Real Analysis

Course MATH 540@UIUC

Reference

- nabla: https://ppasupat.github.io/a9online/wtf-is/nabla.html

Guide

> Took the class with him back in 2018. I would say his exams are pretty similar to the comps, for example: https://math.illinois.edu/system/files/2021-02/MATH 540 - Jan 2021.pdf . The homework from folland's book is kind of easy compare to the exams. I mean, this is a comprehensive exam class for the Math PhD people, so you shouldn't expect it to be any less, and real analysis is known to be hard for many people.

Textbook

Gerald B. Folland, Real Analysis Sol1, Sol2

CH0

Reminder

- Do algebra with $\mu(E)$ carefully, since it can be infinity.

Notation

- X: (in plain text) the universal set.
- $\mathcal{E}$: (mathcal in tex) a collection of subsets.
- A, E: (in tex) a set.
- $\mathcal{P}(X)$: the power set $\{E : E \subset X\}$.
- $\cup A_j$ is by default countable union(or in other symbol, summation/intersection) $\cup_{j=1}^{\infty} A_j$. Arbitrary union will be stressed by using $\cup_{\alpha}^{\infty} A_\alpha$.
- ":=" means this is definition, or can be done by definition.

Set theory

Nota. $A \subset B$: A can be B.

Nota. A set A is smaller than set B is defined as $A \subset B$ but $A \neq B$.

> Def. (**Product set** $X \times Y$)

> Def. (**map**)

> Def. (**todo**) Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of nonempty sets, $X := \prod_{\alpha \in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps.

> $f : A \to \bigcup_{\alpha \in A} X_\alpha$.

> Def. (**Arbitrary infinite sum**) For a set E, $\sum_{x \in E} f(x) := \sup\{\sum_{x \in F} f(x) : \text{finite set } F \subset E\}$.

> Def. (**Set limit**) Given $A_1, A_2, \cdots \in \mathcal{F}$,
>
> $$\limsup_{n\to\infty} A_n := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n = \bigcap_{m=1}^{\infty} B_m = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many n}\}$$

$$\liminf_{n\to\infty} A_n := \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n = \bigcup_{m=1}^{\infty} C_m = \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many n}\}$$

Recall:

1. f is continuous at x if $\forall \{x_n\}, x_n \to x, n \to \infty \implies \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$.
2. $\limsup_n x_n = \lim_{m\to\infty} \sup_{n \geq m} x_n = \lim_{m\to\infty} c_m$, where c_m is monotonic, so that it must converge if we include $\pm\infty$.

Proof. Consider $\omega \in RHS$ or not. If yes, $\omega \in B_m, \forall m$; if not, disappear eventually.

Proof. Consider $\omega \in RHS$ or not. If yes, appear eventually; otherwise fail.

Rmk. $\liminf A_n \subset \limsup A_n$; if equal, we say A_n converges.

E.g. Monotonic set sequence converges (if including ∞).

Elementary real analysis

- Any open set on real line can be expressed as **countable** union of mutually **disjoint** open intervals.
- Compact set: for any open cover of S, there's a finite subcover for S.
- On real line: compact as long as closed + bounded, or sequentially campact.
- Arbitrary union of open set still open, arbitrary intersection of closed set still closed.
- $f : X \to Y$ is continuous on X iff for any open set U in Y, $f^{-1}(U)$ is open in X.

CH1 Measure theory

1.2 σ-algebra/field

> Def. (**Algebra** of sets of X) A non-empty collection $\mathcal{A}$ of subsets of X, that is closed under finite union and complements. In other word,

1. $E_1, E_2 \in \mathcal{A} \to E_1 \cup E_2 \in \mathcal{A}$.
2. $E \in \mathcal{A} \to E^C \in \mathcal{A}$.

Rmk. a) Algebra is closed under finite intersection; b) $\emptyset, X \in \mathcal{A}$. This is important when it comes to covering.

> Def. (**σ-algebra** of sets of X) A non-empty collection $\mathcal{A}$ of subsets of X, that is closed under countable union and complements. E.g. $\mathcal{A} = \{E \in X : \text{E is co-countable}\}$.

Prop. $\mathcal{A}$ is a σ-algebra iff (a) $\mathcal{A}$ is a algebra; (b)

$$E_j \text{ mutually disjoint}, E_j \in \mathcal{A} \to \bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$$

Proof. $\cup E_j = \cup_j [E_j \setminus (\cup_{k<j} E_k)] \in \mathcal{A}$. "This device of replacing a sequence of sets by a disjoint sequence is worth remembering."

Lemma. The intersection of any family of σ-algebras on X is again a σ-algebra.

> Def. (σ**-algebra generated by** $\mathcal{E}$) For $\mathcal{E} \subset \mathcal{P}(X)$, i.e. a collection of subsets of X, there's a **unique smallest** σ-algebra $\mathcal{M(E)}$ containing $\mathcal{E}$, namely, the intersection of all σ-algebras containing $\mathcal{E}$.

Lemma. (1.1) $\mathcal{E} \subset \mathcal{M(F)} \implies \mathcal{M(E)} \subset \mathcal{M(F)}$.

> Def. (**Toplogy** of subsets of X) A non-empty collection $\mathcal{F}$ of subsets of X , satisfying (a) $\emptyset, X \in \mathcal{F}$; (b) closed under arbitrary union; (c) closed under finite intersection.
>
> Def. (**Toplogical space**) A pair $(X, \mathcal{F})$.

Nota. G is the family of open sets in X; F is the family of closed sets; G_δ is the countable intersection of open sets; $F_{\delta\sigma}$ is the countable union of F_δ ... **G is a toplogy**.

> Def. (**Borel** σ**-algebra** of $(X, \mathcal{F})$) The $\mathcal{M}$(G), denoted as $\mathcal{B}_X$, where G is the aforementioned family of open sets.

