Formal Methods（形式化方法）

Lecture 14. Reasoning about Specifications

智能与计算学部 章衡

2021年上学期

Motivation

Features of Z notation

- By using Z notations one can define the specification precisely, which could reduce the misunderstandings in requirement analyses largely

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What can be done by reasoning

- How to assure the specification admitting a desired property?
- How to know whether a program meets the requirements stated in the specification?

Outline

1. Introduction by Example

2. Rigorous Proofs

3. Reasoning about Specifications

Example: Hobby club

- Basic type:

$$[Person]$$

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- Global variable:

  $$Max : \mathbb{N}$$

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  HoClub

  $$s : \mathbb{P}\,\text{Person}$$

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$$\Delta \mathrm{HoClub} \mathrel{\widehat=} \mathrm{HoClub} \wedge \mathrm{HoClub}'$$

$$\Xi \mathrm{HoClub} \mathrel{\widehat=} \Delta \mathrm{HoClub} \mid s' = s$$

Example: Hobby club

EnterClub

$\Delta HoClub$
$p? : Person$

$\#s < Max$
$p? \notin s$
$s' = s \cup \{p?\}$

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$\Delta HoClub$
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LeaveClub

$\Delta HoClub$
$p? : Person$

$p? \in s$
$s' = s \setminus \{p?\}$

Example: Hobby club

$$\text{EnterClub} \mathbin{⨟} \text{LeaveClub} \vDash \#s < \text{Max} \land s' = s$$

Example: Hobby club

$$\mathrm{EnterClub} \fatsemi \mathrm{LeaveClub} \vDash \#s < \mathrm{Max} \land s' = s$$

$$\mathrm{Alpha} \mathrel{\widehat{=}} \mathrm{EnterClub} \fatsemi \mathrm{LeaveClub}$$

Example: Hobby club

$$\mathrm{EnterClub} \fatsemi \mathrm{LeaveClub} \vDash \#s < \mathrm{Max} \land s' = s$$

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Alpha

$s, s' : \mathbb{P}\,\mathrm{Person}$
$p? : \mathrm{Person}$

$$\begin{aligned}
&\exists\, s^{+} : \mathbb{P}\,\mathrm{Person} \bullet \\
&\qquad (\#s \leqslant \mathrm{Max} \land \\
&\qquad \#s^{+} \leqslant \mathrm{Max} \land \\
&\qquad \#s' \leqslant \mathrm{Max} \land \\
&\qquad \#s < \mathrm{Max} \land \\
&\qquad p? \notin s \land \\
&\qquad s^{+} = s \cup \{p?\} \land \\
&\qquad p? \in s^{+} \land \\
&\qquad s' = s^{+} \setminus \{p?\})
\end{aligned}$$

Example: Hobby club

If x does not occur in φ, then $\exists\, x : X \bullet (\varphi \wedge \psi) \equiv \varphi \wedge \exists\, x : X \bullet \psi$

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If x does not occur in φ, then $\exists x : X \bullet (\varphi \land \psi) \equiv \varphi \land \exists x : X \bullet \psi$

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$s, s' : \mathbb{P}\,\mathrm{Person}$
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&\exists s^+ : \mathbb{P}\,\mathrm{Person} \bullet \\
&\qquad (\#s \leqslant \mathrm{Max} \land \\
&\qquad \#s^+ \leqslant \mathrm{Max} \land \\
&\qquad \#s' \leqslant \mathrm{Max} \land \\
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&\qquad p? \notin s \land \\
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&\qquad p? \in s^+ \land \\
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\end{aligned}$$

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$$Alpha \vDash Alpha_1$$

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$$
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\hline
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p? : Person \\
\hline
\#s \leqslant Max \\
\#s' \leqslant Max \\
\#s < Max \\
p? \notin s \\
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\qquad \#s^+ \leqslant Max \wedge \\
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\hline
\end{array}
$$

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By applying the 1-point rule, we have

$$\mathrm{Alpha} \vDash \mathrm{Alpha}_1 \vDash \mathrm{Alpha}_2$$

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s, s' : \mathbb{P}\,Person \\
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\end{array}
$$

