Real Analysis

> The note of UIUC course MATH 540 Real Analysis in Fall 2023, by Zory Zhang. In case of any broken math renderring in Github preview, please open this markdown file using your own markdown editor. PDF version (maybe not up-to-date) can be found here.

Reference

- nabla: https://ppasupat.github.io/a9online/wtf-is/nabla.html

Guide

> Took the class with him back in 2018. I would say his exams are pretty similar to the comps, for example: https://math.illinois.edu/system/files/2021-02/MATH 540 - Jan 2021.pdf . The homework from folland's book is kind of easy compare to the exams. I mean, this is a comprehensive exam class for the Math PhD people, so you shouldn't expect it to be any less, and real analysis is known to be hard for many people.

Textbook

Gerald B. Folland, Real Analysis Sol1, Sol2

Ch0 Preliminaries

High level techniques

1. When we want to show an inequality related to measure, it's usually easier to enlarge it, since the subadditivity is giving you more terms during enlarging. Otherwise, you need to organize your terms and take a pair of it to apply subadditivity. Also, during enlarging, you can directly drop unnecessary terms in intersection, due to monotonicity.
2. When showing general inequality, though, try both. Sometimes either is easier.

Reminder

- Do algebra with $\mu(E)$ carefully, since it can be infinity.

Notation

- X: (in plain text) the universal set.
- $\mathcal{E}$: (mathcal in tex) a collection of subsets.
- A, E: (in tex) a set.
- $\mathcal{P}(X)$: the power set $\{E : E \subset X\}$.
- $\cup A_j$ can be finite, countable, or arbitrary union(same for other symbols like summation/intersection) and should be clear in context. Arbitrary union usually will be stressed by using $\cup_\alpha^\infty A_\alpha$.
- ":=" means this is definition, or can be done by definition.

Set theory

Nota. $A \subset B$: A can be equal to B.

Nota. A set A is called **smaller** than set B, if $A \subset B$ but $A \neq B$.

Def. (Product set $X \times Y$)

Def. (map)

Def. (todo)

Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of nonempty sets, $X := \prod_{\alpha \in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps.

$f : A \to \bigcup_{\alpha \in A} X_\alpha$.

Def. (Arbitrary infinite sum)

For a set E, $\sum_{x \in E} f(x) := \sup\{\sum_{x \in F} f(x) : \text{finite set } F \subset E\}$.

Def. (Set limit)

Given $A_1, A_2, \dots \in \mathcal{F}$,

$$\limsup_{n \to \infty} A_n := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n = \bigcap_{m=1}^{\infty} B_m = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many n}\}$$

$$\liminf_{n \to \infty} A_n := \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n = \bigcup_{m=1}^{\infty} C_m = \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many n}\}$$

Recall:

1. f is continuous at x if $\forall \{x_n\}, x_n \to x, n \to \infty \implies \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$.
2. $\limsup_n x_n = \lim_{m\to\infty} \sup_{n \geq m} x_n = \lim_{m\to\infty} c_m$, where c_m is monotonic, so that it must converge if we include $\pm\infty$.

Proof.

1. Consider $\omega \in RHS$ or not. If yes, $\omega \in B_m, \forall m$; if not, disappear eventually.
2. Consider $\omega \in RHS$ or not. If yes, appear eventually; otherwise fail.

Rmk. $\liminf A_n \subset \limsup A_n$; if equal, we say A_n converges.

E.g. Monotonic set sequence converges (if including ∞).

Elementary real analysis

- Compact set: for any open cover of S, there's a finite subcover for S.

- On real line:
  - A set is compact as long as closed + bounded, or sequentially campact.
  - Any open set can be expressed as **countable** union of mutually **disjoint** open intervals.
- Arbitrary union of open set still open, arbitrary intersection of closed set still closed.
- $f: X \to Y$ is continuous on X iff for any open set U in Y, $f^{-1}(U)$ is open in X.
- $\sum_{j=1}^{n}\sum_{k=1}^{\infty} = \sum_{k=1}^{\infty}\sum_{j=1}^{n}$ is interchangable. Proof by induction on n.

Ch1 Measure theory

1.2 Some algebraic structures

Def. (Algebra of sets of X)

A non-empty collection $\mathcal{A}$ of subsets of X, that is closed under finite union and complement. In other word,

1. $E_1, E_2 \in \mathcal{A} \to E_1 \cup E_2 \in \mathcal{A}$.
2. $E \in \mathcal{A} \to E^C \in \mathcal{A}$.

Rmk. a) Algebra is closed under finite intersection; b) $\emptyset, X \in \mathcal{A}$. This is important when it comes to covering.

Def. (σ-algebra of sets of X)

A non-empty collection $\mathcal{A}$ of subsets of X, that is closed under countable union and complements. E.g. $\mathcal{A} = \{E \in X : \text{E is co-countable}\}$.

> Prop. $\mathcal{A}$ is a σ-algebra iff (a) $\mathcal{A}$ is a algebra; (b)
>
> $$E_j \text{ mutually disjoint}, E_j \in \mathcal{A} \to \bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$$

Proof. $\cup E_j = \cup_j [E_j \setminus (\cup_{k<j} E_k)] \in \mathcal{A}$. "This device of replacing a sequence of sets by a disjoint sequence (yet preserving the union) is worth remembering."

> Lemma. The intersection of any family of σ-algebras on X is again a σ-algebra.

Def. (σ-algebra generated by $\mathcal{E}$)

For $\mathcal{E} \subset \mathcal{P}(X)$, i.e. a collection of subsets of X, there's a **unique smallest** σ-algebra $\mathcal{M}(\mathcal{E})$ containing $\mathcal{E}$, namely, the intersection of all σ-algebras containing $\mathcal{E}$.

> Prop. (1.1) $\mathcal{E} \subset \mathcal{M}(\mathcal{F}) \implies \mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$.

Def. (Toplogy of subsets of X)

A non-empty collection $\mathcal{F}$ of subsets of X , satisfying (a) $\emptyset, X \in \mathcal{F}$; (b) closed under arbitrary union; (c) closed under finite intersection.

Def. (Toplogical space) A pair $(X, \mathcal{F})$.

Nota. G is the family of open sets in X; F is the family of closed sets; G_δ is the countable intersection of open sets; $F_{\delta\sigma}$ is the countable union of F_δ ... **G is a toplogy**.

Def. (Borel σ-algebra of $(X, \mathcal{F})$)

The $\mathcal{M}$(G), denoted as $\mathcal{B}_X$, where G is the aforementioned family of open sets.

> Prop. (1.2) $\mathcal{M}$(G) is the same as $\mathcal{M}(\text{open intervals})$, $\mathcal{M}(F)$, $\mathcal{M}(\text{the open rays } \{(a, \infty)\})$, $\mathcal{M}(\text{the closed rays } \{[a, \infty)\})$, etc. These will be shown in 1.5.

Def. (Borel set)

A Borel set is a member of $\mathcal{B}_X$. E.g. G_δ, F_σ are Borel set. (Many sets look like either one of these two.)

Def. (Product σ-algebra)

We ask for $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{j=1}^n \mathcal{B}_{\mathbb{R}}$. The following definition enables it by: let $\{X_\alpha\}_{\alpha \in A}$ be an indexed collection of nonempty sets, $X := \prod_{\alpha \in A} X_\alpha$, $\pi_\alpha : X \to X_\alpha$ the coordinate maps, and $\mathcal{M}_\alpha$ is a σ-algebra on X_α, then define $\bigotimes \mathcal{A}_\alpha := \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$.

#NotCovered Prop 1.1-1.6

Def. (Elementary family)

A collection $\mathcal{E}$ of subsets of X, s.t.

1. $\emptyset \in \mathcal{E}$;
2. Closed under finite intersection;
3. If $E \in \mathcal{E}$, then E^c is a finite disjoint union of members of $\mathcal{E}$.

> Prop. (1.7) For elementary family $\mathcal{E}$, the collection of finite disjoint union of members of $\mathcal{E}$ is an algebra.

Def. (Ring)

Closed under finite union and differences.

Rmk.

1. A ring $\mathcal{R}$ is closed under finite intersections.
2. A ring is an algebra iff $X \in \mathcal{R}$.

1.3 Measure

Def. (Measure μ on measurable space (X, $\mathcal{A}$))

$\mu : \mathcal{M} \to [0, \infty]$, s.t.

1. $\mu(\emptyset) = 0$;
2. Countable additivity (σ-additivity). If $E_1, E_2, \ldots$ is a collection of **disjoint** members of $\mathcal{M}$, i.e. $E_i \cap E_j = \emptyset$ for all $i \neq j$, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Def. (Finite measure)

$\mu(X) < \infty$.

[!note] Def. (σ**-finite measure**) $X = \bigcup E_j, s.t., \forall j, \mu(E_j) < \infty$.

[!note] Def. (**Semifinite measure**) $\forall E \in \mathcal{M}, \mu(E) = \infty \to (\exists F \subset E, 0 < \mu(F) < \infty)$.

[!note] Def. (**Null set and "almost everywhere (a.e.)"**) E is a null set if $\mu(E) = 0$. Proposition A is true almost everywhere if it is true on all but null set.

E.g. Given $f : X \to [0, \infty]$, we can define a measure by $\mu(E) = \sum_{x \in E} f(x)$.

1. It's semifinite iff $f(x) < \infty$.
2. It's σ-finite iff it's semifinite and $\{x : f(x) > 0\}$ is countable.
3. It's called **counting measure** if for some $x_0 \in X$, $f(x) = \mathbb{1}(x = x_0)$.
4. It's called **point mass or Dirac measure** if $f(x) = 1$.

Thm. Properties of measure:

1. (**Monotone**) $E, F \in \mathcal{M}, E \subset F \implies \mu(E) \leq \mu(F)$.
2. (σ**-subadditive**) $\mu(\cup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$.
3. (**Continuity from below**) $E_1 \subset E_2 \ldots \implies \mu(\cup E_j) = \lim \mu(E_j)$.
4. (**Continuity from above**) $E_1 \supset E_2 \ldots ; \mu(E_1) < \infty \implies \mu(\cap E_j) = \lim \mu(E_j)$. The $\mu(E_1) < \infty$ is to enable $\mu(E_1 \setminus E_j) = \mu(E_1) - \mu(E_j)$, in the convertion between union and intersection.

Ex. (Ch1 q8) $\mu(\liminf E_j) \leq \liminf \mu(E_j)$; If $\mu(\cup E_j) < \infty$, then $\mu(\limsup E_j) \geq \limsup \mu(E_j)$.

> Prop. σ-finite implies semifinite.

Proof. For every E s.t. $\mu(E) = \infty$, given $X = \cup E_j$, define $F_j := E_j \cap E$. By subadditivity, $\infty = \mu(E) = \mu(\bigcup F_j) \leq \sum \mu(F_j)$, then $\exists j, \mu(F_j) > 0$. By monotoncity, $\mu(F_j) \leq \mu(E_j) < \infty$. These two gives the $F := F_j$ as the non-trivial measure subset for each E.

Ex. Given E, define $\mu_E(A) := \mu(A \cap E)$. Then it's a measure.

> **Def. (Complete)**
>
> A measure whose domain contains all subsets of null sets.

#NotCovered THM1.9. Completion of measure.

Continuity of measure (not covered)

> **Def. (Continuity of general measure) μ is continuous if** $\forall\{A_n\}, A_n \to A, n \to \infty \longrightarrow \lim_{n\to\infty}\mu(A_n) = \mu(\lim_{n\to\infty} A_n) := \mu(A)$. **Notice the closeness under union&intersection gives that** $A := \limsup_n A_n \in \mathcal{F}$.

