Vasicek's model. The model analyzed by Vasicek (1977) is one of the earliest models of term structure (see also Brennan and Schwartz (1977c), Richard (1978), and Dothan (1978)). The diffusion process proposed by Vasicek is a mean-reverting version of the Ornstein-Uhlenbeck process. The short-term interest rate r is defined as the unique strong solution of the SDE

$$dr_t = \big(a - br_t\big)\,dt + \sigma\,dW_t^*, \tag{10.5}$$

where a, b and σ are strictly positive constants. It is well-known that the solution to (10.5) is a Markov process with continuous sample paths and Gaussian increments. In fact, equation (10.5) can be solved explicitly, as the following lemma shows.

Lemma 10.1.2 *The unique solution to the SDE (10.5) is given by the formula*

$$r_t = r_s e^{-b(t-s)} + \frac{a}{b}\Big(1 - e^{-b(t-s)}\Big) + \sigma \int_s^t e^{-b(t-u)}\,dW_u^*.$$

For any $s < t$ the conditional distribution of r_t with respect to the σ-field $\mathcal{F}_s$ is Gaussian, with the conditional expected value

$$\mathbb{E}_{\mathbb{P}^*}(r_t \,|\, \mathcal{F}_s) = r_s e^{-b(t-s)} + \frac{a}{b}\Big(1 - e^{-b(t-s)}\Big),$$

and the conditional variance

$$\mathrm{Var}_{\mathbb{P}^*}(r_t \,|\, \mathcal{F}_s) = \frac{\sigma^2}{2b}\Big(1 - e^{-2b(t-s)}\Big).$$

The limits of $\mathbb{E}_{\mathbb{P}^*}(r_t \,|\, \mathcal{F}_s)$ and $\mathrm{Var}_{\mathbb{P}^*}(r_t \,|\, \mathcal{F}_s)$ when t tends to infinity are

$$\lim_{t\to\infty} \mathbb{E}_{\mathbb{P}^*}(r_t \,|\, \mathcal{F}_s) = \frac{a}{b}$$

and

$$\lim_{t\to\infty} \mathrm{Var}_{\mathbb{P}^*}(r_t \,|\, \mathcal{F}_s) = \frac{\sigma^2}{2b}.$$

Proof. Let us fix $s > 0$ and let us consider the process $Y_t = r_t e^{b(t-s)}$ where $t \geq s$. Using Itô's formula and (10.5), we obtain

$$\begin{aligned} dY_t &= e^{b(t-s)}\,dr_t + be^{b(t-s)} r_t\,dt \\ &= e^{b(t-s)}\big(a\,dt - br_t\,dt + \sigma\,dW_t^* + br_t\,dt\big) \\ &= e^{b(t-s)}\big(a\,dt + \sigma\,dW_t^*\big). \end{aligned}$$

Hence we have

$$r_t e^{b(t-s)} - r_s = \int_s^t dY_u = \int_s^t a e^{b(u-s)}\,du + \sigma \int_s^t e^{b(u-s)}\,dW_u^*,$$

and consequently

$$r_t = r_s e^{-b(t-s)} + e^{-b(t-s)} \int_s^t a e^{b(u-s)}\,du + \sigma e^{-b(t-s)} \int_s^t e^{b(u-s)}\,dW_u^*.$$

Finally, we obtain

$$r_t = r_s e^{-b(t-s)} + \frac{a}{b}\Big(1 - e^{-b(t-s)}\Big) + \sigma \int_s^t e^{-b(t-u)}\, dW_u^*$$

since

$$\int_s^t e^{b(u-s)}\, du = \frac{1}{b}\Big(e^{b(t-s)} - 1\Big).$$

It is well-known that for any square-integrable function g the Itô integral $\int_s^t g(u)\, dW_u^*$ is a random variable independent of the σ-field $\mathcal{F}_s$ and has the Gaussian distribution $N(0, \int_s^t g^2(u)\, du)$. In our case, we have that

$$\int_s^t g^2(u)\, du = \int_s^t \sigma^2 e^{-2b(t-u)}\, du = \frac{\sigma^2}{2b}\Big(1 - e^{-2b(t-s)}\Big).$$

We conclude that for any $s < t$ the conditional distribution of r_t with respect to the σ-field $\mathcal{F}_s$ is Gaussian, with the conditional expected value

$$\mathbb{E}_{\mathbb{P}^*}(r_t \,|\, \mathcal{F}_s) = r_s e^{-b(t-s)} + \frac{a}{b}\Big(1 - e^{-b(t-s)}\Big)$$

and the conditional variance

$$\mathrm{Var}_{\mathbb{P}^*}(r_t \,|\, \mathcal{F}_s) = \frac{\sigma^2}{2b}\Big(1 - e^{-2b(t-s)}\Big).$$

This ends the proof of the lemma. □

Note that the solution to (10.5) admits a stationary distribution, namely, the Gaussian distribution with expected value a/b and variance $\sigma^2/2b$.