Alpha_2

$$
\begin{array}{l}
s, s' : \mathbb{P}\,Person \\
p? : Person \\
\hline
\#s \leqslant Max \\
\#s' \leqslant Max \\
\#s < Max \\
p? \notin s \\
\#(s \cup \{p?\}) \leqslant Max \\
p? \in (s \cup \{p?\}) \\
s' = (s \cup \{p?\}) \setminus \{p?\})
\end{array}
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From $p? \notin s$, we know that $(s \cup \{p?\}) \setminus \{p?\} = s$. Consequently,

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s, s' : \mathbb{P}\,Person \\
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p? \in (s \cup \{p?\}) \\
s' = s \\
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\end{array}$$

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- In many cases, mathematicians try to find a weaker form of formal proofs, called rigorous proofs
- They believe that every rigorous proof can be converted into a formal proof
- In a rigorous proof, one is allowed to use the properties in set theory and number theory, as well as the method of induction

Method of induction

Definition (Mathematical induction, 数学归纳法)

To prove "for every natural number n it holds that $P(n)$", it suffices to prove both of the following:

1. $P(0)$ holds;
2. $\forall i : \mathbb{N} \bullet (P(i) \Rightarrow P(i+1))$.

Definition (Structural induction, 结构归纳法)

To prove "for every sequence $s : \operatorname{seq} X$ it holds that $P(s)$", it suffices to prove both of the following:

1. $P(\langle\rangle)$ holds;
2. $\forall x : X; s : \operatorname{seq} X \bullet (P(s) \Rightarrow P(\langle x \rangle \frown s))$.

Method of induction: Example 1

Example

Please prove that, for all sequences $s, t, u : \mathrm{seq}\, X$, we have

$$s \frown (t \frown u) = (s \frown t) \frown u.$$

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$$\begin{aligned} (\langle x \rangle \frown s) \frown (t \frown u) &= \langle x \rangle \frown (s \frown (t \frown u)) \\ &= \langle x \rangle \frown ((s \frown t) \frown u) \end{aligned}$$

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which completes the proof. □

Method of induction: Example 2

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Inductive step: Assume as inductive hypothesis that $\mathrm{rev}(s \frown t) = (\mathrm{rev}\, t) \frown (\mathrm{rev}\, s)$. We need to prove that $\mathrm{rev}((\langle x\rangle \frown s) \frown t) = (\mathrm{rev}\, t) \frown \mathrm{rev}(\langle x\rangle \frown s)$.

Method of induction: Example 2

Example

Please prove that, for all sequences s, t : seq X, we have

$$\mathrm{rev}(s \frown t) = (\mathrm{rev}\, t) \frown (\mathrm{rev}\, s)$$

Proof.

By definition, it is easy to see that $\langle\rangle \frown s = s = s \frown \langle\rangle$ and $\mathrm{rev}(\langle x\rangle \frown t) = (\mathrm{rev}\, t) \frown \langle x\rangle$. Next we prove the desired property by an induction on s.

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$$\mathrm{rev}((\langle x\rangle \frown s) \frown t)$$

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$$\mathrm{rev}((\langle x\rangle \frown s) \frown t) \quad = \quad \mathrm{rev}(\langle x\rangle \frown (s \frown t))$$

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&= \mathrm{rev}(s \frown t) \frown \langle x\rangle
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Method of induction: Example 2

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Example

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&= (\mathrm{rev}\, t) \frown ((\mathrm{rev}\, s) \frown \langle x\rangle) \\
&= (\mathrm{rev}\, t) \frown \mathrm{rev}(\langle x\rangle \frown s),
\end{aligned}$$

which completes the proof. □

Exercise

Prove the following by induction: for every sequence s, we have that $\mathrm{rev}(\mathrm{rev}\,\mathrm{s}) = \mathrm{s}$.

Outline

Example: Fan ID management

- Basic types:

  [Person, ID]

Example: Fan ID management

- Basic types:

  $$[Person, ID]$$

- State space schema:

$$
\begin{array}{|l}
\hline
\textbf{FID} \\
members : ID \rightarrowtail\!\!\!\!\!\!\!\!\to\!\!\!\!\to Person \\
banned : \mathbb{P}\, ID \\
\hline
banned \subseteq \operatorname{dom} members \\
\hline
\end{array}
$$

Example: Fan ID management

- Basic types:

  $[Person, ID]$

- State space schema:

