Math 580 - Combinatorics

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Introduction

Math 318 is one of the nine courses offered for first-year mathematics graduate students at the University of Chicago. It is the second of three courses in the year-long geometry/topology sequence.

These notes are being live-TeXed, though I edit for typos and add diagrams requiring the Ti*k*Z package separately. I am using the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

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Lecture 1

What is Combinatorics?

Combinatorics is the mathematics of counting. Suppose we have some $n \in \mathbb{N}$ and a set S of objects which somehow depend on n. Combinatorics addresses the question "How many objects are in S?" More formally, this is a function $f : \mathbb{N} \to \mathbb{N}$ which counts the number of objects in S as a function of n. What do we know about f?

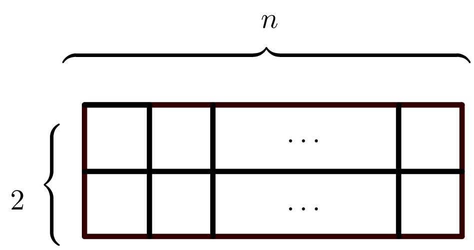
Example. Let S_1 be the set of binary sequences of length n. Then $f(n) = 2^n$.

Example. Let S_2 be the symmetric group on n elements. Then $f(n) = n!$.

Definition. A **derangement** is a permutation in the symmetric group which has no fixed points.

Example. The set of derangements on n elements, $D_n = \{\sigma \in S_n | \ \sigma(k) \neq k \ \forall k \leq n\}$, has size $\#D_n = n! \sum\limits_{i=0}^{n} \frac{(-i)^i}{n!}$.

Example. Suppose we have a $2 \times n$ board, which we want to tile with 2×1 dominoes. How many different ways are there to do this? If $S = \{\textit{proper domino tilings of a } 2 \times n \textit{ board}\}$, then we have $\#S = F_n$, the nth Fibonacci number.



Generating Functions

Definition. A **generating function** corresponding to some counting function f is an element of the ring of formal power series $\mathbb{C}[[x]]$ where the coefficient of the x^n term is $f(n)$.

If F and G are two generating functions, than we have $F(x) = G(x) \iff f(n) = g(n)$ for all $n \in \mathbb{N}$. We can do addition with generating functions, where we add the corresponding coefficients.

$F(x) + G(x) = f(0) + g(0) + (f(1) + g(1))x + (f(2) + g(2))x^2 \dots$. We define multiplication as $F(x) \cdot G(x) = \sum_{n=0}^{\infty} (\sum_{m=0+^n} f(m)g(n-m))x^n$.

Generating functions obey many of the properties of series that we learned in Calculus, except that we don't worry about these things converging. If $f(n) = 1$ for all n, then the generating function is $F(x) = 1 + x + x^2 + \dots$, which equals $\frac{1}{1-x}$. Similarly, if $f(n) = \alpha^n$ for all n, then the generating function is $F(x) = 1 + \alpha x + \alpha^2 x^2 + \dots$, which equals $\frac{1}{1-\alpha x}$. These look like geometric series from Calculus.

Example. Let $F(x)$ be the generating function for the Fibonacci numbers. By the Fibonacci recurrence, we can rewrite this as $F(x) = F_0 + F_1 x + (F_0 + F_1)x^2 + \cdots + (F_{n-2} + F_{n-1})x^n + \cdots = F_0 + F_1 x + F_0 x^2 + F_1 x^2 + \dots$. Factoring out, we can rewrite $F(x) = 1 + xF(x) + x^2 F(x) = \frac{1}{1-x-x^2}$.

Sets and Multisets

Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of n objects. The number of distinct subsets of S of size k is $\binom{n}{k}$, and is called the binomial coefficient.

Theorem. $\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$

Proof. We can think of $\binom{n}{k}$ as the number of subsets of size k on a set of size n, and $k!$ as the number of ways of ordering k objects. Then, $\binom{n}{k} \cdot k!$ represents the number of ordered sequences (without repetition of elements) of length k. We can also think about this as choosing each of the k elements in order. There are n choices for the first, $n-1$ for the second, $n-2$ for the third, and so on, down to $n-k+1$ for the kth. We therefore have $\binom{n}{k} \cdot k! = n(n-1)(n-2)\dots(n-k+1)$. Moving the $k!$ to the denominator of the righthand side completes the proof.

□

Definition. A **multiset** is a collection of objects, like a set, which allows objects to occur with some multiplicity greater than one.

If we denote the natural numbers $1, 2, \dots, n$ as $[n]$, then the number of multisets of size k is denoted $\left(\!\binom{n}{k}\!\right)$.

Theorem. $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$

Proof. Observe that if we have a multiset on $[n]$, we can, without loss of generality, arrange it in increasing order. The set looks like $\{a_1 \le a_2 \le \cdots \le a_k\}$. We can map each such multiset to a unique set by adding 0 to the first element, 1 to the second, 2 to the third, and so on, up to adding $k-1$ to the last element. To see that this is a unique mapping, we can look at the inverse, where we take a set on $[n+k-1]$ and sort it in increasing order $\{b_1 < b_2 < \cdots < b_k\}$, then subtract 0 from the first element, 1 from the second, and so on, up to subtracting $k-1$ from the last element. Since this creates a bijective mapping between multisets of size k on $[n]$ and sets of size k on $[n+k-1]$, we have $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ as desired.

□

Compositions

Definition. A **composition** $\alpha = a_1, a_2, a_3, \dots$ of a natural number n is an ordered multiset of natural numbers such that $\sum \alpha_i = n$.

Definition. A **k-composition** of a natural number n is a composition of n into k parts.

