This document contains solutions for selected problems from "Set Theory: A First Course" by Daniel W. Cunningham.
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Exercises 1.1

1. By definition of $a \notin A/B$ we have $\neg(a \in A/B)$, it means that $a \notin A$ or $a \in B$ and by supposition we know $a \in A$. thus it can not be the case that $a \notin A$, so $a \in B$.

2. By definition of $A \subseteq B$ we know that for all x, $x \in A \Rightarrow x \in B(*)$. Suppose $x \in C \wedge x \notin B$. by $x \notin B$ and contraposition of (*) we have $x \notin A$. by supposition we can make claim that $x \in C \wedge x \notin A$, so we can write for all x ($x \in C \wedge x \notin B \Longrightarrow x \in C \wedge x \notin A$) which is definition of $C/B \subseteq C/A$.

3. For all x we have, $x \in A \wedge x \notin B \Rightarrow x \in C(*)$. Suppose an x that $x \in A \wedge x \notin C$, by $x \notin C$ and (*) using modus tollens we have $\neg(x \in A \wedge x \notin B)$ which is equal to say that $x \notin A \vee x \in B$. by the last sentence and $x \in A$ from supposition, we have $x \in B$. thus we can say $x \in A \wedge x \notin C \Rightarrow x \in B$ which is definition of $A/C \subseteq B$.

4. Suppose an x such that $x \in A$, by $A \subseteq B$ we know $x \in B$ and by $x \subseteq C$, $x \in C$. so we can say that for all x, $x \in A \Rightarrow x \in B \wedge x \in C$ which is definition of $A \subseteq B \cap C$.

5. We prove this by contradiction. Suppose there exist an a such that $a \in A$ but $a \notin B/C$ wich is equal to say that $a \notin B \vee a \in C$. by $a \in A$ from our supposition and $A \subseteq B$ from problem's supposition, we get $a \in B(*)$, thus it can not be the case that $a \notin B$, so it must be the case that $a \in C$ which together with (*) contradict problem's supposition $B \cap C = \emptyset$.

6. Suppose an x such that $x \in A/(B/C)$ which is equal to say that $x \in A \wedge x \notin B/C$. The second conjunct is equal to $x \notin B \vee x \in C$. At least one of the dijuncts must be true, if $x \notin B$, by supposition we have $x \in A$, so we can write $x \in A/B$. we can also say $x \in A/B \cup C$. if $x \in C$ then $x \in C \cup A/B$. We can conclude that $x \in A/(B/C) \Rightarrow x \in A/B \cup C$ which is definition of $A/(B/C) \subseteq A/B \cup C$.

7. $A \not\subseteq C$ means that there exist an a such that $a \in A$ and $a \notin C$. by $a \notin C$ and $A/B \subseteq C$ we know that $a \notin A/B$ which means that $a \notin A$ or $a \in B$. because of $a \in A$ it is only possible $a \in B$. so we have $a \in A$ and $a \in B$ which means $A \cap B \neq \emptyset$.

Exercises 1.5

1. By paring axiom we get the set $\{\{u\}, \{v, w\}\}$. Now by union axiom there exist a set that contains member of member of this set, i.e. $\{u, v, w\}$.

2. By the pairing axiom for every two set there is a set that contains them. take both set A, then we get $\{A\}$.

3. Axiom of regularity says that every non-empty set S contains at least one set x such that $x \cap S = \emptyset$. because the set $\{A\}$ contains just one set A, it must be the case that $A \cap \{A\} = \emptyset$(*). Now suppose that $A \in A$, together with the fact that $A \in \{A\}$, there must be a common object in the two sets which contradict our first result (*).

4. By the axiom of regularity the set $\{A, B\}$ must contain a set which has nothing in common with that (i.e. $\exists(S \in \{A, B\})S \cap \{A, B\}$). Because the set $\{A, B\}$ just contains two set, it must be A or B. it could not be be B because $A \in B$ and $A \in \{A, B\}$. it just remains A, so $A \cap \{A, B\} = \emptyset$. Clearly, $B \notin A$ because it contradicts former claim.

5. According to the regluarity axiom the set $\{A, B, C\}$ must contains a member x which $A \notin x$ and $B \notin x$ and $C \notin x$. x could not be B, because by problem supposition we know that $A \in B$. By the same justification x is not C. it just remains A, therefore the third conjucnt implies that $C \notin A$.

6. By power set axiom we have $\mathcal{P}(A)$. Now by subset axiom we can define $\{x \in \mathcal{P}(A) : x \in B\}$ which is equal to $\mathcal{P}(A) \cap B$.

9. To prove $A = \emptyset$ we must show that for all x $x \in A \Leftrightarrow x \in \emptyset$. the $\Rightarrow$ side is vacuously true because we supposed A to have no member. the $\Leftarrow$ side is true because empty set doesn't have any member.

10. Suppose that for an x $\phi(x, y_0)$ and $\phi(x, y_1)$ are both true, we prove that $y_0 = y_1$. Since $\forall z(z \in y_0 \leftrightarrow z = x)$ and $\forall z(z \in y_1 \leftrightarrow z = x)$ we have $\forall z(z \in y_0 \leftrightarrow z \in y_1)$, thus $y_0 = y_1$ and ϕ describe uniquely such a y. So by $\phi(x, y)$ and replacement axiom for every set A we have a set $\{\{x\} : x \in A\}$.