INTEGRATED LOCAL ENERGY DECAY (MORAWETZ) ESTIMATES

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We consider the Cauchy problem for the wave equation

$$\begin{cases} (-\partial_t^2 + \Delta)u = f \\ u(0) = u_0, \\ \partial_t u(0) = u_1 \end{cases} \tag{1}$$

As we have already seen, its solution satisfies the energy estimate

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2} \lesssim \|(u_0, u_1)\|_{\dot{H}_x^1 \times L_x^2} + \|f\|_{L_t^1 L_x^2}$$

In Section 1, we shall prove another energy estimate that quantifies the dispersive decay of u in terms of L^2-based norms, which is known as the *Morawetz* or the *integrated local energy decay* bound.

One of the reasons why this estimate is useful is that it can be employed as an *intermediary* decay bound, which can enable one to obtain stronger decay bounds under an appropriate asymptotic flatness condition. We shall illustrate this in Section 2 by presenting a result of Rodnianski-Schlag that establishes non-endpoint Strichartz estimates for sub-principal asymptotically flat perturbations of $\Box$ (which involve only first and zeroth order perturbations), assuming the integrated local energy decay condition. We note that this is only the simplest example of a plethora of such results (see [Tataru], [Dafermos-Rodnianski], [Moschidis], [Oliver-Sterbenz], etc).

In section 3 we give a spectral characterization of the integrated local energy decay. This shows that tools from spectral theory (such as Fredholm theory, resonances, semi-classical analysis, etc.) can be used to obtain the integrated local energy decay estimate, whose usefulness is illustrated in Section 2.

1. An integrated local energy decay estimate

We define

$$A_j = \begin{cases} \{x \in \mathbb{R}^d | 2^j \leq |x| < 2^{j+1}\}, j \geq 1 \\ \{x \in \mathbb{R}^d | |x| < 2\}, j = 0 \end{cases}$$

We also define

$$\|u\|_{LE} = \sup_{j \geq 0} \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2 L^2(\mathbb{R}_t \times A_j)}$$
$$\|f\|_{LE^*} = \sum_{j \geq 0} \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2 L^2(\mathbb{R}_t \times A_j)}$$

We consider the equation (1)

Theorem 1.1. *Every solution of* (1) *satisfies*

$$\|\nabla_{t,x} u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{LE^*}.$$

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Proof. Let $X = \varphi(r)x^j\partial_j$, where φ is going to be chosen later. We have

$$\begin{aligned}
\langle Xu, \Delta u\rangle &= \int_{\mathbb{R}^d} \varphi(r)x^j\partial_j u \cdot \Delta u\,dx = \int_{\mathbb{R}^d} \partial_j u \cdot \varphi(r)x^j\Delta u\,dx = -\int_{\mathbb{R}^d} u\cdot\partial_j(\varphi(r)x^j\Delta u)\,dx \\
&= -\int_{\mathbb{R}^d} u\cdot\varphi'(r)\frac{x^jx_j}{r}\Delta u\,dx - \int_{\mathbb{R}^d} u\cdot\varphi(r)\delta_j^j\Delta u\,dx - \int_{\mathbb{R}^d} u\cdot\varphi(r)x^j\partial_j\Delta u\,dx \\
&= -\int_{\mathbb{R}^d} u\cdot r\varphi'(r)\Delta u\,dx - d\int_{\mathbb{R}^d} u\cdot\varphi(r)\Delta u\,dx - \int_{\mathbb{R}^d} u\cdot X\Delta u\,dx \\
&= -\left\langle (r\varphi'(r)+d\varphi(r))u, \Delta u\right\rangle - \langle X(\Delta u), u\rangle \\
&= -\left\langle (r\varphi'(r)+d\varphi(r))u, \Delta u\right\rangle - \langle \Delta(Xu), u\rangle - \langle [X,\Delta]u, u\rangle \\
&= -\left\langle (r\varphi'(r)+d\varphi(r))u, \Delta u\right\rangle - \langle Xu, \Delta u\rangle + \langle [\Delta,X]u, u\rangle
\end{aligned}$$

Thus,

$$\langle Xu, \Delta u\rangle = \frac{1}{2}\langle[\Delta,X]u,u\rangle - \frac{1}{2}\left\langle (r\varphi'(r)+d\varphi(r))u, \Delta u\right\rangle$$

For every index $k\in\{1,2,\ldots,d\}$, we have

$$\begin{aligned}
\partial_k(Xu) &= \partial_k(\varphi(r)x^j\partial_j u) = \varphi'(r)\frac{x_k}{r}x^j\partial_j u + \varphi(r)\partial_k u + \varphi(r)x^j\partial_j\partial_k u \\
\partial_k^2(Xu) &= \varphi''(r)\frac{x_k^2}{r^2}x^j\partial_j u + \varphi'(r)\frac{r - x_k\cdot\frac{x_k}{r}}{r^2}x^j\partial_j u + \varphi'(r)\frac{x_k}{r}\partial_k u + \varphi'(r)\frac{x_k}{r}x^j\partial_j\partial_k u \\
&\quad + \varphi'(r)\frac{x_k}{r}\partial_k u + \varphi(r)\partial_k^2 u \\
&\quad + \varphi'(r)\frac{x_k}{r}x^j\partial_j\partial_k u + \varphi(r)\partial_k^2 u + \varphi(r)x^j\partial_j\partial_k^2 u
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta(Xu) &= \varphi''(r)x^j\partial_j u + \varphi'(r)\frac{d-1}{r}x^j\partial_j u + \varphi'(r)\frac{2}{r}x^j\partial_j u + 2\frac{\varphi'(r)}{r}x^kx^j\partial_j\partial_k u + 2\varphi(r)\Delta u + X(\Delta u) \\
&= \left(\varphi''(r) + \frac{\varphi'(r)}{r}(d+1)\right)x^j\partial_j u + 2\frac{\varphi'(r)}{r}x^kx^j\partial_j\partial_k u + 2\varphi(r)\Delta u + X(\Delta u) \\
[\Delta, X]u &= \left(\varphi''(r) + \frac{\varphi'(r)}{r}(d+1)\right)x^j\partial_j u + 2\frac{\varphi'(r)}{r}x^kx^j\partial_j\partial_k u + 2\varphi(r)\Delta u
\end{aligned}$$

In this case,

$$\langle[\Delta,X]u,u\rangle = \left\langle \left(\varphi''(r) + \frac{\varphi'(r)}{r}(d+1)\right)x^j\partial_j u, u\right\rangle + 2\left\langle \frac{\varphi'(r)}{r}x^kx^j\partial_j\partial_k u, u\right\rangle + 2\langle\varphi(r)\Delta u, u\rangle$$

We also have

$$\begin{aligned}
\left\langle \frac{\varphi'(r)}{r}x_kx_j\partial_j\partial_k u, u\right\rangle &= \int_{\mathbb{R}^d}\partial_j\partial_k u\cdot\frac{\varphi'(r)}{r}x_kx_ju\,dx = -\int_{\mathbb{R}^d}\partial_k u\cdot\partial_j\left(\frac{\varphi'(r)}{r}x_kx_ju\right)\,dx \\
&= -\int_{\mathbb{R}^d}\partial_k u\cdot\frac{\varphi''(r)\frac{x_j}{r}\cdot r - \varphi'(r)\frac{x_j}{r}}{r^2}x_kx_ju\,dx - \int_{\mathbb{R}^d}\partial_k u\cdot\frac{\varphi'(r)}{r}\delta_k^j x_ju\,dx \\
&\quad - \int_{\mathbb{R}^d}\partial_k u\cdot\frac{\varphi'(r)}{r}x_ku\,dx - \int_{\mathbb{R}^d}\partial_k u\cdot\frac{\varphi'(r)}{r}x_kx_j\partial_j u\,dx
\end{aligned}$$

Thus,

$$\begin{aligned}\left\langle \frac{\varphi'(r)}{r} x^k x^j \partial_j \partial_k u, u \right\rangle &= -\int_{\mathbb{R}^d} \frac{\varphi''(r) r - \varphi'(r)}{r} x^j \partial_j u \cdot u \, dx - \int_{\mathbb{R}^d} 2 \frac{\varphi'(r)}{r} x^j \partial_j u \cdot u \, dx \\ &\quad - \int_{\mathbb{R}^d} \frac{\varphi'(r)}{r} (x^j \partial_j u)^2 \, dx \\ &= -\int_{\mathbb{R}^d} \left(\varphi''(r) + \frac{\varphi'(r)}{r} \right) x^j \partial_j u \cdot u \, dx - \int_{\mathbb{R}^d} \frac{\varphi'(r)}{r} (x^j \partial_j u)^2 \, dx\end{aligned}$$

Besides this,

$$\begin{aligned}\langle \varphi(r) \partial_j^2 u, u \rangle &= \int_{\mathbb{R}^d} \partial_j^2 u \cdot \varphi(r) u \, dx = -\int_{\mathbb{R}^d} \partial_j u \cdot \partial_j (\varphi(r) u) \, dx \\ &= -\int_{\mathbb{R}^d} \partial_j u \cdot \frac{\varphi'(r)}{r} x_j u \, dx - \int_{\mathbb{R}^d} \partial_j u \cdot \varphi(r) \partial_j u \, dx\end{aligned}$$

This means that

$$\langle \varphi(r) \Delta u, u \rangle = -\int_{\mathbb{R}^d} \frac{\varphi'(r)}{r} x^j \partial_j u \cdot u \, dx - \int_{\mathbb{R}^d} \varphi(r) |\nabla u|^2 \, dx$$

We have

$$\begin{aligned}\langle [\Delta, X] u, u \rangle &= \left\langle \left(\varphi''(r) + \frac{\varphi'(r)}{r}(d+1) \right) x^j \partial_j u, u \right\rangle + 2 \left\langle \frac{\varphi'(r)}{r} x^k x^j \partial_j \partial_k u, u \right\rangle + 2 \langle \varphi(r) \Delta u, u \rangle \\ &= \left\langle \left(\varphi''(r) + \frac{\varphi'(r)}{r}(d+1) \right) x^j \partial_j u, u \right\rangle - 2 \int_{\mathbb{R}^d} \left(\varphi''(r) + \frac{\varphi'(r)}{r} \right) x^j \partial_j u \cdot u \, dx \\ &\quad - 2 \int_{\mathbb{R}^d} \frac{\varphi'(r)}{r} (x^j \partial_j u)^2 \, dx - 2 \int_{\mathbb{R}^d} \frac{\varphi'(r)}{r} x^j \partial_j u \cdot u \, dx - 2 \int_{\mathbb{R}^d} \varphi(r) |\nabla u|^2 \, dx \\ &= \left\langle \left(-\varphi''(r) + \frac{\varphi'(r)}{r}(d-3) \right) x^j \partial_j u, u \right\rangle - 2 \int_{\mathbb{R}^d} \frac{\varphi'(r)}{r} (x^j \partial_j u)^2 \, dx - 2 \int_{\mathbb{R}^d} \varphi(r) |\nabla u|^2 \, dx\end{aligned}$$

