INTERPOLATION OF LEBESGUE AND LORENTZ SPACES

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ABSTRACT. Given an operator T bounded between two pairs of function spaces, the problem of *interpolation* asks: what can we say about the boundedness of T between function spaces *interpolated* between our original spaces? We set the stage by defining the Lorentz function space $L^{p,q}(X)$ and present both the complex and real methods for interpolating bounds on linear and sub-linear operators. These notes are inspired by [Tao06]; for a textbook treatment, see [Gra14].

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1. LEBESGUE SPACES

Let (X, μ) be a measure space and $1 \leq p \leq \infty$, the L^p SPACE, denoted $L^p(X)$, is the space of measurable functions $f : X \to \mathbb{C}$ such that the norm

$$||f||_{L^p} := \left(\int_X |f|^p d\mu \right)^{1/p}, \qquad \text{when } p \neq \infty,$$
$$||f||_{L^\infty} := \operatorname*{ess\,sup}_{x \in X} |f(x)|, \qquad \text{when } p = \infty,$$

is finite. The quantity above forms a monotone norm, that is, for $f, g \in L^p(X)$ and $\alpha \in \mathbb{C}$, it satisfies the following:

(a) Monotonicity, if $|f| \leq |g|$, then
$$||f||_{L^p} \leq ||g||_{L^p}.$$

(b) Positive definiteness,
$$||f||_{L^p} \geq 0$$
with equality only if $f \equiv 0$.

(c) Absolute homogeneity,
$$||\alpha f||_{L^p} = |\alpha| \, ||f||_{L^p}.$$

(d) The triangle inequality,
$$||f + g||_{L^p} \leq ||f||_{L^p} + ||g||_{L^p}.$$

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1.1. **Complex interpolation.** Let T be an operator mapping a subspace of measurable functions $Y \to \mathbb{C}$ to measurable functions $Y \to \mathbb{C}$, we say it is LINEAR if it satisfies

$$T(\alpha f + g) = \alpha T f + T g$$

for all $\alpha \in \mathbb{C}$ and $f, g : X \to \mathbb{C}$ in the domain of T. For $1 \leq p, q \leq \infty$, a linear operator is STRONG-TYPE (p, q) if it is bounded $T : L^p(X) \to L^q(Y)$, i.e. it satisfies the strong-type (p, q) inequality

$$||Tf||_{L^q} \lesssim ||f||_{L^p}, \qquad \text{uniformly in } f \in L^p(X).$$

Proposition 1. *Let $T : L^p(X) \to L^q(Y)$ be a linear operator. Then the following are equivalent:*
   (a) T is strong-type (p, q).
   (b) T is Lipschitz continuous.
   (c) T is continuous at the origin.

Proof. From linearity we see that (a) $\implies$ (b) and (b) $\implies$ (c) is trivial, so to close the chain of implications we show (c) $\implies$ (a). Let $\varepsilon > 0$, then there exists $\delta > 0$ such that

$$||Tf||_{L^q} < \varepsilon, \qquad \text{whenever } ||f||_{L^p} \leq \delta.$$

For any $f \in L^p(X)$, we use homogeneity and apply the inequality above to the normalised function $\delta f / ||f||_{L^p}$ to obtain the strong-type (p, q) inequality,

$$||Tf||_{L^q} = \frac{||f||_{L^p}}{\delta} \left|\left| T\left(\delta \frac{f}{||f||_{L^p}}\right)\right|\right|_{L^q} \leq \frac{\varepsilon}{\delta} ||f||_{L^p},$$

completing the proof. □

Remark. To avoid any philosophical malaise on whether an operator is well-defined on all L^p-functions, it is convenient to consider a linear operator acting on a sufficiently regular sub-class, namely test functions $C_c^\infty(\mathbb{R}^d)$ or Schwartz functions $\mathcal{S}(\mathbb{R}^d)$. Upon proving a strong-type (p, q) estimate for $p \neq \infty$ for such functions, we can extend the result by density and Lipschitz continuity to all L^p-functions.

Let $1 \leq p_0, p_1 \leq \infty$ and $0 \leq \theta \leq 1$, define $1 \leq p_\theta \leq \infty$ by

$$\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$

The space $L^{p_\theta}(X)$ is known as an *interpolation space* between $L^{p_0}(X)$ and $L^{p_1}(X)$. More precisely, $L^{p_\theta}(X)$ "lives between" $L^{p_0}(X)$ and $L^{p_1}(X)$ in the sense that

$$L^{p_0}(X) \cap L^{p_1}(X) \subseteq L^{p_\theta}(X) \subseteq L^{p_0}(X) + L^{p_1}(X).$$

The first inclusion follows from Holder's inequality, writing $f = f^{1-\theta} f^\theta$,

$$||f||_{L^{p_\theta}} \leq ||f||_{L^{p_0}}^{1-\theta} ||f||_{L^{p_1}}^\theta.$$

The second inclusion follows from decomposing $f = f\mathbb{1}_{|f|>1} + f\mathbb{1}_{|f|\leq 1}$, monotonicity of $|x| \mapsto |x|^p$, and noting that $p_0 < p_\theta < p_1$,

$$||f\mathbb{1}_{|f|>1}||_{L^{p_0}} \leq ||f||_{L^{p_\theta}}, \qquad ||f\mathbb{1}_{|f|\leq 1}||_{L^{p_1}} \leq ||f||_{L^{p_\theta}}.$$

Given $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and a linear operator $T : (L^{p_0} + L^{p_1})(X) \to (L^{q_0} + L^{q_1})(Y)$ of strong-type (p_0, q_0) and (p_1, q_1), we see that T forms a map between the interpolation spaces. *Interpolation* is the problem of establishing the strong-type (p_θ, q_θ) inequality.

Theorem 2 (Riesz-Thorin interpolation). *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and $0 < \theta < 1$, and suppose (X, μ) and (Y, ν) are measure spaces, the latter σ-finite when $q_0 = q_1 = \infty$. If $T : (L^{p_0} + L^{p_1})(X) \to (L^{q_0} + L^{q_1})(Y)$ is a linear operator of strong-type (p_0, q_0) and (p_1, q_1), i.e.*

$$||Tf||_{L^{q_0}} \leq M_0 ||f||_{L^{p_0}}, \qquad \text{uniformly in } f \in L^{p_0}(X)$$

$$||Tf||_{L^{q_1}(Y)} \leq M_1 ||f||_{L^{p_1}}, \qquad \text{uniformly in } f \in L^{p_1}(X)$$

for some $0 < M_0, M_1 < \infty$. Then T satisfies the strong-type (p_θ, q_θ) inequality

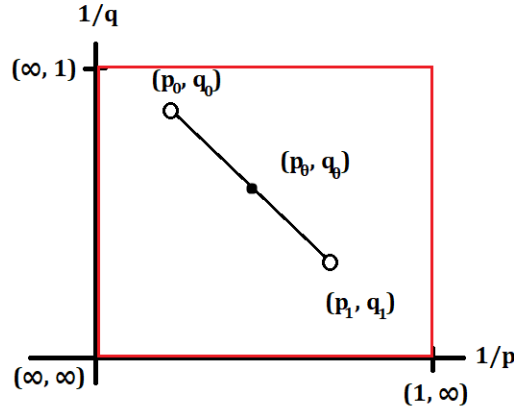$$||Tf||_{L^{q_\theta}} \leq M_0^{1-\theta} M_1^\theta ||f||_{L^{p_\theta}}.$$

FIGURE 1. The *interpolation diagram*; given bounds for exponents (p_0, q_0) and (p_1, q_1), interpolation furnishes bounds for exponents (p_θ, q_θ) on the intermediate line.

