CSEN 703 - Analysis and Design of Algorithms

Lecture 4 - Divide and Conquer II

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In the Previous Lecture

- We looked into designing problems using the D&C strategy.
- We learned how to write recurrences to represent the running time of D&C algorithms.
- We learned about solving recurrences using the recursion tree method.

Outline

1. The Master Method
2. Recap

Solving Recurrences

The Master Method

The Master Theorem

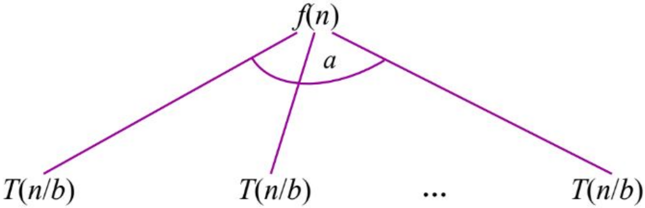
A cookbook for solving recurrences of the form

$$T(n) = aT(\frac{n}{b}) + f(n)$$

where $a \geq 1$, $b > 1$, and $f(n)$ is an asymptotically positive function.

Intuition

$$T(n) = aT(\frac{n}{b}) + f(n)$$



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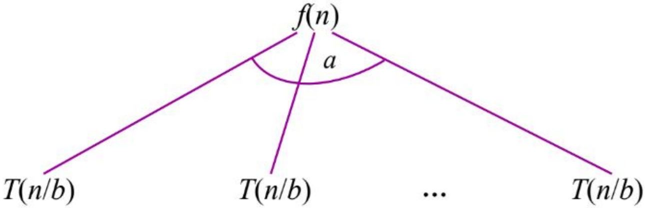

- To open up the next round of recurrences, we substitute $\frac{n}{b}$ in our recurrence.

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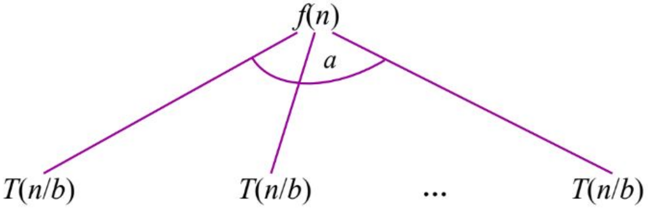

- To open up the next round of recurrences, we substitute $\frac{n}{b}$ in our recurrence.
- We get $T(\frac{n}{b}) = aT(\frac{\frac{n}{b}}{b}) + f(\frac{n}{b}) = aT(\frac{n}{b^2}) + f(\frac{n}{b})$

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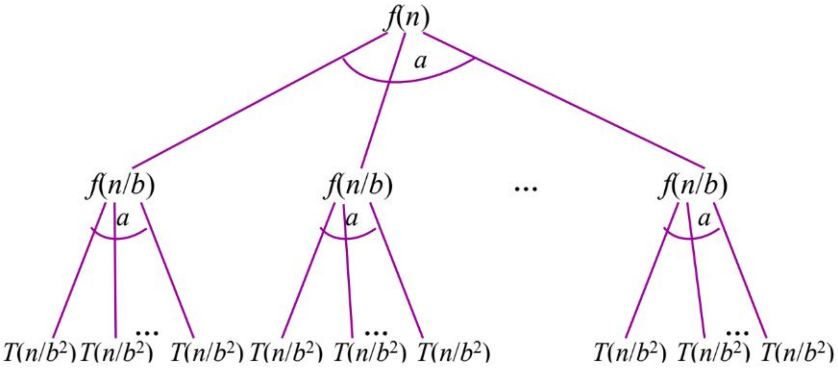

