Bayesian networks (Chapter 14)

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Why Joint Distributions are Important

- Joint distributions gives P(X1…Xn)
- Can calculate probability of any event, then be used for:
  - Classification/Diagnosis… P(X1|X2…Xn)
  - Co-occurrence…P(X1X2)

Specifying probability distributions

- Specifying a probability for every atomic event is impractical
- $P(X_1, \ldots, X_n)$ would need to be specified for **every** combination $x_1, \ldots, x_n$ of values for $X_1, \ldots, X_n$
  - If there are k possible values per variable…
  - … we need to specify $k^n - 1$ probabilities!
- We have already seen it can be easier to specify probability distributions by using (conditional) independence
- **Bayesian networks** allow us
  - to specify any distribution,
  - to specify such distributions concisely if there is (conditional) independence, in a natural way

Conditional Independence

- We say that two variables, A and B, are conditionally independent given C if:
  - $P(A|BC) = P(A|C)$
  - $P(AB|C) = P(A|C)P(B|C)$
- How does this help?
- We **store only a conditional probability table (CPT)** of each variable given its parents

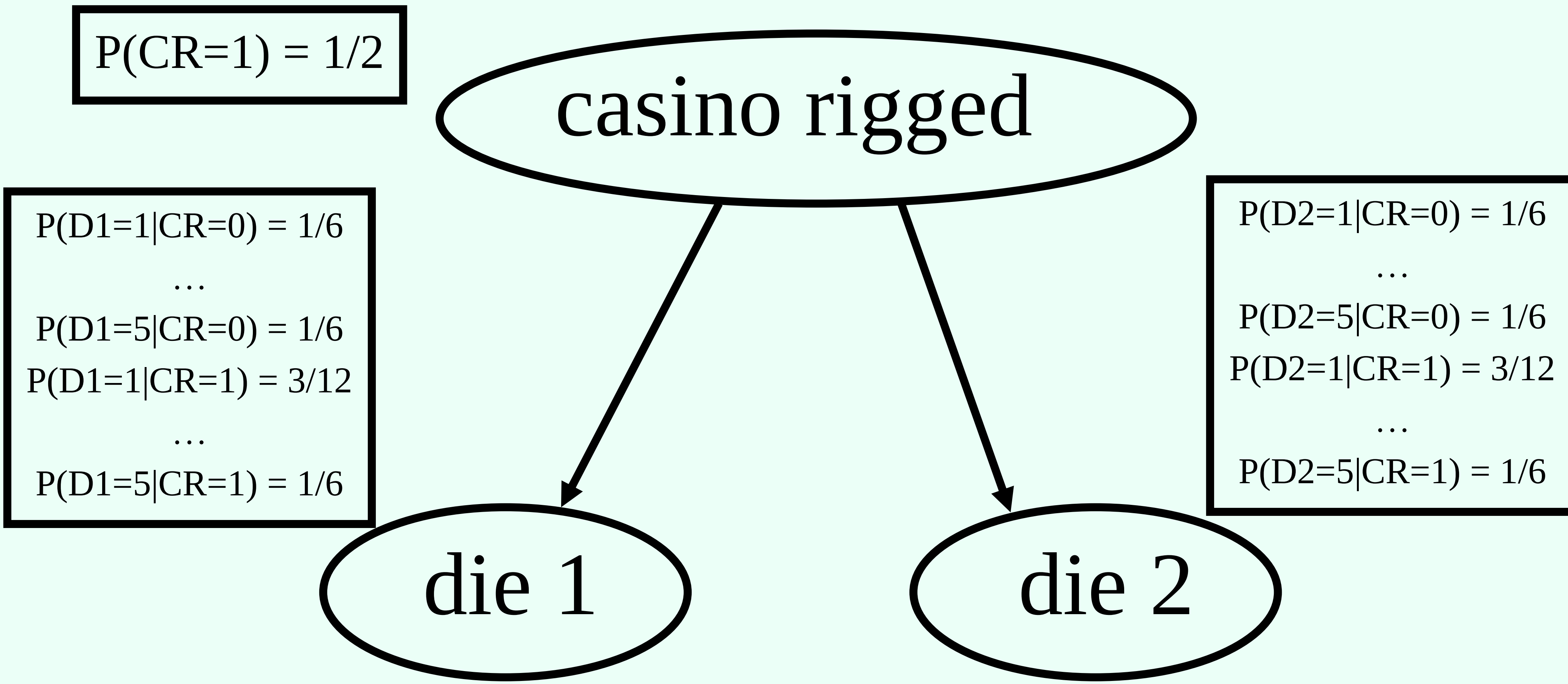
What is Bayes Net?

- A directed acyclic graph (DAG)
- Given parents, each variable is *independent of non-descendants*
- Joint probability decomposes:

$$P(x_1 \ldots x_n) = \prod_i P(x_i \mid \text{parents}(x_i))$$

- For each node X_i, store $P(X_i \mid \text{parents}(X_i))$
- Call this a Conditional Probability Table (CPT)
- CPT size is exponential in number of parents

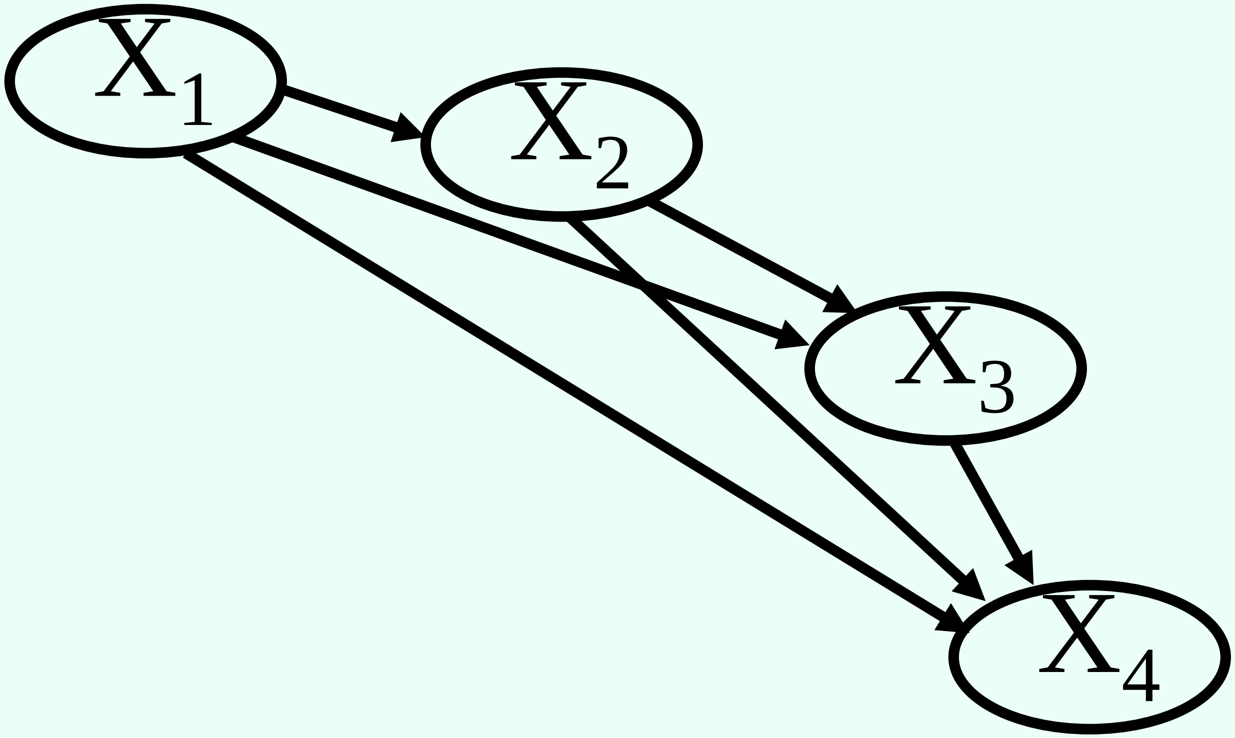
Rigged casino example



A general approach to specifying probability distributions

- Say the variables are $X_1, \ldots, X_n$
- $P(X_1, \ldots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1,X_2)\ldots P(X_n|X_1, \ldots, X_{n-1})$
- or:
- $P(X_1, \ldots, X_n) = P(X_n)P(X_{n-1}|X_n)P(X_{n-2}|X_n,X_{n-1})\ldots P(X_1|X_n, \ldots, X_2)$
- Can specify every component
  - For **every** combination of values for the variables on the right of |, specify the probability over the values for the variable on the left
- If every variable can take k values,
- $P(X_i|X_1, \ldots, X_{i-1})$ requires $(k-1)k^{i-1}$ values
- $\Sigma_{i=\{1,..,n\}}(k-1)k^{i-1} = \Sigma_{i=\{1,..,n\}}k^i-k^{i-1} = k^n - 1$
- Same as specifying probabilities of all atomic events – of course, because we can specify any distribution!

