AN INTRODUCTION TO BUBBLING ANALYSIS

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Abstract. Bubbling analysis, first coined in the work of Sacks and Uhlenbeck (1982), is a method of performing blow-up analysis for conformally invariant elliptic PDE, such as harmonic maps, Einstein manifolds, and elliptic Yang-Mills. We will illustrate the analysis a lá Lin and Rivière (2002) in the case of energy supercritical harmonic maps into spheres.

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1. Harmonic maps

Denote $B^n \subseteq \mathbb{R}^n$ the unit ball, and let N be a closed Riemannian manifold. We say that $u \in H^1(B^n; N)$ is a stationary harmonic map if it is a critical point of the Dirichlet energy,

$$E[u] := \int_M |\nabla u|^2 \, dx.$$

In particular, it satisfies the corresponding Euler-Lagrange equation

$$\Delta u + \mathrm{II}(u)(\nabla u, \nabla u) = 0,$$

where $\mathrm{II}(u)$ denotes the second fundamental form of N at the point u.

1.1. Monotonicity formula.

Stationary harmonic maps also satisfy the conservation law

$$\operatorname{div}\left(|\nabla u|^2 \delta_{ij} - 2\partial_i u \partial_k u\right) = 0.$$

Integrating the above identity, we see that the scale-invariant Dirichlet energy

$$\theta_r(x) := \frac{1}{r^{n-2}} \int_{B^n_r(x)} |\nabla u|^2 \, dx$$

is a monotone quantity $\theta_r \uparrow$ with

$$\frac{d}{dr}\theta_r(x) = \frac{2}{r^{n-2}} \int_{\partial B^n_r(x)} \frac{1}{\rho} \left| \frac{\partial u}{\partial \rho} \right|^2 d\Theta.$$

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1.2. ε-regularity.

Theorem 1. *There exists $\varepsilon(n, N) > 0$ such that if $u : B_r^n(0) \to N$ is a stationary harmonic map and*

$$\theta_r(0) < \varepsilon$$

then u is smooth in a neighborhood of the origin and

$$|\nabla u(0)|^2 \lesssim_{n,N} \frac{\theta_r(0)}{r^2}.$$

Theorem 2. *Let M, N be closed Riemannian manifolds, and suppose $u : M \to N$ is a non-trivial smooth harmonic map. Then there exists $\varepsilon(M, N) > 0$ such that*

$$E[u] \geq \varepsilon(M, N).$$

2. Preliminaries

Let $u_i : B^n \to N$ be a sequence of harmonic maps with uniformly bounded energy, then we can pass to a subsequence converging weakly to u. The motivating question which begins the bubbling analysis is: to what extent does weak convergence fail to be strong? This is captured precisely by the set of energy concentration,

$$\Sigma := \bigcap_{r>0} \left\{ x \in B^n : \liminf_{i\to\infty} \theta(r) \geq \varepsilon(n, N) \right\}.$$

Furthermore, it follows from Fatou's lemma that there exists a non-negative measure ν such that

$$\lim_{i\to\infty} |\nabla u_i|^2 \, dx = |\nabla u|^2 dx + \nu.$$

Lemma 3. *Let $u : B^n \to N$ be a stationary harmonic map, then the $(k-2)$-dimensional Hausdorff measure of the singular set is zero.*

Proof. Suppose $x \in \operatorname{sing} u$, then by the ε-regularity theorem there exists $r_x > 0$ such that

$$\varepsilon \leq \frac{1}{r_x^{n-2}} \int_{B_{r_x}^n(x)} |\nabla u|^2 \, dx.$$

Using the Besocovitch covering lemma, there exists a disjoint collection of balls $B_{r_j}^n(x_j)$ such that

$$\Sigma \subseteq \bigcup_j B_{5r_j}^n(x_j).$$

By disjointness and the estimates above

$$\varepsilon \sum_j r_j^{n-2} \lesssim \int_{\bigcup_j B_{r_j}^n(x_j)} |\nabla u|^2 \, dx.$$

This proves that the singular set has finite $(n-2)$-Hausdorff measure. Note that this further implies that it has Lebesgue measure zero, so by dominated convergence we in fact have that the singular set has $(n-2)$-Hausdorff measure zero. $\square$