We also note that

$$\begin{aligned}\left\langle \left(-\varphi''(r) + \frac{\varphi'(r)}{r}(d-3) \right) x_j \partial_j u, u \right\rangle &= \int_{\mathbb{R}^d} \partial_j u \cdot x_j \psi(r) u \, dx = -\int_{\mathbb{R}^d} u \cdot \partial_j (x_j \psi(r) u) \, dx \\ &= -\int_{\mathbb{R}^d} u \cdot \psi(r) u \, dx - \int_{\mathbb{R}^d} u \cdot x_j \psi'(r) \frac{x_j}{r} u \, dx \\ &\quad - \int_{\mathbb{R}^d} u \cdot x_j \psi(r) \partial_j u \, dx\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^d} \psi(r) x^j \partial_j u \cdot u \, dx = -\int_{\mathbb{R}^d} d\psi(r) u^2 \, dx - \int_{\mathbb{R}^d} r \psi'(r) u^2 \, dx - \int_{\mathbb{R}^d} \psi(r) x^j \partial_j u \cdot u \, dx,$$

which shows that

$$\int_{\mathbb{R}^d} \psi(r) x^j \partial_j u \cdot u \, dx = -\frac{1}{2} \int_{\mathbb{R}^d} (d\psi(r) + r \psi'(r)) u^2 \, dx$$

As $\psi(r) = -\varphi''(r) + \dfrac{\varphi'(r)}{r}(d-3)$, we get that

$$\begin{aligned}\psi'(r) &= -\varphi'''(r) + (d-3)\frac{\varphi''(r)r - \varphi'(r)}{r^2}\\ d\psi(r) + r\psi'(r) &= -d\varphi''(r) + d(d-3)\frac{\varphi'(r)}{r} - r\varphi'''(r) + (d-3)\frac{\varphi''(r)r - \varphi'(r)}{r}\\ &= -r\varphi'''(r) - 3\varphi''(r) + (d-1)(d-3)\frac{\varphi'(r)}{r}\end{aligned}$$

Therefore,

$$\left\langle \left(-\varphi''(r) + \frac{\varphi'(r)}{r}(d-3)\right) x_j\partial_j u, u\right\rangle = -\frac{1}{2}\int_{\mathbb{R}^d}\left(-r\varphi'''(r) - 3\varphi''(r) + (d-1)(d-3)\frac{\varphi'(r)}{r}\right)\cdot u^2\,dx$$

In this case,

$$\begin{aligned}\langle[\Delta, X]u, u\rangle &= -2\int_{\mathbb{R}^d}\frac{\varphi'(r)}{r}(x^j\partial_j u)^2\,dx - 2\int_{\mathbb{R}^d}\varphi(r)|\nabla u|^2\,dx\\ &\quad - \frac{1}{2}\int_{\mathbb{R}^d}\left(-r\varphi'''(r) - 3\varphi''(r) + (d-1)(d-3)\frac{\varphi'(r)}{r}\right)\cdot u^2\,dx\end{aligned}$$

Therefore,

$$\begin{aligned}\left\langle Xu + \frac{r\varphi'(r) + d\varphi(r)}{2}u, \Delta u\right\rangle &= \frac{1}{2}\langle[\Delta, X]u, u\rangle\\ &= -\int_{\mathbb{R}^d}\frac{\varphi'(r)}{r}(x^j\partial_j u)^2\,dx - \int_{\mathbb{R}^d}\varphi(r)|\nabla u|^2\,dx\\ &\quad - \frac{1}{4}\int_{\mathbb{R}^d}\left(-r\varphi'''(r) - 3\varphi''(r) + (d-1)(d-3)\frac{\varphi'(r)}{r}\right)\cdot u^2\,dx\end{aligned}$$

We note that, in general,

$$\begin{aligned}\int_{\mathbb{R}^d}\partial_j h\cdot\varphi(r)x_j h\,dx &= -\int_{\mathbb{R}^d} h\cdot\partial_j(\varphi(r)x_j h)\,dx = -\int_{\mathbb{R}^d} h\cdot\varphi'(r)\frac{x_j}{r}x_j h\,dx - \int_{\mathbb{R}^d} h\cdot\varphi(r)h\,dx\\ &\quad - \int_{\mathbb{R}^d} h\cdot\varphi(r)x_j\partial_j h\,dx \Rightarrow\\ \int_{\mathbb{R}^d}\partial_j h\cdot\varphi(r)x_j h\,d &= -\frac{1}{2}\int_{\mathbb{R}^d} h\cdot\varphi'(r)\frac{x_j^2}{r}h\,dx - \frac{1}{2}\int_{\mathbb{R}^d} h\cdot\varphi(r)h\,dx\end{aligned}$$

$$\langle Xh, h\rangle = -\frac{1}{2}\int_{\mathbb{R}^d} r\varphi'(r)h^2\,dx - \frac{1}{2}\int_{\mathbb{R}^d} d\varphi(r)h^2\,dx$$

We also have

$$\begin{aligned}\int_0^T\left\langle Xu + \frac{r\varphi'(r) + d\varphi(r)}{2}u, -\partial_t^2 u\right\rangle dt &= \left\langle Xu + \frac{r\varphi'(r) + d\varphi(r)}{2}u, -\partial_t u\right\rangle_{|_0^T}\\ &\quad + \int_0^T\left\langle X\partial_t u + \frac{r\varphi'(r) + d\varphi(r)}{2}\partial_t u, \partial_t u\right\rangle dt\\ &= \left\langle Xu + \frac{r\varphi'(r) + d\varphi(r)}{2}u, -\partial_t u\right\rangle_{|_0^T}\\ \langle X\partial_t u, \partial_t u\rangle &= -\int_{\mathbb{R}^d}\frac{r\varphi'(r) + d\varphi(r)}{2}(\partial_t u)^2\end{aligned}$$

We can now write

$$\begin{aligned}
\int_0^T \left\langle Xu + \frac{r\varphi'(r)+d\varphi(r)}{2}u, f\right\rangle\, dt &= \int_0^T \left\langle Xu + \frac{r\varphi'(r)+d\varphi(r)}{2}u, -\partial_t^2 u\right\rangle\, dt \\
&+ \int_0^T \left\langle Xu + \frac{r\varphi'(r)+d\varphi(r)}{2}u, \Delta u\right\rangle\, dt \\
&= \left\langle Xu + \frac{r\varphi'(r)+d\varphi(r)}{2}u, -\partial_t u\right\rangle_{|_0^T} \\
&- \int_0^T \int_{\mathbb{R}^d} \frac{\varphi'(r)}{r}(x^j\partial_j u)^2\, dx\, dt - \int_0^T \int_{\mathbb{R}^d} \varphi(r)|\nabla u|^2\, dx\, dt \\
&- \frac{1}{4}\int_0^T \int_{\mathbb{R}^d} \left(-r\varphi'''(r) - 3\varphi''(r) + (d-1)(d-3)\frac{\varphi'(r)}{r}\right)\cdot u^2\, dx\, dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^d} \frac{\varphi'(r)}{r}(x^j\partial_j u)^2\, dx\, dt - \int_0^T \int_{\mathbb{R}^d} \varphi(r)|\nabla u|^2\, dx\, dt \\
&- \frac{1}{4}\int_0^T \int_{\mathbb{R}^d} \left(-r\varphi'''(r) - 3\varphi''(r) + (d-1)(d-3)\frac{\varphi'(r)}{r}\right)\cdot u^2\, dx\, dt \\
&= \int_0^T \left\langle \varphi(r)x^j\partial_j u + \frac{r\varphi'(r)+d\varphi(r)}{2}u, f\right\rangle\, dt + \left\langle \varphi(r)x^j\partial_j u + \frac{r\varphi'(r)+d\varphi(r)}{2}u, \partial_t u\right\rangle_{|_0^T}
\end{aligned}$$

On the other hand, if w is a smooth function (to be chosen later as well),

$$\begin{aligned}
\int_0^T \langle wu, f\rangle\, dt &= \int_0^T \langle wu, -\partial_t^2 u\rangle\, dt + \int_0^T \langle wu, \Delta u\rangle\, dt \\
&= \langle wu, -\partial_t u\rangle_{|_0^T} + \int_0^T \langle w\partial_t u, \partial_t u\rangle\, dt - \int_0^T \langle \nabla(wu), \nabla u\rangle\, dt \\
&= \langle wu, -\partial_t u\rangle_{|_0^T} + \int_0^T \int_{\mathbb{R}^d} w(r)(\partial_t u)^2\, dx\, dt - \int_0^T \int_{\mathbb{R}^d} w(r)|\nabla u|^2\, dx\, dt \\
&- \int_0^T \int_{\mathbb{R}^d} u\frac{w'(r)}{r}x^j\partial_j u\, dx\, dt \\
&= \langle wu, -\partial_t u\rangle_{|_0^T} + \int_0^T \int_{\mathbb{R}^d} w(r)(\partial_t u)^2\, dx\, dt - \int_0^T \int_{\mathbb{R}^d} w(r)|\nabla u|^2\, dx\, dt \\
&+ \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} \left(w''(r) + \frac{d-1}{r}w'(r)\right) u^2\, dx\, dt
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^T \langle wu, f\rangle\, dt + \langle wu, \partial_t u\rangle_{|_0^T} + \int_0^T \int_{\mathbb{R}^d} w(r)|\nabla u|^2\, dx\, dt - \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} \left(w''(r) + \frac{d-1}{r}w'(r)\right) u^2\, dx\, dt \\
= \int_0^T \int_{\mathbb{R}^d} w(r)(\partial_t u)^2\, dx\, dt
\end{aligned}$$

