5 Page 37

5.1 (a) For any $a, b \in A$, aSb iff $aRb \wedge a \neq b$, but aR^*b iff $aSb \vee a = b$ iff $(aRb \wedge a \neq b) \vee a = b$ iff $aRb \vee a = b$ iff aRb (since R is reflexive $a = b$ implies aRb).

(b) aRb iff $aSb \vee a = b$, but aS^*b iff $aRb \wedge a \neq b$ iff $(aSb \vee a = b) \wedge a \neq b$ iff $(aSb \wedge a \neq b)$ iff aSb (since S is irreflexive).

5.2 a and b are incomparable if $a \neq b$, neither $a < b$ nor $b < a$.

a is maximal in A : $\neg(\exists x \in A)(a < x)$

a is the greatest element of A : $(\forall x \in A)(x < a \vee x = a)$.

a is an upper bound of A: $(\forall x \in A)(x < a \vee x = a)$

a is supremum of A in X: $(\forall x \in A)(x < a \vee x = a) \wedge (\forall a' \in X)(\forall x \in A)[(x < a' \vee x = a') \rightarrow a < a' \vee a = a']$.

5.3 (a) for any $a \in A$ we have aRa, also $aR^{-1}a$, thus R^{-1} is reflexive. Suppose that $aR^{-1}b$ and $bR^{-1}a$, then we have bRa, aRb, by antisymmetricity of R we get $a = b$,therefore R^{-1} is antisymmetric.

Now suppose $aR^{-1}b$ and $bR^{-1}c$, then we get bRa and cRb, but transitivity of R implies cRa, thus $aR^{-1}b$, thus R is transitive.

(a) $(\forall x \in B)(aR^{-1}x)$ iff $(\forall x \in B)(xRa)$.

5.4 Let $R' = R \cap B^2$, then for every $a \in B$ we have $(a, a) \in B^2$, since $B \subseteq A$ and R is an order on A, by reflexitivity $(a, a) \in R$, thus $aR'a$, hence R' is reflexive.

Let $aR'b, bR'a$ then $a, b, c \in B$ and aRb, bRa, because R is antisymmetric we get $a = b$,

5.5 Let $A = \mathcal{P}(\{a, b, c\}) - \{\{a, b, c\}, \emptyset\}$ and $R =\subseteq$.

(a) $B = \{\{a\}, \{b\}\}$.

(b) $B = \{\{a\}, \{b\}\}$.

(c) $B = \{\{a\}, \{b\}\}$.

(d) $B = \{\{a, b\}, \{b, c\}\}$.

5.6 (a) For every $x \in B$ either $x = b$ or $x \in A$, if $x = b$ then $x \notin A$ and both disjunct in the definition of $\prec$ would be false, if $x \in A$ then $x \not< x$ because $<$ is irreflexive, so the first disjunct could not be true, the other disjunct require that $x = b$ but it is impossible because $x \in A$, hence both of them is false, thus $x \not\prec x$.

Now let $x \prec y$, if $x \in A$ and $y = b$ then clearly $x \neq b$ because $b \notin A$, so we can not have $y \in A$ and $x = b$ and also we can not have $y = b, x \in A$ (the first item of first disjunct), thus $y \not\prec x$. Assume that $x \prec y, y \prec z$, if x, y, z all are in A then $x \prec z$ easily follows from transitivity of $<$. but if $x \in A$ and $y = b$ then $y \prec z$ is impossible, because in both disjunnct it requires that $y \in A$, but $y = b$. So the only case we need to check is that when $x, y \in A$ and $x < y$ and $y \in A$ and $z = b$, but from this it easy follows that $x \in A$ and $z = b$, thus $x \prec z$. Notice that $\prec = < \cup (A \times \{b\})$ and $< \subset A^2$, but $(A \times \{b\}) \cap A^2 = \emptyset$, thus $\prec \cap A^2 = (< \cap A^2) \cup (A \times \{b\}) \cap A^2 = < \cup \emptyset = <$.

