INTEGRATED LOCAL ENERGY DECAY FOR THE WAVE EQUATION

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Abstract. The integrated local energy decay estimates (ILED) are a class of inequalities which quantify the dispersive effects of wave equations. Roughly speaking, they capture the outgoing trajectory of wave packets under the flow of the equation. The goal of this note is three-fold. First, we aim to introduce (ILED) and its proof for the usual wave equation using the positive commutator method. Second, we give an application towards proving Strichartz estimates, the cornerstone of well-posedness theory for dispersive equations. Finally, we offer an alternative avenue for proving (ILED) via spectral methods. Our main reference is [MST20].

Contents

1. Introduction

Consider the waves formed by a stone dropped into an infinite still ocean. In the idealised setting, the total energy of the system is conserved for all time, however the water near the initial splash calms as the wave travels further out, so the energy within that region decreases. More precisely, we can control the integral in time of the energy within any compact region by the initial energy times a factor of how long the wave stayed within the region. Such estimates go by the name of *integrated local energy estimates*, and in the setting just described the estimate manifests in the form

$$\int_{\mathbb{R}} \int_K |\nabla_{t,x}\phi(t)|^2 \, dxdt \lesssim R \int_{\mathbb{R}^d} |\nabla_{t,x}\phi(0)|^2 \, dx,$$

where $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a solution to the free wave equation $\Box\phi = 0$ and $K \subseteq \mathbb{R}^d$ is a compact region, for example a ball or annulus, of diameter $R > 0$. For a heuristic proof, if we start with spherically symmetric initial data localised near the origin, the the resulting wave radiates outwards with unit speed, and so the region only sees the bulk of the total energy $\int |\nabla\phi(0)|^2 \, dx$ for time at most equal to the diameter $t \lesssim R$. For large times $t \gg R$ the energy within the region is negligible $\int_K |\nabla\phi(t)|^2 \, dx \approx 0$, so the estimate immediately follows.

The integrated local energy estimates are a quantitative and robust measure of dispersion for linear wave equations. They are quantitative in the sense that they provide L^2_t-integrability for spatially-localised energy, and robust in that they continue to hold for a wide range of wave equations. In particular, we

Date: May 24, 2024.

will consider lower-order perturbations of the Laplacian and thereby d'Alambertian,

$$L := -\Delta + b^j \partial_j + c,$$
$$P := -\partial_t^2 - L,$$

where b^j and c are smooth variable coefficients satisfying the decay conditions

$$\sum_{N \in 2^{\mathbb{N}}} \sup_{\mathbb{R} \times A_N} \langle x \rangle |b(t,x)| + \langle x \rangle^2 |\partial_j b^j(t,x)| + \langle x \rangle^2 |c(t,x)| \leq \kappa. \tag{D}$$

Under suitable additional assumptions on P, we can establish the *integrated local energy decay* estimate,

$$||\nabla_{t,x}\phi||_{\mathsf{LE}_{t,x} \cap L^\infty_t L^2_x} + ||\langle r \rangle^{-1}\phi||_{\mathsf{LE}_{t,x} \cap L^\infty_t L^2_x} \lesssim ||\nabla_{t,x}\phi(0)||_{L^2_x} + ||P\phi||_{\mathsf{LE}^*_{t,x} + L^1_t L^2_x}, \tag{ILED}$$

where $\mathsf{LE}_{t,x}$ is the local energy norm, which, to first approximation, takes the form

$$||\psi||_{\mathsf{LE}_{t,x}} \approx ||\langle x \rangle^{-\frac{1}{2}-}\psi||_{L^2_{t,x}},$$

and $\mathsf{LE}^*_{t,x}$ denotes its dual norm. Observe the spatial weight $\langle x \rangle^{-1/2-\varepsilon}$ is consistent with our earlier heuristic discussion, corresponding to the duration a wave packet stays in any given compact region[1].

Section 2 we introduce the positive commutator method to prove (ILED) for $\Box$, and then establish (ILED) for P in a perturbative setting, namely when (D) holds for $\kappa \ll 1$.

Section 3 illustrates the philosophy that

$$\text{integrated local energy decay} \implies \text{Strichartz}$$

This philosophy was first put forth in the context of Schrodinger equations by Tataru [Tat08], and later for wave equations by Metcalfe-Tataru [MT12].

Section 4 considers the setting of stationary self-adjoint P, where we characterise (ILED) for the wave operator P via resolvent bounds for the elliptic operator L,

$$\text{integrated local energy decay} \iff \mathsf{LE}_x\text{-resolvent bounds}$$

This observation goes back to Kato [Kat65]; we follow the formulation of [Tat13].

Remark. One can also consider perturbations of the metric $P = \partial_\alpha a^{\alpha\beta}\partial_\beta + b^\alpha \partial_\alpha + c$ in the interest of studying decay of waves in general relativity. For simplicity we avoid this setting, since one has to introduce more microlocal ideas to handle the possibility of *trapping*. We point the interested reader to, among many excellent references, [MT12] and [MST20].

Acknowledgments. These notes are heavily based on learning seminar notes written Ovidiu-Neculai Avadanei and Ning Tang; indeed, this is more of a light polishing than anything original. We credit the philosophical commentary to invaluable discussions with Sung-Jin Oh.

2. Integrated local energy decay

To motivate the local energy norm, let us consider the model case of the free wave equation $\Box\phi = 0$ with initial data given by a spherically symmetric Gaussian wave packet localised near the origin. From our heuristic discussion at the beginning of this note, we established the model estimate

$$||\nabla_{t,x}\phi||_{L^2_{t,x}(\mathbb{R}\times A_N)} \lesssim N^{1/2}||\nabla_{t,x}\phi(0)||_{L^2_x},$$

where $A_N \subseteq \mathbb{R}^d$ denotes an annulus adapted to the dyadic scale $N \in 2^{\mathbb{N}}$,

$$A_1 := \{x \in \mathbb{R}^d : |x| \leq 2\},$$
$$A_N := \{x \in \mathbb{R}^d : N \leq |x| \leq 2N\}.$$

[1]The ε-loss in the exponent comes from summing over disjoint regions A_N covering space $\mathbb{R}^d$. Hence this version of the local energy norm does not quite give sharp estimates, though modifications can be made to avoid the ε-loss, e.g. restricting to derivatives tangential to the light cone, L and $\not\nabla$, or introducing a logarithmic weight in the time integration.

This motivates the introduction of the local energy norm and, to account the presence of forcing terms $\Box\phi = f$, its dual norm

$$||\psi||_{\mathsf{LE}_{t,x}} := \sup_{N\in 2^{\mathbb{N}}} ||\langle r\rangle^{-\frac{1}{2}}\psi||_{L^2_{t,x}(\mathbb{R}\times A_N)},$$

$$||f||_{\mathsf{LE}^*_{t,x}} := \sum_{N\in 2^{\mathbb{N}}} ||\langle r\rangle^{\frac{1}{2}} f||_{L^2_{t,x}(\mathbb{R}\times A_N)}.$$

Throughout we will denote by $\langle -,-\rangle$ for the L^2_x-inner product.

