LOCAL WELL-POSEDNESS FOR QUASILINEAR WAVE EQUATIONS (D'APRÉS SMITH-TATARU)

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ABSTRACT. In this note, we outline the work by Smith-Tataru [ST05] concerning the sharp local well-posedness for generic quasi-linear wave equations. That is, given sufficiently regular Lorentzian metrics $\mathbf{g}_{\mu\nu}(\phi)$ and semi-linear terms $\mathcal{N}(\phi)(\partial\phi, \partial\phi)$, we prove that the initial data problem

$$\Box_{\mathbf{g}(\phi)}\phi = \mathcal{N}(\phi)(\partial\phi, \partial\phi),$$
$$(\phi, \partial_t\phi)_{|t=0} = (\phi_0, \phi_1),$$

is locally well-posed in $H^s_x \times H^{s-1}_x(\mathbb{R}^n)$ for $s > \frac{n}{2} + \frac{1}{2}$ when $n = 3, 4, 5$.

CONTENTS

1. INTRODUCTION

In this note, we consider the local well-posedness of *quasilinear wave equations* of the form

$$\begin{aligned} \Box_{\mathbf{g}(\phi)}\phi &= \mathcal{N}(\phi)(\partial\phi, \partial\phi), && \text{on } [0,T] \times \mathbb{R}^n, \\ (\phi, \partial_t\phi) &= (\phi_0, \phi_1), && \text{on } t = 0, \end{aligned} \tag{QNLW}$$

eq:QNLW

where $\mathbf{g}_{\mu\nu}(\phi)$ is a symmetric matrix with signature $(-, +, \cdots +)$, using the convention[1] $\Box_{\mathbf{g}} := \mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$ for its associated wave operator, and $\mathcal{N}(\phi)(\partial\phi, \partial\phi) := \mathcal{N}^{\alpha\beta}(\phi)\partial_\alpha\phi\partial_\beta\phi$ is a bilinear form. Without loss of generality, we can take $t = $ const to be space-like hypersurfaces by reducing to metrics of the form

$$\mathbf{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \mathbf{g}_{ij}dx^i dx^j.$$

We shall also assume sufficient smoothness and boundedness of the metric $\mathbf{g}^{\mu\nu}(\phi)$, its inverse $\mathbf{g}_{\mu\nu}(\phi)$, and of the bilinear form $\mathcal{N}^{\alpha\beta}(\phi)$ as functions of ϕ.

Example. The following can be recast in the form (QNLW),

- the Einstein vacuum equations in wave coordinates,
- the irrotational compressible Euler equations.

For the former, this was observed by Choquet-Bruhat [Fou52], while the later is due to Hughes-Kato-Marsden [HKM77]. The reader may find the lecture notes [Luk] as a more modern reference.

Following the standard set by Hadamard, we say that the initial data problem for the quasi-linear wave equation (QNLW) is *locally well-posed* in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$ if the following hold:

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[1] One can equivalently consider the divergence form of the equation, i.e. using $\partial_\mu\mathbf{g}^{\mu\nu}\partial_\nu$ instead of $\mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$ on the left-hand side, as the lower-order terms are encapsulated by the right-hand side.

item:exist1 (a) *Existence*: for each initial data $\phi[0] \in (H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$, there exists a time $T > 0$ and a solution $\phi[t] \in C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$ to (QNLW).

(b) *(Unconditional) uniqueness*: for each initial data $\phi[0] \in (H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$, the solution $\phi[t]$ to (QNLW) is unique in the space $C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$.

item:cty (c) *Continuity of data-to-solution map*: if $\{\phi_k[0]\}_k$ is a sequence of data converging in the $(H^s_x \times H^{s-1}_x)$-topology to $\phi[0]$, then there exists a common time of existence[2] on which the corresponding sequence of solutions $\{\phi_k[t]\}_k$ to (QNLW) converges to $\phi[t]$ in the $L^\infty_t(H^s_x \times H^{s-1}_x)$-topology,

$$\begin{aligned} &\phi_k[0] \overset{k\to\infty}{\longrightarrow} \phi[0] \qquad \text{in } H^s_x \times H^{s-1}_x \\ \text{implies} \qquad &\phi_k[t] \overset{k\to\infty}{\longrightarrow} \phi[t] \qquad \text{in } L^\infty_t(H^s_x \times H^{s-1}_x). \end{aligned}$$

For the working definition, we will need to slightly modify the existence and uniqueness statements, strengthening the former while weakening the latter, and require an additional property of the data-to-solution map:

item:exist (a+) *(Sub-critical) existence*: the time of existence can be taken to depend only on the size of the data

$$T \equiv T(\|\phi[0]\|_{H^s_x \times H^{s-1}_x}).$$

item:unique (b′) *(Conditional) uniqueness*: uniqueness holds only in the smaller Strichartz space,

$$\left\{\phi[0] \in C^0_t(H^s_x \times H^{s-1}_x) : \partial\phi \in L^2_t L^\infty_x \right\}.$$

item:lipschitz (c+) *Weak Lipschitz continuity of data-to-solution map*: there exists a regularity $s_{\mathrm{Lip}} < s$ such that the data-to-solution map is Lipschitz continuous on bounded sets in $(H^s \times H^{s-1})_x(\mathbb{R}^n)$ with respect to the weaker $(H^{s_{\mathrm{Lip}}} \times H^{s_{\mathrm{Lip}}-1})_x(\mathbb{R}^n)$-topology, i.e. for solutions $\phi[t], \psi[t] \in C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$ to (QNLW) satisfying

$$\|\phi[0]\|_{H^s \times H^{s-1}}, \|\psi[0]\|_{H^s \times H^{s-1}} \leq R,$$

the following stability estimate holds:

$$\|\phi[t] - \psi[t]\|_{L^\infty_t (H^{s_{\mathrm{Lip}}} \times H^{s_{\mathrm{Lip}}-1})_x} \leq C(R) \cdot \|\phi[0] - \psi[0]\|_{(H^{s_{\mathrm{Lip}}} \times H^{s_{\mathrm{Lip}}-1})_x}.$$

In sum, we say that the initial data problem for the quasi-linear wave equation (QNLW) is *locally well-posed* in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$ if (a), (a+), (b′), (c), (c+) hold. This leads us to the following natural question

> *For which values of $s \in \mathbb{R}$ is the initial data problem for the quasi-linear wave equation (QNLW) locally well-posed in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$?*

To simplify the presentation, we will only consider high spatial dimensions, i.e. $n \geq 3$; in dimension $n = 2$, the dispersion of waves is weaker, though analogues of various statements made in this note continue to hold with suitable modifications. In this setting, the sharp result, due to Smith-Tataru, may be stated as follows,

thm:main **Theorem 1.1** (Sharp local well-posedness for (QNLW) [ST05])**.** *In dimensions $n = 3, 4, 5$, the initial data problem for the quasi-linear wave equation (QNLW) is locally well-posed in $(H^s \times H^{s-1})_x(\mathbb{R}^n)$ for $s > \frac{n}{2} + \frac{1}{2}$, with weak Lipschitz continuity of the data-to-solution map with respect to the $(H^1 \times L^2)_x$-topology.*

Remark. The equation (QNLW) is invariant under the scaling symmetry

$$\phi(x^\mu) \mapsto \phi(\tfrac{x^\mu}{\lambda})$$

which also preserves the homogeneous Sobolev norm $\dot{H}^{s_{\mathrm{crit}}}_x \times \dot{H}^{s_{\mathrm{crit}}-1}_x(\mathbb{R}^n)$, where $s_{\mathrm{crit}} := \frac{n}{2}$. It is natural to ask whether one can prove well-posedness up to the critical regularity. Unfortunately, Theorem 1.1 is sharp for generic (QNLW) in dimensions $n = 2, 3$ due to counterexamples of Lindblad [Lin93, Lin96]. On the other hand, it is not difficult to show that the *Nirenberg example*

$$\Box\phi = \partial^\alpha\phi\partial_\alpha\phi,$$

is locally well-posed for $s > s_{\mathrm{crit}}$, thanks to the *null structure* of the non-linearity.

