Formal Methods (形式化方法)

Lecture 14. Reasoning about Specifications

智能与计算学部 章衡

2021年上学期

Motivation

Features of Z notation

- By using Z notations one can define the specification precisely, which could reduce the misunderstandings in requirement analyses largely

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- How to assure the specification admitting a desired property?
- How to know whether a program meets the requirements stated in the specification?

Outline

Example: Hobby club

- Basic type:

$$[Person]$$

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  $$\mathrm{Max} : \mathbb{N}$$

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$$\Delta \mathrm{HoClub} \ \widehat{=}\ \mathrm{HoClub} \wedge \mathrm{HoClub}'$$

$$\Xi \mathrm{HoClub} \ \widehat{=}\ \Delta \mathrm{HoClub} \mid s' = s$$

Example: Hobby club

$$
\begin{array}{|l}
\hline
\textbf{EnterClub} \\
\Delta HoClub \\
p? : Person \\
\hline
\#s < Max \\
p? \notin s \\
s' = s \cup \{p?\} \\
\hline
\end{array}
$$

Example: Hobby club

EnterClub

$\Delta HoClub$
$p? : Person$

$\# s < Max$
$p? \notin s$
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LeaveClub

$\Delta HoClub$
$p? : Person$

$p? \in s$
$s' = s \setminus \{p?\}$

Example: Hobby club

$$\mathrm{EnterClub} \mathbin{;} \mathrm{LeaveClub} \vDash \#s < \mathrm{Max} \land s' = s$$

Example: Hobby club

$$\mathrm{EnterClub} \mathbin{⨾} \mathrm{LeaveClub} \models \#s < \mathrm{Max} \land s' = s$$

$$\mathrm{Alpha} \mathrel{\widehat{=}} \mathrm{EnterClub} \mathbin{⨾} \mathrm{LeaveClub}$$

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Alpha

$s, s' : \mathbb{P}\,\mathrm{Person}$
$p? : \mathrm{Person}$

$$\begin{array}{l}
\exists\, s^{+} : \mathbb{P}\,\mathrm{Person} \bullet \\
\qquad (\#s \leqslant \mathrm{Max} \land \\
\qquad \#s^{+} \leqslant \mathrm{Max} \land \\
\qquad \#s' \leqslant \mathrm{Max} \land \\
\qquad \#s < \mathrm{Max} \land \\
\qquad p? \notin s \land \\
\qquad s^{+} = s \cup \{p?\} \land \\
\qquad p? \in s^{+} \land \\
\qquad s' = s^{+} \setminus \{p?\})
\end{array}$$

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If x does not occur in φ, then $\exists\, x : X \bullet (\varphi \wedge \psi) \equiv \varphi \wedge \exists\, x : X \bullet \psi$

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$\exists s^+ : \mathbb{P}\,Person \bullet$
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$Alpha_1$

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$$\begin{array}{|l}
\hline \mathrm{Alpha}_2 \\
s, s' : \mathbb{P}\,\mathrm{Person} \\
p? : \mathrm{Person} \\
\hline
\#s \leqslant \mathrm{Max} \\
\#s' \leqslant \mathrm{Max} \\
\#s < \mathrm{Max} \\
p? \notin s \\
\#(s \cup \{p?\}) \leqslant \mathrm{Max} \\
p? \in (s \cup \{p?\}) \\
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From $p? \notin s$, we know that $(s \cup \{p?\}) \setminus \{p?\} = s$. Consequently,

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- In many cases, mathematicians try to find a weaker form of formal proofs, called rigorous proofs
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- In a rigorous proof, one is allowed to use the properties in set theory and number theory, as well as the method of induction

Method of induction

Definition (Mathematical induction, 数学归纳法)

To prove "for every natural number n it holds that $P(n)$", it suffices to prove both of the following:

1. $P(0)$ holds;
2. $\forall i : \mathbb{N} \bullet (P(i) \Rightarrow P(i+1))$.

