LINEAR WAVES ON EXTREMAL REISSNER-NORDSTRÖM SPACETIMES (IN SPHERICAL SYMMETRY)

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Abstract. In the series of works [Are11b, Are11a], Aretakis showed that, for generic linear waves evolving on an extremal Reissner-Nordström background, ingoing null derivatives do not decay on the event horizon, or may even grow polynomially. The seed of the *Aretakis instability* stems from the conservation of the *Aretakis charges* along the event horizon. To pass to a growth mechanism, one must complement the conservation laws with a *weak stability* in the form of suitable energy estimates and decay estimates. We provide a simplified account of these results by restricting to the spherically-symmetric setting.

Contents

1. Introduction

The *Reissner-Nordström* space-times arise as an explicit family of solutions to the Einstein-Maxwell equations describing general relativity coupled with electromagnetism. In the Schwarzchild coordinates (t, r), the metrics take the form

$$\mathbf{g}_{M,e} := -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 \not{\mathbf{g}},$$

where $M > 0$ is a positive parameter known as the *mass* and $e \in \mathbb{R}$ is a real parameter known as the *charge*. Under the evolution of the Einstein-Maxwell equations,

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= 2F_{\mu\alpha}F^{\alpha}_{\ \nu} - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}, \\ \nabla_\mu F^{\mu\nu} &= 0. \end{aligned} \tag{E-M}$$

eq:EinsteinMa

the *sub-extremal* sub-family $|e| < M$ is expected to be stable under the *black hole stability conjecture*, while members of the *super-extremal* sub-family $|e| > M$, in view of their naked singularities, are expected to be unstable under the *weak cosmic censorship conjecture*. At the boundary between these two lie the *extremal* members $|e| = M$,

$$\mathbf{g} := -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 \not{\mathbf{g}},$$

which are expected to share aspects of both stability and instability of their non-extremal neighbors in the Reissner-Nordström family.

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In this note, we aim to give a primary exposition into the study of the extremal Reissner-Nordström space-times, which, in view of spherical-symmetry, are the simplest examples within the general class of extremal black holes. For a survey of the broader subject, we refer the interested reader to the brief by Aretakis [Are18] and the recent essay by Dafermos [Daf25]. Keeping the end goal of studying non-linear stability of extremal black holes in mind, we begin by telling an even simpler story – that of the linear wave equation on an extremal Reissner-Nordström exterior background,

$$\Box_{\mathbf{g}}\phi = 0. \tag{W}$$ eq:LW

In particular, we will follow the seminal series of papers by Aretakis [Are11b, Are11a] concerning the stability and instability of these linear scalar perturbations ϕ of extremal Reissner-Nordström $\mathbf{g}$. To emphasise the central ideas and minimise the minor technicalities, we restrict our exposition to the spherically-symmetric setting, in which case the main results may be stated as follows,

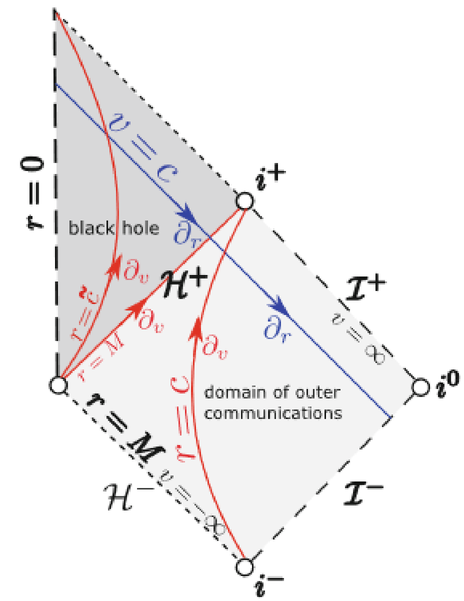

FIGURE 1. Penrose diagram of extremal Reissner-Nordström space-time and the ingoing Eddington-Finkelstein coordinates (v, r) [Are18]. We denote $T := \partial_v$ the Killing vector field and $Y := \partial_r$ the ingoing null vector field.

thm:stability **Theorem 1.1** (Weak stability of extremal Reissner-Nordström). *For solutions ϕ to the linear wave equation (W) on an extremal Reissner-Nordström background, we have the energy decay,*

$$\int_{\Sigma_\tau} |T\phi|^2 + D\cdot|Y\phi|^2 \lesssim \tau^{-2}, \tag{1.1}$$ eq:energydeca

and the pointwise decay,

$$|\phi| \lesssim \begin{cases} \tau^{-3/5}, & \textit{near the horizon,} \\ \tau^{-1}, & \textit{far from the horizon,} \end{cases} \tag{1.2}$$ eq:pointwisec

Theorem 1.2 (Aretakis instability in extremal Reissner-Nordström). *For generic solutions ϕ to the linear wave equation (W) on an extremal Reissner-Nordström background, ingoing null derivatives of ϕ do not decay on the event horizon $\mathcal{H}^+$. More precisely, we have*

$$Y^{m+1}T^m\phi \longrightarrow C\cdot H_0[\phi] \qquad \textit{as } \tau\to\infty \tag{1.3}$$

for some constant $C \neq 0$, and, for $k \geq 2$,

$$\left|Y^{m+k}T^m\phi\right| \gtrsim |H_0[\phi]|\,\tau^{k-1} \qquad \textit{as } \tau\to\infty. \tag{1.4}$$

1.1. **Outline of the strategy.** As a basic caricature of the instability mechanism, rewriting the linear wave equation (W) in ingoing Eddington-Finkelstein coordinates (v, r) for spherically-symmetric solutions yields the following equation on the horizon,

$$\partial_v \left(\partial_r \phi + \frac{1}{M} \phi \right)_{|\mathcal{H}^+} = 0,$$

i.e. the (zero-th) *Aretakis charge*,

$$H_0[\phi] := \partial_r \phi + \frac{1}{M} \phi,$$

is conserved along the event horizon $\mathcal{H}^+$. On the one hand, the weak stability of extremal Reissner-Nordström tells us that the lower-order term ϕ decays along the horizon, while on the other hand, $H_0[\phi] \neq 0$ for generic solutions to the wave equation. It follows then from the conservation law that the ingoing null derivative $\partial_r \phi$ *does not decay* along the horizon. By commuting the wave equation with ∂_r, we can likewise derive an evolution equation for $\partial_r^2 \psi$ along the horizon

$$\partial_v \left(\partial_r^2 \phi \right)_{|\mathcal{H}^+} = -2M^{-2} H_0[\phi] + \text{decaying} \tag{1.5}$$

so upon integrating in v we conclude linear growth in v. In fact we can show in general

$$|Y^k \psi| \gtrsim |H_0[\phi]| v^{k-1}$$

on the horizon for any $k \geq 1$ and v large

To prove decay estimates, we resort to the r^p-method of Dafermos-Rodnianski [DR10], which combines a hierarchy of weighted integrated energy decay estimates, namely, drawing from the later work of Angelopoulos-Aretakis-Gajic [AAG18],

$$\int_{\tau_1}^{\tau_2} \int_{C_\tau} r^{p-1} |\partial_v \psi|^2 \lesssim \int_{C_{\tau_1}} r^p |\partial_v \psi|^2 + \text{lower order}, \tag{1.6}$$

eq:rpfar

$$\int_{\tau_1}^{\tau_2} \int_{\underline{C}_\tau} (r - M)^{3-p} \frac{|\partial_u \psi|^2}{-\partial_u r} \lesssim \int_{\underline{C}_{\tau_1}} (r - M)^{2-p} \frac{|\partial_u \psi|^2}{-\partial_u r} + \text{lower order} \tag{1.7}$$

eq:rpnear

with a dyadic pigeonholing argument to prove energy decay. One can pass these energy decay arguments to pointwise decay estimates via Sobolev embedding.

