INTERPOLATION OF LEBESGUE AND LORENTZ SPACES

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ABSTRACT. Given an operator T bounded between two pairs of function spaces, the problem of *interpolation* asks: what can we say about the boundedness of T between function spaces *interpolated* between our original spaces? We set the stage by defining the Lorentz function space $L^{p,q}(X)$ and present both the complex and real methods for interpolating bounds on linear and sub-linear operators. These notes are inspired by [Tao06]; for a textbook treatment, see [Gra14].

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1. LEBESGUE SPACES

Let (X,μ) be a measure space and $1 \leq p \leq \infty$, the L^p SPACE, denoted $L^p(X)$, is the space of measurable functions $f : X \to \mathbb{C}$ such that the norm

$$||f||_{L^p} := \left(\int_X |f|^p d\mu\right)^{1/p}, \qquad \text{when } p \neq \infty,$$
$$||f||_{L^\infty} := \operatorname*{ess\,sup}_{x\in X} |f(x)|, \qquad \text{when } p = \infty,$$

is finite. The quantity above forms a monotone norm, that is, for $f, g \in L^p(X)$ and $\alpha \in \mathbb{C}$, it satisfies the following:

(a) Monotonicity, if $|f| \leq |g|$, then
$$||f||_{L^p} \leq ||g||_{L^p}.$$
(b) Positive definiteness,
$$||f||_{L^p} \geq 0$$
with equality only if $f \equiv 0$.
(c) Absolute homogeneity,
$$||\alpha f||_{L^p} = |\alpha|\, ||f||_{L^p}.$$
(d) The triangle inequality,
$$||f+g||_{L^p} \leq ||f||_{L^p} + ||g||_{L^p}.$$

1.1. Complex interpolation.

Complex interpolation. Let T be an operator mapping a subspace of measurable functions $Y \to \mathbb{C}$ to measurable functions $Y \to \mathbb{C}$, we say it is LINEAR if it satisfies

$$T(\alpha f + g) = \alpha T f + T g$$

for all $\alpha \in \mathbb{C}$ and $f, g : X \to \mathbb{C}$ in the domain of T. For $1 \leq p, q \leq \infty$, a linear operator is STRONG-TYPE (p,q) if it is bounded $T : L^p(X) \to L^q(Y)$, i.e. it satisfies the strong-type (p,q) inequality

$$||Tf||_{L^q} \lesssim ||f||_{L^p}, \qquad \text{uniformly in } f \in L^p(X).$$

Proposition 1. *Let $T : L^p(X) \to L^q(Y)$ be a linear operator. Then the following are equivalent:*

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(a) T *is strong-type* (p, q).
(b) T *is Lipschitz continuous.*
(c) T *is continuous at the origin.*

Proof. From linearity we see that (a) $\implies$ (b) and (b) $\implies$ (c) is trivial, so to close the chain of implications we show (c) $\implies$ (a). Let $\varepsilon > 0$, then there exists $\delta > 0$ such that

$$||Tf||_{L^q} < \varepsilon, \qquad \text{whenever } ||f||_{L^p} \leq \delta.$$

For any $f \in L^p(X)$, we use homogeneity and apply the inequality above to the normalised function $\delta f / ||f||_{L^p}$ to obtain the strong-type (p, q) inequality,

$$||Tf||_{L^q} = \frac{||f||_{L^p}}{\delta} \Big|\Big| T\Big(\delta \frac{f}{||f||_{L^p}}\Big)\Big|\Big|_{L^q} \leq \frac{\varepsilon}{\delta} ||f||_{L^p},$$

completing the proof. □

Remark. To avoid any philosophical malaise on whether an operator is well-defined on all L^p-functions, it is convenient to consider a linear operator acting on a sufficiently regular sub-class, namely test functions $C_c^\infty(\mathbb{R}^d)$ or Schwartz functions $\mathcal{S}(\mathbb{R}^d)$. Upon proving a strong-type (p, q) estimate for $p \neq \infty$ for such functions, we can extend the result by density and Lipschitz continuity to all L^p-functions.

Let $1 \leq p_0, p_1 \leq \infty$ and $0 \leq \theta \leq 1$, define $1 \leq p_\theta \leq \infty$ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

The space $L^{p_\theta}(X)$ is known as an *interpolation space* between $L^{p_0}(X)$ and $L^{p_1}(X)$. More precisely, $L^{p_\theta}(X)$ "lives between" $L^{p_0}(X)$ and $L^{p_1}(X)$ in the sense that

$$L^{p_0}(X) \cap L^{p_1}(X) \subseteq L^{p_\theta}(X) \subseteq L^{p_0}(X) + L^{p_1}(X).$$

The first inclusion follows from Holder's inequality, writing $f = f^{1-\theta} f^\theta$,

$$||f||_{L^{p_\theta}} \leq ||f||_{L^{p_0}}^{1-\theta} ||f||_{L^{p_1}}^{\theta}.$$

The second inclusion follows from decomposing $f = f\mathbb{1}_{|f|>1} + f\mathbb{1}_{|f|\leq 1}$, monotonicity of $|x| \mapsto |x|^p$, and noting that $p_0 < p_\theta < p_1$,

$$||f\mathbb{1}_{|f|>1}||_{L^{p_0}} \leq ||f||_{L^{p_\theta}}, \qquad ||f\mathbb{1}_{|f|\leq 1}||_{L^{p_1}} \leq ||f||_{L^{p_\theta}}.$$

Given $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and a linear operator $T : (L^{p_0} + L^{p_1})(X) \to (L^{q_0} + L^{q_1})(Y)$ of strong-type (p_0, q_0) and (p_1, q_1), we see that T forms a map between the interpolation spaces. *Interpolation* is the problem of establishing the strong-type (p_θ, q_θ) inequality.

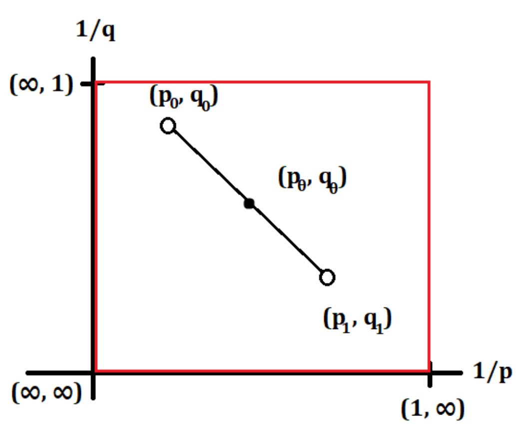

FIGURE 1. The *interpolation diagram*; given bounds for exponents (p_0, q_0) and (p_1, q_1), interpolation furnishes bounds for exponents (p_θ, q_θ) on the intermediate line.

Theorem 2 (Riesz-Thorin interpolation). *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and $0 < \theta < 1$, and suppose (X, μ) and (Y, ν) are measure spaces, the latter σ-finite when $q_0 = q_1 = \infty$. If $T : (L^{p_0} + L^{p_1})(X) \to (L^{q_0} + L^{q_1})(Y)$ is a linear operator of strong-type (p_0, q_0) and (p_1, q_1), i.e.*

$$||Tf||_{L^{q_0}} \leq M_0 ||f||_{L^{p_0}}, \qquad \text{uniformly in } f \in L^{p_0}(X)$$
$$||Tf||_{L^{q_1}(Y)} \leq M_1 ||f||_{L^{p_1}}, \qquad \text{uniformly in } f \in L^{p_1}(X)$$

for some $0 < M_0, M_1 < \infty$. Then T satisfies the strong-type (p_θ, q_θ) inequality

$$||Tf||_{L^{q_\theta}} \leq M_0^{1-\theta} M_1^{\theta} ||f||_{L^{p_\theta}}.$$

Proof. If $p_0 = p_1 = p_\theta$, the result follows from the log convexity of the L^p-norm. That is, by Holder's inequality,

$$||g||_{L^{q_\theta}}^{q_\theta} = \int_Y |g|^{(1-\theta)q_\theta} |g|^{\theta q_\theta} \, d\nu \leq |||g|^{(1-\theta)q_\theta}||_{L^{\frac{q_0}{(1-\theta)q_\theta}}} |||g|^{\theta q_\theta}||_{L^{\frac{q_1}{\theta q_\theta}}} = ||g||_{L^{q_0}}^{(1-\theta)q_\theta} ||g||_{L^{q_1}}^{\theta q_\theta}.$$

Then taking $Tf = g$ and applying the strong type (p_0, q_0) and (p_1, q_1) inequalities,

$$||Tf||_{L^{q_\theta}(Y)} \leq ||Tf||_{L^{q_0}(Y)}^{1-\theta} ||Tf||_{L^{q_1}(Y)}^{\theta} \leq M_0^{1-\theta} M_1^{\theta} ||f||_{L^{p_\theta}(X)}.$$

