Problem Set 7

E522: Macroeconomic Theory I

Question 1

In this problem set, we will numerically compute the value function that solves the Bellman equation

$$V(k) = \max_{k' \in [0, k^\alpha]} \left\{ \log\left(k^\alpha - k'\right) + \beta V\left(k'\right) \right\}$$

that is the recursive representation of the planning problem

$$\max_{\{c_t, k_t\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$c_t + k_{t+1} = k_t^\alpha$$

with $k_0 > 0$ given. Since we know the analytical solution for V, we will be able to check our numerical answer once we have computed it.

Numerical computation requires us to find a good enough approximation to the function V over a discrete set of points $K = \{k_i\}_{i=1}^N$, where we let $k_i > k_{i-1}$ for all i. The set K defines the domain over which we will compute the value function, and so needs to allow for all feasible values of k in the model. In our current setting, it therefore makes sense to set $k_1 = \epsilon$ for ϵ small but positive and $k_N = 1$, where 1 is the largest possible sustainable level of k in the model (attained by setting $c_t = 0$ for all t). Setting $k_1 > 0$ avoids issues with approximating the value function at points where the true value function is not well-defined. The remaining k_i can then be set to form a grid of points spanning $[k_1, k_N]$. A uniformly spaced grid is a useful starting grid.

Given a grid K, we will look for a vector $v \in \mathbb{R}^N$, where $v_i = v(k_i)$ corresponds to the value function when $k = k_i$. In other words, the vector v will approximate the function V on set the K. We will solve for v by considering the "discretized" version of the Bellman equation,

$$v(k_i) = \max_{k_i' \in K} \left\{ \log\left(k_i^\alpha - k_i'\right) + \beta v\left(k_i'\right) \right\}$$

where k_i' is the optimal policy when the state is k_i, which must hold for every $i = 1, ..., N$. Note that the optimal policy choice is restricted to also lie on the grid K. If we stack the $v(k_i)$ into a column vector, this becomes

$$\begin{pmatrix} v(k_1) \\ v(k_2) \\ \vdots \\ v(k_N) \end{pmatrix} = \begin{pmatrix} \max_{k_1' \in K} \left\{ \log\left(k_1^\alpha - k_1'\right) + \beta v\left(k_1'\right) \right\} \\ \max_{k_2' \in K} \left\{ \log\left(k_2^\alpha - k_2'\right) + \beta v\left(k_2'\right) \right\} \\ \vdots \\ \max_{k_N' \in K} \left\{ \log\left(k_N^\alpha - k_N'\right) + \beta v\left(k_N'\right) \right\} \end{pmatrix}$$

Function Iteration Algorithm Recall that the operator on the right-hand side of the Bellman equation is a contraction. Therefore, given an initial guess for the value function vector v^0, we can compute an updated vector v^1 as

$$v^1(k_i) = \max_{k_i' \in K} \left\{ \log\left(k_i^\alpha - k_i'\right) + \beta v^0\left(k_i'\right) \right\}$$

for each i. We also know that repeatedly applying this iteration will result in the updated vectors converging to the best approximation of the value function. Therefore, we can just apply this iteration a sufficient number of times until the change in the vector is sufficiently small.

Therefore, the algorithm is as follows:

1. Set an initial guess for the value function vector approximation, v^0, e.g. the zero vector.
2. Given a vector v^n, compute the updated vector v^{n+1} as

   $$v^{n+1}(k_i) = \max_{k_i' \in K} \left\{ \log\left(k_i^\alpha - k_i'\right) + \beta v^n\left(k_i'\right) \right\}$$

   for all i.
3. Compute the % distance between v^{n+1} and v^n: $d = \max_i \left| \frac{v_i^{n+1} - v_i^n}{v_i^n} \right|$. If $d < \tau$ for some tolerance level $\tau > 0$, set $v = v^{n+1}$, and set the policy vector k' equal to the last computed vector used to update v^n. If $d \geq \tau$, return to step 2, replacing v^n with v^{n+1}.

Set $\beta = 0.96$, and $\alpha = 0.33$. Solve the value function numerically using the function iteration algorithm for a range of grid sizes $N \in \{25, 100, 500\}$, and tolerances, $\tau \in \{0.1, 0.001, 0.00001\}$. In each case, graph your computed vector v, and compare it to the analytical solution

$$V(k) = \frac{\alpha}{1-\alpha\beta} \log k + \frac{1}{1-\beta} \left(\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} \log \alpha\beta \right)$$

(also plotted on the same graph).

Question 2 (Lucas, Stokey, and Prescott 5.7)

Consider the problem of a worker who works for $N > 0$ periods. In each period, she earns income $k_t h_t$, where k_t is her level of human capital and $h_t \in [0,1]$ denotes hours of work (normalized so that the endowment of time is unity). In each period, the worker faces a trade off between devoting time to human capital accumulation and working. Specifically, if she accumulates k_{t+1} units of human capital at the end of period t, then she can spend $h_t = \phi(k_{t+1}/k_t)$ hours working. Assume that ϕ is strictly decreasing, strictly concave, and is continuously differentiable. Furthermore, if the worker spends all her time working, then her human capital depreciates at rate $\delta > 0$ so that $\phi(1-\delta) = 1$, while if she does not work at all, then her human capital grows at a maximal rate $\lambda > 0$ so that $\phi(1+\lambda) = 0$.