2.1. **Model case.** Suppose there exists a smooth, non-constant harmonic map with finite energy $\phi : \mathbb{R}^2 \to N$. By rescaling, we can find a smooth family of harmonic maps $\{\phi_i\}_i$ such that the energy densities concentrate

$$|\nabla \phi_i|^2 \, dx \to c_0 \delta_0.$$

Evidently by undoing the scaling we see that c_0 is precisely given by the energy of ϕ. We can extend this example to $\mathbb{R}^n$ by constants

$$|\nabla \phi_i|^2 \, dx \to c_0 \, d\mathcal{H}^{n-2}|_P.$$

2.2. **Rescaling.** In view of the model example, we want to begin the bubbling analysis by locally approximating the singular set Σ by a plane. Indeed, by rectifiability one has for $\mathcal{L}^{n-2}$-a.e. $x_0 \in \Sigma$ a unique classical tangent space P, i.e. the blow-up at x_0 of the defect measure converges to the tangent measure,

$$\lim_{r\to 0} \frac{1}{r^{n-2}} \mathsf{BlowUp}_{x_0,r} \left(e\, d\mathcal{H}^{n-2}\big|_{\Sigma} \right) = e(x_0)\, d\mathcal{H}^{n-2}\big|_{P},$$

where $\mathsf{BlowUp}_{x_0,r}\mu(A) = \mu(x_0 + rA)$ for any measure μ. On the other hand, Federer-Ziemer established the following well-known Lebesgue differentiation-type result for harmonic maps,

$$\lim_{r\to 0} \theta_r(x_0) = \lim_{r\to 0} \frac{1}{r^{n-2}} \int_{B^n_r(x_0)} |\nabla u|^2\, dx = 0$$

for $\mathcal{H}^{n-2}$-a.e. $x_0 \in B^n$. Collecting the two results, we can write

$$\begin{aligned}\lim_{r\to 0} \lim_{i\to\infty} \frac{1}{r^{n-2}} \mathsf{BlowUp}_{x_0,r} \left(|\nabla u_i|^2\, dx \right) &= \lim_{r\to 0} \frac{1}{r^{n-2}} \mathsf{BlowUp}_{x_0,r} \left(|\nabla u|^2\, dx + e\, d\mathcal{H}^{n-2}\big|_{\Sigma} \right) \\ &= e(x_0)\, d\mathcal{H}^{n-2}\big|_{P}\end{aligned}$$

for $\mathcal{H}^{n-2}$-a.e. $x \in \Sigma$. Thus there exist radii $r_k \downarrow 0$ and indices $i_k \uparrow \infty$ such that

$$\lim_{k\to\infty} |\nabla \widetilde{u}_k|^2\, dx = \lim_{k\to\infty} \frac{1}{r_k^{n-2}} \mathsf{BlowUp}_{x_0,r_k} \left(|\nabla u_{i_k}|^2\, dx \right) = e(x_0)\, d\mathcal{H}^{n-2}\big|_{P},$$

where $\widetilde{u}_k(x) := \mathsf{BlowUp}_{x_0,r_k} u_{i_k}(x) = u_{i_k}(x_0 + r_k x)$. That is, we can write the tangent defect measure as the defect measure of a rescaled sub-sequence of harmonic maps converging weakly to a constant.

Henceforth we pass to this rescaled sequence, and assume without loss of generality that x_0 is the origin and $P = \mathbb{R}^{n-2} \times \{0^2\}$. It follows from the monotonicity formula, c.f. Lin, that one has

$$\lim_{k\to\infty} \int_{B_1^{n-2}(0)\times B_1^2(0)} \sum_{j=1}^{n-2} \left| \frac{\partial u_k}{\partial x_j} \right|^2 dx = 0. \tag{1}$$

3. Bubbling analysis

A bubble $\phi : \mathbb{R}^n \to N$ is a smooth non-constant harmonic map which is invariant under translation with respect to some $(n-2)$-dimensional subspace $P \subseteq \mathbb{R}^n$. We define the energy of ϕ to be

$$E[\phi] := \int_{P^\perp} |\nabla \phi|^2\, d\mathcal{L}^{n-2}.$$

We say that ϕ is a bubble of $\{u_i\}_i$ at $x \in \Sigma$ if there exists a sequence $x_i \to x$ and $r_i \to 0$ such that the blow-ups

$$u_i(x_i + r_i x) \to \phi$$

where the convergence is smooth away from a closed set of finite $(n-2)$ Hausdorff measure. We denote by $\mathcal{B}[x]$ the collection of all bubbles at x.