Proof. If $p_0 = p_1 = p_\theta$, the result follows from the log convexity of the L^p-norm. That is, by Holder's inequality,

$$||g||_{L^{q_\theta}}^{q_\theta} = \int_Y |g|^{(1-\theta)q_\theta} |g|^{\theta q_\theta} \, d\nu \le |||g|^{(1-\theta)q_\theta}||_{L^{\frac{q_0}{(1-\theta)q_\theta}}} |||g|^{\theta q_\theta}||_{L^{\frac{q_1}{\theta q_\theta}}} = ||g||_{L^{q_0}}^{(1-\theta)q_\theta} ||g||_{L^{q_1}}^{\theta q_\theta}.$$

Then taking $Tf = g$ and applying the strong type (p_0, q_0) and (p_1, q_1) inequalities,

$$||Tf||_{L^{q_\theta}(Y)} \le ||Tf||_{L^{q_0}(Y)}^{1-\theta} ||Tf||_{L^{q_1}(Y)}^{\theta} \le M_0^{1-\theta} M_1^{\theta} ||f||_{L^{p_\theta}(X)}.$$

Thus we can assume $p_0 < p_1$; in particular, we avoid the endpoint cases and assume $1 < p_\theta < \infty$. We claim that it suffices to prove the result for simple functions with finite support, i.e. functions of the form

$$f = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$$

for coefficients $a_k \in \mathbb{C}$ and disjoint finite measure sets $A_k \subseteq X$. We can find simple functions with finite measure support $\{f_n\}_n$ such that $f_n \to f$ pointwise and $|f_n| \le |f|$. Assume $f \in L^{p_0}(X) \cap L^{p_1}(X)$, then by log convexity of the L^p-norm we also have $f \in L^{p_\theta}(X)$. By the L^p-dominated convergence theorem, $f_n \to f$ in L^{p_θ} and L^{p_0}. Observe

$$
\begin{aligned}
||Tf||_{L^{q_\theta}(Y)} &\le ||T(f - f_n)||_{L^{q_\theta}(Y)} + ||Tf_n||_{L^{q_\theta}(y)} \\
&\le ||T(f - f_n)||_{L^{q_0}(Y)}^{1-\theta} ||T(f - f_n)||_{L^{q_1}(Y)}^{\theta} + M_0^{1-\theta} M_1^{\theta} ||f_n||_{L^{p_\theta}(X)} \\
&\le M_0^{1-\theta} M_1^{\theta} \left(||f - f_n||_{L^{p_0}(X)}^{1-\theta} ||f - f_n||_{L^{p_1}(X)}^{\theta} + ||f - f_n||_{L^{p_\theta}(X)} + ||f||_{L^{p_\theta}(X)} \right).
\end{aligned}
$$

The first line follows from the triangle inequality and linearity of T, the second follows from log convexity of the L^q-norm and Riesz-Thorin for simple functions, the third follows from the strong type (p_0, q_0) and (p_1, q_1) inequalities and the L^{p_θ}-triangle inequality. Since $|f_n| \le |f|$, we know $||f - f_n||_{L^{p_1}(X)} \le 2||f||_{L^{p_1}(X)}$. This allows us to pass the limit $n \to \infty$ on the right to obtain

$$||Tf||_{L^{q_\theta}(Y)} \le M_0^{1-\theta} M_1^{\theta} ||f||_{L^{p_\theta}(X)},$$

proving the claim.

To prove Riesz-Thorin for simple functions, we argue by duality. For $1 \le q \le \infty$, then

$$||Tf||_{L^{q_\theta}(Y)} = \sup_{||g||_{L^{q_\theta'}(Y)} = 1} \left| \int_Y (Tf) g \, d\nu \right|.$$

Suppose one of $q_0, q_1 \neq 1$, then $1 \leq q'_\theta < \infty$. By density we can take the supremum over simple functions. Fix simple functions with finite measure support f and g, and set

$$F(s) := \int_Y T\left(|f|^{(1-s)\frac{p_\theta}{p_0}+s\frac{p_\theta}{p_1}} \operatorname{sgn}(f)\right) |g|^{(1-s)\frac{q'_\theta}{q'_0}+s\frac{q'_\theta}{q'_1}} \operatorname{sgn}(g)\, dv,$$

with the convention $q'_\theta/q'_0 = q'_\theta/q'_1 = 1$ in the endpoint case $q_0 = q_1 = q_\theta = \infty$. Since we assumed f and g were simple functions and T is linear, the integrand is entire for each fixed $y \in Y$ and satisfies $|F(s)| \lesssim e^{c|z|}$. From the finite measure support assumption, we can apply Fubini-Tonelli and Morera's theorem to conclude F is entire. Hence, the conditions for Hadamard's three lines theorem have been satisfied.

If $q_0 = q_1 = q_\theta = 1$, density of simple functions fails, so we consider instead any $g \in L^\infty(Y)$. Nevertheless, the conditions of Hadamard's theorem continue to be satisfied, since the exponent of $|g|$ reduces to a constant and

$$F(s) = \int_Y T\left(|f|^{(1-s)\frac{p_\theta}{p_0}+s\frac{p_\theta}{p_1}} \operatorname{sgn}(f)\right) g\, dv.$$

By choice of exponents and noting $f = |f| \operatorname{sgn}(f)$,

$$F(\theta) = \int_Y (Tf) g\, dv.$$

Applying Holder's inequality, the strong type (p_0, q_0) inequality, and recalling $|a^{it}| = 1$ for any $a, t \in \mathbb{R}$, we have

$$|F(it)| \leq \left|\left|T\left(|f|^{(1-it)\frac{p_\theta}{p_0}+it\frac{p_\theta}{p_1}} \operatorname{sgn}(f)\right)\right|\right|_{L^{q_0}(Y)} \left|\left||g|^{(1-it)\frac{q'_\theta}{q'_0}+it\frac{q'_\theta}{q'_1}} \operatorname{sgn}(g)\right|\right|_{L^{q'_0}(Y)}$$

$$\leq M_0 \left|\left||f|^{\frac{p_\theta}{p_0}}\right|\right|_{L^{p_0}(Y)} \left|\left||g|^{\frac{q'_\theta}{q'_0}}\right|\right|_{L^{q'_0}(Y)} \leq M_0 ||f||_{L^{p_\theta}(X)}^{\frac{p_\theta}{p_0}} ||g||_{L^{q'_\theta}(Y)}^{\frac{q'_\theta}{q'_0}},$$

and similarly, applying instead the strong type (p_1, q_1) inequality,

$$|F(1+it)| \leq \left|\left|T\left(|f|^{-it\frac{p_\theta}{p_0}+(1+it)\frac{p_\theta}{p_1}} \operatorname{sgn}(f)\right)\right|\right|_{L^{q_1}(Y)} \left|\left||g|^{-it\frac{q'_\theta}{q'_0}+(1+it)\frac{q'_\theta}{q'_1}} \operatorname{sgn}(g)\right|\right|_{L^{q'_1}(Y)}$$

$$\leq M_0 \left|\left||f|^{\frac{p_\theta}{p_1}}\right|\right|_{L^{p_1}(Y)} \left|\left||g|^{\frac{q'_\theta}{q'_1}}\right|\right|_{L^{q'_1}(Y)} \leq M_0 ||f||_{L^{p_\theta}(X)}^{\frac{p_\theta}{p_1}} ||g||_{L^{q'_\theta}(Y)}^{\frac{q'_\theta}{q'_1}}.$$

Hadamard's three lines theorem furnishes the inequality $|F(\theta)| \leq \sup_t |F(it)|^{1-\theta} \sup_t |F(1+it)|^\theta$. Collecting this inequality with the previous two inequalities and comparing exponents, we obtain

$$\left|\int_Y (Tf) g\, dv\right| \leq M_0^{1-\theta} M_1^\theta ||f||_{L^{p_\theta}(X)} ||g||_{L^{q'_\theta}(Y)}.$$

By duality, this finishes the proof. □

2. Lorentz spaces

Given a measure space (X, μ), the two basic quantitative notions of "size" of a function $f : X \to \mathbb{C}$ are the "height" in the range and "width" in the domain. The L^p-norm primarily quantified control over the former; to quantify control over both notions, we introduce for $1 \leq p, q \leq \infty$ the Lorentz space $L^{p,q}(X)$, the space of measurable functions $f : X \to \mathbb{C}$ for which

$$||f||_{L^{p,q}}^* := p^{1/q} \left|\left|\lambda \mu(\{x \in X : |f(x)| > \lambda\})^{1/p}\right|\right|_{L^q((0,\infty),\frac{d\lambda}{\lambda})}$$

$$= p^{1/q} \left(\int_0^\infty \lambda^q \mu(\{x \in X : |f(x)| \geq \lambda\})^{q/p} \frac{d\lambda}{\lambda}\right)^{1/q}$$

is finite. The quantity above forms a monotone quasi-norm, that is, for $f, g \in L^{p,q}(X)$ and $\alpha \in \mathbb{C}$, it satisfies the following properties:

(a) Monotonicity, if $|f| \leq |g|$ then

$$||f||^*_{L^{p,q}} \leq ||g||^*_{L^{p,q}}.$$

(b) Positive definiteness,

$$||f||^*_{L^{p,q}} \geq 0,$$

with equality only if $f \equiv 0$; this is clear.