2.1. **Morawetz estimate.** Our main tool for proving (ILED) will be the positive commutator method. As a primer, we will use it to give a proof of the classical integrated local energy decay statement due to Morawetz [Mor97]. As a general principle, one generates estimates for the linear wave equation by making a judicious choice of multiplier, integrate-by-parts, and apply a duality argument to handle the inhomogeneous setting. For example, the *Noether's theorem* tells us $\Box\phi X\phi$ can be written as a continuity equation for a conserved quantity when X is an infinitesimal generator of a symmetry of $\Box$. For example, taking $X = \partial_t$ corresponding to the time-translation symmetry,

Proposition 1 (Energy identity)**.** *For $\phi : [0,T]\times\mathbb{R}^d \to \mathbb{C}$ with sufficient regularity and decay, we have*

$$\frac{1}{2}||\nabla_{t,x}\phi(T)||^2_{L^2_x} = \frac{1}{2}||\nabla_{t,x}\phi(0)||^2_{L^2_x} - \int_0^T\int_{\mathbb{R}^d} \Box\phi\partial_t\phi\, dxdt.$$

In particular, ϕ obeys the energy estimate

$$||\nabla_{t,x}\phi||_{L^\infty_t L^2_x} \lesssim ||\nabla_{t,x}\phi(0)||_{L^2_x} + ||\Box\phi||_{L^1_t L^2_x}.$$

Proof. We rewrite the product $\Box\phi\,\partial_t\phi$ in divergence form,

$$\begin{aligned}\Box\phi\,\partial_t\phi &= \left(-\partial_t^2\phi + \sum_j \partial_j^2\phi\right)\partial_t\phi \\ &= \partial_t\left(-\frac{1}{2}|\partial_t\phi|^2\right) + \sum_j \partial_j(\partial_j\phi\partial_t\phi) - \partial_j\phi\partial_t\partial_j\phi \\ &= \partial_t\left(-\frac{1}{2}|\partial_t\phi|^2 - \frac{1}{2}\sum_j|\partial_j\phi|^2\right) + \nabla_x\cdot(\partial_t\phi\nabla_x\phi).\end{aligned}$$

Integrating over the region $[0,T]\times\mathbb{R}^d$ and applying the divergence theorem, the boundary terms arising from the first term on the last line furnish the energy, while the second term vanishes provided sufficient decay at spatial infinity. This proves the energy identity. To prove the energy estimate, a duality argument and Cauchy's inequality imply that

$$\left|\int_0^t\int_{\mathbb{R}^d}\Box\phi\,\partial_t\phi\,dxdt\right| \leq ||\Box\phi||_{L^1_tL^2_x}||\partial_t\phi||_{L^\infty_tL^2_x} \leq 2||\Box\phi||^2_{L^1_tL^2_x} + \frac{1}{4}||\partial_t\phi||^2_{L^\infty_tL^2_x},$$

so the second term on the right can be absorbed into the left-hand side of the energy inequality. $\Box$

For the *positive commutator* argument, integration-by-parts is used to generate a commutator term, while one chooses the multiplier such that this commutator has positive sign. Suppose X is stationary and anti-symmetric with respect to the L^2_x-inner product, then

$$\langle X\phi, \Box\phi\rangle = -\partial_t\langle X\phi,\partial_t\phi\rangle + \frac{1}{2}\langle[X,\Delta]\phi,\phi\rangle,$$

since anti-symmetry allows us to write $\langle X\partial_t\phi,\partial_t\phi\rangle = 0$ and $2\langle X\phi,\Delta\phi\rangle = \langle[\Delta,X]\phi,\phi\rangle$. We want to choose X judiciously such that the commutator term has good sign. For example, one can choose the anti-symmetric operator $X = \partial_r + \frac{d-1}{2r}$ and compute

$$[\partial_r + \tfrac{d-1}{2r}, \Delta] = -\frac{2}{r}\frac{1}{r^2}\not{\Delta} + \frac{(d-1)(d-3)}{2r}\frac{1}{r^2},$$

for $d \geq 4$; the modifications for the lower dimensional cases are left to the reader. Integrating-by-parts, we obtain

Theorem 2 (Morawetz estimate). *Let $\phi \in C_t^\infty \mathcal{S}_x(\mathbb{R}\times\mathbb{R}^d)$ be a smooth solution to the free wave equation $\Box\phi = 0$, then*

$$||r^{-\frac{3}{2}} \nabla\!\!\!/ \phi||_{L^2_{t,x}} + ||r^{-\frac{3}{2}}\phi||_{L^2_{t,x}} \lesssim ||\nabla_{t,x}\phi(0)||_{L^2_x}.$$

Proof. Since X is anti-symmetric, multiplying the equation by $X\phi$ gives

$$0 = \langle X\phi, \Box\phi\rangle = -\partial_t\langle X\phi, \partial_t\phi\rangle + \frac{1}{2}\langle [X,\Delta]\phi,\phi\rangle.$$

Integrating the commutator term by parts gives control over $\nabla\!\!\!/\phi$ and ϕ with the appropriate weights. Integrating in time and applying Cauchy-Schwartz, Hardy's inequality, and the energy identity, we conclude the estimate. $\square$

Remark. Another proof using the energy-momentum tensor formalism can be found in the appendix of [SR05].

Remark. When working with vector fields, recall that the commutator $[X, Y]$ is precisely the Lie derivative of X along the flow of Y. In analogy with this picture, the principal symbol of the commutator $[X, \Box]$ is given by the derivative of the symbol of X along the Hamilton flow of $\Box$.

2.2. **(ILED) for $\Box$.** Using the proof of the Morawetz estimate as inspiration, let us outline the strategy for proving the integrated local energy decay estimate. Again, we want to choose our multiplier X to be stationary and anti-symmetric, in which case integration-by-parts leads us to

$$\langle X\phi, \Box\phi\rangle = -\partial_t\langle X\phi, \partial_t\phi\rangle + \frac{1}{2}\langle [X,\Delta]\phi,\phi\rangle.$$

Integrating in time and applying a duality argument,

$$\begin{aligned}\langle [X,\Delta]\phi,\phi\rangle \lesssim{}& ||X\phi(0)||_{L^2_x}||\partial_t\phi(0)||_{L^2_x} + ||X\phi(T)||_{L^2_x}||\partial_t\phi(T)||_{L^2_x}\\ &+ ||X\phi||_{\mathsf{LE}_{t,x}\cap L^\infty_t L^2_x}||\Box\phi||_{\mathsf{LE}^*_{t,x}+L^1_tL^2_x}.\end{aligned}\tag{*}$$

