CIS 610 - Advanced Geometric Methods in Computer Science

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Introduction

This course is an advanced topics course in Geometry and its applications. The current (Spring 2018) offering is focused on Riemannian manifolds, their differential geometry, and the Lie groups and Lie algebras associated with them.

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Lecture 1 (2018-01-11)

Intro

The high-level overview of this course is that we are going to build up the mathematical machinery to talk about *homogeneous manifolds* and their geometry. In a sentence, these are surfaces for which there is some group G which acts transitively on that space.

Suppose that we have a sequence of objects $B = B_0, B_1, B_2, \ldots, B_m$ where the transformation D_i sends B to B_i. We can think about the D_i as coming from some group, so we know how to compose them and each is invertible.

The motion of a rigid body can be described by a curve in this group of transformations of a space E (maybe this is $\mathbb{R}^n$). Given $B \in E$, a **deformation** is a (reasonably smooth) curve $D : [0, 1] \to G$. We write $B_t = D(t)(B)$ to represent the position of B at time t.

Example. If $G = SO(3)$, then we are looking at rigid rotations in $\mathbb{R}^3$. With respect to any orthonormal basis, any rotation has an associated matrix R such that $RR^T = R^T R = I$.

If $G = SE(3)$, we are looking at rigid motions in $\mathbb{R}^3$. If B is a matrix in this group, it can rotate and translate. This is the group of affine maps $\rho \in SE(3)$ has the form $rho(x) = f(x) + u$, where $f(x)$ is a rotation and u a translation. We can write these as transformations in $\mathbb{R}^4$ by doing the following:

$$\begin{pmatrix} R & u \\ \mathbf{0} & 1 \end{pmatrix}$$

Where R is an $n \times n$ rotation, $\mathbf{0}$ is a row of n zeroes, u is a column vector representing the translation, and the last entry is 1. The matrix representations of elements of $SE(3)$ are 4×4 matrices.

If $G = SIM(3)$, we are looking at simple deformations of a non-rigid body - we can grow and shrink as well as rotate and translate. These have matrices which look like

$$\begin{pmatrix} \alpha R & u \\ \mathbf{0} & 1 \end{pmatrix}$$

where $R \in SO(3)$, $\alpha > 0$, and $u \in \mathbb{R}^n$.

$\star$ all of these are *Lie groups* and we will study lots of other things with Lie groups in this course.

The Interpolation Problem

Suppose we have a sequence of deformations $g_0, g_1, \ldots, g_m$, with each $g_i \in G$ and $g_0 = 1 \in G$. We would like to find a smooth curve in G $c : [0, m] \to G$ such that $c(i) = g_i$ for all i.

The naive approach would be to take $(1-t)g_i + tg_{i+1}$, which is a linear/convex interpolation between adjacent points. However, there is no guarantee that all of these intermediate points are actually in G!

What we can do is use Lie groups. These are topological groups, so they come along with a nice manifold where we can do geometric things. At every $g \in G$, there is a tangent space T_gG. The tangent space at $1 \in G$ is special, and it is called the **Lie algebra**. We use Fraktur fonts to denote

Lie algebras. This Lie algebra is denoted $\mathfrak{g}$ and it comes with a multiplication $[\cdot, \cdot]$ called the **Lie bracket**. When G is a matrix, group, $[X, Y] = XY - YX$.

The Lie algebra $\mathfrak{so}(n)$ of $SO(n)$ is the set of skew-symmetric $n \times n$ matrices $B^T = -B$.

The Lie algebra $\mathfrak{se}(n)$ of $SE(n)$ is the set of matrices of the form

$$\begin{pmatrix} B & u \\ 0 & 0 \end{pmatrix}$$

where $B \in \mathfrak{so}(n)$ and $u \in \mathbb{R}^n$.

The Lie algebra $\mathfrak{sim}(n)$ of $SIM(n)$ is the set of matrices of the form

$$\begin{pmatrix} \lambda IB & u \\ 0 & 0 \end{pmatrix}$$

where $B \in \mathfrak{so}(n)$, $\lambda \in \mathbb{R}$, I is the $n \times n$ identity matrix, and $u \in \mathbb{R}^n$.

