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3.1 We must prove that the set $\{x : x \in A \text{ and } x \notin B\}$ exist. Let $P(x, A, B)$ be the property " $x \in A \text{ and } x \notin B$ ", $P(x, A, B)$ implies $x \in A$, because A exist, we have $\{x : x \in A \text{ and } x \notin B\} = \{x \in A : x \in A \text{ and } x \notin B\} = \{x \in A : x \notin B\}$, this set clearly exist by the axiom of comprehension.

3.2 Weak Axiom of Existence implies that some set exist, call one of them A and let $P(x)$ be the property " $x \neq x$ ", by axiom of comprehension the set $X = \{x \in A : x \neq x\}$ exist, it has no element because no object satisfy the property $P(x)$.

3.3 (a) Suppose that V is set of all sets, by Comprehension $X = \{x \in V : x \notin x\}$ exist. Because V is set of all sets, clearly $X \in V$. Now suppose that $X \in X$ then $X \notin X$ by definition, a contradiction. suppose $X \notin X$, then $X \in X$ again by definition.

(b) Assume the contrary, there is a set A that any $x \in A$. then $A = V$ is set of all sets, by previous exercise there is no V .

3.4 By axiom of pairing the set $\{A, B\}$ exist and union axiom implies the existence of $\bigcup\{A, B\}$, let $P(x, A, B) = (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)$ by comprehension there is a set that its elements satisfy $P(x, A, B)$ and $x \in \bigcup\{A, B\}$.

3.5 3.5(a) by axiom of pairing there is $\{A, B\}$ and $\{C\}$. again by pairing $\{\{A, B\}, \{C\}\}$. by axiom of union there is $X = \bigcup\{\{A, B\}, \{C\}\}$. Now $x \in X$ iff $x \in \{A, B\}$ or $x \in \{C\}$ iff $x = A$ or $x = B$ or $x = C$.

(b) Take $\{C, D\}$ instead of $\{C\}$ in the previous exercise.

3.6 Assume that $\mathcal{P}(X) \subseteq X$, Now let $Y = \{x \in X : x \notin x\}$, clearly $Y \subseteq X$, so $Y \in \mathcal{P}(X)$, thus $Y \in X$. also we have either $Y \in Y$ or $Y \notin Y$. if first, $Y \notin Y$, if th second $Y \in Y$, thus $Y \in Y$ iff $Y \notin Y$, a contradiction.

3.7 Let $P(x, A, B)$ be the property " $x = A \vee x = B$ ", apply axiom of comprehension to C , we get the set $X \subseteq C$ such that $x \in X$ iff $x = A$ or $x = B$, so $X = \{A, B\}$.

Let $P'(x, S)$ be the property " $\exists A(A \in S \wedge X \in A)$ ", apply axiom of comprehension to U , we get the set Y such that $x \in Y$ iff for some $A \in S$ we have $x \in A$, thus $Y = \bigcup S$.

Let $P'(x, S)$ be the property " $x \subseteq S$ ", apply axiom of comprehension to P , we get the set Z such that $x \in Z$ iff $x \subseteq S$, thus $Y = \mathcal{P}(S)$.

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4.2 (a) Left to right, assume $A \subseteq B^{(*)}$, and let $x \in A \cap B$, which means that $x \in A$ and $x \in B$, we can conclude $x \in A$, thus $A \cap B \subseteq A^{(**)}$. to prove the other direction, let $x \in A$, by assumption $(*)$ we get $x \in B$, we can conclude $x \in A$ and $x \in B$, which means that $x \in A \cap B$, so we have $A \subseteq A \cap B$, so by this and $(**)$ we have $A = A \cap B$.

Right to left, suppose $A \cap B = A^{(*)}$, let $x \in A$, by $(*)$ $x \in B$, so we have $A \subseteq B$.

Second part, $x \in A \cup B$ iff $x \in B$, it means that there is nothing in A such that is not in B , thus $A - B = \emptyset$.

(b) Left to right, suppose $A \subseteq B \cap C$, let $x \in A$, by previous assumption we have $x \in B \cap C$, which implies that $x \in B$ and $x \in C$, so we have $A \subseteq B$ and $A \subseteq C$.

Right to left, suppose $A \subseteq B$ and $A \subseteq C$, let $x \in A$, by two previous assumption we have both $x \in B$ and $x \in C$ which implies that $x \in B \cap C$, thus we have $A \subseteq B \cap C$.

(c) Suppose $B \cup C \subseteq A$, let $x \in B$, we can get also $x \in B \cup C$, by previous assumption we conclude that $x \in A$, thus $B \subseteq A$. by similar argument we can show $C \subseteq A$.

(d) $x \in A - B$ iff $x \in A \wedge \neg(x \in B)$ iff $x \in A \wedge \neg(x \in B) \vee (x \in B \wedge \neg(x \in B))$ iff $(x \in A \vee x \in B) \wedge \neg(x \in B)$ iff $x \in (A \cup B) - B$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in A))$ iff $x \in A \wedge (\neg(x \in A) \vee \neg(x \in B))$ iff $x \in A - (A \cap B)$.

(e) $x \in A \cap B$ iff $x \in A \wedge x \in B$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge \neg(x \in A))$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge \neg(x \in A))$ iff $x \in A \wedge (x \in B \vee \neg(x \in A))$ iff $x \in A \wedge \neg(\neg(x \in B) \wedge (x \in A))$ iff $x \in A \wedge \neg(x \in A - B)$ iff $x \in A - (A - B)$.

(f) $x \in A - (B - C)$ iff $x \in A \wedge \neg(x \in B - C)$ iff $x \in A \wedge \neg(x \in B \wedge \neg(x \in C))$ iff $x \in A \wedge (\neg(x \in B) \vee (x \in C))$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge x \in C)$ iff $x \in A - B \vee x \in A \cap C$ iff $x \in (A - B) \cup (A \cap C)$.

(g) $(A - B) \cup (B - A) = \emptyset$ iff both $(*) A - B = \emptyset$ and $B - A = \emptyset$, by (a) we get $(*)$ iff $A \subseteq B$ and $B \subseteq A$ iff $A = B$.

4.4 Suppose it exist, then $A' \cup A$ is equal to universal set which does not exist.

4.5 (a) let $x \in A \cap \bigcup S$, then $x \in A$ and $x \in C$ for some $C \in S$, it means that $x \in A \cap C$, clearly $A \cap C \in P(A)$ so $A \cap C \in T_1$ by definition, thus $x \in \bigcup T_1$. (Note that if we take $A \cap C = C$, then we can say that for some $C \in T_1$ we have $x \in C$). Now let $x \in \bigcup T_1$, then there is some $Y \in T_1$ such that $x \in Y$, but by definition of T_1 we know that $Y = A \cap X$ for some $X \in S$, it means that $x \in \bigcup S$ and $x \in A$, thus $x \in A \cap \bigcup S$.

