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5.3 (a) By Lemma 5.4(c) we have $(\omega+1)+\omega = \omega+(1+\omega)$, but $1+\omega = \omega$, thus $(\omega+1)+\omega = \omega+\omega$.

(b) $\omega+\omega^2 = \omega+\omega\cdot\omega = \omega\cdot(1+\omega) = \omega\cdot\omega = \omega^2$.

(c) We first show that $(\omega+1)\cdot n = \omega\cdot n+1$ by induction on n. For $n \neq 0$. For $n = 1$, $(\omega+1)\cdot 1 = \omega\cdot 1+1 = \omega+1$. Assume that it holds for n, now we have $(\omega+1)\cdot(n+1) = (\omega+1)\cdot n+(\omega+1)$ by Exercise 5.2. By induction hypothesis $(\omega+1)\cdot n = \omega\cdot n+1$, therefore $(\omega+1)\cdot n+(\omega+1) = \omega\cdot n+1+(\omega+1)$, by associative law and $1+\omega = \omega$ we get $\omega\cdot n+1+(\omega+1) = \omega\cdot n+(1+\omega)+1 = \omega\cdot n+\omega+1 = \omega\cdot(n+1)+1$, this completes the proof. Now consider $(\omega+1)\cdot\omega\cdot\omega$, we know $(\omega+1)\cdot\omega = \sup\{(\omega+1)\cdot n : n < \omega\} = \sup\{\omega\cdot n+1 : n < \omega\} = \omega\cdot\omega$, thus we have $(\omega+1)\cdot\omega\cdot\omega = \omega\cdot\omega\cdot\omega = \omega^3$.

5.4 We proceed by transfintie induction on α, for $\alpha = 0$, by taking $\beta = n = 0$ this hold obviously. Assume that that it holds for any $\alpha' < \alpha$ and $\alpha = \alpha'+1$, so by induction hypothesis we can write $\alpha' = \beta+n$ where β is a limit ordinal, thus $\alpha = (\beta+n)+1 = \beta+(n+1)$ and we are done.

Now suppose that α is a limit ordinal. Take $\beta = sup\{\gamma < \alpha : \gamma \text{ is a limit}\}$, β is a limit, since otherwise $\beta = \beta'+1$,so $\beta' \in \beta$, which means for some limit $\gamma < \alpha$ we have $\beta' \in \gamma$, but then $\beta' < \gamma < \beta = \beta'+1$, a contradiction. Now we have $\beta \leq \alpha$, if $\alpha = \beta$ we are done. if $\beta < \alpha$ then $\alpha = \beta+\xi$ by Lemma 5.5., But since $\xi \leq \alpha$ by induction hypothesis $\xi = \xi'+n$, thus $\alpha = \beta+(\xi'+n)$, since ξ' is a limit ordinal, by the argument above it can be shown that sum of two limit ordinal is a limit ordinal, thus $\alpha = (\beta+\xi')+n$ and we are done.

5.5 Just take $\alpha = \beta = \omega$ then we have $n+\omega = \omega$ for any $n \in \omega$ (see page 119), so we have infinitely many solutions. Take $\beta = \omega$ and $\alpha = 1$, then $\xi+1 = \omega$ has no solution, since ω is a limit ordinal. Take $\beta = n \in \omega$, then $\alpha < n$ and by the usual arithmetic on N, the equation $\beta = n = \xi+\alpha$ has a unique solution, namely $n-\alpha$.

5.6 Notice that α can not be a successor ordinal greater than $1 :$ $\alpha = \alpha'+1$ then $\xi+\alpha'+1 = \alpha'+1$, take $\xi = \alpha' < \alpha$, then $\alpha'+\alpha'+1 = \alpha'+1$, by Lemma 5.4(b) we get $\alpha'+1 = 1$ which is a contradiction.

First we show that (Fact 1) $k+\omega\cdot n = \omega\cdot n$ for any $0 \neq n \in \omega$. For $n = 1$ it holds trivially. Suppose it holds for n, so $k+\omega\cdot(n+1) = k+\omega\cdot n+\omega$, then by induction hypothesis $k + \omega \cdot n + \omega = \omega \cdot n + \omega = \omega \cdot (n+1)$, thus we are done.