Graphically representing influences



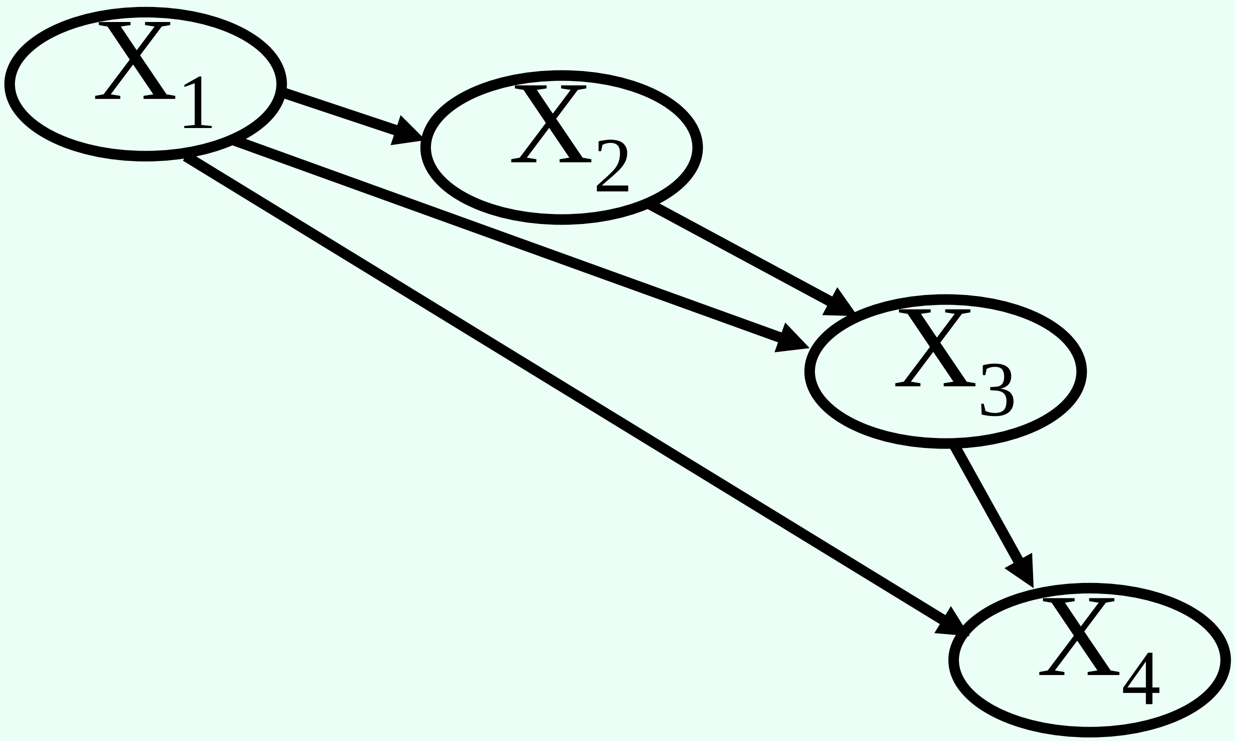
$$P(X_1, \ldots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1,X_2)\ldots P(X_n|X_1, \ldots, X_{n-1})$$

Conditional independence to the rescue!

- Problem: $P(X_i|X_1, \ldots, X_{i-1})$ requires us to specify too many values
- Suppose $X_1, \ldots, X_{i-1}$ partition into two subsets, S and T, so that X_i is conditionally independent from T given S
- $P(X_i|X_1, \ldots, X_{i-1}) = P(X_i|S, T) = P(X_i|S)$
- Requires only $(k-1)k^{|S|}$ values instead of $(k-1)k^{i-1}$ values

