LOCAL WELL-POSEDNESS FOR QUASILINEAR WAVE EQUATIONS (D'APRÉS SMITH-TATARU)

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Abstract. In this note, we outline the work by Smith-Tataru [ST05] concerning the sharp local well-posedness for generic quasi-linear wave equations. That is, given sufficiently regular Lorentzian metrics $\mathbf{g}_{\mu\nu}(\phi)$ and semi-linear terms $\mathcal{N}(\phi)(\partial\phi, \partial\phi)$, we prove that the initial data problem

$$\Box_{\mathbf{g}(\phi)}\phi = \mathcal{N}(\phi)(\partial\phi, \partial\phi),$$
$$(\phi, \partial_t\phi)_{|t=0} = (\phi_0, \phi_1),$$

is locally well-posed in $H^s_x \times H^{s-1}_x(\mathbb{R}^n)$ for $s > \frac{n}{2} + \frac{1}{2}$ when $n = 3, 4, 5$.

Contents

1. Introduction 1
2. Paradifferential decomposition 4
3. Wave packet parametrix 4
4. Dispersive estimates 6
References 8

1. Introduction

In this note, we consider the local well-posedness of *quasilinear wave equations* of the form

$$\begin{aligned} \Box_{\mathbf{g}(\phi)}\phi &= \mathcal{N}(\phi)(\partial\phi, \partial\phi), && \text{on } [0, T] \times \mathbb{R}^n, \\ (\phi, \partial_t\phi) &= (\phi_0, \phi_1), && \text{on } t = 0, \end{aligned} \tag{QNLW}$$

eq:QNLW

where $\mathbf{g}_{\mu\nu}(\phi)$ is a symmetric matrix with signature $(-, +, \cdots +)$, using the convention[1] $\Box_{\mathbf{g}} := \mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$ for its associated wave operator, and $\mathcal{N}(\phi)(\partial\phi, \partial\phi) := \mathcal{N}^{\alpha\beta}(\phi)\partial_\alpha\phi\partial_\beta\phi$ is a bilinear form. Without loss of generality, we can take $t = \text{const}$ to be space-like hypersurfaces by reducing to metrics of the form

$$\mathbf{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \mathbf{g}_{ij}dx^i dx^j.$$

We shall also assume sufficient smoothness and boundedness of the metric $\mathbf{g}^{\mu\nu}(\phi)$, its inverse $\mathbf{g}_{\mu\nu}(\phi)$, and of the bilinear form $\mathcal{N}^{\alpha\beta}(\phi)$ as functions of ϕ.

Example. The following can be recast in the form (QNLW),

- the Einstein vacuum equations in wave coordinates,
- the irrotational compressible Euler equations.

For the former, this was observed by Choquet-Bruhat [Fou52], while the later is due to Hughes-Kato-Marsden [HKM77]. The reader may find the lecture notes [Luk] as a more modern reference.

Following the standards set by Hadamard, we say that the initial data problem for the quasi-linear wave equation (QNLW) is *locally well-posed* in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$ if the following hold:

(a) *Existence*: for each initial data $\phi[0] \in (H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$, there exists a time $T > 0$ and a solution $\phi[t] \in C^0_t(H^s_x \times H^{s-1}_x)([0, T] \times \mathbb{R}^n)$ to (QNLW).

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[1] One can equivalently consider the divergence form of the equation, i.e. using $\partial_\mu\mathbf{g}^{\mu\nu}\partial_\nu$ instead of $\mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$ on the left-hand side, as the lower-order terms are encapsulated by the right-hand side.

(b) *(Unconditional) uniqueness*: for each initial data $\phi[0] \in (H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$, the solution $\phi[t]$ to (QNLW) is unique in the space $C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$.

item:cty (c) *Continuity of data-to-solution map*: if $\{\phi_k[0]\}_k$ is a sequence of data converging in the $(H^s_x \times H^{s-1}_x)$-topology to $\phi[0]$, then there exists a common time of existence[2] on which the corresponding sequence of solutions $\{\phi_k[t]\}_k$ to (QNLW) converges to $\phi[t]$ in the $L^\infty_t(H^s_x \times H^{s-1}_x)$-topology,

$$\begin{aligned} &\phi_k[0] \overset{k\to\infty}{\longrightarrow} \phi[0] && \text{in } H^s_x \times H^{s-1}_x \\ \text{implies} \quad &\phi_k[t] \overset{k\to\infty}{\longrightarrow} \phi[t] && \text{in } L^\infty_t(H^s_x \times H^{s-1}_x). \end{aligned}$$

