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1.2 Suppose that $\kappa = |A|$, then $\kappa^0 = |A^0|$, since $0 = \emptyset$, $A^0 \subseteq \mathcal{P}(\emptyset \times A) = \{\emptyset\}$, but $\emptyset$ is a function from 0 to A: Assume to the contrary that it is not, clearly it is a relation, since $\emptyset \subset \emptyset \times A$. So there must be some $(a, b) \in 0$ and $(a, c) \in 0$ such that $a \neq b$, a clear contradiction. Therefore $\kappa^0 = |A^0| = |\{\emptyset\}| = 1$.

To see $\kappa^1 = \kappa$, define a function $f : A \to A^1$ such that for every $a \in A$, $f(a) = \{(\emptyset, a)\}$, clearly $\{(\emptyset, a)\}$ is function from $1 = \{\emptyset\}$ to A, and any function in A^1 has this form, thus f is one-to-one and onto.

1.3 Since $1^\kappa = |1^A|$, any function $h \in 1^A = \{\emptyset\}^A$ is a constant function, i.e., $h(a) = \emptyset$ for all $a \in A$. but there is only one such function, thus $1^A = \{h\}$, so $1^\kappa = 1$. To see $0^\kappa = 0$, notice that $0^A \subseteq \mathcal{P}(A \times \emptyset) = \{\emptyset\}$, and $\emptyset \notin 0^A$, since otherwise for $a \in A$ (A is non-empty because $\kappa > 1$ by assumption), we must have $(a, b) \in \emptyset$ such that $b \in \emptyset$ (the functions considered in the book are totall by defnition), which is a contradiction. Thus $0^A = \emptyset$ and $0^\kappa = 0$.

1.4 By Cantor theorem we have $\kappa < 2^\kappa$, and by Lemma 1.6(n) we get $\kappa^\kappa < (2^\kappa)^\kappa$. But $(2^\kappa)^\kappa = 2^{\kappa\dot{\kappa}}$ by Theorem 1.7(b), thus $\kappa^\kappa \leq 2^{\kappa\dot{\kappa}}$.

1.5 $|A| \leq |B|$ implies that there is a one-to-one function $f : A \to B$, but then $f^{-1} : ranf \to A$ is an onto function, since $A \neq \emptyset$, take some $a \in A$ and let $h = f^{-1} \cup \{(x, a) : x \in B/ranf\}$, it is easy to check that h is an onto function from B to A.

1.6 Assume that $g : B \to A$ is an onto function, by Theorem 1.9 its enough to show that $\mathcal{P}(A) \leq \mathcal{P}(B)$, define $f : \mathcal{P}(A) \to \mathcal{P}(B)$ such that for every $X \subseteq A$, $f(X) = g^{-1}[X]$, f is one-to-one: assume $f(X) = f(Y)$, thus $g^{-1}[X] = g^{-1}[Y]$, then $g[g^{-1}[X]] = g[g^{-1}[Y]]$ implies $X = Y$.

1.7 Assume that V is the set of all sets, then $\mathcal{P}(V)$ and its members are sets, thus $\mathcal{P}(V) \subseteq V$, then the identity mapping gives us $|\mathcal{P}(V)| \leq |V|$, which contradicts Cantors Theorem which says $|V| < |\mathcal{P}(V)|$.

1.8 Assume that X is finite, then $X = \{a_0, .., a_n\}$. if $f : X \to X$ is a one-to-one mapping, then $f : X \to f[X]$ is onto and one-to-one. Let X_k be the set of indices of a_k in $f[X]$. Then define $h : n \to X_k$ such that $h(i) = j$ if $f(a_i) = a_j$. Since f is one-to-one and onto, h must be so. But we know that $f[X] \subset X$, which implies $X_k \subset n$, this contradicts Lemma 2.2 (page 70).

1.9 Let X be a countbale set, then there is a one-to-one sequence $(a_n : n \in N)$. Define a function $g : X \to X$ as follows: for any $a \in X$, $g(a) = a_{n+1}$. This function is one-to-one because the above sequence is one-to-one, $g[X]$ is proper subset of X, since $a_0 \notin g[X]$.

1.10 Suppose that X' is a countbale subset of X, by previous Exercise X' is a Dedekind set, so there is a one-to-one function f from it, to its proper subset. Define a function $g : X \to X$ such that for any $a \in X'$, $g(a) = f(a)$, otherwise $g(a) = a$. it is clear that g is one-to-one, to see it is into a proper subset of X, notice that $rang = f[X'] \cup X/X'$ and because X' is Dedekind infinite, there is some $x \in X' \subseteq X$ which $x \notin f[X']$, also $x \notin X/X'$, thus $x \notin rang$.

1.11 Let X be a Dedekind set, there is a function f from X to its proper subset, so there is some $a \in X$ such that $a \notin f[X]$. By Recursion Theorem (taking f as g) there is a function $h : N \to X$ such that $h(0) = a$ and for any $n \in N$, $h(n + 1) = f(n)$. It is easy to check that h is one-to-one: suppose $h(n) = h(n)$, it means $f(n) = f(m)$ but f is one-to-one, then $m = n$. clearly h is onto $h[N]$, so it is enumerable and $h[N] \subseteq X$.