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1. If X is an infinite well-orderable set, then by Theorem 1.3 X is equipotent to a unique initial ordinal α, then it is also equipootent to $\alpha+1$. So, there is a one-to-one function from X to $\alpha+1$, the order that defined by $a<b$ iff $f(a)\in f(b)$ induce the well-ordering of $\alpha+1$ on X. So it has another well-ordering, obviously nonisomorphic.

4. By definition we have $|A|<h(A)$, by Lemma 5.4(a) (page 120) we have $|A|+|A|<|A|+h(A)$. We also know that $|A|\leq|A|+|A|$, the conclusion follows.

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2.1 Let $L\subset\omega_\alpha$ be the set of all limit ordinals less than ω_α. Let $g:\omega_\alpha\to L$ such that $g(\beta)=\omega\cdot\beta$. Clearly $\omega\cdot\beta$ is a limit ordinal, by Theorem 5.8 on page 122 there is a bijection from $\omega\cdot\beta$ to $\omega\times\beta$, for finite β, we have $|\omega|\cdot|\beta|<|\omega|\cdot|\omega|=|\omega|$ by Theorem 2.1 and property (i) on page 94, and for infinite β as we have $|\omega|\leq|\beta|$, we get $|\omega|\cdot|\beta|\leq|\beta|\cdot|\beta|=|\beta|$ by Theorem 2.1, thus $\omega\cdot\beta\in\omega_\alpha$. From item (b) of Exercise 5.7 on page 123 it follows that g is a one-to-one function, therefore by Cantor-Bernstein we get $|L|=|\omega_\alpha|$. (the other one-to-one function is identity, as $L\subset\omega_\alpha$).

Let $L'\subset\omega_\alpha$ be the set of all successor ordinals. Define $h:\omega_\alpha\to L'$ by $h(\beta)=\beta+1$. As for infinite β, $|\beta+1|=|\beta|$. For finite β, clearly $\beta+1\in\omega\subseteq\omega_\alpha$, so we have $\beta+1\in L'$. This function is clearly one-to-one, therefore again Cantor-Bernstein implies $|\omega_\alpha|=|L'|$. As L and L' are disjoint and $L\cup L'=\omega_\alpha$, we get $|L|+|L'|=\aleph_\alpha$, or equivalently $\aleph_\alpha+\aleph_\alpha=\aleph_\alpha$.

2.4 By Theorem 5.3 on page 120 we have $|\alpha+\beta|=|\alpha\cup\beta|=|\alpha|+|\beta|$. Without loss of generality assume that $\alpha<\beta$, then Corrolary 2.3 implies $|\alpha+\beta|=|\beta|\leq\aleph_\gamma$. Again suppose $\alpha<\beta$, then $|\alpha\cdot\beta|=|\alpha\times\beta|=|\alpha|\cdot|\beta|=|\beta|<\aleph_\gamma$ by Corollary 2.2.

By Exercise 5.16 on page 123 α^β is isomorphic to $S(\alpha,\beta)$. For every $f\in S(\alpha,\beta)$ the cardinal of the set $\{\gamma<\beta:f(\gamma)\neq0\}=X\subset\beta$ is finite. There are $|\alpha-\{0\}|^{|X|}$ such functions (each function in it can be uniqely extended to a function in α^β, by setting $f(\gamma)=0$ for $\gamma\notin X$). As we can write $S(\alpha,\beta)$ as $\bigcup_{X\in[\beta]^{<\omega}}|\alpha|^{|X|}$ and since $|\alpha|^{|X|=n}=|\alpha|$ by exercise 2.3 (a) and (b) we get $|S(\alpha,\beta)|=|[\beta]^{<\omega}|\cdot|\alpha|=|\beta|\cdot|\alpha|<\aleph_\gamma$.

4.6 We proceed by induction on k, for $k = 1$ it trivially holds. Assume that $m < n$ iff $m \cdot k < n \cdot k$, then by Exercise 4.2 we get $m \cdot k + m < n \cdot k + m$, but as $m < n$ and sum is commutative, again by Exercise 4.2 we get $n \cdot k + m < n \cdot k + n$, which implies $m \cdot k + m < n \cdot k + n$, from definition of multiplication it follows that $m \cdot (k+1) < n \cdot (k+1)$, this completes the proof.

2.6 X is a subset of ω_α, so it has a well-ordering, and therefore has an order type, say β and so is $\omega_\alpha - X$, let suppose it has the order type γ. Also assome for contradiction that $|\gamma| < \aleph_\alpha$, as $|\beta| < \aleph_\alpha$, we can suppose some cardinal $|\gamma|, |\beta| \leq \aleph_\lambda < \aleph_\alpha$. By exercise 2.4 we get $|\beta + \gamma| \leq \aleph_\gamma$. But by theorem 5.3 on page 120 we get $|\beta + \gamma| = |X \cup (\omega_\alpha - X)| = |\omega_\alpha|$, this implies $\aleph_\alpha = \aleph_\gamma$, contradiction.