

Problem Set 7

E522: Macroeconomic Theory I

Question 1

In this problem set, we will numerically compute the value function that solves the Bellman equation

$$V(k) = \max_{k' \in [0, k^\alpha]} \{ \log(k^\alpha - k') + \beta V(k') \}$$

that is the recursive representation of the planning problem

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$c_t + k_{t+1} = k_t^\alpha$$

with $k_0 > 0$ given. Since we know the analytical solution for V , we will be able to check our numerical answer once we have computed it.

Numerical computation requires us to find a good enough approximation to the function V over a discrete set of points $K = \{k_i\}_{i=1}^N$, where we let $k_i > k_{i-1}$ for all i . The set K defines the domain over which we will compute the value function, and so needs to allow for all feasible values of k in the model. In our current setting, it therefore makes sense to set $k_1 = \epsilon$ for ϵ small but positive and $k_N = 1$, where 1 is the largest possible sustainable level of k in the model (attained by setting $c_t = 0$ for all t). Setting $k_1 > 0$ avoids issues with approximating the value function at points where the true value function is not well-defined. The remaining k_i can then be set to form a grid of points spanning $[k_1, k_N]$. A uniformly spaced grid is a useful starting grid.

Given a grid K , we will look for a vector $v \in \mathbb{R}^N$, where $v_i = v(k_i)$ corresponds to the value function when $k = k_i$. In other words, the vector v will approximate the function V on set the K . We will solve for v by considering the “discretized” version of the Bellman equation,

$$v(k_i) = \max_{k'_i \in K} \{ \log(k_i^\alpha - k'_i) + \beta v(k'_i) \}$$

where k'_i is the optimal policy when the state is k_i , which must hold for every $i = 1, \dots, N$. Note that the optimal policy choice is restricted to also lie on the grid K . If we stack the $v(k_i)$ into a column vector, this becomes

$$\begin{pmatrix} v(k_1) \\ v(k_2) \\ \vdots \\ v(k_N) \end{pmatrix} = \begin{pmatrix} \max_{k'_1 \in K} \{ \log(k_1^\alpha - k'_1) + \beta v(k'_1) \} \\ \max_{k'_2 \in K} \{ \log(k_2^\alpha - k'_2) + \beta v(k'_2) \} \\ \vdots \\ \max_{k'_N \in K} \{ \log(k_N^\alpha - k'_N) + \beta v(k'_N) \} \end{pmatrix}$$

Function Iteration Algorithm Recall that the operator on the right-hand side of the Bellman equation is a contraction. Therefore, given an initial guess for the value function vector v^0 , we can compute an updated vector v^1 as

$$v^1(k_i) = \max_{k'_i \in K} \{ \log(k_i^\alpha - k'_i) + \beta v^0(k'_i) \}$$

for each i . We also know that repeatedly applying this iteration will result in the updated vectors converging to the best approximation of the value function. Therefore, we can just apply this iteration a sufficient number of times until the change in the vector is sufficiently small.

Therefore, the algorithm is as follows:

1. Set an initial guess for the value function vector approximation, v^0 , e.g. the zero vector.
2. Given a vector v^n , compute the updated vector v^{n+1} as

$$v^{n+1}(k_i) = \max_{k'_i \in K} \{ \log(k_i^\alpha - k'_i) + \beta v^n(k'_i) \}$$

for all i .

3. Compute the % distance between v^{n+1} and v^n : $d = \max_i \left| \frac{v_i^{n+1} - v_i^n}{v_i^n} \right|$. If $d < \tau$ for some tolerance level $\tau > 0$, set $v = v^{n+1}$, and set the policy vector k' equal to the last computed vector used to update v^n . If $d \geq \tau$, return to step 2, replacing v^n with v^{n+1} .

Set $\beta = 0.96$, and $\alpha = 0.33$. Solve the value function numerically using the function iteration algorithm for a range of grid sizes $N \in \{25, 100, 500\}$, and tolerances, $\tau \in \{0.1, 0.001, 0.00001\}$. In each case, graph your computed vector v , and compare it to the analytical solution

$$V(k) = \frac{\alpha}{1 - \alpha\beta} \log k + \frac{1}{1 - \beta} \left(\log(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta} \log \alpha\beta \right)$$

(also plotted on the same graph).

Question 2 (Lucas, Stokey, and Prescott 5.7)

Consider the problem of a worker who works for $N > 0$ periods. In each period, she earns income $k_t h_t$, where k_t is her level of human capital and $h_t \in [0, 1]$ denotes hours of work (normalized so that the endowment of time is unity). In each period, the worker faces a trade off between devoting time to human capital accumulation and working. Specifically, if she accumulates k_{t+1} units of human capital at the end of period t , then she can spend $h_t = \phi(k_{t+1}/k_t)$ hours working. Assume that ϕ is strictly decreasing, strictly concave, and is continuously differentiable. Furthermore, if the worker spends all her time working, then her human capital depreciates at rate $\delta > 0$ so that $\phi(1 - \delta) = 1$, while if she does not work at all, then her human capital grows at a maximal rate $\lambda > 0$ so that $\phi(1 + \lambda) = 0$.

The worker solves

$$\max_{\{k_{t+1}\}_{t=1}^N} \sum_{t=1}^N \beta^t k_t \phi(k_{t+1}/k_t)$$

subject to

$$k_{t+1} \geq (1 - \delta) k_t$$

$$k_{t+1} \leq (1 + \lambda) k_t$$

with $k_1 > 0$ given.

Let C be the space of continuous and bounded functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and consider the operator T defined on C by

$$(Tf)(k) = \max_{(1-\delta)k \leq y \leq (1+\lambda)k} k\phi(y/k) + \beta f(y)$$

Notice that the sequential problem is only finite dimensional. Therefore, we can define $v_0(k) = 0$ for all $k \geq 0$ as the value of this worker's total future discounted income when she has no more periods left to work. Then, $v_1 = Tv_0$ defines the value when her human capital is k and she has one period left to work. Continuing for each $n \in \{2, \dots, N\}$, we have that $v_N = T^N v_0$ is the maximum value of the worker's sequential problem when she has N periods left to work. In this setting, we care more about the iterations of T than we do about its fixed point.

- (a) Prove by induction that $(T^n v_0)(k) = a_n k$ for all $n \geq 0$ and find a_{n+1} as a function of a_n .
- (b) Suppose that $\beta(1 - \delta) > 1$. Show that $a_1 = 1$, and that $a_{n+1} > a_n$ for all n . Derive a condition on β and λ to ensure that the sequence $\{a_n\}$ is bounded.