ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. We give an exposition of the initial data problem for ordinary differential equations, with a view towards non-linear evolutionary partial differential equations in mind. The material presented borrows a great deal from [Tao06, Chapter 1].

CONTENTS

1. PRELIMINARIES

An ORDINARY DIFFERENTIAL EQUATION is an equation which governs the evolution of a function $u : I \to V$ mapping an interval $I \subseteq \mathbb{R}$ to a finite-dimensional vector space V known as the STATE SPACE. Abstractly, they are equations which take the form

$$G(u, \partial_t u, \dots, \partial_t^k u, t) = 0. \tag{nLin}$$

for some given function $G : V^{k+1} \times I \to W$, where W is another finite-dimensional vector space. We are interested in the INITIAL DATA PROBLEM, also known as the Cauchy problem, in which we aim to find a k-times continuously differentiable solution satisfying INITIAL DATA CONDITIONS,

$$\begin{aligned} u_{|t=0} &= u_0, \\ \partial_t u_{|t=0} &= u_1, \\ &\vdots \\ \partial_t^{k-1} u_{|t=0} &= u_{k-1} \end{aligned}$$

for some $u_0, u_1, \dots, u_{k-1} \in V$.

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1.1. **Types of equations.** The most general form of an ODE is a fully NON-LINEAR equation (nLin). This is a SYSTEM OF EQUATIONS when the codomain of G has more than one dimension, otherwise it is a SCALAR EQUATION. If the number of equations $\dim W$ exceeds the degrees of freedom $\dim V$, then the equation is OVER-DETERMINED, in which case one may require additional constraints on any initial data before a solution can be found. Conversely if there are fewer equations than degrees of freedom, the equation is UNDER-DETERMINED, and so there may be a multiplicity of solutions for any given data.

We will only consider ODE which are FULLY DETERMINED: the number of equations coincides with the degrees of freedom. In this case, assuming some non-degeneracy conditions, any non-linear ODE can be reduced to a QUASI-LINEAR ODE, that is, one which is linear with respect to the highest-order derivatives.

Proposition 1 (Reduction to quasi-linear). *Let $G : V^{k+1} \times I \to V$ is continuously differentiable, and suppose that $u_k \mapsto G(u_0, u_1, \dots, u_{k-1}, u_k, 0)$ is invertible. Then there exists there exists $F : V^{k+1} \times I \to V$ such that*

$$\partial_t^{k+1} u = F(u, \partial_t u, \dots, \partial_t^k u, t). \tag{qLin}$$

Proof. We differentiate the equation (nLin), obtaining by the chain rule

$$\partial_t G + \sum_{j=0}^{k-1} D_{u_j} G \cdot \partial_t^{j+1} u + D_{u_k} G \cdot \partial_t^{k+1} u = 0.$$

Rearranging and inverting the matrix $D_{u_k} G$ gives the result. □

When the function G in (nLin) does not depend on the time-variable t, we say the ODE is AUTONOMOUS, otherwise we say it is NON-AUTONOMOUS. Autonomous ODEs enjoy the property of *time-translation-invariance*, i.e. if $t \mapsto u(t)$ is a solution, then so is $t \mapsto u(t - t_0)$. Every non-autonomous system can be converted into an autonomous one by embedding the time-variable into the state-space.

Proposition 2 (Reduction to autonomy). *Every non-autonomous ODE,*

$$G(u, \partial_t u, \dots, \partial_t^k u, t) = 0, \tag{nAut}$$

can be reduced to an autonomous ODE,

$$\widetilde{G}(\widetilde{u}, \partial_t \widetilde{u}, \dots, \partial_t^k \widetilde{u}) = 0. \tag{Aut}$$

Proof. Define $\widetilde{u} : I \to V \times I$ and $\widetilde{G} : (V \times I)^{k+1} \times I \to W \times I$ by

$$\begin{aligned} \widetilde{u}(t) &:= (u(t), t), \\ \widetilde{G}((u_0, s_0), (u_1, s_1), \dots, (u_k, s_k)) &:= (G(u_0, \dots, u_k), s_1 - 1). \end{aligned}$$

A routine check shows that every solution to (nAut) is a solution to (Aut), and conversely, every solution to (Aut) is a solution to (nAut) modulo time-translation. □

Example. The quasilinear non-autonomous ODE

$$\partial_t u = F(t, u)$$

is equivalent to the *system* of autonomous ODE

$$\begin{aligned} \partial_t u(t) &= F(s, t), \\ \partial_t s(t) &= 1. \end{aligned}$$

The ORDER of an ODE refers to the order of the highest derivative occurring in the equation. Any k-th order system can be reduced to a first-order system at the cost of multiplying the degrees of freedom by k.

Proposition 3 (Reduction of order). *Every k-th order quasi-linear ODE,*

$$\partial_t^k u = F(u, \partial_t u, \dots \partial^{k-1} u), \tag{k^{th}}$$

can be reduced to a first-order quasi-linear ODE,

$$\partial_t \widetilde{u} = \widetilde{F}(u). \tag{1^{st}}$$

Proof. Set $\widetilde{u} : I \to V^k$ and $\widetilde{F} : V^k \to V^k$ to be

$$\widetilde{u} := (u, \partial_t u, \dots, \partial_t^{k-1} u),$$
$$\widetilde{F}(u_0, \dots, u_{k-1}) = (u_1, \dots, u_{k-1}, F(u_0, \dots, u_{k-1})).$$

Then $\widetilde{u}$ is a continuously differentiable solution to the first-order equation (1^{st}) if and only if u is k-times continuously differentiable solution to the k-th order equation (k^{th}). ☐

1.2. **Notions of a solution.** With these reductions at hand, we focus our attention to the study of the initial data problem for first-order quasi-linear systems of equations on $\mathbb{R}^n$,

$$\begin{aligned} \partial_t u &= F(u), \\ u_{|t=0} &= u_0. \end{aligned} \tag{IDP}$$

If we assume the non-linearity is analytic, then one could hope to solve the ODE by differentiating the equation to obtain a recurrence relation for $\partial_t^k u$, and then show the corresponding formal power series for u converges. Indeed, this is the approach of the Cauchy-Kowalevskaya theorem. However, without such strong assumptions on F, it is not clear how one could solve (IDP) by "classical" (read: undergraduate calculus) methods.

To get around this, often times it becomes easier to solve an equation by "weakening" the notion of a solution, giving access to tools from functional analysis when dealing with PDEs, or metric space topology in the case of ODEs. We say that a solution $u : [0, T] \to \mathbb{R}^n$ to (IDP) is a

- a CLASSICAL SOLUTION if $u \in C^1_{\text{loc}}([0, T])$ and satisfies the initial data problem for all $t \in [0, T]$ using the classical notion of the derivative.

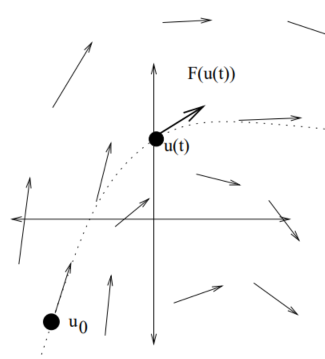

FIGURE 1. The non-linearity $F(u)$ is regarded as a *force* which governs the velocity $\partial_t u$ and thereby the evolution of the solution u.

