Page 11

3.1 We must prove that the set $\{x : x \in A \text{ } and \text{ } x \notin B\}$ exist. Let $P(x, A, B)$ be the property "$x \in A$ *and* $x \notin B$", $P(x, A, B)$ implies $x \in A$, because A exist, we have $\{x : x \in A \text{ } and \text{ } x \notin B\} = \{x \in A : x \in A \text{ } and \text{ } x \notin B\} = \{x \in A : x \notin B\}$, this set clearly exist by the axiom of comprehension.

3.2 Weak Axiom of Existence implies that some set exist, call one of them A and let $P(x)$ be the property "$x \neq x$", by axiom of comprehension the set $X = \{x \in A : x \neq x\}$ exist, it has no element because no object satisfy the property $P(x)$.

3.3 (a) Suppose that V is set of all sets, by Comprehension $X = \{x \in V : x \notin x\}$ exist. Because V is set of all sets, clearly $X \in V$. Now suppose that $X \in X$ then $X \notin X$ by definition, a contradiction. suppose $X \notin X$, then $X \in X$ again by definition.

(b) Assume the contrary, there is a set A that any $x \in A$. then $A = V$ is set of all sets, by previous exercise there is no V.

3.4 By axiom of pairing the set $\{A, B\}$ exist and union axiom implies the existence of $\bigcup\{A, B\}$, let $P(x, A, B) = (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)$ by comprehension there is a set that its elements satisfy $P(x, A, B)$ and $x \in \bigcup\{A, B\}$.

3.5 3.5(a) by axiom of pairing there is $\{A, B\}$ and $\{C\}$. again by pairing $\{\{A, B\}, \{C\}\}$. by axiom of union there is $X = \bigcup\{\{A, B\}, \{C\}\}$. Now $x \in X$ iff $x \in \{A, B\}$ or $x \in \{C\}$ iff $x = A$ or $x = B$ or $x = C$.

(b) Take $\{C, D\}$ instead of $\{C\}$ in the previous exercise.

3.6 Assume that $\mathcal{P}(X) \subseteq X$, Now let $Y = \{x \in X : x \notin x\}$, clearly $Y \subseteq X$, so $Y \in \mathcal{P}(X)$, thus $Y \in X$. also we have either $Y \in Y$ or $Y \notin Y$. if first, $Y \notin Y$, if th second $Y \in Y$, thus $Y \in Y$ iff $Y \notin Y$, a contradiction.

3.7 Let $P(x, A, B)$ be the property "$x = A \vee x = B$", apply axiom of comprehension to C, we get the set $X \subseteq C$ such that $x \in X$ iff $x = A$ or $x = B$, so $X = \{A, B\}$.

Let $P'(x,S)$ be the property "$\exists A(A \in S \land X \in A)$", apply axiom of comprehension to U, we get the set Y such that $x \in Y$ iff for some $A \in S$ we have $x \in A$, thus $Y = \bigcup S$.

Let $P'(x,S)$ be the property "$x \subseteq S$", apply axiom of comprehension to P, we get the set Z such that $x \in Z$ iff $x \subseteq S$, thus $Y = \mathcal{P}(S)$.

Page 15

4.2 (a) Left to right, assume $A \subseteq B$(*), and let $x \in A \cap B$, which means that $x \in A$ and $x \in B$, we can conclude $x \in A$, thus $A \cap B \subseteq A$(**). to prove the other direction, let $x \in A$, by assumption (*) we get $x \in B$, we can conclude $x \in A$ and $x \in B$, which means that $x \in A \cap B$, so we have $A \subseteq A \cap B$, so by this and (**) we have $A = A \cap B$.

Right to left, suppose $A \cap B = A$(*), let $x \in A$, by (*) $x \in B$, so we have $A \subseteq B$.

Second part, $x \in A \cup B$ iff $x \in B$, it means that there is nothing in A such that is not in B, thus $A - B = \emptyset$.

(b) Left to right, suppose $A \subseteq B \cap C$, let $x \in A$, by previous assumption we have $x \in B \cap C$, which implies that $x \in B$ and $x \in C$, so we have $A \subseteq B$ and $A \subseteq C$.

Right to left, suppose $A \subseteq B$ and $A \subseteq C$, let $x \in A$, by two previous assumtion we have both $x \in B$ and $x \in C$ which implies that $x \in B \cap C$, thus we have $A \subseteq B \cap C$.

