A Small Part of POM's RBF

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1 Original Equations

The original equations of the Princeton Ocean Model(POM) are listed following:

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}-fV=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\frac{\partial}{\partial z}\left(K_M\frac{\partial U}{\partial z}\right)+F_x \tag{1}$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}+fU=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\frac{\partial}{\partial z}\left(K_M\frac{\partial V}{\partial z}\right)+F_y \tag{2}$$

$$\frac{\partial p}{\partial z}=-\rho g \tag{3}$$

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0 \tag{4}$$

$$\frac{\partial T}{\partial t}+U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}+W\frac{\partial T}{\partial z}=\frac{\partial}{\partial z}\left(K_H\frac{\partial T}{\partial z}\right)+F_T \tag{5}$$

$$\frac{\partial S}{\partial t}+U\frac{\partial S}{\partial x}+V\frac{\partial S}{\partial y}+W\frac{\partial S}{\partial z}=\frac{\partial}{\partial z}\left(K_H\frac{\partial S}{\partial z}\right)+F_S \tag{6}$$

$$\rho=\rho(T,S,p) \tag{7}$$

where,

$$\begin{aligned}
F_x&=\frac{\partial}{\partial x}\left(2A_M\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left[A_M\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)\right]\\
F_y&=\frac{\partial}{\partial y}\left(2A_M\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial x}\left[A_M\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)\right]\\
F_T&=\frac{\partial}{\partial x}\left(A_H\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_H\frac{\partial T}{\partial y}\right)\\
F_S&=\frac{\partial}{\partial x}\left(A_H\frac{\partial S}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_H\frac{\partial S}{\partial y}\right)
\end{aligned}$$

2 From Cartesian Coordinates to Sigma coordinates

As we can see, POM employs sigma coordinates, which are bottom-following coordinates that map the vertical coordinate from $-H \le z \le \eta$ to $-1 \le \sigma \le 0$ with $\sigma = (z-\eta)/(H+\eta)$, where the bottom is defined by $z=-H(x,y)$ and the surface is defined by $z=\eta(x,y,t)$. The governing equations are transformed from cartesian coordinates (x,y,z,t) to sigma coordinates (x^*,y^*,z^*,t^*) which satisfy

$$x^*=x,\quad y^*=y,\quad \sigma=\frac{z-\eta}{H+\eta},\quad t^*=t \tag{8}$$

The derivatives are then given by the chain rule,

$$\frac{\partial}{\partial x}=\frac{\partial x^*}{\partial x}\frac{\partial}{\partial x^*}+\frac{\partial y^*}{\partial x}\frac{\partial}{\partial y^*}+\frac{\partial \sigma}{\partial x}\frac{\partial}{\partial \sigma}+\frac{\partial t^*}{\partial x}\frac{\partial}{\partial t^*} \tag{9}$$

$$\frac{\partial}{\partial y}=\frac{\partial x^*}{\partial y}\frac{\partial}{\partial x^*}+\frac{\partial y^*}{\partial y}\frac{\partial}{\partial y^*}+\frac{\partial \sigma}{\partial y}\frac{\partial}{\partial \sigma}+\frac{\partial t^*}{\partial y}\frac{\partial}{\partial t^*} \tag{10}$$

$$\frac{\partial}{\partial z}=\frac{\partial x^*}{\partial z}\frac{\partial}{\partial x^*}+\frac{\partial y^*}{\partial z}\frac{\partial}{\partial y^*}+\frac{\partial \sigma}{\partial z}\frac{\partial}{\partial \sigma}+\frac{\partial t^*}{\partial z}\frac{\partial}{\partial t^*} \tag{11}$$

$$\frac{\partial}{\partial t}=\frac{\partial x^*}{\partial t}\frac{\partial}{\partial x^*}+\frac{\partial y^*}{\partial t}\frac{\partial}{\partial y^*}+\frac{\partial \sigma}{\partial t}\frac{\partial}{\partial \sigma}+\frac{\partial t^*}{\partial t}\frac{\partial}{\partial t^*} \tag{12}$$

which can be written in matrix form as

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^*}{\partial x} & \frac{\partial y^*}{\partial x} & \frac{\partial \sigma}{\partial x} & \frac{\partial t^*}{\partial x} \\ \frac{\partial x^*}{\partial y} & \frac{\partial y^*}{\partial y} & \frac{\partial \sigma}{\partial y} & \frac{\partial t^*}{\partial y} \\ \frac{\partial x^*}{\partial z} & \frac{\partial y^*}{\partial z} & \frac{\partial \sigma}{\partial z} & \frac{\partial t^*}{\partial z} \\ \frac{\partial x^*}{\partial t} & \frac{\partial y^*}{\partial t} & \frac{\partial \sigma}{\partial t} & \frac{\partial t^*}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y^*} \\ \frac{\partial}{\partial \sigma} \\ \frac{\partial}{\partial t^*} \end{bmatrix}$$

