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3.1 We must prove that the set $\{x : x \in A \; and \; x \notin B\}$ exist. Let $P(x, A, B)$ be the property "$x \in A \; and \; x \notin B$", $P(x, A, B)$ implies $x \in A$, because A exist, we have $\{x : x \in A \; and \; x \notin B\} = \{x \in A : x \in A \; and \; x \notin B\} = \{x \in A : x \notin B\}$, this set clearly exist by the axiom of comprehension.

3.2 Weak Axiom of Existence implies that some set exist, call one of them A and let $P(x)$ be the property "$x \neq x$", by axiom of comprehension the set $X = \{x \in A : x \neq x\}$ exist, it has no element because no object satisfy the property $P(x)$.

3.3 (a) Suppose that V is set of all sets, by Comprehension $X = \{x \in V : x \notin x\}$ exist. Because V is set of all sets, clearly $X \in V$. Now suppose that $X \in X$ then $X \notin X$ by definition, a contradiction. suppose $X \notin X$, then $X \in X$ again by definition.

(b) Assume the contrary, there is a set A that any $x \in A$. then $A = V$ is set of all sets, by previous exercise there is no V.

3.4 By axiom of pairing the set $\{A, B\}$ exist and union axiom implies the existence of $\bigcup\{A, B\}$, let $P(x, A, B) = (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)$ by comprehension there is a set that its elements satisfy $P(x, A, B)$ and $x \in \bigcup\{A, B\}$.

3.5 3.5(a) by axiom of pairing there is $\{A, B\}$ and $\{C\}$. again by pairing $\{\{A, B\}, \{C\}\}$. by axiom of union there is $X = \bigcup\{\{A, B\}, \{C\}\}$. Now $x \in X$ iff $x \in \{A, B\}$ or $x \in \{C\}$ iff $x = A$ or $x = B$ or $x = C$.

(b) Take $\{C, D\}$ instead of $\{C\}$ in the previous exercise.

3.6 Assume that $\mathcal{P}(X) \subseteq X$, Now let $Y = \{x \in X : x \notin x\}$, clearly $Y \subseteq X$, so $Y \in \mathcal{P}(X)$, thus $Y \in X$. also we have either $Y \in Y$ or $Y \notin Y$. if first, $Y \notin Y$, if th second $Y \in Y$, thus $Y \in Y$ iff $Y \notin Y$, a contradiction.

3.7 Let $P(x, A, B)$ be the property "$x = A \vee x = B$", apply axiom of comprehension to C, we get the set $X \subseteq C$ such that $x \in X$ iff $x = A$ or $x = B$, so $X = \{A, B\}$.

Let $P'(x,S)$ be the property "$\exists A(A \in S \wedge X \in A)$", apply axiom of comprehension to U, we get the set Y such that $x \in Y$ iff for some $A \in S$ we have $x \in A$, thus $Y = \bigcup S$.

Let $P'(x,S)$ be the property "$x \subseteq S$", apply axiom of comprehension to P, we get the set Z such that $x \in Z$ iff $x \subseteq S$, thus $Y = \mathcal{P}(S)$.

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4.2 (a) Left to right, assume $A \subseteq B$(*), and let $x \in A \cap B$, which means that $x \in A$ and $x \in B$, we can conclude $x \in A$, thus $A \cap B \subseteq A$(**). to prove the other direction, let $x \in A$, by assumption (*) we get $x \in B$, we can conclude $x \in A$ and $x \in B$, which means that $x \in A \cap B$, so we have $A \subseteq A \cap B$, so by this and (**) we have $A = A \cap B$.

Right to left, suppose $A \cap B = A$(*), let $x \in A$, by (*) $x \in B$, so we have $A \subseteq B$.

Second part, $x \in A \cup B$ iff $x \in B$, it means that there is nothing in A such that is not in B, thus $A - B = \emptyset$.

(b) Left to right, suppose $A \subseteq B \cap C$, let $x \in A$, by previous assumption we have $x \in B \cap C$, which implies that $x \in B$ and $x \in C$, so we have $A \subseteq B$ and $A \subseteq C$.

Right to left, suppose $A \subseteq B$ and $A \subseteq C$, let $x \in A$, by two previous assumtion we have both $x \in B$ and $x \in C$ which implies that $x \in B \cap C$, thus we have $A \subseteq B \cap C$.

(c) Suppose $B \cup C \subseteq A$, let $x \in B$, we can get also $x \in B \cup C$, by previous assumption we conclude that $x \in A$, thus $B \subseteq A$. by similar argument we can show $C \subseteq A$.

(d) $x \in A - B$ iff $x \in A \wedge \neg(x \in B)$ iff $x \in A \wedge \neg(x \in B) \vee (x \in B \wedge \neg(x \in B))$ iff $(x \in A \vee x \in B) \wedge \neg(x \in B)$ iff $x \in (A \cup B) - B$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in A))$ iff $x \in A \wedge (\neg(x \in A) \vee \neg(x \in B))$ iff $x \in A - (A \cap B)$.

(e) $x \in A \cap B$ iff $x \in A \wedge x \in B$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge \neg(x \in A))$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge \neg(x \in A))$ iff $x \in A \wedge (x \in B \vee \neg(x \in A))$ iff $x \in A \wedge \neg(\neg(x \in B) \wedge (x \in A))$ iff $x \in A \wedge \neg(x \in A - B))$ iff $x \in A - (A - B)$.

(f) $x \in A - (B - C)$ iff $x \in A \wedge \neg(x \in B - C)$ iff $x \in A \wedge \neg(x \in B \wedge \neg(x \in C))$ iff $x \in A \wedge (\neg(x \in B) \vee (x \in C))$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge x \in C)$ iff $x \in A - B \vee x \in A \cap C$ iff $x \in (A - B) \cup (A \cap C)$.

(g) $(A-B)\cup(B-A)=\emptyset$ iff both (*) $A-B=\emptyset$ and $B-A=\emptyset$, by (a) we get (*) iff $A\subseteq B$ and $B\subseteq A$ iff $A=B$.

4.4 Suppose it exist, then $A'\cup A$ is equal to universal set which does not exist.

4.5 (a) let $x\in A\cap\bigcup S$, then $x\in A$ and $x\in C$ for some $C\in S$, it means that $x\in A\cap C$, clearly $A\cap C\in P(A)$ so $A\cap C\in T_1$ by definition, thus $x\in\bigcup T_1$. (Note that if we take $A\cap C=C$, then we can say that for some $C\in T_1$ we have $x\in C$). Now let $x\in\bigcup T_1$, then there is some $Y\in T_1$ such that $x\in Y$, but by definition of T_1 we know that $Y=A\cap X$ for some $X\in S$, it means that $x\in\bigcup S$ and $x\in A$, thus $x\in A\cap\bigcup S$.

(b) Let $x\in A-\bigcup S$, we have $x\in A-\bigcup S$ iff $x\in A$ and $x\notin X$ for any $X\in S$. it equally means that (*) $x\in A-X$ for every $X\in S$. we know that any set in the form of $A-X$ such that $X\in S$ is in T_2, thus (*) means that we have $x\in\bigcap T_2$.

$x\in A-\bigcap S$ iff $x\in A$ and $x\notin C$ for some $C\in S$ iff $x\in A-C$ for some $C\in S$, because any set in the form of $A-X$ such that $X\in S$ is in T_2 we have some $x\in\bigcap T_2$.

4.6 if S is not empty, then there is some $C\in S$, by Axiom Schema of Comprehension the set $\{x\in C:(\forall X)(X\in S\to x\in X)\}$ exist. if it is empty, then we can not apply the axiom of comprehension.

1 Page 18

1.1 1.1 We know that both $\{a\}$ and $\{a,b\}$ are subset of $\{a,b\}$, thus $\{a,b\},\{a\}\in\mathcal{P}(\{a,b\})$, it means that $\{\{a,b\},\{a\}\}\subseteq\mathcal{P}(\{a,b\})$ which implies $\{\{a,b\},\{a\}\}\in\mathcal{P}(\mathcal{P}(\{a,b\}))$.

we have $a,b\in\{a,b\}$, but $(a,b)=\{\{a\},\{a,b\}\}$ which means that there is some $C\in(a,b)$ such tha $a,b\in C$, thus $a,b\in\bigcup(a,b)$.

if $a,b\in A$ then $\{a,b\}$ and $\{a\}$ both are subset of A, thus $\{a,b\},\{a\}\in\mathcal{P}(A)$, again it implies that $\{\{a,b\},\{a\}\}\subseteq\mathcal{P}(A)$, thus $(a,b)=\{\{a,b\},\{a\}\}\in\mathcal{P}(\mathcal{P}(A))$.

