Real Analysis

Course MATH 540@UIUC

Reference

- nabla: https://ppasupat.github.io/a9online/wtf-is/nabla.html

Guide

> Took the class with him back in 2018. I would say his exams are pretty similar to the comps, for example: https://math.illinois.edu/system/files/2021-02/MATH 540 - Jan 2021.pdf . The homework from folland's book is kind of easy compare to the exams. I mean, this is a comprehensive exam class for the Math PhD people, so you shouldn't expect it to be any less, and real analysis is known to be hard for many people.

Textbook

Gerald B. Folland, Real Analysis

CH0

Reminder

- Do algebra with $\mu(E)$ carefully, since it can be infinity.

Notation

- X: the universal set.
- $\mathcal{E}$: a collection of subsets.
- $\mathcal{P}(X)$: the power set $\{E : E \subset X\}$.
- ":=" means can be done by definition.

Set theory

Nota. $A \subset B$: A can be B.

Nota. A set A is smaller than set B is defined as $A \subset B$ but $A \neq B$.

> Def. (**Product set** $X \times Y$)

> Def. (**map**)

> Def. (**todo**) Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of nonempty sets, $X := \prod_{\alpha \in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps.
>
> $f : A \to \bigcup_{\alpha \in A} X_\alpha$.

> Def. (**Arbitrary infinite sum**) For a set E, $\sum_{x \in E} f(x) := \sup\{\sum_{x \in F} f(x) : \text{finite set } F \subset E\}$.

> Def. (**Set limit**) Given $A_1, A_2, \cdots \in \mathcal{F}$,
>
> $$\limsup_{n\to\infty} A_n := \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_n = \bigcap_{m=1}^{\infty} B_m = \{\omega\in\Omega : \omega\in A_n \text{ for infinitely many n}\}$$

$$\liminf_{n\to\infty} A_n := \bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} A_n = \bigcup_{m=1}^{\infty} C_m = \{\omega\in\Omega : \omega\in A_n \text{ for all but finitely many n}\}$$

Recall:

1. f is continuous at x if $\forall\{x_n\}, x_n\to x, n\to\infty \implies \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$.
2. $\limsup_n x_n = \lim_{m\to\infty}\sup_{n\geq m} x_n = \lim_{m\to\infty} c_m$, where c_m is monotonic, so that it must converge if we include $\pm\infty$.

Proof. Consider $\omega\in RHS$ or not. If yes, $\omega\in B_m, \forall m$; if not, disappear eventually.

Proof. Consider $\omega\in RHS$ or not. If yes, appear eventually; otherwise fail.

Rmk. $\liminf A_n \subset \limsup A_n$; if equal, we say A_n converges.

E.g. Monotonic set sequence converges (if including ∞).

CH1 Measure theory

1.2 σ-algebra/field

> Def. (**Algebra** of sets of X) A non-empty collection $\mathcal{A}$ of subsets of X, that is closed under finite union and complements. In other word,

1. $E_1, E_2 \in \mathcal{A} \to E_1\cup E_2\in\mathcal{A}$.
2. $E\in\mathcal{A}\to E^C\in\mathcal{A}$.

Rmk. a) Algebra is closed under finite intersection; b) $\emptyset, X\in\mathcal{A}$. This is important when it comes to covering.

> Def. (σ-**algebra** of sets of X) A non-empty collection $\mathcal{A}$ of subsets of X, that is closed under countable union and complements. E.g. $\mathcal{A} = \{E\in X : \text{E is co-countable}\}$.

Prop. $\mathcal{A}$ is a σ-algebra iff (a) $\mathcal{A}$ is a algebra; (b)

$$E_j \text{ mutually disjoint}, E_j\in\mathcal{A}\to\bigcup_{j=1}^{\infty} E_j\in\mathcal{A}$$

Proof. $\cup E_j = \cup_j[E_j\setminus(\cup_{k<j}E_k)]\in\mathcal{A}$. "This device of replacing a sequence of sets by a disjoint sequence is worth remembering."

Lemma. The intersection of any family of σ-algebras on X is again a σ-algebras.

> Def. (σ-**algebra generated by** $\mathcal{E}$) For $\mathcal{E}\subset\mathcal{P}(X)$, i.e. a collection of subsets of X, there's a **unique smallest** σ-algebra $\mathcal{M}(\mathcal{E})$ containing $\mathcal{E}$, namely, the intersection of all σ-algebras containing $\mathcal{E}$.

