LITTLEWOOD-PALEY THEORY OF FUNCTION SPACES

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Abstract. We present an approach to function spaces based on frequency space localisations. This approach reduces the classical results of Sobolev embedding and traces to elementary Littlewood-Paley theory and interpolation. We draw primarily from [Tri83], [BCD11], [WHHG11], and [Gra14].

Contents

1. Preliminaries

When studying function spaces, such as Lorentz spaces or Sobolev spaces, it is useful to decompose a generic function into simpler pieces, and attempt to prove the desired results for each of those pieces. For example, functions in Lorentz spaces can be decomposed in *physical space* into the sum of *quasi-step functions*. Our approach in these notes will be to decompose into *frequency-localised* pieces and study the various ways these pieces sum.

1.1. **Littlewood-Paley projections.**

We construct a dyadic partition of unity as follows; let $\phi \in C_c^\infty(\mathbb{R}^d)$ satisfy $0 \le \phi \le 1$ and

$$\phi(x) := \begin{cases} 1, & |x| \le 1.4, \\ 0, & |x| > 1.42. \end{cases}$$

Denote the dyadics by $2^{\mathbb{Z}} := \{2^n : n \in \mathbb{Z}\}$. For $N \in 2^{\mathbb{Z}}$, define $\psi, \psi_N, \phi_N \in C_c^\infty(\mathbb{R}^d)$ to be

$$\psi(x) := \phi(x) - \phi(2x), \qquad \psi_N(x) := \psi(x/N), \qquad \phi_N(x) := \phi(x/N).$$

Observe that $\sum_N \psi_N \equiv 1$ since pointwise it forms a telescoping sum. Given a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, we define its Littlewood-Paley projections to frequencies $|\xi| \sim N$ and $|\xi| \lesssim N$ respectively

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by

$$\widehat{u_N} = \widehat{P_N u} = \psi_N \widehat{u}, \qquad \widehat{u_{\le N}} = \widehat{P_{\le N} u} = \phi_N \widehat{u}.$$

Define the Littlewood-Paley projections to frequencies $|\xi| \gtrsim N$ and $N \lesssim |\xi| \lesssim M$ respectively by

$$u_{\ge N} = P_{\ge N} u = (1 - P_{\le N})u, \qquad u_{N \le - \le M} = P_{N \le - \le M} u = \sum_{N \le K \le M} P_K u.$$

Remark. By the Paley-Wiener theorem, the projections are analytic functions in physical space. Thus we can study the Littlewood-Paley projections pointwise without any philosophical consternation.

1.2. **Homogeneous spaces.** We will also need to define a suitable subspace of Schwartz space which behaves better with respect to scaling $u_\lambda(x) := u(x/\lambda)$. Define the HOMOGENEOUS SCHWARTZ SPACE $\dot{\mathcal{S}}(\mathbb{R}^d)$ as the space of Schwartz functions whose Fourier transform vanishes to every order at the origin,

$$\dot{\mathcal{S}}(\mathbb{R}^d) := \{u \in \mathcal{S}(\mathbb{R}^d) : \nabla^k \widehat{u}(0) = 0 \text{ for every } k\}.$$

We remark that the dual space is the space of tempered distributions modulo the space of polynomials $\mathcal{P}(\mathbb{R}^d)$.

Proposition 1. *The dual space of the homogeneous Schwartz space is*

$$\dot{\mathcal{S}}(\mathbb{R}^d)^* = \mathcal{S}(\mathbb{R}^d)^* / \mathcal{P}(\mathbb{R}^d).$$

Proof. Define $T : \mathcal{S}(\mathbb{R}^d)^* \to \dot{\mathcal{S}}(\mathbb{R}^d)^*$ be the restriction map

$$Tu := u_{|\dot{\mathcal{S}}(\mathbb{R}^d)}.$$

We claim that the restriction map is surjective and the kernel is precisely the space of polynomials. The first isomorphism theorem furnishes the result. Surjectivity follows from Hahn-Banach, so it remains to study the kernel. Suppose $u \in \ker T$, then by Plancharel

$$0 = \langle u, \phi \rangle = \langle \widehat{u}, \widehat{\phi} \rangle$$

for all $\phi \in \dot{\mathcal{S}}(\mathbb{R}^d)$. Since $\widehat{\phi}$ vanishes to infinite order at the origin, it follows that $\widehat{u}$ must be supported at the origin. Such distributions are derivatives of the Dirac mass at the origin, which upon applying the Fourier inverse shows that u is a polynomial. □

Remark. One can formulate the homogeneous function spaces as subspaces of $\dot{\mathcal{S}}(\mathbb{R}^d)^*$. This is equivalent to our approach of defining the spaces via metric completion of the homogeneous Schwartz space $\dot{\mathcal{S}}(\mathbb{R}^d)$.

Another way of dealing with the problem of low frequencies is to define the space of tempered distributions $\mathcal{S}_h(\mathbb{R}^d)^*$ such that

$$\lim_{\lambda \to \infty} ||\theta(\lambda \nabla)u||_{L^\infty} = 0$$

for all $\theta \in C^\infty(\mathbb{R}^d)$. Working in this space has the advantage of being a subspace of $\mathcal{S}(\mathbb{R}^d)^*$, rather than a quotient space. However, it has the disadvantage of not being a closed subspace. See Section 1.2.2 and Remark 2.26 in [BCD11] for further commentary.

2. Besov spaces

Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, we define the HOMOGENEOUS BESOV SPACE $\dot{B}^{s,p}_q(\mathbb{R}^d)$ as the completion of the homogeneous Schwartz space $\dot{\mathcal{S}}(\mathbb{R}^d)$ with respect to the norm

$$||u||_{\dot{B}^{s,p}_q} := \left|\left| ||N^s u_N||_{L^p_x} \right|\right|_{\ell^q_N(2^{\mathbb{Z}})} = \left(\sum_{N \in 2^{\mathbb{Z}}} ||N^s u_N||^q_{L^p} \right)^{1/q},$$

with the usual modification at the endpoints $p, q = \infty$. For the same range of exponents, we analogously define the INHOMOGENEOUS BESOV SPACE $B^{s,p}_q(\mathbb{R}^d)$ as the completion of Schwartz space $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm

$$||u||_{B^{s,p}_q} := ||u_{\le 1}||_{L^p} + \left|\left| ||N^s u_N||_{L^p_x} \right|\right|_{\ell^q_N(2^{\mathbb{N}})} = ||u_{\le 1}||_{L^p} + \left(\sum_{N \in 2^{\mathbb{N}}} ||N^s u_N||^q_{L^p} \right)^{1/q}.$$

2.1. **Properties.** We can read off some basic properties of Besov spaces from the definitions and properties of L^p-spaces and ℓ^q-spaces.

Proposition 2 (Monotonicity in s and q). *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$.*

(a) For $1 \leq q_1 \leq q_2 \leq \infty$, we have

$$||u||_{B^{s,p}_{q_2}} \lesssim_{q_1,q_2} ||u||_{B^{s,p}_{q_1}},$$
$$||u||_{\dot{B}^{s,p}_{q_2}} \lesssim_{q_1,q_2} ||u||_{\dot{B}^{s,p}_{q_1}}.$$

In particular, we have the continuous embeddings $B^{s,p}_{q_1}(\mathbb{R}^d) \hookrightarrow B^{s,p}_{q_2}(\mathbb{R}^d)$ and $\dot{B}^{s,p}_{q_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s,p}_{q_2}(\mathbb{R}^d)$.

