DMET 502/701 Computer Graphics

2D Transformations

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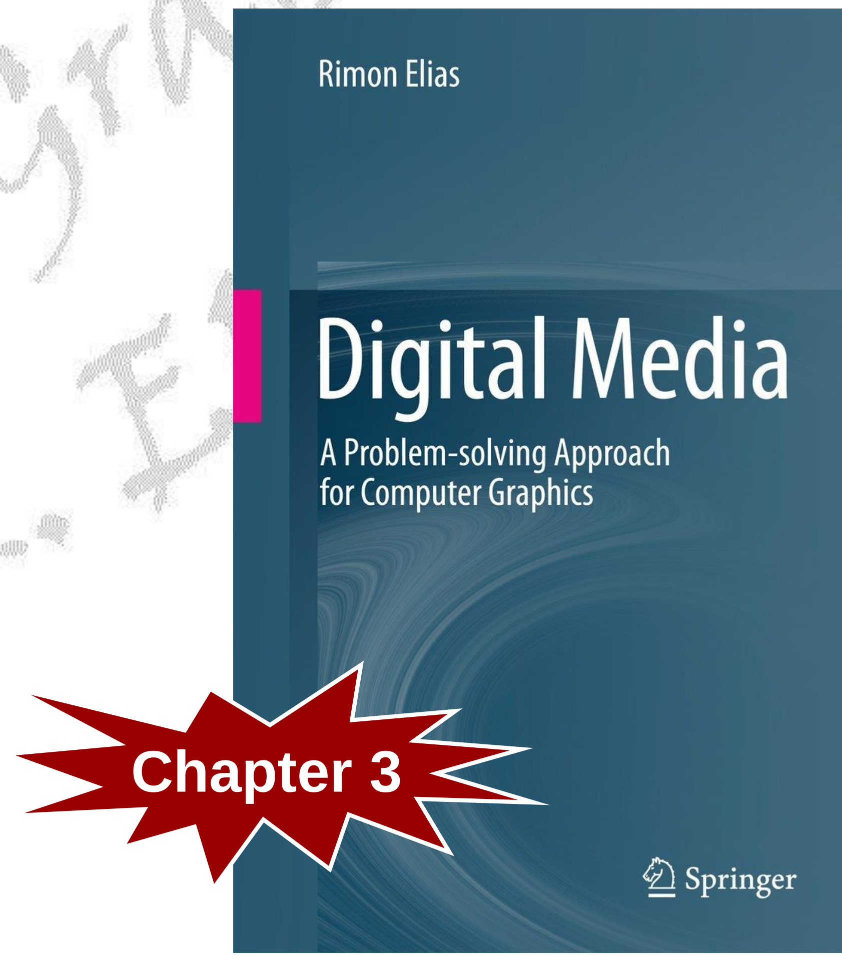


2D Transformations

- Transformation operations in 2D space are a set of geometric operations that can be applied to 2D planar objects/shapes.

- The aim is to alter the status of objects in space.

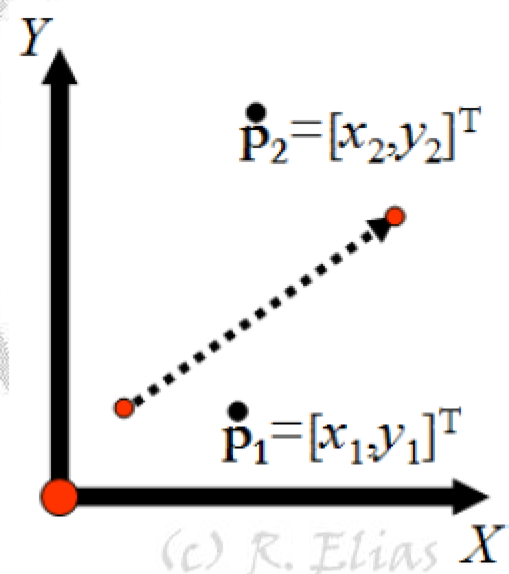
- We will be concerned with object/shape vertices.

- The main 2D transformation operations:
  - Translation
  - Rotation
  - Scaling
  - Reflection
  - Shearing

2D Translation

- The translation operation in 2D space is performed when a 2D point is moved or *translated* from a position $[x_1, y_1]^T$ to another position $[x_2, y_2]^T$.

- The magnitude and direction of translation are characterized by the *translation vector* $\mathbf{t} = [t_x, t_y]^T = [x_2 - x_1, y_2 - y_1]^T$.



Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\mathbf{t}}$$

translation vector

2D Translation

- The same translation operation can be performed on homogeneous points.

- A homogeneous point at $\mathbf{p_1}=[x_1, y_1, 1]^T$ is translated to another position $\mathbf{p_2}= [x_2, y_2, 1]^T$ using

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\mathrm{T}([t_x, t_y]^T)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

translation matrix

2D Translation: An Example

- **Example:** Determine the location of the point $\dot{\mathbf{a}}_1$ = [1, 0]^T after translating it using the translation vector **t** = [2, 3]^T.

- **Answer:**

Inhomogeneous coordinates

$$\dot{\mathbf{a}}_2 = \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\dot{\mathbf{a}}_1} + \underbrace{\begin{bmatrix}2\\3\end{bmatrix}}_{\mathbf{t}} = \underbrace{\begin{bmatrix}3\\3\end{bmatrix}}_{\dot{\mathbf{a}}_2}$$

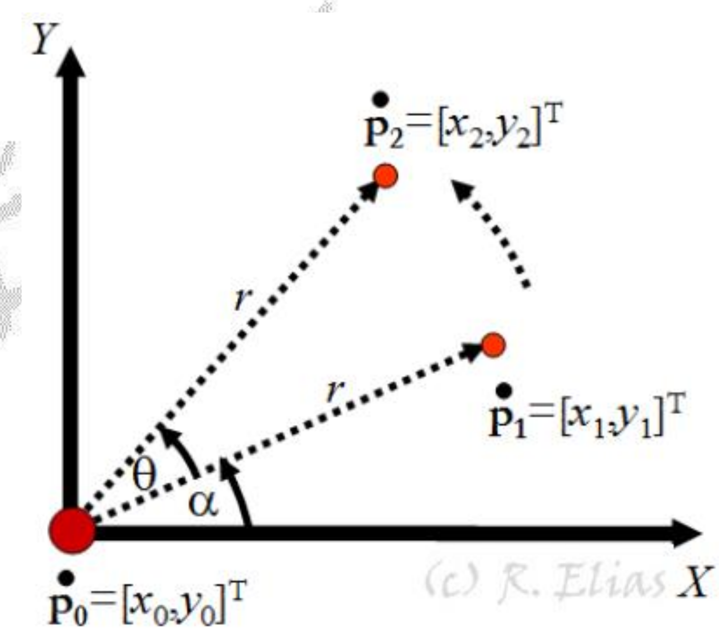
Homogeneous coordinates

$$\mathbf{a}_2 = \underbrace{\begin{bmatrix}1 & 0 & 2\\0 & 1 & 3\\0 & 0 & 1\end{bmatrix}}_{\mathtt{T}([2,3]^T)} \underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\mathbf{a}_1} = \underbrace{\begin{bmatrix}3\\3\\1\end{bmatrix}}_{\mathbf{a}_2}$$

2D Rotation

- A point can be rotated in 2D space if the center of rotation (or the *pivot*), is specified as well as the angle of rotation.