(c) Absolute homogeneity,

$$||\alpha f||^*_{L^{p,q}} = |\alpha| \, ||f||^*_{L^{p,\infty}}.$$

(d) A quasi-triangle inequality,

$$||f + g||^*_{L^{p,q}} \leq 2||f||^*_{L^{p,\infty}} + 2||g||^*_{L^{p,q}}.$$

By convention we set $L^{\infty,\infty}(X) := L^\infty(X)$, and in general when $q = \infty$, the Lorentz space $L^{p,\infty}(X)$ is known as the WEAK L^p SPACE, endowed with the norm

$$||f||^*_{L^{p,\infty}} := \sup_{\lambda > 0} \lambda \mu(\{x \in X : |f(x)| > \lambda\})^{1/p}.$$

Remark. The Lorentz space coincides with the usual Lebesgue space when $p = q$, that is, $L^{p,p}(X) = L^p(X)$. This follows from the *layered cake representation*, writing using the fundamental theorem of calculus

$$|f(x)|^p = \int_0^{|f(x)|} p\lambda^{p-1}\, d\lambda = p \int_0^\infty \lambda^{p-1} \mathbb{1}_{[0,|f(x)|]}(\lambda)\, d\lambda = p \int_0^\infty \lambda^p \mathbb{1}_{|f|>\lambda}(x) \frac{d\lambda}{\lambda}.$$

The representation takes its name from writing $|f(x)|^p$ as the sum of contributions from the "layers" λ below $|f(x)|$. Integrating and applying Fubini's theorem,

$$||f||^p_{L^p} = \int_X |f(x)|^p\, d\mu = p \int_X \int_0^\infty \lambda^p \mathbb{1}_{|f|>\lambda}(x) \frac{d\lambda}{\lambda} = p \int_0^\infty \lambda^p \mu(\{x \in X : |f(x)| > \lambda\}) \frac{d\lambda}{\lambda} = (||f||^*_{L^{p,p}})^p.$$

2.1. **Weak L^p space.** As a primer for studying the general $L^{p,q}$-spaces, we first consider the weak L^p-spaces. The prototypical example of a function in weak L^p-space however not in L^p-space is

$$f(x) := |x|^{-d/p}.$$

In fact, one can think of every weak L^p-function as dominated pointwise by a rearrangement of $|x|^{-d/p}$. Several classical inequalities can be reformulated in terms of weak L^p-space, such as the Hardy-Littlewood maximal inequality and, in probability,

Lemma 3 (Chebyshev's inequality). *Let $f \in L^p(X)$ and $\lambda > 0$, then*

$$\mu(\{x \in X : |f(x)| > \lambda\}) \leq \frac{1}{\lambda^p} \int_{|f|>\lambda} |f|^p\, d\mu.$$

Moreover, $L^p(X) \hookrightarrow L^{p,\infty}(X)$ via the inequality

$$||f||^*_{L^{p,\infty}} \leq ||f||_{L^p}.$$

Proof. We have

$$\mu(\{x \in X : |f(x)| > \lambda\}) = \frac{1}{\lambda^p} \int_{|f|>\lambda} \lambda^p d\mu \leq \frac{1}{\lambda^p} \int_{|f|>\lambda} |f|^p\, d\mu.$$

This proves Chebyshev's inequality. Rearranging and taking the p-th root gives

$$\lambda \mu(\{x \in X : |f(x)| > \lambda\})^{1/p} \leq \left(\int_X |f|^p\, d\mu \right)^{1/p}.$$

Taking the supremum over $\lambda > 0$ on the left furnishes the continuous embedding $L^p(X) \hookrightarrow L^{p,\infty}(X)$. $\square$

Remark. The case $p = 1$ of Chebyshev's inequality is also referred to as Markov's inequality.

Theorem 4 (Weak L^p-duality). *Let $1 < p < \infty$ and $f \in L^{p,\infty}(X)$, then*

$$||f||_{L^{p,\infty}} := \sup_{0 < \mu(A) < \infty} \mu(A)^{-1/p'} \left| \int_X f \mathbb{1}_A \, d\mu \right|$$

defines a norm satisfying

$$||f||^*_{L^{p,\infty}} \leq ||f||_{L^{p,\infty}} \leq \frac{1}{p'} ||f||^*_{L^{p,\infty}}.$$

Proof. It is clear from definition that $|| \cdot ||_{L^{p,\infty}}$ defines a norm, so it remains to show that it is comparable to the quasi-norm $|| \cdot ||^*_{L^{p,\infty}}$. Decomposing into real, imaginary, positive and negative components, we can assume without loss of generality $f \geq 0$. Since $f \in L^{p,\infty}(X)$, we know that the super-level sets $|f| > \lambda$ have finite measure, so rearranging Markov's inequality we obtain

$$\lambda \mu(f > \lambda)^{1/p} \leq \mu(f > \lambda)^{-1/p'} \int_{f > \lambda} f \, d\mu \leq ||f||_{L^{p,\infty}}.$$

Taking the supremum with respect to $\lambda > 0$ on the left gives the desired lower bound on $||f||_{L^{p,\infty}}$. For the upper bound, we apply the layered cake representation and the definition of the weak L^p-quasinorm to write

$$\int_A f \, d\mu = \int_0^\infty \mu(x \in A : f > \lambda) \, d\lambda \leq \int_0^\infty \min\{\mu(f > \lambda), \mu(A)\}$$

$$\leq \int_0^\infty \min\{\lambda^{-p} ||f||^*_{L^{p,\infty}}, \mu(A)\} d\lambda.$$

We split the integral on the right, remarking that $\lambda^{-p} ||f||^*_{L^{p,\infty}} \leq \mu(A)$ if and only if $||f||^*_{L^{p,\infty}} \mu(A)^{-1/p} \leq \lambda$, so by the fundamental theorem of calculus

$$\int_0^\infty \min\{\lambda^{-p} ||f||^*_{L^{p,\infty}}, \mu(A)\} d\lambda = \int_0^{||f||^*_{L^{p,\infty}} \mu(A)^{-1/p}} \mu(A) \, d\lambda + \int_{||f||^*_{L^{p,\infty}} \mu(A)^{-1/p}}^\infty \lambda^{-p} ||f||^*_{L^{p,\infty}} \, d\lambda$$

$$= ||f||^*_{L^{p,\infty}} \mu(A)^{1/p'} + \frac{1}{p-1} ||f||^*_{L^{p,\infty}} \mu(A)^{1/p'} = \frac{1}{p'} ||f||^*_{L^{p,\infty}} \mu(A)^{1/p'}.$$

Rearranging and taking the supremum over $0 < \mu(A) < \infty$ furnishes the desired upper bound. $\square$

Remark. The argument fails in the case $p = 1$; in fact, the weak L^1-space cannot be normed provided that the measure μ is non-zero and non-atomic. Consider for example the Lebesgue measure on $\mathbb{R}$, and assume towards a contradiction that there exists a norm $||| - |||$ such that $||| - ||| \sim || - ||^*_{L^{p,\infty}}$. Define $f_n \in L^{1,\infty}(\mathbb{R})$ by

$$f_n(x) = \frac{1}{|x - n|}. \tag{*}$$

We compute the quasinorms,

$$||f_n||^*_{L^{1,\infty}} = \sup_{\lambda > 0} \lambda \mu(|x| < 1/\lambda) = 2.$$

On the other hand, recall that the harmonic series grows logarithmically, so for $x \in [0, N]$ and $N \gg 1$ we have the pointwise lower bound

$$\log N \lesssim \sum_{n=1}^N f_n.$$

Collecting our results and applying the triangle inequality, we obtain

$$N \log N \lesssim \left|\left| \sum_{n=1}^N f_n \right|\right|^*_{L^{1,\infty}} \sim \left|\left|\left| \sum_{n=1}^N f_n \right|\right|\right| \leq \sum_{n=1}^N |||f_n||| \lesssim \sum_{n=1}^N ||f_n||^*_{L^{1,\infty}} \sim N,$$

a contradiction.

2.2. **Characterisation of $L^{p,q}$.** The $L^{p,q}$-quasinorm can at first glance appears rather mysterious and cumbersome to work with directly, so it will be both enlightening and convenient to introduce characterisations of $L^{p,q}$-functions in terms of simple functions. Much like the L^p-norm, the $L^{p,q}$-quasinorm of a step function is

$$||H\mathbb{1}_E||^*_{L^{p,q}} \sim H\mu(E)^{1/p}.$$

The difference lies in how these quantities are summed in the case of simple functions. For $H_n > 0$ distinct heights and $\{E_n\}_n$ a disjoint family of measurable sets, the L^p-norm of the corresponding simple function is the ℓ^p_n-sum of step function L^p-norms

$$\left|\left| \sum_{n \in \mathbb{Z}} H_n \mathbb{1}_{E_n} \right|\right|_{L^p} = \left|\left| ||H_n \mathbb{1}_{E_n}||_{L^p} \right|\right|_{\ell^p_n} = ||H_n \mu(E_n)^{1/p}||_{\ell^p_n}.$$

The $L^{p,q}$-norm is instead comparable to the ℓ^q-sum of step function L^p-norms. For a generic function $f \in L^{p,q}(X)$, we can decompose and approximate by simple functions in two fashions: vertically into step functions of height $H_n \sim 2^n$ and some width $\mu(E_n)$, or horizontally into step functions of width $\mu(E_n) \sim 2^n$ and some heights H_n.