- a STRONG SOLUTION if $u \in C^0_{\text{loc}}([0, T])$ and solves the initial data problem in the integral sense that

$$u(t) = u_0 + \int_0^t F(u(s))\, ds$$

  for all $t \in [0, T]$.
- a WEAK SOLUTION if $u \in L^\infty([0, T])$ which solves the initial data problem in the sense of distributions, i.e.

$$\int_0^T u(t)\psi(t)\, dt = u_0 \int_0^T \psi(t)\, dt + \int_0^T \psi(t) \left(\int_0^T F(u(s))\, ds \right) dt$$

  for all test functions $\psi \in C_c^\infty([0, T])$.

Proposition 4 (Equivalence of notions). *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $u_0 \in \mathbb{R}^n$, then the notions of classical, strong, and weak solutions to the initial data problem (IDP) are equivalent.*

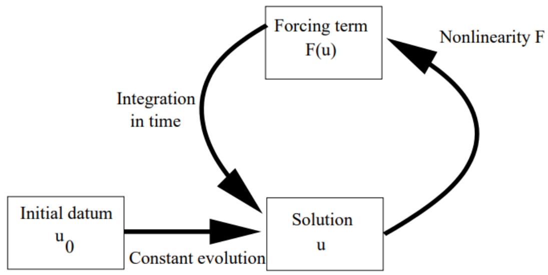

FIGURE 2. The relationship between the solution u and the non-linearity $F(u)$ can be viewed as a *feedback loop* in which the solution influences the non-linearity, and vice versa.

Proof. It is clear that a classical solution is strong by the fundamental theorem of calculus, and that a strong solution is weak. If u is a weak solution, we know that $t \mapsto F(u(t))$ is bounded and measurable and therefore

$$t \mapsto \int_0^t F(u(s))\, ds$$

is Lipschitz continuous. Choose ψ to be approximations to the identity concentrated at t, the weak formulation implies that

$$u(t) = u_0 + \int_0^t F(u(s))\, ds$$

for a.e. t. It follow that u is Lipschitz up to modification on a measure zero set, so it is a strong solution. Then $t \mapsto F(u(t))$ is in fact continuous, and so by the fundamental theorem of calculus $u \in C^1_{\mathrm{loc}}([0,T])$ and solves the initial data problem classically. □

1.3. **Well-posedness.** One should think of the initial data problem as governing the time evolution $\partial_t u$ of a physical state u starting from an initial condition u_0. Not all ODEs are of physical relevance, though if it were to have one, and indeed these are the equations we care about, then we expect it to have the following properties:

- Existence: if a physical phenomenon is governed by an ODE, then it should correspond to a solution.
- Uniqueness: physical reality is deterministic, the past determines the future, so there should only be one solution for each initial data.
- Continuous dependence on initial data: the evolution of a physical phenomenon is stable under perturbations, so the solution should depend continuously on the initial data.

If the initial data problem (IDP) satisfies these three criterion, then we say it is WELL-POSED. The concept of well-posedness was introduced by Hadamard in [Had02] as an attempt to clarify the link between differential equation and physics.

For a more precise statement, we need to specify the space in which we look for solutions to the initial data problem, the time interval of existence, and the topology on the space of solutions. We say well-posedness is

- CONDITIONAL VS. UNCONDITIONAL if well-posedness holds only in a subset $X \subseteq C^0([0,T] \to \mathbb{R}^n)$ vs. the entire space $X = C^0([0,T] \to \mathbb{R}^n)$,
- LOCAL VS GLOBAL if well-posedness holds only for finite time $[0,T]$ vs. infinite time $[0,\infty)$,
- $C^{k,\alpha}$ if the solution map $u_0 \mapsto u$ is $C^{k,\alpha}$-differentiable, where differentiability is taken in a Frechet sense.

Our starting point will be to show local conditional C^0-well-posedness, which states that for any $u_0^* \in \Omega$, there exists a time $T > 0$ and an open ball $B \subseteq \Omega$ containing u_0^* and a subset $X \subseteq C^0([0,T] \to \mathbb{R}^n)$ such that for each $u_0 \in B$ there exists a unique strong solution $u \in X$, and furthermore the solution map $u_0 \mapsto u$ is continuous.

2. Existence

2.1. **Analytic solutions.** Following our earlier remarks, we can formally differentiate the equation to obtain a recurrence relation for the derivatives of u. If one specifies initial data at $t = 0$, these relations fully determine all the derivatives $\partial_t^k u(0)$. In the case where the non-linearity is analytic, the formal power series for u is the natural candidate for the solution to the initial data problem.

Theorem 5 (Cauchy-Kowalevskaya theorem)**.** *Let $F \in C^\omega(\mathbb{R}^n \to \mathbb{R}^n)$ be analytic, then there exists a unique analytic solution to the initial data problem (IDP).*

Proof. By replacing u with $u - u_0$, it is not a loss of generality to prove solve the equation for initial data $u_0 = 0$. Our goal is to obtain an *a priori* estimate on the growth of $\partial_t^m u(0)$ such that the *ansatz*

$$u(t) := \sum_{n=0}^{\infty} \frac{\partial_t^n u(0)}{m!} t^m$$

is shown to be analytic. It would follow by construction and the uniqueness theorem then that $\partial_t u - F(u) \equiv 0$ since the left-hand side is an analytic function which vanishes to every order at $t = 0$. Formally differentiating the equation, it follows from the chain rule and induction that the derivatives of u can be written as a polynomial with non-negative coefficients in the derivatives of F up to order $m - 1$,

$$\partial_t^m u = p_m(F(u), \nabla F(u), \dots, \nabla^{m-1} F(u))$$

for some $p_m \in \mathbb{N}_0[\mathbf{x}]$. Using the initial data $u(0) = 0$, it follows that $\partial_t^m u(0)$ are fully determined by $\nabla^j F(0)$ up to order $j < m$. Note that the polynomial p_m is determined combinatorially, depending only on the order m. To estimate the growth of $\partial_t^m u(0)$, we argue by *analytic majorisation*; non-negativity of the coefficients of p_m imply

$$|\partial_t^m u(0)| \leq \partial_t^m v(0) \tag{*}$$

for any solution to the initial data problem

$$\begin{aligned} \partial_t v &= G(v), \\ v_{|t=0} &= 0, \end{aligned}$$

for some analytic $G \in C^\omega(\mathbb{R}^n \to \mathbb{R}^n)$ such that $|\partial^m F_i(0)| \leq \partial^m G_i(0)$. Our strategy will be to choose G such that the auxiliary ODE above is explicitly solvable for analytic v. The growth estimates on the power series of coefficients of v also hold for those of u via the majoristion (*), completing the proof. To construct G, we know from analyticity of F that there exists $r > 0$ such that

$$|\nabla^m F(0)| \leq \frac{m!}{r^{m+1}}.$$

We construct G such that the m-th order derivatives are precisely the right-hand side, namely

$$G_i(z_1, \dots, z_n) := \frac{1}{r - z_1 - \dots - z_n}.$$

For simplicity, consider the scalar case $n = 1$, the higher dimensional case is similar. By separation of variables, the explicit solution is given by $v(t) = r - r\sqrt{1 - 2t}$, which is clearly analytic in the region $|t| < r/2$. This completes the proof. ☐