- In the figure, a point $\dot{\mathbf{p}}_1 = [x_1, y_1]^T$ is rotated about the origin through an angle θ to reach another point $\dot{\mathbf{p}}_2 = [x_2, y_2]^T$.



- Notice that

$$\dot{\mathbf{p}}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} r\cos(\alpha) \\ r\sin(\alpha) \end{bmatrix}$$

$$\dot{\mathbf{p}}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix}$$

2D Rotation

$$\begin{aligned}\sin(\alpha+\theta) &= \sin(\alpha)\cos(\theta)+\cos(\alpha)\sin(\theta)\\ \cos(\alpha+\theta) &= \cos(\alpha)\cos(\theta)-\sin(\alpha)\sin(\theta)\end{aligned}$$

- Thus,

$$\begin{aligned}
\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} &= \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix} \\
&= \begin{bmatrix} r\left(\cos(\alpha)\cos(\theta)-\sin(\alpha)\sin(\theta)\right) \\ r\left(\sin(\alpha)\cos(\theta)+\cos(\alpha)\sin(\theta)\right) \end{bmatrix} \\
&= \begin{bmatrix} \underbrace{r\cos(\alpha)}_{x_1}\cos(\theta)-\underbrace{r\sin(\alpha)}_{y_1}\sin(\theta) \\ \underbrace{r\sin(\alpha)}_{y_1}\cos(\theta)+\underbrace{r\cos(\alpha)}_{x_1}\sin(\theta) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\dot{\mathbf{R}}(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1}
\end{aligned}$$

Inhomogeneous coordinates

Rotation matrix

2D Rotation

- The same operation is performed on homogeneous points as

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

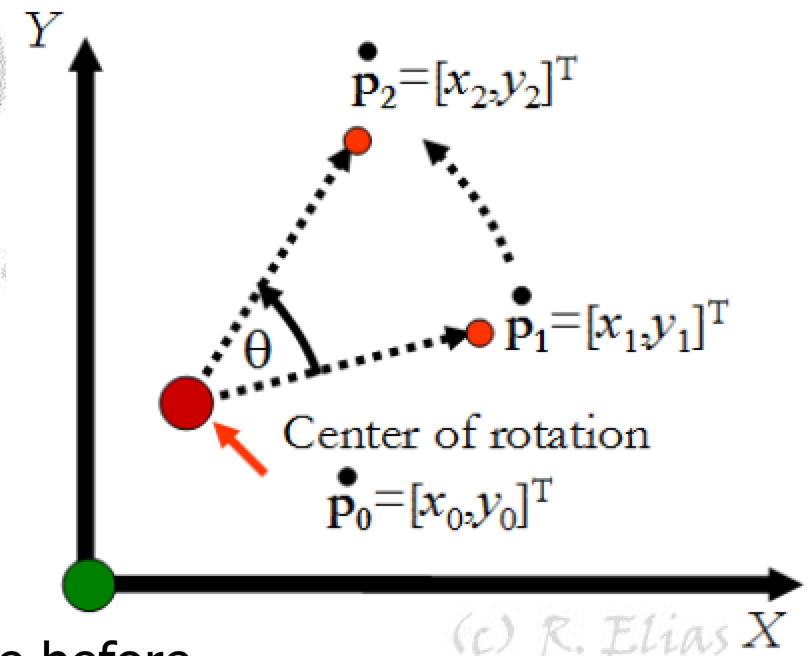
Rotation matrix for homogeneous points

Notice that both rotation matrices assume that the center of rotation is the origin.