(c) Suppose $B \cup C \subseteq A$, let $x \in B$, we can get also $x \in B \cup C$, by previous assumption we conclude that $x \in A$, thus $B \subseteq A$. by similar argument we can show $C \subseteq A$.

(d) $x \in A - B$ iff $x \in A \land \neg(x \in B)$ iff $x \in A \land \neg(x \in B) \lor (x \in B \land \neg(x \in B))$ iff $(x \in A \lor x \in B) \land \neg(x \in B)$ iff $x \in (A \cup B) - B$ iff $(x \in A \land \neg(x \in B)) \lor (x \in A \land \neg(x \in A))$ iff $x \in A \land (\neg(x \in A) \lor \neg(x \in B))$ iff $x \in A - (A \cap B)$.

(e) $x \in A \cap B$ iff $x \in A \land x \in B$ iff $(x \in A \land x \in B) \lor (x \in A \land \neg(x \in A))$ iff $(x \in A \land x \in B) \lor (x \in A \land \neg(x \in A))$ iff $x \in A \land (x \in B \lor \neg(x \in A))$ iff $x \in A \land \neg(\neg(x \in B) \land (x \in A))$ iff $x \in A \land \neg(x \in A - B))$ iff $x \in A - (A - B)$.

(f) $x \in A - (B - C)$ iff $x \in A \land \neg(x \in B - C)$ iff $x \in A \land \neg(x \in B \land \neg(x \in C))$ iff $x \in A \land (\neg(x \in B) \lor (x \in C))$ iff $(x \in A \land \neg(x \in B)) \lor (x \in A \land x \in C)$ iff $x \in A - B \lor x \in A \cap C$ iff $x \in (A - B) \cup (A \cap C)$.

(g) $(A-B)\cup(B-A)=\emptyset$ iff both (*) $A-B=\emptyset$ and $B-A=\emptyset$, by (a) we get (*) iff $A\subseteq B$ and $B\subseteq A$ iff $A=B$.

4.4 Suppose it exist, then $A'\cup A$ is equal to universal set which does not exist.

4.5 (a) let $x\in A\cap\bigcup S$, then $x\in A$ and $x\in C$ for some $C\in S$, it means that $x\in A\cap C$, clearly $A\cap C\in P(A)$ so $A\cap C\in T_1$ by definition, thus $x\in\bigcup T_1$. (Note that if we take $A\cap C=C$, then we can say that for some $C\in T_1$ we have $x\in C$). Now let $x\in\bigcup T_1$, then there is some $Y\in T_1$ such that $x\in Y$, but by definition of T_1 we know that $Y=A\cap X$ for some $X\in S$, it means that $x\in\bigcup S$ and $x\in A$, thus $x\in A\cap\bigcup S$.

(b) Let $x\in A-\bigcup S$, we have $x\in A-\bigcup S$ iff $x\in A$ and $x\notin X$ for any $X\in S$. it equally means that (*) $x\in A-X$ for every $X\in S$. we know that any set in the form of $A-X$ such that $X\in S$ is in T_2, thus (*) means that we have $x\in\bigcap T_2$.

$x\in A-\bigcap S$ iff $x\in A$ and $x\notin C$ for some $C\in S$ iff $x\in A-C$ for some $C\in S$, because any set in the form of $A-X$ such that $X\in S$ is in T_2 we have some $x\in\bigcap T_2$.

4.6 if S is not empty, then there is some $C\in S$, by Axiom Schema of Comprehension the set $\{x\in C:(\forall X)(X\in S\to x\in X)\}$ exist. if it is empty, then we can not apply the axiom of comprehension.

1 Page 18

1.1 1.1 We know that both $\{a\}$ and $\{a,b\}$ are subset of $\{a,b\}$, thus $\{a,b\},\{a\}\in\mathcal{P}(\{a,b\})$, it means that $\{\{a,b\},\{a\}\}\subseteq\mathcal{P}(\{a,b\})$ which implies $\{\{a,b\},\{a\}\}\in\mathcal{P}(\mathcal{P}(\{a,b\}))$.

we have $a,b\in\{a,b\}$, but $(a,b)=\{\{a\},\{a,b\}\}$ which means that there is some $C\in(a,b)$ such tha $a,b\in C$, thus $a,b\in\bigcup(a,b)$.

if $a,b\in A$ then $\{a,b\}$ and $\{a\}$ both are subset of A, thus $\{a,b\},\{a\}\in\mathcal{P}(A)$, again it implies that $\{\{a,b\},\{a\}\}\subseteq\mathcal{P}(A)$, thus $(a,b)=\{\{a,b\},\{a\}\}\in\mathcal{P}(\mathcal{P}(A))$.