Specifically,

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial x^*}+\frac{1}{D}\frac{\partial \eta}{\partial x^*}\right) & 0 \\ 0 & 1 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial y^*}+\frac{1}{D}\frac{\partial \eta}{\partial y^*}\right) & 0 \\ 0 & 0 & \frac{1}{D} & 0 \\ 0 & 0 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial t^*}+\frac{1}{D}\frac{\partial \eta}{\partial t^*}\right) & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y^*} \\ \frac{\partial}{\partial \sigma} \\ \frac{\partial}{\partial t^*} \end{bmatrix}$$

where we have used the following expressions already, because $D=H(x,y)+\eta(x,y,t)$ and $\eta=\eta(x,y,t)$ only relating to x,y,t

$$\frac{\partial D}{\partial x}=\frac{\partial D}{\partial x^*},\quad \frac{\partial \eta}{\partial x}=\frac{\partial \eta}{\partial x^*}$$

$$\frac{\partial D}{\partial y}=\frac{\partial D}{\partial y^*},\quad \frac{\partial \eta}{\partial y}=\frac{\partial \eta}{\partial y^*}$$

$$\frac{\partial D}{\partial t}=\frac{\partial D}{\partial t^*},\quad \frac{\partial \eta}{\partial t}=\frac{\partial \eta}{\partial t^*}$$

3 Gradient Operator and Divergence Operator

Taking equation (1) (2) for instance, it can be changed into an operator form

$$\frac{\partial U}{\partial t}=-(\boldsymbol{U}\cdot\nabla)U+fV-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\frac{\partial}{\partial z}\left(K_M\frac{\partial U}{\partial z}\right)+F_x \tag{13}$$

$$\frac{\partial V}{\partial t}=-(\boldsymbol{U}\cdot\nabla)V-fU-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\frac{\partial}{\partial z}\left(K_M\frac{\partial V}{\partial z}\right)+F_y \tag{14}$$

where

$$\boldsymbol{U}=\begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

$$F_x = \frac{\partial}{\partial x}\left(2A_M\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left[A_M\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right]$$

$$F_y = \frac{\partial}{\partial y}\left(2A_M\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial x}\left[A_M\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right)\right]$$

Also, equation (4) can be changed into an operator form

$$\nabla \cdot \boldsymbol{U} = 0 \tag{15}$$

we define the operator P:

$$P = \begin{bmatrix} 1 & 0 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial x^*} + \frac{1}{D}\frac{\partial \eta}{\partial x^*}\right) \\ 0 & 1 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial y^*} + \frac{1}{D}\frac{\partial \eta}{\partial y^*}\right) \\ 0 & 0 & \frac{1}{D} \end{bmatrix} = \begin{bmatrix} P_x^T \\ P_y^T \\ P_z^T \end{bmatrix}$$

$$\nabla^* = \begin{bmatrix} \frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y^*} \\ \frac{\partial}{\partial \sigma} \end{bmatrix}$$

$$P_t^T = \left[0\ 0\ -\left(\frac{\sigma}{D}\frac{\partial D}{\partial t^*} + \frac{1}{D}\frac{\partial \eta}{\partial t^*}\right)\right]$$

Taking $\frac{\partial}{\partial x}$ for example, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x^*} - \left(\frac{\sigma}{D}\frac{\partial D}{\partial x^*} + \frac{1}{D}\frac{\partial \eta}{\partial x^*}\right)\frac{\partial}{\partial \sigma} = P_x^T \cdot \nabla^*$$

$$\frac{\partial}{\partial y} = P_y^T \cdot \nabla^*$$

$$\frac{\partial}{\partial z} = P_z^T \cdot \nabla^*$$

$$\frac{\partial}{\partial t} = P_t^T \cdot \nabla^* + \frac{\partial}{\partial t^*}$$

Next, the gradient operator appearing in (13) must be constrained so that when it is applied to a scalar, it produces a vector which is in the sigma coordinates. This is done by replacing all occurrences of ∇ in (13) with the operator

$$\nabla = P\nabla^*$$

W is the vertical velocity in z coordinate, ω is the vertical velocity in σ coordinate. $\boldsymbol{U} = (U, V, W)$, $\boldsymbol{U^*} = (U, V, \omega)$. The relationship between W and ω is as below.

$$W = \omega - DP_t^T\nabla^*\sigma - DUP_x^T\nabla^*\sigma - DVP_y^T\nabla^*\sigma \tag{16}$$

since P projects vectors into sigma coordinates. The divergence operator appearing in (15) must also be restricted so that it produces the divergence of a vector field in sigma coordinates. This is also accomplished with P by simply taking the dot product of the vector field with the projected gradient operator $P\nabla$.