Lemma. (1.1) $\mathcal{E}\subset\mathcal{M}(\mathcal{F}) \implies \mathcal{M}(\mathcal{E})\subset\mathcal{M}(\mathcal{F})$.

> Def. (**Toplogy** of subsets of X) A non-empty collection $\mathcal{F}$ of subsets of X , satisfying (a) $\emptyset, X \in \mathcal{F}$; (b) closed under arbitrary union; (c) closed under finite intersection.
>
> Def. (**Toplogical space**) A pair $(X, \mathcal{F})$.

Nota. G is the family of open sets; F is the family of closed sets; G_δ is the countable intersection of open sets; $F_{\delta\sigma}$ is the countable union of F_δ ...

> Def. (**Borel σ-algebra** of $(X, \mathcal{F})$) The $\mathcal{M}$(G), denoted as $\mathcal{B}_X$, where G is the aforementioned family of open sets.

Prop. $\mathcal{M}$(G) is the same as $\mathcal{M}(\text{open intervals})$, $\mathcal{M}(F)$, $\mathcal{M}(\text{the open rays } \{(a, \infty)\})$, etc.

> Def. (**Borel set**) A Borel set is a member of $\mathcal{B}_X$. E.g. G_δ, F_σ are Borel set. (Many sets look like either one of these two.)

> Def. (**Product σ-algebra**) We ask for $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{j=1}^n \mathcal{B}_\mathbb{R}$. This definition enables it by: let $\{X_\alpha\}_{\alpha\in A}$ be an indexed collection of nonempty sets, $X := \prod_{\alpha\in A} X_\alpha$, $\pi_\alpha : X \to X_\alpha$ the coordinate maps, and $\mathcal{M}_\alpha$ is a σ-algebra on X_α, then define $\bigotimes \mathcal{A}_\alpha := \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$.

#NotCovered Prop 1.1-1.6

1.3 Measure

> Def. (**Measure** μ on **measurable space** (X, $\mathcal{A}$)) $\mu : \mathcal{M} \to [0, \infty]$, s.t.
>
> 1. $\mu(\emptyset) = 0$;
> 2. Countable additivity (σ-additivity). If $E_1, E_2, \ldots$ is a collection of **disjoint** members of $\mathcal{M}$, i.e. $E_i \cap E_j = \emptyset$ for all $i \neq j$, then $\mu(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E_i)$.

> Def. (**Finite measure**) $\mu(X) < \infty$.
>
> Def. (**σ-finite measure**) $X = \bigcup E_j, s.t., \forall j, \mu(E_j) < \infty$.
>
> Def. (**Semifinite measure**) $\forall E \in \mathcal{M}, \mu(E) = \infty \to (\exists F \subset E, 0 < \mu(F) < \infty)$.
>
> Def. (**Null set and "almost everywhere (a.e.)"**) E is a null set if $\mu(E) = 0$. Proposition A is true almost everywhere if it is true on all but null set.

E.g. Given $f : X \to [0, \infty]$, we can define a measure by $\mu(E) = \sum_{x\in E} f(x)$.

1. It's semifinite iff $f(x) < \infty$.
2. It's σ-finite iff it's semifinite and $\{x : f(x) > 0\}$ is countable.
3. It's called **counting measure** if for some $x_0 \in X$, $f(x) = \mathbb{1}(x = x_0)$.
4. It's called **point mass or Dirac measure** if $f(x) = 1$.

Thm. Properties of measure:

1. (**Monotone**) $E, F \in \mathcal{M}, E \subset F \implies \mu(E) \leq \mu(F)$.
2. (**σ-subadditive**) $\mu(\cup_{j=1}^\infty E_j) \leq \sum_{j=1}^\infty \mu(E_j)$.
3. (**Continuity from below**) $E_1 \subset E_2 \ldots \implies \mu(\cup E_j) = \lim \mu(E_j)$.

4. (**Continuity from above**) $E_1 \supset E_2 \ldots; \mu(E_1) < \infty \implies \mu(\cap E_j) = \lim \mu(E_j)$. The $\mu(E_1) < \infty$ is to enable $\mu(E_1 \setminus E_j) = \mu(E_1) - \mu(E_j)$, in the convertion between union and intersection.

Prop. σ-finite implies semifinite.