Prop. (1.2) $\mathcal{M}$(G) is the same as $\mathcal{M}(\text{open intervals})$, $\mathcal{M}(F)$, $\mathcal{M}(\text{the open rays } \{(a, \infty)\})$, $\mathcal{M}(\text{the closed rays } \{[a, \infty)\})$, etc. These will be shown in 1.5.

> Def. (**Borel set**) A Borel set is a member of $\mathcal{B}_X$. E.g. G_δ, F_σ are Borel set. (Many sets look like either one of these two.)

> Def. (**Product** σ**-algebra**) We ask for $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{j=1}^n \mathcal{B}_{\mathbb{R}}$. This definition enables it by: let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of nonempty sets, $X := \prod_{\alpha \in A} X_\alpha$, $\pi_\alpha : X \to X_\alpha$ the coordinate maps, and $\mathcal{M}_\alpha$ is a σ-algebra on X_α, then define $\bigotimes \mathcal{A}_\alpha := \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$.

#NotCovered Prop 1.1-1.6

1.3 Measure

> Def. (**Measure** μ on **measurable space** (X, $\mathcal{A}$)) $\mu : \mathcal{M} \to [0, \infty]$, s.t.
>
> 1. $\mu(\emptyset) = 0$;
> 2. Countable additivity (σ-additivity). If $E_1, E_2, \ldots$ is a collection of **disjoint** members of $\mathcal{M}$, i.e. $E_i \cap E_j = \emptyset$ for all $i \neq j$, then $\mu(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E_i)$.

> Def. (**Finite measure**) $\mu(X) < \infty$.
>
> Def. (σ**-finite measure**) $X = \bigcup E_j, s.t., \forall j, \mu(E_j) < \infty$.
>
> Def. (**Semifinite measure**) $\forall E \in \mathcal{M}, \mu(E) = \infty \to (\exists F \subset E, 0 < \mu(F) < \infty)$.

> Def. (**Null set and "almost everywhere (a.e.)"**) E is a null set if $\mu(E) = 0$. Proposition A is true almost everywhere if it is true on all but null set.

E.g. Given $f : X \to [0, \infty]$, we can define a measure by $\mu(E) = \sum_{x \in E} f(x)$.

1. It's semifinite iff $f(x) < \infty$.
2. It's σ-finite iff it's semifinite and $\{x : f(x) > 0\}$ is countable.
3. It's called **counting measure** if for some $x_0 \in X$, $f(x) = \mathbb{1}(x = x_0)$.
4. It's called **point mass or Dirac measure** if $f(x) = 1$.

Thm. Properties of measure:

1. (**Monotone**) $E, F \in \mathcal{M}, E \subset F \implies \mu(E) \leq \mu(F)$.
2. (**σ-subadditive**) $\mu(\cup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$.
3. (**Continuity from below**) $E_1 \subset E_2 \ldots \implies \mu(\cup E_j) = \lim \mu(E_j)$.
4. (**Continuity from above**) $E_1 \supset E_2 \ldots ; \mu(E_1) < \infty \implies \mu(\cap E_j) = \lim \mu(E_j)$. The $\mu(E_1) < \infty$ is to enable $\mu(E_1 \setminus E_j) = \mu(E_1) - \mu(E_j)$, in the convertion between union and intersection.

Cor. $\mu(\liminf E_j) \leq \liminf \mu(E_j), \mu(\limsup E_j) \geq \limsup \mu(E_j)$.

Prop. σ-finite implies semifinite.

Proof. For every E s.t. $\mu(E) = \infty$, given $X = \cup E_j$, define $F_j := E_j \cap E$. By subadditivity, $\infty = \mu(E) = \mu(\bigcup F_j) \leq \sum \mu(F_j)$, then $\exists j, \mu(F_j) > 0$. By monotoncity, $\mu(F_j) \leq \mu(E_j) < \infty$. These two gives the $F := F_j$ as the non-trivial measure subset for each E.

Ex. Given E, define $\mu_E(A) := \mu(A \cap E)$. Then it's a measure.

> Def. (**Complete**) A measure whose domain contains all subsets of null sets.

#NotCovered THM1.9. Completion of measure.

Continuity of measure (not covered)

> Def. (**Continuity of general measure**) μ is continuous if $\forall \{A_n\}, A_n \to A, n \to \infty \longrightarrow \lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n) := \mu(A)$. Notice the closeness under union&intersection gives that $A := \limsup_n A_n \in \mathcal{F}$.

Thm. (**Countable additivity implies continuity**)

Proof. For all convergent sequence $\{A_n\}$, which means

1. Case1: monotonic increasing An $(A_{n-1} \subset A_n)$ Recall countable additivity, construct $D_n = A_n \setminus A_{n-1}$, then

$$\begin{aligned}
\mu(A) = \mu(\lim_{n\to\infty} A_n) &:= \mu(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m) \\
&= \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} D_n) \overset{(*)}{=} \sum_{n=1}^{\infty} \mu(D_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(D_i) = \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} D_i) = \lim_{n\to\infty} \mu(A_n)
\end{aligned}$$

2. Case2: monotonic decreasing An ($A_{n-1} \supset A_n$) Construct $E_n = A_n \setminus A_{n+1}$, then

$$\begin{aligned}\mu(A) &:= \mu(\lim_{n\to\infty} A_n) := \mu(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m) \\ &= \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} E_n) \overset{(*)}{=} \sum_{n=1}^{\infty}\mu(E_n) \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\mu(E_i) = \lim_{n\to\infty}\mu(\bigcup_{i=1}^{n} E_i) = \lim_{n\to\infty}\mu(A_n)\end{aligned}$$