We can think of $\mathfrak{g}$ as a linearization of the group G. There is a map called the **exponential** $\exp : \mathfrak{g} \to G$ such that

$$\exp(X) = e^X = 1 + \frac{X}{1!} + \frac{X^2}{2!} + \dots$$

For the groups we talked about, exp is a surjective map. There is a multivalued 'function' called the **logarithm** $\log : G \to \mathfrak{g}$ such that $\exp(\log(A)) = A \in G$.

We can use log and exp to do our interpolation. First, let $x_0 = \log(g_0)$, $x_1 = \log(g_1)$, and so on. Then find a curve $X : [0, m] \to \mathfrak{g}$ to interpolate the x_i in $\mathfrak{g}$. Finally, the curve in G is given by $c(t) = \exp(X(t))$.

If $\mathfrak{g}$ is a vector space, we can do fancy things like use splines to interpolate.

We still need to worry about actually computing exp and log. There are formulas if we are in $\mathfrak{so}(n), \mathfrak{se}(n)$, and $\mathfrak{sim}(n)$. For $\mathfrak{so}(n)$¡ this is the Rodrigues formula, and there is a variant for $\mathfrak{se}(n)$. Logarithms can be computed for $SO(n), SE(n)$, and $SIM(n)$, but there is an issue when we have an eigenvalue equal to -1 and the logarithm is multivalued.

A real matrix doesn't always have a real logarithm (it does always have a complex one). Let $S(n)$ be the set of real matrices whose eigenvalues $\lambda + \mu i$ live in the horizontal strip of the complex plane $-\pi < \mu < \pi$. Then $\exp : S(n) \to \exp(S(n))$ is a bijection onto the set of real matrices with no negative eigenvalues.

There are efficient algorithms to compute matrix logarithms, which we will discuss later.

Metrics on Lie Groups

Metrics formally define a measurable sense of 'closeness'. How 'close' are two given group elements?

We can give an inner product to $\mathfrak{g} = T_1 G$, then propagate this to $T_g G$ at any g to get a Riemannian metric.

For $G = SO(n)$, $\langle X, Y \rangle = -\frac{1}{2} Tr(XY) = \frac{1}{2} Tr(X^T Y)$ is an inner product on $\mathfrak{so}(n)$.

A curve $\gamma : [0,1] \to G$ has length

$$L(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle^{1/2} \, dt$$

A **geodesic** through I is a curve $\gamma(t)$ such that $\gamma(0) = I$ and $\gamma''(t)$ is normal to the tangent space $T_{\gamma(t)}G$ for all t. It turns out that for all $X \in \mathfrak{so}(n)$ there is a unique geodesic through I such that $\gamma'(0) = X$, namely $\gamma(t) = \exp(tX)$.

If $G = SO(n)$, then we have for all $A \in G$ that there exists some geodesic from I to A. Define the distance

$$d(I, A) = \inf_{\gamma} \{L(\gamma) : \gamma \text{ joins } I \text{ and } A\}$$

The distance $d(A, B)$ in $SO(n)$ is

$$d(A, B) = \sqrt{\theta_1^2 + \theta_2^2 + \cdots + \theta_m^2}$$

where $e^{\pm i\theta_j}$ are the eigenvalues (not equal to 1) of $A^T B$ with $0 < \theta_j \leq \pi$ for all j.

The same inner product $\frac{1}{2}Tr(X^T Y)$ works in $\mathfrak{se}(n)$, but this metric is only left- and not both left- and right-invariant. Consequently, not all geodesics in $\mathfrak{se}(n)$ are given by the exponential. Related to this is that $SE(n)$ is not a compact or a semisimple group.

The Grassmannian

Take $\mathbb{R}^n$ and fix some $k \leq n$. What can we say about the set of subspaces $W \subset \mathbb{R}^n$ of dimension k? This is called the **Grassmannian** and is denoted $G(k, n)$.

We know $G(n, n)$ and $G(0, n)$, as these are just $\mathbb{R}^n$ and $\vec{0}$, respectively.

What about $G(1, 2)$? These are the set of lines through the origin. Each line intersects the unit circle exactly twice, so we can think about this as the set of points on the unit circle with an antipodal identification. This is just the 1-sphere $\mathbb{S}^1$.