(b) Let $x \in A - \bigcup S$, we have $x \in A - \bigcup S$ iff $x \in A$ and $x \notin X$ for any $X \in S$. it equally means that $(*) x \in A - X$ for every $X \in S$. we know that any set in the form of $A - X$ such that $X \in S$ is in T_2 , thus $(*)$ means that we have $x \in \bigcap T_2$.

$x \in A - \bigcap S$ iff $x \in A$ and $x \notin C$ for some $C \in S$ iff $x \in A - C$ for some $C \in S$, because any set in the form of $A - X$ such that $X \in S$ is in T_2 we have some $x \in \bigcap T_2$.

4.6 if S is not empty, then there is some $C \in S$, by Axiom Schema of Comprehension the set $\{x \in C : (\forall X)(X \in S \rightarrow x \in X)\}$ exist. if it is empty, then we can not apply the axiom of comprehension.

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1.1 1.1 We know that both $\{a\}$ and $\{a, b\}$ are subset of $\{a, b\}$, thus $\{a, b\}, \{a\} \in \mathcal{P}(\{a, b\})$, it means that $\{\{a, b\}, \{a\}\} \subseteq \mathcal{P}(\{a, b\})$ which implies $\{\{a, b\}, \{a\}\} \in \mathcal{P}(\mathcal{P}(\{a, b\}))$.

we have $a, b \in \{a, b\}$, but $(a, b) = \{\{a\}, \{a, b\}\}$ which means that there is some $C \in (a, b)$ such tha $a, b \in C$, thus $a, b \in \bigcup(a, b)$.

if $a, b \in A$ then $\{a, b\}$ and $\{a\}$ both are subset of A , thus $\{a, b\}, \{a\} \in \mathcal{P}(A)$, again it implies that $\{\{a, b\}, \{a\}\} \subseteq \mathcal{P}(A)$, thus $(a, b) = \{\{a, b\}, \{a\}\} \in \mathcal{P}(\mathcal{P}(A))$.

1.2 1.2 if a and b exist, then by axiom of pairing and powerset $T = \mathcal{P}(\mathcal{P}(\{a, b\}))$ exist and by previous exercise $(a, b) \in T$. because $(a, b, c) = ((a, b), c)$ by previous argument we have $(a, b, c) \in \mathcal{P}(\mathcal{P}(\{(a, b), c\}))$ which clearly exist.

1.3 if $(a, b) = (b, a)$, it follows from Theorem 1.2 that $a = b$ and $b = a$, so $a = b$.

1.4 if $(a, b, c) = (a', b', c')$ then $((a, b), c) = ((a', b'), c')$, by Theorem 1.2 we have (*) $(a, b) = (a', b')$ and $c = c'$, but again by Theorem 1.2 and (*) we have $a = a'$ and $b = b'$.

1.5 Let $a = \emptyset$, $b = \{a\}$ and $c = \{b\}$, then if $((a, b), c) = (a, (b, c))$ we get $(a, b) = a = \emptyset = \{\{a\}, \{a, b\}\}$ which is a contradiction.

1.6 We first prove that:

(1) $a = c$ or $d = \square$.

(2) $b = d$ or $c = \triangle$.

To prove (1): $\{\{a, \square\}, \{b, \triangle\}\} = \{\{c, \square\}, \{d, \triangle\}\}$ implies either (•) $\{a, \square\} = \{c, \square\}$ or (★) $\{a, \square\} = \{d, \triangle\}$, if (•) then either $a = c$ or $a = \square$, if first we are done, if the second then $\{a, \square\} = \{\square\} = \{c, \square\}$ which means $a = \square = c$, thus in both case $a = c$. if (★) then either $a = d$ or $a = \triangle$, if first then $\{a, \square\} = \{a, \triangle\}$ which implies $\triangle = \square$, contradiction, so we have $a = \triangle$, then $\{\triangle, \square\} = \{d, \triangle\}$ which implies $d = \square$. so we have either $a = c$ or $d = \square$.

To prove (2):

We also have (*) $\{b, \triangle\} = \{c, \square\}$ or (**) $\{b, \triangle\} = \{d, \triangle\}$, if (*) then either $b = c$ or $b = \square$, if first then $\{b, \triangle\} = \{b, \square\}$ which implies a contradiction: $\triangle = \square$, therefore the second case only remains which implies $c = \triangle$. if (**) then either $b = d$ or $b = \square$, if first we are done, if the second then $\{\square, \triangle\} = \{d, \triangle\}$ which implies $b = \square = d$, so in both case we have $b = d$. so we have either (2) $b = d$ or $c = \triangle$.

So we have (1) and (2), assume that $b = d$ from (2), now consider (1), if first case then we are done. if the second then $b = d = \square$, therefore $\{\{a, \square\}, \{\square, \triangle\}\} = \{\{c, \square\}, \{\square, \triangle\}\}$ which implies $a = c$.

Assume the second case of (2), then by first case of (1) we have $a = c = \triangle$, therefore $\{\{\triangle, \square\}, \{b, \triangle\}\} = \{\{\triangle, \square\}, \{d, \triangle\}\}$ which implies $b = d$.

Now consider the second case of (1), then we have $d = \square$ and $c = \triangle$ then $\{\{a, \square\}, \{b, \triangle\}\} = \{\{\triangle, \square\}, \{\square, \triangle\}\} = \{\{\square, \triangle\}\}$, then $a = \triangle = c$ and $b = \square = d$, we are done.

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2.1 Let $(x, y) = \{\{x\}, \{x, y\}\} \in R$, then $\{\{x\}, \{x, y\}\} \subseteq \bigcup R$, thus we have $\{x, y\} \in \bigcup R$ and we know that $x, y \in \{x, y\}$, so for some set $C \in \bigcup R$ we have $x, y \in C$, thus $x, y \in \bigcup \bigcup R$. because the property " $x \in \text{dom } R$ " implies that $(x, y) \in R$ for some y . and because $(x, y) \in R$ implies $x \in A$, the set $\{x \in A : x \in \text{dom } R\}$ exist. Repeat this argument for property " $x \in \text{ran } R$ ".