[2] To be more precise, one can introduce the notion of the *maximal lifespan* $T \equiv T(\phi[0])$ of a solution, and require it to be lower semi-continuous as a function of initial data $\phi[0] \in (H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$.

Remark. The proof contained in [ST05] breaks down in higher dimensions $n \geq 6$ due to a technical failure in the orthogonality argument for the wave packet decomposition; see Section 4.

The basic starting point is the energy estimate, and local well-posedness result for sufficiently smooth initial data,

Theorem 1.2 (Local well-posedness for (QNLW) with smooth data [HKM77])**.** *The initial data problem for the quasi-linear wave equation (QNLW) is (unconditionally) locally well-posed in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$ for $s > \frac{n}{2} + 1$. Furthermore, for all $s \geq 0$, there exists $C \gg 1$ such that any smooth solution ϕ obeys the a priori estimate*

$$\|\partial\phi\|_{L^\infty_t H^{s-1}_x} \lesssim \exp\left(C \int_0^T \|\partial\phi\|_{L^\infty_x}\, dt\right) \|\partial\phi(0)\|_{H^{s-1}_x}. \tag{1.1}$$

Proof. See [IT22] for a modern treatment. □

To prove Theorem 1.1, one essentially needs to close the energy estimate. This is accomplished by proving Strichartz estimates for the linearised equation[3]

$$\begin{aligned} \Box_{\mathbf{g}(\phi)}\psi &= 0, \\ (\psi, \partial_t\psi)_{|t=0} &= (\psi_0, \psi_1), \end{aligned} \tag{LW}$$

eq:LW

and its paradifferential counterpart

$$\begin{aligned} \Box_{\mathbf{g}(\phi)_{<\lambda}}\psi_\lambda &= 0, \\ (\psi, \partial_t\psi)_{|t=0} &= (\psi_0, \psi_1), \end{aligned} \tag{PLW}$$

eq:LWpara

where ϕ is a solution to (QNLW). For flat backgrounds $\mathbf{g}(\phi) \equiv \mathbf{m}$, i.e. when (LW) reduces to the classical linear wave equation, one has the classical Strichartz estimates

$$\|\partial\phi\|_{L^\infty_t H^{s-1}_x} + \|\partial\phi\|_{L^2_t L^\infty_x} \lesssim \|\phi[0]\|_{H^s_x \times H^{s-1}_x}, \qquad s > \tfrac{n}{2} + \tfrac{1}{2},$$

with the $L^2_t L^\infty_x$-control allowing us to close the argument. If one only assumes $\partial\mathbf{g} \in L^2_t L^\infty_x$ in (LW), then the best one can do is to obtain the above Strichartz estimate but with a $\frac{1}{6}$-derivative loss [Tat01a, Tat01b]. The major breakthrough of Smith-Tataru was that actually there is no derivative-loss in Strichartz for (LW) when one uses that ϕ is a solution to (QNLW), which in turn roughly implies $\Box_{\mathbf{g}(\phi)}\mathbf{g}(\phi) \approx 0$,

nm:strichartz **Theorem 1.3** (Loss-less Strichartz estimates for (QNLW) [ST05])**.** *Let $\phi[t] \in C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$ be a solution to (QNLW) as in Theorem 1.1. Then*

(a) Strichartz bounds for non-linear evolution*: the solution satisfies*

$$\|\partial\phi\|_{L^\infty_t H^{s-1}_x} + \|\partial\phi\|_{L^2_t L^\infty_x} \lesssim \|\phi[0]\|_{H^s_x \times H^{s-1}_x}.$$

(b) Strichartz bounds for linear evolution*: for $1 \leq \sigma \leq s+1$ and each $t_0 \in [0,T]$, the initial data problem for the linear equation (LW) is well-posed in $(H^\sigma_x \times H^{\sigma-1}_x)(\mathbb{R}^n)$. Furthermore, the solution satisfies*

$$\|\partial\psi\|_{L^\infty_t H^{\sigma-1}_x} + \|\langle\nabla_x\rangle^{\sigma-\frac{n}{2}-\frac{1}{2}-1-}\partial\psi\|_{L^2_t L^\infty_x} \lesssim \|\psi[t_0]\|_{H^\sigma_x \times H^{\sigma-1}_x}. \tag{1.2}$$

Remark. The dimension of the $L^p_t L^\infty_x$-norm of $\partial\phi$ under the scaling symmetry reads

$$\|\partial\phi\|_{L^p_t L^\infty_x} \approx [t]^{\frac{1}{p}}[x]^{-1} \approx [\partial]^{1-\frac{1}{p}}.$$

Thus, the continuation criterion $L^1_t L^\infty_x$ is scale-invariant, controlling $L^\infty_{t,x}$ via Sobolev embedding incurs a full derivative difference from scaling $1 - \frac{1}{\infty} = 1$, while control of $L^2_t L^\infty_x$ in $n \geq 3$ via Strichartz leads to half-derivative from scaling $1 - \frac{1}{2} = \frac{1}{2}$, and similarly $L^4_t L^\infty_x$ in $n = 2$ leads to three-quarters $1 - \frac{1}{4} = \frac{1}{2} + \frac{1}{4}$.

[3]Strictly speaking, this is not quite the linearisation of (QNLW), as there is an extra order-zero term $(\partial_\phi \mathbf{g})^{\mu\nu}(\phi)\partial_\mu\partial_\nu\phi\,\psi$, however this can be easily treated as perturbative on short time-scales.

	$\mathbf{g}$ in (LW)	Strichartz	Regularity
Hughes-Kato-Marsden [HKM77]	$\partial\mathbf{g} \in L^\infty_{t,x}$	N/A	$s > \frac{n}{2} + 1$
Bahouri-Chemin [BC99]	$\partial\mathbf{g} \in L^2_t L^\infty_x$	$\frac{1}{4}$ loss	$s > \frac{n}{2} + \frac{1}{2} + \frac{1}{4}$
Tataru [Tat01a, Tat01b]	$\partial\mathbf{g} \in L^2_t L^\infty_x$	sharp $\frac{1}{6}$ loss	$s > \frac{n}{2} + \frac{1}{2} + \frac{1}{6}$
Klainerman-Rodnianski [KR03]	$\Box_{\mathbf{g}}\mathbf{g} \approx 0$	$\frac{2-\sqrt{3}}{2}$ loss	$s > \frac{n}{2} + \frac{1}{2} + \frac{2-\sqrt{3}}{2}$
Smith-Tataru [ST05]	$\Box_{\mathbf{g}}\mathbf{g} \approx 0$	lossless	$s > \frac{n}{2} + \frac{1}{2}$

table:history

TABLE 1. A non-exhaustive historical overview of the local well-posedness of quasi-linear wave equations for $n \geq 3$, though one can find results concerning $n = 2$ among the references, and the result of Klainerman-Rodnianski [KR03] works only with $n = 3$.