For the working definition, we will need to slightly modify the existence and uniqueness statements, strengthening the former while weakening the latter, and require an additional property of the data-to-solution map:

item:exist (a+) *(Sub-critical) existence*: the time of existence can be taken to depend only on the size of the data

$$T \equiv T(\|\phi[0]\|_{H^s_x \times H^{s-1}_x}).$$

item:unique (b-) *(Conditional) uniqueness*: uniqueness holds only in the smaller Strichartz space,

$$\left\{\phi[0] \in C^0_t(H^s_x \times H^{s-1}_x) : \partial\phi \in L^2_t L^\infty_x\right\}.$$

tem:lipschitz (c+) *Weak Lipschitz continuity of data-to-solution map*: there exists a regularity $s_{\text{Lip}} < s$ such that the data-to-solution map is Lipschitz continuous on bounded sets in $(H^s \times H^{s-1})_x(\mathbb{R}^n)$ with respect to the weaker $(H^{s_{\text{Lip}}} \times H^{s_{\text{Lip}}-1})_x(\mathbb{R}^n)$-topology, i.e. for solutions $\phi[t], \psi[t] \in C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$ to (QNLW) satisfying

$$\|\phi[0]\|_{H^s \times H^{s-1}}, \ \|\psi[0]\|_{H^s \times H^{s-1}} \leq R,$$

the following stability estimate holds:

$$\|\phi[t] - \psi[t]\|_{L^\infty_t (H^{s_{\text{Lip}}} \times H^{s_{\text{Lip}}-1})_x} \leq C(R) \cdot \|\phi[0] - \psi[0]\|_{(H^{s_{\text{Lip}}} \times H^{s_{\text{Lip}}-1})_x}.$$

In sum, we say that the initial data problem for the quasi-linear wave equation (QNLW) is *locally well-posed* in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$ if (a+), (b-), (c), (c+) hold. This leads us to the following natural question

For which values of $s \in \mathbb{R}$ is the initial data problem for the quasi-linear wave equation (QNLW) locally well-posed in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$?

thm:main **Theorem 1.1** (Sharp local well-posedness for (QNLW) [ST05])**.** *The initial data problem for the quasi-linear wave equation (QNLW) is locally well-posed in $(H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$, provided that*

$$\begin{aligned} s &> \frac{n}{2} + \frac{1}{2} + \frac{1}{4} && \text{if } n = 2, \\ s &> \frac{n}{2} + \frac{1}{2} && \text{if } n = 3, 4, 5. \end{aligned}$$

Furthermore, given a solution $\phi[t] \in C^0_t(H^s_x \times H^{s-1}_x)([0,T] \times \mathbb{R}^n)$ to (QNLW), it satisfies the following:

(a) Energy estimate and dispersive estimate: the solution satisfies the estimate

$$\|\partial\phi\|_{L^\infty_t H^{s-1}_x} + \|\partial\phi\|_{L^2_t L^\infty_x} \lesssim \|\phi[0]\|_{H^s_x \times H^{s-1}_x}.$$

(b) Well-posedness of the linearised equation: for $1 \leq \sigma \leq s+1$ and each $t_0 \in [0,T]$, the initial data problem for the linearisation of (QNLW) about the solution ϕ,

$$\begin{aligned} \Box_{\mathbf{g}(\phi)}\psi &= 0, && \text{on } [0,T] \times \mathbb{R}^n, \\ (\psi, \partial_t\psi) &= (\psi_0, \psi_1), && \text{on } t = t_0, \end{aligned} \tag{LW}$$

eq:linearised

[2] To be more precise, one can introduce the notion of the *maximal lifespan* $T \equiv T(\phi[0])$ of a solution, and require it to be lower semi-continuous as a function of initial data $\phi[0] \in (H^s_x \times H^{s-1}_x)(\mathbb{R}^n)$.

is well-posed in $(H^\sigma_x \times H^{\sigma-1}_x)(\mathbb{R}^n)$*, i.e. for each initial data* $\psi[t_0] \in (H^\sigma_x \times H^{\sigma-1}_x)(\mathbb{R}^n)$*, there exists a unique solution* $\psi[t] \in C^0_t(H^\sigma_x \times H^{\sigma-1}_x)([0,T] \times \mathbb{R}^n)$ *to (LW). Furthermore, the solutions to (LW) satisfy the energy estimate,*

$$\|\psi\|_{L^\infty_t H^\sigma_x} + \|\partial_t \psi\|_{L^\infty_t H^{\sigma-1}_x} \lesssim \|\psi[t_0]\|_{H^\sigma_x \times H^{\sigma-1}_x}, \tag{1.1}$$

and the Strichartz estimate,

$$\|\langle \nabla_x \rangle^\rho \psi\|_{L^2_t L^\infty_x} \lesssim \|\psi[t_0]\|_{H^\sigma_x \times H^{\sigma-1}_x}, \tag{1.2}$$

for $\rho < \sigma - \frac{d-1}{2}$.