Follow this way we could get equation (17) from equation (13)

$$\frac{\partial U}{\partial t^*} = -P_t^T\nabla^*U - (\boldsymbol{U}\cdot P\nabla^*)U + fV - \frac{1}{\rho_0}P_x^T\nabla^*p + P_z^T\nabla^*\left(K_M P_z^T\nabla^*U\right) + F_x^* \tag{17}$$

$$= -(\boldsymbol{U^*}\cdot P\nabla^*)U + D(UP_x^T\nabla^*\sigma + VP_y^T\nabla^*\sigma)P_z^T\nabla^*U + fV - \frac{1}{\rho_0}P_x^T\nabla^*p + P_z^T\nabla^*\left(K_M P_z^T\nabla^*U\right) + F_x \tag{18}$$

Where

$$F_x^* = P_x^T\nabla^*\left(2A_M P_x^T\nabla^*U\right) + P_y^T\nabla^*\left[A_M\left(P_y^T\nabla^*U + P_x^T\nabla^*V\right)\right] \tag{19}$$

$$= \frac{\partial \tau_{xx}}{\partial x^*} - \frac{\partial}{\partial \sigma}\left[\left(\frac{\sigma}{D}\frac{\partial D}{\partial x^*} + \frac{1}{D}\frac{\partial \eta}{\partial x^*}\right)\tau_{xx}\right] + \frac{\partial \tau_{yx}}{\partial y^*} - \frac{\partial}{\partial \sigma}\left[\left(\frac{\sigma}{D}\frac{\partial D}{\partial y^*} + \frac{1}{D}\frac{\partial \eta}{\partial y^*}\right)\tau_{yx}\right]$$

with

$$\tau_{xx} = 2A_M \left[\frac{\partial UD}{\partial x^*} - \frac{\partial}{\partial \sigma} \left(\sigma \frac{\partial D}{\partial x^*} + \frac{\partial \eta}{\partial x^*} \right) U \right] \tag{20}$$

$$\tau_{yx} = \quad A_M \left[\frac{\partial UD}{\partial y^*} - \frac{\partial}{\partial \sigma} \left(\sigma \frac{\partial D}{\partial x^*} + \frac{\partial \eta}{\partial x^*} \right) U + \frac{\partial VD}{\partial x^*} - \frac{\partial}{\partial \sigma} \left(\sigma \frac{\partial D}{\partial x^*} + \frac{\partial \eta}{\partial x^*} \right) V \right] \tag{21}$$

Since Mellor and Blumberg [1985] have shown that the conventional model for horizontal diffusion is incorrect when bottom topographical slopes are large. A new formulation has been suggested which is simpler than the F_x^*, F_y^* and F_ϕ^* equations. They makes it possible to model realistically bottom boundary layers over sharply sloping bottoms. They are defined according to:

$$F_x^* = \frac{\partial}{\partial x^*} \left(H * 2A_M \frac{\partial U}{\partial x^*} \right) + \frac{\partial}{\partial y^*} \left[H * A_M \left(\frac{\partial U}{\partial y^*} + \frac{\partial V}{\partial x^*} \right) \right] \tag{22}$$

Putting all these pieces together, the final equations(We already drop all *) in sigma coordinates is given by

$$\frac{\partial U}{\partial t} = -(\boldsymbol{U}\cdot P\nabla)U + D\left(UP_x^T\nabla\sigma + VP_y^T\nabla\sigma\right)P_z^T\nabla U + fV - \frac{1}{\rho_0}P_x^T\nabla p + P_z^T\nabla\left(K_M P_z^T\nabla U\right) + F_x \tag{23}$$

$$\frac{\partial V}{\partial t} = -(\boldsymbol{U}\cdot P\nabla)V + D\left(UP_x^T\nabla\sigma + VP_y^T\nabla\sigma\right)P_z^T\nabla V - fU - \frac{1}{\rho_0}P_y^T\nabla p + P_z^T\nabla\left(K_M P_z^T\nabla V\right) + F_y \tag{24}$$

$$p = p_{atm} - \rho_0 g\sigma D - gD\int_\sigma^0 \rho' d\sigma' \tag{25}$$

$$P\nabla\ \cdot\ \boldsymbol{U} - DP_z^T\nabla(P_t^T\nabla\sigma + UP_x^T\nabla\sigma + VP_y^T\nabla\sigma) = 0 \tag{26}$$