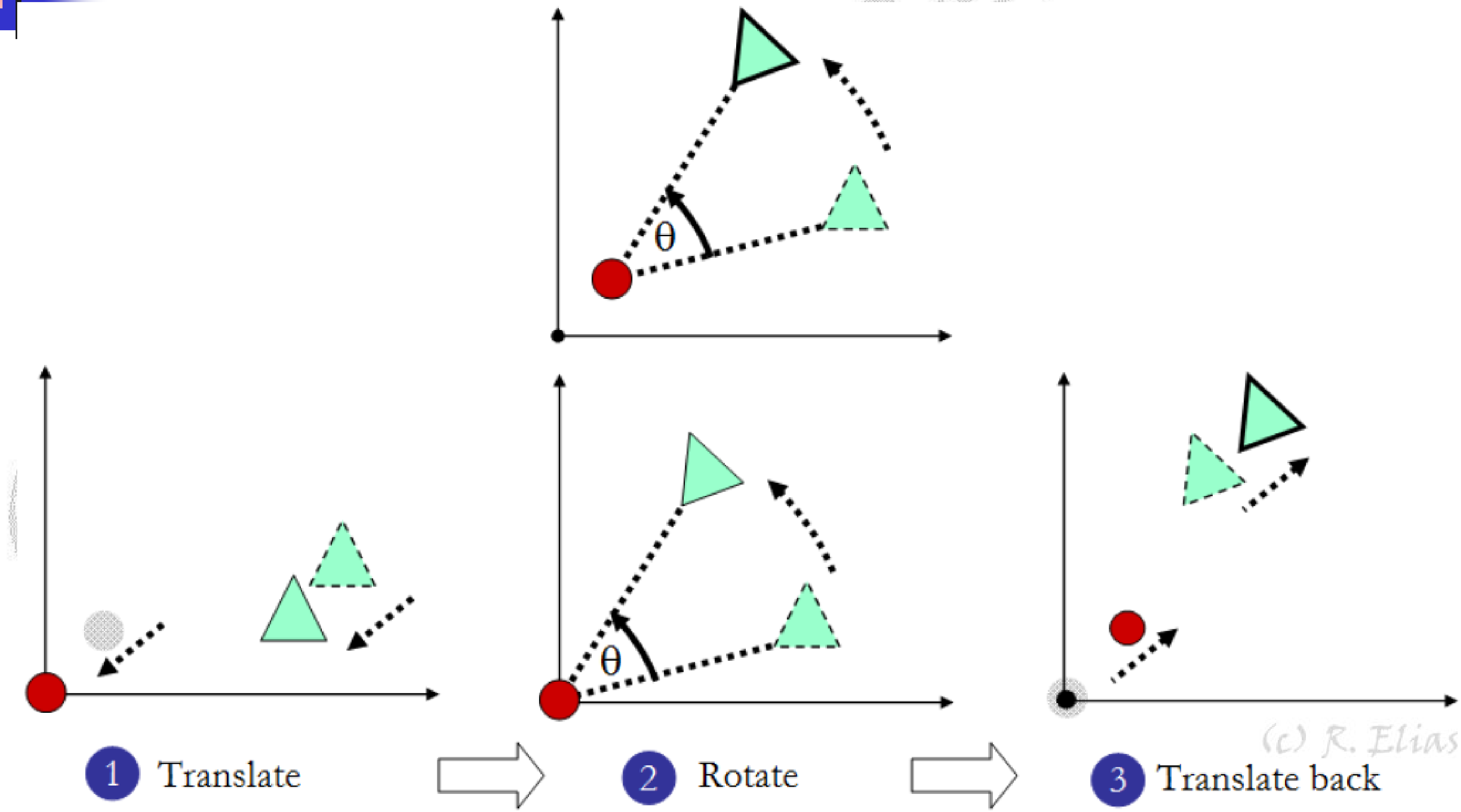
2D Rotation: General Case

- Most of the time, rotation operations are performed about a general point rather than the origin.

- Hence, if a point $\dot{\mathrm{p}}_1$ is rotated about another point $\dot{\mathrm{p}}_0$ to get $\dot{\mathrm{p}}_2$, the following three steps should be performed:
  1. Translate $\dot{\mathrm{p}}_0$ and $\dot{\mathrm{p}}_1$ by a translation vector $\mathbf{t}$ that moves $\dot{\mathrm{p}}_0$ to the origin (i.e., $\mathbf{t} = [-x_0, -y_0]^T$).
  2. Rotate about the origin as done before.
  3. Translate back using the vector $-\mathbf{t}$ or $[x_0, y_0]^T$.



2D Rotation--General Case: An Example



2D Rotation: General Case

- The translation-rotation-translation process can be expressed as

Point to rotate

Center of rotation

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathrm{p}}_2} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\dot{\mathrm{R}}(\theta)} \underbrace{\begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}}_{\dot{\mathrm{p}}_1 - \dot{\mathrm{p}}_0} + \underbrace{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\dot{\mathrm{p}}_0}$$

2 Rotate

1 Translate

3 Translate back

Inhomogeneous coordinates

2D Rotation: General Case

- The same operation is performed on homogeneous points as

3 Translate back   2 Rotate   1 Translate

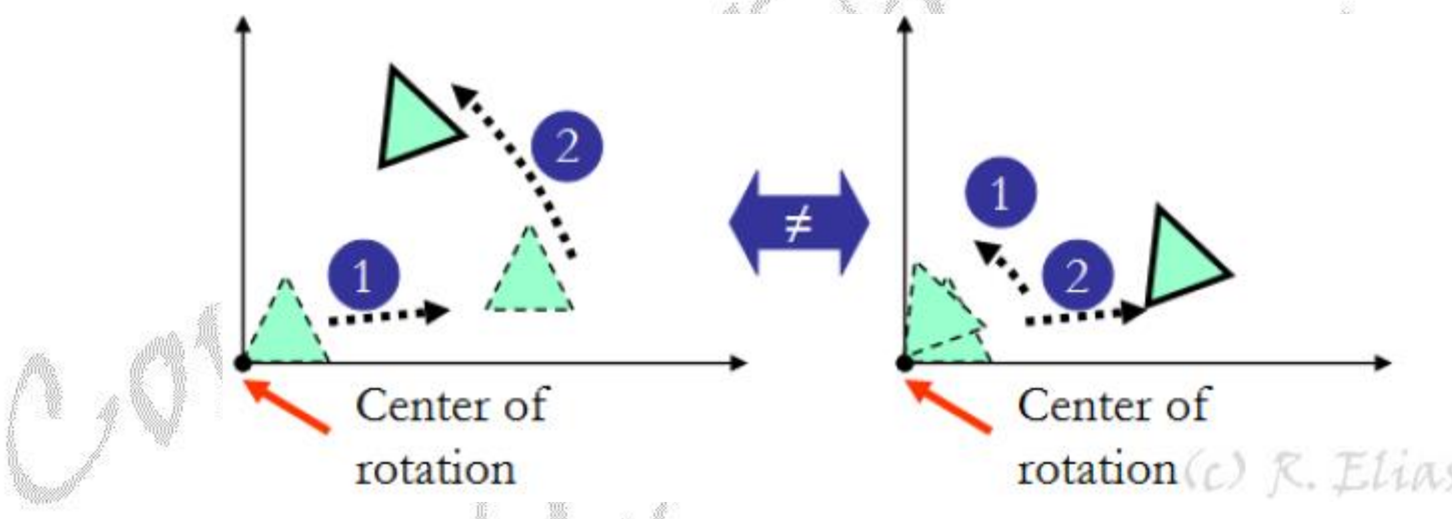
$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{T}([x_0,y_0]^T)} \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{R}(\theta)} \underbrace{\begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{T}([-x_0,-y_0]^T)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

$$= \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & -x_0\cos(\theta) + y_0\sin(\theta) + x_0 \\ \sin(\theta) & \cos(\theta) & -x_0\sin(\theta) - y_0\cos(\theta) + y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{T}([x_0,y_0]^T)\mathtt{R}(\theta)\mathtt{T}([-x_0,-y_0]^T)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

Homogeneous coordinates

2D Rotation: Order of Multiplication

- Transformations are **not** commutative. This means that the order of performing the transformation operations is important.



Translation followed by rotation is not like rotation followed by translation.

2D Rotation: An Example

- **Example:** Derive the overall transformation matrix if an object is translated by a vector $[t_x, t_y]^T$ and then rotated about the origin through an angle θ. Re-estimate the matrix if the order of performing operations is reversed.
- **Answer:**

Rotation → translation

$$\mathrm{TR} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) & t_x \\ \sin(\theta) & \cos(\theta) & t_y \\ 0 & 0 & 1 \end{bmatrix}.$$

$\neq$

$$\mathrm{RT} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) & t_x\cos(\theta) - t_y\sin(\theta) \\ \sin(\theta) & \cos(\theta) & t_x\sin(\theta) + t_y\cos(\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

Translation → rotation

2D Scaling

- In order to scale (i.e., enlarge or shrink) an object in 2D space by a factor s along both directions, the positions of its vertices $[x_i, y_i]^T$ are multiplied by this scaling factor to get $[s\,x_i, s\,y_i]^T$.

- Hence, to scale a point $[x_1, y_1]^T$ using scaling factors s_x and s_y to get $[x_2, y_2]^T$, we may use

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathrm{p}}_2} = \underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\dot{\mathrm{S}}(s_x, s_u)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathrm{p}}_1}$$

Scaling matrix

- Scaling is **uniform** when $s_x = s_y$.
- Scaling is **non-uniform** when $s_x \neq s_y$.