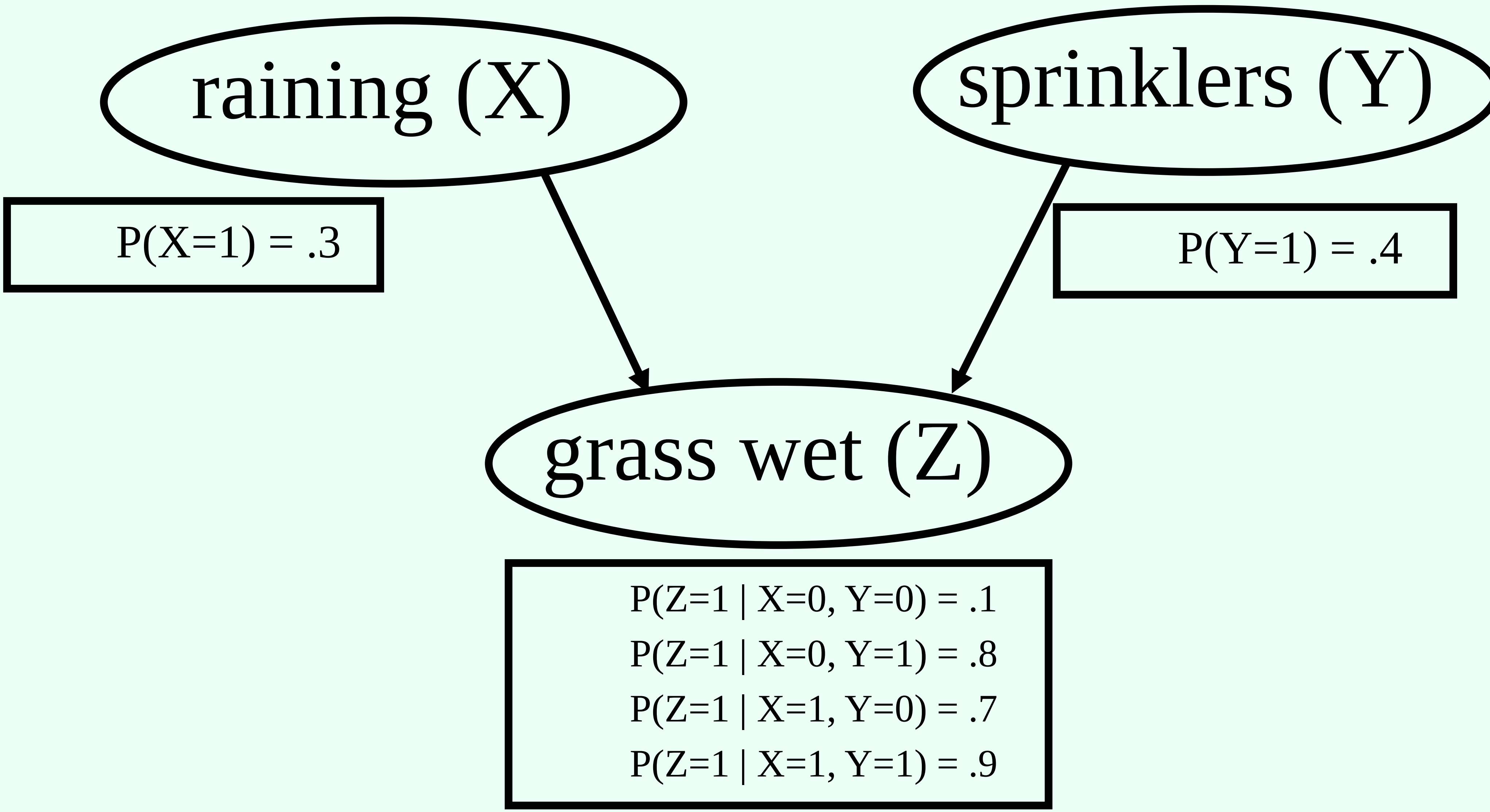
Graphically representing influences

- … if X_4 is conditionally independent from X_2 given X_1 and X_3

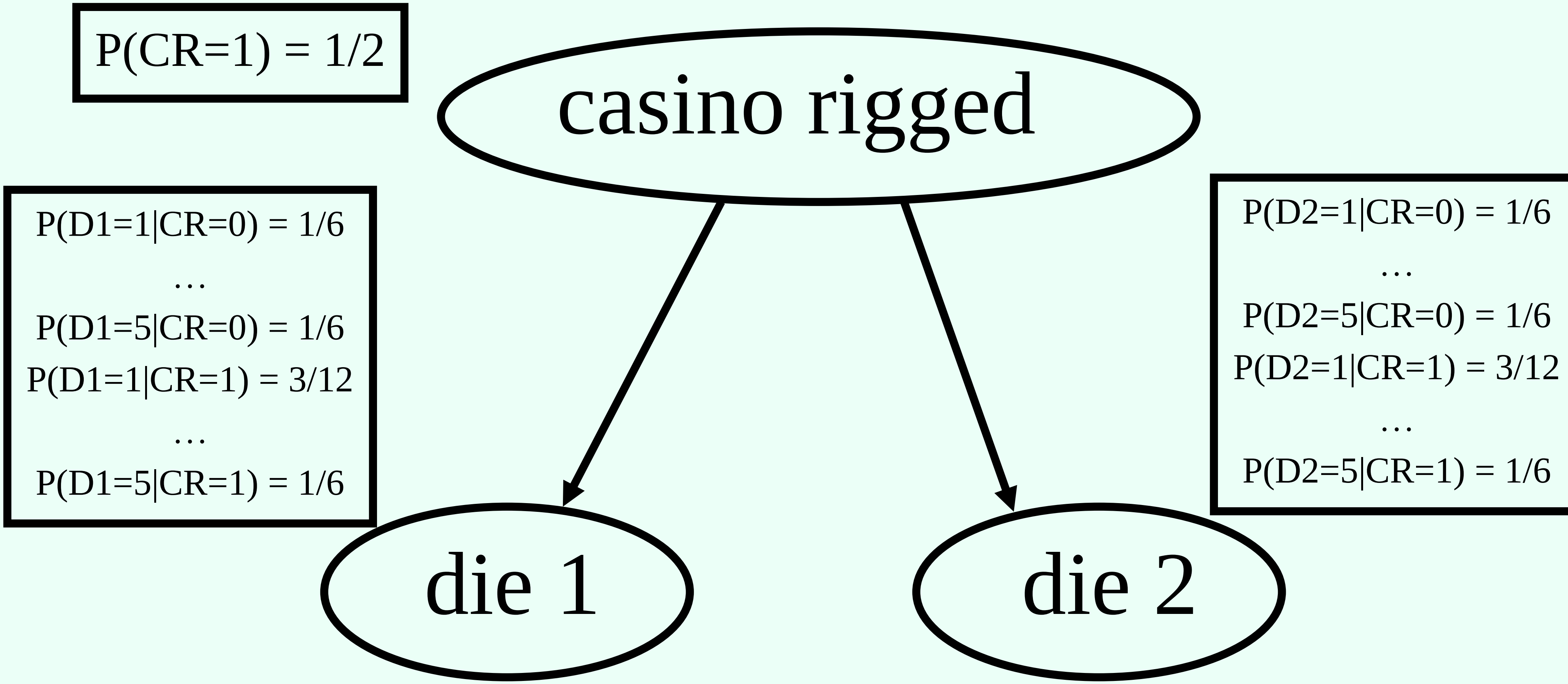


Rain and sprinklers example

sprinklers is independent of raining, so no edge between them

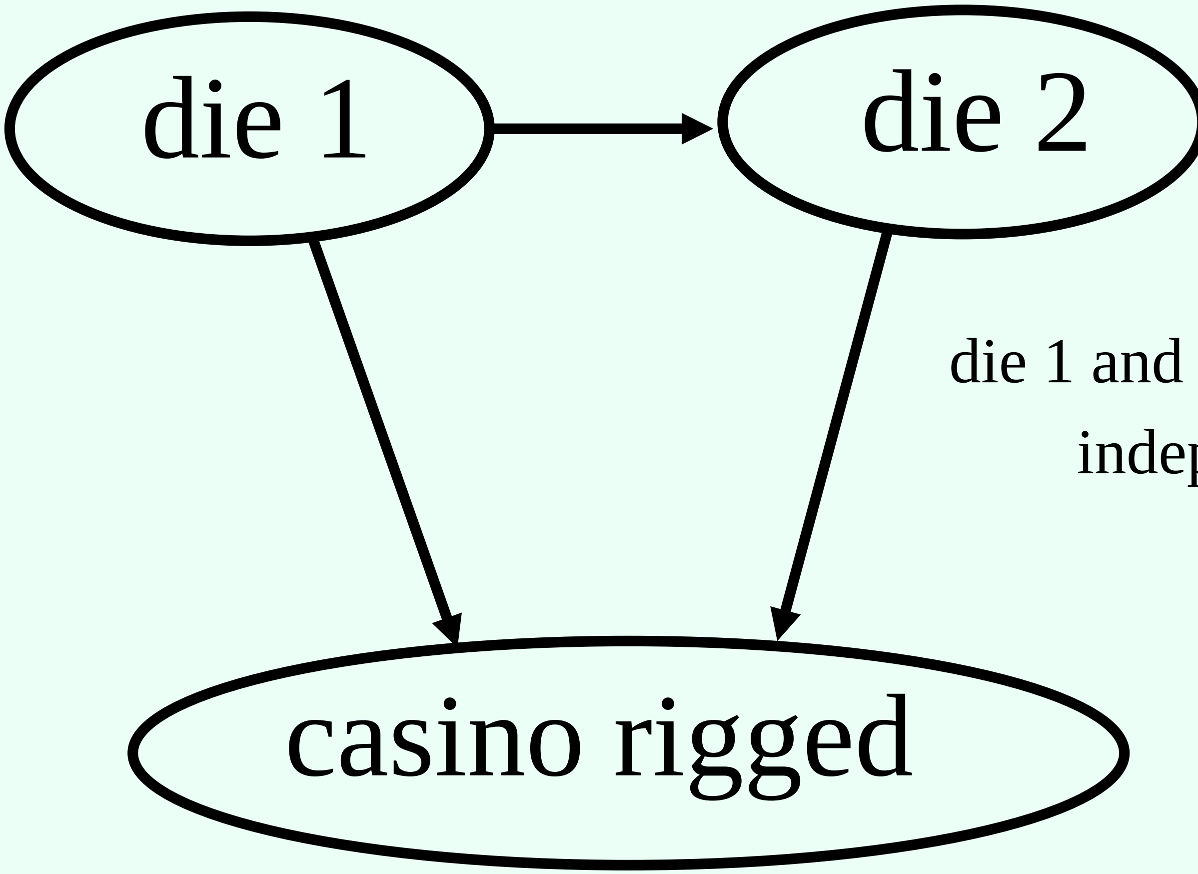


Rigged casino example



die 2 is conditionally independent of die 1 given casino rigged, so no edge between them

