# Exercise 2

**2.** To show $Q^P$ is a lattice, we have to show that for any $f, g \in Q^P$, we have $sup\{f, g\}, inf\{f, g\} \in Q^P$. For any $f, g$ let $h$ be a mapping from $P$ to $Q$, such that for any $a \in P$, $h(a) = sup\{f(a), g(a)\}$, since $Q$ is a lattice this element exists. To see that $h$ is order-preserving, assume that $a, b \in P$ and $a \leq b$, then $f(a) \leq f(b)$ and $g(a) \leq g(b)$. Then it follows that $f(a) \leq sup\{f(b), g(b)\}$ and $g(a) \leq sup\{f(b), g(b)\}$, and finally $sup\{f(a), g(b)\} \leq sup\{f(b), g(b)\}$, which means $h(a) \leq h(b)$. Therefore $h \in Q^P$. By definition we have that $h$ is an upper bound for $\{f, g\}$. To show that it is the least upper bound, consider other $h'$ which is an upper bound for it. Then by definition we have $f(a) \leq h'(a)$ and $g(a) \leq h'(a)$, but $h(a) = sup\{f(a), g(a)\}$, thus $h(a) \leq h'(a)$, which means $h \leq h'$.

**4.** Let $C$ be the set of lower segments of $P$, clearly $\subseteq$ makes a partial order set on $C$. Consider some $p \in P$, if $x \in X \cap Y$ and $p \leq x$, since $X$ and $Y$ are lower segments, we get $p \in X$ and $p \in Y$, thus $p \in X \cap Y$. Therefore $X \cap Y \in C$. Consider some $p \in P$, if $x \in X \cup Y$ and $p \leq x$, then either $x \in X$ or $x \in Y$. Since $X$ and $Y$ are lower segments, we get $p \in X$ or $p \in Y$, thus $p \in X \cup Y$. Therefore $X \cup Y \in C$. It is straightforward to see that $X \cup Y$ and $X \cap Y$ are supremum and infimum, respectively. Suppose that $P$ has the least element $u$, the argument above can be repeated, except for the case that $X$ and $Y$ are disjoint lower segments, but this case can be ruled out because for every $Z \in L(P)$ we have $u \in Z$.

**5.** Let $I, J \in I(L)$, we show that $sup\{I, J\}, inf\{I, J\} \in I(L)$. it is trivial to check that $I \cap J$ is an ideal, thus $I \cap J = inf\{I, J\} \in I(L)$. Let $S_i$ be an ideal such that $I \cup J \subseteq S_i$, and define $sup\{I, J\} = \bigcap S_i$ for all such $S_i$. Now for any ideal $X$ such that $I \cup J \subseteq X$, we have $\bigcap S_i \subseteq X$, therefore it is the least upper bound for $\{I, J\}$. We just need to show that such $S_i$ exists: Let $P = I \cup J \cup \{a \vee b : a, b \in I \cup J\}$ and let $S = P \cup \{a : a \in L \text{ and } a \leq p \text{ for some } p \in P\}$. Clearly $I, J \subseteq S$, we show that $P$ is closed under $\vee$: let $a, b \in S$, if both $a, b$ are in $I$ or $J$ we are done. if $a \in I$ and $b \in J$ then by definition $a \vee b \in P$. if $a = a_I \vee a_J$ and $b = b_I \vee b_J$ for some members (the index shows which set includes the element) then $a \vee b = a_I \vee a_J \vee b_I \vee b_J = (a_I \vee b_I) \vee (a_J \vee b_J)$ but $a_I \vee b_I \in I$ and $a_J \vee b_J \in J$, thus $a \vee b \in P$. Now we show $S$ is ideal since by definition it is closed downward, we just need to show that it is closed under $\vee$: Let $a, b \in S$, we know that $a \leq c$ for some $c \in P$ and $b \leq d$ for some $d \in P$, which means $d \vee c$ is an upper bound for $\{a, b\}$, therefore $a \vee b \leq d \vee c$. But we showed $P$ is closed under $\vee$, thus $d \vee c \in P$, since $S$ is closed downward for $P$, we get $a \vee b \in S$.

**6.** We denote principal ideal generated from $a$ by $p(a) = \{x \in L : x \leq a\}$. Let $PI(L)$ be the set of all principal ideals of $L$, since $I(L)$ is a lattice and $PI(L) \subseteq I(L)$, we need to show $PI(L)$ is closed under $sup$ and $inf$: Let $X, Y \in PI(L)$, then $X = p(a)$ and $Y = p(b)$ for some $a, b$, we show that $sup\{X, Y\} = p(a \vee b)$: let $x \in sup\{X, Y\}$, then $x \leq a_X \vee b_Y$, for some $a_X \in X$ and $b_Y \in Y$, but $a_X \leq a$ and $b_Y \leq b$, thus $a_X \leq a \vee b$ and $b_y \leq a \vee b$, therefore $x \leq a_X \vee b_Y \leq a \vee b$ and $sup\{X, Y\} = p(a \vee b) \in PI(L)$. Now let $x \in inf\{X, Y\} = X \cap Y$, then $x \leq a$ and $x \leq b$, therefore $x \leq a \wedge b$, so $inf\{X, Y\} = p(a \wedge b)$. We proved that $PI(L)$ is a sublattice of $I(L)$. To show that it is isomorphic to $L$, consider the mapping $f : L \to PL(L)$ such that $f(a) = p(a)$. $f$ is order-preserving, since $a \leq b$ implies $p(a) \subseteq p(b)$, so is $f^{-1}$; $p(a) \subseteq p(b)$ implies $a \leq b$.

# Exercise 3

**1.** $Su(X)$ is a partial order by inclusion. Exercise 1.1 implies it is a lattice by operations $\cup$ and $\cap$. The distributivity follows from the simple set-theoretical fact that $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ and $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$.

**2.** Exercise 2.5 implies that $L$ and $PI(L)$ are isomorphic, assume that $f$ is the isomorphism, then $f(a \vee (b \wedge c)) = f(a) \vee [f(b) \wedge f(c)]$ and $f((a \vee b) \wedge (a \vee c)) = [f(a) \vee f(b)] \wedge [f(a) \vee f(c)]$, but since $L$ is distributive, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, therefore $f(a) \vee [f(b) \wedge f(c)] = [f(a) \vee f(b)] \wedge [f(a) \vee f(c)]$.

# Exercise 4

$\subseteq$ forms a partial order relation on any set. Now just notice that the union and intersection of arbitrary set of binary relations is still a binary relation.