2D Scaling

- The same operation is performed on homogeneous points as

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}(s_x,s_y)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

Scaling matrix

Notice that both scaling matrices perform the operation with respect to the origin (i.e., the fixed point).

2D Scaling: General Case

- Scaling operations may performed with respect to a general fixed point.
- As done with general rotation, the following three steps should be performed in case of general scaling:
  1. Translate so that the fixed point coincides with the origin.
  2. Scale as done before.
  3. Translate back.

Fixed point

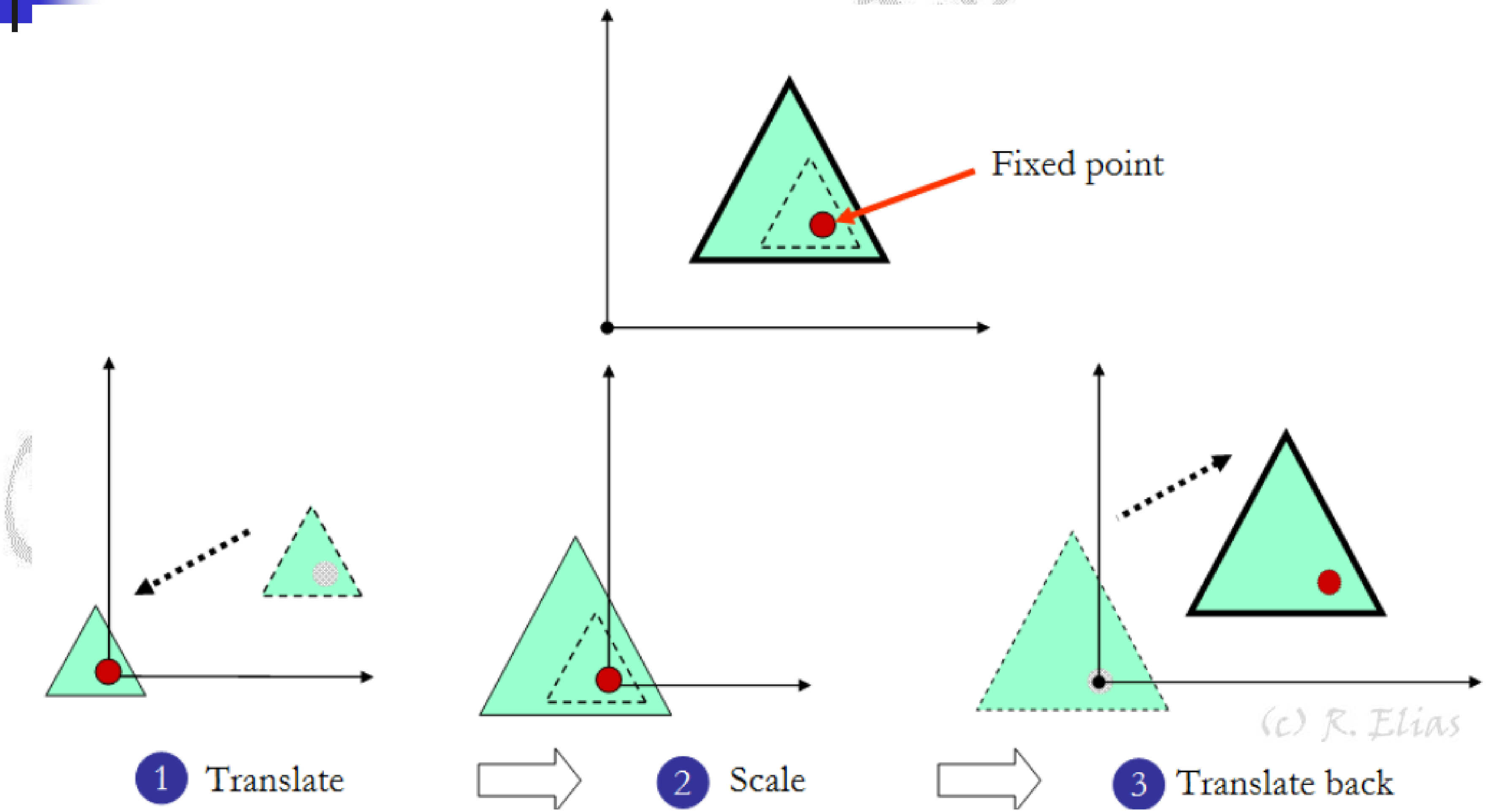
Point to scale

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathrm{p}}_2} = \underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\dot{\mathrm{S}}(s_x, s_y)} \underbrace{\begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}}_{\dot{\mathrm{p}}_1 - \dot{\mathrm{p}}_0} + \underbrace{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\dot{\mathrm{p}}_0}$$

2 Scale 1 Translate 3 Translate back

2D Scaling--General Case: An Example



2D Scaling: General Case

- The same operation is performed on homogeneous points as

3 Translate back   2 Scale   1 Translate

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{T}([x_0,y_0]^T)} \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{S}(s_x,s_y)} \underbrace{\begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{T}([-x_0,-y_0]^T)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

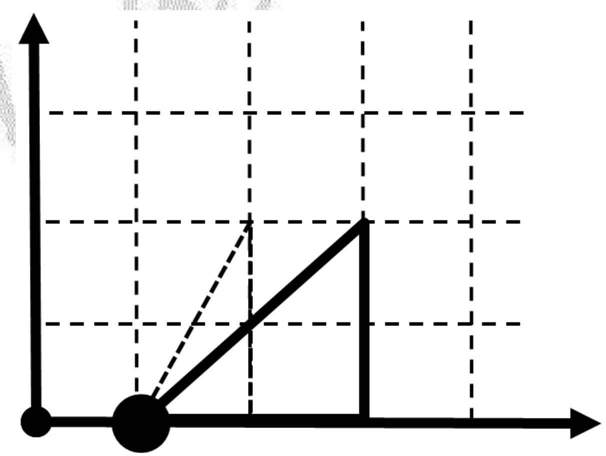
$$\underbrace{\begin{bmatrix} s_x & 0 & -x_0 s_x + x_0 \\ 0 & s_y & -y_0 s_y + y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{T}([x_0,y_0]^T)\mathtt{S}(s_x,s_y)\mathtt{T}([-x_0,-y_0]^T)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

Homogeneous coordinates

2D Scaling: An Example

- **Example:** Derive a single matrix that transforms a triangle bounded by $[1, 0]^T$, $[2, 0]^T$ and $[2, 2]^T$ to another triangle bounded by $[1, 0]^T$, $[3, 0]^T$ and $[3, 2]^T$.

- **Answer:** Sketch it for better visualization.



1. Translate so that the fixed point is at the origin $[0, 0]^T$.
2. Scale the triangle using a factor of 2 in the *x*-direction.
3. Translate back.