Rigged casino example with poorly chosen order

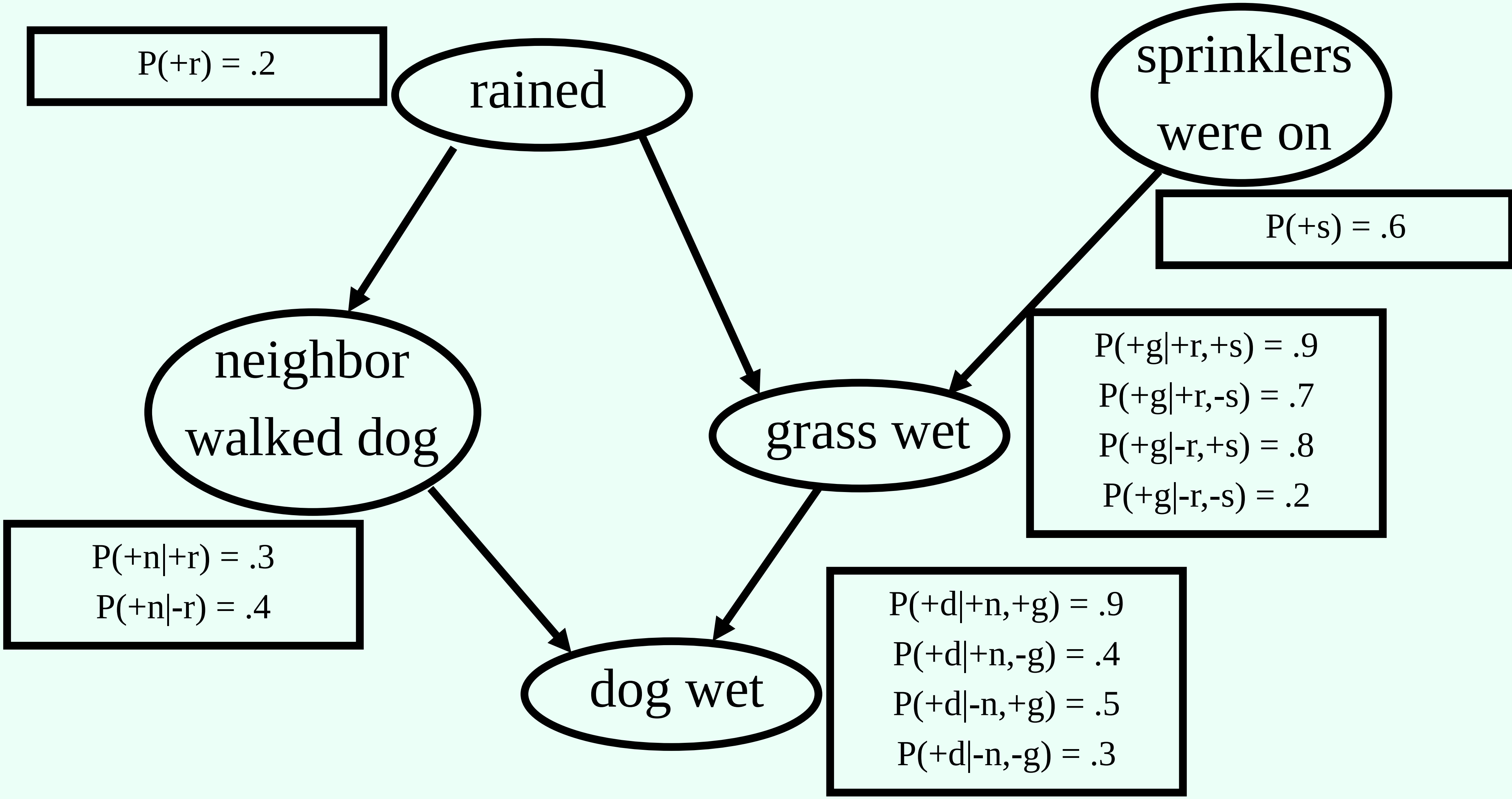


die 1 and die 2 are not independent

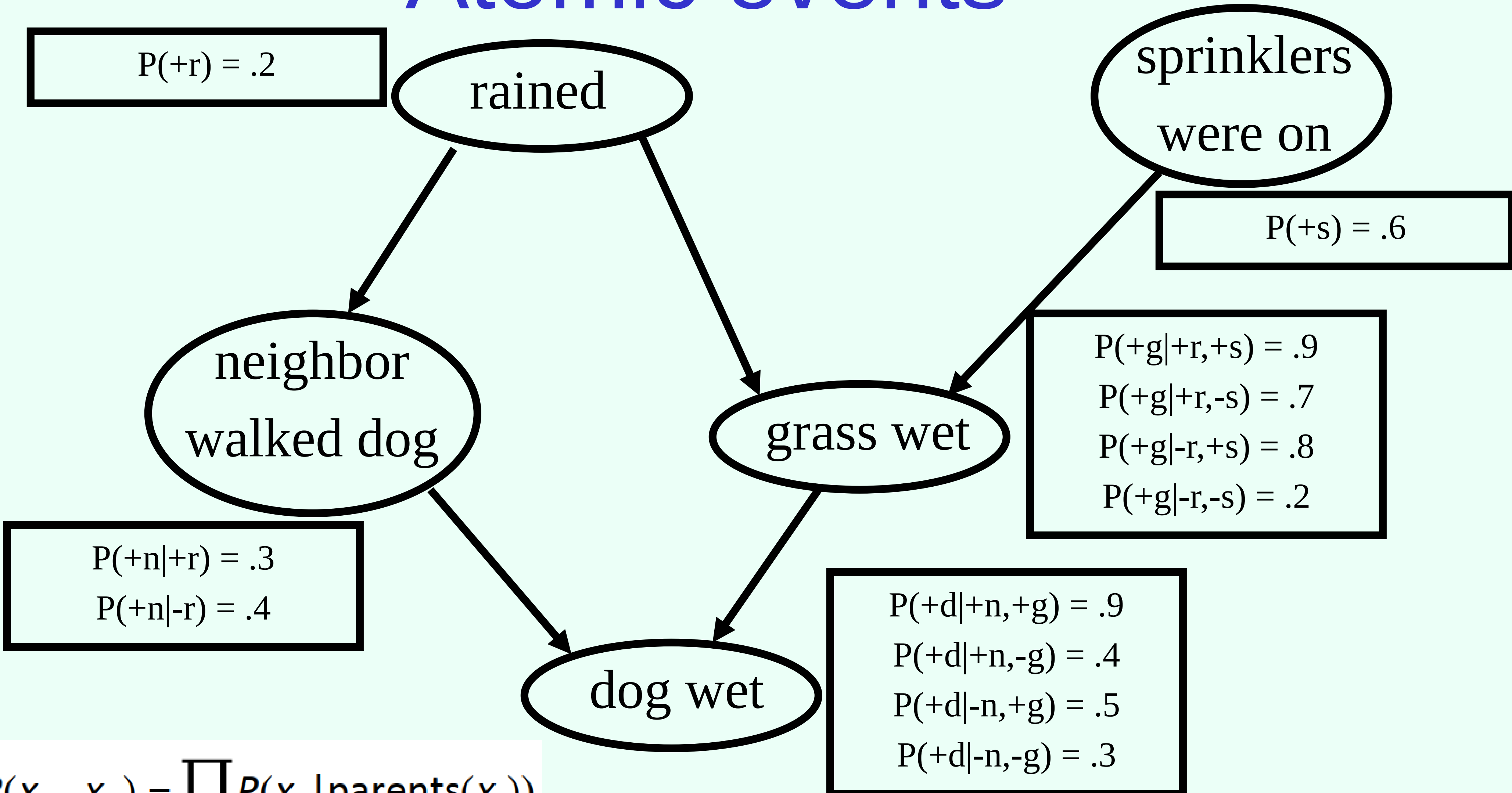
both the dice have relevant information for whether the casino is rigged

need 36 probabilities here!