Thm. (**Countable additivity implies continuity**)

Proof. For all convergent sequence $\{A_n\}$, which means

1. Case1: monotonic increasing An ($A_{n-1} \subset A_n$) Recall countable additivity, construct $D_n = A_n \setminus A_{n-1}$, then

$$\begin{aligned}
\mu(A) &= \mu(\lim_{n\to\infty} A_n) := \mu(\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m) \\
&= \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} D_n) \overset{(*)}{=} \sum_{n=1}^{\infty}\mu(D_n) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\mu(D_i) = \lim_{n\to\infty}\mu(\bigcup_{i=1}^{n} D_i) = \lim_{n\to\infty}\mu(A_n)
\end{aligned}$$

2. Case2: monotonic decreasing An ($A_{n-1} \supset A_n$) Construct $E_n = A_n \setminus A_{n+1}$, then

$$\begin{aligned}
\mu(A) &:= \mu(\lim_{n\to\infty} A_n) := \mu(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m) \\
&= \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} E_n) \overset{(*)}{=} \sum_{n=1}^{\infty}\mu(E_n) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\mu(E_i) = \lim_{n\to\infty}\mu(\bigcup_{i=1}^{n} E_i) = \lim_{n\to\infty}\mu(A_n)
\end{aligned}$$

3. Case3: general An Recall $B_n = \bigcup_{m=n}^{\infty} A_m, C_n = \bigcap_{m=n}^{\infty} A_m$. Clearly $C_n \subset A_n \subset B_n$, and that B_n is monotonic decreasing, C_n is monotonic increasing. From case1, we know that

$$\begin{aligned}
\limsup_{n\to\infty}\mu(A_n) \le \lim_{n\to\infty}\mu(B_n) =& \mu(\lim_{n\to\infty} B_n) \\
=& \mu(B) = \mu(A) = \mu(C) \\
=& \mu(\lim_{n\to\infty} C_n) = \lim_{n\to\infty}\mu(C_n) \le \liminf_{n\to\infty}\mu(A_n)
\end{aligned}$$

However, $\limsup_{n\to\infty} A_n \ge \liminf_{n\to\infty} A_n$, therefore $\lim_{n\to\infty}\mu(A_n) = \limsup_{n\to\infty}\mu(A_n) = \liminf_{n\to\infty}\mu(A_n) = \mu(A)$.

Conclusion: μ is a continuous set function.

Prop. (**Finite additivity + continuity iff countable additivity**) Proof. (only => is needed) Recall continuity: $\forall\{A_n\}, A_n \to A, n \to \infty \longrightarrow \lim_{n\to\infty}\mu(A_n) = \mu(\lim_{n\to\infty} A_n) = \mu(A)$ and (countable additivity) If $A_1, A_2, \ldots$ is a collection of disjoint members of $\mathcal{F}$, then

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(\lim_{n\to\infty}\bigcup_{i=1}^{n} A_i) = \lim_{n\to\infty}\mu(\bigcup_{i=1}^{n} A_i) = \lim_{n\to\infty}\sum_{i=1}^{n}\mu(A_i) = \sum_{i=1}^{\infty}\mu(A_i)$$

1.4 Tools to construct measure

Motiv. In calculus, one defines area by marking grids inside and outside. Approximation from the outside is what we're going to build in the following.

Def. (Outer measure on X)

$\mu^* : \mathcal{P}(X) \to [0,\infty]$, s.t.

1. $\mu^*(\emptyset) = 0$;
2. Monotonicity;
3. (σ-subadditivity) $\mu^*(\cup A_j) \le \sum \mu^*(A_j)$.

> Prop. (1.10) Let $\mathcal{E} \subset \mathcal{P}(X)$ be a family of "elementary sets" that we can later choose, and $\rho : \mathcal{E} \to [0,\infty]$, such that $\emptyset, X \in \mathcal{E}, \rho(\emptyset) = 0$. These elementary sets are enough to define a outer measure:

$$\mu^*(A) := \inf_{\{E_j\}}\{\sum_{j=1}^{\infty}\rho(E_j) : E_j \in \mathcal{E}, A \subset \bigcup_{j=1}^{\infty} E_j\}$$

Proof. The first and the second condition come immediately from the definition of infimum. For the third one, again, consider $\mu^*(A_j)$ as a infimum the largest lowerbound, then for any j and $\epsilon_j > 0$, $\mu^*(A_j) + \epsilon_j$ is not a lowerbound, therefore exists $\sum_{k=1}^{\infty}\rho(E_{j,k}) < \mu^*(A_j) + \epsilon_j$. Suming up LHS gives a value that's less than $\sum \mu^*(A_j) + \sum \epsilon_j$ but greater than $\mu^*(\cup A_j)$. Let $\epsilon_j = \epsilon * 2^{-j}$ and sending ϵ to 0 gives the desired inequality.

Def. (μ^*-measurable)

A set $A \subset X$ is called μ^*-measurable if

$$\forall E \subset X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Motiv. This definition can be understood as, when A is "good", we can use A to evaluate any $E \subset X$, such that the inner measure of A (intersection of two, approximate from inside), $\mu^*(E \cap A)$, is equal to the outer measure of A, $\mu^*(E) - \mu^*(E \cap A^c)$.

Rmk. Notice that to show a set A is μ^*-measurable, due to the subadditivity, it suffices to show

$$\forall E \subset X, s.t. \mu^*(E) < \infty, \mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

> Lemma. For a sequence of disjoint μ^*-measurable sets A_j, let $B_n = \cup_{i=1}^{n} A_i$, then we have:

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ \text{given disjoint, } &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ \text{by induction, } &= \sum_{i=1}^{n}\mu^*(E \cap A_i) \end{aligned}$$

$> The above result can be extended to infinite sums since only one side is needed given subadditivity. > [!importan$

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\begin{aligned}
\mu^{*}(E\cap (A \cup B))
=&\mu^{*}(E\cap (A \cup B)\cap A)+\mu^{*}(E\cap (A \cup B)\cap A^c)\\
=&\mu^{*}(E\cap A)+\mu^{*}(E\cap B\cap A^c)\\
\mu^{*}(E\cap (A \cup B)^c)
=&\mu^{*}(E\cap (A \cup B)^c\cap A)+\mu^{*}(E\cap (A \cup B)^c\cap A^c)\\
=&\mu^{*}(E\cap (A \cup B)^c)=\mu^{*}(E\cap A^c \cap B^c)
\end{aligned}
$$
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2. $\mathcal{M}$ is a σ-algebra: it suffices to prove it's closed under disjoint σ-union, and we only need to check one side of inequality. Define $B_n = \cup_{i=1}^n A_i$,

$$\begin{aligned} \mu^*(E\cap B_n) &= \sum_{i=1}^n \mu^*(E\cap A_i) \\ \mu^*(E) &= \mu^*(E\cap(\cup_{i=1}^n A_i)) + \mu^*(E\cap(\cup_{i=1}^n A_i)^c) \\ &\geq \mu^*(E\cap B_n) + \mu^*(E\cap(\cup_{i=1}^\infty A_i)^c) \\ &\geq \sum_{i=1}^n \mu^*(E\cap A_i) + \mu^*(E\cap(\cup_{i=1}^\infty A_i)^c) \\ \text{take limit, } &\geq \sum_{i=1}^\infty \mu^*(E\cap A_i) + \mu^*(E\cap(\cup_{i=1}^\infty A_i)^c) \\ \text{by subadditivity, } &\geq \mu^*(E\cap(\cup_{i=1}^\infty A_i)) + \mu^*(E\cap(\cup_{i=1}^\infty A_i)^c) \end{aligned}$$

3. $\mu^*|_{\mathcal{M}}$ is a measure: we now know $\cup_{i=1}^\infty A_i \in \mathcal{M}$ is in the domain, which enable us to use the inequality above but with σ-union as E:

$$\mu^*(\cup_{i=1}^\infty A_i) \geq \sum_{i=1}^n \mu^*((\cup_{j=1}^\infty A_j)\cap A_i) + \mu^*((\cup_{i=1}^\infty A_i)\cap(\cup_{i=1}^\infty A_i)^c)$$

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The other side is again by $\sigma$-subadditivity.
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4. $\mu^*|_{\mathcal{M}}$ is complete. Given $B\subset A, \mu^*(A) = \mu^*(B) = 0$, $\forall E\subset X, \mu^*(E) \geq \mu^*(E\cap B^c) = \mu^*(E\cap B) + \mu^*(E\cap B^c)$.

Def. (Premeasure)

$\mu_0 : \mathcal{A} \to [0,\infty]$, $\mathcal{A}$ is a algebra, with:

1. $\mu_0 = 0$;
2. Any $\{A_j\}_{j=0}^\infty \subset A$ that are sequence of disjoint sets s.t. $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$, then $\mu_0(\cup A_j) = \sum_{i=1}^\infty \mu_0(A_j)$.

> Prop. Monotonicity of premeasure.

> Prop. (1.13) By applying Prop. 1.10 with $\rho = \mu_0$, one can construct outermeasure $\mu^* : \mathcal{P}(X) \to [0,\infty]$, which extends the domain of μ_0. Then,

> 1. $\mu^*|_{\mathcal{A}} = \mu_0$;
> 2. $\forall A \in \mathcal{A}$, A is μ^*-measurable.

Proof.

1. (Recall) $\mu^*(D) := \inf_{\{A_j\}}\{\sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A}, D \subset \bigcup_{j=1}^{\infty} A_j\}$;
2. $\mu^*|_{\mathcal{A}} \leq \mu_0$ is true since LHS is a lowerbound of a set containing $\mu_0(A)$ induced by sequence $\{A, \emptyset, \emptyset, \emptyset, \ldots\}$.
3. To show the other side, need to show the RHS is a lowerbound. We only have disjoint complete sequence additivity. For $A \in \mathcal{A}$, covering $\{A_j\}$, construct $B_n = A \cap (A_n \setminus \cup_{i<n} A_i)$, then $\cup B_i = A \in \mathcal{A}$ given covering. Then $\mu_0(A) = \sum_j \mu_0(B_j) \leq \sum_j \mu_0(A_j)$.
4. Want to show: $\forall A \in \mathcal{A}, E \subset X, \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. It suffices to show $\forall \epsilon > 0, \mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. The LHS isn't a lowerbound, therefore exists covering $\{A_j\} \subset \mathcal{A}$ s.t.

$$\begin{aligned}
\mu^*(E) + \epsilon &> \sum_j \mu_0(A_j) \\
\text{By disjoint additivity, } &= \sum_j \mu_0(A_j \cap A) + \mu_0(A_j \cap A^c) \\
&= \sum_j \mu^*(A_j \cap A) + \mu^*(A_j \cap A^c) \\
\text{By subadditivity, } &\geq \mu^*(\cup_j (A_j \cap A)) + \mu^*(\cup_j (A_j \cap A^c)) \\
&= \mu^*(E \cap A) + \mu^*(E \cap A^c)
\end{aligned}$$

Thm. (1.14)

Algebra $\mathcal{A}$, σ-algebra $\mathcal{M} := \mathcal{M}(\mathcal{A})$, premeasure μ_0 on $\mathcal{A}$, and μ^* the outermeasure given in last thm. Then:

1. $\mu := \mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$; (This gives the existence of measure extending μ_0)
2. Any other measure $\tilde{\mu}$ that extends μ_0 has $\forall E \in \mathcal{M}, \tilde{\mu}(E) \leq \mu(E)$, with equality when $\mu(E) < \infty$.
3. If μ_0 is σ-finite, then μ is unique. (This gives the uniqueness of measure extending μ_0 under stronger condition)

Proof.

