GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE ENERGY-CRITICAL DEFOCUSING NON-LINEAR WAVE EQUATION

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Abstract. We give a new proof of the threshold theorem for the energy-critical non-linear wave equation using the energy dispersion + bubbling method of Sterbenz and Tataru.

Contents

1. Introduction

Given initial data in the energy-space $(\phi_0, \phi_1) \in \dot{H}^1_x(\mathbb{R}^d) \times L^2_x(\mathbb{R}^d)$, we are interested in the evolution under the energy-critical defocusing non-linear wave equation,

$$\begin{aligned} \Box\phi &= +|\phi|^{\frac{4}{d-2}}\phi, \\ \phi_{|t=0} &= \phi_0, \\ \partial_t\phi_{|t=0} &= \phi_1. \end{aligned} \tag{NLW}$$

`eq:NLW`

This equation is energy-critical in the sense that the conserved energy,

$$\mathcal{E}[\phi[t]] := \int_{\mathbb{R}^d} \frac{1}{2}|\partial_t\phi|^2 + \frac{1}{2}|\nabla\phi|^2 + \frac{d-2}{2d}|\phi|^{\frac{2d}{d-2}}\, dx,$$

is invariant under the associated scaling symmetry to the equation,

$$\phi(t,x) \mapsto \lambda^{-\frac{d-2}{2}}\phi(t/\lambda, x/\lambda).$$

We are interested in establishing global well-posedness and scattering, that is, the existence of a strong solution $\phi \in C^0_{t,\mathrm{loc}}\dot{H}^1_x \cap \dot{C}^1_{t,\mathrm{loc}}L^2_x(\mathbb{R}\times\mathbb{R}^d)$ which depends continuously on the initial data and is bounded in

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an appropriate Strichartz norm $||\phi||_{S_{t,x}(\mathbb{R}\times\mathbb{R}^d)} < +\infty$. For non-linear wave equations, the main obstruction is the existence of non-trivial steady-states Q, which solve the ground state equation

$$\Delta Q = +|Q|^{\frac{4}{d-2}} Q. \tag{GS}$$

eq:GS

In general, defocusing equations do not admit ground states. For the non-linear wave equation, one can see this by multiplying the equation by Q and integrating by parts. We quickly see from the discrepancy in sign that the only ground state solution is the trivial solution $Q \equiv 0$. The *defocusing conjecture* states that, in view of the lack of ground states, defocusing equations should be globally well-posed and scatter. This conjecture holds (NLW), see [Tao06, Chapter 5.1] and references therein,

m:defocusing **Theorem 1** (Defocusing theorem)**.** *For each finite energy initial data $\phi[0] \in \dot{H}^1_x(\mathbb{R}^d) \times L^2_x(\mathbb{R}^d)$, there exists a unique global solution $\phi \in C^0_t \dot{H}^1_x \cap \dot{C}^1_t L^2_x(\mathbb{R}\times\mathbb{R}^d)$ to the defocusing (NLW) depending continuously on the initial data and scatters.*

The subject of this note will be to establish the conjecture using the energy-dispersion + bubbling argument of Sterbenz and Tataru [ST10a, ST10b].

2. Geometric notation

Geometry of Minkowski space. Let $(\mathbb{R}^{1+d}, \mathbf{m})$ denote $(1+d)$-dimensional Minkowski space with the usual metric, which in rectilinear coordinates $(t, x^1, \dots, x^d)$ takes the diagonal form

$$\mathbf{m} = -(\mathrm{d}t)^2 + (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^d)^2.$$

We will often write $t = x^0$ for the time coordinate and $x = (x^1, \dots, x^d)$ for the spatial coordinates. We reserve Greek indices, such as $\alpha, \beta, \gamma, \dots$ for space-time coordinates $(t, x^1, \dots, x^d)$, while Latin indices, such as $i, j, k, \ell, \dots$ will be reserved for spatial coordinates $(x^1, \dots, x^d)$. Another useful choice are the polar coordinates (t, r, Θ) where $r = |x|$ denotes the radius from the origin, and $\Theta := x/|x|$ denotes the radial projection onto the unit sphere $\mathbb{S}^{d-1}$. In these coordinates, the Minkowski metric takes the form

$$\mathbf{m} = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 g_{\mathbb{S}^{d-1}}.$$

Denote $\partial_r = \frac{x^j}{r}\partial_j$ the radial vector field and $\nabla_{\mathbb{S}^{d-1}}$ for the gradient on the unit sphere $\mathbb{S}^{d-1}$.

Geometry of the light cone. We now introduce notation for the geometry of the light cone and subsets thereof. First and foremost, the forward light cone is defined by

$$C := \{(t,x) \in [0,\infty) \times \mathbb{R}^d : r \le t\}.$$

When studying the light cone, it is convenient to work in null coordinates (u, v, Θ) defined by $u = t - r$ and $v = t + r$. In these coordinates, the Minkowski metric takes the form

$$\mathbf{m} = -\mathrm{d}u\mathrm{d}v + r^2 g_{\mathbb{S}^{d-1}}.$$

The coordinate vector fields $L = \partial_t + \partial_r = 2\partial_v$ and $\underline{L} = \partial_t - \partial_r = 2\partial_u$ are referred to as null vector fields, as they are parallel to the forward and backwards light cones respectively. Observing that the forward light cone is foliated by surfaces

$$\mathbb{H}^d_\rho := \{(t,x) \in [0,\infty) \times \mathbb{R}^d : t^2 + r^2 = \rho^2\},$$

we introduce hyperbolic coordinates (ρ, y, Θ) where $\rho = \sqrt{t^2 - r^2}$ and $y = \tanh^{-1}(r/t)$. Each surface $\mathbb{H}^d_\rho$ is isometric to the simply connected space of constant sectional curvature $-\frac{1}{\rho^2}$. In these coordinates the Minkowski metric takes the form

$$\begin{aligned} \mathbf{m} &= -\mathrm{d}\rho^2 + \rho^2 g_{\mathbb{H}^d_\rho} \\ &= -\mathrm{d}\rho^2 + \rho^2(\mathrm{d}y^2 + \sinh^2(y) g_{\mathbb{S}^{d-1}}). \end{aligned}$$

We refer to the vector field $S = \rho\partial_\rho = x^\mu\partial_\mu$ as the scaling vector field, as it is generated from the scaling symmetry of the linear wave equation.

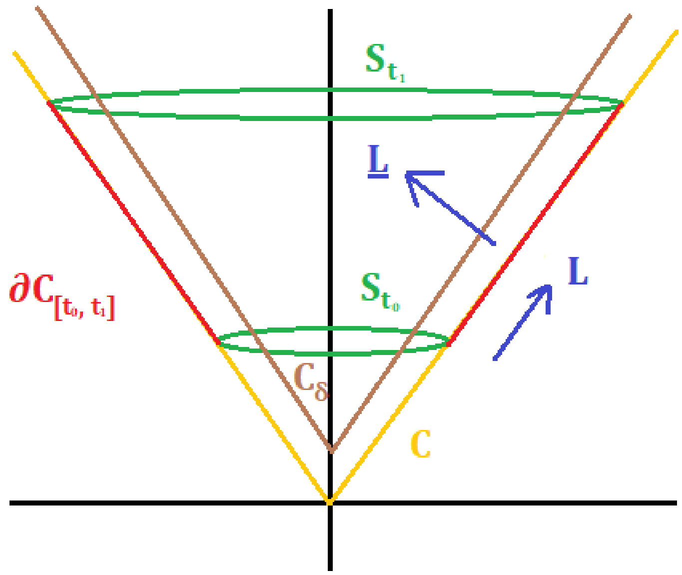

FIGURE 1. The geometry of the light cone. As a wise man once said, a picture is worth $\frac{1}{\varepsilon}$-words for $\varepsilon \ll 1$.

Subsets of $\mathbb{R}^d$ *and* $\mathbb{R}^{1+d}$. Define the restriction of the light cone to a time interval $I \subseteq [0, \infty)$ and a time slice $t \in [0, \infty)$ respectively by

$$C_I := C \cap (I \times \mathbb{R}^d),$$
$$S_t := C \cap (\{t\} \times \mathbb{R}^d).$$

The *null boundary* ∂C_I denotes the boundary of the time-slab C_I modulo the top and bottom time-slices. Due to singularities on the null boundary, we will also consider the shifted light cone

$$C^\delta := (\delta, 0) + C.$$

Accordingly, we have

$$C_I^\delta := C_I \cap C^\delta,$$
$$S_t^\delta := S_t \cap C^\delta.$$

We also define $B_r(x) \subseteq \mathbb{R}^d$ to be the ball of radius r centered at x.