1.2 1.2 if a and b exist, then by axiom of pairing and powerset $T = \mathcal{P}(\mathcal{P}(\{a, b\}))$ exist and by previous exercise $(a, b) \in T$. because $(a, b, c) = ((a, b), c)$ by previous argument we have $(a, b, c) \in \mathcal{P}(\mathcal{P}(\{(a, b), c\}))$ which clearly exist.

1.3 if $(a, b) = (b, a)$, it follows from Theorem 1.2 that $a = b$ and $b = a$, so $a = b$.

1.4 if $(a, b, c) = (a', b', c')$ then $((a, b), c) = ((a', b'), c')$, by Theorem 1.2 we have (*) $(a, b) = (a', b')$ and $c = c'$, but again by Theorem 1.2 and (*) we have $a = a'$ and $b = b'$.

1.5 Let $a = \emptyset$, $b = \{a\}$ and $c = \{b\}$, then if $((a, b), c) = (a, (b, c))$ we get $(a, b) = a = \emptyset = \{\{a\}, \{a, b\}\}$ which is a contradiction.

1.6 We first prove that:

(1) $a = c$ or $d = \square$.

(2) $b = d$ or $c = \triangle$.

To prove (1): $\{\{a, \square\}, \{b, \triangle\}\} = \{\{c, \square\}, \{d, \triangle\}\}$ implies either $(\bullet)$ $\{a, \square\} = \{c, \square\}$ or $(\star)$ $\{a, \square\} = \{d, \triangle\}$, if $(\bullet)$ then either $a = c$ or $a = \square$, if first we are done, if the second then $\{a, \square\} = \{\square\} = \{c, \square\}$ which means $a = \square = c$, thus in both case $a = c$. if $(\star)$ then either $a = d$ or $a = \triangle$, if first then $\{a, \square\} = \{a, \triangle\}$ which implies $\triangle = \square$, contradiction, so we have $a = \triangle$, then $\{\triangle, \square\} = \{d, \triangle\}$ which implies $d = \square$. so we have either $a = c$ or $d = \square$.

To prove (2):

We also have (*) $\{b, \triangle\} = \{c, \square\}$ or (**) $\{b, \triangle\} = \{d, \triangle\}$, if (*) then either $b = c$ or $b = \square$, if first then $\{b, \triangle\} = \{b, \square\}$ which implies a contradiction: $\triangle = \square$, therefore the second case only remains which implies $c = \triangle$. if (**) then either $b = d$ or $b = \square$, if first we are done, if the second then $\{\square, \triangle\} = \{d, \triangle\}$ which implies $b = \square = d$, so in both case we have $b = d$. so we have either (2) $b = d$ or $c = \triangle$.

So we have (1) and (2), assume that $b = d$ from (2), now consider (1), if first case then we are done. if the second then $b = d = \square$, therefore $\{\{a, \square\}, \{\square, \triangle\}\} = \{\{c, \square\}, \{\square, \triangle\}\}$ which implies $a = c$.

Assume the second case of (2), then by first case of (1) we have $a = c = \triangle$, therefore $\{\{\triangle, \square\}, \{b, \triangle\}\} = \{\{\triangle, \square\}, \{d, \triangle\}\}$ which implies $b = d$.

Now consider the second case of (1), then we have $d = \Box$ and $c = \triangle$ then $\{\{a, \Box\}, \{b, \triangle\}\} = \{\{\triangle, \Box\}, \{\Box, \triangle\}\} = \{\{\Box, \triangle\}\}$, then $a = \triangle = c$ and $b = \Box = d$, we are done.

2 Page 22

2.1 Let $(x, y) = \{\{x\}, \{x, y\}\} \in R$, then $\{\{x\}, \{x, y\}\} \subseteq \bigcup R$, thus we have $\{x, y\} \in \bigcup R$ and we know that $x, y \in \{x, y\}$, so for some set $C \in \bigcup R$ we have $x, y \in C$, thus $x, y \in \bigcup\bigcup R$. because the property "$x \in dom\ R$" implies that $(x, y) \in R$ for some y. and because $(x, y) \in R$ implies $x \in A$, the set $\{x \in A : x \in dom\ R\}$ exist. Repeat this argument for property "$x \in ran\ R$".

