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4.1 (a) Because of imporophism there are isomorphic functions $f : A_1 \to B_1$ and $g : A_2 \to B_2$. Define a function $h : A_1 \cup A_2 \to B_1 \cup B_2$ such that for any $a \in A_1$, $h(a) = f(a)$ and otherwise $h(a) = g(a)$. Because $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$ and f and g are one-to-one and onto, h is one-to-one and onto. To see it respect the order consider $a, b \in A_1 \cap A_2$ and $a < b$ ($<$ denotes the order of sum) if both are in A_1 then by isomorphism f, we get $f(a), f(b) \in B_1$ and then $h(a) \prec_1 h(b)$. The case $a, b \in A_2$ can be handled similarly. If $a \in A_1$ and $b \in A_2$, then nessarily we have $a < b$, it means $h(a) = f(a) \in B_1$ and $h(b) = g(b) \in B_2$, then by ordering of sum of B_1 and B_2 we get $h(a) < h(b)$.

(b) Assume that $<_a$ and $<_b$ are the lexographic ordering of A_1, A_2, and B_1, B_2.

Define $h : A_1 \times A_2 \to B_1 \times B_2$ such that, $h((a, b)) = (f(a), g(b))$.

Suppose that $(a_1, a_2) <_a (b_1, b_2)$, then either $a_1 <_1 b_1$:

which implies $f(a_1) \prec_1 f(b_2)$, therefore $(f(a_1), g(a_2)) <_b (f(b_1), g(b_2))$, thus $h((a_1, a_2)) <_b h((b_1, b_2))$.

or: $a_1 = b_1$ and $a_2 <_2 b_2$, which implies $g(a_2) \prec_2 g(b_2)$ and also $f(a_1) = f(b_1)$ therefore therefore $(f(a_1), g(a_2)) <_b (f(b_1), g(b_2))$, thus $h((a_1, a_2)) <_b h((b_1, b_2))$.

The converse can be proved similarly.

4.2 Let $<'$ be the sum of $(N, <)$ and $(N', <^{-1})$ and $<''$ be the sum of $(N', <^{-1})$ and $(N, <)$ (suppose N' is isomorphic to N but non-identical), clearly $<'$ has the least element 0 over $N \cup N'$. But $<''$ has not the least element over $N' \cup N$. So can not be isomorphic. For lexographic ordering, consider the example of antilexographic ordering on page 83, (notice that antilexographic ordering of two A and B is lexographic ordering of B and A). One ordering is isomorphic to the ordering of N, while the other is not.

4.3 Suppose $(A_1, <_1)$ and $(A_2, <_2)$ are two well-ordering (therefore are linear ordering). By Lemma 4.5 $<$ is a linear ordering. To prove that the sum of them is well-ordering, suppose $(A_1 \cup A_2, <)$ and suppose a non-empty subseet $X \subseteq A_1 \cup A_2$, if $X \cap A_1 = \emptyset$ then $X \subseteq A_2$, then X has a least element a on the order $<_2$, therefore it is the least element on the order $<$ (since X contains no element from A_1). if $X \cap A_1 \neq \emptyset$, then $X \cap A_1 \subseteq A_1$, thus it has a least element a in the ordering $<_1$ so it is also the least element in $<$. But we must show it is the least element of X, suppose $b \in X$ but $b \notin A_1$, which implies $b \in A_2$, but then since $a \in A_1$ we get $a < b$, therefore for any $b \in X$ we have $a < b$.

Let $(A_1 \times A_2, <)$ be the lexographic ordering, we are to prove it is a well-ordering. By lemma 4.6 it is a linear ordering. Let $X \subseteq A_1 \times A_2$, suppose a is the least element of the set $\{x : (x, y) \in X\}$ (it exists, because the well-ordering of A) and let b the least of the set $\{y : (a, y) \in X\}$. Now we show (a, b) is the least element of X : let $(x, y) \in A_1 \times A_2$, then either $a <_1 x$ or $x = a$ (because a is the least element), the first case simply implies $(a, b) < (x, y)$. For the second case, we have $a = x$ but either $y = b$ or $b < y$, which means if they are not identical, we have $(a, b) < (x, y)$, thus we are done in both cases.