2. THE BOOTSTRAP ARGUMENT

To simplify presentation, we shall only work in dimensions $n = 3, 4, 5$ for the remainder of this article. We shall also recast the original bootstrap argument of Smith-Tataru in more modern language, following the expository article of Ifrim-Tataru [IT22] (see also [AIT24]).

Let $s > \frac{n}{2} + \frac{1}{2}$, and consider initial data $\phi[0] \in (H^s \times H^{s-1})_x(\mathbb{R}^n)$ which is sufficiently small, i.e.

$$\|\phi[0]\|_{(H^s \times H^{s-1})_x} \leq \varepsilon \ll 1. \tag{2.1}$$

For each $\lambda \in 2^{\mathbb{N}}$, define

$$\begin{aligned}
\phi^{(\lambda)} &:= \text{solution to (QNLW) with } \phi^{(\lambda)}[0] := \mathsf{P}_{\leq\lambda}\phi[0], \\
\psi^{(\lambda)} &:= \text{solution to (QNLW) with } \psi^{(\lambda)}[0] := \mathsf{P}_{\lambda}\phi[0].
\end{aligned}$$

By persistence of regularity, these are smooth solutions. Furthermore, the initial data satisfy the bounds

$$\begin{aligned}
\|\phi^{(\lambda)}[0]\|_{(H^s \times H^{s-1})_x} &\lesssim \varepsilon, \\
\|\psi^{(\lambda)}[0]\|_{(H^1 \times L^2)_x} &\lesssim \lambda^{1-s}\varepsilon.
\end{aligned} \tag{2.2}$$

Theorem 2.1 (Estimates for smooth solutions from regularised data)**.** *The solutions $\phi^{(\lambda)}$ to (QNLW) satisfy the following properties:*

(a) Uniform bounds: *the solutions exist up to unit time-scale $[0, 1]$, and satisfy the uniform bounds*

$$\|\phi^{(\lambda)}[t]\|_{L^\infty_t(H^s \times H^{s-1})_x} \lesssim \varepsilon, \tag{2.3}$$

and higher regularity bounds

$$\|\phi^{(\lambda)}[t]\|_{L^\infty_t(H^{s+1} \times H^s)_x} \lesssim \lambda\varepsilon. \tag{2.4}$$

(b) Difference bounds: *the linearised equations (LW) around $\phi^{(\lambda)}$ are well-posed in $(H^1 \times L^2)_x(\mathbb{R}^n)$, and satisfy the Strichartz estimates*

$$\|\partial\psi\|_{L^\infty_t L^2_x} + \|\partial\psi\|_{L^2_t L^\infty_x} \lesssim \|\partial\psi[0]\|_{L^2_x}, \tag{2.5}$$

uniformly in $\lambda \in 2^{\mathbb{N}}$.

Bootstrap assumptions,

(a) Uniform $(H^s \times H^{s-1})_x$-bounds,

$$\|\phi^{(\lambda)}[t]\|_{L^\infty_t(H^s \times H^{s-1})_x} \leq 1, \tag{2.6}$$

(b) Difference bounds

$$\|\psi^{(\lambda)}[t]\|_{L^\infty_t(H^1 \times L^2)_x} \leq N^{-s}, \tag{2.7}$$

$$\|(\phi^{(\lambda)} - \phi^{(\mu)})[t]\|_{L^\infty_t(H^1 \times L^2)_x} \leq \varepsilon c_\lambda \tag{2.8}$$

3. Geometry and regularity of null hypersurfaces

Let u_θ be the solution the eikonal equation initialised at $t = -2$,

$$\begin{aligned} \mathbf{g}^{\mu\nu}\partial_\mu u \partial_\nu u &= 0, \\ u_{|t=-2} &= \theta \cdot x + 2. \end{aligned} \tag{3.1}$$

By the implicit function theorem, we can write the level sets of u_θ as graphs,

$$\Sigma_{\theta,u} := \{(t,x) \in [-2,2] \times \mathbb{R}^n : x_\theta - \tau_{\theta,u}(t, x'_\theta) = 0\}$$

ec:parametrix

4. Wave packet parametrix

To prove Strichartz estimates for the linear wave equation, we construct a *wave packet parametrix*, that is, a superposition of wave packets which form an approximate solution the initial data problem. Given a null geodesic $\gamma(t)$ contained in a null hypersurface $\Sigma_{\theta,r} = \{x_\theta - \tau_{\theta,r} = 0\}$, we define a *wave packet* $\mathfrak{w}$ localised around γ at scale $\lambda \in 2^{\mathbb{N}}$ to be a function of the form

$$\mathfrak{w} := (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}}\lambda^{\frac{1}{2}-1-1}\mathsf{T}_{<\lambda}\left(w\,\delta(u_{\theta,r})\right),$$

where δ is the Dirac mass at zero, and thus $\delta(u_{\theta,r})$ is a measure[4] supported on the null hypersurface $\Sigma_{\theta,r}$ which takes the form

$$\langle \delta(u_\theta), \varphi\rangle := \int_{[-2,2]}\int_{\mathbb{R}^{n-1}} \varphi(t, x'_\theta, \tau_{\theta,r})\,dx'_\theta\,dt \qquad \text{for } \varphi \in C_c^\infty((-2,2)\times\mathbb{R}^n),$$

and w is a smooth bump function on $\Sigma_{\theta,r}$ localised at scale $(\varepsilon_0\lambda)^{-\frac{1}{2}}$ about the null geodesic γ, i.e.

$$w(t, x'_\theta) = w_0\big((\varepsilon_0\lambda)^{\frac{1}{2}}(x'_\theta - \gamma'_\theta(t))\big), \qquad w_0 \in C_c^\infty(|x'| \le 1),$$

and finally $\mathsf{T}_{<\lambda}$ is a mollification to spatial scales $\triangle x \sim \frac{1}{\lambda}$, taking it with kernel $\chi_{<\lambda}(x) := \lambda^n\chi(\lambda x)$ for some compactly-supported cut-off $\chi \in C_c^\infty(|x| \le \frac{1}{2000})$; morally, one should think of this as a localisation to frequency $|\xi| \lesssim \lambda$.

The parameter $\varepsilon_0 \ll 1$ shall be chosen such that the error as an approximate solution $\Box_{\mathbf{g}}\mathfrak{w}$ is small in $L^1_t L^2_x$, while the choice of amplitude will normalise the wave packet to be of size $O(1)$ in $(H^1 \times L^2)_x$; see Lemmas 4.4 and 4.2 respectively.

