2018～2019 学年第一学期《高等数学 2A》第二次月考参考答案

2018.12.14

一、计算题（每小题 10 分，共 40 分）

1. 计算不定积分 $\int \frac{\sin x}{\cos^2 x - 6\cos x + 5} dx$.

解：$\int \frac{\sin x}{\cos^2 x - 6\cos x + 5} dx = -\int \frac{d\cos x}{\cos^2 x - 6\cos x + 5}$（令 $\cos x = t$）

$$= -\int \frac{dt}{t^2 - 6t + 5} = -\frac{1}{4}\int \left(\frac{1}{t-5} - \frac{1}{t-1}\right) dt$$

$$= -\frac{1}{4}\ln\left|\frac{t-5}{t-1}\right| + C = \frac{1}{4}\ln\left|\frac{t-1}{t-5}\right| + C = \frac{1}{4}\ln\frac{1-\cos x}{5-\cos x} + C.$$

2. 计算不定积分 $\int \frac{\ln(1+e^x)}{e^x} dx$.

解：$\int \frac{\ln(1+e^x)}{e^x} dx = -\int \ln(1+e^x)\, de^{-x}$

$$= -e^{-x}\ln(1+e^x) + \int \frac{1}{1+e^x} dx$$

$$= -e^{-x}\ln(1+e^x) + \int (1 - \frac{e^x}{1+e^x}) dx$$

$$= -e^{-x}\ln(1+e^x) + x - \ln(1+e^x) + C.$$

方法二：令 $u = e^x$，则 $x = \ln u, dx = \frac{1}{u} du,$

$$\int \frac{\ln(1+e^x)}{e^x} dx = \int \frac{\ln(1+u)}{u}\frac{1}{u} du = -\int \ln(1+u)\, d(\frac{1}{u})$$

$$= -\frac{\ln(1+u)}{u} + \int \frac{1}{u(1+u)} du = -\frac{\ln(1+u)}{u} + \ln\frac{u}{1+u} + C$$

$$= -\frac{\ln(1+e^x)}{e^x} + \ln\frac{e^x}{1+e^x} + C.$$

3. 求极限 $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\cdot\left(1+\frac{2}{n}\right)\cdot\left(1+\frac{3}{n}\right)\cdot \cdots \cdot\left(1+\frac{n}{n}\right)\right]^{\frac{2}{n}}$.

解：令 $u_n = \left[\left(1+\frac{1}{n}\right)\cdot\left(1+\frac{2}{n}\right)\cdot \cdots \cdot\left(1+\frac{n}{n}\right)\right]^{\frac{2}{n}}$，

则 $\ln u_n = \frac{2}{n}\left[\ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{2}{n}\right) + \cdots + \ln\left(1+\frac{n}{n}\right)\right]$，

$$\lim_{n\to\infty} \ln u_n = 2\int_0^1 \ln(1+x) dx = 2(1+x)\ln(1+x)\Big|_0^1 - 2\int_0^1 dx = 4\ln 2 - 2.$$

故 $\lim_{n\to\infty} u_n = e^{4\ln 2 - 2} = \frac{16}{e^2}$.

4. 计算定积分 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x + \sin^2 x}{(1+\cos x)^2} dx$.

解：$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x + \sin^2 x}{(1+\cos x)^2} dx = 2\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{(1+\cos x)^2} dx$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)^2 dx = 2\int_0^{\frac{\pi}{2}} \tan^2\frac{x}{2} dx = 2\int_0^{\frac{\pi}{2}} (\sec^2\frac{x}{2} - 1) dx$$

$$= 2\left(2\tan\frac{x}{2} - x\right)\Big|_0^{\frac{\pi}{2}} = 4 - \pi.$$

二、 **解答题（每小题 10 分，共 30 分）**

1. 设函数 $f(x)=\begin{cases} e^x, & -1\le x\le 0, \\ 1, & 0<x\le 2, \\ \sin^2\dfrac{\pi x}{4}, & x>2, \end{cases}$ 求 $F(x)=\int_{-1}^{x} f(t)\,dt\ (-1\le x\le 3)$ 的表达式.