$$\frac{\partial T}{\partial t} = -\ (\boldsymbol{U}\cdot P\nabla)T + D\left(UP_x^T\nabla\sigma + VP_y^T\nabla\sigma\right)P_z^T\nabla T + P_z^T\nabla\left(K_H P_z^T\nabla T\right) + F_T \tag{27}$$

$$\frac{\partial S}{\partial t} = -\ (\boldsymbol{U}\cdot P\nabla)S + D\left(UP_x^T\nabla\sigma + VP_y^T\nabla\sigma\right)P_z^T\nabla S + P_z^T\nabla\left(K_H P_z^T\nabla S\right) + F_S \tag{28}$$

$$\frac{\partial q^2}{\partial t} = -(\boldsymbol{U}\cdot P\nabla)q^2 + D\left(UP_x^T\nabla\sigma + VP_y^T\nabla\sigma\right)P_z^T\nabla q^2 + P_z^T\nabla\left(K_q P_z^T\nabla q^2\right)\ +\ 2K_M\ \left[(P_z^T\nabla U)^2 + (P_z^T\nabla V)^2\right] + \frac{2g}{\rho_0}K_H P_z^T\nabla\rho + \frac{2q^3}{B_1 l} + F_q \tag{29}$$

$$\frac{\partial q^2 l}{\partial t} = -(\boldsymbol{U}\cdot P\nabla)q^2 l + D\left(UP_x^T\nabla\sigma + VP_y^T\nabla\sigma\right)P_z^T\nabla q^2 l + P_z^T\nabla\left(K_q P_z^T\nabla q^2 l\right) + lE_1K_M\left[(P_z^T\nabla U)^2 + (P_z^T\nabla V)^2\right] + \frac{lE_1 g}{\rho_0}K_H P_z^T\nabla\rho - \frac{q^3}{B_1}\tilde{W} + F_l \tag{30}$$

Where

$$\boldsymbol{U} = (U, V, \omega) \tag{31}$$

$$D = \frac{1}{P_z^T\nabla\sigma} \tag{32}$$

$$DF_x \equiv \frac{\partial}{\partial x}(H * 2A_M\frac{\partial U}{\partial x}) + \frac{\partial}{\partial y}\left[H * A_M(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x})\right] \tag{33}$$

$$DF_y \equiv \frac{\partial}{\partial y}(H * 2A_M\frac{\partial V}{\partial y}) + \frac{\partial}{\partial x}\left[H * A_M(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x})\right] \tag{34}$$

Also,

$$DF_\phi \equiv \frac{\partial}{\partial x}(H * A_H\frac{\partial\phi}{\partial x}) + \frac{\partial}{\partial y}(H * A_H\frac{\partial\phi}{\partial y}) \tag{35}$$

where ϕ represents T, S, q^2 and q^2l.

3.1 Continuous formulation of nonhydrostatic primitive equation

$$\frac{\partial \boldsymbol{U}}{\partial t} = -P_t^T \nabla \boldsymbol{U} - (\boldsymbol{U} \cdot P\nabla)\, \boldsymbol{U} - 2\boldsymbol{\Omega} \times \boldsymbol{U} - \frac{1}{\rho} P \nabla p + \boldsymbol{G} + P_z^T \nabla \left(K_M P_z^T \nabla \boldsymbol{U}\right) + \boldsymbol{F} \tag{36}$$

$$\frac{\partial \Phi}{\partial t} = -P_t^T \nabla \Phi \ - (\boldsymbol{U} \cdot P\nabla) \Phi \ + P_z^T \left(K_H P_z^T \nabla \Phi\right) + F_\Phi \tag{37}$$

$$\frac{\partial Q}{\partial t} = -P_t^T \nabla Q - (\boldsymbol{U} \cdot P\nabla)\, Q \ + P_z^T \left(K_Q P_z^T \nabla Q\right) + Q_K \left[(P_z^T \nabla U)^2 + (P_z^T \nabla V)^2\right] + Q_b P_z^T \nabla \rho + Q(B_1) + F_Q \tag{38}$$

$$P\nabla \ \cdot \ \boldsymbol{U} = 0 \tag{39}$$

$$\rho = \rho(T, S, p) \tag{40}$$

Where Φ represents T and S, Q represents q^2 and $q^2 l$. $\boldsymbol{U} = (U\widehat{i}\ , V\widehat{j}\ , W\widehat{k})$, $\boldsymbol{\Omega} = (0\widehat{i}\ , 2|\Omega| \cos\theta \widehat{j}\ , 2|\Omega| \sin\theta \widehat{k})$, $\boldsymbol{G} = (0\widehat{i}\ , 0\widehat{j}\ , -g\widehat{k})$, $\boldsymbol{F} = (F_x \widehat{i}\ , F_y \widehat{j}\ , F_z \widehat{k})$

4 The RBF Method on Sigma Coordinates

5 References