Exercise 1

1. $\subseteq$ is a partial order on any set, to prove it is a lattice, we show that $sup\{a, b\}$ and $inf\{a, b\}$ exits. $sup\{a, b\} = a \cup b$ and clearly $a \cup b \in Su(A)$. Since $a \subseteq a \cup b$ and $b \subseteq a \cup b$ it is an upper bound for $\{a, b\}$. To prove it is the least upper bound, assume that there is some set $c \in Su(A)$ such that $a \subseteq c, b \subseteq c$, and $c \subsetneq a \cup b$, it means there is some k in a or b, such that $k \notin c$, a contradiction. $inf\{a, b\} = a \cap b$ can be proved similarly.

2. L1:(a) $a \vee b = sup\{a, b\} = sup\{b, a\} = b \vee a$.

(b) $a \wedge b = inf\{a, b\} = inf\{b, a\} = b \wedge a$.

L2: (a) Let $t = sup\{a, sup\{b, c\}\}$ and $k = sup\{sup\{a, b\}, c\}$, then t is greater than a and $sup\{b, c\}$, which means t is also greater than b, c, thus $sup\{a, b\} \leq t$, therefore $sup\{sup\{a, b\}, c\} \leq t$, or $k \leq t$. $k = sup\{sup\{a, b\}, c\}$ implies $c \leq k$ and $sup\{a, b\} \leq k$, which means $a \leq k$ and $b \leq k$. Now we can say that $sup\{b, c\} \leq k$, therefore $sup\{a, sup\{b, c\}\} \leq k$. From $t \leq k$ and $k \leq t$ we conclude $t = k$.

(b) Can be proved similarly.

L3: (a) and (b) follows from $inf\{a, a\} = sup\{a, a\} = a$.

L4: (a) Let $t = sup\{a, inf\{a, b\}\}$ then $a \leq t$. Since $inf\{a, b\} \leq a$, we get $sup\{a, inf\{a, b\}\} \leq sup\{a, a\} = a$, thus $t \leq a$ and finally $a = t$.

(b) Let $t = inf\{a, sup\{a, b\}\}$, then $t \leq a$, and since $a \leq sup\{a, b\}$ we have $a = inf\{a, a\} \leq inf\{a, sup\{a, b\}\}$, thus $a \leq t$.

3. From L4(b) we have $x \approx x \wedge (x \vee x)$, replace $x \vee x$ for y in the formula L4(a), we get $x \approx x \vee [x \wedge (x \vee x)]$, replacing the first formula in it, we get $x \approx x \vee x$. Similarly, replace y by $x \wedge x$ in L4(b), we get $x \approx x \wedge [x \vee (x \wedge x)]$, from L4(a) we have $x \approx x \vee (x \wedge x)$. Therefore $x \approx x \wedge x$.

5. Since each item in L1,.., L4 has two symmetric axioms for $\vee$ and $\wedge$, the result trivially follows.

9. (Note: we write $(a, b) \in R$ instead of aRb)

Consider the following collection:

$$O = \{\preceq \subseteq A \times A : \preceq \text{ is a partial order on } A\}$$

Clearly, $(O, \subseteq)$ is a poset and $\leq \in O$, we show that every chain in it has an upper bound. Let $C \subseteq O$ be a chain, clearly $\bigcup C \subseteq A \times A$, we show it is a partial order: *reflexive.* Let $a \in A$, then for every $\preceq \in C$ we have $(a, a) \in \preceq$ (since every partial order is reflexive), thefeore $(a, a) \in \bigcup C$. *antisymmetric.* Consider $(a, b), (b, a) \in \bigcup C$, then for some $\preceq \in C$ we have $(a, b) \in \preceq$ and for some $\preceq' \in C$, $(b, a) \in \preceq$. Since C is a chain we have either $\preceq \subseteq \preceq'$ or $\preceq' \subseteq \preceq$, in either case since they are antisymmetric we get $a = b$. *transitive.* Suppose $(a, b), (b, c) \in \bigcup C$, then for some $\preceq, \preceq' \in C$ we have $(a, b) \in \preceq$ and $(b, c) \in \preceq'$, again, since C is chain one of the orders is superset of another, without loss of generality assume $\preceq' \subseteq \preceq$, then by transitivity of $\preceq$ we get $(a, c) \in \preceq$, thus $(a, c) \in \bigcup C$. We proved any chain C of O, $\bigcup C$ is in O, thus any chain has an upper bound in it. By Zorn's Lemma (see item (e) on page 2) O has a maximal member, denote it by $\leqslant^*$. Since $\leqslant \subseteq \leqslant^*$ we have if $a \leqslant b$ then $a \leqslant^* b$.

Assume that $\leqslant^*$ is not a total order, then there are some a, b such that $a \nleqslant^* b$ and $b \nleqslant^* a$. Define a relation R such that xRy iff $x \leqslant^* a$ and $b \leqslant^* y$. You can check that $\bigcup R$ is partial order on A (tedious but trivial) and $\bigcup R \neq \leqslant^*$ but $\leqslant^* \subseteq \bigcup R$. It contradicts the fact that $\leqslant^*$ is a maximal element.

10. Let S be the element a of L such that there is no $a_1 \vee \cdots \vee a_n$ in L such that $a = a_1 \vee \cdots \vee a_n$ and each a_i is join irreducible. This set is non-empty, let b be a minimal element of it (it exists because L is finite). Notice that b is not join irreducible, otherwise we could have $b = b \vee b$. Since it is not join-irreducible there must be some element $c, d \in L$ such that $b = c \vee d$ but $b \neq c$ and $b \neq d$. On the other hand c and d are less than b, but b is minimal element so $c, d \notin S$, thus we have $c = a_i \vee \cdots \vee a_n$ and $d = a'_i \vee \cdots \vee a'_m$ for some a_i, $i \leq n$ and a'_j, $j \leq m$ such that each a_i and a_j are join irreducible. Now we have $b = c \vee d = a_i \vee \cdots \vee a_n \vee a'_i \vee \cdots \vee a'_m$ which contradicts the fact that $b \in S$.