2.2. **Picard iteration.** From the strong solution perspective, solving (IDP) is equivalent to finding a FIXED POINT $\Phi_{u_0}(u) = u$ of the integral operator $\Phi_{u_0} : C^0([0, T] \to \mathbb{R}^n) \to C^0([0, T] \to \mathbb{R}^n)$ defined by

$$(\Phi_{u_0} u)(t) := u_0 + \int_0^t F(u(s))\, ds.$$

We will argue by *Picard iteration*. Schematically, we start off with an approximate solution u_0, and inductively construct subsequent approximates u_n by inputting u_{n-1} into the integral operator,

$$u_n := \Phi_{u_0}(u_{n-1}) = u_0 + \int_0^t F(u_{n-1}(s))\, ds.$$

These approximate solutions are known as PICARD ITERATES, and the goal then is to show that this sequence $\{u_n\}_n$ converges uniformly to some u. If this were true, then by continuity of Φ_{u_0} and construction of the sequence, the limit u would be a fixed point. A sufficient condition is if Φ_{u_0} is a CONTRACTION, i.e. Lipschitz continuous with constant $L < 1$,

$$||\Phi_{u_0}(u) - \Phi_{u_0}v||_{C^0[0,T]} \leq L||u - v||_{C^0[0,T]}.$$

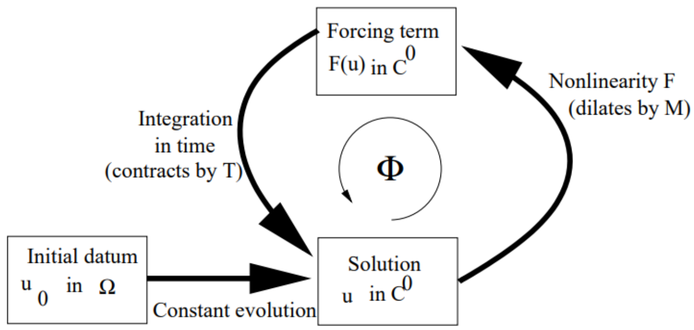

FIGURE 3. The strong solution notion says that u is determined by its past via integration in time. Thus for $T \ll 1$, the initial data u_0 is a good approximate solution. Iterating these approximations through the operator Φ_{u_0} converges provided we have some "gain" at each stage.

Lemma 6 (Contraction mapping principle)**.** *Let (X, d) be a complete non-empty metric space, and let $\Phi : X \to X$ be a contraction on X. Then there exists a unique fixed point $u = \Phi(u)$. Furthermore, if $u_0 \in X$ and we construct the sequence $\{u_n\}_n \subseteq X$ iteratively by*

$$u_{n+1} := \Phi(u_n),$$

then $u_n \to u$.

Proof. To show existence, we aim to show that the iterates $\{u_n\}_n$ form a Cauchy sequence. For $n, k \geq 0$, we obtain

$$\begin{aligned} d(u_n, u_{n+k}) &= d(\Phi^n(u_0), \Phi^n(u_k)) \\ &\leq c^n d(u_0, u_k) \\ &\leq c^n \left(d(u_0, \Phi(u_0)) + \cdots + d(\Phi^{k-1}(u_0), \Phi^k(u_0)) \right), \\ &\leq c^n \sum_{j=0}^{k-1} c^j d(u_0, \Phi(u_0)) \\ &\leq \frac{c^n}{1-c} d(u_0, \Phi(u_0)) \end{aligned} .$$

Since $0 < c < 1$, the above vanishes upon passing the limit $n \to \infty$, proving that the sequence is Cauchy. Since X is complete, we know that there exists a limit $u \in X$. As contractions are continuous,

$$\Phi(u) = \Phi(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} \Phi(u_n) = \lim_{n\to\infty} u_{n+1} = u.$$

To show uniqueness, suppose $v \in X$ is another fixed point, then by definition and the contraction inequality

$$d(u, v) \leq cd(u, v),$$

which, since $0 < c < 1$, can only hold if $d(u, v) = 0$ i.e. $u = v$. $\square$

Theorem 7 (Picard-Lindelof existence theorem). *Let $\Omega \subseteq \mathbb{R}^d$ be a domain, and suppose $F \in \dot{C}^{0,1}_{\text{loc}}(\Omega \to \mathbb{R}^d)$ is locally Lipschitz. For initial data $u_0 \in \Omega$ and $\varepsilon \ll 1$, set*

$$L := ||F||_{\dot{C}^{0,1}(\overline{B_\varepsilon(u_0)})},$$
$$M := ||F||_{C^0(\overline{B_\varepsilon(u_0)})}.$$

Then for $T < \min(\varepsilon/M, 1/L)$, there exists a strong solution $u : [0,T] \to \overline{B_\varepsilon(u_0)}$ to the initial data problem (IDP).

Proof. The contraction mapping principle furnishes a solution provided we show that the integral operator Φ_{u_0} is a contraction mapping on the closed ball $C^0([0,T] \to \overline{B_\varepsilon(u_0)})$. We first show Φ_{u_0} maps the desired space into itself: by choice of T and M, we have

$$||(\Phi_{u_0}u)(t) - u_0||_{C^0[0,T]} \leq \int_0^T |F(u(s))|\, ds \leq TM \leq \varepsilon.$$

Next, we show Φ_{u_0} is a contraction:

$$||\Phi_{u_0}u - \Phi_{u_0}v||_{C^0[0,T]} \leq \int_0^T |F(u(s)) - F(v(s))|\, ds \leq TL||u-v||_{C^0},$$

where $TL < 1$ by construction. This completes the proof. □

Remark. A common theme in solving ODEs and more generally PDEs is that we need to exploit some "smallness" to get an estimate or iteration to close. In this case, we exploit the time of existence T to "defeat" any large oscillations from F which could prevent the iteration from converging.

2.3. **Compactness solutions.** Suppose now we only had continuity of the non-linearity F and no control over the Lipschitz norms. We again argue by approximation, this time approximating the equation and showing the corresponding solutions converge via *compactness*, namely

Lemma 8 (Arzela-Ascoli theorem). *Let (X,d) be a compact metric space, and let $\mathcal{F} \subseteq C(X)$ be a family of continuous functions. Then $\mathcal{F}$ is pre-compact if and only if*

- *$\mathcal{F}$ is uniformly bounded, that is, $||f||_{C(X)} \lesssim 1$ uniformly in $f \in \mathcal{F}$,*
- *$\mathcal{F}$ is equicontinuous, that is, for every $\varepsilon > 0$ there exists a uniform $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$ for all $f \in \mathcal{F}$.*

While we can approximate the non-linearity in the uniform topology by Lipschitz functions, there is no uniform control over the Lipschitz constants. Thus, following the proof of the Picard-Lindelof theorem, we cannot quantitatively show the approximate solutions exist on the same time interval existence. We instead turn to a more qualitative argument relying on *continuous induction on time* to show that the solutions do not "escape" a ball in arbitrarily small time.