(b) For $1 \leq q_1, q_2 \leq \infty$ and $\alpha > 0$, we have

$$||u||_{B^{s,p}_{q_2}} \lesssim_{\alpha} ||u||_{B^{s+\alpha,p}_{q_1}}.$$

In particular, we have the continuous embedding $B^{s+\alpha,p}_{q_1}(\mathbb{R}^d) \hookrightarrow B^{s,p}_{q_2}(\mathbb{R}^d)$.

Proof.

(a) This follows immediately from the embedding $\ell^{q_1} \hookrightarrow \ell^{q_2}$ and the definition of the Besov norms.

(b) By (a), it suffices to prove the result for $q_1 = \infty$ and $q_2 = 1$. We can write

$$\sum_{N\in 2^{\mathbb{N}}} ||N^s u_N||_{L^p} \leq \left|\left| N^{s+\alpha} ||u_N||_{L^p} \right|\right|_{\ell^\infty_N(2^{\mathbb{N}})} \sum_{N\in 2^{\mathbb{N}}} N^{-\alpha} \lesssim \left|\left| N^{s+\alpha} ||u_N||_{L^p} \right|\right|_{\ell^\infty_N(2^{\mathbb{N}})},$$

as desired.

□

The index s measures *regularity*, as such we should expect the Besov norm to measure the norms of derivatives, for example the Riesz potential $|\nabla|^s$ and the Bessel potential $\langle\nabla\rangle^s$. When these differential operators are applied to a Littlewood-Paley piece $P_N u$, we see that in frequency space they are comparable to multiplication by N^s and $\langle N\rangle^s$ respectively. This is made rigorous by the following lemma:

Lemma 3 (Sobolev-Bernstein inequalities). *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. Then*

$$|||\nabla|^s u_N||_{L^p} \sim N^s ||u_N||_{L^p},$$
$$||\langle\nabla\rangle^s u_N||_{L^p} \sim \langle N\rangle^s ||u_N||_{L^p}.$$

Proposition 4 (Lifting property). *Let $1 \leq p, q \leq \infty$ and $s, \sigma \in \mathbb{R}$. Then*

$$|||\nabla|^\sigma u||_{\dot{B}^{s-\sigma,p}_q} \sim ||u||_{\dot{B}^{s,p}_q}$$
$$||\langle\nabla\rangle^\sigma u||_{B^{s-\sigma,p}_q} \sim ||u||_{B^{s,p}_q},$$

That is, $|\nabla|^\sigma : \dot{B}^{s-\sigma,p}_q(\mathbb{R}^d) \to \dot{B}^{s,p}_q(\mathbb{R}^d)$ and $\langle\nabla\rangle^\sigma : B^{s-\sigma,p}_q(\mathbb{R}^d) \to B^{s,p}_q(\mathbb{R}^d)$ isomorphically.

Proof. Immediate from Sobolev-Bernstein. □

The expected duality relations hold for Besov spaces; the proofs are essentially identical to that for Lebesgue spaces and Sobolev spaces. Denote $1 \leq p', q' \leq \infty$ the dual exponents for $1 \leq p, q \leq \infty$, i.e.

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1.$$

Theorem 5 (Duality). *Let $1 \leq p, q < \infty$ and $s \in \mathbb{R}$, then*

$$\dot{B}^{s,p}_q(\mathbb{R}^d)^* \cong \dot{B}^{-s,p'}_{q'}(\mathbb{R}^d),$$
$$B^{s,p}_q(\mathbb{R}^d)^* \cong B^{-s,p'}_{q'}(\mathbb{R}^d).$$

Proof. [Tri83, Theorem 2.11.2] The inclusion $\dot{B}^{-s,p'}_{q'}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s,p}_q(\mathbb{R}^d)^*$ is essentially Holder's inequality while the reverse inclusion follows from the identification $\ell^q L^p(2^{\mathbb{Z}} \times \mathbb{R}^d)^* \cong \ell^{q'} L^{p'}(2^{\mathbb{Z}} \times \mathbb{R}^d)$ and Hahn-Banach. □

2.2. **Embeddings.** As with the Sobolev embedding inequalities, the Besov embeddings boil down to a trade of regularity for integrability. Our main tool will be Bernstein's inequality, which states that frequency localisation allows us to move from L^p-integrability to L^q-integrability at the cost of $(\frac{d}{p}-\frac{d}{q})$-many derivatives.

Lemma 6 (Bernstein's inequalities)**.** *Let $1 \le p \le q \le \infty$, then*

$$||u_N||_{L^q} \lesssim N^{\frac{d}{p}-\frac{d}{q}}||u_N||_{L^p},$$
$$||u_{\le N}||_{L^q} \lesssim N^{\frac{d}{p}-\frac{d}{q}}||u_{\le N}||_{L^p}.$$

Proof. Let $1 \le r \le \infty$ satisfy $\frac{1}{p}+\frac{1}{r}=\frac{1}{q}+1$, then by Young's convolution inequality and a change of variables $Nx = y$, we have the inequality

$$||P_N u||_{L^q} = ||u * \widecheck{\psi_N}||_{L^p} \le ||u||_{L^p}||N^d \check{\psi}(Nx)||_{L^r_x} = N^{d-\frac{d}{r}}||u||_{L^p}||\check{\psi}||_{L^r_y} \sim N^{\frac{d}{p}-\frac{d}{q}}||u||_{L^p}.$$

To obtain u_N instead of u on the right, observe that the same proof holds replacing P_N with the fattened projection $\widetilde{P_N}$. Since $\widetilde{P_N}P_N = P_N$, replacing u with $P_N u$ completes the proof. Arguing similarly furnishes the inequality replacing u_N with $u_{\le N}$. □

Theorem 7 (Homogeneous Besov embedding)**.** *Let $1 \le p_1 \le p_2 \le \infty$ and $1 \le q_1 \le q_2 \le \infty$ and $s_2 \le s_1$, then*

$$||u||_{\dot{B}^{s_2,p_2}_{q_2}} \lesssim ||u||_{\dot{B}^{s_1,p_1}_{q_1}}$$

whenever $\frac{s_1}{d}-\frac{1}{p_1}=\frac{s_2}{d}-\frac{1}{p_2}$. In particular, we have the continuous embedding $\dot{B}^{s_1,p_1}_{q_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_2,p_2}_{q_2}(\mathbb{R}^d)$.

Proof. By Proposition 2 (a), it suffices to prove the result for $q_1 = q_2 = q$. Recall Bernstein's inequality,

$$||u_N||_{L^{p_2}} \lesssim N^{\frac{d}{p_1}-\frac{d}{p_2}}||u_N||_{L^{p_1}}.$$

Applying the identity $\frac{s_1}{d}-\frac{1}{p_1}=\frac{s_2}{d}-\frac{1}{p_2}$, rearranging, and summing in $\ell^q(2^{\mathbb{Z}})$ furnishes the result. □

Theorem 8 (Inhomogeneous embedding)**.** *Let $1 \le p_1 \le p_2 \le \infty$ and $1 \le q_1 \le q_2 \le \infty$ and $s_2 \le s_1$, then*

$$||u||_{B^{s_2,p_2}_{q_2}} \lesssim ||u||_{B^{s_1,p_1}_{q_1}}$$

whenever $\frac{s_1}{d}-\frac{1}{p_1} \ge \frac{s_2}{d}-\frac{1}{p_2}$. In particular, we have the continuous embedding $B^{s_1,p_1}_{q_1}(\mathbb{R}^d) \hookrightarrow B^{s_2,p_2}_{q_2}(\mathbb{R}^d)$.