Example. The compositions of 4 are $(4), (3,1), (1,3), (2,2), (2,1,1), (1,1,2), (1,2,1), (1,1,1,1)$.

How many k-compositions of n are there?

Theorem. *There are $\binom{n-1}{k-1}$ k-compositions of n.*

Proof. We proceed combinatorially. Imagine a string of n 1's. Between each, we can place a plus, indicating we should add those two (or more) adjacent 1's together to make a larger piece, or a comma, indicating that we should separate these two adjacent 1's into separate components. There are $n-1$ spots between the 1's, and we need to place $k-1$ commas to create a composition into k parts. There are clearly $\binom{n-1}{k-1}$ ways to do this, and we are done. $\square$

Lecture 2

The Generating Function for the Binomial Numbers

Let's more closely examine the binomial numbers $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$*. What does it look like for a set of size 1? There is one way to pick a subset of size zero and one way to pick a subset of size one, so the generating function is* $1+x$*. For two elements, we see something similar, where we can choose to either include or not include an element, so our generating function is* $(1+x)(1+x) = (1+x)^2$*. In general, the generating function for size* n *is* $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$*.*

Revisiting Compositions

Definition. A **weak composition** is a composition where some of the parts are allowed to be zero.

How many weak k-compositions are there? Now we are allowed to put more than one comma between two adjacent numbers, as long as we only use $k-1$ of them in total. If we still think about there being a plus between 1's where we don't have a comma, we have a number of weak k-compositions equal to the number of sequences of n 1's and $k-1$ commas. There are $\binom{n+k-1}{k-1}$ of these, and this is in bijection with the set of weak k-compositions, so we are done.

Definition. A **lattice path** from the origin to a point $(a,b) \preceq (0,0)$ is a sequence of steps up $(\uparrow)$ and right $(\rightarrow)$ along the $\mathbb{Z} \times \mathbb{Z}$ lattice beginning at the origin and ending at (a,b).

It's natural to ask how many lattice paths there are. We can think of a path from the origin to (a,b) as requiring a steps right and b steps up. Therefore, we can biject this with the set of binary sequences of length $a+b$ where exactly a of the bits are 1, and there are $\binom{a+b}{a}$ of these (equivalently by the symmetry of the binomial coefficients, we can think of exactly b of the bits being 1 instead and use $\binom{a+b}{b}$).

Multinomial Coefficients

Given some weak k-composition of n, $c_1, c_2, \dots, c_k \geq 0$, we might want to ask about how we can break $[n]$ into k disjoint parts such that each subset $S_i \subset [n]$ has exactly c_i elements. The number of ways to do this is to take c_1 elements from the set of size n, then take c_2 of the remaining $n - c_1$, and so on, giving the expression $\binom{n}{c_1, c_2, \dots, c_k} = \binom{n}{c_1}\binom{n-c_1}{c_2}\dots\binom{n-c_1-c_2-\dots-c_{k-1}}{c_k}$. A little bit of algebra (lots of things cross-cancel) reveals this to be $\frac{n!}{c_1!c_1!\dots c_{k-1}!}$. Using a similar argument to the construction of the generating function for the vanilla binomial numbers, we can think about the multinomial as looking like $(x_1 + x_2 + \dots + x_k)^n$, where picking an x_i from the jth term in the expansion corresponds to putting element i into subset S_j. We can therefore see that the generating function looks like $\sum \binom{n}{c_1 \dots c_k} x_1^{c_1} \dots x_k^{c_k}$.

Permutations

Definition. A **permutation** ω is a bijective map $\omega : [n] \to [n]$.

The number of permutations is $n!$. We can write each permutation in as a product of disjoint cycles (uniquely, up to ordering the cycles), and then look at how many cycles of each length there are.

The function $c(\omega) = (c_1, c_2, c_3 \ldots, c_n)$ is the ordered tuple where c_i is the number of cycles of length i. Obviously, $\sum ic_i = n$ as each element of $[n]$ must appear in exactly one cycle. There are $n!$ total permutations, but there are $c_1 c_2 \ldots c_n$ ways to order the cycles, so we need to divide by this quantity. We also have to worry about rotating the values within a cycle. For example, (123) is the same as (231) and (312), and we don't want to count these multiple times. Each 1-cycle can be written one way. Each 2-cycle two ways, each 3-cycle three ways, and so on. We therefore also have to divide by $1^{c_1} 2^{c_2} \ldots n^{c_n}$ to account for all of the ways to shift the c_k k-cycles. All together, the number of permutations with a particular $c(\omega)$ is equal to $\frac{n!}{c_1 c_2 \ldots c_n 1^{c_1} 2^{c_2} \ldots n^{c_n}}$.

What does the generating function for this thing look like? Let's start by defining a variable $Z_n = \frac{1}{n!} \sum\limits_{\omega \in S_n} t^{c(\omega)}$, where $t^{c(\omega)} = t_1^{c_1} t_2^{c_2} \ldots t_n^{c_n}$. We're interested in $\sum\limits_{n=0}^{\infty} Z_n x^n$. By plugging in what we did above, we get $\sum\limits_{n=0}^{\infty} Z_n x^n = \sum\limits_{c_1, c_2, \ldots, c_n} \frac{t_1^{c_1} \ldots t_n^{c_n} x^n}{1^{c_1} c_1! 2^{c_2} c_2! \ldots n^{c_n} c_n!}$. We can factor this apart as $\prod\limits_{k=1}^{\infty} \sum\limits_{c_k=0}^{\infty} \left(\frac{t_k x^k}{k}\right)^{c_k}$.

Lecture 3