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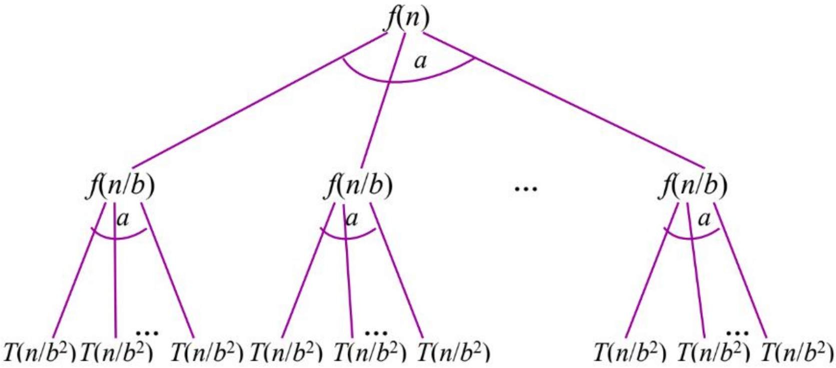

We can continue expanding the recursion tree until the sub-problems bottom out, that is, they reduce to a size of 1.

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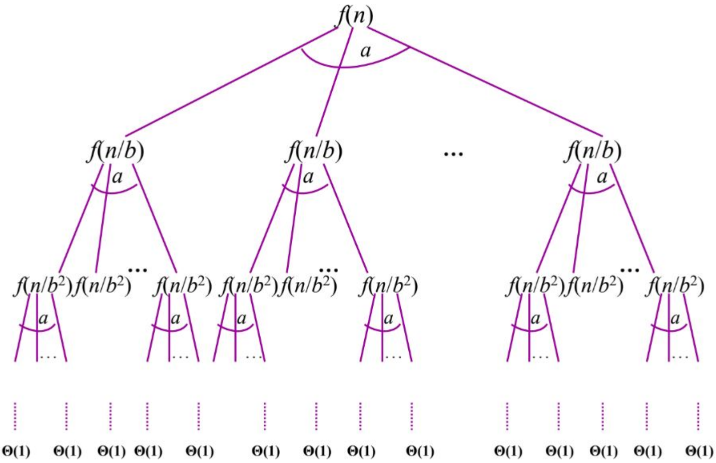

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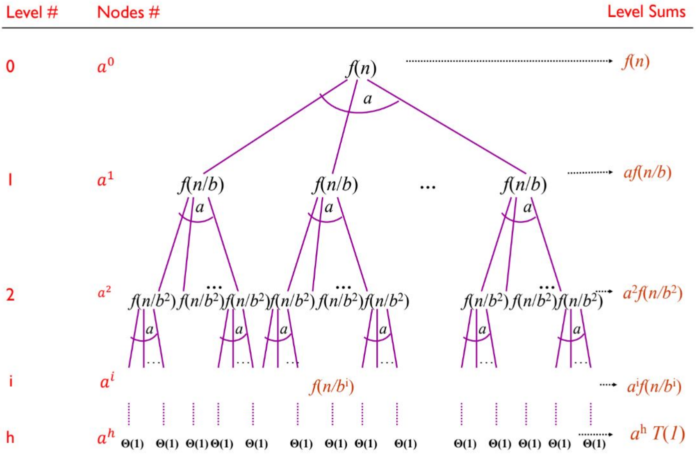

$$T(n) = \Sigma_{i=0}^{h-1} a^i f(\frac{n}{b^i}) + a^h \text{ with } h = log_b(n)$$

Intuition

$$T(n) = \Sigma_{i=0}^{log_b(n)-1} a^i f(\frac{n}{b^i}) + n^{log_b(a)}$$

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We have three cases:

1. The summation is an increasing geometric series.
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   - $T(n) = 3T(\frac{n}{2}) + n \Rightarrow$ Cost is dominated by the leaves.
2. The summation is a constant series.
   - $T(n) = 4T(\frac{n}{2}) + n^2$