1.2 1.2 if a and b exist, then by axiom of pairing and powerset $T = \mathcal{P}(\mathcal{P}(\{a,b\}))$ exist and by previous exercise $(a,b) \in T$. because $(a,b,c) = ((a,b),c)$ by previous argument we have $(a,b,c) \in \mathcal{P}(\mathcal{P}(\{(a,b),c\}))$ which clearly exist.

1.3 if $(a,b) = (b,a)$, it follows from Theorem 1.2 that $a = b$ and $b = a$, so $a = b$.

1.4 if $(a,b,c) = (a',b',c')$ then $((a,b),c) = ((a',b'),c')$, by Theorem 1.2 we have (*) $(a,b) = (a',b')$ and $c = c'$, but again by Theorem 1.2 and (*) we have $a = a'$ and $b = b'$.

1.5 Let $a = \emptyset$, $b = \{a\}$ and $c = \{b\}$, then if $((a,b),c) = (a,(b,c))$ we get $(a,b) = a = \emptyset = \{\{a\},\{a,b\}\}$ which is a contradiction.

1.6 We first prove that:

(1) $a = c$ or $d = \Box$.

(2) $b = d$ or $c = \triangle$.

To prove (1): $\{\{a,\Box\},\{b,\triangle\}\} = \{\{c,\Box\},\{d,\triangle\}\}$ implies either ($\bullet$) $\{a,\Box\} = \{c,\Box\}$ or ($\star$) $\{a,\Box\} = \{d,\triangle\}$, if ($\bullet$) then either $a = c$ or $a = \Box$, if first we are done, if the second then $\{a,\Box\} = \{\Box\} = \{c,\Box\}$ which means $a = \Box = c$, thus in both case $a = c$. if ($\star$) then either $a = d$ or $a = \triangle$, if first then $\{a,\Box\} = \{a,\triangle\}$ which implies $\triangle = \Box$, contradiction, so we have $a = \triangle$, then $\{\triangle,\Box\} = \{d,\triangle\}$ which implies $d = \Box$. so we have either $a = c$ or $d = \Box$.

To prove (2):

We also have (*) $\{b,\triangle\} = \{c,\Box\}$ or (**) $\{b,\triangle\} = \{d,\triangle\}$, if (*) then either $b = c$ or $b = \Box$, if first then $\{b,\triangle\} = \{b,\Box\}$ which implies a contradiction: $\triangle = \Box$, therefore the second case only remains which implies $c = \triangle$. if (**) then either $b = d$ or $b = \Box$, if first we are done, if the second then $\{\Box,\triangle\} = \{d,\triangle\}$ which implies $b = \Box = d$, so in both case we have $b = d$. so we have either (2) $b = d$ or $c = \triangle$.

So we have (1) and (2), assume that $b = d$ from (2), now consider (1), if first case then we are done. if the second then $b = d = \Box$, therefore $\{\{a,\Box\},\{\Box,\triangle\}\} = \{\{c,\Box\},\{\Box,\triangle\}\}$ which implies $a = c$.

Assume the second case of (2), then by first case of (1) we have $a = c = \triangle$, therefore $\{\{\triangle,\Box\},\{b,\triangle\}\} = \{\{\triangle,\Box\},\{d,\triangle\}\}$ which implies $b = d$.

Now consider the second case of (1), then we have $d = \square$ and $c = \triangle$ then $\{\{a, \square\}, \{b, \triangle\}\} = \{\{\triangle, \square\}, \{\square, \triangle\}\} = \{\{\square, \triangle\}\}$, then $a = \triangle = c$ and $b = \square = d$, we are done.

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2.1 Let $(x, y) = \{\{x\}, \{x, y\}\} \in R$, then $\{\{x\}, \{x, y\}\} \subseteq \bigcup R$, thus we have $\{x, y\} \in \bigcup R$ and we know that $x, y \in \{x, y\}$, so for some set $C \in \bigcup R$ we have $x, y \in C$, thus $x, y \in \bigcup \bigcup R$. because the property "$x \in dom\ R$" implies that $(x, y) \in R$ for some y. and because $(x, y) \in R$ implies $x \in A$, the set $\{x \in A : x \in dom\ R\}$ exist. Repeat this argument for property "$x \in ran\ R$".

2.2 (a) by previous argument $ran\ R$ and $dom\ R$ exist, we know that $ran\ R \times dom\ R$ exist, call it A. by comprehension the subset $\{(y, x) \in A : (x, y) \in R\}$ also exist, this set is equal to R^{-1}. again by comprehension the set $\{(x, y) \in dom\ R \times ran\ S : for\ some\ z,\ (x, z) \in R\ and\ (z, y) \in S\}$, this set is equal to $S \circ R$.

(b) Because $A \times B \times C = (A \times B) \times C \subseteq \mathcal{P}((A \times B) \cup C)$, comprehension implies that the set $\{x \in \mathcal{P}((A \times B) \cup C) : x = (y, z)\ for\ some\ y \in A \times B\ and\ z \in C\}$ exist.

2.3 (a) $y \in R[A \cup B]$ iff $(\exists x)(x \in A \cup B \wedge\ xRy)$ iff $(\exists x)((x \in A \vee x \in B) \wedge\ xRy)$ iff $(\exists x)((x \in A \wedge\ xRy) \vee (x \in B \wedge\ xRy))$ iff $(\exists x)(x \in A \wedge\ xRy) \vee (\exists x)(x \in B \wedge\ xRy)$ iff $y \in R[A] \vee y \in R[B]$ iff $y \in R[A] \cup R[B]$.

(b) Let $y \in R[A \cap B]$, then for some $x \in A \cap B$ we have xRy which means that $x \in A$ such that xRy and $x \in B$ such that xRy, thus $x \in R[A] \cap R[B]$.

(c) Suppose that $y \in R[A] - R[B]$, it means there is some $x \in A$ such that xRy but there is no $z \in B$ such that zRy, because xRy holds for x, it can not be in B, thus $x \in A - B$ and xRy which means that $y \in R[A - B]$.

(d) Let $R = \{(a, c), (b, c)\}$ and $A = \{a\}$, $B = \{b\}$ then $R[A] \cap R[B] = \{c\}$ while $R[A \cap B = \emptyset] = \emptyset$. also $R[A - B] = R[\{a\}] = \{c\}$ but $R[A] - R[B] = \{c\} - \{c\} = \emptyset$, so this falsifies converse of both (b) and (c).

(f) Fix $x \in A \cap dom\ R$, then because $x \in dom\ R$ there is some y such that xRy, because $x \in A$ we conclude that $y \in R[A]$, so there is some $y \in R[A]$ such that xRy or equivalently $yR^{-1}x$, thus $x \in R^{-1}[R[A]]$.

Fix $y \in B \cap ran\ R$, since $y \in ran\ R$ for some x we have xRy, but $y \in B$ implies that $x \in R^{-1}[B]$, thus for some $x \in R^{-1}[B]$ we have xRy, therefore

$y \in R[R^{-1}[B]]$.

Let $R = \{(a, c), (b, c), (e, f), (e, g)\}$ and $A = \{a\}$, then $A \cap dom\ R = \{a\}$ but $R[A] = \{c\}$, thus $R^{-1}[R[A]] = R^{-1}[\{c\}] = \{a, b\}$, but $\{a, b\} \not\subseteq \{a\}$.

Let R be as before and $B = \{g\}$, then $R^{-1}[B] = \{e\}$ and $R[R^{-1}[B]] = \{f, g\}$, but $B \cap ranR = \{g\}$.

2.4 $R[X] \subseteq ran\ R$ because for any $y \in R[X]$ we have some $x \in X$ such that xRy, thus $y \in ran\ R$. if $y \in ran\ R$, then for some $x \in dom\ R$ we have xRy, but $dom\ R \subseteq X$, thus $x \in X$, so we get for some $x \in X$, xRy, therefore $y \in R[X]$.

suppose $x \in dom\ R$ then there is some $y \in ran\ R$ such that xRy, but xRy iff $yR^{-1}x$ and $ranR \subseteq Y$, therefore there is some $y \in Y$ such that $yR^{-1}x$ which is equal to say that $x \in R^{-1}[Y]$, left to right is trivial.

(b) Assume $a \notin dom\ R$ but $R[\{a\}] \neq \emptyset$, so for some $y \in R[\{a\}]$ we have aRy which means that $a \in dom\ R$, this contradicts our assumption.

