Arrangement

- Experiments:
  - Exp1: Oct 20, and Oct 25
  - Exp2: Oct 27, and Nov 1
- Review (?)
  - Oct 27
- Exam
  - Nov 3

Artificial Intelligence

Introduction to probability (Chapter 13)

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Uncertainty

- So far in course, everything deterministic
- If I walk with my umbrella, I **will not** get wet
- But: there is some chance my umbrella will break!
- Intelligent systems must take possibility of failure into account…
  - May want to have backup umbrella in city that is often windy and rainy
- … but should not be excessively conservative
  - Two umbrellas not worthwhile for city that is usually not windy
- Need quantitative notion of uncertainty

Probability

- Example: roll two dice
- Random variables:
  - X = value of die 1
  - Y = value of die 2
- Outcome is represented by an ordered pair of values (x, y)
  - E.g., (6, 1): X=6, Y=1
  - Atomic event or sample point tells us the **complete** state of the world, i.e., values of **all** random variables
- Exactly one atomic event will happen; each atomic event has a ≥0 probability; sum to 1
  - E.g., P(X=1 and Y=6) = 1/36

Y						
6	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
1	1/36	1/36	1/36	1/36	1/36	1/36
	1	2	3	4	5	6

X

- An event is a proposition about the state (=subset of states)
  - X+Y = 7
- Probability of event = sum of probabilities of atomic events where event is true

Cards and combinatorics

- Draw a hand of 5 cards from a standard deck with 4*13 = 52 cards (4 suits, 13 ranks each)
- Each of the (52 choose 5) hands has same probability 1/(52 choose 5)
- Probability of event = number of hands in that event / (52 choose 5)
- What is the probability that…
  - no two cards have the same rank?
  - you have a flush (all cards the same suit?)
  - you have a straight (5 cards in order of rank, e.g., 8, 9, 10, J, Q)?
  - you have a straight flush?
  - you have a full house (three cards have the same rank and the two other cards have the same rank)?

Facts about probabilities of events

- If events A and B are disjoint, then
  - P(A or B) = P(A) + P(B)
- More generally:
  - P(A or B) = P(A) + P(B) - P(A and B)
- If events A_1, …, A_n are disjoint and exhaustive (one of them must happen) then $P(A_1) + \ldots + P(A_n) = 1$
  - Special case: for any random variable, $\sum_x P(X=x) = 1$
- Marginalization: $P(X=x) = \sum_y P(X=x \text{ and } Y=y)$

Conditional probability

- We might know something about the world – e.g., "X+Y=6 or X+Y=7" – given this (and **only** this), what is the probability of Y=5?
- Part of the sample space is eliminated; probabilities are renormalized to sum to 1

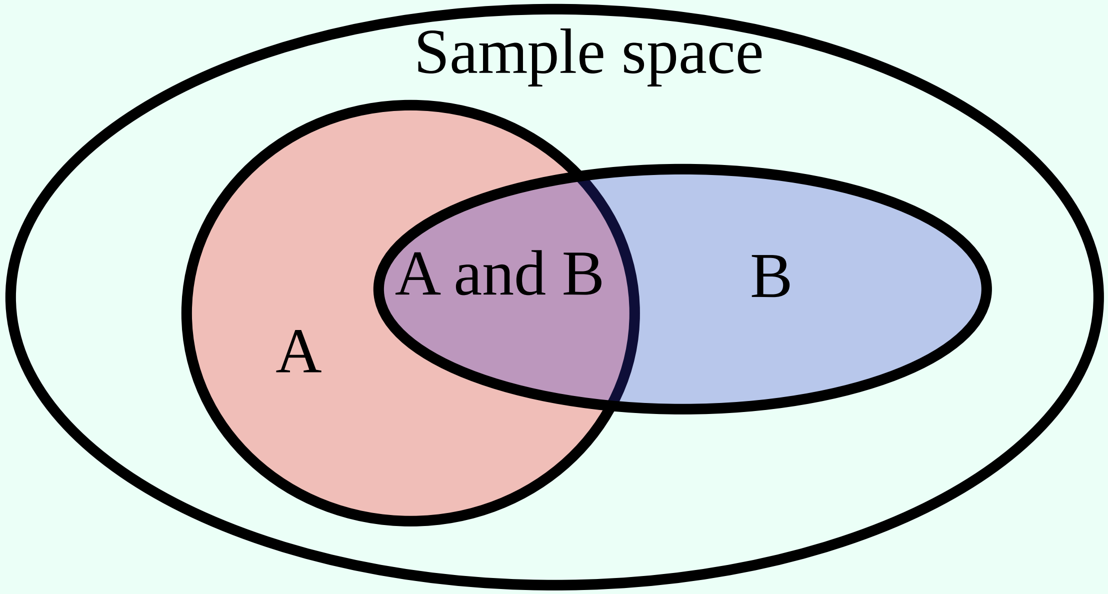
Y						
6	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
1	1/36	1/36	1/36	1/36	1/36	1/36
X	1	2	3	4	5	6