Definition (Structural induction, 结构归纳法)

To prove "for every sequence $s : \mathrm{seq}\, X$ it holds that $P(s)$", it suffices to prove both of the following:

1. $P(\langle\rangle)$ holds;
2. $\forall x : X; s : \mathrm{seq}\, X \bullet (P(s) \Rightarrow P(\langle x \rangle \frown s))$.

Method of induction: Example 1

Example

Please prove that, for all sequences s, t, u : seq X, we have

$$s \frown (t \frown u) = (s \frown t) \frown u.$$

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which completes the proof. □

Method of induction: Example 2

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Inductive step: Assume as inductive hypothesis that $\mathrm{rev}(s \frown t) = (\mathrm{rev}\, t) \frown (\mathrm{rev}\, s)$. We need to prove that $\mathrm{rev}((\langle x\rangle \frown s) \frown t) = (\mathrm{rev}\, t) \frown \mathrm{rev}(\langle x\rangle \frown s)$.

Method of induction: Example 2

Example

Please prove that, for all sequences s, t : seq X, we have

$$\mathrm{rev}(s \frown t) = (\mathrm{rev}\, t) \frown (\mathrm{rev}\, s)$$

Proof.

By definition, it is easy to see that $\langle\rangle \frown s = s = s \frown \langle\rangle$ and $\mathrm{rev}(\langle x\rangle \frown t) = (\mathrm{rev}\, t) \frown \langle x\rangle$. Next we prove the desired property by an induction on s.

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&= ((\mathrm{rev}\, t) \frown (\mathrm{rev}\, s)) \frown \langle x\rangle
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\end{aligned}$$

which completes the proof. □

Exercise

Prove the following by induction: for every sequence s, we have that $\mathrm{rev}(\mathrm{rev}\,\mathrm{s}) = \mathrm{s}$.

Outline

Example: Fan ID management

- Basic types:

  [Person, ID]

Example: Fan ID management

- Basic types:

  $[Person, ID]$

- State space schema:

$$
\begin{array}{|l}
\hline
\textbf{FID} \\
\hline
members : ID \rightarrowtail\!\!\!\!\!\rightarrow Person \\
banned : \mathbb{P}\, ID \\
\hline
banned \subseteq \operatorname{dom} members \\
\hline
\end{array}
$$

Example: Fan ID management

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$$\begin{array}{|l}
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  $\Xi FID \mathrel{\widehat=} \Delta FID \mid members' = members \land banned' = banned$

The initialization theorem (初始化定理)

- Operational schemas: Initialization

$$
\begin{array}{|l}
\hline
\textbf{InitFID} \\
\mathrm{FID}' \\
\hline
\mathrm{members}' = \emptyset \\
\mathrm{banned}' = \emptyset \\
\hline
\end{array}
$$

The initialization theorem (初始化定理)

- Operational schemas: Initialization

InitFID

FID'

$members' = \emptyset$

$banned' = \emptyset$

- The initialization theorem: $\models \exists FID' \bullet InitFID$

The initialization theorem (初始化定理)

- Operational schemas: Initialization

$$\begin{array}{|l}
\hline
\textit{InitFID} \\
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\hline
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\end{array}$$

- The initialization theorem: $\models \exists\, \textit{FID}' \bullet \textit{InitFID}$

  The above is an abbreviation of the following theorem:

  $$\models \exists\, \textit{members}' : \textit{Person} \rightarrowtail\!\!\!\!\!\!\!\!\;\rightarrow\!\!\!+ \textit{ID}; \textit{banned}' : \mathbb{P}\,\textit{ID} \bullet$$
  $$(\textit{banned}' \subseteq \operatorname{dom} \textit{members}' \wedge \textit{members}' = \emptyset \wedge \textit{banned}' = \emptyset)$$

