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2.1 Let $(x, y) = \{\{x\}, \{x, y\}\} \in R$, then $\{\{x\}, \{x, y\}\} \subseteq \bigcup R$, thus we have $\{x, y\} \in \bigcup R$ and we know that $x, y \in \{x, y\}$, so for some set $C \in \bigcup R$ we have $x, y \in C$, thus $x, y \in \bigcup\bigcup R$. because the property "$x \in dom\ R$" implies that $(x, y) \in R$ for some y. and because $(x, y) \in R$ implies $x \in A$, the set $\{x \in A : x \in dom\ R\}$ exist. Repeat this argument for property "$x \in ran\ R$".

2.2 (a) by previous argument $ran\ R$ and $dom\ R$ exist, we know that $ran\ R \times dom\ R$ exist, call it A. by comprehension the subset $\{(y, x) \in A : (x, y) \in R\}$ also exist, this set is equal to R^{-1}. again by comprehension the set $\{(x, y) \in dom\ R \times ran\ S : for\ some\ z,\ (x, z) \in R\ and\ (z, y) \in S\}$, this set is equal to $S \circ R$.

(b) Because $A \times B \times C = (A \times B) \times C \subseteq \mathcal{P}((A \times B) \cup C)$, comprehension implies that the set $\{x \in \mathcal{P}((A \times B) \cup C) : x = (y, z)\ for\ some\ y \in A \times B\ and\ z \in C\}$ exist.

2.3 (a) $y \in R[A \cup B]$ iff $(\exists x)(x \in A \cup B \wedge\ xRy)$ iff $(\exists x)((x \in A \vee x \in B) \wedge\ xRy)$ iff $(\exists x)((x \in A \wedge\ xRy) \vee (x \in B \wedge\ xRy))$ iff $(\exists x)(x \in A \wedge\ xRy) \vee (\exists x)(x \in B \wedge\ xRy)$ iff $y \in R[A] \vee y \in R[B]$ iff $y \in R[A] \cup R[B]$.

(b) Let $y \in R[A \cap B]$, then for some $x \in A \cap B$ we have xRy which means that $x \in A$ such that xRy and $x \in B$ such that xRy, thus $x \in R[A] \cap R[B]$.

(c) Suppose that $y \in R[A] - R[B]$, it means there is some $x \in A$ such that xRy but there is no $z \in B$ such that zRy, because xRy holds for x, it can not be in B, thus $x \in A - B$ and xRy which means that $y \in R[A - B]$.

(d) Let $R = \{(a, c), (b, c)\}$ and $A = \{a\}$, $B = \{b\}$ then $R[A] \cap R[B] = \{c\}$ while $R[A \cap B = \emptyset] = \emptyset$. also $R[A - B] = R[\{a\}] = \{c\}$ but $R[A] - R[B] = \{c\} - \{c\} = \emptyset$, so this falsifies converse of both (b) and (c).

(f) Fix $x \in A \cap dom\ R$, then because $x \in dom\ R$ there is some y such that xRy, because $x \in A$ we conclude that $y \in R[A]$, so there is some $y \in R[A]$ such that xRy or equivalently $yR^{-1}x$, thus $x \in R^{-1}[R[A]]$.

Fix $y \in B \cap ran\ R$, since $y \in ran\ R$ for some x we have xRy, but $y \in B$ implies that $x \in R^{-1}[B]$, thus for some $x \in R^{-1}[B]$ we have xRy, therefore

$y \in R[R^{-1}[B]]$.

Let $R = \{(a, c), (b, c), (e, f), (e, g)\}$ and $A = \{a\}$, then $A \cap dom\ R = \{a\}$ but $R[A] = \{c\}$, thus $R^{-1}[R[A]] = R^{-1}[\{c\}] = \{a, b\}$, but $\{a, b\} \not\subseteq \{a\}$.

Let R be as before and $B = \{g\}$, then $R^{-1}[B] = \{e\}$ and $R[R^{-1}[B]] = \{f, g\}$, but $B \cap ranR = \{g\}$.

2.4 $R[X] \subseteq ran\ R$ because for any $y \in R[X]$ we have some $x \in X$ such that xRy, thus $y \in ran\ R$. if $y \in ran\ R$, then for some $x \in dom\ R$ we have xRy, but $dom\ R \subseteq X$, thus $x \in X$, so we get for some $x \in X$, xRy, therefore $y \in R[X]$.

suppose $x \in dom\ R$ then there is some $y \in ran\ R$ such that xRy, but xRy iff $yR^{-1}x$ and $ranR \subseteq Y$, therefore there is some $y \in Y$ such that $yR^{-1}x$ which is equal to say that $x \in R^{-1}[Y]$, left to right is trivial.