Proof. By Proposition 2 (a), it suffices to prove the result for $q_1 = q_2 = q$. Control over the low frequency term $u_{\le 1}$ follows from Bernstein's inequality. For high frequencies $N \in 2^{\mathbb{N}}$, we can apply the inequality $\frac{s_1}{d}-\frac{1}{p_1} \ge \frac{s_2}{d}-\frac{1}{p_2}$ with Bernstein's inequality to write

$$||u_N||_{L^{p_2}} \lesssim N^{\frac{d}{p_1}-\frac{d}{p_2}}||u_N||_{L^{p_1}} \le N^{s_1-s_2}||u_N||_{L^{p_1}}.$$

Rearranging and summing in $\ell^q(2^{\mathbb{N}})$ furnishes the result. □

2.3. **Traces.** In the interest of solving boundary value problems, we study the boundedness properties of the TRACE operator T which sends a function $u : \mathbb{R}^d \to \mathbb{C}$ to its restriction on a hyperplane $u_{|x_d=0} : \mathbb{R}^{d-1} \to \mathbb{C}$,

$$Tu(x') := u(x', 0) = \langle u, \delta_{x_d=0} \rangle$$

where we use the notation $x = (x_1, \dots, x_d)$ and $x' = (x_1, \dots, x_{d-1})$. We similarly denote $\mathcal{F}'$ the Fourier transform and P'_N the Littlewood-Paley projections on the hyperplane $\mathbb{R}^{d-1}$. As always, the main ingredient for studying the trace operator on Besov spaces is the uncertainty principle. This is encapsulated by the following two lemmas:

Lemma 9. *Given a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, the trace of the Littlewood-Paley projection $P_{\le N}u$ has Fourier support in $|\xi'| \lesssim N$. In particular,*

$$P'_M T P_{\le N} u = 0$$

for $M \gtrsim N$.

Proof. From the uncertainty principle, restricting to a point in physical space is equivalent to integrating in frequency space, that is, $\langle \delta_0, \phi \rangle = \langle 1, \widehat{\phi} \rangle$. Moreover, the composition of the Fourier transform on $\mathbb{R}^{d-1}_{x'}$ with the Fourier transform on $\mathbb{R}_{x_d}$ gives the Fourier transform on $\mathbb{R}^d_x$. Hence, in frequency space

$$\mathcal{F}' T P_{\leq N} u = \mathcal{F}' P_{\leq N} u(\xi', 0) = \int_{\mathbb{R}} \mathcal{F} P_{\leq N} u \, d\xi_d.$$

As $\mathcal{F} P_{\leq N} u$ is supported in the ball $|\xi| \lesssim N$, we see that $TP_N u$ has Fourier support in the ball $|\xi'| \lesssim N$. □

Lemma 10. *Given a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, the trace of the Littlewood-Paley projection $P_{\leq N} u$ satisfies*

$$||TP_{\leq N} u||_{L^p(\mathbb{R}^{d-1}_{x'})} \lesssim N^{\frac{1}{p}} ||P_{\leq N} u||_{L^p(\mathbb{R}^d_x)}$$

Proof. Since $\mathcal{F} P_{\leq N} u$ is supported in $|\xi| \lesssim N$, we see that $\mathcal{F}_{x_d} P_{\leq N} u$ is supported in $|\xi_d| \lesssim N$ for each fixed $x' \in \mathbb{R}^{d-1}$. Thus by Minkowski's integral inequality and Bernstein's inequality in one-dimension,

$$\begin{aligned} ||TP_{\leq N} u||_{L^p(\mathbb{R}^{d-1}_{x'})} &\leq \left|\left| ||P_{\leq N} u||_{L^\infty(\mathbb{R}_{x_d})} \right|\right|_{L^p(\mathbb{R}^{d-1}_{x'})} \\ &\lesssim \left|\left| N^{\frac{1}{p}} ||P_{\leq N} u||_{L^p(\mathbb{R}_{x_d})} \right|\right|_{L^p(\mathbb{R}^{d-1}_{x'})} = N^{\frac{1}{p}} ||P_{\leq N} u||_{L^p(\mathbb{R}^d_x)}. \end{aligned}$$

This completes the proof. □

Theorem 11 (Besov trace theorem)**.** *Let $1 < p < \infty$ and $1 \leq q \leq \infty$ and $s > \frac{1}{p}$, then the trace map satisfies*

$$||Tu||_{\dot{B}^{s-\frac{1}{p},p}_q(\mathbb{R}^{d-1})} \lesssim ||u||_{\dot{B}^{s,p}_q(\mathbb{R}^d)}.$$

In particular, it extends to a bounded linear operator $T : \dot{B}^{s,p}_q(\mathbb{R}^d) \to \dot{B}^{s-\frac{1}{p},p}_q(\mathbb{R}^{d-1})$. The analogous results also hold for the inhomogeneous Besov spaces.

Proof. We perform a Littlewood-Paley decomposition $u = \sum_N P_N u$. By linearity, we can write $P'_M Tu = \sum_N P'_M T P_N u$. Applying the triangle inequality, the previous two lemmas, and boundedness of the projections, we estimate

$$\begin{aligned} ||P'_M Tu||_{L^p(\mathbb{R}^{d-1}_{x'})} &\lesssim \sum_{N : M \lesssim N} ||P'_M T P_N u||_{L^p(\mathbb{R}^{d-1}_{x'})} \\ &\lesssim \sum_{N : M \lesssim N} N^{1/p} ||P_N u||_{L^p(\mathbb{R}^d_x)}. \end{aligned}$$

Multiplying both sides by $M^{s-1/p}$ and taking ℓ^q-norms, this reduces the problem to showing boundedness of the linear operator $A : \ell^q(2^{\mathbb{Z}}) \to \ell^q(2^{\mathbb{Z}})$ defined as

$$A(\{c_N\}_{N \in 2^{\mathbb{Z}}})_M := \sum_{N : M \lesssim N} N^{\frac{1}{p} - s} M^{s - \frac{1}{p}} c_N.$$

Taking $c_N := N^s ||P_N u||_{L^p}$ completes the proof. By Schur's test, we simply need to show that the corresponding kernel of the operator $K(N, M) := \mathbb{1}_{N \lesssim M} N^{1/p - s} M^{s - 1/p}$ is uniformly summable in N and M. Indeed

$$\begin{aligned} \sum_{N \in 2^{\mathbb{Z}}} K(N, M) &\sim \sum_{N : N \lesssim M} N^{\frac{1}{p} - s} M^{s - \frac{1}{p}} \sim 1, \\ \sum_{M \in 2^{\mathbb{Z}}} K(N, M) &\sim \sum_{M : N \lesssim M} N^{\frac{1}{p} - s} M^{s - \frac{1}{p}} \sim 1. \end{aligned}$$

The proof in the inhomogeneous spaces is the same modulo trivial modifications for low frequencies. □

Remark. This proof does not extend to the endpoints $p = 1, \infty$ due to the failure of the Littlewood-Paley decomposition $u = \sum_N P_N u$ to converge in $L^p(\mathbb{R}^d)$. Nonetheless, the result continues to hold at the endpoints, c.f. the maximal function argument of Triebel [Tri83, Theorem 2.7.2].