3. CONSERVATION LAWS

The non-linear wave equation (NLW) is the Euler-Lagrange equation arising when one formally considers solutions as critical points of the Lagrangian

$$\mathcal{S}[\phi] := \int_{\mathbb{R}^{1+d}} \frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi + \frac{d-2}{2d}|\phi|^{\frac{2d}{d-2}}\, dtdx.$$

By Noether's principle, the continuous symmetries of the equation lead to conserved quantities. We present the energy-momentum tensor formalism, which is derived from the diffeomorphism-invariance of the Lagrangian. Define the *energy-momentum tensor* by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - m_{\mu\nu}\left(\frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi + \frac{d-2}{2d}|\phi|^{\frac{2d}{d-2}}\right).$$

Observe that $T_{\mu\nu}$ is a symmetric 2-tensor. Furthermore, if ϕ is a classical solution to (NLW), the energy-momentum tensor is also divergence-free,

$$\nabla^\mu T_{\mu\nu} = 0.$$

We obtain energy identities for the non-linear wave equation by contracting the energy-momentum tensor with well-chosen vector fields, and then integrate over suitable space-time domains. Given a vector field X, we define its deformation tensor to be the Lie derivative of the metric with respect to X, i.e. ${}^{(X)}\pi = \mathcal{L}_X \mathbf{m}$. In coordinates,

$$ {}^{(X)}\pi_{\mu\nu} = \partial_\mu X_\nu + \partial_\nu X_\mu. $$

Define the 1- and 0-currents

$$ \begin{aligned} {}^{(X)}J_\mu[\phi] &:= T_{\mu\nu}[\phi] X^\nu, \\ {}^{(X)}K[\phi] &:= T_{\mu\nu}[\phi] \left(\frac{1}{2} {}^{(X)}\pi^\# \right)^{\mu\nu}. \end{aligned} $$

Then, since T is divergence-free,

$$ \partial^\mu \left({}^{(X)}J_\mu[\phi] \right) = {}^{(X)}K[\phi]. \tag{∇} $$

eq:current

Another way of deriving the divergence identity above would be to multiply the equation (NLW) by $X\phi$ and then integrating-by-parts.[1] More generally, we can apply the same procedure after multiplying the equation by $w\phi$ for a scalar weight w. It follows that the *generalised* 0- *and* 1-*currents*

$$ \begin{aligned} {}^{(X)}J_\mu[\phi] &:= w\phi\partial_\mu\phi - \frac{1}{2}\partial_\mu w |\phi|^2, \\ {}^{(X)}K[\phi] &:= w\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\Box w |\phi|^2 + \kappa |\phi|^{p+1} w, \end{aligned} $$

satisfy the divergence identity

$$ \partial^\mu \left({}^{(w)}J_\mu[\phi] \right) = {}^{(w)}K[\phi]. \tag{∇'} $$

eq:gcurrent

3.1. **Energy conservation identities.** The simplest conservation law arises from the stationarity of the Minkowski metric, that is, the time-like vector field $T = \partial_t$ is a Killing vector field, ${}^{(T)}\pi = 0$. Contracting the energy-momentum tensor with T, the resulting 1-current is precisely the energy density

$$ {}^{(T)}J_\mu[\phi] = \frac{1}{2}|\partial_t\phi|^2 + \frac{1}{2}|\nabla\phi|^2 + \frac{d-2}{2d}|\phi|^{\frac{2d}{d-2}}. $$

Integrating the divergence identity (∇) with $X = T$ over the space-time slab $(t_0, t_1) \times \mathbb{R}^d$ and applying the divergence theorem yields the conservation of energy,

Proposition 2 (Conservation of energy). *Let $\phi \in C^0_{t,\mathrm{loc}}\dot{H}^1_x \cap \dot{C}^1_{t,\mathrm{loc}} L^2_x(I \times \mathbb{R}^d)$ be a strong solution to (NLW), then the energy is conserved, i.e. $\mathcal{E}[\phi[t]] \equiv \mathcal{E}$ for all $t \in I$.*

To make full use of the finite speed of propagation and small data theory, as we will soon see in Section 4, we need to derive an energy conservation law which is local in space. Given an open subset $\Omega \subseteq \mathbb{R}^d$, define the local energy on Ω at time t by

$$ \mathcal{E}_\Omega[\phi[t]] := \int_\Omega \frac{1}{2}|\partial_t\phi|^2 + \frac{1}{2}|\nabla\phi|^2 + \frac{d-2}{2d}|\phi|^{\frac{2d}{d-2}}\, dx. $$

We also write $\mathcal{E}_{S_t}[\phi] := \mathcal{E}_{B_t}[\phi[t]]$. Then, integrating ($\nabla$) over the slab of the light-cone $C_{[t_0,t_1]}$ and applying the divergence theorem, we relate the local energy on the time-slices of the light cone S_{t_0} and S_{t_1} modulo a flux through the null boundary $\partial C_{[t_0,t_1]}$. To compute this flux, we write the 1-current in null coordinates, remarking that $T = \frac{1}{2}(L + \underline{L})$,

$$ \begin{aligned} {}^{(T)}J_L[\phi] &= \frac{1}{2}|L\phi|^2 + \frac{1}{2}|\nabla\!\!\!/\phi|^2 + \frac{d-2}{2d}|\phi|^{\frac{2d}{d-2}}, \\ {}^{(T)}J_{\underline{L}}[\phi] &= \frac{1}{2}|\underline{L}\phi|^2 + \frac{1}{2}|\nabla\!\!\!/\phi|^2 + \frac{d-2}{2d}|\phi|^{\frac{2d}{d-2}}. \end{aligned} $$

[1] Well, technically, "differentiating-by-parts"

Thus, the flux through the null boundary takes the form

$$\mathcal{F}_{\partial C_{[t_0,t_1]}}[\phi] := \int_{\partial C_{[t_0,t_1]}} \left(\frac{1}{2}|L\phi|^2 + \frac{1}{2}|\nabla\!\!\!/\, \phi|^2 + \frac{d-2}{2d}|\phi|^{\frac{2d}{d-2}} \right) dS.$$

In summary, we obtain the following local energy conservation law,

op:energyflux **Proposition 3** (Energy-flux identity). *Let $\phi \in C^0_{t,\mathrm{loc}}\dot{H}^1_x \cap \dot{C}^1_{t,\mathrm{loc}}L^2_x(I \times \mathbb{R}^d)$ be a strong solution to (NLW), then the local energy and flux obey the identity*

$$\mathcal{F}_{\partial C_{[t_0,t_1]}}[\phi] := \mathcal{E}_{S_{t_1}}[\phi] - \mathcal{E}_{S_{t_0}}[\phi].$$

Observe that the flux through the null boundary is non-negative. This implies that the local energy $\mathcal{E}_{S_t}[\phi]$ in the light cone is non-decreasing as $t \uparrow \infty$ and non-increasing as $t \downarrow 0$. Since we are working with finite energy solutions, it follows from monotone convergence that the limits

$$\mathcal{E}_0 := \lim_{t\downarrow 0} \mathcal{E}_{S_t}[\phi],$$
$$\mathcal{E}_\infty := \lim_{t\uparrow \infty} \mathcal{E}_{S_t}[\phi],$$

exist. The former will be relevant for our discussion of the blow-up scenario; we defer the discussion to Section 4. For now, we observe that the existence of the limits implies that

cor:fluxdecay **Corollary 4** (Flux decay property). *Let $\phi \in C^0_{t,\mathrm{loc}}\dot{H}^1_x \cap \dot{C}^1_{t,\mathrm{loc}}L^2_x(I \times \mathbb{R}^d)$ be a strong solution to (NLW), then the flux through the light cone vanishes at the tip and at infinity,*

$$\lim_{t_0,t_1\downarrow 0} \mathcal{F}_{\partial C_{[t_0,t_1]}}[\phi] = 0,$$
$$\lim_{t_0,t_1\uparrow \infty} \mathcal{F}_{\partial C_{[t_0,t_1]}}[\phi] = 0.$$

As a corollary, we can show that the local energy on the exterior annuli $B_{B_{3t}\setminus B_t}[\phi[t]]$ decays as $t \downarrow 0$,

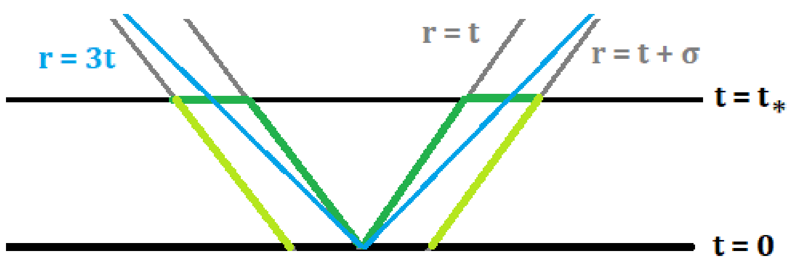

FIGURE 2. We first choose, via flux decay, a time $t = t_*$ below which the flux is small, and then, via monotone convergence, $\sigma \ll 1$ such that the local energy in the annulus $B_{t_*+\sigma} \setminus B_{t_*}$ at time $t = t_*$ is small. The flux on the exterior cone has the correct sign, so we conclude from the energy-flux identity that the local energy in the annulus is small as $t \downarrow 0$.

cor:exterior **Corollary 5** (Exterior energy decay). *Let $\phi \in C^0_t\dot{H}^1_x \cap \dot{C}^1_tL^2_x((0,1] \times \mathbb{R}^d)$ be a finite energy local solution to (NLW), then*

$$\lim_{\sigma\to 0} \limsup_{t\downarrow 0} \mathcal{E}_{B_{t+\sigma}\setminus B_t}[\phi[t]] = 0.$$

In particular,

$$\limsup_{t\downarrow 0} \mathcal{E}_{B_{3t}\setminus B_t}[\phi[t]] = 0.$$

Proof. It is clear that for each $\sigma > 0$, we have $B_{3t} \setminus B_t \subseteq B_{t+\sigma} \setminus B_t$ for $t \ll \sigma$, which shows the latter decay statement is implied by the former. To prove the former, let $\varepsilon > 0$, we choose from flux decay a time $t_* \ll 1$ such that

$$\mathcal{F}_{\partial C_{(0,t_*]}}[\phi] \ll \varepsilon.$$

Then, considering the solution on the time slice $t = t_*$, it follows from monotone convergence that there exists $\sigma \ll 1$ such that

$$\mathcal{E}_{B_{t_*+\sigma}\setminus B_{t_*}}[\phi[t_*]] \ll \varepsilon.$$

It follows then from the energy-flux identity that, for $t \ll t_*$,

$$\begin{aligned}\mathcal{E}_{B_{t+\sigma}\setminus B_t}[\phi[t]] &= \mathcal{E}_{B_{t+\sigma}}[\phi[t]] - \mathcal{E}_{B_t}[\phi[t]] \\ &= \left(\mathcal{E}_{B_{t_*+\sigma}}[\phi[t_*]] - \mathcal{F}_{-\sigma+\partial C_{[t,t_*]}}[\phi]\right) - \left(\mathcal{E}_{B_{t_*}}[\phi[t_*]] - \mathcal{F}_{\partial C_{[t,t_*]}}[\phi]\right).\end{aligned}$$

Since the flux is non-negative, we can throw away the first flux term, while by our choice of $t_* \ll 1$ and $\sigma \ll 1$ the exterior energy and flux at time $t = t_*$ is small,

$$\mathcal{E}_{B_{t+\sigma}\setminus B_t}[\phi[t]] \leq \mathcal{E}_{B_{t_*+\sigma}\setminus B_{t_*}}[\phi[t_*]] + \mathcal{F}_{\partial C_{(0,t_*]}}[\phi] \ll \varepsilon.$$

This proves the result. $\square$

3.2. **Monotonicity formula.** We derive a monotonicity formula for the non-linear wave equation arising from the scaling symmetry. The infinitesimal generator of this symmetry is given by $\Lambda = \partial_\rho + \frac{1}{\rho}$, so, after translating in time $t \mapsto t + \varepsilon$ to handle the degeneracy at the light cone, we define

$$\begin{aligned} X_\varepsilon &= \frac{1}{\rho_\varepsilon}((t+\varepsilon)\partial_t + r\partial_r), \\ w_\varepsilon &= \frac{d-2}{2\rho_\varepsilon}, \end{aligned}$$

where $\rho_\varepsilon := \sqrt{(t+\varepsilon)^2 - r^2}$. Observe that $X_0 = \frac{1}{\rho}S = \partial_\rho$, where S is the scaling vector field.

