Math 500 - Topology and Geometry

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Introduction

Math 500 is a Masters-level first-course in Topology and Geometry. The course follows James Munkres' *Topology, 2ed.* and this set of notes is based on the Fall 2017 offering.

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Please email any corrections or suggestions to ianzach+notes@seas.upenn.edu.

Lecture 1 (2017-08-30)

What is Topology?

Definition. A **topology** is a set X with a collection of subsets $\mathcal{A} \subset \mathcal{P}(X)$ such that:

1 $\varnothing, X \in \mathcal{A}$

2 $\mathcal{A}$ is closed under finite intersection (the intersection of a finite subset of $\mathcal{A}$ is in $\mathcal{A}$)

3 $\mathcal{A}$ is closed under arbitrary union (the union of any (possibly infinite) subset of $\mathcal{A}$ is in $\mathcal{A}$)

The phrases "$\mathcal{A}$ is a topology on X", "X is a topological space with topology $\mathcal{A}$, and the notation $(X, \mathcal{A})$ all refer to the same concept.

Definition. The **standard topology** (also called the euclidean topology or metric topology) on $\mathbb{R}^n$ is the set of subsets $U \subset \mathbb{R}^n$ such that for every U, every point $x \in U$ is interior, meaning that there exists some radius $r > 0$ such that the ball of radius r centered at x is entirely contained in U.

Definition. A set is **open** in a topological space X if it belongs to the topology on X.

Example. The standard topology is a topology over $\mathbb{R}^n$:

1 Every point in the empty set is vacuously interior, and every point of $\mathbb{R}^n$ is trivially interior

2 If we take two open sets and intersect them, any point in the intersection must be an interior point in both constituent sets. The smaller of the two balls witnessing this must lie entirely within both constituent sets, and therefore entirely within the intersection. By induction, we have the finite intersection of open sets being open.

3 Intuitively, taking any union of open sets only creates a bigger set. The ball witnessing any point as interior to some open set clearly lies in any union including that open set.

We can see from this example why it's important to specify closure under *finite* intersection. Singleton sets are not open in the standard topology on $\mathbb{R}^n$, but the Nested Interval Theorem gives us a way to construct a singleton set from the countable intersection of open intervals.

Example. If X is our topological space, $\{\varnothing, X\}$ is a topology, called the **trivial topology**.

Example. Similarly, all of $\mathcal{P}(X)$ is a topology, called the **discrete topology**.

Example. The **Zariski topology** on $\mathbb{R}^n$ is a little more interesting. A set is open in the Zariski topology if it is the complement of the root set of some polynomial. Open sets in the one-dimensional case look like the real line minus a finite number of points. It gets a little more complicated in higher dimensions, as we can have zeroes along entire dimensions of a euclidean space. Let's verify that this is a topology:

1 The empty set is the complement of the root set of the zero function, and the entire space $\mathbb{R}^n$ is the complement of the root set of a polynomial which has no real roots, such as $f(\vec{x}) = 6$.

2 The intersection of two open sets, corresponding to polynomials P and Q is, by DeMorgan's Laws, $\mathbb{R}^n \setminus \{x \mid x \textit{ is a root of } P \textit{ or } Q\}$. Something is a root of P or Q, it must be a root of the product PQ. Since the finite product of polynomials is a polynomial, this set is still the complement of the root set of some polynomial, and is therefore open, and we have closure under finite intersection.

3 Again by DeMorgan's Laws, the union of two open sets corresponding to polynomials P and Q is the set $\mathbb{R}^n \setminus \{x \mid x \ is \ a \ root \ of \ P \ and \ Q\}$. The set of points which are roots of P and Q are the roots of the greatest common polynomial divisor of P and Q. Since this is also a polynomial, our set is the complement of the root set of a polynomial and is therefore open. Since the greatest common polynomial divisor of any set of polynomials has root set no greater than any of the constituent polynomials, we properly have closure under arbitrary union.

The Zariski topology is an object of importance in the area of algebraic geometry.

Definition. If X is a topological space with topology $\mathcal{A}$ and $Y \subset X$, then $\mathcal{B}$ is a topology on Y where a subset $V \subset Y$ is open in $\mathcal{B}$ if and only if there is a U open in $\mathcal{A}$ such that $V = U \cap Y$. This is called the **subset** or **subspace topology**.

Example. Let H^2 denote the closed upper-half plane in $\mathbb{R}^2$. That is, the set of points $(x, y) \in \mathbb{R}^2$ such that $y \geq 0$. Any set which was open in $\mathbb{R}^2$ and does not intersect the x-axis is still open in H^2. However, a set like an open half-disk against the x-axis together with the line segment where it rests up against the x-axis was not an open set in $\mathbb{R}^2$, as the boundary points are not interior, but it is open in H^2 with the subspace topology, as it is the intersection of an open disk in $\mathbb{R}^2$ with the upper half-plane.

Lecture 2 (2017-09-01)

Continuous Maps

Continuous maps are the standard morphisms in topology.

In Analysis, we have a definition of continuity which looks like:

Definition. A function $f : X \to Y$ is **continuous** at $x \in X$ if for any $\delta > 0$ there exists an $\epsilon > 0$ such that $\|x - y\| < \epsilon$ implies $\|f(x) - f(y)\| < \delta$.

The issue with this definition is that we have no natural notion of distance in topology. Instead, we use the definition:

Definition. A function $f : X \to Y$ is **continuous** if the inverse image of an open set in Y is open in X. Equivalently, the inverse image of closed sets are closed.

It turns out that in metric spaces like $\mathbb{R}^n$ with the standard topology, these definitions are equivalent.

Example. Let's consider two topological spaces: $(\mathbb{R}, std)$, the real numbers with the standard topology, and $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$, the real numbers with the discrete topology. The map $f : (\mathbb{R}, \mathcal{P}(\mathbb{R})) \to (\mathbb{R}, std)$, where $f(x) = x$ is continuous. Since every set is open in the discrete topology, the inverse image of any set, in particular any open set, is open. The map $g : (\mathbb{R}, std) \to (\mathbb{R}, \mathcal{P}(\mathbb{R}))$, where $g(x) = x$ is not continuous. To see this, take any set that is closed with respect to the standard topology. This set is open in the discrete topology, but its inverse image is closed.