fig:AAG

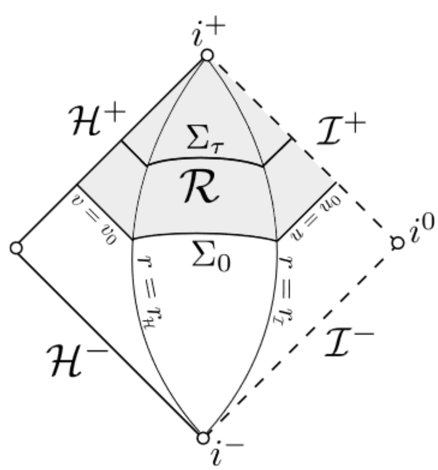

FIGURE 2. Spacelike-null foliation Σ_τ used in [AAG18] which is invariant under the Couch-Torrence conformal symmetry $(u, v) \mapsto (v, u)$. The initial hypersurface Σ_0 consists of $t = 0$ in a region away from both $\mathcal{H}^+$ and $\mathcal{I}^+$ and null hypersurfaces $u = u_0$ and $v = v_0$ in the regions near $\mathcal{H}^+$ and $\mathcal{I}^+$ respectively.

The r^p-hierarchy (1.6) near null infinity $\mathcal{I}^+$ is standard; the novel observation of [Are11a] was that a version of the similar hierarchy (1.7) holds near the horizon $\mathcal{H}^+$. Indeed, this can be expected in view of the Couch-Torrence inversion, which is a discrete conformal symmetry of extremal Reissner-Nordström space-times which "inverts" the domain of outer communication, exchanging $\mathcal{H}^+$ with $\mathcal{I}^+$, given in double null coordinates as

$$\Phi(u,v) = (v,u).$$

We only mention this correspondence as motivation for the r^p-hierarchy near the horizon, which is necessary for decay estimate due to the degeneracy of redshift. See [Are18, Chapter 2.1.4] and [AAG18] for further discussion on this perspective.

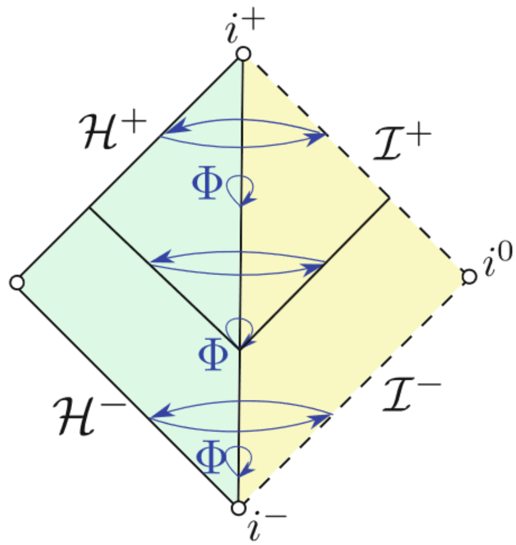

FIGURE 3. The Couch-Torrence inversion is a discrete conformal symmetry of extremal Reissner-Nordström which "inverts" the domain of outer communication, exchanging $\mathcal{H}^+$ with $\mathcal{I}+$. fig:CT

2. PRELIMINARIES

2.1. Geometry of extremal Reissner-Nordström spacetime. The Reissner-Nordström metric with mass $M \geq 0$ and charge $e \in \mathbb{R}$ is wrriten in Schwarzchild coordinates (t,r) as

$$\mathbf{g}_{M,e} = D(r)\,dt^2 + D(r)^{-1}\,dr^2 + r^2 \not{\mathbf{g}},$$

where $\not{\mathbf{g}}$ is the standard metric on the unit sphere $\mathbb{S}^2$ and the function $D(r)$ is given by

$$D(r) := 1 - \frac{2M}{r} + \frac{e^2}{r^2},$$

admitting two distinct roots at $r_\pm = M \pm \sqrt{M^2 - e^2}$. We say $\mathbf{g}_{M,e}$ is *sub-extremal* if $|e| < M$, *super-extremal* if $|e| > M$, and *extremal* if $|e| = M$. In this last case, the function $D(r)$ factors as

$$D(r) = \left(1 - \frac{M}{r}\right)^2,$$

admitting a single double root at $r_\pm = M$.

This metric is spherically-symmetric in the sense that $\mathrm{SO}(3)$ acts by isometry. When $|e| \leq M$, the metric may be analytically extended to describe a black hole space-time of which these coordinates cover the domain of outer communication for $(t,r) \in \mathbb{R} \times (r_+, \infty)$, where $r = r_+$ is the event horizon $\mathcal{H}^+$. To pass to coordinates which cover the event horizon $\mathcal{H}^+$, we introduce the *tortoise coordinate* r_* which arises as the solution to the ODE $\partial_r r^* = \frac{1}{D}$. Note that in the extremal case $|e| = M$, the tortoise coordinate is

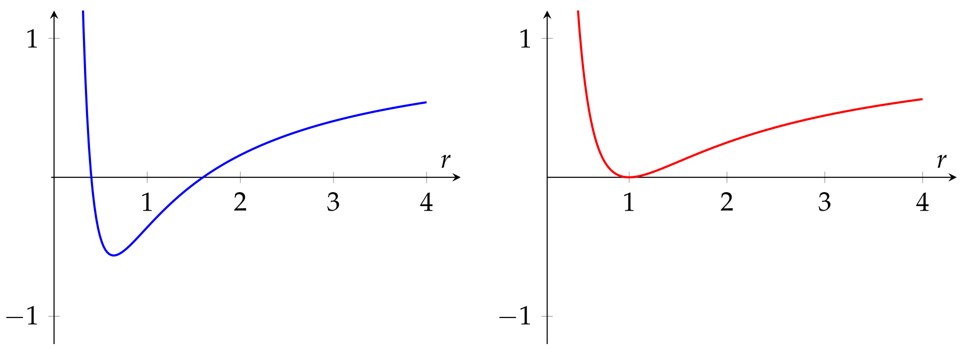

FIGURE 4. Comparison of $D(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}$ with mass $M = 1$ between sub-extremal charge $e = \frac{7}{10}$ on the left and extremal charge $e = 1$ on the right. The former admits two distinct roots $r = r_{\pm}$, while the latter has a double root at $r = M$

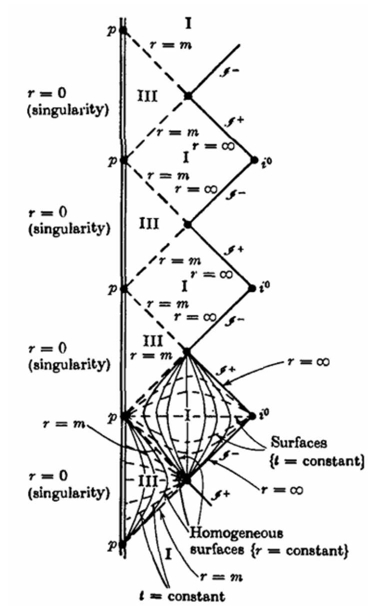

FIGURE 5. The maximal extension of extremal Reissner-Nordström space-time as depicted in Hawking-Ellis [HE23].