3.1. **Good slices.** It follows from Allard's constancy lemma that the energy density $e(x_0)$ of the defect measure can be written as the limit of the energies along slices,

$$\lim_{k\to\infty} \int_{\{X^{n-2}\}\times B_1^2(0)} |\nabla u_k|^2 dX^2 = e, \qquad \text{a.e. } X^{n-2} \in P. \tag{2}$$

It will be convenient to work on "good slices" $\{X^{n-2} = X_k^{n-2}\}$ where certain norms are controlled. Define $f_k : B^{n-2} \to \mathbb{R}$ by

$$f_k(X^{n-2}) := \int_{\{X^{n-2}\}\times B^2(0)} \sum_{j=1}^{n-2} \left| \frac{\partial u_k}{\partial x_j} \right|^2 dX^2.$$

It follows from (1) that $||f_k||_{L^1} \to 0$. Recall the Hardy-Littlewood maximal inequality

$$|\{X^{n-2} : Mf_k(X^{n-2}) \geq \lambda\}| \lesssim \frac{||f_k||_{L^1}}{\lambda} \overset{k\to\infty}{\longrightarrow} 0.$$

This implies that there exists E_k such that $|E_k| > 0.99|B^{n-2}_{1/2}|$ for $k \gg 1$ such that

$$\sup_{0<r<1/2} \frac{1}{r^{n-2}} \int_{B_r^{n-2}(X_k^{n-2})\times B^2(0)} \sum_{j=1}^{n-2} \left| \frac{\partial u_k}{\partial x_j} \right|^2 dx \overset{k\to\infty}{\longrightarrow} 0, \qquad \text{a.e. } X^{n-2} \in E_k \tag{3}$$

by setting $E_k := \{X^{n-2} : Mf_k(X^{n-2}) < \lambda_k^{(1)}\}$ for $\lambda_k := ||f_k||_{L^1}^{1/2}$. Towards controlling the *neck regions*, we would also like the $L^{2,1}$-norm of the gradient to be controlled. Indeed

$$||\nabla u_k||_{L^{2,1}(Q_{2/3})} \lesssim ||u_i||_{W^{2,1}(Q_{2/3})} \lesssim ||\Delta u_k||_{\mathcal{H}_a(Q_{2/3})} \lesssim ||\nabla u_k||_{L^2(Q_1)} \lesssim 1,$$

where the first inequality follows from Sobolev embedding and the third inequality from a lemma of Helein. By Fubini's theorem, there exists F_k such that $|F_k| > 0.99|B_{1/2}^{n-2}|$ and

$$||\nabla u_k||_{L^{2,1}(B^2_{1/2}(0))}(X^{n-2}) \lesssim 1, \qquad \text{a.e. } X^{n-2} \in F_k. \tag{4}$$

Thus, combined with partial regularity, we can choose $\{X_k^{n-2}\}_k \subseteq B_{1/2}^{n-2}(0)$ such that (3), (4) hold, and u_k is smooth in a neighborhood of (X_k^{n-2}, X^2) for all $X^2 \in B^2_{1/2}(0)$.

3.2. **Extracting the bubbles.** To find the first bubble, we need to determine the first characteristic scale and point of energy concentration. To this end, we claim that there exist $0 < \lambda_k^{(1)} < \frac{1}{2}$ and $X_k^2 \in B^2_{1/2}(0)$ achieving the maximum value of

$$\max_{X^2 \in B^2_{1/2}(0)} \frac{1}{\left(\lambda_k^{(1)}\right)^{n-2}} \int_{B^{n-2}_{\lambda_k^{(1)}}(X_k^{n-2}) \times B^2_{\lambda_k^{(1)}}(X^2)} |\nabla u_i|^2 \, dx = \frac{\varepsilon(2,N)}{c(n)}.$$

This follows from ε-regularity and monotonicity. For brevity, we take the concentration points to be the origin $(X_k^{n-2}, X_k^2) = (0^{n-2}, 0^2)$. It is showed in Lin that, upon passing to a subsequence, the blow-up at the characteristic scale centered at the concentration point converges in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ and $H^1_{\text{loc}}(\mathbb{R}^n)$ to the first bubble,

$$\mathsf{BlowUp}_{\lambda_k^{(1)}} u_k \overset{k\to\infty}{\longrightarrow} \phi^{(1)} \qquad \text{in } H^1_{\text{loc}}.$$

The existence of additional bubbles is equivalent to energy concentration at higher scales $\lambda_k^{(2)} \gg \lambda_k^{(1)}$. Assume that there exists $\varepsilon_0 > 0$ such that, upon passing to a subsequence, there exists $\lambda_k^{(2)} \downarrow 0$ such that

$$\frac{1}{\left(\lambda_k^{(2)}\right)^{n-2}} \int_{B^{n-2}_{\lambda_k^{(2)}}(0) \times B^2_{2\lambda_k^{(2)}}(0) \setminus B^2_{\lambda_k^{(2)}}(0)} |\nabla u_k|^2 \, dx \geq \varepsilon_0 \tag{5}$$

and $\lambda_k^{(2)}/\lambda_k^{(1)} \to \infty$. Here, when passing to a subsequence, we get

$$\mathsf{BlowUp}_{\lambda_k^{(2)}} u_k \overset{k\to\infty}{\longrightarrow} \psi \qquad \text{in } L^2_{\text{loc}}.$$