Given a function restricted to frequency λ for all t, we want to construct a wave packet resolution, i.e. show that it arises as a superposition of wave packets. To that end, decompose $\mathbb{R}^n$ into a parallel tiling of rectangles of dimensions $(4\varepsilon_0\lambda)^{-\frac{1}{2}\times(n-1)} \times (8\lambda)^{-1}$ in (x'_θ, x_θ)-coordinates. Then define

$$\begin{aligned} R_{\theta,j} &:= \text{doubles of these rectangles at } t = -2, \\ \Sigma_{\theta,j} &:= \text{null hypersurface centered on } R_{\theta,j}, \\ \gamma_{\theta,j} &:= \text{null geodesic in } \Sigma_{\theta,j} \text{ through center of } R_{\theta,j}, \\ \mathtt{T}_{\theta,j} &:= (32\lambda)^{-1}\text{-neighborhood of } \Sigma_{\theta,j} \cap \{|x'_\theta - \gamma_{\theta,j}(t)| \le (\varepsilon_0\lambda)^{-\frac{1}{2}}\}. \end{aligned}$$

We shall refer to the space-time regions $\mathtt{T}_{\theta,j}$ as slabs; by construction, wave packets are supported in slabs. These slabs satisfy a finite-overlap condition; indeed, those associated to different null hypersurfaces are disjoint, while those associated to the same null hypersurface have finite overlap in the x'_θ-variable. Furthermore, we consider slabs with angles θ taken from

$$\Omega := \text{maximal collection of } \varepsilon_0^{\frac{n-1}{2}}\lambda^{\frac{n-1}{2}}\text{-many unit vectors } \theta \in \mathbb{S}^{n-1} \text{ separated by at least } (\varepsilon_0\lambda)^{-\frac{1}{2}}.$$

[4]Alternatively, one can think of this as the pushforward of the Lebesgue measure on $[-2,2]\times\mathbb{R}^{n-1}$ under the embedding $(t,x') \mapsto (t,x',\tau)$. This convention is common in harmonic analysis in the context of the Fourier extension and restriction problems. Strictly speaking, this is not the same as *surface measure* as it is referred in [ST05].

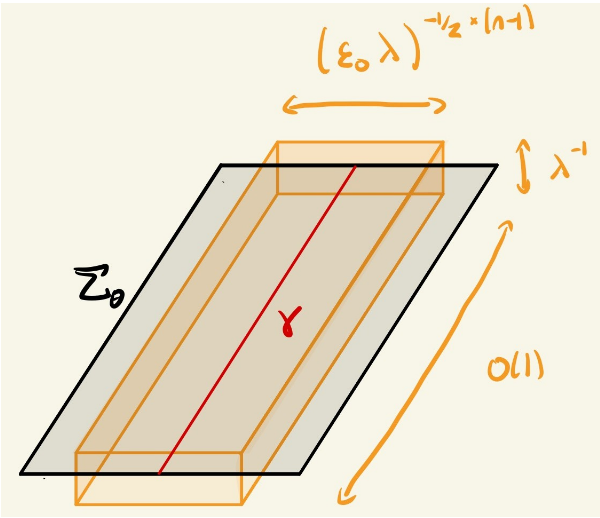

FIGURE 1. Given a null geodesic γ residing in a null hypersurface $\Sigma_{\theta,u}$, the corresponding wave packet is supported in a $1 \times \lambda^{-1} \times (\varepsilon_0\lambda)^{-\frac{1}{2}\times(n-1)}$-slab.

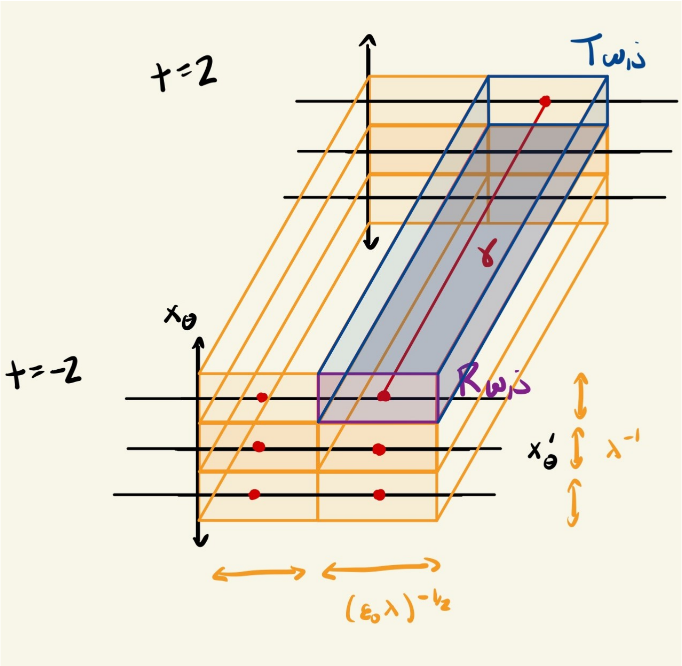

FIGURE 2. Given a fixed angle θ, we cover $[-2,2] \times \mathbb{R}^n$ by slabs localised along null geodesics $\gamma_{\theta,j}$ with dimensions $(\varepsilon_0\lambda)^{-\frac{1}{2}\times(n-1)} \times \lambda^{-1}$. fig:physicalW

op:parametrix **Proposition 4.1** (Existence of wave packet parametrix)**.** *Let $(\phi_0,\phi_1) \in (H^1 \times L^2)_x(\mathbb{R}^n)$ be initial data. Then, in dimensions $n = 2,3,4,5$, there exists a superposition of wave packets*

$$\phi := \sum_{\theta,j} a_{\theta,j}\, \mathfrak{w}^{\theta,j}$$

which is an approximate solution to the paralinearised initial data problem in the sense that

item:WPdata *(a) it matches the initial data at $t=-2$,*

$$\mathsf{P}_\lambda \phi[-2] = (\phi_0,\phi_1) \tag{4.1}$$

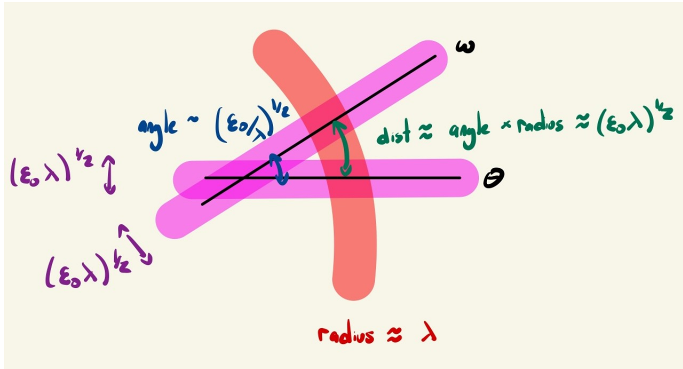

FIGURE 3. In frequency space, the wave packets at time $t = -2$ are effectively localised to $(\varepsilon_0\lambda)^{\frac{1}{2}}$-neighborhoods of the angles θ. These angles are separated by at least $(\frac{\varepsilon_0}{\lambda})^{\frac{1}{2}}$, so when restricted to the dyadic shell $|\xi| \sim \lambda$, there is $O(1)$-overlap between wave packets. fig:frequency

item:WPsize *(b) the size of the coefficients is comparable to the size of the initial data,*

$$\Big(\sum_{\theta,j} |a_{\theta,j}|^2\Big)^{\frac{1}{2}} \lesssim \|\phi[0]\|_{(H^1\times L^2)_x}. \tag{4.2}$$

item:WPenergy *(c) the energy estimate holds,*

$$\|\partial \mathsf{P}_\lambda \phi\|_{L^\infty_t L^2_x} \lesssim \Big(\sum_{\theta,j} |a_{\theta,j}|^2\Big)^{\frac{1}{2}} \tag{4.3}$$ eq:parametrix

item:WPerror *(d) the error on the right-hand side is small,*

$$\|\square_{\mathbf{g}_{<\lambda}} \phi_\lambda\|_{L^1_t L^2_x} \lesssim \varepsilon_0 \Big(\sum_{\theta,j} |a_{\theta,j}|^2\Big)^{\frac{1}{2}}. \tag{4.4}$$ eq:parametrix

