CIS 610 - Advanced Geometric Methods in Computer Science

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Introduction

This course is an advanced topics course in Geometry and its applications. The current (Spring 2018) offering is focused on Riemannian manifolds, their differential geometry, and the Lie groups and Lie algebras associated with them.

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Lecture 1 (2018-01-11)

Intro

The high-level overview of this course is that we are going to build up the mathematical machinery to talk about *homogeneous manifolds* and their geometry. In a sentence, these are surfaces for which there is some group G which acts transitively on that space.

Suppose that we have a sequence of objects $B = B_0, B_1, B_2, \ldots, B_m$ where the transformation D_i sends B to B_i. We can think about the D_i as coming from some group, so we know how to compose them and each is invertible.

The motion of a rigid body can be described by a curve in this group of transformations of a space E (maybe this is $\mathbb{R}^n$). Given $B \in E$, a **deformation** is a (reasonably smooth) curve $D : [0,1] \to G$. We write $B_t = D(t)(B)$ to represent the position of B at time t.

Example. If $G = SO(3)$, then we are looking at rigid rotations in $\mathbb{R}^3$. With respect to any orthonormal basis, any rotation has an associated matrix R such that $RR^T = R^T R = I$.

If $G = SE(3)$, we are looking at rigid motions in $\mathbb{R}^3$. If B is a matrix in this group, it can rotate and translate. This is the group of affine maps $\rho \in SE(3)$ has the form $rho(x) = f(x) + u$, where $f(x)$ is a rotation and u a translation. We can write these as transformations in $\mathbb{R}^4$ by doing the following:

$$\begin{pmatrix} R & u \\ \mathbf{0} & 1 \end{pmatrix}$$

Where R is an $n \times n$ rotation, $\mathbf{0}$ is a row of n zeroes, u is a column vector representing the translation, and the last entry is 1. The matrix representations of elements of $SE(3)$ are 4×4 matrices.

If $G = SIM(3)$, we are looking at simple deformations of a non-rigid body - we can grow and shrink as well as rotate and translate. These have matrices which look like

$$\begin{pmatrix} \alpha R & u \\ \mathbf{0} & 1 \end{pmatrix}$$

where $R \in SO(3)$, $\alpha > 0$, and $u \in \mathbb{R}^n$.

$\star$ all of these are *Lie groups* and we will study lots of other things with Lie groups in this course.

The Interpolation Problem

Suppose we have a sequence of deformations $g_0, g_1, \ldots, g_m$, with each $g_i \in G$ and $g_0 = 1 \in G$. We would like to find a smooth curve in G $c : [0, m] \to G$ such that $c(i) = g_i$ for all i.

The naive approach would be to take $(1-t)g_i + tg_{i+1}$, which is a linear/convex interpolation between adjacent points. However, there is no guarantee that all of these intermediate points are actually in G!

What we can do is use Lie groups. These are topological groups, so they come along with a nice manifold where we can do geometric things. At every $g \in G$, there is a tangent space $T_g G$. The tangent space at $1 \in G$ is special, and it is called the **Lie algebra**. We use Fraktur fonts to denote

Lie algebras. This Lie algebra is denoted $\mathfrak{g}$ and it comes with a multiplication $[\cdot, \cdot]$ called the **Lie bracket**. When G is a matrix, group, $[X, Y] = XY - YX$.

The Lie algebra $\mathfrak{so}(n)$ of $SO(n)$ is the set of skew-symmetric $n \times n$ matrices $B^T = -B$.

The Lie algebra $\mathfrak{se}(n)$ of $SE(n)$ is the set of matrices of the form

$$\begin{pmatrix} B & u \\ 0 & 0 \end{pmatrix}$$

where $B \in \mathfrak{so}(n)$ and $u \in \mathbb{R}^n$.