Assume $b \notin ran\ R$ and $R^{-1}[\{b\}] \neq \emptyset$, so there is some $x \in R^{-1}[\{b\}]$ such that $bR^{-1}x$ or equivalently xRb, it means that $b \in ran\ R$ which contradicts the assumption.

(c) $x \in dom\ R$ iff for some y, xRy iff $yR^{-1}x$ iff $x \in ran\ R^{-1}$.

$y \in ran\ R$ iff for some x, xRy iff $yR^{-1}x$ iff $y \in dom\ R^{-1}$.

(d) $(x, y) \in R$ iff $(y, x) \in R^{-1}$ iff $(x, y) \in (R^{-1})^{-1}$.

(e) if $(x, x) \in Id_{dom\ R}$ then $x \in domR$ which implies that for some y, $(x, y) \in R$, but $(x, y) \in R$ iff $(y, x) \in R^{-1}$, thus we can say that there is some y such that $(x, y) \in R$ and $(y, x) \in R^{-1}$ which is equal to $(x, x) \in R^{-1} \circ R$. the second part can be proved like this.

2.5 $\mathcal{P}(X) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

$\in_Y = \{(\emptyset, \{\emptyset\}), (\emptyset, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \{\{\emptyset\}\})\}$.

$Id_Y = \{(\emptyset, \emptyset), (\{\emptyset\}, \{\emptyset\}), (\{\{\emptyset\}\}, \{\{\emptyset\}\}), (\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\})\}$.

$ran(Id_Y) = dom(Id_Y) = fld(Id_Y) = \mathcal{P}(X)$.

$dom(\in_Y) = \{\emptyset, \{\emptyset\}\}$, $ran(\in_Y) = \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, $fld(\in_Y) = \mathcal{P}(X)$.

2.6 $(x, y) \in T \circ (S \circ R)$ iff $(\exists z)((x, z) \in (S \circ R) \wedge (z, y) \in T)$ iff $(\exists z)((\exists u)[(x, u) \in R \wedge (u, z) \in S] \wedge (z, y) \in T)$ iff $(\exists z)((\exists u)[(x, u) \in R \wedge (u, z) \in S \wedge (z, y) \in T])$

iff $(\exists z)(\exists u)((x, u) \in R \wedge (u, z) \in S \wedge (z, y) \in T)$ iff $(\exists u)((x, u) \in R \wedge (\exists z)[(u, z) \in S \wedge (z, y) \in T])$ iff $(\exists u)((x, u) \in R \wedge (u, y) \in T \circ S)$ iff $(x, y) \in (T \circ S) \circ R$.

2.7 Let $X = \{a\}$ and $Y = \{b, c\}$, $Z = \{d\}$.
(a) $(a, b) \in X \times Y$ but $(a, b) \notin Y \times X$.
(b) $(a, (b, d)) \in X \times (Y \times Z)$ but $(a, (b, d)) \notin (X \times Y) \times Z$.
(c) $((a, a), a) \in X^2 \times X$ but $((a, a), a) \notin X \times X^2$.

2.8 (a) Assume $A \neq \emptyset$ and $B \neq$, then there is some $a \in A$ and $b \in B$, but then $(a, b) \in A \times B$, so $A \times B \neq \emptyset$. Now assume $A \times B \neq \emptyset$, then there is some $x \in A \times B$ such that $x = (a, b)$, but it means that $a \in A$ and $b \in B$, thus $A, B \neq \emptyset$.

(b) $(a, b) \in (A_1 \cup A_2) \times B$ iff $(a \in A_1 \cup A_2) \wedge b \in B$ iff $(a \in A_1 \vee a \in A_2) \wedge b \in B$ iff $(a \in A_1 \wedge b \in B) \vee (a \in A_2 \wedge b \in B)$ iff $(a, b) \in (A_1 \times B) \vee (a, b) \in (A_2 \times B)$ iff $(a, b) \in (A_1 \times B) \cup (A_2 \times B)$.

$(a, b) \in A \times (B_1 \cup B_2)$ iff $a \in A \wedge b \in (B_1 \cup B_2)$ iff $a \in A \wedge (b \in B_1 \vee b \in B_2)$ iff $(a \in A \wedge b \in B_1) \vee (a \in A \wedge b \in B_2)$ iff $(a, b) \in (A \times B_1) \vee (a, b) \in (A \times B_2)$ iff $(a, b) \in (A \times B_1) \cup (A \times B_2)$.

3 Page 28

3.1 if $ran\ f \subseteq dom\ g$, then $f^{-1}[ran\ f] \subseteq f^{-1}[dom\ g]$, but $f^{-1}[ran\ f] = dom\ f$, by Exercise 4.2(a) on Page 15 we have $dom\ f \cap f^{-1}[dom\ g] = dom\ f$, Theorem 3.5 implies $dom\ (g \circ f) = dom\ f$.

3.2 $f_2 \circ f_1 = \{\sqrt{2x - 1} : x > \frac{1}{2}\}$.
$f_1 \circ f_2 = \{2\sqrt{x} - 1 : x > 0\}$
$f_3 \circ f_1 = \{1/(2x - 1) : x \neq \frac{1}{2}\}$
$f_1 \circ f_3 = \{2/x - 1 : x \neq 0\}$

3.3 For f_1: if $f_1(a) = f_1(b)$ then $2a - 1 = 2b - 1$, by adding 1 to each side of equation we get $2a = 2b$, by dividing by 2 we have $a = b$.

For f_2: if $f_1(a) = f_1(b)$ then $\sqrt{a} = \sqrt{b}$, but then $a = \sqrt{a}\ \sqrt{a} = \sqrt{a}\ \sqrt{b} = \sqrt{b}\ \sqrt{b} = b$.

For f_3: if if $f_1(a) = f_1(b)$ then $1/a = 1/b$, because a, b are non-zero multiplying by ab yields $a = b$.

$f_1^{-1} = \{(x + 1)/2 : x\ is\ real\}$
$f_2^{-1} = \{x^2 : x > 0\}$
$f_3^{-1} = \{1/x : x \neq 0\}$

3.4 (a) Assume that f is invertible, let $(a, b) \in f^{-1} \circ f$ then for some z we have (*) $(a, z) \in f$ and $(z, b) \in f^{-1}$, then from (*) we also have $(z, a) \in f^{-1}$, by assumption f^{-1} is a function, so we get $a = b$, because $a \in dom\ f$ we get $(a, b) = (a, a) \in Id_{dom\ f}$. the other side holds by Exercise 2.4(e) on Page 23.

(b) Let $(a, b), (a, c) \in f^{-1}$, then $(b, a), (c, a) \in f$, thus $f(b) = a$ and $f(c) = a$ but (*) $g \circ f = Id_{dom\ f}$ implies $g(f(b)) = b = g(a) = g(f(c)) = c$, therefore $b = c$ and f^{-1} is a function. let $(a, b) \in f^{-1}$ then $(b, a) \in f$, so $f(b) = a$, by (*) we get $g(f(b)) = b = g(a)$, thus $(a, b) \in g$, but we also know that $a \in ran\ f$, therefore $(a, b) \in g \mid ran\ f$. Now let $(a, b) \in g \mid ran\ f$, then $g(a) = b$ and also $a \in ran\ f$, then $f(k) = a$ for some $k \in domf$, but (*) implies $g(f(k)) = g(a) = b = k$ which means that $(b, a) \in f$, $(a, b) \in f^{-1}$.

We give a counter example for the second one, let $f = \{(a, a), (b, a)\}$ and $h = \{(a, a)\}$ then $f \circ h = \{(a, a)\} = Id_{ran\ f}$ but clearly f^{-1} is not a function.

3.5 Let $(g \circ f)(a) = (g \circ f)(b)$, then $g(f(a)) = g(f(b))$ since g is one-to-one we get $f(a) = f(b)$, again because f is one-to-one we have $a = b$.

let $(a, b) \in (f \circ g)^{-1}$, thus $(b, a) \in f \circ g$, it means that for some z we have $(b, z) \in g$ and $(z, a) \in f$, equivalently we have $(a, z) \in f^{-1}$ and $(z, b) \in g^{-1}$ for some z, by definition of composition we get $(a, b) \in g^{-1} \circ f^{-1}$.

3.6 We just need prove right to left of (a) and left to right of (b).

(a) Suppose $x \in f^{-1}[A] \cap f^{-1}[B]$, then for some $y \in A$ we have $yf^{-1}x$ or equivalently $f(x) = y$ and for some $z \in B$, $f(x) = z$, but since f is a function we conclude that $z = y \in A \cap B$, then we can say that for some $y \in A \cap B$, $yf^{-1}x$ holds, therefore $x \in f^{-1}[A \cap B]$.