2.2 (a) by previous argument $\text{ran } R$ and $\text{dom } R$ exist, we know that $\text{ran } R \times \text{dom } R$ exist, call it A . by comprehension the subset $\{(y, x) \in A : (x, y) \in R\}$ also exist, this set is equal to R^{-1} . again by comprehension the set $\{(x, y) \in \text{dom } R \times \text{ran } S : \text{for some } z, (x, z) \in R \text{ and } (z, y) \in S\}$, this set is equal to $S \circ R$.

(b) Because $A \times B \times C = (A \times B) \times C \subseteq \mathcal{P}((A \times B) \cup C)$, comprehension implies that the set $\{x \in \mathcal{P}((A \times B) \cup C) : x = (y, z) \text{ for some } y \in A \times B \text{ and } z \in C\}$ exist.

2.3 (a) $y \in R[A \cup B]$ iff $(\exists x)(x \in A \cup B \wedge xRy)$ iff $(\exists x)((x \in A \vee x \in B) \wedge xRy)$ iff $(\exists x)((x \in A \wedge xRy) \vee (x \in B \wedge xRy))$ iff $(\exists x)(x \in A \wedge xRy) \vee (\exists x)(x \in B \wedge xRy)$ iff $y \in R[A] \vee y \in R[B]$ iff $y \in R[A] \cup R[B]$.

(b) Let $y \in R[A \cap B]$, then for some $x \in A \cap B$ we have xRy which means that $x \in A$ such that xRy and $x \in B$ such that xRy , thus $x \in R[A] \cap R[B]$.

(c) Suppose that $y \in R[A] - R[B]$, it means there is some $x \in A$ such that xRy but there is no $z \in B$ such that zRy , because xRy holds for x , it can not be in B , thus $x \in A - B$ and xRy which means that $y \in R[A - B]$.

(d) Let $R = \{(a, c), (b, c)\}$ and $A = \{a\}$, $B = \{b\}$ then $R[A] \cap R[B] = \{c\}$ while $R[A \cap B = \emptyset] = \emptyset$. also $R[A - B] = R[\{a\}] = \{c\}$ but $R[A] - R[B] = \{c\} - \{c\} = \emptyset$, so this falsifies converse of both (b) and (c).

(f) Fix $x \in A \cap \text{dom } R$, then because $x \in \text{dom } R$ there is some y such that xRy , because $x \in A$ we conclude that $y \in R[A]$, so there is some $y \in R[A]$ such that xRy or equivalently $yR^{-1}x$, thus $x \in R^{-1}[R[A]]$.

Fix $y \in B \cap \text{ran } R$, since $y \in \text{ran } R$ for some x we have xRy , but $y \in B$ implies that $x \in R^{-1}[B]$, thus for some $x \in R^{-1}[B]$ we have xRy , therefore

$y \in R[R^{-1}[B]]$.

Let $R = \{(a, c), (b, c), (e, f), (e, g)\}$ and $A = \{a\}$, then $A \cap \text{dom } R = \{a\}$ but $R[A] = \{c\}$, thus $R^{-1}[R[A]] = R^{-1}[\{c\}] = \{a, b\}$, but $\{a, b\} \not\subseteq \{a\}$.

Let R be as before and $B = \{g\}$, then $R^{-1}[B] = \{e\}$ and $R[R^{-1}[B]] = \{f, g\}$, but $B \cap \text{ran } R = \{g\}$.

2.4 $R[X] \subseteq \text{ran } R$ because for any $y \in R[X]$ we have some $x \in X$ such that xRy , thus $y \in \text{ran } R$. if $y \in \text{ran } R$, then for some $x \in \text{dom } R$ we have xRy , but $\text{dom } R \subseteq X$, thus $x \in X$, so we get for some $x \in X$, xRy , therefore $y \in R[X]$.

suppose $x \in \text{dom } R$ then there is some $y \in \text{ran } R$ such that xRy , but xRy iff $yR^{-1}x$ and $\text{ran } R \subseteq Y$, therefore there is some $y \in Y$ such that $yR^{-1}x$ which is equal to say that $x \in R^{-1}[Y]$, left to right is trivial.

(b) Assume $a \notin \text{dom } R$ but $R[\{a\}] \neq \emptyset$, so for some $y \in R[\{a\}]$ we have aRy which means that $a \in \text{dom } R$, this contradicts our assumption.

Assume $b \notin \text{ran } R$ and $R^{-1}[\{b\}] \neq \emptyset$, so there is some $x \in R^{-1}[\{b\}]$ such that $bR^{-1}x$ or equivalently xRb , it means that $b \in \text{ran } R$ which contradicts the assumption.

(c) $x \in \text{dom } R$ iff for some y , xRy iff $yR^{-1}x$ iff $x \in \text{ran } R^{-1}$.

$y \in \text{ran } R$ iff for some x , xRy iff $yR^{-1}x$ iff $y \in \text{dom } R^{-1}$.

(d) $(x, y) \in R$ iff $(y, x) \in R^{-1}$ iff $(x, y) \in (R^{-1})^{-1}$.

(e) if $(x, x) \in \text{Id}_{\text{dom } R}$ then $x \in \text{dom } R$ which implies that for some y , $(x, y) \in R$, but $(x, y) \in R$ iff $(y, x) \in R^{-1}$, thus we can say that there is some y such that $(x, y) \in R$ and $(y, x) \in R^{-1}$ which is equal to $(x, x) \in R^{-1} \circ R$. the second part can be proved like this.

2.5 $\mathcal{P}(X) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

$\in_Y = \{(\emptyset, \{\emptyset\}), (\emptyset, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \{\{\emptyset\}\})\}$.

$\text{Id}_Y = \{(\emptyset, \emptyset), (\{\emptyset\}, \{\emptyset\}), (\{\{\emptyset\}\}, \{\{\emptyset\}\}), (\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\})\}$.

$\text{ran}(\text{Id}_Y) = \text{dom}(\text{Id}_Y) = \text{fld}(\text{Id}_Y) = \mathcal{P}(X)$.

$\text{dom}(\in_Y) = \{\emptyset, \{\emptyset\}\}$, $\text{ran}(\in_Y) = \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, $\text{fld}(\in_Y) = \mathcal{P}(X)$.

2.6 $(x, y) \in T \circ (S \circ R)$ iff $(\exists z)((x, z) \in (S \circ R) \wedge (z, y) \in T)$ iff $(\exists z)((\exists u)[(x, u) \in R \wedge (u, z) \in S] \wedge (z, y) \in T)$ iff $(\exists z)((\exists u)[(x, u) \in R \wedge (u, z) \in S \wedge (z, y) \in T])$

iff $(\exists z)(\exists u)((x, u) \in R \wedge (u, z) \in S \wedge (z, y) \in T)$ iff $(\exists u)((x, u) \in R \wedge (\exists z)[(u, z) \in S \wedge (z, y) \in T])$ iff $(\exists u)((x, u) \in R \wedge (u, y) \in T \circ S)$ iff $(x, y) \in (T \circ S) \circ R$.