Lemma 9 (Bootstrap argument). *Let $f : [0,T) \to [0,\infty)$ be continuous, and suppose $f(0) \leq C$. Suppose that $f(t) \leq 2C$ implies the stronger bound $f(t) \leq C$, then the stronger bound holds for all time.*

Proof. We argue by connectedness of the interval $[0,T)$. The set of times $A \subseteq [0,T)$ where $f(t) \leq 2C$ holds is

- non-empty, since $0 \in A$,
- closed, since f is continuous,
- open, since if $t \in A$, then we in fact have the stronger bound $f(t) \leq C$, which by continuity allows us to propagate the weaker bound forward in time $f(t^+) \leq 2C$.

Hence $A = [0,T)$. □

Remark. The weaker estimate $f(t) \leq 2C$ is known as a BOOTSTRAP ASSUMPTION. Since it in fact implies a stronger bound $f(t) \leq C$, the estimate "picks itself up by its bootstraps".

Theorem 10 (Peano's existence theorem). *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and suppose $F \in C^0_{\text{loc}}(\Omega \to \mathbb{R}^n)$ is continuous. Then for any $u_0 \in \Omega$, there exists a local solution to the initial data problem (IDP).*

Proof. Assume without loss of generality $u_0 = 0$. There exists $\{F_k\}_k \subseteq C^\infty(\Omega \to \mathbb{R}^n)$ such that $F_k \to F$ uniformly on compact sets. Smooth functions are locally Lipschitz, so, choosing a small closed ball $\overline{B_\varepsilon(0)} \subseteq \Omega$, Picard-Lindelof furnishes local solutions $u_k : [0, T_k) \to B_\varepsilon(0)$ to the initial data problems

$$\begin{aligned} \partial_t u_k &= F_k(u_k), \\ u_{k|t=0} &= 0 \end{aligned}$$

We claim that there exists a uniform time interval $[0, T] \subseteq \bigcap_k [0, T_k)$ on which $\{u_k\}_k$ exist and are pre-compact. By uniform convergence, there exist uniform constant $M > 0$ such that $|F_k(u)| \leq M$ whenever $|u| \leq \varepsilon$. Let $0 < T < \varepsilon/(2M+1)$, we make the bootstrap assumption that $|u_k| \leq \varepsilon$ on the interval $[0, T]$. It follows that

$$|u_k(t)| \leq \int_0^T |F_k(u_k(s))|\, ds \leq TM \leq \frac{\varepsilon}{2}$$

for all $t \in [0, T]$. The local well-posedness theory allows us to continue the solution, thus by continuous induction on time u_k exists on $[0, \varepsilon/(2M+1)$ and satisfies the uniform bound $|u_k| \leq \varepsilon$. Furthermore the equation implies $|\partial_t u_k| \leq M$. This proves $\{u_k\}_k$ is uniformly bounded and equicontinuous, so by Arzela-Ascoli we pass to a subsequence which converges uniformly to a continuous function u. As $F_k(u_k(t)) \to F(u(t))$ uniformly, we conclude u is a desired strong solution, i.e.

$$u(t) = \lim_{k \to \infty} u_k(t) = \lim_{k \to \infty} \int_0^t F_k(u_k(s))\, ds = \int_0^t F(u(s))\, ds$$

for all $t \in [0, \varepsilon/(2M+1)]$. $\square$

Remark. Uniqueness may fail without further assumptions on the regularity of F; for example, letting $0 < \alpha < 1$, consider the initial data problem

$$\begin{aligned} \partial_t u &= |u|^\alpha, \\ u_{|t=0} &= 0. \end{aligned}$$

Then

$$u(t) = \begin{cases} 0, & \text{if } t \leq t_0, \\ \left(\frac{1}{1-\alpha}\right)^{\frac{1}{\alpha-1}} (t - t_0)^{\frac{1}{1-\alpha}}, & \text{if } t > t_0, \end{cases} \tag{*}$$

is an infinite family of solutions indexed by $t_0 \geq 0$. As a general principle, *sub-linear* forcing terms contribute to non-uniqueness.

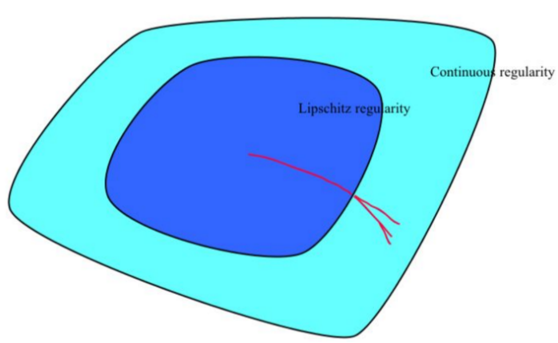

FIGURE 4. Uniqueness fails once the solution leaves the regime of Lipschitz regularity. Approaching the boundary can be viewed as a *blow-up criterion*, see Section 5 for details.

Example. Uniqueness breaks down in that the compactness solution depends on the choice of approximation by smooth functions. Let's consider the previous example for $\alpha = \frac{1}{2}$. Fix $t_0 \geq 0$, and define a uniform approximation of $F(u) := |u|^{1/2}$ by Lipschitz functions $F_k : \mathbb{R} \to \mathbb{R}$ by

$$F_k(u) := \left(\max\left\{ |u| - \frac{1}{k}, 0 \right\} + \frac{1}{k^2 t_0^2} \right)^{1/2}.$$

The solutions $u_k : [0, \infty) \to \mathbb{R}$ to the corresponding initial data problems are given by

$$u_k(t) := \begin{cases} \frac{t}{kt_0}, & \text{if } t \leq t_0, \\ \frac{1}{k} - \frac{1}{k^2 t_0^2} + \frac{(t - t_0 + \frac{2}{kt_0})^2}{4} & \text{if } t > t_0, \end{cases}$$

and converge locally uniformly to (*).

3. Uniqueness

As seen in Peano's theorem, uniqueness breaks down once the solution leaves the regime where the non-linearity is Lipschitz continuous. Thus we restrict our attention to solutions which stay within the state space Ω. In this regime, we have uniqueness for $F \in \dot{C}^{0,1}_{\text{loc}}(\Omega \to \mathbb{R}^n)$, complementing the local existence theory. Our key ingredient will be Gronwall's inequality, which states that linear feedback bounds can at worst lead to exponential growth.

Lemma 11 (Gronwall's integral inequality)**.** *Let $u : [0, T] \to \mathbb{R}^+$ be a continuous and non-negative function, and suppose u obeys the integral inequality*

$$u(t) \leq A + \int_0^t B(s) u(s)\, ds$$

for some $A \geq 0$ and $B : [0, T] \to \mathbb{R}^+$ continuous and non-negative. Then

$$u(t) \leq A \exp\left(\int_0^t B(s)\, ds \right).$$

Moreover, this estimate is sharp, with equality when $u(t) := A \exp(\int_0^t B(s)\, ds)$.