(b) Let $x \in f^{-1}[A - B]$, then there is some $y \in A - B$ such that $yf^{-1}x$ or equivalently (*) $f(x) = y$, clearly $x \in f^{-1}[A]$, we must prove that $x \notin f^{-1}[B]$ or equivalently there is no $z \in B$ such that $zf^{-1}x$, assume to the contrary that it exists, so we get $f(x) = z$, but (*) implies $z = y \in B$, it contradicts our assumption that $y \in A - B$.

3.7 let $f = \{(a, b)\}$ and $A = \{a\}$, then $f \cap A^2 = \emptyset$ but $f|A = f$.

3.8 Let $I = A$ and $S = Id_I$, then $S = (S_i, i \in I)$ is an indexed function such that $S_i = i$.

3.9 (a) Let $f : A \to B$, then $f \subseteq A \times B$, thus $f \in \mathcal{P}(A \times B)$, now let $P(x)$ be the property "$(\forall a, b, c)[(a, b), (a, c) \in x \to b = c] \wedge (\forall a)(a \in A \to (\exists b)[b \in B \wedge (a, b) \in x])$", then $\{x \in \mathcal{P}(A \times B) : P(x)\}$ is the set of all function from A to B.

(b) Let f be a memeber of product of an indexed system $(S_i : i \in I)$, then $f : I \to \bigcup_{i \in I} S_i$ such that for every $i \in I$, $f(i) \in S_i$, then clearly $f \in (\bigcup_{i \in I})^I$, by previous exercise we know that it exists, now by comprehension we have $\prod_{i \in I} S_i = \{f \in (\bigcup_{i \in I})^I : (\forall i \in I)[f(i) \in S_i]\}$, clearly if it is non-empty, every member of it is a function such that satisfies the condition of a product.

3.10 $x \in \bigcup_{a \in \bigcup S} F_a$ iff $(\exists a)[a \in \bigcup S \wedge x \in F_a]$ iff $(\exists a)[(\exists C)(C \in S \wedge a \in C) \wedge x \in Fa]$ iff $(\exists a)[(\exists C)(C \in S \wedge a \in C \wedge x \in F_a)]$ iff $(\exists C)[(\exists a)(C \in S \wedge a \in C \wedge x \in F_a)]$ iff $(\exists C)[C \in S \wedge (\exists a)(a \in C \wedge x \in F_a)]$ iff $(\exists C)[C \in S \wedge x \in \bigcup_{a \in C} F_a]$ iff $x \in \bigcup_{C \in S}(\bigcup_{a \in C} F_a)$.

Let $x \in \bigcap_{a \in \bigcup S} F_a$ then (*) $(\forall a)[a \in \bigcup S \to x \in F_a]$. Now let $C \in S$, then because $C \subseteq \bigcup S$ we get that for every $a \in C$, $x \in F_a$, because C was arbitrary we can conclude that (**) $(\forall C)[C \in S \to (\forall a)(a \in C \to x \in F_a)]$, which is equal to $(\forall C)[C \in S \to x \in \bigcap_{a \in C} F_a]$, thus $x \in \bigcup_{C \in S}(\bigcap_{a \in C} F_a)$. Now let $x \in \bigcup_{C \in S}(\bigcap_{a \in C} F_a)$, then we get (**), let $a \in \bigcup S$, then there is some $C \in S$ such that $a \in C$, but then by (**) we get $(\forall a)(a \in C \to x \in F_a)$ and then $x \in F_a$, because a was arbitary we proved (*), thus $x \in \bigcap_{a \in \bigcup S} F_a$.

3.11 $x \in B - \bigcup_{a \in A} F_a$ then $x \in B$ and for every $a \in A$, $x \notin F_a$, also for every $a \in A$, $x \notin F_a$ and $x \in B$, so for every $a \in A$, $x \in B - F_a$, thus $x \in \bigcap_{a \in A}(B - F_a)$. Now let $x \in \bigcap_{x \in A}(B - F_a)$, then for every $a \in A$, $x \in B$ and $x \notin F_a$,

let $a \in A$, then by above claim $x \notin F_a$, thus $x \notin \bigcup_{a \in A} F_a$, Now assume to the contrary that $x \notin B$, then it implies there is no $a \in A$, $A = \emptyset$ which is a contradiction.

Let $x \in B - \bigcap_{a \in A} F_a$, then (*) $x \in B$ and there is some $a \in A$ such that $x \notin F_a$, by (*) we can claim that there is some $a \in A$ such that $x \in B - F_a$, thus $x \in \bigcup_{a \in A}(B - F_a)$. Now let $x \in \bigcup_{a \in A}(B - F_a)$, then $x \in (B - F_a)$ for some $a \in A$, it follows that there is some $a \in A$ such that $x \in F_a$, thus $x \notin \bigcap_{a \in A} F_a$ and clearly $x \in B$, thus $x \in B - \bigcap_{a \in A} F_a$.

Let $x \in \bigcup_{a \in A} F_a \cap \bigcup_{b \in B} G_b$, then for some $a \in A$, $x \in F_a$ and for some $b \in B$, $x \in G_b$, clearly $(a, b) \in A \times B$, then we can say for some $(a, b) \in A \times B$, $x \in F_a \cap G_b$

$x \in \bigcup_{a\in A} F_a \cap \bigcup_{b\in B} G_b$ iff $(\exists a)(a \in A \wedge x \in F_a) \wedge (\exists b)(b \in B \wedge x \in F_b)$ iff $(\exists a)(\exists b)[(a \in A \wedge x \in F_a) \wedge b \in B \wedge x \in F_b)]$ iff $(\exists a)(\exists b)[(a,b) \in A \times B \wedge x \in F_a \cap F_b]$ iff $x \in \bigcup_{(a,b)\in A\times B}(F_a \cap G_b)$

3.12 (We just prove the first and the third case)

$y \in f[\bigcup_{a\in A} F_a]$ iff $(\exists x)[x \in \bigcup_{a\in A} F_a \wedge f(x) = y]$ iff $(\exists x)[(\exists a)(a \in A \wedge x \in F_a) \wedge f(x) = y]$ iff $(\exists x)[(\exists a)(a \in A \wedge x \in F_a \wedge f(x) = y)]$ iff $(\exists x)(\exists a)[a \in A \wedge x \in F_a \wedge f(x) = y]$ iff $(\exists a)(\exists x)[a \in A \wedge x \in F_a \wedge f(x) = y]$ iff $(\exists a)[a \in A \wedge (\exists x)(x \in F_a \wedge f(x) = y)]$ iff $(\exists a)[a \in A \wedge y \in f[F_a]]$ iff $y \in \bigcup_{a\in A} f[F_a]$.

Let $y \in f[\bigcap_{a\in A} F_a]$, then for some $x \in \bigcap_{a\in A} F_a$, $f(x) = y$, but it means for every $a \in A$, $x \in F_a$ and $f(x) = y$, we can say for every $a \in A$, there is some $x \in F_a$ such that $f(x) = y$ or equally $y \in f[F_a]$, thus $y \in \bigcap_{a\in A} f[F_a]$.

(if f is one-to-one, $\subseteq$ can be replaced by $=$): Now let $y \in \bigcap_{a\in A} f[F_a]$, so for every $a \in A$, there is some $x \in F_a$ such that $f(x) = y$, but because f is one-to-one this x must be unique, name it k, so for every $a \in A$, $k \in F_a$ or equivalently $k \in \bigcap_{a\in A} F_a$, since $f(k) = y$ we get $y \in f[\bigcap_{a\in A} F_a]$.

3.13 Right to left is easy according to Hint, we prove left ro right side:

Let $x \in \bigcap_{a\in A}(\bigcup_{b\in B} F_{a,b})$, define f such that $(a,b) \in f$ iff $x \in F_{a,b}$, we prove $f \in B^A$, let $(x,y),(x,z) \in f$ be two distinct memeber, then $x \in F_{x,y} \cap F_{x,z}$ but because $y \neq z$ we have $F_{x,y} \cap F_{x,z} = \emptyset$, thus it contradicts our assumption, hence f is a function.

From assumption for every $a \in A$ we have $x \in \bigcup_{b\in B} F_{a,b}$, fix arbitrary $a \in A$, then $x \in F_{a,b}$ for some $b \in B$, but by definition of f we have $f(a) = b$, thus $x \in F_{a,f(a)}$, because a was arbitrary we can say $x \in \bigcap_{a\in A} F_{a,f(b)}$ for f, thus $x \in \bigcup_{f\in B^A}(\bigcap_{a\in A} F_{a,f(b)})$.

