Simulation Estimators

Summer School in Structural Estimation

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June 2023

Outline

Pseudo-Random Numbers

- Computers cannot produce truly random numbers, and instead produce what are called "pseudo-random" numbers.
- In general, pseudo-random numbers appear random because they can pass some simple tests of randomness such as a test for serial correlation.
- A very simple example of a pseudo-random number generator is

$$X_r = (kX_{r-1} + c)\,\mathrm{mod}\,m$$

  in which the modulo operator $a\,\mathrm{mod}\,b$ produces the remainder of a/b.
- Actual pseudo-random number generators are more complicated, but they use the same basic principle.

Pseudo-Random Numbers

- A series of (pseudo) random $i.i.d.$ uniformly distributed random numbers on $(0, m)$ $u_r, u_{r+1}, \ldots, r_R$ can be produced as $u_r = X_r/m$.
- Most statistical packages have functions that produce random uniform and random normal vectors.
- These random numbers are in reality deterministic. The periodicity of the cycle is determined by the k, c, and m values.
- For well chosen values, the period is typically high (e.g. 10^9).
- X_0 is referred to as the seed or state number, which determines the sequence of random numbers.
- To use the same sequence of random numbers repeatedly, remember to either save the random number draws from the first use or re-use the same seed number.

Outline

What if . . . you wanted to estimate a mean?

- Suppose you have an $i.i.d.$ sample, y_i, of length N.
- You could use classical method-of-moments by picking an estimate, μ, based on the following moment condition:
$$E(y-\mu)=0$$
- The sample counterpart to this moment condition is
$$\frac{1}{N}\sum_{i=1}^{N} y_i - \hat{\mu} = 0$$
- In this example you have a closed-form expression for the moment condition.
- But you might not . . .

What if . . . you wanted to estimate a mean?

- A different, very convoluted, way to do the same thing would be to proceed as follows.
  - Generate a random vector with a mean, μ. Calculate its average. Call the average from this simulation $\hat{\mu}_1$.
  - Do this S times and then calculate.

$$\tilde{\mu} = \frac{1}{S}\sum_{i=1}^{S}\hat{\mu}_i$$

  - Pick the estimate, μ that sets

$$\tilde{\mu} - \frac{1}{N}\sum_{i=1}^{N} y_i$$

as close to zero as possible.

What if . . . you wanted to estimate a mean?

- This is a simulated method of moments estimator.
- In practice, the estimator is almost the same because you simulate a random vector with a mean μ by simulating a random vector with a mean of zero and then adding μ.
- Its variance will not be the same as the variance of a GMM estimator because of simulation error.
- This example is, of course, silly because we don't need to calculate μ via a simulation. We know it. We just write it down.
- However, there exist applications in which this is not the case.

Outline

SMM is a special case of GMM, so . . .

- Let's briefly review GMM.

- We will come back to this material later in much more detail when we talk about weight matrices for SMM

The Setup

- The following uses the notation in Wooldridge.
- Let
  - Let $\boldsymbol{w}_i$ be an $(M \times 1)$ be an $i.i.d.$ vector of random variables for observation i.
  - $\boldsymbol{\theta}$ be an $(P \times 1)$ vector of unknown coefficients.
  - $\boldsymbol{g}(\boldsymbol{w}_i, \boldsymbol{\theta})$ be an $(L \times 1)$ vector of functions $\boldsymbol{g} : \left(\mathcal{R}^M \times \mathcal{R}^P\right) \to \mathcal{R}^L, \quad L \geq P$
- The function $\boldsymbol{g}(\boldsymbol{w}_i, \boldsymbol{\theta})$ can be nonlinear.
- Let $\boldsymbol{\theta}_0$ be the true value of $\boldsymbol{\theta}$.
- Let $\hat{\boldsymbol{\theta}}$ represent an estimate of $\boldsymbol{\theta}$.
- The “hat” and “naught” notation applies to anything we might want to estimate.

