ME596 Homework 5

Erin Schmidt

Problem 1.

Solve the following problem using the Simplex Search Method:

$$\min \ f(x) = (1 - x_1)^2 + (2 - x_2)^2$$

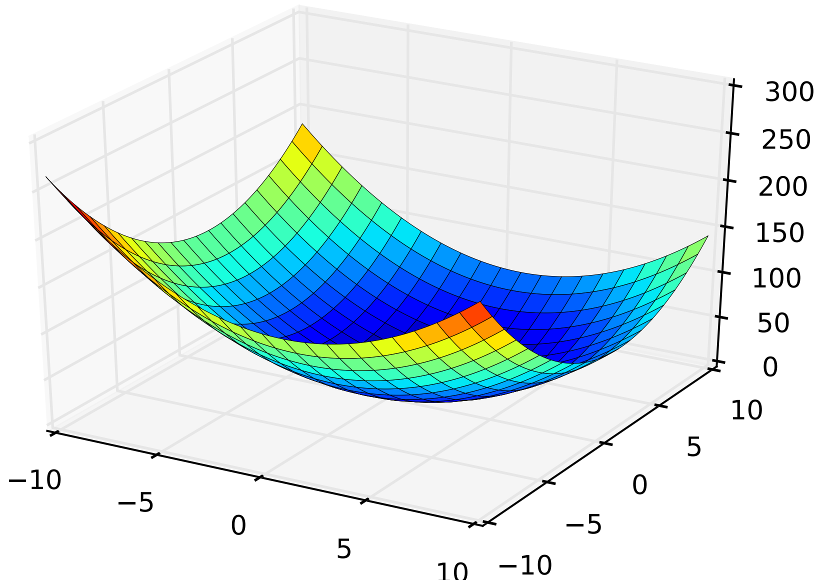
In [2]:
```
#Simplex search

#Erin Schmidt

#for non-linear programming problems ala Nelder and Mead(1965)
#**Note** use Python3 for this script
import math as m
import numpy as np

def simplex_search(f, x_start, max_iter = 100, epsilon = 1E-6, gamma = 5, beta = 0.5):
    """
    parameters of the function:
    f is the function to be optimized
    x_start (numpy array): initial position
    epsilon is the termination criteria
    gamma is the contraction coefficient
    beta is the expansion coefficient
    """

    #init arrays
    N = len(x_start)
    fnew = []
    xnew = []
    x = []

    #generate vertices of initial simplex
    a = .75
    x0 = (x_start)
    x1 = [x0 + [((N + 1)**0.5 + N - 1.)/(N + 1.)*a, 0.]]
    x2 = [x0 + [0., ((N + 1)**0.5 - 1.)/(N + 1.)*a]]
    x3 = [x0 - [0., ((N + 1)**0.5 - 1.)/(N + 1.)*a]]
    x = np.vstack((x1, x2, x3))
    #print(x)

    #simplex iteration
    while True:
        #find best, worst and 2nd worst points --> new center point
        f_run = np.array([f(x[0]), f(x[1]), f(x[2])]).tolist() #func. values at vertices
        #print(f_run)
        xw = x[f_run.index(sorted(f_run)[-1])]          #worst point
```

```python
            xb = x[f_run.index(sorted(f_run)[0])]          #best point
            xs = x[f_run.index(sorted(f_run)[-2])]         #2nd worst point
            xc = (xb + xs)/N #center point
            xr = 2*xc - xw #reflection point

            #check cases
            if f(xr) < f(xb): #expansion
                xnew = 2*xr - xc
                #xnew = (1 - gamma)*xc - gamma*xr
                #print('a', f(xr), f(xb)) #for debugging
            elif f(xr) > f(xw): #contraction 1
                xnew = (1 - beta)*xc + beta*xw
                #print('b', f(xr), f(xw))
            elif f(xs) < f(xr) and f(xr) < f(xw): #contraction 2
                xnew = (1 + beta)*xc - beta*xw
                #print('c', f(xs), f(xr), f(xw))
            else:
                xnew = xr

            #replace vertices
            """
            if f(xnew) < f(xb):
            """
            x[f_run.index(sorted(f_run)[-1])] = xnew
            #x[1] = xb
            #x[2] = xs
            fnew.append(f(xnew))
            print('Current optimum = ', fnew[-1])

            #break is any termination critera satisfied
            if len(fnew) == max_iter or term_check(xb, xc, xs, xnew, N) <= epsilon:
                return {
                    f(x[f_run.index(sorted(f_run)[0])]),
                    x[f_run.index(sorted(f_run)[0])], len(fnew)
                    }

    def term_check(xb, xc, xs, xnew, N): #the termination critera
        return m.sqrt(((f(xb) - f(xc))**2 + (f(xnew) - f(xc))**2 + (f(xs) - f(xc))**2)/(N + 1))

    #testing
    def f(z): #the objective function
        x=z
        return (1 - x[0])*(1 - x[0]) + (2 - x[1])*(2 - x[1])

    #print results
    (f, x, iter) = simplex_search(f, np.array([0,0]))
    #print('\n')
    print('f = ', f)
    print('x = ', x)
    print('iterations = ', iter)
```

```
Current optimum =  1.84872055837
Current optimum =  2.87980947162
Current optimum =  1.91394746769
```

```
Current optimum =  0.132461620311
Current optimum =  1.27236172992
Current optimum =  0.516361455904
Current optimum =  0.139739849491
Current optimum =  0.0947731528218
Current optimum =  0.0218302612897
Current optimum =  0.00723261618504
Current optimum =  0.0234872486365
Current optimum =  0.00648150978763
Current optimum =  0.00501702904282
Current optimum =  8.01402718652e-05
Current optimum =  0.00356397186498
Current optimum =  0.00156201242228
Current optimum =  0.000579639268912
Current optimum =  0.000134229386486
Current optimum =  0.000111407397121
Current optimum =  2.44033741441e-05
Current optimum =  1.44168605522e-05
Current optimum =  7.24841787287e-06
Current optimum =  5.74226451148e-06
Current optimum =  2.61498507089e-06
Current optimum =  2.7872144543e-07
Current optimum =  6.31047342911e-07
Current optimum =  5.03785171206e-07
f =  2.7872144543e-07
x =  [ 1.00022415  1.99952201]
iterations =  27
```