inverse linear as opposed to logarithmic in the non-extremal case $|e| < M$. The metric with respect to the coordinates (t, r^*) takes the form

$$\mathbf{g}_{M,e} = -D(r)\, dt^2 + D(r)\, (dr^*)^2 + r^2 \not{\mathbf{g}}.$$

To extend beyond the event horizon, we introduce the advanced time coordinate $v = t + r^*$. In these *ingoing Eddington-Finkelstein coordinates* (v, r), the metric takes the form

$$\mathbf{g} := -D\, dv^2 + 2dvdr + r^2 \not{\mathbf{g}}.$$

In these coordinates, $T = \partial_v$ is time-translation Killing vector field, which takes the form $T = \partial_t$ in Schwarschild coordinates. In the extremal setting, it is time-like away from the horizon, and null at the horizon, while in the sub-extremal setting, it is space-like within the black hole region. The vector field $Y = \partial_r$ is past-directed null, transverse to $\mathcal{H}^+$, and translation invariant in the sense that $\mathcal{L}_T Y = 0$.

2.2. **The energy-momentum tensor.** Given a vector field V, we define its deformation tensor to be the Lie derivative of the metric with respect to V, i.e.

$$^{(V)}\pi_{\mu\nu} := (\mathcal{L}_V \mathbf{g})^{\mu\nu}.$$

The Lagrangian structure of the wave equation gives rise to the energy-momentum tensor,

$$\mathbf{T}_{\mu\nu}[\phi] := \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\mathbf{g}_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi,$$

which is a symmetric 2-tensor. For general functions, the divergence satisfies

$$\nabla^\mu \mathbf{T}_{\mu\nu}[\phi] = (\Box_{\mathbf{g}}\phi)\,\partial_\nu\phi, \tag{2.1}$$ eq:div

so in particular it is divergence-free when ϕ is a solution to the wave equation (W). A convenient method for obtaining energy identities is to contract the energy-momentum tensor with well-chosen vector fields, integrate over suitable domains and apply Stokes' theorem, which for a general $(0,1)$-current P_μ and region in space-time $\mathcal{R}$ states

$$\int_{\partial\mathcal{R}} P_\mu n^\mu_{\Sigma_0} = \int_{\mathcal{R}} \nabla^\mu P_\mu. \tag{2.2}$$ eq:stokes

Given the vector field V, define the associated 1- and 0-currents

$$\begin{aligned} ^{(V)}J_\mu[\phi] &:= \mathbf{T}_{\mu\nu}[\phi]V^\nu, \\ ^{(V)}K[\phi] &:= \mathbf{T}_{\mu\nu}[\phi]\,^{(V)}\pi^{\mu\nu}, \\ ^{(V)}E[\phi] &:= \left(\nabla^\mu \mathbf{T}_{\mu\nu}[\phi]\right)V^\nu = (\Box_{\mathbf{g}}\phi)(V\phi), \end{aligned}$$

which are related in view of the divergence identity (2.1) by

$$\nabla^\mu\left(^{(V)}J_\mu[\phi]\right) = {}^{(V)}K[\phi] + {}^{(V)}E[\phi]. \tag{2.3}$$ eq:currents

As a particular use of (2.3), the term $^{(V)}E[\phi]$ vanishes for solutions ϕ to the linear wave equation (W), while the term $^{(V)}K[\phi]$ vanishes for Killing vector fields V, giving rise to exact conservation laws for solutions to the wave equation. This is an instance of the well-known Noether's theorem.

2.3. **Hardy inequalities.**

Proposition 2.1 (First Hardy inequality). *For any sufficiently regular ϕ, we have*

$$\int_{\Sigma_\tau} \frac{1}{r^2}|\phi|^2 \lesssim \int_{\Sigma_\tau} D\cdot\left(|\partial_v\phi|^2 + |\partial_r\phi|^2\right) \tag{2.4}$$

Proof. The usual proof of Hardy's inequality a lá completing the square and integrating-by-parts carries through. □

3. Conservation laws on the event horizon

To set expectations on which types of energy estimates and decay estimates are admissible up to the event horizon, let us discuss the conservation laws along the horizon which arise in the extremal setting. Throughout this section we work in ingoing Eddington-Finkelstein coordinates (v, r), in which case the wave operator on the extremal Reissner-Nordström background takes the form

$$\Box_{\mathbf{g}} = D\cdot\partial_r^2 + 2\partial_v\partial_r + \frac{2}{r}\cdot\partial_v + R\cdot\partial_r\phi,$$

where the coefficients D and R are given by

$$\begin{aligned} D(r) &= \left(1 - \frac{M}{r}\right)^2, \\ R(r) &= D'(r) + \frac{2D(r)}{r}. \end{aligned}$$

Observe that $D(M) = D'(M) = 0$, so in particular both coefficients vanish on the event horizon $\mathcal{H}^+$.

Furthermore, the vector field ∂_v is tangential to the event horizon $r = M$, so the wave equation (W) on the horizon reduces to

$$\partial_v \left(\partial_r \phi + \frac{1}{M}\phi \right)_{|\mathcal{H}^+} = 0,$$

i.e. the zero-th Aretakis charge is conserved along the event horizon,

Proposition 3.1 (Conservation of the zero-th Aretakis charge)**.** *Let ϕ be a spherically-symmetric solution to the linear wave equation (W) on the extremal Reissner-Nordström background. Then the quantity*

$$H_0[\phi] := \partial_r \phi + \frac{1}{M}\phi$$

is conserved along the event horizon $\mathcal{H}^+$.

It follows that, for generic spherically-symmetric initial data in the sense that $H_0[\phi[0]] \neq 0$ at the horizon, we *cannot* prove decay estimates for *both* ϕ and $\partial_r\phi$ up to the event horizon. For higher-order derivatives, we can likewise derive suitable identities by commuting into the wave equation (W). As a primer, let us derive an identity for the second-order ingoing null derivative $\partial_r^2\phi$. Differentiating the wave equation (W) with respect to ∂_r gives

$$\begin{aligned}
\partial_r \Box_{\mathbf{g}} \phi &= \partial_r \left(D \cdot \partial_r^2 \phi + 2\partial_v \partial_r \phi + \frac{2}{r} \cdot \partial_v \phi + R \cdot \partial_r \phi \right) \\
&= D \cdot \partial_r^3 \phi + D' \cdot \partial_r^2 \phi + 2\partial_r^2 \partial_v \phi + \frac{2}{r} \cdot \partial_r \partial_v \phi - \frac{2}{r^2} \partial_v \phi + R \cdot \partial_r^2 \phi + R' \cdot \partial_r \phi \\
&= 2\partial_r^2 \partial_v \phi + \frac{2}{M} \cdot \partial_r \partial_v \phi - \frac{2}{M^2} \partial_v \phi + \frac{2}{M^2} \partial_r \phi \qquad \text{evaluated on the horizon } \mathcal{H}^+.
\end{aligned}$$

Rearranging, we obtain the following identity

$$\partial_v \left(\partial_r^2 \phi + \frac{1}{M}\partial_r \phi + \frac{1}{M^2}\phi \right) = -\frac{1}{M^2} H_0[\phi] + M^2 \phi$$

On the other hand, writing instead the wave equation (W) as

$$H_0[\partial_v \phi] = \left(\partial_r + \frac{1}{r} \right) \partial_v \phi|_{\mathcal{H}^+} = 0,$$

and also commuting the tangential ∂_v-derivatives into the equation, we see that the zero-th order Aretakis charge vanishes for all tangential derivatives $H_0[\partial_v^k \phi] = 0$ for $k \geq 1$.