Proposition 5 (Vertically dyadic layer cake decomposition). *Let $1 \le p < \infty$ and $1 \le q \le \infty$, and suppose that $f \in L^{p,q}(X)$. Decomposing*

$$f = \sum_{m \in \mathbb{Z}} f \mathbb{1}_{2^m \le |f| < 2^{m+1}} =: \sum_{m \in \mathbb{Z}} f_m,$$

then

$$||f||^*_{L^{p,q}} \sim_{p,q} \left|\left| ||f_m||_{L^p} \right|\right|_{\ell^q_m}.$$

In particular, $L^{p,q_1}(X) \hookrightarrow L^{p,q_2}(X)$ whenever $q_1 \le q_2$.

Proof. By construction,

$$|f| \sim \sum_{m \in \mathbb{Z}} 2^m \mathbb{1}_{2^m \le |f| < 2^{m+1}}$$

pointwise, so by monotonicity we can consider without loss of generality functions of the form $f = \sum_m 2^m \mathbb{1}_{E_m}$ where $\{E_m\}_m$ is a disjoint family of measurable sets. For such functions, the result takes the form

$$||f||^*_{L^{p,q}} \sim_{p,q} \left|\left| 2^m \mu(E_m)^{1/p} \right|\right|_{\ell^q_m}.$$

We compute

$$(||f||^*_{L^{p,q}})^q = p \sum_{m \in \mathbb{Z}} \int_{2^{m-1}}^{2^m} \lambda^q \mu(|f| > \lambda)^{q/p} \frac{d\lambda}{\lambda} = p \sum_{m \in \mathbb{Z}} \int_{2^{m-1}}^{2^m} \lambda^q \left(\sum_{n \ge m} \mu(E_n) \right)^{q/p} \frac{d\lambda}{\lambda}$$

$$\sim_{p,q} \sum_{m \in \mathbb{Z}} 2^{mq} \left(\sum_{n \ge m} \mu(E_n) \right)^{q/p} \sim \left|\left| 2^m \left(\sum_{n \ge m} \mu(E_n) \right)^{1/p} \right|\right|_{\ell^q_m}^q.$$

Clearly

$$\left|\left| 2^m \mu(E_m)^{1/p} \right|\right|_{\ell^q_m} \le \left|\left| 2^m \left(\sum_{n \ge m} \mu(E_n) \right)^{1/p} \right|\right|_{\ell^q_m}.$$

For the converse inequality, we appeal to a change of indices $n = m + k$ and the triangle inequality,

$$\left|\left| 2^m \left(\sum_{n \ge m} \mu(E_n) \right)^{1/p} \right|\right|_{\ell^q_m} \lesssim \left|\left| 2^n \sum_{n \ge m} \mu(E_n)^{1/p} \right|\right|_{\ell^q_m} \sim \left|\left| 2^m \sum_{k \ge 0} \mu(E_{m+k})^{1/p} \right|\right|_{\ell^q_m}$$

$$\le \sum_{k \ge 0} 2^{-k} \left|\left| 2^{m+k} \mu(E_{m+k})^{1/p} \right|\right|_{\ell^q_m} \sim ||2^m \mu(E_m)^{1/p}||_{\ell^q_m}.$$

This completes the proof. $\square$

Corollary 6 (Density of simple functions). *For $1 \le p, q < \infty$ the space of simple function is dense in $L^{p,q}(X)$.*

Proof. Fix $f \in L^{p,q}(X)$ and $\varepsilon > 0$. Performing a vertically dyadic layer cake decomposition, there exists $n_\varepsilon \in \mathbb{N}$ sufficiently large such that

$$\left|\left|||f_m||_{L^p}\right|\right|_{\ell^q_{|m|\geq n_\varepsilon}} \ll \varepsilon.$$

Fix $N \gg 1$ to be chosen later, define

$$E_{m,k} := \{x \in X : \frac{k}{2^N} \leq |f_m(x)| \leq \frac{k+1}{2^N}\}$$

for $2^{m+N} \leq k \leq 2^{m+N+1} - 1$. Define the simple functions

$$\phi_m := \sum_{k=2^{m+N}}^{2^{m+N+1}} \frac{k}{2^N} \mathbb{1}_{E_{m,k}}, \qquad \phi := \sum_{|m|\leq n_\varepsilon} \phi_m.$$

We claim that $||f - \phi||^*_{L^{p,q}} \lesssim \varepsilon$. By the triangle inequality and choice of n_ε we have

$$||f - \phi||^*_{L^{p,q}} \leq 2\left(\left|\left|\sum_{|m|\leq n_\varepsilon} f_m - \phi_m\right|\right|^*_{L^{p,q}} + \left|\left|\sum_{|m|>n_\varepsilon} f_m\right|\right|^*_{L^{p,q}}\right) \leq 2\left(\left|\left|\sum_{|m|\leq n_\varepsilon} f_m - \phi_m\right|\right|^*_{L^{p,q}} + \varepsilon\right).$$

It remains to control the first term on the right. By the triangle inequality and Holder's inequality,

$$\left|\left|\sum_{|m|\leq n_\varepsilon} f_m - \phi_m\right|\right|^*_{L^{p,q}} \leq 2^{2n_\varepsilon+1} \sum_{|m|\leq n_\varepsilon} ||f_m - \phi_m||^*_{L^{p,q}} \lesssim 2^{2n} \sum_{|m|\leq n_\varepsilon} \left|\left|2^k \mu(2^k \leq |f_m - \phi_m| < 2^{k+1})^{1/p}\right|\right|_{\ell^q_k}$$

$$\lesssim 2^{2n_\varepsilon} \sum_{|m|\leq n_\varepsilon} \left|\left|2^k \mu(2^m \leq |f| < 2^{m+1})^{1/p}\right|\right|_{\ell^q_k} \lesssim 2^{2n} \sum_{|m|\leq n_\varepsilon} 2^{-N} \mu(2^m \leq |f| < 2^{m+1})^{1/p}$$

$$\lesssim 2^{2n_\varepsilon} \sum_{|m|\leq n_\varepsilon} 2^{-N} \frac{||f_m||_{L^p}}{2^m} \lesssim 2^{2n-N} \left|\left|||f_m||_{L^p}\right|\right|_{\ell^q_m} \left|\left|2^{-m}\right|\right|_{\ell^q_{|m|\leq n_\varepsilon}} \lesssim \frac{2^{3n}}{2^N} ||f||^*_{L^{p,q}}.$$

Choosing $N \gg 1$ completes the proof. □

Proposition 7 (Horizontally dyadic layer cake decomposition). *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$, and suppose that $f \in L^{p,q}(X)$. Decomposing,*

$$f = \sum_{m\in\mathbb{Z}} f\mathbb{1}_{H_{m+1}\leq|f|<H_m} =: \sum_{m\in\mathbb{Z}} f_m,$$

where $H_m := \inf\{\lambda > 0 : \mu(|f| > \lambda) \leq 2^{m-1}\}$, then

$$||f||^*_{L^{p,q}} \sim ||H_m 2^{m/p}||_{\ell^q_m}.$$

Proof. It is clear that $H_m \to ||f||_{L^\infty}$ as $m \to -\infty$ and, since $f \in L^{p,q}(X)$, we also have $H_m \to 0$ as $m \to \infty$. Furthermore we have

$$2^{m-1} < \mu(|f| > \lambda) \leq 2^m.$$

whenever $H_{m+1} \leq \lambda < H_m$. For $q \neq \infty$, we compute

$$(||f||^*_{L^{p,q}})^q = p \int_0^\infty \lambda^q \mu(|f| > \lambda)^{q/p} \frac{d\lambda}{\lambda} = p \sum_{m\in\mathbb{Z}} \int_{H_{m+1}}^{H_m} \lambda^q \mu(|f| > \lambda)^{q/p} \frac{d\lambda}{\lambda}$$

$$\sim \sum_{m\in\mathbb{Z}} \int_{H_{m+1}}^{H_m} \lambda^q 2^{mq/p} \frac{d\lambda}{\lambda} \sim \sum_{m\in\mathbb{Z}} 2^{mq/p} \left(H_m^q - H_{m+1}^q\right) \sim \sum_{m\in\mathbb{Z}} 2^{mq/p} H_m^q = ||2^{\frac{m}{p}} H_m||^q_{\ell^q_m}.$$

For the case $q = \infty$ we have

$$||f||^*_{L^{p,\infty}} = \sup_{\lambda>0} \lambda|\{x : |f(x)| > \lambda\}|^{\frac{1}{p}} = \sup_{m\in\mathbb{Z}} \sup_{H_{m+1}\leq\lambda<H_m} \lambda|\{x : |f(x)| > \lambda\}|^{\frac{1}{p}}$$

$$\sim \sup_{m\in\mathbb{Z}} H_m 2^{\frac{m}{p}} = ||H_m 2^{\frac{m}{p}}||_{\ell^\infty_m},$$

completing the proof. □

2.3. **Duality.** Analogous to the L^p-space setting, the continous duals of non-endpoint $L^{p,q}$-spaces are represented by the dual exponent $L^{p',q'}$-spaces. The dual characterisation thereby furnishes a norm on $L^{p,q}(X)$ and moreover a Banach space structure. To this end, we will need the following preliminary lemma, which states that the triangle inequality, up to a uniform constant, continues to hold for quasi-norms provided the component quasi-norms are on different dyadic scales.