3. Case3: general An Recall $B_n = \bigcup_{m=n}^{\infty} A_m, C_n = \bigcap_{m=n}^{\infty} A_m$. Clearly $C_n \subset A_n \subset B_n$, and that B_n is monotonic decreasing, C_n is monotonic increasing. From case1, we know that

$$\begin{aligned}\limsup_{n\to\infty}\mu(A_n) \le \lim_{n\to\infty}\mu(B_n) &= \mu(\lim_{n\to\infty} B_n) \\ &= \mu(B) = \mu(A) = \mu(C) \\ &= \mu(\lim_{n\to\infty} C_n) = \lim_{n\to\infty}\mu(C_n) \le \liminf_{n\to\infty}\mu(A_n)\end{aligned}$$

However, $\limsup_{n\to\infty} A_n \ge \liminf_{n\to\infty} A_n$, therefore $\lim_{n\to\infty}\mu(A_n) = \limsup_{n\to\infty}\mu(A_n) = \liminf_{n\to\infty}\mu(A_n) = \mu(A)$.

Conclusion: μ is a continuous set function.

Prop. (**Finite additivity + continuity iff countable additivity**) Proof. (only => is needed) Recall continuity: $\forall\{A_n\}, A_n \to A, n \to \infty \longrightarrow \lim_{n\to\infty}\mu(A_n) = \mu(\lim_{n\to\infty} A_n) = \mu(A)$ and (countable additivity) If $A_1, A_2, \ldots$ is a collection of disjoint members of $\mathcal{F}$, then

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(\lim_{n\to\infty}\bigcup_{i=1}^{n} A_i) = \lim_{n\to\infty}\mu(\bigcup_{i=1}^{n} A_i) = \lim_{n\to\infty}\sum_{i=1}^{n}\mu(A_i) = \sum_{i=1}^{\infty}\mu(A_i)$$

1.4 Tools to construct measure

Motiv. In calculus, one defines area by marking grids inside and outside. Approximation from the outside is what we're going to build in the following.

> Def. (**Outer measure** on X) $\mu^* : \mathcal{P}(X) \to [0, \infty]$, s.t.
>
> 1. $\mu^*(\emptyset) = 0$;
> 2. Monotonicity;
> 3. (σ-subadditivity) $\mu^*(\cup A_j) \le \sum \mu^*(A_j)$.

Prop. (1.10) Let $\mathcal{E} \subset \mathcal{P}(X)$ be a family of "elementary sets" that we can later choose, and $\rho : \mathcal{E} \to [0, \infty]$, such that $\emptyset, X \in \mathcal{E}, \rho(\emptyset) = 0$. These elementary sets are enough to define a outer measure:

$$\mu^*(A) := \inf_{\{E_j\}}\{\sum_{j=1}^{\infty}\rho(E_j) : E_j \in \mathcal{E}, A \subset \bigcup_{j=1}^{\infty} E_j\}$$

Proof. The first and the second condition come immediately from the definition of infimum. For the third one, again, consider $\mu^*(A_j)$ as a infimum the largest lowerbound, then for any j and $\epsilon_j > 0$, $\mu^*(A_j) + \epsilon_j$ is not a lowerbound, therefore exists $\sum_{k=1}^{\infty}\rho(E_{j,k}) < \mu^*(A_j) + \epsilon_j$. Suming up LHS gives a value that's less than $\sum \mu^*(A_j) + \sum \epsilon_j$ but greater than $\mu^*(\cup A_j)$. Let $\epsilon_j = \epsilon * 2^{-j}$ and sending ϵ to 0 gives the desired inequality.

> Def. (μ^***-measurable**) A set $A \subset X$ is called μ^*-measurable if
>
> $$\forall E \subset X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Motiv. This definition can be understood as, when A is "good", we can use A to evaluate any $E \subset X$, such that the inner measure of A (intersection of two, approximate from inside), $\mu^*(E \cap A)$, is equal to the outer measure of A, $\mu^*(E) - \mu^*(E \cap A^c)$.

Rmk. Notice that to show a set A is μ^*-measurable, due to the subadditivity, it suffices to show

$$\forall E \subset X, s.t. \mu^*(E) < \infty, \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Ex. Let A_j be a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap \bigcup_j A_j) = \sum_j \mu^*(E \cap A_j)$.

Thm. (**Caratheodory's thm**) If μ^* is an outer measure on X, then the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and $\mu^*|_{\mathcal{M}}$ is a complete measure on measurable space $(X, \mathcal{M})$.

Proof.

1. $\mathcal{M}$ is an algebra: the goal is, given $A, B \in \mathcal{M}$, show that $\forall E \subset X, \mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$. Taking the fact that A is μ^*-measurable, and let E be the latter two respectively,

$$\begin{aligned}
\mu^*(E \cap (A \cup B)) =& \mu^*(E \cap (A \cup B) \cap A) + \mu^*(E \cap (A \cup B) \cap A^c) \\
=& \mu^*(E \cap A) + \mu^*(E \cap B \cap A^c) \\
\mu^*(E \cap (A \cup B)^c) =& \mu^*(E \cap (A \cup B)^c \cap A) + \mu^*(E \cap (A \cup B)^c \cap A^c) \\
=& \mu^*(E \cap (A \cup B)^c) = \mu^*(E \cap A^c \cap B^c)
\end{aligned}$$