2.2 (a) by previous argument $ran\ R$ and $dom\ R$ exist, we know that $ran\ R \times dom\ R$ exist, call it A. by comprehension the subset $\{(y, x) \in A : (x, y) \in R\}$ also exist, this set is equal to R^{-1}. again by comprehension the set $\{(x, y) \in dom\ R \times ran\ S : for\ some\ z,\ (x, z) \in R\ and\ (z, y) \in S\}$, this set is equal to $S \circ R$.

(b) Because $A \times B \times C = (A \times B) \times C \subseteq \mathcal{P}((A \times B) \cup C)$, comprehension implies that the set $\{x \in \mathcal{P}((A \times B) \cup C) : x = (y, z)\ for\ some\ y \in A \times B\ and\ z \in C\}$ exist.

2.3 (a) $y \in R[A \cup B]$ iff $(\exists x)(x \in A \cup B \wedge\ xRy)$ iff $(\exists x)((x \in A \vee x \in B) \wedge\ xRy)$ iff $(\exists x)((x \in A \wedge\ xRy) \vee (x \in B \wedge\ xRy))$ iff $(\exists x)(x \in A \wedge\ xRy) \vee (\exists x)(x \in B \wedge\ xRy)$ iff $y \in R[A] \vee y \in R[B]$ iff $y \in R[A] \cup R[B]$.

(b) Let $y \in R[A \cap B]$, then for some $x \in A \cap B$ we have xRy which means that $x \in A$ such that xRy and $x \in B$ such that xRy, thus $x \in R[A] \cap R[B]$.

(c) Suppose that $y \in R[A] - R[B]$, it means there is some $x \in A$ such that xRy but there is no $z \in B$ such that zRy, because xRy holds for x, it can not be in B, thus $x \in A - B$ and xRy which means that $y \in R[A - B]$.

(d) Let $R = \{(a, c), (b, c)\}$ and $A = \{a\}$, $B = \{b\}$ then $R[A] \cap R[B] = \{c\}$ while $R[A \cap B = \emptyset] = \emptyset$. also $R[A - B] = R[\{a\}] = \{c\}$ but $R[A] - R[B] = \{c\} - \{c\} = \emptyset$, so this falsifies converse of both (b) and (c).

(f) Fix $x \in A \cap dom\ R$, then because $x \in dom\ R$ there is some y such that xRy, because $x \in A$ we conclude that $y \in R[A]$, so there is some $y \in R[A]$ such that xRy or equivalently $yR^{-1}x$, thus $x \in R^{-1}[R[A]]$.

Fix $y \in B \cap ran\ R$, since $y \in ran\ R$ for some x we have xRy, but $y \in B$ implies that $x \in R^{-1}[B]$, thus for some $x \in R^{-1}[B]$ we have xRy, therefore

$y \in R[R^{-1}[B]]$.

Let $R = \{(a,c),(b,c),(e,f),(e,g)\}$ and $A = \{a\}$, then $A \cap dom\ R = \{a\}$ but $R[A] = \{c\}$, thus $R^{-1}[R[A]] = R^{-1}[\{c\}] = \{a,b\}$, but $\{a,b\} \not\subseteq \{a\}$.

Let R be as before and $B = \{g\}$, then $R^{-1}[B] = \{e\}$ and $R[R^{-1}[B]] = \{f,g\}$, but $B \cap ranR = \{g\}$.

2.4 $R[X] \subseteq ran\ R$ because for any $y \in R[X]$ we have some $x \in X$ such that xRy, thus $y \in ran\ R$. if $y \in ran\ R$, then for some $x \in dom\ R$ we have xRy, but $dom\ R \subseteq X$, thus $x \in X$, so we get for some $x \in X$, xRy, therefore $y \in R[X]$.

suppose $x \in dom\ R$ then there is some $y \in ran\ R$ such that xRy, but xRy iff $yR^{-1}x$ and $ranR \subseteq Y$, therefore there is some $y \in Y$ such that $yR^{-1}x$ which is equal to say that $x \in R^{-1}[Y]$, left to right is trivial.

(b) Assume $a \notin dom\ R$ but $R[\{a\}] \neq \emptyset$, so for some $y \in R[\{a\}]$ we have aRy which means that $a \in dom\ R$, this contradicts our assumption.

