# A Small Part of POM's RBF

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## 1 Original Equations

The original equations of the Princeton Ocean Model(POM) are listed following:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} - fV = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(K_M\frac{\partial U}{\partial z}\right) + F_x \tag{1}$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} + fU = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(K_M\frac{\partial V}{\partial z}\right) + F_y \tag{2}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{3}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0 \tag{4}$$

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} + W\frac{\partial T}{\partial z} = \frac{\partial}{\partial z}\left(K_H\frac{\partial T}{\partial z}\right) + F_T \tag{5}$$

$$\frac{\partial S}{\partial t} + U\frac{\partial S}{\partial x} + V\frac{\partial S}{\partial y} + W\frac{\partial S}{\partial z} = \frac{\partial}{\partial z}\left(K_H\frac{\partial S}{\partial z}\right) + F_S \tag{6}$$

$$\rho = \rho(T, S, p) \tag{7}$$

where,

$$F_x = \frac{\partial}{\partial x}\left(2A_M\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left[A_M\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right]$$

$$F_y = \frac{\partial}{\partial y}\left(2A_M\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial x}\left[A_M\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right]$$

$$F_T = \frac{\partial}{\partial x}\left(A_H\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial T}{\partial y}\right)$$

$$F_S = \frac{\partial}{\partial x}\left(A_H\frac{\partial S}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial S}{\partial y}\right)$$

## 2 From Cartesian Coordinates to Sigma coordinates

As we can see, POM employs sigma coordinates, which are bottom-following coordinates that map the vertical coordinate from $-H \le z \le \eta$ to $-1 \le \sigma \le 0$ with $\sigma = (z-\eta)/(H+\eta)$, where the bottom is defined by $z = -H(x,y)$ and the surface is defined by $z = \eta(x,y,t)$. The governing equations are transformed from cartesian coordinates $(x, y, z, t)$ to sigma coordinates $(x^*, y^*, z^*, t^*)$ which satisfy

$$x^* = x, \quad y^* = y, \quad \sigma = \frac{z-\eta}{H+\eta}, \quad t^* = t \tag{8}$$

The derivatives are then given by the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial x^*}{\partial x}\frac{\partial}{\partial x^*} + \frac{\partial y^*}{\partial x}\frac{\partial}{\partial y^*} + \frac{\partial \sigma}{\partial x}\frac{\partial}{\partial \sigma} + \frac{\partial t^*}{\partial x}\frac{\partial}{\partial t^*} \tag{9}$$

$$\frac{\partial}{\partial y} = \frac{\partial x^*}{\partial y}\frac{\partial}{\partial x^*} + \frac{\partial y^*}{\partial y}\frac{\partial}{\partial y^*} + \frac{\partial \sigma}{\partial y}\frac{\partial}{\partial \sigma} + \frac{\partial t^*}{\partial y}\frac{\partial}{\partial t^*} \tag{10}$$

$$\frac{\partial}{\partial z} = \frac{\partial x^*}{\partial z}\frac{\partial}{\partial x^*} + \frac{\partial y^*}{\partial z}\frac{\partial}{\partial y^*} + \frac{\partial \sigma}{\partial z}\frac{\partial}{\partial \sigma} + \frac{\partial t^*}{\partial z}\frac{\partial}{\partial t^*} \tag{11}$$

$$\frac{\partial}{\partial t} = \frac{\partial x^*}{\partial t}\frac{\partial}{\partial x^*} + \frac{\partial y^*}{\partial t}\frac{\partial}{\partial y^*} + \frac{\partial \sigma}{\partial t}\frac{\partial}{\partial \sigma} + \frac{\partial t^*}{\partial t}\frac{\partial}{\partial t^*} \tag{12}$$

which can be written in matrix form as

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^*}{\partial x} & \frac{\partial y^*}{\partial x} & \frac{\partial \sigma}{\partial x} & \frac{\partial t^*}{\partial x} \\ \frac{\partial x^*}{\partial y} & \frac{\partial y^*}{\partial y} & \frac{\partial \sigma}{\partial y} & \frac{\partial t^*}{\partial y} \\ \frac{\partial x^*}{\partial z} & \frac{\partial y^*}{\partial z} & \frac{\partial \sigma}{\partial z} & \frac{\partial t^*}{\partial z} \\ \frac{\partial x^*}{\partial t} & \frac{\partial y^*}{\partial t} & \frac{\partial \sigma}{\partial t} & \frac{\partial t^*}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y^*} \\ \frac{\partial}{\partial \sigma} \\ \frac{\partial}{\partial t^*} \end{bmatrix}$$