4.4 Pick a sequence $(a_i : i \in N) \in \Pi_{i \in N} A_i$ such that a_i is not the least element of A_i (it exists, since $2 \leq |A_i|$). Now define a recursive function as follows : $f_0 = (a_i : i \in N)$ and if $f_n = (b_i : i \in N)$, define $f_{n+1} = (c_i : i \in N)$ such that $c_n = a$ such that a is the least elment of A_i, for $i \neq n$ let $c_i = b_i$. Now the set $X = \{f_n : n \in N\}$ has no least element: take any $f_k \in X$, then $f_k = (x_i : i \in N)$ and $f_{k+1} = (y_i : i \in N)$ for some sequences, we know that k is the least element of $diff(f_k, f_{k+1})$ by defnition, and also $y_k < x_k$, which implies $f_{k+1} \prec f_k$.

4.8 Consider $(Seq(A), \prec)$:

Transitivity: Assume that $(a_0 \ldots a_{m-1}) \prec (b_0 \ldots b_{n-1}) \prec (c_0 \ldots c_{j-1})$. Then either for some $k < n$, $a_i = b_i$ for all $i < k$ and either $a_k < b_k$ or a_k is undefined. And there is a $k' < j$ such that $b_i = c_i$ for all $i < k'$ and either $b_{k'} < c_{k'}$ or b_k is undefined. Let $t = min(k', k)$. then $a_i = b_i = c_i$ for all $i < t$, if $t = k$ then $b_k = c_k$, $a_k < b_k$ implies $a_k < c_k$, otherwise a_k is undefined, thus $(a_0 \ldots a_{m-1}) \prec (c_0 \ldots c_{j-1})$. if $t = k'$ then $a_{k'} = b_{k'}$, thus $b_{k'} < c_{k'}$ implies $a_{k'} < c_{k'}$ and $b_{k'}$ is undefined implies $a_{k'}$ is undefined, again $(a_0 \ldots a_{m-1}) \prec (c_0 \ldots c_{j-1})$.

Asymmetry: Assume that $(a_0, \ldots, a_{m-1}) \prec (b_0, \ldots, b_{n-1})$ and $(b_0, \ldots, b_{n-1}) \prec (a_0 \ldots a_{m-1})$. Then there is $k < n$ such that $a_i = b_i$, for all $i < k$, either $a_k < b_k$ or a_k is undefined. And also $j < m$ for which, $b_i = a_i$ for all $i < j$ and either $b_j < a_j$ or b_j is undefined. Assume $j < k$, then $a_j = b_j$ which contradicts both $b_j < a_j$ and b_j is undefined. Assume that $k < j$ then $b_k = a_k$ which contradicts both $a_k < b_k$ and a_k is undefined. The case $k = j$ similarly leads to contradiction.

Linearity: Consider $a = (a_0, \ldots, a_{m-1})$ and $b = (b_0, \ldots b_{n-1})$, let $min(a, b) = \{i \text{ is the least } i \text{ such that } a_i \neq b_i\}$. if $min(a, b) = \emptyset$ then either $a = b$ or $a_i = b_i$ for all $i < m$, which means a_m is undefined, then $a \prec b$. or $a_i = b_i$ for all $i < n$, which means b_n is undefined, thus $b \prec a$.

Well-ordering: Let $X \subseteq Seq(A)$:

Let t_0 be the least element of $T_0 = \{a_0 : (\exists m \in N)(a_0, \ldots, a_{m-1}) \in X\}$. Assume we have defined T_k for $k \leq n$, and t_k is the least element of T_k, define $T_{n+1} = \{a_{k+1} : (\exists m \in N)(t_0, ..., t_k, a_{k+1}, a_{m-1}) \in X\}$. Let $(t_0, ..., t_l)$ be the sequence with the least length l in X.

Let $(x_0, \ldots, x_{k-1}) \in X$, assume to the contrary $(x_0, \ldots, x_{k-1}) \prec (t_0, ..., t_{l-1})$, for some $j < l$, any $x_i = t_i$ for all $i < j$, if $j = k$ then x_j is undefined, so $k < l$, it contradicts our assumption that l is the least length of sequence in X which has t_i's in it. Thus $l \leq k$ if $x_j < t_j$, then t_j is not the least element of T_j, which is a contradiction.

4.13 (a) $\{1 - \frac{1}{n} : n \in N - \{0\}\} \cup \{2 - \frac{1}{n} : n \in N - \{0\}\}$

(b) $\{1 - \frac{1}{n} : n \in N - \{0, 1\}\} \cup \{1 + \frac{1}{n} : n \in N - \{0, 1\}\}$

(c) $\bigcup_{k \in N - \{0\}} \{k - \frac{1}{n} : n \in N - \{0, 1\}\}$