M} = \mathbb{S}^k$ was established in [Tao01] and for general targets in [Tat05]. In this note we turn towards addressing the following questions for large initial data:

- global well-posedness,
- scattering.

Using the symmetries of the equation and the finite speed of propagation, we can reduce the study of the wave map to the forward light cone C. For blow-up, we use time-reversibility of the equation and finite speed of propagation. For scattering, we choose a ball B large so that energy is small outside of this ball, so the small data theory applies. Hence it remains to study the influence cone of B.

Theorem 1 (Bubbling theorem)**.** *Let ϕ be a finite energy solution to the wave maps equation (WM) which either admits time-like energy concentration at the tip of the light cone $(t, x) = (0, 0)$ (respectively at infinity $t = \infty$),*

$$\limsup_t \mathcal{E}_{C_\gamma \cap S_t}[\phi] > 0,$$

where we write the limit $t \searrow 0$ (resp. $t \nearrow \infty$). Then there exists a sequence of concentration points $(t_n, x_n) \in C$ such that $(t_n, x_n) \to (0, 0)$ (resp. $t_n \nearrow \infty$), and scales $r_n > 0$ with the following properties:

(a) time-like concentration,

$$\limsup_{n \to \infty} \frac{x_n}{t_n} = v,$$

for some velocity $v \in \mathbb{R}^2$ with $|v| < 1$,

(b) below self-similar scale,

$$\limsup_{n \to \infty} \frac{r_n}{t_n} = 0,$$

(c) convergence to a soliton,

$$\lim_{n \to \infty} \phi(t_n + r_n t, x_n + r_n x) = L_v Q(t, x)$$

strongly in $H^1_{\mathrm{loc}}([-\frac{1}{2}, \frac{1}{2}] \times \mathbb{R}^2)$ to a Lorentz transformation with velocity v of a non-trivial harmonic map $Q : \mathbb{R}^2 \to \mathbb{M}$ which contains some of the energy concentration,

$$0 < ||Q||_{\dot{H}^1} \leq \lim_t \mathcal{E}_{S_t}[\phi].$$

Remark. The soliton resolution conjecture asks whether the nature of blow-up or non-scattering can be completely characterised by a superposition of solitons. The question of blow-up was partially answered by the thesis of Grinis [Gri16], using the bubble tree argument. In this case, the energy concentration can be completely decomposed into the sum of energies of solitons,

$$\lim_t \mathcal{E}_{C_\gamma \cap S_t}[\phi] = \sum_{Q \text{ bubble}} ||Q||_{\dot{H}^1},$$

sometimes known as the *energy identity*. The remarkable heart of the analysis is showing that no energy is lost between the scales at which the solitons appear.

Theorem 2 (Threshold theorem)**.** *The wave maps equation is globally well-posed for all initial data below the energy threshold and the corresponding solutions scatter in the following sense:*

(a) *(regular data) For regular data $\vec{\phi}_0$, then there exists a unique global regular solution which has Lipschitz dependence on the initial data locally in time in the $\dot{H}^1 \times L^2$ topology.*
(b) *(rough data) The flow map admits an extension to rough data.*
(c) *(weak Lipschitz dependence) The flow map is globally Lipschitz in the $\dot{H}^\sigma$ topology for $\sigma < 1$ close to 1.*
(d) *(scattering) The S-norm is finite.*

Theorem 3 (Dichotomy theorem)**.** *The wave maps equation (WM) is locally well-posed for arbitrary finite energy data. Further, one of the following two properties must hold for the forward maximal solution:*

(a) *the solution is global, scatters at infinity,*
(b) *bubbling off a soliton.*

2. Paradifferential calculus

To identify the frequency interactions between the terms of the wave maps equation (WM), we work instead with the paradifferential formulation, localising to frequencies $|\xi| \sim 2^k$. Decomposing each term in the non-linearity into Littlewood-Paley pieces $\mathbf{S}(\phi)_{k_1}\partial_\alpha\phi_{k_2}\partial^\alpha\phi_{k_3}$, we make the following dichotomy

(a) high-high interactions, e.g. $k_2 \sim k_3 \gg k_1$, or the derivative terms have low frequencies, $k_1 \gg k_2, k_3$,
(b) low-high-low interactions $k_2 \gg k_1, k_3$ or low-low-high interactions $k_3 \gg k_1, k_2$, i.e. one derivative term is high frequency.

The former are good interactions and can always be treated peturbatively, while for the latter one has to identify the non-peturbative part of the interaction. One also wants to use the geometry of the problem, namely $\mathbf{S}(\phi)\partial\phi = 0$. Paralinearising and adding to our paradifferential equation, we obtain

$$\Box\phi_k = -\mathbf{A}(\phi)^\alpha_{\ll k}\partial_\alpha\phi_k + G(\phi)$$

for some non-linear term $G(\phi)$ which we want to treat as perturbative and anti-symmetric matrices

$$\mathbf{A}(\phi)^\alpha_{\ll k} := (\mathbf{S}(\phi) - \mathbf{S}(\phi)^\top)_{\ll k}\partial^\alpha\phi_{\ll k}.$$

2.1. **Renormalisation.** Tao in his work [Tao01] on the small data problem for the case $\mathbb{M} = \mathbb{S}^k$ developed a renormalisation procedure which transforms the non-linearity into a perturbative non-linearity. One seeks a linear transformation $w_k = U_{<k}\psi_k$ to transform the linear paradifferential equation to the constant coefficient equation

$$\Box w_k = \text{perturbative}.$$

Define the anti-symmetric matrix $\mathbf{B}$ by

$$B_k = (\mathbf{S}(\phi) - \mathbf{S}(\phi)^\top)_{<k-10}\phi_k$$

then one wants to solve the ordinary differential equation

$$\begin{aligned} \frac{d}{dk}U_{<k} &= U_{<k}\mathbf{B}_k, \\ \lim_{k\to-\infty} U_{<k} &= I. \end{aligned} \tag{1}$$

We want the solution to have good S and N-estimates, and approximately renormalise the equation $\mathbf{A}_\alpha = \nabla_\alpha\mathbf{B}$,

$$U^\top_{<k}\nabla_\alpha U_{<k} = \nabla_\alpha\mathbf{B}_{<k} - \int_{-\infty}^k [B_{k'}, U^\top_{<k'}, \nabla_\alpha U_{<k'}]dk'. \tag{2}$$

To get good estimates using this *diffusion gauge*, one still needs smallness of the coefficients $\mathbf{A}(\phi)_{\ll k}$ in our paradifferential equation. In the case of the small data problem, this is no issue, however for large data we have to introduce a large *frequency gap* m to compensate.