Theorem 12 (Besov trace extension theorem). *Let $1 \le p, q \le \infty$ and $s > \frac{1}{p}$, then there exists a trace extension operator, i.e. $TE = \mathrm{Id}$, satisfying*

$$||Ev||_{\dot{B}^{s,p}_q(\mathbb{R}^d)} \lesssim ||v||_{\dot{B}^{s-\frac{1}{p},p}_q(\mathbb{R}^{d-1})}.$$

In particular, it extends to a bounded linear operator $E : \dot{B}^{s-\frac{1}{p},p}_q(\mathbb{R}^{d-1}) \to \dot{B}^{s,p}_q(\mathbb{R}^d)$ and the trace operator is surjective. The analogous results hold for the inhomogeneous Besov spaces.

Proof. Let $\psi \in C^\infty_c((1,2))$ such that $\int \psi = 1$, and set $\psi_M(\xi_d) := \psi(M\xi_d)$. Notice then that the inverse Fourier transforms satisfy

$$\widecheck{\psi_M}(0) = \frac{1}{M}, \qquad ||\widecheck{\psi_M}||_{L^p(\mathbb{R})} \sim M^{\frac{1}{p}-1}.$$

Define the extension operator by

$$Ev(x', x_d) := \sum_{M \in 2^{\mathbb{Z}}} MP'_M v(x')\widecheck{\psi_M}(x_d).$$

It is clear from construction that $TE = \mathrm{Id}$, so it remains to show boundedness of the extension operator. Choosing projections P_N appropriately, we have $P_N(P'_M v\widecheck{\psi_M}) = 0$ whenever $N \neq M$. Combining this observation with Fubini's theorem, we obtain

$$N^{s-\frac{1}{p}}||P_N Ev||_{L^p(\mathbb{R}^d)} \lesssim N^{s-\frac{1}{p}+1}||P'_N v||_{L^p(\mathbb{R}^{d-1})}||\widecheck{\psi_N}||_{L^p(\mathbb{R})} \lesssim N^s||P'_N v||_{L^p(\mathbb{R}^{d-1})}.$$

Summing in ℓ^q_N completes the proof. Minor modifications furnishes the inhomogeneous case. □

2.4. **Example: Holder spaces.** The most familiar examples of Besov spaces are the Holder spaces $C^{k,\alpha}(\mathbb{R}^d)$ for $k \in \mathbb{N}_0$ and $0 < \alpha < 1$. As the proofs carry through the same, we will in fact introduce a slightly more general class of Holder spaces; denote the translates $u^h(x) := u(x-h)$, then the homogeneous Holder space $\dot{\Lambda}^{k+\alpha,p}(\mathbb{R}^d)$ is the completion of the homogeneous Schwartz space $\dot{\mathcal{S}}(\mathbb{R}^d)$ with respect to the norm

$$||u||_{\dot{\Lambda}^{k+\alpha,p}} := \sup_{h \neq 0} \frac{||\nabla^k u^h - \nabla^k u||_{L^p}}{|h|^\alpha}.$$

We analogously define the inhomogeneous Holder space $\Lambda^{k+\alpha,p}(\mathbb{R}^d)$ as the completion of Schwartz space $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm

$$||u||_{\Lambda^{k+\alpha,p}} := ||u||_{L^p} + \sup_{0<|h|<1} \frac{||\nabla^k u^h - \nabla^k u||_{L^p}}{|h|^\alpha}.$$

When $p = \infty$, this coincides with the familiar Holder spaces $C^{k,\alpha}(\mathbb{R}^d)$ of functions with α-Holder continuous derivatives up to order k.

Proposition 13. *Let $1 \le p \le \infty$ and $k \in \mathbb{N}_0$ and $0 < \alpha < 1$, then*

$$||u||_{\dot{\Lambda}^{k+\alpha,p}} \sim ||u||_{\dot{B}^{k+\alpha,p}_\infty},$$
$$||u||_{\Lambda^{k+\alpha,p}} \sim ||u||_{B^{k+\alpha,p}_\infty}.$$

In particular, $\dot{\Lambda}^{k+\alpha,p}(\mathbb{R}^d) = \dot{B}^{k+\alpha,p}_\infty(\mathbb{R}^d)$ and $\Lambda^{k+\alpha,p}(\mathbb{R}^d) = B^{k+\alpha,p}_\infty(\mathbb{R}^d)$.

Proof. As usual, we prove the homogeneous case, the inhomogeneous case is a trivial exercise. For simplicity, let us only consider the case $k = 0$. Using $\int \widecheck{\psi_N} = \psi_N(0) = 0$ and a change of variables $Ny = z$, we can write

$$\begin{aligned} u_N(x) = (u * \widecheck{\psi_N})(x) &= N^d \int_{\mathbb{R}^d} u(x-y)\check{\psi}(Ny)dy \\ &= \int_{\mathbb{R}^d} u(x - z/N)\check{\psi}(z)dz = \int_{\mathbb{R}^d} \Big(u(x-z/N) - u(x)\Big)\check{\psi}(z)dz. \end{aligned}$$

Taking the L^p-norm of the above, applying Minkowski's inequality and the Holder condition gives

$$||u_N||_{L^p} \leq \int_{\mathbb{R}^d} ||u^{z/N} - u||_{L^q} |\check{\psi}(z)| dz \lesssim ||u||_{\dot{\Lambda}^{k+\alpha}} N^{-\alpha}.$$

This proves the Holder norm controls the Besov norm. To prove the converse inequality, we decompose the difference $u^h - u$ into high and low frequencies; let $M \in 2^{\mathbb{Z}}$ satisfy $|h|^{-1} \sim M$, it follows from the triangle inequality that

$$||u^h - u||_{L^p} \leq ||u^h_{\leq M} - u_{\leq M}||_{L^p} + ||u^h_{\geq M} + u_{\geq M}||_{L^p}.$$

For the high frequency terms, we crudely estimate by the triangle inequality, noting $||u^h_K||_{L^p} = ||u_K||_{L^p}$,

$$||u^h_{\geq M} - u_{\geq M}||_{L^p} \lesssim \sum_{N \geq M} ||u_N||_{L^p} \lesssim \Big|\Big| ||N^\alpha u_N||_{L^p} \Big|\Big|_{\ell^\infty_N} \sum_{N \geq M} N^{-\alpha} \sim ||u||_{\dot{B}^{\alpha,p}_\infty} |h|^\alpha.$$

This furnishes the desired bound for the high frequency terms. For the low frequency terms, we can write using the fundamental theorem of calculus

$$u^h_{\leq M}(x) - u_{\leq M}(x) = h \cdot \int_0^1 \nabla u_{\leq M}(x - \theta h) d\theta.$$

Taking the L^p-norm of the above and applying the triangle, Minkowski, Sobolev-Bernstein inequalities, we obtain

$$||u^h_{\leq M} - u_{\leq M}||_{L^q} \leq |h| \sum_{N \leq M} ||\nabla u_N||_{L^q_x} \lesssim ||u||_{\dot{B}^{\alpha,p}_\infty} |h| \sum_{N \leq M} N^{1-\alpha} \sim ||u||_{\dot{B}^{\alpha,p}_\infty} |h|^\alpha.$$

This furnishes the desired bound for the low frequency terms, completing the proof. □

3. Triebel-Lizorkin spaces

Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$ and $s \in \mathbb{R}$, we define the HOMOGENEOUS TRIEBEL-LIZORKIN SPACE $\dot{F}^{s,p}_q(\mathbb{R}^d)$ as the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm

$$||u||_{\dot{F}^{s,p}_q} := \Big|\Big| ||N^s u_N||_{\ell^q_N} \Big|\Big|_{L^p_x} = \left|\left| \left(\sum_{N \in 2^{\mathbb{Z}}} |N^s u_N|^q \right)^{1/q} \right|\right|_{L^p}.$$

For the same range of exponents, we analogously define the INHOMOGENEOUS TRIEBEL-LIZORKIN SPACE $F^{s,p}_q(\mathbb{R}^d)$ as the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm

$$||u||_{F^{s,p}_q} := \Big|\Big| ||\langle N \rangle^s u_N||_{\ell^q_N} \Big|\Big|_{L^p_x} = ||u_{\leq 1}||_{L^p} + \left|\left| \left(\sum_{N \in 2^{\mathbb{N}}} |N^s u_N|^q \right)^{1/q} \right|\right|_{L^p}$$

3.1. Properties.

We can read off some basic properties of Triebel-Lizorkin spaces from the definitions and properties of L^p-spaces and ℓ^q-spaces.