In this case,

$$\int_0^T \int_{\mathbb{R}^d} w(r)(\partial_t u)^2\,dx\,dt - \int_0^T \int_{\mathbb{R}^d} \frac{\varphi'(r)}{r}(x^j\partial_j u)^2\,dx\,dt - \int_0^T \int_{\mathbb{R}^d} \varphi(r)|\nabla u|^2\,dx\,dt$$
$$-\frac{1}{4}\int_0^T \int_{\mathbb{R}^d} \left(-r\varphi'''(r) - 3\varphi''(r) + (d-1)(d-3)\frac{\varphi'(r)}{r}\right)\cdot u^2\,dx\,dt$$
$$= \int_0^T \left\langle \varphi(r)x^j\partial_j u + \frac{r\varphi'(r)+d\varphi(r)+2w(r)}{2}u, f\right\rangle dt + \left\langle \varphi(r)x^j\partial_j u + \frac{r\varphi'(r)+d\varphi(r)}{2}u, \partial_t u\right\rangle_{|_0^T}$$
$$+ \langle wu, \partial_t u\rangle_{|_0^T} + \int_0^T \int_{\mathbb{R}^d} w(r)|\nabla u|^2\,dx\,dt - \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} \left(w''(r) + \frac{d-1}{r}w'(r)\right)u^2\,dx\,dt,$$

which means that

$$\int_0^T \int_{\mathbb{R}^d} w(r)(\partial_t u)^2\,dx\,dt - \int_0^T \int_{\mathbb{R}^d} \frac{\varphi'(r)}{r}(x^j\partial_j u)^2\,dx\,dt - \int_0^T \int_{\mathbb{R}^d} (\varphi(r)-w(r))|\nabla u|^2\,dx\,dt$$
$$-\frac{1}{4}\int_0^T \int_{\mathbb{R}^d} \left(-r\varphi'''(r) - 3\varphi''(r) - 2w''(r) - 2\frac{d-1}{r}w'(r) + (d-1)(d-3)\frac{\varphi'(r)}{r}\right)\cdot u^2\,dx\,dt$$
$$= \int_0^T \left\langle \varphi(r)x^j\partial_j u + \frac{r\varphi'(r)+d\varphi(r)+2w(r)}{2}u, f\right\rangle dt + \left\langle \varphi(r)x^j\partial_j u + \frac{r\varphi'(r)+d\varphi(r)}{2}u, \partial_t u\right\rangle_{|_0^T}$$
$$+ \langle wu, \partial_t u\rangle_{|_0^T}$$

We choose $\varphi \in \mathcal{S}(\mathbb{R})$, $\varphi \le -1$ on $[-2,2]$, $|\varphi'(r)| << r|\varphi(r)|$, and $.r|\varphi'(r)| << |\varphi(r)|$. Let $w = -\frac{r\varphi'(r)+d\varphi(r)}{2}$. In this case, $-r\varphi'''(r) - 3\varphi''(r) - 2w''(r) - 2\dfrac{d-1}{r}w'(r) + (d-1)(d-3)\dfrac{\varphi'(r)}{r} = (2d-2)\left(\varphi''(r) + (d-1)\dfrac{\varphi'(r)}{r}\right)$, and we impose $\varphi''(r) + (d-1)\dfrac{\varphi'(r)}{r} \le 0$ on $[0,\infty)$ and $\varphi''(r) + (d-1)\dfrac{\varphi'(r)}{r} \le -\lambda$ on $[0,2]$, where $\lambda > 0$ is a fixed small parameter. We thus have

$$-\int_0^T \int_{\mathbb{R}^d} \frac{r\varphi'(r)+d\varphi(r)}{2}(\partial_t u)^2\,dx\,dt - \int_0^T \int_{\mathbb{R}^d} \frac{\varphi'(r)}{r}(x^j\partial_j u)^2\,dx\,dt$$
$$-\int_0^T \int_{\mathbb{R}^d} \frac{r\varphi'(r)+(d+2)\varphi(r)}{2}|\nabla u|^2\,dx\,dt$$
$$-\frac{d-1}{2}\int_0^T \int_{\mathbb{R}^d} \left(\varphi''(r) + (d-1)\frac{\varphi'(r)}{r}\right)\cdot u^2\,dx\,dt$$
$$= \int_0^T \left\langle \varphi(r)x^j\partial_j u, f\right\rangle dt + \left\langle \varphi(r)x^j\partial_j u, \partial_t u\right\rangle_{|_0^T}$$

Therefore,

$$\|\nabla u\|^2_{L^2L^2([0,T]\times B_1)} + \|\partial_t u\|^2_{L^2L^2([0,T]\times B_1)} + \|u\|^2_{L^2L^2([0,T]\times B_1)}$$
$$\lesssim \left|\int_0^T \left\langle \varphi(r)x^j\partial_j u, f\right\rangle dt\right| + \|\nabla u(T)\|^2_{L^2_x} + \|\nabla u(0)\|^2_{L^2_x}$$
$$+ \|\partial_t u(T)\|^2_{L^2_x} + \|\partial_t u(0)\|^2_{L^2_x}$$

Hardy's inequality implies that

$$\|\nabla_{t,x}u\|^2_{L^2L^2([0,T]\times B_1)} + \|r^{-1}u\|^2_{L^2L^2([0,T]\times B_1)} \lesssim \left|\int_0^T \left\langle \varphi(r)x^j\partial_j u, f\right\rangle dt\right|$$
$$+ \|\nabla_{t,x}u(T)\|^2_{L^2_x} + \|\nabla_{t,x}u(0)\|^2_{L^2_x}$$

By taking $\beta(r) = r\varphi(r)$, we can rewrite the previous inequality in the form

$$\|\nabla_{t,x} u\|^2_{L^2L^2([0,T]\times B_1)} + \|r^{-1}u\|^2_{L^2L^2([0,T]\times B_1)} \lesssim \left|\int_0^T \langle \beta(r)\partial_r u, f\rangle \, dt\right| + \|\nabla_{t,x} u(T)\|^2_{L^2_x} + \|\nabla_{t,x} u(0)\|^2_{L^2_x},$$

with β bounded. We note that for every $k \geq 1$, the function $u^k(t,x) = u(2^k t, 2^k x)$ solves $(-\partial_t^2 + \Delta)u = 2^{2k} f^k$, where $f^k(t,x) = f(2^k t, 2^k x)$. We have

$$\|\nabla_{t,x} u^k\|^2_{L^2L^2([0,T]\times B_1)} + \|r^{-1}u^k\|^2_{L^2L^2([0,T]\times B_1)} \lesssim \left|\int_0^T \left\langle \beta(r)\partial_r u^k, 2^{2k} f^k\right\rangle \, dt\right| + \|\nabla_{t,x} u^k(T)\|^2_{L^2_x} + \|\nabla_{t,x} u^k(0)\|^2_{L^2_x},$$

As $2^{-k}A_k \subset B_1$, we also get that

$$\|\nabla_{t,x} u^k\|^2_{L^2L^2([0,T]\times 2^{-k}A_k)} + \|r^{-1}u^k\|^2_{L^2L^2([0,T]\times 2^{-k}A_k)} \lesssim \left|\int_0^T \left\langle \beta(r)\partial_r u^k, 2^{2k} f^k\right\rangle \, dt\right| + \|\nabla_{t,x} u^k(T)\|^2_{L^2_x} + \|\nabla_{t,x} u^k(0)\|^2_{L^2_x},$$

We can also see that

$$\begin{aligned}
\|\nabla_{t,x} u^k\|^2_{L^2L^2([0,T]\times 2^{-k}A_k)} &= 2^{-k(d-1)}\|\nabla_{t,x} u\|^2_{L^2L^2([0,2^kT]\times A_k)} \\
\|r^{-1}u^k\|^2_{L^2L^2([0,T]\times 2^{-k}A_k)} &= 2^{-k(d-1)}\|r^{-1}u\|^2_{L^2L^2([0,2^kT]\times A_k)} \\
\|\nabla_{t,x} u^k(0)\|^2_{L^2_x} &= 2^{-k(d-2)}\|\nabla_{t,x} u(0)\|^2_{L^2_x} \\
\|\nabla_{t,x} u^k(T)\|^2_{L^2_x} &= 2^{-k(d-2)}\|\nabla_{t,x} u(2^kT)\|^2_{L^2_x} \\
\int_0^T \left\langle \beta(r)\partial_r u^k, 2^{2k} f^k\right\rangle dt &= 2^{-k(d-2)} \int_0^{2^kT} \left\langle \beta\left(\frac{r}{2^k}\right)\partial_r u, f\right\rangle dt
\end{aligned}$$

Therefore,

$$2^{-k}\|\nabla_{t,x} u\|^2_{L^2L^2([0,2^kT]\times A_k)} + 2^{-k}\|r^{-1}u\|^2_{L^2L^2([0,2^kT]\times A_k)} \lesssim \left|\int_0^{2^kT} \left\langle \beta\left(\frac{r}{2^k}\right)\partial_r u, f\right\rangle dt\right| + \|\nabla_{t,x} u(2^kT)\|^2_{L^2_x} + \|\nabla_{t,x} u(0)\|^2_{L^2_x}$$

From the energy identity

$$\|\nabla_{t,x} u^k(2^kT)\|^2_{L^2_x} = \|\nabla_{t,x} u^k(0)\|^2_{L^2_x} + 2\int_0^{2^kT} \langle \partial_t u, f\rangle \, dt$$

we get that

$$\begin{aligned}
2^{-k}\|\nabla_{t,x} u\|^2_{L^2L^2([0,2^kT]\times A_k)} + 2^{-k}\|r^{-1}u\|^2_{L^2L^2([0,2^kT]\times A_k)} &\lesssim \left|\int_0^{2^kT} \left\langle \beta\left(\frac{r}{2^k}\right)\partial_r u, f\right\rangle dt\right| \\
&\quad + \|\nabla_{t,x} u(0)\|^2_{L^2_x} + 2\int_0^{2^kT} |\langle \partial_t u, f\rangle| \, dt \\
&\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}^d} (|\nabla_{t,x} u|)\, |f| \, dx\, dt \\
&\quad + \|\nabla_{t,x} u(0)\|^2_{L^2_x}
\end{aligned}$$