Proposition 3.2 (Identities along the horizon)**.**

$$\partial_r^{m+1} \partial_v^m \phi + \sum_{j=0}^{m} \lambda_j \cdot \partial_v^j \phi = C \cdot H_0[\phi] \tag{3.1}$$

Proof. It follows from the product rule that

$$\begin{aligned}
\partial_r^k \Box_{\mathbf{g}} \phi &= \partial_r^k \left(D \cdot \partial_r^2 \phi + 2\partial_v \partial_r \phi + \frac{2}{r} \cdot \partial_v \phi + R \cdot \partial_r \phi \right) \\
&= D \cdot \partial_r^{k+2} \phi + 2\partial_r^{k+1} \partial_v \phi + \frac{2}{r} \cdot \partial_r^k \partial_v \phi + R \cdot \partial_r^{k+1} \phi \\
&\quad + \sum_{i=1}^{k} \binom{k}{i} \partial_r^i D \cdot \partial_r^{k-i+2} \psi + \sum_{i=1}^{k} \binom{k}{i} \partial_r^i \frac{2}{r} \cdot \partial_r^{k-i} \partial_v \phi \\
&\quad + \sum_{i=1}^{k} \binom{k}{i} \partial_r^i R \cdot \partial_r^{k-i+1} \phi.
\end{aligned}$$

Observe that in the $k = 0$ case we could rewrite $\partial_r \partial_v \phi$ in terms of $\partial_v \phi$. Proceeding inductively gives the result. □

Thus we can rewrite $Y^{m+1} T^m \phi$ in terms of $H_0[\phi]$ and lower-order terms which we aim to prove decay along the horizon.

4. Integrated local energy decay

It will be convenient to introduce various integrated local energy decay estimates to handle various space-time integrals which emerge when correcting the energy fluxes. The strategy will be to correct the fluxes judiciously *near* the horizon and *away* from the horizon, and handle the ensuing error terms in the transition region via the integrated local energy decay.

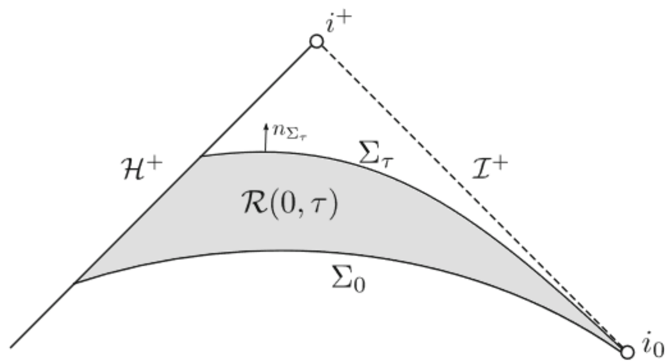

Figure 6. Fix a space-like hypersurface Σ_0, we obtain a folation Σ_τ of the space-time by flowing with respect to the vector field T.

We work in tortoise coordinates (t, r^*), making the ansatz $X = f(r^*)\partial_{r^*}$. Then

$$^{(X)}K[\phi] = F_{tt} \cdot |\partial_t \phi|^2 + F_{r^*r^*} \cdot |\partial_{r^*}\phi|^2,$$

where

$$F_{tt} = \frac{f'}{2D} + \frac{f}{r},$$
$$F_{r_*r_*} = \frac{f'}{2D} - \frac{f}{r}.$$

Proposition 4.1 (Morawetz multiplier in spherical symmetry)**.** *Let ϕ be a spherically symmetric solution to the wave equation (W) on the extremal Reissner-Nordström background, then the Morawetz multiplier $X = -\frac{1}{r^3}\partial_{r^*}$ gives*

$$^{(X)}K = \frac{1}{r^4}|\partial_t \phi|^2 + \frac{5}{r^4}|\partial_{r_*}\phi|^2. \tag{4.1}$$

Lemma 4.2 (Control of boundary terms)**.** *Let $X = f\partial_{r^*}$ for f bounded, and let Σ be space-like or null, then*

$$\left| \int_\Sigma {}^{(X)}J_\mu[\phi]n^\mu \right| \lesssim \int_\Sigma {}^{(T)}J[\phi]n^\mu. \tag{4.2}$$

Proof. Compute and use the triangle inequality. □

Theorem 4.3 (Integrated local energy decay in spherical symmetry)**.** *Let ϕ be a spherically symmetric solution to the wave equation (W) on the extremal Reissner-Nordström background, then for any $\tau_1 < \tau_2$, we have*

$$\int_{\mathcal{R}_{[0,\tau]}} \frac{1}{r^4}|\partial_t \phi|^2 + \frac{1}{r^4}|\partial_{r_*}\phi|^2 \lesssim \int_{\Sigma_0} {}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_0} \tag{4.3}$$

eq:iled

sec:redshift

5. Energy estimates

5.1. Degenerate energy monotonicity formula.

Using the Killing vector field T, we can obtain an exact conservation law for linear waves. Indeed, using Stokes' theorem and the divergence identity (2.1) on the region $\mathcal{R}_{[0,\tau]}$, we obtain

$$\int_{\Sigma_\tau} {}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}^+} {}^{(T)}J_\mu[\phi]n^\mu_{\mathcal{H}^+} = \int_{\Sigma_0} {}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_0}. \tag{5.1}$$

Since T and $n^\mu_{\mathcal{H}^+}$ are null, the second boundary term on the left is non-negative. In fact, one may compute ${}^{(T)}J_\mu[\phi]n^\mu_{\mathcal{H}^+} = |\partial_v\phi|^2$ on the event horizon. This yields the following monotonicity formula,

Proposition 5.1 (Boundedness of T-energy)**.** *Let ϕ be a solution to the wave equation (W) on the extremal Reissner-Nordström background, then*

$$\int_{\Sigma_\tau} {}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_\tau} \leq \int_{\Sigma_0} {}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_0}. \tag{5.2}$$

However, since T degenerates to null approaching the horizon, and in fact $\mathbf{g}(T,T) = -\frac{1}{2}D$ which vanishes to second-order at the horizon, the T-energy along the foliation takes the form

$${}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_\tau} \sim |\partial_v\phi|^2 + D\cdot|\partial_r\phi|^2.$$

In particular, the control over the ingoing null derivative $\partial_v\phi$ degenerates along the horizon.

5.2. **Degeneracy of redshift.** To handle the degeneracy near the horizon, it is standard to use the redshift vector field N introduced by Dafermos-Rodnianski [DR09]. The basic strategy is to choose N to be a time-like vector field, to first approximation taking in double null coordinates (u,v) the form

$$N \approx \frac{\partial_u}{-(\partial_u r)}$$

satisfying

$${}^{(N)J_\mu[\phi]}n^\mu_{\Sigma_\tau} \sim {}^{(N)}K[\phi].$$

In particular, the bulk term ${}^{(N)}K[\phi]$ is non-negative. This construction relies crucially on the positivity of *surface gravity* κ, which measures the strength of the redshift effect. Precisely, since T is null, Killing, and tangential along the horizon, the surface gravity is defined by

$$\nabla_T T = \kappa T.$$

Assuming $\kappa > 0$, one can show both non-degenerate energy boundedness and integrated local energy decay,

$$\int_{\Sigma_\tau} {}^{(N)}J_\mu[\phi]n^\mu_{\Sigma_\tau} + \int_{\mathcal{A}} {}^{(N)}J_\mu[\phi]n^\mu \lesssim \int_{\Sigma_0} {}^{(N)}J_\mu[\phi]n^\mu_{\Sigma_0} \tag{5.3}$$

see [DR09] and also the lecture notes [ERSW13, Lectures on black holes and linear waves].