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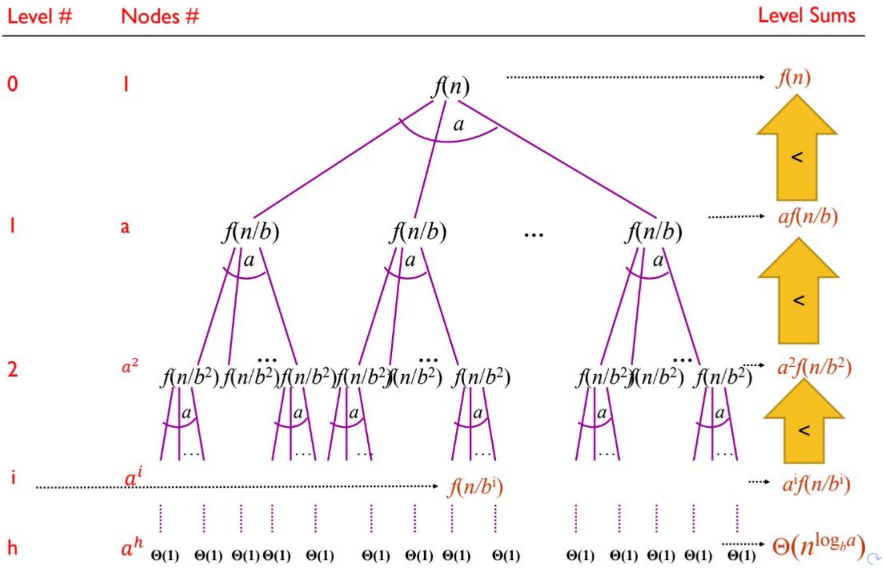
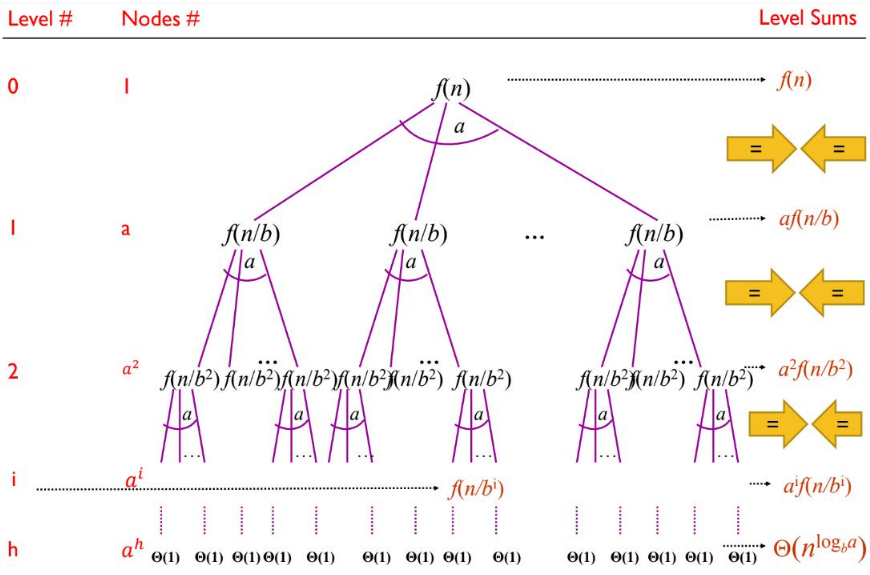
3. The summation is a decreasing geometric series.
   - $T(n) = 2T(\frac{n}{2}) + n^2 \Rightarrow$ Cost is dominated by the root.