Y						
6	1/11	0	0	0	0	0
5	1/11	1/11	0	0	0	0
4	0	1/11	1/11	0	0	0
3	0	0	1/11	1/11	0	0
2	0	0	0	1/11	1/11	0
1	0	0	0	0	1/11	1/11
X	1	2	3	4	5	6

- P(Y=5 | (X+Y=6) or (X+Y=7)) = 2/11

Facts about conditional probability

- $P(A \mid B) = P(A \text{ and } B) / P(B)$



- $P(A \mid B)P(B) = P(A \text{ and } B) = P(B \mid A)P(A)$
- $P(A \mid B) = P(B \mid A)P(A)/P(B)$
  - Bayes' rule

Conditional probability and cards

- Given that your first two cards are Queens, what is the probability that you will get at least three Queens?
- Given that you have at least two Queens (not necessarily the first two), what is the probability that you have at least three Queens?
- Given that you have at least two Queens, what is the probability that you have three Kings?

How can we scale this?

- In principle, we now have a complete approach for reasoning under uncertainty:
  - Specify probability for every atomic event,
  - Can compute probabilities of events simply by summing probabilities of atomic events,
  - Conditional probabilities are specified in terms of probabilities of events: $P(A \mid B) = P(A \text{ and } B) / P(B)$
- If we have n variables that can each take k values, how many atomic events are there?

Independence

- Some variables have nothing to do with each other
- Dice: if X=6, it tells us nothing about Y
- $P(Y=y \mid X=x) = P(Y=y)$
- So: $P(X=x \text{ and } Y=y) = P(Y=y \mid X=x)P(X=x) = P(Y=y)P(X=x)$
  - Usually just write $P(X, Y) = P(X)P(Y)$
  - Only need to specify 6+6=12 values instead of 6*6=36 values
  - Independence among 3 variables: $P(X,Y,Z)=P(X)P(Y)P(Z)$, etc.
- Are the events “you get a flush” and “you get a straight” independent?

An example without cards or dice

	Rain in PlaceA	Sun in PlaceA
Rain in PlaceB	.2	.1
Sun in PlaceB	.2	.5

(disclaimer: no idea if these numbers are realistic)

- What is the probability of
  - Rain in PlaceA? Rain in PlaceB?
  - Rain in PlaceA, given rain in PlaceB?
  - Rain in PlaceB, given rain in PlaceA?
- Rain in PlaceA and rain in PlaceB are correlated

A possibly rigged casino

- With probability ½, the casino is rigged and has dice that come up *6* only 1/12 of the time, and *1* 3/12 of the time

Z=0 (fair casino)

Y \ X	1	2	3	4	5	6
6	1/72	1/72	1/72	1/72	1/72	1/72
5	1/72	1/72	1/72	1/72	1/72	1/72
4	1/72	1/72	1/72	1/72	1/72	1/72
3	1/72	1/72	1/72	1/72	1/72	1/72
2	1/72	1/72	1/72	1/72	1/72	1/72
1	1/72	1/72	1/72	1/72	1/72	1/72

Z=1 (rigged casino)

Y \ X	1	2	3	4	5	6
6	1/96	1/144	1/144	1/144	1/144	1/288
5	1/48	1/72	1/72	1/72	1/72	1/144
4	1/48	1/72	1/72	1/72	1/72	1/144
3	1/48	1/72	1/72	1/72	1/72	1/144
2	1/48	1/72	1/72	1/72	1/72	1/144
1	1/32	1/48	1/48	1/48	1/48	1/96

- What is P(Y=6)?
- What is P(Y=6|X=1)?
- Are they independent?