Remark. The equation (QNLW) is invariant under the scaling symmetry

$$\phi(x^\mu) \mapsto \phi(\tfrac{x^\mu}{\lambda})$$

which also preserves the homogeneous Sobolev norm $\dot{H}^{s_{\text{crit}}}_x \times \dot{H}^{s_{\text{crit}}-1}_x(\mathbb{R}^n)$, where $s_{\text{crit}} := \frac{n}{2}$. Below scaling $s < s_{\text{crit}}$, the common expectation is ill-posedness, leading us to ask the more refined question:

How close can one push the local well-posedness of (QNLW) towards the scaling critical regularity?

Theorem 1.1 is sharp for generic (QNLW) in dimensions $n = 2, 3$ due to counterexamples of Lindblad [Lin93, Lin96]. On the other hand, it is not difficult to show that the *Nirenberg example*

$$\Box\phi = \partial^\alpha \phi \partial_\alpha \phi,$$

is locally well-posed for $s > s_{\text{crit}}$, thanks to the *null structure* of the non-linearity.

Remark. The proof contained in [ST05] breaks down in higher dimensions $n \geq 6$ due to a technical failure in the orthogonality argument for the wave packet decomposition.

The main result: half a derivative above scaling.

$$\begin{aligned} \Box_{\mathbf{g}}\phi &= 0, && \text{on } [0,T] \times \mathbb{R}^n, \\ (\phi, \partial_t \phi) &= (\phi_0, \phi_1), && \text{on } t = 0, \end{aligned} \tag{LW}$$

eq:LW

	Linear equation	**Strichartz**	**Regularity**
Hughes-Kato-Marsden [HKM77]	generic (LW)	N/A	$s > \frac{n}{2} + 1$
Bahouri-Chemin [BC99]	generic (LW)	$\frac{1}{4}$ loss	$s > \frac{n}{2} + \frac{1}{2} + \frac{1}{4}$
Tataru [Tat01a, Tat01b]	generic (LW)	sharp $\frac{1}{6}$ loss	$s > \frac{n}{2} + \frac{1}{2} + \frac{1}{6}$
Klainerman-Rodnianski [KR03]	linearised (QNLW)	$\frac{2-\sqrt{3}}{2}$ loss	$s > \frac{n}{2} + \frac{1}{2} + \frac{2-\sqrt{3}}{2}$
Smith-Tataru [ST05]	linearised (QNLW)	lossless	$s > \frac{n}{2} + \frac{1}{2}$

Table 1. A non-exhaustive historical overview of the local well-posedness of quasi-linear wave equations for $n \geq 3$, though one can find results concerning $n = 2$ among the references, and the result of Klainerman-Rodnianski [KR03] works only with $n = 3$.

table:history

Theorem 1.2 (Local well-posedness for (QNLW) with smooth data [HKM77])**.** *Let* $s \geq 0$*, then there exists* $C \gg 1$ *such that any smooth solution* ϕ *to (QNLW) obeys the a priori estimate*

$$\|\partial\phi\|_{L^\infty_t H^{s-1}_x} \lesssim \exp\left(C \int_0^T \|\partial\phi\|_{L^\infty_x}\, dt\right) \|\partial\phi(0)\|_{H^{s-1}_x}. \tag{1.3}$$

In particular, $\phi[t]$ *may be continued as a smooth solution as long as* $\|\partial\phi\|_{L^1_t L^\infty_x} < \infty$.

Remark. The dimension of the $L^p_t L^\infty_x$-norm of $\partial\phi$ under the scaling symmetry reads

$$\|\partial\phi\|_{L^p_t L^\infty_x} \approx [t]^{\frac{1}{p}} [x]^{-1} \approx [\partial]^{1-\frac{1}{p}}.$$

Thus, the continuation criterion $L_t^1 L_x^\infty$ is scale-invariant, controlling $L_{t,x}^\infty$ via Sobolev embedding incurs a full derivative difference from scaling $1 - \frac{1}{\infty} = 1$, while control of $L_t^2 L_x^\infty$ in $n \geq 3$ via Strichartz leads to half-derivative from scaling $1 - \frac{1}{2} = \frac{1}{2}$, and similarly $L_t^4 L_x^\infty$ in $n = 2$ leads to three-quarters $1 - \frac{1}{4} = \frac{1}{2} + \frac{1}{4}$.