Assume $b \notin ran\ R$ and $R^{-1}[\{b\}] \neq \emptyset$, so there is some $x \in R^{-1}[\{b\}]$ such that $bR^{-1}x$ or equivalently xRb, it means that $b \in ran\ R$ which contradicts the assumption.

(c) $x \in dom\ R$ iff for some y, xRy iff $yR^{-1}x$ iff $x \in ran\ R^{-1}$.

$y \in ran\ R$ iff for some x, xRy iff $yR^{-1}x$ iff $y \in dom\ R^{-1}$.

(d) $(x,y) \in R$ iff $(y,x) \in R^{-1}$ iff $(x,y) \in (R^{-1})^{-1}$.

(e) if $(x,x) \in Id_{dom\ R}$ then $x \in domR$ which implies that for some y, $(x,y) \in R$, but $(x,y) \in R$ iff $(y,x) \in R^{-1}$, thus we can say that there is some y such that $(x,y) \in R$ and $(y,x) \in R^{-1}$ which is equal to $(x,x) \in R^{-1} \circ R$. the second part can be proved like this.

2.5 $\mathcal{P}(X) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

$\in_Y = \{(\emptyset, \{\emptyset\}), (\emptyset, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \{\{\emptyset\}\})\}$.

$Id_Y = \{(\emptyset,\emptyset), (\{\emptyset\},\{\emptyset\}), (\{\{\emptyset\}\},\{\{\emptyset\}\}), (\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset\}\})\}$.

$ran(Id_Y) = dom(Id_Y) = fld(Id_Y) = \mathcal{P}(X)$.

$dom(\in_Y) = \{\emptyset, \{\emptyset\}\}$, $ran(\in_Y) = \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, $fld(\in_Y) = \mathcal{P}(X)$.

2.6 $(x,y) \in T\circ(S\circ R)$ iff $(\exists z)((x,z) \in (S\circ R) \wedge (z,y) \in T)$ iff $(\exists z)((\exists u)[(x,u) \in R \wedge (u,z) \in S] \wedge (z,y) \in T)$ iff $(\exists z)((\exists u)[(x,u) \in R \wedge (u,z) \in S \wedge (z,y) \in T])$

iff $(\exists z)(\exists u)((x,u) \in R \wedge (u,z) \in S \wedge (z,y) \in T)$ iff $(\exists u)((x,u) \in R \wedge (\exists z)[(u,z) \in S \wedge (z,y) \in T])$ iff $(\exists u)((x,u) \in R \wedge (u,y) \in T \circ S)$ iff $(x,y) \in (T \circ S) \circ R$.

2.7 Let $X = \{a\}$ and $Y = \{b, c\}$, $Z = \{d\}$.

(a) $(a, b) \in X \times Y$ but $(a, b) \notin Y \times X$.

(b) $(a, (b, d)) \in X \times (Y \times Z)$ but $(a, (b, d)) \notin (X \times Y) \times Z$.

(c) $((a, a), a) \in X^2 \times X$ but $((a, a), a) \notin X \times X^2$.

2.8 (a) Assume $A \neq \emptyset$ and $B \neq$, then there is some $a \in A$ and $b \in B$, but then $(a, b) \in A \times B$, so $A \times B \neq \emptyset$. Now assume $A \times B \neq \emptyset$, then there is some $x \in A \times B$ such that $x = (a, b)$, but it means that $a \in A$ and $b \in B$, thus $A, B \neq \emptyset$.

(b) $(a, b) \in (A_1 \cup A_2) \times B$ iff $(a \in A_1 \cup A_2) \wedge b \in B$ iff $(a \in A_1 \vee a \in A_2) \wedge b \in B$ iff $(a \in A_1 \wedge b \in B) \vee (a \in A_2 \wedge b \in B)$ iff $(a, b) \in (A_1 \times B) \vee (a, b) \in (A_2 \times B)$ iff $(a, b) \in (A_1 \times B) \cup (A_2 \times B)$.

$(a, b) \in A \times (B_1 \cup B_2)$ iff $a \in A \wedge b \in (B_1 \cup B_2)$ iff $a \in A \wedge (b \in B_1 \vee b \in B_2)$ iff $(a \in A \wedge b \in B_1) \vee (a \in A \wedge b \in B_2)$ iff $(a, b) \in (A \times B_1) \vee (a, b) \in (A \times B_2)$ iff $(a, b) \in (A \times B_1) \cup (A \times B_2)$.