This raises the question: is there any $g : (\mathbb{R}, std) \to (\mathbb{R}, \mathcal{P}(\mathbb{R}))$ which is continuous?

Theorem. *The only continuous functions $g : (\mathbb{R}, std) \to (\mathbb{R}, \mathcal{P}(\mathbb{R}))$ are the constant maps.*

Proof. First, it is easy to see that a constant map is continuous. Without loss of generality, we'll assume that $g(x) = 0$. Let V be open in the discrete topology. If V contains 0, then the inverse image of V is all of $\mathbb{R}$. If V does not contain zero, then the inverse image of V is the empty set. Since both of these are open in the standard topology, the inverse image of any open set is open, and the map is continuous.

To see that such a continuous map must be constant, first observe that $\mathbb{R}$ and $\varnothing$ are the only sets which are both closed and open with respect to the standard topology. Let $g : (\mathbb{R}, std) \to (\mathbb{R}, \mathcal{P}(\mathbb{R}))$ be a continuous map and pick some $x \in \mathbb{R}$. The set $\{g(x)\}$ is both closed and open in the discrete topology (as every set is closed and open), so its inverse image must be, in particular, open. But the inverse image cannot be empty, as we know for sure it contains x, and the only non-empty closed and open set in the standard topology is the entire space. Therefore, for any $x, y \in \mathbb{R}$, we have $g(x) = g(y)$, which is only true for constant maps.

$\square$

Definition. A **homeomorphism** is a continuous bijection between two topological spaces such that the inverse is also continuous.

Under a homeomorphism, we also have the property that the image of open sets is open. This induces a bijection between the open sets of the two topological spaces. In a sense, the existence of a homeomorphism means that two topological spaces are the same.

Example. The two spaces $(-1,1)$ and $(-2,2)$ with the standard topology are homeomorphic under the map $f(x) = 2x$.

Example. The two spaces $(-1,1)$ and $\mathbb{R}$ with the standard topology on each are homeomorphic under the map $f(x) = tan(\frac{\pi}{2}x)$.

Example. Let $S^n = \mathbb{R}^n \cup \{\infty\}$. A set $U \subset S^n$ is open if:

$U = \varnothing$ or $U = S^n$

$U \subset \mathbb{R}^n$ and U is open with respect to the standard topology.

$\infty \in U$ and $U \cap \mathbb{R}^n$ is the complement of a compact subset of $\mathbb{R}^n$. That is, U looks like all of $\mathbb{R}^n$ with a closed and bounded chunk removed, and an additional point ∞.

This forms a topology, and the set S^n is the surface of the n-dimensional sphere. If we think about S^2, there's a natural embedding in $\mathbb{R}^3$, but it turns out that $S^2 \setminus \{(0,0,1)\}$ is homeomorphic to $\mathbb{R}^2$. If we use the (north polar) stereographic projection, which maps points in S^2 to the point in the $\mathbb{R}^2$ plane according to the straight line passing through the north pole and that point, we get a nice homeomorphism, and this is easy to see from the subspace topology that S^2 inherits from $\mathbb{R}^3$. If we then include that the north pole maps to our added point ∞, we get a map from all of S^2 to the set $\mathbb{R}^2 \cup \{\infty\}$ which is a homeomorphism.

The Quotient Topology

Let $(X, \mathcal{A})$ be a topological space and $\sim$ an equivalence relation on X. Then $X/\sim$ inherits a topology, which is that $U' \subset X/\sim$ is open if and only if there is some open set $U \subset X$ such that $U' = U/\sim$.

Definition. This topology is called the **quotient topology**, or the **identification map**.

Example. Take $\mathbb{R}^2$ with the standard topology and define an equivalence relation $(x,y) \sim (x,-y)$. The quotient space looks like the closed (upper or lower) half-plane.

Example. $\mathbb{R}^2$ with the standard topology quotiented by the equivalence relation $(x,y) \sim (-x,-y)$ looks like a cone, and is actually homeomorphic to $\mathbb{R}^2$.

Example. $\mathbb{R}^2$ with the standard topology and the equivalence relation $(x,y) \sim (x+1,y) \sim (x,y+1)$ has a quotient space that looks like the unit square with opposite sides glued together. This is homeomorphic to a (genus 1) torus.

Lecture 3 (09-06-2017)

The Pullback Topology

Let $(X, \mathcal{A})$ be a topological space and Y some set. Given a map $f : X \to Y$, Y inherits a topology from X where $V \subset Y$ is open if and only if $f^{-1}(V) \subset X$ is open.

Definition. This topology on Y is called the **pullback topology**.

The pullback topology is the finest topology on Y which makes f a continuous map.

Example. Take $f : (-1, 1) \to \mathbb{R}$ with $f(x) = x$ and the standard topology on each. The pullback topology on $\mathbb{R}$ has open sets $\varnothing$ and $\mathbb{R}$, whose inverse images are themselves. Also, any set in $\mathbb{R}$ which does not intersect the open interval $(-1, 1)$, as all of these sets have empty inverse image. Finally, any set which is open in $(-1, 1)$ or whose intersection with $(-1, 1)$ is open is also open in the pullback topology.

Group Actions and Fundamental Regions

Lets think about $\mathbb{Z}^2$ as a group action on $\mathbb{R}^2$, where applying $(a, b) \in \mathbb{Z}^2$ to $(x, y) \in \mathbb{R}^2$ means shifting (x, y) right by a and up by b (left, down if a or b is negative, of course). We write this as $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(a, b).(x, y) = (x + a, y + b)$. This establishes an equivalence relation on $\mathbb{R}^2$: $(x_0, y_0)(x_1, y_1)$ if there exists $(a, b) \in \mathbb{Z}^2$ such that $(a, b).(x_0, y_0) = (x_1, y_1)$. This divides $\mathbb{R}^2$ into 1×1 squares, where each square is equivalent to any other, and we identify the left and right edges and the top and bottom edges, but no two points in the interior of any given square are equivalent. We call the squares fundamental regions.