Thus we can assume $p_0 < p_1$; in particular, we avoid the endpoint cases and assume $1 < p_\theta < \infty$. We claim that it suffices to prove the result for simple functions with finite support, i.e. functions of the form

$$f = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$$

for coefficients $a_k \in \mathbb{C}$ and disjoint finite measure sets $A_k \subseteq X$. We can find simple functions with finite measure support $\{f_n\}_n$ such that $f_n \to f$ pointwise and $|f_n| \leq |f|$. Assume $f \in L^{p_0}(X) \cap L^{p_1}(X)$, then by log convexity of the L^p-norm we also have $f \in L^{p_\theta}(X)$. By the L^p-dominated convergence theorem, $f_n \to f$ in L^{p_θ} and L^{p_0}. Observe

$$\begin{aligned} ||Tf||_{L^{q_\theta}(Y)} &\leq ||T(f - f_n)||_{L^{q_\theta}(Y)} + ||Tf_n||_{L^{q_\theta}(y)} \\ &\leq ||T(f - f_n)||_{L^{q_0}(Y)}^{1-\theta} ||T(f - f_n)||_{L^{q_1}(Y)}^{\theta} + M_0^{1-\theta} M_1^{\theta} ||f_n||_{L^{p_\theta}(X)} \\ &\leq M_0^{1-\theta} M_1^{\theta} \left(||f - f_n||_{L^{p_0}(X)}^{1-\theta} ||f - f_n||_{L^{p_1}(X)}^{\theta} + ||f - f_n||_{L^{p_\theta}(X)} + ||f||_{L^{p_\theta}(X)} \right). \end{aligned}$$

The first line follows from the triangle inequality and linearity of T, the second follows from log convexity of the L^q-norm and Riesz-Thorin for simple functions, the third follows from the strong type (p_0, q_0) and (p_1, q_1) inequalities and the L^{p_θ}-triangle inequality. Since $|f_n| \leq |f|$, we know $||f - f_n||_{L^{p_1}(X)} \leq 2||f||_{L^{p_1}(X)}$. This allows us to pass the limit $n \to \infty$ on the right to obtain

$$||Tf||_{L^{q_\theta}(Y)} \leq M_0^{1-\theta} M_1^{\theta} ||f||_{L^{p_\theta}(X)},$$

proving the claim.

To prove Riesz-Thorin for simple functions, we argue by duality. For $1 \leq q \leq \infty$, then

$$||Tf||_{L^{q_\theta}(Y)} = \sup_{||g||_{L^{q'_\theta}(Y)} = 1} \left| \int_Y (Tf) g \, d\nu \right|.$$

Suppose one of $q_0, q_1 \neq 1$, then $1 \leq q'_\theta < \infty$. By density we can take the supremum over simple functions. Fix simple functions with finite measure support f and g, and set

$$F(s) := \int_Y T\left(|f|^{(1-s)\frac{p_\theta}{p_0} + s\frac{p_\theta}{p_1}} \operatorname{sgn}(f) \right) |g|^{(1-s)\frac{q'_\theta}{q'_0} + s\frac{q'_\theta}{q'_1}} \operatorname{sgn}(g) \, d\nu,$$

with the convention $q'_\theta / q'_0 = q'_\theta / q'_1 = 1$ in the endpoint case $q_0 = q_1 = q_\theta = \infty$. Since we assumed f and g were simple functions and T is linear, the integrand is entire for each fixed $y \in Y$ and satisfies $|F(s)| \lesssim e^{c|z|}$. From the finite measure support assumption, we can apply Fubini-Tonelli and Morera's theorem to conclude F is entire. Hence, the conditions for Hadamard's three lines theorem have been satisfied.

If $q_0 = q_1 = q_\theta = 1$, density of simple functions fails, so we consider instead any $g \in L^\infty(Y)$. Nevertheless, the conditions of Hadamard's theorem continue to be satisfied, since the exponent of $|g|$ reduces to a constant and

$$F(s) = \int_Y T\left(|f|^{(1-s)\frac{p_\theta}{p_0} + s\frac{p_\theta}{p_1}} \operatorname{sgn}(f) \right) g \, d\nu.$$

By choice of exponents and noting $f = |f| \operatorname{sgn}(f)$,

$$F(\theta) = \int_Y (Tf) g \, d\nu.$$

Applying Holder's inequality, the strong type (p_0, q_0) inequality, and recalling $|a^{it}| = 1$ for any $a, t \in \mathbb{R}$, we have

$$\begin{aligned} |F(it)| &\leq \left|\left| T\left(|f|^{(1-it)\frac{p_\theta}{p_0} + it\frac{p_\theta}{p_1}} \operatorname{sgn}(f)\right)\right|\right|_{L^{q_0}(Y)} \left|\left| |g|^{(1-it)\frac{q'_\theta}{q'_0} + it\frac{q'_\theta}{q'_1}} \operatorname{sgn}(g) \right|\right|_{L^{q'_0}(Y)} \\ &\leq M_0 \left|\left| |f|^{\frac{p_\theta}{p_0}} \right|\right|_{L^{p_0}(Y)} \left|\left| |g|^{\frac{q'_\theta}{q'_0}} \right|\right|_{L^{q'_0}(Y)} \leq M_0 ||f||_{L^{p_\theta}(X)}^{\frac{p_\theta}{p_0}} ||g||_{L^{q'_\theta}(Y)}^{\frac{q'_\theta}{q'_0}}, \end{aligned}$$

and similarly, applying instead the strong type (p_1, q_1) inequality,

$$\begin{aligned} |F(1+it)| &\leq \left|\left| T\left(|f|^{-it\frac{p_\theta}{p_0} + (1+it)\frac{p_\theta}{p_1}} \operatorname{sgn}(f)\right)\right|\right|_{L^{q_1}(Y)} \left|\left| |g|^{-it\frac{q'_\theta}{q'_0} + (1+it)\frac{q'_\theta}{q'_1}} \operatorname{sgn}(g) \right|\right|_{L^{q'_1}(Y)} \\ &\leq M_0 \left|\left| |f|^{\frac{p_\theta}{p_1}} \right|\right|_{L^{p_1}(Y)} \left|\left| |g|^{\frac{q'_\theta}{q'_1}} \right|\right|_{L^{q'_1}(Y)} \leq M_0 ||f||_{L^{p_\theta}(X)}^{\frac{p_\theta}{p_1}} ||g||_{L^{q'_\theta}(Y)}^{\frac{q'_\theta}{q'_1}}. \end{aligned}$$

Hadamard's three lines theorem furnishes the inequality $|F(\theta)| \leq \sup_t |F(it)|^{1-\theta} \sup_t |F(1+it)|^\theta$. Collecting this inequality with the previous two inequalities and comparing exponents, we obtain

$$\left| \int_Y (Tf) g \, d\nu \right| \leq M_0^{1-\theta} M_1^\theta ||f||_{L^{p_\theta}(X)} ||g||_{L^{q'_\theta}(Y)}.$$

By duality, this finishes the proof. ☐

2. Lorentz spaces

Given a measure space (X, μ), the two basic quantitative notions of "size" of a function $f : X \to \mathbb{C}$ are the "height" in the range and "width" in the domain. The L^p-norm primarily quantified control over the former; to quantify control over both notions, we introduce for $1 \leq p, q \leq \infty$ the Lorentz space $L^{p,q}(X)$, the space of measurable functions $f : X \to \mathbb{C}$ for which

$$\begin{aligned} ||f||_{L^{p,q}}^* &:= p^{1/q} \left|\left| \lambda \mu(\{x \in X : |f(x)| > \lambda\})^{1/p} \right|\right|_{L^q((0,\infty), \frac{d\lambda}{\lambda})} \\ &= p^{1/q} \left(\int_0^\infty \lambda^q \mu(\{x \in X : |f(x)| \geq \lambda\})^{q/p} \frac{d\lambda}{\lambda} \right)^{1/q} \end{aligned}$$

is finite. The quantity above forms a monotone quasi-norm, that is, for $f, g \in L^{p,q}(X)$ and $\alpha \in \mathbb{C}$, it satisfies the following properties:

(a) Monotonicity, if $|f| \leq |g|$ then

$$||f||_{L^{p,q}}^* \leq ||g||_{L^{p,q}}^*.$$

(b) Positive definiteness,

$$||f||_{L^{p,q}}^* \geq 0,$$

with equality only if $f \equiv 0$; this is clear.