Lemma 8. *Let $||-||$ denote a quasinorm on a topological vector space X. Let $f_1, \ldots, f_N \in X$ satisfy*

$$||f_n|| \leq 2^{-\varepsilon n}$$

for some $\varepsilon > 0$. Then

$$\left\|\sum_{n=1}^{N} f_n\right\| \lesssim_{\varepsilon} 1,$$

where the implicit constant is independent of N.

Proof. There exists a constant $C > 1$ such that for functions f and g the quasinorm satisfies the following quasi-triangle inequality

$$||f + g|| \leq C||f|| + C||g||.$$

Let $\eta > 0$ such that $C = 2^{\eta}$. We consider first the case where $\varepsilon > \eta$, then

$$\left\|\sum_{n=1}^{N} f_n\right\| \leq C||f_1|| + C\left\|\sum_{n=2}^{N} f_n\right\| \leq \cdots \leq C||f_1|| + \cdots + C^N||f_N||$$

$$\leq \sum_{n=1}^{N} C^n||f_n|| \leq \sum_{n=1}^{N} 2^{(\eta-\varepsilon)n} \leq \frac{1}{1 - 2^{\eta-\varepsilon}} \lesssim_{\varepsilon} 1.$$

In the first line we apply the quasi-triangle inequality N-times and the second line we note that we obtain a convergent geometric series. If $\varepsilon \leq \eta$, choose $M_{\varepsilon} \in \mathbb{N}$ such that $\varepsilon M_{\varepsilon} > \eta$. For notational convenience, let $d \in \mathbb{N}$ such that $dM_{\varepsilon} > N$ and set $f_{N+1} = \cdots = f_{dM_{\varepsilon}} = 0$. Define

$$g_k = \sum_{n=kM_{\varepsilon}+1}^{(k+1)M_{\varepsilon}} f_n$$

for $k = 0, \ldots, d - 1$. Trivially $\sum_n f_n = \sum_k g_k$, so to reduce to the previous case it remains to prove analogous bounds $||g_k|| \lesssim_{\varepsilon} 2^{-\varepsilon M_{\varepsilon} k}$ where the implicit constant is uniform in k. We apply the quasi-triangle inequality M_{ε}-times to obtain the desired inequality for each k,

$$\left\|\sum_{n=kM_{\varepsilon}+1}^{(k+1)M_{\varepsilon}} f_n\right\| \leq C||f_{kM_{\varepsilon}+1}|| + \cdots + C^{M_{\varepsilon}}||f_{(k+1)M_{\varepsilon}}||$$

$$\leq C^{M_{\varepsilon}}\left(2^{-\varepsilon(kM_{\varepsilon}+1)} + \cdots + 2^{-\varepsilon(k+1)M_{\varepsilon}}\right) \leq C^{M_{\varepsilon}} M_{\varepsilon} 2^{-\varepsilon k M_{\varepsilon}}.$$

Arguing as we did in the first case, we conclude

$$\left\|\sum_{n=1}^{N} f_n\right\| = \left\|\sum_{k=0}^{d-1} g_k\right\| \leq \frac{C^{M_{\varepsilon}} M_{\varepsilon}}{1 - 2^{\eta-\varepsilon M_{\varepsilon}}} \lesssim_{\varepsilon} 1.$$

This completes the proof. □

Theorem 9 ($L^{p,q}$-Holder's inequality)**.** *Let $1 \leq p, p_1, p_2 < \infty$ and $1 \leq q, q_1, q_2 \leq \infty$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$||fg||_{L^{p,q}}^* \lesssim ||f||_{L^{p_1,q_1}}^* ||g||_{L^{p_2,q_2}}^*.$$

Proof. By absolute homogeneity we can renormalize, replacing f and g with $f / ||f||^*_{L^{p_1,q_1}}$ and $g / ||g||^*_{L^{p_2,q_2}}$. It suffices then to prove the case where $||f||^*_{L^{p_1,q_1}} = ||g||^*_{L^{p_2,q_2}} = 1$. Set

$$H_{n,f} = \inf\{\lambda > 0 : |\{x : |f(x)| > \lambda\}| \leq 2^{n-1}\},$$
$$H_{n,f} = \inf\{\lambda > 0 : |\{x : |g(x)| > \lambda\}| \leq 2^{n-1}\},$$
$$E_{n,f} = \{x : H_{n+1,g} \leq |f(x)| < H_{n,f}\},$$
$$E_{n,g} = \{x : H_{n+1,g} \leq |g(x)| < H_{n,g}\},$$

where $\{E_{n,f}\}_n$ and $\{E_{n,g}\}_n$ are families of disjoint measurable sets. The horizontal layered cake decomposition furnishes the estimates

$$||H_{n,f} 2^{\frac{n}{p_1}}||_{\ell^{q_1}(\mathbb{Z})} \sim ||H_{n,g} 2^{\frac{n}{p_2}}||_{\ell^{q_2}(\mathbb{Z})} \sim 1.$$

By construction,

$$|fg| \leq \sum_k \sum_n H_{n,f} H_{n+k,g} \mathbb{1}_{E_{n,f} \cap E_{n+k,g}}.$$

From monotonicity of the quasi-norm and the quasi-triangle inequality, it remains to show

$$\left|\left|\sum_{k\geq 1} \sum_n H_{n,f} H_{n+k,g} \mathbb{1}_{E_{n,f}\cap E_{n+k,g}}\right|\right|^*_{L^{p,q}}, \left|\left|\sum_{k\leq 0} \sum_n H_{n,f} H_{n+k,g} \mathbb{1}_{E_{n,f}\cap E_{n+k,g}}\right|\right|^*_{L^{p,q}} \lesssim 1.$$

We prove the former; the latter is symmetric. In the proof of the horizontal dyadic layered cake decomposition, we showed that $\mu(E_{n,f}) \sim \mu(E_{n,g}) \sim 2^n$. In particular, $\mu(E_{n,f} \cap E_{n+k,g}) \lesssim 2^n$ uniformly in n and k, so by Holder's inequality in $\ell^q_n(\mathbb{Z})$ we obtain

$$\left|\left|\sum_n H_{n,f} H_{n+k,g} \mathbb{1}_{E_{n,f}\cap E_{n+k,g}}\right|\right|^*_{L^{p,q}} \lesssim ||H_{n,f} H_{n+k,g} 2^{\frac{n}{p}}||_{\ell^q_n(\mathbb{Z})} = 2^{-\frac{k}{p_2}} ||H_{n,f} 2^{\frac{n}{p_1}} H_{n+k,g} 2^{\frac{n+k}{p_2}}||_{\ell^q_n(\mathbb{Z})}$$

$$\lesssim 2^{-\frac{k}{p_2}} ||H_{n,f} 2^{\frac{n}{p_1}}||_{\ell^{q_1}_n(\mathbb{Z})} ||H_{n,f} 2^{\frac{n}{p_2}}||_{\ell^{q_2}_n(\mathbb{Z})} \lesssim 2^{-\frac{k}{p_2}}.$$

Note the second line we make the change of variables $n + k \mapsto n$. Summing in $k \geq 1$ and applying the previous lemma, we conclude the desired inequality. A remark on the symmetric proof for the sum over terms $k \leq 0$, we simply replace $n + k$ with $n - k$ and $2^{-k/p_2}$ with $2^{k/p_2}$. $\qquad\square$

For brevity we denote the dyadic numbers $2^{\mathbb{Z}}$, that is, the numbers of the form 2^n for $n \in \mathbb{Z}$. In a similar spirit to Lemma 8, we can show that the q-th power of a convergent sum of dyadics is comparable to the sum of the q-th powers and the maximum of the q-th powers for $1 \leq q < \infty$. This result will be convenient algebraically,

Lemma 10. *Let $1 \leq q < \infty$ and suppose $\mathcal{F} \subseteq 2^{\mathbb{Z}}$ is finite. Then*

$$\sum_{N\in\mathcal{F}} N^q \sim \left(\sum_{N\in\mathcal{F}} N\right)^q \sim \max_{N\in\mathcal{F}} N^q.$$

Proof. We have

$$\sum_{N\in\mathcal{F}} N^q \leq \left(\sum_{N\in\mathcal{F}} N\right)^q \leq \left(\max_{N\in\mathcal{F}} 2N\right)^q \leq 2^q \sum_{N\in\mathcal{F}} N^q$$

$\qquad\square$

Theorem 11 ($L^{p,q}$-duality). *Let $1 < p < \infty$ and $1 \leq q \leq \infty$, and suppose $f \in L^{p,q}(X)$, then*

$$||f||^*_{L^{p,q}} \sim_{p,q} \sup_{||g||^*_{L^{p',q'}} \leq 1} \left|\int_X f\bar{g}\, d\mu\right|.$$

Furthermore, if $q \neq \infty$, then $L^{p,q}(X)$ forms a Banach space whose continuous dual is $L^{p',q'}(X)$.