Definition. A **fundamental region** of a group action and is the (closure of) largest region such that no two interior points are identified with respect to the induced equivalence relation.

Example. The fundamental region described above, the square with opposite edges identified, defines a torus.

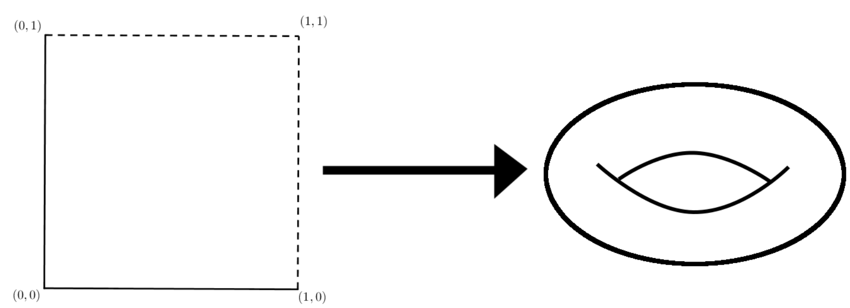

Figure 1:

Open sets in the torus $\mathbb{R}$ are $\varnothing$ and $\mathbb{T}$, and the intersection of any standard open set with the fundamental region. This is the same as the inherited subspace topology from $\mathbb{R}^2$.

Example. If we consider $\mathbb{R}^2/\mathbb{Z}$, where $a \in \mathbb{Z}$ acts on $\mathbb{R}^2$ by $a.(x, y) = (x + a, y)$. A fundamental domain of this action is a vertical strip of unit width. Again this inherits a subspace topology from $\mathbb{R}^2$.

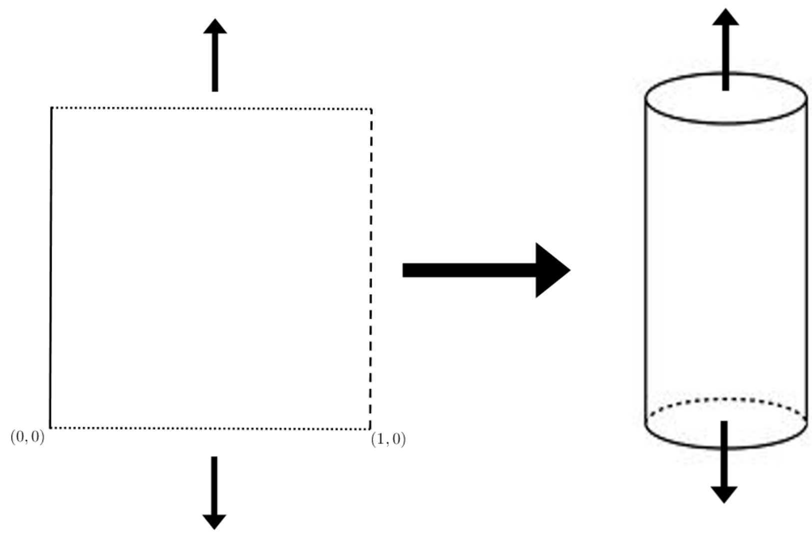

Figure 2:

Example. Consider again $\mathbb{R}^2/\mathbb{Z}$ but this time with the group action $a.(x, y) = (x + a, (-1)^a y)$. The fundamental region is still a strip of unit width, but this time instead of identifying points on the boundary with their horizontal translation, we identify them with their horizontal translation composed with reflection about the x-axis. This space is homeomorphic to an infinite Moebius strip, which is difficult to draw.

Example. Consider the equivalence relation on $\mathbb{R}^2$ described by $(x, y) \sim (x, y)$, $(0, y) \sim (1, y)$, and $(x, 0) \sim (1 - x, 1)$. The fundamental region again is a square with the left and right edges identified by simple translation, but the top and bottom edges are now identified by translation plus a flip across the squares vertical axis of symmetry. This is homeomorphic to the Klein bottle, which is, again, hard to draw.

Example. The previous example where we also identify the left and right edges by translation and a flip is called the real projective plane, denoted $\mathbb{R}P^2$. Both vertical and horizontal strips of this space look like Moebius strips. This is, once again, not easy to draw.

Example. This one we can draw! Take the unit square as the fundamental region, but identify the top and left edge with each other by symmetry about the corner where they intersect, and do the same for the bottom and right edge. This space is homeomporhic to the 2-sphere $\mathbb{S}^2$.

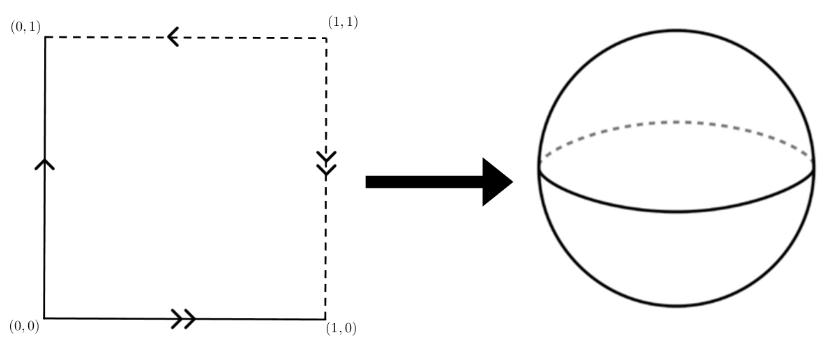

Figure 3:

Lecture 4 (09-08-2017)

Mapping Cylinders and Tori

Definition. Let I denote the closed unit interval $[0,1]$. The **mapping cylinder** of a continuous map : $fX \to Y$ is the quotient space defined by $X \times I/\sim$, where $(x,0) \sim (f(x),1)$.