Proof. By a limiting argument we can assume $A > 0$. Differentiating the right-hand side of the integral inequality, the fundamental theorem of calculus and the inequality imply

$$\frac{d}{dt} \left(A + \int_0^t B(s) u(s)\, ds \right) \leq B(t) \left(A + \int_0^t B(s) u(s)\, ds \right).$$

Hence by the chain rule

$$\frac{d}{dt} \log \left(A + \int_0^t B(s) u(s)\, ds \right) \leq B(t).$$

Integrating, we obtain

$$\log \left(A + \int_0^t B(s) u(s)\, ds \right) \leq \log A + \int_0^t B(s)\, ds,$$

which upon exponentiating completes the proof. $\square$

Theorem 12 (Picard-Lindelof uniqueness theorem)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and suppose $F \in \dot{C}^{0,1}_{\text{loc}}(\Omega \to \mathbb{R}^n)$ is locally Lipschitz. If $u, v \in C^1_{\text{loc}}([0, T] \to \Omega)$ are solutions to the initial data problem (IDP), then $u \equiv v$.*

Proof. Since $[0, T]$ is a compact interval, u and v range over a compact subset of Ω. Therefore by local Lipschitz continuity there exists $L > 0$ such that $|F(u) - F(v)| \leq L|u - v|$. The difference of the two solutions satisfy

$$\partial_t (u - v) = F(u) - F(v).$$

Integrating and applying the triangle inequality gives

$$|u(t) - v(t)| \leq \int_0^t |F(u(s)) - F(v(s))|\, ds \leq L \int_0^t |u(s) - v(s)|\, ds.$$

We conclude from Gronwall's inequality that $u \equiv v$. $\square$

4. Continuous dependence on initial data

Studying the proof of the Picard-Lindelof existence theorem, we see that the corresponding solutions to initial data in any domain $\Omega \subseteq \mathbb{R}^n$ can be taken to exist on the same time-interval $[0, T]$, provided there exists a uniform bound and Lipschitz constant on an ε-neighborhood $\overline{B_\varepsilon(\Omega)} \subseteq \mathbb{R}^n$. Combined with uniqueness, the SOLUTION OPERATOR $S : \Omega \to C^0([0, T] \to \mathbb{R}^n)$ mapping initial data to solutions is well-defined. Our goal in this section will be to study the regularity of this operator.

4.1. $C^{0,1}$-dependence.

We begin with Lipschitz dependence on initial data. This is a simple consequence of retracing the proof of the Picard-Lindelof theorem and tracking down the constants.

Theorem 13 ($C^{0,1}$-dependence on data)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and suppose $F \in C^{0,1}(\overline{B_\varepsilon(\Omega)} \to \mathbb{R}^n)$ is Lipschitz and bounded with constants*

$$L := ||F||_{\dot{C}^{0,1}(\overline{B_\varepsilon(\Omega)})},$$
$$M := ||F||_{C^0(\overline{B_\varepsilon(\Omega)})}.$$

Then for $T < \min(\varepsilon/M, 1/L)$, the solution operator $S : \Omega \to C^0([0, T] \to \mathbb{R}^n)$ is well-defined and Lipschitz continuous with constant $\frac{1}{1-TL}$.

Proof. The solution Su_0 is a fixed point of Φ_{u_0}, so we can write

$$Su_0 - Sv_0 = \Phi_{u_0}(Su_0) - \Phi_{v_0}(Sv_0) + u_0 - v_0.$$

Following the Picard-Lindelof existence proof, we showed that the integral operators Φ_{u_0} are contractions on $C^0([0, T] \to \Omega)$ for every initial data $u_0 \in \Omega$ with Lipschitz constant TL. Thus, taking norms above and applying the triangle inequality, we obtain

$$||Su_0 - Sv_0||_{C^0[0,T]} \le ||\Phi_{u_0}(Su_0) - \Phi(Sv_0)||_{C^0[0,T]} + |u_0 - v_0| \le TL||Su_0 - Sv_0||_{C^0[0,T]} + |u_0 - v_0|.$$

Rearranging,

$$||Su_0 - Sv_0||_{C^0[0,T]} \le \frac{1}{1 - TL}|u_0 - v_0|,$$

as desired. □

4.2. C^1-dependence.

Assume the non-linearity is continuously differentiable, we want to show that the solution operator is continuously differentiable. This is meant in the *Frechet* sense, i.e. there exists a linear operator $DS(u_0) : \mathbb{R}^n \to C^0([0, T] \to \mathbb{R}^n)$ such that

$$\lim_{v \to 0} \left|\left| \frac{S(u_0 + v) - S(u_0) - DS(u_0)}{|v|} \right|\right|_{C^0[0,T]} = 0$$

and $u_0 \to DS(u_0)$ is continuous. As in the finite-dimensional codomain setting, this is equivalent to the existence and continuity of "partial" Frechet derivatives, so we will consider smooth one-parameter families of initial data $h \mapsto u_0(h)$ for $|h| \ll 1$, and show that the solution $u(t, h) := Su_0(h)(t)$ is continuously differentiable in h. The total derivative DS can be reconstructed from the partial derivatives $\partial_h u$.

The difference quotient in h satisfies

$$\partial_t \left(\frac{u(t, h) - u(t, 0)}{h} \right) = \frac{F(u(t, h)) - F(u(t, h_0))}{h},$$

then, assuming u is continuously differentiable in h and F is continuously differentiable, taking $h \to 0$ gives

$$\partial_t \partial_h u(t, 0) = \nabla F(u(t, 0)) \cdot \partial_h u(t, 0).$$

This shows that $\partial_h u$, if it exists, satisfies the LINEARISED EQUATION,

$$\begin{aligned} \partial_t A &= \nabla F(u) \cdot A, \\ A_{|t=0} &= \partial_h u_0(0). \end{aligned} \tag{Lin}$$

A priori, we do not know if u is continuously differentiable with respect to its initial data, so we instead work backwards by studying the linearised initial data problem. Heuristically, the *dynamics* of the original equation are dominated by the linearised equation, as the higher-order terms in the non-linearity are negligible.

Theorem 14 (C^1-dependence on data). *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and suppose $F \in C^1(\overline{B_\varepsilon(\Omega)} \to \mathbb{R}^n)$ is continuously differentiable and bounded with constants*

$$L := ||F||_{\dot{C}^1(\overline{B_\varepsilon(\Omega)})},$$
$$M := ||F||_{C^0(\overline{B_\varepsilon(\Omega)})}.$$

Then for $T < \min(\varepsilon/M, 1/L)$, there exists a unique solution $u \in C^2_{\mathrm{loc}}([0,T] \to \mathbb{R}^n)$ to (IDP), and the solution operator $S : \Omega \to C^0([0,T] \to \mathbb{R}^n)$ is well-defined and continuously differentiable.