1. Let $\mathcal{B}$ the collection of μ^*-measurable sets. By C-thm, $\mathcal{B}$ is a σ-algebra and $\mu^*|_{\mathcal{B}}$ is a measure. By Prop 1.13, $\mathcal{A} \subset \mathcal{B}$, and $\mathcal{M}$ is the smallest σ-algebra containing A, therefore $\mathcal{M} \subset \mathcal{B}$, $\mu^*|_{\mathcal{M}}$ is a measure.
2. Goal: $\forall E \in \mathcal{M}, \tilde{\mu}(E) \leq \mu(E)$. Notice that for any covering $\{A_j\} \subset \mathcal{A}$ of E, $\tilde{\mu}(E) \leq \tilde{\mu}(\cup A_j) \leq \sum \tilde{\mu}(A_j) = \sum \mu_0(A_j) = \sum \mu(A_j)$, therefore a lowerbound, which is not greater than the greatest lowerbound μ^*.
3. Claim $\mu^*(\cup A_j) = \tilde{\mu}(\cup A_j)$: since both are measure extending μ_0 defined on $\mathcal{A}$ where finite union is closed, consider using contituity by

$$\begin{aligned}\mu^*(\cup A_j) &= \lim \mu^*(\cup_{j=1}^{\infty} A_j) = \lim \mu^*(\cup_{j=1}^{\infty} A_j) \\ &= \lim \mu_0(\cup_{j=1}^{\infty} A_j) = \lim \tilde{\mu}(\cup_{j=1}^{\infty} A_j) = \tilde{\mu}(\cup_{j=1}^{\infty} A_j)\end{aligned}$$

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4. Goal: $\forall E \in \mathcal{M}, \tilde{\mu}(E) \geq \mu(E)$ when $\mu(E) < \infty$. Notice that for any covering $\{A_j\} \subset \mathcal{A}$ of E, $\mu^*(E) \leq \mu^*(\cup A_j) = \tilde{\mu}(\cup A_j) = \tilde{\mu}(E) + \tilde{\mu}(\cup A_j \setminus E)$. It suffices to show that $\tilde{\mu}(\cup A_j \setminus E) \leq \epsilon$ for any $\epsilon > 0$, and further more, $\mu^*(\cup A_j \setminus E) \leq \epsilon$, given part 2. Consider adding ϵ to the infimum, i.e. $\forall \epsilon > 0$, there's a covering $\{A_j\} \subset \mathcal{A}$ of E, s.t. $\mu^*(E) + \mu^*(\cup A_j \setminus E) = \mu^*(\cup A_j) \leq \sum \mu_0(A_j) < \mu^*(E) + \epsilon$. When $\mu(E) < \infty$, subtracting it on both sides gives the desired.
5. Goal: $\forall E \in \mathcal{M}, \tilde{\mu}(E) = \mu(E)$. Recall definition, $X = \cup A_j$, s.t. $A_j \in \mathcal{A}, \mu_0(A_j) < \infty$. Make it disjoint by $B_j := A_j \setminus (\cup_{k<j} A_k)$ to have a partition of E. Then $\tilde{\mu}(E) = \sum \tilde{\mu}(E \cap B_j) = \sum \mu(E \cap B_j) = \mu(E)$, given part 4.

> Ex. (Ch1 q18)
>
> 1. If $\mu^*(E) < \infty$, then E is μ^*-measurable iff $\exists B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$;
> 2. If μ_0 is $\sigma-$finite, the restriction of $\mu^*(E) < \infty$ is superfluous.

Proof.

1. (<=) By showing both $B, B \setminus E$ are μ^*-measurable and within the $\mathcal{M}$. Or by enlarging $\mu^*(A \cap E) + \mu^*(A \cap E^c)$ into $\mu^*(A \cap B) + \mu^*(A \cap B^c)$ to bound above $\mu^*(A)$. (=>) Just take intersection of open set covering.
2. (=>) Partition E into disjoint finite pieces, find $B_{n,j} \in \mathcal{A}_\sigma$, s.t., $\mu^*(B_{n,j}) \leq \mu^*(E_j) + \frac{1}{n} 2^{-j}, i.e. \mu^*(B_{n,j} \setminus E) \leq \epsilon_{n,j}$. Then we can show $\mu^*(B \setminus E) \leq \frac{1}{n}$.

> Ex. (Ch1 q19) For an outer measure induced from a finite premeasure, then $\mu^*(E) + \mu^*(E^c) = \mu^*(X)$ implies E is μ^*-measurable.

Proof. As usual, take $\epsilon = \frac{1}{n}$ so that there's $B_n \in A_\sigma, B_n \supset E, \mu^*(B_n) \leq \mu^*(E) + \frac{1}{n}$, therefore $B := \cap B_n \in A_{\sigma\delta}, \mu^*(B) = \mu^*(E)$. Notice B is measurable, therefore we can have $\mu^*(B) + \mu^*(B^c) = \mu^*(X)$, and further, $\mu^*(B^c) = \mu^*(E^c)$. To get $B \setminus E$ to apply ex. ch1q18, consider, $\mu^*(B^c) = \mu^*(E^c) = \mu^*(E^c \cap B) + \mu^*(E^c \cap B^c) = \mu^*(B \setminus E) + \mu^*(B^c)$.

> Ex. (Ch1 q24) μ is a finite measure. Suppose that $E \subset X, E \notin \mathcal{M}$ satisfies $\mu^*(E) = \mu^*(X)$.
>
> 1. If $A, B \in \mathcal{M}, A \cap E = B \cap E$, then $\mu(A) = \mu(B)$;
> 2. Let $\mathcal{M}_E := \{A \cap E : A \in \mathcal{M}\}$, and define function v as $v(A \cap E) = \mu(A)$. Then $\mathcal{M}_E$ is a σ-algebra on E and v is a measure on $\mathcal{M}_E$.

1.5 Borel measure on $\mathbb{R}$

Recall. $\mathcal{B}_{\mathbb{R}} := \mathcal{M}(G)$.

Def.

1. **Open invervals** $\mathcal{A}_0 := \{(a, b) : -\infty \leq a < b \leq +\infty\}$;
2. **h-intervals** $\mathcal{A}_h := \{(a, b] : -\infty \leq a < b < +\infty\} \cup \{(a, \infty) : -\infty \leq a < +\infty\} \cup \{\emptyset\}$

3. $\mathcal{A}_2 :=$ finite union of disjoint h-intervals.

> Prop. $\mathcal{M}(\mathcal{A}_0) = \mathcal{M}(\mathcal{A}_h) = \mathcal{M}(\mathcal{A}_2) = \mathcal{M}(G) := \mathcal{B}_{\mathbb{R}}$.

Proof. By lemma 1.1, it suffices to show that
$\mathcal{A}_0, \mathcal{A}_h, \mathcal{A}_2 \subset \mathcal{M}(G), \mathcal{A}(G) \subset \mathcal{M}(\mathcal{A}_0) \cap \mathcal{M}(\mathcal{A}_h) \cap \mathcal{M}(\mathcal{A}_2)$.

> Prop. $\mathcal{A}_2$ is a algebra.

Thm.

$F : \mathbb{R} \to \mathbb{R}$ (non-strictly) increasing and right-continuous. We can construct premeasure μ_0 by $\mu_0(\emptyset) = 0$ and $\mu_0(\cup_{j=1}^n (a_j, b_j]) = \sum_{i=1}^n F(b_j) - F(a_j)$ where $(a_j, b_j]$ are disjoint.

Proof.

1. Goal: μ_0 is well-defined (consistent with different union partition). Draw diagram.
2. Goal: For any disjoint sequence s.t. $\cup_{j=1}^\infty I_j \in \mathcal{A}_2$, we have $\mu_0(\cup_{j=1}^\infty I_j) = \sum_{j=1}^\infty \mu_0(I_j)$. Since the union is in $\mathcal{A}_2$, it can be expressed in a finite union of disjoint h-intervals. By considering each h-interval as a trunk, the sequence can be partitioned into **finitely many subsequences**, each is with a trunk and disjoint to others. With finite additivity and relabelling, consider each trunk and corresponding subsequence seperately, WOLG, say $I := \cup_{j=1}^\infty I_j := (a, b]$. For $I_j = (a_j, b_j]$, discard contained ones to get disjoint intervals.
3. Goal: For $I := \cup_{j=1}^\infty I_j := (a, b]$, show $\mu_0(\cup_{j=1}^\infty I_j) \le \sum_{j=1}^\infty \mu_0(I_j)$. It's obvious given monotonicity.
4. Goal: For $I := \cup_{j=1}^\infty I_j := (a, b]$, show $\mu_0(\cup_{j=1}^\infty I_j) \ge \sum_{j=1}^\infty \mu_0(I_j)$.
   - First suppose a and b are finite. Recall that any open set on real line can be expressed as countable union of disjoint open intervals, and a open cover of a compact set on real line (closed) can be reduced to a finite yet valid subcover.
   - To have open interval and compact set from h-interval, we make use of right-continuity, which gives us that $\forall \epsilon > 0, \exists \delta > 0, F(a + \delta) - F(a) < \epsilon$, and further more, $\forall j, \exists \delta_j, F(b_j + \delta_j) - F(b_j) < \epsilon \cdot 2^{-j}$. Now we can adjust the boundary of sets.
   - Extend I from $(a, b]$ into $[a + \delta, b]$, which is compact, and extend I_j from $(a_j, b_j]$ into $(a_j, b_j + \delta_j)$. To simpify, we can adjust so that we have $b_j + \delta_j \in (a_{j+1}, b_{j+1})$. Now that we have an open cover $I \subset \cup_{j=1}^\infty (a_j, b_j + \delta_j)$, we obtain a finite subcover (with relabelling) $I \subset \cup_{j=1}^n (a'_j, b'_j + \delta_j)$. Summing up

$$
\begin{aligned}
\mu_0((a'_j, b'_j]) &= F(b'_j) - F(a'_j) \\
&\ge F(b'_j + \delta_j) - F(a'_j) - \epsilon \cdot 2^{-j} \\
&\ge F(a'_{j+1}) - F(a'_j) - \epsilon \cdot 2^{-j}
\end{aligned}
$$

, we get

$$\begin{aligned}\sum_{j=1}^{\infty}\mu_0(I_j) \geq \sum_{j=1}^{n}\mu_0((a_j', b_j']) \\ &\geq F(b_n'+\delta_n) - F(a_1') - \epsilon \\ &\geq F(b) - F(a+\delta) - \epsilon \\ &\geq F(b) - F(a) - 2\epsilon \\ &= \mu_0(I) - 2\epsilon\end{aligned}$$

.

- Corner case of a, b being infinite is omitted.

Thm.

Given F increasing and right-continuous, then

1. There's a unique Borel measure μ_F on $\mathbb{R}$ s.t. $\mu_F((a,b]) = F(b) - F(a)$. To be explicit, $\mu_F = \inf\{\sum_{j=1}^{\infty}\mu_0((a_j,b_j]) : E \subset \cup_{j=1}^{\infty}(a_j,b_j]\}$.
2. If other distribution function G, then $\mu_F = \mu_G$ iff $F - G$ is constant.
3. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, and we define $F(x) = \mu((0,x]), x > 0$, $F(0) = 0$, $F(x) = -\mu((x,0]), x < 0$, then F is increasing and right continuous, and $\mu = \mu_F$.

Proof.

1. The constructed μ_0 is σ-finite, since $\mathbb{R} = \cup_{-\infty}^{\infty}(j, j+1]$. Then it follows from the Thm 1.14;
2. $\mu_F = \mu_G \iff \forall a, b, F(b) - G(b) = F(a) - G(a)$.
3. Take $x > 0$ as example. The monotonicity is from the monotonicity of F, and the right-continuous can be get from the continuity. μ and μ_F is the same on $\mathcal{A}_2$, therefore the same on $\mathcal{B}_{\mathbb{R}}$.

Rmk.

1. The collection $\mathcal{M}_\mu$ of μ^*-measurable in Caratheodory's thm is the largest σ-algebra (in fact strictly larger than $\mathcal{B}_{\mathbb{R}}$, denoted as $\mathcal{E}$) and gives the domain of the completion $\bar{\mu}_F$ of μ_F (Ex22a), which is called the **Lebesgue-Stieltjes measure** associated to F.
2. When $F(x) = x$, the Lebesgue-Stieltjes measure associated is called the **Lebesgue measure** m. The domain is denoted as $\mathcal{L}$.

> Lemma. (1.17) $\mu|_{\mathcal{E}}(E) = \inf\{\sum_{j=1}^{\infty}\mu((a_j,b_j)) : E \subset \cup_{j=1}^{\infty}(a_j,b_j)\}$.

Proof. Say the RHS is $\tilde{\mu}(E)$.