Conditional independence

- Intuition:
  - the only reason that X tells us something about Y,
  - is that X tells us something about Z,
  - and Z tells us something about Y
- If we already know Z, then X tells us nothing about Y
- $P(Y \mid Z, X) = P(Y \mid Z)$ or
- $P(X, Y \mid Z) = P(X \mid Z)P(Y \mid Z)$
- "X and Y are conditionally independent given Z"

Medical diagnosis

- X: does patient have flu?
- Y: does patient have headache?
- Z: does patient have fever?
- $P(Y,Z|X) = P(Y|X)P(Z|X)$
- $P(X=1) = .2$
- $P(Y=1 \mid X=1) = .5$, $P(Y=1 \mid X=0) = .2$
- $P(Z=1 \mid X=1) = .4$, $P(Z=1 \mid X=0) = .1$
- What is $P(X=1|Y=1,Z=0)$?

(disclaimer: no idea if these numbers are realistic)

Conditioning can also **introduce** dependence

- X: is it raining?
  - P(X=1) = .3
- Y: are the sprinklers on?
  - P(Y=1) = .4
  - X and Y are independent
- Z: is the grass wet?
  - P(Z=1 | X=0, Y=0) = .1
  - P(Z=1 | X=0, Y=1) = .8
  - P(Z=1 | X=1, Y=0) = .7
  - P(Z=1 | X=1, Y=1) = .9

Not wet

	Raining	Not raining
Sprinklers	.012	.056
No sprinklers	.054	.378

Wet

	Raining	Not raining
Sprinklers	.108	.224
No sprinklers	.126	.042

- Conditional on Z=1, X and Y are **not** independent
- If you know Z=1, rain seems likely; then if you also find out Y=1, this “explains away” the wetness and rain seems less likely

Monty Hall problem

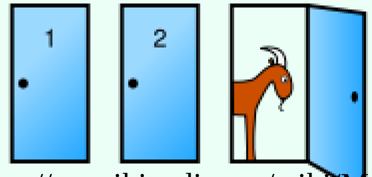


image taken from http://en.wikipedia.org/wiki/Monty_Hall_problem

- Game show participants can choose one of three doors
- One door has a car, two have a goat
  - Assumption: car is preferred to goat
- Participant chooses door, but not opened yet
- At least one of the other doors contains a goat; the (knowing) host will open one such door (flips coin to decide if both have goats)
- Participant is asked whether she wants to switch doors (to the other closed door) – should she?

Monty Hall problem

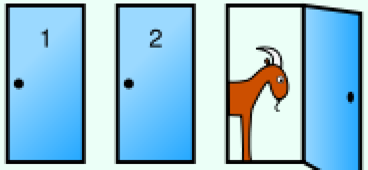


image taken from http://en.wikipedia.org/wiki/Monty_Hall_problem

- Ci : car is behand door i; Xi : player chose door i;

  Hi : host opened door i;

- Consider X1 and H3

$$P(H3|C1, X1) = \frac{1}{2}$$
$$P(H3|C2, X1) = 1$$
$$P(H3|C3, X1) = 0$$
$$P(Ci) = \frac{1}{3}$$
$$P(Ci, Xi) = P(Ci)P(Xi)$$
$$P(H3|X1) = \frac{1}{2}$$

$$P(C2|H3, X1) = \frac{P(C2, H3, X1)}{P(H3, X1)} = \frac{P(H3|C2, X1)P(C2, X1)}{P(H3, X1)}$$

$$= \frac{P(C2)P(X1)}{P(H3|X1)P(X1)} = \frac{1/3}{1/2} = \frac{2}{3}$$

Expected value

- If Z takes numerical values, then the expected value of Z is $E(Z) = \sum_z P(Z=z)*z$
  - Weighted average (weighted by probability)
- Suppose Z is sum of two dice
- $E(Z) = (1/36)*2 + (2/36)*3 + (3/36)*4 + (4/36)*5 + (5/36)*6 + (6/36)*7 + (5/36)*8 + (4/36)*9 + (3/36)*10 + (2/36)*11 + (1/36)*12 = 7$
- Simpler way: $E(X+Y)=E(X)+E(Y)$ (always!)
  - Linearity of expectation
- $E(X) = E(Y) = 3.5$

Linearity of expectation…

- If a is used to represent an atomic state, then $E(X) = \sum_x P(X=x)*x = \sum_x (\sum_{a:X(a)=x} P(a))*x = \sum_a P(a)*X(a)$
- $E(X+Y) = \sum_a P(a)*(X(a)+Y(a)) = \sum_a P(a)*X(a) + \sum_a P(a)*Y(a) = E(X)+E(Y)$