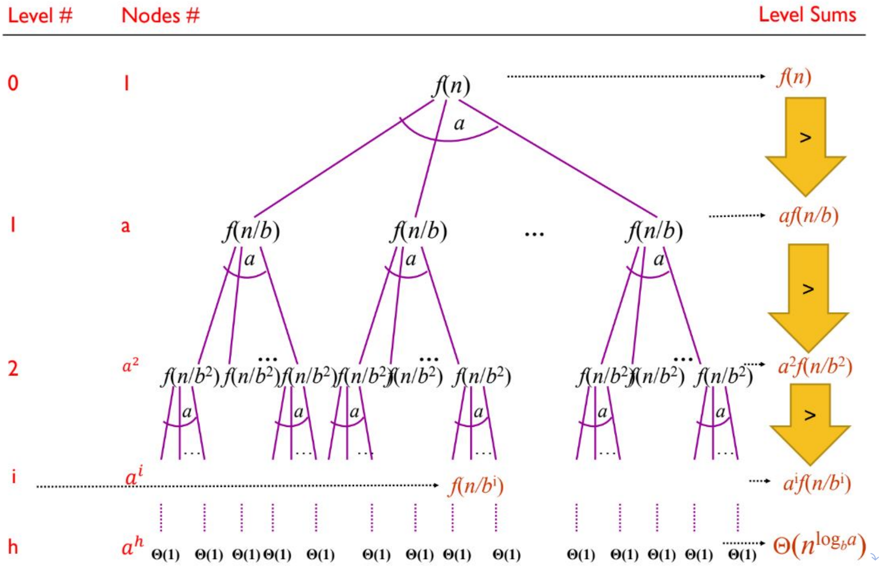
Case 1 - Cost Dominated by Leaves



Case 2 - Cost Distributed along the Tree



Case 3 - Cost Dominated by Root



The Master Method

The Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, $f(n)$ be an asymptotically positive function, and $T(n) = aT(\frac{n}{b}) + f(n)$.
$T(n)$ can be asymptotically bounded in 3 cases:

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1. If $f(n) = O(n^{log_b(a)-\varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(n^{log_b(a)})$.

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2. If $f(n) = \Theta(n^{log_b(a)})$, then $T(n) = \Theta(n^{log_b(a)} log(n))$.

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2. If $f(n) = \Theta(n^{log_b(a)})$, then $T(n) = \Theta(n^{log_b(a)} log(n))$.
3. If $f(n) = \Omega(n^{log_b(a)+\varepsilon})$ for some $\varepsilon > 0$ and $af(\frac{n}{b}) \leq cf(n)$ for $0 < c < 1$, then $T(n) = \Theta(f(n))$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

1. $T(n) = T(\frac{n}{2}) + \Theta(1)$.

The Master Theorem - Examples

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1. $T(n) = T(\frac{n}{2}) + \Theta(1)$.
   $a = 1, b = 2, n^{log_b(a)} = n^{log_2(1)} = 1$
   Cost of root $= c$, Cost of leaves $= c$.
   This is Case 2 of the master theorem since $c = \Theta(1)$.
   Hence, $T(n) = \Theta(log(n))$.
2. $T(n) = 2T(\frac{n}{2}) + \Theta(n)$.

The Master Theorem - Examples

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   Cost of root $= c$, Cost of leaves $= c$.
   This is Case 2 of the master theorem since $c = \Theta(1)$.
   Hence, $T(n) = \Theta(log(n))$.
2. $T(n) = 2T(\frac{n}{2}) + \Theta(n)$.
   $a = 2, b = 2, n^{log_b(a)=n}$
   Cost of root $= n$, Cost of leaves $= n$
   This is Case 2 of the master theorem since $n = \Theta(n)$.
   Hence, $T(n) = \Theta(nlog(n))$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

3. $T(n) = 16T(\frac{n}{4}) + \Theta(n)$.

The Master Theorem - Examples

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Use the master theorem to obtain a bound on $T(n)$.