2.7 Let $X = \{a\}$ and $Y = \{b, c\}$, $Z = \{d\}$.

- (a) $(a, b) \in X \times Y$ but $(a, b) \notin Y \times X$.
- (b) $(a, (b, d)) \in X \times (Y \times Z)$ but $(a, (b, d)) \notin (X \times Y) \times Z$.
- (c) $((a, a), a) \in X^2 \times X$ but $((a, a), a) \notin X \times X^2$.

2.8 (a) Assume $A \neq \emptyset$ and $B \neq \emptyset$, then there is some $a \in A$ and $b \in B$, but then $(a, b) \in A \times B$, so $A \times B \neq \emptyset$. Now assume $A \times B \neq \emptyset$, then there is some $x \in A \times B$ such that $x = (a, b)$, but it means that $a \in A$ and $b \in B$, thus $A, B \neq \emptyset$.

(b) $(a, b) \in (A_1 \cup A_2) \times B$ iff $(a \in A_1 \cup A_2) \wedge b \in B$ iff $(a \in A_1 \vee a \in A_2) \wedge b \in B$ iff $(a \in A_1 \wedge b \in B) \vee (a \in A_2 \wedge b \in B)$ iff $(a, b) \in (A_1 \times B) \vee (a, b) \in (A_2 \times B)$ iff $(a, b) \in (A_1 \times B) \cup (A_2 \times B)$.

$(a, b) \in A \times (B_1 \cup B_2)$ iff $a \in A \wedge b \in (B_1 \cup B_2)$ iff $a \in A \wedge (b \in B_1 \vee b \in B_2)$ iff $(a \in A \wedge b \in B_1) \vee (a \in A \wedge b \in B_2)$ iff $(a, b) \in (A \times B_1) \vee (a, b) \in (A \times B_2)$ iff $(a, b) \in (A \times B_1) \cup (A \times B_2)$.

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3.1 if $\text{ran } f \subseteq \text{dom } g$, then $f^{-1}[\text{ran } f] \subseteq f^{-1}[\text{dom } g]$, but $f^{-1}[\text{ran } f] = \text{dom } f$, by Exercise 4.2(a) on Page 15 we have $\text{dom } f \cap f^{-1}[\text{dom } g] = \text{dom } f$, Theorem 3.5 implies $\text{dom } (g \circ f) = \text{dom } f$.

3.2 $f_2 \circ f_1 = \{\sqrt{2x-1} : x > \frac{1}{2}\}$.

$$f_1 \circ f_2 = \{2\sqrt{x} - 1 : x > 0\}$$

$$f_3 \circ f_1 = \{1/(2x-1) : x \neq \frac{1}{2}\}$$

$$f_1 \circ f_3 = \{2/x - 1 : x \neq 0\}$$

3.3 For f_1 : if $f_1(a) = f_1(b)$ then $2a - 1 = 2b - 1$, by adding 1 to each side of equation we get $2a = 2b$, by dividing by 2 we have $a = b$.

For f_2 : if $f_1(a) = f_1(b)$ then $\sqrt{a} = \sqrt{b}$, but then $a = \sqrt{a} \sqrt{a} = \sqrt{a} \sqrt{b} = \sqrt{b} \sqrt{b} = b$.

For f_3 : if $f_1(a) = f_1(b)$ then $1/a = 1/b$, because a, b are non-zero multiplying by ab yields $a = b$.

$$f_1^{-1} = \{(x+1)/2 : x \text{ is real}\}$$

$$f_2^{-1} = \{x^2 : x > 0\}$$

$$f_3^{-1} = \{1/x : x \neq 0\}$$

3.4 (a) Assume that f is invertible, let $(a, b) \in f^{-1} \circ f$ then for some z we have (*) $(a, z) \in f$ and $(z, b) \in f^{-1}$, then from (*) we also have $(z, a) \in f^{-1}$, by assumption f^{-1} is a function, so we get $a = b$, because $a \in \text{dom } f$ we get $(a, b) = (a, a) \in \text{Id}_{\text{dom } f}$. the other side holds by Exercise 2.4(e) on Page 23.

(b) Let $(a, b), (a, c) \in f^{-1}$, then $(b, a), (c, a) \in f$, thus $f(b) = a$ and $f(c) = a$ but (*) $g \circ f = \text{Id}_{\text{dom } f}$ implies $g(f(b)) = b = g(a) = g(f(c)) = c$, therefore $b = c$ and f^{-1} is a function. let $(a, b) \in f^{-1}$ then $(b, a) \in f$, so $f(b) = a$, by (*) we get $g(f(b)) = b = g(a)$, thus $(a, b) \in g$, but we also know that $a \in \text{ran } f$, therefore $(a, b) \in g \mid \text{ran } f$. Now let $(a, b) \in g \mid \text{ran } f$, then $g(a) = b$ and also $a \in \text{ran } f$, then $f(k) = a$ for some $k \in \text{dom } f$, but (*) implies $g(f(k)) = g(a) = b = k$ which means that $(b, a) \in f$, $(a, b) \in f^{-1}$.

We give a counter example for the second one, let $f = \{(a, a), (b, a)\}$ and $h = \{(a, a)\}$ then $f \circ h = \{(a, a)\} = \text{Id}_{\text{ran } f}$ but clearly f^{-1} is not a function.

3.5 Let $(g \circ f)(a) = (g \circ f)(b)$, then $g(f(a)) = g(f(b))$ since g is one-to-one we get $f(a) = f(b)$, again because f is one-to-one we have $a = b$.

let $(a, b) \in (f \circ g)^{-1}$, thus $(b, a) \in f \circ g$, it means that for some z we have $(b, z) \in g$ and $(z, a) \in f$, equivalently we have $(a, z) \in f^{-1}$ and $(z, b) \in g^{-1}$ for some z , by definition of composition we get $(a, b) \in g^{-1} \circ f^{-1}$.

3.6 We just need prove right to left of (a) and left to right of (b).

(a) Suppose $x \in f^{-1}[A] \cap f^{-1}[B]$, then for some $y \in A$ we have $yf^{-1}x$ or equivalently $f(x) = y$ and for some $z \in B$, $f(x) = z$, but since f is a function we conclude that $z = y \in A \cap B$, then we can say that for some $y \in A \cap B$, $yf^{-1}x$ holds, therefore $x \in f^{-1}[A \cap B]$.