2. $\mathcal{M}$ is a σ-algebra: it suffices to prove it's closed under disjoint σ-union, and we only need to check one side of inequality. Define $B_n = \cup_{i=1}^n A_n$,

$$\begin{aligned}
\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\
\text{given disjoint, } &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\
\text{by induction, } &= \sum_{i=1}^n \mu^*(E \cap A_i) \\
\mu^*(E) &= \mu^*(E \cap (\cup_{i=1}^n A_i)) + \mu^*(E \cap (\cup_{i=1}^n A_i)^c) \\
&\geq \mu^*(E \cap B_n) + \mu^*(E \cap (\cup_{i=1}^\infty A_i)^c) \\
&\geq \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap (\cup_{i=1}^\infty A_i)^c) \\
\text{take limit, } &\geq \sum_{i=1}^\infty \mu^*(E \cap A_i) + \mu^*(E \cap (\cup_{i=1}^\infty A_i)^c) \\
\text{by subadditivity, } &\geq \mu^*(E \cap (\cup_{i=1}^\infty A_i)) + \mu^*(E \cap (\cup_{i=1}^\infty A_i)^c)
\end{aligned}$$

3. $\mu^*|_{\mathcal{M}}$ is a measure: we now know $\cup_{i=1}^\infty A_i \in \mathcal{M}$ is in the domain, which enable us to use the inequality above but with σ-union as E:

$$\mu^*(\cup_{i=1}^\infty A_i) \geq \sum_{i=1}^n \mu^*((\cup_{j=1}^\infty A_j) \cap A_i) + \mu^*((\cup_{i=1}^\infty A_i) \cap (\cup_{i=1}^\infty A_i)^c)$$

   The other side is again by σ-subadditivity.

4. $\mu^*|_{\mathcal{M}}$ is complete. Given $B \subset A, \mu^*(A) = \mu^*(B) = 0$, $\forall E \subset X, \mu^*(E) \geq \mu^*(E \cap B^c) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$.

> Def. (**Premeasure**) $\mu_0 : \mathcal{A} \to [0, \infty]$, $\mathcal{A}$ is a algebra, with:
>
> 1. $\mu_0 = 0$;
> 2. Any $\{A_j\}_{j=0}^{\infty} \subset A$ that are sequence of disjoint sets s.t. $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$, then $\mu_0(\cup A_j) = \sum_{i=1}^{\infty} \mu_0(A_j)$.

Prop. Monotonicity of premeasure.

Prop. (1.13) By applying Prop. 1.10 with $\rho = \mu_0$, one can construct outermeasure $\mu^* : \mathcal{P}(X) \to [0, \infty]$, which extends the domain of μ_0. Then,

1. $\mu^*|_{\mathcal{A}} = \mu_0$;
2. $\forall A \in \mathcal{A}$, A is μ^*-measurable.

Proof.

1. (Recall) $\mu^*(D) := \inf_{\{A_j\}}\{\sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A}, D \subset \bigcup_{j=1}^{\infty} A_j\}$;
2. $\mu^*|_{\mathcal{A}} \leq \mu_0$ is true since LHS is a lowerbound of a set containing $\mu_0(A)$ induced by sequence $\{A, \emptyset, \emptyset, \emptyset, \ldots\}$.
3. To show the other side, need to show the RHS is a lowerbound. We only have disjoint complete sequence additivity. For $A \in \mathcal{A}$, covering $\{A_j\}$, construct $B_n = A \cap (A_n \setminus \cup_{i<n} A_i)$, then $\cup B_i = A \in \mathcal{A}$ given covering. Then $\mu_0(A) = \sum_j \mu_0(B_j) \leq \sum_j \mu_0(A_j)$.
4. Want to show: $\forall A \in \mathcal{A}, E \subset X, \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. It suffices to show $\forall \epsilon > 0, \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. The LHS isn't a lowerbound, therefore exists covering $\{A_j\} \subset \mathcal{A}$ s.t.

$$\begin{aligned}
\mu^*(E) + \epsilon &> \sum_j \mu_0(A_j) \\
\text{By disjoint additivity, } &= \sum_j \mu_0(A_j \cap A) + \mu_0(A_j \cap A^c) \\
&= \sum_j \mu^*(A_j \cap A) + \mu^*(A_j \cap A^c) \\
\text{By subadditivity, } &\geq \mu^*(\cup_j (A_j \cap A)) + \mu^*(\cup_j (A_j \cap A^c)) \\
&= \mu^*(E \cap A) + \mu^*(E \cap A^c)
\end{aligned}$$

Thm. (1.14) Algebra $\mathcal{A}$, σ-algebra $\mathcal{M} := \mathcal{M}(\mathcal{A})$, premeasure μ_0 on $\mathcal{A}$, and μ^* the outermeasure given in last thm. Then:

1. $\mu := \mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$; (This gives the existence of measure extending μ_0)
2. Any other measure $\tilde{\mu}$ that extends μ_0 has $\forall E \in \mathcal{M}, \tilde{\mu}(E) \leq \mu(E)$, with equality when $\mu(E) < \infty$.
3. If μ_0 is σ-finite, then μ is unique. (This gives the uniqueness of measure extending μ_0 under stronger condition)

Proof.

1. Let $\mathcal{B}$ the collection of μ^*-measurable sets. By C-thm, $\mathcal{B}$ is a σ-algebra and $\mu^*|_{\mathcal{B}}$ is a measure. By Prop 1.13, $\mathcal{A} \subset \mathcal{B}$, and $\mathcal{M}$ is the smallest σ-algebra containing A, therefore $\mathcal{M} \subset \mathcal{B}$, $\mu^*|_{\mathcal{M}}$ is a measure.
2. Goal: $\forall E \in \mathcal{M}, \tilde{\mu}(E) \leq \mu(E)$. Notice that for any covering $\{A_j\} \subset \mathcal{A}$ of E, $\tilde{\mu}(E) \leq \tilde{\mu}(\cup A_j) \leq \sum \tilde{\mu}(A_j) = \sum \mu_0(A_j) = \sum \mu(A_j)$, therefore a lowerbound, which is not greater

than the greatest lowerbound μ^*.

