ASYMPTOTIC STABILITY OF HARMONIC MAPS FOR THE LANDAU-LIFSHITZ EQUATION

JASON ZHAO

Abstract. Following [GNT10], we exposit the proof of asymptotic stability for harmonic maps under the Schrödinger maps equation in m-equivariant symmetry for $m \geq 3$.

Contents

1. Introduction

In [LL35], Landau and Lifshitz proposed a description for the dynamics of the magnetisation vector[1] in an isotropic ferromagnet. The eponymous *Landau-Lifshitz* equation is given by

$$\partial_t u = -\alpha\big(u \times (u \times \Delta u)\big) + \beta\big(u \times \Delta u\big), \tag{LL}$$

eq:LL

where $u : I \times \mathbb{R}^2 \to \mathbb{S}^2$ is the *magnetisation vector*. The parameter $\alpha \geq 0$ is the *Gilbert damping*, representing the strength of dissipation in the model, while the parameter $\beta \in \mathbb{R}$ is the *exchange constant*, representing the strength of dispersion. In the physics literature, the equation is often rescaled so that the parameters are balanced such that $\alpha^2 + \beta^2 = 1$, though mathematically this will be inconsequential for our discussion.

From the perspective of geometric equations, the family of Landau-Lifshitz equations interpolates between the dispersive *Schrödinger maps*, $(\alpha, \beta) = (1, 0)$, and the dissipative *harmonic maps heat flow*, $(\alpha, \beta) = (0, 1)$,

$$\partial_t u = \Delta u + |\nabla u|^2 u, \tag{HMHF}$$

$$\partial_t u = u \times \Delta u. \tag{SM}$$

eq:schrodinge

The equation is naturally associated to the Dirichlet energy,

$$\mathcal{E}[u] := \frac{1}{2} \int_{\mathbb{R}^2} |\partial_1 u|^2 + |\partial_2 u|^2 \, dx,$$

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[1] This is the direction in which the magnetic moment of a ferromagnet "prefers" to align.

which, for finite-energy (strong) solutions $u \in C_t^0 \dot{H}_x^1(I \times \mathbb{R}^2)$ to the Landau-Lifshitz equation (LL), satisfies the energy-balance identity

$$\mathcal{E}[u(t)] + 2\alpha \int_0^t \int_{\mathbb{R}^2} |u \times (u \times \Delta u)|^2 \, dxds = \mathcal{E}[u_0]. \tag{E}$$

eq:energy

In particular, the energy is non-increasing, and in fact conserved in the dispersive case. Both the equation (LL) and the Dirichlet energy are invariant under the rescaling

$$u(t,x) \mapsto u(t/\lambda^2, x/\lambda),$$

i.e. (LL) is *energy-critical*. It is then natural to study the dynamics of the Landau-Lifshitz equation (LL) in the energy topology $\dot{H}^1(\mathbb{R}^2)$. The energy space further decomposes into infinitely-many connected components $\dot{H}^1_m(\mathbb{R}^2)$ indexed by their topological class $\deg(u) \equiv m$. The degree of a finite-energy map $u \in \dot{H}^1(\mathbb{R}^2)$ is given by

$$\deg(u) := \frac{1}{4\pi} \int_{\mathbb{R}^2} u \cdot (\partial_1 u \times \partial_2 u) \, dx.$$

For continuous maps, the degree captures the number of times the plane wraps around the sphere under $u : \mathbb{R}^2 \to \mathbb{S}^2$ (see Appendix A.2).

The Landau-Lifshitz equation admits stationary solutions in the form of *harmonic maps*. Remarkably, maps from the plane to the sphere $u : \mathbb{R}^2 \to \mathbb{S}^2$ are *self-dual* in the sense that the second-order harmonic maps equation admits a reduction to the first-order *Cauchy-Riemann* equations. One can read off the reduction from the Bogomoln'yi identity (see Appendix A.2),

$$\mathcal{E}[u] = \frac{1}{2} \int_{\mathbb{R}^2} |\partial_1 u - u \times \partial_2 u|^2 + 4\pi \deg(u). \tag{B}$$

