Revista Matemática Iberoamericana
Vol. 4, n.º 1, 1988

Lipschitz and bi-Lipschitz Functions

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Let $Q_0 = [0,1]^n$ be the unit cube in $\mathbb{R}^n$ and let $f: Q_0 \to \mathbb{R}^m$, $m \geqslant n$, have Lipschitz norm bounded by one,

$$(1) \qquad \|f\|_{\mathrm{Lip}} = \sup_{\substack{x,y \in Q_0 \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|} \leqslant 1.$$

Then classical results (see *e.g.* Federer [2] or Stein [5]) assert that

$$Df = \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j}$$

is defined almost everywhere on Q_0, and f may be recovered from Df via integration along line segments parallel to the axes. We also recall two classical qualitative results. Let $\mathcal{H}(\cdot)$ denote n dimensional Hausdorff measure and let h denote n dimensional Hausdorff content,

$$h(E) = \inf \sum_{j=1}^{\infty} c_n r_j^n$$

where the infimum is taken over all coverings of E by balls $B(x_j, r_j)$ with no restrictions on the radii r_j. Then $h(E) \leqslant \mathcal{H}(E)$ and if $E \subset \mathbb{R}^n$, $h(E) = \mathcal{H}(E)$. Sard's theorem asserts that $\mathcal{H}(\{f(x): \mathrm{Rank}\,(Df(x)) < n\}) = 0$. A slightly stronger result is that one can decompose

$$Q_0 = G \cup \bigcup_{j=1}^{\infty} K_j$$

where $\mathcal{H}(f(G)) = 0$ and f is bi-Lipschitz on each K_j, *i.e.* there are constants c_j such that

$$|f(x) - f(y)| \geqslant c_j |x - y|, \qquad x, y \in K_j.$$

A qualitative version of the last result was first given by Guy David in [1].

Theorem (Guy David [1]). *Suppose $f\colon Q_0 \to \mathbb{R}^n$ satisfies $\|f\|_{\text{Lip}} = 1$ and $\mathcal{H}(f(Q_0)) \geqslant \epsilon > 0$. Then there is $\delta = \delta(\epsilon) > 0$ and $K \subset Q_0$ such that $\mathcal{H}(K)$, $\mathcal{H}(f(K)) \geqslant \delta$ and*

$$|f(x) - f(y)| \geqslant \delta |x - y|, \qquad x, y \in K.$$

David's result was used to prove boundedness properties for singular integrals on certain surfaces $\mathcal{S} \subset \mathbb{R}^m$. If $\mathcal{S} = f(\mathbb{R}^n)$ where f is Lipschitz and satisfies some other criterion, the above theorem can be used to show that for all $x_0 \in \mathcal{S}$ and all $r > 0$, $\mathcal{S} \cap \{x \in \mathbb{R}^m\colon |x - x_0| \leqslant r\}$ contains a subset $K = K(x_0, r)$ such that $\mathcal{H}(K) \geqslant cr^n$ and such that singular integrals are known to be bounded operators on $L^2(K)$. Real variales methods are then used to show that singular integrals are bounded on $L^2(\mathcal{S})$. In this note we present a generalization and strengthening of David's theorem. Our proof is also shorter than David's.

Theorem. *Suppose $f\colon Q_0 \to \mathbb{R}^m$ satisfies $\|f\|_{\text{Lip}} = 1$. Then for each $\delta > 0$ there is $M(\delta) < \infty$ and there are closed sets $K_1, \ldots, K_M \subset Q_0$, $M \leqslant M(\delta)$, such that*

$$h\left(f\left(Q_0 \Big\backslash \bigcup_{j=1}^{M} K_j\right)\right) < \delta$$

and such that

$$|f(x) - f(y)| \geqslant \frac{\delta}{2} |x - y|, \qquad x, y \in K_j, \qquad 1 \leqslant j \leqslant M.$$

By using truncation methods, the theorem can be seen to have L^p analogues.

Corollary. *Suppose $f = (f_1, \ldots, f_m)\colon Q_0 \to \mathbb{R}^m$ is such that each f_j is in the Sobolev space $W^{1,n+\epsilon}$ (one derivative in $L^{n+\epsilon}$) with $\|f_j\|_{W^{1,n+\epsilon}} \leqslant 1$, $1 \leqslant j \leqslant m$. Then the conclusions of the theorem hold with $M = M(\epsilon, \delta)$.*

PROOF. Fix $N < \infty$ and build F such that $\|F\|_{\text{Lip}} \leqslant N$ and $\mathcal{H}(\{x\colon F(x) \neq f(x)\}) \leqslant cN^{-(n+\epsilon)}$. Then use the theorem plus the fact that for any $G \subset Q_0$, $\mathcal{H}(f(G)) \leqslant C\mathcal{H}(G)^{(\epsilon/n)/(1+\epsilon/n)}$. $\square$

The proof of the theorem is given in section 2. The main tool is a Littlewood-Paley inequality. The theorem can be used to obtain a different approach to Guy David's results on singular integrals; this will appear elsewhere.