In [3]:
```
#graphically verify the minimum
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm
import matplotlib.pyplot as plt
%config InlineBackend.figure_formats=['svg']
%matplotlib inline

x = np.arange(-10,10,.1)
y = np.arange(-10,10,.1)
(x, y) = np.meshgrid(x, y)
z = (1 - x)**2 + (2 - y)**2
fig = plt.figure()
ax = fig.gca(projection='3d')
ax.plot_surface(x, y, z, label='parametric curve', cmap=cm.jet, linewidth=0.2)
#ax.legend()
plt.show()
```



Quite sensibly the expansion and contraction coefficients, γ and β, should be larger than one and between zero and one respectively. If values outside of this range are used, for instance by using a β value greater than one, and the search will march to the extrema and return large function values, depending on the initial guess of the design vector x_0. The parameter ϵ mostly determines the precision of the final result, that is the number of decimals away from the absolute optima point.

Problem 2.

Consider the constrained optimization problem

$$\text{Design variable}\, x = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0 \end{cases}$$

$$\min\ f(x) = (x_1 - 3)^3 + (x_2 - 3)^2$$

Subject to

$$\begin{aligned} x_1 &\leq 2 \\ x_1 x_2 &= 8 \end{aligned}$$

Based on an exterior penalty function method, employing $r_p = 5$ transform the problem into an unconstrained one by creating an unconstrained pseudo-objective function Φ. If a steepest descent method is used to minimize Φ starting from the point $x = [3, 5]^T$ determine the search direction d that should be used

Using an exterior penalty function our constraints should have the form

$$r_p(\max(0, g_1)^2 + |g_2|^2).$$

Therefore the pseudo-objective function will have the form

$$\Phi = (x_1 - 3)^3 + (x_2 - 3)^2 + r_p(\max(0, x_1 - 2)^2 + (|x_1 x_2| - 2)^2).$$

Taking the gradient of the objective pseudo-function we have

$$
\begin{aligned}
\frac{\partial \Phi}{\partial x_1} &= \frac{\partial f}{\partial x_1} + r_p \left(2g_1 \frac{\partial g_1}{\partial x_1} + 2g_2 \frac{\partial g_2}{\partial x_1} \right) \\
\frac{\partial \Phi}{\partial x_2} &= \frac{\partial f}{\partial x_2} + r_p \left(2g_1 \frac{\partial g_1}{\partial x_2} + 2g_2 \frac{\partial g_2}{\partial x_2} \right),
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{\partial f}{\partial x_1} = 3(x_1 - 3)^2 && \frac{\partial f}{\partial x_2} = 2(x_2 - 3) \\
&\frac{\partial g_1}{\partial x_1} = 1 && \frac{\partial g_1}{\partial x_2} = 0 \\
&\frac{\partial g_2}{\partial x_1} = |x_1 x_2 - 8| x_2 && \frac{\partial g_2}{\partial x_2} = |x_1 x_2 - 8| x_1.
\end{aligned}
$$

.

