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3.1 We must prove that the set $\{x : x \in A \text{ } and \text{ } x \notin B\}$ exist. Let $P(x, A, B)$ be the property "$x \in A \text{ } and \text{ } x \notin B$", $P(x, A, B)$ implies $x \in A$, because A exist, we have $\{x : x \in A \text{ } and \text{ } x \notin B\} = \{x \in A : x \in A \text{ } and \text{ } x \notin B\} = \{x \in A : x \notin B\}$, this set clearly exist by the axiom of comprehension.

3.2 Weak Axiom of Existence implies that some set exist, call one of them A and let $P(x)$ be the property "$x \neq x$", by axiom of comprehension the set $X = \{x \in A : x \neq x\}$ exist, it has no element because no object satisfy the property $P(x)$.

3.3 (a) Suppose that V is set of all sets, by Comprehension $X = \{x \in V : x \notin x\}$ exist. Because V is set of all sets, clearly $X \in V$. Now suppose that $X \in X$ then $X \notin X$ by definition, a contradiction. suppose $X \notin X$, then $X \in X$ again by definition.

(b) Assume the contrary, there is a set A that any $x \in A$. then $A = V$ is set of all sets, by previous exercise there is no V.

3.4 By axiom of pairing the set $\{A, B\}$ exist and union axiom implies the existence of $\bigcup\{A, B\}$, let $P(x, A, B) = (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)$ by comprehension there is a set that its elements satisfy $P(x, A, B)$ and $x \in \bigcup\{A, B\}$.

3.5 3.5(a) by axiom of pairing there is $\{A, B\}$ and $\{C\}$. again by pairing $\{\{A, B\}, \{C\}\}$. by axiom of union there is $X = \bigcup\{\{A, B\}, \{C\}\}$. Now $x \in X$ iff $x \in \{A, B\}$ or $x \in \{C\}$ iff $x = A$ or $x = B$ or $x = C$.

(b) Take $\{C, D\}$ instead of $\{C\}$ in the previous exercise.

3.6 Assume that $\mathcal{P}(X) \subseteq X$, Now let $Y = \{x \in X : x \notin x\}$, clearly $Y \subseteq X$, so $Y \in \mathcal{P}(X)$, thus $Y \in X$. also we have either $Y \in Y$ or $Y \notin Y$. if first, $Y \notin Y$, if th second $Y \in Y$, thus $Y \in Y$ iff $Y \notin Y$, a contradiction.

3.7 Let $P(x, A, B)$ be the property "$x = A \vee x = B$", apply axiom of comprehension to C, we get the set $X \subseteq C$ such that $x \in X$ iff $x = A$ or $x = B$, so $X = \{A, B\}$.

Let $P'(x,S)$ be the property "$\exists A(A \in S \wedge X \in A)$", apply axiom of comprehension to U, we get the set Y such that $x \in Y$ iff for some $A \in S$ we have $x \in A$, thus $Y = \bigcup S$.

Let $P'(x,S)$ be the property "$x \subseteq S$", apply axiom of comprehension to P, we get the set Z such that $x \in Z$ iff $x \subseteq S$, thus $Y = \mathcal{P}(S)$.

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4.2 (a) Left to right, assume $A \subseteq B$(*), and let $x \in A \cap B$, which means that $x \in A$ and $x \in B$, we can conclude $x \in A$, thus $A \cap B \subseteq A$(**). to prove the other direction, let $x \in A$, by assumption (*) we get $x \in B$, we can conclude $x \in A$ and $x \in B$, which means that $x \in A \cap B$, so we have $A \subseteq A \cap B$, so by this and (**) we have $A = A \cap B$.

Right to left, suppose $A \cap B = A$(*), let $x \in A$, by (*) $x \in B$, so we have $A \subseteq B$.

Second part, $x \in A \cup B$ iff $x \in B$, it means that there is nothing in A such that is not in B, thus $A - B = \emptyset$.

(b) Left to right, suppose $A \subseteq B \cap C$, let $x \in A$, by previous assumption we have $x \in B \cap C$, which implies that $x \in B$ and $x \in C$, so we have $A \subseteq B$ and $A \subseteq C$.

Right to left, suppose $A \subseteq B$ and $A \subseteq C$, let $x \in A$, by two previous assumtion we have both $x \in B$ and $x \in C$ which implies that $x \in B \cap C$, thus we have $A \subseteq B \cap C$.