It is also clear that we have the same inequality for $k = 0$. By taking the supremum with respect to $k \geq 0$, we deduce that for every $\delta > 0$,

$$\begin{aligned}\|\nabla_{t,x} u\|_{LE}^2 + \|r^{-1} u\|_{LE}^2 &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\nabla_{t,x} u||f| \, dx \, dt + \|\nabla_{t,x} u(0)\|_{L_x^2}^2 \\ &\lesssim \|\nabla_{t,x} u(0)\|_{L_x^2}^2 + \delta \|\nabla_{t,x} u\|_{LE}^2 + \frac{1}{\delta} \|f\|_{LE^*}^2\end{aligned}$$

By choosing $\delta > 0$ sufficiently small, we deduce that

$$\|\nabla_{t,x} u\|_{LE}^2 + \|r^{-1} u\|_{LE}^2 \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}^2 + \|f\|_{LE^*}^2$$

We can now immediately see that

$$\|\nabla_{t,x} u\|_{LE} + \|\langle r \rangle^{-1} u\|_{LE} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{LE^*}.$$

□

Corollary 1.2. *Every solution of* (1) *satisfies*

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE} + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE} \lesssim \|\nabla_{t,x} u(0)\|_{L_x^2} + \|f\|_{L_t^1 L_x^2 + LE^*}.$$

Proof. As we have already seen,

$$\|\nabla_{t,x} u\|_{LE}^2 + \|\langle r \rangle^{-1} u\|_{LE}^2 \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\nabla_{t,x} u||f| \, dx \, dt + \|\nabla_{t,x} u(0)\|_{L_x^2}^2$$

From the energy identity

$$\|\nabla_{t,x} u(T)\|_{L_x^2}^2 = \|\nabla_{t,x} u(0)\|_{L_x^2}^2 + 2 \int_0^T \langle \partial_t u, f \rangle \, dt$$

we immediately get that

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 \lesssim \|\nabla_{t,x} u(0)\|_{L_x^2}^2 + \int_{\mathbb{R}} |\partial_t u||f| \, dt$$

Thus,

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE}^2 + \|\langle r \rangle^{-1} u\|_{LE}^2 \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\nabla_{t,x} u||f| \, dx \, dt + \|\nabla_{t,x} u(0)\|_{L_x^2}^2$$

From Hardy's inequality ($d \geq 3$), we get that

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE}^2 + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE}^2 \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}^d} |\nabla_{t,x} u||f| \, dx \, dt + \|\nabla_{t,x} u(0)\|_{L_x^2}^2$$

Thus, for every $\delta > 0$

$$\begin{aligned}\|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE}^2 + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE}^2 &\lesssim \delta \|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE}^2 + \frac{1}{\delta} \|f\|_{L_t^1 L_x^2 + LE^*}^2 \\ &\quad + \|\nabla_{t,x} u(0)\|_{L_x^2}^2\end{aligned}$$

By choosing $\delta > 0$ small enough, we get that

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE}^2 + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE}^2 \lesssim \|\nabla_{t,x} u(0)\|_{L_x^2}^2 + \|f\|_{L_t^1 L_x^2 + LE^*}^2$$

Therefore,

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE} + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE} \lesssim \|\nabla_{t,x} u(0)\|_{L_x^2} + \|f\|_{L_t^1 L_x^2 + LE^*}.$$

□

We now consider the perturbed equation

$$\begin{cases} (-\partial_t^2 + L)u = 0 \\ u(0) = u_0, \partial_t u(0) = u_1 \end{cases} \tag{2}$$

where $L = -\Delta + b^k \partial_k + c$ satisfies the decay condition

$$\sum_{j=0}^{\infty} \sup_{\mathbb{R}_t \times A_j} \langle x \rangle |b| + \langle x \rangle^2 |\partial_l b^l| + \langle x \rangle^2 |c| < K, \tag{3}$$

where $K > 0$ is a positive constant.

Corollary 1.3. *Let u be a solution of* (2)*. If K is small enough, then u satisfies*

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE} + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE} \lesssim \|\nabla_{t,x} u(0)\|_{L_x^2} + \|f\|_{L_t^1 L_x^2 + LE^*}.$$

Proof. We write $Bu = b^l \partial_l u + cu$. We rewrite the equation as $(-\partial_t^2 + \Delta)u = f - Bu$. Thus,

$$\begin{aligned} \|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE} + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE} &\lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f - Bu\|_{LE^*} \\ &\lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{LE^*} + \|Bu\|_{LE^*} \end{aligned}$$

For every $k \geq 0$, we have

$$\begin{aligned} \|Bu\|_{LE^*} = \sum_{k=0}^{\infty} 2^{\frac{k}{2}} \|Bu\|_{L^2 L^2(\mathbb{R}_t \times A_k)} &= \sum_{k=0}^{\infty} 2^{\frac{k}{2}} \|b^l \partial_l u\|_{L^2 L^2(\mathbb{R}_t \times A_k)} + 2^{\frac{k}{2}} \|cu\|_{L^2 L^2(\mathbb{R}_t \times A_k)} \\ &\lesssim K(\|\nabla_{t,x} u\|_{LE} + \|\langle r \rangle^{-1} u\|_{LE}) \end{aligned}$$

Thus,

$$\begin{aligned} \|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE} + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE} &\lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f - Bu\|_{LE^*} \\ &\lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{LE^*} + \|Bu\|_{LE^*} \\ &\lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{LE^*} \\ &\quad + K(\|\nabla_{t,x} u\|_{LE} + \|\langle r \rangle^{-1} u\|_{LE}) \end{aligned}$$

If $K > 0$ is sufficiently small, we deduce that

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2 \cap LE} + \|\langle r \rangle^{-1} u\|_{L_t^\infty L_x^2 \cap LE} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{LE^*},$$

□

Remark 1.4. *We note that the integrated local energy decay estimated also takes place under perturbations of the metric (see* [Metcalfe-Tataru]*).*

2. Strichartz estimates

Definition 2.1. *A pair (p, q) is said to be **wave-admissible** in dimension $d + 1$ if*

$$p \in [2, \infty], \frac{1}{p} + \frac{d-1}{2q} \leq \frac{d-1}{4}, (p, q, d) \neq (2, \infty, 3)$$

We recall the following result:

Theorem 2.2. *Let $u_0, u_1 \in \mathcal{S}(\mathbb{R}^d)$, and let u be the solution to* (1) *with this initial data. Let (p, q) and $(\tilde{p}, \tilde{q})$ be pairs of wave-admissible exponents, which also obey the scaling conditions*

$$\frac{1}{p} + \frac{d}{q} = -2 + \frac{1}{\tilde{p}'} + \frac{d}{\tilde{q}'} = -1 + \frac{d}{2},$$

where $\tilde{p}'$ and $\tilde{q}'$ are the Lebesgue duals to $\tilde{p}$ and $\tilde{q}$, i.e $\frac{1}{\tilde{p}'}+\frac{1}{\tilde{p}}=\frac{1}{\tilde{q}'}+\frac{1}{\tilde{q}}=1$*, and $(\hat{p},\hat{q})$ another pair of wave-admissible exponents satisfying*

$$\frac{1}{\hat{p}}+\frac{d}{\hat{q}}=\frac{d}{2}.$$

In addition, we assume that they also satisfy the ***non-endpoint*** *condition $p,\tilde{p},\hat{p}>2$. Then,*

$$\|\nabla_{t,x}u\|_{L^\infty L^2}+\|u\|_{L^pL^q}+\|\nabla_{t,x}u\|_{L^{\hat{p}}L^{\hat{q}}}\lesssim_{p,q,\tilde{p},\tilde{q},\hat{p},\hat{q}}\|(u_0,u_1)\|_{\dot{H}^1\times L^2}+\|f\|_{L^{\tilde{p}'}L^{\tilde{q}'}}$$

We are also going to need the following theorem

Theorem 2.3. *Let X and Y be Banach spaces, and let $T:L^p(\mathbb{R};X)\to L^q(\mathbb{R};Y)$ $(1\le p,q\le\infty)$ be of the form*

$$Tf(t)=\int_{-\infty}^{\infty}K(t,s)f(s)\,ds$$

for some kernel $K:\mathbb{R}\times\mathbb{R}\to\mathcal{B}(X\to Y)$. Provided that $p<q$, the truncated operator

$$\tilde{T}f(t)=\int_{-\infty}^{t}K(t,s)f(s)\,ds$$

defines a bounded operator from $L^p(\mathbb{R};X)$ to $L^q(\mathbb{R};Y)$.

We are now going to prove the following result:

Theorem 2.4. *(Rodnianski-Schlag) We assume that the coefficients of* (2) *satisfy* (3) *for some $K>0$ (in particular, K can be large). We also assume that (ILED) holds for* (2).

Let $u_0,u_1\in\mathcal{S}(\mathbb{R}^d)$, and let u be the solution to (2) *with this initial data. Let (p,q) and $(\hat{p},\hat{q})$, be pairs of wave-admissible exponents, which also obey the scaling conditions*

$$\frac{1}{p}+\frac{d}{q}=\frac{1}{\hat{p}}+\frac{d}{\hat{q}}-1=-1+\frac{d}{2}.$$

In addition, we assume that they also satisfy the ***non-endpoint*** *condition $p,\hat{p}>2$. Then,*

$$\|\nabla_{t,x}u\|_{L^\infty L^2}+\|u\|_{L^pL^q}+\|\nabla_{t,x}u\|_{L^{\hat{p}}L^{\hat{q}}}\lesssim_{p,q,\hat{p},\hat{q}}\|(u_0,u_1)\|_{\dot{H}^1\times L^2}+\|f\|_{L-t^1L^2_x+LE^*}$$

Proof. We recall some results concerning the linear wave equation. The first one says that the solution of the homogeneous problem

$$\begin{aligned}&(-\partial_t^2+\Delta)u=0\\&u(0)=u_0,\\&\partial_t u(0)=u_1\end{aligned}$$

is given by the formula

$$u(t)=\cos(t\sqrt{-\Delta})u_0+\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1$$

Besides this, the purely inhomogeneous problem

$$\begin{aligned}&(-\partial_t^2+\Delta)u=F\\&u(0)=0,\\&\partial_t u(0)=0\end{aligned}$$

has a solution given by the formula

$$u(t)=-\int_{-\infty}^{t}\frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}F(s)\,ds$$

We also note that when u is a solution of the homogeneous problem,

$$\|\nabla_{t,x}u\|_{L^\infty_t L^2_x \cap LE} \lesssim \|(u_0,u_1)\|_{\dot H^1_x \times L^2_x}$$