Proposition 14 (Mononoticity in q and s). *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$.*

(a) For $1 \leq q_1 \leq q_2 \leq \infty$, we have

$$||u||_{F^{s,p}_{q_2}} \lesssim_{q_1,q_2} ||u||_{F^{s,p}_{q_1}},$$
$$||u||_{\dot{F}^{s,p}_{q_2}} \lesssim_{q_1,q_2} ||u||_{\dot{F}^{s,p}_{q_1}}.$$

In particular, we have the continuous embeddings $F^{s,p}_{q_1}(\mathbb{R}^d) \hookrightarrow F^{s,p}_{q_2}(\mathbb{R}^d)$ and $\dot{F}^{s,p}_{q_1}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s,p}_{q_2}(\mathbb{R}^d)$.

(b) For $1 \leq q_1, q_2 \leq \infty$ and $\alpha > 0$, we have

$$||u||_{F^{s,p}_{q_2}} \lesssim_\alpha ||u||_{F^{s+\alpha,p}_{q_1}}.$$

In particular, we have the continuous embedding $B^{s+\alpha,p}_{q_1}(\mathbb{R}^d) \hookrightarrow B^{s,p}_{q_2}(\mathbb{R}^d)$.

Proof.

(a) This follows immediately from the embedding $\ell^{q_1} \hookrightarrow \ell^{q_2}$ and the definition of the Triebel-Lizorkin norms.

(b) By (a), it suffices to prove the result for $q_1 = \infty$ and $q_2 = 1$. We can write

$$\sum_{N\in 2^{\mathbb{N}}} |N^s u_N| \leq \left|\left|N^{s+\alpha}|u_N|\right|\right|_{\ell^\infty_N(2^{\mathbb{N}})} \sum_{N\in 2^{\mathbb{N}}} N^{-\alpha} \lesssim \left|\left|N^{s+\alpha}|u_N|\right|\right|_{\ell^\infty_N(2^{\mathbb{N}})}.$$

Taking the L^p-norm of both sides, we conclude the result.

□

Proposition 15 (Lifting property). *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$ and $s, \sigma \in \mathbb{R}$. Then*

$$||\,|\nabla|^\sigma u||_{\dot{F}^{s-\sigma,p}_q} \sim ||u||_{\dot{F}^{s,p}_q}$$
$$||\langle\nabla\rangle^\sigma u||_{F^{s-\sigma,p}_q} \sim ||u||_{B^{s,p}_q},$$

That is, $|\nabla|^\sigma : \dot{F}^{s-\sigma,p}_q(\mathbb{R}^d) \to \dot{F}^{s,p}_q(\mathbb{R}^d)$ and $\langle\nabla\rangle^\sigma : F^{s-\sigma,p}_q(\mathbb{R}^d) \to F^{s,p}_q(\mathbb{R}^d)$ isomorphically.

Proof. Maximal function argument. □

The expected duality relations hold for Triebel-Lizorkin spaces; the proofs are essentially identical to that for Lebesgue spaces and Sobolev spaces.

Theorem 16 (Duality). *Let $1 \leq p < \infty$ and $1 < q < \infty$ and $s \in \mathbb{R}$, then*

$$\dot{F}^{s,p}_q(\mathbb{R}^d)^* \cong \dot{F}^{-s,p'}_{q'}(\mathbb{R}^d),$$
$$F^{s,p}_q(\mathbb{R}^d)^* \cong F^{-s,p'}_{q'}(\mathbb{R}^d).$$

Proof. [Tri83, Theorem 2.11.2] The inclusion $\dot{F}^{-s,p'}_{q'}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s,p}_q(\mathbb{R}^d)^*$ is essentially Holder's inequality while the reverse inclusion follows from the identification $\ell^q L^p(2^{\mathbb{Z}} \times \mathbb{R}^d)^* \cong \ell^{q'} L^{p'}(2^{\mathbb{Z}} \times \mathbb{R}^d)$ and Hahn-Banach. □

3.2. **Embeddings.** As the Triebel-Lizorkin norm is defined by first summing pointwise, we cannot easily apply Bernstein's inequality as we did in the case of the Besov embedding inequalities. Instead, we will rely on an interpolation argument due to [BM01].

Theorem 17 (Homogeneous Triebel-Lizorkin embedding). *Let $1 \leq p_1 < p_2 < \infty$ and $1 \leq q_1, q_2 \leq \infty$ and $s_2 < s_1$, then*

$$||u||_{\dot{F}^{s_2,p_2}_{q_2}} \lesssim ||u||_{\dot{F}^{s_1,p_1}_{q_1}}$$

whenever $\frac{s_1}{d} - \frac{1}{p_1} = \frac{s_2}{d} - \frac{1}{p_2}$. In particular, we have the continuous embedding $\dot{F}^{s_1,p_1}_{q_1}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s_2,p_2}_{q_2}(\mathbb{R}^d)$.

Proof. By Proposition 14 (a), it suffices to consider the case $q_1 = \infty$ and $q_2 = 1$. We argue by interpolating the $\dot{F}^{s_2,p_2}_1$-norm between the $\dot{B}^{s_1,\infty}_\infty$-norm and $\dot{F}^{s_1,p_1}_\infty$-norm. Set $s_0 := s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}$, then $s_2 = s_\theta$ is the interpolation exponent between s_0 and s_1 for $\theta = \frac{p_1}{p_2}$, i.e. $(1-\theta)s_0 + \theta s_1 = s_2$. We control each Littlewood-Paley projection pointwise depending on whether it is high frequency or low frequency,

$$|u_N| \leq \min\left\{N^{-s_0}\left|\left|A^{s_0}|u_A|\right|\right|_{\ell^\infty_A(2^{\mathbb{Z}})}, N^{-s_1}\left|\left|B^{s_1}|u_B|\right|\right|_{\ell^\infty_B(2^{\mathbb{Z}})}\right\}$$

where the transition occurs when $M^{-s_0}\left|\left|A^{s_0}|u_A|\right|\right|_{\ell^\infty_A(2^{\mathbb{Z}})} \sim M^{-s_1}\left|\left|B^{s_1}|u_B|\right|\right|_{\ell^\infty_B(2^{\mathbb{Z}})}$. We can therefore write