Moment Restrictions

- GMM is based on what are generally called moment restrictions and sometimes called orthogonality conditions (The latter terminology comes from the rational expectations literature.)

$$E\left(\boldsymbol{g}\left(\boldsymbol{w}_i, \boldsymbol{\theta}_0\right)\right) = 0$$

- This condition is expressed in terms of the population. The corresponding sample moment restriction is

$$\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{g}\left(\boldsymbol{w}_i, \boldsymbol{\theta}\right) = 0$$

- What we want to do is choose $\hat{\boldsymbol{\theta}}$ to get $N^{-1}\sum_{i=1}^{N} \boldsymbol{g}\left(\boldsymbol{w}_i, \boldsymbol{\theta}\right)$ as close to zero as possible.

Criterion Function

- The estimator, $\hat{\boldsymbol{\theta}}$ minimizes a quadratic form:

$$\boldsymbol{Q}_N(\boldsymbol{\theta}) = \underset{(1\times L)}{\left[N^{-1}\sum_{i=1}^{N}\boldsymbol{g}(\boldsymbol{w}_i,\boldsymbol{\theta})\right]'} \underset{(L\times L)}{\widehat{\boldsymbol{\Xi}}} \underset{(L\times 1)}{\left[N^{-1}\sum_{i=1}^{N}\boldsymbol{g}(\boldsymbol{w}_i,\boldsymbol{\theta})\right]}$$

  where $\widehat{\boldsymbol{\Xi}}$ is a positive definite matrix that converges in probability to $\boldsymbol{\Xi}_0$

- In this case, $\boldsymbol{Q}_N$ converges in probability to

$$\{E[\boldsymbol{g}(\boldsymbol{w}_i,\boldsymbol{\theta})]\}'\boldsymbol{\Xi}\{E[\boldsymbol{g}(\boldsymbol{w}_i,\boldsymbol{\theta})]\}$$

Outline

Setup

- Let w_i be an $i.i.d.$ data vector, $i = 1, \ldots, n$.
- Let $y_{is}(\theta)$ be an $i.i.d.$ simulated vector from simulation s, $i = 1, \ldots, n$, and $s = 1, \ldots, S$.
- The simulated data vector, $y_{is}(\theta)$, depends on a vector of structural parameters, θ.
- The goal is to estimate θ by matching a set of *simulated moments*, denoted as $g(y_{is}(\theta))$, with the corresponding set of actual *data moments*, denoted as $\mathbb{E}(g(w_i))$.
- The simulated moments, $g(y_{is}(\theta))$ are functions of the parameter vector θ because the moments will differ depending on the choice of θ.

Moment Matching

- The first step is to estimate $\mathbb{E}\left(g\left(w_i\right)\right)$ using the actual data.
- The second step is to construct S simulated data sets based on a given parameter vector.
- For each of these data sets, compute a simulated moment, $g\left(y_{is}\left(\theta\right)\right)$.
- Note that you have to make the *exact* same calculations on the simulated data as you do on the real data.
- $Sn > n$.
- Michaelides and Ng (2000) find that good finite sample performance requires a simulated sample that is approximately ten times as large as the actual data sample.

Now let's figure out how to match the moments

- Define
$$g_n(\theta) = n^{-1} \sum_{i=1}^{n} \left[g(w_i) - S^{-1} \sum_{s=1}^{S} g(y_{is}(\theta)) \right].$$

- The simulated moments estimator of θ is then defined as the solution to the minimization of
$$\hat{\theta} = \arg\min_{\theta} Q(\theta, n) \equiv g_n(\theta)' \hat{W}_n g_n(\theta),$$

- $\hat{W}_n$ is a positive definite matrix that converges in probability to a deterministic positive definite matrix W.

Weight Matrix

- In most applications, one can use the **optimal** weight matrix, which we will call Λ, and which can be calculated **easily** as the inverse of the variance covariance matrix of $\sum_{i=1}^{n} g(\boldsymbol{w}_i)$.
- **This weight matrix can be calculated without any knowledge of the model!!!**
- Either use GMM or stack influence functions.
  - Computationally identical.
  - We will do influence functions tomorrow!