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4.1 (a) transitive.

(b) reflexive, transitive.

(c) symmetric.

(d) $\subseteq$: reflexive, transitive. $\subset$: transitive.

(e) reflexive, transitive, symmetric.

(f) symmetric and transitive.

4.2 (a) for every $a \in A$, $f(a) = f(a)$ since f is a function, thus aEa and E is reflexive. Let aEb, then for some $a, b \in A$ we have $f(a) = f(b)$ but also $f(b) = f(a)$, thus bEa, therefore E is symmetric. Suppose that aEb and bEc, then we get $f(a) = f(b)$ and $f(b) = f(c)$, since f is a function we have $f(a) = f(c)$, thus aEc, E is transitive.

(b) We define $\phi : A/E \to B$ such that $\phi([a]_E) = f(a)$ for every $[a]_E \in A/E$, if $[a]_E = [a']_E$ then aEa', by definition we get $f(a) = f(a')$ which means that $\phi([a]_E) = \phi([a']_E)$.

(c) for every $a \in A$ we have $\phi \circ j(x) = \phi(j(a)) = \phi([a]_E)) = f(a)$, because f and j have the same domain we can conclude $\phi \circ j = f$.

4.2 Because for every $(r, \gamma) \in P$ we have $r = r$ and $\gamma - \gamma = 0 = 2\pi \times 0$ which 0 is an integer multiple of 2π, we get $(r, \gamma) \sim (r, \gamma)$. Now let $(r, \gamma) \sim (r', \gamma')$, then $r = r'$ and $\gamma - \gamma' = 2\pi k$ is an integer, because $\gamma' - \gamma = -(\gamma - \gamma') = 2\pi(-k)$ is also an integer, together with symmetricity of $=$ we get $r' = r$, so we conclude that $(r', \gamma') \sim (r, \gamma)$, thus $\sim$ is symmetric.

Let $(r, \gamma) \sim (r', \gamma')$ and $(r', \gamma') \sim (r'', \gamma'')$, by transitivity of identity we simply get $r = r''$, also $\gamma - \gamma' = 2\pi k$ and $\gamma' - \gamma'' = 2\pi k'$ such that k, k' are both integer, but then $\gamma - \gamma'' = (\gamma - \gamma') + (\gamma' - \gamma'') = 2\pi k + 2\pi k' = 2\pi(k + k')$ clearly is an integer , thus $(r, \gamma) \sim (r'', \gamma'')$.

Consider (r, γ), then there is some (r, γ') such that $\gamma - \gamma' = 2\pi k$ and is an integer. then $\gamma' = \gamma - 2\pi k$, we argue that for some integer k' we have $0 \leq \gamma - 2\pi k' \leq 2\pi$, if there is no such k' that satisfies last inequality then we also do not have $-\gamma \leq -2\pi k' \leq 2\pi - \gamma$ and also $\gamma - 2\pi \leq 2\pi k' \leq \gamma$, dividing by 2π yields that there is no $\gamma/2\pi - 1 \leq k' \leq \gamma/2\pi$, but it contradicts the fact tht for any real number X there is an integer $X - 1 \leq k' \leq X$, so we can take $(r, \gamma - 2\pi k')$.

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5.1 (a) For any $a, b \in A$, aSb iff $aRb \wedge a \neq b$, but aR^*b iff $aSb \vee a = b$ iff $(aRb \wedge a \neq b) \vee a = b$ iff $aRb \vee a = b$ iff aRb (since R is reflexive $a = b$ implies aRb).

(b) aRb iff $aSb \vee a = b$, but aS^*b iff $aRb \wedge a \neq b$ iff $(aSb \vee a = b) \wedge a \neq b$ iff $(aSb \wedge a \neq b)$ iff aSb (since S is irreflexive).

5.2 a and b are incomparable if $a \neq b$, neither $a < b$ nor $b < a$.

a is maximal in A : $\neg(\exists x \in A)(a < x)$

a is the greatest element of A : $(\forall x \in A)(x < a \vee x = a)$.

a is an upper bound of A: $(\forall x \in A)(x < a \vee x = a)$

a is supremum of A in X: $(\forall x \in A)(x < a \vee x = a) \wedge (\forall a' \in X)(\forall x \in A)[(x < a' \vee x = a') \to a < a' \vee a = a']$.

5.3 (a) for any $a \in A$ we have aRa, also $aR^{-1}a$, thus R^{-1} is reflexive. Suppose that $aR^{-1}b$ and $bR^{-1}a$, then we have bRa, aRb, by antisymmetricity of R we get $a = b$,therefore R^{-1} is antisymmetric.

Now suppose $aR^{-1}b$ and $bR^{-1}c$, then we get bRa and cRb, but transitivity of R implies cRa, thus $aR^{-1}b$, thus R is transitive.

(a) $(\forall x \in B)(aR^{-1}x)$ iff $(\forall x \in B)(xRa)$.

5.4 Let $R' = R \cap B^2$, then for every $a \in B$ we have $(a, a) \in B^2$, since $B \subseteq A$ and R is an order on A, by reflexitivity $(a, a) \in R$, thus $aR'a$, hence R' is reflexive.

Let $aR'b, bR'a$ then $a, b, c \in B$ and aRb, bRa, because R is antisymmetric we get $a = b$,

5.5 Let $A = \mathcal{P}(\{a, b, c\}) - \{\{a, b, c\}, \emptyset\}$ and $R = \subseteq$.

(a) $B = \{\{a\}, \{b\}\}$.

(b) $B = \{\{a\}, \{b\}\}$.

(c) $B = \{\{a\}, \{b\}\}$.

(d) $B = \{\{a, b\}, \{b, c\}\}$.

5.6 (a) For every $x \in B$ either $x = b$ or $x \in A$, if $x = b$ then $x \notin A$ and both disjunct in the definition of $\prec$ would be false, if $x \in A$ then $x \not< x$ because $<$ is irreflexive, so the first disjunct could not be true, the other disjunct require that $x = b$ but it is impossible because $x \in A$, hence both of them is false, thus $x \not\prec x$.

Now let $x \prec y$, if $x \in A$ and $y = b$ then clearly $x \neq b$ because $b \notin A$, so we can not have $y \in A$ and $x = b$ and also we can not have $y = b, x \in A$ (the first item of first disjunct), thus $y \not\prec x$. Assume that $x \prec y, y \prec z$, if x, y, z all are in A then $x \prec z$ easily follows from transitivity of $<$. but if $x \in A$ and $y = b$ then $y \prec z$ is impossible, because in both disjunnct it requires that $y \in A$, but $y = b$. So the only case we need to check is that when $x, y \in A$ and $x < y$ and $y \in A$ and $z = b$, but from this it easy follows that $x \in A$ and $z = b$, thus $x \prec z$. Notice that $\prec = < \cup (A \times \{b\})$ and $< \subset A^2$, but $(A \times \{b\}) \cap A^2 = \emptyset$, thus $\prec \cap A^2 = (< \cap A^2) \cup (A \times \{b\}) \cap A^2 = < \cup \emptyset = <$.

5.7 Because R is reflexive for every $a \in A, aRa$ and also aRa which implies that aEa. Now let aEb then aRb and bRa, also bRa and aRb, thus bEa. Let aEb, bEc then aRb, bRa and bRc, cRb, by transitivity of R we get aRc and cRa thus aEc, hence E is transitive.

Assume aRb, then $[a]_E R/E [b]_E$, now let $b' \in [b]_E$, then bEb', hence bRb', by transitivity of R we get aRb', hence $[a]_E R/E [b']_E$. we can repeat this argument for a.

Because R is reflexive for every a, we have aRa, also $[a]_E R/E [a]_E$. Assume that $[a]_E R/E [b]_E$ and $[b]_E R/E [a]_E$, then we get aRb and bRa, hence aEb which means that $[a]_E = [b]_E$.

To prove that R/E is transitive, assume $[a]_E R/E [b]_E$ and $[b]_E R/E [c]_E$, then we get aRb and bRc, by transitivity of R it follows that aRc, hence $[a]_E R/E [c]_E$.

5.8 (a) Let $S \subseteq A$, then every $x \in S$, $x \subseteq \bigcup S$, thus $\bigcup S$ is an upper bound of S, to prove that it is the least upper bound assume we prove that it is subset of every upper bound a, i.e $\bigcup S \subseteq a$, let $x \in \bigcup S$, then for some $C \in S$, $x \in C$, but a is an upper bound for S, thus $C \subseteq a$, hence $x \in a$, thus $\bigcup S \subseteq a$.