(c) Absolute homogeneity,

$$||\alpha f||_{L^{p,q}}^* = |\alpha| \, ||f||_{L^{p,\infty}}^*.$$

(d) A quasi-triangle inequality,

$$||f+g||_{L^{p,q}}^* \leq 2||f||_{L^{p,\infty}}^* + 2||g||_{L^{p,q}}^*.$$

By convention we set $L^{\infty,\infty}(X) := L^\infty(X)$, and in general when $q = \infty$, the Lorentz space $L^{p,\infty}(X)$ is known as the weak L^p space, endowed with the norm

$$||f||_{L^{p,\infty}}^* := \sup_{\lambda > 0} \lambda \mu(\{x \in X : |f(x)| > \lambda\})^{1/p}.$$

Remark. The Lorentz space coincides with the usual Lebesgue space when $p = q$, that is, $L^{p,p}(X) = L^p(X)$. This follows from the *layered cake representation*, writing using the fundamental theorem of calculus

$$|f(x)|^p = \int_0^{|f(x)|} p\lambda^{p-1}\,d\lambda = p\int_0^\infty \lambda^{p-1}\mathbb{1}_{[0,|f(x)|]}(\lambda)\,d\lambda = p\int_0^\infty \lambda^p \mathbb{1}_{|f|>\lambda}(x)\frac{d\lambda}{\lambda}.$$

The representation takes its name from writing $|f(x)|^p$ as the sum of contributions from the "layers" λ below $|f(x)|$. Integrating and applying Fubini's theorem,

$$||f||_{L^p}^p = \int_X |f(x)|^p\,d\mu = p\int_X\int_0^\infty \lambda^p \mathbb{1}_{|f|>\lambda}(x)\frac{d\lambda}{\lambda} = p\int_0^\infty \lambda^p \mu(\{x \in X : |f(x)| > \lambda\})\frac{d\lambda}{\lambda} = (||f||_{L^{p,p}}^*)^p\,.$$

2.1. **Weak L^p space.** As a primer for studying the general $L^{p,q}$-spaces, we first consider the weak L^p-spaces. The prototypical example of a function in weak L^p-space however not in L^p-space is

$$f(x) := |x|^{-d/p}.$$

In fact, one can think of every weak L^p-function as dominated pointwise by a rearrangement of $|x|^{-d/p}$. Several classical inequalities can be reformulated in terms of weak L^p-space, such as the Hardy-Littlewood maximal inequality and, in probability,

Lemma 3 (Chebyshev's inequality)**.** *Let $f \in L^p(X)$ and $\lambda > 0$, then*

$$\mu(\{x \in X : |f(x)| > \lambda\}) \le \frac{1}{\lambda^p}\int_{|f|>\lambda} |f|^p\,d\mu.$$

Moreover, $L^p(X) \hookrightarrow L^{p,\infty}(X)$ via the inequality

$$||f||_{L^{p,\infty}}^* \le ||f||_{L^p}.$$

Proof. We have

$$\mu(\{x \in X : |f(x)| > \lambda\}) = \frac{1}{\lambda^p}\int_{|f|>\lambda} \lambda^p d\mu \le \frac{1}{\lambda^p}\int_{|f|>\lambda} |f|^p\,d\mu.$$

This proves Chebyshev's inequality. Rearranging and taking the p-th root gives

$$\lambda\mu(\{x \in X : |f(x)| > \lambda\})^{1/p} \le \left(\int_X |f|^p\,d\mu\right)^{1/p}.$$

Taking the supremum over $\lambda > 0$ on the left furnishes the continuous embedding $L^p(X) \hookrightarrow L^{p,\infty}(X)$. $\square$

Remark. The case $p = 1$ of Chebyshev's inequality is also referred to as Markov's inequality.

Theorem 4 (Weak L^p-duality)**.** *Let $1 < p < \infty$ and $f \in L^{p,\infty}(X)$, then*

$$||f||_{L^{p,\infty}} := \sup_{0<\mu(A)<\infty} \mu(A)^{-1/p'}\left|\int_X f\mathbb{1}_A\,d\mu\right|$$

defines a norm satisfying

$$||f||_{L^{p,\infty}}^* \le ||f||_{L^{p,\infty}} \le \frac{1}{p'}||f||_{L^{p,\infty}}^*.$$

Proof. It is clear from definition that $||\cdot||_{L^{p,\infty}}$ defines a norm, so it remains to show that it is comparable to the quasi-norm $||\cdot||_{L^{p,\infty}}^*$. Decomposing into real, imaginary, positive and negative components, we can assume without loss of generality $f \ge 0$. Since $f \in L^{p,\infty}(X)$, we know that the super-level sets $|f| > \lambda$ have finite measure, so rearranging Markov's inequality we obtain

$$\lambda\mu(f > \lambda)^{1/p} \le \mu(f > \lambda)^{-1/p'}\int_{f>\lambda} f\,d\mu \le ||f||_{L^{p,\infty}}.$$

Taking the supremum with respect to $\lambda > 0$ on the left gives the desired lower bound on $||f||_{L^{p,\infty}}$. For the upper bound, we apply the layered cake representation and the definition of the weak L^p-quasinorm to write

$$\begin{aligned}\int_A f\,d\mu = \int_0^\infty \mu(x \in A : f > \lambda)\,d\lambda &\le \int_0^\infty \min\{\mu(f > \lambda), \mu(A)\}\\ &\le \int_0^\infty \min\{\lambda^{-p}||f||_{L^{p,\infty}}^*, \mu(A)\}d\lambda.\end{aligned}$$

We split the integral on the right, remarking that $\lambda^{-p}||f||^*_{L^{p,\infty}} \leq \mu(A)$ if and only if $||f||^*_{L^{p,\infty}}\mu(A)^{-1/p} \leq \lambda$, so by the fundamental theorem of calculus

$$\begin{aligned}\int_0^\infty \min\{\lambda^{-p}||f||^*_{L^{p,\infty}}, \mu(A)\} d\lambda &= \int_0^{||f||^*_{L^{p,\infty}}\mu(A)^{-1/p}} \mu(A)\, d\lambda + \int_{||f||^*_{L^{p,\infty}}\mu(A)^{-1/p}}^\infty \lambda^{-p}||f||^*_{L^{p,\infty}}\, d\lambda \\ &= ||f||^*_{L^{p,\infty}}\mu(A)^{1/p'} + \frac{1}{p-1}||f||^*_{L^{p,\infty}}\mu(A)^{1/p'} = \frac{1}{p'}||f||^*_{L^{p,\infty}}\mu(A)^{1/p'}.\end{aligned}$$

Rearranging and taking the supremum over $0 < \mu(A) < \infty$ furnishes the desired upper bound. ☐

Remark. The argument fails in the case $p = 1$; in fact, the weak L^1-space cannot be normed provided that the measure μ is non-zero and non-atomic. Consider for example the Lebesgue measure on $\mathbb{R}$, and assume towards a contradiction that there exists a norm $|||-|||$ such that $|||-||| \sim ||-||^*_{L^{p,\infty}}$. Define $f_n \in L^{1,\infty}(\mathbb{R})$ by

$$f_n(x) = \frac{1}{|x-n|}. \tag{*}$$

We compute the quasinorms,

$$||f_n||^*_{L^{1,\infty}} = \sup_{\lambda>0} \lambda\mu(|x| < 1/\lambda) = 2.$$

On the other hand, recall that the harmonic series grows logarithmically, so for $x \in [0, N]$ and $N \gg 1$ we have the pointwise lower bound

$$\log N \lesssim \sum_{n=1}^N f_n.$$

Collecting our results and applying the triangle inequality, we obtain

$$N \log N \lesssim \Big|\Big| \sum_{n=1}^N f_n \Big|\Big|^*_{L^{1,\infty}} \sim \Big|\Big|\Big| \sum_{n=1}^N f_n \Big|\Big|\Big| \leq \sum_{n=1}^N |||f_n||| \lesssim \sum_{n=1}^N ||f_n||^*_{L^{1,\infty}} \sim N,$$

a contradiction.

2.2. **Characterisation of $L^{p,q}$.** The $L^{p,q}$-quasinorm can at first glance appears rather mysterious and cumbersome to work with directly, so it will be both enlightening and convenient to introduce characterisations of $L^{p,q}$-functions in terms of simple functions. Much like the L^p-norm, the $L^{p,q}$-quasinorm of a step function is

$$||H\mathbb{1}_E||^*_{L^{p,q}} \sim H\mu(E)^{1/p}.$$

The difference lies in how these quantities are summed in the case of simple functions. For $H_n > 0$ distinct heights and $\{E_n\}_n$ a disjoint family of measurable sets, the L^p-norm of the corresponding simple function is the ℓ^p_n-sum of step function L^p-norms

$$\Big|\Big| \sum_{n\in\mathbb{Z}} H_n\mathbb{1}_{E_n} \Big|\Big|_{L^p} = \Big|\Big| ||H_n\mathbb{1}_{E_n}||_{L^p} \Big|\Big|_{\ell^p_n} = ||H_n\mu(E_n)^{1/p}||_{\ell^p_n}.$$

The $L^{p,q}$-norm is instead comparable to the ℓ^q-sum of step function L^p-norms. For a generic function $f \in L^{p,q}(X)$, we can decompose and approximate by simple functions in two fashions: vertically into step functions of height $H_n \sim 2^n$ and some width $\mu(E_n)$, or horizontally into step functions of width $\mu(E_n) \sim 2^n$ and some heights H_n.

