Exercise 5

4. (C1): Let $a \in A$, then every $b \in u(A)$, $a \leq b$, which means a is a lower bound for $u(A)$, thus $a \in l(u(A))$, therefore $A \subseteq C(A)$.

(C2): First note that $X \subseteq u(l(X))$: let $x \in X$, then every $b \in l(X)$, $b \leq x$ which means that x is an upper bound for $l(X)$, thus $x \in u(l(X))$, from this we get (*) $u(A) \subseteq u(l(u(A)))$. Now, let $a \in C^2(A) = l(u(l(u(A))))$ which means $a \leq b$ for any $b \in u(l(u(A)))$, from (*) we get $a \leq c$ for any $c \in u(A)$, which means $a \in l(u(A)) = C(A)$, therefore $C^2(A) \subseteq C(A)$. $C(A) \subseteq C^2(A)$ follows from the first item.

(C3): Suppose that $A \subseteq B$, then $u(B) \subseteq u(A)$: let $x \in u(B)$, then $b \leq x$ for any $b \in B$ so for any $b \in A$, thus $x \in u(A)$. By a similar argument, we can show that $l(u(A)) \subseteq l(u(B))$, which means $C(A) \subseteq C(B)$.

Let $f(a) = C(\{a\})$, we show that $f(a \wedge b) = C(\{a\}) \cap C(\{b\})$. Notice that $C(\{a\}) = l(\{a\})$. Let $x \in f(a \wedge b) = C(\{a \wedge b\})$ then $x \leq a \wedge b$ then $x \leq a$ and $x \leq b$, which means $x \in C(\{a\}) \cap C(\{b\})$. For the converse, let $x \in C(\{a\}) \cap C(\{b\})$, then $x \leq a$ and $x \leq b$, therefore $x \leq a \wedge b$, thus $x \in C(\{a \wedge b\})$.

Now we show that $f(a \vee b) = C(\{a \vee b\}) = C(\{a, b\})$: Let $x \in C(\{a \vee b\})$, then $x \leq a \vee b$.

Now let $t \in u(\{a, b\})$ then $a \leq t$, $b \leq t$, so $a \vee b \leq t$, so $x \leq t$, therefore $x \in l(u(\{a, b\}))$. For the converse, let $x \in C(\{a, b\})$, then $x \leq a$ and $x \leq b$, so $x \leq a \vee b$, which means $x \in C(\{a \vee b\})$.

5. Assume that C is a closure operator on A such that $A_i \in K$ iff $A_i = C(A_i)$ for any $A_i \subseteq A$. K is closed under arbitrary intersection, i.e $C(\bigcap A_i) = \bigcap A_i$, this follows from Theorem 5.2. Now suppose that K is closed under arbitrary intersections, define this:

$$C(Y) = \bigcap \{X \in K : Y \subseteq X\}$$

We show C is a closure: (C1): Since $Y \subseteq X$ for any $X \in \{X \in K : Y \subseteq X\}$, we have $Y \subseteq C(Y)$.

(C2): let $x \in C^2(Y)$, then $x \in X$ for any $X \in K$ such that $C(Y) \subseteq X$, but $C(Y) \in K$ (because it is closed under intersection), therefore for one $X \in K$ we have $X = C(Y)$, thus $x \in C(Y)$ and thereby $C^2(Y) \subseteq C(Y)$.

(C3): Suppose that $X \subseteq Y$, let $x \in C(X)$ then $x \in Z$ for any $Z \in K$ such that $X \subseteq Z$, we know that $X \subseteq Y \subseteq C(Y)$ and $C(Y) \in K$, thus for some $Z \in K$ we have $Z = C(Y)$, therefore $x \in C(Y)$, thereby $C(X) \subseteq C(Y)$.