$$\begin{aligned}
\sum_{N\in 2^{\mathbb{Z}}} N^{s_2}|u_N(x)| &\lesssim \sum_{N\in 2^{\mathbb{Z}}} N^{s_2}\min\left\{N^{-s_0}\left|\left|A^{s_0}|u_A|\right|\right|_{\ell^\infty_A(2^{\mathbb{Z}})}, N^{-s_1}\left|\left|B^{s_1}|u_B|\right|\right|_{\ell^\infty_B(2^{\mathbb{Z}})}\right\} \\
&\leq \left|\left|A^{s_0}|u_A|\right|\right|_{\ell^\infty_A(2^{\mathbb{Z}})} \sum_{N\lesssim M} N^{s_2-s_0} + \left|\left|B^{s_1}|u_B|\right|\right|_{\ell^\infty_B(2^{\mathbb{Z}})} \sum_{N\gtrsim M} N^{s_2-s_1} \\
&\lesssim \left|\left|A^{s_0}|u_A|\right|\right|_{\ell^\infty_A(2^{\mathbb{Z}})} M^{s_2-s_0} + \left|\left|B^{s_1}|u_B|\right|\right|_{\ell^\infty_B(2^{\mathbb{Z}})} M^{s_2-s_1} \\
&\lesssim \left|\left|A^{s_0}|u_A|\right|\right|_{\ell^\infty_A(2^{\mathbb{Z}})} \left(\frac{\left|\left|B^{s_1}|u_B|\right|\right|_{\ell^\infty_B(2^{\mathbb{Z}})}}{\left|\left|A^{s_0}|u_A|\right|\right|_{\ell^\infty_A(2^{\mathbb{Z}})}}\right)^{\frac{s_2-s_0}{s_1-s_0}} + \left|\left|B^{s_1}|u_B|\right|\right|_{\ell^\infty_B(2^{\mathbb{Z}})} \left(\frac{\left|\left|B^{s_1}|u_B|\right|\right|_{\ell^\infty_B(2^{\mathbb{Z}})}}{\left|\left|A^{s_0}|u_A|\right|\right|_{\ell^\infty_A(2^{\mathbb{Z}})}}\right)^{\frac{s_2-s_1}{s_1-s_0}} \\
&\lesssim \left|\left|A^{s_0}|u_A|\right|\right|^{1-\theta}_{\ell^\infty_A(2^{\mathbb{Z}})} \left|\left|B^{s_1}|u_B|\right|\right|^{\theta}_{\ell^\infty_B(2^{\mathbb{Z}})}.
\end{aligned}$$

Taking the L^{p_2}-norm of both sides and applying Holder, we obtain

$$||u||_{\dot{F}^{s_2,p_2}_1} \lesssim \left|\left|A^{s_0}||u_A||_{L^\infty_x}\right|\right|^{1-\theta}_{\ell^\infty_A(2^{\mathbb{Z}})} \left|\left|\left|\left|B^{s_1}|u_B|\right|\right|^{\theta}_{\ell^\infty_B(2^{\mathbb{Z}})}\right|\right|_{L^{p_2}_x}.$$

By definition $\theta p_2 = p_1$, so the second term on the right can be written as the $\dot{F}^{s_1,p_1}_\infty$-norm. The first term on the right is the $\dot{B}^{s_1,\infty}_\infty$-norm which is controlled by the $\dot{F}^{s_1,p_1}_\infty$-norm via Bernstein's inequality,

$$\left|\left|A^{s_0}||u_A||_{L^\infty_x}\right|\right|_{\ell^\infty_A(2^{\mathbb{Z}})} \lesssim \left|\left|A^{s_1}||u_A||_{L^{p_1}}\right|\right|_{\ell^\infty_A(2^{\mathbb{Z}})} \leq ||u||_{\dot{F}^{s_1,p_1}_\infty}.$$

This completes the proof. □

Theorem 18 (Inhomogeneous Triebel-Lizorkin embedding)**.** *Let $1 \leq p_1 < p_2 < \infty$ and $1 \leq q_1, q_2 \leq \infty$ and $s_2 < s_1$, then*

$$||u||_{F^{s_2,p_2}_{q_2}} \lesssim ||u||_{F^{s_1,p_1}_{q_1}}$$

whenever $\frac{s_1}{d} - \frac{1}{p_1} \geq \frac{s_2}{d} - \frac{1}{p_2}$. In particular, we have the continuous embedding $F^{s_1,p_1}_{q_1}(\mathbb{R}^d) \hookrightarrow F^{s_2,p_2}_{q_2}(\mathbb{R}^d)$.

Proof. By Proposition 14 (a), it suffices to consider the case $q_1 = \infty$ and $q_2 = 1$, and by (b) we can assume without loss of generality the critical exponents $\frac{s_1}{d} - \frac{1}{p_1} = \frac{s_2}{d} - \frac{1}{p_2}$. Control over the low frequency term $u_{\leq 1}$ follows from Bernstein's inequality. For the high frequency terms $N \in 2^{\mathbb{N}}$, mimicking the proof of the homogeneous embedding inequality furnishes the result. □

Remark. Comparing with the Besov embedding inequalities, note the Triebel-Lizorking embeddings do not have the restriction $q_1 \leq q_2$ on the range of summability indices.

3.3. **Traces.** As with the embedding inequalities, studying the properties of the trace operator on Triebel-Lizorkin spaces is more complicated than that of Besov spaces. For conciseness of presentation, we will only consider the sub-critical problem.

Theorem 19 (Triebel-Lizorkin trace theorem)**.** *Let $1 < p < \infty$ and $1 \leq q \leq \infty$ and $s > \sigma > \frac{1}{p}$, then the trace map satisfies*

$$||Tu||_{B^{s-\sigma,p}_p(\mathbb{R}^{d-1})} \lesssim ||u||_{F^{s,p}_q(\mathbb{R}^d)}.$$

In particular, it extends to a bounded linear operator $T : F^{s,p}_q(\mathbb{R}^d) \to B^{s-\sigma,p}_p(\mathbb{R}^{d-1})$.

Proof. It suffices by Proposition 14 (a) to prove the result for $q = \infty$. We perform a Littlewood-Paley decomposition $u = \sum_N P_N u$. By linearity, we can write $P'_M Tu = \sum_N P'_M T P_N u$. Applying the triangle inequality, the previous two lemmas, and boundedness of the projections, we estimate

$$\begin{aligned} ||P'_M Tu||_{L^p(\mathbb{R}^{d-1}_{x'})} &\lesssim \sum_{N:M\lesssim N} ||P'_M T P_N u||_{L^p(\mathbb{R}^{d-1}_{x'})} \\ &\lesssim \sum_{N:M\lesssim N} N^{\frac{1}{p}} ||P_N u||_{L^p(\mathbb{R}^d_x)}. \end{aligned}$$

Multiplying both sides by $M^{s-\sigma}$ and taking ℓ^p-norms, we get

$$\begin{aligned} \Big|\Big| ||M^{s-\sigma} P_M Tu||_{L^p(\mathbb{R}^d_x)} \Big|\Big|_{\ell^p_M(2^{\mathbb{N}})} &\lesssim \Big|\Big| M^{s-\sigma} \sum_{N:M\lesssim N} N^{1/p} ||P_N u||_{L^p(\mathbb{R}^d_x)} \Big|\Big|_{\ell^p_M(2^{\mathbb{N}})} \\ &\lesssim \Big|\Big| \sum_{N:M\lesssim N} M^{s-\sigma} N^{\frac{1}{p}-s} \Big|\Big|_{\ell^p_M(2^{\mathbb{N}})} \Big|\Big| ||N^s P_N u||_{\ell^\infty_N} \Big|\Big|_{L^p_x} \lesssim \Big|\Big| ||N^s P_N u||_{\ell^\infty_N} \Big|\Big|_{L^p_x} \end{aligned}$$

This completes the proof. □

Remark. The trace theorem also holds for the critical exponent $\sigma = \frac{1}{p}$ and endpoints $p = 1, \infty$. Furthermore, the extension operator as defined in Theorem 12 admits the desired boundedness properties. See [Tri83, Theorem 2.7.2] for details.