4.1. **Properties of wave packets.** As a first step, we prove that a singular wave packet is an approximate solution, in the sense that it satisfies the energy bound and error estimate.

lem:energyWP **Lemma 4.2** (Energy estimate for $\mathfrak{w}$)**.** *Wave packets have $O(1)$-energy,*

$$\|\partial \mathsf{P}_\lambda \mathfrak{w}\|_{L^\infty_t L^2_x} \lesssim 1. \tag{4.5}$$ eq:WPenergy

Proof for $\partial = \partial_x$. The estimate for the full space-time gradient will follow from the usual energy estimate once one has the error estimate (4.9), though it is instructive to see how to read off the result for the spatial derivatives from the construction. Roughly speaking[5],

$$\begin{aligned} \text{support of } \mathsf{T}_{<\lambda}(w\delta(u_{\theta,r})) &\approx 1 \times \lambda^{-1} \times (\varepsilon_0\lambda)^{-\frac{1}{2}\times(n-1)}, \\ \text{amplitude of } \mathsf{T}_{<\lambda}(w\delta(u_{\theta,r})) &\approx \lambda. \end{aligned}$$

The former is fairly clear; to say a word about the latter, $\delta(x_\theta - \tau)$ is a unit point mass, while the mollification spreads it out in the θ-direction to scale λ^{-1}, so dimensional analysis tells us the resulting amplitude is λ. Using the usual heuristic that $\partial_x \approx \lambda$ at frequencies comparable to λ, we arrive at

$$\|\partial_x \mathsf{P}_\lambda \mathfrak{w}\|_{L^\infty_t L^2_x} \lessapprox \lambda \cdot (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}} \lambda^{\frac{1}{2}-1-1} |\text{amplitude}| \cdot |\text{support}|^{\frac{1}{2}} \lessapprox 1.$$

This gives (4.5), modulo time-derivative control. □

Lemma 4.3 (Wave packet error decomposition)**.** *Let $\mathfrak{w}$ be a wave packet, then the error decomposes as*

$$\square_{\mathbf{g}_{<\lambda}} \mathsf{P}_\lambda \mathfrak{w} = \mathcal{L}(\partial \mathbf{g}, \partial \widetilde{\mathsf{P}_\lambda} \tilde{\mathfrak{w}}) + (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}} \lambda^{\frac{1}{2}-1-1} \mathsf{P}_\lambda \mathsf{T}_{<\lambda} \sum_{j=0,1,2} \psi_j \delta^{(j)}(x_\theta - \tau_{\theta,r}), \tag{4.6}$$ eq:errordecom

[5]Here, we write $\lessapprox$ and $\approx$ to mean that we aren't making any rigorous claims.

where $\mathcal{L}$ is a bilinear form which obeys Hölder's inequality, $\widetilde{\mathsf{P}_\lambda}$ is a fattened frequency projection, $\widetilde{\mathfrak{w}}$ is another wave packet, $\delta^{(j)}$ are derivatives of the Dirac mass, and $\psi_j(t,x')$ obey

$$\|\psi_j((\varepsilon_0\lambda)^{-\frac{1}{2}}x'_\theta)\|_{L^2_t H^{s-1}_{x'_\theta}} \lesssim \varepsilon_0\lambda^{1-j}. \tag{4.7}$$ eq:errorcoeff

lem:errorWP **Corollary 4.4** (Error estimate for $\mathfrak{w}$). *Each wave packet has small error,*

$$\|\Box_{\mathbf{g}_{<\lambda}}\mathsf{P}_\lambda\mathfrak{w}\|_{L^1_tL^2_x} \lesssim \varepsilon_0, \tag{4.8}$$ eq:L1error

$$\|\Box_{\mathbf{g}_{<\lambda}}\mathsf{P}_\lambda\mathfrak{w}\|_{L^2_{t,x}} \lesssim \varepsilon_0. \tag{4.9}$$ eq:L2error

Proof. Obviously (4.9) is stronger than (4.8), so we focus on proving an $L^2_{t,x}$-error estimate. For the first term on the right-hand side of (4.6), we place $\partial\mathbf{g}$ in $L^2_tL^\infty_x$, gaining smallness from our bootstrap assumption, and $\partial\widetilde{\mathfrak{w}}$ in $L^\infty_tL^2_x$, in which it is unit size by[6] (4.5), yielding

$$\|\mathcal{L}(\partial\mathbf{g},\partial\widetilde{\mathsf{P}_\lambda}\widetilde{\mathfrak{w}})\|_{L^2_{t,x}} \lesssim \|\partial\mathbf{g}\|_{L^2_tL^\infty_x}\|\partial\widetilde{\mathfrak{w}}\|_{L^\infty_tL^2_x} \lesssim \varepsilon_2.$$

Taking $\varepsilon_2 \ll \varepsilon_0$ is an acceptable contribution towards (4.9).

For the second term on the right-hand side of (4.6), dimensional analysis yields

$$\text{support of } \mathsf{P}_\lambda\mathsf{T}_{<\lambda}(\psi_j\delta^{(j)}) \approx 1\times\lambda^{-1}\times(\varepsilon_0\lambda)^{-\frac{1}{2}\times(n-1)},$$
$$\|\text{amplitude of } \mathsf{P}_\lambda\mathsf{T}_{<\lambda}(\psi_j\delta^{(j)})\|_{L^2_t} \lessapprox \|\text{amplitude of } \psi_j\|_{L^2_t}\cdot\lambda^{1+j} \lessapprox \varepsilon_0\lambda^2.$$

Roughly speaking, $s-1>\frac{n-1}{2}$ under the assumptions of Theorem 1.1, we can use Sobolev embedding on $\mathbb{R}^{n-1}$ to read off the amplitude bounds using the scaled Sobolev estimate (4.7), while derivatives of the Dirac mass contribute amplitude λ. By dimensional analysis, we arrive at (4.9). □

Remark. Note that we used the Sobolev embedding for $s-1>\frac{n-1}{2}$.