Definition. The **mapping torus** of a map $f : X \to X$ is similar, except that we require that the map be from a space to a copy of itself and we define the equivalence relation as $(x,0) \sim (f(x),0)$.

Example. The mapping cylinder of $f : \mathbb{S}^1 \to \mathbb{S}^1$ where $f(x) = x$ is a regular old cylinder. The mapping torus is a regular old torus.

Example. We can equivalently think of $\mathbb{S}^1$ as $\{(x,y)|x^2 + y^2 = 1\}$ in Euclidean space or as $\{(r,\theta)|r = 1\}$ in polar coordinates. Using this second formulation, consider the map $f : \mathbb{S}^1 \to \mathbb{S}^1$ where $f(\theta) = 2\theta$. This is a two-to-one map which maps antipodal points to each other. The mapping cylinder of f is a Moebius strip.

Example. What about the three-to-one map $f(\theta) = 3\theta$? The mapping cylinder of this looks like a three-bladed wing with a one-third twist and the ends glued together.

Boundaries and Exteriors

Let $(X, \mathcal{A})$ be a topological space and let $K \subseteq X$.

Definition. The **interior** of K is the largest open subset contained in K. That is, it is the union of all $U \subset K$ such that $U \in \mathcal{A}$.

Definition. The **closure** of K is the smallest closed subset containing K. That is, it is the intersection of all $V \supset K$ such that $(X - V) \in \mathcal{A}$.

Definition. The **boundary** of K is the intersection of the closure of K with the closure of the complement of K, that is $Bd(K) = \overline{K} \cap \overline{X - K}$. If K is open, then $Bd(K) = \overline{K} - K$. If K is closed, then $Bd(K) = \varnothing$.

Example. Take $K = \mathbb{Q} \cap [0,1] \subset \mathbb{R}$ with the standard topology on $\mathbb{R}$. The interior of this set is empty, as there is no open interval which doesn't contain an irrational number, so $\varnothing$ is the largest open subset in K. The closure of K is the entire interval $[0,1]$, as there is no smaller closed set which contains all of the rationals in that interval. We also have that the boundary $Bd(K) = [0,1]$.

Example. Take $K = \mathbb{R} - \{0\}$ with the Zariski topology on $\mathbb{R}$. The interior of K is K, as K is open. The closure of K is all of $\mathbb{R}$, and the boundary is $\{0\}$.

Definition. A point x is a **limit point** of $K \subset X$ if every open set containing x has non-empty intersection with K. Equivalently, x is a limit point of K if $x \in \overline{K - \{x\}}$.

Example. Take $\mathbb{R}$ with the Zariski topology. If U is an open set, then every $x \in \mathbb{R}$ is a limit point of U. In fact, for any infinite subset of $\mathbb{R}$, every point in $\mathbb{R}$ is a limit point.

Definition. A **neighborhood** of a point $x \in X$ is an open set containing x.

Example. In the either-or topology on $[-1,1] \subset \mathbb{R}$ has open sets $\varnothing$, $[-1,1]$, a set is open if and only if it does not contain 0 or it contains $(-1,1)$. If U is an open set and $0 \in U$, then

$U = (-1,1),\ [-1,1),\ (-1,1],\ \textit{or}\ [-1,1]$. Closed sets are subsets of $\{-1,1\}$, all of $[-1,1]$, $\varnothing$, and any set that contains 0.

What are the continuous functions? From EO to std, constant functions are continuous. Anything else? From std to EO, continuous functions look like $f(x) = \frac{1}{2}sgn(x)$.

Example. Take $\frac{1}{2} \in [-1,1]$ with the either-or topology. Is $\frac{1}{2}$ a limit point of $[-1,1] - \{\frac{1}{2}\}$? No! That set is already closed, so it contains all of its limit points.

Is 0 a limit point of $(\frac{1}{2}, \frac{3}{4})$? Yes! Every open set containing 0 contains $(-1,1)$, so it obviously also contains $(\frac{1}{2}, \frac{3}{4})$.

Lecture 5 (09-11-2017)

Topological Bases

Definition. Let X be a set. A collection $\mathcal{B}$ of subsets of X is called a **base** (or **basis**) of X if:

1 If $x \in X$ then there is a $B \in \mathcal{B}$ such that $x \in B$. Equivalently, $\mathcal{B}$ covers X.

2 If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$ and $x \in B_3$.

This is a weaker concept than a topology; we don't require that the union of base elements is a base element and we only require that the intersection of base elements contains another base element.

Example. Consider $\mathbb{R}^2$ with the standard topology. Define $\mathcal{B} = \{B_x(r) | x \in \mathbb{R}^2, r > 0\}$ as the set of open balls in $\mathbb{R}^2$. This is a base. It is easy to see the first criterion is satisfied. To see the second, consider two balls which both contain some point x. Then there is a small ball centered at x which is fully contained in the intersection of the two balls. This smaller ball is also a base element, so we are done.

Lemma. *If $\mathcal{B}$ is a base and $B_1, B_2, \ldots, B_n \in \mathcal{B}$, and $x \in B_1 \cap B_2 \cap \cdots \cap B_n$ then there exists a base element $B' \subset B_1 \cap \cdots \cap B_n$ which contains x.*

Proof. We proceed by induction. Since $x \in B_1 \cap B_2$, there exists some $D_1 \in \mathcal{B}$ such that $x \in D_1 \subset B_1 \cap B_2$. Then $x \in D_1 \cap B_3$, so there exists some $D_2 \in \mathcal{B}$ with $x \in D_2$. We proceed iteratively like this to find there is some $D_{n-1} \in \mathcal{B}$ with $x \in D_{n-1}$, and we set $B' = D_{n-1}$.

$\square$

Definition. A **topology generated by a base** is the collection of sets which are unions of base elements.