eq:B

It follows that solutions to the Cauchy-Riemann equations

$$\partial_1 u = u \times \partial_2 u, \tag{HM}$$

eq:CR

are minimisers of Dirichlet energy within each topological class, i.e. the space of finite energy configurations with fixed degree $\deg(u) \equiv N$. The equations (HM) can be identified with usual formulation of the Cauchy-Riemann equations after stereographic projection $\mathbb{S}^2 \to \mathbb{C}$, under which the solutions to (HM) correspond precisely to rational functions on $\mathbb{C}$. The basic m-equivariant solutions correspond to the maps $z \mapsto z^m$, taking the form

$$Q^m(r,\theta) := e^{m\theta R} Q^m(r), \qquad Q^m(r) := \begin{pmatrix} h_1^m \\ 0 \\ h_3^m \end{pmatrix},$$

where

$$h_1^m(r) := \frac{2r^m}{r^{2m}+1}, \qquad h_2^m(r) := \frac{r^{2m}-1}{r^{2m}+1},$$

and R is the generator for rotation about the z-axis of the sphere $\mathbb{S}^2$.

In view of the energy-balance identity (E), variational heuristics suggest that harmonic maps are stable under the Landau-Lifshitz equation (LL) in the energy topology within each topological class. This is the main subject of this article.

1.1. **Main result.** Following [GNT10], we give an exposition on the *asymptotic stability* of the *harmonic maps* under the Landau-Lifshitz equation (LL) in *equivariant symmetry*.

To set the stage, the m-equivariant maps take the form,

$$u(t,x) = e^{m\theta R} u(t,r),$$

where R is the generator for rotation about the z-axis of the sphere $\mathbb{S}^2$. For such maps, the energy norm reduces to

$$||u||^2_{\dot{H}^1_m} := ||\partial_r u||^2_{L^2_x} + ||\tfrac{u}{r}||^2_{L^2_x}.$$

The Landau-Lifshitz equation (LL) is invariant under rotation, scaling, and translation. Observe that working in equivariant symmetry kills the last symmetry, so the *moduli space of solitons* $\mathcal{Q}_m \subseteq \dot{H}^1_m(\mathbb{R}^2)$ reduces to the two-dimensional family

$$\mathcal{Q}_m := \{Q^m_{\alpha,\lambda} \in \dot{H}^1_m(\mathbb{R}^2) : \alpha \in \mathbb{S}^1 \text{ and } \lambda \in (0,\infty)\},$$

parametrised by the scaling parameter $\lambda \in (0,\infty)$ and rotation parameter $\alpha \in \mathbb{S}^1$ as follows,

$$Q^m_{\alpha,\lambda}(r,\theta) := e^{\alpha R} Q^m(r/\lambda, \theta).$$

The problem of stability for $\mathcal{Q}_m$ under (LL) concerns the dynamics of m-equivariant solutions to (LL) with energy close to the ground state,

$$\mathcal{E}[u] - \mathcal{E}[Q^m] \ll 1. \tag{1.1}$$ eq:close

In [GKT07], Gustafson-Kang-Tsai appled standard variational arguments to show that the family of harmonic maps $\mathcal{Q}_m$ is *orbitally stable* under the equation (LL) in the sense that solutions with energy close to the ground state, i.e. satisfying (1.1), are in fact confined within a small neighborhood of the moduli space of solitons. More precisely,

$$\inf_{\alpha,\lambda} ||Q^m_{\alpha,\lambda} - u||^2_{\dot{H}^1} \lesssim \mathcal{E}[u] - \mathcal{E}[Q^m] \ll 1. \tag{1.2}$$ eq:orbital

Projecting the solution onto $\mathcal{Q}_m$, the solution schematically takes the form

$$u(t) = \underbrace{Q_{\alpha(t),\lambda(t)}}_{\text{projection to } \mathcal{Q}_m} + \underbrace{\varepsilon(t)}_{\text{small error in } \dot{H}^1} \tag{1.3}$$ eq:decomp

The orbital stability result (1.2) does not say much about the precise global dynamics of near solitons since scaling is a non-compact symmetry. Possibilities to consider include

- Blow-up in either finite-time or infinite time, e.g. concentration into small scales,
$$\lambda(t) \to 0,$$
Finite-time blow-up is possible in the case $m = 1$, see [MRR11, Per14]. Infinite-time blow-up is possible in the case $m = 2$, see [GNT10, Theorem 2].
- Breather-type solutions, e.g. oscillation between disparate scales,
$$\liminf \lambda(t) < \limsup \lambda(t).$$
This is possible in the case $m = 2$, see [GNT10, Theorem 2].
- Global existence and asymptotic stability, e.g. the modulation parameters converge and the error in (1.3) disperses,
$$(\alpha(t), \lambda(t)) \to (\alpha_\infty, \lambda_\infty), \qquad \text{and} \qquad ||\varepsilon||_{\mathcal{S}_{t,x}([0,\infty)\times\mathbb{R}^2)} < \infty.$$
This holds for $m \geq 3$, see [GNT10, Theorem 1].