(b) Let $x \in f^{-1}[A - B]$, then there is some $y \in A - B$ such that $yf^{-1}x$ or equivalently (*) $f(x) = y$, clearly $x \in f^{-1}[A]$, we must prove that $x \notin f^{-1}[B]$ or equivalently there is no $z \in B$ such that $zf^{-1}x$, assume to the contrary that it exists, so we get $f(x) = z$, but (*) implies $z = y \in B$, it contradicts our assumption that $y \in A - B$.

3.7 let $f = \{(a, b)\}$ and $A = \{a\}$, then $f \cap A^2 = \emptyset$ but $f|A = f$.

3.8 Let $I = A$ and $S = \text{Id}_I$, then $S = (S_i, i \in I)$ is an indexed function such that $S_i = i$.

3.9 (a) Let $f : A \rightarrow B$, then $f \subseteq A \times B$, thus $f \in \mathcal{P}(A \times B)$, now let $P(x)$ be the property " $(\forall a, b, c)[(a, b), (a, c) \in x \rightarrow b = c] \wedge (\forall a)(a \in A \rightarrow (\exists b)[b \in B \wedge (a, b) \in x])$ ", then $\{x \in \mathcal{P}(A \times B) : P(x)\}$ is the set of all function from A to B.

(b) Let f be a member of product of an indexed system $(S_i : i \in I)$, then $f : I \rightarrow \bigcup_{i \in I} S_i$ such that for every $i \in I$, $f(i) \in S_i$, then clearly $f \in (\bigcup_{i \in I} S_i)^I$, by previous exercise we know that it exists, now by comprehension we have $\prod_{i \in I} S_i = \{f \in (\bigcup_{i \in I} S_i)^I : (\forall i \in I)[f(i) \in S_i]\}$, clearly if it is non-empty, every member of it is a function such that satisfies the condition of a product.

3.10 $x \in \bigcup_{a \in \bigcup S} F_a$ iff $(\exists a)[a \in \bigcup S \wedge x \in F_a]$ iff $(\exists a)[(\exists C)(C \in S \wedge a \in C) \wedge x \in F_a]$ iff $(\exists a)[(\exists C)(C \in S \wedge a \in C \wedge x \in F_a)]$ iff $(\exists C)[(\exists a)(C \in S \wedge a \in C \wedge x \in F_a)]$ iff $(\exists C)[C \in S \wedge (\exists a)(a \in C \wedge x \in F_a)]$ iff $(\exists C)[C \in S \wedge x \in \bigcup_{a \in C} F_a]$ iff $x \in \bigcup_{C \in S} (\bigcup_{a \in C} F_a)$.

Let $x \in \bigcap_{a \in \bigcup S} F_a$ then $(*) (\forall a)[a \in \bigcup S \rightarrow x \in F_a]$. Now let $C \in S$, then because $C \subseteq \bigcup S$ we get that for every $a \in C$, $x \in F_a$, because C was arbitrary we can conclude that $(**) (\forall C)[C \in S \rightarrow (\forall a)(a \in C \rightarrow x \in F_a)]$, which is equal to $(\forall C)[C \in S \rightarrow x \in \bigcap_{a \in C} F_a]$, thus $x \in \bigcup_{C \in S} (\bigcap_{a \in C} F_a)$. Now let $x \in \bigcup_{C \in S} (\bigcap_{a \in C} F_a)$, then we get $(**)$, let $a \in \bigcup S$, then there is some $C \in S$ such that $a \in C$, but then by $(**)$ we get $(\forall a)(a \in C \rightarrow x \in F_a)$ and then $x \in F_a$, because a was arbitrary we proved $(*)$, thus $x \in \bigcap_{a \in \bigcup S} F_a$.

3.11 $x \in B - \bigcup_{a \in A} F_a$ then $x \in B$ and for every $a \in A$, $x \notin F_a$, also for every $a \in A$, $x \notin F_a$ and $x \in B$, so for every $a \in A$, $x \in B - F_a$, thus $x \in \bigcap_{a \in A} (B - F_a)$. Now let $x \in \bigcap_{a \in A} (B - F_a)$, then for every $a \in A$, $x \in B$ and $x \notin F_a$,

let $a \in A$, then by above claim $x \notin F_a$, thus $x \notin \bigcup_{a \in A} F_a$, Now assume to the contrary that $x \notin B$, then it implies there is no $a \in A$, $A = \emptyset$ which is a contradiction.

Let $x \in B - \bigcap_{a \in A} F_a$, then $(*) x \in B$ and there is some $a \in A$ such that $x \notin F_a$, by $(*)$ we can claim that there is some $a \in A$ such that $x \in B - F_a$, thus $x \in \bigcup_{a \in A} (B - F_a)$. Now let $x \in \bigcup_{a \in A} (B - F_a)$, then $x \in (B - F_a)$ for some $a \in A$, it follows that there is some $a \in A$ such that $x \in F_a$, thus $x \notin \bigcap_{a \in A} F_a$ and clearly $x \in B$, thus $x \in B - \bigcap_{a \in A} F_a$.

Let $x \in \bigcup_{a \in A} F_a \cap \bigcup_{b \in B} G_b$, then for some $a \in A$, $x \in F_a$ and for some $b \in B$, $x \in G_b$, clearly $(a, b) \in A \times B$, then we can say for some $(a, b) \in A \times B$, $x \in F_a \cap G_b$

$x \in \bigcup_{a \in A} F_a \cap \bigcup_{b \in B} G_b$ iff $(\exists a)(a \in A \wedge x \in F_a) \wedge (\exists b)(b \in B \wedge x \in F_b)$ iff $(\exists a)(\exists b)[(a \in A \wedge x \in F_a) \wedge b \in B \wedge x \in F_b]$ iff $(\exists a)(\exists b)[(a, b) \in A \times B \wedge x \in F_a \cap F_b]$ iff $x \in \bigcup_{(a,b) \in A \times B} (F_a \cap G_b)$

3.12 (We just prove the first and the third case)

$y \in f[\bigcup_{a \in A} F_a]$ iff $(\exists x)[x \in \bigcup_{a \in A} F_a \wedge f(x) = y]$ iff $(\exists x)[(\exists a)(a \in A \wedge x \in F_a) \wedge f(x) = y]$ iff $(\exists x)[(\exists a)(a \in A \wedge x \in F_a \wedge f(x) = y)]$ iff $(\exists x)(\exists a)[a \in A \wedge x \in F_a \wedge f(x) = y]$ iff $(\exists a)(\exists x)[a \in A \wedge x \in F_a \wedge f(x) = y]$ iff $(\exists a)[a \in A \wedge (\exists x)(x \in F_a \wedge f(x) = y)]$ iff $(\exists a)[a \in A \wedge y \in f[F_a]]$ iff $y \in \bigcup_{a \in A} f[F_a]$.