(b) The set of all lower bounds of $\emptyset$ is the set of all $a \in A$ such that for every $x \in \emptyset$, $a \subseteq x$, so all member of A satisfy this condition because $x \notin \emptyset$, the greatest element of A is X, since $Y \subseteq X$ for any $Y \in A = \mathcal{P}(X)$.

5.9 (a) $\subseteq$ is reflexive, antisymmetric and transitive on any set, thus it is an ordering.

(b) Let $F \subseteq Fn(X, Y)$, assume that *sup* F exist, and F is not compatible, then there are some $g, f \in F$ such that for some $x \in dom\ f \cap dom\ g, f(x) \neq g(x)$, hence there are distinct $a, b \in Y$ such that $(x, a) \in f$ and $(x, b) \in g$, but then since $f, g \subseteq sup\ F$, hence $(x, b), (x, a) \in sup\ F$, it contradicts the fact that *sup* F is a function. Now assume that F is a compatible system of functions. then by Theorem 3.12 $\bigcup F$ is a function and clearly $\bigcup F \in Fn(X, Y)$, it follows from a similar argument to Exercise 5.8(a) that $\bigcup F = sup\ F$.

5.10 (a) Because for every $S \in Pt(A)$ we have that for all $C \in S$, there is some some $D \in S$ such that $C \subseteq D$, namely C itself, thus $S \preccurlyeq S$ for all $S \in Pt(A)$ and it is reflexive.

Assume that $S_1 \preccurlyeq S_2$ and $S_2 \preccurlyeq S_1$, then for every $C \in S_1$, $C \subseteq D$ for some $D \in S_2$, but because $S_2 \preccurlyeq S1$ and $D \in S_2$, we have some $E \in S_1$ such that $D \subseteq E$, we show that $E = C$. Assume it is not the case, then $C \subseteq D \subseteq E$ implies $C \cap E \neq \emptyset$, contrary to the assumption that S is a partition, thus the relation is symmetric.

Let $S_1 \preccurlyeq S_2$ and $S_2 \prec S_3$, then for every $C \in S_1, C \subseteq D$ for some $D \in S_2$, but because $S_2 \preccurlyeq S_3$, there is some $E \in S3$ such that $D \subseteq E$, thus for every $C \in S_1, C \subseteq E$ for some $E \in S3$, therefore $S_1 \preccurlyeq S_2$ and $\preccurlyeq$ is transitive.

(b) Let $S = \{C \cap D : C \in S_1 \wedge D \in S_2\}$, clearly S is a partition and $S \preccurlyeq S1$, $S \preccurlyeq S_2$, thus S is a lower bound for $\{S_1, S_2\}$, we prove that any lower bound $S' \preceq S$. Assume S' is a lower bound, then for every $C \in S'$, $C \subseteq D$ for some $D \in S_1$ and also $C \subseteq D'$ for some $D' \in S_2$ but then there is some $X \in S$ and $C \subseteq X$, namely $X = D \cap D'$, so we proved that for every $C \in S'$, $C \subseteq X$ for some $X \in S$, thus $S' \preccurlyeq S$.

aE_Sb implies $aE_{S_1}b, aE_{S_2}b$

(c) Let $T = (T_i : i \in I)$ and $S = \{(\bigcap_{i \in I} f_i) : f \in \prod T_i\}$, fix some $T_k \in T$, we want to prove that $S \preccurlyeq T_k$. let $C \in S$ and $x \in C$ then for some f, $x \in f_i$ for all $i \in I$, but $f_i = D$ for some $D \in T_i$, from this it follows that $x \in f_k$, thus for some $D \in T_k$ we have $x \in D$, thus $C \subseteq D$, we conclude that $S \preccurlyeq T_k$. We prove S is greatest lower bound, assume S' is another lower bound for T, it means that for every $C \in S'$ and for every T_i there is some $D_i \in T_i$ such that $C \subseteq D_i$, define $f : I \to \bigcup T$ by $f_i = D_i$, then clearly $C \subseteq \bigcap_{i \in I} f_i$, but $\bigcap_{i \in I} f_i \in S$, thus $S' \preccurlyeq S$.

(d) Let $T' = \{S \in Pt(A) : (\forall i \in I)(T_i \preccurlyeq S)\}$ clearly it is the set of upper bounds of T, by previous exercise $Inf\ T'$ exist, we prove that $Inf\ T' \in T'$, fix some $T_k \in T$, and let $C \in T_k$, we know that for every $S \in T'$ we have $T_k \preccurlyeq S$, it means that for every $S \in T'$ there is some $D \in S$ such that $C \subseteq D$, if we index T' by J, we have for every $T'_j \in T'$ there is some $D_{T'_j} \in T'_j$ such that $C \subseteq D_{T'_j}$, define $f : J \to \bigcup T'$ by $f_j = D_{T'_j}$ then clearly $C \subseteq \bigcap_{j \in J} f_j \in Inf\ T'$, thus we proved for arbitrary $T_k, T_k \preccurlyeq Inf\ T'$, thus $InfT' \in T'$ and it is the least element of it, the least element among upper bound of T, thus $sup\ T = inf\ T'$.

5.11 Let f be the isomorphism, let $y_1, y_2 \in Q$ then there is some $x_1, x_2 \in P$ such that $f(x_1) = y_1, f(x_2) = y_2$ but because $<$ is linearly ordered we have either $x_1 = x_2$ or $x_1 < x_2$ or $x_2 < x_1$, but because f is an isomorphism we get either $f(x_1) = f(x_2)$ or $f(x_1) \prec f(x_2)$ or $f(x_2) \prec f(x_1)$, rewrite this for y_1 and y_2.

5.12 Suppose that $x <_1 y$ for some $x, y \in P_1$ then we have $f(x) <_2 f(y)$, but since $f(x), f(y) \in P_2$ we get $g(f(x)) <_3 g(f(y))$, thus $g \circ f(x) <_3 g \circ f(y)$. Now let $u <_3 z$ for some $u, z \in P_3$, because g is an isomorphism there are some $t, v \in P_2$ such that $f(t) = u, f(v) = z$ and $t <_2 v$, but because f is isomorphism we get $f(x) = t <_2 v = f(y)$ which implies $x <_1 y$.

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2.1 Assume that for some $k \in N$ such that $n < k < n + 1$, by Lemma 2.1(ii) $k < n + 1$ implies either $k < n$ or $k = n$, if $k < n$ by transitivity of $<$ on N and our assumption that $n < k$ we get $n < n$, if $k = n$ again by assumption $n < n$, but $n < n$ contradicts Theorem 2.2.

2.2 Assume to the contrary that $m < n$ but $n < m + 1$, but it means that there is some n such that $m < n < m + 1$ which contradicts previous exercise. Assume $m < n$, by previous argument $m + 1 \leq n$, but $n < n + 1$, thus $m + 1 < n + 1$. assume two distinct natural number m, n then either $m < n$ or $n < m$, so we get $S(m) < S(n)$ or $S(n) < S(m)$, in both case $S(n) \neq S(m)$.

2.3 For every $n \in N$ let $f(n) = S(n)$, therefore $ran\ f = N - \{0\}$ (since otherwise for some k, $0 = S(k) = k + 1 = k \cup \{k\}$ implies $k \in 0$) which is a proper subset of N, by previous exercise f is one-to-one because $S(n)$ is one-to-one.

2.4 if $n \in N, n \neq 0$ then $n \in ranf$ in previous exercise, then there is some $k \in N$ such that $f(k) = S(k) = k + 1 = n$, because f is one-to-one, k is unique.

2.5 Define function g on N by $g(n) = S(S(n)) = (n+1)+1$, like previous argument we can prove that g is one-to-one and onto $N - \{0,1\}$, so for ever $n \in N - \{0,1\}$ we get unique $k \in N$ such that $(k+1)+1 = n$.

2.6 if $m \in N$ and $m < n$ then clearly $m \in n$. we prove it by induction on n that if $m \in n$ then $m \in N$, this is trivially true for $n = 0$. assume the hypothesis and that $m \in n+1$ then either $m = n$ or $m \in n$, if $m = n$ then $m \in N$, since $n \in N$. if $m \in n$ then by induction hypothesis we get $m \in N$.

2.7 Let $x \in m$, since n is the set of natural number less that n and $x < m < n$, we get $x \in n$, also $m < n$ implies $m \in n$ but $m \notin m$, thus $m \subset n$. Now assume $m \subseteq n$, then there is some $q \in n$ such that $q \notin m$, but q is a natural number, thus $q < n$ and $q \not< m$ or equivalently $m < q$, by transitivity $m < n$ which means $m \in n$.