Aiming towards (ILED), we want to choose X such that it is bounded from $\dot{H}^1_x(\mathbb{R}^d)$ to $L^2_x(\mathbb{R}^d)$, bounded from $\nabla_x\mathsf{LE}_{t,x}\cap r^{-1}\mathsf{LE}_{t,x}$ to $\mathsf{LE}_{t,x}$, and has positive commutator with the Laplacian,

$$||X\phi||_{L^2_x} \lesssim ||\nabla_x\phi||_{L^2_x}, \tag{L2}$$

$$||X\phi||_{\mathsf{LE}_{t,x}} \lesssim ||\nabla_{t,x}\phi||_{\mathsf{LE}_{t,x}} + ||r^{-1}\phi||_{\mathsf{LE}_{t,x}}, \tag{LE}$$

$$\langle [X,\Delta]\phi,\phi\rangle \gtrsim ||\nabla_{t,x}\phi||^2_{\mathsf{LE}_{t,x}} + ||r^{-1}\phi||^2_{\mathsf{LE}_{t,x}}. \tag{C}$$

Indeed, inserting the inequalities above into (*), we obtain

$$\begin{aligned}||\nabla_{t,x}\phi||^2_{\mathsf{LE}_{t,x}} + ||r^{-1}\phi||^2_{\mathsf{LE}_{t,x}} \lesssim{}& ||\nabla_{t,x}\phi(0)||^2_{L^2_x} + ||\nabla_{t,x}\phi(T)||^2_{L^2_x}\\ &+ \left(||\nabla_{t,x}\phi||_{\mathsf{LE}_{t,x}\cap L^\infty_tL^2_x} + ||r^{-1}\phi||_{\mathsf{LE}_{t,x}\cap L^\infty_tL^2_x}\right)||\Box\phi||_{\mathsf{LE}^*_{t,x}+L^1_tL^2_x}.\end{aligned}$$

In view of the energy identity from Lemma 1 and duality, the $\mathsf{LE}_{t,x}$-norms on the left-hand side can be replaced by $\mathsf{LE}_{t,x}\cap L^\infty_tL^2_x$-norms, while on the right-hand side the energy at time $t = T$ can be controlled by the energy at time $t = 0$ and the last term on the right. Thus, we can write

$$\begin{aligned}||\nabla_{t,x}\phi||^2_{\mathsf{LE}_{t,x}\cap L^\infty_tL^2_x} + ||r^{-1}\phi||^2_{\mathsf{LE}_{t,x}\cap L^\infty_tL^2_x} \lesssim{}& ||\nabla_{t,x}\phi(0)||^2_{L^2_x}\\ &+ \left(||\nabla_{t,x}\phi||_{\mathsf{LE}_{t,x}\cap L^\infty_tL^2_x} + ||r^{-1}\phi||_{\mathsf{LE}_{t,x}\cap L^\infty_tL^2_x}\right)||\Box\phi||_{\mathsf{LE}^*_{t,x}+L^1_tL^2_x}\\ \lesssim{}& ||\nabla_{t,x}\phi(0)||^2_{L^2_x} + \frac{2}{\delta}||\Box\phi||^2_{\mathsf{LE}^*_{t,x}+L^1_tL^2_x}\\ &+ \delta\left(||\nabla_{t,x}\phi||_{\mathsf{LE}_{t,x}\cap L^\infty_tL^2_x} + ||r^{-1}\phi||_{\mathsf{LE}_{t,x}\cap L^\infty_tL^2_x}\right)^2,\end{aligned}$$

using Cauchy's inequality. The choice of $\delta > 0$ here is arbitrary, so taking $\delta \ll 1$ we can absorb the final term on the second line into the left-hand side to conclude

Theorem 3 ((ILED) for $\Box$)**.** *The integrated local energy decay estimate* (ILED) *holds for the d'Alembertian* $P = \Box$.

It remains to choose X satisfying the boundedness properties (L2) and (LE), and the positivity of the commutator (C). The multiplier $X = \partial_r + \frac{d-1}{2r}$ is a good start for establishing (C), however it does not furnish control over the radial derivatives or time derivatives, so some suitable modifications need to be made. In fact, the anti-symmetry of X and the estimate (C) should not be taken too literally; as we detail the proof, some suitable substitutes will be introduced. Many computations will be omitted for brevity; to see details, we refer to [MS06].

"Proof" of positive commutator estimates (C). Our goal is to construct an anti-symmetric operator X which has positive commutator with Δ. Fix a smooth radial function $\alpha(r)$ to be chosen later, and set

$$X = \alpha \frac{x^j}{r}\partial_j + \partial_j \frac{x^j}{r}\alpha.$$

The commutator is given by (exercise!)

$$[X, \Delta] = -\partial_k \frac{x^k}{r}\alpha' \frac{x^j}{r}\partial_j - \partial_\ell \left(\delta^{\ell k} - \frac{x^k x^\ell}{r^2}\right)\frac{\alpha}{r}\left(\delta^{jk} - \frac{x^j x^k}{r^2}\right)\partial_j - (\Delta\partial_j)\left(\frac{x^j}{r}\alpha\right).$$

Integrating by parts,

$$\langle [X,\Delta]\phi, \phi\rangle = \int_{\mathbb{R}^d} \left(\alpha' |\partial_r \phi|^2 + \frac{\alpha}{r}\frac{1}{r^2}|\nabla\!\!\!/ \phi|^2 - (\Delta\partial_j)\left(\frac{x^j}{r}\alpha\right)|\phi|^2\right) dx. \tag{†}$$

We want to choose α such that we have non-negative terms $\alpha, \alpha', (\Delta\partial_j)(\frac{x^j}{r}\alpha) > 0$ which behave like the weights $\frac{\alpha}{r}, \alpha' \approx r^{-1}$ and $(\Delta\partial_j)(\frac{x^j}{r}\alpha) \approx r^{-3}$. Fix a dyadic integer $N > 0$, then set

$$\alpha(r) := \frac{r}{r+N}.$$

We compute

$$\begin{aligned}
\alpha'(r) &= \frac{N}{(r+N)^2},\\
-(\Delta\partial_j)\left(\frac{x^j}{r}\alpha\right) &= -\frac{1}{r^{d-1}}\partial_r r^{d-1}\partial_r\left((d-1)\frac{\alpha}{r} + \alpha'\right)\\
&= \frac{1}{r(N+r)^3}\left((d-3)r + 3(d-3)\frac{Nr}{N+r} + \frac{3N^2(d-1)}{N+r}\right).
\end{aligned}$$

We see that the weights are non-negative and have the desired size on the dyadic annulus A_N,

$$\begin{aligned}
\frac{\alpha(r)}{r} \sim \alpha'(r) &\sim N^{-1}, \qquad \text{when } r \sim N,\\
-(\Delta\partial_j)\left(\frac{x^j}{r}\alpha\right) &\sim N^{-3}, \qquad \text{when } r \sim N.
\end{aligned}$$

The non-negativity allows us to restrict our attention to the dyadic annulus, in which case we can bound the commutator (†) below by

$$\langle [X,\Delta]\phi,\phi\rangle \gtrsim ||r^{-\frac{1}{2}}\nabla_x \phi||^2_{L^2_x(A_N)} + ||r^{-\frac{1}{2}} r^{-1}\phi||^2_{L^2_x(A_N)}.$$

This estimate is uniform in N, so we can insert this into the left-hand side of (*) and take the supremum over N to recover the full local energy norm of $\nabla_x\phi$ and $r^{-1}\phi$. $\square$

Remark. The Morawetz multiplier $X = \partial_r + \frac{d-2}{r}$ corresponds to the choice $\alpha(r) = \frac{1}{2r}$.

