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1.1 Consider the order of rational numbers $(Q, <)$, consider the set $Q[a] \cup \{a\}$ for some $a \in Q$.

1.2 Notice that ω is the initial segment of $\omega + 1$, since $\omega = (\omega + 1)[\omega]$ (compare to the notation $W[a]$), then it follows from Corollary 1.5 (a) that they can not be isomorphic.

1.3 (We assume the question is about the well-ordering of order type ω, since otherwise, the answer is $\aleph_1$).

Intuitively, each well ordering on N can be seen as a permutation of elements of N, we are just changing the position of objects in the well-ordering. Let $B \subseteq N^N$ be the set of all bijections and X be the set of all well-ordering on N. We prove that they have the same cardinality: define $F : X \to B$ as follows: $F((N, R)) = f$, such that f is the isomorphism between $(N, <)$ and (N, R), clearly each order has a unique isomorphism function, thus the function is one-to-one. Conversely, define $G : B \to X$ for any $f \in B$ such that $G(f) = (N, R)$ in which, R is defined as follows: for each $x, y \in N$, $x < y$ iff $f(x)Rf(y)$, it is easy to see that R is a well-ordering on N and for each f such a well-ordering is unique, thus G is one-to-one. Having F and G, Cantor-Bernstein theorem implies $|X| = |B|$. But Theorem 2.5(c) (page 100) implies $|B| = 2^{\aleph_0}$.

1.4 let k be the least element of A, define a recursive function f such that $f_0 = k$ and for each n, $f_{n+1} = t$ such that t is the least element of $A - \{f_0, \ldots, f_n\}$. To show f is one-to-one consider $f_n = f_m$, it means that f_n is the least element of $A - \{f_0, \ldots, f_{n-1}\}$ and f_m is the least element of $A - \{f_0, \ldots, f_{m-1}\}$, assume that $n < m$ (or vice versa) then $n \leq m - 1$, thus $f_n \in \{f_0, \ldots, f_{m-1}\}$ and $f_n \notin A - \{f_0, \ldots, f_{m-1}\}$ but it contaradicts that f_m is in this set.

1.5 Let $(W_1 \cup W_2, \prec)$ be the sum of the two ordering, f and g be the two isomorphic functions from N to W_1 and W_2, and define $F : W_1 \cup W_2 \to (\omega + \omega)$, if $a \in W_1$ let $F(a) = f(a)$, otherwise $F(a) = \omega + g(a)$. F is one-to-one and onto. To see it preserves order, Assume $F(a) = F(b)$ for some $a, b \in domF$, if a, b both are W_1 or both are in W_2 we are done by the isomorphism of either f or g. Suppose that $a \in W_1$ and $b \in W_2$ (or vice versa). Then $F(a) = f(a) \in \omega$, and $F(b) = \omega + g(b)$, thus $f(a) \in \omega + g(b)$ which means $F(a) \in F(b)$.

1.6 Define $f : N \times N \to \omega \cdot \omega$ such that $f(n,m) = \omega \cdot n + m$, since every $x \in \omega \cdot \omega$ has the form $\omega \cdot n + m$ (see the definition of it in page 104) the function is onto. To see it preserves order, assume that $(n,m) < (j,k)$ then either $n < m$: from it follows that $\omega \cdot n \in \omega \cdot m$, thus $\omega \cdot n + m \in \omega \cdot j + k$, thus $f(m,n) < f(j,k)$. or $m = n$ and $j < k$, which means $\omega \cdot n = \omega \cdot j$, but since $m < k$ we get $\omega \cdot n + m < \omega \cdot j + k$ (k-th successor of anything contains m-th (where $m < k$) successor of it), thus $f(m,n) < f(j,k)$.

1.7 Because $x < a$ for any $x \in W' = W \cup \{a\}$ such that $x \in W$, we get $W = W'[a]$ which means W is an initial segment of W, thus has a smaller order type.

1.8 One order type is ω, while the other one is $\omega + \omega$, so they are noniso-morphic.