And $G(1, 3)$? This is the 2-sphere with an antipodal identification, which we know to be homeomorphic to $\mathbb{R}\mathrm{P}^2$, the real projective plane.

In general, $G(1, n)$ is $\mathbb{R}\mathrm{P}^{k-1}$. Also, by duality, $G(k, n) \cong G(n - k, n)$.

Lecture 2 (2018-01-16)

Manifolds Induced by Actions of $SO(n)$

What do we mean by the 'distance' $d(V, W)$ between two subspaces V, W of the Grassmannian $G(k, n)$? We know something about distances in $SO(n)$, so what happens if we let $SO(n)$ act on $G(k, n)$?

We can specify a k-dimensional subspace V with k orthonormal (column) vectors in $\mathbb{R}^n$ and write this as an $n \times k$ matrix A which satisfies $A^T A = I_k$.

A rotation $R \in SO(n)$ acts on V by rotating each vector in V, that is R applied to $A \in V$ can be described as $(R, A) \mapsto RA$, with respect to the ordinary matrix product. Here, RA will also have k orthonormal columns (We're hiding an equivalence relation here).

The action $\cdot : SO(n) \times G(k, n) \to G(k, n)$ is transitive! We're not going to prove this here, but it should be intuitively pretty clear: pick some V and a W we want to send it to. Then we just need to show that the map which sends the ith column of V to the ith column of W is in $SO(n)$.

Since it is transitive, we can look at the **stabilizer** of a subspace V, which is the subgroup $K \subset SO(n)$ such that $R \cdot V = V$ for all $R \in K$. It can be shown that $G(k, n) \cong SO(n)/K$, where we think of $G(k, n)$ as the cosets RK with $R \in SO(n)$ modulo the equivalence relation $R_1 \sim R_2$ if and only if $R_1^{-1} R_2 \in K$, and then taking the canonical projection onto these equivalence classes.

The stabilizer of the first k columns of I_n is $K = S(O(k) \times O(n-k))$. Here, we can write K as the set

$$k = \left\{ \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} : P \in O(k),\ Q \in O(n-k),\ \det(P)\det(Q) = 1 \right\}$$

This is a Lie group, and its associated Lie algebra is

$$\mathfrak{k} = \left\{ \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} : S \in \mathfrak{so}(k),\ T \in \mathfrak{so}(n-k) \right\}$$

We call the $X \in \mathfrak{k}$ the 'vertical' tangent vectors and the $X \in \mathfrak{m}$ the 'horizontal' tangent vectors.

The tangent space $T_I SO(n) = \mathfrak{so}(n)$ splits as the direct sum of $\mathfrak{k} \oplus \mathfrak{m}$ where

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} : A \in M_{n-k,n} \right\}$$

The tangent space $T_o(SO(n)/K)$ to $SO(n)/K$ at o is isomorphic to $\mathfrak{m}$ (a coset of K).

Using the metric on $\mathfrak{so}(n)$, $\frac{1}{2} Tr(X^T Y)$, $\mathfrak{k}$ and $\mathfrak{m}$ are orthogonal complements and $SO(n)/K$ is what is called a *naturally reductive homogeneous space*.

Geodesics in $G(k, n)$ are projections of horizontal geodesics in $SO(n)$.

Theorem. *The distance between two subspaces $V, W \in G(k, n)$ specified by matrices A and B, respectively, is*

$$d(V, W) = \sqrt{\theta_1^2 + \theta_2^2 + \cdots + \theta_k^2}$$

where $\cos\theta_i$ are the singular values of $A^T B$ with $0 \le \theta_i \le \frac{\pi}{2}$. These θ_i are called the ***principle angles****.*

Another interesting group is $SPD(n)$, which are the symmetric positive definite $n \times n$ matrices. We can write this as $SPD(n) \cong GL^+(n)/SO(n)$, where $GL^+(n)$ is the set of real $n \times n$ matrices with strictly positive determinant. Here, geodesics can be computed, but this involves a big ugly integral.

Explicit computation of geodesics in $G(k, n)$ allows for the generalization of optimization methods like gradient descent. Whether we can do it in $SE(n)$ and Grassmannians of affine spaces is an open problem.