Second, notice that (Fact 2) $L = \xi + \omega \cdot \omega = sup\{\xi + \alpha : \alpha < \omega \cdot \omega\} = sup\{\xi + \omega \cdot n : n < \omega\} = R$: take $x \in L$, then for some $\alpha < \omega \cdot \omega$, $x \in \xi + \alpha$. But $\alpha < \omega \cdot \omega$ implies $\alpha \in \omega \cdot k$ for some $k \in \omega$, By Lemma 5.4 we get $\xi + \alpha \in \xi + \omega \cdot k$, thus $x \in R$. For converse, take $x \in R$, then for some n, $x \in \xi + \omega \cdot n$, but $\omega \cdot n < \omega \cdot \omega$ (taking $\omega \cdot n$ as α), thus trivially $x \in L$.

The first limit ordinal after ω is $\omega + \omega = \omega \cdot 2$, we show that α can not be $\omega \cdot n$ for any $n > 1$, assume to the contrary $\xi + \omega \cdot n = \omega \cdot n$ for some $\xi < \omega \cdot n$. but $n = k+1$, thus $\xi + \omega \cdot (k+1) = \xi + \omega \cdot k + \omega = \omega \cdot (k+1)$, take $\xi = \omega \cdot k$ (clearly $\omega \cdot k < \omega \cdot (k+1)$), but then we get $\omega \cdot k + \omega \cdot k + \omega = \omega \cdot k + \omega$, by Lemma 5.4(b) we get $\omega \cdot k + \omega = \omega \cdot n = \omega$ which is contradiction for $n > 1$.

Now take the first limit ordinal after all $\omega \cdot n$, namely $\omega \cdot \omega$. Consider $\xi < \omega \cdot \omega$, but then $\xi = \beta + k$ for some $k \in \omega$ and β is a limit ordinal (By Exercise 5.4), but every limit ordinal less than $\omega \cdot \omega$ has a form $\omega \cdot n$, thus we have $\xi = \omega \cdot n + k$. Now consider $\xi + \omega \cdot \omega = \omega \cdot n + (k + \omega \cdot \omega)$. Buy from Fact 1 and 2 above we get $k + \omega \cdot \omega = sup\{k + \omega \cdot n : n < \omega\} = sup\{\omega \cdot n : n < \omega\} = \omega \cdot \omega$, thus $\xi + \omega \cdot \omega = \omega \cdot n + \omega \cdot \omega$. Again $\omega \cdot n + \omega \cdot \omega = sup\{\omega \cdot n + \omega \cdot m : m < \omega\} = sup\{\omega \cdot (n+m) : m < \omega\} = \omega \cdot \omega$, therefore $\xi + \omega \cdot \omega = \omega \cdot \omega$.

$\alpha < n$ and by the usual arithmetic on N, the equation $\beta = n = \xi + \alpha$ has a unique solution, namely $n - \alpha$.

5.7 (a) We proceed by induction on α_2, for $\alpha_2 = 0$ the statement trivially holds. Assume that α_2 is a successor ordinal, then $\alpha_2 = \alpha + 1$, suppose it holds for any ordinal less than α_2, from $\alpha_1 < \alpha_2$ we get either $\alpha_1 = \alpha$ or $\alpha_1 < \alpha$, in both cases we get $\beta \cdot \alpha_1 < \beta \cdot \alpha + \beta = \beta \cdot (\alpha + 1)$, thus $\beta \cdot \alpha_1 < \beta \cdot \alpha_2$.

Now assume that α_2 is a limit, suppose $\alpha_1 < \alpha_2$, then since α_2 is a limit, we get some $\alpha \in \alpha_2$ such that $\alpha_1 < \alpha$, but then by induction hypothesis we get $\beta \cdot \alpha_1 < \beta \cdot \alpha$, but $\beta \cdot \alpha_2 = \sup\{\beta \cdot \xi : \xi \in \alpha_2\}$, which means $\beta \cdot \alpha < \beta \cdot \alpha_2$ (since $\alpha \in \alpha_2$), thus $\beta \cdot \alpha_1 < \beta \cdot \alpha_2$ (by transitivity of ordinals).