Proposition 4 (Gauge covariant S-estimate)**.** *Let ψ_k be a frequency-localised solution to the linear equation*

$$\Box\psi_k = 2\mathbf{A}(\phi)^\alpha_{<k-m}\partial_\alpha\psi_k + G \tag{3}$$

where $\mathbf{A}(\phi)^{\alpha}_{<k-m} : I \times \mathbb{R}^2 \to \mathfrak{so}(\mathbb{R}^2)$ *is the anti-symmetric matrix*

$$\mathbf{A}(\phi)^{\alpha}_{<k-m} := \Big(\mathbf{S}(\phi) - \mathbf{S}^{\top}(\phi)\Big)^{\alpha}_{<k-m} \partial_\alpha \phi_{<k-m},$$

and ϕ *is a smooth wave map on* I *with bounds*

$$||\phi||_{\underline{\mathsf{E}}[I]} + ||\phi||_{\underline{\mathsf{X}}[I]} + ||\phi||_{\mathsf{S}[I]} \leq \mathcal{F}. \tag{4}$$

There exists then a frequency gap $m \geq m(\mathcal{F}) \geq 20$ *of logarithmic growth* $m(\mathcal{F}) \sim \log \mathcal{F}$ *such that the following energy estimate holds*

$$||\psi_k||_{\mathsf{S}[I]} \lesssim_{\mathcal{F}} ||\psi_k[0]||_{\dot{H}^1 \times L^2} + ||G||_{\mathsf{N}}. \tag{5}$$

Proof. Standard energy estimates

$$||\psi_k||_{\underline{\mathsf{E}}[I]} \lesssim_{\mathcal{F}} ||\psi_k[0]||_{\dot{H}^1 \times L^2} + 2^{\delta m}||G||_{\mathsf{N}[I]} + 2^{-\delta m}||\psi_k||_{\mathsf{S}[I]}. \tag{6}$$

□

3. Energy dispersion method

Consider the paradifferential formulation of the wave maps problem. To apply the gauge covariant estimates, one needs to show that terms with high-high interactions are indeed perturbative. For the small data problem, this follows immediately from the multi-linear and energy estimates. Define the *energy dispersion norm* by

$$||\phi||_{\mathsf{ED}[I]} = \sup_{k \in \mathbb{Z}} ||P_k \phi||_{L^\infty_{t,x}[I]}.$$

Theorem 5 (Energy-dispersed regularity theorem)**.** *There exist functions* $F(\mathcal{E}) \gg 1$ *and* $\varepsilon(\mathcal{E}) \ll 1$ *of energy such that if* $\phi : [t_0, t_1] \times \mathbb{R}^2 \to \mathbb{M}$ *is a solution to the wave maps equation (WM) with finite energy* $\mathcal{E}[\phi] \equiv \mathcal{E}$ *and energy dispersion*

$$||\phi||_{\mathsf{ED}[I]} \leq \varepsilon(\mathcal{E}), \tag{7}$$

then

$$||\phi||_{\mathsf{S}[I]} \leq \mathcal{F}(\mathcal{E}). \tag{8}$$

In addition, there exists a polynomial $K(\mathcal{F})$ *such that if* $\{c_\kappa\}_\kappa$ *is any* (δ_0, δ_1)*-admissible frequency envelope for* $\vec{\phi}_0$*, we have the bound*

$$||\phi||_{\mathsf{S}_c[I]} \leq K(\mathcal{F}(E)).$$

In particular, one may extend ϕ *to a finite energy wave-map on the interval* $(t_0 - T, t_1 + T)$ *for some* $T \ll_{\mathcal{E},c,\varepsilon} 1$.

3.1. **Induction on energy.**

To illustrate the induction on energy scheme, we will aim for a qualitative statement, though as we detail the proof we will arrive at the full quantitative energy-dispersion theorem. We say that an energy $\mathcal{E}$ is *regular* if there exists parameters $\varepsilon \ll 1$ sufficiently small and $\mathcal{F} \gg 1$ sufficiently large such that for every wave map $\phi : I \times \mathbb{R}^2 \to \mathbb{M}$ with energy $\mathcal{E}[\phi] = \mathcal{E}$ we have

$$||\phi||_{\mathsf{ED}[I]} \leq \varepsilon \text{ implies } ||\phi||_{\mathcal{S}[I]} \leq \mathcal{F}.$$

We remark that we are free to choose these parameters ε and $\mathcal{F}$, though as we will soon see in the proof we can give a quantitative dependence on $\mathcal{E}[\phi]$. Let us denote the set of regular energies by

$$\mathcal{R} := \{\mathcal{E} \in [0, \infty) : \mathcal{E} \text{ is a regular energy}\}.$$

The global well-posedness theorem for small energy furnishes the base case for our induction on energy, $[0, \mathcal{E}_0] \subseteq \mathcal{R}$ for $\mathcal{E}_0 \ll 1$ sufficiently small. Assume then for induction that energies are regular up to some $\mathcal{E}_0$. Our goal is to construct a positive non-increasing function of energy $e(\mathcal{E}) > 0$ to push the induction forward by showing that $\mathcal{E}_0 + e$ is a regular energy for any $e \leq e(\mathcal{E}_0)$. This induction step allows us to conclude the usual continuous induction argument, as it shows

- $\mathcal{R}$ is open: if $[0, \mathcal{E}_0] \subseteq \mathcal{R}$ then $[0, \mathcal{E}_0 + e(\mathcal{E}_0)] \subseteq \mathcal{R}$,
- $\mathcal{R}$ is closed: if we have a sequence of regular energies $\{\mathcal{E}_n\}_n \subseteq \mathcal{R}$ such that $\mathcal{E}_n \nearrow \mathcal{E}$, then since $e : [0, \infty) \to (0, \infty)$ is positive non-increasing, $e(\mathcal{E}_n) > e(\mathcal{E}) > 0$. Taking n large, we have $\mathcal{E} \leq \mathcal{E}_n + e(\mathcal{E}_n)$ and therefore by the induction step $\mathcal{E}$ is regular.

By connectedness, we conclude $\mathcal{R} = [0, \infty)$, i.e. all energies are regular.

Remark. If instead $e(\mathcal{E}_n) \to 0$ as $\mathcal{E}_n \nearrow \mathcal{E}$, e.g. if we were not precise and allowed e to depend also on ε or $\mathcal{F}$, then we would not be able to prove closedness of the set of regular energies. The Kenig-Merle strategy is to assume towards a contradiction that there exists a *critical element*, i.e. a minimal energy blow-up solution, and then attempt to eliminate this possibility.