Our next goal is to find the price of a bond in Vasicek's model. To this end, we shall use the risk-neutral valuation formula directly. An alternative derivation of bond pricing formula, based on a PDE approach, is given in Proposition 10.1.3.

Proposition 10.1.2 *The price at time t of a zero-coupon bond in Vasicek's model equals:*

$$B(t,T) = e^{m(t,T) - n(t,T) r_t}, \tag{10.6}$$

where

$$n(t,T) = \frac{1}{b}\Big(1 - e^{-b(T-t)}\Big) \tag{10.7}$$

and

$$m(t,T) = \frac{\sigma^2}{2}\int_t^T n^2(u,T)\, du - a\int_t^T n(u,T)\, du. \tag{10.8}$$

The bond price volatility is a function $b(\cdot, T) : [0,T] \to \mathbb{R}$, specifically, $b(t,T) = -\sigma n(t,T)$, and thus the dynamics of the bond price under $\mathbb{P}^*$ are

$$dB(t,T) = B(t,T)\big(r_t\, dt - \sigma n(t,T)\, dW_t^*\big). \tag{10.9}$$

Proof. We shall first evaluate $B(0,T)$ using the formula:

$$B(0,T) = \mathbb{E}_{\mathbb{P}^*}(B_T^{-1}) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^T r_t\,dt}\right).$$

We know already that

$$r_t = r_0 e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right) + \sigma \int_0^t e^{-b(t-u)}\,dW_u^*,$$

so that

$$\int_0^T r_t\,dt = r_0 \int_0^T e^{-bt}\,dt + \frac{a}{b}T - \frac{a}{b}\int_0^T e^{-bt}\,dt + \sigma \int_0^T\!\!\int_0^t e^{-b(t-u)}\,dW_u^*\,dt.$$

Observe that

$$\int_0^T e^{-bt}\,dt = \frac{1}{b}\left(1 - e^{-bT}\right) = n(0,T).$$

Using the stochastic Fubini theorem (see Theorem IV.45 in Protter (2003)), we obtain

$$\begin{aligned}\int_0^T \int_0^t e^{-b(t-u)}\,dW_u^*\,dt &= \int_0^T \int_0^t e^{-bt}e^{bu}\,dW_u^*\,dt = \int_0^T \int_u^T e^{-bt}e^{bu}\,dt\,dW_u^* \\ &= \int_0^T e^{bu} \int_u^T e^{-bt}\,dt\,dW_u^* = \frac{1}{b}\int_0^T \left(1 - e^{-b(T-u)}\right)dW_u^*.\end{aligned}$$

This means that

$$\int_0^T r_t\,dt = r_0 n(0,T) + \frac{a}{b}T - an(0,T) + \sigma \int_0^T n(u,T)\,dW_u^*.$$

It is easy to check that

$$\int_0^T n(u,T)\,du = \frac{T}{b} - n(0,T),$$

and thus

$$\xi_T \stackrel{\text{def}}{=} \int_0^T r_t\,dt = n(0,T)r_0 + a\int_0^T n(u,T)\,du + \sigma \int_0^T n(u,T)\,dW_u^*.$$

The random variable ξ_T has under $\mathbb{P}^*$ the Gaussian distribution with expected value $n(0,T)r_0 + a\int_0^T n(u,T)\,du$ and variance $\sigma^2 \int_0^T n^2(u,T)\,du$. In view of Lemma 10.1.1, we obtain

$$B(0,T) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\xi_T}\right) = e^{-n(0,T)r_0 - a\int_0^T n(u,T)\,du + \frac{\sigma^2}{2}\int_0^T n^2(u,T)\,du}.$$

