Formal Methods（形式化方法）

Lecture 11. Bag（包）

智能与计算学部 章衡

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Outline

Why need to use bags?

- The multiple occurrences of elements are unable to be expressed in a set.

$$\{\text{Alice}, \text{Bob}, \text{Jone}\} = \{\text{Alice}, \text{Bob}, \text{Jone}, \text{Bob}\}$$

- The multiple occurrences of elements can be expressed in a sequence, but accompanied with an order among the elements

$$\langle \text{Alice}, \text{Bob}, \text{Jone}, \text{Bob} \rangle$$

- In many applications, however, we need a structure to record the multiple occurrences without an order among the elements; such a structure is called a bag(包), a.k.a. multiset

$[\![\cdot]\!]$-representation of a bag

Definition ($[\![\cdot]\!]$-representation)

Given a set X, a bag on X can be represented by an expression of the following form:

$$[\![a_1, a_2, \ldots, a_n]\!]$$

where $a_1, a_2, \ldots, a_n \in X$. In particular, we call it an empty bag (denoted $[\![]\!]$) if $n = 0$.

- Elements in a bag must be of the same type
- The order among elements plays no role in the representation

Example

$$\begin{aligned} [\![\text{Alice}, \text{Bob}, \text{Jone}, \text{Bob}]\!] &= [\![\text{Alice}, \text{Bob}, \text{Bob}, \text{Jone}]\!] \\ &\neq [\![\text{Alice}, \text{Bob}, \text{Jone}]\!] \end{aligned}$$

Limitation of $[\![\cdot]\!]$-representation

- A bag might have an infinite number of elements (noting that a set is also a bag), however, the $[\![\cdot]\!]$-representation is not able to express an infinite bag.

Function-representation of a bag

Example

In order to evaluate the quality of online courses, we not only care about which students participate in the learning activities, but also how often a student participates in the learning activities:

$$\text{Alice} : 1, \text{Bob} : 2, \text{Jone} : 0, \text{Mike} : 3$$

Such information can be expressed by the function

$$\{\text{Alice} \mapsto 1, \text{Bob} \mapsto 2, \text{Mike} \mapsto 3\}$$

Definition

Given a set X, let

$$\text{bag}\, X == X \nrightarrow \mathbb{N}_1,$$

and call each function in bag X a bag on X.

Moreover, we use $b : \text{bag}\, X$ to declare that b is a bag on X.

Example

$$\text{studyRecord} : \text{bag}\, \text{Student}$$

Outline

Operators: Bag count (包计数)

Definition

Suppose B : bag X. For all $a \in X$, let

$$\textcolor{red}{\mathrm{count}\, B\, a} == \begin{cases} n & \text{if } a \mapsto n \in B \\ 0 & \text{otherwise} \end{cases}$$

Example

$$L = [\![Alice, Alice, Jone, Bob, Bob, Bob]\!].$$
$$\{Alice \mapsto 2, Jone \mapsto 1, Bob \mapsto 3\}.$$
$$\mathrm{count}\, L\, Alice = 2, \mathrm{count}\, L\, Bob = 3,$$
$$\mathrm{count}\, L\, Jone = 1, \mathrm{count}\, L\, Mike = 0.$$

Operators: Bag scaling (包扩大)

Definition

Suppose B : bag X. For all $k \in \mathbb{N}$, let

$$k \otimes B == \{a : X; n : \mathbb{N}_1 \mid a \mapsto n \in B \land k > 0 \bullet a \mapsto k * n\},$$

and call it the bag scaled from B by k.

Example

$$L = [\![Alice, Alice, Jone, Bob, Bob, Bob]\!].$$
$$\{Alice \mapsto 2, Jone \mapsto 1, Bob \mapsto 3\}.$$
$$2 \otimes L = \{Alice \mapsto 4, Jone \mapsto 2, Bob \mapsto 6\}.$$

count and $\otimes$: Generic definitions

count: Bag count

$\otimes$: Bag scaling

Definition

$$
\begin{array}{l}
[X] \\
\hline
\mathrm{count} : \mathrm{bag}\, X \rightarrowtail (X \rightarrow \mathbb{N}) \\
\_ \otimes \_ : \mathbb{N} \times \mathrm{bag}\, X \rightarrow \mathrm{bag}\, X \\
\hline
\forall B : \mathrm{bag}\, X \bullet \\
\qquad \mathrm{count}\, B = (\lambda\, x : X \bullet 0) \oplus B \\
\forall n : \mathbb{N}; B : \mathrm{bag}\, X; x : X \bullet \\
\qquad \mathrm{count}\, (n \otimes B)\, x = n * \mathrm{count}\, B\, x
\end{array}
$$

Recall that $f \oplus g == ((\mathrm{dom}\ g) \lhd\ f) \cup g$.

