Macroeconomic Theory II (Spring 2022)
Assignment 3 Solution

Consider the stochastic neoclassical growth model we studied in class. Assume that $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}, f(z,k,n) = zk^{\alpha}n^{1-\alpha}$ and $\log(z') = \rho\log(z) + \epsilon$ where ϵ is normally distributed with mean zero and variance σ_ϵ^2. Set $\sigma = 2, \rho = 0.95, \sigma_e = 0.01$.

1. Assume a period in the model corresponds to a quarter in the data. Calibrate δ, β and α to match investment-output ratio of 25%, share of labor cost in the production as 2/3, capital-to-quarterly output ratio of 12, and consumption-output ratio of 75%.

Parameter	Moment to match	Value
α	$\frac{\omega N}{Y} = 1-\alpha$	0.33
δ	$1 = (1-\delta) + \frac{I}{K} = (1-\delta) + \frac{I}{Y}\frac{Y}{K}$	0.02
β	$1/\beta = \alpha\frac{Y}{K} + (1-\delta)$	0.993

2. Compute the deterministic steady-state values for capital, consumption, output and investment.

Solve Euler equation:

$$u_c = \beta\mathbb{E}u_c'(z'\alpha k'^{\alpha-1} + 1 - \delta)$$

Solve steady state values:

$$K^* = \left[\frac{1-\beta(1-\delta)}{\alpha\beta}\right]^{\frac{1}{\alpha-1}}$$
$$Y^* = (K^*)^{\alpha}$$
$$C^* = Y^* - \delta K^*, \quad I^* = \delta K^*$$

3. Linearize the model around the steady-state (if you feel more comfortable you can use the log-linearization method). Show your equations.

Log-linearize the following three equations.

$$C^{-\sigma} = \beta\mathbb{E}_{z'|z}C'^{-\sigma}(z'\alpha(K')^{\alpha-1} + 1 - \delta) \tag{1}$$
$$C + K' = zK^{\alpha} + (1-\delta)K \tag{2}$$
$$\log(z') = \rho\log(z) + \epsilon' \tag{3}$$

$y = f(x) \approx f(x^*) + f'(x^*)(x - x^*)$

$\Rightarrow \frac{y - f(x^*)}{y^*} \approx \frac{f'(x^*)x^*}{y^*}\frac{x-x^*}{x^*}$

$\Rightarrow \frac{y-y^*}{y^*} = \hat{y} \approx \frac{f'(x^*)x^*}{y^*}\hat{x} = \frac{d\ln y}{d\ln x}\Big|_{x^*}\hat{x}$

$\frac{d\ln y}{d\ln x}\Big|_{x^*} = \frac{dy^*}{dx^*}\Big/\left(\frac{y^*}{x^*}\right)$

We get:

$$
\begin{aligned}
&-\sigma\hat{c} = -\sigma\mathbb{E}\hat{c}' + \rho\hat{z} + (\alpha-1)\hat{k}' \\
&C^*\hat{c} + K^*\hat{k}' = z^*(K^*)^\alpha(\hat{z} + \alpha\hat{k}) + (1-\delta)K^*\hat{k} \\
&\hat{z}' = \rho\hat{z} + \epsilon'
\end{aligned}
$$

Fit the system of equations into the matrix representation

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha-1 & -\sigma \\ 0 & K^* & 0 \end{bmatrix}
\begin{bmatrix} \hat{z}' \\ \hat{k}' \\ \mathbb{E}\hat{c}' \end{bmatrix}
=
\begin{bmatrix} \rho & 0 & 0 \\ -\rho & 0 & -\sigma \\ z^*(K^*)^\alpha & \alpha z^*(K^*)^\alpha + (1-\delta)K^* & -C^* \end{bmatrix}
\begin{bmatrix} \hat{z} \\ \hat{k} \\ \hat{c} \end{bmatrix}
+
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \epsilon_z'
$$

4. Solve the policy function for capital and consumption using linearized (or log-linearized) model. The solution of linearized model will give you the rules for consumption and capital as deviations from steady-states. To find the policy functions in levels, convert these rules using the definitions for the deviations from steady-state.

Let $\hat{k}' = a_1\hat{k} + a_2\hat{z}$, then

$$
\begin{aligned}
&-\sigma\hat{c} = -\sigma\mathbb{E}\hat{c}' + \rho\hat{z} + (\alpha-1)(a_1\hat{k} + a_2\hat{z}) \\
&C^*\hat{c} + K^*(a_1\hat{k} + a_2\hat{z}) = z^*(K^*)^\alpha(\hat{z} + \alpha\hat{k}) + (1-\delta)K^*\hat{k} \\
&\hat{z}' = \rho\hat{z} + \epsilon'
\end{aligned}
$$