By duality, we get that

$$\left\|\int_{-\infty}^{\infty} \cos(t\sqrt{-\Delta})F(t)\,dt\right\|_{L^2_x} \lesssim \|F\|_{L^1_t L^2_x + LE^*}$$

$$\left\|\int_{-\infty}^{\infty} \sin(t\sqrt{-\Delta})F(t)\,dt\right\|_{L^2_x} \lesssim \|F\|_{L^1_t L^2_x + LE^*}$$

By applying the homogeneous Strichartz estimate, we deduce that

$$\begin{aligned}
\left\|\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\int_{-\infty}^{\infty} \cos(s\sqrt{-\Delta})F(s)\,ds\right\|_{L^p_t L^q_x} &+ \left\|\nabla_{t,x}\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\int_{-\infty}^{\infty} \cos(s\sqrt{-\Delta})F(s)\,ds\right\|_{L^{\hat p}_t L^{\hat q}_x} \\
&\lesssim \left\|\int_{-\infty}^{\infty} \cos(s\sqrt{-\Delta})F(s)\,ds\right\|_{L^2_x} \\
\left\|\cos(t\sqrt{-\Delta})\int_{-\infty}^{\infty} \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}}F(s)\,dt\right\|_{L^p_t L^q_x} &+ \left\|\nabla_{t,x}\cos(t\sqrt{-\Delta})\int_{-\infty}^{\infty} \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}}F(s)\,dt\right\|_{L^{\hat p}_t L^{\hat q}_x} \\
&\lesssim \left\|\int_{-\infty}^{\infty} \sin(s\sqrt{-\Delta})F(s)\,ds\right\|_{L^2_x}
\end{aligned}$$

Thus,

$$\left\|\int_{-\infty}^{\infty} \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}F(s)\,ds\right\|_{L^p_t L^q_x} + \left\|\nabla_{t,x}\int_{-\infty}^{\infty} \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}F(s)\,ds\right\|_{L^{\hat p}_t L^{\hat q}_x} \lesssim \|F\|_{L^1_t L^2_x + LE^*}$$

As $p > 2$, the Christ-Kiselev lemma enables us to deduce that

$$\left\|\int_{-\infty}^{t} \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}F(s)\,ds\right\|_{L^p_t L^q_x} + \left\|\nabla_{t,x}\int_{-\infty}^{t} \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}F(s)\,ds\right\|_{L^{\hat p}_t L^{\hat q}_x} \lesssim \|F\|_{L^1_t L^2_x + LE^*}$$

Along with the homogeneous Strichartz estimate, this immediately implies that

$$\|u\|_{L^p_t L^q_x} + \|\nabla_{t,x}u\|_{L^{\hat p}_t L^{\hat q}_x} \lesssim \|(u_0,u_1)\|_{\dot H^1_x \times L^2_x} + \|F\|_{L^1_t L^2_x + LE^*}$$

We now return to the problem

$$\begin{aligned}
&(-\partial_t^2 - L)u = f \\
&u(0) = u_0, \\
&\partial_t u(0) = u_1
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
&(-\partial_t^2 + \Delta)u = f + Bu \\
&u(0) = u_0, \\
&\partial_t u(0) = u_1
\end{aligned}$$

As in the proof of Corollary 1.3, we have

$$\|Bu\|_{LE^*} \lesssim K(\|\nabla_{t,x}u\|_{LE} + \|\langle r\rangle^{-1}u\|_{LE}) \lesssim \|(u_0,u_1)\|_{\dot H^1_x \times L^2_x} + \|f\|_{L^1_t L^2_x + LE^*}$$

We deduce that

$$\begin{aligned}\|u\|_{L^pL^q} + \|\nabla_{t,x}u\|_{L_t^{\hat{p}}L_x^{\hat{q}}} &\lesssim \|(u_0,u_1)\|_{\dot{H}_x^1\times L_x^2} + \|f+Bu\|_{L_t^1L_x^2+LE^*}\\ &\lesssim \|(u_0,u_1)\|_{\dot{H}_x^1\times L_x^2} + \|f\|_{L_t^1L_x^2+LE^*} + \|Bu\|_{L_t^1L_x^2+LE^*}\\ &\lesssim \|(u_0,u_1)\|_{\dot{H}_x^1\times L_x^2} + \|f\|_{L_t^1L_x^2+LE^*}\end{aligned}$$

This finishes the proof. □

Remark 2.5. *We note that we can further generalize the previous Strichartz estimates to general non-endpoint wave-admissible pairs $(\tilde{p},\tilde{q})$ satisfying the scaling condition $\frac{1}{\tilde{p}}+\frac{d}{\tilde{q}}=\frac{d}{2}$ if we replace the integrated local energy decay estimate by the condition*

$$\|\nabla_{t,x}u\|_{LE} + \|\langle r\rangle^{-1}u\|_{LE} \lesssim \|(u_0,u_1)\|_{\dot{H}_x^1\times L_x^2} + \|f\|_{L_t^{\tilde{p}'}L_x^{\tilde{q}'}+LE^*}.$$

Remark 2.6. *For the endpoint case see the argument of* [Keel-Tao] *or* [Ionescu-Kenig].

Remark 2.7. *If (b,c) don't satisfy* (3)*, then the previous result can fail in some instances. For example, if $c=\alpha r^{-2}$, then the wave admissible pairs (p,q), $(\hat{p},\hat{q})$, and $(\tilde{p},\tilde{q})$ for which Strichartz estimates hold depend sensitively on α (see* [Burq Planchon Stalker Tahvildar-Zadeh]*).*

3. A spectral characterization

We assume that $L=-\Delta+b^k\partial_k+c$, where b is purely imaginary and c is real, and both of them are time-independent and satisfy the decay condition (3) for some(possibly large) $K\in(0,\infty)$. These conditions imply that L is self-adjoint. It is clear that $\langle Lu,u\rangle$ is bounded on $\dot{H}^1$. We also assume that L is coercive, in the sense that $\langle Lu,u\rangle \gtrsim \|u\|_{\dot{H}^1}^2$. We define the wave *resolvents* of L as

$$\mathbf{R}_z=(z^2-L)^{-1},$$

which are well-defined as bounded operators from $\dot{H}^1$ to $D(L)$ as long as $z^2\notin\sigma(L)$.The spectral theorem implies the following bound for $\mathbf{R}_z$:

Lemma 3.1. *For any $\tau\in\mathbb{R}$ and $\varepsilon>0$, we have*

$$|\tau|\|\mathbf{R}_{\tau\pm i\varepsilon}g\|_{L^2}\lesssim\varepsilon^{-1}\|g\|_{L^2}$$

Proof. From the spectral theorem, we have (we keep in mind that $\sigma(L)\subset\mathbb{R}$) (for every $\tau\neq 0$)

$$\begin{aligned}\|\mathbf{R}_{\tau\pm i\varepsilon}g\|_{L^2} &\lesssim d((\tau\pm i\varepsilon)^2,\sigma(L))^{-1}\|g\|_{L^2} = d(\tau^2-\varepsilon^2\pm 2\tau i\varepsilon,\sigma(L))^{-1}\|g\|_{L^2}\\ &\lesssim |\tau|^{-1}\varepsilon^{-1}\|g\|_{L^2}\end{aligned}$$

Thus,

$$|\tau|\|\mathbf{R}_{\tau\pm i\varepsilon}g\|_{L^2}\lesssim\varepsilon^{-1}\|g\|_{L^2}$$

Here we have used the bound

$$d((\tau\pm i\varepsilon)^2,\sigma(L))\geq d((\tau\pm i\varepsilon)^2,\mathbb{R})=d(\tau^2-\varepsilon^2\pm 2\tau i\varepsilon,\mathbb{R})=2|\tau|\varepsilon,$$

which follows from the fact that $\sigma(L)\subset\mathbb{R}$.

The same conclusion is clearly true for $\tau=0$. □

Lemma 3.2. *For any $\tau\in\mathbb{R}$ and $\varepsilon>0$, we have*

$$\|\nabla_x\mathbf{R}_{\tau\pm i\varepsilon}g\|_{L^2}+\varepsilon\|\mathbf{R}_{\tau\pm i\varepsilon}g\|_{L^2}\lesssim\varepsilon^{-1}\|g\|_{L^2}$$

Proof. Let $u = \mathbf{R}_{\tau\pm i\varepsilon} g$ By definition, $((\tau \pm i\varepsilon)^2 - L)u = g$. We have

$$\Re\langle u, g\rangle = \Re\langle u, (\tau^2 - \varepsilon^2)u \pm 2i\tau\varepsilon u - Lu\rangle = \langle u, (\tau^2 - \varepsilon^2)u - Lu\rangle$$

Thus, for every $\delta > 0$,

$$\begin{aligned}\langle Lu, u\rangle + \varepsilon^2\|u\|_{L^2}^2 &= \tau^2\|u\|_{L^2}^2 - \Re\langle u, g\rangle \lesssim \tau^2\|u\|_{L^2}^2 + |\langle u, g\rangle| \\ &\lesssim \tau^2\|u\|_{L^2}^2 + \|u\|_{L^2}\|g\|_{L^2} \lesssim \varepsilon^{-2}\|g\|_{L^2}^2 + \delta^{-2}\varepsilon^{-2}\|g\|_{L^2}^2 + \delta^2\varepsilon^2\|u\|_{L^2}^2\end{aligned}$$

By choosing $\delta > 0$ sufficiently small, we deduce that

$$\langle Lu, u\rangle + \varepsilon^2\|u\|_{L^2}^2 \lesssim \varepsilon^{-2}\|g\|_{L^2}^2$$

As $\langle Lu, u\rangle \gtrsim \|u\|_{\dot H^1}^2$, we get that

$$\|u\|_{\dot H^1}^2 + \varepsilon^2\|u\|_{L^2}^2 \lesssim \varepsilon^{-2}\|g\|_{L^2}^2,$$

hence

$$\|u\|_{\dot H^1} + \varepsilon\|u\|_{L^2}^2 \lesssim \varepsilon^{-1}\|g\|_{L^2}^2,$$

The conclusion immediately follows. ☐

We are also going to define the spatial counterparts of LE and LE^*:

$$\|u\|_{\mathcal{LE}} = \sup_{j\geq 0}\|\langle r\rangle^{-\frac{1}{2}}u\|_{L^2(A_j)}$$

$$\|f\|_{\mathcal{LE}^*} = \sum_{j\geq 0}\|\langle r\rangle^{\frac{1}{2}}f\|_{L^2(A_j)}$$

We shall prove the following result (for further reference, see [Metcalfe-Sterbenz-Tataru]):

Theorem 3.3. *The following are equivalent:*

1. *Every solution u to* (2) *satisfies ILED*
2. *For every $\tau \in \mathbb{R}$ and $\varepsilon > 0$, we have*

$$|\tau \pm i\varepsilon|\|\mathbf{R}_{\tau\pm i\varepsilon}g\|_{\mathcal{LE}} + \|\nabla_x \mathbf{R}_{\tau\pm i\varepsilon}g\|_{\mathcal{LE}} + \|\langle r\rangle^{-1}\mathbf{R}_{\tau\pm i\varepsilon}g\|_{\mathcal{LE}} \leq C\|g\|_{\mathcal{LE}^*},$$

where $C > 0$ is an universal constant.