解：(1) $-1\le x\le 0, F(x)=\int_{-1}^{x} e^t dt = e^t\big|_{-1}^{x} = e^x-\dfrac{1}{e}$,

(2) $0<x\le 2,\ F(x)=F(0)+\int_0^x dt = 1-\dfrac{1}{e}+x = 1-\dfrac{1}{e}+x$,

(3) $2<x\le 3,\ \ F(x)=F(2)+\int_2^x \sin^2\dfrac{\pi t}{4}dt$

$$=3-\frac{1}{e}+\frac{1}{2}\int_2^x(1-\cos\frac{\pi t}{2})dt$$

$$=3-\frac{1}{e}+\frac{1}{2}\left(x-2-\frac{2}{\pi}\sin\frac{\pi t}{2}\bigg|_2^x\right)$$

$$=2-\frac{1}{e}+\frac{1}{2}x-\frac{1}{\pi}\sin\frac{\pi x}{2}.$$

$$F(x)=\begin{cases} e^x-\dfrac{1}{e}, & -1\le x\le 0, \\ 1-\dfrac{1}{e}+x, & 0<x\le 2, \\ 2-\dfrac{1}{e}+\dfrac{x}{2}-\dfrac{1}{\pi}\sin\dfrac{\pi x}{2}, & 2<x\le 3. \end{cases}$$

2. 求曲线段 $y=\dfrac{1}{4}x^2-\dfrac{1}{2}\ln x\ \ (1\le x\le e)$ 的弧长.

解：$s=\int_1^e\sqrt{1+(\dfrac{x}{2}-\dfrac{1}{2x})^2}dx$

$$=\int_1^e(\frac{x}{2}+\frac{1}{2x})dx$$

$$=\left(\frac{1}{4}x^2+\frac{1}{2}\ln x\right)\bigg|_1^e$$

$$=\frac{1}{4}(e^2-1)+\frac{1}{2}=\frac{1}{4}(e^2+1).$$

3. 求极限 $\lim\limits_{x\to 0}\dfrac{x\int_0^{x^2}\arctan u\,du}{\sqrt[4]{1+x^5}-1}$.

解：$\lim\limits_{x\to 0}\dfrac{x\int_0^{x^2}\arctan u\,du}{\sqrt[4]{1+x^5}-1}$

$$=\lim_{x\to 0}\frac{x\int_0^{x^2}\arctan u\,du}{\frac{1}{4}x^5}=\lim_{x\to 0}\frac{\int_0^{x^2}\arctan u\,du}{\frac{1}{4}x^4}$$

$$=\lim_{x\to 0}\frac{2x\arctan x^2}{x^3}=\lim_{x\to 0}\frac{2x^3}{x^3}=2.$$

三、解答题（每小题 12 分，共 24 分）

1.设函数$\varphi(x)=\int_0^{\sin x} f(tx^2)\mathrm{d}t$，其中$f(x)$是连续函数,且$f(0)=2$. 求$\varphi'(x)$的表达式.

解：当$x\neq 0$时，令$u=tx^2$，$t=\dfrac{u}{x^2},(x\neq 0)$，$\mathrm{d}t=\dfrac{\mathrm{d}u}{x^2}$，

$$\varphi(x)=\int_0^{\sin x} f(tx^2)\mathrm{d}t=\int_0^{x^2\sin x} f(u)\frac{1}{x^2}\mathrm{d}u=\frac{1}{x^2}\int_0^{x^2\sin x} f(u)\mathrm{d}u,$$

$\varphi(0)=0.$

$x\neq 0$时，$\varphi'(x)=-\dfrac{2}{x^3}\displaystyle\int_0^{x^2\sin x} f(u)\mathrm{d}u+\dfrac{f(x^2\sin x)}{x^2}(2x\sin x+x^2\cos x)$，

