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3.1 Let $f : A_1 \to B_1$ and $g : A_2 \to B_2$ be two one-to-one and onto mapping:

(a) define $h : A_1 \cup A_2 \to B_1 \cup B_2$ such that for every $a \in A_1, h(a) = f(a)$ and for every $b \in B_1, h(b) = g(b)$, since A_2 and A_1 are disjoint and f and g are one-to-one and onto, it follows that h is one-to-one and onto.

(b) for every $(x, y) \in A_1 \times A_2$ let $h((x, y)) = (f(x), g(y))$.

(c) For every $(a_1, ..., a_n) \in Seq(A_1)$, define $F : Seq(A_1) \to Seq(B_1)$ by $F(\ (a_1, ..., a_n)\ ) =\ (f(a_1), ..., f(a_n))$, F is one-to-one and onto.

3.2 Let B be a countable set then there is an enumaration of it $(b_n : n \in N)$. Let A be a finte set, then there is a one-to-one and onto sequence $(a_1, ..., a_k)$ of $A - B$, define $(c_n : n \in N)$ as follows: for every $i < k, c_i = a_i$ and for other $c_{i+k} = b_i$. we prove it is onto: let $y \in A \cup B$ then it is either $y = a_i$ for some $i < k$ or $y = b_j$ for some $j \in N$, in first case we have $c_i = a_i$, in other case we have $c_{k+j} = b_j$. To prove that it is one-to-one let $c_n = c_m$ for some $n, m \in N$ if $n < k$ and $m < k$ then $a_n = a_m$ and since it is one-to-one we get $m = n$, if n, m are both greater or equal to k we have $b_m = b_n$ and again $m = n$. other case is when $m < k$ and $k \leq n$ which is impssible since $c_m = c_n$ and A and $A - B$ are disjoint.

3.3 Since B is countable there is an enumration of it $(b_n : n \in N)$ and let $a \in A$, for every $n \in N$ let $f(n) = (a, b_n)$, $f : N \to A \times B$ and is one-to-one. Let $g : A \times B \to N$ as follows: for every $(a_i, b_j) \in A \times B$, $g((a_i, b_j)) = 2^i 3^j$. Since both g and h are one-to-one, Cantor-Bernestein Theorem implies $|A \times B| = |N|$.

3.4 Since A is finite we have a one-to-one and onto mapping $f : A \to k$ for some $k \in N$.

Let $(a_1, ..., a_n)$ be sequence of length n in $Seq(A)$, let $p_1, ..., p_n$ be the first n prime number, define $F((a_1, ..., a_n)) = p_1^{f(a_1)+1} \cdot p_2^{f(a_2)+1} ... \cdot p_n^{f(a_n)+1}$, this function is one-to-one because of unique factorization.

Now we define $h : N \to Seq(A)$, for each $n \in N$, there are $|A|^n = k^n$ distinct sequence of length n which is finite, so we have a one-to-one and onto mapping $g : A^n \to k^n$, let $h(n) = x, x \in A^n$ which has the least $g(x)$ among other sequences in A^n.

Since h and F are on-to-one, by Cantor-Bernestein we have $|Seq(A)| = |N|$.

3.5 $[A]^n$ is subset of all finite subset of A, but the set of all finite subset of a countable set is countable by Corollary 3.11, We show that it is infinite, assume that it is finite so we have a finite sequence of $S_1, ..., S_k$ of S, each of S_i is fintie, but union of finite system of fintie set is finite (Theorem 2.7), so

$A - \bigcup_{i=0}^{i=k} S_i$ is infinite, call it X, let $a \in X$ and $S \in [A]^n$, pick some $s \in S$, then $|S - \{s\} \cup \{a\}| = n$ but clearly $S - \{s\} \cup \{a\} \notin [A]^n$ (since $a \notin S_i$ for any $S_i \in [A]^n$).

Since $[A]^n$ is infinite, Theorem 3.2 implies that it is countable.

3.6 Let $X \subset N^N$ be the set of eventually constant sequences of natural numbers and let $N^{\in N}$ be the set of all finite sequence of natural numbers. define $f : X \to N^{\in N}$ as follows: for each sequences $(s_n)_{n=0}^{\infty} \in X$ such that for some $n_0 \in N$, $s_n = s_{n_0}$ for all $n \geq n_0$, let $f((s_n)_{n=0}^{\infty}) = (s_0, ..., s_{n_0})$, this function is one-to-one and onto, therefore by Theorem 10 $|X| = |N^{\in N}| = |N|$.

3.9 To prove the function is injective, suppose that $f(s) = f(s')$ for some $s, s' \in Seq(N - \{0\})$, then $f(s) = p_0^{s_0} \cdot p_1^{s_1} ... \cdot p_k^{s_k} = f(s') = p_0^{s'_0} \cdot p_1^{s'_1} ... \cdot p_k^{s'_k}$, this implies $k = k'$ since otherwise one has a prime factor which is not in the factorization of other. so we $p_i^{s_i} = p_i^{s'_i}$ for every $i \leq k$, but since $s_i \neq 0$ this implies $s_i = s'_i$, it means that $s = s'$.

$f[Seq(N - \{0\})]$ is infinite since for every $p_0^{s_0} \cdot p_1^{s_1} ... \cdot p_k^{s_k}$ in it we have $p_0^{s_0} \cdot p_1^{s_1} ... \cdot p_k^{s_k} < p_0^{s_0} \cdot p_1^{s_1} ... \cdot p_k^{s_k+1}$, so since $f[Seq(N - \{0\})] \subset N$ and is infinite, by Theorem 3.2 it is countable, therefore $|f[Seq(N - \{0\})]| = |N|$, but f is one-to-one function from $Seq(N - \{0\})$ to $f[Seq(N - \{0\})]$ so $Seq(N - \{0\})$ is countable.

3.10 Since A_n is finite there are some bijective mapping $f : |A_n| \to A_n$, pick the one mapping that respects the ordering of $<$ on A_n, so if $a, b \in |A_n|$ and $a < b$ then $f(a) < f(b)$. so for every A_n we can construct an enumeration $(a_n(k) : k \in |A_n|)$ such that $a_n(k) = f(k)$, therefore $\bigcup_{n=0}^{\infty} A_n$ is at most countable.

3.11 Let X be a set that is at most countable, so we can write it as a finite sequence or infinite $\{x_0, x_1 ...\}$, let P a partition on it, for every $A \in P$ pick x_j such that j is least in enumeration of X that is in A, this is a representation of A.