2D Scaling: An Example

1 Translate

$$M_1 = T([-1,0]^T) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2 Scale

$$M_2 = S(2,1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 Translate back

$$M_3 = T([1,0]^T) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The overall transformation

$$\begin{aligned} M &= M_3 M_2 M_1 \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

2D Reflection

- The reflection operation in 2D space mirrors an object about an axis.
- There are two basic reflection operations:
  1. About the x-axis.
  2. About the y-axis.

2D Reflection: About the *x*-axis

- This operation flips the *y*-values.

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\dot{\mathrm{Ref}}_x} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1}$$

$= \dot{\mathrm{S}}(1,-1)$

Reflection matrices

$= \mathrm{S}(1,-1)$

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathrm{Ref}_x} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

2D Reflection: About the *y*-axis

- This operation flips the *x*-values.

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix}x_2\\y_2\end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix}-1&0\\0&1\end{bmatrix}}_{\dot{\mathtt{R}}\mathtt{ef}_y}\underbrace{\begin{bmatrix}x_1\\y_1\end{bmatrix}}_{\dot{\mathbf{p}}_1}$$

$=\dot{\mathtt{S}}(-1,1)$

Reflection matrices

$=\mathtt{S}(-1,1)$

Homogeneous coordinates

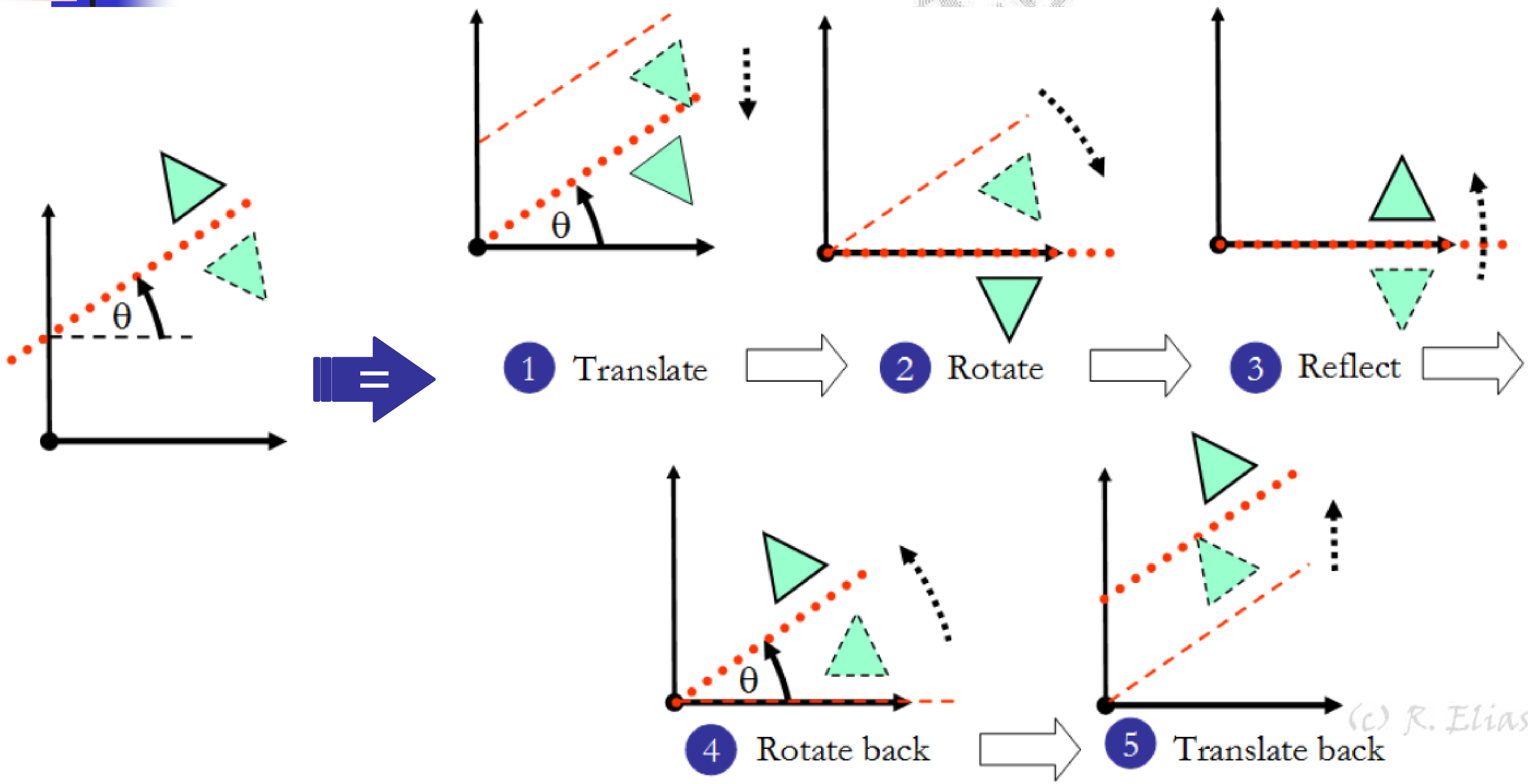
$$\underbrace{\begin{bmatrix}x_2\\y_2\\1\end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{\mathtt{Ref}_y}\underbrace{\begin{bmatrix}x_1\\y_1\\1\end{bmatrix}}_{\mathbf{p}_1}$$

2D Reflection: General Case

- In the general case, the reflection operation can be performed about any arbitrary axis.