Modifying the proof of (C) to control ∂_t. Our proof of the positive commutator bound does not control the time-derivatives of ϕ, since for anti-symmetric multipliers $\langle X\partial_t\phi, \partial_t\phi\rangle = 0$. On the other hand, if we were to choose a positive symmetric multiplier, then one could hope to recover the full estimate provided the lower-order error terms can be handled. Let $\beta(r)$ be a smooth radial function to be chosen later, set

$$Y := X + \beta.$$

Then multiplying the equation by $Y\phi$, we obtain

$$\begin{aligned}\langle Y\phi, \Box\phi\rangle &= -\partial_t\langle X\phi, \partial_t\phi\rangle + \frac{1}{2}\langle [X,\Delta]\phi, \phi\rangle \\ &\quad - \partial_t\langle \beta\phi, \partial_t\phi\rangle + \langle \beta\partial_t\phi, \partial_t\phi\rangle + \langle \beta\phi, \Delta\phi\rangle.\end{aligned}$$

Evidently the term $\langle \beta\partial_t\phi, \partial_t\phi\rangle = \int \beta|\partial_t\phi|^2$ gives the desired control over the time derivatives, provided we choose $\beta > 0$ and $\beta \approx r^{-1}$. However, we must contend with the final term having bad sign, indeed, integrating-by-parts and using symmetry,

$$\begin{aligned}\langle \beta\phi, \Delta\phi\rangle &= \langle -\beta\nabla\phi, \nabla\phi\rangle + \langle -\nabla\beta\phi, \nabla\phi\rangle \\ &= -\langle \beta\nabla\phi, \nabla\phi\rangle + \frac{1}{2}\langle \Delta\beta\phi, \phi\rangle \\ &= \int_{\mathbb{R}^d} -\beta|\partial_r\phi|^2 - \frac{\beta}{r^2}|\nabla\!\!\!/\phi|^2 + \frac{1}{2}\Delta\beta|\phi|^2\,dx.\end{aligned} \tag{††}$$

We choose β carefully such that the contributions of bad sign to (††) are dominated by the good signs arising from the positive commutator identity (†). Choose for example

$$\beta(r) := \frac{1}{2}\alpha'(r) = \frac{1}{2}\frac{N}{(r+N)^2},$$

then the coefficient in front of $|\partial_r\phi|^2$ is killed, the coefficient for $\frac{1}{r^2}|\nabla\!\!\!/\phi|^2$ is dominated since $\alpha' < \frac{\alpha}{r}$, and the coefficients for $|\phi|^2$ are related by

$$-\frac{1}{2}(\Delta\partial_j)\left(\frac{x^j}{r}\alpha\right) = -\frac{1}{2}\Delta\beta - \frac{d-1}{2}\Delta\left(\frac{\alpha}{r}\right),$$

so it suffices to show that $\Delta\frac{\alpha}{r} < 0$. Indeed,

$$\Delta\left(\frac{\alpha}{r}\right) = \frac{-(d-3)r - (d-1)N}{r(r+N)^3}.$$

This shows that sign is not an issue. Furthermore, β has the correct size on the dyadic annulus A_N,

$$\beta \sim N^{-1}, \qquad \text{when } r \sim N.$$

The non-negativity allows us to restrict our attention to the dyadic annulus, in which case we can bound from below

$$\langle \beta\partial_t\phi, \partial_t\phi\rangle + \frac{1}{2}\langle [X,\Delta]\phi, \phi\rangle + \langle \beta\phi, \Delta\phi\rangle \gtrsim ||r^{-\frac{1}{2}}\partial_t\phi||^2_{L^2_x(A_N)}.$$

This estimate is uniform in N, taking the supremum over N recovers the local energy norm of $\partial_t\phi$. ☐

Proof of L^2-bounds (L2) and $\mathsf{LE}_{t,x}$-bounds (LE) and concluding (ILED)*.* To control the terms $X\phi$ and $\beta\phi$, observe that the pointwise bounds on α and α' imply

$$|X\phi| + |\beta\phi| \lesssim |\nabla_x\phi| + \frac{1}{r}|\phi|.$$

The $\mathsf{LE}_{t,x}$-bounds follow immediately by definition,

$$||X\phi||_{\mathsf{LE}_{t,x}} + ||\beta\phi||_{\mathsf{LE}_{t,x}} \lesssim ||\nabla_x\phi||_{\mathsf{LE}_{t,x}} + ||r^{-1}\phi||_{\mathsf{LE}_{t,x}},$$

while one needs an application of Hardy's inequality to conclude the L^2-bounds,

$$||X\phi||_{L^2_x} + ||\beta\phi||_{L^2_x} \lesssim ||\nabla_x\phi||_{L^2_x} + ||r^{-1}\phi||_{L^2_x} \lesssim ||\nabla_x\phi||_{L^2_x}.$$

Collecting all our bounds, this completes the proof of (ILED). ☐

Remark. The integrated local energy decay estimate as stated in (ILED) fails in dimensions $d = 1, 2$, though from our proof it is clear that one simply has to drop the Hardy term $||\langle r\rangle^{-1}\phi||_{\mathsf{LE}_{t,x}}$ to recover an admissible estimate.

2.3. **(ILED) for small perturbations of $\Box$.** To illustrate the robustness of the integrated local energy decay estimates, we prove the analogous results for lower-order perturbations of the wave operator. In the case where the decay condition (D) for sufficiently small $\kappa \ll 1$, these perturbations can be easily absorbed by the left-hand side of (ILED) for $\Box$.