If X is a set with a base $\mathcal{B}$, then there is a smallest (coarsest) topology on X containing $\mathcal{B}$, which is the topology generated by $\mathcal{B}$. Open sets are the base elements, arbitrary unions of base elements, and $\varnothing$ and X by definition. Do we get the intersection property as well?

Claim. Yes.

Proof. If $B_1, \ldots, B_n$ are base elements, then we can write the intersection $\bigcap_{i \in [n]} B_i$ as the union of base elements just by taking neighborhoods of each point in the intersection. If we have $U_1, \ldots, U_n$ open in X and $x \in \bigcap_{i \in [n]} U_i$, then there is some base element in the intersection containing x. If we do this for all points in the intersection, we can write the intersection as an arbitrary union of base elements, and we are done. $\square$

Definition. Let $(X, \mathcal{A}$ be a topological space. Take $\mathcal{B} \subset \mathcal{A}$ a collection of sets such that $\varnothing, X \in \mathcal{B}$ and if $x \in U \in \mathcal{A}$, then there is some $B \in \mathcal{B}$ such that $x \in B \subset U$. We call $\mathcal{B}$ a **base for the topology $\mathcal{A}$**.

Lecture 6 (09-13-2017)

Bases, Separability, Hausdorff

Definition. If $(X, \mathcal{A})$ is a topological space and p a point in X, then a **base at the point p** is a collection $\mathcal{U}$ of open sets such that whenever p is in some $V \in \mathcal{A}$, there exists a $U \in \mathcal{U}$ with $p \in U \in \mathcal{U}$.

Example. The set of open balls centered at p form a base at p in $\mathbb{R}^n$ with the standard topology.

Example. The set of open balls forms a base for $\mathbb{R}^n$. So does the set of all rectangular prisms. So does the set of all cubes. The set of cubes is obviously a subset of the set of prisms, but neither of these is a subset or a superset of the set of balls. We can also have a base where we have balls/prisms/cubes with rational centers and rational radii/side lengths. The cardinality of these bases is the same as the cardinality of $\mathbb{Q}$.

Definition. If a set has a base which has cardinality in bijection with some subset of $\mathbb{N}$ or $\mathbb{Q}$, then it has a **countable base**.

Which topologies have countable bases? We have seen that $\mathbb{R}^n$ with the standard topology does. How about $\mathbb{R}^n$ with the discrete topology? The answer is no. Since every singleton set is open in the discrete topology, any base must contain every singleton. Since the number of singleton subsets of $\mathbb{R}$ is uncountable, there cannot be a countable base.

Definition. A subset is **dense** if every open set in the space contains some element of the subset.

Definition. A space is **separable** if it has a countable, dense subset.

If a set has a countable base, it is separable, one countable, dense subset is just a single element from each base element.

Observe that any topology on a finite or countable set is separable, as the set of singletons is countable.

Definition. A topological space is **Hausdorff** if whenever we have two points $p, q \in X$ with $p \neq q$, there exist disjoint open sets such that p belongs to one and q belongs to the other.

Example. $\mathbb{R}^n$ with the standard topology is separable. If $p \neq q$, we can take small open balls around p and q with radius less than half the distance between them. These balls are disjoint and open, so we are done.

Example. The set $[-1, 1]$ with the either-or topology is not Hausdorff. If we take p to be any non-zero point in $(-1, 1)$ and $q = 0$, then any open set containing zero must also contain p.

Example. The Zariski topology is not Hausdorff (this is on the homework).

Example. The line with a double point, defined as $\mathbb{R} \sqcup \mathbb{R}/\sim$ with $x \sim y$ if $x = y$ and $x, y \neq 0$ looks like the real line with two zeros, call them 0_1 and 0_2. The open sets in this topology are the empty set, the whole space, and anything that kind of looks like a standard open set. This space is not Hausdorff. Any open set containing 0_1 necessarily contains a neighborhood of 0_2, and vice versa. This space is separable, however. Rational balls will form a countable base, for example.

Theorem. *If $\mathcal{A}$ is a topology on X and $\mathcal{B}$ is a base for $\mathcal{A}$, then $\mathcal{B}$ is a base.*

Proof. Clearly we have $\varnothing, X \in \mathcal{B}$. We only need to show that any point in the intersection of two base elements B_1, B_2 is in a third base element $B_3 \subset B_1 \cap B_2$. We have $B_1 \cap B_2$ open because $\mathcal{B} \subset \mathcal{A}$. So by the definition of a topology, there must e a $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$, and we're done.

□

Theorem. *If $\mathcal{B}$ is a base for a set X and $\mathcal{A}$ is the topology generated by $\mathcal{B}$, then $\mathcal{B}$ is a base for the topology $\mathcal{A}$.*

Proof. This proof is also straightforward. If $B_1, B_2 \in \mathcal{B}$ are basis elements, and we take an $x \in B_1 \cap B_2$ then there is some $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$. Since $\mathcal{B}$ generates $\mathcal{A}$, if x is in some open set $U \in \mathcal{A}$, then it is in the union of some collection $\{B_i\} \subset \mathcal{B}$, so $x \in B_j$ for some $B_j \in \mathcal{B}$ with $B_j \subset U$, and we are done. □

Definition. A **subbase** (or subbasis) for a set X is a collection $\mathcal{S}$ of sets such that $\bigcup_{S \in \mathcal{S}} S = X$.

Lecture 7 (09-18-2017)

Definition. A topological space X is **first-countable** if there exists a countable base at every point $x \in X$.

Definition. A topological space X is **second-countable** if it has a countable base.

Example. Consider $\mathbb{R} \sqcup \mathbb{R}/\sim$, where $x \sim y$ if $x = y$ and $x, y < 1$. This looks like the line but with two copies of the point $\{1\}$ and a second copy of every point greater than 1. This space is not Hausdorff. If we take an open set at the first copy of 1, it must intersect any open neighborhood of the other copy of 1.