Remark. Let us also point the interested reader to the results of Bejenaru-Tataru [BT14] on the case $m = 1$, and the recent preprint of Bejenaru-Tataru-Pillai [BPT24] which deals with the most delicate case $m = 2$.

We are primarily interested in the asymptotic stabilty result of [GNT10, Theorem 1] for Schrödinger maps (SM), i.e. the purely dispersive case of the Landau-Lifshitz equation (LL). The dissipative case is an easy modification;

thm:main **Theorem 1.1** (Asymptotic stability of harmonic maps in equivariant symmetry)**.** *Let $m \geq 3$, then given initial data $u_0 \in \dot{H}^1_m(\mathbb{R}^2)$ with energy close to the ground state*

$$\mathcal{E}[u_0] - \mathcal{E}[Q^m] \ll 1,$$

there exists a global solution $u \in C^0_t(\dot{H}^1_m)_x([0,\infty) \times \mathbb{R}^2)$ to the Schrödinger maps equation (SM) converging to a fixed soliton $Q^m_{\alpha_\infty,\lambda_\infty} \in \mathcal{Q}_m$ in the sense that

$$||u(t) - Q_{\alpha_\infty,\lambda_\infty}||_{L^\infty_x} \xrightarrow{t\to\infty} 0. \tag{1.4}$$ eq:convergenc

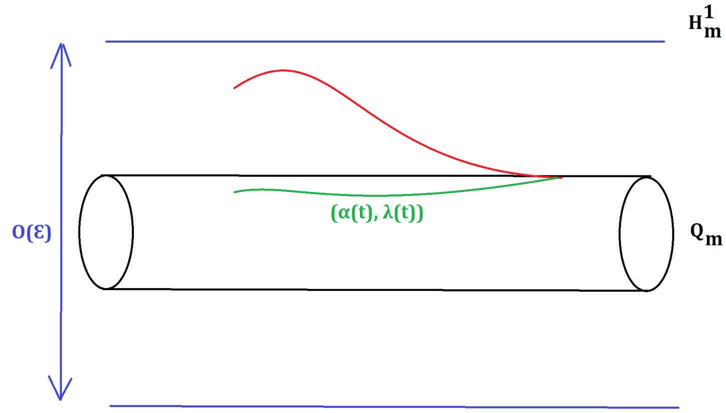

FIGURE 1. The near soliton dynamics of (LL) can be decomposed into an ODE for the dynamics on the moduli space and a PDE for the error.

1.2. **Outline of the argument.** Our overview of the argument will follow the exposition of [GGKT08], which in turn outlines the stability result for $m \geq 4$ in [GKT07]; we will leave the modifications of the argument in [GNT10] to handle the $m = 3$ case to the details. For our notation, we will instead borrow from [KTV14, Geometric Wave Equations] and [BT14, BPT24].

We decompose the solution into a soliton profile and a dispersive error,

$$u(t,r) = \underbrace{Q_{\alpha(t),\lambda(t)}(r)}_{\text{modulated soliton}} + \underbrace{\varepsilon(t,r)}_{\text{dispersive error}} . \tag{1.5}$$ eq:decomp2

To prove the main theorem, we want construct appropriate modulation parameters (α, λ) and correction term ε such that

item:goal1 (a) the error obeys global-in-time dispersive bounds, $\varepsilon \in \mathsf{S}_{t,x}([0,\infty) \times \mathbb{R}^2)$, to obtain the dispersive decay, and

item:goal2 (b) integrability bounds on the derivative of the modulation parameters, $(\dot{\alpha}, \dot{\lambda}) \in L^1_t([0,\infty))$, to conclude convergence to a soliton.