Proof. Using the equation and the chain rule, we see that the solution obtained from Picard-Lindelof iteration has regularity $u \in C^2_{\mathrm{loc}}([0,T] \to \mathbb{R}^n)$. By local well-posedness, the linearised equation (Lin) admits a solution in $[0,T]$. We claim that $\partial_h u(t,0)$ exists and $A = \partial_h u(t,0)$. Translating, this argument shows that $\partial_h u(t,h)$ exists for all $t \in [0,T]$ and $|h| \ll 1$. Set

$$B(t) := \frac{u(t,h) - u(t,0)}{h} - A(t),$$

our goal is to show that $B(t) \to 0$ as $h \to 0$ for each fixed t. Differentiating and using the equations,

$$\begin{aligned}\partial_t B(t) &= \frac{F(u(t,h)) - F(u(t,0))}{h} - \nabla F(u(t,0)) \cdot A(t) \\ &= \int_0^1 \frac{u(t,h) - u(t,0)}{h} \cdot \nabla F(s \cdot u(t,h) + (1-s) \cdot u(t,0))\, ds - \nabla F(u(t,0)) \cdot A(t) \\ &= C_1(t) \cdot B(t) + C_2(t) \cdot A(t),\end{aligned}$$

where

$$C_1(t) := \int_0^1 \nabla F(s \cdot u(t,h) + (1-s) \cdot u(t,0))\, ds,$$
$$C_2(t) := \int_0^1 \left(\nabla F(s \cdot u(t,h) + (1-s) \cdot u(t,0)) - \nabla F(u(t,0))\right)\, ds.$$

From our regularity assumptions, we see that $|A|, |C_1|, |C_2| \leq N$ for some uniform constant $N \gg 1$. Thus, integrating the expression for $\partial_t B$ and applying the bounds above, we obtain

$$|B(t)| \leq |B(0)| + TN||C_2||_{C^0[0,T]} + N \int_0^t |B(s)|\, ds.$$

Using Gronwall's inequality, this implies

$$|B(t)| \leq e^t \left(|B(0)| + TN||C_2||_{C^0[0,T]}\right)$$

Since $h \mapsto u(0,h)$ and F are continuously differentiable, $|B(0)|$ and $||C_2||_{C^0[0,T]}$ vanish as $h \to 0$. We conclude from construction of B that $\partial_h u$ exists and is given by A.

It remains to show that $\partial_h u$ is continuous in h uniformly in t. Again, by translation it is not a loss of generality to show the result for $h = 0$. Integrating the linearised equation, we obtain

$$\begin{aligned}|\partial_h u(t,h) - \partial_h u(t,0)| &\leq |\partial_h u(0,h) - \partial_h u(0,0)| + \int_0^t |\nabla F(u(s,h)) \cdot \partial_h u(s,h) - \nabla F(u(s,0)) \cdot \partial_h u(s,0)|\, ds \\ &\leq |\partial_h u(0,h) - \partial_h u(0,0)| + \int_0^t |\nabla F(u(s,h)) - \nabla F(u(s,0))| \cdot |\partial_h u(s,h)|\, ds \\ &\quad + \int_0^t |\nabla F(u(s,0)) \cdot |\partial_h u(s,h) - \partial_h u(s,0)|\, ds\end{aligned}$$

By Gronwall's inequality.

$$|\partial_h u(t,h) - \partial_h u(t,0)| \leq e^{Lt} \left(|\partial_h u_0(h) - \partial_h u_0(0)| + C \int_0^t |\nabla F(u(s,h)) - \nabla F(u(s,0))|\, ds\right)$$

for some $L, C > 0$. Since $\partial_h u_0$ and ∇F are continuous, the right-hand side vanishes uniformly in t as $h \to 0$. $\square$

Corollary 15 (C^k-dependence on data). *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and suppose $F \in C^k(\overline{B_\varepsilon(\Omega)} \to \mathbb{R}^n)$ is continuously differentiable and bounded with constants*

$$L := ||F||_{C^k(\overline{B_\varepsilon(\Omega)})},$$
$$M := ||F||_{C^0(\overline{B_\varepsilon(\Omega)})}.$$

Then for $T < \min(\varepsilon/M, 1/L)$, the solution operator $S : \Omega \to C^0([0,T] \to \mathbb{R}^n)$ is well-defined and continuously differentiable.

Proof. Induction on k and the C^1-wellposedness theory. $\square$

5. Maximal solutions

In the previous sections, we have studied the *local* well-posedness theory of the initial data problem. Inductively applying the local theory to the end-time of existence for any solution u, we can show that there exists a MAXIMAL TIME OF EXISTENCE $T_{\max}$, i.e. there does not exist a solution $v : [0, T^+] \to \mathbb{R}^n$ to (IDP) such that $u \equiv v$ on $[0, T_{\max})$. We refer to the solution defined on $[0, T_{\max})$ as the MAXIMAL SOLUTION.

Theorem 16 (Continuation criterion). *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and suppose $F \in \dot{C}^{0,1}_{\mathrm{loc}}(\Omega \to \mathbb{R}^n)$ is locally Lipschitz. If $u : [0,T] \to \Omega$ is a solution to the initial data problem (IDP) which does not approach the boundary of Ω or blow-up, i.e.*

$$\lim_{t \to T} \operatorname{dist}(u(t), \partial\Omega) > 0, \qquad \textit{and} \qquad \lim_{t \to T} |u(t)| < \infty,$$

then u can be extended to a solution on $[0, T^+]$.

Proof. Since u does not approach the boundary or blow-up, there exists a closed ball $\overline{B_\varepsilon(u(T))} \subseteq \Omega$ on which we can continue the solution via the local theory. $\square$

Corollary 17 (Existence of maximal solutions). *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and suppose $F \in \dot{C}^{0,1}_{\mathrm{loc}}(\Omega \to \mathbb{R}^n)$ is locally Lipschitz. Then there exists a maximal solution defined on a half-open interval $[0, T_{max})$.*

Proof. The maximal time interval of existence must be half-open, since if u is defined on $[0,T]$, then by local well-posedness it can be extended to $[0, T^+]$. Define the maximal time interval of existence $[0, T_{\max})$ as the union of all intervals $[0,T]$ for which one has a solution to the initial data problem (IDP). By the uniqueness theorem, we may glue these solutions together to obtain a maximal solution. $\square$

Corollary 18 (Blow-up criterion). *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and suppose $F \in \dot{C}^{0,1}_{\mathrm{loc}}(\Omega \to \mathbb{R}^n)$ is locally Lipschitz. If u is a maximal solution to the initial data problem (IDP) and $T_{max} < \infty$, then either*

$$\lim_{t \to T_{max}} \operatorname{dist}(u(t), \partial\Omega) = 0, \qquad \textit{or} \qquad \lim_{t \to T_{max}} |u(t)| = \infty.$$

Proof. Suppose u does not blow-up, then there exists a compact subset $K \subseteq \Omega$ on which $u(t) \in K$ for all $[0, T_{\max})$, and denote

$$L := ||F||_{\dot{C}^{0,1}(\overline{B_\varepsilon(K)})},$$
$$M := ||F||_{C^0(\overline{B_\varepsilon(K)})}.$$

Then for all t, local well-posedness theory allows us to extend u to a solution on $[t, t+T]$ for $T < \min(\varepsilon/M, 1/L)$. Taking $t \uparrow T_{\max}$ contradicts maximality of $T_{\max}$. $\square$

Remark. The proposition does not apply to global solutions, e.g. the constant function $u \equiv u_0$ is a global solution to $u' = 0$ and $u(0) = u_0$, and it does not exhibit blow-up.