Weight Matrix

- In many applications one can calculate the weight matrix as the inverse of the variance covariance matrix of $n^{-1}\sum_{i=1}^{n} g(w_i)$. This weight matrix has an exact analogy with GMM.
- In some cases this type of weight matrix is not feasible; for example, in the multinomial probit model.
- In such cases you use a two stage procedure.
  - In the first stage, minimize $Q(\theta, n)$ using the identity as the weight matrix.
  - Use the resulting estimate, $\hat{\theta}$ to construct the weight matrix as the inverse of the variance of g_n.

Inference

- The simulated moments estimator is asymptotically normal for fixed S!! (This is not the case for SMLE.)
- The asymptotic distribution of θ is given by

$$\sqrt{n}\left(\hat{\theta}-\theta\right) \xrightarrow{d} \mathcal{N}\left(0, \operatorname{avar}(\hat{\theta})\right)$$

  in which

$$\operatorname{avar}(\hat{\theta}) \equiv \left(1+\frac{1}{S}\right)\left[\frac{\partial g_n\left(\theta\right)}{\partial \theta} W \frac{\partial g_n\left(\theta\right)}{\partial \theta'}\right]^{-1}.$$

  and W is the efficient weight matrix
- Note larger S (more simulated samples) $\Rightarrow$ smaller standard errors

Inference (numerical derivative)

In previous formula, how to compute Jacobian $\partial g_n(\theta)/\partial\theta$?

1. Get the SMM estimate, $\hat{\theta}$
2. Remember the dimensions:
   - Parameters θ: $k \times 1$
   - Moments g: $m \times 1$
   - Jacobian $\partial g_n(\theta)/\partial\theta$: $k \times m$
3. Choose step size $h_i > 0$ for each parameter $i = 1, ..., k$
   - Each parameter gets its own step size
   - Must be chosen with care! Smaller is better ... except when it's not
   - Good initial guess: 1% of parameter's estimate
4. Compute two-sided difference. Element $\{i, j\}$ of $\partial g_n(\theta)/\partial\theta$ is

$$\frac{\partial g_{n,j}\,(\theta)}{\partial\theta_i} \approx \frac{g_{n,j}(\hat{\theta} + h_i) - g_{n,j}(\hat{\theta} - h_i)}{2h_i},$$

where "$+h_i$" means perturb parameter i upward by h_i

Test of Overidentifying Restrictions

- As in the case of plain vanilla GMM, one can perform a test of the overidentifying restrictions of the model

$$\frac{nS}{1+S}Q(\theta, n)$$

- This statistic converges in distribution to a χ^2 with degrees of freedom equal to the dimension of g_n minus the dimension of θ.

So How Do you Actually DO SMM? Data Step

- The data steps:
  - Choose moments to match. Can include means, variances, covariances, regression slopes, etc. Need at least as many moments as parameters.
  - Extra moments provide a test of overidentifying restrictions.
  - Compute moments in the actual data and stack them in a vector
  - Estimate the covariance matrix of this vector. Invert it. This is your efficient SMM/GMM weighting matrix.
  - Calculate many more moments than you think you will need. You **will** change your moment vector at some point.

So How Do You Actually DO SMM?

- Choose an optimizer. These usually have two inputs, a set of parameters over which to optimize and a function to optimize.
- Write a function to be optimized. It will input a parameter vector and output a GMM objective function. This will involve reading in data moments and a weight matrix and then calculating simulated moments and then forming a quadratic form.
- The function subroutine will have to call a model solving routine, a model simulating routine, and a moment calculating routine. The goal in this chain is to eat up parameters and spit back moments.
- When you are done, calculate the Jacobian matrix and the standard errors.

Pseudo Code

```
function SMM_Objective_Function(in double parameters[numberParameters], out double objectiveFunctionValue)
        read weightMatrix[numberMoments, numberMoments]
        read dataMoments[numberMoments]
        call function solveTheModel( in double parameters[numberParameters],
                out double valueFunction[stateSpaceSize],
                out int policyFunction[stateSpaceSize])
        call function simulateFirms( in double valueFunction[stateSpaceSize],
                in int policyFunction[stateSpaceSize],
                out double simulatedFirms[numberOfFirms, numberOfVariables])
        call function calculateMoments(in double simulatedFirms[numberOfFirms, numberOfVariables],
                out double SimulatedMoments[numberMoments])
        momentError = dataMoments − SimulatedMoments
        objectiveFunctionValue = momentError′ * weightMatrix * momentError
```