Proof of error decomposition (4.6). Suppressing subscripts for clarity $u\equiv u_{\theta,r}$ and $\tau\equiv\tau_{\theta,r}$, we compute

$$\begin{aligned}
\Box_{\mathbf{g}_{<\lambda}}\mathsf{P}_\lambda\mathfrak{w} &= (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}}\lambda^{-\frac{1}{2}+1}\Big([\Box_{\mathbf{g}_{<\lambda}},\mathsf{P}_\lambda\mathsf{T}_{<\lambda}]+\mathsf{P}_\lambda\mathsf{T}_{<\lambda}\Box_{\mathbf{g}_{<\lambda}}\Big)w\,\delta(u)\\
&= (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}}\lambda^{-\frac{1}{2}+1}[\Box_{\mathbf{g}_{<\lambda}},\mathsf{P}_\lambda\mathsf{T}_{<\lambda}]w\,\delta(u)\\
&\quad+(\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}}\lambda^{-\frac{1}{2}+1}\mathsf{P}_\lambda\mathsf{T}_{<\lambda}\left(\Box_{\mathbf{g}_{<\lambda}}w\cdot\delta(u)+2\overline{\mathbf{g}}^{\alpha\beta}_{<\lambda}\partial_\alpha w\cdot\partial_\beta\delta(u)+w\cdot\Box_{\mathbf{g}_{<\lambda}}\delta(u)\right)\\
&=: \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV},
\end{aligned}$$

commuting $\Box$ with the frequency projections and then applying the product rule. Here we have denoted $\overline{\mathbf{g}}^{\alpha\beta}_{<\lambda}=\frac{1}{2}(\mathbf{g}^{\alpha\beta}_{<\lambda}+\mathbf{g}^{\beta\alpha}_{<\lambda})$.

<u>*Term* I</u>. Since the metric is cut-off to frequencies much lower than λ, the commutator clearly projects to frequencies $|\xi|\sim\lambda$. Thus, one can harmlessly insert fattened projections $\widetilde{\mathsf{P}}_\lambda\widetilde{\mathsf{T}}_{<\lambda}$ in front of the commutator. Furthermore, while two derivatives fall on the wave packet, standard commutator arguments[7] allow us to move one derivative onto the metric. In total, we can rewrite

$$\mathrm{I}=[\mathbf{g}^{\alpha\beta}_{<\lambda},\mathsf{P}_\lambda\mathsf{T}_{<\lambda}]\partial_\alpha\partial_\beta\widetilde{\mathfrak{w}}=\mathcal{L}(\partial\mathbf{g},\partial\widetilde{\mathfrak{w}})$$

for another wave packet $\widetilde{\mathfrak{w}}$ and some translation-invariant bilinear operator $\mathcal{L}(-,-)$.

<u>*Term* II</u>. We compute two derivatives of the bump function on $\mathbb{R}^{n-1}$ localised to the null geodesic γ,

$$\partial_\alpha\partial_\beta w=\begin{cases}O(\varepsilon_0\lambda) & \text{if two spatial derivatives,}\\ O((\varepsilon_0\lambda)^{\frac{1}{2}}\ddot{\gamma}) & \text{if two time derivatives,}\\ O(\varepsilon_0\lambda\dot{\gamma})\end{cases}$$

Since $\|\ddot{\gamma}\|_{L^2_t}\lesssim\varepsilon_1$, this is acceptable for (4.7) when placed into ψ_0.

[6] To be perfectly rigorous, the energy estimate and the error estimate should be proved in conjunction.

[7] In a word, the principal symbol of the commutator is given by the Poisson bracket, so one can, to leading order, write $[\mathbf{g}(x),\chi(\nabla/\lambda)]\approx\{\mathbf{g}(x),\chi(\xi/\lambda)\}\approx\partial_x\mathbf{g}\cdot\partial_\xi\chi(\xi/\lambda)\approx\frac{1}{\lambda}\partial_x\mathbf{g}$.

<u>*Term* III</u>. We compute, schematically,

$$\begin{aligned}\overline{\mathbf{g}}_{<\lambda} \cdot \partial w \cdot \partial \delta(u) &= \overline{\mathbf{g}_{<\lambda}} \cdot \partial w \cdot \partial u \cdot \delta^{(1)}(u) \\ &= (\overline{\mathbf{g}}_{<\lambda})_{|\Sigma} \cdot \partial w \cdot \partial u \cdot \delta^{(1)}(u) + (\partial \overline{\mathbf{g}}_{<\lambda})_{|\Sigma} \cdot \partial w \cdot \partial u \cdot \delta(u).\end{aligned}$$

<u>*Term* IV</u>. We compute

$$\Box_{\mathbf{g}_{<\lambda}} \delta(u) = \mathbf{g}^{\alpha\beta}_{<\lambda} \left(\partial_\alpha u \, \partial_\beta u \, \delta^{(2)}(u) + \partial_\alpha \partial_\beta \tau \, \delta^{(1)}(u) \right).$$

Applying the distributional product rule, we can rewrite the terms above schematically as

$$\begin{aligned}\mathbf{g}_{<\lambda} \cdot \partial\partial\tau \cdot \delta^{(1)}(x_\theta - \tau) &= (\partial \mathbf{g})_{|\Sigma} \cdot \partial\partial\tau \cdot \delta(x_\theta - \tau) \\ \mathbf{g}_{<\lambda} \cdot (\partial u)^2 \cdot \delta^{(2)}(x_\theta - \tau) &= (\partial\partial \mathbf{g}_{<\lambda})_{|\Sigma} \cdot (\partial u)^2 \cdot \delta(x_\theta - \tau) \\ &\quad + 2(\partial \mathbf{g}_{<\lambda})_{|\Sigma} \cdot (\partial u)^2 \cdot \delta^{(1)}(x_\theta - \tau) \\ &\quad + (\mathbf{g}_{>\lambda})_{|\Sigma} (\partial u)^2 \cdot \delta^{(2)}(x_\theta - \tau)\end{aligned}$$

Here, the derivatives on the metric are in the θ-direction; observe that ∂u depends only on t and x'_θ so no x^θ-derivatives fall on these terms. In the last line, we have freely replaced $\mathbf{g}_{<\lambda}(\partial u)^2$ by $\mathbf{g}_{>\lambda}(\partial u)^2$, since u is optical, i.e. $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$.

□

Remark. It is instructive to compare the computation above against the flat case. When u is optical,

$$\Box \delta(u) = \partial_\alpha u \partial^\alpha u \, \delta^{(2)}(u) + \Box u \, \delta^{(1)}(u) = 0.$$

Thus, the measures $\delta(u)$ on null hypersurfaces are exact solutions to the linear wave equation on flat backgrounds.

4.2. **Orthogonality of wave packets.** We want to show that, given any superposition of wave packets, the energy estimate (4.3) and the error estimate (4.4) hold. In view of the corresponding estimate for a single wave packet, we want to show that the interactions between wave packets are negligible, i.e. they are almost orthogonal. In place of the $L^\infty_t L^2_x$-bound (4.3), it is convenient to prove the weaker $L^2_{t,x}$-bound,

$$\|\partial \mathsf{P}_\lambda \phi\|_{L^2_{t,x}} \lesssim \Big(\sum_{\theta,j} |a_{\theta,j}|^2 \Big)^{\frac{1}{2}}. \tag{4.10}$$ eq:L2L2orthog

Then (4.10) and (4.4) imply (4.3) by the energy estimate.

