Formal Methods（形式化方法）

Lecture 9. Sequence（序列）

智能与计算学部 章衡

2021年上学期

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Research

- General game playing (通用博弈)
- Knowledge representation & reasoning (知识表示与推理)

For more information about my research, please visit

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Outline

Why we need sequences?

Example

Below is a list of customers visited a shop (ordered by visiting time):

$$\text{Tom}, \text{Mike}, \text{Jone}, \text{Jone}, \text{Mary}$$

How to represent such a list in language Z?

- In a set, we do not care about the order among elements; however, in many applications, the order among elements is important
- Multiple occurrences of an element is not able to be recorded in a set

How to represent a sequence

Definition ($\langle\cdot\rangle$-representation)

In language Z, a sequence (序列) is a finite enumerated collection of elements of the same type:

$$\langle a_1, a_2, \ldots, a_n \rangle$$

In particular, we call it the empty sequence (denoted $\langle\rangle$) if $n = 0$.

Example

$\langle \mathrm{Tom}, \mathrm{Mike}, \mathrm{Jone}, \mathrm{Jone}, \mathrm{Mary} \rangle$

$\langle \mathrm{Tom}, \mathrm{Mike}, \mathrm{Jone}, \mathrm{Mary} \rangle$

Example

$\langle \mathrm{Tom}, \mathrm{Mike}, \mathrm{Jone}, \mathrm{Mary} \rangle \neq \langle \mathrm{Mike}, \mathrm{Tom}, \mathrm{Jone}, \mathrm{Mary} \rangle$

Alternative definition: Sequence as function

Definition (Sequence as function)

One can represent the sequence

$$\langle a_1, a_2, \ldots, a_n \rangle$$

as $f : \mathbb{N}_1 \rightarrow\!\!\!\!\!\!+\; \{a_1, a_2, \ldots, a_n\}$, defined as bellows:

$$f == \{1 \mapsto a_1, 2 \mapsto a_2, \ldots, n \mapsto a_n\}$$

Formally, in language Z, given a set X, a sequence on X is defined as a function $f : 1..n \rightarrow X$ for some natural number n.

In particular, we have

$$\langle\rangle = \emptyset$$

Example

$\langle \mathrm{Tom}, \mathrm{Mike}, \mathrm{Jone}, \mathrm{Mary} \rangle = \{1 \mapsto \mathrm{Tom}, 2 \mapsto \mathrm{Mike}, 3 \mapsto \mathrm{Jone}, 4 \mapsto \mathrm{Mary}\}$.

Declaration of a sequence

Definition (Notations)

Given a set X, let

$$\mathrm{seq}\, X$$

denote the set of all sequences on X.
Furthermore, we use

$$s : \mathrm{seq}\, X$$

to declare that s is a sequence on X。

Definition

$$\begin{aligned}
\mathrm{seq}\, X &== \{f : \mathbb{N} \nrightarrow X \mid \mathrm{dom}\, f = 1..\sharp f\}.\\
\mathrm{seq}_1\, X &== \{f : \mathrm{seq}\, X \mid \sharp f > 0\}\\
\mathrm{iseq}\, X &== \mathrm{seq}\, X \cap (\mathbb{N} \rightarrowtail X).
\end{aligned}$$

More notations

Definition

Given a sequence s and suppose $i \in \mathrm{dom}\, s$, let $s\ i$ denote the i-th element of s.

Example

Let customers $==$ $\langle \mathrm{Tom}, \mathrm{Mike}, \mathrm{Jone}, \mathrm{Mary}\rangle$. Then customers $2 = \mathrm{Mike}$.

Definition (Length of a sequence)

Let s be a sequence. Then the length (长度) of s is defined as the number of elements of s, i.e., $\#s$.

Example

$\#\langle \mathrm{Tom}, \mathrm{Mike}, \mathrm{Jone}, \mathrm{Mary}\rangle = 4$,
$\#\langle \mathrm{Tom}, \mathrm{Mike}, \mathrm{Jone}, \mathrm{Jone}, \mathrm{Mary}\rangle = 5$.