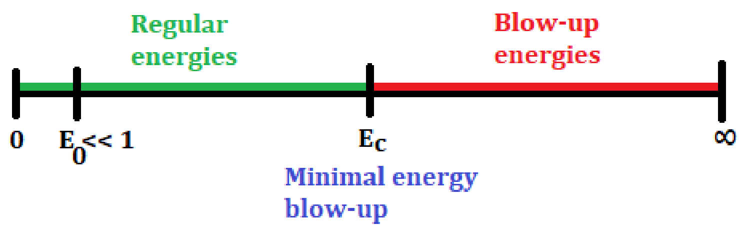

FIGURE 1. If the set of regular energies $\mathcal{R}$ was not closed, i.e. $\mathcal{R} = [0, \mathcal{E}_c)$, then there would exist a minimal energy blow-up wave map $\mathcal{E}[\phi_c] = \mathcal{E}_c$.

We end this subsection by getting the proof of the induction step started. Suppose for induction $\mathcal{E}_0$ is a regular energy and $e \leq e(\mathcal{E}_0) \ll 1$ a small energy increment to be chosen later. Let $\phi : I \times \mathbb{R}^2 \to \mathbb{M}$ be a wave map with energy $\mathcal{E}[\phi] = \mathcal{E}_0 + e$ and energy dispersion

$$||\phi||_{\mathsf{ED}[I]} \leq \varepsilon.$$

We want to compare with a wave map $\widetilde{\phi} : I \times \mathbb{R}^2 \to \mathbb{M}$ with an energy by hypothesis assumed to be regular $\mathcal{E}[\widetilde{\phi}] = \mathcal{E}_0$. Choose then a *cut frequency* $k_* \in \mathbb{R}$ to truncate the initial data such that its projection Π back onto $T\mathbb{M}$ has regular energy,

$$\mathcal{E}[\Pi P_{\leq k_*}\phi[0]] = \mathcal{E}_0.$$

Such a projection is well-defined since energy-dispersed solutions stay close to the manifold. Local well-posedness guarantees that there exists a solution $\widetilde{\phi} : J \times \mathbb{R}^2 \to \mathbb{M}$ with initial data $\widetilde{\phi}[0] = \Pi P_{\leq k_*}\phi[0]$ on some small time interval $J \subseteq I$. To make use of the fact $\mathcal{E}_0$ is a regular energy, we need to pass the energy dispersion of ϕ to $\widetilde{\phi}$. The heuristic is that $\widetilde{\phi}[0] \approx P_{\leq k_*}\phi[0]$ up to higher-order errors. Indeed,

$$||P_k(P_{\leq k_*}\phi[0] - \widetilde{\phi}[0])||_{\dot{H}^1 \times L^2} \lesssim_{\mathcal{E}_0} \varepsilon^{\frac{1}{4}} 2^{-\frac{1}{2}|k-k_*|}. \tag{9}$$

The proof uses standard Moser-type estimates, i.e. chain rule and Bernstein's inequality, see [ST10b, Section 11]. The gain on the right-hand side allows us to use the Sobolev-Bernstein inequality to estimate the energy-dispersion of $\widetilde{\phi}$ at time $t = 0$ by

$$\begin{aligned} ||P_k\widetilde{\phi}[0]||_{L^\infty_x} &\leq ||P_k(P_{\leq k_*}\phi[0] - \widetilde{\phi}[0])||_{L^\infty_x} + ||P_k(P_{\leq k_*}\phi[0]||_{L^\infty_x} \\ &\lesssim_{\mathcal{E}_0} 2^{\frac{k}{2}}||P_k(P_{\leq k_*}\phi[0] - \widetilde{\phi}[0])||_{\dot{H}^1 \times L^2} + ||P_k\phi[0]||_{L^\infty_x} \lesssim_{k^*} \varepsilon^{\frac{1}{4}} + \varepsilon \lesssim \varepsilon^{\frac{1}{4}}. \end{aligned}$$

Choosing then $\varepsilon^{\frac{1}{4}} \ll \varepsilon(\mathcal{E}_0)$, the local well-posedness theory guarantees that $\widetilde{\phi}$ is energy-dispersed in that, after possibly choosing a smaller time interval $J_0 \subseteq J \subseteq I$,

$$||\widetilde{\phi}||_{\mathsf{ED}[J_0]} \leq \varepsilon(\mathcal{E}_0).$$

Then the induction hypothesis guarantees that we have the dispersive bound

$$||\widetilde{\phi}||_{\mathsf{S}[J_0]} \leq \mathcal{F}(\mathcal{E}_0).$$

3.2. **Bootstrap argument.** We want to propagate S-control for the evolution of low frequencies $\widetilde{\phi}$ to our original solution ϕ. We also want to propagate our estimates from the sub-interval $J \subseteq I$ to the full time interval I. To this end, suppose we have suitable choices of $e, \varepsilon, \mathcal{F}$, and let ϕ be a wave map such that

$$||\phi||_{\mathsf{ED}[I]} \leq \varepsilon. \tag{10}$$

We make the following bootstrap assumptions on the sub-interval $J \subseteq I$,

$$||\phi||_{\mathsf{S}[J]} \leq 2\mathcal{F}, \tag{11}$$

$$||\widetilde{\phi}||_{\mathsf{ED}[J]} \leq \widetilde{\varepsilon}. \tag{12}$$

The second bootstrap assumption along with the induction on energy hypothesis imply

$$||\widetilde{\phi}||_{\mathsf{S}[J]} \leq \widetilde{\mathcal{F}}. \tag{13}$$

We aim to improve the bootstrap assumptions to

$$||\phi||_{\mathsf{S}[J]} \leq \mathcal{F}, \tag{14}$$

$$||\widetilde{\phi}||_{\mathsf{ED}[J]} \leq \frac{1}{2}\widetilde{\varepsilon}. \tag{15}$$

To conclude the proof of Theorem 5, we need some technical but nonetheless easy results which allow us to use the usual bootstrap argument. First, we need to check that the improved bounds (14) and (15) on some base case interval J_0. This is done by propagating our good energy bounds at time $t = 0$.

Lemma 6 (Seed bound)**.** *Let $\phi : I \times \mathbb{R}^2 \to \mathbb{M}$ be an affinely Schwartz function and $\{c_k\}_k$ a frequency envelope. If $I_n \subseteq I$ is a decreasing sequence of intervals converging to $t = 0$, then*

$$\lim_{n\to\infty} ||\phi||_{\mathsf{S}[I_n]} \lesssim ||\phi[0]||_{\dot{H}^1 \times L^2}, \tag{16}$$

$$\lim_{n\to\infty} ||\phi||_{\mathsf{S}_c[I_n]} \lesssim ||\phi[0]||_{(\dot{H}^1 \times L^2)_c}. \tag{17}$$

Proof. Follows from the energy estimate. □

Next, we show that the norms in question depend continuously on the endpoints of the intervals. This allows us to use the improved bounds (14) and (15) to push the continuous induction forward.