Lemma 6 (Local monotonicity formula). *Let ϕ be a smooth solution to (NLW) on an open subset $\mathcal{O} \subseteq C_{(0,\infty)}$ of the forward light cone. Then the 1-current defined in null coordinates by*

$$\begin{aligned} {}^{(X_\varepsilon)}P_L[\phi] &= \frac{1}{2}\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{1/2}\left|r^{-\frac{d-2}{2}}L\left(r^{\frac{d-2}{2}}\phi\right)\right|^2 + \frac{1}{2}\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{\frac{1}{2}}\left(|\nabla\!\!\!/\phi|^2 + \frac{(d-2)^2}{4}\frac{1}{r^2}|\phi|^2 + \frac{d-2}{d}|\phi|^{\frac{2d}{d-2}}\right), \\ {}^{(X_\varepsilon)}P_{\underline{L}}[\phi] &= \frac{1}{2}\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{1/2}\left|r^{-\frac{d-2}{2}}\underline{L}\left(r^{\frac{d-2}{2}}\phi\right)\right|^2 + \frac{1}{2}\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{\frac{1}{2}}\left(|\nabla\!\!\!/\phi|^2 + \frac{(d-2)^2}{4}\frac{1}{r^2}|\phi|^2 + \frac{d-2}{d}|\phi|^{\frac{2d}{d-2}}\right), \end{aligned}$$

setting the other components to zero, obeys the divergence identity

$$\partial^\mu\left({}^{(X_\varepsilon)}P_\mu[\phi]\right) = \frac{1}{\rho_\varepsilon}\left|\left(\partial_{\rho_\varepsilon} + \frac{1}{\rho_\varepsilon}\right)\phi\right|^2.$$

Proof. See Appendix A $\square$

Suppose the flux $\mathcal{F}_{\partial C_{[t_0,t_1]}}[\phi]$ through the null boundary vanishes, then it follows that ϕ must also vanish along the null boundary. This kills the null boundary terms when we integrate the local monotonicity formula over the cone $C_{[t_0,t_1]}$ and apply the divergence theorem, thereby furnishing the identity

$$\mathcal{M}_{S_{t_1}}[\phi] + \iint_{C_{[t_0,t_1]}} \frac{1}{\rho}\left|\left(\partial_\rho + \frac{1}{\rho}\right)\phi\right|^2 dtdx = \mathcal{M}_{S_{t_0}}[\phi],$$

for the weighted energy

$$\begin{aligned}
\mathcal{M}^{\varepsilon}_{S_t}[\phi] &:= \int_{S_t} {}^{(X_\varepsilon)}P_T[\phi]\,dx \\
&= \frac{1}{2}\int_{S_t} \left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{1/2} \left| r^{-\frac{d-2}{2}} \underline{L}\left(r^{\frac{d-2}{2}}\phi\right)\right|^2 + \left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{1/2} \left| r^{-\frac{d-2}{2}} L\left(r^{\frac{d-2}{2}}\phi\right)\right|^2 \\
&\quad + \left(\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{\frac{1}{2}} + \left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{\frac{1}{2}}\right)\left(|\nabla\phi|^2 + \frac{(d-2)^2}{4}\frac{1}{r^2}|\phi|^2 + \frac{d-2}{d}|\phi|^{\frac{2d}{d-2}}\right)dx.
\end{aligned}$$

Since the second term is non-negative, this implies monotonocity of $\mathcal{M}_{S_t}[\phi]$. In particular, the second term vanishes as $t_0, t_1 \to 0$, implying that rescalings of ϕ are asymptotically self-similar. On the other hand, the weights $(\frac{v}{u})^{1/2}$ blow-up at the null boundary, so there cannot be null concentration of energy. In the case where the flux does not vanish, we can still prove an "almost" monotonicity formula, leveraging the flux decay and a local Hardy's inequality to control the $\frac{|\phi|^2}{r^2}$ terms in the weighted energy. Before stating the almost monotonicity formula, we state and prove this Hardy decay. Define

$$\mathcal{G}_{\partial S_t}[\phi] := \frac{1}{t}\int_{\partial S_t} |\phi|^2.$$

This is well-defined for finite-energy solutions by the Sobolev trace theorem; in fact, $\phi \in H^{1/2}(\partial S_t)$.

Proposition 7 (Local Hardy's inequality)**.** *Let $\phi \in C^0_{t,\mathrm{loc}}\dot{H}^1_x \cap \dot{C}^1_{t,\mathrm{loc}}L^2_x(I\times\mathbb{R}^d)$, then*

$$\mathcal{G}_{\partial S_{t_0}}[\phi] + \int_{t_0}^{t_1} \mathcal{G}_{\partial S_t}[\phi]\frac{dt}{t} \le \mathcal{G}_{\partial S_{t_1}}[\phi] + \mathcal{F}_{\partial C_{[t_0,t_1]}}[\phi].$$

and

$$\mathcal{G}_{\partial S_t}[\phi] \le \mathcal{E}_{\{t\}\times\mathbb{R}^d\setminus S_t}[\phi].$$

Proof. Standard proof of Hardy's inequality adapted to the null cone. □

Corollary 8 (Decay of Hardy term)**.** *Let $\phi \in C^0_{t,\mathrm{loc}}\dot{H}^1_x \cap \dot{C}^1_{t,\mathrm{loc}}L^2_x(I\times\mathbb{R}^d)$ be a strong solution to (NLW), then $\mathcal{G}_{\partial S_t}[\phi]$ vanishes as $t\to 0$ and $t\to\infty$,*

$$\begin{aligned}
\lim_{t\downarrow 0}\mathcal{G}_{\partial S_t}[\phi] &= 0,\\
\lim_{t\uparrow\infty}\mathcal{G}_{\partial S_t}[\phi] &= 0.
\end{aligned}$$

Proof. The local Hardy's inequality and finite energy implies that $\int \frac{1}{t}\mathcal{G}_{\partial S_t}[\phi] < \infty$. We claim that if $\mathcal{G}_{\partial S_t}[\phi]$ does not decay as $t\downarrow 0$ or $t\uparrow\infty$, then we can extract a uniform lower bound for either $t \ll 1$ or $t \gg 1$ respectively, a contradiction since $\int \frac{1}{t}$ diverges logarithmically. Suppose the $t\downarrow 0$ case fails; the case $t\uparrow\infty$ is similar, then local Hardy implies

$$0 < \limsup_{t\downarrow 0}\mathcal{G}_{\partial S_t}[\phi] \le \mathcal{G}_{S_t}[\phi] + \mathcal{F}_{\partial C_{[0,t]}}.$$

By flux decay, the flux term can be absorbed into the left-hand side for $t \ll 1$, proving the claim. □

thm:monotone1 **Theorem 9** (Almost monotonicity formula I)**.** *Let ϕ be a strong solution to (NLW) on $[\varepsilon,1]\times\mathbb{R}^d$ with negligible flux through the null boundary, finite energy and negligible Hardy term at $t=1$,*

$$\begin{aligned}
\mathcal{E}_{S_1}[\phi] &\le \mathcal{E},\\
\mathcal{F}_{\partial C_{[\varepsilon,1]}}[\phi] &\le \varepsilon^{1/2}\mathcal{E},\\
\mathcal{G}_{\partial S_1}[\phi] &\le \varepsilon^{1/2}\mathcal{E}.
\end{aligned}$$

Then

$$\mathcal{M}^{\varepsilon}_{S_1}[\phi] + \iint_{C_{[\varepsilon,1]}} \frac{1}{\rho_\varepsilon}\left|\left(\partial_{\rho_\varepsilon} + \frac{1}{\rho_\varepsilon}\right)\phi\right|^2 dtdx \lesssim \mathcal{E}.$$

Proof. Approximating a strong solution in the energy topology, it suffices to consider smooth solutions. We integrate the local monotonicity formula over the cone $C_{[\varepsilon,1]}$ and apply the divergence theorem,

$$\mathcal{M}^{\varepsilon}_{S_1}[\phi] + \iint_{C_{[\varepsilon,1]}} \frac{1}{\rho_\varepsilon} \left| \left(\partial_{\rho_\varepsilon} + \frac{1}{\rho_\varepsilon} \right) \phi \right|^2 dtdx = \mathcal{M}^{\varepsilon}_{S_\varepsilon}[\phi] + \frac{1}{2} \int_{\partial C_{[\varepsilon,1]}} {}^{(X_\varepsilon)}P_L[\phi]\, dS.$$

We claim that the right-hand side is controlled by $\mathcal{E}$.