FID

$$members : ID \rightarrowtail\!\!\!\!\!\leftrightarrow Person$$
$$banned : \mathbb{P}\, ID$$

$$banned \subseteq \mathrm{dom}\, members$$

FID′

$$members' : ID \rightarrowtail\!\!\!\!\!\leftrightarrow Person$$
$$banned' : \mathbb{P}\, ID$$

$$banned' \subseteq \mathrm{dom}\, members'$$

Example: Fan ID management

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  $[Person, ID]$

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\qquad
\begin{array}{|l}
\hline
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banned' : \mathbb{P}\, ID \\
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banned' \subseteq \operatorname{dom} members' \\
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- $\Delta FID \mathrel{\widehat{=}} FID \wedge FID'$

Example: Fan ID management

- Basic types:

  $[Person, ID]$

- State space schema:

$$\begin{array}{|l}
\multicolumn{1}{l}{\underline{\quad FID \quad\quad\quad\quad\quad\quad\quad\quad}} \\
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\hline
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\hline
\end{array}
\qquad
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- $\Delta FID \mathrel{\widehat{=}} FID \land FID'$

  $\Xi FID \mathrel{\widehat{=}} \Delta FID \mid members' = members \land banned' = banned$

The initialization theorem (初始化定理)

- Operational schemas: Initialization

$$
\begin{array}{|l}
\hline
\textit{InitFID} \\
\mathrm{FID}' \\
\hline
\mathrm{members}' = \emptyset \\
\mathrm{banned}' = \emptyset \\
\hline
\end{array}
$$

The initialization theorem (初始化定理)

- Operational schemas: Initialization

$$
\begin{array}{|l}
\hline
\textit{InitFID} \\
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- The initialization theorem: $\models \exists \mathit{FID}' \bullet \mathit{InitFID}$

The initialization theorem (初始化定理)

- Operational schemas: Initialization

InitFID
FID'
$members' = \emptyset$ $banned' = \emptyset$

- The initialization theorem: $\models \exists FID' \bullet InitFID$

  The above is an abbreviation of the following theorem:

$$\models \exists members' : Person \rightarrowtail ID; banned' : \mathbb{P}\, ID \bullet$$
$$(banned' \subseteq \operatorname{dom} members' \land members' = \emptyset \land banned' = \emptyset)$$

Prove the initialization theorem

$$\vDash \exists\, members' : Person \rightarrowtail\!\!\!\!\!\rightarrow ID; banned' : \mathbb{P}\, ID \bullet \\ (banned' \subseteq \operatorname{dom} members' \wedge members' = \emptyset \wedge banned' = \emptyset) \tag{1}$$

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$$\models \exists\, members' : Person \rightarrowtail\!\!\!\!\!\rightarrow ID; banned' : \mathbb{P}\, ID \bullet \\ (banned' \subseteq \operatorname{dom} members' \land members' = \emptyset \land banned' = \emptyset) \tag{1}$$

1-point rule (bidirection)

$$\frac{\Sigma \models \exists\, x : S \bullet (\varphi \land x = t)}{\Sigma \models t \in S \land \varphi[t/x]} \quad [\text{1-point}] \quad \text{<x does not occur in t>}$$

Prove the initialization theorem

$$\vDash \exists\, members' : Person \rightarrowtail\!\!\!\!\twoheadrightarrow ID; banned' : \mathbb{P}\, ID \bullet \\ (banned' \subseteq \operatorname{dom} members' \wedge members' = \emptyset \wedge banned' = \emptyset) \quad (1)$$

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- By applying the above rule, (1) can be simplified as

$$\vDash \emptyset \in Person \rightarrowtail\!\!\!\!\twoheadrightarrow ID \wedge \emptyset \in \mathbb{P}\, ID \wedge \emptyset \subseteq \operatorname{dom} \emptyset \quad (2)$$

Prove the initialization theorem

$$\models \exists \mathrm{members}' : \mathrm{Person} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathrm{ID}; \mathrm{banned}' : \mathbb{P}\,\mathrm{ID} \bullet \\ (\mathrm{banned}' \subseteq \mathrm{dom}\,\mathrm{members}' \wedge \mathrm{members}' = \emptyset \wedge \mathrm{banned}' = \emptyset) \quad (1)$$

1-point rule (bidirection)