(b) Assume that $\beta \cdot \alpha_1 = \beta \cdot \alpha_2$ and $\alpha_1 \neq \alpha_2$, thereofere either $\alpha_1 < \alpha_2$ or $\alpha_2 < \alpha_1$, then by previous item we get $\beta \cdot \alpha_1 < \beta \cdot \alpha_2$, which is a contradiction. $\alpha_2 < \alpha_1$ can be handled similarly.

5.8 (a) We prove by induction on γ: It trivially holds for $\gamma = 0$. Suppose hat γ is a successor, thus $\gamma = \xi + 1$, by induction hypothesis $\alpha + \xi \leq \beta + \xi$, then $(\alpha+\xi)+1 \leq (\beta+\xi)+1$, associative law implies $\alpha+\xi+1 \leq \beta+(\xi+1)$, which means $\alpha+\gamma \leq \beta+\gamma$.

Now, assume γ is a limit ordinal: let $x \in \alpha+\gamma$, then for $x \in \alpha+\xi$ for some $\xi \in \gamma$, by induction hypothesis $\alpha+\xi \leq \beta+\xi$, by transitivity, $x \in \beta+\xi$, which means that $x \in sup\{\beta+\xi : \xi < \gamma\} = \beta+\gamma$ thus $\alpha+\gamma \subseteq \beta+\gamma$, by Lemma 2.9 it $\alpha+\gamma \leq \beta+\gamma$.

(b) Again, we proceed by induction on γ: For $\gamma = 0$ it holds trivially. Assume $\gamma = \xi+1$, then by induction hypothesis $\alpha\cdot\xi \leq \beta\cdot\xi$, by previous item we get $\alpha\cdot\xi+\alpha \leq \beta\cdot\xi+\alpha$, but since $\alpha < \beta$, by Lemma 5.4(a) $\beta\cdot\xi+\alpha < \beta\cdot\xi+\beta$. By transitivity $\alpha\cdot\xi+\alpha < \beta\cdot\xi+\beta$, thus $\alpha\cdot(\xi+1) \leq \beta\cdot(\xi+1)$.

Assume that γ is a limit ordinal, let $x \in \alpha\cdot\gamma$, so for some $\xi \in \gamma$ we have $x \in \alpha\cdot\xi$, by induction hypothesis $\alpha\cdot\xi \leq \beta\cdot\xi$, thus $x \in sup\{\beta\cdot\xi : \xi < \gamma\} = \beta\cdot\gamma$, Lemma 2.9 implies $\alpha\cdot\gamma \leq \beta\cdot\gamma$.

$\leq$ can not be replaced by $<$, just consider $1 < 2$, then $1+\omega = 2+\omega$ and $1\cdot\omega = 2\cdot\omega$.

5.9 (a) $1+\omega = 2+\omega$, but $1 \neq 2$.

(b) $1\cdot\omega = 2\cdot\omega$, but $1 \neq 2$.

(c) $(1+1)\cdot\omega = 2\cdot\omega = \omega$, but $1\cdot\omega + 1\cdot\omega = \omega+\omega$

5.10 We proceed by induction on α. if $\alpha = 0$, then $\alpha = \omega\cdot 0$. Now assume α is a limit, by Exercise 5.4, $\alpha = \beta + n$, since α is a limit ordinal n must be zero. By the construction of β in the proof and induction hypothesis we know that $\alpha = \beta = sup\{\gamma < \alpha : \gamma \text{ is limit}\} = sup\{\gamma < \alpha : \gamma = \omega\cdot\beta \text{ for some } \beta\}$.

Let $C = \{\beta : \xi < \alpha, \xi \text{ is limit and } \xi = \omega\cdot\beta\}$and $\kappa = supC$. We show that $\alpha = \omega\cdot\kappa :$ let $x \in \alpha$, then $x \in \omega\cdot\beta$, but $\beta \leq \kappa$, which means $\omega\cdot\beta \leq \omega\cdot\kappa$ (by Exercise 5.7), thus $x \in \omega\cdot\kappa$. Conversely, let $x \in \omega\cdot\kappa$: either $\kappa = \beta$ such that β is maximum of C, which means there is $\xi = \omega\cdot\beta$ and $\xi < \alpha$, thus $x \in \alpha$. Or for any $\zeta < \kappa$, there is a $\beta \in C$ such that $\zeta < \beta$, which means κ is a limit, thus $x \in \omega\cdot\zeta$ for some $\zeta < \kappa$, but then there is a $\beta \in C$ such that $\zeta < \beta$ and $\omega\cdot\beta < \alpha$, therefore $\omega\cdot\zeta < \omega\cdot\beta < \alpha$, thus $x \in \alpha$.