Properties of bags and $\otimes$

Let $B : \text{bag}\, X$ and $a_1, \ldots, a_n : X$. Then we have

1. $\text{dom}\ [\![a_1, \ldots, a_n]\!] = \{a_1, \ldots, a_n\}$
2. $n \otimes [\![\,]\!] = 0 \otimes B = [\![\,]\!]$
3. $1 \otimes B = B$
4. $(n * m) \otimes B = n \otimes (m \otimes B)$

Operators: Bag membership and sub-bag relation

Definition

Suppose $B : \mathrm{bag}\,X$. For all $c \in X$, we say that c is a member of B, denoted $c \text{ in } B$, if $c \in \mathrm{dom}\,B$.

Definition

Suppose $A, B : \mathrm{bag}\,X$. We say that B is a sub-bag of A, denoted $B \sqsubseteq A$, if for all $c \in X$, we have $\mathrm{count}\,B\,c \leqslant \mathrm{count}\,A\,c$.

Example

$$L = [\![Alice, Alice, Jone, Bob, Bob, Bob]\!].$$
$$T = [\![Alice, Alice, Alice, Jone, Bob, Bob, Bob, Mike]\!].$$
$$Alice \text{ in } L, Bob \text{ in } L, \neg(Mike \text{ in } L), Mike \text{ in } T.$$
$$L \sqsubseteq T, T \not\sqsubseteq L.$$

in and $\sqsubseteq$: Generic definitions

in: Bag member

$\sqsubseteq$: Sub-bag relation

Definition

$$
\begin{array}{l}
[X] \\
\_ \mathrm{in} \_ : X \leftrightarrow \mathrm{bag}\, X \\
\_ \sqsubseteq \_ : \mathrm{bag}\, X \leftrightarrow \mathrm{bag}\, X \\
\hline
\forall x : X; B : \mathrm{bag}\, X \bullet \\
\quad (x \;\mathrm{in}\; B \Leftrightarrow x \in \mathrm{dom}\, B) \\
\forall A, B : \mathrm{bag}\, X \bullet \\
\quad B \sqsubseteq A \Leftrightarrow (\forall x : X \bullet \mathrm{count}\, B\, x \leq \mathrm{count}\, A\, x)
\end{array}
$$

Properties of in and $\sqsubseteq$

Suppose $B, C, D : \text{bag}\, X$. Then we have

1. $x \text{ in } B \Leftrightarrow \text{count}\, B\, x > 0$
2. $B \sqsubseteq C \Rightarrow \text{dom}\, B \subseteq \text{dom}\, C$
3. $[\![\,]\!] \sqsubseteq B$
4. $B \sqsubseteq B$
5. $B \sqsubseteq C \wedge C \sqsubseteq B \Rightarrow B = C$
6. $B \sqsubseteq C \wedge C \sqsubseteq D \Rightarrow B \sqsubseteq D$

Operators: Bag union and bag difference

Example

Suppose $B, C : \text{bag}\, X$, and

$$B = \{a \mapsto 2, b \mapsto 3\} \qquad \text{count}\, B\, a = 2, \quad \text{count}\, B\, b = 3, \quad \text{count}\, B\, c = 0$$
$$C = \{a \mapsto 1, c \mapsto 2\} \qquad \text{count}\, C\, a = 1, \quad \text{count}\, C\, b = 0, \quad \text{count}\, C\, c = 2$$

How to define the bag union of B and C?

$$B \uplus C = \{a \mapsto 3, b \mapsto 3, c \mapsto 2\}$$

How to define the bag difference of B and C?

$$B \mathbin{\cup\!\!\!\!-} C = \{a \mapsto 1, b \mapsto 3\}$$

- Bag union: $\text{count}\, B \uplus C\, a = \text{count}\, B\, a + \text{count}\, C\, a$
- Bag difference: $\text{count}\, B \mathbin{\cup\!\!\!\!-} C\, a = \max\{\text{count}\, B\, a - \text{count}\, C\, a, 0\}$