Specifically,

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial x^*} + \frac{1}{D}\frac{\partial \eta}{\partial x^*}\right) & 0 \\ 0 & 1 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial y^*} + \frac{1}{D}\frac{\partial \eta}{\partial y^*}\right) & 0 \\ 0 & 0 & \frac{1}{D} & 0 \\ 0 & 0 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial t^*} + \frac{1}{D}\frac{\partial \eta}{\partial t^*}\right) & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y^*} \\ \frac{\partial}{\partial \sigma} \\ \frac{\partial}{\partial t^*} \end{bmatrix}$$

where we have used the following expressions already, because $D = H(x, y) + \eta(x, y, t)$ and $\eta = \eta(x, y, t)$ only relating to $x, y, t$

$$\frac{\partial D}{\partial x} = \frac{\partial D}{\partial x^*}, \quad \frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial x^*}$$

$$\frac{\partial D}{\partial y} = \frac{\partial D}{\partial y^*}, \quad \frac{\partial \eta}{\partial y} = \frac{\partial \eta}{\partial y^*}$$

$$\frac{\partial D}{\partial t} = \frac{\partial D}{\partial t^*}, \quad \frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial t^*}$$

## 3 Gradient Operator and Divergence Operator

Taking equation (1) (2) for instance, it can be changed into an operator form

$$\frac{\partial U}{\partial t} = -(\boldsymbol{U} \cdot \nabla)U + fV - \frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(K_M \frac{\partial U}{\partial z}\right) + F_x \tag{13}$$

$$\frac{\partial V}{\partial t} = -(\boldsymbol{U} \cdot \nabla)V - fU - \frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(K_M \frac{\partial V}{\partial z}\right) + F_y \tag{14}$$

where

$$\boldsymbol{U} = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

$$F_x = \frac{\partial}{\partial x}\left(2A_M\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left[A_M\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right]$$

$$F_y = \frac{\partial}{\partial y}\left(2A_M\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial x}\left[A_M\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right)\right]$$

Also, equation (4) can be changed into an operator form

$$\nabla \cdot \boldsymbol{U} = 0 \tag{15}$$

we define the operator $P$:

$$P = \begin{bmatrix} 1 & 0 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial x^*} + \frac{1}{D}\frac{\partial \eta}{\partial x^*}\right) \\ 0 & 1 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial y^*} + \frac{1}{D}\frac{\partial \eta}{\partial y^*}\right) \\ 0 & 0 & \frac{1}{D} \end{bmatrix} = \begin{bmatrix} P_x^T \\ P_y^T \\ P_z^T \end{bmatrix}$$

$$\nabla^* = \begin{bmatrix} \frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y^*} \\ \frac{\partial}{\partial \sigma} \end{bmatrix}$$

$$P_t^T = \begin{bmatrix} 0 & 0 & -\left(\frac{\sigma}{D}\frac{\partial D}{\partial t^*} + \frac{1}{D}\frac{\partial \eta}{\partial t^*}\right) \end{bmatrix}$$

Taking $\frac{\partial}{\partial x}$ for example, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x^*} - \left(\frac{\sigma}{D}\frac{\partial D}{\partial x^*} + \frac{1}{D}\frac{\partial \eta}{\partial x^*}\right)\frac{\partial}{\partial \sigma} = P_x^T \cdot \nabla^*$$

$$\frac{\partial}{\partial y} = P_y^T \cdot \nabla^*$$

$$\frac{\partial}{\partial z} = P_z^T \cdot \nabla^*$$

$$\frac{\partial}{\partial t} = P_t^T \cdot \nabla^* + \frac{\partial}{\partial t^*}$$

$$\omega = D\left(\frac{\partial \sigma}{\partial t} + U\frac{\partial \sigma}{\partial x} + V\frac{\partial \sigma}{\partial y} + W\frac{\partial \sigma}{\partial z}\right)$$

Where $\omega$ represents vertical velocity in sigma coordinte. Next, the gradient operator appearing in (13) must be constrained so that when it is applied to a scalar, it produces a vector which is in the sigma coordinates. This is done by replacing all occurrences of $\nabla$ in (13) with the operator

$$\nabla = P\nabla^*$$

since P projects vectors into sigma coordinates. The divergence operator appearing in (15) must also be restricted so that it produces the divergence of a vector field in sigma coordinates. This is also accomplished with P by simply taking the dot product of the vector field with the projected gradient operator $P\nabla$.