There are two possibilities as detailed in the first section. First, the convergence is strong, in which case we obtain another bubble $\phi^{(2)} := \psi$. Second, the convergence is weak, in which case this is measured precisely by a defect measure,

$$|\nabla \mathsf{BlowUp}_{\lambda_k^{(2)}} u_k|^2 \, dx \to |\nabla \psi|^2 \, dx + \sum_{j=1}^{\ell} c_j \, d\mathcal{H}^{n-2}|_{P_j}$$

for some parallel planes P_j, and we perform the bubbling analysis as with the first bubble.

3.3. **Neck regions.** Arguing inductively by Fatou's lemma, we see that the energy density $e(x)$ bounds above the energy of the bubbles. *A priori*, it is possible that the inequality is strict, as, following the bubble construction, some of the energy could be lost in *neck regions* between characteristic scales. For example, in the case of a single bubble $m = 1$, we know that its energy is given by

$$\int_{\mathbb{R}^2} |\nabla \phi|^2 \, dX^2 = \lim_{R\to\infty} \int_{\{X_i^{n-2}\} \times B^4_R(X_2^i)} |\nabla \phi|^2 \, dX^2 = \lim_{R\to\infty} \lim_{i\to\infty} \int_{\{X_i^{n-2}\} \times B^2_{R\lambda_k^{(1)}}(X_2^i)} |\nabla u_i|^2 \, dX^2.$$

Furthermore, we chose a slice such that (2) holds,

$$e(x) = \lim_{i\to\infty} \int_{\{X_i^{n-2}\} \times B^2_1(0)} |\nabla u_i|^2 \, dX^2,$$

so to conclude the energy identity, it suffices to show

$$\lim_{R\to\infty} \lim_{i\to\infty} \int_{\{X_i^{n-2}\} \times B^2_1(0) \setminus B^2_{R\lambda_k^{(1)}}(X_2^i)} |\nabla u_i|^2 \, dX^2 = 0.$$

From the inductive construction of the bubbles, we see that in the case $m = 1$ there cannot be any energy concentration in the intermediate region at smaller scales. More precisely, for every $\varepsilon > 0$, there exists $R \gg 1$ and $i_0 \in \mathbb{N}$ such that

$$\frac{1}{r^{n-2}} \int_{B_r^{n-2}(0) \times B_{2r}^2(0) \setminus B_r^2(0)} |\nabla u_i|^2 \, dx \leq \sqrt{\varepsilon}$$

for all $R\lambda_k^{(1)} \leq r \leq \frac{1}{2}$ and $i \geq i_0$. Choosing $\varepsilon \ll 1$, we can apply ε-regularity to deduce that

$$|X^2| \, |\nabla u_i(0, X^2)| \lesssim \varepsilon.$$

We can view this as a Lorentz space estimate. Indeed, suppose $X^2 \in B_{1/2}^2(0) \setminus B^2_{R\lambda_k^{(1)}}(0)$ satisfies $|\nabla u_i(0, X^2)| > t$, then the inequality above implies that $|X^2| \lesssim \sqrt{\varepsilon}/t$. In particular, $t|\{X^2 : |\nabla u_i(0, X^2)| > t\}|^{1/2} \lesssim \sqrt{\varepsilon}$, i.e.

$$||\nabla u||_{L^{2,\infty}(B_{1/2}^2(0) \setminus B^2_{R\lambda_k^{(1)}})} \lesssim \sqrt{\varepsilon}.$$

By Holder's inequality we are done.

In the case of two bubbles, $m = 2$, it suffices to show that

$$\lim_{R \to \infty} \lim_{k \to \infty} \int_{\{0^{n-2}\} \times B^2_{\lambda_k^{(2)}/R}(0) \setminus B^2_{R\lambda_k^{(1)}}(0)} |\nabla u_k|^2 \, dx = 0$$

and

$$\lim_{R \to \infty} \lim_{k \to \infty} \int_{\{0^{n-2}\} \times B_1^2(0) \setminus B^2_{R\lambda_k^{(2)}}(0)} |\nabla u_k|^2 \, dx = 0$$