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3.1 if *ran* $f \subseteq$ *dom* g, then $f^{-1}[ran\ f] \subseteq f^{-1}[dom\ g]$, but $f^{-1}[ran\ f] = dom\ f$, by Exercise 4.2(a) on Page 15 we have $dom\ f \cap f^{-1}[dom\ g] = dom\ f$, Theorem 3.5 implies $dom\ (g \circ f) = dom\ f$.

3.2 $f_2 \circ f_1 = \{\sqrt{2x-1} : x > \frac{1}{2}\}$.
$f_1 \circ f_2 = \{2\sqrt{x} - 1 : x > 0\}$
$f_3 \circ f_1 = \{1/(2x-1) : x \neq \frac{1}{2}\}$
$f_1 \circ f_3 = \{2/x - 1 : x \neq 0\}$

3.3 For f_1: if $f_1(a) = f_1(b)$ then $2a - 1 = 2b - 1$, by adding 1 to each side of equation we get $2a = 2b$, by dividing by 2 we have $a = b$.

For f_2: if $f_1(a) = f_1(b)$ then $\sqrt{a} = \sqrt{b}$, but then $a = \sqrt{a}\ \sqrt{a} = \sqrt{a}\ \sqrt{b} = \sqrt{b}\ \sqrt{b} = b$.

For f_3: if if $f_1(a) = f_1(b)$ then $1/a = 1/b$, because a, b are non-zero multiplying by ab yields $a = b$.

$f_1^{-1} = \{(x+1)/2 : x\ is\ real\}$
$f_2^{-1} = \{x^2 : x > 0\}$
$f_3^{-1} = \{1/x : x \neq 0\}$

3.4 (a) Assume that f is invertible, let $(a, b) \in f^{-1} \circ f$ then for some z we have (*) $(a, z) \in f$ and $(z, b) \in f^{-1}$, then from (*) we also have $(z, a) \in f^{-1}$, by assumption f^{-1} is a function, so we get $a = b$, because $a \in dom\ f$ we get $(a, b) = (a, a) \in Id_{dom\ f}$. the other side holds by Exercise 2.4(e) on Page 23.

(b) Let $(a, b), (a, c) \in f^{-1}$, then $(b, a), (c, a) \in f$, thus $f(b) = a$ and $f(c) = a$ but (*) $g \circ f = Id_{dom\ f}$ implies $g(f(b)) = b = g(a) = g(f(c)) = c$, therefore $b = c$ and f^{-1} is a function. let $(a, b) \in f^{-1}$ then $(b, a) \in f$, so $f(b) = a$, by (*) we get $g(f(b)) = b = g(a)$, thus $(a, b) \in g$, but we also know that $a \in ran\ f$, therefore $(a, b) \in g \mid ran\ f$. Now let $(a, b) \in g \mid ran\ f$, then $g(a) = b$ and also $a \in ran\ f$, then $f(k) = a$ for some $k \in domf$, but (*) implies $g(f(k)) = g(a) = b = k$ which means that $(b, a) \in f$, $(a, b) \in f^{-1}$.

We give a counter example for the second one, let $f = \{(a, a), (b, a)\}$ and $h = \{(a, a)\}$ then $f \circ h = \{(a, a)\} = Id_{ran\ f}$ but clearly f^{-1} is not a function.

3.5 Let $(g \circ f)(a) = (g \circ f)(b)$, then $g(f(a)) = g(f(b))$ since g is one-to-one we get $f(a) = f(b)$, again because f is one-to-one we have $a = b$.

let $(a, b) \in (f \circ g)^{-1}$, thus $(b, a) \in f \circ g$, it means that for some z we have $(b, z) \in g$ and $(z, a) \in f$, equivalently we have $(a, z) \in f^{-1}$ and $(z, b) \in g^{-1}$ for some z, by definition of composition we get $(a, b) \in g^{-1} \circ f^{-1}$.

3.6 We just need prove right to left of (a) and left to right of (b).

(a) Suppose $x \in f^{-1}[A] \cap f^{-1}[B]$, then for some $y \in A$ we have $yf^{-1}x$ or equivalently $f(x) = y$ and for some $z \in B$, $f(x) = z$, but since f is a function we conclude that $z = y \in A \cap B$, then we can say that for some $y \in A \cap B$, $yf^{-1}x$ holds, therefore $x \in f^{-1}[A \cap B]$.