2. Paradifferential decomposition

Proposition 2.1.

$$\|\psi\|_{L_t^2 L_x^\infty} \lesssim \varepsilon_0^{-\frac{1}{2}} \lambda^{\sigma-1} \|\psi[-2]\|_{H_x^1 \times L_x^2}, \tag{2.1}$$

3. Wave packet parametrix

Let $\gamma \equiv \gamma(t)$ be a null geodesic, and let $\Sigma_{\theta,u}$ be the null surface containing γ,

$$\Sigma_{\theta,u} := \{(t,x) \in [-2,2] \times \mathbb{R}^n : x_\theta - \tau_{\theta,u}(t, x'_\theta) = 0\}.$$

$$\mathfrak{w} := (\varepsilon_0 \lambda)^{\frac{1}{2}\frac{n-1}{2}} \lambda^{\frac{1}{2}-1} \mathsf{T}_{<\lambda} \left(w \, \mathrm{dS}_{\Sigma_{\theta,u}} \right)$$

where w is a smooth bump function on $\Sigma_{\theta,u}$ localised at scale $(\varepsilon_0\lambda)^{-\frac{1}{2}}$ about the null geodesic γ, i.e.

$$w = w_0\big((\varepsilon_0\lambda)^{\frac{1}{2}}(x'_\theta - \gamma'_\theta(t))\big)$$

for some $w_0 \in C_c^\infty(|x'| \leq 1)$.

The direction $\theta \in \mathbb{S}^{n-1}$ varies over maximal collection of approximately $\varepsilon_0^{\frac{n-1}{2}} \lambda^{\frac{n-1}{2}}$ unit vectors separated by at least $\varepsilon_0^{\frac{1}{2}} \lambda^{-\frac{1}{2}}$

Decompose $\mathbb{R}^n$ into parallel tiling of rectangles, length λ^{-1} in parallel x_θ and $(\varepsilon_0\lambda)^{-\frac{1}{2}}$ in the remainin x'_θ directions.

op:parametrix **Proposition 3.1** (Existence of wave packet parametrix)**.** *Let $(\phi_0, \phi_1) \in (H^1 \times L^2)_x(\mathbb{R}^n)$ be initial data. Then, in dimensions $n = 2,3,4,5$, there exists a superposition of wave packets*

$$\phi := \sum_{\theta,j} a_{\theta,j} \, \mathfrak{w}^{\theta,j}$$

which is an approximate solution to the paralinearised initial data problem in the sense that

item:WPdata *(a) it matches the initial data at $t = -2$,*

$$\mathsf{P}_\lambda \phi[-2] = (\phi_0, \phi_1) \tag{3.1}$$

item:WPsize *(b) the size of the coefficients is comparable to the size of the initial data,*

$$\Big(\sum_{\theta,j} |a_{\theta,j}|^2\Big)^{\frac{1}{2}} \lesssim \|\phi[0]\|_{(H^1 \times L^2)_x}. \tag{3.2}$$

item:WPenergy *(c) the energy estimate holds,*

$$\|\partial \mathsf{P}_\lambda \phi\|_{L_t^\infty L_x^2} \lesssim \Big(\sum_{\theta,j} |a_{\theta,j}|^2\Big)^{\frac{1}{2}} \tag{3.3}$$

item:WPerror *(d) the error on the right-hand side is small,*

$$\|\Box_{\mathbf{g}_{<\lambda}} \phi_\lambda\|_{L_t^1 L_x^2} \lesssim \varepsilon_0 \Big(\sum_{\theta,j} |a_{\theta,j}|^2\Big)^{\frac{1}{2}}. \tag{3.4}$$

3.1. **Wave packets as approximate solutions.**

Lemma 3.2 (Wave packet error decomposition)**.**

Lemma 3.3 (Energy estimate for $\mathfrak{w}$)**.** *Wave packets have $O(1)$-energy,*

$$\|\partial \mathsf{P}_\lambda \mathfrak{w}\|_{L_t^\infty L_x^2} \lesssim 1. \tag{3.5}$$

eq:WPenergy

Proof. By construction, the wave packet has amplitude

$$\|\partial \mathsf{P}_\lambda \mathfrak{w}\|_{L_t^\infty L_x^2} \lesssim \lambda \cdot |\text{amplitude}| \cdot |\text{support}|^{\frac{1}{2}} \lesssim 1$$