Theorem 4 ((ILED) for P)**.** *Let $L := -\Delta + b^j\partial_j + c$ be a lower-order perturbation of the Laplacian. Suppose the coefficients b^j and c satisfy the decay condition (D) for sufficiently small $\kappa \ll 1$, i.e.*

$$\sum_{N\in 2^{\mathbb{N}}} \sup_{\mathbb{R}\times A_N} \langle r\rangle |b| + \langle r\rangle^2 |\partial_j b^j| + \langle r\rangle^2 |c| \leq \kappa \ll 1,$$

then the wave operator $P := -\partial_t^2 - L$ satisfies (ILED)*.*

Proof. We want to treat the lower-order terms perturbatively, so to that end we write $\Box\phi =: P\phi + B\phi$ and apply (ILED) for $\Box$ along with the triangle inequality to obtain

$$||\nabla_{t,x}\phi||_{\mathsf{LE}_{t,x}\cap L^\infty_t L^2_x} + ||\langle r\rangle^{-1}\phi||_{\mathsf{LE}_{t,x}\cap L^\infty_t L^2_x} \lesssim ||\nabla_{t,x}\phi(0)||_{L^2_x} + ||P\phi||_{\mathsf{LE}^*_{t,x}+L^1_t L^2_x} + ||B\phi||_{\mathsf{LE}^*_{t,x}}.$$

It remains to show the $B\phi$ term can be absorbed into the left-hand side. Indeed, the triangle inequality, Holder's inequality and the decay (D) respectively imply

$$\begin{aligned}
||B\phi||_{\mathsf{LE}^*_{t,x}} &\leq \sum_{N\in 2^{\mathbb{N}}} ||\langle r\rangle^{\frac{1}{2}} b^j\partial_j\phi||_{L^2_{t,x}(\mathbb{R}\times A_N)} + ||\langle r\rangle^{\frac{1}{2}} c\phi||_{L^2_{t,x}(\mathbb{R}\times A_N)} \\
&\leq \left(\sup_{N\in 2^{\mathbb{N}}} ||\langle r\rangle^{-\frac{1}{2}}\partial_j\phi||_{L^2_{t,x}(\mathbb{R}\times A_N)}\right)\left(\sum_{N\in 2^{\mathbb{N}}}\sup_{\mathbb{R}\times A_N}\langle r\rangle|b|\right) \\
&\quad + \left(\sup_{N\in 2^{\mathbb{N}}} ||\langle r\rangle^{-\frac{1}{2}}\langle r\rangle^{-1}\phi||_{L^2_{t,x}(\mathbb{R}\times A_N)}\right)\left(\sum_{N\in 2^{\mathbb{N}}}\sup_{\mathbb{R}\times A_N}\langle r\rangle^2|c|\right) \\
&\leq \kappa\left(||\nabla_{t,x}\phi||_{\mathsf{LE}_{t,x}} + ||\langle r\rangle^{-1}\phi||_{\mathsf{LE}_{t,x}}\right).
\end{aligned}$$

Choosing κ smaller than the implicit constant in (ILED) for $\Box$ completes the proof. □

3. Strichartz estimates

In the study of non-linear wave equations, we are interested in the low regularity well-posedness and long-time dynamics of solutions. The classical approach using the energy method, while robust, e.g. the work of Kato [Kat75] on quasi-linear equations, often fails to obtain optimal results as it does not fully take into account the linear dispersive effects of the equation. Thus one is led to introduce another quantitative measure of dispersion, the Strichartz estimates,

$$||\phi||_{L^q_t L^p_x} + ||\nabla_{t,x}\phi||_{L^q_t L^p_x} \lesssim ||\nabla_{t,x}\phi(0)||_{L^2_x} + ||P\phi||_{L^1_t L^2_x + \mathsf{LE}^*_{t,x}} \tag{S}$$

where $2 \leq p, q \leq \infty$ are wave-admissible

$$\frac{1}{q} + \frac{d-1}{2p} \leq \frac{d-1}{4}$$

and obey the scaling condition

$$\frac{1}{q} + \frac{d}{p} = \frac{d}{2}.$$

The classic Strichartz estimate for $\Box$ is proved using a TT^*-argument, however this is often difficult to adapt in the setting of more general wave operators P. Instead, our strategy will be to assume integrated local energy decay (ILED) holds for P, and show that Strichartz for P follows as a corollary of Strichartz for $\Box$.

Lemma 5 (Strichartz for $\Box$)**.** *The Strichartz estimates (S) hold for $P = \Box$.*

Proof. Standard references include [SS00] and [Sog95]. □

Theorem 6 (Strichartz for P)**.** *Let $L := -\Delta + b^j\partial_j + c$ be a lower-order perturbation of the Laplacian such that the coefficients b^j and c satisfy the decay condition (D). Then* (ILED) *implies (S) for the wave operator $P = -\partial_t^2 - L$.*

Proof. The decay of the coefficients and integrated local energy decay assumption allow us to control the lower order terms $\Box\phi =: P\phi + B\phi$ by the right-hand side of Strichartz. Indeed, applying (S) for $\Box$ and the triangle inequality gives us

$$||\phi||_{L_t^q L_x^p} + ||\nabla_{t,x}\phi||_{L_t^q L_x^p} \lesssim ||\nabla_{t,x}\phi(0)||_{L_x^2} + ||P\phi||_{L_t^1 L_x^2 + \mathsf{LE}_{t,x}^*} + ||B\phi||_{\mathsf{LE}_{t,x}^*}.$$

The decay of the coefficients and (ILED) respectively give the inequalities

$$||B\phi||_{\mathsf{LE}_{t,x}^*} \lesssim ||\nabla_{t,x}\phi||_{\mathsf{LE}_{t,x}} + ||\langle r\rangle^{-1}\phi||_{\mathsf{LE}_{t,x}} \lesssim ||\nabla_{t,x}\phi(0)||_{L_x^2} + ||P\phi||_{\mathsf{LE}_{t,x}^* + L_t^1 L_x^2}.$$

This completes the proof. $\square$

Remark. For $\Box$, one actually has a wider range of exponents for the $L_t^{q'} L_x^{p'}$-norm on the right-hand side of (S). The full statement of Strichartz estimates for the wave equation is

$$||\phi||_{L_t^q L_x^p} + ||\nabla_{t,x}\phi||_{L_t^q L_x^p} \lesssim ||\nabla_{t,x}\phi(0)||_{L_x^2} + ||P\phi||_{L_t^{q'} L_x^{p'} + \mathsf{LE}_{t,x}^*}$$

where (q, p) and (q', p') are wave-admissible and obey the scaling condition

$$\frac{1}{q} + \frac{d}{p} = \frac{d}{2} = \frac{1}{q'} + \frac{d}{p'} - 2.$$

To prove the more general estimate for P, one needs (ILED) on the dual problem.

Remark. The decay condition (D) is crucial; in the case of inverse square potential $L = -\Delta + \frac{\alpha}{r^2}$, the wave admissible pairs for which Strichartz holds depends on the coefficient α; see [BPST04].