(c) Suppose $B \cup C \subseteq A$, let $x \in B$, we can get also $x \in B \cup C$, by previous assumption we conclude that $x \in A$, thus $B \subseteq A$. by similar argument we can show $C \subseteq A$.

(d) $x \in A - B$ iff $x \in A \wedge \neg(x \in B)$ iff $x \in A \wedge \neg(x \in B) \vee (x \in B \wedge \neg(x \in B))$ iff $(x \in A \vee x \in B) \wedge \neg(x \in B)$ iff $x \in (A \cup B) - B$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in A))$ iff $x \in A \wedge (\neg(x \in A) \vee \neg(x \in B))$ iff $x \in A - (A \cap B)$.

(e) $x \in A \cap B$ iff $x \in A \wedge x \in B$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge \neg(x \in A))$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge \neg(x \in A))$ iff $x \in A \wedge (x \in B \vee \neg(x \in A))$ iff $x \in A \wedge \neg(\neg(x \in B) \wedge (x \in A))$ iff $x \in A \wedge \neg(x \in A - B))$ iff $x \in A - (A - B)$.

(f) $x \in A - (B - C)$ iff $x \in A \wedge \neg(x \in B - C)$ iff $x \in A \wedge \neg(x \in B \wedge \neg(x \in C))$ iff $x \in A \wedge (\neg(x \in B) \vee (x \in C))$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge x \in C)$ iff $x \in A - B \vee x \in A \cap C$ iff $x \in (A - B) \cup (A \cap C)$.

(g) $(A - B) \cup (B - A) = \emptyset$ iff both (*) $A - B = \emptyset$ and $B - A = \emptyset$, by (a) we get (*) iff $A \subseteq B$ and $B \subseteq A$ iff $A = B$.

4.4 Suppose it exist, then $A' \cup A$ is equal to universal set which does not exist.

4.5 (a) let $x \in A \cap \bigcup S$, then $x \in A$ and $x \in C$ for some $C \in S$, it means that $x \in A \cap C$, clearly $A \cap C \in P(A)$ so $A \cap C \in T_1$ by definition, thus $x \in \bigcup T_1$. (Note that if we take $A \cap C = C$, then we can say that for some $C \in T_1$ we have $x \in C$). Now let $x \in \bigcup T_1$, then there is some $Y \in T_1$ such that $x \in Y$, but by definition of T_1 we know that $Y = A \cap X$ for some $X \in S$, it means that $x \in \bigcup S$ and $x \in A$, thus $x \in A \cap \bigcup S$.

(b) Let $x \in A - \bigcup S$, we have $x \in A - \bigcup S$ iff $x \in A$ and $x \notin X$ for any $X \in S$. it equally means that (*) $x \in A - X$ for every $X \in S$. we know that any set in the form of $A - X$ such that $X \in S$ is in T_2, thus (*) means that we have $x \in \bigcap T_2$.

$x \in A - \bigcap S$ iff $x \in A$ and $x \notin C$ for some $C \in S$ iff $x \in A - C$ for some $C \in S$, because any set in the form of $A - X$ such that $X \in S$ is in T_2 we have some $x \in \bigcap T_2$.

4.6 if S is not empty, then there is some $C \in S$, by Axiom Schema of Comprehension the set $\{x \in C : (\forall X)(X \in S \to x \in X)\}$ exist. if it is empty, then we can not apply the axiom of comprehension.

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1.1 1.1 We know that both $\{a\}$ and $\{a, b\}$ are subset of $\{a, b\}$, thus $\{a, b\}, \{a\} \in \mathcal{P}(\{a, b\})$, it means that $\{\{a, b\}, \{a\}\} \subseteq \mathcal{P}(\{a, b\})$ which implies $\{\{a, b\}, \{a\}\} \in \mathcal{P}(\mathcal{P}(\{a, b\}))$.

we have $a, b \in \{a, b\}$, but $(a, b) = \{\{a\}, \{a, b\}\}$ which means that there is some $C \in (a, b)$ such tha $a, b \in C$, thus $a, b \in \bigcup(a, b)$.

if $a, b \in A$ then $\{a, b\}$ and $\{a\}$ both are subset of A, thus $\{a, b\}, \{a\} \in \mathcal{P}(A)$, again it implies that $\{\{a, b\}, \{a\}\} \subseteq \mathcal{P}(A)$, thus $(a, b) = \{\{a, b\}, \{a\}\} \in \mathcal{P}(\mathcal{P}(A))$.