To control the weighted energy $\mathcal{M}^{\varepsilon}_{S_\varepsilon}[\phi]$ on the time slice S_ε, we first note that the weights can be treated as constants $|\frac{v_\varepsilon}{u_\varepsilon}| \sim |\frac{u_\varepsilon}{v_\varepsilon}| \sim 1$. It follows that we can estimate the integrand of the weighted energy $\mathcal{M}^{\varepsilon}_{S_\varepsilon}[\phi]$ by the integrand of the local energy $\mathcal{E}_{S_\varepsilon}[\phi]$ and the Hardy term $\frac{|\phi|^2}{r^2}$, e.g.

$$\begin{aligned}\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{1/2} \left| r^{-\frac{d-2}{2}} \underline{L}\left(r^{\frac{d-2}{2}}\phi\right)\right|^2 + \left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{1/2} \left| r^{-\frac{d-2}{2}} L\left(r^{\frac{d-2}{2}}\phi\right)\right|^2 &\lesssim \left| r^{-\frac{d-2}{2}} \underline{L}\left(r^{\frac{d-2}{2}}\phi\right)\right|^2 + \left| r^{-\frac{d-2}{2}} L\left(r^{\frac{d-2}{2}}\phi\right)\right|^2 \\ &\lesssim |\partial_t\phi|^2 + |\partial_r\phi|^2 + \frac{|\phi|^2}{r^2},\end{aligned}$$

and similarly for the other terms, so in total

$$\mathcal{M}^{\varepsilon}_{S_\varepsilon}[\phi] \lesssim \mathcal{E}_{S_\varepsilon}[\phi] + \int_{S_\varepsilon} \frac{|\phi|^2}{r^2}\, dx \lesssim \mathcal{E}$$

by finite energy, which controls $\mathcal{E}_{S_\varepsilon}[\phi]$, and negligible flux and Hardy term, which controls the right-hand side of the local Hardy's inequality.

To control the weighted flux term on the null boundary $\partial C_{[\varepsilon,1]}$, we observe that the weights are bounded above $|\frac{u_\varepsilon}{v_\varepsilon}| \leq 1$ and $|\frac{v_\varepsilon}{u_\varepsilon}| \lesssim \frac{1}{\varepsilon}$ in this region. It follows that we can estimate the integrand of the weighted flux by the integrand of the usual flux $\mathcal{F}_{[\varepsilon,1]}[\phi]$ and the Hardy term $\frac{|\phi|^2}{r^2}$, e.g.

$${}^{(X_\varepsilon)}P_L[\phi] \lesssim \varepsilon^{-1/2} \left(|L\phi|^2 + \frac{|\phi|^2}{r^2} \right) + {}^{(T)}J_L[\phi],$$

on the null boundary $\partial C_{[\varepsilon,1]}$, so integrating and applying the negligible flux and Hardy assumptions,

$$\int_{\partial C_{[\varepsilon,1]}} {}^{(X_\varepsilon)}P_L[\phi]\, dS \lesssim \mathcal{E}.$$

This completes the proof. $\square$

thm:monotone2 **Corollary 10** (Almost monotonicity formula II). *Let ϕ be a strong solution to (NLW) on $[\varepsilon,1]\times\mathbb{R}^d$ satisfying the hypotheses of Theorem 9. Then*

$$\mathcal{M}_{S_1^{\delta_1}}[\phi] \lesssim \mathcal{M}_{S_{t_0}^{\delta_0}}[\phi] + \left(\left(\frac{\delta_1}{t_0}\right)^{1/2} + \frac{1}{|\log(\delta_1/\delta_0)|} \right) \mathcal{E}$$

whenever $2\varepsilon \leq \delta_0 \leq \delta_1 \leq t_0$.

Proof. For any $\delta \in [\delta_0, \delta_1]$, we have that $S_1^{\delta_1} \subseteq S_1^{\delta}$ and $S_{t_0}^{\delta_0} \subseteq S_{t_0}^{\delta}$. Thus, integrating the local monotonicity formula over the translated cone $C^{\delta}_{[t_0,1]}$ and applying the divergence theorem, we see that it suffices to control the flux term

$$\int_{\partial C^{\delta}_{[t_0,1]}} {}^{(X)}P_L[\phi]\, dS \lesssim \left(\left(\frac{\delta_1}{t_0}\right)^{1/2} + \frac{1}{|\log(\delta_1/\delta_0)|} \right) \mathcal{E}$$

for appropriately chosen δ. We control the term with $\frac{u}{v}$ weight by localised Hardy and local conservation of energy,

$$\begin{aligned}&\int_{\partial C^{\delta}_{[t_0,1]}} \left(\frac{u}{v}\right)^{\frac{1}{2}} \left(|\nabla\!\!\!/ \phi|^2 + \frac{(d-2)^2}{4}\frac{1}{r^2}|\phi|^2 + \frac{d-2}{d}|\phi|^{\frac{2d}{d-2}} \right) dS \\ &\lesssim \left(\frac{\delta_1}{t_0}\right)^{1/2} \left(\mathcal{F}_{\partial C^{\delta}_{[t_0,1]}}[\phi] + \mathcal{E}_{S_1 \setminus S_1^{\delta}}[\phi] + \mathcal{G}_{S_1}[\phi] \right) \lesssim \left(\frac{\delta_1}{t_0}\right)^{1/2} \mathcal{E}.\end{aligned}$$

To treat the term with $\frac{v}{u}$ weight, we compute

$$r^{-\frac{d-2}{2}} L\left(r^{\frac{d-2}{2}}\phi\right) = \left(L + \frac{d-2}{2}\frac{1}{r}\right)\phi.$$

Finite energy plus pigeonhole principle. □

sec:local

4. Localising to the light cone

Since the wave equation is time reversible, it suffices to consider the problem in one direction of time. To unify the notation, we consider the global well-posedness problem starting at time $t = 1$ backwards-in-time and the scattering problem starting at time $t = 0$ forwards-in-time. Assume towards a contradiction that the theorem fails, e.g. ϕ is a solution on $t \in (0, 1]$ blowing up as $t \downarrow 0$ or a global solution on $t \in [0, \infty)$ which fails to scatter $||\phi||_{S[0,\infty)} = +\infty$. We claim that, after translating and rescaling, it suffices to consider solution which are regular outside of the forward light cone C, e.g.

$$\mathcal{E}_{(\{t\}\times\mathbb{R}^3)\setminus S_t}[\phi[t]] \leq \varepsilon_*. \qquad (\varepsilon_*)$$ eq:exterior

This follows from the small data theory, smooth localisation of the energy, finite speed of propagation, and energy-flux decay.

4.1. Non-scattering scenario.

Suppose towards a contradiction that given finite energy data at time $t = 0$ the solution is global yet fails to scatter as $t \to \infty$. We localise by remarking that the energy-flux identity implies the energy exterior to a large light cone remains small for all time, and so the solution can be regarded as regular in view of small data theory. Indeed, by monotone convergence we can find a large ball $B_{R_*} \subseteq \mathbb{R}^d$ outside of which the energy of the initial data is arbitrarily small,

$$\mathcal{E}_{\mathbb{R}^d\setminus B_{R_*}}[\phi[0]] \ll \varepsilon_*.$$

Recall that the energy-flux identity implies that the local energy $\mathcal{E}_{S_t}[\phi]$ is non-decreasing, or, equivalently, the exterior energy $\mathcal{E}_{\mathbb{R}^d\setminus S_t}[\phi]$ is non-increasing, as $t \to \infty$. Translating in time so that our solution starts at time $t = R_*$, we may assume the solution obeys

$$\mathcal{E}_{(\{t\}\times\mathbb{R}^3)\setminus S_t}[\phi] \ll \varepsilon_*,$$

for all $t \in [R_*, \infty)$. To localise the solution to this exterior region, we will need finite speed of propagation and the following smooth cut-off lemma,

lem:ext

Lemma 11 (Exterior energy localisation)**.** *Fix a cut-off $\chi \in C_c^\infty(\mathbb{R}^d)$ supported on the unit ball $|x| \leq 2$ and such that $\chi \equiv 1$ on the ball $|x| \leq 1$. Denote the rescalings $\chi_\lambda(x) := \chi(x/\lambda)$, then*

$$||\nabla((1-\chi_\lambda)\phi)||_{L^2(\mathbb{R}^d)} \lesssim_{d,||\chi||_{L^\infty},||\nabla\chi||_{L^\infty}} ||\phi||_{L^{\frac{2d}{d-2}}(|x|\geq\lambda)} + ||\nabla\phi||_{L^2(|x|\geq\lambda)}$$

uniformly in $\lambda > 0$.

Proof. Computing the gradient of $\nabla((1-\chi_\lambda)\phi)$ using the product and chain rules, and then applying the triangle inequality to the L^2-norm gives

$$||\nabla((1-\chi_\lambda))\phi)||_{L^2(\mathbb{R}^d)} \leq \lambda^{-1}||\nabla\chi||_{L^\infty}||\phi||_{L^2(\lambda\leq|x|\leq2\lambda)} + ||\chi||_{L^\infty}||\nabla\phi||_{L^2(|x|\geq\lambda)}.$$

Here we have observed that the gradient of χ is supported in the annulus $1 \leq |x| \leq 2$. To estimate the lower order term, we apply Holder's inequality,

$$\lambda^{-1}||\phi||_{L^2(\lambda\leq|x|\leq2\lambda)} \leq |A|\,||\phi||_{L^{\frac{2d}{d-2}}(\lambda\leq|x|\leq2\lambda)},$$

where $|A|$ denotes the volume of the annulus $1 \leq |x| \leq 2$. This completes the proof. □

Cutting off the data at time $t = R_*$ to this exterior region, we obtain a global small-energy solution ϕ^{reg} to (NLW) which by finite speed of propagation agrees with ϕ in this exterior region. In particular, as will be relevant for our rescaling argument in Section 5, the energy dispersion norm must be large within the light cone.

4.2. **Blow-up scenario.** Suppose towards a contradiction that given data at time $t = 1$ the solution blows-up backwards in time as one approaches $t \downarrow 0$. To localise this scenario, we show that blow-up occurs only if energy concentrates within a light cone,

prop:blowup **Proposition 12** (Blow-up implies energy concentration). *Suppose $\phi \in C_t^0 \dot{H}_x^1 \cap \dot{C}_t^1 L_x^2((0,1] \times \mathbb{R}^d)$ is a solution to (NLW) which blows-up as $t \downarrow 0$. Then there exists $x_* \in \mathbb{R}^d$ such that*

$$\limsup_{t \downarrow 0} \mathcal{E}_{B_t(x_*)}[\phi[t]] \geq \varepsilon_*$$

for some absolute constant $\varepsilon_* > 0$.