4. A SPECTRAL CHARACTERISATION

We now turn to the problem of proving (ILED) for lower-order yet large perturbations of the d'Alembertian. The setting we are interested in is as follows,

$$L := -\Delta + b^j\partial_j + c,$$
$$P := -\partial_t^2 - L,$$

where b^j and c are smooth variable coefficients which we now take to be *stationary*, i.e. time-independent, and satisfy the usual decay conditions (D), though in contrast to the setting in Section 2.3 we allow for large $\kappa > 0$. Instead, we will rely on bounds on the resolvent, which formally is given by

$$R_\omega := (\omega^2 - L)^{-1},$$

where $\omega \in \mathbb{C}$ is the spectral parameter. One can loosely relate the wave operator P and the resolvent via the Fourier transform in time, which maps $\partial_t \mapsto i\omega$. We can define spatial local energy spaces by

$$||\psi||_{\mathsf{LE}_x} := \sup_{N\in 2^{\mathbb{N}}} ||\langle r\rangle^{-\frac{1}{2}}\psi||_{L_x^2(A_N)},$$
$$||g||_{\mathsf{LE}_x^*} := \sum_{N\in 2^{\mathbb{N}}} ||\langle r\rangle^{\frac{1}{2}} g||_{L_x^2(A_N)}.$$

Then, under suitable spectral assumptions on L, we can establish the LE_x*-resolvent bounds*

$$|\omega|||R_\omega g||_{\mathsf{LE}_x} + ||\nabla_x R_\omega g||_{\mathsf{LE}_x} + ||\langle r\rangle^{-1} R_\omega g||_{\mathsf{LE}_x} \lesssim ||g||_{\mathsf{LE}_x^*}, \qquad \operatorname{Im}\omega < 0. \tag{LER}$$

Our goal for this section is to show that in fact the resolvent bounds are equivalent to the integrated local energy decay. As a primer, we show equivalence of the classical L_x^2-energy estimates with the L_x^2-resolvent bounds, and then establish the resolvent bounds under self-adjointness and coercivity of L.

Remark. We omit discussion of how to prove LE_x-resolvent bounds, but it essentially boils down to the existence of zero resonances/eigenvalues. One proceeds by contradiction, extracting from the failure of (LER) a non-trivial zero resonance.

4.1. **Energy estimates and L^2_x-resolvent bounds.** To connect the resolvent with the corresponding wave equation, we remark that the resolvent can be defined using the Fourier-Laplace transform of a solution to the wave equation. Precisely, consider the initial data problem

$$\begin{aligned} P\phi &= 0, \\ \phi_{|t=0} &= 0, \\ \partial_t \phi_{|t=0} &= g. \end{aligned}$$

Then

$$R_\omega g = \int_0^\infty e^{-it\omega} \phi(t)\, dt.$$

Standard energy estimates, e.g. applying the energy identity and Gronwall's inequality, imply that ϕ obeys an exponential growth bound of the form

$$||\nabla_{t,x}\phi(t)||_{L^2_x} \lesssim e^{\beta t} ||g||_{L^2_x}. \tag{E}$$

This implies that the resolvent is well-defined in the half-space $\operatorname{Im}(\omega) < -\beta$ with uniform bound

$$|\omega| ||R_\omega g||_{L^2_x} + ||\nabla_x R_\omega g||_{L^2_x} \lesssim |\operatorname{Im}(\omega) + \beta|^{-1} ||g||_{L^2_x}. \tag{LR}$$

Conversely, one can invert the Fourier transform and express the wave evolution in terms of the resolvent,

$$\phi(t) = \frac{1}{2\pi} \int_{\operatorname{Im}(\omega)=\sigma} e^{i\omega t} R_\omega g \, d\omega$$

for $\sigma < -\beta$. Thus we see that the energy bounds for the forward evolution are intimately connected with the resolvent bounds. In fact, we claim that uniform energy estimates (E), i.e. with $\beta = 0$, are equivalent to L^2_x-resolvent bounds (LR), again with $\beta = 0$.

Theorem 7. *Let $L := -\Delta + b^j \partial_j + c$ be a lower order perturbation of the Laplacian such that the coefficients b^j and c are stationary. Then uniform L^2_x-energy estimates are equivalent to L^2_x-resolvent bounds.*

Proof. The forward follows from directly from the wave solution formulation of the resolvent as discussed earlier. For the converse, apply Plancharel's theorem to the resolvent bounds. □

Corollary 8. *Let $L := -\Delta + b^j \partial_j + c$ be a lower order perturbation of the Laplacian such that the coefficients b^j and c are stationary. Furthermore, we require L to satisfy*

- *self-adjointness; b is imaginary and divergence-free, and c is real,*
- *coercivity;*

$$\langle L\phi, \phi \rangle \gtrsim ||\nabla_x \phi||_{L^2_x}.$$

Then the L^2_x-energy estimates hold for the wave operator $P := -\partial_t^2 - L$.

Proof. By Theorem 7, it suffices to prove the L^2_x-resolvent bounds. We argue by spectral methods. Since L is self-adjoint, the spectrum is located on the real axis $\sigma(L) \subseteq \mathbb{R}$. The coercivity estimate implies that the spectrum is in fact uniformly positive. Let

$$(\omega^2 - L)\psi = g.$$

Multiplying the equation by ψ and taking the real part, we obtain

$$\begin{aligned} \operatorname{Re}\langle \psi, g \rangle &= \operatorname{Re}\langle \psi, (|\operatorname{Re}(\omega)|^2 - |\operatorname{Im}(\omega)|^2)\psi + 2i \operatorname{Re}(\omega) \operatorname{Im}(\omega)\psi - L\psi \rangle \\ &= |\operatorname{Re}(\omega)|^2 ||\psi||^2_{L^2_x} - |\operatorname{Im}(\omega)|^2 ||\psi||^2_{L^2_x} - \langle L\psi, \psi \rangle. \end{aligned}$$

Rearranging and applying Cauchy-Schwartz and Cauchy's inequality,

$$\langle L\psi, \psi \rangle + |\operatorname{Im}(\omega)|^2 ||\psi||^2_{L^2_x} \leq |\operatorname{Re}(\omega)|^2 ||\psi||^2_{L^2_x} + \frac{1}{2} |\operatorname{Im}(\omega)|^2 ||\psi||_{L^2_x} + \frac{4}{|\operatorname{Im}(\omega)|^2} ||g||_{L^2_x}.$$

We can absorb the second term on the right into the left-hand side. Thus, combined with the coercivity estimate, we conclude

$$||\nabla_x \psi||_{L^2_x} + |\operatorname{Im}(\omega)| ||\psi||_{L^2_x} \lesssim |\operatorname{Im}(\omega)|^{-1} ||g||_{L^2_x}.$$

It remains to control the L^2_x-norm of ψ with weight $|\operatorname{Re}(\omega)|$. By the spectral theorem, keeping in mind that the spectrum is located in the real axis,

$$||R_\omega g||_{L^2_x} \lesssim \operatorname{dist}(\omega^2, \sigma(L))^{-1}||g||_{L^2_x} \lesssim |\operatorname{Re}(\omega)|^{-1}|\operatorname{Im}(\omega)|^{-1}||g||_{L^2_x}.$$

Rearranging gives the result. This completes the proof. □

4.2. **(ILED) and LE_x-resolvent bounds.** Following the proof of Theorem 7, we prove the equivalence of the integrated local energy decay and the LE_x-resolvent bounds. Using Corollary 8 as inspiration, this sets the stage for proving (ILED) via the spectral properties of L.