Proof. The right-hand side is controlled by the left-hand side by Holder's inequality, we will show the converse. By absolute homogeneity, we can assume without loss of generality $||f||^*_{L^{p,q}} = 1$, and using the vertically dyadic layered cake decomposition we can assume f takes the form

$$f = \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_n},$$

for $\{F_n\}_n$ a family of disjoint measurable sets. Then

$$1 = \left(||f||^*_{L^{p,q}} \right)^q = ||2^n \mu(F_n)^{1/p}||^q_{\ell^q_n} = \sum_{n \in \mathbb{Z}} 2^{nq} \mu(F_n)^{q/p}.$$

We want to construct $g \in L^{p',q'}(X)$ such that $\int_X f\overline{g}\,d\mu \sim 1$ and $||g||^*_{L^{p',q'}} \lesssim 1$. To this end, set

$$g := \sum_{n \in \mathbb{Z}} \left(2^n \mu(F_n)^{1/p} \right)^{q-1} \mu(F_n)^{-1/p'} \mathbb{1}_{F_n} = \sum_{n \in \mathbb{Z}} 2^{n(q-1)} \mu(F_n)^{(q-p)/p} \mathbb{1}_{F_n}.$$

Then

$$\int_X f\overline{g}\,d\mu = \sum_{n \in \mathbb{Z}} \left(2^n \mu(F_n)^{1/p} \right)^{q-1} 2^n \mu(F_n)^{1/p} = \sum_{n \in \mathbb{Z}} \left(2^n \mu(F_n)^{1/p} \right)^q = (||f||^*_{L^{p,q}})^q = 1.$$

Denoting $S_N \subseteq \mathbb{Z}$ the collection of integers $n \in \mathbb{Z}$ satisfying $2^{n(q-1)} \mu(F_n)^{(q-p)/p} \sim N$, we apply the vertical layered cake decomposition and Lemma 10 to estimate the $L^{p',q'}$-norm of g by

$$\left(||g||^*_{L^{p',q'}} \right)^{q'} \sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \left(\sum_{n \in S_N} \mu(F_n) \right)^{q'/p'} \sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \left(\sum_{n \in S_N} N^{\frac{p}{q-p}} 2^{-n(q-1)\frac{p}{q-p}} \right)^{q'/p'}$$

$$\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'(1 + \frac{p}{p'}\frac{1}{q-p})} \sum_{n \in S_N} N^{\frac{p}{q-p}} 2^{-n(q-1)q'\frac{p}{q-p}\frac{p}{p'}} \sim \sum_{N \in 2^{\mathbb{Z}}} \left(2^{n(q-1)} \mu(F_n)^{\frac{q-p}{p}} \right) 2^{-nq\frac{p-1}{q-p}}$$

$$\sim \sum_{n \in \mathbb{Z}} \mu(F_n)^{\frac{q'}{p}(q-1)} 2^{n(q-1)q'\frac{q-1}{q-p} - nq\frac{p-1}{q-p}} \sim \sum_{n \in \mathbb{Z}} 2^{nq} \mu(F_n)^{q/p} \sim ||2^n \mu(F_n)^{1/p}||^{q/q'}_{\ell^q_n} \sim 1,$$

as desired.

To show duality for $1 < p < \infty$ and $1 \le q < \infty$, consider $\ell : L^{p,q}(X) \to \mathbb{C}$ a bounded linear functional, we want to show that there exists $g \in L^{p',q'}(X)$ such that

$$\ell(f) = \int_X f\overline{g}\,d\mu.$$

Observe that $E \mapsto \ell(\mathbb{1}_E)$ forms a complex measure absolutely continuous with respect to μ. Thus by Radon-Nikodym, there exists $g \in L^1_{\text{loc}}(X)$ such that

$$\ell(\mathbb{1}_E) = \int_X \mathbb{1}_E \overline{g}\,d\mu.$$

This extends by linearity and density replacing $\mathbb{1}_E$ with any $f \in L^{p,q}(X)$. By the dual characterisation of the $L^{p,q}$-norm and boundedness of ℓ, we conclude $g \in L^{p',q'}(X)$, completing the proof. $\qquad\square$

Remark. Modifying the example in the case $q = \infty$ (*), one can show that $L^{1,q}(X)$ cannot admit a norm for all $1 < q < \infty$.

2.4. **Real interpolation.** Let T be an operator mapping a subspace of measurable functions $X \to \mathbb{C}$ to measurable functions $Y \to \mathbb{C}$, we say it is SUB-LINEAR if it satisfies

- absolute homogeneity, $|T(\alpha f)| = |\alpha|\,|Tf|$,
- sub-linearity, $|T(f+g)| \le |Tf| + |Tg|$,

for all $\alpha \in \mathbb{C}$ and $f, g : X \to \mathbb{C}$ in the domain of T. For $1 \le p, q \le \infty$, a sub-linear operator T is

- STRONG-TYPE (p,q) if it is bounded $T : L^p(X) \to L^q(Y)$, i.e. it satisfies the strong-type (p,q) inequality

$$||Tf||_{L^q} \lesssim ||f||_{L^p}, \qquad \text{uniformly in } f \in L^p(X)$$

- **WEAK-TYPE** (p, q) if it is bounded $T : L^p(X) \to L^{q,\infty}(Y)$, i.e. it satisfies the weak-type (p, q) inequality
$$||Tf||^*_{L^{q,\infty}} \lesssim ||f||_{L^p}, \qquad \text{uniformly in } f \in L^p(X),$$
- **RESTRICTED WEAK-TYPE** (p, q) for $q \neq \infty$ if the weak-type (p, q) inequality holds for step functions $f = \mathbb{1}_F$.

Remark. Strong-type implies weak-type, while weak-type implies restricted weak-type, however the converses generally fail. For example, the Hardy-Littlewood maximal function is weak-type $(1, 1)$ however not strong-type $(1, 1)$. The operator

$$Tf(x) := |x|^{-d/q} \int_{\mathbb{R}^d} f(y)|y|^{-d/p'} \, dy$$

is restricted weak-type (p, q) however not weak-type (p, q).

A typical technique to prove boundedness of a linear operator is to show the desired inequality $||Tf|| \lesssim ||f||$ for an elementary class of functions, such as simple functions or test functions. In the case of linear operators, boundedness implies Lipschitz continuity $||Tf - Tg|| \lesssim ||f - g||$, so we can extend the inequality by density to the entire space. This fails for sub-linear operators, so we will instead take a more technical approach in the spirit of Lemma 8.

Lemma 12. *Let $f : X \to [0, \infty)$ be measurable, then there exists sequence of measurable simple functions $f_k = \sum_n 2^n \mathbb{1}_{F_{k,n}}$ such that $f_k \leq 2^{1-k} f$ and f can be decomposed pointwise as*

$$f = \sum_{k \in \mathbb{N}} f_k.$$

Furthermore, if $|| - ||$ is a monotonic norm, then

$$||f|| \sim \sum_{k \in \mathbb{N}} ||f_k||.$$

We conclude that if T is a sub-linear operator bounded on simple functions, then T is bounded.