$\uplus$ and $\uminus$: Generic definitions

$\uplus$: Bag union

$\uminus$: Bag difference

Definition

$$
\begin{array}{l}
[X] \\
\hline
\_ \uplus \_, \_ \uminus \_ : \mathrm{bag}\, X \times \mathrm{bag}\, X \rightarrow \mathrm{bag}\, X \\
\hline
\forall B, C : \mathrm{bag}\, X; x : X \bullet \\
\qquad \mathrm{count}\, (B \uplus C)\, x = \mathrm{count}\, B\, x + \mathrm{count}\, C\, x \land \\
\qquad \mathrm{count}\, (B \uminus C)\, x = \max \{\mathrm{count}\, B\, x - \mathrm{count}\, C\, x, 0\} \\
\hline
\end{array}
$$

Properties of bag union and difference

Suppose $B, C, D : \mathrm{bag}\, X; m, n : \mathbb{N}$. Then we have

1. $\mathrm{dom}(B \uplus C) = \mathrm{dom}\, B \cup \mathrm{dom}\, C$
2. $[\![\,]\!] \uplus B = B \uplus [\![\,]\!] = B$
3. $B \uplus C = C \uplus B$
4. $(B \uplus C) \uplus D = B \uplus (C \uplus D)$
5. $B \uminus [\![\,]\!] = B$
6. $[\![\,]\!] \uminus B = [\![\,]\!]$
7. $(B \uplus C) \uminus C = B$
8. $(n + m) \otimes B = n \otimes B \uplus m \otimes B$
9. $n \geq m \Rightarrow (n - m) \otimes B = n \otimes B \uminus m \otimes B$
10. $n \otimes (B \uplus C) = n \otimes B \uplus n \otimes C$
11. $n \otimes (B \uminus C) = n \otimes B \uminus n \otimes C$

Operators: function items

Example

$$\langle a, a, b, b, b, c \rangle$$

$$[\![a, a, b, b, b, c]\!]$$

$$\text{items}\langle a, a, b, b, b, c \rangle = \{a \mapsto 2, b \mapsto 3, c \mapsto 1\}.$$

items: Generic definition

Definition

$$
\begin{array}{l}
[X] \\
\hline
\mathrm{items} : \mathrm{seq}\, X \to \mathrm{bag}\, X \\
\hline
\forall\, s : \mathrm{seq}\, X;\ x : X \bullet \\
\qquad \mathrm{count}\,(\mathrm{items}\, s)\, x = \#(s \rhd \{x\})
\end{array}
$$

Properties of items

Suppose $s, t : \mathrm{seq}\, X$. Then we have

1. $\mathrm{dom}(\mathrm{items}\, s) = \mathrm{ran}\, s$
2. $\mathrm{items}\, \langle a_1, \ldots, a_n \rangle = [\![a_1, \ldots, a_n]\!]$
3. $\mathrm{items}(s \frown t) = \mathrm{items}\, s \uplus \mathrm{items}\, t$
4. $\mathrm{items}\, s = \mathrm{items}\, t \Leftrightarrow (\exists\, f : \mathrm{dom}\, s \rightarrowtail\!\!\!\!\!\twoheadrightarrow \mathrm{dom}\, t \bullet s = f \mathbin{;} t)$

Exercies

Ex. 1

Let

$$\begin{aligned} \text{balls} &== [\![\text{blue}, \text{green}, \text{white}, \text{green}, \text{blue}, \text{red}, \text{blue}]\!] \\ \text{balls}_1 &== \{\text{white} \mapsto 2, \text{red} \mapsto 1, \text{blue} \mapsto 2, \text{green} \mapsto 2\} \\ \text{balls}_2 &== \{\text{green} \mapsto 2, \text{blue} \mapsto 2, \text{white} \mapsto 1\} \end{aligned}$$

What are the bags represented by the following expressions?

1. $\text{balls} \uplus \text{balls}_1$;
2. $\text{balls} \uplus \text{balls}_2$;
3. $\text{balls} \cup\!\!\!\!\,\smile \text{balls}_1$;
4. $\text{balls} \cup\!\!\!\!\,\smile \text{balls}_2$.

Ex. 2

Let $S == \langle \text{red}, \text{white}, \text{red}, \text{green}, \text{green}, \text{red}, \text{blue} \rangle$. What is the bag represented by items S?

Ex. 3

Let $s, t : \text{seq}\, X$. Prove $\text{items}(s \frown t) = \text{items}\, s \uplus \text{items}\, t$.