# Exercise 1

**1.** $\subseteq$ is a partial order on any set, to prove it is a lattice, we show that $sup\{a, b\}$ and $inf\{a, b\}$ exits. $sup\{a, b\} = a \cup b$ and clearly $a \cup b \in Su(A)$. Since $a \subseteq a \cup b$ and $b \subseteq a \cup b$ it is an upper bound for $\{a, b\}$. To prove it is the least upper bound, assume that there is some set $c \in Su(A)$ such that $a \subseteq c, b \subseteq c$, and $c \subsetneq a \cup b$, it means there is some $k$ in $a$ or $b$, such that $k \notin c$, a contradiction. $inf\{a, b\} = a \cap b$ can be proved similarly.

**2.** L1:(a) $a \vee b = sup\{a, b\} = sup\{b, a\} = b \vee a$.

(b) $a \wedge b = inf\{a, b\} = inf\{b, a\} = b \wedge a$.

L2: (a) Let $t = sup\{a, sup\{b, c\}\}$ and $k = sup\{sup\{a, b\}, c\}$, then $t$ is greater than $a$ and $sup\{b, c\}$, which means $t$ is also greater than $b, c$, thus $sup\{a, b\} \leq t$, therefore $sup\{sup\{a, b\}, c\} \leq t$, or $k \leq t$. $k = sup\{sup\{a, b\}, c\}$ implies $c \leq k$ and $sup\{a, b\} \leq k$, which means $a \leq k$ and $b \leq k$. Now we can say that $sup\{b, c\} \leq k$, therefore $sup\{a, sup\{b, c\}\} \leq k$. From $t \leq k$ and $k \leq t$ we conclude $t = k$.

(b) Can be proved similarly.

L3: (a) and (b) follows from $inf\{a, a\} = sup\{a, a\} = a$.

L4: (a) Let $t = sup\{a, inf\{a, b\}\}$ then $a \leq t$. Since $inf\{a, b\} \leq a$, we get $sup\{a, inf\{a, b\}\} \leq sup\{a, a\} = a$, thus $t \leq a$ and finally $a = t$.

(b) Let $t = inf\{a, sup\{a, b\}\}$, then $t \leq a$, and since $a \leq sup\{a, b\}$ we have $a = inf\{a, a\} \leq inf\{a, sup\{a, b\}\}$, thus $a \leq t$.

**3.** From L4(b) we have $x \approx x \wedge (x \vee x)$, replace $x \vee x$ for $y$ in the formula L4(a), we get $x \approx x \vee [x \wedge (x \vee x)]$, replacing the first formula in it, we get $x \approx x \vee x$. Similarly, replace $y$ by $x \wedge x$ in L4(b), we get $x \approx x \wedge [x \vee (x \wedge x)]$, from L4(a) we have $x \approx x \vee (x \wedge x)$. Therefore $x \approx x \wedge x$.

**5.** Since each item in L1,.., L4 has two symmetric axioms for $\vee$ and $\wedge$, the result trivially follows.

**9.** (Note: we write $(a, b) \in R$ instead of $aRb$)

Consider the following collection:

$$O = \{\preceq \subseteq A \times A : \preceq \text{ is a partial order on } A\}$$

Clearly, $(O, \subseteq)$ is a poset and $\leq \in O$, we show that every chain in it has an upper bound. Let $C \subseteq O$ be a chain, clearly $\bigcup C \subseteq A \times A$, we show it is a partial order: *reflexive.* Let $a \in A$, then for every $\preceq \in C$ we have $(a, a) \in \preceq$ (since every partial order is reflexive), thefeore $(a, a) \in \bigcup C$. *antisymmetric.* Consider $(a, b), (b, a) \in \bigcup C$, then for some $\preceq \in C$ we have $(a, b) \in \preceq$ and for some $\preceq' \in C$, $(b, a) \in \preceq$. Since $C$ is a chain we have either $\preceq \subseteq \preceq'$ or $\preceq' \subseteq \preceq$, in either case since they are antisymmetric we get $a = b$. *transitive.* Suppose $(a, b), (b, c) \in \bigcup C$, then for some $\preceq, \preceq' \in C$ we have $(a, b) \in \preceq$ and $(b, c) \in \preceq'$, again, since $C$ is chain one of the orders is superset of another, without loss of generality assume $\preceq' \subseteq \preceq$, then by transitivity of $\preceq$ we get $(a, c) \in \preceq$, thus $(a, c) \in \bigcup C$. We proved any chain $C$ of $O$, $\bigcup C$ is in $O$, thus any chain has an upper bound in it. By Zorn's Lemma (see item (e) on page 2) $O$ has a maximal member, denote it by $\leqslant^*$. Since $\leqslant \subseteq \leqslant^*$ we have if $a \leqslant b$ then $a \leqslant^* b$.

Assume that $\leqslant^*$ is not a total order, then there are some $a, b$ such that $a \nleqslant^* b$ and $b \nleqslant^* a$. Define a relation $R$ such that $xRy$ iff $x \leqslant^* a$ and $b \leqslant^* y$. You can check that $\bigcup R$ is partial order on $A$ (tedious but trivial) and $\bigcup R \neq \leqslant^*$ but $\leqslant^* \subseteq \bigcup R$. It contradicts the fact that $\leqslant^*$ is a maximal element.

**10.** Let $S$ be the set of elements $a$ of $L$ such that there is no $a_1 \vee \cdots \vee a_n$ in $L$ such that $a = a_1 \vee \cdots \vee a_n$ and each $a_i$ is join irreducible. This set is non-empty, let $b$ be a minimal element of it (it exists because $L$ is finite). Notice that $b$ is not join irreducible, otherwise we could have $b = b \vee b$. Since it is not join-irreducible there must be some element $c, d \in L$ such that $b = c \vee d$ but $b \neq c$ and $b \neq d$. On the other hand $c$ and $d$ are less than $b$, but $b$ is minimal element so $c, d \notin S$, thus we have $c = a_i \vee \cdots \vee a_n$ and $d = a'_i \vee \cdots \vee a'_m$ for some $a_i$, $i \leq n$ and $a'_j$, $j \leq m$ such that each $a_i$ and $a_j$ are join irreducible. Now we have $b = c \vee d = a_i \vee \cdots \vee a_n \vee a'_i \vee \cdots \vee a'_m$ which contradicts the fact that $b \in S$.