Example. Consider the initial data problem

$$\partial_t u = u^2,$$
$$u_{|t=0} = u_0.$$

The map $u \mapsto u^2$ is locally Lipschitz, however the solution

$$u(t) = \frac{1}{1/u_0 - t}$$

admits blow-up in finite time, namely $t = 1/u_0$. Following the proof of Picard-Lindelof, we see that the length of time in which a solution exists depends inversely with the local Lipschitz constant, and so the blow-up in this case coincides with the blow-up of the local Lipschitz constant of $u \mapsto u^2$.

This example suggests that if we had uniform control over the Lipschitz constant, then we could obtain a global solution, which are obviously maximal.

Proposition 19 (Existence of a global solution)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and suppose $F \in C^{0,1}(\Omega \to \mathbb{R}^n)$ is Lipschitz continuous, then there exists a unique global solution to the initial data problem (IDP).*

Proof. Denote

$$L := ||F||_{\dot{C}^{0,1}(\Omega)}$$

and let $0 < T < 1/L$. It follows that the integral operator Φ_{u_0} as defined in the proof of the Picard-Lindelof existence theorem is a contraction on the space $C^0([0,T] \to \Omega)$,

$$||\Phi_{u_0} u - \Phi_{u_0} v||_{C^0[0,T]} \leq \int_0^T |F(u(s)) - F(v(s))| \, ds \leq TL||u - v||_{C^0[0,T]}.$$

The contraction mapping principle furnishes a unique solution $u : [0,T] \to \Omega$ to the initial data problem. We iterate the Picard-Lindelof scheme at each endpoint of the previous construction, supposing we had a solution $u_k : [0,kT] \to \Omega$ to the initial data problem for some $k \in \mathbb{N}$, and solving

$$u'_{k+1}(t) = F(u_{k+1}(t)),$$
$$u_{k+1}(kT) = u_k(kT).$$

By Picard-Lindelof iteration, we obtain a solution $u_{k+1} : [0,(k+1)T] \to \Omega$ satisfying the original problem. Proceeding inductively in k furnishes a global solution $u : [0,\infty) \to \Omega$. □

6. Semi-linear equations

Suppose the quasi-linear equation admits the *vacuum solution* $u \equiv 0$, i.e. the non-linearity satisfies $F(0) = 0$. If the non-linearity is continuously differentiable, then we can Taylor expand and reformulate the initial data problem as a SEMI-LINEAR equation,

$$\begin{aligned} \partial_t u - Lu &= N(u), \\ u_{|t=0} &= u_0, \end{aligned} \tag{sLin}$$

where $L \in \mathrm{End}(\mathbb{R}^n)$ is a linear operator and $N : \mathbb{R}^n \to \mathbb{R}^n$ is a non-linear operator which vanishes faster than linearly at zero, i.e.

$$\lim_{|u| \to 0} \frac{|N(u)|}{|u|} = 0.$$

6.1. Linear theory.

Linear theory. The case $N = 0$ corresponds to the HOMOGENEOUS LINEAR equation,

$$\begin{aligned} \partial_t u - Lu &= 0, \\ u_{|t=0} &= u_0. \end{aligned} \tag{0Lin}$$

The evolution of this equation is driven by the LINEAR PROPAGATOR $e^{tL} \in \mathrm{End}(\mathbb{R}^n)$. Observe that the propagator satisfies the *group law* $e^{tL}e^{sL} = e^{(t+s)L}$ and $e^{0L} = \mathrm{Id}$. Linearity of the propagator $e^{tL}(u_0 + v_0) = e^{tL}u_0 + e^{tL}v_0$ shows that the solutions to the homogeneous equation forms a vector space. This is known as the PRINCIPLE OF SUPERPOSITION.

Theorem 20 (Linear propagators). *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be linear, then the initial data problem for the homogeneous linear equation (0Lin) is globally well-posed and admits the solution formula*

$$u(t) = e^{tL}u_0.$$

Proof. In one-dimension $n = 1$, this can be derived from separation of variables. In arbitrary dimensions, we can multiply the equation by an integrating factor e^{-tL}, allowing us to rewrite it as $\partial_t(e^{-tL}u) = 0$. Applying the fundamental theorem of calculus furnishes the formula. □

Remark. If $u_0 \in \mathbb{R}^n$ is an eigenvector of L, i.e. $Lu_0 = \lambda u_0$ for some $\lambda \in \mathbb{C}$, then the unique global solution is given by $u(t) = e^{t\lambda}u_0$. Thus, if all the eigenvalues of L have negative real part, then the equation is *dissipative*, decaying exponentially as $t \to +\infty$, and if all the eigenvalues have positive real part, then the equation is *anti-dissipative*, growing exponentially as $t \to +\infty$.

Suppose now at every time t, the evolution of the linear equation is influenced by an external *forcing term* $f \in C^0(\mathbb{R} \to \mathbb{R}^n)$. This corresponds to the study of the INHOMOGENEOUS LINEAR equation

$$\begin{aligned} \partial_t u - Lu &= f, \\ u_{|t=0} &= u_0. \end{aligned} \tag{fLin}$$

By linearity, we can solve the equation without loss of generality starting with zero initial data $u_0 = 0$. One can think of the inhomogeneous problem as a family of homogeneous problems at each time $t = t_0$ with initial data given by the infinitesimal force $f(s)ds$. Superimposing the corresponding solutions to each homogeneous problem furnishes the solution to the inhomogeneous problem.

Theorem 21 (Duhamel's formula). *Let $L \in \mathrm{End}(\mathbb{R}^n)$ be a linear operator and $f \in C^0(\mathbb{R} \to \mathbb{R}^n)$ be continuous. Then the initial data problem for the inhomogeneous linear equation (fLin) is globally well-posed and admits the solution formula*

$$u(t) = e^{tL}u_0 + \int_0^t e^{(t-s)L} f(s)\, ds.$$

Proof. We use the method of integrating factors, making the ansatz $u(t) = e^{tL}v(t)$ for some $v \in C^1(\mathbb{R} \to \mathbb{R}^n)$ such that $v(0) = u_0$. This allows us to write the inhomogeneous equation as

$$\partial_t v = e^{tL} f.$$

By the fundamental theorem of calculus, this is equivalent to

$$v(t) = u_0 + \int_0^t e^{-sL} f(s)\, ds.$$

Multiplying both sides by e^{tL} and using the group law furnishes Duhamel's formula. □

6.2. **Duhamel iteration.** In view of Duhamel's formula, we see that solving the semi-linear problem (sLin) is equivalent to solving the integral equation

$$u(t) = u_{\text{lin}} + DNu(t) \tag{sLin'}$$

where $u_{\text{lin}} \in C^0([0,T] \to \mathbb{R}^n)$ is the linear evolution of initial data and $D : C^0([0,T] \to \mathbb{R}^n) \to C^0([0,T] \to \mathbb{R}^n)$ is the Duhamel operator,

$$u_{\text{lin}} := e^{tL}u_0, \qquad DF(t) := \int_0^t e^{(t-s)L} F(s)\, ds.$$

We will argue by *Duhamel iteration*. Schematically, we start off with an approximate solution $u^{(0)} := u_{\text{lin}}$, and inductively construct subsequent approximates $u^{(n)}$ be inputting $u^{(n-1)}$ into Duhamel's formula,

$$u^{(n)} := u_{\text{lin}} + DNu^{(n-1)}.$$

These approximate solutions are known as DUHAMEL ITERATES, and the goal then is to show that this sequence $\{u^{(n)}\}_n$ converges uniformly to some u. To this end, we modify the contraction mapping scheme.