1. Goal: $\mu(E) \leq \tilde{\mu}(E)$. Since $(a,b) = \cup_{n=1}^{\infty}(a, b-\frac{1}{n}]$, $E \subset \cup(a_j,b_j) \subset \cup\cup(a_j, b_j - \frac{1}{n})$, therefore the set in left contains the set in right;

2. Goal: $\mu(E)+\epsilon \geq \tilde{\mu}(E), \forall \epsilon>0$. Use right-continuity.

$$\begin{aligned} \exists\{(a_j,b_j]\}, \mu(E)+\epsilon \geq \sum \mu_0((a_j,b_j]) &= \sum F(b_j)-F(a_j) \\ &\geq \sum F(b_j+\delta_j)-F(a_j)-\epsilon\cdot 2^{-j} \\ &= -\epsilon+\sum \mu_0((a_j,b_j+\delta_j]) \\ &\geq -\epsilon+\sum \mu((a_j,b_j+\delta_j)) \\ &\geq -\epsilon+\tilde{\mu}(E) \end{aligned}$$

Thm. (1.18)

$\mu|_{\mathcal{E}}(E)=\inf\{\mu(U): U\supset E, U \text{ } open\}$, and $\mu|_{\mathcal{E}}(E)=\sup\{\mu(K): K\subset E, K \text{ } compact\}$. This is so important that it's used as definition in some textbooks.

Proof. Say, to show $\mu(E)=\tilde{\mu}(E)=\mu'(E)$, with the formula given in lemma 1.17:

1. Goal: $\mu(E)\leq\tilde{\mu}(E)$. This is because $\mu(E)\leq\mu(U), \forall U$;
2. Goal: $\mu(E)+\epsilon\geq\tilde{\mu}(E), \forall\epsilon>0$. Again, $\exists\{(a_j,b_j]\}, \mu(E)+\epsilon\geq\sum\mu((a_j,b_j))\geq\mu(\cup(a_j,b_j))\geq\tilde{\mu}(E)$.
3. Goal: $\mu(E)\geq\mu'(E)$. The same as 1.
4. Goal: $\mu(E)\leq\mu'(E)$. Use the first equality.
   1. If E is bounded:
      1. Subcase: If E is compact. Just take K:=E.
      2. Subcase: If otherwise. Consider $\bar{E}\setminus E$, then by the first equality, $\exists open\ U\supset\bar{E}\setminus E, s.t. \mu(\bar{E}\setminus E)+\epsilon>\mu(U)$. Let $K=\bar{E}\setminus U$, then it's compact and $K\subset E$.

$$\begin{aligned} \mu(K)=\mu(E)-\mu(E\cap U) &= \mu(E)-(\mu(U)-\mu(U\setminus E)) \\ &= \mu(E)-\mu(U)+\mu(U\setminus E) \\ &\geq \mu(E)-\mu(U)+\mu(\bar{E}\setminus E)\geq\mu(E)-\epsilon \end{aligned}$$

   2. If E is unbounded, partition it as $E_j=E\cap(j,j+1]$. By case 1, $\forall\epsilon>0, \exists K_j\subset E_j, s.t. \mu(K_j)\geq\mu(E_j)-\epsilon\cdot 2^{-|j|}$. $\mu'(E)\geq\mu(\cup_{-n}^{n}K_j)\geq\mu(\cup_{-n}^{n}E_j)-\epsilon$.

Thm. (1.19)

If $E\subset\mathbb{R}$, then TFAE:

1. $E\in\mathcal{E}$;
2. $E=V\setminus N_1$, where $V\in G_\delta, \mu(N_1)=0$;
3. $E=H\cup N_2$, where $H\in F_\sigma, \mu(N_2)=0$.

Proof. We know $V, H\in\mathcal{E}$. Since μ is complete on $\mathcal{E}$, all $N_1, N_2\in\mathcal{E}$, and σ-algebra is closed under countable union and intersection, (2) and (3) each imply (1). Now to show the converse,

1. Suppose $\mu(E)<\infty$. Based on thm 1.18, for $j\in\mathbb{N}$, we can have open $U_j\supset E$ and compact $K_j\subset E$, s.t. $\mu(U_j)-2^{-j}\leq\mu(E)\leq\mu(K_j)+2^{-j}$. Let $V:=\cap U_j, H:=\cup K_j$, then $H\subset E\subset V$. While $\mu(E)\leq\mu(V)\leq\mu(U_j)\leq\mu(E)+2^{-j}, \forall j$ and $\mu(E)-2^{-j}\leq\mu(K_j)\leq\mu(H)\leq\mu(E)$, we can have

$\mu(H)=\mu(E)=\mu(V)<\infty$. $N_1:=V\setminus E, N_2:=E\setminus H$, then $\mu(N_1)=\mu(V)-\mu(E)=0, \mu(N_2)=\mu(E)-\mu(H)=0$.

2. Otherwise. Again, the constructed μ_0, is σ-finite, since $\mathbb{R}=\cup_{-\infty}^{\infty}(j,j+1]$, and therefore $E_j:=E\cap(j,j+1], \mu(E_j)<\infty, E=\cup E_j$.
    1. Notice that (1)->(3) implies (1)->(2). So we only need to show the former.
    2. Consider the partition, in which we have $E_j=H_j\cup N_j$. Let $H:=\cup H_j, N=\cup N_j$. Then $E=\cup(H_j\cup N_j)=H\cup N$.

Prop. (1.20)

If $E\in\mathcal{E}, \mu(E)<\infty$, then $\forall\epsilon>0, \exists A$ that is a finite union of open intervals such that $\mu(E\triangle A)<\epsilon$.

Proof. Based on thm 1.18, we can have open $U\supset E$ and compact $K\subset E$, s.t. $\mu(U)\le\mu(E)\le\mu(K)+\epsilon$. Since $U=\cup_{j=1}^{\infty}(a_j,b_j)$ gives a open cover of compact set K, we can have the subcover $A:=\cup_{j=1}^{n}(a_j,b_j)\supset K$. Then $\mu(E)-\epsilon\le\mu(K)\le\mu(A)\le\mu(U)\le\mu(E)+\epsilon$ and $\mu(E)-\epsilon\le\mu(K)\le\mu(A\cap E)\le\mu(U)\le\mu(E)+\epsilon$. Then $|\mu(E)-\mu(A)|\le\epsilon, |\mu(E)-\mu(A\cap E)|\le\epsilon$, which means

$$
\begin{aligned}
&\mu(E\triangle A)\\
&\le\mu(E\setminus A)+\mu(A\setminus E)\\
&=\mu(A)-\mu(E)+\mu(E)-\mu(A\cap E)+\mu(E)-\mu(A\cap E)\\
&\le 3\epsilon
\end{aligned}
$$

.

> Prop. For any $E\in\mathcal{L}$, we have $E+s, rE\in\mathcal{L}$, and $\mu(E+s)=\mu(E), \mu(rE)=|r|\mu(E)$.

Proof. They agree on the algebra, and by uniqueness of thm 1.14 (3), they also agree on $\mathcal{B}_{\mathbb{R}}$. Further more, since Lebesgue measure zero is preserved by translations and diluations, by thm 1.19, they agree on $\mathcal{L}$. #TODO

E.g. (**Cantor set** C) Repeadly remove the middle thirds open interval, starting from $[0,1]$. It's compact, totally disconnected, no where dense, no isolated points, $m(C)=0$, and $0,1\in C$. Moreover, it's uncountable and with the cardinality of $\mathbb{R}$. This can be proved by constructing $f:C\to[0,1]$ and let it onto. $f:\sum a_j3^{-j}\mapsto\sum\frac{a_j}{2}2^{-j}$.

Thm.

If $F\subset\mathbb{R}$, s.t. $\forall G\subset F, G\in\mathcal{L}$, then $m(F)=0$.

Cor. (**Existence of non-measurable set**) For F that $m(F)>0, \exists G\subset F, G\notin\mathcal{L}$.

Def. (Coset)

A coset of $\mathbb{Q}$ in additive group $(\mathbb{R},+)$ is $\mathbb{Q}+x$, where $x\in\mathbb{R}$.

Proof of Thm.

1. Let E be the set that contains exactly one point from each coset. The existence of E is given by the axiom of choice.
2. Claim: $\forall r_1, r_2 \in \mathbb{Q}, r_1 \neq r_2 \rightarrow (E+r_1) \cap (E+r_2) = \emptyset$. Otherwise, that means $e_1, e_2 \in E, e_1 \neq e_2, e_1 - e_2 \in \mathbb{Q}$, contradicts with the "exactly one point".
3. Claim: $\mathbb{R} = \cup_{r\in\mathbb{Q}}(E+r)$. For any $x \in \mathbb{R}$, there's a coset $\mathbb{Q}+x$, in which E contains exactly an element $q+x$. Then $x = q+x+(-q)$ will be contained in $E+(-q)$, which is when $r=-q$, in the union.
4. Now $F = F \cap \mathbb{R} = \cup_{r\in\mathbb{Q}}(F \cap (E+r)) = \cup F_r$, it suffices to show $m(F_r) = 0$. Given that $m(F_r) = \sup\{m(K) : compact\ K \subset F_r\}$, this holds iff $\forall compact\ K \subset F_r, m(K) = 0$. We're going to use the fact that K is bounded.
5. Suppose not, i.e. there's a K, s.t. $m(K) > 0$. Due to to the same reason as 2, we have $\forall r_1, r_2 \in \mathbb{Q}, r_1 \neq r_2 \rightarrow (K+r_1) \cap (K+r_2) = \emptyset$. Note that it's still bounded after translation. Further more, let's bound the translation scale. Let $H = \cup_{r\in\mathbb{Q}\cap[0,1]}(K+r)$, which is a disjoint union of bounded set and should be bounded as a whole (within the union of $(-M_r, M_r)$). Yet since every summand in this σ-additivity (infinite) summation, $m(K+r) > 0$, we have $m(H) = \infty$, contradict.

> Ex. (Ch1 q30) If $E \in \mathcal{L}, m(E) < 0$, then $\forall \alpha < 1, \exists open\ interval\ I$, s.t. $m(E \cap I) > \alpha m(I)$.

Thm. (Steinhams' thm)

If $E \in \mathcal{L}, m(E) < 0$, the set $E - E = \{x - y : x, y \in E\}$ contains an interval centered at 0.

Proof.

1. Based on ex.ch1q30, let $F := E \cap I = (a, b)$, then $F - F = (E-E) \cap (I-I)$, therefore $F - F \subset E - E$. To show $(-\delta, \delta) \subset E - E$, it suffices to show that $\forall x, |x| < \delta, (F+x) \cap F \neq \emptyset$.
2. Suppose not, then $2m(F) > 2\alpha m(I)$ according to ex.ch1q30. OTAH, $2m(F) = m((F+x) \cup F) \leq m((I+x) \cup I) = b - a + |x| = b - a + \delta$, if $|x| < m(I)$. But if we let $\delta = (2\alpha - 1)(b-a)$, then $2m(F) \leq 2\alpha(b-a)$, contradicts.

Ch2 Integration

2.1 Measurable function

Motiv. $f^{-1}(E) = \{x \in X : f(x) \in E\}$ preserves unions, intersections, and complements on E. Thus $f^{-1}(\mathcal{B})$ and $\mathcal{H} = \{E \in \mathcal{B} : f^{-1}(E) \in \mathcal{A}\}$ are σ-algebra.

Def. (Measurable function)

Measurable spaces $(X, \mathcal{A}), (Y, \mathcal{B})$, we say $f : X \rightarrow Y$ is $(\mathcal{A}, \mathcal{B})$-measurable if $f^{-1}(\mathcal{B}) \subset \mathcal{A}$, or equivalently, $\mathcal{H} \supset \mathcal{B}$.

Rmk.

1. Random variables are special cases of measurable function.
2. Composition of measurable mappings are measurable.

3. For $E \in \mathcal{A}$, the indicator function $\mathbb{1}_E$ or $\mathcal{X}_E$ is $\mathcal{A}$-measurable.

> Prop. (2.1) $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable iff $\mathcal{H} \supset \mathcal{B}_0$, where $\mathcal{M}(B_0) = \mathcal{B}$.

Proof. $\mathcal{H}$ is a σ-algebra, then $\mathcal{H} \supset \mathcal{B}_0 \implies \mathcal{H} \supset \mathcal{M}(\mathcal{B}_0) = \mathcal{B}$ by prop1.1 (Real Analysis > ^db1186).

> Cor. Let X, Y be metric or topological spaces, then every continuous $f : X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measuable.

Proof. Recall that $f : X \to Y$ is continuous on X iff for any open set U in Y, $f^{-1}(U)$ is open in X. By prop 2.1, it follows.