□

Lemma 3.4 (Error estimate for $\mathfrak{w}$)**.** *Each wave packet has small error,*

$$\|\Box_{\mathbf{g}_{<\lambda}} \mathsf{P}_\lambda \mathfrak{w}\|_{L_t^1 L_x^2} \lesssim \varepsilon_0, \tag{3.6}$$

eq:L1error

$$\|\Box_{\mathbf{g}_{<\lambda}} \mathsf{P}_\lambda \mathfrak{w}\|_{L_{t,x}^2} \lesssim \varepsilon_0. \tag{3.7}$$

eq:L2error

Obviously (3.7) is stronger than (3.6), so we focus on proving an $L_{t,x}^2$-error estimate. We write

$$\begin{aligned}
\Box_{\mathbf{g}_{<\lambda}} \mathsf{P}_\lambda \mathfrak{w} &= (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}} \lambda^{-\frac{1}{2}+1} \Big([\Box_{\mathbf{g}_{<\lambda}}, \mathsf{P}_\lambda \mathsf{T}_{<\lambda}] + \mathsf{P}_\lambda \mathsf{T}_{<\lambda} \Box_{\mathbf{g}_{<\lambda}} \Big) w \, \mathrm{dS}_\Sigma \\
&= (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}} \lambda^{-\frac{1}{2}+1} [\Box_{\mathbf{g}_{<\lambda}}, \mathsf{P}_\lambda \mathsf{T}_{<\lambda}] w \, \mathrm{dS}_{\Sigma_{\theta,u}} \\
&\quad + (\varepsilon_0\lambda)^{\frac{1}{2}\frac{n-1}{2}} \lambda^{-\frac{1}{2}+1} \mathsf{P}_\lambda \mathsf{T}_{<\lambda} \left(\Box_{\mathbf{g}_{<\lambda}} w \cdot \mathrm{dS}_\Sigma + 2\overline{\mathbf{g}}_{<\lambda}^{\alpha\beta} \partial_\alpha w \cdot \partial_\beta \mathrm{dS}_\Sigma + w \cdot \Box_{\mathbf{g}_{<\lambda}} \mathrm{dS}_\Sigma \right) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},
\end{aligned}$$

commuting $\Box$ with the frequency projections and then applying the product rule. Here we have denoted $\overline{\mathbf{g}}_{<\lambda}^{\alpha\beta} = \frac{1}{2}(\mathbf{g}_{<\lambda}^{\alpha\beta} + \mathbf{g}_{<\lambda}^{\beta\alpha})$.

Estimating the term I. Since the metric is cut-off to frequencies much lower than λ, the commutator clearly projects to frequencies $|\xi| \sim \lambda$. Thus, one can harmlessly insert fattened projections $\widetilde{\mathsf{P}}_\lambda \widetilde{\mathsf{T}}_{<\lambda}$ in front of the commutator. Furthermore, while two derivatives fall on the wave packet, standard commutator arguments[3] allow us to move one derivative onto the metric. In total, we can rewrite

$$\mathrm{I} = [\mathbf{g}_{<\lambda}^{\alpha\beta}, \mathsf{P}_\lambda \mathsf{T}_{<\lambda}] \partial_\alpha \partial_\beta \widetilde{\mathfrak{w}} = \mathcal{L}(\partial \mathbf{g}, \partial \widetilde{\mathfrak{w}})$$

for another wave packet $\widetilde{\mathfrak{w}}$ and some translation-invariant bilinear operator $\mathcal{L}(-,-)$ with finite-measure kernel. To estimate in $L_{t,x}^2$, we place $\partial \mathbf{g}$ in $L_t^2 L_x^\infty$, gaining smallness from our bootstrap assumption, and $\partial \widetilde{\mathfrak{w}}$ in $L_t^\infty L_x^2$, in which it is unit size by construction (3.5), yielding

$$\|\mathrm{I}\|_{L_{t,x}^2} \lesssim \|\partial \mathbf{g}\|_{L_t^2 L_x^\infty} \|\partial \widetilde{\mathfrak{w}}\|_{L_t^\infty L_x^2} \lesssim \varepsilon_2.$$

Taking $\varepsilon_2 \leq \varepsilon_0$ is an acceptable contribution towards (3.7). □