5.11 (a) $\{1-\frac{1}{n} : n \in N-\{0\}\} \cup \{1\}$

(b) $\{1-\frac{1}{n} : n \in N-\{0\}\} \cup \{2-\frac{1}{n} : n \in N-\{0\}\}$

(c) $\{k-\frac{1}{n} : n \in N-\{0\}, k \in \{1,2,3\}\}$

(d) let $F(a,b) = \{b-\frac{b-a}{2^n} : n \in N\}$, clearly $F(a,b)$ is isomorphic to N for any $a < b \in Q$. Let $F(a,b)(n)$ denote the n-th element of $F(a,b)$.

Now consider the set $\bigcup_{k,n\in N} F(F(n,n+1)(k), F(n,n+1)(k+1))$ which is isomorphic to ω^ω.

(e) Denote $F(F(n,n+1)(k), F(n,n+1)(k+1))$ by $F_{k,n}$ and it's m-th element by $F_{k,n}(m)$, then the set $\bigcup_{k,n,m\in N} F(F_{k,n}(m), F_{k,n}(m+1))$ is isomorphic to ϵ.

5.12 $\omega^2 \cdot 2^2 = \omega^2 \cdot 4$, but $(\omega \cdot 2)^2 = (\omega \cdot 2) \cdot (\omega \cdot 2) = \omega \cdot (2 \cdot \omega) \cdot 2 = \omega^2 \cdot 2$ where $< \omega^2 \cdot 4$.

5.15 (a) Define $\xi_0 = 0$ and $\xi_{n+1} = \omega + \xi_n$, $\xi = \sup\{\xi_n : n \in \omega\}$. We show that $\omega + \xi = \xi$: let $x \in \omega + \xi$ then for some ξ_n, $x \in \omega + \xi_n$, which means $x \in \xi_{n+1}$ (notice that $\omega + \xi = sup\{\omega + \alpha : \alpha < \xi\}$), thus $x \in \xi$ by definition of ξ. Conversely, let $x \in \xi$, again by definition of ξ, $x \in \omega + \xi_k = \xi_n$ such that $k + 1 = n$, since $\xi_0 = \emptyset$, therefore $x \in \omega + \xi$. To prove that it is the least ordinal with this property, first we show $\xi_n < \xi_{n+1}$. Clearly for $n = 0$ it holds, since $0 < \omega + 0$. Assume that it holds for n: thus $\xi_n < \xi_{n+1}$, then by Lemma 5.4(b) we get $\omega + \xi_n < \omega + \xi_n$, which means $\xi_{n+1} < \xi_{n+2}$, we are done.

Now just consider $\alpha < \xi$ and assume $\omega + \alpha = \alpha$. it means $\omega + \alpha = \alpha < \omega + \xi_n$ for some n, but by Lemma 5.4 (b) we get $\alpha = \xi_n$, then $\omega + a = \omega + \xi_n$ which means $\xi_n = \xi_{n+1}$, a contradiction.

(b) Let $\xi_0 = 0$, $\xi_{n+1} = \omega \cdot \xi_n$ and $\xi = sup\{\xi_n : n \in \omega\}$, we show that $\omega \cdot \xi = \xi$: let $x \in \omega \cdot \xi$, then for some n, $x \in \omega \cdot \xi_n = \xi_{n+1}$, thus $x \in \xi$. Conversely, let $x \in \xi$, then $x \in \xi_{n+1}$ (since $\xi_0 = \emptyset$), thus $x \in \omega \cdot \xi_k$ and $x \in \omega \cdot \xi$.

(c) Define $\xi_0 = 0$ and $\xi_{n+1} = \omega^{\xi_n}$ and $\xi = sup\{\xi_n : n \in \omega\}$, the proposition $\omega^\xi = \xi$ can be proved similar to the previous items.