Proposition 5 (Vertically dyadic layer cake decomposition)**.** *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$, and suppose that $f \in L^{p,q}(X)$. Decomposing*

$$f = \sum_{m\in\mathbb{Z}} f\mathbb{1}_{2^m \leq |f| < 2^{m+1}} =: \sum_{m\in\mathbb{Z}} f_m,$$

then

$$||f||^*_{L^{p,q}} \sim_{p,q} \Big|\Big| ||f_m||_{L^p} \Big|\Big|_{\ell^q_m}.$$

In particular, $L^{p,q_1}(X) \hookrightarrow L^{p,q_2}(X)$ whenever $q_1 \leq q_2$.

Proof. By construction,

$$|f| \sim \sum_{m\in\mathbb{Z}} 2^m \mathbb{1}_{2^m \leq |f| < 2^{m+1}}$$

pointwise, so by monotonicity we can consider without loss of generality functions of the form $f = \sum_m 2^m\mathbb{1}_{E_m}$ where $\{E_m\}_m$ is a disjoint family of measurable sets. For such functions, the result takes the form

$$||f||^*_{L^{p,q}} \sim_{p,q} \Big|\Big| 2^m\mu(E_m)^{1/p} \Big|\Big|_{\ell^q_m}.$$

We compute

$$(||f||^*_{L^{p,q}})^q = p\sum_{m\in\mathbb{Z}}\int_{2^{m-1}}^{2^m}\lambda^q\mu(|f|>\lambda)^{q/p}\frac{d\lambda}{\lambda} = p\sum_{m\in\mathbb{Z}}\int_{2^{m-1}}^{2^m}\lambda^q\Big(\sum_{n\geq m}\mu(E_n)\Big)^{q/p}\frac{d\lambda}{\lambda}$$
$$\sim_{p,q}\sum_{m\in\mathbb{Z}}2^{mq}\Big(\sum_{n\geq m}\mu(E_n)\Big)^{q/p}\sim\Big|\Big|2^m\Big(\sum_{n\geq m}\mu(E_n)\Big)^{1/p}\Big|\Big|^q_{\ell^q_m}.$$

Clearly

$$\Big|\Big|2^m\mu(E_m)^{1/p}\Big|\Big|_{\ell^q_m}\leq\Big|\Big|2^m\Big(\sum_{n\geq m}\mu(E_n)\Big)^{1/p}\Big|\Big|_{\ell^q_m}.$$

For the converse inequality, we appeal to a change of indices $n = m+k$ and the triangle inequality,

$$\Big|\Big|2^m\Big(\sum_{n\geq m}\mu(E_n)\Big)^{1/p}\Big|\Big|_{\ell^q_m}\lesssim\Big|\Big|2^n\sum_{n\geq m}\mu(E_n)^{1/p}\Big|\Big|_{\ell^q_m}\sim\Big|\Big|2^m\sum_{k\geq 0}\mu(E_{m+k})^{1/p}\Big|\Big|_{\ell^q_m}$$
$$\leq\sum_{k\geq 0}2^{-k}\Big|\Big|2^{m+k}\mu(E_{m+k})^{1/p}\Big|\Big|_{\ell^q_m}\sim||2^m\mu(E_m)^{1/p}||_{\ell^q_m}.$$

This completes the proof. □

Corollary 6 (Density of simple functions)**.** *For $1\leq p,q<\infty$ the space of simple function is dense in $L^{p,q}(X)$.*

Proof. Fix $f\in L^{p,q}(X)$ and $\varepsilon>0$. Performing a vertically dyadic layer cake decomposition, there exists $n_\varepsilon\in\mathbb{N}$ sufficiently large such that

$$\Big|\Big|||f_m||_{L^p}\Big|\Big|_{\ell^q_{|m|\geq n_\varepsilon}}\ll\varepsilon.$$

Fix $N\gg 1$ to be chosen later, define

$$E_{m,k} := \{x\in X : \frac{k}{2^N}\leq|f_m(x)|\leq\frac{k+1}{2^N}\}$$

for $2^{m+N}\leq k\leq 2^{m+N+1}-1$. Define the simple functions

$$\phi_m := \sum_{k=2^{m+N}}^{2^{m+N+1}}\frac{k}{2^N}\mathbb{1}_{E_{m,k}},\qquad \phi := \sum_{|m|\leq n_\varepsilon}\phi_m.$$

We claim that $||f-\phi||^*_{L^{p,q}}\lesssim\varepsilon$. By the triangle inequality and choice of n_ε we have

$$||f-\phi||^*_{L^{p,q}}\leq 2\left(\Big|\Big|\sum_{|m|\leq n_\varepsilon}f_m-\phi_m\Big|\Big|^*_{L^{p,q}}+\Big|\Big|\sum_{|m|>n_\varepsilon}f_m\Big|\Big|^*_{L^{p,q}}\right)\leq 2\left(\Big|\Big|\sum_{|m|\leq n_\varepsilon}f_m-\phi_m\Big|\Big|^*_{L^{p,q}}+\varepsilon\right).$$

It remains to control the first term on the right. By the triangle inequality and Holder's inequality,

$$\Big|\Big|\sum_{|m|\leq n_\varepsilon}f_m-\phi_m\Big|\Big|^*_{L^{p,q}}\leq 2^{2n_\varepsilon+1}\sum_{|m|\leq n_\varepsilon}||f_m-\phi_m||^*_{L^{p,q}}\lesssim 2^{2n}\sum_{|m|\leq n_\varepsilon}\Big|\Big|2^k\mu(2^k\leq|f_m-\phi_m|<2^{k+1})^{1/p}\Big|\Big|_{\ell^q_k}$$
$$\lesssim 2^{2n_\varepsilon}\sum_{|m|\leq n_\varepsilon}\Big|\Big|2^k\mu(2^m\leq|f|<2^{m+1})^{1/p}\Big|\Big|_{\ell^q_k}\lesssim 2^{2n}\sum_{|m|\leq n_\varepsilon}2^{-N}\mu(2^m\leq|f|<2^{m+1})^{1/p}$$
$$\lesssim 2^{2n_\varepsilon}\sum_{|m|\leq n_\varepsilon}2^{-N}\frac{||f_m||_{L^p}}{2^m}\lesssim 2^{2n-N}\Big|\Big|||f_m||_{L^p}\Big|\Big|_{\ell^q_m}\Big|\Big|2^{-m}\Big|\Big|_{\ell^q_{|m|\leq n_\varepsilon}}\lesssim\frac{2^{3n}}{2^N}||f||^*_{L^{p,q}}.$$

Choosing $N\gg 1$ completes the proof. □

Proposition 7 (Horizontally dyadic layer cake decomposition)**.** *Let $1\leq p<\infty$ and $1\leq q\leq\infty$, and suppose that $f\in L^{p,q}(X)$. Decomposing,*

$$f=\sum_{m\in\mathbb{Z}}f\mathbb{1}_{H_{m+1}\leq|f|<H_m}=:\sum_{m\in\mathbb{Z}}f_m,$$

where $H_m := \inf\{\lambda>0 : \mu(|f|>\lambda)\leq 2^{m-1}\}$, then

$$||f||^*_{L^{p,q}}\sim||H_m2^{m/p}||_{\ell^q_m}.$$

Proof. It is clear that $H_m \to ||f||_{L^\infty}$ as $m \to -\infty$ and, since $f \in L^{p,q}(X)$, we also have $H_m \to 0$ as $m \to \infty$. Furthermore we have

$$2^{m-1} < \mu(|f| > \lambda) \leq 2^m.$$

whenever $H_{m+1} \leq \lambda < H_m$. For $q \neq \infty$, we compute

$$\begin{aligned}(||f||_{L^{p,q}}^*)^q &= p\int_0^\infty \lambda^q \mu(|f| > \lambda)^{q/p}\frac{d\lambda}{\lambda} = p\sum_{m\in\mathbb{Z}}\int_{H_{m+1}}^{H_m}\lambda^q\mu(|f|>\lambda)^{q/p}\frac{d\lambda}{\lambda}\\ &\sim \sum_{m\in\mathbb{Z}}\int_{H_{m+1}}^{H_m}\lambda^q 2^{mq/p}\frac{d\lambda}{\lambda} \sim \sum_{m\in\mathbb{Z}}2^{mq/p}\left(H_m^q - H_{m+1}^q\right) \sim \sum_{m\in\mathbb{Z}}2^{mq/p}H_m^q = ||2^{\frac{m}{p}}H_m||_{\ell_m^q}^q.\end{aligned}$$

For the case $q = \infty$ we have

$$\begin{aligned}||f||_{L^{p,\infty}}^* &= \sup_{\lambda>0}\lambda|\{x : |f(x)| > \lambda\}|^{\frac{1}{p}} = \sup_{m\in\mathbb{Z}}\sup_{H_{m+1}\leq\lambda<H_m}\lambda|\{x : |f(x)|>\lambda\}|^{\frac{1}{p}}\\ &\sim \sup_{m\in\mathbb{Z}} H_m 2^{\frac{m}{p}} = ||H_m 2^{\frac{m}{p}}||_{\ell_m^\infty},\end{aligned}$$

completing the proof. □

2.3. **Duality.** Analogous to the L^p-space setting, the continous duals of non-endpoint $L^{p,q}$-spaces are represented by the dual exponent $L^{p',q'}$-spaces. The dual characterisation thereby furnishes a norm on $L^{p,q}(X)$ and moreover a Banach space structure. To this end, we will need the following preliminary lemma, which states that the triangle inequality, up to a uniform constant, continues to hold for quasi-norms provided the component quasi-norms are on different dyadic scales.