To identify the main enemies, let us consider the linearised equation about a fixed soliton Q, which one might naively expect to govern, to leading order, the dynamics of the dispersive error ε. To formulate the linearised equation, which is solved by fields $u_{\text{lin}} : I \times \mathbb{R}^2 \to T_Q\mathbb{S}^2$, it is convenient to introduce an orthonormal frame $\{\mathbf{v}_Q, \mathbf{w}_Q\} \subseteq Q^*T\mathbb{S}^2$. Throughout we will use the *Coulomb gauge*,

$$\mathbf{v}_Q = \begin{pmatrix} h_3 \\ 0 \\ -h_1 \end{pmatrix}, \qquad \mathbf{w}_Q = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The orthonormal frame induces an isomorphism between the pullback bundle and the trivial complex bundle $Q^*T\mathbb{S}^2 \cong (I \times \mathbb{R}^2) \times \mathbb{C}$, so we can rewrite the linearised equation in coordinates and study an equation for complex scalar fields $\phi_{\text{lin}} : I \times \mathbb{R}^2 \to \mathbb{C}$, which represents u_{lin} in these coordinates

$$\phi_{\text{lin}} = \langle u_{\text{lin}}, \mathbf{v}_Q \rangle + i \langle u_{\text{lin}}, \mathbf{w}_Q \rangle.$$

In these coordinates, the *linearised Schrödinger maps equation* takes the form

$$(i\partial_t - \mathsf{H}_Q)\phi_{\text{lin}} = 0, \tag{1.6}$$ eq:linearised

where H_Q is the *linearised operator*, which from the self-dual structure of the equation (compare with (3.1)) factors into

$$\mathsf{H}_Q := \mathsf{L}_Q^* \mathsf{L}_Q, \qquad \mathsf{L}_Q := h_1 \partial_r {h_1}^{-1}.$$

From the linearised equation (1.6), we see that the main enemy to decay of the error ε in the decomposition (1.5) of a solution satisfying the original equation (SM) would be the kernel of the linearised operator H_Q, as such elements would lead to non-decaying, constant-in-time solutions. One can easily read off from the factorisation above that the kernel is generated by

$$\ker \mathsf{H}_Q = \operatorname{span} h_1.$$

One can also read off the generator of the kernel from the symmetries of the equation. Since the equation is invariant under scaling and rotation, differentiating the modulated soliton $Q_{\alpha,\lambda}$ in these parameters generates elements of the kernel,

$$\begin{aligned} \frac{\partial Q_{\alpha,\lambda}}{\partial \alpha}\Big|_{(\alpha,\lambda)=(0,1)} &= h_1 \mathbf{v}_Q, \\ \frac{\partial Q_{\alpha,\lambda}}{\partial \lambda}\Big|_{(\alpha,\lambda)=(0,1)} &= h_1 \mathbf{w}_Q. \end{aligned} \tag{1.7}$$

eq:generators

To kill these enemies, our first take would be to impose the condition that the error "$\varepsilon \approx \phi_{\text{lin}}$" is orthogonal to the kernel of the linearised operator H_Q,

$$\phi_{\text{lin}} \perp \ker \mathsf{H_Q} \qquad \text{"i.e."} \qquad \int_0^\infty \phi_{\text{lin}} \overline{h_1}\, r dr = 0. \tag{1.8}$$

eq:orthogonal

Since the error is in the energy space, $\varepsilon \in \dot{H}^1_m(\mathbb{R}^2)$, the orthgonality condition on the right is well-defined provided that $rh_1 \in L^2_r([0,\infty))$ by Cauchy-Schwartz. The spatial asymptotics of the kernel elements are precisely $h_1(r) = O(r^{-m})$, so one must restrict to the high equivariance case $m \geq 3$ to make sense of the orthogonality condition.

To enforce the orthogonality condition (1.8), we impose the condition initially and solve the corresponding ODE arising from differentiating (1.8) in time. Rewriting the resulting system using the original equation (SM) and the decomposition (1.5), we obtain a first-order *modulation equation* for the parameters (α, λ). The equation itself is quite complicated, however the key point is that, in view of the orthogonality of the error (1.8) to the generators of the kernel (1.7), the forcing terms which are *linear* in the error ε are killed, so the derivatives of the parameters $(\dot{\alpha}, \dot{\lambda})$ only see *quadratic* forcing terms and higher,

$$|\dot{\alpha}| + |\dot{\lambda}| = O(|\varepsilon|^2). \tag{1.9}$$

eq:quadratic

Thus, ignoring any technical problems with spatial asymptotics of ε, an L^2_t-dispersive estimate for ε would imply an L^1_t-estimate for the parameters. This caricature is accurate for $m > 3$, but due to some slow spatial asymptotics which do not allow one to close certain Hardy-type bounds, one has to alter the orthogonality condition (1.8) in the case $m = 3$.