In the extremal setting, the surface gravity vanishes, in which case one cannot choose N such that the bulk term is non-negative on the horizon. Indeed, taking the ansatz $N = N^v\partial_v + N^r\partial_r$ we compute

$${}^{(N)}K[\phi] = F_{vv}\cdot|\partial_v\phi|^2 + F_{rr}\cdot|\partial_r\phi|^2 + F_{vr}\cdot(\partial_v\phi\partial_r\phi)$$

where the coefficients are given by

$$\begin{aligned} F_{vv} &= \partial_r N^v, \\ F_{rr} &= D\cdot\left(\frac{\partial_r N^r}{2} - \frac{N^r}{r}\right) - \frac{N^r D'}{2}, \\ F_{vr} &= D\cdot\partial_r N^v - \frac{2N^r}{r}. \end{aligned}$$

Since N is necessarily time-like, we need $N^r(M) \neq 0$ since $\mathbf{g}(N,N) = -D\cdot|N^v|^2 + 2N^vN^r$. It follows that $F_{rr}(M) = 0$, wherease the sign-indefinite coefficient F_{vr} is non-zero. It follows that the bulk term is linear in $\partial_r\phi$ on the horizon and manifestly sign indefinite.

5.3. **Generalised currents.** To rectify this issue, we introduce an extra Lagrangian term $w \cdot \phi \nabla_\mu \phi$ in the current; this corresponds to multiplying the wave equation by $w\phi$. Define the generalised currents by

$$
\begin{aligned}
{}^{(N,w)}J_\mu[\phi] &:= {}^{(N)}J_\mu[\phi] + w \cdot \psi \nabla_\mu \phi, \\
{}^{(N,w)}K[\phi] &:= {}^{(N)}K[\phi] + \nabla^\mu w \cdot \phi \nabla_\mu \phi + w \cdot \nabla^\alpha \phi \nabla_\alpha \phi.
\end{aligned}
$$

Then for solutions to the wave equation, we have the divergence identity

$$
\nabla^\mu \left({}^{(N,w)}J_\mu[\phi] \right) ={}^{(N,w)} K[\phi]. \tag{5.4}
$$

eq:div2

Taking

$$
w = \frac{N^r(M)}{M} = -\frac{1}{2}, \qquad N = 16r\partial_v + \left(\frac{3}{2}r + M \right) \partial_r,
$$

we obtain

Proposition 5.2 (Coercivity of N-bulk term)**.** *Let ϕ be a solution to the wave equation (W) on the extremal Reissner-Nordström background, then*

$$
{}^{(N,-\frac{1}{2})}K[\phi] \gtrsim |\partial_v \phi|^2 + \sqrt{D} \cdot |\partial_r \phi|^2 \tag{5.5}
$$

in a neighborhood of the event horizon $\mathcal{A} = \{M \leq r \leq \frac{9}{8}M\}$

Proof. We compute

$$
{}^{N,w}K[\phi] = F_{vv} \cdot |\partial_v \phi|^2 + (F_{rr} + wD) \cdot |\partial_r \phi|^2 + (F_{vr} + 2w) \cdot (\partial_v \phi \partial_r \phi).
$$

Then

$$
\begin{aligned}
F_{vv} &= 16, \\
F_{rr} + wD &= F_{rr} - \frac{1}{2}D, \\
F_{vr} - 1 &= 16D + 2\sqrt{D}.
\end{aligned}
$$

Note the second coefficient is positive since the leading term D' has a positive coefficient. Suitable application of Cauchy-Schwarz implies that the sign-indefinite term F_{vr} can be controlled by the sign-definite terms. □

5.4. **Cut-off generalised currents.** Since the bulk term ${}^{(N,-\frac{1}{2})}K[\phi]$ cannot be chosen to be non-negative far away from the horizon, we simply modify the current by choosing appropriate cut-offs. To this end, we extend the redshift vector field by

$$
\begin{aligned}
N^v &= 1, \qquad r \geq \tfrac{8}{7}M, \\
N^r &= 0, \qquad r \geq \tfrac{8}{7}M.
\end{aligned}
$$

Then N is future-directed time-like, and we define the cut-off currents by

$$
\begin{aligned}
{}^{(N,\delta,-\frac{1}{2})}J_\mu[\phi] &:= {}^{(N)}J_\mu[\phi] - \frac{1}{2}\delta \phi \nabla_\mu \phi, \\
{}^{(N,\delta,-\frac{1}{2})}K[\phi] &:= \nabla^\mu \left({}^{(N,\delta,-\frac{1}{2})}J_\mu[\phi] \right)
\end{aligned}
$$

where δ is a cut-off satisfying $\delta \equiv 1$ for $M \leq r \leq \frac{9}{8}M$ and $\delta \equiv 0$ for $\frac{8}{7}M < r < \infty$. We arrive at the divergence identity,

$$
{}^{(N,\delta,-\frac{1}{2})}K[\phi] = \nabla^\mu \left({}^{(N,\delta,-\frac{1}{2})}J_\mu[\phi] \right) \tag{5.6}
$$

eq:div3

By Hardy's inequality, one can verify

Lemma 5.3 (Comparable N-energies)**.** *For sufficiently regular ϕ, we have*

$$
\int_{\Sigma_\tau} {}^{(N,\delta,-\frac{1}{2})}J_\mu[\phi] n^\mu \sim \int_{\Sigma_\tau} {}^{(N)}J_\mu[\phi] n^\mu_{\Sigma_\tau}. \tag{5.7}
$$

Proposition 5.4 (Uniform boundedness of N-energy)**.** *Let ϕ be a solution to the wave equation (W) on the extremal Reissner-Nordström background, then*

$$\int_{\Sigma_\tau} {}^{(N)}J_\mu[\phi]n^\mu_{\Sigma_\tau} \lesssim \int_{\Sigma_0} {}^{(N)}J_\mu[\phi]n^\mu_{\Sigma_0}. \tag{5.8}$$ eq:Nbound

Proof. By Stokes' theorem (2.2) and the divergence identity (5.6), we have

$$\int_{\Sigma_\tau} {}^{(N,\delta,-\frac{1}{2})}J_\mu[\phi]n^\mu + \int_{\mathcal{R}} {}^{(N,\delta,-\frac{1}{2})}K[\phi] + \int_{\mathcal{H}^+} {}^{(N,\delta,-\frac{1}{2})}J_\mu[\phi]n^\mu = \int_{\Sigma_0} {}^{(N,\delta,-\frac{1}{2})}J_\mu[\phi]n^\mu.$$

The space-time integral has the correct sign close to the horizon, vanishes away from the horizon, while the intermediate region can be handled by a degenerate integrated local energy decay (4.3). The boundary term on the horizon has the correct sign. □