3. Claim $\mu^*(\cup A_j) = \tilde{\mu}(\cup A_j)$: since both are measure extending μ_0 defined on $\mathcal{A}$ where finite union is closed, consider using contituity by

$$\mu^*(\cup A_j) = \lim \mu^*(\cup_{j=1}^{\infty} A_j) = \lim \mu^*(\cup_{j=1}^{\infty} A_j)$$
$$= \lim \mu_0(\cup_{j=1}^{\infty} A_j) = \lim \tilde{\mu}(\cup_{j=1}^{\infty} A_j) = \tilde{\mu}(\cup_{j=1}^{\infty} A_j)$$

.

4. Goal: $\forall E \in \mathcal{M}, \tilde{\mu}(E) \geq \mu(E)$ when $\mu(E) < \infty$. Notice that for any covering $\{A_j\} \subset \mathcal{A}$ of E, $\mu^*(E) \leq \mu^*(\cup A_j) = \tilde{\mu}(\cup A_j) = \tilde{\mu}(E) + \tilde{\mu}(\cup A_j \setminus E)$. It suffices to show that $\tilde{\mu}(\cup A_j \setminus E) \leq \epsilon$ for any $\epsilon > 0$, and further more, $\mu^*(\cup A_j \setminus E) \leq \epsilon$, given part 2. Consider adding ϵ to the infimum, i.e. $\forall \epsilon > 0$, there's a covering $\{A_j\} \subset \mathcal{A}$ of E, s.t. $\mu^*(E) + \mu^*(\cup A_j \setminus E) = \mu^*(\cup A_j) \leq \sum \mu_0(A_j) < \mu^*(E) + \epsilon$. When $\mu(E) < \infty$, subtracting it on both sides gives the desired.
5. Goal: $\forall E \in \mathcal{M}, \tilde{\mu}(E) = \mu(E)$. Recall definition, $X = \cup A_j$, s.t. $A_j \in \mathcal{A}, \mu_0(A_j) < \infty$. Make it disjoint by $B_j := A_j \setminus (\cup_{k<j} A_k)$ to have a partition of E. Then $\tilde{\mu}(E) = \sum \tilde{\mu}(E \cap B_j) = \sum \mu(E \cap B_j) = \mu(E)$, given part 4.

Ex. (Ch1 q18)

1. If $\mu^*(E) < \infty$, then E is μ^*-measurable iff $\exists B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$;
2. If μ_0 is $\sigma-$finite, the restriction of $\mu^*(E) < \infty$ is superfluous.

Ex. (Ch1 q24) μ is a finite measure. Suppose that $E \subset X, E \notin \mathcal{M}$ satisfies $\mu^*(E) = \mu^*(X)$.

1. If $A, B \in \mathcal{M}, A \cap E = B \cap E$, then $\mu(A) = \mu(B)$;
2. Let $\mathcal{M}_E := \{A \cap E : A \in \mathcal{M}\}$, and define function v as $v(A \cap E) = \mu(A)$. Then $\mathcal{M}_E$ is a σ-algebra on E and v is a measure on $\mathcal{M}_E$.

1.5 Borel measure on $\mathbb{R}$

Recall. $\mathcal{B}_{\mathbb{R}} := \mathcal{M}(G)$.

> Def.
>
> 1. **Open invervals** $\mathcal{A}_0 := \{(a, b) : -\infty \leq a < b \leq +\infty\}$;
> 2. **h-intervals** $\mathcal{A}_h := \{(a, b] : -\infty \leq a < b < +\infty\} \cup \{(a, \infty) : -\infty \leq a < +\infty\} \cup \{\emptyset\}$
> 3. $\mathcal{A}_2 :=$ finite union of disjoint h-intervals.

Prop. $\mathcal{M}(\mathcal{A}_0) = \mathcal{M}(\mathcal{A}_h) = \mathcal{M}(\mathcal{A}_2) = \mathcal{M}(G) := \mathcal{B}_{\mathbb{R}}$.

Proof. By lemma 1.1, it suffices to show that $\mathcal{A}_0, \mathcal{A}_h, \mathcal{A}_2 \subset \mathcal{M}(G), \mathcal{A}(G) \subset \mathcal{M}(\mathcal{A}_0) \cap \mathcal{M}(\mathcal{A}_h) \cap \mathcal{M}(\mathcal{A}_2)$.

Prop. $\mathcal{A}_2$ is a algebra.

Thm. $F : \mathbb{R} \to \mathbb{R}$ (non-strictly) increasing and right-continuous. We can construct premeasure μ_0 by $\mu_0(\emptyset) = 0$ and $\mu_0(\cup_{j=1}^{n}(a_j, b_j]) = \sum_{i=1}^{n} F(b_j) - F(a_j)$ where $(a_j, b_j]$ are disjoint.

Proof.