Lemma 7 (Continuity properties)**.** *Let $\phi : I \times \mathbb{R}^2 \to \mathbb{M}$ be an affinely Schwartz function and $\{c_k\}_k$ a frequency envelope. For each sub-interval $J \subseteq I$, we have that $\phi \in \mathsf{S}[J] \cap \mathsf{S}_c[J]$, and its S-norm $||\phi||_{\mathsf{S}[J]}$, its S_c-norm $||\phi||_{\mathsf{S}_c[J]}$, and its energy-dispersion norm $||\phi||_{\mathsf{ED}[J]}$ all depend continuously on the endpoints of J.*

Proof. For the S-norm, use scale invariance and the fact that convergence in the Schwartz topology is stronger that convergence in S-norm. For the frequency envelope modification S_c, the proof for each dyadic piece uses the same argument as before, while smallness of the tail frequencies allows us to conclude. A similar argument holds for the ED-norm. □

Finally, we show that the bootstrap assumptions (11) and (12) form a closed condition, and good estimates allow us to extend the wave map, so the improvement really is an open condition.

Lemma 8 (Closure and extension property)**.** *Let $\phi : I \times \mathbb{R}^2 \to \mathbb{M}$ be a classical wave map and $\{c_k\}_k$ a frequency envelope. If I_n is an increasing sequence of intervals such that $\bigcup_n I_n = I$ and*

$$||\phi||_{\mathsf{S}[I_n]} \leq \mathcal{F},$$

$$||\phi||_{\mathsf{ED}[I_n]} \leq \varepsilon,$$

then $\phi \in \mathsf{S}[I]$. Furthermore, it can be extended to a classical wave map in a larger interval.

Proof. Use the frequency envelope local well-posedness result. □

3.3. **Comparing ϕ and $\widetilde{\phi}$.** It remains to improve our bootstrap assumptions (11), (12) to the estimates (14), (15). We divide the analysis between comparing low frequencies and comparing high frequencies.

3.3.1. *Low frequencies.* We use the *a priori* space-time control (11) for our solution ϕ to pass good energy dispersion estimates (10) for ϕ back down to the truncated solution $\widetilde{\phi}$, improving (12) to (15). Towards this end it suffices to compare $\widetilde{\phi}$ against the frequencies of ϕ below the cut frequency k_*. We claim that there exists a non-decreasing positive function $K_1(\mathcal{F}) > 0$ of polynomial growth such that

$$||\widetilde{\phi} - P_{\leq k_*}\phi||_{\mathsf{S}[J]} \leq K_1(\mathcal{F})\varepsilon^{\delta_0}.$$

Given this estimate, choosing $\varepsilon \ll \widetilde{\varepsilon} \ll 1$ such that $K_1(\mathcal{F})\varepsilon^{\delta_0} \ll \widetilde{\varepsilon}$, it follows that

$$||\widetilde{\phi}||_{\mathsf{ED}[J]} \lesssim ||\widetilde{\phi} - P_{\leq k_*}\phi||_{\mathsf{S}[J]} + ||\phi||_{\mathsf{ED}[J]} \leq K_1(\mathcal{F})\varepsilon^{\delta_0} + \varepsilon \leq \frac{1}{2}\widetilde{\varepsilon},$$

improving our energy dispersion bound (12) to (15), as desired.

The difference $\psi := \widetilde{\phi} - \phi_{\leq k_*}$ satisfies the equation

$$\Box\psi = -\mathbf{S}(\widetilde{\phi})\partial^\alpha\widetilde{\phi}\partial_\alpha\widetilde{\phi} + P_{\leq k_*}(\mathbf{S}(\phi)\partial^\alpha\phi\partial_\alpha\phi).$$

Proposition 9 (Low frequency evolution)**.** *Let ϕ be a wave map with energy $\mathcal{E}[\phi] = \mathcal{E} + e$, and denote $\widetilde{\phi}$ the wave map with initial data $\widetilde{\phi}[0] = \Pi P_{\leq k_*}\phi[0]$ and energy $\mathcal{E}[\widetilde{\phi}] = \mathcal{E}$. If ϕ and $\widetilde{\phi}$ are defined on the time interval J with bounds*

$$||\phi||_{\mathsf{ED}[J]} \leq \varepsilon, \tag{18}$$

$$||\phi||_{\mathsf{S}[J]} \leq \mathcal{F}, \tag{19}$$

and

$$||\widetilde{\phi}||_{\mathsf{ED}[J]} \leq \widetilde{\varepsilon}, \tag{20}$$

$$||\widetilde{\phi}||_{\mathsf{S}[J]} \leq \widetilde{\mathcal{F}}, \tag{21}$$

for appropriate choices of $\varepsilon, \widetilde{\varepsilon}, \mathcal{F}, \widetilde{\mathcal{F}}$, then there exists a non-decreasing positive function $K_1(\mathcal{F}) > 0$ of polynomial growth such that

$$||\widetilde{\phi} - P_{\leq k_*}\phi||_{\mathsf{S}[J]} \leq K_1(\mathcal{F})\varepsilon^{\delta_0}. \tag{22}$$

Proof. Set $\psi := \widetilde{\phi} - P_{\leq k_*}\phi$ and fix the frequency envelope $c_k = \varepsilon^{\delta_0}2^{-\delta_0|k-k_*|}$. We aim to show the stronger frequency envelope estimate

$$||\psi||_{\mathsf{S}_c[J]} \lesssim_{\mathcal{F}} 1. \tag{23}$$

The initial data satisfies the estimate in the sense of (9), furnishing the base case for an induction. We therefore make the bootstrap assumption

$$||\psi||_{\mathsf{S}_c[J]} \lesssim_{\mathcal{F}} 2, \tag{24}$$

and aim to improve. The difference satisfies the equation

$$\Box\psi = -\mathfrak{D}(\widetilde{\phi}, \psi) + \mathfrak{C}(\phi) \tag{25}$$

where the difference $\mathfrak{D}$ and the generalised commutator $\mathfrak{C}$ are defined as

$$\begin{aligned} \mathfrak{D}(\widetilde{\phi}, \psi) &= \mathbf{S}(\widetilde{\phi})\partial^\alpha\widetilde{\phi}\partial_\alpha\widetilde{\phi} - \mathbf{S}(\widetilde{\phi} + \psi)\partial^\alpha(\widetilde{\phi} + \psi)\partial_\alpha(\widetilde{\phi} + \psi) \\ \mathfrak{C}(\phi) &= P_{<k_*}(\mathbf{S}(\phi)\partial^\alpha\phi\partial_\alpha\phi) - \mathbf{S}(P_{<k_*}\phi)\partial^\alpha P_{<k_*}\phi\partial_\alpha P_{<k_*}\phi. \end{aligned}$$

