Taking logarithms of this and rearranging slightly we get

$$\int_{t^*}^{T} \eta^*(s)(T-s)ds = -\log(Z_M(t^*;T)) - r^*(T-t^*) + \tfrac{1}{6}c^2(T-t^*)^3. \tag{17.1}$$

Observe that I am carrying around in the notation today's date t^*. This is a constant but I want to emphasize that we are doing the calibration to *today's* yield curve. If we calibrate again tomorrow, the market yield curve will have changed.

Differentiate (17.1) twice with respect to T to get

$$\eta^*(t) = c^2(t-t^*) - \frac{\partial^2}{\partial t^2}\log(Z_M(t^*;t)).$$

With this choice for the time-dependent parameter $\eta(t)$ the theoretical and actual market prices of zero-coupon bonds are the same. It also follows that

$$A(t;T) = \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right) - (T-t)\frac{\partial}{\partial t}\log(Z_M(t^*;t)) - \tfrac{1}{2}c^2(t-t^*)(T-t)^2.$$

17.3 THE EXTENDED VASICEK MODEL OF HULL & WHITE

The Ho & Lee model isn't the only one that can be calibrated, it's just the easiest. Most one-factor models have the potential for fitting, but the more tractable the model the easier the fitting. If the model is not at all tractable, having no nice explicit zero-coupon bond price formula, then we can always resort to numerical methods.

The next easiest model to fit is the Vasicek model. The Vasicek model has the following stochastic differential equation for the risk-neutral spot rate

$$dr = (\eta - \gamma r)dt + c\,dX.$$

Hull & White extend this to include a time-dependent parameter

$$dr = (\eta(t) - \gamma r)dt + c\,dX.$$

Assuming that γ and c have been estimated statistically, say, we choose $\eta = \eta^*(t)$ at time t^* so that our theoretical and the market prices of bonds coincide.

Under this risk-neutral process the value of a zero-coupon bonds

$$Z(r,t;T) = e^{A(t;T) - rB(t;T)},$$

where

$$A(t;T) = -\int_t^T \eta^*(s)B(s;T)ds + \frac{c^2}{2\gamma^2}\left(T - t + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right).$$

and

$$B(t;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right).$$

To fit the yield curve at time t^* we must make $\eta^*(t)$ satisfy

$$A(t^*;T) = -\int_{t^*}^{T} \eta^*(s)B(s;T)ds + \frac{c^2}{2\gamma^2}\left(T - t^* + \frac{2}{\gamma}e^{-\gamma(T-t^*)} - \frac{1}{2\gamma}e^{-2\gamma(T-t^*)} - \frac{3}{2\gamma}\right)$$
$$= \log(Z_M(t^*;T)) + r^*B(t^*,T). \qquad (17.2)$$

This is an integral equation for $\eta^*(t)$ if we are given all of the other parameters and functions, such as the market prices of bonds, $Z_M(t^*;T)$.

Although (17.2) may be solved by Laplace transform methods, it is particularly easy to solve by differentiating the equation twice with respect to T. This gives

$$\eta^*(t) = -\frac{\partial^2}{\partial t^2}\log(Z_M(t^*;t)) - \gamma\frac{\partial}{\partial t}\log(Z_M(t^*;t)) + \frac{c^2}{2\gamma}\left(1 - e^{-2\gamma(t-t^*)}\right). \qquad (17.3)$$

From this expression we can now find the function $A(t;T)$,

$$A(t;T) = \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right) - B(t;T)\frac{\partial}{\partial t}\log(Z_M(t^*;t))$$
$$-\frac{c^2}{4\gamma^3}\left(e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)}\right)^2\left(e^{2\gamma(t-t^*)} - 1\right).$$

17.4 YIELD-CURVE FITTING: FOR AND AGAINST

17.4.1 For

The building blocks of the bond pricing equation are delta hedging and no arbitrage. If we are to use a one-factor model correctly then we must abide by the delta-hedging assumptions. We must buy and sell instruments to remain delta neutral. The buying and selling of instruments must be done at the market prices. We *cannot* buy and sell at a theoretical price. But we are not modeling the bond prices directly; we model the spot rate and bond prices are then derivatives of the spot rate. This means that there is a real likelihood that our output bond prices will differ markedly from the market prices. This is useless if we are to hedge with these bonds. The model thus collapses and cannot be used for pricing other instruments, unless we can find a way to generate the correct prices for our hedging instruments from the model; this is yield curve fitting.

Once we have fitted the prices of traded products we then dynamically or statically hedge with these products. The idea being that even if the model is wrong so that we lose money on the contract we are pricing then we should make that money back on the hedging instruments.

17.4.2 Against

If the market prices of simple bonds were correctly given by a model, such as Ho & Lee or Hull & White, fitted at time t^* then, when we come back a week later, t^* + one week, say, to refit the function $\eta^*(t)$, we would find that this function *had not changed* in the