Our strategy will be to show that if energy does not concentrate, then we can extend the solution uniformly backwards in time to $[-\delta, 1] \times \mathbb{R}^d$ by gluing together a finite number of regions where this can be done. We first start with an exterior region, choosing $R_* \gg 1$ such that

$$\mathcal{E}_{\mathbb{R}^d \setminus B_{R_*}}[\phi[1]] \ll \varepsilon_*.$$

Using the exterior energy localisation from Lemma 11, we obtain a global small-energy solution ϕ^{reg} to (NLW) which by finite speed of propagation agrees with ϕ outside the ball B_{R_*+1} at time $t = 0$. In particular, ϕ can be extended backwards in the domain of dependence of the exterior region. If we can find an open cover of B_{R_*+2} at time $t = 0$ by regions where ϕ can be extended, then we can conclude the result by extracting a finite cover by compactness and finite speed of propagation.

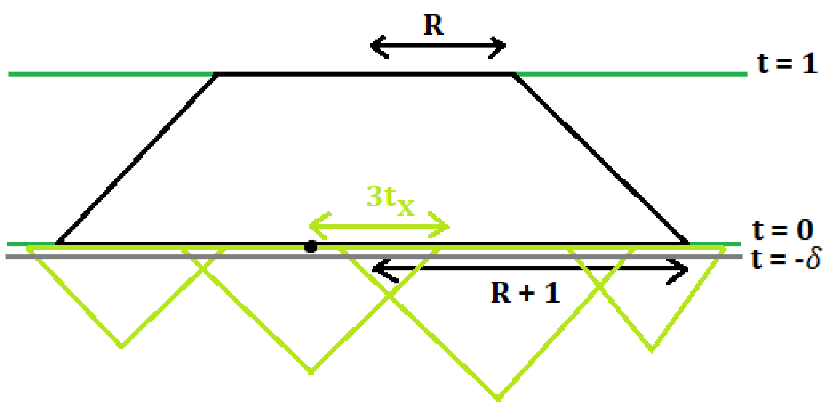

FIGURE 3. The solution has small exterior energy and thus can be continued for some small time. If there is no energy concentration, we can cover the interior by a finite collection of light cones with small energy. Gluing together the local solutions, we can extend the whole solution uniformly in time.

By the exterior energy decay proved in Corollary 5, it suffices to prove the result replacing the local energy in the light cone $\mathcal{E}_{B_t(x_*)}[\phi[t]]$ with the energy in the larger cone $\mathcal{E}_{B_{3t}(x_*)}[\phi[t]]$. Then, if energy does not concentrate, for each $x \in \mathbb{R}^d$ there exists $t_x \ll 1$ such that

$$\mathcal{E}_{B_{3t_x}(x)}[\phi[t_x]] < \varepsilon_*.$$

To use the small data theory to rule out blow-up within the domain of dependence of, say, $B_{2t_x}(x)$ at time $t = t_x$, we need to localise in space,

lem:local **Lemma 13** (Localisation of energy). *Fix a cut-off $\chi \in C_c^\infty(\mathbb{R}^d)$ supported on the unit ball $|x| \leq 1$ and such that $\chi \equiv 1$ on the ball $|x| \leq \frac{1}{2}$. Denote the rescalings $\chi_\lambda(x) := \chi(x/\lambda)$, then*

$$||\nabla(\chi_\lambda \phi)||_{L^2(\mathbb{R}^d)} \lesssim_{d, ||\chi||_{L^\infty}, ||\nabla \chi||_{L^\infty}} ||\phi||_{L^{\frac{2d}{d-2}}(|x| \leq \lambda)} + ||\nabla \phi||_{L^2(|x| \leq \lambda)}.$$

Proof. Computing the gradient of $\nabla(\chi_\lambda \phi)$ using the product and chain rules, and then applying the triangle inequality to the L^2-norm gives

$$||\nabla(\chi_\lambda\phi)||_{L^2(\mathbb{R}^d)} \leq \lambda^{-1}||\nabla\chi||_{L^\infty}||\phi||_{L^2(|x|\leq\lambda)} + ||\chi||_{L^\infty}||\nabla\phi||_{L^2(|x|\leq\frac{1}{2}\lambda)}.$$

To estimate the lower order term, we apply Holder's inequality,

$$\lambda^{-1}||\phi||_{L^2(|x|\leq\lambda)} \leq |B|\,||\phi||_{L^{\frac{2d}{d-2}}(|x|\leq\lambda)},$$

where $|B|$ denotes the volume of the unit ball. This proves the result. □

Applying the localisation lemma, we can apply a cut-off to obtain data $\phi^{\mathrm{reg}}[t_x]$ which has small energy and agrees with $\phi[t_x]$ on the ball $B_{2t_x}(x)$. By the small-data theory, the localised data admits a global solution ϕ^{reg}, while finite speed of propagation tells us that this solution agrees with ϕ in the domain of dependence of $B_{2t_x}(x)$ at time $t = t_x$. In particular, ϕ is regular in the ball $B_{t_x}(x)$ at time $t = 0$ and can be continued past this time. This completes the proof of Proposition 12. That is, after translating in space, the solution blows up by concentrating energy inside a light cone towards the origin,

$$\limsup_{t\downarrow 0} \mathcal{E}_{S_t}[\phi] \geq \varepsilon_*.$$

Following the proof of exterior energy decay, Corollary 5, we can truncate the data at time $t_0 \ll 1$ so that the solution remains unchanged in the interior of the light cone $C_{[0,t_0]}$ and has small energy exterior to the light cone. We leave this as an exercise.

sec:ED

5. Concentration of energy

The bubbling analysis of a singularity begins by locating points $x(t)$ and scales $\lambda(t)$ at which the energy concentrates as $t \downarrow 0$ in the blow-up scenario or $t \uparrow \infty$ in the non-scattering scenario. For our purposes, it will suffice to perform the bubbling analysis along a sequence of times, though a continuous in time bubbling analysis is needed if one wants to show soliton resolution; see [JL23] and related works.

5.1. **Energy-dispersed regularity.** We first locate the points and scales of concentration by showing that, if the energy does not concentrate, then the solution is regular. Define the energy dispersion norm

$$||\phi||_{\mathsf{ED}[I]} := \sup_{N\in 2^{\mathbb{Z}}} \left(N^{-\frac{d-2}{2}}||P_N\phi||_{L^\infty_{t,x}(I\times\mathbb{R}^d)} + N^{-\frac{d}{2}}||\partial_t P_N\phi||_{L^\infty_{t,x}(I\times\mathbb{R}^d)} \right).$$

Proving well-posedness theory amounts to controlling the non-linear term in an appropriate dual Strichartz norm. One typically argues by Sobolev embedding to place each of the factors of the non-linearity in a judicious choice of Strichartz norm; in fact, one can be more efficient via

Lemma 14 (Refined Sobolev inequality)**.** *Let $\phi \in \mathcal{S}(\mathbb{R}^d)$, then*

$$||\phi||_{L_x^{\frac{2d}{d-2}}} \lesssim ||\phi||_{\dot{H}^1}^{\frac{d-2}{d}} \left(\sup_{N\in 2^{\mathbb{Z}}} N^{-\frac{d-2}{2}} ||\phi_N||_{L^\infty_x} \right)^{\frac{2}{d}}. \tag{1}$$

Proof. See harmonic analysis notes. See also [BCD11, Theorem 2.43]. □

Theorem 15 (Energy-dispersed regularity theorem)**.** *There exist functions $\mathcal{F}(\mathcal{E}) \gg 1$ and $\varepsilon(\mathcal{E}) \ll 1$ of energy such that if $\phi \in C^1_{t,\mathrm{loc}}\dot{H}^1_x \cap C^0_{t,\mathrm{loc}}L^2_x(I\times\mathbb{R}^d)$ is a solution to the energy-critical non-linear wave equation (NLW) with sub-threshold energy $\mathcal{E}[\phi] \equiv \mathcal{E}$ and energy dispersion*

$$||\phi||_{\mathsf{ED}[I]} \leq \varepsilon(\mathcal{E}),$$

then

$$||\phi||_{\mathsf{S}[I]} \leq \mathcal{F}(\mathcal{E}).$$

In addition, we can continue the solution to a larger interval $I \subset J$.

Proof. By local well-posedness and the Strichartz estimates, there exists $T \ll 1$ such that

$$||\phi||_{\mathsf{S}[0,T]} \leq C\mathcal{E}^{1/2}.$$

To control the Strichartz norms, we argue by a continuity argument. Suppose we want to control a Strichartz norm $L_t^p L_x^q$. The refined Sobolev inequality with coercivity of energy gives control over the endpoint exponent $L_t^\infty L_x^{\frac{2d}{d-2}}$ in terms of energy and energy dispersion,

$$||\phi||_{L_t^\infty L_x^{\frac{2d}{d-2}}} \leq C\mathcal{E}^{\frac{d-2}{2d}} ||\phi||_{\mathsf{ED}}^{\frac{2}{d}}.$$

Assume for a bootstrap assumption that every non-endpoint Strichartz norm is controlled by $2C\mathcal{E}^{1/2}$. Then, choosing the energy dispersion smaller than $\frac{1}{1000}\mathcal{E}^{1/2}$, we interpolate

$$||\phi||_{L_t^p L_x^q} \leq ||\phi||^\theta_{L_t^\infty L_x^{\frac{2d}{d-2}}} ||\phi||^{1-\theta}_{L_t^{2+} L_{t,x}^{\frac{2d}{d-3}-}} \leq C\mathcal{E}^{1/2}.$$

This proves the result. Then a standard continuation argument using you favourite Strichartz norm, e.g. $L_{t,x}^{\frac{2(d+1)}{d-1}}$, [KM08, Theorem 2.7], furnishes the conclusion. □

The energy-dispersed regularity theorem allows us to locate points and *frequency* scales of energy concentration in the event of blow-up or non-scattering. Of course, the uncertainty principle would allow us to convert these frequency scales into physical scales, c.f. [ST10a, Section 6.2], though it will be more convenient for our physical space arguments to work with the following physical space version of energy dispersion as introduced in [OT16, Section 8.1],

$$||\phi||_{\widetilde{\mathsf{ED}}[I]} := \sup_{N\in 2^{\mathbb{Z}}} \left(N^{-\frac{d-2}{2}} ||\chi_{1/N} * \phi||_{L^\infty_{t,x}(I\times\mathbb{R}^d)} + N^{-\frac{d}{2}} ||\partial_t \chi_{1/N} * \phi||_{L^\infty_{t,x}(I\times\mathbb{R}^d)} \right)$$

where $\chi \in C_c^\infty(\mathbb{R}^d)$ is a cut-off supported in the unit ball $B \subseteq \mathbb{R}^d$ with unit mass $\int \chi = 1$, and the physical space scaling $\chi_{1/N}(x) := N^d \chi(Nx)$ is the dual to the frequency space scaling $\widehat{\chi}_N(\xi) := \widehat{\chi}(\xi/N)$. Observe that the rescaled cut-off $\chi_{1/N}$ is supported in the ball $B_{1/N}$ and has unit mass $\int \chi_{1/N} = 1$.

