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3.1 We must show that the set $\{x : x \in A \text{ and } x \notin B\}$ exists. Let $P(x, A, B)$ be the property "$x \in A$ and $x \notin B$", then $P(x, A, B)$ implies $x \in A$, because A exists, we get $\{x : x \in A \text{ and } x \notin B\} = \{x \in A : x \in A \text{ and } x \notin B\} = \{x \in A : x \notin B\}$, this set exists by the axiom of comprehension, and is unique according to the Lemma 3.4.

3.2 By the Weak Axiom of Existence some set exists, call let A. Take the property $P(x)$ to be $x \neq x$, then the set $\{x \in A : x \neq x\}$ exists by comprehension. But this is $\emptyset$, since every element of any set is self-identical.

3.3 (a) Assume that V the set of all sets exists, then by comprehension the set $X = \{x \in V : x \notin x\}$ exists. Now, $x \in X$ iff $x \in V$ and $x \notin x$. Clearly, X is a set, thus $X \in V$, if $X \in X$, then it satisfies the property of being a member of X, thus $X \notin X$. if $X \notin X$, and since $X \in V$ we get by definition $X \in X$, a contradiction.

(b) Assume to the contrary that there is one A such that every x, $x \in A$, then every set is in A, which means that $A = V$, which is impossible by the previous argument.

3.4 By the axiom of pair the set $\{A, B\}$ exist and the union axiopm implies the existence of $\bigcup\{A, B\}$. Now take the property $\mathbf{P}(x, A, B) =$ "$(x \in A \wedge x \notin B) \vee (x \notin B \wedge x \in A)$", now by the comprehension axiom the set $C = \{x \in \bigcup\{A, B\} : \mathbf{P}(x, A, B)\}$ exist. Now it is easy to check $x \in C$ iff either $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin B$.

3.5 (a) By the axiom of pair there are the sets $\{A, B\}$ and $\{C\}$. Again by pairing the set $\{\{A, B\}, \{C\}\}$ exists. Now the axiom union implies the existence of $P = \bigcup\{\{A, B\}, \{C\}\}$. But now, $x \in P$ iff $x \in \{A, B\}$ or $x \in, \{C\}$ iff $x = A$ or $x = B$, or $x = C$.

(b) Just repeat the above argument and take $\{C, D\}$ instead of $\{C\}$.

3.6 Assume that $\mathcal{P}(X) \subseteq X$ and let $Y = \{x \in X : x \notin x\}$. Clearly $Y \subseteq X$, thus by our assumption $Y \in X$, if $Y \notin Y$ then $Y \in Y$. if $Y \in Y$ then $Y \notin Y$, which is a contradiction.

3.7 (Weak Axiom Of Pair) Take the property $\mathbf{P}(x, A, B) =$ "$x = A \vee x = B''$, then by comprehension the set $X = \{x \in C : \mathbf{P}(x, A, B)\}$ exists. Now it is easy to see $x \in X$ iff $x = A$ or $x = B$.

(Weak Axiom Of Union) Take $\mathbf{P}(x, S) =$ "$\exists A(A \in S \wedge x \in A)''$, then by comprehension we get a set Y such that $x \in Y$ iff for some $A \in S$, $x \in S$, thus $Y = \bigcup S$.

(Weak Axiom Of Power Set) Consider the property "$x \subseteq S$", then by applying the comprehension to the set P we get the set Z such that $x \in Z$ iff $x \subseteq S$, thus $Z = \mathcal{P}(S)$.