Let $y \in f[\bigcap_{a \in A} F_a]$, then for some $x \in \bigcap_{a \in A} F_a$, $f(x) = y$, but it means for every $a \in A$, $x \in F_a$ and $f(x) = y$, we can say for every $a \in A$, there is some $x \in F_a$ such that $f(x) = y$ or equally $y \in f[F_a]$, thus $y \in \bigcap_{a \in A} f[F_a]$.

(if f is one-to-one, \subseteq can be replaced by $=$): Now let $y \in \bigcap_{a \in A} f[F_a]$, so for every $a \in A$, there is some $x \in F_a$ such that $f(x) = y$, but because f is one-to-one this x must be unique, name it k , so for every $a \in A$, $k \in F_a$ or equivalently $k \in \bigcap_{a \in A} F_a$, since $f(k) = y$ we get $y \in f[\bigcap_{a \in A} F_a]$.

3.13 Right to left is easy according to Hint, we prove left to right side:

Let $x \in \bigcap_{a \in A} (\bigcup_{b \in B} F_{a,b})$, define f such that $(a, b) \in f$ iff $x \in F_{a,b}$, we prove $f \in B^A$, let $(x, y), (x, z) \in f$ be two distinct member, then $x \in F_{x,y} \cap F_{x,z}$ but because $y \neq z$ we have $F_{x,y} \cap F_{x,z} = \emptyset$, thus it contradicts our assumption, hence f is a function.

From assumption for every $a \in A$ we have $x \in \bigcup_{b \in B} F_{a,b}$, fix arbitrary $a \in A$, then $x \in F_{a,b}$ for some $b \in B$, but by definition of f we have $f(a) = b$, thus $x \in F_{a,f(a)}$, because a was arbitrary we can say $x \in \bigcap_{a \in A} F_{a,f(a)}$ for f , thus $x \in \bigcup_{f \in B^A} (\bigcap_{a \in A} F_{a,f(a)})$.

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4.1 (a) transitive.

(b) reflexive, transitive.

(c) symmetric.

(d) \subseteq : reflexive, transitive. \subset : transitive.

(e) reflexive, transitive, symmetric.

(f) symmetric and transitive.

4.2 (a) for every $a \in A$, $f(a) = f(a)$ since f is a function, thus aEa and E is reflexive. Let aEb , then for some $a, b \in A$ we have $f(a) = f(b)$ but also $f(b) = f(a)$, thus bEa , therefore E is symmetric. Suppose that aEb and bEc , then we get $f(a) = f(b)$ and $f(b) = f(c)$, since f is a function we have $f(a) = f(c)$, thus aEc , E is transitive.

(b) We define $\phi : A/E \rightarrow B$ such that $\phi([a]_E) = f(a)$ for every $[a]_E \in A/E$, if $[a]_E = [a']_E$ then aEa' , by definition we get $f(a) = f(a')$ which means that $\phi([a]_E) = \phi([a']_E)$.

(c) for every $a \in A$ we have $\phi \circ j(x) = \phi(j(a)) = \phi([a]_E) = f(a)$, because f and j have the same domain we can conclude $\phi \circ j = f$.

4.2 Because for every $(r, \gamma) \in P$ we have $r = r$ and $\gamma - \gamma = 0 = 2\pi \times 0$ which 0 is an integer multiple of 2π , we get $(r, \gamma) \sim (r, \gamma)$. Now let $(r, \gamma) \sim (r', \gamma')$, then $r = r'$ and $\gamma - \gamma' = 2\pi k$ is an integer, because $\gamma' - \gamma = -(\gamma - \gamma') = 2\pi(-k)$ is also an integer, together with symmetricity of \sim we get $r' = r$, so we conclude that $(r', \gamma') \sim (r, \gamma)$, thus \sim is symmetric.

Let $(r, \gamma) \sim (r', \gamma')$ and $(r', \gamma') \sim (r'', \gamma'')$, by transitivity of identity we simply get $r = r''$, also $\gamma - \gamma' = 2\pi k$ and $\gamma' - \gamma'' = 2\pi k'$ such that k, k' are both integer, but then $\gamma - \gamma'' = (\gamma - \gamma') + (\gamma' - \gamma'') = 2\pi k + 2\pi k' = 2\pi(k + k')$ clearly is an integer, thus $(r, \gamma) \sim (r'', \gamma'')$.

Consider (r, γ) , then there is some (r, γ') such that $\gamma - \gamma' = 2\pi k$ and is an integer. then $\gamma' = \gamma - 2\pi k$, we argue that for some integer k' we have $0 \leq \gamma - 2\pi k' \leq 2\pi$, if there is no such k' that satisfies last inequality then we also do not have $-\gamma \leq -2\pi k' \leq 2\pi - \gamma$ and also $\gamma - 2\pi \leq 2\pi k' \leq \gamma$, dividing by 2π yields that there is no $\gamma/2\pi - 1 \leq k' \leq \gamma/2\pi$, but it contradicts the fact tht for any real number X there is an integer $X - 1 \leq k' \leq X$, so we can take $(r, \gamma - 2\pi k')$.

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5.1 (a) For any $a, b \in A$, aSb iff $aRb \wedge a \neq b$, but aR^*b iff $aSb \vee a = b$ iff $(aRb \wedge a \neq b) \vee a = b$ iff $aRb \vee a = b$ iff aRb (since R is reflexive $a = b$ implies aRb).

(b) aRb iff $aSb \vee a = b$, but aS^*b iff $aRb \wedge a \neq b$ iff $(aSb \vee a = b) \wedge a \neq b$ iff $(aSb \wedge a \neq b)$ iff aSb (since S is irreflexive).

5.2 a and b are incomparable if $a \neq b$, neither $a < b$ nor $b < a$.

a is maximal in A : $\neg(\exists x \in A)(a < x)$
 a is the greatest element of A : $(\forall x \in A)(x < a \vee x = a)$.
 a is an upper bound of A : $(\forall x \in A)(x < a \vee x = a)$
 a is supremum of A in X : $(\forall x \in A)(x < a \vee x = a) \wedge (\forall a' \in X)(\forall x \in A)[(x < a' \vee x = a') \rightarrow a < a' \vee a = a']$.

5.3 (a) for any $a \in A$ we have aRa , also $aR^{-1}a$, thus R^{-1} is reflexive. Suppose that $aR^{-1}b$ and $bR^{-1}a$, then we have bRa, aRb , by antisymmetry of R we get $a = b$, therefore R^{-1} is antisymmetric.