Since $m(0,T)$ is given by

$$m(0,T) = \frac{\sigma^2}{2}\int_0^T n^2(u,T)\,du - a\int_0^T n(u,T)\,du,$$

we conclude that

$$B(0,T) = e^{m(0,T)-n(0,T)r_0}.$$

The general valuation formula (10.6) is an easy consequence of the Markov property of r. Finally, to establish (10.9), it suffices to apply Itô's rule. □

By combining (10.7) with (10.8), we obtain a more explicit representation for $m(t,T)$

$$\begin{aligned} m(t,T) &= \Big(\frac{a}{b} - \frac{\sigma^2}{2b^2}\Big)\big(n(t,T) - T + t\big) - \frac{\sigma^2}{4b} n^2(t,T) \\ &= \Big(\frac{a}{b} - \frac{\sigma^2}{2b^2}\Big)\Big(\frac{1}{b}\Big(1 - e^{-b(T-t)}\Big) - T + t\Big) - \frac{\sigma^2}{4b^3}\Big(1 - e^{-b(T-t)}\Big)^2. \end{aligned}$$

Let us consider any security whose payoff depends on the short-term rate r as the only state variable. More specifically, we assume that this security is of European style, pays dividends continuously at a rate $h(r_t,t)$, and yields a terminal payoff $G_T = g(r_T)$ at time T. Its arbitrage price $\pi_t(X)$ is given by the following version of the risk-neutral valuation formula:

$$\pi_t(X) = \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T h(r_u,u)e^{-\int_t^u r_v\,dv}\,du \,+ g(r_T)\,e^{-\int_t^T r_u\,du}\,\Big|\,\mathcal{F}_t\Big) = v(r_t,t),$$

where v is some function $v : \mathbb{R} \times [0,T^*] \to \mathbb{R}$. From the well-known connection[1] between diffusion processes and partial differential equations, it follows that the function $v : \mathbb{R} \times [0,T^*] \to \mathbb{R}$ solves the following valuation PDE

$$\frac{\partial v}{\partial t}(r,t) + \frac{1}{2}\sigma^2\,\frac{\partial^2 v}{\partial r^2}(r,t) + (a - br)\,\frac{\partial v}{\partial r}(r,t) - rv(r,t) + h(r,t) = 0$$

subject to the terminal condition $v(r,T) = g(r)$. Solving this equation with $h = 0$ and $g(r) = 1$, Vasicek (1977) proved the following result (recall that we have already derived formulae (10.10)–(10.12) through the probabilistic approach; see Proposition 10.1.2).

Proposition 10.1.3 *The price at time t of a zero-coupon bond in Vasicek's model equals*

$$B(t,T) = v(r_t,t,T) = e^{m(t,T)-n(t,T)r_t}, \tag{10.10}$$

where

$$n(t,T) = \frac{1}{b}\Big(1 - e^{-b(T-t)}\Big) \tag{10.11}$$

and

$$m(t,T) = \frac{\sigma^2}{2}\int_t^T n^2(u,T)\,du - a\int_t^T n(u,T)\,du. \tag{10.12}$$

[1] Recall that this result is known as the *Feynman-Kac formula* (see Sect. A.17 or Theorem 5.7.6 in Karatzas and Shreve (1998a)).

Proof. To establish the bond pricing formula through the PDE approach, it is enough to assume that the bond price is given by (10.10), with the functions m and n satisfying $m(T,T) = n(T,T) = 0$, and to make use of the valuation PDE. By separating terms that do not depend on r, and those that are linear in r, we arrive at the following system of differential equations:

$$m_t(t,T) = an(t,T) - \tfrac{1}{2}\sigma^2 n^2(t,T) \tag{10.13}$$

and

$$n_t(t,T) = bn(t,T) - 1 \tag{10.14}$$

with $m(T,T) = n(T,T) = 0$, that in turn yields easily the expressions (10.11)–(10.12). Using Itô's formula, one can check that

$$dB(t,T) = B(t,T)\big(r_t\,dt + \sigma n(t,T)\,dW_t^*\big), \tag{10.15}$$

so that the bond price volatility equals $b(t,T) = -\sigma n(t,T)$, where $n(t,T)$ is given by (10.11). □

If the bond price admits representation (10.10), then obviously

$$Y(t,T) = \frac{n(t,T)r_t - m(t,T)}{T-t},$$

and thus the bond's yield, $Y(t,T)$, is an affine function of the short-term rate r_t. For this reason, Markovian models of the short-term rate in which the bond price satisfies (10.10) for some functions m and n are termed *affine models of the term structure.*