Outline

Concatenation (连接)

Example

$$\begin{aligned} record &= \langle Tom, Mike, Jone, Mary \rangle \\ newrecord &= \langle Alice, Bob \rangle \\ record \frown newrecord &= \langle Tom, Mike, Jone, Mary, Alice, Bob \rangle \end{aligned}$$

Definition (Concatenation)

Let $s, t : \mathrm{seq}\, X$. The concatenation of s and t, denoted $s \frown t$, is a sequence on X obtained by chaining together s and t.

Formally, $s \frown t$ is a sequence on X of length $\#s + \#t$ such that

$$(s \frown t)\, i = \begin{cases} s\, i & \text{if } i \in 1..\#s \\ t(i - \#s) & \text{if } i \in (\#s + 1)..(\#s + \#t) \end{cases}$$

Definition (Alternative definition)

$s \frown t == s \cup (\lambda\, n : \mathbb{N} \mid n > \#s \bullet n - \#s) \fatsemi t.$

Reverse (逆置)

Example

$$\begin{aligned} \text{record} &= \langle \text{Tom}, \text{Mike}, \text{Jone}, \text{Mary} \rangle \\ \text{rev record} &= \langle \text{Mary}, \text{Jone}, \text{Mike}, \text{Tom} \rangle \end{aligned}$$

Definition (Reverse)

Let $s : \text{seq}\, X$. The reverse of s, denoted rev s, is a sequence on X obtained from s by reversing the order of elements in s.

Formally, rev s is a sequence on X of length # such that

$$(\text{rev}\, s)\, i = s(\#s - i + 1).$$

- rev is a total function from seq X to seq X.

Concatenation and reverse: Generic definitions

$\frown$: Concatenation
rev: Reverse

Definition

$$
\begin{array}{l}
[X] \\
\hline
\_ \frown \_ : \operatorname{seq} X \times \operatorname{seq} X \to \operatorname{seq} X \\
\operatorname{rev} : \operatorname{seq} X \to \operatorname{seq} X \\
\hline
\forall s, t : \operatorname{seq} X \bullet s \frown t = s \cup \{ n : \operatorname{dom} t \bullet n + \# s \mapsto t(n) \} \\
\forall s : \operatorname{seq} X \bullet \operatorname{rev} s = \{ n : \mathbb{N} \mid n \in \operatorname{dom} s \bullet (n, s(\# s - n + 1)) \}
\end{array}
$$

Properties of concatenation and reverse

Let $s, t, u : \mathrm{seq}\, X$. Then we have

1. $(s \frown t) \frown u = s \frown (t \frown u)$
2. $\langle\rangle \frown s = s$
3. $s \frown \langle\rangle = s$
4. $\#(s \frown t) = \#s + \#t$
5. $\mathrm{rev}\, \langle\rangle = \langle\rangle$
6. $\mathrm{rev}\, \langle x\rangle = \langle x\rangle$
7. $\mathrm{rev}(s \frown t) = (\mathrm{rev}\, t) \frown (\mathrm{rev}\, s)$
8. $\mathrm{rev}(\mathrm{rev}\, s) = s$

head, last, tail and front

Example

$$
\begin{aligned}
\mathrm{head}\langle a, b, c\rangle &= a \\
\mathrm{tail}\langle a, b, c\rangle &= \langle b, c\rangle \\
\mathrm{last}\langle a, b, c\rangle &= c \\
\mathrm{front}\langle a, b, c\rangle &= \langle a, b\rangle
\end{aligned}
$$

Definition

Let $s : \mathrm{seq}_1\, X$. We define

- head s: the first element of s
- tail s: the sequence obtained from s by removing the first element
- front s: the sequence obtained from s by removing the last element
- last s: the last element of s

head, last, tail, front: Generic definitions

head, last
tail, front

Definition (head, last, tail, front)

$$[X]$$

$$\mathrm{head}, \mathrm{last} : \mathrm{seq}_1\, X \to X$$
$$\mathrm{tail}, \mathrm{front} : \mathrm{seq}_1\, X \to \mathrm{seq}\, X$$

$$\forall\, s : \mathrm{seq}_1\, X \bullet$$
$$\quad \mathrm{head}\, s = s\, 1 \land$$
$$\quad \mathrm{last}\, s = s\, \#s \land$$
$$\quad \mathrm{tail}\, s = \{n : \mathbb{N} \mid n \in 1 \mathrel{..} (\#s - 1) \bullet n \mapsto s\,(n + 1)\} \land$$
$$\quad \mathrm{front}\, s = 1 \mathrel{..} (\#s - 1) \lhd s$$