Projecting to each frequency $|\xi| \sim 2^k$, we obtain the following linear paradifferential form for the equation

$$\Box\psi_k + 2\widetilde{\mathbf{A}}^\alpha(\widetilde{\phi})_{<k-m}\partial_\alpha\psi_k = \mathfrak{L}_k^m(\widetilde{\phi}, \psi) + \mathfrak{D}_k^m(\widetilde{\phi}, \psi) + \mathfrak{C}_k^m(\phi). \tag{26}$$

where

$$\begin{aligned} \widetilde{\mathbf{A}}^\alpha(\widetilde{\phi})_{<k-m} &:= \left(\mathbf{S}(\widetilde{\phi})_{<k-m} - \mathbf{S}^\top(\widetilde{\phi})_{<k-m}\right)\partial^\alpha\widetilde{\phi}_{<k-m}, \\ \mathfrak{L}_k^m &:= 2\left(\mathbf{A}^\alpha_{<k-m}(\widetilde{\phi}) - \mathbf{A}^\alpha_{<k-m}(\widetilde{\phi} + \psi)\right)\partial_\alpha(\widetilde{\phi}_k + \psi_k), \end{aligned}$$

and $\mathfrak{D}_k^m$ are matched frequency difference terms and $\mathfrak{C}_k^m$ are matched frequency commutator terms. The prototypical terms of $\mathfrak{L}_k^m$ are of the form

$$\mathbf{R}_{1;k}(\widetilde{\phi}, \psi) := \widetilde{\phi}_{<k-m}\partial^\alpha \psi_{<k-m}\partial_\alpha \widetilde{\phi}_k,$$
$$\mathbf{R}_{2;k}(\widetilde{\phi}, \psi) := \psi_{<k-m}\partial^\alpha \widetilde{\phi}_{<k-m}\partial_\alpha \widetilde{\phi}_k.$$

To close the bootstrap using the gauge covariant S-estimate (5), we need to show that the right-hand side of the paradifferential equation (26) is perturbative. The matched frequency terms can be dealt with easily, while the separated frequencies must be treated more carefully. The latter will be our focus.

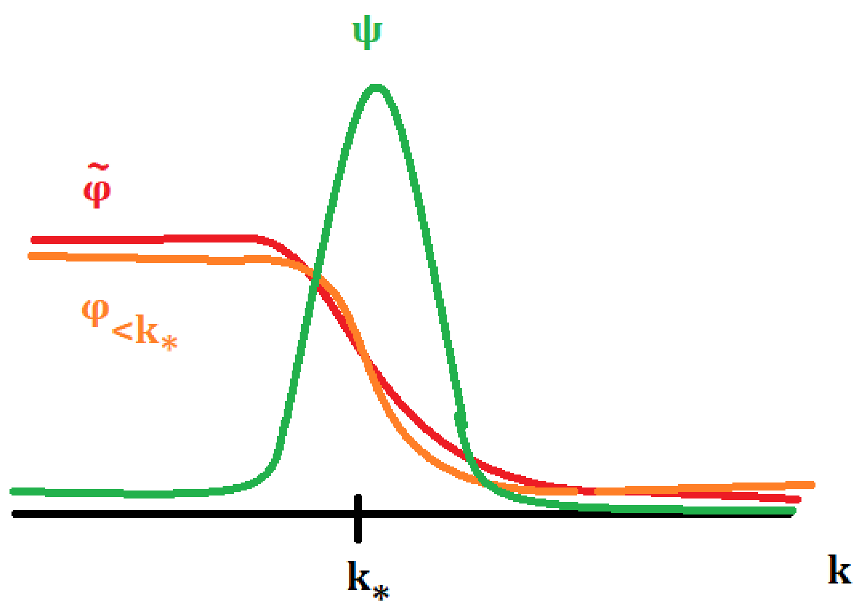

FIGURE 2. Frequency envelopes of $\widetilde{\phi}$ and ψ. Both $\widetilde{\phi}$ and $P_{\leq k_*}\phi$ have the bulk of their norms below $|\xi| \sim 2^{k_*}$, and decay exponentially above, while ψ decays exponentially in both directions.

Using our bootstrap assumption, Moser estimates and product bounds, we can prove the following estimates

$$||\phi_{>k_*+10}||_{\mathsf{S}_c[J]} \lesssim 1, \tag{27}$$
$$||(\mathbf{S}'(\widetilde{\phi})\psi)_k||_{\mathsf{S}[I]} \lesssim_{\widetilde{\mathcal{F}}} c_k. \tag{28}$$

To estimate $\mathbf{R}_{1;k}$, we freely apply the $\mathsf{S} \times \mathsf{N} \to \mathsf{N}$ bound and bilinear null form estimate to write

$$||\mathbf{R}_{1;k}(\widetilde{\phi}, \psi)||_{\mathsf{N}[J]} \lesssim ||\widetilde{\phi}_{<k-m}||_{\mathsf{S}[J]} \sum_{j<k-m} ||\psi_j||_{\mathsf{S}[J]}||\widetilde{\phi}_k||_{\mathsf{S}[J]}.$$

In the range $k < k_* + 10$, note that $k - m < k_*$, hence $\widetilde{\psi}$ is favourable, the rest can be discarded,

$$||\mathbf{R}_{1;k}(\widetilde{\phi}, \psi)||_{\mathsf{N}[J]} \lesssim_{\widetilde{\mathcal{F}}} \sum_{j<k-m} c_j \lesssim_{\widetilde{\mathcal{F}}} c_{k-m} \lesssim_{\widetilde{\mathcal{F}}} 2^{-\delta_0 m} c_k.$$

In the range $k \geq k_* + 10$, the $\widetilde{\psi}_j$ term is no longer always favourable, however we gain some decay from the high frequency improvements of $\widetilde{\phi}_k$. $\square$

3.3.2. *High frequencies.* We now turn towards comparing the solution with truncated data $\widetilde{\phi}$ with the original solution ϕ to improve the space-time control (11) to (14). We claim that there exists a non-decreasing function $K_2(\mathcal{F}) > 0$ of polynomial growth such that

$$||\widetilde{\phi} - \phi||_{\mathsf{S}[J]} \leq K_2(\mathcal{F}).$$

In view of the previous section, in which we showed that $\widetilde{\phi} - P_{\leq k_*}\phi$ is negligible, one can view this as an estimate on the evolution of the high frequencies of ϕ. Given this estimate, choosing $\mathcal{F} \gg \widetilde{\mathcal{F}} \gg 1$ such that $K_2(\widetilde{\mathcal{F}}) \ll \mathcal{F}$, it follows from the triangle inequality that

$$||\phi||_{\mathsf{S}[J]} \leq ||\widetilde{\phi}||_{\mathsf{S}[J]} + ||\phi - \widetilde{\phi}||_{\mathsf{S}[J]} \leq \widetilde{\mathcal{F}} + K_2(\widetilde{\mathcal{F}}) \ll \mathcal{F},$$

improving our space-time control (11) to (14), completing the bootstrap argument.