Proof. For every E s.t. $\mu(E) = \infty$, given $X = \cup E_j$, define $F_j := E_j \cap E$. By subadditivity, $\infty = \mu(E) = \mu(\bigcup F_j) \le \sum \mu(F_j)$, then $\exists j, \mu(F_j) > 0$. By monotoncity, $\mu(F_j) \le \mu(E_j) < \infty$. These two gives the $F := F_j$ as the non-trivial measure subset for each E.

> Def. (**Complete**) A measure whose domain contains all subsets of null sets.

#NotCovered THM1.9. Completion of measure.

Continuity

> Def. (**Continuity of general measure**) μ is continuous if $\forall \{A_n\}, A_n \to A, n \to \infty \longrightarrow \lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n) := \mu(A)$. Notice the closeness under union&intersection gives that $A := \limsup_n A_n \in \mathcal{F}$.

Thm. (**Countable additivity implies continuity**)

Proof. For all convergent sequence $\{A_n\}$, which means

1. Case1: monotonic increasing An ($A_{n-1} \subset A_n$) Recall countable additivity, construct $D_n = A_n \setminus A_{n-1}$, then

$$\begin{aligned}
\mu(A) = \mu(\lim_{n\to\infty} A_n) &:= \mu(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m) \\
&= \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} D_n) \overset{(*)}{=} \sum_{n=1}^{\infty} \mu(D_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(D_i) = \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} D_i) = \lim_{n\to\infty} \mu(A_n)
\end{aligned}$$

2. Case2: monotonic decreasing An ($A_{n-1} \supset A_n$) Construct $E_n = A_n \setminus A_{n+1}$, then

$$\begin{aligned}
\mu(A) := \mu(\lim_{n\to\infty} A_n) &:= \mu(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m) \\
&= \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} E_n) \overset{(*)}{=} \sum_{n=1}^{\infty} \mu(E_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) = \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} E_i) = \lim_{n\to\infty} \mu(A_n)
\end{aligned}$$

3. Case3: general An Recall $B_n = \bigcup_{m=n}^{\infty} A_m, C_n = \bigcap_{m=n}^{\infty} A_m$. Clearly $C_n \subset A_n \subset B_n$, and that B_n is monotonic decreasing, C_n is monotonic increasing. From case1, we know that

$$\begin{aligned}
\limsup_{n\to\infty} \mu(A_n) \le \lim_{n\to\infty} \mu(B_n) =& \mu(\lim_{n\to\infty} B_n) \\
=& \mu(B) = \mu(A) = \mu(C) \\
=& \mu(\lim_{n\to\infty} C_n) = \lim_{n\to\infty} \mu(C_n) \le \liminf_{n\to\infty} \mu(A_n)
\end{aligned}$$

However, $\limsup_{n\to\infty} A_n \ge \liminf_{n\to\infty} A_n$, therefore $\lim_{n\to\infty} \mu(A_n) = \limsup_{n\to\infty} \mu(A_n) = \liminf_{n\to\infty} \mu(A_n) = \mu(A)$.

Conclusion: μ is a continuous set function.

Prop. (**Finite additivity + continuity iff countable additivity**) Proof. (only => is needed) Recall continuity: $\forall\{A_n\}, A_n \to A, n \to \infty \longrightarrow \lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n) = \mu(A)$ and (countable additivity) If $A_1, A_2, \ldots$ is a collection of disjoint members of $\mathcal{F}$, then

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(\lim_{n\to\infty} \bigcup_{i=1}^{n} A_i) = \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} A_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

1.4 Tools to construct measure

Motiv. In calculus, one defines area by marking grids inside and outside. Approximation from the outside is what we're going to build in the following.

> Def. (**Outer measure** on X) $\mu^* : \mathcal{P}(X) \to [0, \infty]$, s.t.
>
> 1. $\mu^*(\emptyset) = 0$;
> 2. Monotonicity;
> 3. (σ-subadditivity) $\mu^*(\cup A_j) \leq \sum \mu^*(A_j)$.