Proposition 5.5 (Integrated local energy decay)**.** *Let ϕ be a solution to the wave equation (W) on the extremal Reissner-Nordström background, then for any $\tau_1 < \tau_2$, we have*

$$\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} {}^{(N,-\frac{1}{2})}K[\phi] \lesssim \int_{\Sigma_{\tau_1}} {}^{(N)}J_\mu[\phi]n^\mu_{\Sigma_{\tau_1}} \tag{5.9}$$

6. Energy decay via r^pmethod

To obtain energy decay estimates, we employ the r^p-method of Dafermos-Rodnianski [DR10] in neighborhoods of future null infinity $\mathcal{I}^+$ *and* the event horizon $\mathcal{H}^+$. As the argument near $\mathcal{I}^+$ is standard, applying to a larger class of spacetimes, we will only give a brief account as a primer for the argument near $\mathcal{H}^+$, which is a novelty of the extremal Reissner-Nordström background.

fig:spacenull

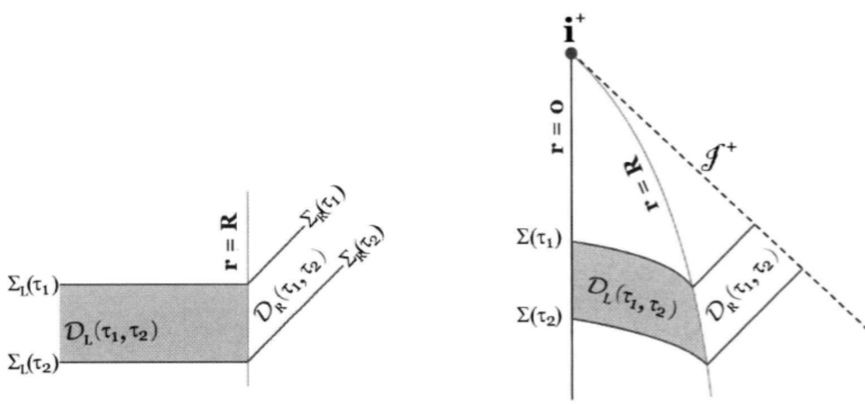

Figure 7. Spacelike-null folation of Minkowski space, taken from [Kla11]; on the left is $\mathbb{R}^{1+3}$ and on the right is the compactified Penrose diagram.

To capture the late-time asymptotics of linear waves, it is convenient to work with a spacelike-null foliation of spacetime, see Figures 6 and 6, rather than a spacelike foliation. As an extreme example, for compactly supported initial data in 3 spatial dimensions, the strong Huygen's principle implies that the solution remains supported within the *wave zone*. A space-like foliation Σ_τ will intersect the wave zone for all τ and thus cannot decay, while the space-like null foliation $\widetilde{\Sigma}_\tau$ does not see the solution at all for large τ. Working in double null coordinates (u,v), the multiplier $V = r^p\partial_v$ yields the following hierarchy of integrated energy estimates for the radiation field $\psi := r\phi$,

Proposition 6.1 ($\mathcal{I}^+$-localised r^p-hierarchy)**.** *For $p < 3$, the spherically-symmetric solutions ϕ to the wave equation (W) on the extremal Reissner-Nordström background satisfy the following r^p-weighted estimates,*

$$\int_{\widetilde{\mathcal{N}}_{\tau_2}} r^p\frac{|\partial_v\psi|^2}{r^2} + \int_{\widetilde{\mathcal{D}}_{[\tau_1,\tau_2]}} r^{p-1}(p+2)\frac{|\partial_v\psi|^2}{r^2} \lesssim \int_{\widetilde{\Sigma}_{\tau_1}} {}^{(T)}J_\mu[\psi]n^\mu_{\widetilde{\Sigma}_{\tau_1}} + \int_{\widetilde{\mathcal{N}}_{\tau_1}} r^p\frac{|\partial_v\psi|^2}{r^2}. \tag{6.1}$$ eq:rp1

pacelikenull2

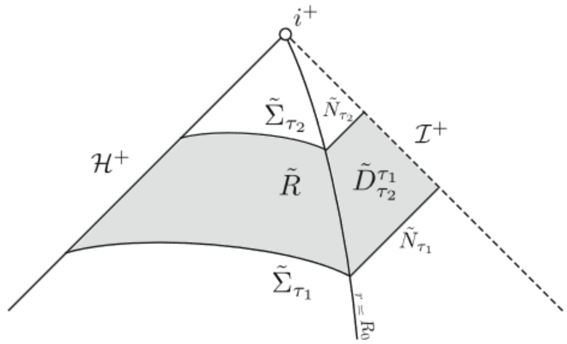

FIGURE 8. The spacelike-null foliation $\widetilde{\Sigma}_\tau$ in [Are11b] of extremal Reissner-Nordström, which is used to derive the r^p-hierarchy of integrated energy estimates.

Given an energy which is uniformly bounded, $\mathcal{E}[\phi[\tau]] \lesssim \mathcal{E}[\phi[0]]$, to prove decay it suffices to estimate the energy on a *discrete* sequence of dyadic times $\tau_n \sim C^n$. And indeed, using the pigeonhole principle or a variant of the mean value theorem, one can leverage the hierarchy of integrated energy estimates (6.1) along with uniform boundedness of the r^p-weighted energies to obtain such a sequence such that

$$\int_{\widetilde{\mathcal{N}}_{\tau_n}} r^{p-2}\frac{|\partial_v\psi|^2}{r^2} \lesssim \frac{1}{\tau_{n-1}}\int_{\widetilde{\mathcal{N}}_{\tau_{n-1}}} r^{p-1}\frac{|\partial_v\psi|^2}{r^2} \lesssim \frac{1}{\tau_{n-1}}\frac{1}{\tau_{n-2}}\int_{\widetilde{\mathcal{N}}_0} r^p\frac{|\partial_v\psi|^2}{r^2}.$$

As the sequence of times was chosen dyadically, this two-fold appliation of the hierarchy immediately implies a τ^{-2} decay rate. For more details, see [Are11b, Section 5], while for the general scheme, see the original paper of Dafermos-Rodnianski [DR10] or the lecture notes of Klainerman [Kla11, Section 9].

6.1. ***T*-*P*-*N*-hierarchy in a neighborhood of $\mathcal{H}^+$.** In view of the Couch-Torrence correspondence between $\mathcal{I}^+$ and $\mathcal{H}^+$ (see Figure 3 and the discussion therein), one might expect that the r^p-hierarchy near future null infinity corresponds to an analogous $(r-M)^{-p}$-hierarchy near the event horizon. We will not explicitly adopt this perspective, as it is technically unnecessary for our purposes, however we point the interested reader to [AAG18] which studied precise late-time asymptotics using a Couch-Torrence invariant spacelike-null foliation as depicted in Figure 1.1.