(b) Let $x \in f^{-1}[A - B]$, then there is some $y \in A - B$ such that $yf^{-1}x$ or equivalently (*) $f(x) = y$, clearly $x \in f^{-1}[A]$, we must prove that $x \notin f^{-1}[B]$ or equivalently there is no $z \in B$ such that $zf^{-1}x$, assume to the contrary that it exists, so we get $f(x) = z$, but (*) implies $z = y \in B$, it contradicts our assumption that $y \in A - B$.

3.7 let $f = \{(a, b)\}$ and $A = \{a\}$, then $f \cap A^2 = \emptyset$ but $f|A = f$.

3.8 Let $I = A$ and $S = Id_I$, then $S = (S_i, i \in I)$ is an indexed function such that $S_i = i$.

3.9 (a) Let $f : A \to B$, then $f \subseteq A \times B$, thus $f \in \mathcal{P}(A \times B)$, now let $P(x)$ be the property "$(\forall a, b, c)[(a, b), (a, c) \in x \to b = c] \wedge (\forall a)(a \in A \to (\exists b)[b \in B \wedge (a, b) \in x])$", then $\{x \in \mathcal{P}(A \times B) : P(x)\}$ is the set of all function from A to B.

(b) Let f be a memeber of product of an indexed system $(S_i : i \in I)$, then $f : I \to \bigcup_{i \in I} S_i$ such that for every $i \in I$, $f(i) \in S_i$, then clearly $f \in (\bigcup_{i \in I}^{S_i})^I$, by previous exercise we know that it exists, now by comprehension we have $\prod_{i \in I} S_i = \{f \in (\bigcup_{i \in I}^{S_i})^I : (\forall i \in I)[f(i) \in S_i]\}$, clearly if it is non-empty, every member of it is a function such that satisfies the condition of a product.

3.10 $x \in \bigcup_{a \in \bigcup S} F_a$ iff $(\exists a)[a \in \bigcup S \wedge x \in F_a]$ iff $(\exists a)[(\exists C)(C \in S \wedge a \in C) \wedge x \in Fa]$ iff $(\exists a)[(\exists C)(C \in S \wedge a \in C \wedge x \in F_a)]$ iff $(\exists C)[(\exists a)(C \in S \wedge a \in C \wedge x \in F_a)]$ iff $(\exists C)[C \in S \wedge (\exists a)(a \in C \wedge x \in F_a)]$ iff $(\exists C)[C \in S \wedge x \in \bigcup_{a \in C} F_a]$ iff $x \in \bigcup_{C \in S}(\bigcup_{a \in C} F_a)$.

Let $x \in \bigcap_{a \in \bigcup S} F_a$ then (*) $(\forall a)[a \in \bigcup S \to x \in F_a]$. Now let $C \in S$, then because $C \subseteq \bigcup S$ we get that for every $a \in C$, $x \in F_a$, because C was arbitrary we can conclude that (**) $(\forall C)[C \in S \to (\forall a)(a \in C \to x \in F_a)]$, which is equal to $(\forall C)[C \in S \to x \in \bigcap_{a \in C} F_a]$, thus $x \in \bigcap_{C \in S}(\bigcap_{a \in C} F_a)$. Now let $x \in \bigcap_{C \in S}(\bigcap_{a \in C} F_a)$, then we get (**), let $a \in \bigcup S$, then there is some $C \in S$ such that $a \in C$, but then by (**) we get $(\forall a)(a \in C \to x \in F_a)$ and then $x \in F_a$, because a was arbitary we proved (*), thus $x \in \bigcap_{a \in \bigcup S} F_a$.

3.11 $x \in B - \bigcup_{a \in A} F_a$ then $x \in B$ and for every $a \in A$, $x \notin F_a$, also for every $a \in A$, $x \notin F_a$ and $x \in B$, so for every $a \in A$, $x \in B - F_a$, thus $x \in \bigcap_{a \in A}(B - F_a)$. Now let $x \in \bigcap_{a \in A}(B - F_a)$, then for every $a \in A$, $x \in B$ and $x \notin F_a$,

let $a \in A$, then by above claim $x \notin F_a$, thus $x \notin \bigcup_{a \in A} F_a$, Now assume to the contrary that $x \notin B$, then it implies there is no $a \in A$, $A = \emptyset$ which is a contradiction.