Prove the initialization theorem

$$\vDash \exists\, \mathrm{members}' : \mathrm{Person} \rightarrowtail\!\!\!\!\!\!\!\!\;\,\, \mapsto \mathrm{ID}; \mathrm{banned}' : \mathbb{P}\,\mathrm{ID} \bullet \\ (\mathrm{banned}' \subseteq \mathrm{dom}\,\mathrm{members}' \wedge \mathrm{members}' = \emptyset \wedge \mathrm{banned}' = \emptyset) \tag{1}$$

Prove the initialization theorem

$$\models \exists\, members' : Person \rightarrowtail\!\!\!\!\rightarrow ID; banned' : \mathbb{P}\, ID \bullet \\ (banned' \subseteq \operatorname{dom} members' \land members' = \emptyset \land banned' = \emptyset) \quad (1)$$

1-point rule (bidirection)

$$\frac{\Sigma \models \exists x : S \bullet (\varphi \land x = t)}{\Sigma \models t \in S \land \varphi[t/x]} \quad [1\text{-point}] \quad \langle x \text{ does not occur in } t \rangle$$

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- By applying the above rule, (1) can be simplified as

$$\vDash \emptyset \in Person \rightarrowtail\!\!\!\!\twoheadrightarrow ID \land \emptyset \in \mathbb{P}\, ID \land \emptyset \subseteq \operatorname{dom} \emptyset \qquad (2)$$

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$$\models \emptyset \in Person \rightarrowtail\!\!\!\!\!\!\rightarrow ID \land \emptyset \in \mathbb{P}\, ID \land \emptyset \subseteq \mathrm{dom}\, \emptyset \quad (2)$$

- To prove this, it is equivalent to prove all of the following:

$$\models \emptyset \in Person \rightarrowtail\!\!\!\!\!\!\rightarrow ID,$$
$$\models \emptyset \in \mathbb{P}\, ID,$$
$$\models \emptyset \subseteq \mathrm{dom}\, \emptyset.$$

Precondition of an operation

AddMember

$$\Delta \mathrm{FID}$$
$$\mathrm{applicant?} : \mathrm{Person}$$
$$\mathrm{id!} : \mathrm{ID}$$

$$\mathrm{applicant?} \notin \mathrm{ran}\ \mathrm{members}$$
$$\mathrm{id!} \notin \mathrm{dom}\ \mathrm{members}$$
$$\mathrm{members}' = \mathrm{members} \cup \{\mathrm{id!} \mapsto \mathrm{applicant?}\}$$
$$\mathrm{banned}' = \mathrm{banned}$$

Precondition of an operation

AddMember

ΔFID
$applicant? : Person$
$id! : ID$

$applicant? \notin \operatorname{ran} members$
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$members' = members \cup \{id! \mapsto applicant?\}$
$banned' = banned$

- We need to know when the operation can be executed.

Precondition of an operation

AddMember

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$applicant? : Person$

$id! : ID$

$applicant? \notin \operatorname{ran} members$

$id! \notin \operatorname{dom} members$

$members' = members \cup \{id! \mapsto applicant?\}$

$banned' = banned$

- We need to know when the operation can be executed.
- If such a condition is not true, we need to report an error.

Precondition of an operation

PreAddMember

FID
$applicant? : Person$

$\exists FID'; id! : ID \bullet$
$(applicant? \notin \operatorname{ran} members \land$
$id! \notin \operatorname{dom} members \land$
$members' = members \cup \{id! \mapsto applicant?\} \land$
$banned' = banned)$

Precondition of an operation

PreAddMember

FID

$applicant? : Person$

$$\exists FID'; id! : ID \bullet$$
$$(applicant? \notin \operatorname{ran} members \wedge$$
$$id! \notin \operatorname{dom} members \wedge$$
$$members' = members \cup \{id! \mapsto applicant?\} \wedge$$
$$banned' = banned)$$

- Unfolding the predicate of the above schema, we have

$$\exists members' : ID \rightarrowtail\!\!\!\!\!\rightarrow Person; banned' : \mathbb{P}\, ID; id! : ID \bullet$$
$$(banned' \subseteq \operatorname{dom} members' \wedge applicant? \notin \operatorname{ran} members \wedge$$
$$id! \notin \operatorname{dom} members \wedge members' = members \cup \{id! \mapsto applicant?\} \wedge$$
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Simplification of precondition