The Lie algebra $\mathfrak{sim}(n)$ of $SIM(n)$ is the set of matrices of the form

$$\begin{pmatrix} \lambda IB & u \\ 0 & 0 \end{pmatrix}$$

where $B \in \mathfrak{so}(n)$, $\lambda \in \mathbb{R}$, I is the $n \times n$ identity matrix, and $u \in \mathbb{R}^n$.

We can think of $\mathfrak{g}$ as a linearization of the group G. There is a map called the **exponential** $\exp : \mathfrak{g} \to G$ such that

$$\exp(X) = e^X = 1 + \frac{X}{1!} + \frac{X^2}{2!} + \dots$$

For the groups we talked about, exp is a surjective map. There is a multivalued 'function' called the **logarithm** $\log : G \to \mathfrak{g}$ such that $\exp(\log(A)) = A \in G$.

We can use log and exp to do our interpolation. First, let $x_0 = \log(g_0)$, $x_1 = \log(g_1)$, and so on. Then find a curve $X : [0, m] \to \mathfrak{g}$ to interpolate the x_i in $\mathfrak{g}$. Finally, the curve in G is given by $c(t) = \exp(X(t))$.

If $\mathfrak{g}$ is a vector space, we can do fancy things like use splines to interpolate.

We still need to worry about actually computing exp and log. There are formulas if we are in $\mathfrak{so}(n), \mathfrak{se}(n)$, and $\mathfrak{sim}(n)$. For $\mathfrak{so}(n)$¡ this is the Rodrigues formula, and there is a variant for $\mathfrak{se}(n)$. Logarithms can be computed for $SO(n), SE(n)$, and $SIM(n)$, but there is an issue when we have an eigenvalue equal to -1 and the logarithm is multivalued.

A real matrix doesn't always have a real logarithm (it does always have a complex one). Let $S(n)$ be the set of real matrices whose eigenvalues $\lambda + \mu i$ live in the horizontal strip of the complex plane $-\pi < \mu < \pi$. Then $\exp : S(n) \to \exp(S(n))$ is a bijection onto the set of real matrices with no negative eigenvalues.

There are efficient algorithms to compute matrix logarithms, which we will discuss later.

Metrics on Lie Groups

Metrics formally define a measurable sense of 'closeness'. How 'close' are two given group elements?

We can give an inner product to $\mathfrak{g} = T_1 G$, then propagate this to $T_g G$ at any g to get a Riemannian metric.

For $G = SO(n)$, $\langle X, Y \rangle = -\frac{1}{2} Tr(XY) = \frac{1}{2} Tr(X^T Y)$ is an inner product on $\mathfrak{so}(n)$.

A curve $\gamma : [0,1] \to G$ has length

$$L(\gamma) = \int_0^1 \left\langle \gamma'(t), \gamma'(t) \right\rangle^{1/2} dt$$

A **geodesic** through I is a curve $\gamma(t)$ such that $\gamma(0) = I$ and $\gamma''(t)$ is normal to the tangent space $T_{\gamma(t)}G$ for all t. It turns out that for all $X \in \mathfrak{so}(n)$ there is a unique geodesic through I such that $\gamma'(0) = X$, namely $\gamma(t) = \exp(tX)$.

If $G = SO(n)$, then we have for all $A \in G$ that there exists some geodesic from I to A. Define the distance

$$d(I, A) = \inf_{\gamma} \{L(\gamma) : \gamma \text{ joins } I \text{ and } A\}$$

The distance $d(A, B)$ in $SO(n)$ is

$$d(A, B) = \sqrt{\theta_1^2 + \theta_2^2 + \cdots + \theta_m^2}$$

where $e^{\pm i\theta_j}$ are the eigenvalues (not equal to 1) of $A^T B$ with $0 < \theta_j \leq \pi$ for all j.

The same inner product $\frac{1}{2}Tr(X^T Y)$ works in $\mathfrak{se}(n)$, but this metric is only left- and not both left- and right-invariant. Consequently, not all geodesics in $\mathfrak{se}(n)$ are given by the exponential. Related to this is that $SE(n)$ is not a compact or a semisimple group.