We prove (4.10) by showing a fixed-time $t = \text{const}$ orthogonality estimate. This will be a consequence of the following two "properties" of wave packets which are "morally" true,

- Parallel wave packets, i.e. fixing θ, have finitely-overlapping supports $\mathtt{T}^t_{\theta,j}$ in physical space, see Figure 2
- Non-parallel wave packets, i.e. distinct $\theta, \omega \in \Omega$, at scale λ are effectively localised in frequency space to the dual rectangles $(\mathtt{T}^t_{\theta,j})^*$ and $(\mathtt{T}^t_{\omega,k})^*$, which are contained in $(\varepsilon_0\lambda)^{\frac{1}{2}}$-neighborhoods of θ and ω respectively. Since we have chosen angles separated by at least $(\frac{\varepsilon_0}{\lambda})^{\frac{1}{2}}$, the supports are effectively disjoint on the dyadic shell $|\xi| \sim \lambda$, see Figure 3.

Evidently these hold in the flat case, we want to show these persist for the variable background.

Lemma 4.5. *WE effectively need to show the reverse Sobolev trace theorem*

$$\|\mathsf{T}_{<\lambda}(\psi^{\theta,j} \cdot \delta(u_{\theta,j}))\|_{L^2 H^{\frac{n-1}{2}+}} \lesssim \lambda^{\frac{1}{2}} \|\psi^{\theta,j}\|_{H^{\frac{n-1}{2}+}(\mathbb{R}^{n-1})}. \tag{4.11}$$

Lemma 4.6 (Orthogonality at "good" t)**.** *Set*

$$\Phi := (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}} \lambda^{\frac{1}{2}-1} \mathsf{P}_\lambda \mathsf{T}_{<\lambda} \sum_{\theta,j} \psi^{\theta,j} \, \delta(u_{\theta,j}).$$

Suppose $t \in [-2,2]$ is a time such that $\|\partial \mathbf{g}(t)\|_{C^{0,0+}_x} \leq \varepsilon_0$, then

$$\|\Phi\|^2_{L^2_x} \lesssim \sum_{\theta,j} \left\| \psi^{\theta,j}((\varepsilon_0\lambda)^{-\frac{1}{2}} x) \right\|^2_{H^{\frac{n-1}{2}+}(\mathbb{R}^{n-1})} \tag{4.12}$$

Proof. Decompose

$$\begin{aligned}
\Phi &:= \sum_{\theta\in\Omega} \Phi_\theta, \\
\Phi_\theta &:= \mathsf{P}_\lambda \sum_j \mathfrak{v}^{\theta,j}, \\
\mathfrak{v}^{\theta,j} &:= (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}} \lambda^{\frac{1}{2}-1} \mathsf{T}_{<\lambda}\big(\psi^{\theta,j}\delta(u_{\theta,j})\big).
\end{aligned}$$

It is easy to check

$$\|\Phi_\theta\|^2_{L^2_{y_\theta} H^{\frac{n-1}{2}+}_{y'_\theta}(\mathbb{R}^n)} \lesssim (\varepsilon_0)^{\frac{n-1}{2}} \sum_j \|\psi^{\theta,j}\|^2_{H^{\frac{n-1}{2}+}(\mathbb{R}^{n-1})}.$$

after rescaling.

By Plancharel's theorem,

$$\|\Phi(t)\|^2_{L^2_x} \sim \int_{|\xi|\sim\lambda} \left| \sum_{\theta\in\Omega} \widehat{v_\theta} \right|^2 d\xi$$

Fourier transform of surface measure □

Lemma 4.7 (Orthogonality at "bad" t)**.** *Set*

$$\Phi := (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}} \lambda^{\frac{1}{2}-1} \mathsf{P}_\lambda \mathsf{T}_{<\lambda} \sum_{\theta,j} \psi^{\theta,j}\,\delta(u_{\theta,j}).$$

Suppose $t \in [-2,2]$ is a time such that $\|\partial \mathbf{g}(t)\|_{C^{0,0+}_x} \geq \varepsilon_0$. Then

$$\|\Phi(t)\|^2_{L^2_x} \lesssim \left(\frac{1}{\varepsilon_0} \|\partial \mathbf{g}(t)\|_{C^{0,0+}_x} \right)^{\frac{n-1}{2}} \sum_{\theta,j} \|\psi^{\theta,j}\|^2_{H^{\frac{n-1}{2}+}(\mathbb{R}^{n-1})}. \tag{4.13}$$

Proof. □

Proof of Proposition 4.1 (c). It would follow that

$$\|\partial_x \mathsf{P}_\lambda \phi(t)\|^2_{L^2_x} \lesssim \left(1 + \left(\frac{1}{\varepsilon_0} \|\mathbf{g}(t)\|_{C^{0,\delta}_x} \right)^{\frac{n-1}{2}} \right) \sum_{\theta,j} |a_{\theta,j}|^2$$

□

Proof of Proposition 4.1 (d). □

4.3. **Matching wave packets to initial data.** It remains to show Proposition 4.1 (a)-(b); put loosely, any initial data $(\phi_0, \phi_1) \in (H^1 \times L^2)_x(\mathbb{R}^n)$ can be matched at time $t = -2$ to a superposition of wave packets.

Maximal collection of θ. We decompose

$$\phi[0] = \sum_{\theta\in\Omega} \phi^\theta[0],$$

where

$$\phi^\theta := \frac{1}{2}\left(\phi^\theta_0(x + t\theta) + u_0 \right)$$

Approximate solution for $\Box_{\mathbf{g}_{<\lambda}}$. Fourier transform trick, u^ω_0 compact support in frequency, take Fourier transform in x_θ, then extend periodically the Fourier transform with period $\lambda\theta$,

$$\widehat{\phi^\theta} = \sum_{k\in\mathbb{Z}}$$

5. Dispersive estimates

The analysis in the previous sections tell us that the geometry of slabs is approximately that of Minkowski space. Thus, one can expect that the same harmonic analysis counting arguments used to prove Strichartz estimate (or, alternatively, Fourier restriction estimates) hold in this setting.

5.1. **Dispersive decay.**

$$\mathrm{dist}(x_2, C^t_{P_1}) = \inf_{\theta \in \mathbb{S}^{n-1}} |x_2 - \gamma_\theta(t_2)|$$

$$\delta u(P_1, P_2) := \sup_{\theta \in \mathbb{S}^{n-1}} |u_\theta(P_2) - u_\theta(P_1)|$$

Lemma 5.1 (Properties of δu)**.** *The parameter δu is negative if P_2 is inside the cone, positive in the exterior. Furthermore, $\delta u \approx \mathrm{dist}(x_2, C^t_1)$*

Let

$$\#_\lambda(\mathtt{P}_1, \mathtt{P}_2) := \#\text{ of slabs at scale } \lambda \text{ containing } \mathtt{P}_1 \text{ and } \mathtt{P}_2.$$

Proposition 5.2 (Dispersive decay, I)**.** *The number of slabs at scale λ containing a pair of points $\mathtt{P}_1 = (t_1, x_1)$ and $\mathtt{P}_2 = (t_2, x_2)$ is bounded by*

$$\#_\lambda(\mathtt{P}_1, \mathtt{P}_2) \lesssim \begin{cases} \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}} \left(1 + \lambda\, \mathrm{dist}(x_2, C^{t_2}_1)\right)^{\frac{n-3}{2}} (1 + \lambda|t_1 - t_2|)^{-\frac{n-1}{2}} & \text{if } \delta u \in I_1, \\ \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}} \left(1 + \lambda\, \mathrm{dist}(x_2, C^{t_2}_1)\right)^{-1} & \text{if } \delta u \in I_2, \\ 0 & \text{otherwise}, \end{cases} \tag{5.1}$$

where

$$I_1 := \left\{-4\lambda^{-1} \le \delta u(\mathtt{P}_1, \mathtt{P}_2) \le \min(2|t_1 - t_2|, C(\lambda\varepsilon_0)^{-1}|t_1 - t_2|^{-1})\right\}$$
$$I_2 := \left\{2|t_1 - t_2| \le \delta u(\mathtt{P}_1, \mathtt{P}_2) \le C(\varepsilon_0\lambda)^{-\frac{1}{2}}\right\}.$$

In actuality we will only need the worst case, i.e. $\delta u \in I_2$.