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- By applying the above rule, (1) can be simplified as

$$\models \emptyset \in \mathrm{Person} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathrm{ID} \wedge \emptyset \in \mathbb{P}\,\mathrm{ID} \wedge \emptyset \subseteq \mathrm{dom}\,\emptyset \quad (2)$$

- To prove this, it is equivalent to prove all of the following:

$$\models \emptyset \in \mathrm{Person} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathrm{ID},$$
$$\models \emptyset \in \mathbb{P}\,\mathrm{ID},$$
$$\models \emptyset \subseteq \mathrm{dom}\,\emptyset.$$

Precondition of an operation

AddMember

$$\Delta FID$$
$$applicant? : Person$$
$$id! : ID$$

$$applicant? \notin \operatorname{ran} members$$
$$id! \notin \operatorname{dom} members$$
$$members' = members \cup \{id! \mapsto applicant?\}$$
$$banned' = banned$$

Precondition of an operation

$$
\begin{array}{|l}
\textit{AddMember} \\
\hline
\Delta FID \\
applicant? : Person \\
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\hline
\end{array}
$$

- We need to know when the operation can be executed.

Precondition of an operation

AddMember

ΔFID

$applicant? : Person$

$id! : ID$

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$id! \notin \operatorname{dom} members$

$members' = members \cup \{id! \mapsto applicant?\}$

$banned' = banned$

- We need to know when the operation can be executed.
- If such a condition is not true, we need to report an error.

Precondition of an operation

$$
\begin{array}{l}
\textbf{PreAddMember} \\
\hline
FID \\
applicant? : Person \\
\hline
\exists\, FID'; id! : ID \bullet \\
\qquad (applicant? \notin \operatorname{ran} members \land \\
\qquad\ id! \notin \operatorname{dom} members \land \\
\qquad\ members' = members \cup \{id! \mapsto applicant?\} \land \\
\qquad\ banned' = banned) \\
\hline
\end{array}
$$

Precondition of an operation

PreAddMember

FID
$applicant? : Person$

$$\exists FID'; id! : ID \bullet$$
$$(applicant? \notin \operatorname{ran} members \land$$
$$id! \notin \operatorname{dom} members \land$$
$$members' = members \cup \{id! \mapsto applicant?\} \land$$
$$banned' = banned)$$

- Unfolding the predicate of the above schema, we have

$$\exists members' : ID \rightarrowtail\!\!\!\!\!\!\twoheadrightarrow Person; banned' : \mathbb{P}\, ID; id! : ID \bullet$$
$$(banned' \subseteq \operatorname{dom} members' \land applicant? \notin \operatorname{ran} members \land$$
$$id! \notin \operatorname{dom} members \land members' = members \cup \{id! \mapsto applicant?\} \land$$
$$banned' = banned)$$

Simplification of precondition

Most often used rules for precondition simplification

$$\frac{\Sigma \vDash \exists x : S \bullet (\varphi \wedge x = t)}{\Sigma \vDash t \in S \wedge \varphi[t/x]} \quad [\text{1-point}] \quad <x \text{ does not occur in } t>$$

$$\frac{\Sigma \vDash \varphi \wedge \psi}{\Sigma \vDash \varphi} \quad [\wedge] \quad <\Sigma, \varphi \vDash \psi>$$

$$\frac{\Sigma \vDash \varphi}{\Sigma \vDash \varphi'} \quad [=] \quad <\Sigma \vDash t_1 = t_2 \text{ and } \varphi' \text{ is obtained from } \varphi \text{ by substituting } t_2 \text{ for some occurrence of } t_1>$$

Simplification of precondition

Most often used rules for precondition simplification

$$\frac{\Sigma \vDash \exists x : S \bullet (\varphi \wedge x = t)}{\Sigma \vDash t \in S \wedge \varphi[t/x]} \quad \text{[1-point]} \quad \langle x \text{ does not occur in } t\rangle$$

$$\frac{\Sigma \vDash \varphi \wedge \psi}{\Sigma \vDash \varphi} \quad [\wedge] \quad \langle \Sigma, \varphi \vDash \psi \rangle$$