1. Goal: μ_0 is well-defined (consistent with different union partition). Draw diagram.

2. Goal: For any disjoint sequence s.t. $\cup_{j=1}^{\infty} I_j \in \mathcal{A}_2$, we have $\mu_0(\cup_{j=1}^{\infty} I_j) = \sum_{j=1}^{\infty} \mu_0(I_j)$. Since the union is in $\mathcal{A}_2$, it can be expressed in a finite union of disjoint h-intervals. By considering each h-interval as a trunk, the sequence can be partitioned into **finitely many subsequences**, each is with a trunk and disjoint to others. With finite additivity and relabelling, consider each trunk and corresponding subsequence seperately, WOLG, say $I := \cup_{j=1}^{\infty} I_j := (a, b]$. For $I_j = (a_j, b_j]$, discard contained ones to get disjoint intervals.
3. Goal: For $I := \cup_{j=1}^{\infty} I_j := (a, b]$, show $\mu_0(\cup_{j=1}^{\infty} I_j) \leq \sum_{j=1}^{\infty} \mu_0(I_j)$. It's obvious given monotonicity.
4. Goal: For $I := \cup_{j=1}^{\infty} I_j := (a, b]$, show $\mu_0(\cup_{j=1}^{\infty} I_j) \geq \sum_{j=1}^{\infty} \mu_0(I_j)$.
   - First suppose a and b are finite. Recall that any open set on real line can be expressed as countable union of disjoint open intervals, and a open cover of a compact set on real line (closed) can be reduced to a finite yet valid subcover.
   - To have open interval and compact set from h-interval, we make use of right-continuity, which gives us that $\forall \epsilon > 0, \exists \delta > 0, F(a + \delta) - F(a) < \epsilon$, and further more, $\forall j, \exists \delta_j, F(b_j + \delta_j) - F(b_j) < \epsilon \cdot 2^{-j}$. Now we can adjust the boundary of sets.
   - Extend I from $(a, b]$ into $[a + \delta, b]$, which is compact, and extend I_j from $(a_j, b_j]$ into $(a_j, b_j + \delta_j)$. To simpify, we can adjust so that we have $b_j + \delta_j \in (a_{j+1}, b_{j+1})$. Now that we have an open cover $I \subset \cup_{j=1}^{\infty} (a_j, b_j + \delta_j)$, we obtain a finite subcover (with relabelling) $I \subset \cup_{j=1}^{n} (a'_j, b'_j + \delta_j)$. Summing up

$$\begin{aligned}\mu_0((a'_j, b'_j]) &= F(b'_j) - F(a'_j) \\ &\geq F(b'_j + \delta_j) - F(a'_j) - \epsilon \cdot 2^{-j} \\ &\geq F(a'_{j+1}) - F(a'_j) - \epsilon \cdot 2^{-j}\end{aligned}$$

     , we get

$$\begin{aligned}\sum_{j=1}^{\infty} \mu_0(I_j) &\geq \sum_{j=1}^{n} \mu_0((a'_j, b'_j]) \\ &\geq F(b'_n + \delta_n) - F(a'_1) - \epsilon \\ &\geq F(b) - F(a + \delta) - \epsilon \\ &\geq F(b) - F(a) - 2\epsilon \\ &= \mu_0(I) - 2\epsilon\end{aligned}$$

     .
   - Corner case of a, b being infinite is omitted.

Thm. Given F increasing and right-continuous, then

1. There's a unique Borel measure μ_F on $\mathbb{R}$ s.t. $\mu_F((a, b]) = F(b) - F(a)$. To be explicit, $\mu_F = \inf\{\sum_{j=1}^{\infty} \mu_0((a_j, b_j]) : E \subset \cup_{j=1}^{\infty} (a_j, b_j]\}$.
2. If other distribution function G, then $\mu_F = \mu_G$ iff $F - G$ is constant.
3. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, and we define $F(x) = \mu((0, x]), x > 0, F(0) = 0, F(x) = -\mu((x, 0]), x < 0$, then F is increasing and right continuous, and $\mu = \mu_F$.

Proof.

1. The constructed μ_0 is σ-finite, since $\mathbb{R} = \cup_{-\infty}^{\infty} (j, j + 1]$. Then it follows from the Thm 1.14;
2. $\mu_F = \mu_G \iff \forall a, b, F(b) - G(b) = F(a) - G(a)$.

3. Take $x > 0$ as example. The monotonicity is from the monotonicity of F, and the right-continuous can be get from the continuity. μ and μ_F is the same on $\mathcal{A}_2$, therefore the same on $\mathcal{B}_\mathbb{R}$.

Rmk.

1. The collection $\mathcal{M}_\mu$ of μ^*-measurable in Caratheodory's thm is the largest (in fact strictly larger than $\mathcal{B}_\mathbb{R}$, denoted as $\mathcal{E}$) gives the domain of the completion of μ_F (Ex22a), which is called the **Lebesgue-Stieltjes measure** associated to F.
2. When $F(x) = x$, the Lebesgue-Stieltjes measure associated is called the **Lebesgue measure** m. The domain is denoted as $\mathcal{L}$.

Lemma. (1.17) $\mu|_\mathcal{E}(E) = \inf\{\sum_{j=1}^\infty \mu((a_j, b_j)) : E \subset \cup_{j=1}^\infty (a_j, b_j)\}$.

Proof. Say the RHS is $\tilde{\mu}(E)$.

1. Goal: $\mu(E) \leq \tilde{\mu}(E)$. Since $(a, b) = \cup_{n=1}^\infty (a, b - \frac{1}{n}]$, $E \subset \cup(a_j, b_j) \subset \cup\cup\, (a_j, b_j - \frac{1}{n})$, therefore the set in left contains the set in right;
2. Goal: $\mu(E) + \epsilon \geq \tilde{\mu}(E), \forall \epsilon > 0$. Use right-continuity.

$$
\begin{aligned}
\exists\{(a_j, b_j]\}, \mu(E) + \epsilon &\geq \sum \mu_0((a_j, b_j]) = \sum F(b_j) - F(a_j) \\
&\geq \sum F(b_j + \delta_j) - F(a_j) - \epsilon \cdot 2^{-j} \\
&= -\epsilon + \sum \mu_0((a_j, b_j + \delta_j]) \\
&\geq -\epsilon + \sum \mu((a_j, b_j + \delta_j)) \\
&\geq -\epsilon + \tilde{\mu}(E)
\end{aligned}
$$

Thm. (1.18)

$\mu|_\mathcal{E}(E) = \inf\{\mu(U) : U \supset E, U\ open\}$, and $\mu|_\mathcal{E}(E) = \sup\{\mu(K) : K \subset E, K\ compact\}$. This is so important that it's used as definition in some textbooks.