Outline

You need to worry about identification

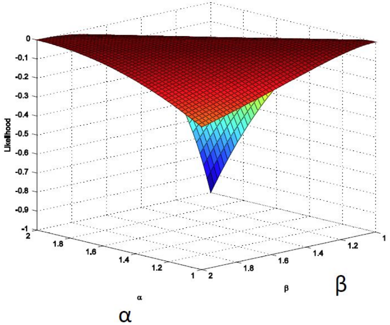
- Identification means different things in different contexts. Sigh.
- What does “identification” usually mean in reduced-form work?
  - Does x affect y?
  - Does y affect x?
  - Does some omitted variable z affect both x and y?
- Exogenous variation is very useful for answering this kind of question
- “Identification” in these papers usually means that the researcher has an experimental design that establishes causality and is identifying a direction.

We will use the formal statistical definition of identification

- Econometrician defines an objective function over parameters and data
- Goal: Select parameter values that minimize this objective function
  - Example: Find slope coefficient that minimize sum of squared errors
- A parameter is (point) *identified* if there is a unique minimum for the objective function at parameter's true value in the population
- In what follows, I'll use the formal statistical definition of identification

Example of an unidentified model

- Suppose we want to estimate parameters α and β by MLE
- Parameters α and β appear in the likelihood function only in the form α/β
- $\Rightarrow \alpha/\beta$ is identified, but α and β are not separately identified
- Likelihood function is flat at its max:



Identification and SMM

- The success of the SMM procedure relies on picking moments g that can identify the structural parameters θ
- The conditions for global identification of a simulated moments estimator are similar to those for GMM:
  - The expected value of the difference between the simulated and actual moments equals zero iff the structural parameters are at their true value
  - A sufficient condition for identification is a one-to-one mapping between the structural parameters and a subset of the data moments of the same dimension

Identification and SMM

- Let $g_\theta\left(y_{is}\left(\theta\right)\right)$ be a subvector of $g\left(y_{is}\left(\theta\right)\right)$ with the same dimension as θ
- Local identification implies that the Jacobian determinant,

$$\det\left(\partial g_\theta\left(y_{is}\left(\theta\right)\right)/\partial\theta\right),$$

  is non-zero. i.e., Jacobian has full rank.
- This condition can be interpreted loosely as saying that the moments, g_θ, are informative about the structural parameters, θ
- That is, the sensitivity of moments to parameters is high

How to choose moments in SMM/GMM

- Best-case scenario: Each moment depends on just one model parameter
  - "Moment #1 identifies parameter #1, moment #2 identifies parameter #2..."
  - This is how many macro papers are (misleadingly) written
- More realistic: Every moment depends on every parameter
- All parameters can affect all moments, but the mapping has to be one-to-one and onto
- Do comparative statics to understand how each moment moves with each parameter. Make sure you understand the economics behind each comparative static.
- Need enough moments, and moments that move in different directions for different parameters, to obtain identification
- Try targeting empirical policy functions (more details later)

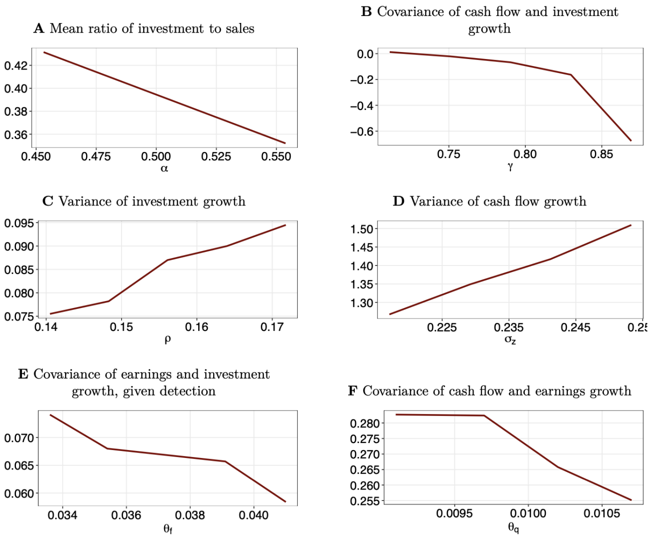
Identification

- Don't start estimating until you're sure the model is identified, or you are in for a world of grief.
- It's usually impossible to prove formally that your model is identified
- How do you ensure that the model is identified?