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Simplification of precondition

$$\begin{aligned}
&\exists\, \mathrm{members}' : \mathrm{ID} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathrm{Person}; \mathrm{banned}' : \mathbb{P}\,\mathrm{ID}; \mathrm{id!} : \mathrm{ID} \bullet \\
&\qquad (\mathrm{banned}' \subseteq \operatorname{dom} \mathrm{members}' \land \mathrm{applicant?} \notin \operatorname{ran} \mathrm{members} \land \\
&\qquad\ \mathrm{id!} \notin \operatorname{dom} \mathrm{members} \land \mathrm{members}' = \mathrm{members} \cup \{\mathrm{id!} \mapsto \mathrm{applicant?}\} \land \\
&\qquad\ \mathrm{banned}' = \mathrm{banned})
\end{aligned} \tag{3}$$

Simplification of precondition

$$\begin{aligned}&\exists\, members' : ID \rightarrowtail\!\!\!\!\!\!\twoheadrightarrow Person; banned' : \mathbb{P}\, ID; id! : ID \bullet\\ &\quad (banned' \subseteq \mathrm{dom}\, members' \land applicant? \notin \mathrm{ran}\, members \land\\ &\quad id! \notin \mathrm{dom}\, members \land members' = members \cup \{id! \mapsto applicant?\} \land\\ &\quad banned' = banned)\end{aligned} \tag{3}$$

- By applying 1-point rule for variable $banned'$, (3) can be simplified as

$$\begin{aligned}&\exists\, members' : ID \rightarrowtail\!\!\!\!\!\!\twoheadrightarrow Person; id! : ID \bullet\\ &\quad (banned \subseteq \mathrm{dom}\, members' \land applicant? \notin \mathrm{ran}\, members \land\\ &\quad id! \notin \mathrm{dom}\, members \land members' = members \cup \{id! \mapsto applicant?\} \land\\ &\quad banned \in \mathbb{P}\, ID)\end{aligned} \tag{4}$$

Simplification of precondition

$$\begin{aligned} &\exists\, members' : ID \rightarrowtail\!\!\!\!\!\twoheadrightarrow Person; banned' : \mathbb{P}\, ID; id! : ID \bullet \\ &\quad (banned' \subseteq \mathrm{dom}\, members' \land applicant? \notin \mathrm{ran}\, members \land \\ &\quad id! \notin \mathrm{dom}\, members \land members' = members \cup \{id! \mapsto applicant?\} \land \\ &\quad banned' = banned) \end{aligned} \tag{3}$$

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$$\begin{aligned} &\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \land applicant? \notin \mathrm{ran}\, members \land \\ &\quad id! \notin \mathrm{dom}\, members \land members \cup \{id! \mapsto applicant?\} \in ID \rightarrowtail\!\!\!\!\!\twoheadrightarrow Person \land \\ &\quad banned \in \mathbb{P}\, ID) \end{aligned} \tag{5}$$

Simplification of precondition

$$\begin{aligned}\exists\, id! : ID \bullet (&banned \subseteq \operatorname{dom}(members \cup \{id! \mapsto applicant?\}) \land applicant? \notin \operatorname{ran} members \land \\ &id! \notin \operatorname{dom} members \land members \cup \{id! \mapsto applicant?\} \in ID \rightarrowtail\!\!\!\twoheadrightarrow Person \land \\ &banned \in \mathbb{P}\, ID)\end{aligned} \tag{6}$$

Simplification of precondition

$$\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \wedge applicant? \notin \mathrm{ran}\, members \wedge id! \notin \mathrm{dom}\, members \wedge members \cup \{id! \mapsto applicant?\} \in ID \rightarrowtail\!\!\!\!\!\!\twoheadrightarrow Person \wedge banned \in \mathbb{P}\, ID) \tag{6}$$

- By the declaration $banned : \mathbb{P}\, ID$ we know $banned \in \mathbb{P}\, ID$. Consequently, (6) can be equivalently rewritten as

$$\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \wedge applicant? \notin \mathrm{ran}\, members \wedge id! \notin \mathrm{dom}\, members \wedge members \cup \{id! \mapsto applicant?\} \in ID \rightarrowtail\!\!\!\!\!\!\twoheadrightarrow Person) \tag{7}$$

Simplification of precondition

$$\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \land applicant? \notin \mathrm{ran}\, members \land \\ id! \notin \mathrm{dom}\, members \land members \cup \{id! \mapsto applicant?\} \in ID \rightarrowtail\!\!\!\!\twoheadrightarrow Person \land \\ banned \in \mathbb{P}\, ID) \tag{6}$$