Thm. (2.3)

Based on prop 1.2 and 2.1, T.F.A.E:

1. $f : X \to \mathbb{R}$ is $\mathcal{A}$-measuable.
2. $\{(a, \infty) : \forall a \in \mathbb{R}\} \subset \mathcal{H}$;
3. $\{[a, \infty) : \forall a \in \mathbb{R}\} \subset \mathcal{H}$;
4. $\{(-\infty, a) : \forall a \in \mathbb{R}\} \subset \mathcal{H}$;
5. $\{(-\infty, a] : \forall a \in \mathbb{R}\} \subset \mathcal{H}$;
6. (ch2q4) $\{(r, \infty) : \forall r \in \mathbb{Q}\} \subset \mathcal{H}$;
7. If $f : X \to \bar{\mathbb{R}}$, change the above intervals as including infinity.

> Ex. (ch2q1) $f : X \to \bar{\mathbb{R}}$ is measurable iff $f^{-1}(\{-\infty\}), f^{-1}(\{+\infty\}) \in \mathcal{A}$ and f is measurable on $f^{-1}(\mathbb{R})$.

> Ex. (ch2q3) $\{f_n\}$ measurable, then $\{x : \lim f_n(x) \text{ exists}\} \in \mathcal{M}$.

Def. (Measurable on subset of X)

f is measurable on $E \in \mathcal{M}$, if $f|_E$ is $\mathcal{M}_E$-measurable, i.e. $\forall B \in \mathcal{B}_{\mathbb{R}}, f^{-1}(B) \cap E \in \mathcal{M}$.

Prop. (2.4)

Let $(X, \mathcal{M})$ and $(Y_\alpha, \mathcal{N}_\alpha)$ be measurable spaces, and $Y := \prod_\alpha Y_\alpha, \mathcal{N} := \otimes \mathcal{N}_\alpha$, and π_α the coordinate maps. Then $f : X \to Y$ is measurable iff $f_\alpha := \pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable for all α.

Cor. $f : X \to \mathbb{C}$ is measurable iff Ref, Imf are measurable.

Thm. (2.6)

If $f, g : X \to \mathbb{C}$ are $\mathcal{A}$-measurable, then $f + g$, fg are $\mathcal{A}$-measurable.

Proof. Work on level sets. Consider $fg = \frac{1}{2}((f + g)^2 - f^2 - g^2)$.

Def.

$\mathcal{B}_{\bar{\mathbb{R}}} = \{E \subset \bar{\mathbb{R}} : E \cap \mathbb{R} \subset \mathcal{B}_{\mathbb{R}}\}.$

Prop. (2.7, 2.8)

In $\bar{\mathbb{R}}$, lim inf, lim sup exist. Let $\{f_j\}$ be a sequence of $\mathcal{A}$-measurable function, then $g_1 = \sup f_j, g_2 = \inf f_j, g_3 = \limsup f_j, g_4 = \liminf g_j$ are measurable. Thus, if f, g are measurable, then $\max(f, g), \min(f, g)$ are measurable.

Proof. $\limsup_j = \inf_k \sup_{j>k}$.

Rmk. We prefer to work on $\bar{\mathbb{R}}$ from now on since the limsup and liminf always exists.

Def. (Decomposition of real-valued func)

$f^+ := \max(f(x), 0), f^- := \max(-f(x), 0)$, then $f = f^+ - f^-, |f| = f^+ + f^-$. By Prop. 2.7, if f is measurable, so are f^+, f^-.

Def. (Polar decomposition of complex-valued func)

$f = (sgn \circ f)|f|$, where $sgn(z) = \frac{z}{|z|}\mathbb{1}(z \neq 0)$. If f is measurable, so are $|f|, sgn \circ f$.

Rmk. Note that $|sgn(z)| = 1$. Conversely, to decompose absolute value of complex number, consider $|z| = \frac{z \cdot \bar{z}}{|z|} = \frac{1}{sgn(z)} \cdot z$.

Proof. Note that the map $z \mapsto |z|$ is continuous except at the origin. Thus if $U \subset \mathbb{C}$ is open, $sgn^{-1}(U) \setminus \{0\}$ is open. By prop 2.1, sgn is measurable. Therefore $|f| = |\cdot| \circ f, sgn f = sgn \circ f$ are also measurable.

Def. (Simple function)

A finite linear combination of $\mathbb{1}_E$ of sets in A as the E, i.e.

$$f = \sum_{j=1}^{m} a_j \mathbb{1}_{E_j}, E_j \in \mathcal{A}$$

Prop. (Standard representation)

Any simply function can be written so that disjoint $\cup E_j = X$.

Proof. Simply function takes finite many values, say $z_1, z_2, \ldots z_n$. Let $E_j := f^{-1}(\{z_j\})$, which is disjoint. Let $E_0 = X \setminus (\cup E_j)$ and $a_0 = 0$, then $f(x) = \sum_{j=1}^{n} z_j \mathbb{1}_{E_j}(x)$ gives the existence of standard representation.

Thm. (2.10)

1. If f is $\mathcal{A}$-measurable, $f: X \to [0, \infty]$, then $\exists simple \ \{\phi_j\}$ s.t. $\phi_j \nearrow, \forall x, \phi_j(x) \to f(x)$, and $\phi_j \to f$ uniformly on any set on which f is bounded;
2. If f is $\mathcal{A}$-measurable, $f: X \to \mathbb{C}$, then $\exists simple \ \{\phi_j\}$ s.t. $|\phi_j| \nearrow, \forall x, \phi_j(x) \to f(x)$, and $\phi_j \to f$ uniformly on any set on which f is bounded.

Proof.

1. Consider n as a parameter to control a partition over the codomain. For any n, define **Dyatic intervals** $I_{k,n} := [k2^{-n}, (k+1)2^{-n})$ for $k = 0 \dots 2^{2n}$, and also let $F_n := [2^n, \infty]$. Then $[0, \infty] = (\cup_k I_{k,n}) \cup F_n$.
2. Define an approximation from below of f on each intervals, i.e. $E_{k,n} := f^{-1}(I_{k,n}), \phi_n(E_{k,n}) := k2^{-n}$. The same for F_n. Then in summary,

$$\phi_n = \sum_{k=1}^{2^{2n}-1} k2^{-n} \mathcal{X}_{E_{k,n}} + 2^n \mathcal{X}_{F_n}$$

3. Claim $\phi_n \le f$.
4. Claim $\phi_n \le \phi_{n+1}$.
5. Claim $\forall x \in [0, \infty] \setminus F_n, 0 \le f(x) - \phi_n(x) \le 2^{-n}$.
6. Claim $\forall x, \phi_n(x) \to f(x)$.
7. Claim $\forall A \subset X, s.t. f(A) \subset [0, \infty), \sup_{x \in A} |f - \phi_n| \to 0$.

Prop. (2.11)

Each of the following is valid iff μ is complete:

1. If f is measurable and $f = g$ a.e., then g is measurable;
2. If f_n is measurable and $f_n(x) \to f(x)$ a.e., then f is measurable.

Proof.

1. (1<=) Note that $\mu(\{x : f(x) \ne g(x)\}) = 0$. $\forall B \in \mathcal{B}_{\mathbb{R}}, g^{-1}(B) = \{x : g(x) \in B\}$, i.e., $(\{x : f(x) \in B\} \cap \{f = g\}) \cup (\{x : g(x) \in B\} \cap \{f \ne g\}) \in \mathcal{A}$;
2. (2<=) Note that $E := \{x : f_n(x) \to f(x) \text{ as } n \to \infty\}, \mu(E^c) = 0$. Then $f|_E$ is measurable given prop2.7 (Real Analysis > ^bfc106), thus $f|_E^{-1}(B) \cap E \in \mathcal{A}$. Apply the same trick of partition.
3. (1=>) For any null set E and its any subset F, construct $f := \mathcal{X}_E, g := 2\mathcal{X}_F$, then $\mu(\{f \ne g\}) = \mu(E) = 0$. Now g is measurable, then $F := g^{-1}(\{2\}) \in \mathcal{A}$.
4. (2=>) For any null set E and its any subset F, construct $f_n := 0, f := \mathcal{X}_F$, then $\mu(\{f_n(x) \to f(x)\}^c) = \mu(F) = 0$. Now g is measurable, then $F := g^{-1}(\{1\}) \in \mathcal{A}$.

2.2 Integration of non-negative func.

Def. (Integral)

1. $L^+ := \{\text{all } \mathcal{A} - \text{measurable } f : X \to [0, \infty]\}$;

2. For a simple func $\phi(x) = \sum_{j=1}^n a_j \mathbb{1}_{E_j}(x)$ with standard representation, define $\int_X \phi \, d\mu := \sum_{j=1}^n a_j \mu(E_j)$; (Here $0 \cdot \infty = 0$)
3. $\int_A \phi := \int_X \phi \mathcal{X}_A \, d\mu$;
4. $\int_X f \, d\mu := \sup_\phi \{\int_X \phi \, d\mu : 0 \leq \phi \leq f, \textit{simple } \phi\}$.

Prop. (2.13)

For simple functions ϕ, φ:

1. If $c > 0$, then $\int c\phi \, d\mu = c \int \phi \, d\mu$;
2. $\int(\phi + \varphi) = \int \phi + \int \varphi$;
3. If $\varphi \leq \phi$, then $\int \varphi \leq \int \phi$;
4. $v : A \mapsto \int_A \phi \, d\mu$ is a measure on $\mathcal{A}$.

Proof.

1. 3. Can be easily shown and easily extended to general L^+ function case.
2. In standard form, $\phi = \sum_{j=1}^n a_j \mathbb{1}_{E_j}, \varphi = \sum_{k=1}^m b_k \mathbb{1}_{F_k}$. Since $E_j = \cup_k (E_j \cap F_k)$, we can have $\phi = \sum_{j=1}^n a_j \sum_k \mathbb{1}_{E_j \cap F_k}$, and similarly for φ. Then

$$\begin{aligned} \int(\phi + \varphi) &= \int (\sum_{j,k} (a_j + b_k) \mathcal{X}_{E_j \cap F_k}) \\ &= \sum_{j,k} (a_j + b_k) \mu(E_j \cap F_k) = \int \phi + \int \varphi \end{aligned}$$

.

3. For disjoint $\{A_k\}$,

$$\begin{aligned} v(\cup_{k=1}^\infty A_k) = \int_{\cup A_k} \phi &= \sum_{j=1}^n a_j \mu(\cup A_k) = \sum_{j=1}^n a_j \sum_{k=1}^\infty \mu(A_k) \\ &= \sum_{k=1}^\infty (\sum_{j=1}^n a_j \mu(A_k)) = \sum_{k=1}^\infty \int_{A_k} \phi = \sum_{k=1}^\infty v(A_k) \end{aligned}$$

.

Thm. Monotone convergence thm (MCT)

If $f_n \in L^+, \{f_n\} \nearrow$ and $\forall x, f_n(x) \to f(x)$, then $\int f = \lim_{n \to \infty} \int f_n$.

Rmk. This relex the computation of $\int f$ from supremum of all simple func to a monotone sequence of func, which exists by thm 2.10 (Real Analysis > ^d33331).

Proof.

1. Note that $f(x) = \sup_n f_n(x)$. For any n, $\int f_n \leq \int f$, and $\{\int f_n\} \nearrow$ by prop2.13 (4), thus $\lim \int f_n \leq \int f$. To the other direction, try to take a fraction of RHS.

2. For simple func $\phi, s.t., 0 \leq \phi \leq f$, for any $\alpha \in (0,1)$, let $E_n = \{x : f_n(x) \geq \alpha\phi(x)\}$, we claim that (a) $X = \cup E_n$; (b) $E_n \nearrow$; (c) $v_\phi(X) := \int \phi = \lim \int_{E_n} \phi =: \lim v_\phi(E_n)$.
3. (c) is the continuity from below property of measure.
4. Then, $\lim_n \int f_n \geq \lim \int_{E_n} f_n \geq \lim \int_{E_n} \alpha\phi = \alpha \int \phi$. By taking limit on α, we get $\lim_n \int f_n \geq \int \phi$.
5. Take supremum on ϕ, $\lim_n \int f_n \geq \sup\{\int_X \phi \, \mathrm{d}\mu\} =: \int_X f \, \mathrm{d}\mu$.