Proof. For fixed $x \in X$, consider the binary expansion of $f(x) \in [0, \infty)$. We construct recursively

$$f_k(x) := 2^{n_k(x)}, \qquad n_k(x) := \sup\{n \in \mathbb{Z} : 2^n \leq f(x) - \sum_{i=1}^{k-1} f_i(x)\},$$

arguing inductively gives $f_k \leq 2^{1-k} f$. Informally, $f_k(x)$ corresponds to the k-th non-zero entry in the binary expansion of $f(x)$. Under this interpretation, it is clear that $f = \sum_k f$ pointwise, f_k is measurable since the map from $y \in [0, \infty)$ to the k-th non-zero entry in its binary expansion is continuous, and f_k are simple,

$$f_k := \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_{k,n}}, \qquad F_{k,n} = \{x \in X : n_k(x) = n\}.$$

It follows from the triangle inequality and pointwise bound on f_k that

$$||f|| \leq \sum_{k \in \mathbb{N}} ||f_k|| \leq \sum_{k \in \mathbb{N}} 2^{1-k} ||f|| \lesssim ||f||.$$

Suppose T is sub-linear and bounded on simple functions, then by sub-linearity and the triangle inequality

$$||Tf|| \leq \sum_{n \in \mathbb{N}} ||Tf_k|| \lesssim \sum_{n \in \mathbb{N}} ||f_k|| \sim ||f||$$

completing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 13 (Characterisations of restricted weak-type (p, q)). *Let $1 < p, q < \infty$ and suppose T is a sub-linear operator. Then the following are equivalent:*

(a) *T is of restricted weak-type (p, q), i.e.*

$$||T\mathbb{1}_F||^*_{L^{q,\infty}} \lesssim \mu(F)^{1/p}$$

uniformly in $F \subseteq X$ measurable.

(b) *T satisfies the inequality*

$$\int_Y |T\mathbb{1}_F|\mathbb{1}_E \, d\nu \lesssim \mu(F)^{1/p}\nu(E)^{1/q'}$$

uniformly in $F \subseteq X$ and $E \subseteq Y$ measurable.

(c) *T forms a bounded operator $T : L^{p,1}(X) \to L^{q,\infty}(Y)$, i.e.*

$$||Tf||^*_{L^{q,\infty}} \lesssim ||f||^*_{L^{p,1}}$$

uniformly in $f \in L^{p,1}(X)$.

Proof. (a) $\implies$ (b) using Holder's inequality,

$$\int_Y |T\mathbb{1}_F|\mathbb{1}_E \, d\nu \leq |||T\mathbb{1}_F|\,\mathbb{1}_E||^*_{L^{1,1}} \lesssim ||T\mathbb{1}_F||^*_{L^{q,\infty}}||\mathbb{1}_E||_{L^{q',1}} \lesssim \mu(F)^{1/p}\nu(E)^{1/q'}.$$

(c) $\implies$ (a) follows from taking $f = \mathbb{1}_F$,

$$||T\mathbb{1}_F||^*_{L^{q,\infty}} \lesssim ||\mathbb{1}_F||^*_{L^{p,1}} \sim \mu(F)^{1/p}.$$

(b) $\implies$ (c) using Lemma 12, the dual characterisation of the $L^{p,q}$-norm and density of simple functions, we can take a vertically dyadic layered cake decomposition and assume $f \in L^{p,1}(X)$ and $g \in L^{q',1}(Y)$ take the form

$$f = \sum_{n\in\mathbb{Z}} 2^n \mathbb{1}_{F_n}, \qquad g = \sum_{m\in\mathbb{Z}} 2^m \mathbb{1}_{E_m}$$

for $\{F_n\}_n$ and $\{E_m\}_m$ families of disjoint measurable sets in X and Y respectively. Suppose $||g||^*_{L^{q',1}} \lesssim 1$, then it follows from (b) that

$$\left|\int_X Tf\,\overline{g} \, d\nu\right| \lesssim \sum_{m,n\in\mathbb{Z}} 2^{m+n} \int_Y |T\mathbb{1}_{F_n}|\,\mathbb{1}_{E_m} \, d\nu$$

$$\lesssim \sum_{m,n\in\mathbb{Z}} 2^{m+n}\mu(F_n)^{1/p}\nu(E_m)^{1/q'} \leq ||2^m\nu(E_m)^{1/q'}||_{\ell^1_m}||2^n\mu(F_n)^{1/p}||_{\ell^1_n} \lesssim ||f||^*_{L^{p,q}}.$$

This completes the proof. $\qquad\square$

Remark. If $p = q = 1$, the equivalence fails; in this case, (c) states that T is weak type $(1,1)$. For the non-equivalence of (a) and (c), Hagelstein and Jones [HJ05] exhibited an operator which is restricted weak-type $(1,1)$ however not weak-type $(1,1)$. For an example of the non-equivalence of (b) and (c), consider

$$(Tf)(x) := |x|^{-d} \int_{\mathbb{R}^d} f(y) \, dy,$$

which is weak-type $(1,1)$ however does not satisfy (b).

Theorem 14 (Hunt interpolation [Hun64]). *Let $1 \leq p_0 \neq p_1 \leq \infty$ and $1 \leq q_0 \neq q_1 \leq \infty$, and suppose T is a sub-linear operator which is bounded $L^{p_0,1}(X) \to L^{q_0,\infty}(Y)$ and $L^{p_1,1}(X) \to L^{q_1,\infty}(Y)$, i.e.*

$$||Tf||^*_{L^{q_0,\infty}} \lesssim ||f||^*_{L^{p_0,1}}, \qquad \text{uniformly in } f \in L^{p_0,1}(X)$$

$$||Tf||^*_{L^{q_1,\infty}} \lesssim ||f||^*_{L^{p_1,1}}, \qquad \text{uniformly in } f \in L^{p_1,1}(X).$$

Then for all $1 \leq r \leq \infty$ and $0 < \theta < 1$ the operator $T : L^{p_\theta,r}(X) \to L^{q_\theta,r}(Y)$ is bounded, i.e.

$$||Tf||^*_{L^{q_\theta,r}} \lesssim ||f||^*_{L^{p_\theta,r}}.$$

Proof. We first reduce the endpoint cases to the non-endpoint cases. From the restriction $p_0 \neq p_1$ and $q_0 \neq q_1$, the interpolated exponents (p_θ, q_θ) must lie in the interior of the interpolation square, so it suffices to interpolate between non-endpoint exponents $(p_{\theta_0}, q_{\theta_0})$ and $(p_{\theta_1}, q_{\theta_1})$ for some $0 < \theta_0 < \theta < \theta_1 < 1$. Thus by Proposition 13 we need to show the restricted weak-type (p_θ, q_θ) estimate holds for all $0 < \theta < 1$. Indeed,

$$||T\mathbb{1}_F||^*_{L^{q_\theta,\infty}} = \sup_{\lambda>0} \lambda\nu(|T\mathbb{1}_F| > \lambda)^{1/q_\theta} = \sup_{\lambda>0} \left(\lambda\nu(|T\mathbb{1}_F| > \lambda)^{1/q_0}\right)^{1-\theta} \left(\lambda\nu(|T\mathbb{1}_F| > \lambda)^{1/q_1}\right)^{\theta}$$

$$\lesssim \left(||T\mathbb{1}_F||^*_{L^{q_0,\infty}}\right)^{1-\theta} \left(||T\mathbb{1}_F||^*_{L^{q_1,\infty}}\right)^{\theta} \lesssim \mu(F)^{1/q_\theta}.$$

Assume then $1 < p_0 \neq p_1 < \infty$ and $1 < q_0 \neq q_1 < \infty$. In this regime, we can use Lemma 12, the dual characterisation of the $L^{p,q}$-norm, and density of simple functions to take without loss of generality $f \in L^{p_\theta, r}(X)$ and $g \in L^{q'_\theta, r'}(Y)$ of the form

$$f = \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_n}, \qquad g = \sum_{m \in \mathbb{Z}} 2^m \mathbb{1}_{E_m}$$

for $\{F_n\}_n$ and $\{E_m\}_m$ families of disjoint measurable sets in X and Y respectively. Using sub-linearity and the previous remarks, we need to estimate the quantity

$$||T||^*_{L^{q_\theta,\infty}} \sim \sup \left\{ \left| \int_Y Tf\overline{g}\, d\nu \right| : ||g||^*_{L^{q'_\theta,r'}} \lesssim 1 \right\} \lesssim \left\{ \sum_{m,n \in \mathbb{Z}} \int_Y |T\mathbb{1}_{F_n}|\mathbb{1}_{E_m}\, d\nu : ||2^m \nu(E_m)^{1/q'_\theta}||_{\ell^{r'}_m} \lesssim 1 \right\}.$$