3. $T(n) = 16T(\frac{n}{4}) + \Theta(n)$.
   $a = 16, b = 4, n^{log_b(a)} = n^{log_4(16)} = n^2$
   Cost of root $= cn$, Cost of leaves $= cn^2$.
   Looks like Case 1 of the master theorem.
   $n \leq \frac{n^2}{n^\varepsilon}$ for $\varepsilon = 1$. Case 1 proved. Hence, $T(n) = \Theta(n^2)$.
4. $T(n) = 2T(\frac{n}{2}) + \Theta(n^4)$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

3. $T(n) = 16T(\frac{n}{4}) + \Theta(n)$.
   $a = 16, b = 4, n^{log_b(a)} = n^{log_4(16)} = n^2$
   Cost of root $= cn$, Cost of leaves $= cn^2$.
   Looks like Case 1 of the master theorem.
   $n \leq \frac{n^2}{n^\varepsilon}$ for $\varepsilon = 1$. Case 1 proved. Hence, $T(n) = \Theta(n^2)$.
4. $T(n) = 2T(\frac{n}{2}) + \Theta(n^4)$.
   $a = 2, b = 2, n^{log_b(a)} = n$
   Cost of root $= cn^4$, Cost of leaves $= cn$
   Looks like Case 3 of the master theorem.
   $n^4 \geq n.n^\varepsilon$ for $\varepsilon = 1$.
   Regularity condition: $2(\frac{n}{2})^4 \leq cn^4$, $\frac{n^4}{2^3} \leq cn^4$ for $c = \frac{1}{8}$.
   Case 3 proved. Hence, $T(n) = \Theta(n^4)$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

5. $T(n) = 2T(\frac{n}{2}) + \Theta(\frac{n}{log(n)})$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

5. $T(n) = 2T(\frac{n}{2}) + \Theta(\frac{n}{log(n)})$.
   $a = 2, b = 2, n^{log_b(a)} = n$
   Cost of root $= c\frac{n}{log(n)}$, Cost of leaves $= cn$
   Looks like Case 1 of the master theorem.
   $\frac{n}{log(n)} \leq \frac{n}{n^{\varepsilon}}$, $log(n) \geq n^{\varepsilon} \Rightarrow$ does not hold!
   This is a gap between cases 1 and 2 and can not be solved using the master theorem.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

6. $T(n) = 2T(\frac{n}{2}) + \Theta(nlog(n))$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

6. $T(n) = 2T(\frac{n}{2}) + \Theta(nlog(n))$.
   $a = 2, b = 2, n^{log_b(a)} = n^{log_2(2)} = n$
   Cost of root $= cnlog(n)$, Cost of leaves $= cn$.
   Looks like Case 3 of the master theorem.
   $nlog(n) \geq n.n^{\varepsilon}$, $log(n) \geq n^{\varepsilon} \Rightarrow$ does not hold!
   This is a gap between cases 2 and 3 and can not be solved using the master theorem.
7. $T(n) = T(n-1) + \Theta(n)$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

6. $T(n) = 2T(\frac{n}{2}) + \Theta(nlog(n))$.
   $a = 2, b = 2, n^{log_b(a)} = n^{log_2(2)} = n$
   Cost of root $= cnlog(n)$, Cost of leaves $= cn$.
   Looks like Case 3 of the master theorem.
   $nlog(n) \geq n.n^{\varepsilon}$, $log(n) \geq n^{\varepsilon} \Rightarrow$ does not hold!
   This is a gap between cases 2 and 3 and can not be solved using the master theorem.
7. $T(n) = T(n-1) + \Theta(n)$.
   Not in the form of the master theorem.

Outline

1. The Master Method
2. Recap

Points to Take Home

1. The Master Theorem.
2. Reading Material:
   - Introduction to Algorithms, Chapter 4: Section 4.5 and Chapter 7: Sections 7.1 and 7.2.

Next Lecture: Quick Sort!

Due Credits

The presented material is based on:

1. Previous editions of the course at the GUC due to Dr. Wael Aboulsaadat, Dr. Haythem Ismail, Dr. Amr Desouky, and Dr. Carmen Gervet.
2. Stony Brook University's Analysis of Algorithms Course.
3. MIT's Introduction to Algorithms Course.
4. Stanford's Design and Analysis of Algorithms Course.