3 Page 28

3.1 if $ran\ f \subseteq dom\ g$, then $f^{-1}[ran\ f] \subseteq f^{-1}[dom\ g]$, but $f^{-1}[ran\ f] = dom\ f$, by Exercise 4.2(a) on Page 15 we have $dom\ f \cap f^{-1}[dom\ g] = dom\ f$, Theorem 3.5 implies $dom\ (g \circ f) = dom\ f$.

3.2 $f_2 \circ f_1 = \{\sqrt{2x - 1} : x > \frac{1}{2}\}$.

$f_1 \circ f_2 = \{2\sqrt{x} - 1 : x > 0\}$

$f_3 \circ f_1 = \{1/(2x - 1) : x \neq \frac{1}{2}\}$

$f_1 \circ f_3 = \{2/x - 1 : x \neq 0\}$

3.3 For f_1: if $f_1(a) = f_1(b)$ then $2a - 1 = 2b - 1$, by adding 1 to each side of equation we get $2a = 2b$, by dividing by 2 we have $a = b$.

For f_2: if $f_1(a) = f_1(b)$ then $\sqrt{a} = \sqrt{b}$, but then $a = \sqrt{a}\ \sqrt{a} = \sqrt{a}\ \sqrt{b} = \sqrt{b}\ \sqrt{b} = b$.

For f_3: if if $f_1(a) = f_1(b)$ then $1/a = 1/b$, because a, b are non-zero multiplying by ab yields $a = b$.

$f_1^{-1} = \{(x + 1)/2 : x\ is\ real\}$

$f_2^{-1} = \{x^2 : x > 0\}$

$f_3^{-1} = \{1/x : x \neq 0\}$

3.4 (a) Assume that f is invertible, let $(a,b) \in f^{-1} \circ f$ then for some z we have (*) $(a,z) \in f$ and $(z,b) \in f^{-1}$, then from (*) we also have $(z,a) \in f^{-1}$, by assumption f^{-1} is a function, so we get $a = b$, because $a \in dom\ f$ we get $(a,b) = (a,a) \in Id_{dom\ f}$. the other side holds by Exercise 2.4(e) on Page 23.

(b) Let $(a,b),(a,c) \in f^{-1}$, then $(b,a),(c,a) \in f$, thus $f(b) = a$ and $f(c) = a$ but (*) $g \circ f = Id_{dom\ f}$ implies $g(f(b)) = b = g(a) = g(f(c)) = c$, therefore $b = c$ and f^{-1} is a function. let $(a,b) \in f^{-1}$ then $(b,a) \in f$, so $f(b) = a$, by (*) we get $g(f(b)) = b = g(a)$, thus $(a,b) \in g$, but we also know that $a \in ran\ f$, therefore $(a,b) \in g \mid ran\ f$. Now let $(a,b) \in g \mid ran\ f$, then $g(a) = b$ and also $a \in ran\ f$, then $f(k) = a$ for some $k \in domf$, but (*) implies $g(f(k)) = g(a) = b = k$ which means that $(b,a) \in f$, $(a,b) \in f^{-1}$.

We give a counter example for the second one, let $f = \{(a,a),(b,a)\}$ and $h = \{(a,a)\}$ then $f \circ h = \{(a,a)\} = Id_{ran\ f}$ but clearly f^{-1} is not a function.

3.5 Let $(g \circ f)(a) = (g \circ f)(b)$, then $g(f(a)) = g(f(b))$ since g is one-to-one we get $f(a) = f(b)$, again because f is one-to-one we have $a = b$.

let $(a,b) \in (f \circ g)^{-1}$, thus $(b,a) \in f \circ g$, it means that for some z we have $(b,z) \in g$ and $(z,a) \in f$, equivalently we have $(a,z) \in f^{-1}$ and $(z,b) \in g^{-1}$ for some z, by definition of composition we get $(a,b) \in g^{-1} \circ f^{-1}$.

3.6 We just need prove right to left of (a) and left to right of (b).

(a) Suppose $x \in f^{-1}[A] \cap f^{-1}[B]$, then for some $y \in A$ we have $yf^{-1}x$ or equivalently $f(x) = y$ and for some $z \in B$, $f(x) = z$, but since f is a function we conclude that $z = y \in A \cap B$, then we can say that for some $y \in A \cap B$, $yf^{-1}x$ holds, therefore $x \in f^{-1}[A \cap B]$.