It is with great sorrow that I dedicate this paper to the memory of my good friend José-Luis Rubio de Francia.

2. Proof of the Theorem

Let $F(x)$ be a real valued function on $\mathbb{R}^n$ and let $F(x,y)$ denote its Poisson (harmonic) extension to $\mathbb{R}^{n+1}_+ = \{(x,y): x \in \mathbb{R}^n, y > 0\}$. Also let

$$\nabla F(x,y) = (F_{x_1}(x,y), \dots, F_{x_n}(x,y), F_y(x,y))$$

denote the gradient of F. Then if $Q \subset \mathbb{R}^n$ is any cube with sidelength $\ell(Q)$, and if $R(Q) = Q \times (0, \ell(Q)]$, ∇F satisfies the well-known BMO type estimate

$$\iint_{R(Q)} |\nabla F|^2 y \, dx \, dy \leqslant C\mathcal{H}(Q) \|F\|^2_{L^\infty(\mathbb{R}^n)}. \tag{2.1}$$

See Fefferman-Stein [3] or Garnett's book [4] page 240 for the proof.

We denote by $\mathfrak{D}$ the collection of all dyadic cubes in $\mathbb{R}^n$, *i.e.* the collection of all cubes Q of form $\prod_{j=1}^n [a_j 2^{-k}, (a_j+1)2^{-k}]$ where a_j and k lie in $\mathbb{Z}$. For such a cube Q we denote by $\ell(Q) = 2^{-k}$ the sidelength of Q. From now on, all cubes will be dyadic. We also let

$$T(Q) = Q \times \left[\frac{1}{2}\ell(Q), \ell(Q)\right]$$

denote the top half of $R(Q)$. If $Q, Q' \in \mathfrak{D}$, we say that Q and Q' are semi-adjacent if $\ell(Q) = \ell(Q')$, $Q \cap Q' = \varnothing$, and there is $Q'' \in \mathfrak{D}$ with $\ell(Q'') = \ell(Q)$, such that $Q \cap Q'' \neq \varnothing$, $Q' \cap Q'' \neq \varnothing$. Then

(2.2) For each $Q \in \mathfrak{D}$ there are exactly $5^n - 3^n$ semi-adjacent cubes Q'.

Let f satisfy the hypotheses of the theorem; by Whitney's extension theorem (see [2] or [5]) we may assume f is defined on all of $\mathbb{R}^n$ and $\|f\|_{\mathrm{Lip}} \leqslant 1$ there. Write $f = (f_1, \dots, f_m)$ and $Df = \left(\dfrac{\partial f_j}{\partial x_k}\right)$. Let $F_{j,k}$ be the harmonic extension of $\dfrac{\partial f_j}{\partial x_k}$ to $\mathbb{R}^{n+1}_+$ and let

$$|\nabla Df| = \Big(\sum_{j,k} |\nabla F_{j,k}|^2\Big)^{1/2}.$$

Our next lemma says that if Q and Q' are semi-adjacent, $h(f(Q))$ is large but $f(Q)$ and $f(Q')$ are not well separated, then $|\nabla Df|$ must be large somewhere in $T(Q)$.

Lemma 2.1. *Suppose Q and Q' are semi-adjacent and $h(f(Q)) \geqslant \delta\mathcal{H}(Q)$. If there are $x \in Q$, $x' \in Q'$ such that*

$$|f(x) - f(x')| \leqslant \frac{\delta}{2}|x - x'|,$$

then

$$\iint_{T(Q)} |\nabla Df|^2 y\, dx\, dy \geqslant c(\delta)\mathcal{H}(Q),$$

where $c(\delta) > 0$ is a constant depending only on δ.