Follow this way we could get equation (16) from equation (13)

$$\frac{\partial U}{\partial t^*} = -P_t^T\nabla^* U - (\boldsymbol{U}\cdot P\nabla^*)U + fV - \frac{1}{\rho_0}P_x^T\nabla^* p + P_z^T\nabla^*\left(K_M P_z^T\nabla^* U\right) + F_x^* \tag{16}$$

Where

$$F_x^* = P_x^T\nabla^*\left(2A_M P_x^T\nabla^* U\right) + P_y^T\nabla^*\left[A_M\left(P_y^T\nabla^* U + P_x^T\nabla^* V\right)\right] \tag{17}$$

$$= \frac{\partial \tau_{xx}}{\partial x^*} - \frac{\partial}{\partial \sigma}\left[\left(\frac{\sigma}{D}\frac{\partial D}{\partial x^*} + \frac{1}{D}\frac{\partial \eta}{\partial x^*}\right)\tau_{xx}\right] + \frac{\partial \tau_{yx}}{\partial y^*} - \frac{\partial}{\partial \sigma}\left[\left(\frac{\sigma}{D}\frac{\partial D}{\partial y^*} + \frac{1}{D}\frac{\partial \eta}{\partial y^*}\right)\tau_{yx}\right]$$

with

$$\tau_{xx} = 2A_M \left[ \frac{\partial UD}{\partial x^*} - \frac{\partial}{\partial \sigma} \left( \sigma \frac{\partial D}{\partial x^*} + \frac{\partial \eta}{\partial x^*} \right) U \right] \tag{18}$$

$$\tau_{yx} = \quad A_M \left[ \frac{\partial UD}{\partial y^*} - \frac{\partial}{\partial \sigma} \left( \sigma \frac{\partial D}{\partial x^*} + \frac{\partial \eta}{\partial x^*} \right) U + \frac{\partial VD}{\partial x^*} - \frac{\partial}{\partial \sigma} \left( \sigma \frac{\partial D}{\partial x^*} + \frac{\partial \eta}{\partial x^*} \right) V \right] \tag{19}$$

Since Mellor and Blumberg [1985] have shown that the conventional model for horizontal diffusion is incorrect when bottom topographical slopes are large. A new formulation has been suggested which is simpler than the $F_x^*$, $F_y^*$ and $F_\phi^*$ equations. They makes it possible to model realistically bottom boundary layers over sharply sloping bottoms. They are defined according to:

$$F_x^* = \frac{\partial}{\partial x^*} \left( H * 2A_M \frac{\partial U}{\partial x^*} \right) + \frac{\partial}{\partial y^*} \left[ H * A_M \left( \frac{\partial U}{\partial y^*} + \frac{\partial V}{\partial x^*} \right) \right] \tag{20}$$

Putting all these pieces together, the final equations(We already drop all *) in sigma coordinates is given by

$$\frac{\partial U}{\partial t} = -P_t^T \nabla U - (\boldsymbol{U} \cdot P\nabla) U + fV - \frac{1}{\rho_0} P_x^T \nabla p + P_z^T \nabla \left( K_M P_z^T \nabla U \right) + F_x \tag{21}$$

$$\frac{\partial V}{\partial t} = -P_t^T \nabla V - (\boldsymbol{U} \cdot P\nabla) V - fU - \frac{1}{\rho_0} P_y^T \nabla p + P_z^T \nabla \left( K_M P_z^T \nabla V \right) + F_y \tag{22}$$

$$P\nabla \cdot \boldsymbol{U} = 0 \tag{23}$$

$$\frac{\partial T}{\partial t} = -P_t^T \nabla T - (\boldsymbol{U} \cdot P\nabla) T + P_z^T \nabla \left( K_H P_z^T \nabla T \right) + F_T \tag{24}$$

$$\frac{\partial S}{\partial t} = -P_t^T \nabla S - (\boldsymbol{U} \cdot P\nabla) S + P_z^T \nabla \left( K_H P_z^T \nabla S \right) + F_S \tag{25}$$