(b) Let $x \in f^{-1}[A - B]$, then there is some $y \in A - B$ such that $yf^{-1}x$ or equivalently (*) $f(x) = y$, clearly $x \in f^{-1}[A]$, we must prove that $x \notin f^{-1}[B]$ or equivalently there is no $z \in B$ such that $zf^{-1}x$, assume to the contrary that it exists, so we get $f(x) = z$, but (*) implies $z = y \in B$, it contradicts our assumption that $y \in A - B$.

3.7 let $f = \{(a,b)\}$ and $A = \{a\}$, then $f \cap A^2 = \emptyset$ but $f|A = f$.

3.8 Let $I = A$ and $S = Id_I$, then $S = (S_i, i \in I)$ is an indexed function such that $S_i = i$.

3.9 (a) Let $f : A \to B$, then $f \subseteq A \times B$, thus $f \in \mathcal{P}(A \times B)$, now let $P(x)$ be the property "$(\forall a, b, c)[(a, b), (a, c) \in x \to b = c] \wedge (\forall a)(a \in A \to (\exists b)[b \in B \wedge (a, b) \in x])$", then $\{x \in \mathcal{P}(A \times B) : P(x)\}$ is the set of all function from A to B.

(b) Let f be a memeber of product of an indexed system $(S_i : i \in I)$, then $f : I \to \bigcup_{i \in I} S_i$ such that for every $i \in I$, $f(i) \in S_i$, then clearly $f \in (\bigcup_{i \in I})^I$, by previous exercise we know that it exists, now by comprehension we have $\prod_{i \in I} S_i = \{f \in (\bigcup_{i \in I})^I : (\forall i \in I)[f(i) \in S_i]\}$, clearly if it is non-empty, every member of it is a function such that satisfies the condition of a product.

3.10 $x \in \bigcup_{a \in \bigcup S} F_a$ iff $(\exists a)[a \in \bigcup S \wedge x \in F_a]$ iff $(\exists a)[(\exists C)(C \in S \wedge a \in C) \wedge x \in Fa]$ iff $(\exists a)[(\exists C)(C \in S \wedge a \in C \wedge x \in F_a)]$ iff $(\exists C)[(\exists a)(C \in S \wedge a \in C \wedge x \in F_a)]$ iff $(\exists C)[C \in S \wedge (\exists a)(a \in C \wedge x \in F_a)]$ iff $(\exists C)[C \in S \wedge x \in \bigcup_{a \in C} F_a]$ iff $x \in \bigcup_{C \in S}(\bigcup_{a \in C} F_a)$.

Let $x \in \bigcap_{a \in \bigcup S} F_a$ then (*) $(\forall a)[a \in \bigcup S \to x \in F_a]$. Now let $C \in S$, then because $C \subseteq \bigcup S$ we get that for every $a \in C$, $x \in F_a$, because C was arbitrary we can conclude that (**) $(\forall C)[C \in S \to (\forall a)(a \in C \to x \in F_a)]$, which is equal to $(\forall C)[C \in S \to x \in \bigcap_{a \in C} F_a]$, thus $x \in \bigcup_{C \in S}(\bigcap_{a \in C} F_a)$. Now let $x \in \bigcup_{C \in S}(\bigcap_{a \in C} F_a)$, then we get (**), let $a \in \bigcup S$, then there is some $C \in S$ such that $a \in C$, but then by (**) we get $(\forall a)(a \in C \to x \in F_a)$ and then $x \in F_a$, because a was arbitary we proved (*), thus $x \in \bigcap_{a \in \bigcup S} F_a$.

3.11 $x \in B - \bigcup_{a \in A} F_a$ then $x \in B$ and for every $a \in A$, $x \notin F_a$, also for every $a \in A$, $x \notin F_a$ and $x \in B$, so for every $a \in A$, $x \in B - F_a$, thus $x \in \bigcap_{a \in A}(B - F_a)$. Now let $x \in \bigcap_{x \in A}(B - F_a)$, then for every $a \in A$, $x \in B$ and $x \notin F_a$,

let $a \in A$, then by above claim $x \notin F_a$, thus $x \notin \bigcup_{a \in A} F_a$, Now assume to the contrary that $x \notin B$, then it implies there is no $a \in A$, $A = \emptyset$ which is a contradiction.