Estimating the term II. We are left to compute two derivatives of the bump function on $\mathbb{R}^{n-1}$ localised to the null geodesic γ,

$$\partial_\alpha \partial_\beta w = \begin{cases} O(\varepsilon_0 \lambda) & \text{if two spatial derivatives,} \\ O((\varepsilon_0\lambda)^{\frac{1}{2}} \ddot{\gamma}) & \text{if two time derivatives,} \\ O(\varepsilon_0 \lambda \dot{\gamma}) & \end{cases}$$

Since $\|\ddot{\gamma}\|_{L_t^2} \lesssim \varepsilon_1$ this is acceptable. □

Estimating the term III. □

[3] In a word, the principal symbol of the commutator is given by the Poisson bracket, so one can, to leading order, write $[\mathbf{g}(x), \chi(\nabla/\lambda)] \approx \{\mathbf{g}(x), \chi(\xi/\lambda)\} \approx \partial_x \mathbf{g} \cdot \partial_\xi \chi(\xi/\lambda) \approx \frac{1}{\lambda} \partial_x \mathbf{g}$.

Estimating the term IV. Surface measure is an approximate solution.

$$\begin{aligned}\langle \Box_{\mathbf{g}_{<\lambda}} \mathrm{dS}_\Sigma, \varphi\rangle &= \int_\Sigma \partial_\alpha\partial_\beta \left(\mathbf{g}^{\alpha\beta}_{<\lambda}\cdot\varphi\right)\mathrm{dS}_\Sigma \\ &= \int_\Sigma \partial_\alpha\partial_\beta \mathbf{g}^{\alpha\beta}_{<\lambda}\cdot\varphi + \left(\partial_\alpha \mathbf{g}^{\alpha\beta}_{<\lambda}\cdot\partial_\beta\varphi + \partial_\beta \mathbf{g}^{\alpha\beta}_{<\lambda}\cdot\partial_\alpha\varphi\right) + \mathbf{g}^{\alpha\beta}_{<\lambda}\cdot\partial_\alpha\partial_\beta\varphi\,\mathrm{dS}_\Sigma \\ &= \int_\Sigma (\mathbf{g}-\mathbf{g}_{>\lambda})^{\alpha\beta}\,\partial_\alpha\partial_\beta\varphi\,\mathrm{dS}_\Sigma \\ &= \int_{\mathbb{R}^n} \mathbf{g}^{\alpha\beta}_{<\lambda}\partial_\alpha\partial_\beta\varphi\,\sqrt{1+\partial^{\mathbf{a}}\partial_{\mathbf{a}}\tau}\,dx'dt\end{aligned}$$

$(t,x')\mapsto(t,x',\tau(t,x'))$ □

3.2. **Almost orthogonality of wave packets.**

$$\|\partial \mathrm{P}_\lambda\phi\|_{L^2_{t,x}} \lesssim \Big(\sum_{\theta,j}|a_{\theta,j}|^2\Big)^{\frac{1}{2}}. \tag{3.8}$$

Lemma 3.5 (Orthogonality at "good" t)**.**

$$\|\phi(t)\|^2_{L^2_x} \lesssim \sum_{\theta,j}\|\psi^{\theta,j}\|^2_{H^{\frac{n-1}{2}+}(\mathbb{R}^{n-1})} \tag{3.9}$$

Lemma 3.6 (Orthogonality at "bad" t)**.**

$$\|\Phi(t)\|^2_{L^2_x} \lesssim \Big(\frac{1}{\varepsilon_0}\|\partial\mathbf{g}(t)\|_{C^{0,\delta}_x}\Big)^{\frac{n-1}{2}}\sum_{\theta,j}\|\psi^{\theta,j}\|^2_{H^{\frac{n-1}{2}+}(\mathbb{R}^{n-1})} \tag{3.10}$$

It would follow that

$$\|\partial_x \mathrm{P}_\lambda\phi(t)\|^2_{L^2_x} \lesssim \Big(1+\Big(\frac{1}{\varepsilon_0}\|\mathbf{g}(t)\|_{C^{0,\delta}_x}\Big)^{\frac{n-1}{2}}\Big)\sum_{\theta,j}|a_{\theta,j}|^2$$

3.3. **Matching wave packets to initial data.** It remains to show Proposition 3.1 (a)-(b); put loosely, any initial data $(\phi_0,\phi_1)\in(H^1\times L^2)_x(\mathbb{R}^n)$ can be matched at time $t=-2$ to a superposition of wave packets.