Proof. Step 1: Reduction to forward solutions. We first prove that every solution of $(-\partial_t^2 - L)u = f$ with $u(0) = u_0$ and $u_t(0) = u_1$ satisfies

$$\|\nabla u\|_{LE} + \|\langle r\rangle^{-1}u\|_{LE} \lesssim \|(u_0, u_1)\|_{\dot H^1\times L^2} + \|f\|_{LE^*}$$

if and only if every solution of the forward problem $(-\partial_t^2 - L)u = f$, with f supported away from $\{t = -\infty\}$ satisfies

$$\|\nabla u\|_{LE} + \|\langle r\rangle^{-1}u\|_{LE} \lesssim \|f\|_{LE^*}$$

When f is supported away from $\{t = -\infty\}$, and $u(t) = 0$ for t sufficiently negative, we deduce that there exists t_0 such that $u(t) = 0, \forall t \leq t_0$. Thus, $\partial_t u(t_0) = 0$. By applying ILED to $\tilde u(t) = u(t + t_0)$ (which satisfies $\tilde u(0) = 0$, $\partial_t \tilde u(0) = 0$), and by using the time-invariance of the LE and LE^* norms, we deduce that

$$\|\nabla u\|_{LE} + \|\langle r\rangle^{-1}u\|_{LE} \lesssim \|f\|_{LE^*}$$

For the converse, we use a method that is similar to the one employed by Rodnianski and Schlag. We consider v to be the solution of the problem

$$\begin{aligned}&(-\partial_t^2 + \Delta)v = f \\ &v(0) = u_0 \\ &\partial_t v(0) = u_1\end{aligned}$$

By 1.2, we have

$$\|\nabla_{t,x}v\|_{LE} + \|\langle r\rangle^{-1}v\|_{LE} \lesssim \|(u_0,u_1)\|_{\dot{H}^1_x\times L^2_x} + \|f\|_{LE^*}$$

We can now immediately see that $u-v$ satisfies

$$\begin{aligned}&(-\partial_t^2 + L)(u-v) = -Bv\\ &(u-v)(0) = 0\\ &\partial_t(u-v)(0) = 0\end{aligned}$$

Let v_+ be the forward solution of

$$\begin{aligned}&(-\partial_t^2 + L)(v_+) = \mathbf{1}_{[0,\infty)}(-Bv)\\ &(v_+)(0) = 0\\ &\partial_t(v_+)(0) = 0\end{aligned}$$

and v_- be the backward solution of

$$\begin{aligned}&(-\partial_t^2 + L)(v_-) = \mathbf{1}_{(-\infty,0)}(-Bv)\\ &(v_-)(0) = 0\\ &\partial_t(v_-)(0) = 0\end{aligned}$$

Thus,

$$\|\nabla_{t,x}v_+\|_{LE} + \|\langle r\rangle^{-1}v_+\|_{LE} \lesssim \|\mathbf{1}_{[0,\infty)}(-Bv)\|_{LE^*} \lesssim \|Bv\|_{LE^*},$$

and by using the time symmetry of the LE and LE^* norms,

$$\|\nabla_{t,x}v_-\|_{LE} + \|\langle r\rangle^{-1}v_-\|_{LE} \lesssim \|\mathbf{1}_{(-\infty,0)}(-Bv)\|_{LE^*} \lesssim \|Bv\|_{LE^*},$$

As in the proof of Corollary 1.3, we get that

$$\|Bv\|_{LE^*} \lesssim K(\|\nabla_{t,x}v\|_{LE} + \|\langle r\rangle^{-1}v\|_{LE}) \lesssim \|(u_0,u_1)\|_{\dot{H}^1_x\times L^2_x} + \|f\|_{LE^*}$$

We note that $u-v-v_+-v_-$ is a finite energy solution of

$$\begin{aligned}&(-\partial_t^2 + L)(u-v-v_+-v_-) = 0\\ &(u-v-v_+-v_-)(0) = 0\\ &\partial_t(u-v-v_+-v_-)(0) = 0\end{aligned}$$

Thus, $u = v+v_++v_-$. This means that

$$\begin{aligned}\|\nabla_{t,x}u\|_{LE} + \|\langle r\rangle^{-1}u\|_{LE} &\lesssim \|\nabla_{t,x}v\|_{LE} + \|\langle r\rangle^{-1}v\|_{LE} + \|\nabla_{t,x}v_+\|_{LE} + \|\langle r\rangle^{-1}v_+\|_{LE}\\ &\quad + \|\nabla_{t,x}v_-\|_{LE} + \|\langle r\rangle^{-1}v_-\|_{LE}\\ &\lesssim \|(u_0,u_1)\|_{\dot{H}^1_x\times L^2_x} + \|f\|_{LE^*} + \|Bv\|_{LE^*}\\ &\lesssim \|(u_0,u_1)\|_{\dot{H}^1_x\times L^2_x} + \|f\|_{LE^*},\end{aligned}$$

as desired.

Step 2: Reduction to damped forward solutions. Now we prove that the condition

$$\|\nabla_{t,x}u\|_{LE} + \|\langle r\rangle^{-1}u\|_{LE} \lesssim \|f\|_{LE^*}$$

for forward solutions is equivalent to the condition

$$\|e^{-\varepsilon t}\nabla_{t,x}u\|_{LE} + \|\langle r\rangle^{-1}e^{-\varepsilon t}u\|_{LE} \lesssim \|e^{-\varepsilon t}f\|_{LE^*}, \forall\varepsilon > 0$$

We first show that the latter implies the former.

Let u_k be the forward solution corresponding to $\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)f$. As the operator L is elliptic, we have

$$u_k(t) = -\int_{-\infty}^{t} \frac{\sin(t-s)L}{L}(\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}f)(s)\,ds,$$

which shows that u_k is supported in $\{t \in [k,\infty)\}$. We also have $u = \sum_{k\in\mathbb{Z}} u_k$.

We note that

$$\begin{aligned}
\|e^{-\varepsilon t}\nabla_{t,x}u\|_{LE} &\simeq \sup_{j\geq 0} 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)} \\
&\simeq \sup_{j\geq 0}\left(\sum_{k\in\mathbb{Z}} 2^{-j}\|e^{-\varepsilon t}\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)\nabla_{t,x}u\|^2_{L^2L^2(\mathbb{R}\times A_j)}\right)^{\frac{1}{2}} \\
&\simeq \sup_{j\geq 0}\left(\sum_{k\in\mathbb{Z}} 2^{-j}e^{-2k}\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)\nabla_{t,x}u\|^2_{L^2L^2(\mathbb{R}\times A_j)}\right)^{\frac{1}{2}} \\
&\simeq \|2^{-\frac{j}{2}}e^{-k}\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l^\infty_j l^2_k} \\
&\lesssim \|2^{-\frac{j}{2}}e^{-k}\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l^2_k l^\infty_j}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\langle r\rangle^{-1}e^{-\varepsilon t}u\|_{LE} &\simeq \sup_{j\geq 0} 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}\times A_j)} \\
&\simeq \|2^{-\frac{3j}{2}}e^{-k}\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l^\infty_j l^2_k} \\
&\lesssim \|2^{-\frac{3j}{2}}e^{-k}\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l^2_k l^\infty_j}
\end{aligned}$$

We also note that

$$\begin{aligned}
\|f\|_{LE^*} &\simeq \sum_{l\geq 0} 2^{\frac{l}{2}}\|f\|_{L^2L^2(\mathbb{R}\times A_l)} \simeq \sum_{l\geq 0} 2^{\frac{l}{2}}\left(\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)f\|^2_{L^2L^2(\mathbb{R}\times A_l)}\right)^{\frac{1}{2}} \\
&\simeq \|2^{\frac{l}{2}}\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)f\|_{L^2L^2(\mathbb{R}\times A_l)}\|_{l^1_l l^2_k} \gtrsim \|2^{\frac{l}{2}}\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)f\|_{L^2L^2(\mathbb{R}\times A_l)}\|_{l^2_k l^1_l}
\end{aligned}$$

This means that it suffices to prove the inequality

$$\begin{aligned}
\|2^{-\frac{j}{2}}e^{-k}\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l^2_k l^\infty_j} + \|2^{-\frac{3j}{2}}e^{-k}\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l^2_k l^\infty_j} \\
\lesssim \|2^{\frac{l}{2}}\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)f\|_{L^2L^2(\mathbb{R}\times A_l)}\|_{l^2_k l^1_l}
\end{aligned}$$

As u_k is supported in $[k,\infty)$, for every $j\geq 0$ we have

$$\begin{aligned}
\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)} &= \|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)\sum_{k'\in\mathbb{Z}}\nabla_{t,x}u_{k'}\|_{L^2L^2(\mathbb{R}\times A_j)} \\
&= \|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)\sum_{k'\leq k}\nabla_{t,x}u_{k'}\|_{L^2L^2(\mathbb{R}\times A_j)} \\
\|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)u\|_{L^2L^2(\mathbb{R}\times A_j)} &= \|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)\sum_{k'\in\mathbb{Z}}u_{k'}\|_{L^2L^2(\mathbb{R}\times A_j)} \\
&= \|\mathbf{1}_{[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon})}(t)\sum_{k'\leq k}u_{k'}\|_{L^2L^2(\mathbb{R}\times A_j)}
\end{aligned}$$