PROOF. Since the hypotheses and conclusions are dilation invariant, it is sufficient to treat the case where $Q = Q_0$ is the unit cube. Suppose that the lemma is false, so that there is a sequence of functions f_j satisfying the hypotheses of the lemma but such that

$$\iint_{T(Q_0)} |\nabla Df|^2 y\, dx\, dy \leqslant 2^{-j}.$$

By Arzelà-Ascoli we may assume the f_j converge uniformly to f on compacta. Then $\|f\|_{\mathrm{Lip}} \leqslant 1$, and since $h(f(Q_0)) \geqslant \liminf h(f_j(Q_0))$, f satisfies the hypotheses of the lemma. On the other hand,

$$\iint_{T(Q_0)} |\nabla Df|^2 y\, dx\, dy = 0,$$

so by the uniqueness principle for harmonic functions Df is constant a.e. on $\mathbb{R}^n$, and consequently f is linear on $\mathbb{R}^n$.

However, there is $x' \in Q'$ such that

$$|f(x) - f(x')| \leqslant \frac{\delta}{2}|x - x'|$$

for some $x \in Q_0$. This is not possible for a linear map satisfying $h(f(Q_0)) \geqslant \delta$ and $\|f\|_{\mathrm{Lip}} \leqslant 1$. $\square$

Let

$$G_1 = \left\{ Q \in \mathfrak{D} \colon Q \subset Q_0, h(f(Q)) \leqslant \frac{\delta}{2}\mathcal{H}(Q) \right\}$$

and let

$$G_1 = \bigcup_{Q \in \mathcal{G}_1} Q$$

so that

$$h(f(G_1)) \leqslant \frac{\delta}{2}. \tag{2.3}$$

The set G_1 is a «garbage» set to be thrown out. Let $\mathfrak{F}$ denote the collection of all unordered pairs of semi-adjacent cubes Q, Q' such that both $Q, Q' \notin G_1$, and such that Q, Q' satisfy the hypotheses of Lemma 2.1. We enumerate the collection $\mathfrak{F}$ by $\{(Q_1^1, Q_2^1), (Q_1^2, Q_2^2), (Q_1^3, Q_2^3), \dots\}$ where $\ell(Q_1^j) \geqslant \ell(Q_1^{j+1})$. Let $c(\delta)$ be the constant of Lemma 2.1, and let

$$G_2 = \left\{ x\colon \sum_{j=1}^{\infty} \sum_{k=1}^{2} \chi_{Q_k^j}(x) \geqslant C_1 \delta^{-1} c(\delta)^{-1} \right\},$$

where C_1 is a constant to be fixed later.

Lemma 2.2. $\mathcal{H}(G_2) \leqslant \delta/2$.

PROOF. By Chebychev's inequality,

$$\begin{aligned}
\mathcal{H}(G_2) &\leqslant C_1^{-1} \delta c(\delta) \sum_{j=1}^{\infty} \sum_{k=1}^{2} \mathcal{H}(Q_k^j) \\
&\leqslant C_1^{-1} \delta \sum_{j=1}^{\infty} \sum_{k=1}^{2} \iint_{T(Q_k^j)} |\nabla Df|^2 y \, dx \, dy \\
&= C_1^{-1} \delta \iint |\nabla Df|^2 \sum_{j,k} \chi_{T(Q_k^j)}(x, y) y \, dx \, dy \\
&\leqslant C_1^{-1} \delta C(n) \iint_{R(Q_0)} |\nabla Df|^2 y \, dx \, dy \\
&\leqslant C_1^{-1} \delta C(n) C.
\end{aligned}$$

The penultimate inequality follows from remark (2.2), while the final inequality results from Lemma 2.1. The proof is concluded by choosing $C_1 \geqslant 2C(n)C$. □

Setting $G = G_1 \cup G_2$, it follows from (2.3) and Lemma 2.2 that $h(f(G)) \leqslant \delta$. We now divide $Q_0 \setminus G$ into M disjoint compact a K_α so that f is bi-Lipschitz on each K_α. To do this we define inductively indices α_j. To each x will correspond $\alpha_j(x)$ and we will define

$$K_\alpha = \left\{x \in Q_0 \setminus G \colon \lim_{j \to \infty} \alpha_j(x) = \alpha\right\}.$$

Each $\alpha_j(x)$ will be a finite string of zeros and ones, $\alpha_j(x) = \{\epsilon_1^j(x), \dots, \epsilon_k^j(x)\}$, where $\epsilon_l^j = 0$ or 1.