Substituting up we have

$$
\begin{aligned}
\frac{\partial \Phi}{\partial x_1} &= 3(x_1 - 3)^2 = r_p(2\max(0, x_1 - 2) + 2|x_1 x_2 - 8|) \\
\frac{\partial \Phi}{\partial x_2} &= 2(x_2 - 3) + r_p(2|x_1 x_2 - 8| x_2).
\end{aligned}
$$

The direction of steepest descent, d, is $-\nabla \Phi = \langle \frac{\partial \Phi}{\partial x_1}, \frac{\partial \Phi}{\partial x_2} \rangle$. At $x = [3, 5]^T$ we can evaluate the direction of steepest decent as

$$
\begin{aligned}
\left. \frac{\partial \Phi}{\partial x_1} \right|_{x=(3,5)} &= 5(2(1) + 2(7)(6)) = 360 \\
\left. \frac{\partial \Phi}{\partial x_2} \right|_{x=(3,5)} &= 2(2) + 5(2(7)(3)) = 218.
\end{aligned}
$$

Normalizing by the gradient so that we have a unit vector, d, in the direction of steepest descent

$$
\frac{\langle 360, 218 \rangle}{\sqrt{360^2 + 218^2}} = \langle 0.855, 0.518 \rangle.
$$

The next design point, assuming that a single variable search has been performed to optimize the step size (yielding $\alpha = 0.002$), will be

$$
\begin{aligned}
x^{(i+1)} &= x^{(i)} + \alpha_i d \\
x^{(i+1)} &= x^{(i)} - \alpha_i \nabla \Phi
\end{aligned}
$$

$$
\begin{aligned}
x^{(i+1)} &= (3, 5) - 0.002 \langle 0.855, 0.518 \rangle \\
&= (2.998, 4.99).
\end{aligned}
$$

Addendum

The following is a summary and collection of notes of the presentation given by Dr. Christina Ivler, of the U.S. Army Research, Development and Engineering Command (RDECOM), during the April, 22nd seminar session.

0.0.1 Dyanmics and Control of Fly-by-Wire Helicopters

- There are engineering challenges pertaining to helicopter cargo ops.: a two-body dynamic system, non-collocated control problem which is "notoriously hard", there is no-direct control of the payload by the pilot.
- Design goals: a control system with cable angle as an output. State-space coupling is key to the problem.
- Tradeoffs: handling quality vs. load damping.
- Other issues: pilot induced oscillations and feedback
- Two constrained optimization operations performed on-the-fly (one for each mode): when the stick is active (e.g. the pilot is in control), and when the stick is fixed.
- The optimization constraints include stability constraints, handling quality constraints, load damping constraints, minimize actuator activity.
- System comparisons c.f. optimized baselines (e.g. other modern fly-by-wire systems): 1/2 the settling time for a 1000 kg test (but with only a single data point).

0.0.2 Controls Allocation

- The basic question is how to distribute control usage to get desired performance.
- There are 4 basic allocation approaches:
  1. Control Ganging
  2. Cascaded Pseudo-Inverse
  3. Linear Programming
  4. Quadratic Programming
- The goal is to solve the control positions that give the desired moment $B_n = d$ where B is $n \times m$ and $\dot{x} = Ax + B_n$.
- There are issues with the legacy approaches to this problem
  1. Pilot induced oscillations (more controls available than desired moments)
  2. $B^T(BB^T)^{-1}$ minimizes $u^T u$ if there is no saturation.
- New methods use convex optimization (e.g. min J, the cost function, subject to $B_n = d$ and

$$\begin{aligned} u_{min} \leq \quad & u \quad \leq u_{max} \\ \dot{u}_{min} \leq \quad & \dot{u} \quad \leq \dot{u}_{max} \end{aligned}$$

- The hard problem is doing the optimization in a real-time controller (64 Hz) with 50 iterations per time step.
- Frequency sweeps were used to excite the system.
- The optimization improved various metrics: phase delay, model following cost, percent time saturated, cost.
- This approach <u>does</u> have some drawbacks: the controller is non-deterministic, the optimization is computationally intensive.
- During piloted test using quadratic programming control was reported to be "crisper" and with less over-shoot.

0.0.3 UAV Research

- Flight control challenges for UAV's:
  1. Poor models
  2. Dynamics very unstable without feedback control
  3. Control system design requirements poorly understood
- The research has useful applications, especially UAV control in GPS denied environments.
- Tried to identify frequency domains by exciting aircraft at a variety of frequencies.
- Used dynamic inverse control laws with large bandwidth.