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2.1 Since S is finite, we can prove it by induction on n in $|S| = n$. for $|S| = 0$ it trivially holds. Assume that it holds for every set $|S| = n$, then consider $|S| = n+1$, we would get $\bigcup S = \bigcup_{i=0}^{i=n-1} X_i \cup X_n$, but from induction hypothesis we have $|\bigcup_{i=0}^{i=n-1} S| = \Sigma_{i=0}^{n-1}|X_i|$, with Lemma 2.6 we get $|\bigcup S| = |\bigcup_{i=0}^{i=n-1} X_i \cup X_n| = |\bigcup_{i=0}^{i=n-1} X_i| + |X_n| = \Sigma_{i=0}^{n-1}|X_i| + |X_n| = \Sigma_{i=0}^{n}|X_i|$.

2.2 Fix some X, assume that $|X| = m$, consider Y, if $|Y| = 0$ then $Y = \emptyset$, thus $|X \times Y| = m \cdot 0 = 0$, we are done. Assume that it holds for every set $|S| = n$, thus for it we have $|X \times S| = m \cdot n$, we must prove it for $|X \times Y| = m \cdot (n+1)$. Now let $|Y| = n+1$ then $Y = \{y_0, ..., y_n\}$, but $X \times Y = X \times \{y_0, ..., y_{n-1}\} \cup X \times \{y_n\}$, from induction hypothesis we have $|X \times \{y_0, ..., y_{n-1}\}| = m \cdot n$, also we know that $|X \times \{y_n\}| = |X| = m$, so by Lemma 2.6 $|X \times Y| = |X \times \{y_0, ..., y_{n-1}\}| + |X \times \{y_n\}| = m \cdot n + m = m \cdot (n+1)$, since X was arbitary finite set, it holds for all finite set.

2.3 We proceed by induction on cardinal of X, if $|X| = 0$ then $X = \emptyset$, thus $|\mathcal{P}(X)| = |\{\emptyset\}| = 2^0 = 1$. Now assume that it holds for $|X| = n$, consider when $|X| = n+1$, let $X' = \{x_0, ..., x_{n-1}\}$, then $\mathcal{P}(X) = \mathcal{P}(X') \cup \{K \cup \{x_n\} : K \in \mathcal{P}(X')\}$, by induction hypothesis $|\mathcal{P}(X')| = 2^{|X'|} = 2^n$ and so the other set, they are also disjoint set, thus we get $|\mathcal{P}(X) = |\mathcal{P}(X')| + |\{K \cup \{x_n\} : K \in \mathcal{P}(X')\}| = 2^n + 2^n = 2^n \cdot 2 = 2^{n+1}$ that completes the proof.

2.4 Fix some X, we proceed by induction on $|Y|$, if $|Y| = 0$ then $Y = \emptyset$, then $|X^{\emptyset}| = |\{\emptyset\}| = |X|^0 = 1$. Now assume that it holds for all set S such that $|S| = n$. if $|Y| = n + 1$ then $X^Y = X^{\{y_0,...,y_{n-1}\} \cup \{y_n\}} = \{f \cup \{(y_n, x)\} : (f, x) \in X^{\{y_0,...,y_{n-1}\}} \times X\}$, (Why they are equal: clearly each $f \cup \{(y_n, x)\} \in X^Y$, since $y_n \notin \{y_0, ..., y_{n-1}\}$, it is a function with domain Y and range X. Now let $g \in X^Y$, then for some $x \in X$ we have $g(y_n) = x$, clearly $(g - \{(y_n, x)\}) \in X^{\{y_0,...,y_{n-1}\}}$, also $(g - \{(y_n, x)\}, x) \in X^{\{y_0,...,y_{n-1}\}} \times X$, for it we have $(g - \{(y_n, x)\}) \cup \{(y_n, x)\} = g$). from exercise 2.2 and induction hypothesis we can conclude that $|X^Y| = |\{f \cup \{(y_n, x)\} : (f, x) \in X^{\{y_0,...,y_{n-1}\}} \times X\}| = |X^{\{y_0,...,y_{n1}\}} \times X| = |X^{\{y_0,...,y_{n-1}\}}| \cdot |X| = |X|^n \cdot |X| = |X|^{n+1} = |X|^{|Y|}$, therefore it holds for $|Y| = n + 1$, we are done.

2.7 (we just prove some of these properties) Consider two disjoint set X, Y, Z such that $|X| = m$ and $|Y| = n, |Z| = p$, commutativity of addition: by Lemma 2.6 we have $|X \cup Y| = |X| + |Y| = m + n$, but we know that $X \cup Y = Y \cup X$ so $|X \cup Y| = |Y \cup X|$ it implies $m + n = n + m$.

associativity of addition: since $(X \cup Y) \cup Z = X \cup (Y \cup Z)$ we get $|(X \cup Y) \cup Z| = |X \cup (Y \cup Z)| = |(X \cup Y)| + |Z| = |X| + |(Y \cup Z)| = m + (n + p) = (m + n) + p$.

distributivity of multiplication over addition: since $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$ (by exercise 2.8 on page 23) then by previous exercises it easily follows that $|X| \cdot (|Y| + |Z|) = (|X| \cdot |Y|) + (|X| \cdot |Z|) = m \cdot (n + p) = (m \cdot n) + (m \cdot p)$.