Example. Consider $\mathbb{R} \sqcup \mathbb{R}/\sim$, where $x \sim y$ if $x = y$ and $x, y \leq 1$. This looks like the line with one branch extending towards $-\infty$ and two branches extending towards $+\infty$ from 1. This space is Hausdorff. If we take an open neighborhood around 1, we can see that the inverse image is open only if that neighborhood contains pieces of all three branches, so it isn't possible to have an open set that contains $1 + \epsilon_1$ for all $\epsilon_1 > 0$ on one branch without also having some small neighborhood which also contains $1 + \epsilon_2$ for some $\epsilon_2 > 0$ on the second.

Back to Subbases

Recall that a subbase $\mathcal{S}$ of a set X is a collection of subsets such that $\bigcup \mathcal{S} = X$.

Definition. If $\mathcal{S}$ is a subbase, then **the topology generated by $\mathcal{S}$** is the set of all arbitrary unions and finite intersections of elements of $\mathcal{S}$.

The proof that this is in fact a topology is trivial.

Definition. If $\mathcal{S}$ is a subbase, then a base $\mathcal{B}$ formed by the set of all finite intersections of elements of $\mathcal{S}$ plus the set X itself is **the base generated by $\mathcal{S}$**.

Again, the proof that this is a proper base is trivial.

Definition. Let $(X, \mathcal{A}_1)$ and $(Y, \mathcal{A}_2)$ be topological spaces with respective bases $\mathcal{B}_1 \subset X$ and $\mathcal{B}_2 \subset Y$. Then $X \times Y$ has a topology called the **product topology** which is generated by a base $\mathcal{B}_3$ where $W \in \mathcal{B}_3$ if and only if $W = U \times V$ for some $U \in \mathcal{B}_1$ and $V \in \mathcal{B}_2$. That is, base elements in the product topology are products of the base elements in the factor topologies.

Example. Consider $\mathbb{R} \times \mathbb{R}$ and let $\mathcal{B}_1 = \mathcal{B}_2 = \{(a, b) | -\infty \leq a < b \leq +\infty\}$ be bases for $\mathbb{R}$ which consist of all open intervals. Then the base for $\mathbb{R}^2$, $\mathcal{B}_3 = \{(a, b) \times (c, d) | (a, b) \in \mathcal{B}_1, (c, d) \in \mathcal{B}_2\}$ is the set of open rectangles in $\mathbb{R}^2$. We showed last time (in the general case of prisms in $\mathbb{R}^n$) that this is indeed a base.

Definition. Let $(X, \mathcal{A}_1)$ and $(Y, \mathcal{A}_2)$ be topological spaces with respective bases $\mathcal{B}_1 \subset X$ and $\mathcal{B}_2 \subset Y$. The the subbase $\mathcal{S} = \{U \times Y | U \in \mathcal{B}_1\} \cup \{X \times V | V \in \mathcal{B}_2\}$ is the **standard subbase on $\boldsymbol{X \times Y}$**.

The proof that the standard subbase generates the product topology extends naturally to all product topologies generated by a finite number of factor spaces.

We can think of the product of k copies of X as the set of functions from a set of size k to X, $X^k = \{f : [k] \to X\}$.

Example. We can think of $\mathbb{R}^3$ as $\{f : \{1,2,3\} \to \mathbb{R}\}$. $(f(1), f(2), f(3))$ is an ordered triple in $\mathbb{R}^3$, and it also completely specifies a function. We can think of $\mathbb{R}^n$ as $\bigtimes_{[n]} \mathbb{R} = \{f : [n] \to \mathbb{R}\}$. Similarly, the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ is $\mathbb{R}^{\mathbb{R}} = \bigtimes_{\mathbb{R}} \mathbb{R}$.

Lecture 8 (09-20-2017)

The Cantor set

The Cantor set is one of those pathological examples in mathematics. Consider the interval $C_0 = [0,1] \subset \mathbb{R}$. Given C_k, define C_{k+1} as C_k with the middle third of each constituent interval removed. So

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

and so on.

While we don't have a precise definition for what it means to take a limit to C_∞, we can define $C_\infty = \bigcap C_i$ and this is totally fine from a topological point of view.

Definition. The **Cantor set** is the name given to $C_\infty = \mathcal{C}$.

The Cantor set is not empty, it's actually uncountable. To see this, consider all reals in $[0,1]$ expressed in their ternary expansion. The Cantor set contains all numbers which do not have any 1s in this representation. This is obviously an uncountable set. We also note that the Cantor set doesn't contain any intervals.

The Cantor set inherits a subspace topology from $\mathbb{R}$. It is an exercise on the homework to show that the map $f : C_\infty \to [0,1]$ where we replace all of the 2s in the ternary representation with 1s and interpret it as the binary representation of a real number is a continuous function.

The Cantor set has no interior points, so the complement of the Cantor set is dense and open.

Definition. Denote the **measure** of the set C_k as mC_k. Here we'll use measure as 'total length', although in a proper, measure-theoretic sense, this definition isn't quite correct.

What is the measure of the Cantor set? The measure of $C_0 = [0,1]$ is 1. We can define a recurrence, where $mC_k = 1 - \frac{1}{2}\sum_{i=1}^{k}(\frac{2}{3})^i$. Then, $mC_\infty = 1 - \frac{1}{2}\sum_{i=1}^{\infty}(\frac{2}{3})^i = 1 - \frac{1}{2}\frac{\frac{2}{3}}{1-\frac{2}{3}} = 0$.

What if at each stage we remove a little less than $\frac{1}{3}$ of each interval? Say we remove $\frac{\alpha}{3}$, where $0 < \alpha < 1$. We still get a Cantor-like set, but here we get a recurrence which looks like $mC_k = mC_{k-1} - 2^{k-1}\alpha(\frac{1}{3})^k$. Here, the measure of C_∞ is $1-\alpha$, but its interior is still empty, so its complement is open and dense.

We can even make a Cantor-like set of full measure by putting a smaller copy of the Cantor set into each gap created by removing an interval. This set is uncountable, has measure 1, and is nowhere dense.