Approximate solution for $\Box$. Maximal collection of θ. We decompose

$$\phi[0] = \sum_{\theta\in\Omega}\phi^\theta[0],$$

where

$$\phi^\theta := \frac{1}{2}\left(\phi^\theta_0(x+t\theta)+u_0\right)$$

Approximate solution for $\Box_{\mathbf{g}_{<\lambda}}$. Fourier transform trick, u^ω_0 compact support in frequency, take Fourier transform in x_θ, then extend periodically the Fourier transform with period $\lambda\theta$,

$$\widehat{\phi^\theta} = \sum_{k\in\mathbb{Z}}$$

4. Dispersive estimates

The analysis in the previous sections tell us that the geometry of slabs is approximately that of Minkowski space. Thus, one can expect that the same harmonic analysis counting arguments used to prove Strichartz estimate (or, alternatively, Fourier restriction estimates) hold in this setting.

4.1. **Dispersive decay.**

$$\operatorname{dist}(x_2, C^t_{P_1}) = \inf_{\theta \in \mathbb{S}^{n-1}} |x_2 - \gamma_\theta(t_2)|$$

$$\delta u(P_1, P_2) := \sup_{\theta \in \mathbb{S}^{n-1}} |u_\theta(P_2) - u_\theta(P_1)|$$

Lemma 4.1 (Properties of δu)**.** *The parameter δu is negative if P_2 is inside the cone, positive in the exterior. Furthermore, $\delta u \approx \operatorname{dist}(x_2, C^t_1)$*

Let

$$\#_\lambda(\mathtt{P}_1, \mathtt{P}_2) := \text{\# of slabs at scale } \lambda \text{ containing } \mathtt{P}_1 \text{ and } \mathtt{P}_2.$$

Proposition 4.2 (Dispersive decay, I)**.** *The number of slabs at scale λ containing a pair of points $\mathtt{P}_1 = (t_1, x_1)$ and $\mathtt{P}_2 = (t_2, x_2)$ is*

$$\#_\lambda(\mathtt{P}_1, \mathtt{P}_2) \lesssim \begin{cases} \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}} \left(1 + \lambda \operatorname{dist}(x_2, C^{t_2}_1)\right)^{\frac{n-3}{2}} (1 + \lambda|t_1 - t_2|)^{-\frac{n-1}{2}} & \text{if } m \in I_1, \\ \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}} \left(1 + \lambda \operatorname{dist}(x_2, C^{t_2}_1)\right)^{-1} & \text{if } m \in I_2, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where

$$I_1 := \left\{ -4\lambda^{-1} \le \delta u(\mathtt{P}_1, \mathtt{P}_2) \le \min(2|t_1 - t_2|, C(\lambda\varepsilon_0)^{-1}|t_1 - t_2|^{-1}) \right\}$$
$$I_2 := \left\{ 2|t_1 - t_2| \le \delta u(\mathtt{P}_1, \mathtt{P}_2) \le C(\varepsilon_0\lambda)^{-\frac{1}{2}} \right\}.$$

In actuality we will only need the worst case, i.e. $\delta u \in I_2$.

Corollary 4.3 (Dispersive decay, II)**.** *For all pairs of points P_1, P_2 in space-time, the number of slabs at scale λ containing both is*

$$\#_\lambda(P_1, P_2) \lesssim \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}} (\lambda|t_1 - t_2|)^{-1} \tag{4.2}$$

4.2. **Strichartz estimates.**

Proposition 4.4. *Let*

$$\phi := \sum_{T \in \mathcal{T}} a_T \mathbb{1}_T$$

then

$$\|\phi\|_{L^2_t L^\infty_x} \lesssim (\varepsilon_0\lambda)^{-\frac{1}{2}\frac{n-1}{2}} \lambda^{\frac{1}{2}} \tag{4.3}$$

We proceed by discretising the problem. Dividing $[0, 1]$ into $O(\lambda)$-many sub-intervals I_j of length $2\lambda^{-1}$, we can find points $\mathtt{P}_j = (t_j, x_j)$ nearly maximising $|\phi|$ on each space-time region $I_j \times \mathbb{R}^n$. It follows that

$$\|\phi\|_{L^2_t L^\infty_x} \lesssim \Big(\sum_j \int_{I_j} \|\phi(t)\|^2_{L^\infty_x}\, dt\Big)^{\frac{1}{2}} \lesssim \Big(\sum_j \lambda^{-1} |\phi(t_j, x_j)|^2\Big)^{\frac{1}{2}} \lesssim \lambda^{-\frac{1}{2}} \sum_{T \in \mathcal{T}} \Big(\sum_j |a_{\mathtt{T}}|^2 \cdot |\mathbb{1}_{\mathtt{T}}(t_j, x_j)|^2\Big)^{\frac{1}{2}}.$$

