Linear algebra

Metadata

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Course MATH 416 Honor@UIUC

Textbook

- Linear Algebra via Exterior Products (2020)
- Linear Algebra Done Right (2023)
- Linear Algebra Done Wrong (2021)
- Linear Algebra (Stephen H. Friedberg, Arnold J. Insel etc.) (2021) [main]

Reminder

1. Carefully look at "dependent" or "independent".

Ch1 Vector Spaces

1.2 Vector Space

> Def. (**Vector space** V on field F) A non-empty set with vector addition and scalar multiplication, with the following axioms:
>
> 1. Additive commutativity;
> 2. Additive and scalar multiplicative associativity;
> 3. Additive identity and scalar multiplicative identity;
> 4. Additive inverse;
> 5. Vector and scalar additive distributivity.

Rmk. This definition gives rise to a few special vector space, e.g. $\mathbb{R}^n$ and $\mathcal{P}^n$, which will compose others by standard procedure introduced later.

Thm. (1.1 Cancellation law for vector addition) By playing inverse (rule 4).

Cor. a) $\exists! \underline{0}$; b) $\exists! \underline{-x}$; c) $0 \cdot \underline{x} = \underline{0}$; d) $(-\lambda) \cdot \underline{x} = -(\lambda \underline{x}) = \lambda \cdot \underline{-x}$; e) $\lambda \cdot \underline{0} = \underline{0}$.

1.3 Subspaces

> Def. (**Subspace** W of vector space V) A non-empty subset of V, such that:
>
> 1. $\underline{0} \in W$;
> 2. Closed under vector addition and scalar multiplication.

Thm. (1.4) Subspace is closed under arbitrary intersection.

1.4 Linear combination

> Def. (**Span**) For a set $S \subset V$, $span(S) := \bigcap_{S \subset \text{subspace } W \subset V} W$.

Prop. $span(S)$ is the set of linear combination of elements in S.

1.5 Linear independence

> Def. (**Linear dependent**) n distinct s_i, exists $\lambda_1 \dots \lambda_n$ are not all zero, such that $\sum \lambda_i s_i = 0$.

Thm. S are linear independent set of vectors, $v \in V \setminus S$, then $S \cup \{v\}$ are linear dep. iff $v \in span(S)$.

1.6 Bases and dimention

> Def. (**Basis**) Minimal (defined in the subset inclusion sence, not in size sence) spanning set.

Cor. $span(S) = V$, then it's basis iff it's linear indep.

Thm. (**Replacement thm**) V has a basis $s_1, \dots, s_n$ of size n, let $\{x_1, \dots, x_i\}$ of size i be linear indep. and $i \leq n$, then $\{x_1, \dots, x_i, s_{i+1}, \dots, s_n\}$ (some of si is replaced by xi) is a basis.

Cor. card(linear indep) <= card(basis) <= card(spanning set)

Cor. Basis has the same cardinality.

Cor. If $|S| = \dim V$, then TFAE: a) spanning; b) linear indep; c) basis.

Thm. (1.11) $W \subset V, \dim W \leq \dim V$, then $\dim W = \dim V$ iff $W = V$.

Cor. $\dim V < \infty, W \subset V$, then W posseses a compliment.

> Def. (**Quotient space**) Given subspace W, define $x \sim y$ if $x - y \in W$, $[x] := \{y : x \sim y\} =: \{x + w | w \in W\} =: x + W$, and $\{[x]\} := V_{/W}$ is a vector space called quotient space, by the intuitive definition of addition and scalar multiplication: $[v] = [\sum \lambda_i s_i] := \sum \lambda_i s_i$ and $\lambda[x] := [\lambda x]$, e.g. $-[x] = [-x]$.

Prop. $\dim(V_{/W}) = \dim V - \dim W$.

Thm. Given subspace W, there's a bijection between $\{H : subspace\ H, W \subset H\}$ and $\{\bar{H} \in V_{/W} : subspace\ H\}$. This together with the usage of flags give another proof for Cor 1.11.

> Def. (**Direct sum**) $W_1 \oplus W_2$ if $W_1 + W_2 = V$ and $W_1 \cap W_2 = \emptyset$.

Cor. $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$, by showing $\dim \bar{V} = \dim \bar{W}_1 + \dim \bar{W}_2$.

1.7 Maximal linear independent subset

> Def. (**Chain / nest / tower**) A collection of elements that are totally ordered.

Thm. (**Hausdorff maximal principle / the axiom of choice**) Every partially ordered set has a maximal linearly / totally ordered subset. It's the same as the next thm.

Thm. (**Zorn's lemma**) For a partially ordered set $(X, \leq)$, for any $C \subset X$ be totally ordered. Suppose $\exists x_c \in X, s.t., \forall x \in C, x \leq x_c$ (every chain has a top), then $\exists x_m, s.t., \forall y \in X, x_m \leq y \to x_m = y$ (maximum exists).

> Def. (**Maximal linear independent set**) Again, maximal with respect to set inclusion.

Lemma. A set is a maximal linear independent set iff it's a basis.