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- By $members : ID \rightarrowtail\!\!\!\!\twoheadrightarrow Person; id! : ID; applicant? : Person$ and $id! \notin \mathrm{dom}\, members$, we have that $members \cup \{id! \mapsto applicant?\} \in ID \twoheadrightarrow\!\!\!\!\!+\, Person$. By $applicant? \notin \mathrm{ran}\, members$, we obtain that $members \cup \{id! \mapsto applicant?\} \in ID \rightarrowtail\!\!\!\!\twoheadrightarrow Person$. Thus, (7) can be simplified as

$$\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \land applicant? \notin \mathrm{ran}\, members \land \\ id! \notin \mathrm{dom}\, members) \tag{8}$$

Simplification of precondition

$$\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \land applicant? \notin \mathrm{ran}\, members \land id! \notin \mathrm{dom}\, members) \tag{9}$$

Simplification of precondition

$$\exists \mathit{id!} : ID \bullet (\mathit{banned} \subseteq \mathrm{dom}(\mathit{members} \cup \{\mathit{id!} \mapsto \mathit{applicant?}\}) \land \mathit{applicant?} \notin \mathrm{ran}\,\mathit{members} \land \mathit{id!} \notin \mathrm{dom}\,\mathit{members}) \tag{9}$$

- By properties $\mathrm{dom}(A \cup B) = \mathrm{dom}\,A \cup \mathrm{dom}\,B$ and $\mathrm{dom}\{\mathit{id!} \mapsto \mathit{applicant?}\} = \{\mathit{id!}\}$, we conclude that $\mathrm{dom}(\mathit{members} \cup \{\mathit{id!} \mapsto \mathit{applicant?}\} = (\mathrm{dom}\,\mathit{members}) \cup \{\mathit{id!}\}$. Thus, (9) can be simplified as

$$\exists \mathit{id!} : ID \bullet (\mathit{banned} \subseteq (\mathrm{dom}\,\mathit{members}) \cup \{\mathit{id!}\} \land \mathit{applicant?} \notin \mathrm{ran}\,\mathit{members} \land \mathit{id!} \notin \mathrm{dom}\,\mathit{members}) \tag{10}$$

Simplification of precondition

$$\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \wedge applicant? \notin \mathrm{ran}\, members \wedge id! \notin \mathrm{dom}\, members) \quad (9)$$

- By properties $\mathrm{dom}(A \cup B) = \mathrm{dom}\, A \cup \mathrm{dom}\, B$ and $\mathrm{dom}\{id! \mapsto applicant?\} = \{id!\}$, we conclude that $\mathrm{dom}(members \cup \{id! \mapsto applicant?\} = (\mathrm{dom}\, members) \cup \{id!\}$. Thus, (9) can be simplified as

$$\exists\, id! : ID \bullet (banned \subseteq (\mathrm{dom}\, members) \cup \{id!\} \wedge applicant? \notin \mathrm{ran}\, members \wedge id! \notin \mathrm{dom}\, members) \quad (10)$$

- By the definition of FID we know that $banned \subseteq \mathrm{dom}\, members$. Thus, (10) can be simplified as

$$\exists\, id! : ID \bullet (applicant? \notin \mathrm{ran}\, members \wedge id! \notin \mathrm{dom}\, members)$$

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$$\begin{aligned} & \exists\, id! : ID \bullet (applicant? \notin \operatorname{ran} members \land id! \notin \operatorname{dom} members) \\ \equiv\ & applicant? \notin \operatorname{ran} members \land \exists\, id! : ID \bullet id! \notin \operatorname{dom} members \end{aligned} \tag{11}$$

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$$\equiv \quad applicant? \notin \mathrm{ran}\, members \wedge \exists\, id! : ID \bullet id! \notin \mathrm{dom}\, members \tag{12}$$
$$\equiv \quad applicant? \notin \mathrm{ran}\, members \wedge \mathrm{dom}\, members \neq ID \tag{13}$$

Simplification of precondition

$$
\begin{array}{|l}
\hline
\textit{PreAddMember} \\
\hline
\textit{FID} \\
\textit{applicant?} : \textit{Person} \\
\hline
\textit{applicant?} \notin \operatorname{ran} \textit{members} \\
\operatorname{dom} \textit{members} \neq \textit{ID} \\
\hline
\end{array}
$$

Simplified precondition schema PreAddMember