Generally, one would have to study the dynamics of the error coupled with the dynamics of the modulation parameters. However, the Schrödinger maps equation (SM) admits *self-dual* structure, in that it can be written as

$$\partial_t u = \mathbf{J}\mathbf{D}_z \partial_{\bar{z}} u. \tag{1.10}$$

Differentiating the equation, we obtain the dispersive-elliptic system

$$\begin{aligned} \mathbf{D}_t \varepsilon' &= \mathbf{J}\mathbf{D}_{\bar{z}}\mathbf{D}_z \varepsilon', \\ \varepsilon' &= \partial_{\bar{z}} u. \end{aligned} \tag{1.11}$$

This is known as the *generalised Hasimoto transformation*. The Cauchy-Riemann operator $u \mapsto \partial_{\bar{z}} u$ kills the harmonic maps component of the solution, isolating the error term, thus the variable $\varepsilon' = \partial_{\bar{z}} u$ is at least linear in the error ε.

2. Geometric preliminaries

2.1. **Geometric structure.** We begin with some geometric generalities; for more details, see the monograph of Tao [Tao06, Chapter 6.2] and the notes of Tataru [KTV14, Geometric Dispersive Equations].

Given a map $u : I \times \mathbb{R}^2 \to \mathbb{S}^2$, we will use $\mathbf{D}_j$ to denote the pullback of the connection on the sphere, and $\mathbf{J}$ the complex structure given by $\frac{\pi}{2}$-rotation. Using this notation, the equation (SM) takes the form

$$\partial_t u = \mathbf{J}(\mathbf{D}_1 \partial_1 u + \mathbf{D}_2 \partial_2 u). \tag{2.1}$$

eq:geoschrod

To reveal the linear Schrödinger structure of the equation, it is convenient to fix an orthonormal frame $\{\mathbf{v}, \mathbf{w}\} \subseteq u^* T\mathbb{S}^2$ on the pullback bundle. Using this frame, we can identify each tangent space $T_{u(t,x)}\mathbb{S}^2$ with $\mathbb{C}$ via the isomorphism

$$\begin{aligned} T_{u(t,x)}\mathbb{S}^2 &\longrightarrow \mathbb{C}, \\ X^1 \mathbf{v} + X^2 \mathbf{w} &\longmapsto X^1 + iX^2. \end{aligned}$$

Thus we can identify the field $v : I \times \mathbb{R}^2 \to u^* T\mathbb{S}^2$ with a complex scalar field $\psi : I \times \mathbb{R}^2 \to \mathbb{C}$,

$$\psi := \langle v, \mathbf{v} \rangle \, \mathbf{v} + i \langle v, \mathbf{w} \rangle \, \mathbf{w}.$$

In particular, we will denote ψ_α for the differentiated fields $\partial_\alpha u$. In this frame, the action of the covariant derivative on the field v corresponds to the following operation on the complex scalar field ψ,

$$\mathbf{D}_\alpha v \longmapsto (\partial_\alpha + iA_\alpha)\psi,$$

where the frame coefficients $A_\alpha := \langle \mathbf{e}_2, \partial_\alpha \mathbf{e}_1 \rangle$ track the change in the frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ relative to the vector field ∂_α. We abuse notation by denoting the operator on the right by $\mathbf{D}_\alpha \psi$. *A priori*, the connection coefficients satisfy the curl system

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha = \operatorname{Im}(\psi_\alpha \overline{\psi_\beta}), \tag{2.2}$$

eq:curvature

Note that thus far we have not fixed a frame; indeed, any other frame could be obtained from rotating by $e^{i\chi}$. Varying the gauge choice leads to the gauge invariance

$$\psi \mapsto e^{i\chi}\psi, \qquad A_\alpha \mapsto A_\alpha + \partial_\alpha \chi.$$