Properties of head, last, tail and front

Let $s, t : \mathrm{seq}\, X$. Then we have

1. $\mathrm{head}\,\langle x \rangle = \mathrm{last}\,\langle x \rangle = x$
2. $\mathrm{tail}\,\langle x \rangle = \mathrm{front}\,\langle x \rangle = \langle\rangle$
3. $s \neq \langle\rangle \Rightarrow \mathrm{head}(s \frown t) = \mathrm{head}\, s \land \mathrm{tail}(s \frown t) = (\mathrm{tail}\, s) \frown t$
4. $t \neq \langle\rangle \Rightarrow \mathrm{last}(s \frown t) = \mathrm{last}\, t \land \mathrm{front}(s \frown t) = s \frown (\mathrm{front}\, t)$
5. $s \neq \langle\rangle \Rightarrow \langle \mathrm{head}\, s \rangle \frown (\mathrm{tail}\, s) = s$
6. $s \neq \langle\rangle \Rightarrow (\mathrm{front}\, s) \frown \langle \mathrm{last}\, s \rangle = s$
7. $s \neq \langle\rangle \Rightarrow \mathrm{head}(\mathrm{rev}\, s) = \mathrm{last}\, s \land \mathrm{tail}(\mathrm{rev}\, s) = \mathrm{rev}(\mathrm{front}\, s)$
8. $s \neq \langle\rangle \Rightarrow \mathrm{last}(\mathrm{rev}\, s) = \mathrm{head}\, s \land \mathrm{front}(\mathrm{rev}\, s) = \mathrm{rev}(\mathrm{tail}\, s)$

Squash (压缩)

Example

$$s == \{1 \mapsto x, 2 \mapsto y, 3 \mapsto z, 4 \mapsto y, 5 \mapsto y, 6 \mapsto x\} = \langle x, y, z, y, y, x \rangle$$
$$4 \mathinner{..} 6 \lhd s = \{4 \mapsto y, 5 \mapsto y, 6 \mapsto x\}$$
$$\mathrm{squash}\,(4 \mathinner{..} 6 \lhd s) = \{1 \mapsto y, 2 \mapsto y, 3 \mapsto x\} = \langle y, y, x \rangle$$

- The operator squash converts every finite injective partial function $f : \mathbb{N}_1 \rightarrowtail\!\!\!\!\!\!\!+\; X$ into a sequence on X.

Squash: Generic definition

Definition (squash)

$$
\begin{array}{l}
[X] \\
\hline
\mathrm{squash} : (\mathbb{N}_1 \twoheadrightarrow\!\!\!\!\!\!\mapsto X) \nrightarrow \mathrm{seq}\, X \\
\hline
\mathrm{squash}\{\} = \langle\rangle \\
\forall\, f : \mathbb{N}_1 \twoheadrightarrow\!\!\!\!\!\!\mapsto X; i : \mathbb{N}_1 \mid (\exists\, n : \mathbb{N}_1 \bullet \# f = n) \land i = \min(\mathrm{dom}\, f) \bullet \\
\qquad \mathrm{squash}\, f = \langle f\, i \rangle \frown \mathrm{squash}\, (\{i\} \mathbin{\lhd\!\!\!-} f)
\end{array}
$$

- where $\min(\mathrm{dom}\, f)$ denotes the minimum number in $\mathrm{dom}\, f$.