Lemma 8. *Let $||-||$ denote a quasinorm on a topological vector space X. Let $f_1, \ldots, f_N \in X$ satisfy*

$$||f_n|| \leq 2^{-\varepsilon n}$$

for some $\varepsilon > 0$. Then

$$\left|\left|\sum_{n=1}^N f_n\right|\right| \lesssim_\varepsilon 1,$$

where the implicit constant is independent of N.

Proof. There exists a constant $C > 1$ such that for functions f and g the quasinorm satisfies the following quasi-triangle inequality

$$||f + g|| \leq C||f|| + C||g||.$$

Let $\eta > 0$ such that $C = 2^\eta$. We consider first the case where $\varepsilon > \eta$, then

$$\begin{aligned}\left|\left|\sum_{n=1}^N f_n\right|\right| &\leq C||f_1|| + C\left|\left|\sum_{n=2}^N f_n\right|\right| \leq \cdots \leq C||f_1|| + \cdots + C^N||f_N||\\ &\leq \sum_{n=1}^N C^n||f_n|| \leq \sum_{n=1}^N 2^{(\eta-\varepsilon)n} \leq \frac{1}{1-2^{\eta-\varepsilon}} \lesssim_\varepsilon 1.\end{aligned}$$

In the first line we apply the quasi-triangle inequality N-times and the second line we note that we obtain a convergent geometric series. If $\varepsilon \leq \eta$, choose $M_\varepsilon \in \mathbb{N}$ such that $\varepsilon M_\varepsilon > \eta$. For notational convenience, let $d \in \mathbb{N}$ such that $dM_\varepsilon > N$ and set $f_{N+1} = \cdots = f_{dM_\varepsilon} = 0$. Define

$$g_k = \sum_{n=kM_\varepsilon+1}^{(k+1)M_\varepsilon} f_n$$

for $k = 0, \ldots, d-1$. Trivially $\sum_n f_n = \sum_k g_k$, so to reduce to the previous case it remains to prove analogous bounds $||g_k|| \lesssim_\varepsilon 2^{-\varepsilon M_\varepsilon k}$ where the implicit constant is uniform in k. We apply the quasi-triangle inequality M_ε-times to obtain the desired inequality for each k,

$$\begin{aligned}\left|\left|\sum_{n=kM_\varepsilon+1}^{(k+1)M_\varepsilon} f_n\right|\right| &\leq C||f_{kM_\varepsilon+1}|| + \cdots + C^{M_\varepsilon}||f_{(k+1)M_\varepsilon}||\\ &\leq C^{M_\varepsilon}\left(2^{-\varepsilon(kM_\varepsilon+1)} + \cdots + 2^{-\varepsilon(k+1)M_\varepsilon}\right) \leq C^{M_\varepsilon}M_\varepsilon 2^{-\varepsilon k M_\varepsilon}.\end{aligned}$$

Arguing as we did in the first case, we conclude

$$\left|\left|\sum_{n=1}^{N} f_n\right|\right| = \left|\left|\sum_{k=0}^{d-1} g_k\right|\right| \leq \frac{C^{M_\varepsilon} M_\varepsilon}{1-2^{\eta-\varepsilon M_\varepsilon}} \lesssim_\varepsilon 1.$$

This completes the proof. $\square$

Theorem 9 ($L^{p,q}$-Holder's inequality)**.** *Let $1 \leq p, p_1, p_2 < \infty$ and $1 \leq q, q_1, q_2 \leq \infty$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$||fg||^*_{L^{p,q}} \lesssim ||f||^*_{L^{p_1,q_1}} ||g||^*_{L^{p_2,q_2}}.$$

Proof. By absolute homogeneity we can renormalize, replacing f and g with $f/||f||^*_{L^{p_1,q_1}}$ and $g/||g||^*_{L^{p_2,q_2}}$. It suffices then to prove the case where $||f||^*_{L^{p_1,q_1}} = ||g||^*_{L^{p_2,q_2}} = 1$. Set

$$\begin{aligned} H_{n,f} &= \inf\{\lambda > 0 : |\{x : |f(x)| > \lambda\}| \leq 2^{n-1}\}, \\ H_{n,f} &= \inf\{\lambda > 0 : |\{x : |g(x)| > \lambda\}| \leq 2^{n-1}\}, \\ E_{n,f} &= \{x : H_{n+1,g} \leq |f(x)| < H_{n,f}\}, \\ E_{n,g} &= \{x : H_{n+1,g} \leq |g(x)| < H_{n,g}\}, \end{aligned}$$

where $\{E_{n,f}\}_n$ and $\{E_{n,g}\}_n$ are families of disjoint measurable sets. The horizontal layered cake decomposition furnishes the estimates

$$||H_{n,f} 2^{\frac{n}{p_1}}||_{\ell^{q_1}(\mathbb{Z})} \sim ||H_{n,g} 2^{\frac{n}{p_2}}||_{\ell^{q_2}(\mathbb{Z})} \sim 1.$$

By construction,

$$|fg| \leq \sum_k \sum_n H_{n,f} H_{n+k,g} \mathbb{1}_{E_{n,f} \cap E_{n+k,g}}.$$

From monotonicity of the quasi-norm and the quasi-triangle inequality, it remains to show

$$\left|\left|\sum_{k\geq 1} \sum_n H_{n,f} H_{n+k,g} \mathbb{1}_{E_{n,f}\cap E_{n+k,g}}\right|\right|^*_{L^{p,q}}, \left|\left|\sum_{k\leq 0} \sum_n H_{n,f} H_{n+k,g} \mathbb{1}_{E_{n,f}\cap E_{n+k,g}}\right|\right|^*_{L^{p,q}} \lesssim 1.$$

We prove the former; the latter is symmetric. In the proof of the horizontal dyadic layered cake decomposition, we showed that $\mu(E_{n,f}) \sim \mu(E_{n,g}) \sim 2^n$. In particular, $\mu(E_{n,f} \cap E_{n+k,g}) \lesssim 2^n$ uniformly in n and k, so by Holder's inequality in $\ell^q_n(\mathbb{Z})$ we obtain

$$\begin{aligned} \left|\left|\sum_n H_{n,f} H_{n+k,g} \mathbb{1}_{E_{n,f}\cap E_{n+k,g}}\right|\right|^*_{L^{p,q}} &\lesssim ||H_{n,f} H_{n+k,g} 2^{\frac{n}{p}}||_{\ell^q_n(\mathbb{Z})} = 2^{-\frac{k}{p_2}} ||H_{n,f} 2^{\frac{n}{p_1}} H_{n+k,g} 2^{\frac{n+k}{p_2}}||_{\ell^q_n(\mathbb{Z})} \\ &\lesssim 2^{-\frac{k}{p_2}} ||H_{n,f} 2^{\frac{n}{p_1}}||_{\ell^{q_1}_n(\mathbb{Z})} ||H_{n,f} 2^{\frac{n}{p_2}}||_{\ell^{q_2}_n(\mathbb{Z})} \lesssim 2^{-\frac{k}{p_2}}. \end{aligned}$$

Note the second line we make the change of variables $n + k \mapsto n$. Summing in $k \geq 1$ and applying the previous lemma, we conclude the desired inequality. A remark on the symmetric proof for the sum over terms $k \leq 0$, we simply replace $n + k$ with $n - k$ and $2^{-k/p_2}$ with $2^{k/p_2}$. $\square$

For brevity we denote the dyadic numbers $2^{\mathbb{Z}}$, that is, the numbers of the form 2^n for $n \in \mathbb{Z}$. In a similar spirit to Lemma 8, we can show that the q-th power of a convergent sum of dyadics is comparable to the sum of the q-th powers and the maximum of the q-th powers for $1 \leq q < \infty$. This result will be convenient algebraically,

Lemma 10. *Let $1 \leq q < \infty$ and suppose $\mathcal{F} \subseteq 2^{\mathbb{Z}}$ is finite. Then*

$$\sum_{N\in\mathcal{F}} N^q \sim \left(\sum_{N\in\mathcal{F}} N\right)^q \sim \max_{N\in\mathcal{F}} N^q.$$

Proof. We have

$$\sum_{N\in\mathcal{F}} N^q \leq \left(\sum_{N\in\mathcal{F}} N\right)^q \leq \left(\max_{N\in\mathcal{F}} 2N\right)^q \leq 2^q \sum_{N\in\mathcal{F}} N^q.$$

$\square$

Theorem 11 ($L^{p,q}$-duality)**.** *Let $1 < p < \infty$ and $1 \leq q \leq \infty$, and suppose $f \in L^{p,q}(X)$, then*

$$||f||^*_{L^{p,q}} \sim_{p,q} \sup_{||g||^*_{L^{p',q'}} \leq 1} \left| \int_X f \overline{g} \, d\mu \right|.$$

Furthermore, if $q \neq \infty$, then $L^{p,q}(X)$ forms a Banach space whose continuous dual is $L^{p',q'}(X)$.