2.8 Assume that there is such function f, then $ranf \subseteq N$ must have a least element u, thus $u = f(k)$ for some $k \in N$, but then definition of f implies $f(k) > f(k+1)$ which contradicts the assumption that $f(k)$ is the least element of $ranf$.

2.9 Let $Y \subseteq X$, but then $X \subseteq N$ implies $Y \subseteq N$ so Y have a least element on order $<$, it means there is some $u \in Y$ such that for every $n \in Y, u < n$, but since $< \cap X^2 \subseteq <$ and $Y \subseteq X$ we conclude that for every $n \in Y, u < \cap X^2 n$.

2.10 Let $X \subseteq A$, then either $X \subseteq N$ or $N \in X$, if $X \subseteq N$ then $\prec$ is ordering of N so it has a least element, if $N \in X$, consider $X - \{N\}$, clearly it has the least element u, because $u \prec N$ it is the least element of X too.

2.11 Assume $P(n)$ does not hold for some $k \leq n$, let X be the set of these elements, by well-ordering it has least element u, (*) for every $k \leq v < u$ we have $P(v)$, if $u = 0$ then $k = 0$ by assumption so it is ordinary induction and we are done, if $u \neq 0$ then for some successor element l, $u = u' + 1$, but since $k < u$, we get $k \leq u'$ then it follows from (*) $P(u')$, but then by (b) $P(u'+1) = P(u)$ holds which contradicts our assumption.

2.12 Assume to the contrary that for some $n \in N, n \leq K$ the property $P(n)$ does not hold, thus the set $X = \{\neg P(n) : (\exists n \in N)(n \leq k)\}$ is non-empty, by well-ordering there is an element $u \in X$ such that is the least element of X. u could not be 0 because $P(0)$ holds, so it is a successor element, thus $u = u' + 1$ for some $u' \in N$. since it is the least element, for every $t < u$, $P(t)$ holds, since $u' < u \leq k$, $P(u')$ holds, then by (b) $P(u' + 1) = P(u)$ holds, a contradiction.

2.13 Assume that for all $l < n, P(m, l)$

fix m_0, we prove $P(m_0, n)$ for all n. assume that for all $l < n, P(m_0, l)$, since for all $l < n$ when $k = m_0$, $P(k, l)$ holds then $P(m_0, n)$ also holds by (**).

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3.1 Fix some $n \in N$, We prove the claim $P(m)$="$n < m \rightarrow f_n \prec f_m$" for all $k \leq m$ such that $k = n+1$, $P(k)$ holds since $n < n+1 \rightarrow f_n \prec f_{n+1}$ holds by assumption (has a true consequence). Now assume that for an m, $k \leq m$, $P(m)$ holds, thus $k \leq m$ and $n < m \rightarrow f_n \prec f_m$. assume that $n < m + 1$ then either $n < m$ or $n = m$, if $n < m$ then by induction hypothesis $f_n \prec f_m$, since $f_m \prec f_{m+1}$ is true by assumption, we get $f_n \prec f_{m+1}$ by transitivity of $\prec$. if $m = n$ then trivially $f_n \prec f_{n+1} = f_{m+1}$ holds, so we proved if $P(m)$ then $n < m + 1 \rightarrow f_n \prec f_{m+1}$ which is $P(m + 1)$, thus $P(m)$ holds for all $k = n + 1 \leq m$, since n was arbitrary it holds for all $m, n \in N$.

3.2 Let $g : A \times N \rightarrow A$ be the function that $g(x, n)$ is the successor of x. Let u be the $\prec$-least element of A, by recursion theorem there is a function $f : N \rightarrow A$ such that $f_0 = u$ and $f_{n+1} = g(f_n, n)$ = successor of f_n, the function is total since by (a) every element of A has a successor. if p is successor of q then $q \prec p$, thus we have $f_n \prec f_{n+1}$. by previus exercise for every $m < n$ we have $f_n \prec f_m$ thus f is one-to-one. To prove it is onto, assume that there is some $a \in A$ such for no $n \in N, f(n) = a$ (not in $ranf$), let a be the least of them, clearly $a \neq u$ since $f_0 = u$, thus by (c) a is successor of some $q \in A$, because a was the least element that is not in range of f, and because $q \prec a$, q must be in range of f, thus for some $k \in N, f_k = q$, but then $f_{k+1} = g(f_k, k)$= successor of $f_k = q$ which is a, thus $a \in ranf$, a contradiction.

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4.1 Fix some $k, m \in N$, we proceed by induction on n, for $n = 0$, $(k+m)+0 = k+(m+0)$ implies $k+m = k+m$ so we are done. Now assume (*) $(k+m)+n = k+(m+n)$ holds for n, then $(k+m)+(n+1) = [(k+m)+n]+1$ by 4.3, by induction hypothesis (*) it follows that $(k+m)+(n+1) = [k+(m+n)]+1$, hence 4.3 implies $[k+(m+n)]+1 = k+((m+n)+1) = k+(m+(n+1))$, thus $(k+m)+(n+1) = k+(m+(n+1))$, this completes the proof. since k, m were arbitrary the propositions holds for all $k, m, n \in N$.

4.2 We prove by induction on k, Let $k = 0$ then $m < n$ iff $m+0 < n+0$ trivially. Now assume that $m < n$ iff $m+k < n+k$ holds for k, we must prove that $m < n$ implies $m+(k+1) < n+(k+1)$. Assume $m < n$, then $m+k < n+k$ by induction hypothesis, exercise 2.2 (page 45) implies that $(m+k)+1 < (n+k)+1$, by previous exercise $m+(k+1) < n+(k+1)$, this completes the induction. the other side can be done by the fact that $S(n)$ is one-to-one.

4.3 We prove it by induction on m, if $m = 0$ then $0 \leq n$ iff there is some $k \in N$ such that $n = 0+k$, namely $k = n$. Let $m+1 \leq n$, since $m < m+1$ we have (*) $m < n$ and $m \leq n$, by induction hypothesis we get $n = m+k$ for some $k \in N$, $k \neq 0$ since otherwise we get $n = m$ which contradicts (*), hence $k = k'+1$ for some $k' \in N$, thus $n = m+k = m+(k'+1) = m+(1+k') = (m+1)+k'$ for some $k' \in N$, thus we are done.

4.4 We use parametric version of Recursion Theorem, let $P = A = N$ and $a : P \to N$ be such that $a(p) = 0$ for every $p \in P = N$ and $g : N \times N \times \times N \to N$ such that $g(p, x, n) = x + p$, then there is a function $\cdot$ such that $m \cdot 0 = \cdot(m, 0) = a(m) = 0$ for all $m \in N$ and $m \cdot (n+1) = \cdot(m, n+1) = g(m, \cdot(m, n), n) = \cdot(m, n) + m = m \cdot n + m$.

4.5 We should prove that $m \cdot n = n \cdot m$ for every $m, n \in N$. we proceed by induction on n, if $n = 0$ then $m \cdot 0 = 0$, we need to show that $0 = 0 \cdot m$ for all m, if $m = 0$ then $0 = 0 \cdot 0$, now assume that $0 = 0 \cdot m$ holds then $0 \cdot (m+1) = 0 \cdot m + 0$, by induction hypothesis we get $0 \cdot (m+1) = 0+0 = 0$, thus for every m we have $m \cdot 0 = 0 \cdot m$. Now assume that $m \cdot n = n \cdot m$ holds for n, we should prove (*) $m \cdot (n+1) = (n+1) \cdot m$ for all $m \in N$, we proceed by induction on m, if $m = 0$ it trivially holds. Now assume that (*) holds for m, we should prove that $(m+1)\cdot(n+1) = (n+1)\cdot(m+1)$. but we know that $(m+1)\cdot(n+1) = [(m+1)\cdot n] + (m+1) = [n\cdot(m+1)] + (m+1) =$

$= (n\cdot m + n) + (m+1)$
$= (n\cdot +m) + (n+1)$
$= (m\cdot n + m) + (n+1)$
$= (m\cdot(n+1)) + (n+1)$
$= (n+1)\cdot m + (n+1)$
$= (n+1)\cdot(m+1)$ and this completes the proof.