3.4. **Example: Sobolev spaces.** The most familiar examples of Triebel-Lizorkin spaces are the Lebesgue spaces $L^p(\mathbb{R}^d)$ and the Sobolev spaces $W^{s,p}(\mathbb{R}^d)$. Let $1 < p < \infty$ and $s \in \mathbb{R}$, we define the HOMOGENEOUS SOBOLEV SPACE, also known as the Riesz potential space, $\dot{W}^{s,p}(\mathbb{R}^d)$, as the completion of the homogeneous Schwartz space $\dot{\mathcal{S}}(\mathbb{R}^d)$ with respect to the norm

$$||u||_{\dot{W}^{s,p}} := |||\nabla|^s u||_{L^p}.$$

Analogously, we define the INHOMOGENEOUS SOBOLEV SPACE, also known as the Bessel potential space, $W^{s,p}(\mathbb{R}^d)$, as the closure of Schwartz space $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm

$$||u||_{W^{k,p}} := ||\langle\nabla\rangle^s u||_{L^p}.$$

The realisations of these spaces on the Triebel-Lizorkin scale of function spaces follows from the *almost orthogonality* of the projection operators in form of the Littlewood-Paley square function estimate.

Lemma 20 (Littlewood-Paley square function estimate)**.** *For $1 < p < \infty$, we have*

$$||u||_{L^p} \sim \Big|\Big| \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s} |u_N|^2 \Big)^{\frac{1}{2}} \Big|\Big|_{L^p}.$$

In particular, $L^p(\mathbb{R}^d) = \dot{F}^{0,p}_2(\mathbb{R}^d) = F^{0,p}_2(\mathbb{R}^d)$.

Proof. We refer the interested reader to [Ste16, Chapter IV] or [Duo01, Theorem 8.4]. □

Remark. The estimate fails at the endpoints. For $p = \infty$, taking $f = 1$, we have $f_N = 0$ and therefore $S(f) = 0$. For $p = 1$, take $f = \phi_\varepsilon$ where $\{\phi_\varepsilon\}_\varepsilon \subseteq C^\infty_c(\mathbb{R}^d)$ is an approximation to the identity.

Theorem 21 (Triebel-Lizorkin characterisation of Sobolev spaces)**.** *Let $1 < p < \infty$ and $s \in \mathbb{R}$, then*

$$\begin{aligned} ||u||_{\dot{W}^{s,p}} &\sim ||u||_{\dot{F}^{s,p}_2}, \\ ||u||_{W^{s,p}} &\sim ||u||_{F^{s,p}_2}. \end{aligned}$$

In particular, $\dot{W}^{s,p}(\mathbb{R}^d) = \dot{F}^{s,p}_2(\mathbb{R}^d)$ and $W^{s,p}(\mathbb{R}^d) = F^{s,p}_2(\mathbb{R}^d)$.

Proof. We prove the homogeneous case, leaving the inhomogeneous case as an exercise. The symbol of the Littlewood-Paley projections $P_N = \psi_N(\nabla)$ is a smooth cut-off $\psi_N \in C^\infty_c(\mathbb{R}^d - 0)$. Note that the square function estimate continues to hold replacing it with any other smooth cut-off $\chi \in C^\infty_c(\mathbb{R}^d - 0)$, for example

$$\chi(\xi) = |\xi|^{-s}\psi(\xi), \qquad \chi_N(\xi) = \chi(\xi/N) = N^s \psi_N(\xi)|\xi|^{-s}.$$

Observe that $\widetilde{\chi_N} * g = N^s P_N |\nabla|^{-s} g$. Taking $g = |\nabla|^s u$, it follows from the square function estimate that

$$\Big|\Big|\Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}|u_N|^2\Big)^{\frac{1}{2}}\Big|\Big|_{L^p} \lesssim |||\nabla|^s u||_{L^p}.$$

For the converse, we argue by duality and the direction above;

$$\begin{aligned}
|||\nabla|^s u||_{L^p} &= \sup_{||g||_{L^{p'}}=1} \langle |\nabla|^s u, g\rangle = \sup_{||g||_{L^{p'}}=1} \sum_{N\in 2^{\mathbb{Z}}} \langle \widetilde{P_N} P_N |\nabla|^s u, g\rangle = \sup_{||g||_{L^{p'}}=1} \sum_{N\in 2^{\mathbb{Z}}} \langle N^s P_N |\nabla|^s u, N^{-s}\widetilde{P_N} g\rangle \\
&\le \sup_{||g||_{L^{p'}}=1} \int_{\mathbb{R}^d} \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}|u_N|^2\Big)^{\frac{1}{2}} \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{-2s}|\widetilde{P_N}|\nabla|^s g|^2\Big)^{\frac{1}{2}}\, dx \\
&\le \sup_{||g||_{L^{p'}}=1} \Big|\Big|\Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}|u_N|^2\Big)^{\frac{1}{2}}\Big|\Big|_{L^p} \Big|\Big|\Big(\sum_{N\in 2^{\mathbb{Z}}} N^{-2s}|\widetilde{P_N}|\nabla|^s g|^2\Big)^{\frac{1}{2}}\Big|\Big|_{L^{p'}} \lesssim \Big|\Big|\Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}|u_N|^2\Big)^{\frac{1}{2}}\Big|\Big|_{L^p},
\end{aligned}$$

where the first line we use self-adjointness, the second line follows from Cauchy-Schwartz, the third line follows from Holder's inequality. □

4. Embeddings between $B^{s,p}_q(\mathbb{R}^d)$ and $F^{s,p}_q(\mathbb{R}^d)$

The Besov and Triebel-Lizorkin spaces can be embedded into one another via applications of the sequence space embeddings and Minkowski's inequalities,

$$\Big|\Big|\sum_N |a_N|\Big|\Big|_{L^p} \le \sum_N ||a_N||_{L^p}, \qquad \text{when } 1 \le p \le \infty,$$

$$\Big|\Big|\sum_N |a_N|\Big|\Big|_{L^p} \ge \sum_N ||a_N||_{L^p} \qquad \text{when } 0 < p < 1.$$

Proposition 22. *Let $1 \le p < \infty$ and $1 \le q \le \infty$ and $s \in \mathbb{R}$, then*

$$||u||_{\dot{B}^{s,p}_{\max(p,q)}} \lesssim ||u||_{\dot{F}^{s,p}_q} \lesssim ||u||_{\dot{B}^{s,p}_{\min(p,q)}}.$$

In particular, we have $\dot{B}^{s,p}_{\min(p,q)}(\mathbb{R}^d) \hookrightarrow \dot{F}^{s,p}_q(\mathbb{R}^d) \hookrightarrow \dot{B}^{s,p}_{\max(p,q)}(\mathbb{R}^d)$. The inhomogeneous variants also hold.