We now note that

$$
\begin{aligned}
&\|2^{-\frac{j}{2}}e^{-k}\|\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l_k^2l_j^\infty} = \left(\sum_{k\in\mathbb{Z}} e^{-2k}\sup_{j\geq 0} 2^{-j}\|\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)\nabla_{t,x}u\|^2_{L^2L^2(\mathbb{R}\times A_j)}\right)^{\frac{1}{2}}\\
&= \left(\sum_{k\in\mathbb{Z}} e^{-2k}\sup_{j\geq 0} 2^{-j}\|\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)\sum_{k'\leq k}\nabla_{t,x}u_{k'}\|^2_{L^2L^2(\mathbb{R}\times A_j)}\right)^{\frac{1}{2}}\\
&\lesssim \left(\sum_{k\in\mathbb{Z}} e^{-2k}\left(\sum_{k'\leq k}\sup_{j\geq 0} 2^{-\frac{j}{2}}\|\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)\nabla_{t,x}u_{k'}\|_{L^2L^2(\mathbb{R}\times A_j)}\right)^2\right)^{\frac{1}{2}}\\
&\lesssim \left(\sum_{k\in\mathbb{Z}} e^{-2k}\left(\sum_{k'\leq k}\sup_{j\geq 0} 2^{-\frac{j}{2}}\|\nabla_{t,x}u_{k'}\|_{L^2L^2(\mathbb{R}\times A_j)}\right)^2\right)^{\frac{1}{2}}\\
&\lesssim \left(\sum_{k\in\mathbb{Z}} e^{-2k}\left(\sum_{k'\leq k}\sum_{l\geq 0} 2^{\frac{l}{2}}\|\mathbf{1}_{\left[\frac{k'}{\varepsilon},\frac{k'+1}{\varepsilon}\right)}(t)f\|_{L^2L^2(\mathbb{R}\times A_l)}\right)^2\right)^{\frac{1}{2}}\\
&\lesssim \left(\sum_{k\in\mathbb{Z}}\left(\sum_{k'\leq k} e^{-(k-k')}\sum_{l\geq 0} 2^{\frac{l}{2}}\|e^{-\varepsilon t}\mathbf{1}_{\left[\frac{k'}{\varepsilon},\frac{k'+1}{\varepsilon}\right)}(t)f\|_{L^2L^2(\mathbb{R}\times A_l)}\right)^2\right)^{\frac{1}{2}}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\|2^{-\frac{3j}{2}}e^{-k}\|\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l_k^2l_j^\infty}\\
&\quad\lesssim \left(\sum_{k\in\mathbb{Z}}\left(\sum_{k'\leq k} e^{-(k-k')}\sum_{l\geq 0} 2^{\frac{l}{2}}\|e^{-\varepsilon t}\mathbf{1}_{\left[\frac{k'}{\varepsilon},\frac{k'+1}{\varepsilon}\right)}(t)f\|_{L^2L^2(\mathbb{R}\times A_l)}\right)^2\right)^{\frac{1}{2}}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|2^{-\frac{j}{2}}e^{-k}\|\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l_k^2l_j^\infty} + \|2^{-\frac{3j}{2}}e^{-k}\|\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l_k^2l_j^\infty}\\
&\quad\lesssim \left(\sum_{k\in\mathbb{Z}}\left(\sum_{k'\leq k} e^{-(k-k')}\sum_{l\geq 0} 2^{\frac{l}{2}}\|e^{-\varepsilon t}\mathbf{1}_{\left[\frac{k'}{\varepsilon},\frac{k'+1}{\varepsilon}\right)}(t)f\|_{L^2L^2(\mathbb{R}\times A_l)}\right)^2\right)^{\frac{1}{2}}
\end{aligned}
$$

However, the sequence $(c_k)_{k\geq 0}$ given by $c_k = e^{-k}$ is l^1_k, and $\|c_k\|_{l^1_k} = (1-e^{-1})^{-1}$. By Young's inequality, we get

$$\begin{aligned}
&\|2^{-\frac{j}{2}}e^{-k}\|\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l^2_k l^\infty_j} + \|2^{-\frac{3j}{2}}e^{-k}\|\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)u\|_{L^2L^2(\mathbb{R}\times A_j)}\|_{l^2_k l^\infty_j}\\
&\lesssim \left(\sum_{k\in\mathbb{Z}}\left(\sum_{k'\leq k} e^{-(k-k')}\sum_{l\geq 0}2^{\frac{l}{2}}\|e^{-\varepsilon t}\mathbf{1}_{\left[\frac{k'}{\varepsilon},\frac{k'+1}{\varepsilon}\right)}(t)f\|_{L^2L^2(\mathbb{R}\times A_l)}\right)^2\right)^{\frac{1}{2}}\\
&\lesssim \|c_k\|_{l^1_k}\left(\sum_{k\in\mathbb{Z}}\left(\sum_{l\geq 0}2^{\frac{l}{2}}\|e^{-\varepsilon t}\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)f\|_{L^2L^2(\mathbb{R}\times A_l)}\right)^2\right)^{\frac{1}{2}}\\
&\lesssim \left(\sum_{k\in\mathbb{Z}}\left(\sum_{l\geq 0}2^{\frac{l}{2}}\|e^{-\varepsilon t}\mathbf{1}_{\left[\frac{k}{\varepsilon},\frac{k+1}{\varepsilon}\right)}(t)f\|_{L^2L^2(\mathbb{R}\times A_l)}\right)^2\right)^{\frac{1}{2}}
\end{aligned}$$

Along with the previous discussion, this implies that

$$\|e^{-\varepsilon t}\nabla_{t,x}u\|_{LE} + \|\langle r\rangle^{-1}e^{-\varepsilon t}u\|_{LE} \lesssim \|e^{-\varepsilon t}f\|_{LE^*}, \forall\varepsilon > 0$$

We now prove the converse. As f is supported away from $\{t = -\infty\}$, there exists $m \in \mathbb{Z}$ such that $\operatorname{supp} f \subseteq \{t \in [m,\infty)\}$. For every $\varepsilon > 0$, and $j \geq 0$, we have

$$2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}\times A_j)} \lesssim \|e^{-\varepsilon t}f\|_{LE^*} \lesssim e^{-\varepsilon m}\|f\|_{LE^*}$$

For every $n \in \mathbb{Z}$, we get that

$$\begin{aligned}
&2^{-\frac{j}{2}}e^{-\varepsilon n}\|\nabla_{t,x}u\|_{L^2L^2((-\infty,n]\times A_j)} + 2^{-\frac{j}{2}}e^{-\varepsilon n}\|\langle r\rangle^{-1}u\|_{L^2L^2((-\infty,n]\times A_j)}\\
&\lesssim 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2((-\infty,n]\times A_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}e^{-\varepsilon t}u\|_{L^2L^2((-\infty,n]\times A_j)}\\
&\lesssim 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}\times A_j)}\\
&\lesssim e^{-\varepsilon m}\|f\|_{LE^*}
\end{aligned}$$

Thus,

$$\begin{aligned}
&2^{-\frac{j}{2}}e^{-\varepsilon n}\|\nabla_{t,x}u\|_{L^2L^2((-\infty,n]\times A_j)} + 2^{-\frac{j}{2}}e^{-\varepsilon n}\|\langle r\rangle^{-1}u\|_{L^2L^2((-\infty,n]\times A_j)}\\
&\lesssim e^{-\varepsilon m}\|f\|_{LE^*}
\end{aligned}$$

We now take the limit at 0 with respect to ε, and we get that

$$2^{-\frac{j}{2}}\|\nabla_{t,x}u\|_{L^2L^2((-\infty,n]\times A_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}u\|_{L^2L^2((-\infty,n]\times A_j)} \lesssim \|f\|_{LE^*}$$

By taking the limit at ∞ with respect to n, we deduce that

$$2^{-\frac{j}{2}}\|\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}\times A_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}u\|_{L^2L^2(\mathbb{R}\times A_j)} \lesssim \|f\|_{LE^*}$$

We now immediately get that

$$\|\nabla_{t,x}u\|_{LE} + \|\langle r\rangle^{-1}u\|_{LE} \lesssim \|f\|_{LE^*},$$

as claimed.

Step 3: Reduction to a form for which Plancherel's Theorem can be applied. We define $A'_0 = A_0$, and $A'_j = \{x \in \mathbb{R}^d | 2^{j-1} < |x| < 2^{j+1}\}$. We note that $A'_k \subset A_{k-1} \cup A_k$ for every $k \geq 1$. We also consider a smooth partition of unity $(\chi_k)_{k\geq 0}$, with $0 \leq \chi_k \leq 1$, $\displaystyle\sum_{k=0}^{\infty}\chi_k = 1$, and $\operatorname{supp}\chi_k \subseteq A'_k$ for every $k \geq 0$.

We claim that the condition

$$\|e^{-\varepsilon t}\nabla_{t,x}u\|_{LE} + \|\langle r\rangle^{-1}e^{-\varepsilon t}u\|_{LE} \lesssim \|e^{-\varepsilon t}f\|_{LE^*}$$

is equivalent to

$$2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)} + 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)} \lesssim 2^{\frac{k}{2}}\|e^{-\varepsilon t}f\|_{L^2L^2(\mathbb{R}_t\times A'_k)}$$

For one implication, we note that if we fix $j, k \geq 0$, and if f is supported on A'_k, we have (when $j \geq 1$)

$$\begin{aligned}
&2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)} + 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)}\\
&\quad\lesssim 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A_{j-1})} + 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A_{j-1})}\\
&\quad+ 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A_j)} + 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A_j)}\\
&\quad\lesssim 2^{\frac{k}{2}}\|e^{-\varepsilon t}f\|_{L^2L^2(\mathbb{R}_t\times A_k)} \lesssim 2^{\frac{k}{2}}\|e^{-\varepsilon t}f\|_{L^2L^2(\mathbb{R}_t\times A'_k)}
\end{aligned}$$

and

$$\begin{aligned}
&\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A'_0)} + 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A'_0)}\\
&\quad\lesssim 2^{\frac{k}{2}}\|e^{-\varepsilon t}f\|_{L^2L^2(\mathbb{R}_t\times A_k)} \lesssim 2^{\frac{k}{2}}\|e^{-\varepsilon t}f\|_{L^2L^2(\mathbb{R}_t\times A'_k)}
\end{aligned}$$

when $j = 0$.