Most often used rules for precondition simplification

$$\frac{\Sigma \vDash \exists x : S \bullet (\varphi \wedge x = t)}{\Sigma \vDash t \in S \wedge \varphi[t/x]} \quad [\text{1-point}] \quad <\text{x does not occur in t}>$$

$$\frac{\Sigma \vDash \varphi \wedge \psi}{\Sigma \vDash \varphi} \quad [\wedge] \quad <\Sigma, \varphi \vDash \psi>$$

$$\frac{\Sigma \vDash \varphi}{\Sigma \vDash \varphi'} \quad [=] \quad <\Sigma \vDash t_1 = t_2 \text{ and } \varphi' \text{ is obtained from } \varphi \text{ by substituting } t_2 \text{ for some occurrence of } t_1>$$

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$$\begin{aligned}
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\end{aligned} \tag{3}$$

Simplification of precondition

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- By applying 1-point rule for variable $banned'$, (3) can be simplified as

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\end{aligned} \tag{5}$$

Simplification of precondition

$$
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\end{aligned} \tag{6}
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- By the declaration $banned : \mathbb{P}\, ID$ we know $banned \in \mathbb{P}\, ID$. Consequently, (6) can be equivalently rewritten as

$$\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \land applicant? \notin \mathrm{ran}\, members \land \\ id! \notin \mathrm{dom}\, members \land members \cup \{id! \mapsto applicant?\} \in ID \rightarrowtail\!\!\!\!\rightarrow Person) \tag{7}$$

Simplification of precondition

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- By $members : ID \rightarrowtail\!\!\!\!\twoheadrightarrow Person; id! : ID; applicant? : Person$ and $id! \notin \mathrm{dom}\, members$, we have that $members \cup \{id! \mapsto applicant?\} \in ID \twoheadrightarrow Person$. By $applicant? \notin \mathrm{ran}\, members$, we obtain that $members \cup \{id! \mapsto applicant?\} \in ID \rightarrowtail\!\!\!\!\twoheadrightarrow Person$. Thus, (7) can be simplified as

$$\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \land applicant? \notin \mathrm{ran}\, members \land \\ id! \notin \mathrm{dom}\, members) \tag{8}$$

Simplification of precondition

$$\exists\, id! : ID \bullet (banned \subseteq \operatorname{dom}(members \cup \{id! \mapsto applicant?\}) \land applicant? \notin \operatorname{ran} members \land id! \notin \operatorname{dom} members) \tag{9}$$

Simplification of precondition

$$\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \land applicant? \notin \mathrm{ran}\, members \land id! \notin \mathrm{dom}\, members) \tag{9}$$

- By properties $\mathrm{dom}(A \cup B) = \mathrm{dom}\, A \cup \mathrm{dom}\, B$ and $\mathrm{dom}\{id! \mapsto applicant?\} = \{id!\}$, we conclude that $\mathrm{dom}(members \cup \{id! \mapsto applicant?\} = (\mathrm{dom}\, members) \cup \{id!\}$. Thus, (9) can be simplified as

$$\exists\, id! : ID \bullet (banned \subseteq (\mathrm{dom}\, members) \cup \{id!\} \land applicant? \notin \mathrm{ran}\, members \land id! \notin \mathrm{dom}\, members) \tag{10}$$

Simplification of precondition

$$\exists\, id! : ID \bullet (banned \subseteq \mathrm{dom}(members \cup \{id! \mapsto applicant?\}) \wedge applicant? \notin \mathrm{ran}\, members \wedge id! \notin \mathrm{dom}\, members) \tag{9}$$