The difference $\psi := \widetilde{\phi} - \phi$ satisfies the equation

$$\Box\psi = -\mathbf{S}(\widetilde{\phi})\partial^\alpha\widetilde{\phi}\partial_\alpha\widetilde{\phi} + \mathbf{S}(\widetilde{\phi}+\psi)\partial^\alpha(\widetilde{\phi}+\psi)\partial_\alpha(\widetilde{\phi}+\psi).$$

As in the previous section, our strategy will be to reduce the problem to a perturbation of the gauge covariant equation (3). We want to apply the paradifferential argument as in the preceding proof for low frequencies, however the coefficients for this equation $\widetilde{\phi}$ are not small. To remedy this lack of smallness, we need three intermediate steps as outlined in the following diagram:

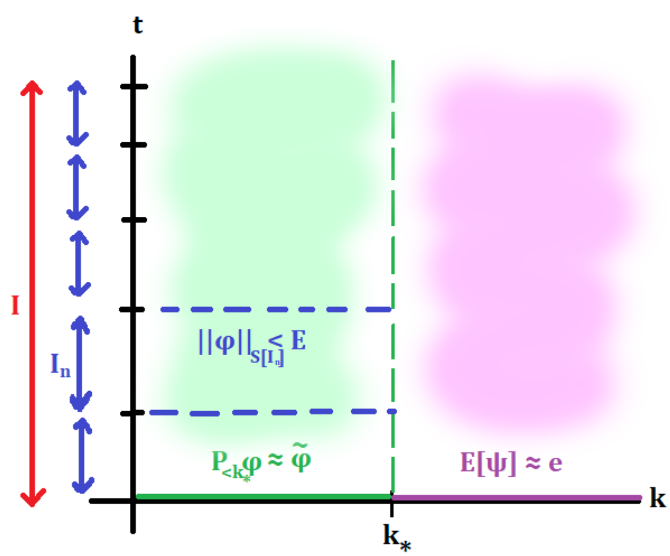

FIGURE 3. Proposition 9 shows that the low frequencies are close. The high frequencies are controlled by the energy step via conservation of energy. The size of $\widetilde{\phi}$ in the S-norm is large, so we divide into $O_{\widetilde{\mathcal{F}}}(1)$-many sub-intervals on which the norm is comparable to the energy. We conclude the argument using perturbation theory on each sub-interval.

We first establish uniform energy bounds for ψ. As ϕ and $\widetilde{\phi}$ are wave maps, they have conserved energy $\mathcal{E}[\phi] = \mathcal{E}$ and $\mathcal{E}[\widetilde{\phi}] \equiv \mathcal{E} + e$ respectively. Thus, using Proposition 9, one can conclude their difference has energy on the order $\mathcal{E}[\psi] \lesssim e$.

Lemma 10 (Almost energy conservation)**.** *Let ϕ be a wave map with energy $\mathcal{E}[\phi] = \mathcal{E} + e$, and denote $\widetilde{\phi}$ the wave map with initial data $\widetilde{\phi}[0] = \Pi P_{\leq k_*}\phi[0]$ and energy $\mathcal{E}[\widetilde{\phi}] = \mathcal{E}$. If ϕ and $\widetilde{\phi}$ are defined on the time interval J, their difference $\psi := \phi - \widetilde{\phi}$ satisfies the almost conservation of energy*

$$\mathcal{E}[\psi(t)] \lesssim e \tag{29}$$

for all $t \in J$.

Proof. Decomposing into high and low frequencies $\phi = P_{>k_*}\phi + P_{\leq k_*}\phi$ and viewing the energy as arising from the inner product on $\dot{H}^1 \times L^2$, we can write

$$\mathcal{E}[\phi] = \mathcal{E}[P_{>k_*}\phi] + \mathcal{E}[P_{\leq k_*}\phi] + 2\langle P_{>k_*}\phi, P_{\leq k_*}\phi\rangle_{\dot{H}^1\times L^2}.$$

The Littlewood-Paley projections are non-negative operators, so the inner product on the right is non-negative. On the other hand, the estimate (22) from Proposition 9 states that the low frequency terms $\widetilde{\phi}$ and $P_{\leq k_*}\phi$ are close, so, in particular, the reverse triangle inequality implies

$$\left|\mathcal{E}[P_{\leq k_*}\phi] - \mathcal{E}[\widetilde{\phi}]\right| + |\mathcal{E}[\psi] - \mathcal{E}[P_{>k_*}\phi]| \leq 100K_1(\mathcal{F})\varepsilon^{\delta_0}.$$

Collecting our results, we obtain

$$\begin{aligned}\mathcal{E}[\psi] &\leq \mathcal{E}[P_{>k_*}\phi] + 100\varepsilon^{\delta_0}K(\mathcal{F})\\ &\leq \mathcal{E}[\phi] - \mathcal{E}[P_{\leq k_*}\phi] + 100\varepsilon^{\delta_0}K(\mathcal{F})\\ &\leq \mathcal{E}[\phi] - \mathcal{E}[\widetilde{\phi}] + 200\varepsilon^{\delta_0}K(\mathcal{F})\\ &\leq e + 200\varepsilon^{\delta_0}K(\mathcal{F})\end{aligned}$$

Making appropriate choices of ε and $\mathcal{F}$ completes the proof. □

Next, we prove partial divisibility for the S-norm. For functions with finite L_t^p-norm for $1 \leq p < \infty$, we can divide the interval into sub-intervals on which the L_t^p-norms are small. However, since the S-norm contains L_t^∞-norms, one cannot hope for divisibility into arbitrarily small norms, but one can divide into norms on the order of the energy.

Lemma 11 (Partial divisibility)**.** *Let $\widetilde{\phi}$ be a wave map on the interval J with energy $\mathcal{E}[\widetilde{\phi}] = \mathcal{E}$ and space-time control $||\phi||_{\mathsf{S}[J]} = \widetilde{\mathcal{F}}$. Then there exists a non-decreasing positive function $K_2(\widetilde{\mathcal{F}}) > 0$ of polynomial growth such that we can partition the time interval into $K_2(\widetilde{\mathcal{F}})$-many sub-intervals $J = \bigsqcup J_k$ such that*

$$||\widetilde{\phi}||_{\mathsf{S}[J_k]} \lesssim \mathcal{E}. \tag{30}$$

Proof. See [ST10b, Section 10.2] □

Finally, we can use the perturbation theory as in the previous section to obtain good estimates for the S-norm of ψ on each sub-interval. The coefficients $\widetilde{\phi}$ have size on the order of the energy, we can choose the energy step $e \ll 1$ sufficiently small, depending *only* on energy, to close the continuity argument.

