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2.1 Assume that for some $k \in N$ such that $n < k < n+1$, by Lemma 2.1(ii) $k < n+1$ implies either $k < n$ or $k = n$, if $k < n$ by transitivity of $<$ on N and our assumption that $n < k$ we get $n < n$, if $k = n$ again by assumption $n < n$, but $n < n$ contradicts Theorem 2.2.

2.2 Assume to the contrary that $m < n$ but $n < m+1$, but it means that there is some n such that $m < n < m+1$ which contradicts previous exercise. Assume $m < n$, by previous argument $m+1 \leq n$, but $n < n+1$, thus $m+1 < n+1$. assume two distinct natural number m, n then either $m < n$ or $n < m$, so we get $S(m) < S(n)$ or $S(n) < S(m)$, in both case $S(n) \neq S(m)$.

2.3 For every $n \in N$ let $f(n) = S(n)$, therefore $ran\ f = N - \{0\}$ (since otherwise for some k, $0 = S(k) = k+1 = k \cup \{k\}$ implies $k \in 0$) which is a proper subset of N, by previous exercise f is one-to-one because $S(n)$ is one-to-one.

2.4 if $n \in N, n \neq 0$ then $n \in ranf$ in previous exercise, then there is some $k \in N$ such that $f(k) = S(k) = k+1 = n$, because f is one-to-one, k is unique.

2.5 Define function g on N by $g(n) = S(S(n)) = (n+1)+1$, like previous argument we can prove that g is one-to-one and onto $N - \{0, 1\}$, so for ever $n \in N - \{0, 1\}$ we get unique $k \in N$ such that $(k+1)+1 = n$.

2.6 if $m \in N$ and $m < n$ then clearly $m \in n$. we prove it by induction on n that if $m \in n$ then $m \in N$, this is trivially true for $n = 0$. assume the hypothesis and that $m \in n+1$ then either $m = n$ or $m \in n$, if $m = n$ then $m \in N$, since $n \in N$. if $m \in n$ then by induction hypothesis we get $m \in N$.

2.7 Let $x \in m$, since n is the set of natural number less that n and $x < m < n$, we get $x \in n$, also $m < n$ implies $m \in n$ but $m \notin m$, thus $m \subset n$. Now assume $m \subseteq n$, then there is some $q \in n$ such that $q \notin m$, but q is a natural number, thus $q < n$ and $q \not< m$ or equivalently $m < q$, by transitivity $m < n$ which means $m \in n$.

2.8 Assume that there is such function f, then $ranf \subseteq N$ must have a least element u, thus $u = f(k)$ for some $k \in N$, but then definition of f implies $f(k) > f(k+1)$ which contradicts the assumption that $f(k)$ is the least element of $ranf$.

2.9 Let $Y \subseteq X$, but then $X \subseteq N$ implies $Y \subseteq N$ so Y have a least element on order $<$, it means there is some $u \in Y$ such that for every $n \in Y, u < n$, but since $< \cap X^2 \subseteq <$ and $Y \subseteq X$ we conclude that for every $n \in Y, u < \cap X^2 n$.

2.10 Let $X \subseteq A$, then either $X \subseteq N$ or $N \in X$, if $X \subseteq N$ then $\prec$ is ordering of N so it has a least element, if $N \in X$, consider $X - \{N\}$, clearly it has the least element u, because $u \prec N$ it is the least element of X too.

2.11 Assume $P(n)$ does not hold for some $k \leq n$, let X be the set of these elements, by well-ordering it has least element u, (*) for every $k \leq v < u$ we have $P(v)$, if $u = 0$ then $k = 0$ by assumption so it is ordinary induction and we are done, if $u \neq 0$ then for some successor element l, $u = u' + 1$, but since $k < u$, we get $k \leq u'$ then it follows from (*) $P(u')$, but then by (b) $P(u'+1) = P(u)$ holds which contradicts our assumption.

2.12 Assume to the contrary that for some $n \in N, n \leq K$ the property $P(n)$ does not hold, thus the set $X = \{\neg P(n) : (\exists n \in N)(n \leq k)\}$ is non-empty, by well-ordering there is an element $u \in X$ such that is the least element of X. u could not be 0 because $P(0)$ holds, so it is a successor element, thus $u = u' + 1$ for some $u' \in N$. since it is the least element, for every $t < u$, $P(t)$ holds, since $u' < u \leq k$, $P(u')$ holds, then by (b) $P(u' + 1) = P(u)$ holds, a contradiction.

2.13 Assume that for all $l < n, P(m, l)$

fix m_0, we prove $P(m_0, n)$ for all n. assume that for all $l < n, P(m_0, l)$, since for all $l < n$ when $k = m_0$, $P(k, l)$ holds then $P(m_0, n)$ also holds by (**).