Prop. (1.10) Let $\mathcal{E} \subset \mathcal{P}(X)$ be a family of "elementary sets" that we can later choose, and $\rho : \mathcal{E} \to [0, \infty]$, such that $\emptyset, X \in \mathcal{E}, \rho(\emptyset) = 0$. These elementary sets are enough to define a outer measure:

$$\mu^*(A) := \inf_{\{E_j\}} \{\sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{E}, A \subset \bigcup_{j=1}^{\infty} E_j\}$$

Proof. The first and the second condition come immediately from the definition of infimum. For the third one, again, consider $\mu^*(A_j)$ as a infimum the largest lowerbound, then for any j and $\epsilon_j > 0$, $\mu^*(A_j) + \epsilon_j$ is not a lowerbound, therefore exists $\sum_{k=1}^{\infty} \rho(E_{j,k}) < \mu^*(A_j) + \epsilon_j$. Suming up LHS gives a value that's less than $\sum \mu^*(A_j) + \sum \epsilon_j$ but greater than $\mu^*(\cup A_j)$. Let $\epsilon_j = \epsilon * 2^{-j}$ and sending ϵ to 0 gives the desired inequality.

> Def. (**μ^*-measuable**) A set $A \subset X$ is called μ^*-measuable if
>
> $$\forall E \subset X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Motiv. This definition can be understood as, when A is "good", we can use A to evaluate any $E \subset X$, such that the inner measure of A (intersection of two, approximate from inside), $\mu^*(E \cap A)$, is equal to the outer measure of A, $\mu^*(E) - \mu^*(E \cap A^c)$.

Rmk. Notice that to show a set is μ^*-measuable, due to the subadditivity, it suffices to show

$$\forall E \subset X, s.t. \mu^*(E) < \infty, \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Thm. (**Caratheodory's thm**) If μ^* is an outer measure on X, then the collection $\mathcal{M}$ of μ^*-measuable sets is a σ-algebra, and $\mu^*|_{\mathcal{M}}$ is a complete measure on measuable space $(X, \mathcal{M})$.

Proof.

1. $\mathcal{M}$ is an algebra: the goal is, given $A, B \in \mathcal{M}$, show that $\forall E \subset X, \mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$. Taking the fact that A is μ^*-measuable, and let E be the latter two respectively,

$$
\begin{aligned}
\mu^*(E\cap(A\cup B)) =&\mu^*(E\cap(A\cup B)\cap A)+\mu^*(E\cap(A\cup B)\cap A^c)\\
=&\mu^*(E\cap A)+\mu^*(E\cap B\cap A^c)\\
\mu^*(E\cap(A\cup B)^c) =&\mu^*(E\cap(A\cup B)^c\cap A)+\mu^*(E\cap(A\cup B)^c\cap A^c)\\
=&\mu^*(E\cap(A\cup B)^c)=\mu^*(E\cap A^c\cap B^c)
\end{aligned}
$$

2. $\mathcal{M}$ is a σ-algebra: it suffices to prove it's closed under disjoint σ-union, and we only need to check one side of inequality. Define $B_n=\cup_{i=1}^n A_n$,

$$
\begin{aligned}
\mu^*(E\cap B_n) &= \mu^*(E\cap B_n\cap A_n)+\mu^*(E\cap B_n\cap A_n^c)\\
\text{given disjoint,} &= \mu^*(E\cap A_n)+\mu^*(E\cap B_{n-1})\\
\text{by induction,} &= \sum_{i=1}^n \mu^*(E\cap A_i)\\
\mu^*(E) &= \mu^*(E\cap(\cup_{i=1}^n A_i))+\mu^*(E\cap(\cup_{i=1}^n A_i)^c)\\
&\geq \mu^*(E\cap B_n)+\mu^*(E\cap(\cup_{i=1}^\infty A_i)^c)\\
&\geq \sum_{i=1}^n \mu^*(E\cap A_i)+\mu^*(E\cap(\cup_{i=1}^\infty A_i)^c)\\
\text{take limit,} &\geq \sum_{i=1}^\infty \mu^*(E\cap A_i)+\mu^*(E\cap(\cup_{i=1}^\infty A_i)^c)\\
\text{by subadditivity,} &\geq \mu^*(E\cap(\cup_{i=1}^\infty A_i))+\mu^*(E\cap(\cup_{i=1}^\infty A_i)^c)
\end{aligned}
$$

3. $\mu^*|_{\mathcal{M}}$ is a measure: we now know $\cup_{i=1}^\infty A_i\in\mathcal{M}$ is in the domain, which enable us to use the inequality above but with σ-union as E:

$$
\mu^*(\cup_{i=1}^\infty A_i)\geq\sum_{i=1}^n \mu^*((\cup_{j=1}^\infty A_j)\cap A_i)+\mu^*((\cup_{i=1}^\infty A_i)\cap(\cup_{i=1}^\infty A_i)^c)
$$

> The other side is again by σ-subadditivity.