Proof. The right-hand side is controlled by the left-hand side by Holder's inequality, we will show the converse. By absolute homogeneity, we can assume without loss of generality $||f||^*_{L^{p,q}} = 1$, and using the vertically dyadic layered cake decomposition we can assume f takes the form

$$f = \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_n},$$

for $\{F_n\}_n$ a family of disjoint measurable sets. Then

$$1 = \left(||f||^*_{L^{p,q}} \right)^q = ||2^n \mu(F_n)^{1/p}||^q_{\ell^q_n} = \sum_{n \in \mathbb{Z}} 2^{nq} \mu(F_n)^{q/p}.$$

We want to construct $g \in L^{p',q'}(X)$ such that $\int_X f \overline{g} \, d\mu \sim 1$ and $||g||^*_{L^{p',q'}} \lesssim 1$. To this end, set

$$g := \sum_{n \in \mathbb{Z}} \left(2^n \mu(F_n)^{1/p} \right)^{q-1} \mu(F_n)^{-1/p'} \mathbb{1}_{F_n} = \sum_{n \in \mathbb{Z}} 2^{n(q-1)} \mu(F_n)^{(q-p)/p} \mathbb{1}_{F_n}.$$

Then

$$\int_X f \overline{g} \, d\mu = \sum_{n \in \mathbb{Z}} \left(2^n \mu(F_n)^{1/p} \right)^{q-1} 2^n \mu(F_n)^{1/p} = \sum_{n \in \mathbb{Z}} \left(2^n \mu(F_n)^{1/p} \right)^q = (||f||^*_{L^{p,q}})^q = 1.$$

Denoting $S_N \subseteq \mathbb{Z}$ the collection of integers $n \in \mathbb{Z}$ satisfying $2^{n(q-1)} \mu(F_n)^{(q-p)/p} \sim N$, we apply the vertical layered cake decomposition and Lemma 10 to estimate the $L^{p',q'}$-norm of g by

$$\begin{aligned} \left(||g||^*_{L^{p',q'}} \right)^{q'} &\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \Big(\sum_{n \in S_N} \mu(F_n) \Big)^{q'/p'} \sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \Big(\sum_{n \in S_N} N^{\frac{p}{q-p}} 2^{-n(q-1)\frac{p}{q-p}} \Big)^{q'/p'} \\ &\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'(1 + \frac{p}{p'} \frac{1}{q-p})} \sum_{n \in S_N} N^{\frac{p}{q-p}} 2^{-n(q-1)q' \frac{p}{q-p} \frac{p}{p'}} \sim \sum_{N \in 2^{\mathbb{Z}}} \left(2^{n(q-1)} \mu(F_n)^{\frac{q-p}{p}} \right) 2^{-nq \frac{p-1}{q-p}} \\ &\sim \sum_{n \in \mathbb{Z}} \mu(F_n)^{\frac{q'}{p}(q-1)} 2^{n(q-1)q' \frac{q-1}{q-p} - nq \frac{p-1}{q-p}} \sim \sum_{n \in \mathbb{Z}} 2^{nq} \mu(F_n)^{q/p} \sim ||2^n \mu(F_n)^{1/p}||^{q/q'}_{\ell^q_n} \sim 1, \end{aligned}$$

as desired.

To show duality for $1 < p < \infty$ and $1 \leq q < \infty$, consider $\ell : L^{p,q}(X) \to \mathbb{C}$ a bounded linear functional, we want to show that there exists $g \in L^{p',q'}(X)$ such that

$$\ell(f) = \int_X f \overline{g} \, d\mu.$$

Observe that $E \mapsto \ell(\mathbb{1}_E)$ forms a complex measure absolutely continuous with respect to μ. Thus by Radon-Nikodym, there exists $g \in L^1_{\text{loc}}(X)$ such that

$$\ell(\mathbb{1}_E) = \int_X \mathbb{1}_E \overline{g} \, d\mu.$$

This extends by linearity and density replacing $\mathbb{1}_E$ with any $f \in L^{p,q}(X)$. By the dual characterisation of the $L^{p,q}$-norm and boundedness of ℓ, we conclude $g \in L^{p',q'}(X)$, completing the proof. □

Remark. Modifying the example in the case $q = \infty$ (*), one can show that $L^{1,q}(X)$ cannot admit a norm for all $1 < q < \infty$.

2.4. **Real interpolation.** Let T be an operator mapping a subspace of measurable functions $X \to \mathbb{C}$ to measurable functions $Y \to \mathbb{C}$, we say it is SUB-LINEAR if it satisfies

- absolute homogeneity, $|T(\alpha f)| = |\alpha| \, |Tf|$,
- sub-linearity, $|T(f+g)| \leq |Tf| + |Tg|$,

for all $\alpha \in \mathbb{C}$ and $f, g : X \to \mathbb{C}$ in the domain of T. For $1 \leq p, q \leq \infty$, a sub-linear operator T is

- STRONG-TYPE (p,q) if it is bounded $T : L^p(X) \to L^q(Y)$, i.e. it satisfies the strong-type (p,q) inequality

$$||Tf||_{L^q} \lesssim ||f||_{L^p}, \qquad \text{uniformly in } f \in L^p(X)$$

- WEAK-TYPE (p,q) if it is bounded $T : L^p(X) \to L^{q,\infty}(Y)$, i.e. it satisfies the weak-type (p,q) inequality
$$||Tf||^*_{L^{q,\infty}} \lesssim ||f||_{L^p}, \qquad \text{uniformly in } f \in L^p(X),$$
- RESTRICTED WEAK-TYPE (p,q) for $q \neq \infty$ if the weak-type (p,q) inequality holds for step functions $f = \mathbb{1}_F$.

Remark. Strong-type implies weak-type, while weak-type implies restricted weak-type, however the converses generally fail. For example, the Hardy-Littlewood maximal function is weak-type $(1,1)$ however not strong-type $(1,1)$. The operator
$$Tf(x) := |x|^{-d/q} \int_{\mathbb{R}^d} f(y)|y|^{-d/p'}\, dy$$
is restricted weak-type (p,q) however not weak-type (p,q).

A typical technique to prove boundedness of a linear operator is to show the desired inequality $||Tf|| \lesssim ||f||$ for an elementary class of functions, such as simple functions or test functions. In the case of linear operators, boundedness implies Lipschitz continuity $||Tf - Tg|| \lesssim ||f - g||$, so we can extend the inequality by density to the entire space. This fails for sub-linear operators, so we will instead take a more technical approach in the spirit of Lemma 8.

Lemma 12. *Let $f : X \to [0,\infty)$ be measurable, then there exists sequence of measurable simple functions $f_k = \sum_n 2^n \mathbb{1}_{F_{k,n}}$ such that $f_k \leq 2^{1-k} f$ and f can be decomposed pointwise as*
$$f = \sum_{k \in \mathbb{N}} f_k.$$
Furthermore, if $|| - ||$ is a monotonic norm, then
$$||f|| \sim \sum_{k \in \mathbb{N}} ||f_k||.$$
We conclude that if T is a sub-linear operator bounded on simple functions, then T is bounded.

Proof. For fixed $x \in X$, consider the binary expansion of $f(x) \in [0,\infty)$. We construct recursively
$$f_k(x) := 2^{n_k(x)}, \qquad n_k(x) := \sup\{n \in \mathbb{Z} : 2^n \leq f(x) - \sum_{i=1}^{k-1} f_i(x)\},$$
arguing inductively gives $f_k \leq 2^{1-k} f$. Informally, $f_k(x)$ corresponds to the k-th non-zero entry in the binary expansion of $f(x)$. Under this interpretation, it is clear that $f = \sum_k f$ pointwise, f_k is measurable since the map from $y \in [0,\infty)$ to the k-th non-zero entry in its binary expansion is continuous, and f_k are simple,
$$f_k := \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_{k,n}}, \qquad F_{k,n} = \{x \in X : n_k(x) = n\}.$$
It follows from the triangle inequality and pointwise bound on f_k that
$$||f|| \leq \sum_{k \in \mathbb{N}} ||f_k|| \leq \sum_{k \in \mathbb{N}} 2^{1-k} ||f|| \lesssim ||f||.$$
Suppose T is sub-linear and bounded on simple functions, then by sub-linearity and the triangle inequality
$$||Tf|| \leq \sum_{n \in \mathbb{N}} ||Tf_k|| \lesssim \sum_{n \in \mathbb{N}} ||f_k|| \sim ||f||$$
completing the proof. ◻

Proposition 13 (Characterisations of restricted weak-type (p,q))**.** *Let $1 < p, q < \infty$ and suppose T is a sub-linear operator. Then the following are equivalent:*

(a) T is of restricted weak-type (p,q), i.e.
$$||T\mathbb{1}_F||^*_{L^{q,\infty}} \lesssim \mu(F)^{1/p}$$
uniformly in $F \subseteq X$ measurable.