Write $\hat{c}$ and $\mathbb{E}\hat{c'}$ in terms of $\hat{k}$ and $\hat{z}$:

$$
\begin{aligned}
\hat{c} &= \frac{z^*(K^*)^\alpha(\hat{z} + \alpha\hat{k}) + (1-\delta)K^*\hat{k} - K^*(a_1\hat{k} + a_2\hat{z})}{C^*} \\
\hat{c'} &= \frac{z^*(K^*)^\alpha(\hat{z'} + \alpha\hat{k'}) + (1-\delta)K^*\hat{k'} - K^*(a_1\hat{k'} + a_2\hat{z'})}{C^*} \\
\mathbb{E}\hat{c'} &= \frac{z^*(K^*)^\alpha(\rho\hat{z} + \alpha(a_1\hat{k} + a_2\hat{z})) + (1-\delta)K^*(a_1\hat{k} + a_2\hat{z}) - K^*(a_1(a_1\hat{k} + a_2\hat{z}) + a_2\rho\hat{z})}{C^*}
\end{aligned}
$$

Plug it back:

$$
\begin{aligned}
&-\sigma(z^*(K^*)^\alpha(\hat{z} + \alpha\hat{k}) + (1-\delta)K^*\hat{k} - K^*(a_1\hat{k} + a_2\hat{z})) \\
&= -\sigma(z^*(K^*)^\alpha(\rho\hat{z} + \alpha(a_1\hat{k} + a_2\hat{z})) + (1-\delta)K^*(a_1\hat{k} + a_2\hat{z}) - K^*(a_1(a_1\hat{k} + a_2\hat{z}) + a_2\rho\hat{z})) \\
&+ C^*(\rho\hat{z} + (\alpha-1)(a_1\hat{k} + a_2\hat{z}))
\end{aligned}
$$

a_1 and a_2 can be solved as:

$$
\begin{aligned}
-\sigma(z^*(K^*)^\alpha\alpha + (1-\delta)K^* - K^*a_1) &= -\sigma(z^*(K^*)^\alpha(\alpha a_1 + (1-\delta)K^*a_1 - K^*a_1^2) + C^*(\alpha-1)a_1 \\
-\sigma(z^*(K^*)^\alpha - K^*a_2) &= -\sigma(z^*(K^*)^\alpha(\rho + \alpha a_2) + (1-\delta)K^*a_2 - K^*(a_1a_2 + a_2\rho)) \\
&+ C^*(\rho + (\alpha-1)a_2)
\end{aligned}
$$

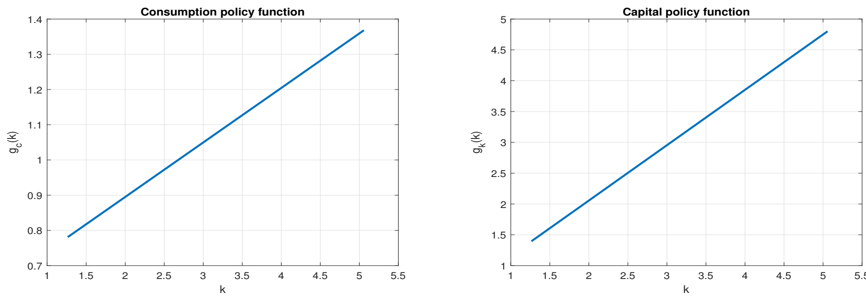

Figure 1: Policy function

5. Solve the model using value function iteration method.

   (a) To do this you need to convert the AR(1) process into Markov process. I uploaded a code which does it for you. The method is called tauchen method. It takes the inputs N, number of grid points for the shock, μ, mean value of the shock, ρ, persistency of the shock,σ, standard deviation of the shock, and m, the maximum distance to the mean in terms of multiples of the standard deviation. You can set $N = 7$ and $m = 3$. The other inputs are the parameters of the model. The output of the code is Z, the discretized values for the shock, $Zprob$, the transition matrix for the shock, where entry (i, j) of the matrix is showing the transition probability from state z_i to state z_j. Be careful that the levels you obtained using the matlab code, Z, are the log values of the shock. To convert them into actual levels, take the exponent of Z.

   (b) When discreizing the capital, center the grid points around k^*, steady-state capital level, and set the lower bound of capital $\underline{k} = (1 - \kappa)k^*$ and upper bound to $\overline{k} = (1 + \kappa)k^*$. You can start by using $N_k = 200$ grid points for k and set $\kappa = 0.6$.

(c) Solve the model using value function iteration method. Compute the associated policy functions and value functions. Plot them against capital for different values of the shock.

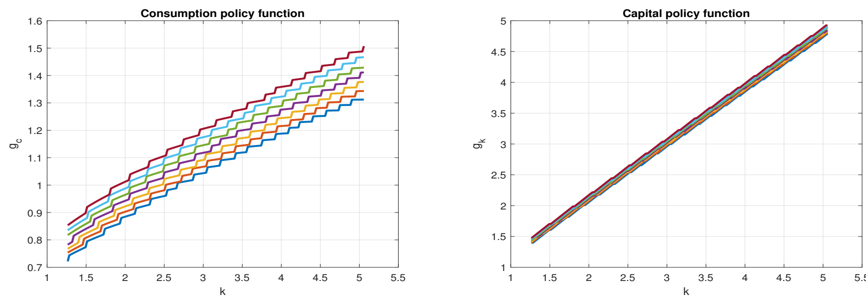

Figure 2: Policy function

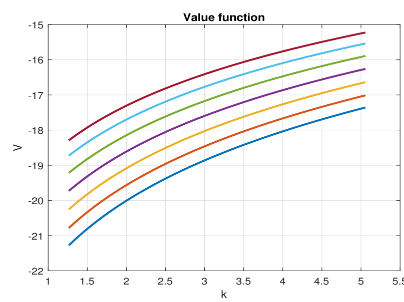

Figure 3: Value function

(d) Report the robustness of your results with respect to N_k and κ. Try two other values, say $N_k \in \{100, 500\}$ and $\kappa \in \{0.3, 0.8\}$ and plot the policy function for capital for the mean value of the shock.