Jamshidian (1989a) obtained closed-form solutions for the prices of a European option written on a zero-coupon and on a coupon-bearing bond for Vasicek's model. He showed that the arbitrage price at time t of a call option on a U-maturity zero-coupon bond, with strike price K and expiry $T \le U$, equals (let us mention that Jamshidian implicitly used the *forward measure* technique, presented in Sect. 9.6)

$$C_t = B(t,T)\mathbb{E}_{\mathbb{Q}}\big((\xi\eta - K)^+ \big| \mathcal{F}_t\big),$$

where $\eta = B(t,U)/B(t,T)$ and $\mathbb{Q}$ stands for some probability measure equivalent to the spot martingale measure $\mathbb{P}^*$.

The random variable ξ is independent of the σ-field $\mathcal{F}_t$ under $\mathbb{Q}$, and has under $\mathbb{Q}$ a lognormal distribution such that the variance $\mathrm{Var}_{\mathbb{Q}}\,(\ln \xi)$ equals $v_U(t,T)$, where

$$v_U^2(t,T) = \int_t^T |b(t,T) - b(t,U)|^2\,du,$$

or explicitly

$$v_U^2(t,T) = \frac{\sigma^2}{2b^3}\big(1 - e^{-2b(T-t)}\big)\big(1 - e^{-b(U-t)}\big)^2.$$

The bond option valuation formula established in Jamshidian (1989a) reads as follows

$$C_t = B(t,U)N\big(h_1(t,T)\big) - KB(t,T)N\big(h_2(t,T)\big), \tag{10.16}$$

where for every $t \le T \le U$

$$h_{1,2}(t,T) = \frac{\ln\big(B(t,U)/B(t,T)\big) - \ln K \pm \frac{1}{2}\, v_U^2(t,T)}{v_U(t,T)}. \tag{10.17}$$

As in the case of Merton's model, the valuation formula for a bond option in Vasicek's model is a special case of the general result of Proposition 11.3.1. This is due to the fact that the bond volatility $b(t,T)$ is deterministic.

It is important to observe that the coefficient a that is present in the dynamics (10.5) of r under $\mathbb{P}^*$, does not enter the bond option valuation formula. This suggests that the actual value of the risk premium has no impact whatsoever on the bond option price (at least if it is deterministic); the only relevant quantities are in fact the bond price volatilities $b(t,T)$ and $b(t,U)$. To account for the risk premium, it is enough to make an equivalent change of the probability measure in (10.15). Since the volatility of the bond price is invariant with respect to such a transformation of the underlying probability measure, the bond option price is independent of the risk premium, provided that the bond price volatility is deterministic.

The determination of the risk premium may thus appear irrelevant, if we concentrate on the valuation of derivatives. This is not the case, however, if our aim is to model the actual behavior of bond prices. Let us stress that stochastic term structure models presented in this text have derivative pricing as a primary goal (as opposed to, for instance, asset management).

Remarks. Let us analyze the instantaneous forward rate $f(t,T)$. Note that

$$f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T} = -\frac{\partial m(t,T)}{\partial T} + r_t \frac{\partial n(t,T)}{\partial T},$$

where $n(t,T)$ and $m(t,T)$ are given by (10.7) and (10.8) respectively. Let us denote $g(t) = 1 - e^{-bt}$. Elementary calculations show that

$$\frac{\partial m(t,T)}{\partial T} = g(T-t)\big(\tfrac{1}{2}\sigma^2 b^{-2} g(T-t) - ab^{-1}\big)$$

and, of course,

$$\frac{\partial n(t,T)}{\partial T} = e^{-b(T-t)}.$$

Consequently, we find that

$$f(t,T) = g(T-t)\big(ab^{-1} - \tfrac{1}{2}\sigma^2 b^{-2} g(T-t)\big) + r_t e^{-b(T-t)}.$$

It is interesting to note that the instantaneous forward rate $f(t,T)$ tends to 0 when maturity date T tends to infinity (the current date t being fixed).

Dothan's model. It is evident that Vasicek's model, as indeed any Gaussian model, allows for negative values of (nominal) interest rates. This property is manifestly incompatible with no-arbitrage in the presence of cash in the economy. This important