Squash: Examples

Example

$$\text{squash}\,\{\} = \langle\rangle$$
$$\text{squash}\,\{3 \mapsto \text{x}, 6 \mapsto \text{y}, 10 \mapsto \text{x}\} = \langle \text{x}, \text{y}, \text{x}\rangle$$
$$\text{squash}\,\langle \text{x}, \text{x}, \text{y}\rangle = \langle \text{x}, \text{x}, \text{y}\rangle$$

$\mathrm{squash}\, f = \langle f\, i \rangle \frown \mathrm{squash}\, (\{i\} \lhd f)$

Example

Let $f == \{2 \mapsto x, 4 \mapsto y\}$. Then we have

$$
\begin{aligned}
\mathrm{squash}\, f &= \mathrm{squash}\, \{2 \mapsto x, 4 \mapsto y\} \\
&= \langle x \rangle \frown \mathrm{squash}\, (\{2\} \lhd \{2 \mapsto x, 4 \mapsto y\}) \\
&= \langle x \rangle \frown \mathrm{squash}\, \{4 \mapsto y\} \\
&= \langle x \rangle \frown (\langle y \rangle \frown \mathrm{squash}\, (\{4\} \lhd \{4 \mapsto y\})) \\
&= \langle x \rangle \frown (\langle y \rangle \frown \mathrm{squash}\, (\{\})) \\
&= \langle x \rangle \frown (\langle y \rangle \frown \langle \rangle) \\
&= \langle x \rangle \frown \langle y \rangle \\
&= \langle x, y \rangle
\end{aligned}
$$

Extraction (抽取)

Example

$$\{1, 3\} \upharpoonleft \langle a, b, c\rangle = \langle a, c\rangle$$
$$\{3, 4\} \upharpoonleft \langle Tom, Mike, Jone, Jone, Mary\rangle = \langle Jone, Jone\rangle$$

Definition (Extraction)

Let $s : \mathrm{seq}\, X$ and $I \subseteq \mathrm{dom}\, s$. Then the extraction of s by I, denoted $I \upharpoonleft s$, is defined as the sequence on X obtained from s by removing all elements at positions $i \in \mathrm{dom}\, s \setminus I$.

Definition (Alternative definition)

$$I \upharpoonleft s == squash(I \lhd s).$$

Filter (过滤)

Example

$$\langle a, b, c \rangle \upharpoonright \{a, c\} = \langle a, c \rangle$$
$$\langle Tom, Mike, Jone, Jone, Mary \rangle \upharpoonright \{Tom, Jone\} = \langle Tom, Jone, Jone \rangle$$

Definition (Filter)

Let $s : \mathrm{seq}\, X$ and $V \subseteq X$. Then the filtered sequence of s by V, denoted $s \upharpoonright V$, is defined as the sequence on X obtained from s by removing all elements $e \in X \setminus V$.

Definition (Alternative definition)

$$s \upharpoonright V == squash(s \rhd V).$$

Extraction and Filter: Generic definitions

$$\upharpoonleft : \text{ Extraction}$$
$$\restriction : \text{ Filter}$$

Definition

$$[X]$$

$$\_ \upharpoonleft \_ : \mathbb{P}\,\mathbb{N}_1 \times \operatorname{seq} X \to \operatorname{seq} X$$
$$\_ \restriction \_ : \operatorname{seq} X \times \mathbb{P}\,X \to \operatorname{seq} X$$

$$\forall I : \mathbb{P}\,\mathbb{N}_1; s : \operatorname{seq} X \bullet I \upharpoonleft s = \operatorname{squash}(I \lhd s)$$

$$\forall s : \operatorname{seq} X; V : \mathbb{P}\,X \bullet s \restriction V = \operatorname{squash}(s \rhd V)$$

Properties of Extraction and Filter

Let $s, t : \mathrm{seq}\, X$, $I \subseteq \mathrm{dom}\, s$ and $V, W : \mathbb{P}\, X$. Then we have

1. $\langle\rangle \upharpoonright V = I \upharpoonleft \langle\rangle = \langle\rangle$
2. $(s \frown t) \upharpoonright V = (s \upharpoonright V) \frown (t \upharpoonright V)$
3. $\mathrm{ran}\, s \subseteq V \Leftrightarrow s \upharpoonright V = s$
4. $s \upharpoonright \emptyset = \emptyset \upharpoonleft s = \langle\rangle$
5. $\#(s \upharpoonright V) \leq \#s$
6. $(s \upharpoonright V) \upharpoonright W = s \upharpoonright (V \cap W)$

Distributed catenation (分布连接) / flattening (平展)

Example

$$s == \langle\langle a, b, c\rangle, \langle d, e, f, g\rangle, \langle h, i\rangle\rangle$$
$$\frown/\, s = \langle a, b, c, d, e, f, g, h, i\rangle$$

Definition

Let $s : \operatorname{seq}(\operatorname{seq} X)$. Then the distributed catenation of s, denoted $\frown/\, s$, is defined as the sequence on X consisting of the constituent sequences of s concatenated in order.