Let's see the N_k case at first. $N_k = 500$ uses finer grid than $N_k = 100$.

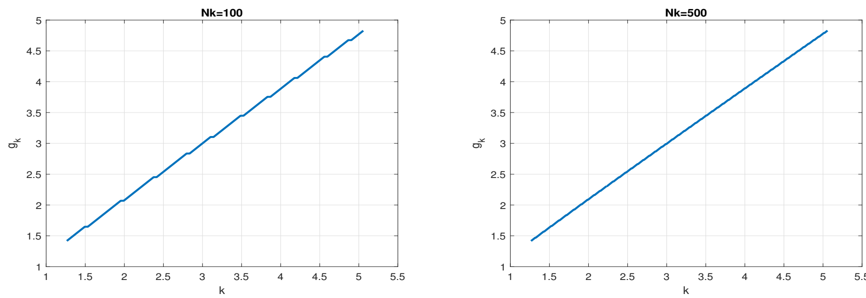

Figure 4: N_k comparison

If we use fine grids, we can obtain more robust policy function. However, it takes more time to compute the policy function. Now let's look at κ case. When κ increases, the gap between lower bound and upper bound becomes bigger. This will become coarse grids because N_k is same in this case. Therefore, we can obtain more robust policy function in $\kappa = 0.3$ case.

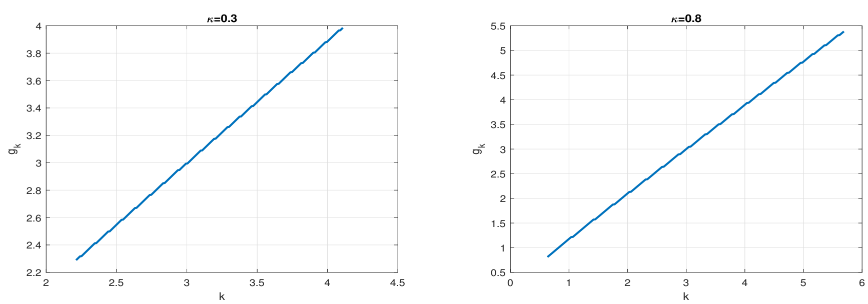

Figure 5: κ comparison

(e) Compare the policy function for capital obtained using linearized model and value function iteration method. Discuss the goodness of approximation in the linearized model.

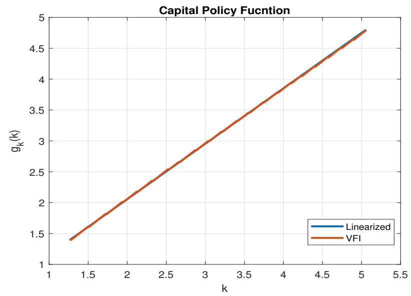

Figure 6: Linearized and VFI

If you compare the both methods, you can know that the results are very similar. Therefore, the fit of the linearized model is pretty good.

(f) Do the same comparison of policy functions for $\sigma = 5$ and $\sigma = 10$. Discuss the sensitivity of approximation with respect to σ.

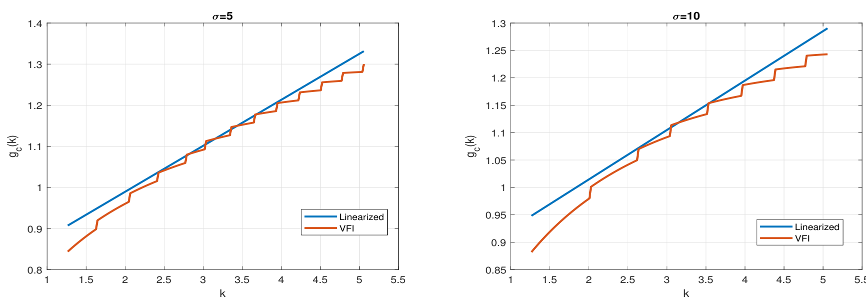

Figure 7: σ comparison

I used the consumption policy function instead of the capital policy function because we can recognize the difference better. As you can see, the approximation grows worse as σ gets bigger. If the utility function is log function, we can obtain the following policy function for consumption.

$$c_t = (1 - \alpha\beta)k_t^{\alpha}$$

The consumption function does not include σ parameter. Therefore, the gap between linearized model and VFI model are small. However, when σ gets bigger, the policy function becomes more concave, so we can get worse results. But we can know that the fit is good near the steady state.