> Cor. (2.15) $\{f_n\} \subset L^+$, then $\int f := \int \sum_n f_n = \sum_n \int f_n$.

Proof.

1. When there's only two, use prop 2.10 to get two increasing sequence of each, then use MCT to express w/ simply func, then use prop 2.13(2), then use MCT to transform simple func back to f.
2. Induction on n, then take limit with MCT on partial sum sequence to show infinite sum.

> Cor. (2.16) If $f \in L^+$, then $\int f = 0$ iff $f = 0$ a.e.

> Cor. (Null set contribute nothing) If $f \in L^+, \mu(E) = 0$, then $\int_E f = 0$.

(Fatou's lemma, equivalent to MCT)

If $\{f_n\} \subset L^+$, then $\int \liminf_n f_n \leq \liminf_n \int f_n$.

Proof. Recall $\liminf_n a_n := \lim_n(\inf_{k\geq n} a_k)$. Then $\forall n, \forall j \geq n, \inf_{k\geq n} f_k \leq f_j$, thus $\int \inf_{k\geq n} f_k \leq \int f_j$. Take infimum on j, we have $\int \inf_{k\geq n} \leq \inf_{j\geq n} \int f_j$. By MCT, $\int \liminf_n f_n = \lim_n \int \inf_{k\geq n} f_k \leq \lim_n \inf_{j\geq n} \int f_j = \liminf_n \int f_n$.

Rmk. (Show MCT using Fatou's lemma) If $f_n \in L^+, \{f_n\} \nearrow$ and $\forall x, f_n(x) \to f(x)$, then apply Fatou's lemma to $\{f_n\}$, we get $\int f = \int \liminf f_n \leq \liminf_n \int f_n$. OTAH, apply Fatou's lemma to $\{f - f_n\}$, we get $0 = \int \liminf(f - f_n) \leq \liminf_n \int(f - f_n) = \int f - \limsup_n \int f_n$, i.e. $\limsup \int f_n \leq \int f$. Thus $\{\int f_n\}_n$ converges to $\int f$.

> Cor. If $\{f_n\} \subset L^+, f \in L^+$, and $f_n(x) \to f(x)$ a.e., then $\int f \leq \liminf \int f_n$.

> Ex. (ch2q13) If $\{f_n\} \subset L^+, f_n(x) \to f(x)$ pointwise, and $\int f = \lim \int f_n < \infty$, then $\forall E \in \mathcal{M}, \int_E f = \lim \int_E f_n$. This need not be true if $\int f = \lim \int f_n = \infty$.

Proof.

1. By Fatou's lemma, $\int_E f = \int_E \liminf f_n \leq \liminf \int_E f_n$.
2. OTAH, note that we can also apply Fatou's lemma on E^c since $\int_{E^c} f = \int f - \int_E f$: $\int_{E^c} \liminf f_n \leq \liminf \int_{E^c} f_n$, then $LHS = \int f - \int_E f, RHS = \int f - \limsup \int_E f_n$, which gives $\int_E f \geq \limsup \int_E f_n$, thus $\{\int_E f_n\}_n$ converges to $\int_E f$.
3. Counter example on $\int f = \lim \int f_n = \infty$: consider $f_n := \mathcal{X}_{[n,n+1)} + \mathcal{X}_{(-\infty,0)}$, then $f := \mathcal{X}_{(-\infty,0)}; \forall x, f_n(x) \to f(x)$. Then $\int f = \lim \int f_n = \infty$, yet for $E = [0,\infty), \int_E f = 0, \lim \int_E f_n = 1$.

Prop. (2.20)

If $f \in L^+, \int f < \infty$, then $H := \{x : f(x) = \infty\}$ is a null set and $F := \{x : f(x) > 0\}$ is $\sigma-$finite.

Proof.

1. H is measurable. Suppose H is not null set, then $\int f \geq \int f \mathcal{X}_H = \infty \cdot m(H) = \infty$, contradict.
2. $F_n := \{x : f(x) > \frac{1}{n}\}$ is measurable and $F = \cup F_n$. Suppose $\mu(F_n) = \infty$, then again, $\int f \geq \int f \mathcal{X}_{F_n} \geq \frac{1}{n} \cdot m(F_n) = \infty$, contradict.

> Cor. If $f \in L^+, \int f < \infty$, then $\forall \epsilon > 0, \exists E \in \mathcal{A}$, s.t. $\mu(E) < \infty, \int_E f > (\int f) - \epsilon$.

2.3 Integration of complex func

Def. (Integral)

If either $\int f^+, \int f^- < \infty$, the integral is defined as $\int f := \int f^+ - \int f^-$.

Def. (Integrable)

$f : X \to \mathbb{C}$ is integrable if $\int |f| < \infty$. The set (vector space) of integrable function is called $L^1(X, \mu)$.

Thm.

If $f, g \in L^1$, then

1. (Linearity) $\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$;
2. $|f| \leq |g|\ a.e. \implies \int |f| \leq \int |g|$;
3. $\forall A \in \mathcal{M}, \lambda_f : A \mapsto \int_A |f| \, \mathrm{d}\mu$ is a measure.

Prop. (2.22)

$|\int f| \leq \int |f|$.

Proof.

1. In the case of $f : X \to \mathbb{R}$, $|\int f| = |\int f^+ - \int f^-| \leq |\int f^+ + |\int f^-| = \int |f|$.
2. In the case of $f : X \to \mathbb{C}$, use polar decomposition. $|\int f| = \frac{1}{sgn(\int f)}(\int f) := \int \alpha f$, where $\alpha = \frac{1}{sgn(\int f)}, |\alpha| = 1$. Since $|\int f| \in \bar{\mathbb{R}}$, we know $\int \alpha f = Re(\int \alpha f) = \int Re(\alpha f) \leq \int |\alpha f| = \int |\alpha||f| = \int |f|$.

> Cor. (Null set contributes nothing) If $f \in L^1, \mu(E) = 0$, then $\int_E f = 0$. Follows from $|\int_E f| \leq \int_E |f| = 0$.

> Prop. (2.23)

> 1. If $f \in L^1$, then $F := \{x : f(x) \neq 0\}$ is $\sigma-$finite.
> 2. $(\forall E \in \mathcal{A}, \int_E f = \int_E g) \iff \int |f-g| = 0 \iff f = g\, a.e.$

Rmk. For the purposes of integration, it makes no difference if we alter functions on null sets. Thus we can treat $\bar{\mathbb{R}}$-valued function that are finite a.e. as real-valued funtions. It's more convenient to redefine L^1 as the equivalence classes of a.e.-defined integrable func, where f and g are considered equivalent iff $f = g$ a.e. This new vector space with distance $\rho(f,g) = \int |f-g|$ is a metric space.

Proof.

1. Follows from prop 2.20(Real Analysis > ^49150d);
2. The second equivalence follows from prop 2.16;
3. (<=) $|\int_E f - \int_E g| \leq \mathcal{X}_E |f-g| = 0$;
4. (=>) It suffices to show under real-valued. $\forall E \in \mathcal{M}, \int_E (f-g) = 0$. Suppose $f = g$ not a.e., then one of $(f-g)^+, (f-g)^-$ is nonzero on a set of positive measure. Let E be that set, then $\int_E (f-g) \neq 0$. Contradict.

> **Thm. (Dominated convergence thm, DCT)**
>
> $\{f_n\} \subset L^1 s.t.$
>
> 1. $f_n(x) \to f(x)$ a.e.;
> 2. $\exists g \in L^1, g : X \to \mathbb{R}, s.t., \forall n, |f_n| \leq g$ a.e.;
>
> Then $\int f = \lim \int f_n$.

Proof.

1. $\int |f| \leq \int g < \infty$, thus $f \in L^1$;
2. Case1: $f_n : X \to \bar{\mathbb{R}}$. Since $|f_n| \leq g$, we have $f_n + g, g - f_n \geq 0$ a.e. Applying Fatou's lemma on them gives two inequality: $\int f \leq \liminf \int f_n$ and $\int f \geq \limsup f_n$.
3. Case2: $f_n : X \to \mathbb{C}$. Trivial.

> Ex. (2.3.20, **Generalized DCT**) If $f_n, g_n, f, g \in L^1$, $f_n \to f$ a.e., $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$. Proof is similar to DCT.

> Ex. (2.3.21) If $f_n, f \in L^1$, $f_n \stackrel{a.e.}{\to} f$, then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$. Based on ex2.3.20.

> Thm. (2.25, **Exchange summation and integral**) $\{f_n\} \subset L^1, s.t. \sum \int |f_j| < \infty$, then $\exists f \in L^1, \sum f_j \to f$ a.e. and $\int f = \sum \int f_j$.

Proof. With cor2.15(Real Analysis > ^12cbd9), $g := \sum |f_j|, \int g = \sum \int |f_j| < \infty$, so $g \in L^1$. With prop2.20(Real Analysis > ^49150d), g is finite a.e., thus the series $\sum f_j(x)$ converges a.e. with bound g. Apply DCT.

> Thm. (2.26, **Simple functions are dense in** L^1)
>
> 1. Let $f \in L^1, \forall \epsilon > 0, \exists simple := \sum a_j \mathcal{X}_{E_j}\ \phi, s.t. ||f - \phi||_1 \leq \epsilon$;
> 2. If μ is a L-S measure on $\mathbb{R}$, then the sets E_j can be taken to be finite unions of open intervals;

> 3. Moreover, there is a continuous func g that vanishes outside a bounded interval such that $\int |f - g| \, \mathrm{d}\mu < \epsilon$.

Proof.

1. With prop2.10(Real Analysis > ^d33331), $\exists simple \{\phi_j\}$ s.t. $|\phi_j| \nearrow, \forall x, \phi_j(x) \to f(x)$. Then $|f - \phi_n| \leq |f| + |\phi_n| \leq 2|f|$, thus $f - \phi_n \in L^1$. By DCT, $\lim \int (f - \phi_n) = \int 0 = 0$.
2. If $E, F \in \mathcal{M}$, then $\mu(E \triangle F) = \int |\mathcal{X}_E - \mathcal{X}_F|$. With prop1.20(Real Analysis > ^47ce71), that means when $\mu(E_j) < \infty$, we can approximate $\mathcal{X}_{E_j}$ with a finite sum of functions $\mathcal{X}_{I_k}$, where I_k's are open intervals, arbitrarily close in the L^1 metric. In this case, $\mu(E_j) = |a_j|^{-1} \int_{E_j} |\phi_n| \leq \infty$. The ϵ_j's for each E_j can be chosen as 2^{-j}.
3. We can approximate $\mathcal{X}_{I_k}$ with continuous func in the L^1 metric. #TODO

Thm. (**Exchange differentiation and integral**) #NotCovered

Def. (Riemann sum)

Let $\{x_0, x_1, x_2 \dots\}, a = x_0 < x_1 \dots < x_n = b$ be a partition, with mesh $||P|| := \max_{j=1}^n (x_j - x_{j-1})$. Define upperbound func $U_P = \sum_{j=1}^n \sup_{x \in I_j} f \cdot \mathcal{X}_{I_j}$, where $I_j := [x_{j-1}, x_j)$. Then the **Riemann sum** is $\sum_{j=1}^n \sup_{x \in I_j} f \cdot (x_j - x_{j-1}) = \int_{[a,b]} U_P \, \mathrm{d}m =: U(f, P)$, and similarly $L(f, P)$.

Def. (Riemann integral)

f is **Riemann integrable** if for any sequence of partition $P_1 \subset P_2 \dots$ with $\lim_n ||P_n|| = 0$, we have $L(f) := \lim L(f, P_n) = \lim U(f, P_n) =: U(f)$. And $\int_a^b f \, \mathrm{d}x := L(f)$.

Thm. (Riemann integrable)

Let $f : [a, b] \to \mathbb{R}$ be bounded, $D_f := \{x : f \text{ is discontinuous at } x\}$, then

1. If f is Riemann integrable, then it is Lebesgue measurable;
2. f is Riemann integrable iff $m(D_f) = 0$.

Proof.