More elaborate rain and sprinklers example



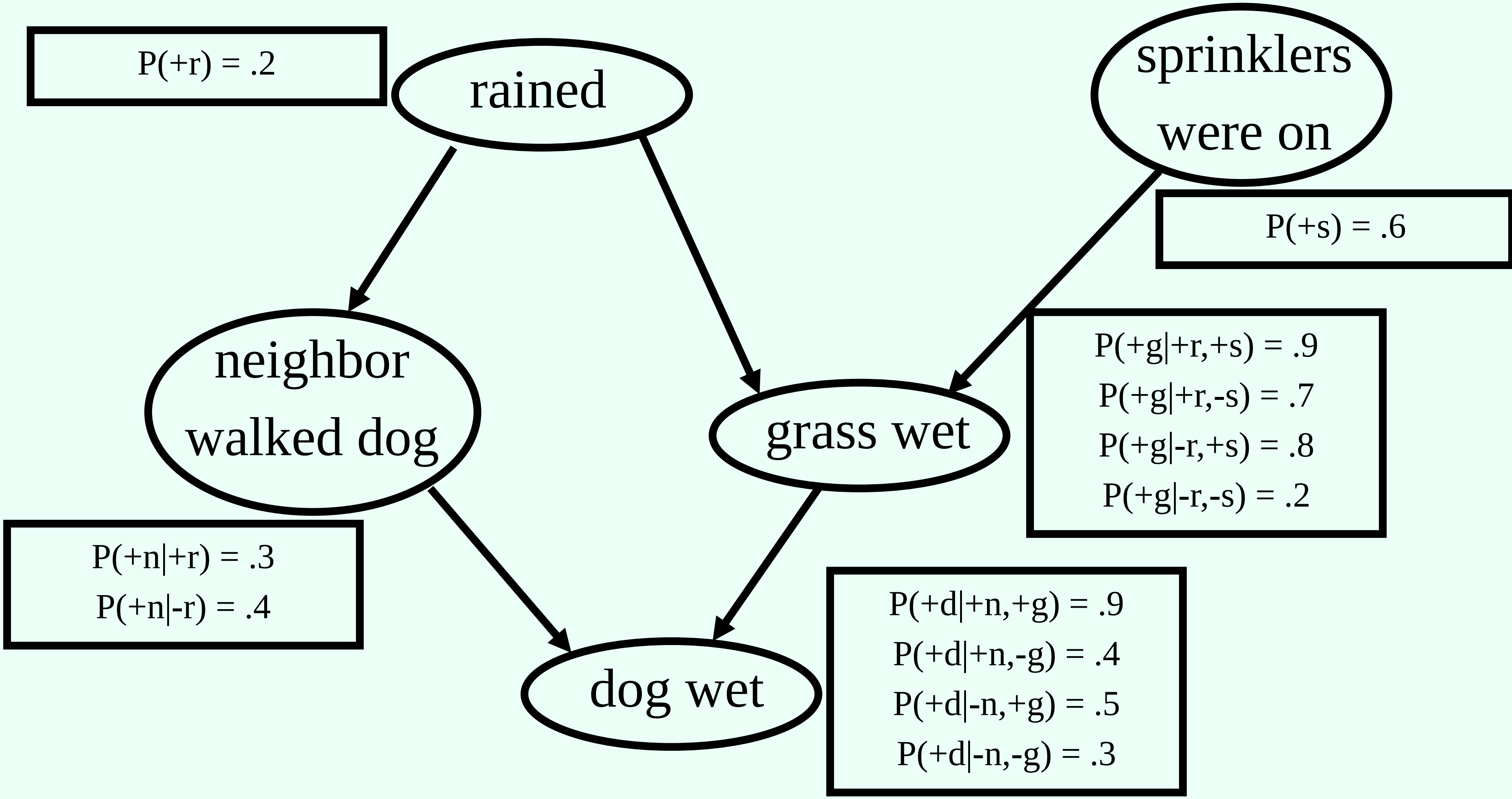
Atomic events



$$P(x_1 \ldots x_n) = \prod_i P(x_i \mid \text{parents}(x_i))$$

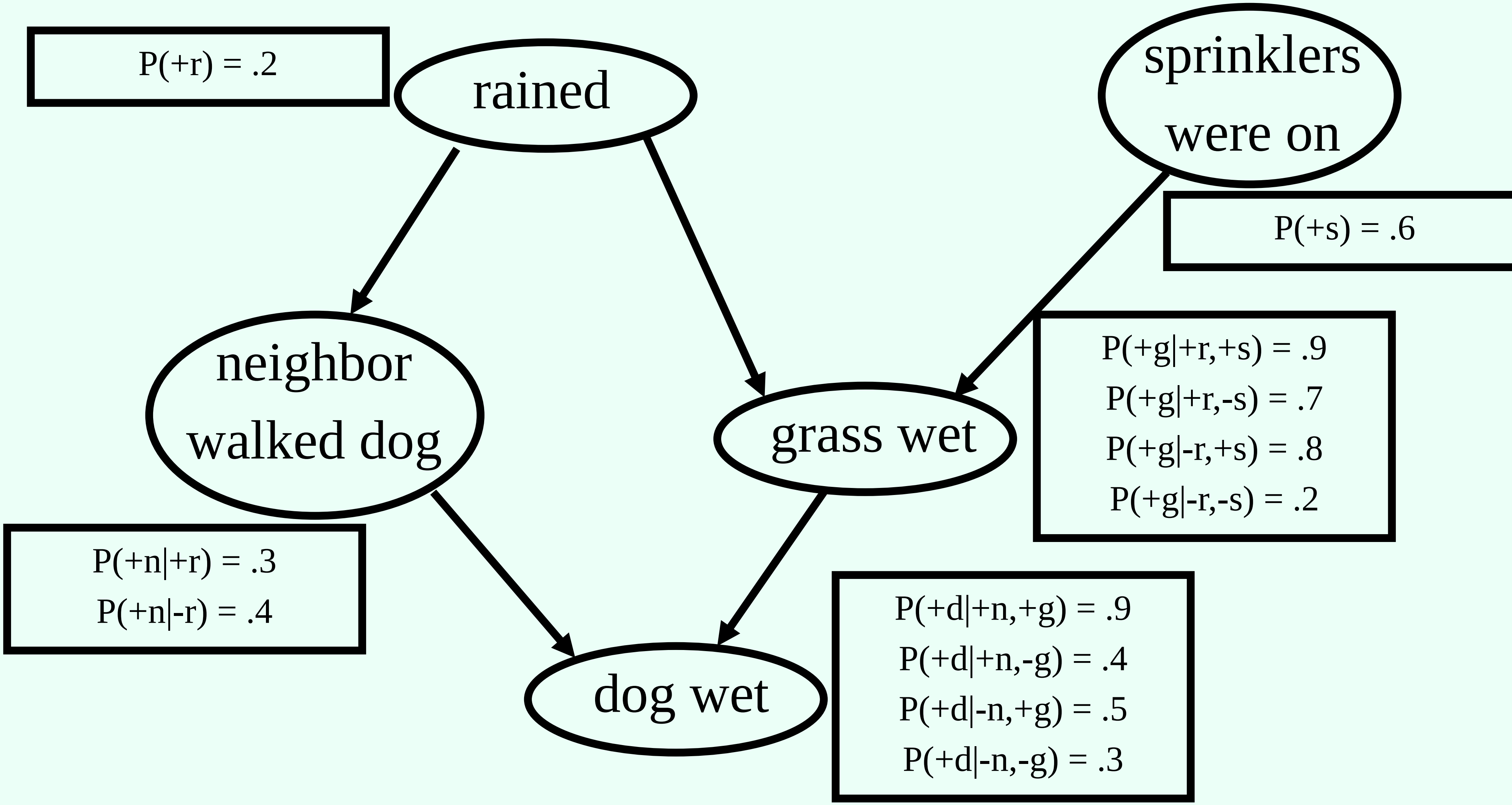
- Can easily calculate the probability of any atomic event
- P(+r,+s,+n,+g,+d) = P(+r)P(+s)P(+n|+r)P(+g|+r,+s)P(+d|+n,+g)
- Can also sample atomic events easily

Inference



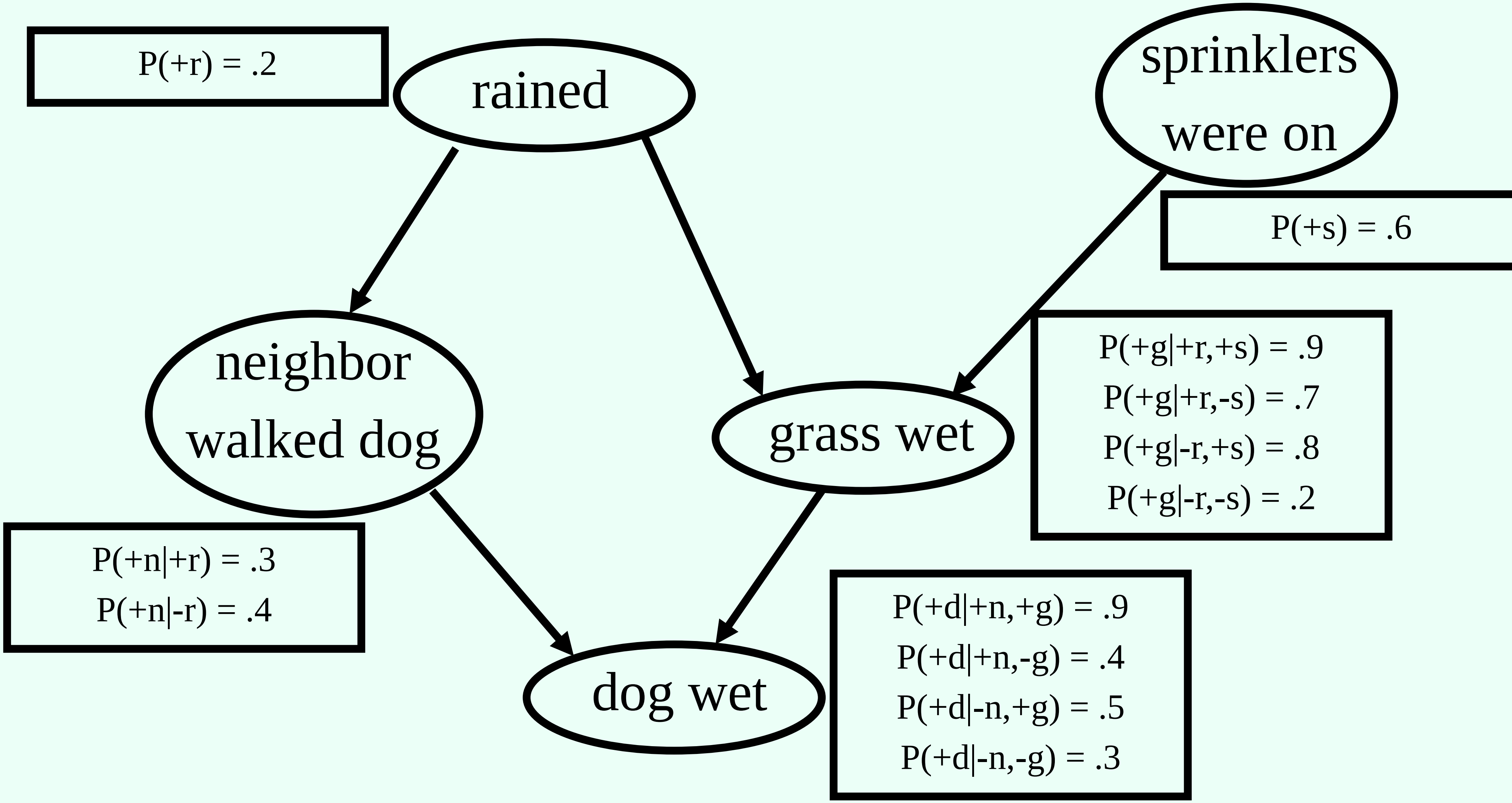
- Want to know: $P(+r|+d) = P(+r,+d)/P(+d)$
- Let's compute $P(+r,+d)$