Fixing a gauge fully determines the connection coefficients in terms of the original field u. We will work with Coulomb gauge,

$$\partial_1 A_1 + \partial_2 A_2 = 0, \tag{2.3}$$

eq:coulomb

which, upon differentiating the curvature relation (2.2) and inverting the Laplacian, we see that the connection coefficients are given precisely by

$$A_\alpha[\psi] = -\frac{1}{2}\Delta^{-1}\partial^\beta \operatorname{Im}(\psi_\alpha \overline{\psi_\beta}). \tag{2.4}$$

eq:coefficien

In general, this has unfavourable high $\times$ high $\to$ low interactions, as schematically the right-hand side takes the form $D^{-1}(\psi\overline{\psi})$, though in our setting of equivariant symmetry, the situation is much better. In any case, thinking of the connection coefficients as quadratic terms, we can rewrite the Hasimoto-transformed system (**??**) in these coordinates as the cubic-type non-linear Schrödinger equation

$$i\partial_t \psi_{\overline{z}} - \Delta \psi_{\overline{z}} = A_t[\psi]\psi_{\overline{z}} + i\partial_{\overline{z}} A_z[\psi]\psi_{\overline{z}} + iA_z[\psi]\partial_{\overline{z}}\psi_{\overline{z}} + iA_{\overline{z}}[\psi]\partial_z \psi_{\overline{z}} - A_z[\psi]A_{\overline{z}}[\psi]\psi_{\overline{z}}. \tag{2.5}$$

$$\begin{tikzcd} (I \times \mathbb{R}^2) \times \mathbb{C} \arrow[r, "\mathbf{e}"] & u^* T\mathbb{S}^2 \arrow[r, "u"] \arrow[d, "v"] & T\mathbb{S}^2 \arrow[d] \\ I \times \mathbb{R}^2 \arrow[u, "\psi"] \arrow[r] & I \times \mathbb{R}^2 \arrow[r, "u"] & \mathbb{S}^2 \end{tikzcd}$$

FIGURE 2. The commutative diagram connecting the trivial bundle $(I \times \mathbb{R}^2) \times \mathbb{C}$, the pullback bundle $u^* T\mathbb{S}^2$, and the tangent bundle $T\mathbb{S}^2$.

Proposition 2.1 (Schrödinger maps in Coulomb gauge)**.** *Let $m \in \mathbb{N}$ and $\mathbf{m}$ be a solution to Landau-Lifshitz. Then the equation in Coulomb gauge*

2.2. **Splitting and orthogonality.** To identify our enemies, let us detail the derivation of the linearised equation (1.6). Starting from decomposition of the solution into soliton profile and error (1.5), we further decompose the error into a component perpindicular to the soliton (i.e. in the tangent space $T_Q\mathbb{S}^2$) and a component parallel to the soliton,

$$
\begin{aligned}
u(t,r) &= \underbrace{Q(r)}_{\text{soliton profile}} + \underbrace{\varepsilon(t,r)}_{\text{perturbation}} \\
&= \underbrace{Q(r)}_{\text{soliton profile}} + \underbrace{u_{\text{lin}}(t,r)}_{\text{perpindicular perturbation}} + \underbrace{\gamma(t,r)Q(r)}_{\text{parallel perturbation}}
\end{aligned} \tag{2.6}
$$

For small perturbation $|\varepsilon| \ll 1$, the main term is the perpindicular component in view of the constraint $|u| \equiv |Q| \equiv 1$; indeed one can compute that the parallel component is of quadratic order,

$$\gamma = \sqrt{1 - |u_{\text{lin}}|^2} - 1 = O(|u_{\text{lin}}|^2).$$

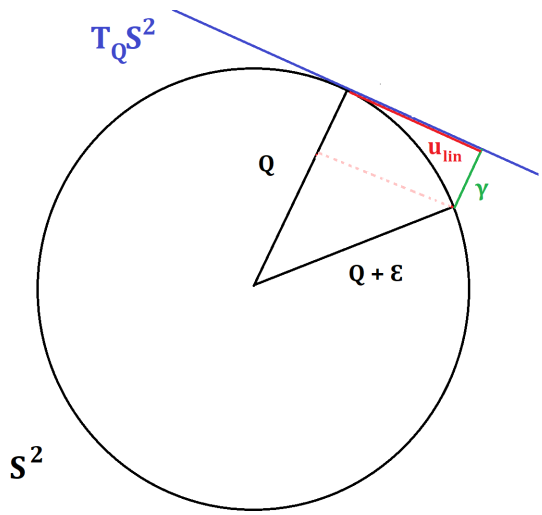

FIGURE 3. Decomposition of the solution

$$u = Q + (u_{\text{lin}} + \gamma),$$

into a component parallel to the soliton $\gamma \in \operatorname{span} Q$ and a component in the tangent space $u_{\text{lin}} \in T_Q\mathbb{S}^2$.

