Linear algebra

Metadata

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Textbook

- Linear Algebra via Exterior Products (2020)
- Linear Algebra Done Right (2023)
- Linear Algebra Done Wrong (2021)
- Linear Algebra (Stephen H. Friedberg, Arnold J. Insel etc.) (2021) [main]

Reminder

1. Carefully look at "dependent" or "independent".

Ch1 Vector Spaces

Def. (**Vector space** V on field F) A non-empty set with vector addition and scalar multiplication, with the following axioms:

1. Additive commutativity;
2. Additive and scalar multiplicative associativity;
3. Additive identity and scalar multiplicative identity;
4. Additive inverse;
5. Vector and scalar additive distributivity.

Rmk. This definition gives rise to a few special vector space, e.g. $\mathbb{R}^n$ and $\mathcal{P}^n$, which will compose others by standard procedure introduced later.

Prop. Cancellation rule. By playing inverse (rule 4).

Cor. $\exists!\underline{0}$.

Cor. $\exists!\underline{-x}$.

Cor. a) $0 \cdot \underline{x} = \underline{0}$; b) $(-\lambda) \cdot \underline{x} = -(\lambda \underline{x}) = \lambda \cdot \underline{-x}$; c) $\lambda \cdot \underline{0} = \underline{0}$.

Def. (**Subspace** W of vector space V) A non-empty subset of V, such that:

1. $\underline{0} \in W$;
2. Closed under vector addition and scalar multiplication.

Prop. Subspace is closed under arbitrary intersection.

Def. (**Span**) For a set $S \subset V$, $span(S) := \bigcap_{S \subset \text{subspace } W \subset V} W$.

Prop. $span(S)$ is the set of linear combination of elements in S.

Eigenvalue

Eigenvalue, eigenvector and movement

For a matrix $A_{n\times n}$, consider all $(\vec{u}, \lambda)$ pair such that: $A\vec{u} = \lambda\vec{u}$ We call them **eigenvalues** and **eigenvectors** of matrix A. There're totally n pairs of $(\vec{u}_i, \lambda_i)$ for diagonalizable linear transformation, and the eigenvectors form a basis(some λ_i might be the same).

If we regard matrix/transformation W as a space movement in Euclidean space, we need to apply it on certain vector to examine its feature. What if we try to apply it multiple times?

$$\vec{v} = \sum_i \alpha_i \vec{u}_i$$
$$W^k\vec{v} = \sum_i \alpha_i W^k \vec{u}_i = \sum_i \alpha_i \lambda_i^k \vec{u}_i$$

We find out that the largest eigenvalue corresponding eigenvector will eventually dominate as k getting larger and larger. That's why we would like to conclude:

- first principle eigenvalue(largest) indicates the movement speed
- first principle eigenvector indicates the movement direction

e.g. When A is the adjacency matrix, $(A\vec{v})_i = \frac{1}{deg_i}\sum_{j\in N(i)} v_j$ When $L = I - D^{-1}A$, the Laplacian matrix, $(L\vec{v})_i = \frac{1}{deg_i}\sum_{j\in N(i)}(v_i - v_j)$

How to find them?

When the transformation A is normal operator, which means orthogonal diagonalizable, then:

$$A = P\Lambda P^{-1}$$

where Λ stretchs (eigenvalues), P rotates (orthonormal eigenvectors). Further more, when A is symmetric real matrix(e.g. adjacency and Laplacian matrix), then it is hermitian/self-adjoint, which means all eigenvalues are real.

THM

Symmetric real matrix M $M := \sum_i \lambda_i v_i v_i^T$, #TODO (upd) dim V=K We may use the same eigenvectors in M^k, such that $M^k := \sum_i \lambda_i^k v_i v_i^T$ claim: $M^{-1} := \sum_i \frac{1}{\lambda_i} v_i v_i^T$, $M^{-1}M = I$ proof: substitute

thm2: $tr(M) = \sum_i \lambda_i$ https://courses.cs.washington.edu/courses/cse521/16sp/521-lecture-8.pdf

Variational Characterization of Eigenvalues

$$\text{symmetric real } M_{n\times n},\ \text{eigenvalue } \lambda_1 \le \lambda_2 \ldots \le \lambda_n$$
$$\text{Rayleigh quotient } R_M(\boldsymbol{x}) = \frac{\boldsymbol{x}^T M \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}}$$
$$\lambda_k = \min_{\forall V, \dim V = k} \max_{\boldsymbol{x}\in V - \{\boldsymbol{0}\}} R_M(\boldsymbol{x})$$

proof: https://blog.csdn.net/a358463121/article/details/100166818

证明V里面一定存在向量使得Rayleigh quotient时，只需要取$\lambda_1, \lambda_2, ..., \lambda k$对应的$v1, v_2, ..., v_k$组成的空间$V$即可。 https://zhuanlan.zhihu.com/p/80817719

Hilbert space

conjucture symmetric

Reference

Basic knowledge in Spectral Theory.