Theorem 9. *Let $L := -\Delta + b^j\partial_j + c$ be a lower order perturbation of the Laplacian such that the coefficients b^j and c are stationary and satisfy the decay condition (D). Then* (ILED) *holds for the wave operator $P := -\partial_t^2 - L$ if and only if* (LER) *holds for the resolvent $R_\omega := (\omega^2 - L)^{-1}$.*

Proof of (ILED) $\implies$ (LER). Applying the integrated local energy decay estimate to solutions to the free wave equation

$$P\phi = 0,$$

gives the uniform energy estimate

$$||\nabla_{t,x}\phi||_{L^\infty_t L^2_x} \lesssim ||\nabla_{t,x}\phi(0)||_{L^2_x}.$$

By Theorem 7, the L^2_x-resolvent bounds follow from the above. Observe that by construction we have the embedding $L^2_x(\mathbb{R}^d) \hookrightarrow \mathsf{LE}_x(\mathbb{R}^d)$, so we can conclude the LE_x-resolvent bounds from the L^2_x-resolvent bounds in the case $\operatorname{Im}(\omega) \leq -1$. Thus, we are left with proving the case $-1 < \operatorname{Im}(\omega) < 0$. Let $\psi \in \dot{H}^1_x(\mathbb{R}^d)$ and consider the equation

$$(\omega^2 - L)\psi = g.$$

The corresponding inhomogeneous P equation is given by

$$(-\partial_t^2 - L)(e^{i\omega t}\psi) = e^{i\omega t}g.$$

Inserting the above into the integrated local energy decay,

$$|\omega|\,||e^{i\omega t}\psi||_{\mathsf{LE}_{t,x}} + ||e^{i\omega t}\nabla_x\psi||_{\mathsf{LE}_{t,x}} + ||e^{i\omega t}\langle r\rangle^{-1}\psi||_{\mathsf{LE}_{t,x}} \lesssim |\omega|\,||e^{i\omega T}\psi||_{L^2_x} + ||e^{i\omega T}\nabla_x\psi||_{L^2_x} + ||e^{i\omega t}g||_{\mathsf{LE}^*_{t,x}},$$

where we think of our solution to the wave equation as starting from $t = -T$ and we integrate on the interval $[-T, 0]$ rather than the entire real line[2]. Taking $T \to \infty$, the initial data terms on the right vanish, while integrating out in time $t \mapsto e^{i\omega t}$ recovers the desired spatial LE_x-norm on both sides. □

For the converse, the obvious approach would be to apply the Fourier transform in time to the wave equation

$$(-\partial_t^2 - L)\phi = f$$

to obtain the elliptic equation

$$(\omega^2 - L)\widehat{u} = \widehat{f},$$

apply the resolvent bounds, integrate in the spectral parameter, and conclude the integrated local energy decay via Plancharel's theorem. However, one must take care to justify the integration, thus we proceed by making two reductions. First, we claim that if the following integrated local energy decay holds

$$||\nabla_{t,x}\phi||_{\mathsf{LE}_{t,x}} + ||\langle r\rangle^{-1}\phi||_{\mathsf{LE}_{t,x}} \lesssim ||f||_{\mathsf{LE}^*_{t,x}} \tag{FILED}$$

for forward solutions $\operatorname{supp} f \subseteq \{t \geq 0\}$, then (ILED) holds. In this scenario, our solution has adequate time decay as $t \to -\infty$, however we still need to handle the time decay for $t \to +\infty$. Thus, we make a second reduction, claiming that if integrated local energy decay holds for damped forward solutions,

$$||e^{-\varepsilon t}\nabla_{t,x}\phi||_{\mathsf{LE}_{t,x}} + ||e^{-\varepsilon t}\langle r\rangle^{-1}\phi||_{\mathsf{LE}_{t,x}} \lesssim ||e^{-\varepsilon t}f||_{\mathsf{LE}^*_{t,x}} \tag{DILED}$$

uniformly in $\varepsilon > 0$, then the result holds for forward solutions (FILED). We conclude by showing (LER) implied the (FILED), completing the proof.

[2] One can easily modify our proof to show that this version of (ILED) holds.

Proof of (FILED) $\implies$ (ILED). Our strategy will be to decompose ϕ into a solution to the classical wave equation and backwards/forwards solutions to the P equation. Patching together the integrated local energy decay statements for $\Box$ and forward solutions for P, we can conclude (ILED) for P. Let ψ be the solution to the inhomogeneous classical wave equation with the same forcing and initial data, that is,

$$\begin{aligned}(-\partial_t^2 + \Delta)\psi &= f,\\ \psi[0] &= \phi[0].\end{aligned}$$

The difference of ϕ and ψ satisfies an inhomogeneous P equation with zero initial data,

$$\begin{aligned}(-\partial_t^2 - L)(\phi - \psi) &= B\psi,\\ (\phi - \psi)[0] &= 0,\end{aligned}$$

where as usual $B\psi$ denotes the lower order terms of L. Truncating the forcing term $B\psi$ forwards and backwards in time, we define the corresponding forward ψ_+ and backwards solutions ψ_- with zero initial data by

$$\begin{aligned}(-\partial_t^2 - L)\psi_\pm &= \mathbb{1}_{[0,\pm\infty)} B\psi,\\ \psi_\pm[0] &= 0,\end{aligned}$$

where we abused notation to write $(-\infty, 0] = [0, -\infty)$. By construction, $\phi - \psi - \psi_+ - \psi_-$ is a solution to a homogeneous wave equation with zero initial data, so standard theory implies that we have the decomposition

$$\phi = \psi + \psi_+ + \psi_-.$$

The integrated local energy decay holds for the classical wave equation, while we have assumed it holds for forward/backwards solutions, so we have

$$\begin{aligned}||\nabla_{t,x}\psi||_{\mathsf{LE}_{t,x}} + ||\langle r\rangle^{-1}\psi||_{\mathsf{LE}_{t,x}} &\lesssim ||\nabla_{t,x}\phi(0)||_{L^2_x} + ||f||_{\mathsf{LE}^*_{t,x}},\\ ||\nabla_{t,x}\psi_\pm||_{\mathsf{LE}_{t,x}} + ||\langle r\rangle^{-1}\psi_\pm||_{\mathsf{LE}_{t,x}} &\lesssim ||\mathbb{1}_{[0,\pm\infty)} B\psi||_{\mathsf{LE}^*_{t,x}} \lesssim ||B\psi||_{\mathsf{LE}^*_{t,x}},\end{aligned}$$

The forcing term for the forward and backwards solution can be estimated by the decay assumption (D) and (ILED) for the classical wave equation,

$$||B\psi||_{\mathsf{LE}^*_{t,x}} \lesssim ||\nabla_{t,x}\psi||_{\mathsf{LE}_{t,x}} + ||\langle r\rangle^{-1}\psi||_{\mathsf{LE}_{t,x}} \lesssim ||\nabla_{t,x}\phi(0)||_{L^2_x} + ||f||_{\mathsf{LE}^*_{t,x}}.$$

In view of the decomposition of ϕ, we conclude its (ILED) estimate via the triangle inequality. $\square$

From here on, we assume the forcing term is supported forward in time $\operatorname{supp} f \subseteq \{t \geq 0\}$.