The Coulomb frame

$$\mathbf{v}_Q(r) = \begin{pmatrix} h_3(r) \\ 0 \\ -h_1(r) \end{pmatrix}, \qquad \mathbf{w}_Q(r) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

3. GENERALISED HASIMOTO TRANSFORMATION

In view of the Bogomoln'yi identity (B) and the Hamiltonian structure of the equation (see Appendix A.1), the equation can be put in self-dual form,

$$\partial_t u = \mathbf{J}\mathbf{D}_z \partial_{\bar{z}} u, \tag{3.1}$$ eq:selfdual

where

$$
\begin{aligned}
\partial_{\bar{z}} u &:= \partial_1 u - \mathbf{J}\partial_2 u, \\
\mathbf{D}_z v &:= \mathbf{D}_1 v + \mathbf{J}\mathbf{D}_2 v.
\end{aligned}
$$

Then, applying the covariant Cauchy-Riemann operator to the self-dual formuation (3.1), we obtain the generalised Hasimoto-transformed Schrödinger maps equation, which is an elliptic-dispersive system,

$$\begin{aligned} \mathbf{D}_t \varepsilon' &= \mathbf{J} \mathbf{D}_{\bar{z}} \mathbf{D}_z \varepsilon', \\ \varepsilon' &= \partial_{\bar{z}} u. \end{aligned} \tag{3.2}$$

$$(i\partial_t - \widetilde{\mathsf{H}}_Q)\psi = \text{non-linear} \tag{3.3}$$

where

$$\begin{aligned} \widetilde{\mathsf{H}}_Q &:= -\Delta + \widetilde{V}_Q, \\ \widetilde{V}_Q(r) &:= \frac{2m(1-h_3(r))}{r^2} = \frac{4m}{r^2(r^2+1)} \end{aligned}$$

4. Elliptic estimates

Proposition 4.1. *Given the decomposition*

$$\mathbf{m} = \mathbf{Q} + \varepsilon$$

we have the estimates

$$\begin{aligned} &||\varepsilon||_{\dot{H}^1_x} \lesssim ||\partial_{\bar{z}} \mathbf{m}||_{L^2_x} + ||\varepsilon||_{L^\infty_x} ||\varepsilon||_{\dot{H}^1_x}, \\ &||\tfrac{\varepsilon}{r}||_{L^\infty_p} \lesssim ||\partial_{\bar{z}} \mathbf{m}||_{L^\infty_x} + ||\varepsilon||_{L^\infty_x} ||\varepsilon||_{L^\infty_p}, \\ &||\varepsilon||_{L^\infty_x} \lesssim ||\varepsilon||_{\dot{H}^1_x}. \end{aligned}$$

Proposition 4.2 (Hardy-Sobolev inequality)**.** *Let $m \in \mathbb{Z}$ be a non-zero integer. Then for any finite energy m-equivariant $f \in \dot{H}^1_m(\mathbb{R}^2)$, we have*

$$||\tfrac{f}{r}||_{L^2_x} + ||f||_{L^\infty_x} \lesssim ||f||_{\dot{H}^1_m}. \tag{4.1}$$

eq:hardy

Proof. This follows immediately. □

5. Dispersive estimates

Denote the evolution of ε_1 by

$$(i\partial_t + \mathsf{H}_{\mathbf{Q}})\varepsilon_1 = N_1 + N_2. \tag{5.1}$$

We want to derive Strichartz estimates for the linearised operator and treat the right-hand side perturbatively.

Proposition 5.1 (Endpoint Strichartz for $\mathsf{H}_{\mathbf{Q}_{\alpha,\lambda}}$)**.** *Let $m \geq 2$, then*

$$\begin{aligned} ||e^{-i\mathsf{H}_{\mathbf{Q}_{\alpha,\lambda}} t}\phi||_{L^\infty_t L^2_x \cap L^2_t L^\infty_2} &\lesssim ||\phi||_{L^2_x}, \\ \Big|\Big| \int_{-\infty}^t e^{-i\mathsf{H}_{\mathbf{Q}_{\alpha,\lambda}}} f(t')dt' \Big|\Big|_{L^\infty_t L^2_x \cap L^2_t L^\infty_2} &\lesssim ||f||_{L^1_t L^2_+ L^2_t + L^1_2}. \end{aligned} \tag{5.2}$$