Now suppose $aR^{-1}b$ and $bR^{-1}c$, then we get bRa and cRb , but transitivity of R implies cRa , thus $aR^{-1}b$, thus R is transitive.

(a) $(\forall x \in B)(aR^{-1}x) \text{ iff } (\forall x \in B)(xRa)$.

5.4 Let $R' = R \cap B^2$, then for every $a \in B$ we have $(a, a) \in B^2$, since $B \subseteq A$ and R is an order on A , by reflexivity $(a, a) \in R$, thus $aR'a$, hence R' is reflexive.

Let $aR'b, bR'a$ then $a, b, c \in B$ and aRb, bRa , because R is antisymmetric we get $a = b$,

5.5 Let $A = \mathcal{P}(\{a, b, c\}) - \{\{a, b, c\}, \emptyset\}$ and $R = \subseteq$.

- (a) $B = \{\{a\}, \{b\}\}$.
- (b) $B = \{\{a\}, \{b\}\}$.
- (c) $B = \{\{a\}, \{b\}\}$.
- (d) $B = \{\{a, b\}, \{b, c\}\}$.

5.6 (a) For every $x \in B$ either $x = b$ or $x \in A$, if $x = b$ then $x \notin A$ and both disjunct in the definition of \prec would be false, if $x \in A$ then $x \not\prec x$ because $<$ is irreflexive, so the first disjunct could not be true, the other disjunct require that $x = b$ but it is impossible because $x \in A$, hence both of them is false, thus $x \not\prec x$.

Now let $x \prec y$, if $x \in A$ and $y = b$ then clearly $x \neq b$ because $b \notin A$, so we can not have $y \in A$ and $x = b$ and also we can not have $y = b, x \in A$ (the first item of first disjunct), thus $y \not\prec x$. Assume that $x \prec y, y \prec z$, if x, y, z all are in A then $x \prec z$ easily follows from transitivity of $<$. but if $x \in A$ and $y = b$ then $y \prec z$ is impossible, because in both disjunct it requires that $y \in A$, but $y = b$. So the only case we need to check is that when $x, y \in A$ and $x < y$ and $y \in A$ and $z = b$, but from this it easy follows

that $x \in A$ and $z = b$, thus $x \prec z$. Notice that $\prec = \prec \cup (A \times \{b\})$ and $\prec \subset A^2$, but $(A \times \{b\}) \cap A^2 = \emptyset$, thus $\prec \cap A^2 = (\prec \cap A^2) \cup (A \times \{b\}) \cap A^2 = \prec \cup \emptyset = \prec$.

5.7 Because R is reflexive for every $a \in A$, aRa and also aRa which implies that aEa . Now let aEb then aRb and bRa , also bRa and aRb , thus bEa . Let aEb, bEc then aRb, bRa and bRc, cRb , by transitivity of R we get aRc and cRa thus aEc , hence E is transitive.

Assume aRb , then $[a]_E R/E [b]_E$, now let $b' \in [b]_E$, then bEb' , hence bRb' , by transitivity of R we get aRb' , hence $[a]_E R/E [b']_E$. we can repeat this argument for a .

Because R is reflexive for every a , we have aRa , also $[a]_E R/E [a]_E$. Assume that $[a]_E R/E [b]_E$ and $[b]_E R/E [a]_E$, then we get aRb and bRa , hence aEb which means that $[a]_E = [b]_E$.

To prove that R/E is transitive, assume $[a]_E R/E [b]_E$ and $[b]_E R/E [c]_E$, then we get aRb and bRc , by transitivity of R it follows that aRc , hence $[a]_E R/E [c]_E$.

5.8 (a) Let $S \subseteq A$, then every $x \in S$, $x \subseteq \bigcup S$, thus $\bigcup S$ is an upper bound of S , to prove that it is the least upper bound assume we prove that it is subset of every upper bound a , i.e $\bigcup S \subseteq a$, let $x \in \bigcup S$, then for some $C \in S$, $x \in C$, but a is an upper bound for S , thus $C \subseteq a$, hence $x \in a$, thus $\bigcup S \subseteq a$.

(b) The set of all lower bounds of \emptyset is the set of all $a \in A$ such that for every $x \in \emptyset$, $a \subseteq x$, so all member of A satisfy this condition because $x \notin \emptyset$, the greatest element of A is X , since $Y \subseteq X$ for any $Y \in A = \mathcal{P}(X)$.

5.9 (a) \subseteq is reflexive, antisymmetric and transitive on any set, thus it is an ordering.

(b) Let $F \subseteq Fn(X, Y)$, assume that $\sup F$ exist, and F is not compatible, then there are some $g, f \in F$ such that for some $x \in \text{dom } f \cap \text{dom } g$, $f(x) \neq g(x)$, hence there are distinct $a, b \in Y$ such that $(x, a) \in f$ and $(x, b) \in g$, but then since $f, g \subseteq \sup F$, hence $(x, b), (x, a) \in \sup F$, it contradicts the fact that $\sup F$ is a function. Now assume that F is a compatible system of functions. then by Theorem 3.12 $\bigcup F$ is a function and clearly $\bigcup F \in Fn(X, Y)$, it follows from a similar argument to Exercise 5.8(a) that $\bigcup F = \sup F$.

5.10 (a) Because for every $S \in Pt(A)$ we have that for all $C \in S$, there is some $D \in S$ such that $C \subseteq D$, namely C itself, thus $S \preceq S$ for all $S \in Pt(A)$ and it is reflexive.

Assume that $S_1 \preceq S_2$ and $S_2 \preceq S_1$, then for every $C \in S_1$, $C \subseteq D$ for some $D \in S_2$, but because $S_2 \preceq S_1$ and $D \in S_2$, we have some $E \in S_1$ such that $D \subseteq E$, we show that $E = C$. Assume it is not the case, then $C \subseteq D \subseteq E$ implies $C \cap E \neq \emptyset$, contrary to the assumption that S is a partition, thus the relation is symmetric.

Let $S_1 \preceq S_2$ and $S_2 \preceq S_3$, then for every $C \in S_1$, $C \subseteq D$ for some $D \in S_2$, but because $S_2 \preceq S_3$, there is some $E \in S_3$ such that $D \subseteq E$, thus for every $C \in S_1$, $C \subseteq E$ for some $E \in S_3$, therefore $S_1 \preceq S_3$ and \preceq is transitive.