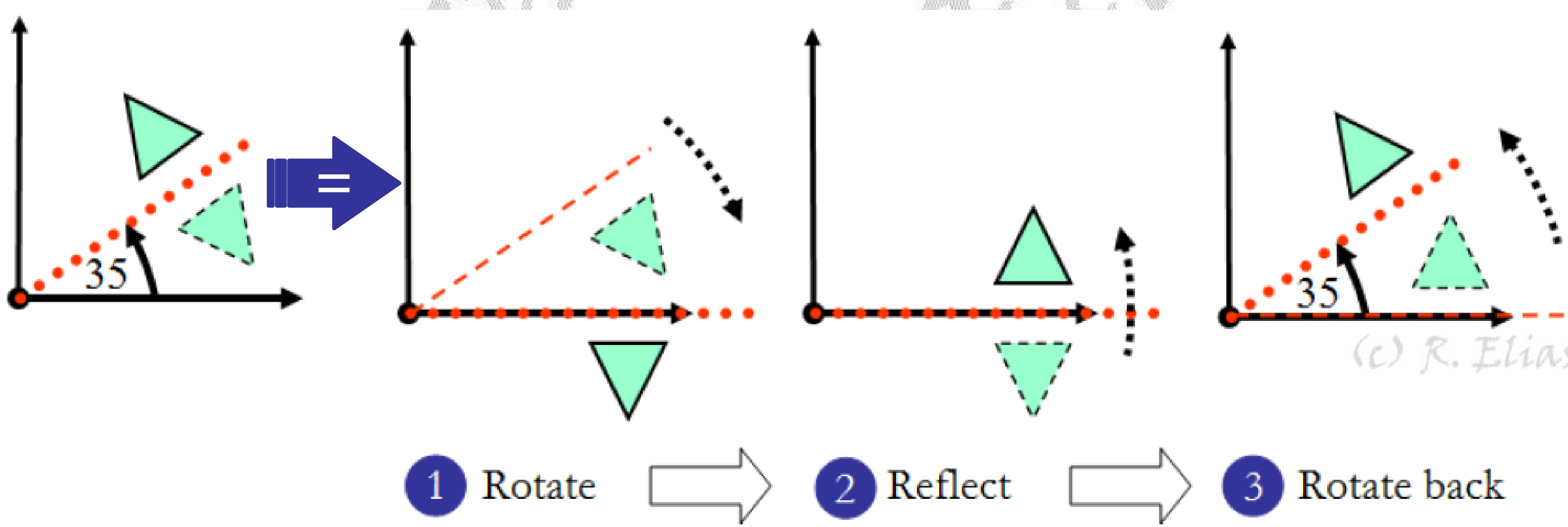
- We may perform the following steps in case of general reflection:
  1. Translate so that the reflection axis passes through the origin.
  2. Rotate so that the reflection axis gets aligned with one of the coordinate axes.
  3. Reflect about that coordinates axis as done before.
  4. Rotate back through the same angle of Step 2 in the opposite direction.
  5. Translate back using the same vector of Step 1 in the opposite direction.

2D Reflection--General Case: An Example



2D Reflection: An Example

- **Example:** Derive the transformation matrix to reflect an object about a line passing through the origin with an inclination angle of 35°.
- **Answer:** Sketch it for better visualization. No translation is needed.



2D Reflection: An Example

- The steps:
  1. Rotate the object through an angle of -35° about the origin.
  2. Reflect the object about the x-axis.
  3. Rotate the object back through an angle of 35° about the origin.

2D Reflection: An Example

1 Rotate

$$M_1 = R(-35) = \begin{bmatrix} \cos(-35) & -\sin(-35) & 0 \\ \sin(-35) & \cos(-35) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2 Reflect

$$M_2 = \mathtt{Ref}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 Rotate back

$$M_3 = R(35) = \begin{bmatrix} \cos(35) & -\sin(35) & 0 \\ \sin(35) & \cos(35) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The overall transformation

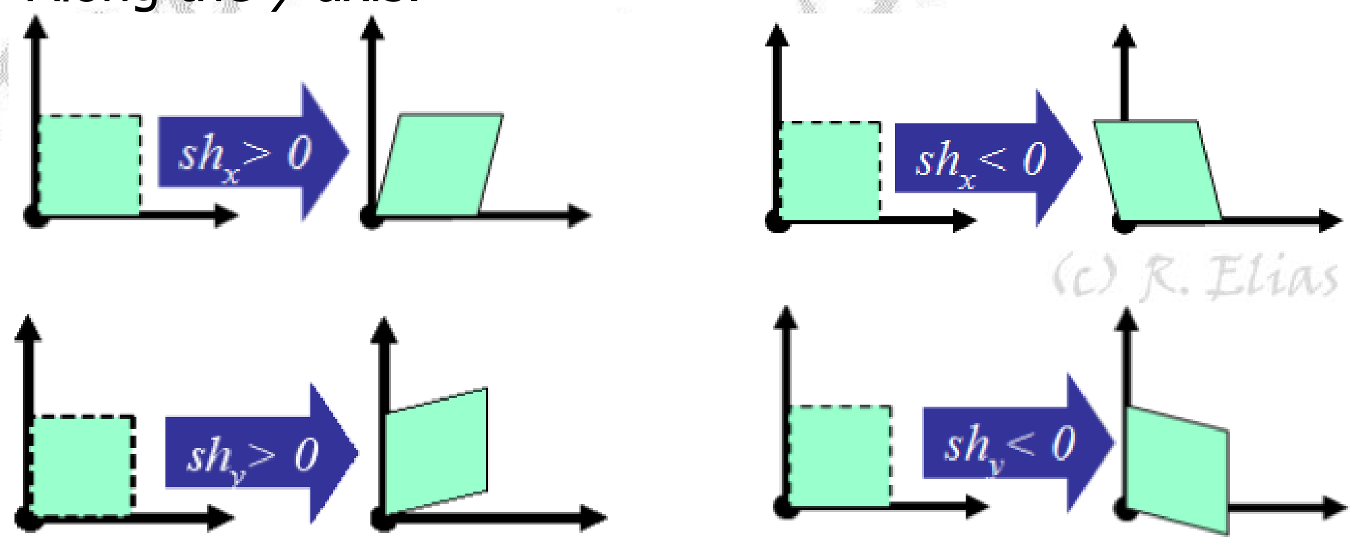
$$M = M_3 M_2 M_1$$

$$= \begin{bmatrix} \cos(35) & -\sin(35) & 0 \\ \sin(35) & \cos(35) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-35) & -\sin(-35) & 0 \\ \sin(-35) & \cos(-35) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.3420 & 0.9397 & 0 \\ 0.9397 & -0.3420 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2D Shearing

- The 2D shearing operation results in shape distortion of the object by stretching the object with respect to an axis.

- There are two basic shearing operations:
  1. Along the *x*-axis.
  2. Along the *y*-axis.



2D Shearing: Along the *x*-axis

- This operation alters the *x*-values.

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} 1 & sh_x \\ 0 & 1 \end{bmatrix}}_{\dot{\mathbf{S}}\mathbf{h}_x(sh_x)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1}$$

sh_x is a factor

Shearing matrices

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}\mathbf{h}_x(sh_x)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

2D Shearing: Along the *y*-axis

- This operation alters the *y*-values.

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} 1 & 0 \\ sh_y & 1 \end{bmatrix}}_{\dot{\mathbf{S}}\mathbf{h}_y(sh_y)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1}$$

sh_y is a factor

Shearing matrices

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}\mathbf{h}_y(sh_y)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

2D Shearing: General Case

- In the general case, the shearing operation can be performed along any arbitrary axis.

- We may perform the following steps in case of general shearing:
  1. Translate the object so that the arbitrary axis passes through the origin.
  2. Rotate so that the axis gets aligned with one of the coordinate axes.
  3. Shear along that coordinate axis as done before.
  4. Rotate back through the same angle in the opposite direction.
  5. Translate back using the same vector in the opposite direction.

