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3.1 We must prove that the set $\{x : x \in A \; and \; x \notin B\}$ exist. Let $P(x, A, B)$ be the property "$x \in A \; and \; x \notin B$", $P(x, A, B)$ implies $x \in A$, because A exist, we have $\{x : x \in A \; and \; x \notin B\} = \{x \in A : x \in A \; and \; x \notin B\} = \{x \in A : x \notin B\}$, this set clearly exist by the axiom of comprehension.

3.2 Weak Axiom of Existence implies that some set exist, call one of them A and let $P(x)$ be the property "$x \neq x$", by axiom of comprehension the set $X = \{x \in A : x \neq x\}$ exist, it has no element because no object satisfy the property $P(x)$.

3.3 (a) Suppose that V is set of all sets, by Comprehension $X = \{x \in V : x \notin x\}$ exist. Because V is set of all sets, clearly $X \in V$. Now suppose that $X \in X$ then $X \notin X$ by definition, a contradiction. suppose $X \notin X$, then $X \in X$ again by definition.

(b) Assume the contrary, there is a set A that any $x \in A$. then $A = V$ is set of all sets, by previous exercise there is no V.

3.4 By axiom of pairing the set $\{A, B\}$ exist and union axiom implies the existence of $\bigcup\{A, B\}$, let $P(x, A, B) = (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)$ by comprehension there is a set that its elements satisfy $P(x, A, B)$ and $x \in \bigcup\{A, B\}$.

3.5 3.5(a) by axiom of pairing there is $\{A, B\}$ and $\{C\}$. again by pairing $\{\{A, B\}, \{C\}\}$. by axiom of union there is $X = \bigcup\{\{A, B\}, \{C\}\}$. Now $x \in X$ iff $x \in \{A, B\}$ or $x \in \{C\}$ iff $x = A$ or $x = B$ or $x = C$.

(b) Take $\{C, D\}$ instead of $\{C\}$ in the previous exercise.

3.6 Assume that $\mathcal{P}(X) \subseteq X$, Now let $Y = \{x \in X : x \notin x\}$, clearly $Y \subseteq X$, so $Y \in \mathcal{P}(X)$, thus $Y \in X$. also we have either $Y \in Y$ or $Y \notin Y$. if first, $Y \notin Y$, if th second $Y \in Y$, thus $Y \in Y$ iff $Y \notin Y$, a contradiction.

3.6 Let $P(x, A, B)$ be the property "$x = A \vee x = B$", apply axiom of comprehension to C, we get the set $X \subseteq C$ such that $x \in X$ iff $x = A$ or $x = B$, so $X = \{A, B\}$.

Let $P'(x, S)$ be the property "$\exists A(A \in S \wedge X \in A)$", apply axiom of comprehension to U, we get the set Y such that $x \in Y$ iff for some $A \in S$ we have $x \in A$, thus $Y = \bigcup S$.

Let $P'(x, S)$ be the property "$x \subseteq S$", apply axiom of comprehension to P, we get the set Z such that $x \in Z$ iff $x \subseteq S$, thus $Y = \mathcal{P}(S)$.

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4.2 (a) Left to right, assume (*) $A \subseteq B$, and let $x \in A \cap B$, which means that $x \in A$ and $x \in B$, we can conclude $x \in A$, thus $A \cap B \subseteq A$(**). to prove the other direction, let $x \in A$, by assumption (*) we get $x \in B$, we can conclude $x \in A$ and $x \in B$, which means that $x \in A \cap B$, so we have $A \subseteq A \cap B$, so by this and (**) we have $A = A \cap B$.

Right to left, suppose $A \cap B = A$(*), let $x \in A$, by (*) $x \in B$, so we have $A \subseteq B$.

Second part, $x \in A \cup B$ iff $x \in B$, it means that there is nothing in A such that is not in B, thus $A - B = \emptyset$.

(b) Left to right, suppose $A \subseteq B \cap C$, let $x \in A$, by previous assumption we have $x \in B \cap C$, which implies that $x \in B$ and $x \in C$, so we have $A \subseteq B$ and $A \subseteq C$.

Right to left, suppose $A \subseteq B$ and $A \subseteq C$, let $x \in A$, by two previous assumtion we have both $x \in B$ and $x \in C$ which implies that $x \in B \cap C$, thus we have $A \subseteq B \cap C$.

(c) Suppose $B \cup C \subseteq A$, let $x \in B$, we can get also $x \in B \cup C$, by previous assumption we conclude that $x \in A$, thus $B \subseteq A$. by similar argument we can show $C \subseteq A$.

(d) $x \in A - B$ iff $x \in A \wedge \neg(x \in B)$ iff $x \in A \wedge \neg(x \in B) \vee (x \in B \wedge \neg(x \in B))$ iff $(x \in A \vee x \in B) \wedge \neg(x \in B)$ iff $x \in (A \cup B) - B$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in A))$ iff $x \in A \wedge (\neg(x \in A) \vee \neg(x \in B))$ iff $x \in A - (A \cap B)$.

(e) $x \in A \cap B$ iff $x \in A \wedge x \in B$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge \neg(x \in A))$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge \neg(x \in A))$ iff $x \in A \wedge (x \in B \vee \neg(x \in A))$ iff $x \in A \wedge \neg(\neg(x \in B) \wedge (x \in A))$ iff $x \in A \wedge \neg(x \in A - B))$ iff $x \in A - (A - B)$.

(f) $x \in A - (B - C)$ iff $x \in A \wedge \neg(x \in B - C)$ iff $x \in A \wedge \neg(x \in B \wedge \neg(x \in C))$ iff $x \in A \wedge (\neg(x \in B) \vee (x \in C))$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge x \in C)$ iff $x \in A - B \vee x \in A \cap C$ iff $x \in (A - B) \cup (A \cap C)$.

(g) $(A-B)\cup(B-A)=\emptyset$ iff both (*) $A-B=\emptyset$ and $B-A=\emptyset$, by (a) we get (*) iff $A\subseteq B$ and $B\subseteq A$ iff $A=B$.

4.4 Suppose it exist, then $A'\cup A$ is equal to universal set which does not exist.

4.5 (a) let $x\in A\cap\bigcup S$, then $x\in A$ and $x\in C$ for some $C\in S$, it means that $x\in A\cap C$, clearly $A\cap C\in P(A)$ so $A\cap C\in T_1$ by definition, thus $x\in\bigcup T_1$. (Note that if we take $A\cap C=C$, then we can say that for some $C\in T_1$ we have $x\in C$). Now let $x\in\bigcup T_1$, then there is some $Y\in T_1$ such that $x\in Y$, but by definition of T_1 we know that $Y=A\cap X$ for some $X\in S$, it means that $x\in\bigcup S$ and $x\in A$, thus $x\in A\cap\bigcup S$.

(b) Let $x\in A-\bigcup S$, we have $x\in A-\bigcup S$ iff $x\in A$ and $x\notin X$ for any $X\in S$. it equally means that (*) $x\in A-X$ for every $X\in S$. we know that any set in the form of $A-X$ such that $X\in S$ is in T_2, thus (*) means that we have $x\in\bigcap T_2$.

$x\in A-\bigcap S$ iff $x\in A$ and $x\notin C$ for some $C\in S$ iff $x\in A-C$ for some $C\in S$, because any set in the form of $A-X$ such that $X\in S$ is in T_2 we have some $x\in\bigcap T_2$.

4.6 if S is not empty, then there is some $C\in S$, by Axiom Schema of Comprehension the set $\{x\in C:(\forall X)(X\in S\to x\in X)\}$ exist. if it was empty, then we could not apply the axiom of comprehension.