Recall that in Section 5, the stationary vector field T, which is null on the horizon and thus does not see any of the redshift effect, yielded a *degenerate* energy, while the redshift vector field N yielded a *non-degenerate* energy. The key idea of Aretakis [Are11a, Section 5.4] was to introduce a vector field P which captures less of the degenerate redshift effect near the horizon N but more when compared to T, in the sense that we have the following hierarchy of energies,

$$\begin{aligned}
{}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_\tau} &\sim |\partial_v\phi|^2 + D\cdot|\partial_r\phi|^2,\\
{}^{(P)}J_\mu[\phi]n^\mu_{\Sigma_\tau} &\sim |\partial_v\phi|^2 + \sqrt{D}\cdot|\partial_r\phi|^2,\\
{}^{(N)}J_\mu[\phi]n^\mu_{\Sigma_\tau} &\sim |\partial_v\phi|^2 + |\partial_r\phi|^2.
\end{aligned}$$

To first approximation, one should think of the P-vector field as time-like in the domain of outer communication and degenerating to null in a linear fashion as one approaches the event horizon,

$$P \approx T - \sqrt{D}\cdot Y$$

Proposition 6.2 (Microscopic T-P-N-hierarchy). *There exists a T-invariant time-like vector field P such that*

$$ {}^{(T)}J_\mu[\phi]n^\mu_\Sigma \lesssim {}^{(P)}K[\phi], \tag{6.2}$$ eq:TPmicro

and

$$ {}^{(P)}J_\mu[\phi]n^\mu_\Sigma \lesssim {}^{(N,\delta,-\frac{1}{2})}K[\phi], \tag{6.3}$$ eq:PNmicro

in an appropriate neighborhood $\mathcal{A}$ of the horizon.

Proof. Let us now turn to the derivation of the P-vector field, taking the ansatz $P = P^v\partial_v + P^r\partial_r$ for coefficients P^v and P^r to be determined later. Fix radii r_0 and r_1 between the horizon and photon sphere $M < r_0 < r_1 < 2M$, we divide our analysis between the region near the horizon $r \le r_0$ and the region far from the horizon $r \ge r_1$, and smoothly interpolate between the two. Away from the horizon, we only care for causality, so we take $P^v = 1$ and $P^r = 0$. It remains to choose P near the horizon to satisfy (6.2)-(6.3).

We would like the vector field P to be time-like and to capture the redshift effect in a sense between T and N. To that end, we take $P^r := -\sqrt{D}$, in which case the 0-current takes the form

$$^{(P)}K[\phi] = F_{vv}\cdot|\partial_v\phi|^2 + F_{rr}\cdot|\partial_r\phi|^2 + F_{vr}\cdot(\partial_v\phi\partial_r\phi)$$

where the coefficients of the sign-definite terms are

$$F_{vv} = \partial_r P^v,$$
$$F_{rr} = D\cdot\left(\frac{M}{2r^2} + \frac{\sqrt{D}}{r}\right),$$

and the coefficient of the sign-indefinite term is

$$F_{vr} = \sqrt{D}\left(\sqrt{D}\cdot\partial_r P^v + \frac{2}{r}\right).$$

We need the former to control the latter. Using Cauchy's inequality on F_{vr} yields

$$F_{vr} \le \varepsilon D + \frac{1}{\varepsilon}\left(\sqrt{D}\cdot\partial_r P^v + \frac{2}{r}\right)^2.$$

From our computation we see that $F_{rr} \sim D$ near the horizon, so choosing $\varepsilon \ll 1$ this term in the 0-current is favourable for controlling the first term on the right. On the other hand, in view of the degeneracy $\sqrt{D(M)} = 0$ at the horizon, we can choose the coefficient P^v such that F_{vv} dominates the second term on the right, i.e. $\frac{1}{\varepsilon}(\sqrt{D}\cdot\partial_r P^v + \frac{2}{r})^2 < \partial_r P^v$.[1] Collecting these judicious choices of coefficients and parameters, we see that the 0-current of P is comparable to the T-energy,

$$^{(P)}K[\phi] \sim |\partial_v\phi|^2 + D\cdot|\partial_r\phi|^2 \sim {}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_\tau},$$

furnishing (6.2).

Near the horizon, we compute $-\mathbf{g}(P,P) \sim \sqrt{D}$, so the P-energy takes the form

$$^{(P)}J_\mu[\phi]n^\mu_{\Sigma_\tau} \sim |\partial_v\phi|^2 + \sqrt{D}\cdot|\partial_r\phi|^2 \sim {}^{(N,\delta,-\frac{1}{2})}K[\phi],$$

which proves (6.3). $\square$

With these at hand, we are now ready to establish the integrated T-P-N-hierarchy. Let us recall boundedness of T-energy and N-energy; we also have boundedness of the P-energy,

Proposition 6.3 (Uniform boundedness of P-energy)**.** *Let ϕ be a solution to the wave equation (W) on the extremal Reissner-Nordström background, then*

$$\int_{\tilde{\Sigma}_\tau} {}^{(P)}J_\mu[\phi]n^\mu_{\tilde{\Sigma}_\tau} \lesssim \int_{\tilde{\Sigma}_0} {}^{(P)}J_\mu[\phi]n^\mu_{\tilde{\Sigma}_0}. \tag{6.4}$$ eq:boundP

Proof. By Stokes' theorem (2.2) and the divergence identity (2.3), we have

$$\int_{\tilde{\Sigma}_\tau} {}^{(P)}J_\mu[\phi]n^\mu + \int_{\mathcal{H}^+} {}^{(P)}J_\mu[\phi]n^\mu + \int_{\mathcal{I}^+} {}^{(P)}J_\mu[\phi]n^\mu + \int_{\mathcal{R}} {}^{(P)}K = \int_{\tilde{\Sigma}_0} {}^{(P)}J_\mu[\phi]n^\mu.$$

Since P is time-like, the boundary integrals over $\mathcal{H}^+$ and $\mathcal{I}^+$ are non-negative. For the space-time integral, (6.2) implies $^{(P)}K$ is non-negative near the horizon $r \le r_0$, while by construction of P it vanishes far away from the horizon $r \ge r_1$. In the intermediate region $r_0 \le r \le r_1$, we can estimate it by the right-hand side using integrated local energy decay (4.3). $\square$

[1] We leave this as an exercise.

Proposition 6.4 ($\mathcal{H}^+$-localised T-P-N-hierarchy)**.** *Let ϕ be a spherically-symmetric solution to the wave equation (W) on the extremal Reissner-Nordström background, then*

$$\int_{\tau_1}^{\tau_2}\left(\int_{\widetilde{\Sigma}_\tau\cap\{M\leq r\leq r_0\}} {}^{(T)}J_\mu[\phi]n^\mu_{\widetilde{\Sigma}_\tau}\right)d\tau \lesssim \int_{\widetilde{\Sigma}_{\tau_1}} {}^{(P)}J_\mu[\phi]n^\mu_{\widetilde{\Sigma}_{\tau_1}} \tag{6.5}$$ eq:TP

and

$$\int_{\tau_1}^{\tau_2}\left(\int_{\widetilde{\Sigma}_\tau\cap\{M\leq r\leq r_0\}} {}^{(P)}J_\mu[\phi]n^\mu_{\widetilde{\Sigma}_\tau}\right)d\tau \lesssim \int_{\widetilde{\Sigma}_{\tau_1}} {}^{(N)}J_\mu[\phi]n^\mu_{\widetilde{\Sigma}_{\tau_1}} \tag{6.6}$$ eq:PN

in an appropriate neighborhood $\mathcal{A}$ of the horizon.