Composite Transformations

- A sequence of transformations of any type can be defined as a sequence of matrices; M_1, M_2, ..., M_n.

p_1 is transformed by M_1 to get **p_2**.

p_2 is transformed by M_2 to get **p_3**.

$$
\begin{aligned}
p_2 &= M_1 p_1, \\
p_3 &= M_2 \underbrace{p_2}_{M_1 p_1} &= M_2 M_1 p_1, \\
p_4 &= M_3 \underbrace{p_3}_{M_2 p_2} &= M_3 M_2 M_1 p_1, \\
\ldots &= \ldots\ldots
\end{aligned}
$$

- The general equation to transform a point $\mathbf{p}_1$ to another point $\mathbf{p}_2$ using a series of *n* transformation operations can be expressed as

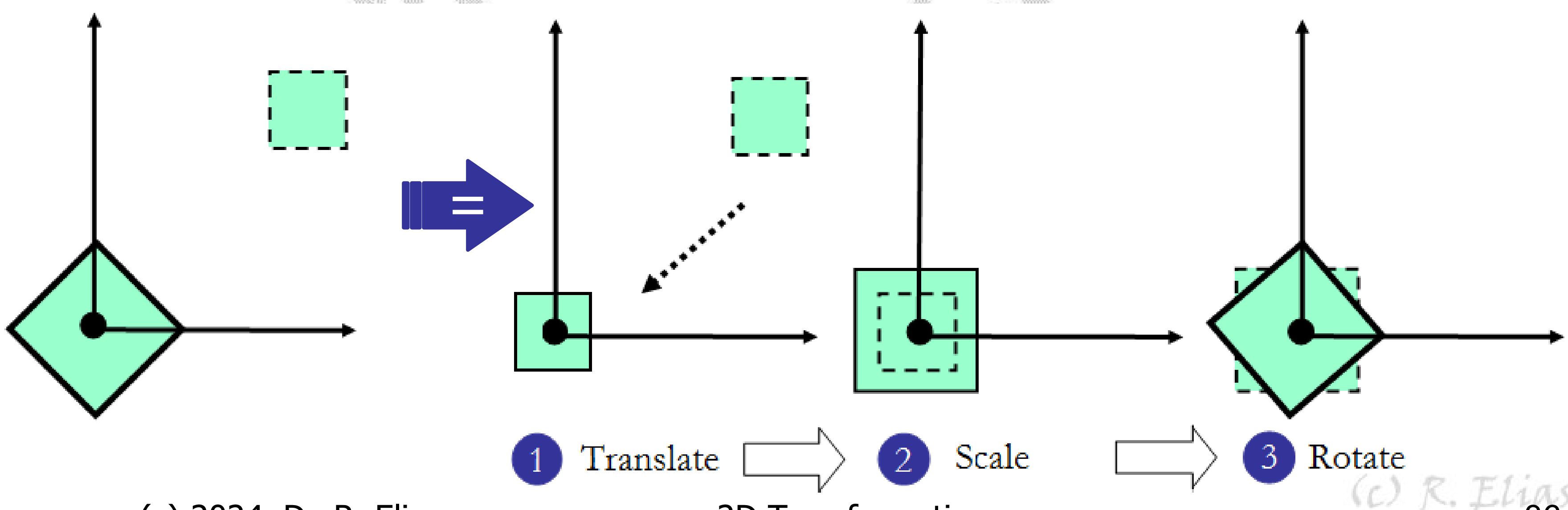
$$\boxed{\mathbf{p}_2 = M_n \ldots M_3 M_2 M_1 \mathbf{p}_1}$$

Notice that the order of multiplication is reversed.

Composite Transformations: An Example

- **Example:** Consider a unit square centered at the point $[5, 5]^T$ with sides parallel to the two coordinate axes. Find the 2D transformation matrix that transforms that square into another square whose vertices are at $[1, 0]^T$, $[0, 1]^T$, $[-1, 0]^T$ and $[0, -1]^T$.

- **Answer:** Sketch it for better visualization.



Composite Transformations: An Example

- The steps:
  1. Translate the square using a translation vector of $[-5, -5]^T$ so that its center coincides with the origin.
  2. Scale the square by a scaling factor of $\sqrt{2}$.
  3. Rotate the square through an angle of 45° about the origin.

Composite Transformations: An Example

1 Translate

$$\mathtt{M}_1 = \mathtt{T}([-5,-5]^T) = \begin{bmatrix} 1 & 0 & -5 \\ 1 & 0 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

2 Scale

$$\mathtt{M}_2 = \mathtt{S}\left(\sqrt{2},\sqrt{2}\right) = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 Rotate

$$\mathtt{M}_3 = \mathtt{R}(45) = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The overall transformation

$$\mathtt{M} = \mathtt{M}_3\mathtt{M}_2\mathtt{M}_1$$

$$= \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 1 & 0 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix}$$

Axes Transformations

- All the previous transformation operations discussed above are applied to objects while fixing the coordinate axes.

- Similarly, the coordinate axes may be transformed while fixing the objects in 2D space.

- In this case, although the objects in space are not transformed; however, their vertex coordinates get affected by axes transformation.

Axes Translation

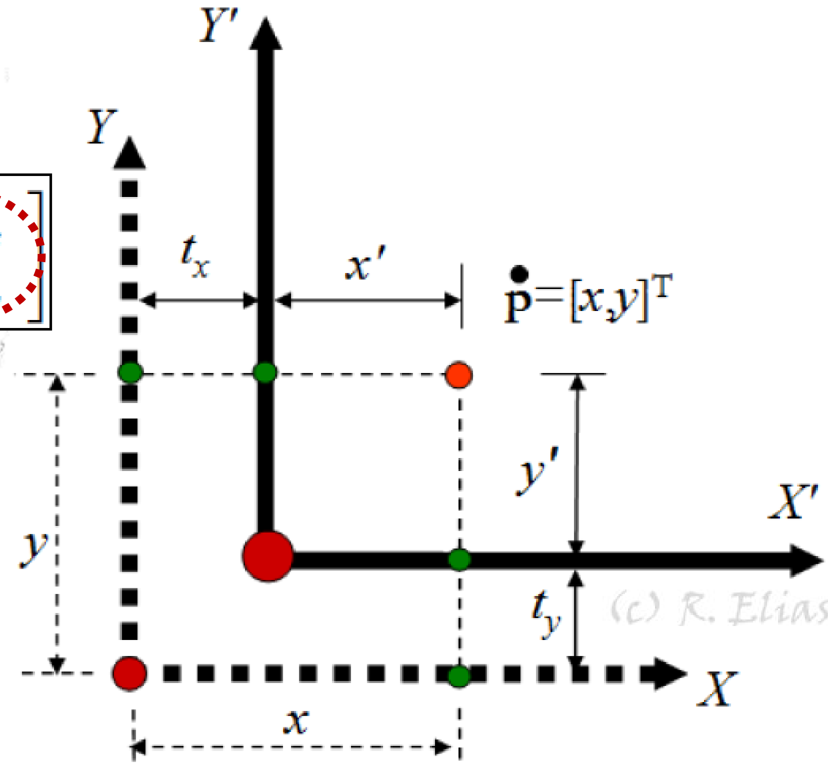
- When the x- and y-axes are translated to the x'- and y'-axes using a vector $[t_x, t_y]^T$, a point $[x, y]^T$ will have new coordinates $[x', y']^T$ expressed as