At stage zero we define $\alpha_0(x) = \{0\}$ for all $x \in Q_0$. At stage one, let Q_1^1 and Q_2^1 be the first pair of cubes in $\mathfrak{F}$. Define

$$\begin{aligned} \alpha_1(x) &= \{0, 0\} && \text{on} \quad Q_1^1 \\ \alpha_1(x) &= \{0, 1\} && \text{on} \quad Q_2^1 \\ \alpha_1(x) &= \{0\} && \text{on} \quad Q_0 \setminus (Q_1^1 \cup Q_2^1). \end{aligned}$$

We suppose by induction that $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ have been defined and that α_{k-1} is constant on each $Q \in \mathfrak{D}$ with $\ell(Q) \leqslant \ell(Q_1^{k-1}) = \ell(Q_2^{k-1})$. Let Q_1^k and Q_2^k be the k^{th} pair of cubes in $\mathfrak{F}$, and suppose $\alpha_{k-1}(x) = \{\epsilon_1, \dots, \epsilon_s\}$ on Q_1^k, $\alpha_{k-1}(x) = \{\epsilon_1', \dots, \epsilon_t'\}$ on Q_2^k.

Case 1. $s = t$. Define

$$\begin{aligned} \alpha_k(x) &= \{\epsilon_1, \dots, \epsilon_s, 0\} && \text{on} \quad Q_1^k \\ \alpha_k(x) &= \{\epsilon_1', \dots, \epsilon_t', 1\} && \text{on} \quad Q_2^k \\ \alpha_k(x) &= \alpha_{k-1}(x) && \text{on} \quad Q_0 \setminus (Q_1^k \cup Q_2^k). \end{aligned}$$

Case 2. $s > t$. Define $\epsilon_{t+1}' = 2 - 2^{1+\epsilon_t}$ and

$$\begin{aligned} \alpha_k(x) &= \alpha_{k-1}(x) && \text{on} \quad Q_0 \setminus Q_2^k \\ \alpha_k(x) &= \{\epsilon_1', \dots, \epsilon_t', \epsilon_{t+1}'\} && \text{on} \quad Q_2^k. \end{aligned}$$

Case 3. $s < t$. Reverse the roles of Q_1^k and Q_2^k and apply Case 2.

The procedure guarantees that $\alpha_l(x)$ distinguishes between points in Q_1^k and Q_2^k whenever $l \geqslant k$. More precisely, if $l \geqslant k$, $Q_1 \subset Q_1^k$, $Q_2 \subset Q_2^k$, $\alpha_l(x) = \{\epsilon_1, \dots, \epsilon_u\}$ on Q_1, $\alpha_l(x) = \{\epsilon_1', \dots, \epsilon_v'\}$ on Q_2, then $u, v \geqslant k + 1$ and there is $j \leqslant k + 1$ such that

$$\epsilon_j \neq \epsilon_j'. \tag{2.4}$$

Let C_1 be the constant in the definition of G_2, and for $\alpha = \{\epsilon_1, \dots, \epsilon_s\}$ define $\rho(\alpha) = s$ to be the «length» of α. Then by the definition of G_2 and the α_j's,

$$\rho(\alpha_j(x)) \leqslant 1 + C_1 \delta^{-1} c(\delta)^{-1}$$

for all $x \in Q_0 \setminus G_2$. If $x \in Q_0 \setminus G$ we see therefore that $\alpha(x) = \lim_{j\to\infty} \alpha_j(x)$ is well defined and

$$\rho(\alpha(x)) \leqslant 1 + C_1 \delta^{-1} c(\delta)^{-1}.$$

For $s \leqslant 1 + C_1 \delta^{-1} c(\delta)^{-1}$ and $\alpha = \{\epsilon_1, \ldots, \epsilon_s\}$ a string of zeros and ones, define

$$K_\alpha = \{x \in Q_0 \setminus G: \alpha(x) = \alpha\}.$$

Then there are at most $M(\delta)$ sets K_α, so we need only check that f is bi-Lipschitz on each K_α. To this end, suppose $x, y \in K_\alpha$, but

$$|f(x) - f(y)| < \frac{\delta}{2} |x - y|.$$

Then $x \in Q$, $y \in Q'$ where Q and Q' are semi-adjacent. Since $x, y \notin G_1$, $h(f(Q)), h(f(Q')) \geqslant \ell(Q)$. Therefore the pair (Q, Q') must show up in $\mathfrak{F}$ as a pair (Q_1^k, Q_2^k). By (2.4), $\alpha_l(x) \neq \alpha_l(y)$ whenever $l \geqslant k$, so $\alpha(x) \neq \alpha(y)$. This contradiction completes the proof of the theorem. $\square$

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N.S.F. Grant DMS-86-02500.