To prove that multiplication is distributive over addition we must show that $m\cdot(n+k) = m\cdot n + m\cdot k$. we proceed by induction on k, if $k = 0$ then $m\cdot(n+0) = m\cdot n$ on the other hand $m\cdot n = m\cdot n + 0 = m\cdot n + m\cdot 0$, thus $m\cdot(n+0) = m\cdot n + m\cdot 0$. Now assume that $m\cdot(n+k) = m\cdot n + m\cdot k$ holds for k, then $m\cdot(n+(k+1))$

$= m\cdot((n+k)+1)$
$= m\cdot(n+k) + m$
$= (m\cdot n + m\cdot k) + m$ (by induction hypothesis)
$= m\cdot n + (m\cdot k + m)$
$= m\cdot n + m\cdot(k+1)$

this completes the induction.

To prove that it is associative we need to prove $(m\cdot n)\cdot k = m\cdot(n\cdot k)$ for all $m, n, k \in N$. Fix some $m, n \in N$, we proceed by induction on k, for $k = 0$, $(m\cdot n)\cdot 0 = m\cdot(n\cdot 0) = 0$ holds, since $m\cdot 0 = 0$. Assume that $(m\cdot n)\cdot k = m\cdot(n\cdot k)$ holds for k, then: $(m\cdot n)\cdot(k+1)$

$= (m\cdot n)\cdot k + m\cdot n$
$= m\cdot(n\cdot k) + m\cdot n$ by induction hypothesis
$= m\cdot((n\cdot k) + n)$ by distributive property
$= m\cdot(n\cdot(k+1))$

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1.4 (a) for every $(a, b) \in A \times B$, let $f((a, b)) = (b, a)$.
(b) $f(((a, b), c)) = (a, (b, c))$.
(c) Since $B \neq \emptyset$ there is some $b \in B$, let $f(a) = (a, b)$ for every $a \in A$.

1.5 for every $s \in S$ let $f(s) = \{s\}$, clearly $f(s) \in \mathcal{P}(S)$ and for every $s \in S$ there is a unique $\{s\}$, thus f is one-to-one.

1.6 We need to show there is a one-to-one mapping $f : A \to A^S$. if $A = \emptyset$ then $A^S = \emptyset$ and this case is trivial, so assume that it is non-empty, for every $a \in A$, let $f(a) = h_a$ such that $h : S \to A$ is a function such that for every $s \in S$, $h_a(s) = a$, clearly there is just one function for each $a \in A$, therefore f is one-to-one.

1.7 Like previous exercise for empty A the proof is trivial, assume that it is non-emty, so there is some $a \in A$. for every $f \in A^S$ define $F(f) = f'$ such that $f' \in A^T$, $f'|S = f$ and for every $t \in T - S, f'(t) = a$, clearly $F : A^S \to A^T$, to prove that it is one-to-one assume $F(f) = F(g)$, then there are two function $f', g' \in A^T$ such that $f' = g'$ and $f'|S = f$ and $g'|S = g$, it means that $g = f$.

1.8 Since $2 \leq |S|$ there are at least two distinct element $a, b \in S$. define F as follows: for every $t \in T$ let $f_t \in S^T$ such that $f_t(t) = a$, for every $t \neq x \in T$, $f_t(x) = b$, clearly this is function in A^T. To prove it is one-to-one, assume that $F(t) = F(t')$, then $f_t = f_{t'}$, it means that $f_t(t) = f_{t'}(t)$, but $f_t(t) = a$, therefore $f_{t'}(t) = a$ but the only value for which $f_{t'}(x)$ is equal to a is when $x = t'$, from this we get $t = t'$.

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2.1 Since S is finite, we can prove it by induction on n in $|S| = n$. for $|S| = 0$ it trivially holds. Assume that it holds for every set $|S| = n$, then consider $|S| = n+1$, we would get $\bigcup S = \bigcup_{i=0}^{i=n-1} X_i \cup X_n$, but from induction hypothesis we have $|\bigcup_{i=0}^{i=n-1} S| = \Sigma_{i=0}^{n-1}|X_i|$, with Lemma 2.6 we get $|\bigcup S| = |\bigcup_{i=0}^{i=n-1} X_i \cup X_n| = |\bigcup_{i=0}^{i=n-1} X_i| + |X_n| = \Sigma_{i=0}^{n-1}|X_i| + |X_n| = \Sigma_{i=0}^{n}|X_i|$.

2.2 Fix some X, assume that $|X| = m$, consider Y, if $|Y| = 0$ then $Y = \emptyset$, thus $|X \times Y| = m \cdot 0 = 0$, we are done. Assume that it holds for every set $|S| = n$, thus for it we have $|X \times S| = m \cdot n$, we must prove it for $|X \times Y| = m \cdot (n + 1)$. Now let $|Y| = n + 1$ then $Y = \{y_0, ..., y_n\}$, but $X \times Y = X \times \{y_0, ..., y_{n-1}\} \cup X \times \{y_n\}$, from induction hypothesis we have $|X \times \{y_0, ..., y_{n-1}\}| = m \cdot n$, also we know that $|X \times \{y_n\}| = |X| = m$, so by Lemma 2.6 $|X \times Y| = |X \times \{y_0, ..., y_{n-1}\}| + |X \times \{y_n\}| = m \cdot n + m = m \cdot (n+1)$, since X was arbitary finite set, it holds for all finite set.

2.3 We proceed by induction on cardinal of X, if $|X| = 0$ then $X = \emptyset$, thus $|\mathcal{P}(X)| = |\{\emptyset\}| = 2^0 = 1$. Now assume that it holds for $|X| = n$, consider when $|X| = n+1$, let $X' = \{x_0, ..., x_{n-1}\}$, then $\mathcal{P}(X) = \mathcal{P}(X') \cup \{K \cup \{x_n\} : K \in \mathcal{P}(X')\}$, by induction hypothesis $|\mathcal{P}(X')| = 2^{|X'|} = 2^n$ and so the other set, they are also disjoint set, thus we get $|\mathcal{P}(X) = |\mathcal{P}(X')| + |\{K \cup \{x_n\} : K \in \mathcal{P}(X')\}| = 2^n + 2^n = 2^n \cdot 2 = 2^{n+1}$ that completes the proof.

2.4 Fix some X, we proceed by induction on $|Y|$, if $|Y| = 0$ then $Y = \emptyset$, then $|X^{\emptyset}| = |\{\emptyset\}| = |X|^0 = 1$. Now assume that it holds for all set S such that $|S| = n$. if $|Y| = n+1$ then $X^Y = X^{\{y_0,...,y_{n-1}\} \cup \{y_n\}} = \{f \cup \{(y_n, x)\} : (f, x) \in X^{\{y_0,...,y_{n-1}\}} \times X\}$, (Why they are equal: clearly each $f \cup \{(y_n, x)\} \in X^Y$, since $y_n \notin \{y_0, ..., y_{n-1}\}$, it is a function with domain Y and range X. Now let $g \in X^Y$, then for some $x \in X$ we have $g(y_n) = x$, clearly $(g - \{(y_n, x)\}) \in X^{\{y_0,...,y_{n-1}\}}$, also $(g - \{(y_n, x)\}, x) \in X^{\{y_0,...,y_{n-1}\}} \times X$, for it we have $(g - \{(y_n, x)\}) \cup \{(y_n, x)\} = g$). from exercise 2.2 and induction hypothesis we can conclude that $|X^Y| = |\{f \cup \{(y_n, x)\} : (f, x) \in X^{\{y_0,...,y_{n-1}\}} \times X\}| = |X^{\{y_0,...,y_{n1}\}} \times X| = |X^{\{y_0,...,y_{n-1}\}}| \cdot |X| = |X|^n \cdot |X| = |X|^{n+1} = |X|^{|Y|}$, therefore it holds for $|Y| = n+1$, we are done.

2.7 (we just prove some of these properties) Consider two disjoint set X, Y, Z such that $|X| = m$ and $|Y| = n, |Z| = p$, commutativity of addition: by Lemma 2.6 we have $|X \cup Y| = |X| + |Y| = m + n$, but we know that $X \cup Y = Y \cup X$ so $|X \cup Y| = |Y \cup X|$ it implies $m + n = n + m$.

associativity of addition: since $(X \cup Y) \cup Z = X \cup (Y \cup Z)$ we get $|(X \cup Y) \cup Z| = |X \cup (Y \cup Z)| = |(X \cup Y)| + |Z| = |X| + |(Y \cup Z)| = m + (n + p) = (m + n) + p$.

distributivity of multiplication over addition: since $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$ (by exercise 2.8 on page 23) then by previous exercises it easily follows that $|X| \cdot (|Y| + |Z|) = (|X| \cdot |Y|) + (|X| \cdot |Z|) = m \cdot (n + p) = (m \cdot n) + (m \cdot p)$.