Thm. For any linearly independent subset S of a vector space V, there's a basis that contains S.

Proof. Construct X to be the collection of independent sets containing S. For any chain C in X, we need to find a top of it in X. This can be done by taking union of sets in C, which means it's a top and therefore containing S. Also, it's independent, since for any u_i for $i = 1 \ldots n$, we can find a set in C such that it contains all these vectors, therefore they're linearly independent.

Cor. Every vector space has basis.

Thm. Subspace $W \subset V$, then $\exists W', s.t. V = W \oplus W'$.

Ch2. Linear Transformations and Matrices

> Def. (**Linear map**) $T: V \to W$.

Rmk.

1. $T(0) = 0$.
2. $Ker(T) \subset V, Ran(T) \subset W$ are subspaces, called **null space / kernel** and **range / image**, and their dimention is called **nullity** and **rank**.
3. (2.4) T is 1-1 iff $Ker(T) = \{0\}$.

Thm. (**Dimension thm**) For linear $T: V \to W$, and V is finite-dimensional, then $nullity(T) + rank(T) = \dim(V)$.

Thm. T is isomorphic iff $\exists T^{-1}, s.t., T \circ T^{-1} = id_V. T^{-1} \circ T = id_W, T^{-1}\ linear$.

Lemma. Isomorphic T, $\dim V = n$, then $\dim W = n$.

Cor. Subspace $V' \subset V$, then $T|_{V'}: V' \to T(V')$ is still isomorphic.

Thm. $T: V \to W$ induces linear $\bar{T}: V_{/KerT} \to R(T)$ by letting $\bar{T} := [x] \mapsto T(x)$.

Cor. $\dim V < \infty, \dim KerT + \dim R(T) = \dim V$.

Ch3. Elementary Matrix Operations and Systems of Linear Equations

Eigenvalue

Eigenvalue, eigenvector and movement

For a matrix $A_{n\times n}$, consider all $(\vec{u}, \lambda)$ pair such that: $A\vec{u} = \lambda\vec{u}$ We call them **eigenvalues** and **eigenvectors** of matrix A. There're totally n pairs of $(\vec{u}_i, \lambda_i)$ for diagonalizable linear transformation, and the eigenvectors form a basis(some λ_i might be the same).

If we regard matrix/transformation W as a space movement in Euclidean space, we need to apply it on certain vector to examine its feature. What if we try to apply it multiple times?

$$\vec{v} = \sum_i \alpha_i \vec{u}_i$$
$$W^k \vec{v} = \sum_i \alpha_i W^k \vec{u}_i = \sum_i \alpha_i \lambda_i^k \vec{u}_i$$

We find out that the largest eigenvalue corresponding eigenvector will eventually dominate as k getting larger and larger. That's why we would like to conclude:

- first principle eigenvalue(largest) indicates the movement speed
- first principle eigenvector indicates the movement direction

e.g. When A is the adjacency matrix, $(A\vec{v})_i = \frac{1}{deg_i} \sum_{j \in N(i)} v_j$ When $L = I - D^{-1}A$, the Laplacian matrix, $(L\vec{v})_i = \frac{1}{deg_i} \sum_{j \in N(i)} (v_i - v_j)$

How to find them?

When the transformation A is normal operator, which means orthogonal diagonalizable, then:

$$A = P \Lambda P^{-1}$$

where Λ stretchs (eigenvalues), P rotates (orthonormal eigenvectors). Further more, when A is symmetric real matrix(e.g. adjacency and Laplacian matrix), then it is hermitian/self-adjoint, which means all eigenvalues are real.

THM

Symmetric real matrix M $M := \sum_i \lambda_i v_i v_i^T$, #TODO (upd) dim V=K We may use the same eigenvectors in M^k, such that $M^k := \sum_i \lambda_i^k v_i v_i^T$ claim: $M^{-1} := \sum_i \frac{1}{\lambda_i} v_i v_i^T$, $M^{-1}M = I$ proof: substitute

thm2: $tr(M) = \sum_i \lambda_i$ https://courses.cs.washington.edu/courses/cse521/16sp/521-lecture-8.pdf

Variational Characterization of Eigenvalues

$$\text{symmetric real } M_{n\times n}, \text{ eigenvalue } \lambda_1 \le \lambda_2 \ldots \le \lambda_n$$
$$\text{Rayleigh quotient } R_M(\boldsymbol{x}) = \frac{\boldsymbol{x}^T M \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}}$$
$$\lambda_k = \min_{\forall V, \dim V = k} \max_{\boldsymbol{x} \in V - \{\boldsymbol{0}\}} R_M(\boldsymbol{x})$$

proof: https://blog.csdn.net/a358463121/article/details/100166818

证明V里面一定存在向量使得Rayleigh quotient时，只需要取$\lambda_1, \lambda_2, \ldots, \lambda k$对应的$v1, v_2, \ldots, v_k$组成的空间$V$即可。 https://zhuanlan.zhihu.com/p/80817719

Hilbert space

conjucture symmetric

Reference

Basic knowledge in Spectral Theory.