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4.2 (a) $A \subseteq B$ iff $A \cap B = A$: Assume that $A \subseteq B$ (*), it is clear that $A \cap B \subset A$, let $x \in A$, then by (*) $x \in B$, thus $x \in A \cap B$, therefore $A \cap B = A$. For the converse, assume $A = A \cap B$ and let $x \in A$, then $x \in A \cap B$, which means $x \in B$, thus $A \subseteq B$.

$A \cap B = A$ iff $A \cup B = B$: Suppose $A \cap B = A$ (*), it is clear that $B \subseteq A \cup B$, let $x \in A \cup B$, if $x \in B$ we are done, if $x \in A$, then by (*) and previous item we get $A \subseteq B$, thus in either case $x \in B$, therefore $A \cup B \subseteq B$, hence $A \cup B = B$. Conversely, suppose $A \cup B = B$, and let $x \in A$, then $x \in B$, thus $A \subseteq B$, by previous item we get $A \cap B = A$.

$A \cup B = B$ iff $A - B = \emptyset$: Assume $A \cup B = B$, and some $x \in A - B$, thus $x \in A$ but $x \notin B$, but $x \in A$ implies $x \in A \cup B$, by assumption $x \in B$ which is a contradiction. For the converse, suppose $A - B = \emptyset$, we show $A \cup B \subseteq B$ (the other side is clear). let $x \in A \cup B$, if $x \in B$ we are done, let $x \in A$, and to the contrary $x \notin B$, then we get $x \in A - B$ which is a contradiction.

(b) Suppose that $A \subseteq B \cap C$, let $x \in A$, then $x \in B$, thus $A \subseteq B$. This also implies $x \in C$, thus $A \subseteq C$. Conversely, suppose $A \subseteq C$ and $A \subseteq B$, then $x \in A$ implies $x \in B$ and $x \in C$, thus $x \in B \cap C$.

(c) Assume $B \cup C \subseteq A$, let $x \in B$, then $x \in B \cup C$, thus $x \in A$, therefore $B \subseteq A$. if $x \in C$, again, $x \in B \cup C$, thus we get $x \in A$, thus $C \subseteq A$. For the converse, assume that $B \subseteq A$ and $C \subseteq A$. Let $x \in B \cup C$, then either $x \in B$ or $x \in C$, in either case by the assumption we get $x \in A$, therefore $B \cup C \subseteq A$.

(d) $x \in A - B$ iff $x \in A \wedge \neg(x \in B)$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in B \wedge \neg(x \in B))$ iff $(x \in A \vee x \in B) \wedge \neg(x \in B)$ iff $x \in (A \cup B) - B$ iff $(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in A)$ iff $x \in A \wedge (\neg(x \in A)) \vee \neg(x \in B)$ iff $x \in A - (A \cap B)$.

(e) $x \in A \cap B$ iff $x \in A \wedge x \in B$ iff $(x \in A \wedge x \in B) \vee (x \in A \wedge \neg(x \in A))$ iff $x \in A \wedge (x \in B \vee \neg(x \in A))$ iff $x \in A \wedge \neg(\neg(x \in B) \wedge x \in A)$ iff $x \in A \wedge \neg(x \in A - B)$ iff $x \in A - (A - B)$.

(f) $x \in A - (B - C)$ iff $x \in A \wedge \neg(x \in B - C)$ iff $\in A \wedge \neg(x \in B \wedge \neg(x \in C))$ iff $x \in A \wedge (\neg(x \in B) \vee (x \in C))$ iff $(x \in A \neg(x \in B)) \vee (x \in A \wedge x \in C)$ iff $(x \in A - B) \vee (x \in A \cap C)$ iff $x \in (A - B) \cup (A \cap C)$.

(g) $(A - B) \cup (B - A) = \emptyset$ iff both $A - B = \emptyset$ and $B - A = \emptyset$. But by item (a), $A - B = \emptyset$ and $B - A = \emptyset$ iff $A \subseteq B$ and $B \subseteq A$, thus $A = B$.

4.4 Assume that the complement of A, the set A' exists. Let x be an arbitrary set, then we have either $x \in A$ or $x \notin A$ or equivalently $A \cup A = V$, V is the universal set which does not exist.

4.5 (a) Let $a \in A \cap \bigcup S$, then for some $X \in S$, $a \in X$ and also $a \in A$, therefore $a \in A \cap X$, name it Y, clearly $Y \subseteq A$, thus $Y \in \mathcal{P}(A)$. Now we have some $Y \in \mathcal{P}(A)$ such that $Y = A \cap X$ for some $X \in S$, thus $Y \in T_1$, which means $a \in \bigcup T_1$. Conversely, let $a \in \bigcup T_1$, then $a \in Y$ for some $Y \in T_1$, but by definition of T_1, $Y = A \cap X$ for some $X \in S$, which means $a \in A$ and $a \in X$ thus $a \in A \cap \bigcup S$.

(b) $A - \bigcup S = \bigcap T_1$: let $x \in A - \bigcup S$, then $x \in A$ and for every $X \in S$, $x \notin X$. Take an arbitrary $X \in S$, by the previous observation, we have $x \in A$ but $x \notin X$, therefore $x \in A - X = Y$ and clearly $Y \in \mathcal{P}(A)$, thus $Y \in T_2$, since X was arbitrary we have $x \in A - X$ for any $X \in S$, but any $A - X$ is in T_2, thus for any member $Y \in T_2$, $x \in Y$, which means $x \in \bigcap T_2$. Conversely, let $x \in \bigcap T_2$, then $x \in Y$ for every $Y \in T_2$, but by definition of T_2, we have $Y = A - X$ for some $X \in S$. Since for arbitrary $X \in S$, we have $A - X \in Y$, the previous proposition means that for arbitrary $X \in S$ we have $x \notin X$, thus $x \notin \bigcup S$, thereby $x \in A - \bigcup S$.

$A - \bigcap S = \bigcup T_2$: Let $x \in A - \bigcap S$, then $x \in A$ and $x \notin X$ for some $X \in S$, thus $x \in A - X$. But $A - X \in \mathcal{P}(A)$, thus $A - X \in T_2$, which means for some $Y \in T_2$ (namely $Y = A - X$), we have $x \in Y$, thus $x \in \bigcup T_2$.