ME596 Homework 2

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Problem 1.

Minimize f and find the optimum, x^*

$$f(x,u) = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right)$$

subject to the constraint

$$g(x,u) = c - xu = 0.$$

Using the Lagrangian multipliers method we have

$$\begin{aligned} H(x,u,\lambda) &= f(x,u) + \lambda g(x,u) \\ &= \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right) + \lambda(c - xu). \end{aligned}$$

The necessary conditions are

$$\frac{\partial H}{\partial x} = \frac{x}{a^2} - \lambda u = 0 \tag{1a}$$

$$\frac{\partial H}{\partial u} = \frac{u}{b^2} - \lambda x = 0 \tag{1b}$$

Solving equations 1a and 1b for λ we have

$$u = \frac{x}{\lambda a^2} \tag{2}$$

and substituting equation 2 into equation 1b and simplifying we find the value of the Lagrangian multiplier

$$\lambda = \frac{1}{ab}$$

which, we can use to find an expression for x,

$$x = u\frac{a}{b}. \tag{3}$$

To satisfy the sufficient conditions, and utilizing subscript notation for the derivatives, we have

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{u=0} = 0 \tag{4a}$$

$$\begin{aligned} \left.\frac{\partial^2 f}{\partial x^2}\right|_{u=0} &= H_{xx} - H_{xu}g_u^{-1}g_x - g_x^T(g_u^{-1})^T H_{ux} \\ &+ g_x^T(g_u^{-1})^T H_{uu} g_u^{-1} g_x \end{aligned} \tag{4b}$$

Supplying values for the derivatives of H and g we have

$$\frac{\partial^2 H}{\partial x^2} = \frac{1}{a^2} \qquad \frac{\partial g}{\partial x} = -u$$
$$\frac{\partial^2 H}{\partial u^2} = \frac{1}{b^2} \qquad \frac{\partial g}{\partial u} = -x$$
$$\frac{\partial^2 H}{\partial(xu)} = \frac{\partial^2 H}{\partial(ux)} = -\lambda = -\frac{1}{ab}$$

Substituting these values into equation 4b we have

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{u=0} = \frac{1}{a^2} + 2\frac{u}{abx} + \frac{u^2}{b^2x^2} \tag{5}$$

which, substituting equation 2 for u and canceling terms we have

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{u=0} = \frac{4}{a^2}$$

This must be a positive number, thus the sufficient conditions are satisfied for a minimum. If we substitute equation 3 into the constraint g we find that the minimum is at

$$x^* = \sqrt{\frac{ac}{b}}, \qquad u^* = \sqrt{\frac{bc}{a}}$$

and

$$f^* = \frac{c}{ab}$$

Problem 2.

Maximize V, where

$$V(x, y, z) = 8xyz$$

subject to the constraint

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

We will use a direct substitution approach to turn this into an unconstrained problem of two variables. First solving for z in the constraint we have

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

If we substitute this value for z into the objective function, and note that we will minimize the negative to find the maximum we have

$$\min \;\; V(x, y) = -8xyc\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

To satisfy the necessary conditions of the problem, and with some simplifications, we have

$$\begin{aligned}
\frac{\partial V}{\partial x} &= -8yc\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2}\left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right) = 0 \\
\frac{\partial V}{\partial y} &= -8xc\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2}\left(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right) = 0
\end{aligned}$$

which we can simplify as

$$
\begin{aligned}
1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2} = 0 \\
1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2} = 0.
\end{aligned}
$$

Substituting one into the other we have

$$x^* = \frac{a}{\sqrt{3}}, \quad y^* = \frac{b}{\sqrt{3}},$$

and by extension

$$z^* = \frac{c}{\sqrt{3}}, \quad \text{and} \quad V^* = \frac{8a^2b^2c^2}{3\sqrt{3}}.$$

To satisfy the sufficient condition for a minimum we have

$$
\begin{aligned}
\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x^*, y=y^*} &= \frac{8cxy(3a^2b^2 - 2b^2x^2 - 3a^2y^2)}{a^4b^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{3/2}} = \frac{-32cb}{a} \\
\left.\frac{\partial^2 V}{\partial y^2}\right|_{x=x^*, y=y^*} &= \frac{-32cb}{a} \\
\left.\frac{\partial^2 V}{\partial x \partial y}\right|_{x=x^*, y=y^*} &= \frac{-8c(a^4b^4 - 3a^4b^2y^2 + 2a^4y^4 - 3a^2b^4x^2 + 3a^2b^2x^2y^2 + 2b^4x^4)}{a^4b^4\left(\frac{-x^2}{a^2} - \frac{y^2}{b^2} + 1\right)^{3/2}} \\
&= \frac{112c}{27}\left(\frac{1}{3}\right)^{-3/2}
\end{aligned}
$$

If we assume that the coefficients a, b, c are positive and real then

$$\frac{\partial^2 V}{\partial x^2} < 0$$

and given the sufficient condition for a minimum that the determinant of the Hessian matrix

$$\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y)}\right)^2 > 0$$

then we have a maximum (or the minimum of the negative of the objective function) when

$$\frac{b^2}{a^2} > \frac{9}{32^2}\left(\frac{112}{27}\right)^2 \approx 0.151$$