4. $\mu^*|_{\mathcal{M}}$ is complete. Given $B\subset A, \mu^*(A)=\mu^*(B)=0$,
$\forall E\subset X, \mu^*(E)\geq\mu^*(E\cap B^c)=\mu^*(E\cap B)+\mu^*(E\cap B^c)$.

> Def. (**Premeasure**) $\mu_0:\mathcal{A}\to[0,\infty]$, $\mathcal{A}$ is a algebra, with:
>
> 1. $\mu_0=0$;
> 2. Any $\{A_j\}_{j=0}^\infty\subset A$ that are sequence of disjoint sets s.t. $\bigcup_{j=1}^\infty A_j\in\mathcal{A}$, then $\mu_0(\cup A_j)=\sum_{i=1}^\infty \mu_0(A_j)$.

Prop. Monotonicity of premeasure.

Prop. (1.13) By applying Prop. 1.10 with $\rho=\mu_0$, one can construct outermeasure $\mu^*:\mathcal{P}(X)\to[0,\infty]$, which extends the domain of μ_0. Then,

1. $\mu^*|_{\mathcal{A}}=\mu_0$;
2. $\forall A\in\mathcal{A}$, A is μ^*-measuable.

Proof.

1. (Recall) $\mu^*(D):=\inf_{\{A_j\}}\{\sum_{j=1}^\infty \mu_0(A_j):A_j\in\mathcal{A}, D\subset\bigcup_{j=1}^\infty A_j\}$;
2. $\mu^*|_{\mathcal{A}}\leq\mu_0$ is true since LHS is a lowerbound of a set containing $\mu_0(A)$ induced by sequence $\{A,\emptyset,\emptyset,\emptyset,\ldots\}$.

3. To show the other side, need to show the RHS is a lowerbound. We only have disjoint complete sequence additivity. For $A \in \mathcal{A}$, covering $\{A_j\}$, construct $B_n = A \cap (A_n \setminus \cup_{i<n} A_i)$, then $\cup B_i = A \in \mathcal{A}$ given covering. Then $\mu_0(A) = \sum_j \mu_0(B_j) \leq \sum_j \mu_0(A_j)$.
4. Want to show: $\forall A \in \mathcal{A}, E \subset X, \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. It suffices to show $\forall \epsilon > 0, \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. The LHS isn't a lowerbound, therefore exists covering $\{A_j\} \subset \mathcal{A}$ s.t.

$$\begin{aligned}
\mu^*(E) + \epsilon &> \sum_j \mu_0(A_j) \\
\text{By disjoint additivity, } &= \sum_j \mu_0(A_j \cap A) + \mu_0(A_j \cap A^c) \\
&= \sum_j \mu^*(A_j \cap A) + \mu^*(A_j \cap A^c) \\
\text{By subadditivity, } &\geq \mu^*(\cup_j (A_j \cap A)) + \mu^*(\cup_j (A_j \cap A^c)) \\
&= \mu^*(E \cap A) + \mu^*(E \cap A^c)
\end{aligned}$$

Thm. Algebra $\mathcal{A}$, σ-algebra $\mathcal{M} := \mathcal{M}(\mathcal{A})$, premeasure μ_0 on $\mathcal{A}$, and μ^* the outermeasure given in last thm. Then:

1. $\mu := \mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$. This gives the existence of measure extending μ_0;
2. Any other measure $\tilde{\mu}$ that extends μ_0 has $\forall E \in \mathcal{M}, \tilde{\mu}(E) \leq \mu(E)$, with equality when $\mu(E) < \infty$.
3. If μ_0 is σ-finite, then μ is unique. This gives the uniqueness of measure extending μ_0 under stronger condition;

Proof.

1. Let $\mathcal{B}$ the collection of μ^*-measuable sets. By C-thm, $\mathcal{B}$ is a σ-algebra and $\mu^*|_{\mathcal{B}}$ is a measure. By Prop 1.13, $\mathcal{A} \subset \mathcal{B}$, and $\mathcal{M}$ is the smallest σ-algebra containing A, therefore $\mathcal{M} \subset \mathcal{B}$, $\mu^*|_{\mathcal{M}}$ is a measure.
2. Goal: $\forall E \in \mathcal{M}, \tilde{\mu}(E) \leq \mu(E)$.
3. Goal: $\forall E \in \mathcal{M}, \tilde{\mu}(E) \geq \mu(E)$ when $\mu(E) < \infty$.
4.