1. f is Riemann integrable iff $\forall \{P_n\}, \lim_n ||P_n|| = 0 \to U(f) = L(f)$.
2. Note that $L_{P_1} \leq L_{P_2} \dots \leq f \leq \dots \leq U_{P_2} \leq U_{P_1}$. Let $L(x) = \lim_n L_{P_n}(x)$ be the pointwise limit due to monotone sequence. Then $L \leq f \leq U$. Since $|L| \leq |U| \leq \sup |f(x)| < \infty$, by DCT, $\int L = \int \lim L_{P_n} = \lim \int L_{P_n} = \lim L(f, P_n) = L(f)$. Therefore, Riemann integrable iff $L(f) = U(f) = \int f$ iff $\int L = \int U$ iff $L = U$ a.e.
3. If Riemann integrable, then $L = f$ a.e since $m(L \neq f) \leq m(L \neq U) = 0$. Then f is measurable according to prop2.11.
4. Define $\omega_f(A) := \sup_{x \in A} f(x) - \inf_{x \in A} f(x)$ the **oscillation** of f on A. $\lim_{\delta \to 0} \omega_f(x_0 - \delta, x_0 + \delta) := \lim G(\delta) := \Omega_f^{(x_0)}$, where $G(\delta) \searrow \ as\ \delta \to 0$. We claim that f is continuous at x_0 iff $\Omega_f^{(x_0)} = 0$.

5. Let $x \in [a,b] \setminus (\cup_{n=1}^{\infty} P_n)$, define $I_n(x) := (x_{j-1}, x_j)$ s.t. $x \in I_n(x)$. Note that $m(\cup_{n=1}^{\infty} P_n) = 0$. Then $\Omega_f^{(x_0)} = \lim_n \omega_f(I_n(x)) = \lim(U_{P_n}(x) - L_{P_n}(x)) = U(x) - L(x)$. Then $D_f = \{x : \Omega_f^{(x_0)} \neq 0\} = \{U \neq L\}$.

E.g. The Cantor set $\mathcal{C}$. Then both $\mathcal{X}_{\mathbb{Q}}$ and $\mathcal{X}_{\mathcal{C}}$ are Riemann integrable since both are discontinuous everywhere and $\int_{[0,1]} \mathcal{X}_{\mathbb{Q}} = 0 = \mathcal{X}_{\mathcal{C}}$.

> Ex (Ch2q26). Show that if $f \in L^1(\mathbb{R}, \mathcal{L}, m)$, $F(x) = \int_{-\infty}^{x} f(t)\, \mathrm{d}t$, then F is continuous on $\mathbb{R}$.

Proof. F cts on $\mathbb{R}$ iff $\forall \{x_n\} \subset \mathbb{R}$, #TODO

Def. (Gamma function Γ)

$$\Gamma(z) := \int_{(0,\infty)} t^{z-1} e^{-t}\, \mathrm{d}t,$$

where $z \in \mathbb{C}, Re(z) > 0, t^{z-1} := \exp[(z-1)\log t]$.

Rmk.

1. $f_z(t) := t^{z-1}e^{-t}, |f_z(t)| \leq |t^{z-1}| = t^{Re(z)-1}$ and $\forall t \geq 1, |f_z(t)| \leq C_z \exp(\frac{-t}{2})$. For $a > -1, \int_{(0,1)} t^a\, \mathrm{d}t < \infty$; also $\int_1^{\infty} \exp(\frac{-t}{2}) < \infty$. Thus $f_z \in L^1((0,\infty))$ for $Re(z) > 0$.
2. $\forall z \in \mathbb{C}, Re(z) > 0 \to \Gamma(z+1) = z\Gamma(z)$;
3. Use (2) to extend. By induction on n, define $\Gamma(z) := \Gamma(z+1)/z$ for $Re(z) > -n-1$;
4. $\Gamma(1) = 1, \Gamma(n+1) = n!$.

2.4 Modes of convergence

Thm.

1. $(f_n \rightrightarrows f) \implies (f_n \to f) \implies (f_n \xrightarrow{a.e.} f)$;
2. $(f_n \xrightarrow{L_1} f) \implies (f_n \xrightarrow{\mu} f)$;
3. $(f_n \xrightarrow{L_r} f) \implies (f_n \xrightarrow{L_s} f)$ if $r \geq s \geq 1$.
4. If $f_n \xrightarrow{a.e.} f, |f_n| \leq g \in L^1$, then $f_n \xrightarrow{1} f$.

Proof. #TODO

Def. (Converge in measure)

$\forall \epsilon > 0, \lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| > \epsilon\}) = 0$.

Def. (Cauchy in measure)

$\forall \sigma, \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m > N, \mu(\{x : |f_n(x) - f_m(x)| > \epsilon\}) < \sigma$. It's usually denoted as $\mu(\{x : |f_n(x) - f_m(x)| > \epsilon\} \to 0$ as $m, n \to \infty$.

♁ **Thm.**

Suppose $\{f_n\}$ is Cauchy in measure, then

1. $\exists measurable\ f, s.t., f_n \xrightarrow{\mu} f$;
2. There is a subsequence $\{f_{n_j}\} s.t.\ f_{n_j} \xrightarrow{a.e.} f$;
3. If also $f_n \xrightarrow{\mu} g$, then $g = f$ a.e.

Proof.

1. Given Cauchy in measure, pick subsequence s.t. if $E_j := \{x : |f_{n_j}(x) - f_{n_{j+1}}(x)| > 2^{-j}\}$, then $\mu(E_j) < 2^{-j}$. Let $F_k := \cup_{j=k}^{\infty} E_j$, then $\mu(F_k) < 2^{1-k}$ by subadditivity. OTAH, for $x \notin F_k, i \geq j \geq k, |f_{n_i}(x) - f_{n_j}(x)| \leq 2^{1-j} \leq 2^{1-k} (*)$. Thus $\{f_{n_j}\}_{j=k}^{\infty}$ is pointwise Cauchy and therefore convergent on F_k^c.
2. Let $F := \cap F_k = \limsup E_j$, then $\mu(F) = 0$. If we take $f(x) := \mathcal{X}_{F^c} \lim_j f_{n_j}(x)$, then f is measurable and $f_{n_j} \to f$ a.e. (the 2)
3. Then for $x \in F^c, \exists N, \forall j > N, |f_{n_j}(x) - f(x)| \leq 2^{1-j}$. Then $f_{n_j} \xrightarrow{\mu} f$. #TODO
4. Since $\{|f_n - f| \geq \epsilon\} \subset \{|f_n - f_{n_j}| \geq \frac{\epsilon}{2}\} \cup \{|f_{n_j} - f| \geq \frac{\epsilon}{2}\}$, and the second term is small due to Cauchy in measure, we get $f_n \xrightarrow{\mu} f$ (the 1);
5. If also $f_n \xrightarrow{\mu} g$, and $\{|f - g| \geq \epsilon\} \subset \{|f - f_n| \geq \frac{\epsilon}{2}\} \cup \{|f_n - g| \geq \frac{\epsilon}{2}\}$, then $\forall \epsilon > 0, \mu(\{|f - g| > \epsilon\}) = 0$. Let $\epsilon \to 0$ using some sequence, then $f = g$ a.e. (the 3)

♁ **Cor. (Cauchy Criterion)**

$\{f_n\}$ is Cauchy in measure iff $\exists measurable\ f, s.t., f_n \xrightarrow{\mu} f$.

Proof. (<=) $\{|f_n - f_m| \geq \epsilon\} \subset \{|f_n - f| \geq \frac{\epsilon}{2}\} \cup \{|f - f_m| \geq \frac{\epsilon}{2}\}$, take $n, m \to \infty$, and the two measures on RHS diminish.

♁ **Cor.**

If $f_n \xrightarrow{L_1} f$, then there is a subsequence $\{f_{n_j}\} s.t.\ f_{n_j} \xrightarrow{a.e.} f$.

> Ex. (ch2q33) Replace the condition of "$f := \liminf f_n$" into "$f_n \xrightarrow{\mu} f$" in Fatou's lemma, it still holds.

✎ **(Almost uniform convergence)**

$f_n \overset{a}{\rightrightarrows} f$ on F if $\forall \epsilon > 0, \exists E \subset F, s.t., \mu(F \setminus E) < \epsilon, f_n \rightrightarrows f$ on E.

♁ **Thm. (Egorov's thm)**

$\mu(X) < \infty, f_n(x) \to f(x)$ a.e. on X, then $f_n \overset{a}{\rightrightarrows} f$ on X.

Proof.

1. $\forall \epsilon > 0$, we want to find $E \subset F$. It suffices to find $E \subset \{x \in F : f_n(x) \to f(x) \text{ as } n \to \infty\}$, since the difference is a null set. Then WLOG, $f_n(x) \to f(x)$ everywhere.
2. Define $E_n(k) := \cup_{m=n}^{\infty}\{x : |f_m(x) - f(x)| > \frac{1}{k}\} =: \cup_{m=n}^{\infty} F_m(k)$.
3. Claim: $\forall k, n, E_n(k) \supset E_{n+1}(k)$.
4. Claim: $\cap_n E_n(k) = \limsup_m F_m = \emptyset$. Otherwise, $\exists x \in \cap_n \cup_{m=n}^{\infty} F_m(k)$, i.e., $\forall n, \exists m \geq n, |f_m(x) - f(x)| > \frac{1}{k}$, i.e. $\exists \{m_t\}_{t=1}^{\infty}, s.t. |f_{m_t}(x) - f(x)| > \frac{1}{k}$. Contradict with (1).
5. Since $\mu(E_1(k)) \leq \mu(X) < \infty$, by continuity from above, $\mu(E_n(k)) \to \mu(\emptyset) = 0$.
6. Then let k varies, then by definition, $\forall k, \forall \epsilon > 0, \exists N_k \in \mathbb{N}$, s.t. $\forall n \geq N_k, \mu(E_n(k)) < \epsilon 2^{-k}$. Thus $\forall \epsilon > 0, H := \cup_{k=1}^{\infty} E_{N_k}(k), \mu(H) < \epsilon$. The goal is to show $f_n \overset{\rightarrow}{\rightarrow} f$ on $E := H^c$.
7. $\forall x \in H^c, \forall k, x \in E_{N_k}^c(k)$. Then $\forall m \geq N_k, x \in F_m(k), i.e. |f_m(x) - f(x)| \leq \frac{1}{k}$. Thus $\forall k, \forall m \geq N_k, \sup_{H^c} |f_m - f| \leq \frac{1}{k}$. Let $k \to \infty$, then it follows.

Rmk. #TODO

> Ex. (ch2q40) The condition "$\mu(X) < \infty$" in Egorov's thm can be changed into "$|f_n| \leq g \in L^1$".

Proof.

1. Step 1-4 remain valid. If we can fix step 5, then the following steps are still valid.
2. Note that $y \in E_1(k) \iff \exists m \geq 1, |f_m(y) - f(y)| > \frac{1}{k}$, which is $\sup_m |f_m(y) - f(y)| > \frac{1}{k}$. OTAH, $2g(y) \geq \sup_m |f_m(y) - f(y)|$. Then $E_1(k) \subset \{2g(x) > \frac{1}{k}\}$.
$\mu(E_1(k)) \leq \mu(\{2g(x) > \frac{1}{k}\}) \leq 2k \int g < \infty$.

Lemma. (Measurable function is almost simple func)

$f : E \to \mathbb{C}, \mu(E) < \infty$, then $\forall \epsilon > 0, \exists simple\ \phi, \exists F \subset E, F \in \mathcal{M}$, s.t. $\mu(E \setminus F) < \epsilon, \forall x \in F, |f(x) - \phi(x)| < \epsilon$.

Proof. #TODO

Cor. (Almost everywhere bounded)

$f : E \to \mathbb{C}, \mu(E) < \infty$, then $\forall \epsilon > 0, \exists M \in \mathbb{R}, E \in \mathcal{M}$, s.t., $\mu(E^c) < \epsilon, |f(x)| < M$ on E.

Proof. Lemma, and then $|f(x)| \leq |f(x) - \phi(x)| + |\phi(x)| =: M$.