(b) Let $S = \{C \cap D : C \in S_1 \wedge D \in S_2\}$, clearly S is a partition and $S \preceq S_1$, $S \preceq S_2$, thus S is a lower bound for $\{S_1, S_2\}$, we prove that any lower bound $S' \preceq S$. Assume S' is a lower bound, then for every $C \in S'$, $C \subseteq D$ for some $D \in S_1$ and also $C \subseteq D'$ for some $D' \in S_2$ but then there is some $X \in S$ and $C \subseteq X$, namely $X = D \cap D'$, so we proved that for every $C \in S'$, $C \subseteq X$ for some $X \in S$, thus $S' \preceq S$.

$aE_S b$ implies $aE_{S_1} b, aE_{S_2} b$

(c) Let $T = (T_i : i \in I)$ and $S = \{(\bigcap_{i \in I} f_i) : f \in \prod T_i\}$, fix some $T_k \in T$, we want to prove that $S \preceq T_k$. let $C \in S$ and $x \in C$ then $x \in f_i$ for all $i \in I$, but $f_i = D$ for some $D \in T_i$, from this it follows that $x \in f_k$, thus for some $D \in T_k$ we have $x \in D$, thus $C \subseteq D$, we conclude that $S \preceq T_k$. We prove S is greatest lower bound, assume S' is another lower bound for T , it means that for every $C \in S'$ and for every T_i there is some $D_i \in T_i$ such that $C \subseteq D_i$, define $f : I \rightarrow \bigcup T$ by $f_i = D_i$, then clearly $C \subseteq \bigcap_{i \in I} f_i$, but $\bigcap_{i \in I} f_i \in S$, thus $S' \preceq S$.

(d) Let $T' = \{S \in Pt(A) : (\forall i \in I)(T_i \preceq S)\}$ clearly it is the set of upper bounds of T , by previous exercise $\inf T'$ exist, we prove that $\inf T' \in T'$, fix some $T_k \in T$, and let $C \in T_k$, we know that for every $S \in T'$ we have $T_k \preceq S$, it means that for every $S \in T'$ there is some $D \in S$ such that $C \subseteq D$, if we index T' by J , we have for every $T'_j \in T'$ there is some $D_{T'_j} \in T'_j$ such that $C \subseteq D_{T'_j}$, define $f : J \rightarrow \bigcup T'$ by $f_j = D_{T'_j}$ then clearly $C \subseteq \bigcap_{j \in J} f_j \in \inf T'$, thus we proved for arbitrary $T_k, T_k \preceq \inf T'$, thus $\inf T' \in T'$ and it is the least element of it, the least element among upper bound of T , thus $\sup T = \inf T'$.

5.11 Let f be the isomorphism, let $y_1, y_2 \in Q$ then there is some $x_1, x_2 \in P$ such that $f(x_1) = y_1, f(x_2) = y_2$ but because $<$ is linearly ordered we have either $x_1 = x_2$ or $x_1 < x_2$ or $x_2 < x_1$, but because f is an isomorphism we get either $f(x_1) = f(x_2)$ or $f(x_1) \prec f(x_2)$ or $f(x_2) \prec f(x_1)$, rewrite this for y_1 and y_2 .

5.12 Suppose that $x <_1 y$ for some $x, y \in P_1$ then we have $f(x) <_2 f(y)$, but since $f(x), f(y) \in P_2$ we get $g(f(x)) <_3 g(f(y))$, thus $g \circ f(x) <_3 g \circ f(y)$. Now let $u <_3 z$ for some $u, z \in P_3$, because g is an isomorphism there are some $t, v \in P_2$ such that $f(t) = u, f(v) = z$ and $t <_2 v$, but because f is isomorphism we get $f(x) = t <_2 v = f(y)$ which implies $x <_1 y$.

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2.1 Assume that for some $k \in N$ such that $n < k < n + 1$, by Lemma 2.1(ii) $k < n + 1$ implies either $k < n$ or $k = n$, if $k < n$ by transitivity of $<$ on N and our assumption that $n < k$ we get $n < n$, if $k = n$ again by assumption $n < n$, but $n < n$ contradicts Theorem 2.2.

2.2 Assume to the contrary that $m < n$ but $n < m + 1$, but it means that there is some n such that $m < n < m + 1$ which contradicts previous exercise. Assume $m < n$, by previous argument $m + 1 \leq n$, but $n < n + 1$, thus $m + 1 < n + 1$. assume two distinct natural number m, n then either $m < n$ or $n < m$, so we get $S(m) < S(n)$ or $S(n) < S(m)$, in both case $S(n) \neq S(m)$.

2.3 For every $n \in N$ let $f(n) = S(n)$, therefore $\text{ran } f = N - \{0\}$ (since otherwise for some $k, 0 = S(k) = k + 1 = k \cup \{k\}$ implies $k \in 0$) which is a proper subset of N , by previous exercise f is one-to-one because $S(n)$ is one-to-one.

2.4 if $n \in N, n \neq 0$ then $n \in \text{ran } f$ in previous exercise, then there is some $k \in N$ such that $f(k) = S(k) = k + 1 = n$, because f is one-to-one, k is unique.

2.5 Define function g on N by $g(n) = S(S(n)) = (n+1) + 1$, like previous argument we can prove that g is one-to-one and onto $N - \{0, 1\}$, so for ever $n \in N - \{0, 1\}$ we get unique $k \in N$ such that $(k+1) + 1 = n$.

2.6 if $m \in N$ and $m < n$ then clearly $m \in n$. we prove it by induction on n that if $m \in n$ then $m \in N$, this is trivially true for $n = 0$. assume the hypothesis and that $m \in n+1$ then either $m = n$ or $m \in n$, if $m = n$ then $m \in N$, since $n \in N$. if $m \in n$ then by induction hypothesis we get $m \in N$.

2.7 Let $x \in m$, since n is the set of natural number less than n and $x < m < n$, we get $x \in n$, also $m < n$ implies $m \in n$ but $m \notin m$, thus $m \subset n$. Now assume $m \subseteq n$, then there is some $q \in n$ such that $q \notin m$, but q is a natural number, thus $q < n$ and $q \not\in m$ or equivalently $m < q$, by transitivity $m < n$ which means $m \in n$.

2.8 Assume that there is such function f , then $\text{ran} f \subseteq N$ must have a least element u , thus $u = f(k)$ for some $k \in N$, but then definition of f implies $f(k) > f(k+1)$ which contradicts the assumption that $f(k)$ is the least element of $\text{ran} f$.

2.9 Let $Y \subseteq X$, but then $X \subseteq N$ implies $Y \subseteq N$ so Y have a least element on order $<$, it means there is some $u \in Y$ such that for every $n \in Y, u < n$, but since $< \cap X^2 \subseteq <$ and $Y \subseteq X$ we conclude that for every $n \in Y, u < \cap X^2 n$.