$$\frac{\partial q^2}{\partial t} = -P_t^T \nabla q^2 - (\boldsymbol{U} \cdot P\nabla) q^2 + P_z^T \nabla \left( K_q P_z^T \nabla q^2 \right) + 2K_M \left[ (P_z^T \nabla U)^2 + (P_z^T \nabla V)^2 \right] + \frac{2g}{\rho_0} K_H P_z^T \nabla \rho + \frac{2q^3}{B_1 l} + F_q \tag{26}$$

$$\frac{\partial q^2 l}{\partial t} = -P_t^T \nabla q^2 l - (\boldsymbol{U} \cdot P\nabla) q^2 l + P_z^T \nabla \left( K_q P_z^T \nabla q^2 l \right) + lE_1 K_M \left[ (P_z^T \nabla U)^2 + (P_z^T \nabla V)^2 \right] + \frac{lE_1 g}{\rho_0} K_H P_z^T \nabla \rho - \frac{q^3}{B_1} \tilde{W} + F_l \tag{27}$$

$$p = p_{atm} - \rho_0 g \sigma D - gD \int_\sigma^0 \rho' d\sigma' \tag{28}$$

Where

$$W = \omega + U \left( \sigma \frac{\partial D}{\partial x^*} + \frac{\partial \eta}{\partial x^*} \right) + V \left( \sigma \frac{\partial D}{\partial y^*} + \frac{\partial \eta}{\partial y^*} \right) + \sigma \frac{\partial D}{\partial t^*} + \frac{\partial \eta}{\partial t^*} \tag{29}$$

$$DF_x \equiv \frac{\partial}{\partial x} (H * 2A_M \frac{\partial U}{\partial x}) + \frac{\partial}{\partial y} \left[ H * A_M ( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} ) \right] \tag{30}$$

$$DF_y \equiv \frac{\partial}{\partial y} (H * 2A_M \frac{\partial V}{\partial y}) + \frac{\partial}{\partial x} \left[ H * A_M ( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} ) \right] \tag{31}$$

Also,

$$DF_\phi \equiv \frac{\partial}{\partial x} (H * A_H \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y} (H * A_H \frac{\partial \phi}{\partial y}) \tag{32}$$

where $\phi$ represents $T$, $S$, $q^2$ and $q^2 l$.

### 3.1 Continuous formulation of nonhydrostatic primitive equation

$$\frac{\partial \boldsymbol{U}}{\partial t}=-P_t^T\nabla\boldsymbol{U}-(\boldsymbol{U}\cdot P\nabla)\,\boldsymbol{U}-2\boldsymbol{\Omega}\times\boldsymbol{U}-\frac{1}{\rho}P\nabla p+\boldsymbol{G}+P_z^T\nabla\left(K_M P_z^T\nabla\boldsymbol{U}\right)+\boldsymbol{F} \tag{33}$$

$$\frac{\partial \Phi}{\partial t}=-P_t^T\nabla\Phi\;-(\boldsymbol{U}\cdot P\nabla)\Phi\;+P_z^T\left(K_H P_z^T\nabla\Phi\right)+F_\Phi \tag{34}$$

$$\frac{\partial Q}{\partial t}=-P_t^T\nabla Q-(\boldsymbol{U}\cdot P\nabla)\,Q\;+P_z^T\left(K_Q P_z^T\nabla Q\right)+Q_K\left[(P_z^T\nabla U)^2+(P_z^T\nabla V)^2\right]+Q_b P_z^T\nabla\rho+Q(B_1)+F_Q \tag{35}$$

$$P\nabla\;\cdot\;\boldsymbol{U}=0 \tag{36}$$

$$\rho=\rho(T,S,p) \tag{37}$$

Where $\Phi$ represents $T$ and $S$, $Q$ represents $q^2$ and $q^2 l$. $\boldsymbol{U}=(U\widehat{i}\;,V\widehat{j}\;,W\widehat{k})$, $\boldsymbol{\Omega}=(0\widehat{i}\;,2|\Omega|\cos\theta\widehat{j}\;,2|\Omega|\sin\theta\widehat{k})$, $\boldsymbol{G}=(0\widehat{i}\;,0\widehat{j}\;,-g\widehat{k})$, $\boldsymbol{F}=(F_x\widehat{i}\;,F_y\widehat{j}\;,F_z\widehat{k})$

# 4 The RBF Method on Sigma Coordinates

# 5 References