Definition. Call the function $\mathfrak{c} : \mathcal{C} \to [0,1]$ such that if $a = 0.a_1a_2a_3\ldots_t$ is the ternary expansion of an element of the cantor set, $\mathfrak{c}(a) = 0.\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\ldots_b$ where we interpret $\mathfrak{c}(a)$ as a binary number, the **Cantor map**.

The Cantor map is pretty obviously surjective, but it is not bijective. To see this, consider $0.0222\ldots_t$ and 0.2_t. These are not equal (they are representations of $\frac{1}{3}$ and $\frac{2}{3}$, respectively) but the first maps to $0.0111\ldots_b$ and the second to 0.1_b, and both of these are representations of $\frac{1}{2}$, so the map is clearly not injective. This map is continuous, as we showed in the homework.

We can also think of extending this map to $\tilde{\mathfrak{c}} : [0,1] \to [0,1]$ where we use the original map on elements of the Cantor set and linear interpolation on points not in the Cantor set.

This leads to a similar map $f : [0,1] \to [0,1] \times [0,1]$ where if $a = 0.a_1a_2a_3a_4\ldots_t$, $f(a) = (0, \frac{a_1}{2}\frac{a_3}{2}\ldots_b, 0.\frac{a_2}{2}\frac{a_4}{2}\ldots_b)$. This map is also continuous and surjective, but not bijective. We're now really close to the idea that there could exist a homeomorphism $g : [0,1] \to [0,1] \times [0,1]$. In particular, this would mean that dimension is not invariant under homeomorphism and that things like $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ for all m, n. Can such a homeomorphism exist? The answer is no, thankfully.

Lecture 9 (09-22-2017)

Back To Bases

How can we determine if two bases generate the same topology?

Theorem. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two topologies on the same underlying set X generated by bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. Then the following are equivalent:*

1. *$\mathcal{A}'$ is finer than $\mathcal{A}$*
2. *If x is in X and x is in some base element $B \in \mathcal{B}$, then there exists a $B' \in \mathcal{B}'$ such that $x \in B'$ and $B' \subset B$.*

Proof. Assume that $\mathcal{A}'$ is finer than $\mathcal{A}$ and take some $x \in X$ and some $B \in \mathcal{B}$ such that $x \in B$. Since B is an open set in $\mathcal{A}$, B is also an open set in $\mathcal{A}'$, so B can be written as the union $\bigcup B_i'$ for some $\{B_i'\} \subset \mathcal{B}'$. Since $x \in B$, x is in the union of these B_i', so x must be in at least one of the B_i' which by construction is a subset of B.

Now, assume that $x \in B \in \mathcal{B}$ implies the existence of some $B' \in \mathcal{B}'$ with $x \in B' \subset B$. By taking intersections and small neighborhoods, we can necessarily write any such B as a union of some collection of B_i'. But since we can do this, any open set built from elements of $\mathcal{B}$ can be built from elements of $\mathcal{B}'$, so any open set in $\mathcal{A}$ is also open in $\mathcal{A}'$, hence $\mathcal{A}'$ is finer than $\mathcal{A}$.

□

The Product Topology

Remark. The book uses $\mathbb{R}^\omega$ to denote the set of all sequences in $\mathbb{R}$ indexed by the natural numbers and $\mathbb{R}^\infty$ to denote those sequences which are eventually all zeros. We'll do our best to be consistent with this.

Definition. Given a product of topological spaces $\prod X_\alpha$, the **box topology** is the one with open sets that can be written as a product $\prod U_\alpha$ where U_alpha is open in X_α.

Under infinite products, this topology is too fine. Consider the function $f : \mathbb{R} \to \mathbb{R}^\omega$, where $f(t) = (t, t, t, \dots)$. This function is not continuous from the standard topology to the box topology. To see this, consider the set $(-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{-1}{3}) \times \dots$ in $\mathbb{R}^\omega$. This is open in the box topology. Under f, the inverse image of this set is $\{0\}$, which is not open.

Definition. Let $\pi_\alpha(x)$ be the function which sends x in the product to its α coordinate. This extends to sets by saying that $\pi_\alpha(U)$ is the set of elements $y \in X_\alpha$ such that there exists some $x \in U$ where $\pi_\alpha(x) = y$. The **product topology** is defined by the base consisting of the sets $\pi_\alpha^{-1}(U_\alpha)$ where U_α is a base element (or any open set) in X_α. Equivalently, an open set in the product topology is one which is a product of open sets in the factor spaces where all but finitely many are equal to the whole space.

Lecture 10 (09-25-2017)

Metric Topologies

Definition. A **metric** on a space X is a function $d : X \times X \to R_+$ which satisfies:

1. $d(x, y) = 0$ if and only if $x = y$ (positivity)
2. $d(x, y) = d(y, x)$ for all x, y (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Definition. If X is a space with metric d, and $x \in X$, $r > 0$, **the ball of radius r centered at x** is the set $\{y | d(x, y) < r\}$. This is denoted $B_r(x)$ (or $B_x(r)$, $B(x, r)$, $B(r, x) \ldots$).

Definition. The **topology generated by a metric** is the topology on a metric space generated by the base of all balls of finite radius centered at all points.

If x is an element of X and x is in some set U which is open in the metric topology, then there is a sufficiently small r such that $B_r(x)$ is entirely contained in U. Hence all points of U are interior.

Let's note that while such a base does indeed generate the topology we want, it's not always the best or smallest base that does so. For example, the set of all balls of rational radius with rational center generates the standard topology on $\mathbb{R}^n$.

In fact, every metric space is at least first-countable, as we can take a base at a point consisting of the balls centered at that point of rational radius.

Metric spaces are also Hausdorff. To see this, consider two distinct points x, y. Since they are not identical, the distance between them is positive, say 3ϵ. Then $B_\epsilon(x)$ and $B_\epsilon(y)$ are disjoint open sets separating x and y.