Inference…



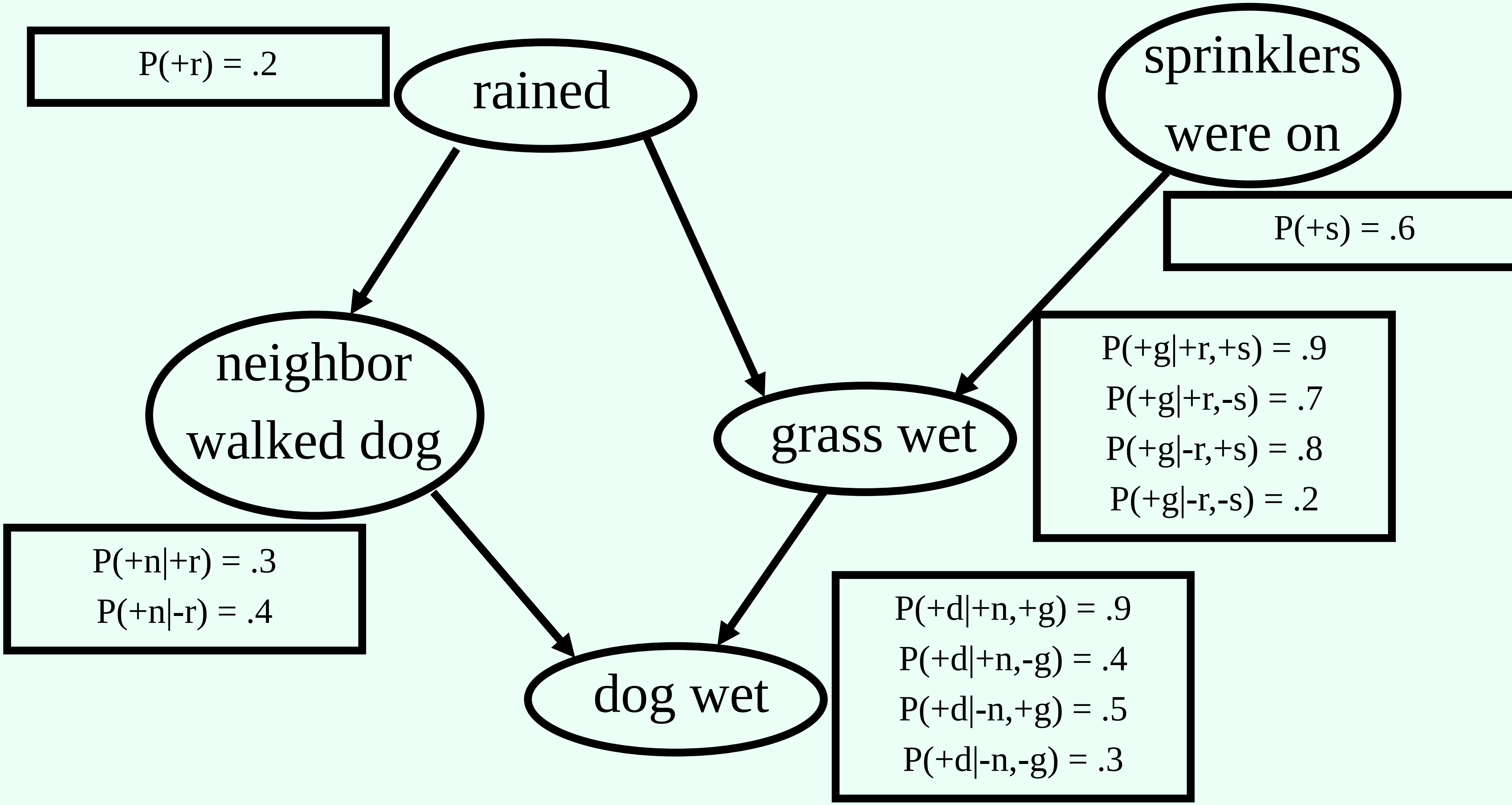
- $P(+r,+d) = \Sigma_s \Sigma_g \Sigma_n P(+r)P(s)P(n|+r)P(g|+r,s)P(+d|n,g) =$
  $P(+r)\Sigma_s P(s)\Sigma_g P(g|+r,s)\Sigma_n P(n|+r)P(+d|n,g)$

Variable elimination



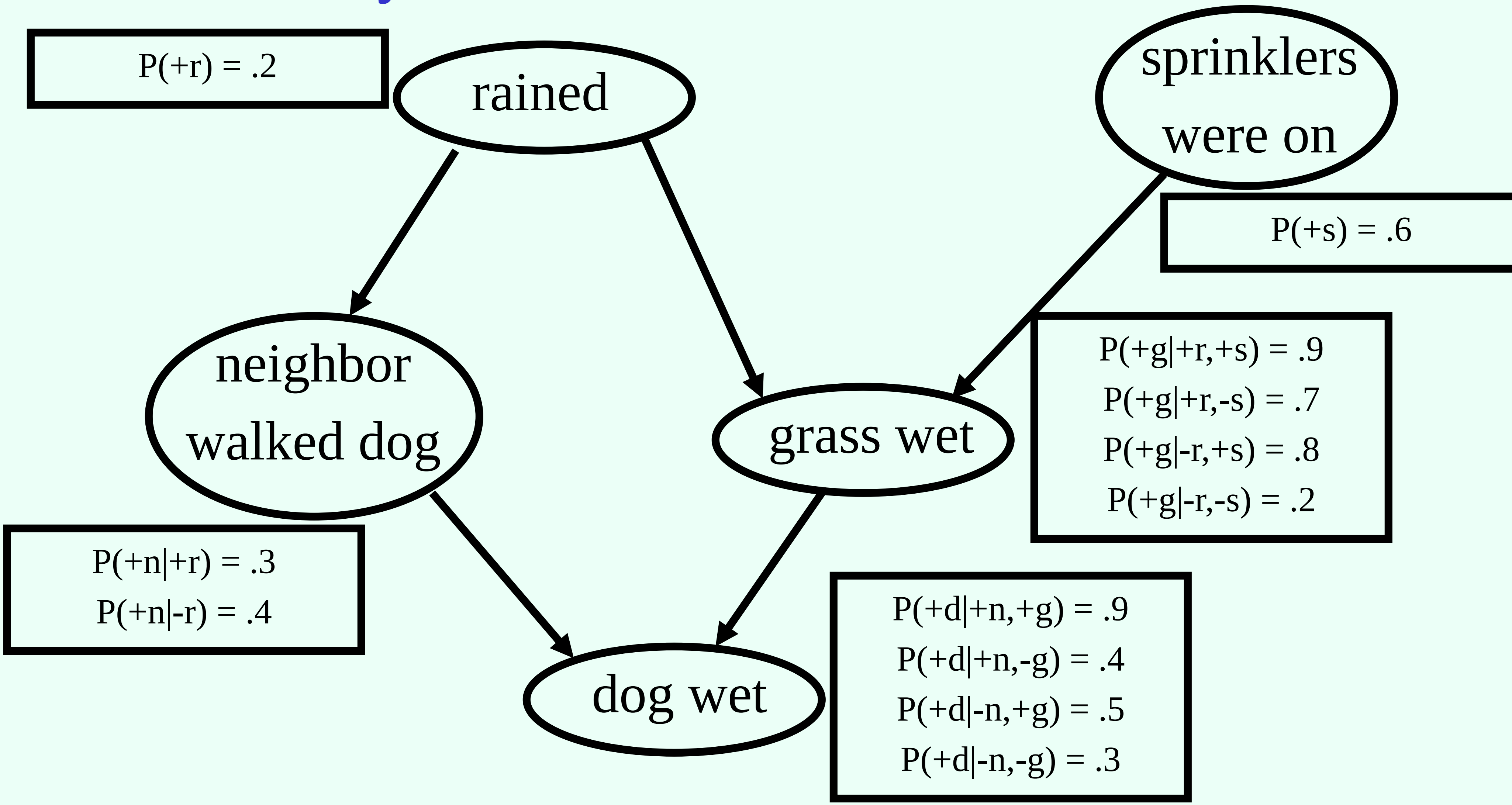
- From the factor Σ_n P(n|+r)P(+d|n,g) we sum out n to obtain a factor only depending on g
- [Σ_n P(n|+r)P(+d|n,+g)] = P(+n|+r)P(+d|+n,+g) + P(-n|+r)P(+d|-n,+g) = .3*.9+.7*.5 = .62
- [Σ_n P(n|+r)P(+d|n,-g)] = P(+n|+r)P(+d|+n,-g) + P(-n|+r)P(+d|-n,-g) = .3*.4+.7*.3 = .33
- Continuing to the left, g will be summed out next, etc.