6. Modulation theory

Appendix A.

ndix:hasimoto

A.1. **Generalised Hasimoto transform.** It is convenient to write the Schrödinger maps equation (SM) in geometric formulation,

$$\partial_t u = \mathbf{J}(\mathbf{D}_1 \partial_1 u + \mathbf{D}_2 \partial_2 u).$$

Then

$$\begin{aligned} \mathbf{J}\mathbf{D}_z \partial_{\bar{z}} u &= \mathbf{J}(\mathbf{D}_1 + \mathbf{J}\mathbf{D}_2)(\partial_1 u - \mathbf{J}\partial_2 u) \\ &= \mathbf{J}(\mathbf{D}_1 \partial_1 u - \mathbf{J}\mathbf{J}\mathbf{D}_2 \partial_2 u) + \mathbf{J}\mathbf{J}(\mathbf{D}_1 \partial_2 u - \mathbf{D}_2 \partial_1 u) \\ &= \mathbf{J}(\mathbf{D}_1 \partial_1 u + \mathbf{D}_2 \partial_2 u). \end{aligned}$$

appendix:bogo

A.2. **Bogomoln'yi identity.** We compute

$$\begin{aligned} |\partial_1 u - u \times \partial_2 u|^2 &= (\partial_1 u - u \times \partial_2 u) \cdot (\partial_1 u - u \times \partial_2 u) \\ &= |\partial_1 u|^2 + |u \times \partial_2 u|^2 - 2(\partial_1 u) \cdot (u \times \partial_2 u). \end{aligned}$$

In the last line, the first two terms are exactly the Dirichlet energy density, while the last term is the pull-back of the volume form on $\mathbb{S}^2$ to $\mathbb{R}^2$ under u. Indeed, since the almost complex structure acts isometrically on the tangent space,

$$|\partial_1 u|^2 + |u \times \partial_2 u|^2 = |\partial_1 u|^2 + |\partial_2 u|^2.$$

To see the pull-back of the volume form, recall that

$$d\,\mathrm{Vol}_{\mathbb{S}^2} := u^1 du^2 \wedge du^3 - u^2 du^1 \wedge du^3 + u^3 du^1 \wedge du^2.$$

Then, pulling back by $u : \mathbb{R}^2 \to \mathbb{S}^2$, we obtain

$$\begin{aligned} u^* d\,\mathrm{Vol}_{\mathbb{S}^2} &= u^1(\partial_1 u^2 dx^1 + \partial_2 u^2 dx^2) \wedge (\partial_1 u^3 dx^1 + \partial_2 u^3 dx^2) \\ &\quad - u^2(\partial_1 u^1 dx^1 + \partial_2 u^1 dx^2) \wedge (\partial_1 u^3 dx^1 + \partial_2 u^3 dx^2) \\ &\quad + u^3(\partial_1 u^1 dx^1 + \partial_2 u^1 dx^2) \wedge (\partial_1 u^2 dx^1 + \partial_2 u^2 dx^2) \\ &= u \cdot (\partial_1 u \times \partial_2 u)\, dx^1 \wedge dx^2 \\ &= (\partial_1 u) \cdot (u \times \partial_2 u)\, dx^1 \wedge dx^2, \end{aligned}$$

where the last line we have used the scalar triple product identity. By the degree theorem,

$$\int_{\mathbb{R}^2} (\partial_1 u) \cdot (u \times \partial_2 u)\, dx = \int_{\mathbb{R}^2} u^* d\,\mathrm{Vol} = \deg(u) \int_{\mathbb{S}^2} d\,\mathrm{Vol} = 4\pi \deg(u)$$

This completes the proof of the Bogomoln'yi identity (B).

A.3. **Harmonic maps from (LL).** Here we show that

$$\partial_t u = \alpha(\Delta u + |\nabla u|^2 u) + \beta(u \times \Delta u)$$

is equivalent to

$$\partial_t u = -\alpha(u \times (u \times \Delta u)) + \beta(u \times \Delta u).$$

Consider the vector cross product formula

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c.$$

Then

$$u \times (u \times \Delta u) = (u \cdot \Delta u)u - |u|^2 \Delta u.$$

Observe that

$$u \cdot \nabla u = 0.$$

Thus

$$u \cdot \Delta u = -|\nabla u|^2.$$

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