(b) T satisfies the inequality
$$\int_Y |T\mathbb{1}_F| \mathbb{1}_E \, d\nu \lesssim \mu(F)^{1/p} \nu(E)^{1/q'}$$
uniformly in $F \subseteq X$ and $E \subseteq Y$ measurable.

(c) T forms a bounded operator $T : L^{p,1}(X) \to L^{q,\infty}(Y)$, i.e.

$$||Tf||^*_{L^{q,\infty}} \lesssim ||f||^*_{L^{p,1}}$$

uniformly in $f \in L^{p,1}(X)$.

Proof. (a) $\implies$ (b) using Holder's inequality,

$$\int_Y |T\mathbb{1}_F|\mathbb{1}_E \, d\nu \le |||T\mathbb{1}_F|\,\mathbb{1}_E||^*_{L^{1,1}} \lesssim ||T\mathbb{1}_F||^*_{L^{q,\infty}}||\mathbb{1}_E||_{L^{q',1}} \lesssim \mu(F)^{1/p}\nu(E)^{1/q'}.$$

(c) $\implies$ (a) follows from taking $f = \mathbb{1}_F$,

$$||T\mathbb{1}_F||^*_{L^{q,\infty}} \lesssim ||\mathbb{1}_F||^*_{L^{p,1}} \sim \mu(F)^{1/p}.$$

(b) $\implies$ (c) using Lemma 12, the dual characterisation of the $L^{p,q}$-norm and density of simple functions, we can take a vertically dyadic layered cake decomposition and assume $f \in L^{p,1}(X)$ and $g \in L^{q',1}(Y)$ take the form

$$f = \sum_{n\in\mathbb{Z}} 2^n \mathbb{1}_{F_n}, \qquad g = \sum_{m\in\mathbb{Z}} 2^m \mathbb{1}_{E_m}$$

for $\{F_n\}_n$ and $\{E_m\}_m$ families of disjoint measurable sets in X and Y respectively. Suppose $||g||^*_{L^{q',1}} \lesssim 1$, then it follows from (b) that

$$\begin{aligned}\left|\int_X Tf\,\overline{g}\,d\nu\right| &\lesssim \sum_{m,n\in\mathbb{Z}} 2^{m+n}\int_Y |T\mathbb{1}_{F_n}|\,\mathbb{1}_{E_m}\,d\nu \\ &\lesssim \sum_{m,n\in\mathbb{Z}} 2^{m+n}\mu(F_n)^{1/p}\nu(E_m)^{1/q'} \le ||2^m\nu(E_m)^{1/q'}||_{\ell^1_m}||2^n\mu(F_n)^{1/p}||_{\ell^1_n} \lesssim ||f||^*_{L^{p,q}}.\end{aligned}$$

This completes the proof. □

Remark. If $p = q = 1$, the equivalence fails; in this case, (c) states that T is weak type $(1,1)$. For the non-equivalence of (a) and (c), Hagelstein and Jones [HJ05] exhibited an operator which is restricted weak-type $(1,1)$ however not weak-type $(1,1)$. For an example of the non-equivalence of (b) and (c), consider

$$(Tf)(x) := |x|^{-d}\int_{\mathbb{R}^d} f(y)\,dy,$$

which is weak-type $(1,1)$ however does not satisfy (b).

Theorem 14 (Hunt interpolation [Hun64]). *Let $1 \le p_0 \ne p_1 \le \infty$ and $1 \le q_0 \ne q_1 \le \infty$, and suppose T is a sub-linear operator which is bounded $L^{p_0,1}(X) \to L^{q_0,\infty}(Y)$ and $L^{p_1,1}(X) \to L^{q_1,\infty}(Y)$, i.e.*

$$\begin{aligned}||Tf||^*_{L^{q_0,\infty}} &\lesssim ||f||^*_{L^{p_0,1}}, \qquad \text{uniformly in } f \in L^{p_0,1}(X)\\ ||Tf||^*_{L^{q_1,\infty}} &\lesssim ||f||^*_{L^{p_1,1}}, \qquad \text{uniformly in } f \in L^{p_1,1}(X).\end{aligned}$$

Then for all $1 \le r \le \infty$ and $0 < \theta < 1$ the operator $T : L^{p_\theta,r}(X) \to L^{q_\theta,r}(Y)$ is bounded, i.e.

$$||Tf||^*_{L^{q_\theta,r}} \lesssim ||f||^*_{L^{p_\theta,r}}.$$

Proof. We first reduce the endpoint cases to the non-endpoint cases. From the restriction $p_0 \ne p_1$ and $q_0 \ne q_1$, the interpolated exponents (p_θ, q_θ) must lie in the interior of the interpolation square, so it suffices to interpolate between non-endpoint exponents $(p_{\theta_0}, q_{\theta_0})$ and $(p_{\theta_1}, q_{\theta_1})$ for some $0 < \theta_0 < \theta < \theta_1 < 1$. Thus by Proposition 13 we need to show the restricted weak-type (p_θ, q_θ) estimate holds for all $0 < \theta < 1$. Indeed,

$$\begin{aligned}||T\mathbb{1}_F||^*_{L^{q_\theta,\infty}} &= \sup_{\lambda>0}\lambda\nu(|T\mathbb{1}_F| > \lambda)^{1/q_\theta} = \sup_{\lambda>0}\left(\lambda\nu(|T\mathbb{1}_F| > \lambda)^{1/q_0}\right)^{1-\theta}\left(\lambda\nu(|T\mathbb{1}_F| > \lambda)^{1/q_1}\right)^{\theta}\\ &\lesssim \left(||T\mathbb{1}_F||^*_{L^{q_0,\infty}}\right)^{1-\theta}\left(||T\mathbb{1}_F||^*_{L^{q_1,\infty}}\right)^{\theta} \lesssim \mu(F)^{1/q_\theta}.\end{aligned}$$

Assume then $1 < p_0 \ne p_1 < \infty$ and $1 < q_0 \ne q_1 < \infty$. In this regime, we can use Lemma 12, the dual characterisation of the $L^{p,q}$-norm, and density of simple functions to take without loss of generality $f \in L^{p_\theta,r}(X)$ and $g \in L^{q'_\theta,r'}(Y)$ of the form

$$f = \sum_{n\in\mathbb{Z}} 2^n \mathbb{1}_{F_n}, \qquad g = \sum_{m\in\mathbb{Z}} 2^m \mathbb{1}_{E_m}$$

for $\{F_n\}_n$ and $\{E_m\}_m$ families of disjoint measurable sets in X and Y respectively. Using sub-linearity and the previous remarks, we need to estimate the quantity

$$||T||^*_{L^{q_\theta,\infty}} \sim \sup\Big\{\Big|\int_Y Tf\overline{g}\,d\nu\Big| : ||g||^*_{L^{q'_\theta,r'}} \lesssim 1\Big\} \lesssim \Big\{\sum_{m,n\in\mathbb{Z}} \int_Y |T\mathbb{1}_{F_n}|\mathbb{1}_{E_m}\,d\nu : ||2^m\nu(E_m)^{1/q'_\theta}||_{\ell^{r'}_m} \lesssim 1\Big\}.$$

From Proposition 13, the restricted weak-type inequalities imply

$$\begin{aligned}\int_Y |T\mathbb{1}_{F_n}|\mathbb{1}_{E_m}\,d\nu &\lesssim 2^{n+m}\min\{\mu(F_n)^{1/p_0}\nu(E_m)^{1/q'_0}, \mu(F_n)^{1/p_1}\nu(E_m)^{1/q'_1}\}\\ &\lesssim 2^{n+m}N^{\frac{1}{p_\theta}}M^{\frac{1}{q'_\theta}}\min\{N^{\frac{1}{p_0}-\frac{1}{p_\theta}}M^{\frac{1}{q'_0}-\frac{1}{q'_\theta}}, N^{\frac{1}{p_1}-\frac{1}{p_\theta}}M^{\frac{1}{q'_1}-\frac{1}{q'_\theta}}\} =: 2^{n+m}N^{\frac{1}{p_\theta}}M^{\frac{1}{q'_\theta}}A(N,M),\end{aligned}$$

where for algebraic convenience we replace the measures $\mu(F_n)$ and $\nu(E_m)$ with dyadics $N, M \in 2^{\mathbb{Z}}$ satisfying $N \sim \mu(F_n)$ and $M \sim \nu(E_m)$. We claim that

$$\sum_{N\in 2^{\mathbb{Z}}} A(N,M) \sim \sum_{M\in 2^{\mathbb{Z}}} A(N,M) \sim 1$$

uniformly in $N, M \in 2^{\mathbb{Z}}$. This would complete the proof, as summing in n and m, applying Holder's inequality in $N, M \in 2^{\mathbb{Z}}$, and Lemma 10, we can conclude