Corollary 5.3 (Dispersive decay, II)**.** *The number of slabs at scale λ containing a pair of points $\mathtt{P}_1$ and $\mathtt{P}_2$ is bounded by*

$$\#_\lambda(P_1, P_2) \lesssim \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}} \left(\lambda|t_1 - t_2|\right)^{-1} \tag{5.2}$$

Proof for $\delta u < -4\lambda^{-1}$ case. There are no such slabs,

$$\#_\lambda(\mathtt{P}_1, \mathtt{P}_2) = 0.$$

□

Proof for $|\delta u| \le 4\lambda^{-1}$ and $|t_1 - t_2| \le 2\lambda^{-1}$ case. Here we use the trivial bound

$$\#_\lambda(\mathtt{P}_1, \mathtt{P}_2) \lesssim \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}}$$

□

Proof for $|\delta u| \le 4\lambda^{-1}$ and $|t_1 - t_2| > 2\lambda^{-1}$ case. □

Proof for $4\lambda^{-1} < \delta u \le 2|t_1 - t_2|$ case. □

5.2. **Strichartz estimates.** Wave packets have size $(\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}}\lambda^{-\frac{1}{2}}$

Proposition 5.4. *Let*

$$\phi := \sum_{\mathtt{T}\in\mathcal{T}} a_{\mathtt{T}}\mathbb{1}_{\mathtt{T}}$$

then

$$\|\phi\|_{L_t^2L_x^\infty} \lesssim (\varepsilon_0\lambda)^{-\frac{1}{2}\frac{n-1}{2}}\lambda^{\frac{1}{2}}\|a_{\mathtt{T}}\|_{\ell^2_{\mathtt{T}}} \tag{5.3}$$

We proceed by discretising the problem. Dividing $[0,1]$ into $O(\lambda)$-many sub-intervals I_j of length $2\lambda^{-1}$, we can find points $\mathtt{P}_j = (t_j, x_j)$ nearly maximising $|\phi|$ on each space-time region $I_j \times \mathbb{R}^n$. It follows that

$$\|\phi\|_{L_t^2L_x^\infty} \lesssim \Big(\sum_j \int_{I_j} \|\phi(t)\|^2_{L_x^\infty}\,dt\Big)^{\frac{1}{2}} \lesssim \Big(\sum_j \lambda^{-1}|\phi(t_j,x_j)|^2\Big)^{\frac{1}{2}} \lesssim \lambda^{-\frac{1}{2}} \sum_{T\in\mathcal{T}} \Big(\sum_j |a_{\mathtt{T}}|^2\cdot|\mathbb{1}_{\mathtt{T}}(t_j,x_j)|^2\Big)^{\frac{1}{2}}.$$

After passing to an $O(\lambda)$ subset of points, we can choose t_j to be λ^{-1}-separated and such that the inequality above continues to hold. Next, we dyadically decompose the sum over slabs $\mathtt{T}$ with respect to the size $N^{-\frac{1}{2}}$ of the coefficients a_T, and similarly the sum over points $\mathtt{P}_j$ with respect to L the number of slabs containing them; we denote the number of such points by $m(L)$. It follows that

$$\begin{aligned}
\|\phi\|_{L_t^2L_x^\infty} &\lesssim \lambda^{-\frac{1}{2}} \sum_{\substack{N\in 2^{\mathbb{N}}\\ N\lesssim(\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}}} \sum_{\substack{\mathtt{T}\in\mathcal{T}\\ |a_{\mathtt{T}}|\sim N^{-\frac{1}{2}}}} \Big(\sum_{\substack{L\in 2^{\mathbb{N}}\\ L\lesssim(\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}}} \sum_{\substack{j\\ \#\{\mathtt{T}\in\mathcal{T}:\mathtt{P}_j\in\mathtt{T}\}\sim L}} |a_{\mathtt{T}}|^2\cdot|\mathbb{1}_{\mathtt{T}}(t_j,x_j)|^2\Big)^{\frac{1}{2}}\\
&\lesssim \lambda^{-\frac{1}{2}}\Big(\sum_{L\lesssim(\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}} m(L)\Big|\sum_{N\lesssim(\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}} N^{-\frac{1}{2}}\cdot L\Big|^2\Big)^{\frac{1}{2}}\\
&\lesssim \lambda^{-\frac{1}{2}} \sum_{N\lesssim(\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}} \Big(\sum_{L\lesssim(\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}} m(L)^2N^{-1}L^2\Big)^{\frac{1}{2}}
\end{aligned}$$

In the above, we needed only to sum over a finite range of scales, since each point lies in $O((\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}})$-many slabs. This restricts us to a finite range of scales for L, and allows us to regard the contribution of slabs with small coefficients $|a_T| \lesssim (\frac{\varepsilon_0}{\lambda})^{\frac{n-1}{2}}$ as $O(1)$.

We conclude with a counting argument. To summarise notation and introduce new ones,

$$\begin{aligned}
N &:= \text{size of coefficient},\\
L &:= \#\text{ of slabs},\\
m(L) &:= \#\text{ of points intersecting } L\text{-many slabs},\\
K &:= \#\text{ of pairs } (i,j) \text{ for which } \mathtt{P}_i, \mathtt{P}_j \text{ lie in a common slab w/ multiplicity},\\
n(\mathtt{T}) &:= \#\text{ of points in slab } \mathtt{T}.
\end{aligned}$$

Observe that

$$\sum_{\substack{\mathtt{T}\in\mathcal{T}_N\\ n(\mathtt{T})\geq 2}} |n(\mathtt{T})|^2 \sim K,$$
$$\sum_{\mathtt{T}\in\mathcal{T}_N} n(\mathtt{T}) \sim m(L)\cdot L.$$

and Cauchy-Schwartz, assuming $|\mathcal{T}_N| \sim N$, This is because $\sum|a_T|^2 \sim \sum_N \sum_{T\in\mathcal{T}_N} N^{-1} \sim 1$

$$\sum_{\substack{\mathtt{T}\in\mathcal{T}_N\\ n(\mathtt{T})\geq 2}} |n(\mathtt{T})|^2 \gtrsim N^{-1}\Big(\sum_{\substack{\mathtt{T}\in\mathcal{T}_N\\ n(\mathtt{T})\geq 2}} n(\mathtt{T})\Big)^2$$

$$K \lesssim \sum_{i,j} \#_\lambda(\mathtt{P}_i, \mathtt{P}_j) \lesssim \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}} \sum_{1 \le i < j \le M} |t_i - t_j|^{-1} \lesssim m(L) \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}} \log \lambda$$

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