Proof. Say, to show $\mu(E) = \tilde{\mu}(E) = \mu'(E)$, with the formula given in lemma 1.17:

1. Goal: $\mu(E) \leq \tilde{\mu}(E)$. This is because $\mu(E) \leq \mu(U), \forall U$;
2. Goal: $\mu(E) + \epsilon \geq \tilde{\mu}(E), \forall \epsilon > 0$. Again, $\exists\{(a_j, b_j]\}, \mu(E) + \epsilon \geq \sum \mu((a_j, b_j)) \geq \mu(\cup(a_j, b_j)) \geq \tilde{\mu}(E)$.
3. Goal: $\mu(E) \geq \mu'(E)$. The same as 1.
4. Goal: $\mu(E) \leq \mu'(E)$. Use the first equality.
    1. If E is bounded:
        1. Subcase: If E is compact. Just take K:=E.
        2. Subcase: If otherwise. Consider $\bar{E} \setminus E$, then by the first equality, $\exists open\ U \supset \bar{E} \setminus E, s.t. \mu(\bar{E} \setminus E) + \epsilon > \mu(U)$. Let $K = \bar{E} \setminus U$, then it's compact and $K \subset E$.

$$
\begin{aligned}
\mu(K) = \mu(E) - \mu(E \cap U) &= \mu(E) - (\mu(U) - \mu(U \setminus E)) \\
&= \mu(E) - \mu(U) + \mu(U \setminus E) \\
&\geq \mu(E) - \mu(U) + \mu(\bar{E} \setminus E) \geq \mu(E) - \epsilon
\end{aligned}
$$

2. If E is unbounded, partition it as $E_j = E \cap (j, j+1]$. By case 1, $\forall \epsilon > 0, \exists K_j \subset E_j, s.t. \mu(K_j) \geq \mu(E_j) - \epsilon \cdot 2^{-|j|}$. $\mu'(E) \geq \mu(\cup_{-n}^{n} K_j) \geq \mu(\cup_{-n}^{n} E_j) - \epsilon$.

Thm. (1.19)

If $E \subset \mathbb{R}$, then TFAE:

1. $E \in \mathcal{E}$;
2. $E = V \setminus N_1$, where $V \in G_\delta, \mu(N_1) = 0$;
3. $E = H \cup N_2$, where $H \in F_\sigma, \mu(N_2) = 0$.

Proof. We know $V, H \in \mathcal{E}$. Since μ is complete on $\mathcal{E}$, all $N_1, N_2 \in \mathcal{E}$, and σ-algebra is closed under countable union and intersection, (2) and (3) each imply (1). Now to show the converse,

1. Suppose $\mu(E) < \infty$. Based on thm 1.18, for $j \in \mathbb{N}$, we can have open $U_j \supset E$ and compact $K_j \subset E$, s.t. $\mu(U_j) - 2^{-j} \leq \mu(E) \leq \mu(K_j) + 2^{-j}$. Let $V := \cap U_j, H := \cup K_j$, then $H \subset E \subset V$. While $\mu(E) \leq \mu(V) \leq \mu(U_j) \leq \mu(E) + 2^{-j}, \forall j$ and $\mu(E) - 2^{-j} \leq \mu(K_j) \leq \mu(H) \leq \mu(E)$, we can have $\mu(H) = \mu(E) = \mu(V) < \infty$. $N_1 := V \setminus E, N_2 := E \setminus H$, then $\mu(N_1) = \mu(V) - \mu(E) = 0, \mu(N_2) = \mu(E) - \mu(H) = 0$.
2. Otherwise. Again, the constructed μ_0, is σ-finite, since $\mathbb{R} = \cup_{-\infty}^{\infty}(j, j+1]$, and therefore $E_j := E \cap (j, j+1], \mu(E_j) < \infty, E = \cup E_j$.
   1. Notice that (1)->(3) implies (1)->(2). So we only need to show the former.
   2. Consider the partition, in which we have $E_j = H_j \cup N_j$. Let $H := \cup H_j, N = \cup N_j$. Then $E = \cup(H_j \cup N_j) = H \cup N$.

Prop. (1.20) If $E \in \mathcal{E}, \mu(E) < \infty$, then $\forall \epsilon > 0, \exists A$ that is a finite union of open intervals such that $\mu(E \triangle A) < \epsilon$.

Proof. Based on thm 1.18, we can have open $U \supset E$ and compact $K \subset E$, s.t. $\mu(U) \leq \mu(E) \leq \mu(K) + \epsilon$. Since $U = \cup_{j=1}^{\infty}(a_j, b_j)$ gives a open cover of compact set K, we can have the subcover $A := \cup_{j=1}^{n}(a_j, b_j) \supset K$. Then $\mu(E) - \epsilon \leq \mu(K) \leq \mu(A) \leq \mu(U) \leq \mu(E) + \epsilon$ and $\mu(E) - \epsilon \leq \mu(K) \leq \mu(A \cap E) \leq \mu(U) \leq \mu(E) + \epsilon$. Then $|\mu(E) - \mu(A)| \leq \epsilon, |\mu(E) - \mu(A \cap E)| \leq \epsilon$, which means

$$
\begin{aligned}
&\mu(E \triangle A) \\
&\leq \mu(E \setminus A) + \mu(A \setminus E) \\
&= \mu(A) - \mu(E) + \mu(E) - \mu(A \cap E) + \mu(E) - \mu(A \cap E) \\
&\leq 3\epsilon
\end{aligned}
$$

.

Prop. For any $E \in \mathcal{L}$, we have $E + s, rE \in \mathcal{L}$, and $\mu(E + s) = \mu(E), \mu(rE) = |r|\mu(E)$.