Elimination order matters



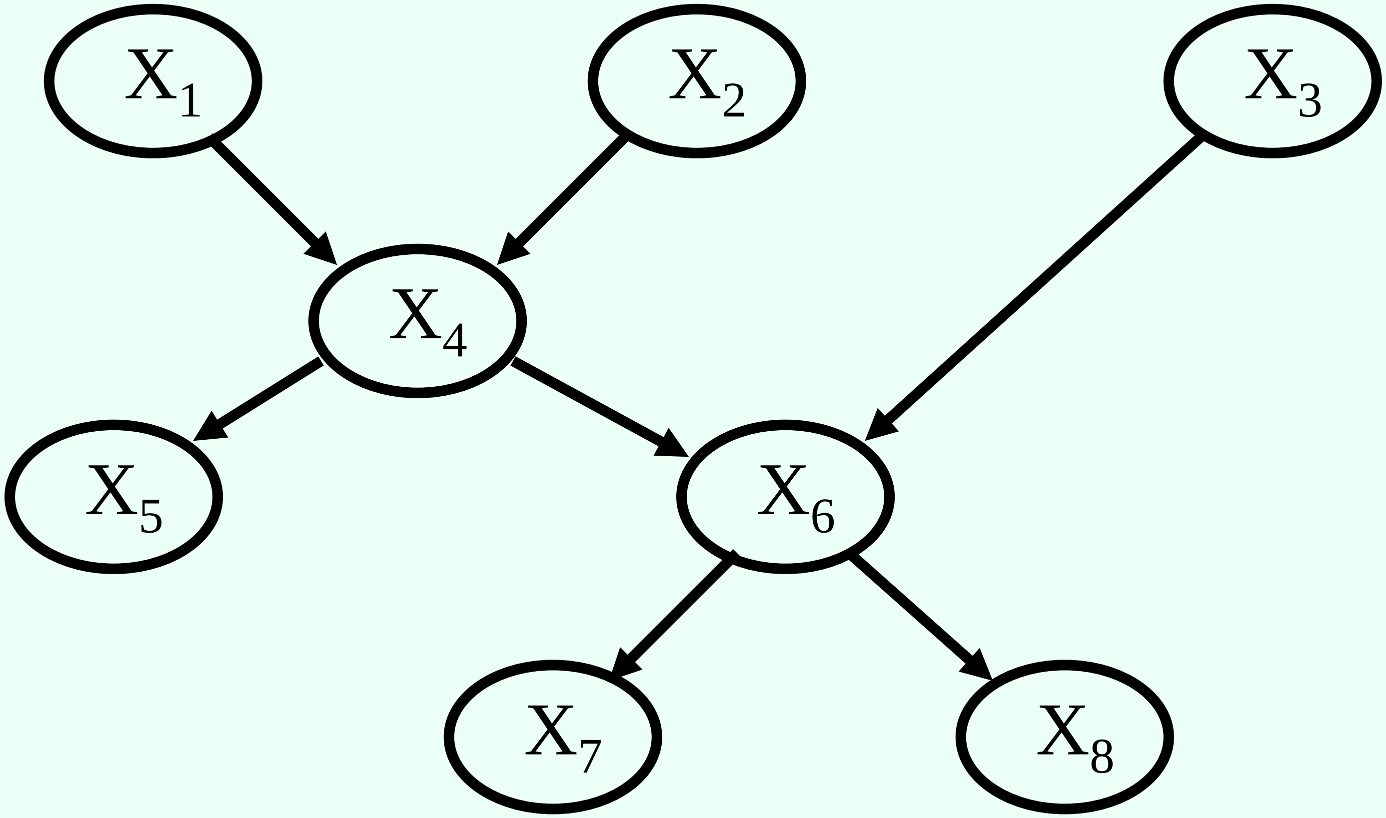
- $P(+r,+d) = \Sigma_n \Sigma_s \Sigma_g\ P(+r)P(s)P(n|+r)P(g|+r,s)P(+d|n,g) = P(+r)\Sigma_n P(n|+r)\Sigma_s P(s)\Sigma_g\ P(g|+r,s)P(+d|n,g)$
- Last factor will depend on two variables in this case!

Don't always *need* to sum over **all** variables



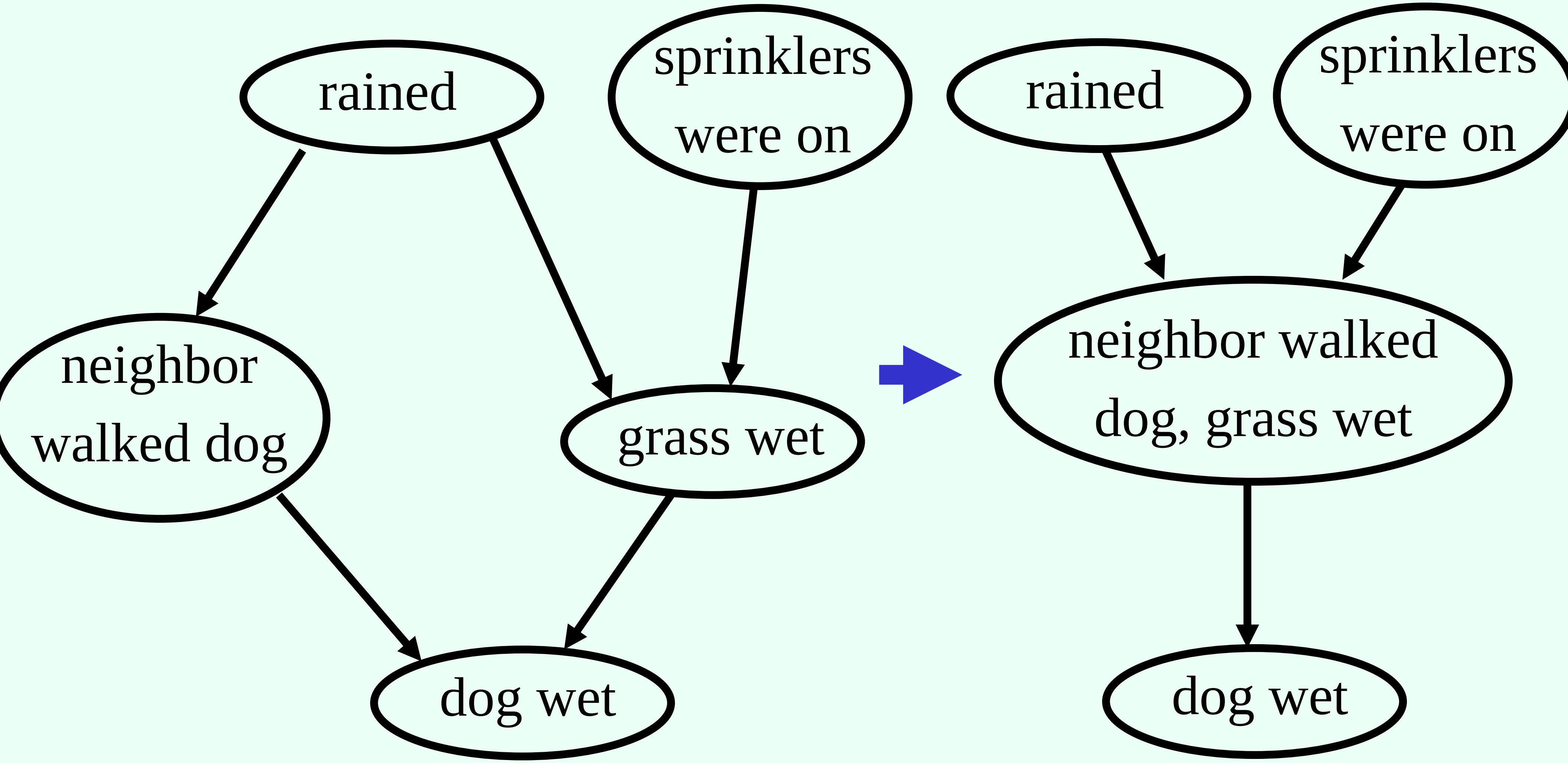
- Can drop parts of the network that are irrelevant
- $P(+r, +s) = P(+r)P(+s) = .6*.2 = .12$
- $P(+n, +s) = \Sigma_r P(r, +n, +s) = \Sigma_r P(r)P(+n|r)P(+s) = P(+s)\Sigma_r P(r)P(+n|r) =$
  $P(+s)(P(+r)P(+n|+r) + P(-r)P(+n|-r)) = .6*(.2*.3 + .8*.4) = .6*.38 = .228$
- $P(+d \mid +n, +g, +s) = P(+d \mid +n, +g) = .9$

Trees are easy



- Choose an extreme variable to eliminate first
- Its probability is “absorbed” by its neighbor
- $\ldots\Sigma_{x4}\, P(x_4|x_1,x_2)\ldots\Sigma_{x5}\, P(x_5|x_4) = \ldots\Sigma_{x4}\, P(x_4|x_1,x_2)[\Sigma_{x5}\, P(x_5|x_4)]\ldots$

Clustering algorithms

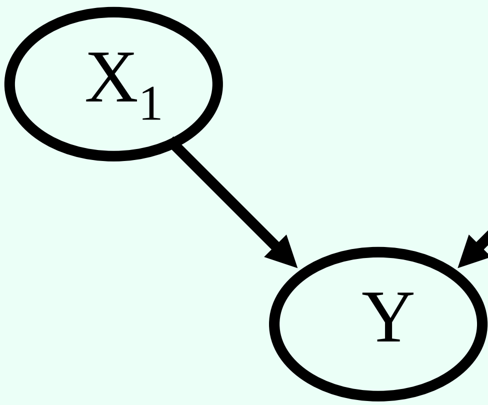


- Merge nodes into “meganodes” until we have a tree
  - Then, can apply special-purpose algorithm for trees
- Merged node has values {+n+g,+n-g,-n+g,-n-g}
  - Much larger CPT