1.2 1.2 if a and b exist, then by axiom of pairing and powerset $T = \mathcal{P}(\mathcal{P}(\{a,b\}))$ exist and by previous exercise $(a,b) \in T$. because $(a,b,c) = ((a,b),c)$ by previous argument we have $(a,b,c) \in \mathcal{P}(\mathcal{P}(\{(a,b),c\}))$ which clearly exist.

1.3 if $(a,b) = (b,a)$, it follows from Theorem 1.2 that $a = b$ and $b = a$, so $a = b$.

1.4 if $(a,b,c) = (a',b',c')$ then $((a,b),c) = ((a',b'),c')$, by Theorem 1.2 we have (*) $(a,b) = (a',b')$ and $c = c'$, but again by Theorem 1.2 and (*) we have $a = a'$ and $b = b'$.

1.5 Let $a = \emptyset$, $b = \{a\}$ and $c = \{b\}$, then if $((a,b),c) = (a,(b,c))$ we get $(a,b) = a = \emptyset = \{\{a\},\{a,b\}\}$ which is a contradiction.

1.6 We first prove that:

(1) $a = c$ or $d = \Box$.

(2) $b = d$ or $c = \triangle$.

To prove (1): $\{\{a,\Box\},\{b,\triangle\}\} = \{\{c,\Box\},\{d,\triangle\}\}$ implies either ($\bullet$) $\{a,\Box\} = \{c,\Box\}$ or ($\star$) $\{a,\Box\} = \{d,\triangle\}$, if ($\bullet$) then either $a = c$ or $a = \Box$, if first we are done, if the second then $\{a,\Box\} = \{\Box\} = \{c,\Box\}$ which means $a = \Box = c$, thus in both case $a = c$. if ($\star$) then either $a = d$ or $a = \triangle$, if first then $\{a,\Box\} = \{a,\triangle\}$ which implies $\triangle = \Box$, contradiction, so we have $a = \triangle$, then $\{\triangle,\Box\} = \{d,\triangle\}$ which implies $d = \Box$. so we have either $a = c$ or $d = \Box$.

To prove (2):

We also have (*) $\{b,\triangle\} = \{c,\Box\}$ or (**) $\{b,\triangle\} = \{d,\triangle\}$, if (*) then either $b = c$ or $b = \Box$, if first then $\{b,\triangle\} = \{b,\Box\}$ which implies a contradiction: $\triangle = \Box$, therefore the second case only remains which implies $c = \triangle$. if (**) then either $b = d$ or $b = \Box$, if first we are done, if the second then $\{\Box,\triangle\} = \{d,\triangle\}$ which implies $b = \Box = d$, so in both case we have $b = d$. so we have either (2) $b = d$ or $c = \triangle$.

So we have (1) and (2), assume that $b = d$ from (2), now consider (1), if first case then we are done. if the second then $b = d = \Box$, therefore $\{\{a,\Box\},\{\Box,\triangle\}\} = \{\{c,\Box\},\{\Box,\triangle\}\}$ which implies $a = c$.

Assume the second case of (2), then by first case of (1) we have $a = c = \triangle$, therefore $\{\{\triangle,\Box\},\{b,\triangle\}\} = \{\{\triangle,\Box\},\{d,\triangle\}\}$ which implies $b = d$.

Now consider the second case of (1), then we have $d = \Box$ and $c = \triangle$ then $\{\{a, \Box\}, \{b, \triangle\}\} = \{\{\triangle, \Box\}, \{\Box, \triangle\}\} = \{\{\Box, \triangle\}\}$, then $a = \triangle = c$ and $b = \Box = d$, we are done.

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2.1 Let $(x, y) = \{\{x\}, \{x, y\}\} \in R$, then $\{\{x\}, \{x, y\}\} \subseteq \bigcup R$, thus we have $\{x, y\} \in \bigcup R$ and we know that $x, y \in \{x, y\}$, so for some set $C \in \bigcup R$ we have $x, y \in C$, thus $x, y \in \bigcup \bigcup R$. because the property "$x \in dom\ R$" implies that $(x, y) \in R$ for some y. and because $(x, y) \in R$ implies $x \in A$, the set $\{x \in A : x \in dom\ R\}$ exist. Repeat this argument for property "$x \in ran\ R$".