Let $x \in B - \bigcap_{a \in A} F_a$, then (*) $x \in B$ and there is some $a \in A$ such that $x \notin F_a$, by (*) we can claim that there is some $a \in A$ such that $x \in B - F_a$, thus $x \in \bigcup_{a \in A}(B - F_a)$. Now let $x \in \bigcup_{a \in A}(B - F_a)$, then $x \in (B - F_a)$ for some $a \in A$, it follows that there is some $a \in A$ such that $x \in F_a$, thus $x \notin \bigcap_{a \in A} F_a$ and clearly $x \in B$, thus $x \in B - \bigcap_{a \in A} F_a$.

Let $x \in \bigcup_{a \in A} F_a \cap \bigcup_{b \in B} G_b$, then for some $a \in A$, $x \in F_a$ and for some $b \in B$, $x \in G_b$, clearly $(a, b) \in A \times B$, then we can say for some $(a, b) \in A \times B$, $x \in F_a \cap G_b$

$x \in \bigcup_{a\in A} F_a \cap \bigcup_{b\in B} G_b$ iff $(\exists a)(a \in A \wedge x \in F_a) \wedge (\exists b)(b \in B \wedge x \in F_b)$ iff $(\exists a)(\exists b)[(a \in A \wedge x \in F_a) \wedge b \in B \wedge x \in F_b)]$ iff $(\exists a)(\exists b)[(a,b) \in A \times B \wedge x \in F_a \cap F_b]$ iff $x \in \bigcup_{(a,b)\in A\times B}(F_a \cap G_b)$

3.12 (We just prove the first and the third case)

$y \in f[\bigcup_{a\in A} F_a]$ iff $(\exists x)[x \in \bigcup_{a\in A} F_a \wedge f(x) = y]$ iff $(\exists x)[(\exists a)(a \in A \wedge x \in F_a) \wedge f(x) = y]$ iff $(\exists x)[(\exists a)(a \in A \wedge x \in F_a \wedge f(x) = y)]$ iff $(\exists x)(\exists a)[a \in A \wedge x \in F_a \wedge f(x) = y]$ iff $(\exists a)(\exists x)[a \in A \wedge x \in F_a \wedge f(x) = y]$ iff $(\exists a)[a \in A \wedge (\exists x)(x \in F_a \wedge f(x) = y)]$ iff $(\exists a)[a \in A \wedge y \in f[F_a]]$ iff $y \in \bigcup_{a\in A} f[F_a]$.

Let $y \in f[\bigcap_{a\in A} F_a]$, then for some $x \in \bigcap_{a\in A} F_a$, $f(x) = y$, but it means for every $a \in A$, $x \in F_a$ and $f(x) = y$, we can say for every $a \in A$, there is some $x \in F_a$ such that $f(x) = y$ or equally $y \in f[F_a]$, thus $y \in \bigcap_{a\in A} f[F_a]$.

(if f is one-to-one, $\subseteq$ can be replaced by $=$): Now let $y \in \bigcap_{a\in A} f[F_a]$, so for every $a \in A$, there is some $x \in F_a$ such that $f(x) = y$, but because f is one-to-one this x must be unique, name it k, so for every $a \in A$, $k \in F_a$ or equivalently $k \in \bigcap_{a\in A} F_a$, since $f(k) = y$ we get $y \in f[\bigcap_{a\in A} F_a]$.

3.13 Right to left is easy according to Hint, we prove left ro right side:

Let $x \in \bigcap_{a\in A}(\bigcup_{b\in B} F_{a,b})$, define f such that $(a,b) \in f$ iff $x \in F_{a,b}$, we prove $f \in B^A$, let $(x,y),(x,z) \in f$ be two distinct memeber, then $x \in F_{x,y} \cap F_{x,z}$ but because $y \neq z$ we have $F_{x,y} \cap F_{x,z} = \emptyset$, thus it contradicts our assumption, hence f is a function.

From assumption for every $a \in A$ we have $x \in \bigcup_{b\in B} F_{a,b}$, fix arbitrary $a \in A$, then $x \in F_{a,b}$ for some $b \in B$, but by definition of f we have $f(a) = b$, thus $x \in F_{a,f(a)}$, because a was arbitrary we can say $x \in \bigcap_{a\in A} F_{a,f(b)}$ for f, thus $x \in \bigcup_{f\in B^A}(\bigcap_{a\in A} F_{a,f(b)})$.