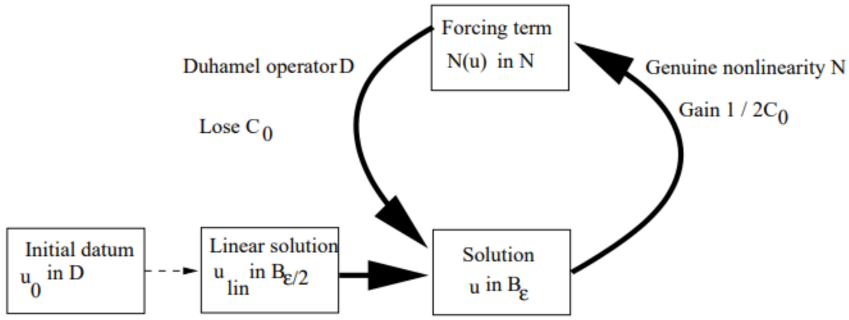

FIGURE 5. The abstract Duhamel iteration scheme.

Lemma 22 (Abstract Duhamel iteration)**.** *Let $\mathcal{N}$ and $\mathcal{S}$ be Banach spaces. Suppose $D : \mathcal{N} \to \mathcal{S}$ is a bounded linear operator such that*

$$||DF||_{\mathcal{S}} \leq C||F||_{\mathcal{N}},$$

and $N : \mathcal{S} \to \mathcal{N}$ is a Lipschitz non-linear operator such that $N(0) = 0$ and

$$||Nu - Nv||_{\mathcal{N}} \leq \frac{1}{2C}||u - v||_{\mathcal{S}},$$

for all $u, v \in \overline{B_\varepsilon(0)}$. Then for all $u_{lin} \in \overline{B_{\varepsilon/2}(0)}$, there exists a unique solution $u \in \overline{B_\varepsilon(0)}$ to the equation (sLin'), with the map $u_{lin} \mapsto u$ Lipschitz with constant at most 2.

Proof. Solving the equation (sLin') is equivalent to finding a fixed point of the operator $\Phi : \mathcal{S} \to \mathcal{S}$ defined by

$$\Phi(u) := u_{\text{lin}} + DNu.$$

We claim this map is a contraction on the closed ball $\overline{B_\varepsilon(0)} \subseteq \mathcal{S}$ with Lipschitz constant $\frac{1}{2}$; the contraction mapping principle completes the proof. To show Φ maps the ball into itself, observe by the triangle inequality

$$||\Phi(u)||_{\mathcal{S}} \leq ||u_{\text{lin}}||_{\mathcal{S}} + ||DNu||_{\mathcal{S}} \leq \varepsilon$$

for any $u \in \overline{B_\varepsilon(0)}$. To show Φ is a contraction, for any $u, v \in \mathcal{S}$, we can write $\Phi(u) - \Phi(v) = D(Nu - Nv)$. Taking norms, we obtain

$$||\Phi(u) - \Phi(v)||_{\mathcal{S}} \leq C||Nu - Nv||_{\mathcal{N}} \leq \frac{1}{2}||u - v||_{\mathcal{S}},$$

as desired. Similarly, to show that the linear-to-nonlinear solution map is Lipschitz continuous, we use the equation and the triangle inequality to write

$$||u - v||_{\mathcal{S}} \leq ||u_{\text{lin}} - v_{\text{lin}}||_{\mathcal{S}} + ||D(Nu - Nv)||_{\mathcal{S}} \leq ||u_{\text{lin}} - v_{\text{lin}}||_{\mathcal{S}} + \frac{1}{2}||u - v||_{\mathcal{S}}.$$

Rearranging shows that the Lipschitz constant is at most 2. □

Remark. One should think of the space $\mathcal{S}$ as the *solution space* and $\mathcal{N}$ as the space where the non-linearity resides. Making a judicious choice of these spaces so that the iteration scheme converges is a central problem in the study of semi-linear partial differential equations for small data.

Theorem 23 (Linear stability implies non-linear stability)**.** *Let $L \in \text{End}(\mathbb{R}^n)$ be a dissipative linear operator in that*

$$\langle Lu, u\rangle \leq -\sigma\langle u, u\rangle$$

for some $\sigma > 0$, and suppose $N \in C^2_{\text{loc}}(\mathbb{R}^n \to \mathbb{R}^n)$ vanishes faster than linearly at the origin. Then for initial data sufficiently close to the origin $|u_0| \ll 1$, there exists a unique global solution $u \in C^1([0, \infty) \to \mathbb{R}^n)$ to the semi-linear equation (sLin) obeying the exponential decay,

$$|u(t)| \leq 2e^{-\sigma t}|u_0|.$$

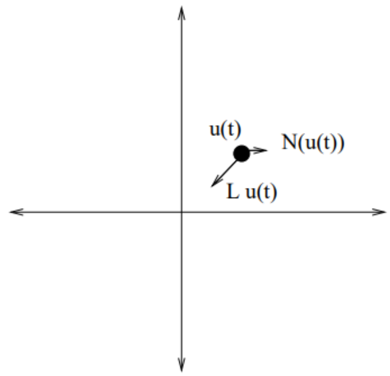

FIGURE 6. For small data, the dissipative effect of the linear term Lu dominates the non-linear term Nu, leading to global well-posedness and decay.

Proof. Define the solution space $\mathcal{S}$ and the non-linearity space $\mathcal{N}$ as subsets of the space of continuous functions $C^0([0,\infty) \to \mathbb{R}^n)$ such that the corresponding norms,

$$||u||_{\mathcal{S}} := \sup_{t\geq 0} e^{\sigma t}|u(t)|,$$
$$||u||_{\mathcal{N}} := \sup_{t\geq 0} e^{\sigma t}|u(t)|$$

are finite. By Gronwall's inequality, $||u_{\text{lin}}||_{\mathcal{S}} \leq |u_0|$, so the triangle inequality implies

$$||DF||_{\mathcal{S}} \leq \frac{1}{\sigma}||F||_{\mathcal{N}}.$$

Since N vanishes super-linearly at the origin, Taylor expanding gives $|Nu - Nv| \ll |u-v|(|u|+|v|)$ for all $|u|, |v| \ll 1$. In particular,

$$|Nu(t) - Nv(t)| \ll \varepsilon e^{-2\sigma t}||u-v||_{\mathcal{S}}.$$

for $||u||_{\mathcal{S}}, ||v||_{\mathcal{S}} \leq \varepsilon \ll 1$. Rearranging,

$$||Nu - Nv||_{\mathcal{N}} \leq \frac{1}{2}\sigma||u-v||_{\mathcal{S}}.$$

Applying the abstract Duhamel iteration for $|u_0| \leq \frac{\varepsilon}{2}$ furnishes the desired solution. □

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