2.2 (a) by previous argument $ran\ R$ and $dom\ R$ exist, we know that $ran\ R \times dom\ R$ exist, call it A. by comprehension the subset $\{(y, x) \in A : (x, y) \in R\}$ also exist, this set is equal to R^{-1}. again by comprehension the set $\{(x, y) \in dom\ R \times ran\ S : for\ some\ z,\ (x, z) \in R\ and\ (z, y) \in S\}$, this set is equal to $S \circ R$.

(b) Because $A \times B \times C = (A \times B) \times C \subseteq \mathcal{P}((A \times B) \cup C)$, comprehension implies that the set $\{x \in \mathcal{P}((A \times B) \cup C) : x = (y, z)\ for\ some\ y \in A \times B\ and\ z \in C\}$ exist.

2.3 (a) $y \in R[A \cup B]$ iff $(\exists x)(x \in A \cup B \wedge\ xRy)$ iff $(\exists x)((x \in A \vee x \in B) \wedge\ xRy)$ iff $(\exists x)((x \in A \wedge\ xRy) \vee (x \in B \wedge\ xRy))$ iff $(\exists x)(x \in A \wedge\ xRy) \vee (\exists x)(x \in B \wedge\ xRy)$ iff $y \in R[A] \vee y \in R[B]$ iff $y \in R[A] \cup R[B]$.

(b) Let $y \in R[A \cap B]$, then for some $x \in A \cap B$ we have xRy which means that $x \in A$ such that xRy and $x \in B$ such that xRy, thus $x \in R[A] \cap R[B]$.

(c) Suppose that $y \in R[A] - R[B]$, it means there is some $x \in A$ such that xRy but there is no $z \in B$ such that zRy, because xRy holds for x, it can not be in B, thus $x \in A - B$ and xRy which means that $y \in R[A - B]$.

(d) Let $R = \{(a, c), (b, c)\}$ and $A = \{a\}$, $B = \{b\}$ then $R[A] \cap R[B] = \{c\}$ while $R[A \cap B = \emptyset] = \emptyset$. also $R[A - B] = R[\{a\}] = \{c\}$ but $R[A] - R[B] = \{c\} - \{c\} = \emptyset$, so this falsifies converse of both (b) and (c).

(f) Fix $x \in A \cap dom\ R$, then because $x \in dom\ R$ there is some y such that xRy, because $x \in A$ we conclude that $y \in R[A]$, so there is some $y \in R[A]$ such that xRy or equivalently $yR^{-1}x$, thus $x \in R^{-1}[R[A]]$.

Fix $y \in B \cap ran\ R$, since $y \in ran\ R$ for some x we have xRy, but $y \in B$ implies that $x \in R^{-1}[B]$, thus for some $x \in R^{-1}[B]$ we have xRy, therefore

$y \in R[R^{-1}[B]]$.

Let $R = \{(a,c),(b,c),(e,f),(e,g)\}$ and $A = \{a\}$, then $A \cap dom\ R = \{a\}$ but $R[A] = \{c\}$, thus $R^{-1}[R[A]] = R^{-1}[\{c\}] = \{a,b\}$, but $\{a,b\} \not\subseteq \{a\}$.

Let R be as before and $B = \{g\}$, then $R^{-1}[B] = \{e\}$ and $R[R^{-1}[B]] = \{f,g\}$, but $B \cap ranR = \{g\}$.

2.4 $R[X] \subseteq ran\ R$ because for any $y \in R[X]$ we have some $x \in X$ such that xRy, thus $y \in ran\ R$. if $y \in ran\ R$, then for some $x \in dom\ R$ we have xRy, but $dom\ R \subseteq X$, thus $x \in X$, so we get for some $x \in X$, xRy, therefore $y \in R[X]$.

suppose $x \in dom\ R$ then there is some $y \in ran\ R$ such that xRy, but xRy iff $yR^{-1}x$ and $ranR \subseteq Y$, therefore there is some $y \in Y$ such that $yR^{-1}x$ which is equal to say that $x \in R^{-1}[Y]$, left to right is trivial.

(b) Assume $a \notin dom\ R$ but $R[\{a\}] \neq \emptyset$, so for some $y \in R[\{a\}]$ we have aRy which means that $a \in dom\ R$, this contradicts our assumption.