Distributed catenation / flattening: Generic definition

Definition

$$[X]$$

$$\frown/ : \mathrm{seq}(\mathrm{seq}\,X) \to \mathrm{seq}\,X$$

$$\frown/\langle\rangle = \langle\rangle$$

$$\forall\, s : \mathrm{seq}\,X \bullet \frown/\langle s\rangle = s$$

$$\forall\, q, r : \mathrm{seq}(\mathrm{seq}\,X) \bullet \frown/(q \frown r) = (\frown/\, q) \frown (\frown/\, r)$$

Distributed catenation / flattening: Examples

Example

$$\frown/\langle\langle a, b, c\rangle, \langle d, e\rangle\rangle = \langle a, b, c, d, e\rangle$$
$$\frown/\langle\langle\rangle, \langle\rangle, \langle\rangle, \langle\rangle\rangle = \langle\rangle$$
$$\frown/\langle\langle a\rangle, \langle a\rangle, \langle a\rangle\rangle = \langle a, a, a\rangle$$
$$\frown/\langle\langle\langle a, b, c\rangle\rangle, \langle\langle d, e\rangle\rangle, \langle\langle f, g, h\rangle\rangle, \langle\langle i, j, k\rangle\rangle\rangle =$$
$$\langle\langle a, b, c\rangle, \langle d, e\rangle, \langle f, g, h\rangle, \langle i, j, k\rangle\rangle$$

Exercises

1. What are the function representations of the following sequences?
    1. $\langle 1, 1, 1 \rangle$;
    2. $\langle 3, 2, 1 \rangle$;
    3. $\langle \{Ben, Kate\}, \{Alice, Mike\} \rangle$.
2. What are the $\langle \cdot \rangle$-representations of the following sequences?
    1. $\{1 \mapsto Kate, 2 \mapsto Kate\}$;
    2. $\{3 \mapsto Alice, 1 \mapsto Alice, 4 \mapsto Mike, 2 \mapsto Mike\}$;
    3. $\{x : \mathbb{N} \mid 0 < x < 5 \bullet x \mapsto x^2\}$.
3. Let $s == \langle Ben \rangle$ and $t == \langle Kate, Alice, Mike \rangle$. What are the sequences defined by the following expressions?
    1. $(\text{tail}\, t) \frown s$;
    2. $\langle \text{last}\, t \rangle \frown (\text{front}\, t)$.
4. Let $s, t : \text{seq}\, X$. Prove $\text{rev}(s \frown t) = (\text{rev}\, t) \frown (\text{rev}\, s)$.

Prove $\mathrm{rev} s \frown t = (\mathrm{rev}\, t) \frown (\mathrm{rev}\, s)$.

Proof.

By definition, we know that

$$(\mathrm{rev}\, s)\, i = s\,(\#s - i + 1).$$

For $1 \leqslant i \leqslant \#s + \#t$, we have $(\mathrm{rev}\, s \frown t)\, i = s \frown t\,(\#s + \#t - i + 1)$.
Case 1 $(1 \leqslant i \leqslant \#t)$:
$(\mathrm{rev}\, s \frown t)\, i = s \frown t\,(\#s + \#t - i + 1) = t\,(\#t - i + 1) = (\mathrm{rev}\, t)\, i =$
$(\mathrm{rev}\, t) \frown (\mathrm{rev}\, s)\, i$.
Case 2 $(\#t < i \leqslant \#s + \#t)$:
$(\mathrm{rev}\, s \frown t)\, i = s \frown t\,(\#s + \#t - i + 1) = s\,(\#s + \#t - i + 1) =$
$(\mathrm{rev}\, s)\,(\#s - (\#s + \#t - i + 1) + 1) = (\mathrm{rev}\, s)\,(i - \#t) =$
$(\mathrm{rev}\, t) \frown (\mathrm{rev}\, s)\,(i - \#t + \#t) = (\mathrm{rev}\, t) \frown (\mathrm{rev}\, s)\, i$. $\square$