Corollary 16 (Physical-space version of energy-dispersed regularity)**.** *The physical space version of energy-dispersion controls the usual energy-dispersion norm*

$$||\phi||_{\mathsf{ED}[I]} \lesssim ||\phi||_{\widetilde{\mathsf{ED}}[I]}.$$

In particular, the energy-dispersed regularity theorem continues to hold replacing small ED*-norm with small* $\widetilde{\mathsf{ED}}$*-norm.*

Proof. Fix a large parameter $M_0 \in 2^{\mathbb{Z}}$, we compute

$$N^{-\frac{d-2}{2}} ||P_N\phi||_{L_x^\infty} \leq N^{-\frac{d-2}{2}} ||\chi_{1/M_0N} * P_N\phi||_{L_x^\infty} + N^{-\frac{d-2}{2}} ||P_N\phi - \chi_{1/M_0N} * P_N\phi||_{L_x^\infty}.$$

The first term can be controlled by boundedness of the Littlewood-Paley projections,

$$N^{-\frac{d-2}{2}} ||\chi_{1/M_0N} * P_N\phi||_{L_x^\infty} \leq M_0^{\frac{d-2}{2}} (M_0N)^{-\frac{d-2}{2}} ||\chi_{1/M_0N}\phi||_{L_x^\infty}.$$

We claim that the second term can be absorbed into the left-hand side after choosing $M_0 \gg 1$. Indeed, using the fact that $\int \chi_{1/M_0N} = 1$, the fundamental theorem of calculus, and Sobolev-Bernstein, we have

$$\begin{aligned} \left|P_N\phi(x) - (\chi_{1/M_0N} * P_N\phi)(x)\right| &\leq \int_{\mathbb{R}^d} \chi_{1/M_0N}(x-y)|P_N\phi(y) - P_N\phi(x)|\,dy \\ &\leq \int_{\mathbb{R}^d} \chi_{1/M_0N}(x-y) ||\nabla P_N\phi||_{L_x^\infty} |x-y|\,dy \\ &\leq \left(\int_{\mathbb{R}^d} \chi(x)|x|dx\right) M_0^{-1}N^{-1} ||\nabla P_N\phi||_{L_x^\infty} \lesssim_\chi M_0^{-1} ||P_N\phi||_{L_x^\infty}. \end{aligned}$$

Arguing similarly for $\partial_t\phi$, we conclude the result. □

5.2. **Rescaling.** It follows from the energy-dispersed regularity theorem that there exist $(t_n, \widetilde{x_n})$, approaching either $t_n \downarrow 0$ in the blow-up scenario or $t_n \uparrow \infty$ in the non-scattering scenario, and scales $\widetilde{N_n} \in 2^{\mathbb{Z}}$ at which the energy-dispersion is non-negligible,

$$\widetilde{N_n}^{-\frac{d-2}{2}} |\chi_{1/\widetilde{N_n}} * \phi(t_n, \widetilde{x_n})| + \widetilde{N_n}^{-\frac{d}{2}} |\partial_t \chi_{1/\widetilde{N_n}} * \phi(t_n, \widetilde{x_n})| > \varepsilon(\mathcal{E}). \tag{$\widetilde{\text{C}}$}$$

Due to flux decay, there exists a sequence $\varepsilon_n \to 0$ such that

$$\mathcal{F}_{\partial C_{[\varepsilon_n t_n, t_n]}}[\phi] \leq \varepsilon_n^{1/2} \mathcal{E}. \tag{$\widetilde{\text{F}}$}$$

Note in the non-scattering case, we also require that $\varepsilon_n t_n \to \infty$. Furthermore, recall that in Section 4 we have truncated our solution so that the energy exterior to the light cone is negligible in the sense that

$$\mathcal{E}_{(\{t\} \times \mathbb{R}^d) \setminus S_t}[\phi] \leq \varepsilon^{1000} \mathcal{E}. \tag{$\widetilde{\text{E}}$}$$

Rescaling $\phi_n(t, x) := t_n^{\frac{d-2}{2}} \phi(t_n t, t_n x)$, we obtain a sequence of finite-energy solutions to (NLW) on $[\varepsilon_n, 1]$ which admit points of concentration $x_n := \widetilde{x_n}/t_n$ at time $t = 1$ and scales $N_n := t_n \widetilde{N_n}$ such that

$$N_n^{-\frac{d-2}{2}} |\chi_{1/N_n} * \phi_n(1, x_n)| + N_n^{-\frac{d}{2}} |\partial_t \chi_{1/N_n} * \phi_n(1, x_n)| > \varepsilon(\mathcal{E}), \tag{C}$$

eq:concentrat

flux decay on the null boundary,

$$\mathcal{F}_{\partial C_{[\varepsilon_n, 1]}}[\phi_n] \leq \varepsilon_n^{1/2} \mathcal{E}, \tag{F}$$

eq:fluxdecay

negligible energy exterior to the light cone,

$$\mathcal{E}_{(\{t\} \times \mathbb{R}^d) \setminus S_t}[\phi_n] \leq \varepsilon^{1000} \mathcal{E}, \tag{E}$$

eq:exteriorer

for all $t \in [\varepsilon_n, 1]$, and uniformly bounded energy,

$$\mathcal{E}[\phi_n] \equiv \mathcal{E}. \tag{B}$$

eq:boundedene

From this point on, we do not distinguish between the blow-up scenario and non-scattering scenario, and work with the sequence of finite-energy solutions $\phi_n : [\varepsilon_n, 1] \times \mathbb{R}^d \to \mathbb{R}$ satisfying the properties (C), (F), (E), (B).

5.3. **Dichotomy.** By the uncertainty principle, $P_N \phi$ is localised at physical scales $\frac{1}{N}$. Since the energy exterior to the light cone is negligible a la (E), the pointwise concentration at a point x and frequency scale N must be negligible if the ball $B_{1/N}(x)$ sees a large portion of the exterior to the cone. Thus, for there to be concentration (C), our frequencies must be sufficiently large and our points of concentration cannot be too far outside the cone,

$$N_n \geq M_{\varepsilon, \mathcal{E}},$$
$$|x_n| \leq 1 + N_n^{-1} \delta_{\varepsilon, \mathcal{E}}$$

for some uniform frequency $M_{\varepsilon, \mathcal{E}} \in 2^{\mathbb{Z}}$ and parameter $\delta_{\varepsilon, \mathcal{E}} > 0$. We are interested in the fastest concentrating scales, so we pass to a subsequence witnessing $\limsup_n N_n$. We obtain the following trichotomy:

(a) self-similar concentration, $N_n \sim 1$,
(b) time-like concentration, $N_n \to \infty$ and $|x_n| \leq \gamma < 1$,
(c) null concentration, $N_n \to \infty$ and $|x_n| \to 1$.

In fact, only the first two cases are admissible. We eliminate the null concentration scenario using the weighted monotonicity formula.

Lemma 17 (No null concentration)**.** *Let ϕ_n be a sequence of solutions to the energy-critical non-linear wave equation (NLW) on $[\varepsilon_n, 1]$ satisfying* (C)*,* (F)*,* (E) *and* (B)*. There exists $M(\mathcal{E}) \gg 1$ and $0 < \gamma(\mathcal{E}) < 1$ such that if $N_n \geq M$ and $|x_n| \geq \gamma$, then energy dispersion is small.*

6. Bubbling off a soliton

Proposition 18. *There exists a constant $\varepsilon_0 > 0$ such that for every sequence $\{\phi_n\}_n$ of solutions to the defocusing (NLW) on $(-2,2) \times 4B$ with uniformly small energy*

$$\mathcal{E}_{\{0\}\times 4B}[\phi_n] \ll \varepsilon_0^2, \tag{2}$$

and is asymptotically stationary,

$$\iint_{(-2,2)\times 4B} |(Y+b)\phi_n|^2 \, dx \overset{n\to\infty}{\longrightarrow} 0, \tag{3}$$

for some uniformly time-like vector field Y and a smooth function b, then, after passing to a subsequence, we can extract a solution $\Phi \in H^{3/2}_{t,x}((-1,1)\times B)$ to the defocusing (NLW) to which the sequence converges strongly in $H^1((-1,1)\times B)$ and

$$(Y+b)\Phi = 0. \tag{4}$$

Proof. □

7. Rigidity

In our blow-up analysis, we showed that there are only two possibilities for energy concentration: either time-like concentration or self-similar concentration. Using the compactness lemma, we can extract either a stationary solution or a self-similar solution. Both are impossible, leading to a contradiction.