Assume $b \notin ran\ R$ and $R^{-1}[\{b\}] \neq \emptyset$, so there is some $x \in R^{-1}[\{b\}]$ such that $bR^{-1}x$ or equivalently xRb, it means that $b \in ran\ R$ which contradicts the assumption.

(c) $x \in dom\ R$ iff for some y, xRy iff $yR^{-1}x$ iff $x \in ran\ R^{-1}$.

$y \in ran\ R$ iff for some x, xRy iff $yR^{-1}x$ iff $y \in dom\ R^{-1}$.

(d) $(x,y) \in R$ iff $(y,x) \in R^{-1}$ iff $(x,y) \in (R^{-1})^{-1}$.

(e) if $(x,x) \in Id_{dom\ R}$ then $x \in domR$ which implies that for some y, $(x,y) \in R$, but $(x,y) \in R$ iff $(y,x) \in R^{-1}$, thus we can say that there is some y such that $(x,y) \in R$ and $(y,x) \in R^{-1}$ which is equal to $(x,x) \in R^{-1} \circ R$. the second part can be proved like this.

2.5 $\mathcal{P}(X) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

$\in_Y = \{(\emptyset, \{\emptyset\}), (\emptyset, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \{\{\emptyset\}\})\}$.

$Id_Y = \{(\emptyset, \emptyset), (\{\emptyset\}, \{\emptyset\}), (\{\{\emptyset\}\}, \{\{\emptyset\}\}), (\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\})\}$.

$ran(Id_Y) = dom(Id_Y) = fld(Id_Y) = \mathcal{P}(X)$.

$dom(\in_Y) = \{\emptyset, \{\emptyset\}\}$, $ran(\in_Y) = \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, $fld(\in_Y) = \mathcal{P}(X)$.

2.6 $(x,y) \in T \circ (S \circ R)$ iff $(\exists z)((x,z) \in (S \circ R) \wedge (z,y) \in T)$ iff $(\exists z)((\exists u)[(x,u) \in R \wedge (u,z) \in S] \wedge (z,y) \in T)$ iff $(\exists z)((\exists u)[(x,u) \in R \wedge (u,z) \in S \wedge (z,y) \in T])$

iff $(\exists z)(\exists u)((x,u) \in R \wedge (u,z) \in S \wedge (z,y) \in T)$ iff $(\exists u)((x,u) \in R \wedge (\exists z)[(u,z) \in S \wedge (z,y) \in T])$ iff $(\exists u)((x,u) \in R \wedge (u,y) \in T \circ S)$ iff $(x,y) \in (T \circ S) \circ R$.

2.7 Let $X = \{a\}$ and $Y = \{b, c\}$, $Z = \{d\}$.

(a) $(a, b) \in X \times Y$ but $(a, b) \notin Y \times X$.

(b) $(a, (b, d)) \in X \times (Y \times Z)$ but $(a, (b, d)) \notin (X \times Y) \times Z$.

(c) $((a, a), a) \in X^2 \times X$ but $((a, a), a) \notin X \times X^2$.

2.8 (a) Assume $A \neq \emptyset$ and $B \neq$, then there is some $a \in A$ and $b \in B$, but then $(a, b) \in A \times B$, so $A \times B \neq \emptyset$. Now assume $A \times B \neq \emptyset$, then there is some $x \in A \times B$ such that $x = (a, b)$, but it means that $a \in A$ and $b \in B$, thus $A, B \neq \emptyset$.

(b) $(a, b) \in (A_1 \cup A_2) \times B$ iff $(a \in A_1 \cup A_2) \wedge b \in B$ iff $(a \in A_1 \vee a \in A_2) \wedge b \in B$ iff $(a \in A_1 \wedge b \in B) \vee (a \in A_2 \wedge b \in B)$ iff $(a, b) \in (A_1 \times B) \vee (a, b) \in (A_2 \times B)$ iff $(a, b) \in (A_1 \times B) \cup (A_2 \times B)$.

$(a, b) \in A \times (B_1 \cup B_2)$ iff $a \in A \wedge b \in (B_1 \cup B_2)$ iff $a \in A \wedge (b \in B_1 \vee b \in B_2)$ iff $(a \in A \wedge b \in B_1) \vee (a \in A \wedge b \in B_2)$ iff $(a, b) \in (A \times B_1) \vee (a, b) \in (A \times B_2)$ iff $(a, b) \in (A \times B_1) \cup (A \times B_2)$.