Proof. They agree on the algebra, and by uniqueness of thm 1.14 (3), they also agree on $\mathcal{B}_\mathbb{R}$. Further more, since Lebesgue measure zero is preserved by translations and diluations, by thm 1.19, they agree on $\mathcal{L}$. #TODO

E.g. (**Cantor set** C) Repeadly remove the middle thirds open interval, starting from $[0, 1]$. It's compact, totally disconnected, no where dense, no isolated points, $m(C) = 0$, and $0, 1 \in C$. Moreover, it's uncountable and with the cardinality of $\mathbb{R}$. This can be proved by constructing $f: C \to [0,1]$ and let it onto. $f: \sum a_j 3^{-j} \mapsto \sum \frac{a_j}{2} 2^{-j}$.

Thm.

If $F \subset \mathbb{R}$, s.t. $\forall G \subset F, G \in \mathcal{L}$, then $m(F) = 0$.

Cor. (**Existence of non-measurable set**) For F that $m(F) > 0, \exists G \subset F, G \notin \mathcal{L}$.

> Def. (**Coset**) A coset of $\mathbb{Q}$ in additive group $(\mathbb{R}, +)$ is $\mathbb{Q} + x$, where $x \in \mathbb{R}$.

Proof of Thm.

1. Let E be the set that contains exactly one point from each coset. The existence of E is given by the axiom of choice.
2. Claim: $\forall r_1, r_2 \in \mathbb{Q}, r_1 \neq r_2 \to (E + r_1) \cap (E + r_2) = \emptyset$. Otherwise, that means $e_1, e_2 \in E, e_1 \neq e_2, e_1 - e_2 \in \mathbb{Q}$, contradicts with the "exactly one point".
3. Claim: $\mathbb{R} = \cup_{r \in \mathbb{Q}} (E + r)$. For any $x \in \mathbb{R}$, there's a coset $\mathbb{Q} + x$, in which E contains exactly an element $q + x$. Then $x = q + x + (-q)$ will be contained in $E + (-q)$, which is when $r = -q$, in the union.
4. Now $F = F \cap \mathbb{R} = \cup_{r \in \mathbb{Q}} (F \cap (E + r)) = \cup F_r$, it suffices to show $m(F_r) = 0$. Given that $m(F_r) = \sup\{m(K) : compact\ K \subset F_r\}$, this holds iff $\forall compact\ K \subset F_r, m(K) = 0$. We're going to use the fact that K is bounded.
5. Suppose not, i.e. there's a K, s.t. $m(K) > 0$. Due to to the same reason as 2, we have $\forall r_1, r_2 \in \mathbb{Q}, r_1 \neq r_2 \to (K + r_1) \cap (K + r_2) = \emptyset$. Note that it's still bounded after translation. Further more, let's bound the translation scale. Let $H = \cup_{r \in \mathbb{Q} \cap [0,1]} (K + r)$, which is a disjoint union of bounded set and should be bounded as a whole (within the union of $(-M_r, M_r)$). Yet since every summand in this σ-additivity (infinite) summation, $m(K + r) > 0$, we have $m(H) = \infty$, contradict.

Ex. (Ch1 q30) If $E \in \mathcal{L}, m(E) < 0$, then $\forall \alpha < 1, \exists open\ interval\ I$, s.t. $m(E \cap I) > \alpha m(I)$.

Thm. (**Steinhams' thm**) If $E \in \mathcal{L}, m(E) < 0$, the set $E - E = \{x - y : x, y \in E\}$ contains an interval centered at 0.

Ch2 Integration

2.1 Measurable function

Motiv. $f^{-1}(E) = \{x \in X : f(x) \in E\}$ preserves unions, intersections, and complements. $f^{-1}(\mathcal{B})$ and $\mathcal{M} = \{E \in \mathcal{B} : f^{-1}(E) \in \mathcal{A}\}$ are σ-algebra.

> Def. (**Measurable function**) Measurable space $(X, \mathcal{A}), (Y, \mathcal{B})$, we say $f: X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable if $f^{-1}(\mathcal{B}) \subset \mathcal{A}$, or equivalently, $\mathcal{M} \supset \mathcal{B}$.

Rmk.

1. Random variables are special cases of measurable function.
2. Composition of measurable mappings are measurable.

Prop. (2.1) $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable iff $\mathcal{M} \supset \mathcal{B}_0$, where $\mathcal{M}(B_0) = \mathcal{B}$.

Proof. $\mathcal{M}$ is a σ-algebra, and $\mathcal{M} \supset \mathcal{B}_0 \implies \mathcal{M} \supset \mathcal{M}(\mathcal{B}_0) = \mathcal{B}$.

Cor. Let X, Y be metric or topological spaces, then every continuous $f : X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measuable.

Proof. Recall that $f : X \to Y$ is continuous on X iff for any open set U in Y, $f^{-1}(U)$ is open in X. By prop 2.1, it follows.

Thm. Based on prop 1.2 and 2.1, T.F.A.E:

1. $f : X \to \mathbb{R}$ is $\mathcal{A}$-measuable.
2. $\{(a, \infty) : \forall a \in \mathbb{R}\} \subset \mathcal{M}$.
3. $\{[a, \infty) : \forall a \in \mathbb{R}\} \subset \mathcal{M}$.
4. $\{(-\infty, a) : \forall a \in \mathbb{R}\} \subset \mathcal{M}$.
5. $\{(-\infty, a] : \forall a \in \mathbb{R}\} \subset \mathcal{M}$.
6. If $f : X \to \bar{\mathbb{R}}$, change the above as including infinity.

Thm. If f, g are $\mathcal{A}$-measurable, then $f + g, fg$ are $\mathcal{A}$-measurable.

Proof. Work on level sets. Consider $fg = \frac{1}{2}((f + g)^2 - f^2 - g^2)$.

Def. $\mathcal{B}_{\bar{\mathbb{R}}} = \{E \subset \bar{\mathbb{R}} : E \cap \mathbb{R} \subset \mathcal{B}_{\mathbb{R}}\}$.