$x=0$时，$\varphi'(0)=\displaystyle\lim_{x\to 0}\frac{\varphi(x)-\varphi(0)}{x-0}=\lim_{x\to 0}\frac{\int_0^{x^2\sin x} f(u)\mathrm{d}u}{x^3}$

$$=\lim_{x\to 0}\frac{f(x^2\sin x)(2x\sin x+x^2\cos x)}{3x^2}=f(0)=2.$$

$$\therefore \varphi'(x)=\begin{cases}-\dfrac{2}{x^3}\displaystyle\int_0^{x^2\sin x} f(u)\mathrm{d}u+\dfrac{f(x^2\sin x)}{x}(2\sin x+x\cos x), & x\neq 0,\\ 2, & x=0.\end{cases}$$

2. 过曲线$L: y=x^{\frac{1}{3}}\ (x\geq 0)$上点$A(1,1)$作切线，使该切线与曲线$L$以及$x$轴所围的平面图形为$D$. (1)求$D$的面积； (2)求$D$绕$x$轴旋转一周所得立体的体积.

解：(1) $y'|_{x=1}=\dfrac{1}{3}x^{-\frac{2}{3}}|_{x=1}=\dfrac{1}{3}$，

切线方程：$y-1=\dfrac{1}{3}(x-1)$，即：$y=\dfrac{1}{3}x+\dfrac{2}{3}$.

方法一：$S_D=\dfrac{3}{2}-\displaystyle\int_0^1 x^{\frac{1}{3}}\mathrm{d}x=\dfrac{3}{2}-\dfrac{3}{4}x^{\frac{4}{3}}\Big|_0^1=\dfrac{3}{4}$.

方法二：$S_D=\dfrac{1}{2}\cdot 2\cdot\dfrac{2}{3}+\displaystyle\int_0^1(\dfrac{1}{3}x+\dfrac{2}{3}-x^{\frac{1}{3}})\mathrm{d}x=\dfrac{3}{4}$.

(2) $V=\dfrac{\pi}{3}\cdot 1^2\cdot 3-\pi\displaystyle\int_0^1 x^{\frac{2}{3}}\mathrm{d}x=\pi-\dfrac{3}{5}\pi x^{\frac{5}{3}}\Big|_0^1=\dfrac{2\pi}{5}$.

四、证明题（6 分）

设函数$f(x)$在$[0,1]$上连续，在$(0,1)$上可导，且$f(0)=0$，$0<f'(x)\leq 1$.

证明 $\left(\int_0^1 f(x)\mathrm{d}x\right)^2\geq\int_0^1 f^3(x)\mathrm{d}x$.

证明：令$F(t)=(\int_0^t f(x)\mathrm{d}x)^2-\int_0^t f^3(x)\mathrm{d}x$，则

$F'(t)=2f(t)\int_0^t f(x)\mathrm{d}x-f^3(t)=f(t)\left(2\int_0^t f(x)\mathrm{d}x-f^2(t)\right).$

$f(0)=0$，$f'(t)>0$，所以当$t>0$，$f(t)>f(0)>0$.

令$G(t)=2\int_0^t f(x)\mathrm{d}x-f^2(t)$，

则$G(0)=0$，且 $G'(t)=2f(t)\left(1-f'(t)\right)$，

$\because 0<f'(t)\leq 1, \therefore G'(t)\geq 0.$

所以当$t>0$，$G(t)$单调增加，$G(t)\geq G(0)=0$.

于是$F'(t)=f(t)G(t)\geq 0$，

故当$t>0$，$F(t)$单调增加，$F(t)\geq F(0)=0$.

从而$F(1)\geq F(0)$，即 $\left(\int_0^1 f(x)\mathrm{d}x\right)^2\geq\int_0^1 f^3(x)\mathrm{d}x$.