After passing to an $O(\lambda)$ subset of points, we can choose t_j to be λ^{-1}-separated and the inequality above continues to hold. Next, we dyadically decompose the sum over slabs $\mathtt{T}$ with respect to the size $N^{-\frac{1}{2}}$ of the coefficients a_T, and similarly the sum over points $\mathtt{P}_j$ with respect to L the number of slabs containing

them; we denote the number of such points by $m(L)$. It follows that

$$\begin{aligned}
\|\phi\|_{L_t^2 L_x^\infty} &\lesssim \lambda^{-\frac{1}{2}} \sum_{\substack{N\in 2^{\mathbb{N}} \\ N\lesssim (\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}}} \sum_{\substack{\mathtt{T}\in\mathcal{T} \\ |a_{\mathtt{T}}|\sim N^{-\frac{1}{2}}}} \Big(\sum_{\substack{L\in 2^{\mathbb{N}} \\ L\lesssim (\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}}} \sum_{\substack{j \\ \#\{\mathtt{T}\in\mathcal{T}:\mathtt{P}_j\in\mathtt{T}\}\sim L}} |a_{\mathtt{T}}|^2 \cdot |\mathbb{1}_{\mathtt{T}}(t_j,x_j)|^2 \Big)^{\frac{1}{2}} \\
&\lesssim \lambda^{-\frac{1}{2}} \Big(\sum_{L\lesssim (\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}} m(L) \Big| \sum_{N\lesssim (\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}} N^{-\frac{1}{2}} \cdot L \Big|^2 \Big)^{\frac{1}{2}} \\
&\lesssim \lambda^{-\frac{1}{2}} \sum_{N\lesssim (\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}} \Big(\sum_{L\lesssim (\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}}} m(L)^2 N^{-1} L^2 \Big)^{\frac{1}{2}}
\end{aligned}$$

In the above, we needed only to sum over a finite range of scales, since each point lies in $O((\frac{\lambda}{\varepsilon_0})^{\frac{n-1}{2}})$-many slabs. This restricts us to a finite range of scales for L, and allows us to regard the contribution of slabs with small coefficients $|a_T| \lesssim (\frac{\varepsilon_0}{\lambda})^{\frac{n-1}{2}}$ as $O(1)$.

We conclude with a counting argument. To summarise notation and introduce new ones,

$$\begin{aligned}
N &:= \text{size of coefficient}, \\
L &:= \#\text{ of slabs}, \\
m(L) &:= \#\text{ of points intersecting } L\text{-many slabs}, \\
K &:= \#\text{ of pairs } (i,j) \text{ for which } \mathtt{P}_i, \mathtt{P}_j \text{ lie in a common slab w/ multiplicity}, \\
n(\mathtt{T}) &:= \#\text{ of points in slab } \mathtt{T}.
\end{aligned}$$

Observe that

$$\sum_{\substack{\mathtt{T}\in\mathcal{T}_N \\ n(\mathtt{T})\geq 2}} |n(\mathtt{T})|^2 \sim K,$$

$$\sum_{\mathtt{T}\in\mathcal{T}_N} n(\mathtt{T}) \sim m(L)\cdot L.$$

and Cauchy-Schwartz, assuming $|\mathcal{T}_N| \sim N$, This is because $\sum |a_T|^2 \sim \sum_N \sum_{T\in\mathcal{T}_N} N^{-1} \sim 1$

$$\sum_{\substack{\mathtt{T}\in\mathcal{T}_N \\ n(\mathtt{T})\geq 2}} |n(\mathtt{T})|^2 \gtrsim N^{-1} \Big(\sum_{\substack{\mathtt{T}\in\mathcal{T}_N \\ n(\mathtt{T})\geq 2}} n(\mathtt{T}) \Big)^2$$

$$K \lesssim \sum_{i,j} \#_\lambda(\mathtt{P}_i,\mathtt{P}_j) \lesssim \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}} \sum_{1\leq i<j\leq M} |t_i - t_j|^{-1} \lesssim m(L) \left(\frac{\lambda}{\varepsilon_0}\right)^{\frac{n-1}{2}} \log\lambda$$

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