- By properties $\mathrm{dom}(A \cup B) = \mathrm{dom}\, A \cup \mathrm{dom}\, B$ and $\mathrm{dom}\{id! \mapsto applicant?\} = \{id!\}$, we conclude that $\mathrm{dom}(members \cup \{id! \mapsto applicant?\} = (\mathrm{dom}\, members) \cup \{id!\}$. Thus, (9) can be simplified as

$$\exists\, id! : ID \bullet (banned \subseteq (\mathrm{dom}\, members) \cup \{id!\} \wedge applicant? \notin \mathrm{ran}\, members \wedge id! \notin \mathrm{dom}\, members) \tag{10}$$

- By the definition of FID we know that $banned \subseteq \mathrm{dom}\, members$. Thus, (10) can be simplified as

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Simplification of precondition

$$\begin{array}{|l}
\multicolumn{1}{l}{\text{PreAddMember}} \\
\hline
FID \\
applicant? : Person \\
\hline
applicant? \notin \operatorname{ran} members \\
\operatorname{dom} members \neq ID \\
\hline
\end{array}$$

Simplified precondition schema PreAddMember

Properties of the specification

BanMember

ΔFID

$ban? : ID$

$ban? \in \operatorname{dom} members$

$banned' = banned \cup \{ban?\}$

$members' = members$

Properties of the specification

$$
\begin{array}{|l}
\hline
\textbf{BanMember} \\
\Delta FID \\
ban? : ID \\
\hline
ban? \in \operatorname{dom} members \\
banned' = banned \cup \{ban?\} \\
members' = members \\
\hline
\end{array}
$$

- Property to be verified: To execute the operation BanMember on some banned member, the state of the system will not changed.

Properties of the specification

$$
\begin{array}{|l}
\hline
\text{BanMember} \\
\Delta FID \\
ban? : ID \\
\hline
ban? \in \operatorname{dom} members \\
banned' = banned \cup \{ban?\} \\
members' = members \\
\hline
\end{array}
$$

- Property to be verified: To execute the operation BanMember on some banned member, the state of the system will not changed.
- Such a property can be stated as follows:

$$BanMember \mid ban? \in banned \models \Xi FID$$

Properties of the specification

- Be definition, the above statement is equivalent to the following one:

$$\begin{aligned}\Delta FID; ban? : ID \mid (&ban? \in \operatorname{dom} members \wedge \\ &banned' = banned \cup \{ban?\} \wedge members' = members \wedge \\ &ban? \in banned) \\ \models \\ &\Delta FID \mid members' = members \wedge banned' = banned\end{aligned}$$

Properties of the specification

- Be definition, the above statement is equivalent to the following one:

$$
\begin{aligned}
&\Delta FID;\, ban? : ID \mid (ban? \in \operatorname{dom} members \wedge \\
&\qquad banned' = banned \cup \{ban?\} \wedge members' = members \wedge \\
&\qquad ban? \in banned) \\
&\models \\
&\qquad \Delta FID \mid members' = members \wedge banned' = banned
\end{aligned}
$$

- From $ban? \in banned$ and $banned' = banned \cup \{ban?\}$, we know $banned' = banned$, which completes the proof.

Exercises

SM

$\mathrm{dir} : B \rightarrow\!\!\!\!\!\rightarrow\!\!\!\!\!\!\!\!\!\!\!\!\; U$

$\mathrm{free} : \mathbb{P}\ B$

$\mathrm{free} = B \setminus (\mathrm{dom}\ \mathrm{dir})$

InitSM

SM'

$\mathrm{dir}' = \{\}$

$\mathrm{free}' = B$

$\mathrm{Release}_0$

$\Delta\ SM$

$u? : U$

$b? : B$

$r! : \mathrm{Report}$

$(b? \mapsto u?) \in \mathrm{dir}$

$\mathrm{free}' = \mathrm{free} \cup \{b?\}$

$\mathrm{dir}' = \{b?\} \lhd \mathrm{dir}$

$r! = \text{“Okay”}$

Ex. 1

What is the initialization theorem of the above specification? Write it down, and prove it.

Ex. 2

What is the schema of precondition of $\mathrm{Release}_0$? Write it down, and simplify it.