Lemma. (cts func are dense in L^1)

μ is Lebesgue-Stieltjes measure, $f : E \to \mathbb{C}$ where $\mu(E) < \infty$. Then $\forall \epsilon > 0, \exists$ continuous func $g, F \in \mathcal{M}, F \subset E$, s.t. $m(E \setminus F) < \epsilon, ||f - g||_1 < \epsilon$.

Proof. #TODO thm2.26

Thm. (Lusin's thm)

μ is Lebesgue-Stieltjes measure, $f : E \to \mathbb{C}$ where $\mu(E) < \infty$. Then $\forall \epsilon > 0, \exists$ continuous func $g, F \in \mathcal{M}, F \subset E$, s.t. $m(E \setminus F) < \epsilon, \forall x, |f(x) - g(x)| < \epsilon$.

Proof. #TODO

Rmk. (Littlewood's three principles of real analysis)

1. Every measurable set in $\mathbb{R}$ is nearly a finite union of intervals (prop1.20 #TODO);
2. Every function (of class Lp) on $\mathbb{R}$ is nearly continuous (Egorov's);
3. Every convergent sequence of measurable functions on finite measure set is nearly uniformly convergent (Lusin's).

2.5 Product measure

Def. (Rectangle)

Consider measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, v)$, a rectangle is in the form $A \times B := \{(x, y) : x \in A, y \in B\}$, where $A \in \mathcal{A}, B \in \mathcal{B}$. Let $\mathcal{R}_0$ be the collection of **finite disjoint union of rectangles**.

Rmk.

1. $(A \times B) \cap (E \times F) = (A \cap E) \times (B \cap F)$ is still rectangle;
2. $(A \times B)^c = (X \times B^c) \cup (A^c \times B) \in \mathcal{R}_0$;
3. The rectangle set is a elementary family;
4. By prop1.7(Real Analysis > ^668d67), $\mathcal{R}_0$ is an algebra that can generate the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$, which is defined in ch1.2 (Real Analysis > ^f3e312).

> Prop. (**Product measure**)
>
> 1. $\forall E = \cup_{j=1}^m (A_j \times B_j) \in \mathcal{R}_0$, define $\pi(E) := \sum_{j=1}^m \mu(A_j) v(B_j)$. (Again, $0 \cdot \infty = 0$)
> 2. $\pi(E)$ is well-defined;
> 3. $\pi(E)$ is a premeasure;
> 4. The induced outermeasure $\pi^*(E) := \inf_{\{R_j\}} \{\sum_{j=1}^\infty \pi(R_j) : R_j \in \mathcal{R}_0, \cup R_j \supset E\}$;
> 5. The **product measure** $\mu \times v$ on $\mathcal{A} \otimes \mathcal{B}$ can be defined as $\pi^*|_{\mathcal{A} \otimes \mathcal{B}}$.

> Prop. If μ, v are σ-finite, then 1) $\mu \times v$ is σ-finite; 2) therefore, by thm1.14 (Real Analysis > ^63f1c6), $\mu \times v$ is the unique one among those that extend $\pi|_{\mathcal{R}_0}$.

#TODO extend results in 2.1-2.3

Def.

1. x-session of E: $E_x := \{y \in Y : (x, y) \in E\}$, y-session of E: E^y.
2. x-session of f: $f_x := y \mapsto f(x, y)$, y-session of f: f^y.
3. In other word, say $f : E \to H$, then $f_x : E_x \to H$.

> Prop.
>
> 1. If $E \in \mathcal{A} \otimes \mathcal{B}$, then $E_x \in \mathcal{B}, E^y \in \mathcal{A}$;
> 2. If f is $\mathcal{A} \otimes \mathcal{B}$-measurable, then f_x is $\mathcal{B}$-measurable, f^y is $\mathcal{A}$-measurable.

Proof. #TODO

Def. (Monotone class)

$\mathcal{C} \subset P(X)$, s.t.,

1. Closed under countable increasing union;
2. Closed under countable decreasing intersection.

The motivation is to make use of the continuity of measure.

Lemma. (The monotone class lemma)

If $\mathcal{A}$ is an algebra, then the generated monotone class coincides with the generated σ-algebra.

Proof. #TODO

> Prop. If μ_1, μ_2 are finite measure, $E \in \mathcal{A} \otimes \mathcal{B}$, and $f(x) = \mu_2(E_x)$ are $\mathcal{A}$-measurable, $g(x) = \mu_1(E^y)$ are $\mathcal{B}$-measurable, then
>
> $$\begin{aligned}(\mu_1 \times \mu_2)(E) &= \int_X \mu_2(E) \,\mathrm{d}\mu_1 = \int_X \left(\int_Y \mathcal{X}_{E_x} \,\mathrm{d}\mu_2 \right) \mathrm{d}\mu_1 \\ &= \int_Y \mu_1(E) \,\mathrm{d}\mu_2 = \int_Y \left(\int_X \mathcal{X}_{E^y} \,\mathrm{d}\mu_1 \right) \mathrm{d}\mu_2\end{aligned}$$

Proof. #TODO

> Prop. If μ_1, μ_2 are σ-finite measure, $E \in \mathcal{A} \otimes \mathcal{B}$, and $f := x \mapsto \mu_2(E_x)$ are $\mathcal{A}$-measurable, $g := y \mapsto \mu_1(E^y)$ are $\mathcal{B}$-measurable, then
>
> $$\begin{aligned}(\mu_1 \times \mu_2)(E) &= \int_X \mu_2(E) \,\mathrm{d}\mu_1 = \int_X \left(\int_Y \mathcal{X}_{E_x} \,\mathrm{d}\mu_2 \right) \mathrm{d}\mu_1 \\ &= \int_Y \mu_1(E) \,\mathrm{d}\mu_2 = \int_Y \left(\int_X \mathcal{X}_{E^y} \,\mathrm{d}\mu_1 \right) \mathrm{d}\mu_2\end{aligned}$$

Proof. #TODO

Thm. (Funibi-Tonelli thm)

If μ_1, μ_2 are σ-finite measure,

1. (Tonelli) If $f \in L^+(X \times Y)$, then $x \mapsto \int f_x \,\mathrm{d}\mu_2 \in L^+(X), y \mapsto \int f^y \,\mathrm{d}\mu_1 \in L^+(Y)$, and

$$\begin{aligned}\int f \,\mathrm{d}(\mu_1 \times \mu_2) &= \int_X \mu_2(E) \,\mathrm{d}\mu_1 = \int_X \left(\int_Y \mathcal{X}_{E_x} \,\mathrm{d}\mu_2 \right) \mathrm{d}\mu_1 \\ &= \int_Y \mu_1(E) \,\mathrm{d}\mu_2 = \int_Y \left(\int_X \mathcal{X}_{E^y} \,\mathrm{d}\mu_1 \right) \mathrm{d}\mu_2\end{aligned}$$

2. (Fubini) #TODO

E.g. Show that $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$. Note that $\frac{1}{x} = \int_0^\infty \exp(-xt)\,\mathrm{d}t, \forall x \geq 0$. #TODO

2.6 Lebesgue Differential Thm, LDT

In this part, remember that n refers to the dimension. We may use c_n to denote a constant that only depends on n.

Def.

1. (**Locally integrable**) $f : \mathbb{R}^n \to \mathbb{C}, s.t., \int_K |f|\,\mathrm{d}m < \infty$ for any compact K. The set of it is denoted as $L^1_{loc}(\mathbb{R}^n)$, which is a superset of $L^1(\mathbb{R}^n)$;
2. (**Hardy-littlewood max function**) $M_f := x \mapsto \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f|\,\mathrm{d}m$;
3. (**Variants**) **Uncentered max func** $\tilde{M}_f := x \mapsto \sup_B\{\frac{1}{m(B)} \int_B |f|\,\mathrm{d}m : x \in B\}$. **Rectangle version** $M_f^* := x \mapsto \sup_R\{\cdots : x \in R, R \text{ is rectangle with any direction}\}$. **Kakeya/Besicovitch set**: a set containing unit line segments pointing all possible directions.

Rmk.

1. $m(B_r(x)) = r^n \cdot m(B_1(0)) =: c_n \cdot m(B_1(0))$;
2. $M_f \leq \tilde{M}_f \leq 2^n M_f$;
3. There is Kakeya set with zero measure, resulting failure of Vitali Covering lemma, thus LDT, on the rectangle version.

> Lemma. (**Vitali Covering lemma**) Suppose measurable $E \subset \cup_{\alpha\in A} B_\alpha$, where $\sup_{\alpha\in A} r(B_\alpha) < \infty$. Then there is a disjoint countable subcollection $\alpha_1, \ldots \alpha_k \ldots$, s.t. $m(E) \leq c_n \sum_{k=1}^\infty m(B_{\alpha_k})$.

Proof.

1. When RHS is infinite, we are done. WLOG, $\sum_{k=1}^\infty m(B_{\alpha_k}) < \infty$, which means $\lim_{k\to\infty} m(B_{\alpha_k}) = 0$, i.e. $\lim_{k\to\infty} r(B_{\alpha_k}) = 0$.
2. It sufficient to show that $\forall \alpha, B_\alpha \subset \cup_{k=1}^\infty 5B_{\alpha_k}$. Here the dilution of ball $5B_r(x) := B_{5r}(x)$. The heuristic is that we keep choosing the largest available ball and then cross out all overlapping balls to get a disjoint collection. Then we need to show the dilution of these balls can still cover the whole set;
3. The largest ball may not exists. An operational construction makes use of supremum. Pick α_{k+1}, s.t. $r(B_{\alpha_{k+1}}) > \frac{1}{2}\sup_\beta\{r(B_\beta) : B_\beta \cap (\cup_{j=1}^k B_{\alpha_j}) = \emptyset\}$, which is finite.
4. Now $\forall \alpha$, find the first $k, s.t. r(B_{\alpha_{k+1}}) < \frac{1}{2} r(B_\alpha)$, i.e. $\forall j \leq k, r(B_{\alpha_{k+1}}) \geq \frac{1}{2} r(B_\alpha)$. If we can find an overlaping one, we are done, because the dilution of it will cover B_α.
5. Note that B_α is itself not a potential option for supremum when getting α_{k+1}, otherwise won't get a smaller one: $B_{\alpha_{k+1}}$. Hence we know $B_\alpha \cap (\cup_{j=1}^k B_{\alpha_j}) \neq \emptyset$. Find any $j, s.t. B_\alpha \cap B_{\alpha_j} \neq \emptyset$,then we are done.

Thm.

The H-L max func is of weak(1,1). In other word, let $E_\lambda := \{x \in \mathbb{R}^n : M_f(x) > \lambda\}$, then $\exists c_n \in \mathbb{R}^{>0}, \forall \lambda > 0, \forall f \in L^1, m(E_\lambda) \leq \frac{c_n \cdot ||f||_1}{\lambda}$.

Proof.

1. $\forall x \in E_\lambda, M_f(x) := \sup_r \frac{1}{m(B_x(r))} \int_{B_x(r)} |f| > \lambda$. Then $\exists B_x, s.t. \frac{1}{m(B_x)} \int_{B_x} |f| > \lambda$ and $E_\lambda \subset \cup_{x \in E_\lambda} B_x$.
2. Note that $m(B_x) \leq \frac{1}{\lambda} \int_{B_x} |f| \leq \frac{||f||_1}{\lambda} < \infty$, we can take supremum. Then by covering lemma, $m(E_\lambda) \leq c_n \sum_{k=1}^{\infty} m(B_{x_k}) \leq c_n \sum_{k=1}^{\infty} \frac{1}{\lambda} \int_{B_x} |f| = \frac{c_n}{\lambda} \int_{\cup B_x} |f| \leq c_n \frac{||f||_1}{\lambda}$.

Thm. (Lebesgue Differential Thm, LDT)

For $f \in L^1_{loc}(\mathbb{R}^n)$, $\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f \, \mathrm{d}m = f(x)$ a.e.

Proof. #TODO

> Cor.
>
> 1. **Lebesgue set** $L_f := \{x : \lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(y)| \, \mathrm{d}y = 0\}$, then $\mu(L_f^c) = 0$;
> 2. For n=1, $\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(y) \, \mathrm{d}y = f(x)$ a.e. When f is continuous, we can drop "a.e", and get the fundamental thm of calculus.