For the converse, we denote by u^k be the forward solution corresponding to $\chi_k f$. Thus, $u = \sum_k u^k$. For every $j \geq 0$ we have

$$\begin{aligned}
&2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A_j)}\\
&\quad\lesssim 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A_j)} + 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A_j)}\\
&\quad\lesssim 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)} + 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)}\\
&\quad\lesssim \sum_{k\geq 0} 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u^k\|_{L^2L^2(\mathbb{R}_t\times A'_j)} + 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u^k\|_{L^2L^2(\mathbb{R}_t\times A'_j)}\\
&\quad\lesssim \sum_{k\geq 0} 2^{\frac{k}{2}}\|e^{-\varepsilon t}\chi_k f\|_{L^2L^2(\mathbb{R}_t\times A'_k)}\\
&\quad\lesssim \|e^{-\varepsilon t}\chi_k f\|_{L^2L^2(\mathbb{R}_t\times A'_0)} + \sum_{k\geq 1} 2^{\frac{k}{2}}\|e^{-\varepsilon t}\chi_k f\|_{L^2L^2(\mathbb{R}_t\times A'_k)}\\
&\quad\lesssim \|e^{-\varepsilon t}\chi_k f\|_{L^2L^2(\mathbb{R}_t\times A_0)} + \sum_{k\geq 1} 2^{\frac{k}{2}}(\|e^{-\varepsilon t}\chi_k f\|_{L^2L^2(\mathbb{R}_t\times A_k)} + \|e^{-\varepsilon t}\chi_k f\|_{L^2L^2(\mathbb{R}_t\times A_{k-1})})\\
&\quad\lesssim \|e^{-\varepsilon t}f\|_{LE^*}
\end{aligned}$$

By taking the supremum in $j \geq 0$, we immediately deduce that

$$\|e^{-\varepsilon t}\nabla_{t,x}u\|_{LE} + \|\langle r\rangle^{-1}e^{-\varepsilon t}u\|_{LE} \lesssim \|e^{-\varepsilon t}f\|_{LE^*},$$

as claimed.

Step 4: Concluding the proof. We now prove that the condition

$$2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)} + 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)} \lesssim 2^{\frac{k}{2}}\|e^{-\varepsilon t}f\|_{L^2L^2(\mathbb{R}_t\times A'_k)}, \forall j, k \geq 0$$

is equivalent to the spectral characterization.

Let now f be a function supported in A'_k, and u the associated forward solution. For every $j \geq 0$, we have

$$\begin{aligned}
&2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)} + 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\langle r\rangle^{-1}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)}\\
&\lesssim 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla_{t,x}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)} + 2^{-\frac{3j}{2}}\|e^{-\varepsilon t}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)}\\
&\lesssim 2^{\frac{k}{2}}\|e^{-\varepsilon t}f\|_{L^2L^2(\mathbb{R}_t\times A'_k)}
\end{aligned}$$

We take the Fourier transform in time and obtain (via Plancherel's Theorem) that

$$\begin{aligned}
&2^{-\frac{j}{2}}\|(|\tau-i\varepsilon|,\nabla_x)\widehat{u}(\tau-i\varepsilon)\|_{L^2L^2(\mathbb{R}_\tau\times A'_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}\widehat{u}(\tau-i\varepsilon)\|_{L^2L^2(\mathbb{R}_t\times A'_j)}\\
&\qquad\lesssim 2^{\frac{k}{2}}\|\widehat{f}(\tau-i\varepsilon)\|_{L^2L^2(\mathbb{R}_\tau\times A'_k)}
\end{aligned}$$

In particular, this holds for every function f of the form $f(t,x)=\phi(t)g(x)$, with ϕ supported away from $\{t=-\infty\}$, and g supported in A'_k. In this case, $\widehat{u}(\tau-i\varepsilon)=\mathbf{R}_{\tau-i\varepsilon}(\widehat{\phi}(\tau-i\varepsilon)g(x)) = \widehat{\phi}(\tau-i\varepsilon)\mathbf{R}_{\tau-i\varepsilon}(g(x))$. Thus, for every $j,k\geq 0$,

$$\begin{aligned}
&2^{-\frac{j}{2}}\|\widehat{\phi}(\tau-i\varepsilon)\|(|\tau-i\varepsilon|,\nabla_x)\mathbf{R}_{\tau-i\varepsilon}g(x)\|_{L^2(A'_j)}\|_{L^2(\mathbb{R}_\tau)}\\
&\qquad+2^{-\frac{j}{2}}\|\widehat{\phi}(\tau-i\varepsilon)\|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}g(x)\|_{L^2(A'_j)}\|_{L^2(\mathbb{R}_\tau)}\\
&\qquad\lesssim 2^{\frac{k}{2}}\|\widehat{\phi}(\tau-i\varepsilon)\|g(x)\|_{L^2(A'_k)}\|_{L^2(\mathbb{R}_\tau)}
\end{aligned}$$

This shows that

$$2^{-\frac{j}{2}}\|(|\tau-i\varepsilon|,\nabla_x)\mathbf{R}_{\tau-i\varepsilon}g(x)\|_{L^2(A'_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}g(x)\|_{L^2(A'_j)} \lesssim 2^{\frac{k}{2}}\|g(x)\|_{L^2(A'_k)}$$

For arbitrary g, we take a partition of unity $(\chi_k)_{k\geq 0}$, with $0\leq\chi_k\leq 1$, $\sum_{k=0}^{\infty}\chi_k=1$, and $\operatorname{supp}\chi_k\subseteq A_k$ for every $k\geq 0$. The previous discussion implies that for every $k\geq 0$,

$$\|(|\tau-i\varepsilon|,\nabla_x)\mathbf{R}_{\tau-i\varepsilon}(\chi_k g)\|_{\mathcal{LE}} + \|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}(\chi_k g)\|_{\mathcal{LE}} \lesssim 2^{\frac{k}{2}}\|\chi_k g\|_{L^2L^2(\mathbb{R}_t\times A'_k)}.$$

Therefore,

$$\begin{aligned}
\|(|\tau-i\varepsilon|,\nabla_x)\mathbf{R}_{\tau-i\varepsilon}g\|_{\mathcal{LE}} + \|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}g\|_{\mathcal{LE}} &\leq \sum_{k=0}^{\infty}\|(|\tau-i\varepsilon|,\nabla_x)\mathbf{R}_{\tau-i\varepsilon}(\chi_k g)\|_{\mathcal{LE}}\\
&\quad+\sum_{k=0}^{\infty}\|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}(\chi_k g)\|_{\mathcal{LE}} \lesssim \sum_{k=0}^{\infty}2^{\frac{k}{2}}\|\chi_k g\|_{L^2L^2(\mathbb{R}_t\times A'_k)}\\
&\lesssim \|\chi_0 g\|_{L^2L^2(\mathbb{R}_t\times A'_0)} + \sum_{k=1}^{\infty}2^{\frac{k}{2}}(\|\chi_k g\|_{L^2L^2(\mathbb{R}_t\times A_{k-1})} + \|\chi_k g\|_{L^2L^2(\mathbb{R}_t\times A_k)})\\
&\lesssim \|g\|_{L^2L^2(\mathbb{R}_t\times A_0)} + \sum_{k=1}^{\infty}2^{\frac{k}{2}}(\|g\|_{L^2L^2(\mathbb{R}_t\times A_{k-1})} + \|g\|_{L^2L^2(\mathbb{R}_t\times A_k)})\\
&\lesssim \|g\|_{\mathcal{LE}^*}
\end{aligned}$$

Conversely, when f is a function that is supported away from $\{t=-\infty\}$ and in A'_k, then $\widehat{f}(\tau-i\varepsilon)$ is supported in A'_k as well, so the previous inequality implies that

$$\begin{aligned}
\|(|\tau-i\varepsilon|,\nabla_x)\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau-i\varepsilon)\|_{\mathcal{LE}} + \|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau-i\varepsilon)\|_{\mathcal{LE}} &\lesssim \|\widehat{f}(\tau-i\varepsilon)\|_{\mathcal{LE}^*}\\
&= 2^{\frac{k}{2}}\|\widehat{f}(\tau-i\varepsilon)\|_{L^2(A'_k)}
\end{aligned}$$

Thus, for every $j \geq 1$,

$$\begin{aligned}
&2^{-\frac{j}{2}}\|(|\tau - i\varepsilon|, \nabla_x)\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A'_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A'_j)} \\
&\quad\lesssim 2^{-\frac{j}{2}}\|(|\tau - i\varepsilon|, \nabla_x)\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A_j)} \\
&\quad+ 2^{-\frac{j}{2}}\|(|\tau - i\varepsilon|, \nabla_x)\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A_{j-1})} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A_{j-1})} \\
&\quad\lesssim 2^{\frac{k}{2}}\|\widehat{f}(\tau - i\varepsilon)\|_{L^2(A_k)} \lesssim 2^{\frac{k}{2}}\|\widehat{f}(\tau - i\varepsilon)\|_{L^2(A'_k)},
\end{aligned}$$

while for $j = 0$,

$$\begin{aligned}
&\|(|\tau - i\varepsilon|, \nabla_x)\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A'_0)} + \|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A'_0)} \\
&\quad\lesssim \|(|\tau - i\varepsilon|, \nabla_x)\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A_0)} + \|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A_0)} \\
&\quad\lesssim 2^{\frac{k}{2}}\|\widehat{f}(\tau - i\varepsilon)\|_{L^2(A_k)} \lesssim 2^{\frac{k}{2}}\|\widehat{f}(\tau - i\varepsilon)\|_{L^2(A'_k)},
\end{aligned}$$

Thus, for every $j, k \geq 0$

$$\begin{aligned}
&2^{-\frac{j}{2}}\|(|\tau - i\varepsilon|, \nabla_x)\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A'_j)} + 2^{-\frac{j}{2}}\|\langle r\rangle^{-1}\mathbf{R}_{\tau-i\varepsilon}\widehat{f}(\tau - i\varepsilon)\|_{L^2(A'_j)} \\
&\quad\lesssim 2^{\frac{k}{2}}\|\widehat{f}(\tau - i\varepsilon)\|_{L^2(A'_k)}
\end{aligned}$$

By taking the L^2-norm in τ and using Plancherel's identity, we deduce that

$$2^{-\frac{j}{2}}\|e^{-\varepsilon t}\nabla u\|_{L^2L^2(\mathbb{R}_t\times A'_j)} + 2^{-\frac{j}{2}}\|e^{-\varepsilon t}\langle r\rangle^{-1}u\|_{L^2L^2(\mathbb{R}_t\times A'_j)} \lesssim 2^{\frac{k}{2}}\|e^{-\varepsilon t}f\|_{L^2L^2(\mathbb{R}_t\times A'_k)}$$

This finishes the proof. □

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