Proposition 12 (High frequency evolution)**.** *] Let ϕ be a wave map with energy $\mathcal{E}[\phi] = \mathcal{E} + e$, and denote $\widetilde{\phi}$ the wave map with initial data $\widetilde{\phi}[0] = \Pi P_{\leq k_*}\phi[0]$ and energy $\mathcal{E}[\widetilde{\phi}] = \mathcal{E}$. If ϕ and $\widetilde{\phi}$ are defined on the time interval J with bounds*

$$||\phi||_{\mathsf{ED}[J]} \leq \varepsilon, \tag{31}$$

$$||\phi||_{\mathsf{S}[J]} \leq \mathcal{F}, \tag{32}$$

and

$$||\widetilde{\phi}||_{\mathsf{ED}[J]} \leq \widetilde{\varepsilon}, \tag{33}$$

$$||\widetilde{\phi}||_{\mathsf{S}[J]} \leq \widetilde{\mathcal{F}}, \tag{34}$$

for appropriate choices of $\varepsilon, \widetilde{\varepsilon}, \mathcal{F}, \widetilde{\mathcal{F}}$, then there exists a function of energy $e(\mathcal{E})$ and a non-decreasing positive function $K_2(\widetilde{\mathcal{F}}) > 0$ of polynomial growth such that if $e \leq e(\mathcal{E})$ then

$$||\phi - \widetilde{\phi}||_{\mathsf{S}[J]} \leq K_2(\widetilde{\mathcal{F}}). \tag{35}$$

Proof. By the previous two lemmas, it suffices to consider the case where one has space-time control of $\widetilde{\phi}$ by the energy (30) and prove an estimate of the form

$$||\psi||_{\mathsf{S}[J]} \leq 1 \tag{36}$$

for $\psi := \phi - \widetilde{\phi}$. Indeed, by partitioning our overall interval, it follows from the triangle inequality that

$$||\psi||_{\mathsf{S}[J]} \leq \sum_{k=1}^{K_2(\widetilde{\mathcal{F}})} ||\psi||_{\mathsf{S}[J_k]} \leq K_2(\widetilde{\mathcal{F}}),$$

as desired. Using the estimate (29) for $e(\mathcal{E}) \ll 1$ and the energy estimate, there exists a small time interval $J_0 \subseteq J$ on which the conclusion (36) holds. We propagate this base case by making the bootstrap assumption

$$||\psi||_{\mathsf{S}[I]} \leq 2 \tag{37}$$

for some sub-interval $I \subseteq J$, and aim to improve to (36). The difference satisfies the equation

$$\Box\psi = -\mathbf{S}(\widetilde{\phi})\partial^\alpha\widetilde{\phi}\partial_\alpha\widetilde{\phi} + \mathbf{S}(\widetilde{\phi}+\psi)\partial^\alpha(\widetilde{\phi}+\psi)\partial_\alpha(\widetilde{\phi}+\psi). \tag{38}$$

To close the bootstrap argument, we project to each frequency $|\xi| \sim 2^k$, obtaining the linear paradifferential form for the equation, and apply the gauge covariant estimate (5)

$$\Box\psi_k + 2\mathbf{A}^\alpha(\phi)_{<k-m}\partial_\alpha\psi_k = \mathfrak{T}_k^m(\widetilde{\phi}) + \mathfrak{T}_k^m(\widetilde{\phi}+\psi) + \mathfrak{L}_k^m(\widetilde{\phi},\psi) \tag{39}$$

where $\mathfrak{T}_k^m$ denotes a matched frequency trilinear expression, and $\mathfrak{L}_k^m$ contains the low-high-low and low-low-high interactions,

$$\begin{aligned}\mathfrak{L}_k^m(\widetilde{\phi},\psi) &= -\left(\mathbf{A}^\alpha(\widetilde{\phi}+\psi)_{<k-m} - \mathbf{A}^\alpha(\widetilde{\phi})_{<k-m}\right)\partial_\alpha\widetilde{\phi}_k \\ &= -\mathbf{S}(\widetilde{\phi})_{<k-m}\partial^\alpha\psi_{<k-m}\partial_\alpha\widetilde{\phi}_k + \left(\mathbf{S}'(\widetilde{\phi})\psi\right)_{<k-m}\partial^\alpha\widetilde{\phi}_{<k-m}\partial_\alpha\widetilde{\phi}_k = \mathbf{R}_{1;k}(\widetilde{\phi},\psi) + \mathbf{R}_{2;k}(\widetilde{\phi},\psi).\end{aligned}$$

Here we have abused notation to write $\mathbf{S}$ for the anti-symmetrisation of the second fundamental form. It remains to show that the right-hand side of our paradifferential equation is indeed perturbative. The matched frequencies $\mathfrak{T}_k^m$ can be dealt with easily by the energy dispersion estimates, while the separated frequency interactions in $\mathfrak{L}_k^m$ need to be treated more carefully. The latter will be our focus. In view of the low frequency results from Proposition 9, we can restrict our attention to high frequencies $k \geq k_* - 10$.

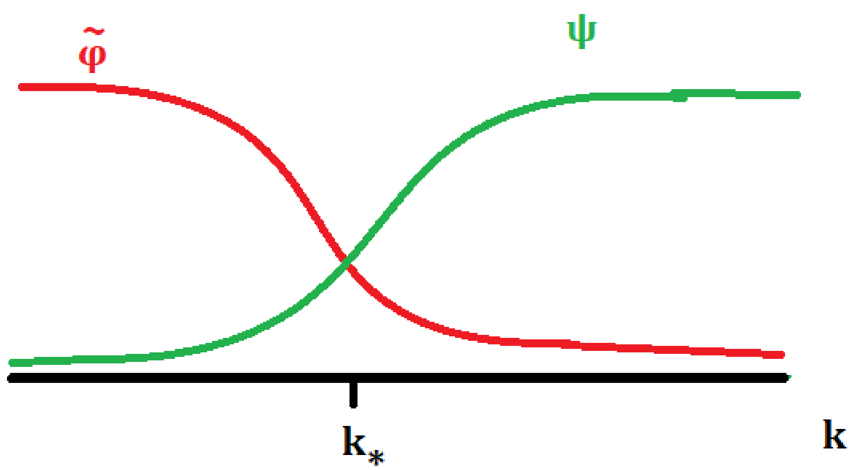

FIGURE 4. Frequency envelopes of $\widetilde{\phi}$ and ψ. The former has the bulk of its norm below $|\xi| \sim 2^{k_*}$ and decays exponentially above, while the latter has the bulk of its norm above and decays exponentially below.