5.7 Because R is reflexive for every $a \in A, aRa$ and also aRa which implies that aEa. Now let aEb then aRb and bRa, also bRa and aRb, thus bEa. Let aEb, bEc then aRb, bRa and bRc, cRb, by transitivity of R we get aRc and cRa thus aEc, hence E is transitive.

Assume aRb, then $[a]_E R/E [b]_E$, now let $b' \in [b]_E$, then bEb', hence bRb', by transitivity of R we get aRb', hence $[a]_E R/E [b']_E$. we can repeat this argument for a.

Because R is reflexive for every a, we have aRa, also $[a]_E R/E [a]_E$. Assume that $[a]_E R/E [b]_E$ and $[b]_E R/E [a]_E$, then we get aRb and bRa, hence aEb which means that $[a]_E = [b]_E$.

To prove that R/E is transitive, assume $[a]_E R/E [b]_E$ and $[b]_E R/E [c]_E$, then we get aRb and bRc, by transitivity of R it follows that aRc, hence $[a]_E R/E [c]_E$.

5.8 (a) Let $S \subseteq A$, then every $x \in S$, $x \subseteq \bigcup S$, thus $\bigcup S$ is an upper bound of S, to prove that it is the least upper bound assume we prove that it is subset of every upper bound a, i.e $\bigcup S \subseteq a$, let $x \in \bigcup S$, then for some $C \in S$, $x \in C$, but a is an upper bound for S, thus $C \subseteq a$, hence $x \in a$, thus $\bigcup S \subseteq a$.

(b) The set of all lower bounds of $\emptyset$ is the set of all $a \in A$ such that for every $x \in \emptyset$, $a \subseteq x$, so all member of A satisfy this condition because $x \notin \emptyset$, the greatest element of A is X, since $Y \subseteq X$ for any $Y \in A = \mathcal{P}(X)$.

5.9 (a) $\subseteq$ is reflexive, antisymmetric and transitive on any set, thus it is an ordering.

(b) Let $F \subseteq Fn(X, Y)$, assume that $sup\ F$ exist, and F is not compatible, then there are some $g, f \in F$ such that for some $x \in dom\ f \cap dom\ g, f(x) \neq g(x)$, hence there are distinct $a, b \in Y$ such that $(x, a) \in f$ and $(x, b) \in g$, but then since $f, g \subseteq sup\ F$, hence $(x, b), (x, a) \in sup\ F$, it contradicts the fact that $sup\ F$ is a function. Now assume that F is a compatible system of functions. then by Theorem 3.12 $\bigcup F$ is a function and clearly $\bigcup F \in Fn(X, Y)$, it follows from a similar argument to Exercise 5.8(a) that $\bigcup F = sup\ F$.

5.10 (a) Because for every $S \in Pt(A)$ we have that for all $C \in S$, there is some some $D \in S$ such that $C \subseteq D$, namely C itself, thus $S \preccurlyeq S$ for all $S \in Pt(A)$ and it is reflexive.

Assume that $S_1 \preccurlyeq S_2$ and $S_2 \preccurlyeq S_1$, then for every $C \in S_1$, $C \subseteq D$ for some $D \in S_2$, but because $S_2 \preccurlyeq S1$ and $D \in S_2$, we have some $E \in S_1$ such that $D \subseteq E$, we show that $E = C$. Assume it is not the case, then $C \subseteq D \subseteq E$ implies $C \cap E \neq \emptyset$, contrary to the assumption that S is a partition, thus the relation is symmetric.

Let $S_1 \preccurlyeq S_2$ and $S_2 \prec S_3$, then for every $C \in S_1, C \subseteq D$ for some $D \in S_2$, but because $S_2 \preccurlyeq S_3$, there is some $E \in S3$ such that $D \subseteq E$, thus for every $C \in S_1, C \subseteq E$ for some $E \in S3$, therefore $S_1 \preccurlyeq S_2$ and $\preccurlyeq$ is transitive.