Proposition 19 (No non-trivial finite-energy stationary solutions)**.** *Let ϕ be a finite-energy smooth solution to the defocusing (NLW) on $\mathbb{R}^{1+d}$ which is stationary, i.e. there exists a unit constant time-like vector field Y such that $Y\phi = 0$. Then $\phi \equiv 0$.*

Proof. Any unit constant time-like vector field can be Lorentz transformed into the vector field $T = \partial_t$, so in particular we can write $Y\phi = \partial_t L_v \phi$ for some Lorentz transformation L_v. As these transformations commute with the wave operator $\Box$, it follows from stationarity that the Lorentz transformed solution satisfies the elliptic equation

$$\Delta L_v\phi = \Box L_v \phi = L_v \Box \phi = L_v\left(|\phi|^{p-1}\phi\right) = |L_v\phi|^{p-1}L_v\phi.$$

Writing $Q = L_v\phi$, multiply the equation by Q and integrate to obtain

$$\int_{\mathbb{R}^d} Q\Delta Q \, dx = \int_{\mathbb{R}^d} |Q|^{p+1}\, dx.$$

Integrating the left-hand side by parts, we see that it is non-positive, while the right-hand side is non-negative. We conclude $Q \equiv 0$. □

Proposition 20 (No non-trivial finite-energy self-similar solutions)**.** *Let ϕ be a finite-energy smooth solution to the defocusing (NLW) on the forward light cone $C_{(0,\infty)}$ which is self-similar, i.e. $(\partial_\rho + \frac{1}{\rho})\phi = 0$. Then $\phi \equiv 0$.*

Proof. Writing in hyperbolic coordinates $\Box = -\rho^{-d}\partial_\rho \rho^d \partial_\rho + \rho^{-2}\Delta_{\mathbb{H}^d}$, we compute

$$\begin{aligned}
\Box\phi &= -\rho^{-d}\partial_\rho\left(\rho^d\partial_\rho\phi\right) + \rho^{-2}\Delta_{\mathbb{H}^d}\phi \\
&= \rho^{-d}\partial_\rho(\rho^{d-1}\phi) + \rho^{-2}\Delta_{\mathbb{H}^d}\phi \\
&= \left((d-1)\rho^{-2}\phi + \rho^{-1}\partial_\rho\phi\right) + \rho^{-2}\Delta_{\mathbb{H}^d}\phi = \rho^{-2}\left(d-2+\Delta_{\mathbb{H}^d}\right)\phi.
\end{aligned}$$

Thus ϕ solves the elliptic equation

$$(d-2+\Delta_{\mathbb{H}^d})\phi = \rho^2|\phi|^{p-1}\phi.$$

Since ϕ is finite-energy, we know that $\phi \in \dot{H}^1(\mathbb{H}^d_\rho)$ on each hyperboloid; we leave this as an exercise. As with the case of stationary solutions, multiplying the equation by ϕ, integrating on $\mathbb{H}^d$, and integrating-by-parts furnishes the result. □

Remark. Recall self-similar coordinates for the non-linear wave equation are given by

$$\rho = \sqrt{t^2 - r^2}, \qquad \xi = \frac{|x|}{|t|}.$$

The Minkowski metric can be expressed in these coordinates as

$$\mathbf{m} = -d\rho^2 + \rho^2 \left(\frac{d\xi^2}{(1-\xi^2)^2} + \frac{\xi^2}{1-\xi^2} g_{\mathbb{S}^{d-1}} \right).$$

In particular $(\partial_\rho - \frac{1}{\rho})\phi = 0$ implies that

$$\phi(t,x) = t^{-\frac{2}{p-1}} W(x/t)$$

for some self-similar profile W. In view of the scaling symmetries, this implies that the flux on any portion of the null boundary is the same.

Appendix A. Derivation of monotonicity formula

app:A

We derive the key monotonicity formula associated to the scaling vector field. We follow a computation similar to that for Maxwell-Klein-Gordon [OT16, Section 5]. In view of the degeneracy of the light cone, we make a translation in time $t \mapsto t + \varepsilon$, defining

$$\begin{aligned} X_\varepsilon &= \frac{1}{\rho_\varepsilon^k}((t+\varepsilon)\partial_t + r\partial_r), \\ w_\varepsilon &= \frac{d-k-1}{2\rho_\varepsilon^k}, \end{aligned}$$

where $\rho_\varepsilon := \sqrt{(t+\varepsilon)^2 - r^2}$. In view of scaling, $k = d - 1 - \frac{4}{p-1}$; we are interested in the NLW energy critical exponents, so $k = 1$. Observe that $X_0 = \frac{1}{\rho} S = \partial_\rho$, where S is the scaling vector field. To simplify the discussion, we first restrict to the case $\varepsilon = 0$. Translating in time, we will conclude

Theorem 21 (Monotonicity formula)**.** *Let ϕ be a smooth solution to (NLW) on an open subset $\mathcal{O} \subseteq C_{(0,\infty)}$ of the forward light cone. Then the* 1*-current defined by*

$$^{(X_\varepsilon)}P_\mu[\phi] = {}^{(X_\varepsilon)}J_\mu[\phi] + {}^{(w_\varepsilon)}J_\mu[\phi] + {}^{(\mathcal{H}_\varepsilon)}J_\mu[\phi] + {}^{(\mathcal{N}_\varepsilon)}J_\mu[\phi] \tag{5}$$

satisfies the divergence identity

$$\partial^\mu \left({}^{(X_\varepsilon)}P_\mu[\phi] \right) = \frac{1}{\rho_\varepsilon} \left| \left(\partial_{\rho_\varepsilon} + \frac{1}{\rho_\varepsilon} \right) \phi \right|^2. \tag{6}$$

In null coordinates,

$$\begin{aligned} {}^{(X_\varepsilon)}P_L[\phi] &= \frac{1}{2}\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{1/2} \left| r^{-\frac{d-2}{2}} L \left(r^{\frac{d-2}{2}} \phi \right) \right|^2 + \frac{1}{2}\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{\frac{1}{2}} \left(|\nabla\!\!\!/ \phi|^2 + \frac{(d-2)^2}{4}\frac{1}{r^2}|\phi|^2 + \frac{2\kappa}{p+1}|\phi|^{p+1} \right) \\ {}^{(X_\varepsilon)}P_{\underline{L}}[\phi] &= \frac{1}{2}\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{1/2} \left| r^{-\frac{d-2}{2}} \underline{L} \left(r^{\frac{d-2}{2}} \phi \right) \right|^2 + \frac{1}{2}\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{\frac{1}{2}} \left(|\nabla\!\!\!/ \phi|^2 + \frac{(d-2)^2}{4}\frac{1}{r^2}|\phi|^2 + \frac{2\kappa}{p+1}|\phi|^{p+1} \right). \end{aligned} \tag{7}$$

A.0.1. 0*-current.* We first work in hyperbolic coordinates. Writing the metric in these coordinates, the Lie derivative of the metric along the coordinate vector field ∂_ρ is given by $\mathcal{L}_{\partial_\rho}\mathbf{m} = 2\rho(\mathrm{d}y^2 + \sinh^2(y) g_{\mathbb{S}^{d-1}})$. Thus, we can write the deformation tensor along X_0 as

$$\begin{aligned} \frac{1}{2}{}^{(X_0)}\pi &= \rho(\mathrm{d}y^2 + \sinh^2(y) g_{\mathbb{S}^{d-1}} \\ &= \frac{1}{\rho}\mathrm{d}\rho^2 + \frac{1}{\rho}\mathbf{m}, \end{aligned}$$

with metric dual

$$\begin{aligned}\frac{1}{2}{}^{(X_0)}\pi^{\#} &= \rho(\partial_y \odot \partial_y + \sinh^2(y) g^{-1}_{\mathbb{S}^{d-1}}) \\ &= \frac{1}{\rho}\partial_\rho \odot \partial_\rho + \frac{1}{\rho}\mathbf{m}^{-1}.\end{aligned}$$

The 0-current is given by

$$\begin{aligned}{}^{(X_0)}K[\phi] &= T[\phi]\left(\frac{1}{2}{}^{(X_0)}\pi^{\#}\right) \\ &= \frac{1}{\rho}\left(T_{\rho\rho} + \operatorname{tr} T\right) \\ &= \frac{1}{\rho}\left(|\partial_\rho\phi|^2 + \frac{2-d}{2}|\nabla\phi|^2 - \kappa\frac{d-2}{2}|\phi|^{\frac{2d}{d-2}}\right).\end{aligned}$$

For $d \geq 3$, the second term has bad sign. Furthermore, we want to rewrite the first term in terms of the infinitesimal generator of the scaling symmetry $\Lambda = \partial_\rho + \frac{1}{\rho}$. To remove the term with bad sign, we introduce the generalised current. The d'Alembertian of our weight is $-\frac{1}{2}\Box\frac{1}{\rho} = -\frac{d-2}{2}\frac{1}{\rho^3}$, so the generalised 0-current takes the form

$$\begin{aligned}{}^{(w_0)}K[\phi] &= \frac{d-2}{2\rho}|\nabla\phi|^2 - \frac{(d-2)^2}{4}\frac{1}{\rho^3}|\phi|^2 + \kappa\frac{d-2}{2\rho}|\phi|^{\frac{2d}{d-2}} \\ &= \frac{1}{\rho}\left(\frac{d-2}{2}|\nabla\phi|^2 - \frac{(d-2)^2}{4}\left|\frac{1}{\rho}\phi\right|^2 + \kappa\frac{d-2}{2}|\phi|^{\frac{2d}{d-2}}\right).\end{aligned}$$

It remains to handle the first term of ${}^{(X_0)}K[\phi]$ and the second term coming from ${}^{(w_0)}K[\phi]$. This will be done by Hardy's inequality, or, more descriptively, completing-the-square with integration-by-parts. Define the Hardy 0- and 1-currents

$$\begin{aligned}{}^{(\mathcal{H}_0}J_\rho[\phi] &:= -\frac{d-2}{2\rho^2}|\phi|^2, \\ {}^{(\mathcal{H}_0)}K[\phi] &:= -\frac{(d-2)^2}{2\rho^3}|\phi|^2 + \frac{d-2}{2\rho^2}|\phi|^2 \\ &= \frac{1}{\rho}\left(\frac{(d-2)^2}{2}\left|\frac{1}{\rho}\phi\right|^2 + \frac{d-2}{\rho^2}\phi\partial_\rho\phi\right).\end{aligned} \tag{8}$$

We set the other components of ${}^{(\mathcal{H}_0)}J[\phi$ to be zero. Then, using $\nabla^\mu({}^{(\mathcal{H}_0)}J_\mu = -\rho^{-d}\partial_\rho(\rho^d({}^{(\mathcal{H}_0)}J_\rho))$, we obtain the divergence identity

$$\nabla^\mu\left({}^{(\mathcal{H}_0)}J_\mu[\phi]\right) = {}^{(\mathcal{H}_0)}K[\phi]. \tag{9}$$