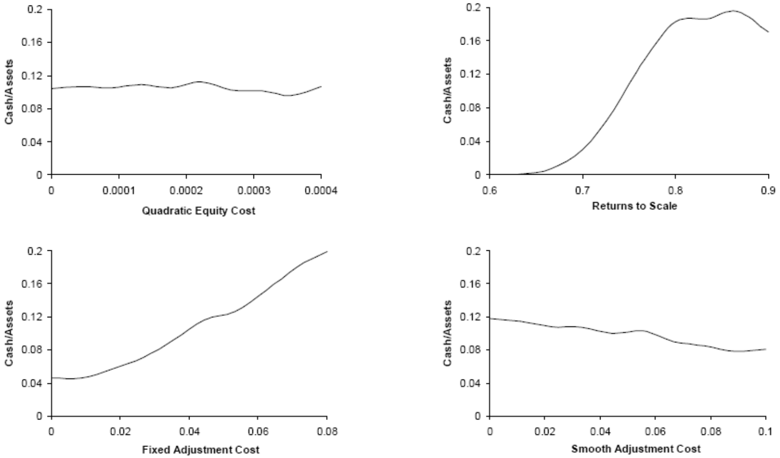
Use Economics (more on this later!)

- **PLAY WITH YOUR MODEL UNTIL YOU UNDERSTAND HOW IT WORKS!!!!!!!!!**
- Do comparative statics: plot the simulated moments as functions of the parameters.
- You want to find steep, monotonic relationships.
- You want moments that move in different directions for different parameters.

Example One: Terry, Whited and Zakolyukina (2023)



Example Two: Riddick and Whited (2009)



Outline

Dynamic models require the estimation of dynamics

- A large fraction of dynamic models have driving processes that follow autoregressive processes.
- One statistic that needs to be matched in many structural estimations is an $AR(1)$ coefficient.

Consistent estimation of a first-order autoregressive coefficient with fixed effects

- Suppose you have a variable

$$y_{it}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T.$$

  that follows a process

$$y_{it} = \alpha_i + \rho y_{it-1} + u_{it},$$

  in which u_{it} is possibly correlated with α_i.

- OLS will not work.
- You cannot do firm-level deviations from means with a lagged dependent variable.
- Dynamic panel models suffer from weak instrument problems.

Han and Phillips (2010)

- Han and Phillips (2010) use double differencing to remove the fixed effect.
- Some ugly but easy algebra shows that a consistent estimate of ρ is given by

$$\hat{\rho} = \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} \Delta y_{it-1} \left(2\Delta y_{it} + \Delta y_{it-1}\right)}{\sum_{i=1}^{N} \sum_{t=2}^{T} \left(\Delta y_{it-1}\right)^{2}}$$

  where Δ is the first difference operator.
- This estimator is clearly obtained from regressing $2\Delta y_{it} + \Delta y_{it-1}$ on Δy_{it-1}.

Sample Julia code for a balanced panel

Consider a panel with dimensions T and N:

```
k = size(biginflnc,2);
vmx = zeros(k,k);
for ic in 1:capN;
    global phii = biginflnc[(ic-1)*capT+1:ic*capT,:];
    phii = sum(phii,dims=1);
    global vmx = vmx + phii'*phii;
end

vmx = vmx./((capN*capT)^2);
```

Han, C.„ Phillips, P.C.B., 2010. GMM estimation for dynamic panels with fixed effects and strong instruments at unity. Econometric Theory 26, 119–151.

Michaelides, A.„ Ng, S., 2000. Estimating the rational expectations model of speculative storage: A Monte Carlo comparison of three simulation estimators. Journal of Econometrics 96, 231–266.

Riddick, L.A.„ Whited, T.M., 2009. The corporate propensity to save. Journal of Finance 64, 1729–1766.

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