The worker solves

$$\max_{\{k_{t+1}\}_{t=1}^N} \sum_{t=1}^N \beta^t k_t \phi(k_{t+1}/k_t)$$

subject to

$$k_{t+1} \geq (1-\delta) k_t$$
$$k_{t+1} \leq (1+\lambda) k_t$$

with $k_1 > 0$ given.

Let C be the space of continuous and bounded functions $f : \mathbb{R}_+ \to \mathbb{R}_+$, and consider the operator T defined on C by

$$(Tf)(k) = \max_{(1-\delta)k \leq y \leq (1+\lambda)k} k\phi(y/k) + \beta f(y)$$

Notice that the sequential problem is only finite dimensional. Therefore, we can define $v_0(k) = 0$ for all $k \geq 0$ as the value of this worker's total future discounted income when she has no more periods left to work. Then, $v_1 = Tv_0$ defines the value when her human capital is k and she has one period left to work. Continuing for each $n \in \{2, ..., N\}$, we have that $v_N = T^N v_0$ is the maximum value of the worker's sequential problem when she has N periods left to work. In this setting, we care more about the iterations of T than we do about its fixed point.

(a) Prove by induction that $(T^n v_0)(k) = a_n k$ for all $n \geq 0$ and find a_{n+1} as a function of a_n.

Solution Check the base case $n = 0$: by definition, $T^0 v_0 = v_0$, and $v_0(k) = 0$, so that $a_0 = 0$.

Now suppose that $(T^n v_0)(k) = a_n k$ for some $n \geq 0$, and consider the case for $n+1$: we have

$$\left(T^{n+1} v_0\right) = T\left(T^n v_0\right)$$

$$\left(T^{n+1} v_0\right) = T\left(a_n k\right)$$

Use the definition of T to obtain

$$(Tf)(k) = \max_{(1-\delta)k \leq y \leq (1+\lambda)k} k\phi(y/k) + \beta f(y)$$

$$T(a_n k) = \max_{(1-\delta)k \leq y \leq (1+\lambda)k} k\phi(y/k) + \beta a_n y$$

Suppose y is interior. Then the FOC implies

$$\phi'(y/k) + \beta a_n = 0$$

Since ϕ is strictly concave, we can invert ϕ' to obtain

$$y = k\left(\phi'\right)^{-1}(-\beta a_n)$$

Hence

$$T(a_n k) = \left(\phi\left(\left(\phi'\right)^{-1}(-\beta a_n)\right) + \beta a_n \left(\phi'\right)^{-1}(-\beta a_n)\right) k$$

so that we can define $a_{n+1} = \phi\left(\left(\phi'\right)^{-1}(-\beta a_n)\right) + \beta a_n \left(\phi'\right)^{-1}(-\beta a_n)$. If y is not interior, then

$$T(a_n k) = \left(\phi(1+x) + \beta a_n (1+x)\right) k$$

where $x \in (-\delta, \lambda)$. Hence $a_{n+1} = \phi(1+x) + \beta a_n (1+x)$.

Therefore, if $(T^n v_0)(k) = a_n k$, then $\left(T^{n+1} v_0\right)(k) = a_{n+1} k$. Since it is true for $n = 0$, it is true for all $n \geq 0$.

(b) Suppose that $\beta(1-\delta) > 1$. Show that $a_1 = 1$, and that $a_{n+1} > a_n$ for all n. Derive a condition on β and λ to ensure that the sequence $\{a_n\}$ is bounded.

Solution We have $(Tv_0)(k) = a_1 k$ for some a_1. Using the definition of T, we obtain

$$(Tv_0)(k) = \max_{(1-\delta)k \le y \le (1+\lambda)k} k\phi(y/k)$$

Since ϕ is decreasing, the optimal choice is $y = (1-\delta)k$ so that

$$(Tv_0)(k) = k\phi(1-\delta) = k$$

Hence $a_1 = 1$.

To prove $a_{n+1} > a_n$, suppose that $a_n > 0$ and write

$$a_{n+1}k - a_n k = \max_{(1-\delta)k \le y \le (1+\lambda)k} \{k\phi(y/k) + \beta a_n y\} - a_n k$$

Recall that $\beta(1-\delta) > 1$, and consider a candidate choice $y = (1-\delta)k$,

$$a_{n+1}k - a_n k \ge k\phi(1-\delta) + \beta a_n (1-\delta)k - a_n k$$

$$a_{n+1} - a_n \ge 1 + (\beta(1-\delta) - 1)a_n > 0$$

Therefore if $a_n > 0$, then $a_{n+1} > a_n$. Since $a_1 = 1 > 0$, $a_{n+1} > a_n$ for all n.

To prove boundedness, consider the interior optimality condition

$$y = k(\phi')^{-1}(-\beta a_n)$$

where $(\phi')^{-1}$ is decreasing on $\mathbb{R}_-$. As $n \to \infty$, $(\phi')^{-1}(-\beta a_n)$ gets larger so that there exists an N such that $y = (1+\lambda)k$ is optimal for all $n \ge N$. Hence for such n,

$$a_{n+1}k = k\phi(1+\lambda) + \beta a_n (1+\lambda)k$$

$$a_{n+1} = \beta(1+\lambda)a_n$$

Therefore, we require $\beta(1+\lambda) \le 1$ for the sequence $\{a_n\}$ to be bounded.