Collecting the 0-curents,

$${}^{(X_0)}K[\phi] = \frac{1}{\rho}\left(|\partial_\rho\phi|^2 + \frac{2-d}{2}|\nabla\phi|^2 - \kappa\frac{d-2}{2}|\phi|^{\frac{2d}{d-2}}\right), \tag{10}$$

$${}^{(w_0)}K[\phi] = \frac{1}{\rho}\left(\frac{d-2}{2}|\nabla\phi|^2 - \frac{(d-2)^2}{4}\left|\frac{1}{\rho}\phi\right|^2 + \kappa\frac{d-2}{2}|\phi|^{\frac{2d}{d-2}}\right), \tag{11}$$

$${}^{(\mathcal{H}_0)}K[\phi] = \frac{1}{\rho}\left(\frac{(d-2)^2}{2}\left|\frac{1}{\rho}\phi\right|^2 + \frac{d-2}{\rho^2}\phi\partial_\rho\phi\right), \tag{12}$$

we write

$${}^{(X_0)}K[\phi] + {}^{(w_0)}K[\phi] + {}^{(\mathcal{H}_0)}K[\phi] = \frac{1}{\rho}\left|\left(\partial_\rho + \frac{d-2}{2\rho}\right)\phi\right|^2. \tag{13}$$

A.0.2. 1-*current*. Ideally we wold like for the flux term to have a sign. It is convenient here to work in null coordinates, so we perform a null decomposition. At each point $p = (t_0, x_0)$, consider an orthonormal frame $\{e_{\mathfrak{a}}\}_{\mathfrak{a}=1,\ldots,d-1}$ for the sphere $\{t_0\} \times \partial B_{\rho_0}(0)$. Then $\{L, \underline{L}, e_1, \ldots, e_{d-1}\}$ forms a null frame at p. We decompose $\nabla\phi$ with respect to the null frame into $L\phi$, $\underline{L}\phi$ and $\nabla\!\!\!/_{\mathfrak{a}}\phi$,

$$\begin{aligned} T[\phi](L, \underline{L}) &= |L\phi|^2, \\ T[\phi](\underline{L}, \underline{L}) &= |\underline{L}\phi|^2, \\ T[\phi](L, \underline{L}) &= |\nabla\!\!\!/\phi|^2 + \kappa\frac{d-2}{d}|\phi|^{\frac{2d}{d-2}}. \end{aligned} \tag{14}$$

Furthermore, recall that

$$\rho^2 = uv, \qquad X_0 = \frac{1}{2}\left(\frac{v}{u}\right)^{1/2} L + \frac{1}{2}\left(\frac{u}{v}\right)^{1/2} \underline{L}.$$

Then the 1-currents in these coordinates take the form

$$\begin{aligned} {}^{(X_0)}J_L[\phi] &= \frac{1}{2}\left(\frac{v}{u}\right)^{1/2} |L\phi|^2 + \frac{1}{2}\left(\frac{u}{v}\right)^{1/2}\left(|\nabla\!\!\!/\phi|^2 + \kappa\frac{d-2}{d}|\phi|^{\frac{2d}{d-2}}\right), \\ {}^{(X_0)}J_{\underline{L}}[\phi] &= \frac{1}{2}\left(\frac{u}{v}\right)^{1/2} |\underline{L}\phi|^2 + \frac{1}{2}\left(\frac{v}{u}\right)^{1/2}\left(|\nabla\!\!\!/\phi|^2 + \kappa\frac{d-2}{d}|\phi|^{\frac{2d}{d-2}}\right) \end{aligned} \tag{15}$$

and

$$\begin{aligned} {}^{(w_0)}J_L[\phi] &= \frac{d-2}{2\rho}\phi L\phi + \frac{d-2}{4}\left(\frac{u}{v}\right)^{1/2}\frac{1}{\rho^2}|\phi|^2, \\ {}^{(w_0)}J_{\underline{L}}[\phi] &= \frac{d-2}{2\rho}\phi \underline{L}\phi + \frac{d-2}{4}\left(\frac{v}{u}\right)^{1/2}\frac{1}{\rho^2}|\phi|^2 \end{aligned} \tag{16}$$

and, writing $\mathrm{d}\rho = \frac{1}{2}\left(\frac{u}{v}\right)^{1/2}\mathrm{d}v + \frac{1}{2}\left(\frac{v}{u}\right)^{1/2}\mathrm{d}u$,

$$\begin{aligned} {}^{(\mathcal{H}_0)}J_L[\phi] &= -\frac{d-2}{2}\left(\frac{u}{v}\right)^{1/2}\frac{1}{\rho^2}|\phi|^2, \\ {}^{(\mathcal{H}_0)}J_L[\phi] &= -\frac{d-2}{2}\left(\frac{v}{u}\right)^{1/2}\frac{1}{\rho^2}|\phi|^2. \end{aligned} \tag{17}$$

Evidently, adding these three expressions does not give a good sign. We complete the square again, computing

$$\begin{aligned} \left|r^{-\frac{d-2}{2}} L\left(r^{\frac{d-2}{2}}\phi\right)\right|^2 &= |L\phi|^2 + \frac{d-2}{r}\phi L\phi + \frac{(d-2)^2}{4}\frac{1}{r^2}|\phi|^2, \\ \left|r^{-\frac{d-2}{2}} \underline{L}\left(r^{\frac{d-2}{2}}\phi\right)\right|^2 &= |\underline{L}\phi|^2 - \frac{d-2}{r}\phi L\phi + \frac{(d-2)^2}{4}\frac{1}{r^2}|\phi|^2. \end{aligned}$$

The game then is to rewrite

$$\begin{aligned} \frac{d-2}{2\rho}\phi L\phi - \frac{1}{2}\left(\frac{v}{u}\right)^{1/2}\frac{d-2}{r}\phi L\phi &= -\frac{d-2}{2}\frac{t}{r\rho}\phi L\phi, \\ \frac{d-2}{2\rho}\phi \underline{L}\phi + \frac{1}{2}\left(\frac{u}{v}\right)^{1/2}\frac{d-2}{r}\phi \underline{L}\phi &= +\frac{d-2}{2}\frac{t}{r\rho}\phi \underline{L}\phi, \\ -\frac{d-2}{4}\left(\frac{u}{v}\right)^{1/2}\frac{1}{\rho^2}|\phi|^2 - \frac{1}{2}\left(\frac{v}{u}\right)^{1/2}\frac{(d-2)^2}{4}\frac{1}{r^2}|\phi|^2 &= -\left(\frac{d-2}{4}\frac{1}{\rho v} + \frac{1}{2}\frac{(d-2)^2}{4}\frac{v}{r^2\rho}\right)|\phi|^2, \\ -\frac{d-2}{4}\left(\frac{v}{u}\right)^{1/2}\frac{1}{\rho^2}|\phi|^2 - \frac{1}{2}\left(\frac{u}{v}\right)^{1/2}\frac{(d-2)^2}{4}\frac{1}{r^2}|\phi|^2 &= -\left(\frac{d-2}{4}\frac{1}{\rho u} + \frac{1}{2}\frac{(d-2)^2}{4}\frac{u}{r^2\rho}\right)|\phi|^2. \end{aligned}$$

To kill these terms and replace them with ones with non-negative sign, we introduce a divergence-free 1-current known as the *null current* (see for example [OT16] for the Maxwell-Klein-Gordon case). Define

$$
\begin{aligned}
{}^{(N_0)}J_L[\phi] &:= +\frac{d-2}{4}\frac{1}{r^{d-1}}L\left(r^{d-1}\frac{t}{\rho r}|\phi|^2\right),\\
{}^{(N_0)}J_{\underline{L}}[\phi] &:= -\frac{d-2}{4}\frac{1}{r^{d-1}}\underline{L}\left(r^{d-1}\frac{t}{\rho r}|\phi|^2\right),
\end{aligned}
$$

and the other components are set to zero. This is divergence-free since mixed partials commute, $L\underline{L} = \underline{L}L$. Expanding via the product rule, we obtain

$$
\begin{aligned}
{}^{(N_0)}J_L[\phi] &= +\frac{d-2}{2}\frac{t}{r\rho}\phi L\phi + \left(\frac{d-2}{4}\frac{1}{\rho v} + \frac{(d-2)^2}{4}\frac{v}{r^2\rho} - \frac{(d-2)^2}{4}\frac{1}{r\rho}\right)|\phi|^2,\\
{}^{(N_0)}J_{\underline{L}}[\phi] &= -\frac{d-2}{2}\frac{t}{r\rho}\phi \underline{L}\phi + \left(\frac{d-2}{4}\frac{1}{\rho u} + \frac{(d-2)^2}{4}\frac{u}{r^2\rho} + \frac{(d-2)^2}{4}\frac{1}{r\rho}\right)|\phi|^2.
\end{aligned}
$$

Adding the currents together, we see that the $\phi L\phi$ terms cancel. It remains to examine the coefficient in front of $|\phi|^2$. Factoring out $\frac{d-2}{4}$, we consider

$$
\begin{aligned}
\frac{v}{r^2\rho} - \frac{1}{2}\frac{v}{r^2\rho} - \frac{1}{r\rho} &= \frac{1}{2}\frac{v}{r^2\rho} - \frac{r}{r^2\rho} = \frac{1}{2}\frac{u}{r^2\rho} = \frac{1}{2}\left(\frac{u}{v}\right)^{1/2}\frac{1}{r^2},\\
\frac{u}{r^2\rho} - \frac{1}{2}\frac{u}{r^2\rho} + \frac{1}{r\rho} &= \frac{1}{2}\frac{u}{r^2\rho} + \frac{r}{r^2\rho} = \frac{1}{2}\frac{v}{r^2\rho} = \frac{1}{2}\left(\frac{v}{u}\right)^{1/2}\frac{1}{r^2}.
\end{aligned}
$$

This completes the derivation of the monotonicity formula.

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