Proof. Using Stokes' theorem (2.2) on the divergence identity (2.3) and using boundedness (6.4) to handle the boundary terms, we have

$$\int_{\mathcal{A}} {}^{(P)}K \lesssim \int_{\Sigma_{\tau_1}} {}^{(P)}J_\mu[\phi]n^\mu_{\Sigma_0}.$$

Then integral inequality (6.5) follows from the microscopic inequality (6.2). Similarly, the integral inequality (6.6) follows from the divergence identity for the modified N-current ${}^{(N,\delta,-\frac{1}{2})}J_\mu[\phi]$, boundedness of the N-energy, and the microscopic inequality (6.3). □

6.2. **Energy decay via dyadic pigeonholing.** With the T-P-N-hierarchy of estimates localised near the horizon, we are ready to establish energy decay. Using the P-N estimate (6.6), boundedness of the P-energy (6.4), and the pigeonhole principle, we have a dyadic sequence τ_n such that

$$\int_{\Sigma_{\tau_n}\cap\mathcal{A}} {}^{(P)}J_\mu[\phi]n^\mu_{\Sigma_{\tau_n}} \lesssim \frac{1}{\tau_n}\int_{\Sigma_0\cap\mathcal{A}} {}^{(N)}J_\mu[\phi]n^\mu_{\Sigma_{\tau_n}} \tag{6.7}$$

By the mean value theorem, we can find $\tau_* \in [\tau_n, \tau_{n+1}]$ such that

$$\int_{\Sigma_{\tau_*}\cap\mathcal{A}} {}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_{\tau_*}} = \frac{1}{\tau_{n+1}-\tau_n}\int_{\tau_n}^{\tau_{n+1}}\int_{\Sigma_\tau\cap\mathcal{A}} {}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_\tau}. \tag{6.8}$$

By the T-P estimate,

$$\int_{\tau_n}^{\tau_{n+1}}\int_{\Sigma_\tau\cap\mathcal{A}} {}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_\tau} \lesssim \int_{\Sigma_{\tau_n}\cap\mathcal{A}} {}^{(P)}J_\mu[\phi]n^\mu_{\Sigma_{\tau_n}}. \tag{6.9}$$

Collecting the inequalities and using the dyadic property $\tau_n \sim \tau_{n+1} \sim \tau_{n+1}-\tau_n$, we conclude decay of the T-energy,

$$\int_{\Sigma_{\tau_{n+1}}\cap\mathcal{A}} {}^{(T)}J_\mu[\phi]n^\mu_{\Sigma_{\tau_{n+1}}} \lesssim \frac{1}{\tau^2_{n+1}}. \tag{6.10}$$

This completes the proof of the energy decay statement (1.1) of Theorem 1.1.

7. Pointwise estimates

Obtaining pointwise boundedness and decay estimates from the energy boundedness and decay estimates follow from standard applications of Sobolev embedding. That is, we can write ϕ using the fundamental theorem of calculus on the interval $[r,\infty)$ and applying Cauchy-Schwartz to obtain the estimate

$$|\phi(r)|^2 \leq \frac{1}{r}\int_r^\infty |\partial_\rho\phi|^2\rho^2 d\rho. \tag{7.1}$$ eq:sobolev

Far away from the horizon $r \geq R_0$, we have $|\partial_\rho\phi|^2 \lesssim {}^{(T)}J_\mu[\phi]n^\mu_{\widetilde{\Sigma}_\tau}$, so applying decay of the T-energy far away from the horizon furnishes

Proposition 7.1 (Decay away from the horizon)**.** *Let ϕ be a spherically-symmetric solution to the linear wave equation (W) on an extremal Reissner-Nordström background, then*

$$|\phi(r)|^2 \lesssim \frac{1}{\tau^2} \tag{7.2}$$

away from the horizon.

Proof. We have

$$|\phi(r)|^2 \lesssim \frac{1}{r} \int_{\tilde{\Sigma}_\tau} {}^{(T)}J_\mu[\phi] n^\mu_{\tilde{\Sigma}_\tau}.$$

Using decay of the T-energy (1.1) completes the proof. □

This completes the proof of the pointwise decay away from the horizon (1.2) from Theorem 1.1.

Close to the horizon $M \leq r \leq R_0$, the control over the transverse derivatives in the T-energy degenerates, so we are left with instead

Lemma 7.2 (Degenerate decay near horizon)**.** *Let ϕ be a spherically-symmetric solution to the linear wave equation (W) on an extremal Reissner-Nordström background, then*

$$|\phi(r)|^2 \lesssim \frac{1}{(r-M)^2} \frac{1}{\tau^2}. \tag{7.3}$$

eq:degenerate

Proof. By Sobolev embedding (7.1) and pointwise coercivity, we have

$$\begin{aligned} |\phi(r)|^2 &\lesssim \frac{1}{r} \int_{\Sigma_\tau \cap \{r' \geq r\}} {}^{(N)}J_\mu[\phi] n^\mu \\ &\lesssim \frac{1}{rD(r)} \int_{\Sigma_\tau} {}^{(T)}J_\mu[\phi] n^\mu. \end{aligned}$$

The result follows then from decay of the T-energy (1.1). □

Nonetheless, we can interpolate between the degenerate pointwise decay *near* the horizon and integrated decay *up to* the horizon to obtain decay *at* the horizon. That is, using the fundamental theorem of calculus on the interval $[r_0, r_0 + \tau^{-\alpha}]$, see Figure 7, for α to be chosen later, we have

Lemma 7.3 (Interpolation)**.** *Let ϕ be spherically-symmetric, then*

$$|\phi(r)|^2 \lesssim |\phi(r + \tau^{-\alpha})|^2 + 2 \int_{\Sigma_\tau \cap \{r_0 \leq r \leq r_0 + \tau^{-\alpha}\}} \phi \partial_\rho \phi. \tag{7.4}$$

eq:interpolat

g:interpolate

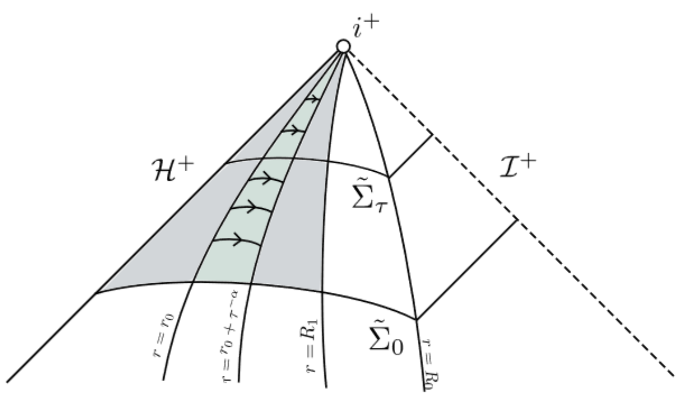

FIGURE 9. Interpolation between the degenerate decay near the horizon and the integrated decay up to the horizon by using the fundamental theorem of calculus.

The first term on the right has favourable decay as it concerns a region away from the horizon, while the second term is favourable since $\partial_\rho \phi$ is uniformly bounded, ϕ decays in an integrated sense, and the region of integration decays.

Proposition 7.4 (Pointwise decay near the horizon)**.** *Let ϕ be a spherically-symmetric solution to the linear wave equation (W) on an extremal Reissner-Nordström background, then*

$$|\phi(r)|^2 \lesssim \frac{1}{\tau^{6/5}} \tag{7.5}$$

near the horizon.

Proof. Continuing from (7.4), we estimate the first term on the right using the degenerate decay (7.3), and the second term on the right by placing $\partial_\rho \phi$ in L^∞ and ϕ in L^2. Controlling the latter using Hardy's inequality and T-energy decay (1.1), we obtain

$$|\phi(r)|^2 \lesssim \tau^{-2+2\alpha} + \tau^{-\frac{\alpha}{2}-1}.$$

Optimising by taking $\alpha = \frac{2}{5}$ furnishes the desired decay rate. $\square$

This completes the proof of the pointwise decay (1.2) near the horizon from Theorem 1.1.

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