The Matrix Exponential

We can think of Lie groups (naively) as groups of symmetries of geometric and topological objects, and Lie algebras are kind of like the 'infinitesimal' transformations of these objects.

We can look at $SO(n)$ as rotations of $\mathbb{R}^n$ and $\mathfrak{so}(n)$ as the set of real skew-symmetric matrices.

The Lie algebra at the identity of the Lie group can be thought of as a 'linearization' of the group. The exponential is a way of 'delinearizing'.

Recall we defined the matrix exponential as the map

$$e^A \mapsto I_n + \sum_{k \geq 1} \frac{A^k}{k!} = \sum_{k \geq 0} \frac{A^k}{k!}$$

where matrix powers are repeated products and A^0 is the identity matrix.

Lemma. *If the $n \times n$ real or complex matrix A has entries $A = (a_{ij})$ and we let μ denote the a_{ij} of maximum absolute value (or modulus), then the absolute value (or modulus) of any element of A^p (denoted $a_{ij}^{(p)}$) is no greater than $(n\mu)^p$.*

Thus the series $\sum_{p \geq 0} \frac{a_{ij}^{(p)}}{p!}$ converges absolutely, so the matrix exponential as an infinite sum is well-defined.

What is the exponential of the matrix $A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$? We can write $A = \theta J$ where J is the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We know what the powers of J look like. $J^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = I^2$. It's easy to show that $J^3 = -J$.

We can compute a few powers of A: $A = \theta J$, $A^2 = \theta^2 J^2 = -\theta^2 I$, $A^3 = -\theta^3 J$, $A^4 = \theta^4 I$.

From here, we can write

$$e^A = I + \frac{\theta J}{1!} - \frac{\theta^2 I}{2!} - \frac{\theta^3 J}{3!} + \frac{\theta^4 I}{4!} + \dots$$

We can pull this apart into two sums, one with Is and the other with Js:

$$e^A = I\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + J\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

which we recognize as $I\cos\theta + J\sin\theta$, so we get

$$e^A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

In fact, every rotation matrix looks like the exponential of some skew-symmetric matrix.

The matrix exponential is not always surjective. Let $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Note that the trace of A is 0. We have that $A^2 = (a^2 + bc)I = -\det(A)I$.

If $a^2 + bc = 0$, we have $e^A = I + A$.

If $a^2 + bc < 0$, let $\omega > 0$ be such that $\omega^2 = -(a^2 + bc)$. Then $e^A = \cos\omega I + \frac{\sin\omega}{\omega}$.

If $a^2 + bc > 0$, let $\omega > 0$ be such that $\omega^2 = a^2 + bc$. Then $e^A = \cosh\omega I + \frac{\sinh\omega}{\omega}$. In all cases, $\det(A) = 1$ and $Tr(e^A) \geq -2$. But $B = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ is not the exponential of any matrix with trace 0, as $Tr(e^B) < -2$.

A fundamental property of the matrix exponential is that if λ_i are the eigenvalues of the matrix A, then e^{λ_i} are the eigenvalues of e^A, and the eigenvector associated with λ_i is the same as the one associated with e^{λ_i}.

Lemma. *Let A and U be real or complex matrices, with U invertible. Then $e^{UAU^{-1}} = Ue^AU^{-1}$.*

Lemma. *Given any $n \times n$ matrix A, there exists an invertible P and upper triangular T such that $A = PTP^{-1}$.*

Lemma (Schur)**.** *Given any $n \times n$ matrix A, there exists a unitary U and upper triangular T such that $A = UTU^*$, where U^* is the conjugate-transpose of U.*

This tells us that if A is Hermitian, there exists a unitary U and a real diagonal matrix D such that $A = UDU^*$.

If $A = PTP^{-1}$ where T is upper triangular, the diagonal entries of T are the eigenvalues of A, and A and T have the same characteristic polynomial.

Lemma. *Given any complex $n \times n$ matrix A with eigenvalues λ_i, the eigenvalues of e^A are e^{λ_i}, and the eigenvectors are the same. Thus,* $\det(e^A) = e^{Tr(A)}$.