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4.2 (a) $A \subseteq B$ iff $A \cap B = A$: Assume that $A \subseteq B$ (*), it is clear that $A \cap B \subset A$, let $x \in A$, then by (*) $x \in B$, thus $x \in A \cap B$, therefore $A \cap B = A$. For the converse, assume $A = A \cap B$ and let $x \in A$, then $x \in A \cap B$, which means $x \in B$, thus $A \subseteq B$.

$A \cap B = A$ iff $A \cup B = B$: Suppose $A \cap B = A$ (*), it is clear that $B \subseteq A \cup B$, let $x \in A \cup B$, if $x \in B$ we are done, if $x \in A$, then by (*) and previous item we get $A \subseteq B$, thus in either case $x \in B$, therefore $A \cup B \subseteq B$, hence $A \cup B = B$. Conversely, suppose $A \cup B = B$, and let $x \in A$, then $x \in B$, thus $A \subseteq B$, by previous item we get $A \cap B = A$.

$A \cup B = B$ iff $A - B = \emptyset$: Assume $A \cup B = B$, and some $x \in A - B$, thus $x \in A$ but $x \notin B$, but $x \in A$ implies $x \in A \cup B$, by assumption $x \in B$ which is a contradiction. For the converse, suppose $A - B = \emptyset$, we show $A \cup B \subseteq B$ (the other side is clear). let $x \in A \cup B$, if $x \in B$ we are done, let $x \in A$, and to the contrary $x \notin B$, then we get $x \in A - B$ which is a contradiction.

(b) Suppose that $A \subseteq B \cap C$, let $x \in A$, then $x \in B$, thus $A \subseteq B$. This also implies $x \in C$, thus $A \subseteq C$. Conversely, suppose $A \subseteq C$ and $A \subseteq B$, then $x \in A$ implies $x \in B$ and $x \in C$, thus $x \in B \cap C$.

(c) Assume $B \cup C \subseteq A$, let $x \in B$, then $x \in B \cup C$, thus $x \in A$, therefore $B \subseteq A$. if $x \in C$, again, $x \in B \cup C$, thus we get $x \in A$, thus $C \subseteq A$. For the converse, assume that $B \subseteq A$ and $C \subseteq A$. Let $x \in B \cup C$, then either $x \in B$ or $x \in C$, in either case by the assumption we get $x \in A$, therefore $B \cup C \subseteq A$.

(d) $x \in A - B$ iff $x \in A \wedge \neg(x \in B)$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in B \wedge \neg(x \in B))$ iff $(x \in A \vee x \in B) \wedge \neg(x \in B)$ iff $x \in (A \cup B) - B$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in A)$ iff $x \in A \wedge (\neg(x \in A)) \vee \neg(x \in B)$ iff $x \in A - (A \cap B)$.

(e) $x \in A \cap B$ iff $x \in A \wedge x \in B$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge \neg(x \in A))$ iff $x \in A \wedge (x \in B \vee \neg(x \in A))$ iff $x \in A \wedge \neg(\neg(x \in B) \wedge x \in A)$ iff $x \in A \wedge \neg(x \in A - B)$ iff $x \in A - (A - B)$.

(f) $x \in A - (B - C)$ iff $x \in A \wedge \neg(x \in B - C)$ iff $\in A \wedge \neg(x \in B \wedge \neg(x \in C))$ iff $x \in A \wedge (\neg(x \in B) \vee (x \in C))$ iff $(x \in A \neg(x \in B)) \vee (x \in A \wedge x \in C)$ iff $(x \in A - B) \vee (x \in A \cap C)$ iff $x \in (A - B) \cup (A \cap C)$.

(g) $(A - B) \cup (B - A) = \emptyset$ iff both $A - B = \emptyset$ and $B - A = \emptyset$. But by item (a), $A - B = \emptyset$ and $B - A = \emptyset$ iff $A \subseteq B$ and $B \subseteq A$, thus $A = B$.