Proof. It is clear that the norms coincide when $p = q$, so suppose otherwise. If $q < p$, then

$$\Big|\Big|\,||a_N||_{L^p_x}\Big|\Big|_{\ell^p_N} = \Big|\Big|\,||a_N||_{\ell^p_N}\Big|\Big|_{L^p_x} \lesssim \Big|\Big|\,||a_N||_{\ell^q_N}\Big|\Big|_{L^p_x} = \Big|\Big|\sum_N |a_N|^q\Big|\Big|^{\frac{1}{q}}_{L^{p/q}_x} \le \left(\sum_N |||a_N|^q||_{L^{p/q}_x}\right)^{\frac{1}{q}} = \Big|\Big|\,||a_N||_{L^p_x}\Big|\Big|_{\ell^q_N},$$

and if $p < q$, then

$$\Big|\Big|\,||a_N||_{L^p_x}\Big|\Big|_{\ell^q_N} = \left(\sum_N |||a_N|^q||_{L^{p/q}_x}\right)^{\frac{1}{q}} \le \Big|\Big|\sum_N |a_N|^q\Big|\Big|^{\frac{1}{q}}_{L^{p/q}_x} = \Big|\Big|\,||a_N||_{\ell^q_N}\Big|\Big|_{L^p_x} \le \Big|\Big|\,||a_N||_{\ell^p_N}\Big|\Big|_{L^p_x} = \Big|\Big|\,||a_N||_{L^p_N}\Big|\Big|_{\ell^p_N}.$$

Taking $a_N = N^s u_N$ completes the proof. □

5. Dimensional analysis

We now turn to the problem of showing that the Besov and Triebel-Lizorkin spaces in fact represent a diversity of function spaces via dimensional analysis. This is done by testing the norms against sums of Schwartz functions that are *modulated, rescaled, and translated*,

$$u(x) := \sum_{j=1}^n a_j e^{2\pi i x\cdot\xi_j} \chi\Big(\frac{x - x_j}{R_j}\Big).$$

Regularity. Let's first study the *regularity* exponent s by modulating a single Schwartz function χ which is frequency-localised to $|\xi| \ll 1$. Choose $|\xi_M| \sim M$ for $M \in 2^{\mathbb{N}}$, and define, equivalently in frequency space or physical space,

$$\widehat{f_M}(\xi) := \widehat{\chi}(\xi - \xi_M), \qquad f_M(x) := e^{2\pi i x\cdot\xi_M}\chi(x).$$

Observe that f_M is frequency-localised to $|\xi| \sim N$. It follows that $P_N f_M = f_M$ if and only if $N = M$, and vanishes otherwise, thus

$$||f_M||_{\dot{B}^{s,p}_q} = M^s ||\chi||_{L^p},$$
$$||f_N||_{\dot{F}^{s,p}_q} = M^s ||\chi||_{L^p}.$$

Integrability. To study the *integrability* exponent p, we argue by rescaling by $R \in 2^{\mathbb{Z}}$. Fix any Schwartz function $\chi \in \mathcal{S}(\mathbb{R}^d)$, and denote its rescaling by $\chi_R(x) := \chi(x/R)$. Since the Fourier transform satisfies $\mathcal{F}\chi_R = R^d(\mathcal{F}\chi)_{1/R}$ and similarly for its inverse, the Littlewood-Paley projections of the rescalings are $P_N\chi_R(x) = P_{RN}\chi(x/R)$. Making a change of variables $x/R \mapsto x$ and $RN \mapsto N$, we obtain

$$||\chi_R||_{\dot{B}^{s,p}_q} = R^{\frac{d}{p}-s}||\chi||_{\dot{B}^{s,p}_q},$$
$$||\chi_R||_{\dot{F}^{s,p}_q} = R^{\frac{d}{p}-s}||\chi||_{\dot{F}^{s,p}_q}.$$

Summability. The *summability* exponent q gives control over the number of frequency scales, thus it is a "logarithmic" quantity which is of lesser importance compared to regularity and integrability. Again, let $\chi \in \mathcal{S}(\mathbb{R}^d)$ frequency-localised to $|\xi| \ll 1$, and frequencies $|\xi_K| = K$ for dyadic integers $K \in 2^{\mathbb{N}}$. We define, equivalently in frequency space or physical space, g_N to be a superposition of frequency-localised pieces,

$$\widehat{g_N}(\xi) := \sum_{K=1}^{N} \widehat{\chi}(\xi - \xi_K), \qquad g_N(x) := \sum_{K=1}^{N} e^{2\pi i x\cdot\xi_K}\chi(x).$$

It follows that

$$||g_N||_{\dot{B}^{s,p}_q} = |\log_2 N|^{\frac{1}{q}}||\chi||_{L^p},$$
$$||g_N||_{\dot{F}^{s,p}_q} = |\log_2 N|^{\frac{1}{q}}||\chi||_{L^p}.$$

So far the dimensional analyses for the Besov and Triebel-Lizorkin spaces have been identical. To distinguish the two, we introduce spatial translation, which we remark commutes with the Littlewood-Paley projections. Let $|x_K| \ll_p |x_{2K}|$ be sufficiently separated, and define

$$\widehat{h_N}(\xi) := \sum_{K=1}^{N} e^{2\pi i \xi\cdot x_K}\widehat{\chi}(\xi - \xi_K), \qquad h_N(x) := \sum_{K=1}^{N} e^{2\pi i x\cdot\xi_K}\chi(x - x_K).$$

Compared to the previous scenario, the Besov norm is unchanged since taking the L^p-norm of each Littlewood-Paley piece is translation-invariant. However, the pieces constituting h_N are separated not only in frequency but also pointwise spatially. Thus its Triebel-Lizorkin norm is, up to a negligible error, the L^p-norm of $\log_2 N$-many disjoint identical masses. That is,

$$||h_N||_{\dot{B}^{s,p}_q} = |\log_2 N|^{\frac{1}{q}}||\chi||_{L^p},$$
$$||h_N||_{\dot{F}^{s,p}_q} \sim |\log_2 N|^{\frac{1}{p}}||\chi||_{L^p}.$$

Heuristically then, the q exponent only controls the number of frequency scales for the Triebel-Lizorkin norm when the scales are also physically localised, while the Besov norm is agnostic about where these scales are physically located due to the translation-invariance.

Proposition 23. *Let $1 \le q_1, q_2 \le \infty$ and $s_1, s_2 \in \mathbb{R}$.*

(a) If $1 \le p_1, p_2 \le \infty$, then $\dot{B}^{s_1,p_1}_{q_1}(\mathbb{R}^d) = \dot{B}^{s_2,p_2}_{q_2}(\mathbb{R}^d)$ if and only if $s_1 = s_2$, $p_1 = p_2$, and $q_1 = q_2$.

(b) If $1 \le p_1, p_2 < \infty$, then $\dot{F}^{s_1,p_1}_{q_1}(\mathbb{R}^d) = \dot{F}^{s_2,p_2}_{q_2}(\mathbb{R}^d)$ if and only if $s_1 = s_2$, $p_1 = p_2$, and $q_1 = q_2$.

(c) If $1 \leq p_1 < \infty$ *and* $1 \leq p_2 \leq \infty$*, then* $\dot{F}^{s_1,p_1}_{q_1}(\mathbb{R}^d) = \dot{B}^{s_2,p_2}_{q_2}(\mathbb{R}^d)$ *if and only if* $s_1 = s_2$ *and* $p_1 = p_2 = q_1 = q_2$*.*

The analogous statements also hold for the inhomogeneous spaces.

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