Definition. The **discrete metric** is the metric $d(x, y) = 0$ if $x = y$ and 1 if $x \neq y$.

Claim. The discrete metric generates the discrete topology.

Proof. To see this, consider the ball $B_{\frac{1}{2}}(x)$. This is just the set $\{x\}$, and since the singletons are open, the corresponding topology must be the discrete one. □

Claim. The trivial topology is not generated by any metric.

Proof. The trivial topology is not Hausdorff, so it cannot be a metric topology. □

The Euclidean metric on $\mathbb{R}^n$, $d(x, y) = \sqrt{\sum (x_i - y_i)^2}$ generates the standard topology.

Definition. The **ℓ^p metric** on $\mathbb{R}^n$ is $d(x, y) = (\sum |x_i - y_i|^p)^{\frac{1}{p}}$.

Definition. The **ℓ^∞ metric** on $\mathbb{R}^n$ is $d(x, y) = \sup\{|x_i - y_i|\}$.

Claim. All of the ℓ^p metrics generate the same topology on $\mathbb{R}^n$.

Proof. Pick two metrics ℓ^p and ℓ^q with $p < q$. We'll argue that base elements (balls) in one topology contain base elements of the other. Containment one way is trivial. For a fixed radius r, the ℓ^p ball of radius r sits inside of the ℓ^q ball of radius r. To see containment the other way, consider the ℓ^q ball of radius r_1 and think of it as an open set in the ℓ^p topology. There is some point of maximum distance from the center, and if we pick some radius r_2 less than this, the ball will sit inside of the ℓ^q ball, and we're done. □

We call $\ell^p(\mathbb{R}^\omega)$ the set of sequences whose ℓ^p norm is finite. This forms a vector space.

Lecture 11 (09-25-2017)

Metrics, Continued

Recall that ℓ^p is a proper subset of $\mathbb{R}^\omega$.

Definition. The $\boldsymbol{\ell^p}$ **norm** on $\mathbb{R}^n$ is $d(x,y) = (\sum |x_i|^p)^{\frac{1}{p}}$.

Definition. The $\boldsymbol{\ell^\infty}$ **norm** on $\mathbb{R}^n$ is $d(x,y) = \sup\{|x_i|\}$.

If X is a vector space, then a function $||\cdot|| : X \to \mathbb{R}$ is a norm if:

1. $||\vec{x}|| \geq 0$ for all $x \in X$ with equality if and only if $x = \vec{0}$
2. $||c\vec{x}|| = |c| \cdot ||\vec{x}||$ for all scalars c and vectors $\vec{x}$
3. $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$ for all vectors $\vec{x}, \vec{y}$

Definition. We call such an X a **normed vector space**.

Metrics do not need to come from norms, but norms induce metrics. A normed vector space is a metric space, but a metric space need not be a normed vector space.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ with a norm

$$||f||_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}}$$

Definition. This is called the $\boldsymbol{L^p}$ **pseudonorm**.

Why is this a pseudonorm? There are functions which are non-zero which have zero integral, such as functions which are zero everywhere except on a set of measure zero. We can resolve this by defining an equivalence relation $\sim$ where $f \sim g$ if and only if $||f - g||_p = 0$. These equivalence classes form the vector space of L^p functions on $\mathbb{R}$, denoted $L^p(\mathbb{R})$. On these classes, the L^p pseudonorm $||\cdot||_p$ is a proper norm. As it turns out, these are actually *complete* normed vector spaces, i.e. Banach spaces.

Definition. Given a metric $d(x,y)$, we can create a **bounded metric** $\bar{d}(x,y)$ by defining $\bar{d}(x,y) = \min(d(x,y), 1)$. It is easy to see that this forms a proper metric.

When do two metrics generate the same topology? Metric balls form a base. If we can show that balls in one metric contain balls in the other, and vice versa, then the metrics generate the same topology.

Definition. The **Hilbert cube** is the set $I^\omega \subset \mathbb{R}^\omega$. That is, it is the set of all sequences with entries from the interval $[0,1]$.

The Hilbert cube has finite volume, but the diagonal has infinite length. That is, the point $(0,0,0,\dots)$ is infinitely far from the point $(1,1,1,\dots)$ with respect to any ℓ^p metric (for finite p). This space is homeomorphic to the ball of radius 1 in the ℓ^∞ metric.

A subbase for the Hilbert cube is the set $\{\pi_i^{-1}(U) | U \text{ } open \text{ } in \text{ } [0,1]\}$.

Definition. A topological space is **metrizable** if there exists a metric on X which generates that topology.

As an example, the standard topology on $\mathbb{R}^n$ is metrizable as it is generated by the ℓ^2 metric, for example.

Since metric spaces are first-countable, first-countability is a necessary condition for a space to be metrizable.

Example. $\mathbb{R}^{\mathbb{R}}$ with the product topology is not first-countable, therefore there does not exist a metric on $\mathbb{R}^{\mathbb{R}}$ which generates the product topology.

Theorem. *A countable product of metric spaces (with the product topology) is metrizable.*

Proof. Let X be a metric space with metric d and consider the product $X^{\mathbb{N}}$. Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be points in $X^{\mathbb{N}}$. Next, define $D(x, y) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$ where $\bar{d}$ is the bounded version of d. It's easy to see that D is a proper metric.

Under D, the ball of radius r centered at x is

$$B_r(x) = \{y \in X^{\mathbb{N}} | D(x, y) < r\}$$

$$= \left\{y \in X^{\mathbb{N}} | \bar{d}(x_1, y_1) < r,\ \bar{d}(x_2, y_2) < 2r,\ \bar{d}(x_3, y_3) < 3r, \dots\right\}$$

Since eventually nr becomes larger than 1, everything beyond the nth point must be inside the ball. Thus, only finitely many (x_i, y_i) need to be checked, so these balls look like the product of finitely many open balls in X and infinitely many copies of X itself, which is a base for the product topology.

$\square$