Proof of (DILED) $\implies$ (FILED). It suffices to prove the result on each dyadic annulus A_N. Fixing an annulus, the integrated local energy decay for damped forward solutions takes the form

$$||\langle r\rangle^{-\frac{1}{2}} e^{-\varepsilon t}\nabla_{t,x}\phi||_{L^2_{t,x}(\mathbb{R}\times A_N)} + ||\langle r\rangle^{-\frac{1}{2}} e^{-\varepsilon t}\langle r\rangle^{-1}\phi||_{L^2_{t,x}(\mathbb{R}\times A_N)} \lesssim ||e^{-\varepsilon t} f||_{\mathsf{LE}^*_{t,x}}.$$

Since f is supported on $t \geq 0$, we can easily remove the damping. To remove the damping on the left-hand side, we restrict to the time interval $(-\infty, T]$. In this time interval, we can naively bound the damping from below,

$$e^{-\varepsilon T}\left(||\langle r\rangle^{-\frac{1}{2}}\nabla_{t,x}\phi||_{L^2_{t,x}(\mathbb{R}\times A_N)} + ||\langle r\rangle^{-\frac{1}{2}}\langle r\rangle^{-1}\phi||_{L^2_{t,x}(\mathbb{R}\times A_N)}\right) \lesssim ||f||_{\mathsf{LE}^*_{t,x}}.$$

Taking $\varepsilon \to 0$ and then $T \to +\infty$ proves the result. $\square$

Proof of (LER) $\implies$ (DILED) *via Plancherel.* We are now ready to conclude (ILED) by proxy of proving (DILED). We would like to apply Plancherel's theorem in the spectral parameter to recover the $\mathsf{LE}_{t,x}$-norm from the LE_x-norms, however it is not the case that $\mathsf{LE}_{t,x} = L^2_t\mathsf{LE}_x$. Nevertheless, we can obtain a function space of this form by localising to a dyadic annulus. Fix a partition of unity

$$1 \equiv \sum_{N\in 2^{\mathbb{N}}} \chi_N$$

subordinate to the cover $\{A_N \cup A_{2N}\}_N$. Truncating our forcing term in space by $f_N := \chi_N f$, we obtain a decomposition of ϕ in terms of the corresponding forward solutions ϕ_N via

$$\phi = \sum_{N \in 2^{\mathbb{N}}} \phi_N.$$

Then we can conclude (DILED) by the triangle inequality provided we can show

$$||\langle r\rangle^{-\frac{1}{2}} e^{-\varepsilon t} \nabla_{t,x} \phi_N||_{L^2_{t,x}(\mathbb{R}\times A_M)} + ||\langle r\rangle^{-\frac{1}{2}} e^{-\varepsilon t} \langle r\rangle^{-1} \phi_N||_{L^2_{t,x}(\mathbb{R}\times A_M)} \lesssim ||\langle r\rangle^{\frac{1}{2}} e^{-\varepsilon t} f_N||_{L^2_{t,x}(\mathbb{R}\times A_N)}$$

for each $M, N \in 2^{\mathbb{N}}$. Indeed, in view of the ℓ^∞-structure of $\mathsf{LE}_{t,x}$ and ℓ^1-structure of $\mathsf{LE}^*_{t,x}$, the bound above implies the full $\mathsf{LE}_{t,x}$-bound. By Plancherel's theorem, this is equivalent to

$$\begin{aligned}|\tau - i\varepsilon|\, ||\langle r\rangle^{-\frac{1}{2}} \widehat{\phi}_N(\tau - i\varepsilon)||_{L^2_{t,x}(\mathbb{R}\times A_M)} + ||\langle r\rangle^{-\frac{1}{2}} \nabla_x \widehat{\phi}_N(\tau - i\varepsilon)||_{L^2_{t,x}(\mathbb{R}\times A_M)} \\ + ||\langle r\rangle^{-\frac{1}{2}} \langle r\rangle^{-1} \widehat{\phi}_N(\tau - i\varepsilon)||_{L^2_{\tau,x}(\mathbb{R}\times A_M)} \lesssim ||\langle r\rangle^{\frac{1}{2}} \widehat{f}_N(\tau - i\varepsilon)||_{L^2_{\tau,x}(\mathbb{R}\times A_N)}.\end{aligned}$$

Noting $\widehat{\phi}_N = R_{\tau - i\varepsilon} \widehat{f}_N$, this is precisely the LE_x-resolvent bound after taking the L^2_τ-norm. □

References

[BPST04] Nicolas Burq, Fabrice Planchon, John G. Stalker, and A. Shadi Tahvildar-Zadeh. Strichartz Estimates for the Wave and Schrödinger Equations with Potentials of Critical Decay. *Indiana University Mathematics Journal*, 53(6):1665–1680, 2004.

[Kat65] Tosio Kato. Wave operators and similarity for some non-selfadjoint operators. *Mathematische Annalen*, 162:258–279, 1965.

[Kat75] Tosio Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Archive for Rational Mechanics and Analysis*, 58(3):181–205, September 1975.

[Mor97] Cathleen S. Morawetz. Time decay for the nonlinear Klein-Gordon equation. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 306(1486):291–296, January 1997.

[MS06] Jason Metcalfe and Christopher D. Sogge. Long-Time Existence of Quasilinear Wave Equations Exterior to Star-Shaped Obstacles via Energy Methods. *SIAM Journal on Mathematical Analysis*, 38(1):188–209, January 2006.

[MST20] Jason Metcalfe, Jacob Sterbenz, and Daniel Tataru. Local energy decay for scalar fields on time dependent non-trapping backgrounds. *American Journal of Mathematics*, 142(3):821–883, 2020.

[MT12] Jason Metcalfe and Daniel Tataru. Global parametrices and dispersive estimates for variable coefficient wave equations. *Mathematische Annalen*, 353(4):1183–1237, August 2012.

[Sog95] Christopher Donald Sogge. *Lectures on Nonlinear Wave Equations*. International Press, 1995.

[SR05] Jacob Sterbenz and Igor Rodnianski. Angular Regularity and Strichartz Estimates for the Wave Equation, March 2005.

[SS00] Jalal M. Ihsan Shatah and Michael Struwe. *Geometric Wave Equations*. Courant Lecture Notes in Mathematics. American Mathematical Soc., 2000.

[Tat08] Daniel Tataru. Parametrices and Dispersive Estimates for Schrödinger Operators with Variable Coefficients. *American Journal of Mathematics*, 130(3):571–634, 2008.

[Tat13] Daniel Tataru. Local decay of waves on asymptotically flat stationary space-times. *American Journal of Mathematics*, 135(2):361–401, April 2013.