Inhomogeneous coordinates

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -t_x \\ -t_y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous coordinates



Axes Rotation

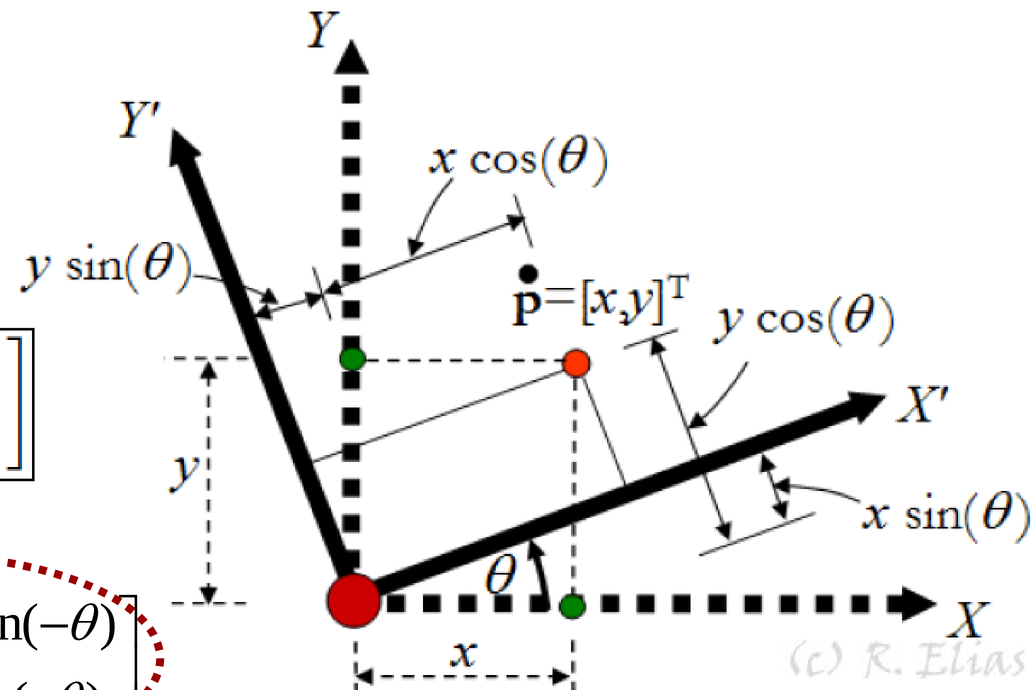
- When the *x*- and *y*-axes are rotated through an angle θ to the x'- and y'-axes, a point $[x, y]^T$ will have new coordinates $[x', y']^T$ expressed as

$$
\begin{aligned}
x' &= x\cos(\theta) + y\sin(\theta) \\
y' &= y\cos(\theta) - x\sin(\theta)
\end{aligned}
$$

$$
\boxed{\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}
$$

$$
\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}
$$

Compare with matrix on Slide 8.



Axes Rotation

Inhomogeneous coordinates

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Axes rotation matrices

Homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Axes Scaling

- The axes scaling process affects the units of the coordinate system.
- When the *x*- and *y*-axes are scaled to the *x*′- and *y*′-axes, a point $[x, y]^T$ will have new coordinates $[x', y']^T$ expressed as

Inhomogeneous coordinates

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{s_x} & 0 \\ 0 & \frac{1}{s_y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Axes scaling matrices

Homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Axes Reflection

- The 2D axes may be reflected about the *x*-axis where *y*-coordinates are affected.

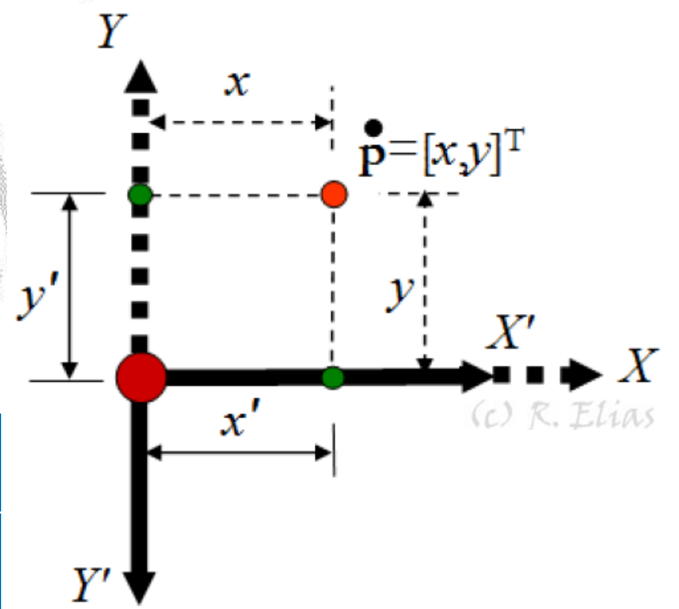
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Inhomogeneous coordinates

Axes reflection matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous coordinates



Axes Reflection

- The 2D axes may be reflected about the *y*-axis where *x*-coordinates are affected.

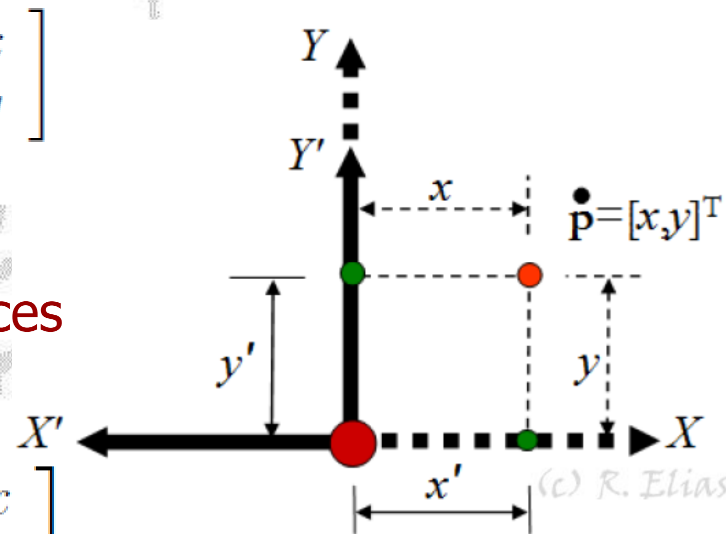
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Inhomogeneous coordinates

Axes reflection matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous coordinates



Summary

- 2D transformation operations
  - Translation
  - Rotation
  - Scaling
  - Reflection
  - Shearing
  - Composite transformations
  - Axes transformations