(b) Assume $a \notin dom\ R$ but $R[\{a\}] \neq \emptyset$, so for some $y \in R[\{a\}]$ we have aRy which means that $a \in dom\ R$, this contradicts our assumption.

Assume $b \notin ran\ R$ and $R^{-1}[\{b\}] \neq \emptyset$, so there is some $x \in R^{-1}[\{b\}]$ such that $bR^{-1}x$ or equivalently xRb, it means that $b \in ran\ R$ which contradicts the assumption.

(c) $x \in dom\ R$ iff for some y, xRy iff $yR^{-1}x$ iff $x \in ran\ R^{-1}$.

$y \in ran\ R$ iff for some x, xRy iff $yR^{-1}x$ iff $y \in dom\ R^{-1}$.

(d) $(x, y) \in R$ iff $(y, x) \in R^{-1}$ iff $(x, y) \in (R^{-1})^{-1}$.

(e) if $(x, x) \in Id_{dom\ R}$ then $x \in domR$ which implies that for some y, $(x, y) \in R$, but $(x, y) \in R$ iff $(y, x) \in R^{-1}$, thus we can say that there is some y such that $(x, y) \in R$ and $(y, x) \in R^{-1}$ which is equal to $(x, x) \in R^{-1} \circ R$. the second part can be proved like this.

2.5 $\mathcal{P}(X) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

$\in_Y = \{(\emptyset, \{\emptyset\}), (\emptyset, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \{\{\emptyset\}\})\}$.

$Id_Y = \{(\emptyset, \emptyset), (\{\emptyset\}, \{\emptyset\}), (\{\{\emptyset\}\}, \{\{\emptyset\}\}), (\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\})\}$.

$ran(Id_Y) = dom(Id_Y) = fld(Id_Y) = \mathcal{P}(X)$.

$dom(\in_Y) = \{\emptyset, \{\emptyset\}\}$, $ran(\in_Y) = \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, $fld(\in_Y) = \mathcal{P}(X)$.

2.6 $(x, y) \in T \circ (S \circ R)$ iff $(\exists z)((x, z) \in (S \circ R) \wedge (z, y) \in T)$ iff $(\exists z)((\exists u)[(x, u) \in R \wedge (u, z) \in S] \wedge (z, y) \in T)$ iff $(\exists z)((\exists u)[(x, u) \in R \wedge (u, z) \in S \wedge (z, y) \in T])$

iff $(\exists z)(\exists u)((x, u) \in R \wedge (u, z) \in S \wedge (z, y) \in T)$ iff $(\exists u)((x, u) \in R \wedge (\exists z)[(u, z) \in S \wedge (z, y) \in T])$ iff $(\exists u)((x, u) \in R \wedge (u, y) \in T \circ S)$ iff $(x, y) \in (T \circ S) \circ R$.

2.7 Let $X = \{a\}$ and $Y = \{b, c\}$, $Z = \{d\}$.

(a) $(a, b) \in X \times Y$ but $(a, b) \notin Y \times X$.

(b) $(a, (b, d)) \in X \times (Y \times Z)$ but $(a, (b, d)) \notin (X \times Y) \times Z$.

(c) $((a, a), a) \in X^2 \times X$ but $((a, a), a) \notin X \times X^2$.

2.8 (a) Assume $A \neq \emptyset$ and $B \neq$, then there is some $a \in A$ and $b \in B$, but then $(a, b) \in A \times B$, so $A \times B \neq \emptyset$. Now assume $A \times B \neq \emptyset$, then there is some $x \in A \times B$ such that $x = (a, b)$, but it means that $a \in A$ and $b \in B$, thus $A, B \neq \emptyset$.

(b) $(a, b) \in (A_1 \cup A_2) \times B$ iff $(a \in A_1 \cup A_2) \wedge b \in B$ iff $(a \in A_1 \vee a \in A_2) \wedge b \in B$ iff $(a \in A_1 \wedge b \in B) \vee (a \in A_2 \wedge b \in B)$ iff $(a, b) \in (A_1 \times B) \vee (a, b) \in (A_2 \times B)$ iff $(a, b) \in (A_1 \times B) \cup (A_2 \times B)$.

$(a, b) \in A \times (B_1 \cup B_2)$ iff $a \in A \wedge b \in (B_1 \cup B_2)$ iff $a \in A \wedge (b \in B_1 \vee b \in B_2)$ iff $(a \in A \wedge b \in B_1) \vee (a \in A \wedge b \in B_2)$ iff $(a, b) \in (A \times B_1) \vee (a, b) \in (A \times B_2)$ iff $(a, b) \in (A \times B_1) \cup (A \times B_2)$.