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}}\int_Y |T\mathbb{1}_{F_n}|\mathbb{1}_{E_m}\,d\nu &\lesssim \sum_{N,M\in 2^{\mathbb{Z}}} A(N,M)\Big(\sum_{\substack{n\in\mathbb{Z}\\ \mu(F_n)\sim N}} 2^n N^{\frac{1}{p_\theta}}\Big)\Big(\sum_{\substack{m\in\mathbb{Z}\\ \nu(E_m)\sim M}} 2^m M^{\frac{1}{p_\theta}}\Big)\\ &\lesssim \Big(\sum_{N,M\in 2^{\mathbb{Z}}} A(N,M)\Big(\sum_{\substack{n\in\mathbb{Z}\\ \mu(F_n)\sim N}} 2^n N^{\frac{1}{p_\theta}}\Big)^r\Big)^{1/r}\Big(\sum_{N,M\in 2^{\mathbb{Z}}} A(N,M)\Big(\sum_{\substack{m\in\mathbb{Z}\\ \nu(E_m)\sim M}} 2^m M^{\frac{1}{p_\theta}}\Big)^{r'}\Big)^{1/r'}\\ &\lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}}\Big(\sum_{\substack{n\in\mathbb{Z}\\ \mu(F_n)\sim N}} 2^n \mu(F_n)^{\frac{1}{p_\theta}}\Big)^r\Big)^{1/r}\Big(\sum_{M\in 2^{\mathbb{Z}}}\Big(\sum_{\substack{m\in\mathbb{Z}\\ \nu(E_m)\sim M}} 2^m \nu(E_m)^{\frac{1}{p_\theta}}\Big)^{r'}\Big)^{1/r'}\\ &\lesssim \Big(\sum_{n\in\mathbb{Z}} 2^{nr}\mu(F_n)^{\frac{r}{p_\theta}}\Big)^{1/r}\Big(\sum_{m\in\mathbb{Z}} 2^{mr'}\nu(E_m)^{\frac{r'}{p_\theta}}\Big)^{1/r'}\\ &\lesssim \Big|\Big|2^n\mu(F_n)^{1/p_\theta}\Big|\Big|_{\ell^r_n}\Big|\Big|2^m\nu(E_m)^{1/q'_\theta}\Big|\Big|_{\ell^r_n} \lesssim ||f||^*_{L^{p_\theta,r}}.\end{aligned}$$

We prove the claim for summing in N uniformly in M; the other case is symmetric. Write

$$N^{\frac{1}{p_0}-\frac{1}{p_\theta}}M^{\frac{1}{q'_0}-\frac{1}{q'_\theta}} = \left(N^{\frac{1}{p_1}-\frac{1}{p_0}}M^{\frac{1}{q'_1}-\frac{1}{q'_0}}\right)^{-\theta}, \qquad N^{\frac{1}{p_1}-\frac{1}{p_\theta}}M^{\frac{1}{q'_1}-\frac{1}{q'_\theta}} = \left(N^{\frac{1}{p_1}-\frac{1}{p_0}}M^{\frac{1}{q'_1}-\frac{1}{q'_0}}\right)^{1-\theta}.$$

Fixing M, the transition for the minimum defining $A(N,M)$ occurs at $N = N_0$ for which the two quantities above are equal. Assuming without loss of generality that $p_0 < p_\theta < p_1$, splitting the regimes of summation with respect to the transition, and summing dyadically,

$$\begin{aligned}\sum_{N\in 2^{\mathbb{Z}}} A(N,M) &= \sum_{N\geq N_0}\left(N^{\frac{1}{p_1}-\frac{1}{p_0}}M^{\frac{1}{q'_1}-\frac{1}{q'_0}}\right)^{-\theta} + \sum_{N<N_0}\left(N^{\frac{1}{p_1}-\frac{1}{p_0}}M^{\frac{1}{q'_1}-\frac{1}{q'_0}}\right)^{1-\theta}\\ &= \left(N_0^{\frac{1}{p_1}-\frac{1}{p_0}}M^{\frac{1}{q'_1}-\frac{1}{q'_0}}\right)^{-\theta} + \left(N_0^{\frac{1}{p_1}-\frac{1}{p_0}}M^{\frac{1}{q'_1}-\frac{1}{q'_0}}\right)^{1-\theta} = 2,\end{aligned}$$

as $B^{-\theta} = B^{1-\theta}$ if and only if $B = 1$. ☐

Corollary 15 (Marcinkiewicz interpolation). *Let $1 \leq p_0 \leq p_1 < \infty$ and $1 \leq q_1 \neq q_2 \leq \infty$ such that $p_i \leq q_i$, and suppose T is a sub-linear operator of weak-type (p_0, q_0) and (p_1, q_1), i.e.*

$$\begin{aligned}||Tf||^*_{L^{q_0,\infty}} &\lesssim ||f||^*_{L^{p_0}}, &&\text{uniformly in } f \in L^{p_0}(X),\\ ||Tf||^*_{L^{q_1,\infty}} &\lesssim ||f||^*_{L^{p_1}}, &&\text{uniformly in } f \in L^{p_1}(X).\end{aligned}$$

Then T satisfies a strong-type (p_θ, q_θ) inequality, i.e.

$$||Tf||_{L^{q_\theta}} \lesssim ||f||_{L^{p_\theta}}.$$

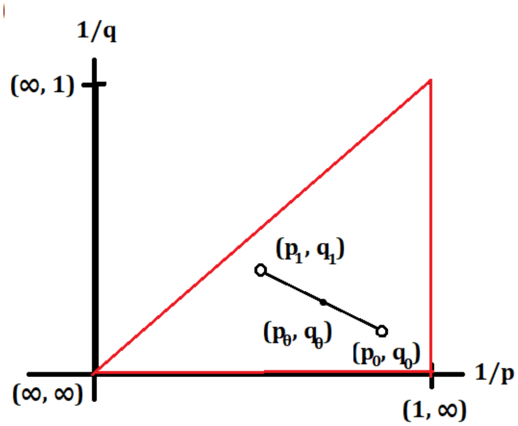

FIGURE 2. The *interpolation diagram*; the restriction $p_0 \leq q_0$ and $p_1 \leq q_1$ guarantees that $p_\theta \leq q_\theta$, so the intermediate line lies on the lower triangle.

Proof. Suppose $p_0 < p_1$, then recalling weak-type implies restricted weak-type, we apply Hunt interpolation for $r = q_\theta$ and use the embedding $L^{p_\theta,q_\theta}(X) \subseteq L^{p_\theta,p_\theta}(X)$ which holds since $p_\theta \leq q_\theta$,

$$||Tf||_{L^{q_\theta}} = ||Tf||^*_{L^{q_\theta,q_\theta}} \lesssim ||f||^*_{L^{p_\theta,q_\theta}} \lesssim ||f||^*_{L^{p_\theta,p_\theta}} = ||f||_{L^{p_\theta}}.$$

Assume now $p := p_0 = p_\theta = p_1$, we have that T is weak-type (p, q_0) and (p, q_1), i.e.

$$\nu(|Tf| > \lambda) \lesssim \left(\frac{||f||_{L^p}}{\lambda}\right)^{q_0},$$
$$\nu(|Tf| > \lambda) \lesssim \left(\frac{||f||_{L^p}}{\lambda}\right)^{q_1}.$$

Taking a layered cake decomposition, and assume without loss of generality $q_0 < q_1$, then

$$||Tf||^{q_\theta}_{L^{q_\theta}} \sim \int_0^\infty \lambda^{q_\theta} \nu(|Tf| > \lambda) \frac{d\lambda}{\lambda} \lesssim \int_0^\infty \lambda^{q_\theta} \min\left\{ \left(\frac{||f||_{L^p}}{\lambda}\right)^{q_0}, \left(\frac{||f||_{L^p}}{\lambda}\right)^{q_1} \right\} \frac{d\lambda}{\lambda}$$
$$\lesssim \int_0^{||f||_{L^p}} \lambda^{q_\theta} \left(\frac{||f||_{L^p}}{\lambda}\right)^{q_0} \frac{d\lambda}{\lambda} + \int_{||f||_{L^p}}^\infty \lambda^{q_\theta} \left(\frac{||f||_{L^p}}{\lambda}\right)^{q_1} \frac{d\lambda}{\lambda} \lesssim ||f||^q_{L^p}.$$

This completes the proof. □

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