Let $x \in B - \bigcap_{a \in A} F_a$, then (*) $x \in B$ and there is some $a \in A$ such that $x \notin F_a$, by (*) we can claim that there is some $a \in A$ such that $x \in B - F_a$, thus $x \in \bigcup_{a \in A}(B - F_a)$. Now let $x \in \bigcup_{a \in A}(B - F_a)$, then $x \in (B - F_a)$ for some $a \in A$, it follows that there is some $a \in A$ such that $x \in F_a$, thus $x \notin \bigcap_{a \in A} F_a$ and clearly $x \in B$, thus $x \in B - \bigcap_{a \in A} F_a$.

Let $x \in \bigcup_{a \in A} F_a \cap \bigcup_{b \in B} G_b$, then for some $a \in A$, $x \in F_a$ and for some $b \in B$, $x \in G_b$, clearly $(a, b) \in A \times B$, then we can say for some $(a, b) \in A \times B$, $x \in F_a \cap G_b$

$x \in \bigcup_{a\in A} F_a \cap \bigcup_{b\in B} G_b$ iff $(\exists a)(a \in A \wedge x \in F_a) \wedge (\exists b)(b \in B \wedge x \in F_b)$ iff $(\exists a)(\exists b)[(a \in A \wedge x \in F_a) \wedge b \in B \wedge x \in F_b)]$ iff $(\exists a)(\exists b)[(a, b) \in A \times B \wedge x \in F_a \cap F_b]$ iff $x \in \bigcup_{(a,b)\in A\times B}(F_a \cap G_b)$

3.12 (We just prove the first and the third case)

$y \in f[\bigcup_{a\in A} F_a]$ iff $(\exists x)[x \in \bigcup_{a\in A} F_a \wedge f(x) = y]$ iff $(\exists x)[(\exists a)(a \in A \wedge x \in F_a) \wedge f(x) = y]$ iff $(\exists x)[(\exists a)(a \in A \wedge x \in F_a \wedge f(x) = y)]$ iff $(\exists x)(\exists a)[a \in A \wedge x \in F_a \wedge f(x) = y]$ iff $(\exists a)(\exists x)[a \in A \wedge x \in F_a \wedge f(x) = y]$ iff $(\exists a)[a \in A \wedge (\exists x)(x \in F_a \wedge f(x) = y)]$ iff $(\exists a)[a \in A \wedge y \in f[F_a]]$ iff $y \in \bigcup_{a\in A} f[F_a]$.

Let $y \in f[\bigcap_{a\in A} F_a]$, then for some $x \in \bigcap_{a\in A} F_a$, $f(x) = y$, but it means for every $a \in A$, $x \in F_a$ and $f(x) = y$, we can say for every $a \in A$, there is some $x \in F_a$ such that $f(x) = y$ or equally $y \in f[F_a]$, thus $y \in \bigcap_{a\in A} f[F_a]$.

(if f is one-to-one, $\subseteq$ can be replaced by $=$): Now let $y \in \bigcap_{a\in A} f[F_a]$, so for every $a \in A$, there is some $x \in F_a$ such that $f(x) = y$, but because f is one-to-one this x must be unique, name it k, so for every $a \in A$, $k \in F_a$ or equivalently $k \in \bigcap_{a\in A} F_a$, since $f(k) = y$ we get $y \in f[\bigcap_{a\in A} F_a]$.

3.13 Right to left is easy according to Hint, we prove left ro right side:

Let $x \in \bigcap_{a\in A}(\bigcup_{b\in B} F_{a,b})$, define f such that $(a, b) \in f$ iff $x \in F_{a,b}$, we prove $f \in B^A$, let $(x, y), (x, z) \in f$ be two distinct memeber, then $x \in F_{x,y} \cap F_{x,z}$ but because $y \neq z$ we have $F_{x,y} \cap F_{x,z} = \emptyset$, thus it contradicts our assumption, hence f is a function.

From assumption for every $a \in A$ we have $x \in \bigcup_{b\in B} F_{a,b}$, fix arbitrary $a \in A$, then $x \in F_{a,b}$ for some $b \in B$, but by definition of f we have $f(a) = b$, thus $x \in F_{a,f(a)}$, because a was arbitrary we can say $x \in \bigcap_{a\in A} F_{a,f(b)}$ for f, thus $x \in \bigcup_{f\in B^A}(\bigcap_{a\in A} F_{a,f(b)})$.