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6.1 Assume that N^N is countable, then there is an enumeration $(a_n : n \in N)$ of it such that $a_n \in N^N$. define $d : N \to N$ such that $d(n) = a_n(n) + 1$. we prove that $d \neq a_n$ for any $n \in N$, assume that for some $k \in N$, $d = a_k$ then for every $x \in N, d(x) = a_k(x)$, but then $d(k) = a_k(k) = a_k(k) + 1$, a contradiction.

6.2 Let $f \in N^N$, then $f : N \to N$, so $f \subseteq N \times N$, so $f \in \mathcal{P}(N \times N)$, thus (*) $N^N \subseteq \mathcal{P}(N \times N)$, we also know that $2^N \subset N^N$, so let $F : 2^N \to N^N$ be the identity map $F(f) = f$, clearly it is one-to-one. We also know that $|2^N| = |\mathcal{P}(N)|$ and by next exercise $|\mathcal{P}(N)| = |\mathcal{P}(N \times N)|$ (since $|N \times N| = |N|$), thus we have $|2^N| = |\mathcal{P}(N \times N)$, from this and (*) it follows that there is an injective function $G : N^N \to 2^N$

Since F and G are one-to-one, it follows from Cantor-Bernestein that $|2^N| = |N^N|$.

6.3 Let f be a bijective between A and B, define $g : P(A) \to P(B)$ by $g(X) = f[X]$ for each $X \subset A$. We prove it is one-to-one, Let $g(X) = g(X')$ then $f[X] = f[X']$ but since f is one-to-one we have $f^{-1}[f[X]] = f^{-1}[f[X']]$, so $X = X'$. Let $Y \subseteq B$, then $f^{-1}[B] \subset A$, but it means that for some $X \in P(A)$, such that $X = f^{-1}[B]$ we have $f[X] = f[f^{-1}[B]] = B$, so it is onto.