Logic gates in Bayes nets

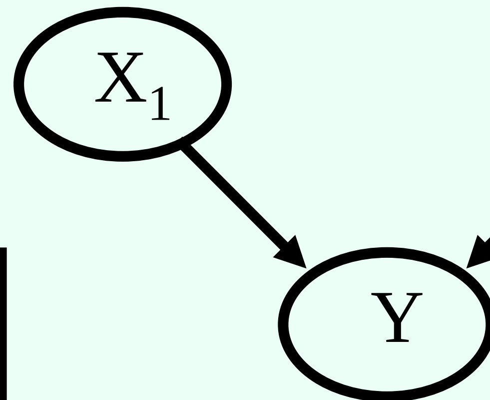
- Not everything needs to be random…

AND gate



$P(+y|+x_1,+x_2) = 1$
$P(+y|-x_1,+x_2) = 0$
$P(+y|+x_1,-x_2) = 0$
$P(+y|-x_1,-x_2) = 0$

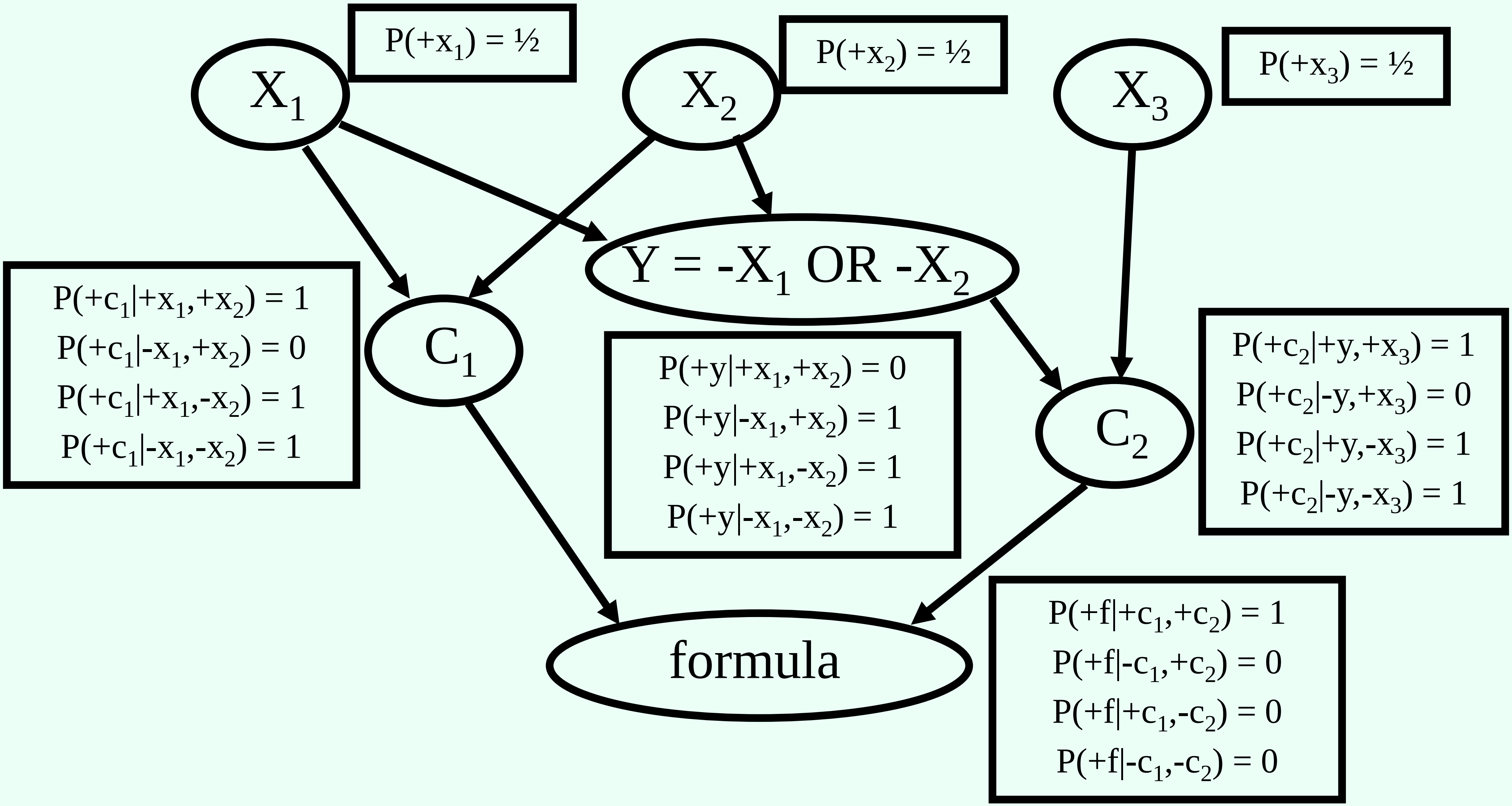
OR gate



$P(+y|+x_1,+x_2) = 1$
$P(+y|-x_1,+x_2) = 1$
$P(+y|+x_1,-x_2) = 1$
$P(+y|-x_1,-x_2) = 0$

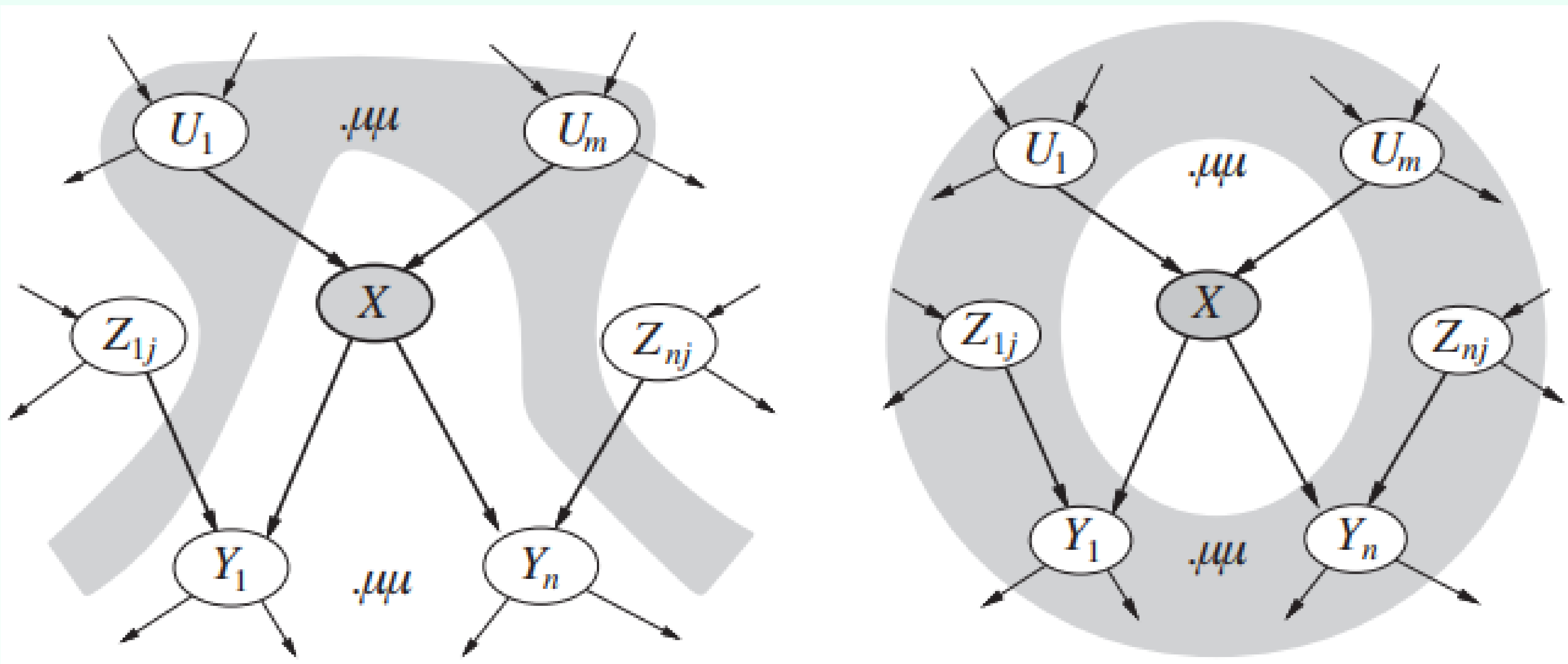
Modeling satisfiability as a Bayes Net

- $(+X_1 \text{ OR } -X_2) \text{ AND } (-X_1 \text{ OR } -X_2 \text{ OR } -X_3)$



More about conditional independence

- A node is conditionally independent of its non-descendants, given its parents
- A node is conditionally independent of everything else in the graph, given its parents, children, and children’s parents (its Markov blanket)



(a) (b)