From Proposition 13, the restricted weak-type inequalities imply

$$\int_Y |T\mathbb{1}_{F_n}|\mathbb{1}_{E_m}\, d\nu \lesssim 2^{n+m} \min\{\mu(F_n)^{1/p_0}\nu(E_m)^{1/q'_0}, \mu(F_n)^{1/p_1}\nu(E_m)^{1/q'_1}\}$$

$$\lesssim 2^{n+m} N^{\frac{1}{p_\theta}} M^{\frac{1}{q'_\theta}} \min\{N^{\frac{1}{p_0}-\frac{1}{p_\theta}} M^{\frac{1}{q'_0}-\frac{1}{q'_\theta}}, N^{\frac{1}{p_1}-\frac{1}{p_\theta}} M^{\frac{1}{q'_1}-\frac{1}{q'_\theta}}\} =: 2^{n+m} N^{\frac{1}{p_\theta}} M^{\frac{1}{q'_\theta}} A(N,M),$$

where for algebraic convenience we replace the measures $\mu(F_n)$ and $\nu(E_m)$ with dyadics $N, M \in 2^{\mathbb{Z}}$ satisfying $N \sim \mu(F_n)$ and $M \sim \nu(E_m)$. We claim that

$$\sum_{N \in 2^{\mathbb{Z}}} A(N,M) \sim \sum_{M \in 2^{\mathbb{Z}}} A(N,M) \sim 1$$

uniformly in $N, M \in 2^{\mathbb{Z}}$. This would complete the proof, as summing in n and m, applying Holder's inequality in $N, M \in 2^{\mathbb{Z}}$, and Lemma 10, we can conclude

$$\sum_{m,n \in \mathbb{Z}} \int_Y |T\mathbb{1}_{F_n}|\mathbb{1}_{E_m}\, d\nu \lesssim \sum_{N,M \in 2^{\mathbb{Z}}} A(N,M) \left(\sum_{\substack{n \in \mathbb{Z} \\ \mu(F_n) \sim N}} 2^n N^{\frac{1}{p_\theta}} \right) \left(\sum_{\substack{m \in \mathbb{Z} \\ \nu(E_m) \sim M}} 2^m M^{\frac{1}{p_\theta}} \right)$$

$$\lesssim \left(\sum_{N,M \in 2^{\mathbb{Z}}} A(N,M) \left(\sum_{\substack{n \in \mathbb{Z} \\ \mu(F_n) \sim N}} 2^n N^{\frac{1}{p_\theta}} \right)^r \right)^{1/r} \left(\sum_{N,M \in 2^{\mathbb{Z}}} A(N,M) \left(\sum_{\substack{m \in \mathbb{Z} \\ \nu(E_m) \sim M}} 2^m M^{\frac{1}{p_\theta}} \right)^{r'} \right)^{1/r'}$$

$$\lesssim \left(\sum_{N \in 2^{\mathbb{Z}}} \left(\sum_{\substack{n \in \mathbb{Z} \\ \mu(F_n) \sim N}} 2^n \mu(F_n)^{\frac{1}{p_\theta}} \right)^r \right)^{1/r} \left(\sum_{M \in 2^{\mathbb{Z}}} \left(\sum_{\substack{m \in \mathbb{Z} \\ \nu(E_m) \sim M}} 2^m \nu(E_m)^{\frac{1}{p_\theta}} \right)^{r'} \right)^{1/r'}$$

$$\lesssim \left(\sum_{n \in \mathbb{Z}} 2^{nr} \mu(F_n)^{\frac{r}{p_\theta}} \right)^{1/r} \left(\sum_{m \in \mathbb{Z}} 2^{mr'} \nu(E_m)^{\frac{r'}{p_\theta}} \right)^{1/r'}$$

$$\lesssim \left|\left| 2^n \mu(F_n)^{1/p_\theta} \right|\right|_{\ell^r_n} \left|\left| 2^m \nu(E_m)^{1/q'_\theta} \right|\right|_{\ell^r_n} \lesssim ||f||^*_{L^{p_\theta,r}}.$$

We prove the claim for summing in N uniformly in M; the other case is symmetric. Write

$$N^{\frac{1}{p_0}-\frac{1}{p_\theta}} M^{\frac{1}{q'_0}-\frac{1}{q'_\theta}} = \left(N^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_0}} \right)^{-\theta}, \qquad N^{\frac{1}{p_1}-\frac{1}{p_\theta}} M^{\frac{1}{q'_1}-\frac{1}{q'_\theta}} = \left(N^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_0}} \right)^{1-\theta}.$$

Fixing M, the transition for the minimum defining $A(N,M)$ occurs at $N = N_0$ for which the two quantities above are equal. Assuming without loss of generality that $p_0 < p_\theta < p_1$, splitting the regimes of summation with respect to the transition, and summing dyadically,

$$\sum_{N \in 2^{\mathbb{Z}}} A(N,M) = \sum_{N \geq N_0} \left(N^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_0}} \right)^{-\theta} + \sum_{N < N_0} \left(N^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_0}} \right)^{1-\theta}$$

$$= \left(N_0^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_0}} \right)^{-\theta} + \left(N_0^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_0}} \right)^{1-\theta} = 2,$$

as $B^{-\theta} = B^{1-\theta}$ if and only if $B = 1$. $\qquad \square$

Corollary 15 (Marcinkiewicz interpolation). *Let $1 \le p_0 \le p_1 < \infty$ and $1 \le q_1 \ne q_2 \le \infty$ such that $p_i \le q_i$, and suppose T is a sub-linear operator of weak-type (p_0, q_0) and (p_1, q_1), i.e.*

$$||Tf||^*_{L^{q_0,\infty}} \lesssim ||f||^*_{L^{p_0}}, \qquad \text{uniformly in } f \in L^{p_0}(X),$$
$$||Tf||^*_{L^{q_1,\infty}} \lesssim ||f||^*_{L^{p_1}}, \qquad \text{uniformly in } f \in L^{p_1}(X).$$

Then T satisfies a strong-type (p_θ, q_θ) inequality, i.e.

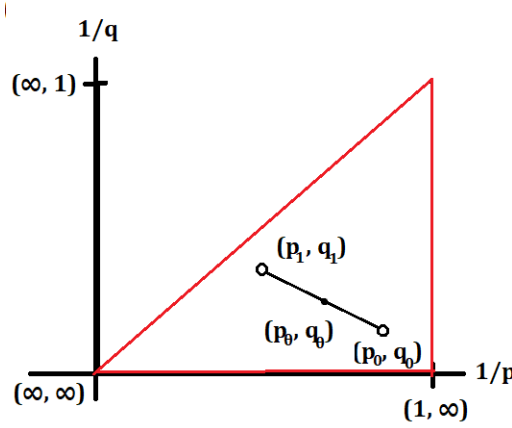$$||Tf||_{L^{q_\theta}} \lesssim ||f||_{L^{p_\theta}}.$$

FIGURE 2. The *interpolation diagram*; the restriction $p_0 \le q_0$ and $p_1 \le q_1$ guarantees that $p_\theta \le q_\theta$, so the intermediate line lies on the lower triangle.

Proof. Suppose $p_0 < p_1$, then recalling weak-type implies restricted weak-type, we apply Hunt interpolation for $r = q_\theta$ and use the embedding $L^{p_\theta, q_\theta}(X) \subseteq L^{p_\theta, p_\theta}(X)$ which holds since $p_\theta \le q_\theta$,

$$||Tf||_{L^{q_\theta}} = ||Tf||^*_{L^{q_\theta, q_\theta}} \lesssim ||f||^*_{L^{p_\theta, q_\theta}} \lesssim ||f||^*_{L^{p_\theta, p_\theta}} = ||f||_{L^{p_\theta}}.$$

Assume now $p := p_0 = p_\theta = p_1$, we have that T is weak-type (p, q_0) and (p, q_1), i.e.

$$\nu(|Tf| > \lambda) \lesssim \left(\frac{||f||_{L^p}}{\lambda} \right)^{q_0},$$
$$\nu(|Tf| > \lambda) \lesssim \left(\frac{||f||_{L^p}}{\lambda} \right)^{q_1}.$$

Taking a layered cake decomposition, and assume without loss of generality $q_0 < q_1$, then

$$||Tf||^{q_\theta}_{L^{q_\theta}} \sim \int_0^\infty \lambda^{q_\theta} \nu(|Tf| > \lambda) \frac{d\lambda}{\lambda} \lesssim \int_0^\infty \lambda^{q_\theta} \min \left\{ \left(\frac{||f||_{L^p}}{\lambda} \right)^{q_0}, \left(\frac{||f||_{L^p}}{\lambda} \right)^{q_1} \right\} \frac{d\lambda}{\lambda}$$
$$\lesssim \int_0^{||f||_{L^p}} \lambda^{q_\theta} \left(\frac{||f||_{L^p}}{\lambda} \right)^{q_0} \frac{d\lambda}{\lambda} + \int_{||f||_{L^p}}^\infty \lambda^{q_\theta} \left(\frac{||f||_{L^p}}{\lambda} \right)^{q_1} \frac{d\lambda}{\lambda} \lesssim ||f||^q_{L^p}.$$

This completes the proof. □

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