(b) Let $S = \{C \cap D : C \in S_1 \wedge D \in S_2\}$, clearly S is a partition and $S \preccurlyeq S1$, $S \preccurlyeq S_2$, thus S is a lower bound for $\{S_1, S_2\}$, we prove that any lower bound $S' \preceq S$. Assume S' is a lower bound, then for every $C \in S'$, $C \subseteq D$ for some $D \in S_1$ and also $C \subseteq D'$ for some $D' \in S_2$ but then there is some $X \in S$ and $C \subseteq X$, namely $X = D \cap D'$, so we proved that for every $C \in S'$, $C \subseteq X$ for some $X \in S$, thus $S' \preccurlyeq S$.

$aE_S b$ implies $aE_{S_1}b, aE_{S_2}b$

(c) Let $T = (T_i : i \in I)$ and $S = \{(\bigcap_{i \in I} f_i) : f \in \prod T_i\}$, fix some $T_k \in T$, we want to prove that $S \preccurlyeq T_k$. let $C \in S$ and $x \in C$ then for some f, $x \in f_i$ for all $i \in I$, but $f_i = D$ for some $D \in T_i$, from this it follows that $x \in f_k$, thus for some $D \in T_k$ we have $x \in D$, thus $C \subseteq D$, we conclude that $S \preccurlyeq T_k$. We prove S is greatest lower bound, assume S' is another lower bound for T, it means that for every $C \in S'$ and for every T_i there is some $D_i \in T_i$ such that $C \subseteq D_i$, define $f : I \to \bigcup T$ by $f_i = D_i$, then clearly $C \subseteq \bigcap_{i \in I} f_i$, but $\bigcap_{i \in I} f_i \in S$, thus $S' \preccurlyeq S$.

(d) Let $T' = \{S \in Pt(A) : (\forall i \in I)(T_i \preccurlyeq S)\}$ clearly it is the set of upper bounds of T, by previous exercise $Inf\ T'$ exist, we prove that $Inf\ T' \in T'$, fix some $T_k \in T$, and let $C \in T_k$, we know that for every $S \in T'$ we have $T_k \preccurlyeq S$, it means that for every $S \in T'$ there is some $D \in S$ such that $C \subseteq D$, if we index T' by J, we have for every $T'_j \in T'$ there is some $D_{T'_j} \in T'_j$ such that $C \subseteq D_{T'_j}$, define $f : J \to \bigcup T'$ by $f_j = D_{T'_j}$ then clearly $C \subseteq \bigcap_{j \in J} f_j \in Inf\ T'$, thus we proved for arbitrary $T_k, T_k \preccurlyeq Inf\ T'$, thus $InfT' \in T'$ and it is the least element of it, the least element among upper bound of T, thus $sup\ T = inf\ T'$.

5.11 Let f be the isomorphism, let $y_1, y_2 \in Q$ then there is some $x_1, x_2 \in P$ such that $f(x_1) = y_1, f(x_2) = y_2$ but because $<$ is linearly ordered we have either $x_1 = x_2$ or $x_1 < x_2$ or $x_2 < x_1$, but because f is an isomorphism we get either $f(x_1) = f(x_2)$ or $f(x_1) \prec f(x_2)$ or $f(x_2) \prec f(x_1)$, rewrite this for y_1 and y_2.

5.12 Suppose that $x <_1 y$ for some $x, y \in P_1$ then we have $f(x) <_2 f(y)$, but since $f(x), f(y) \in P_2$ we get $g(f(x)) <_3 g(f(y))$, thus $g \circ f(x) <_3 g \circ f(y)$. Now let $u <_3 z$ for some $u, z \in P_3$, because g is an isomorphism there are some $t, v \in P_2$ such that $f(t) = u, f(v) = z$ and $t <_2 v$, but because f is isomorphism we get $f(x) = t <_2 v = f(y)$ which implies $x <_1 y$.