Using the frequency envelope refinements for the low frequency evolution estimates and Moser estimates for the second fundamental form, we have the following estimates

$$||\widetilde{\phi}_k||_{\mathsf{S}[J]} \lesssim 2^{-\delta_0(k-k_*)_+}, \tag{40}$$

$$||\psi_k||_{\mathsf{S}[J]} \lesssim 2^{-\delta_0(k_*-k)_+}, \tag{41}$$

$$||(\mathbf{S}'(\widetilde{\phi})\psi)_k||_{\mathsf{S}[J]} \lesssim 2^{-\delta_0(k_*-k)_+}, \tag{42}$$

$$||\mathbf{S}'(\widetilde{\phi})\psi||_{\mathsf{S}[J]} \lesssim 1. \tag{43}$$

To estimate $\mathbf{R}_{1;k}$, we freely apply the $\mathsf{S} \times \mathsf{N} \to \mathsf{N}$ bound and bilinear null form estimate to write

$$\begin{aligned} ||\mathbf{R}_{1;k}(\widetilde{\phi}, \psi)||_{\mathsf{N}[J]} &\lesssim ||\widetilde{\phi}_{<k-m}||_{\mathsf{S}[J]} \sum_{j<k-m} ||\psi_j||_{\mathsf{S}[J]} ||\widetilde{\phi}_k||_{\mathsf{S}[J]} \\ &\lesssim_{\mathcal{E}} \sum_{j<k-m} 2^{-\delta(k_*-j)_+} 2^{-\delta(k-k_*)_+}. \end{aligned}$$

When $k_* - 10 \leq k \leq k_* + m$, the right-hand side is controlled by $2^{-\delta m}$. When $k > k_* + m$, the right-hand side is controlled by $|k - k_*| 2^{-\delta(k-k_*)}$. We conclude

$$||\mathbf{R}_{1;k}(\widetilde{\phi}, \psi)||_{\mathsf{N}[J]} \lesssim 2^{-\frac{1}{4}\delta m} 2^{-\frac{1}{2}\delta(k-k_*)}.$$

The first factor on the right guarantees smallness, the second factor on the right guarantees summability. The estimate for $\mathbf{R}_{2;k}$ is similar. □

4. Bubbling

4.1. **Monotonicity formula.**

If we instead integrated the stress-energy tensor over the light cone, we can obtain a monotonicity formula for the energy when restricted to slices of the light cone in time. Before we state the formula, we will need to introduce some notation. We denote the forward light cone by

$$C := \{(t, x) \in \mathbb{R}^{1+2} : r \leq t\}$$

and its restrictions to some time interval $I \subseteq [0, \infty)$ as well as time-slices by

$$\begin{aligned} C_I &:= C \cap (I \times \mathbb{R}^2), \\ S_t &:= C \cap (\{t\} \times \mathbb{R}^2). \end{aligned}$$

The *null boundary* ∂C_I denotes the boundary of the time-slab C_I modulo the top and bottom time-slices. Due to singularities on the null boundary, we will also consider the shifted light cone

$$C^\delta := (\delta, 0) + C.$$

Accordingly, we have

$$\begin{aligned} C_I^\delta &:= C_I \cap C^\delta, \\ S_t^\delta &:= S_t \cap C^\delta, \end{aligned}$$

In view of the null boundary, we define the null frame $\{L, \underline{L}, \not\partial\}$ to be the vector fields given by

$$\begin{aligned} L &:= \partial_t + \partial_r, \\ \underline{L} &:= \partial_t - \partial_r, \\ \not\partial &:= \frac{1}{r}\partial_\theta. \end{aligned}$$

Contracting the stress-energy tensor $T_{\alpha\beta}$ with the vector field ∂_t and then integrating then over the slab of the light cone $C_{[t_0,t_1]}$, we obtain in view of the divergence-free property and Stokes' theorem the *monotonicity formula*

$$\mathcal{E}_{S_{t_1}}[\phi] = \mathcal{F}_{[t_0,t_1]}[\phi] + \mathcal{E}_{S_{t_0}}[\phi], \tag{$\uparrow$}$$

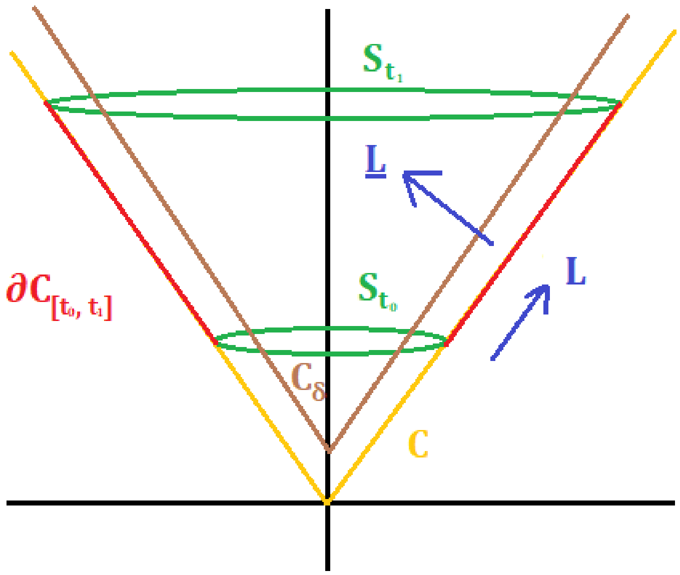

FIGURE 5. The light cone C, its vertical shift C^δ, the time-slices S_t, the null boundary $\partial C_{[t_0,t_1]}$, and the null frame $\{L, \underline{L}, \not\partial\}$.

where $\mathcal{E}_{S_{t_0}}[\phi]$ denotes the energy on the time-slice S_{t_0},

$$\mathcal{E}_{S_{t_0}}[\phi] := \int_{S_{t_0}} |\partial_t \phi|^2 + |\nabla_x \phi|^2 \, dx,$$

and $\mathcal{F}_{[t_0,t_1]}[\phi]$ denotes the *flux* of the wave map on the null-boundary of the light cone,

$$\mathcal{F}_{[t_0,t_1]}[\phi] := \int_{\partial C_{[t_0,t_1]}} \left(\frac{1}{4}|L\phi|^2 + \frac{1}{2}|r^{-1}\partial_\theta \phi|^2 \right) dA.$$

The key observation is that the flux is non-negative, hence (†) is indeed a monotonicity formula for the energy on time-slices of the light cone $\mathcal{E}_{S_t}[\phi] \nearrow$.

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