Project Report on

A Change of Variables Theorem for the Riemann Integral

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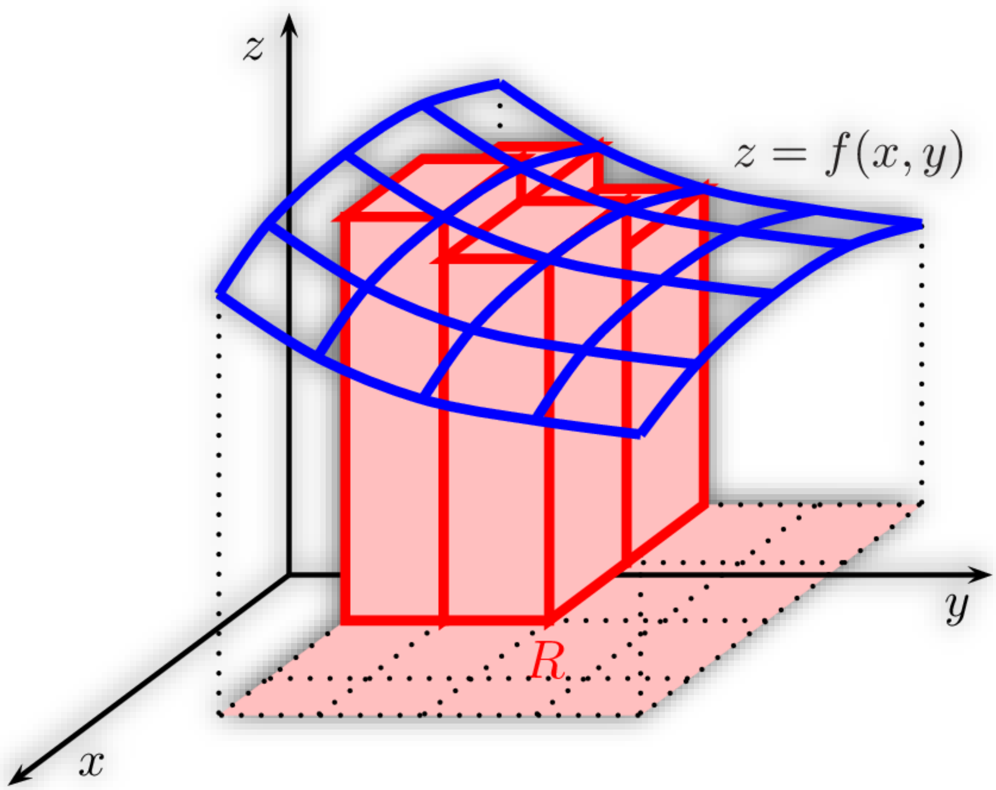

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Certificate

I hereby certify that the project dissertation titled “A Change of Variables Theorem for the Riemann Integral” which is submitted by Aneesh Panchal (2K20/MC/21) and Ayushi Sagar (2K20/MC/35) of Mathematics and Computing Department, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Bachelor of Technology, is a record of the project work carried out by the students. To the best of my knowledge this work has not been submitted in part or full for any Degree or Diploma to this university or elsewhere.

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Introduction:

A In the branch of mathematics known as real analysis, the Riemann integral, created by Bernhard Riemann, was the first rigorous definition of the integral of a function on an interval.
It was presented to the faculty at the University of Göttingen in 1854, but not published in a journal until 1868.
For many functions and practical applications, the Riemann integral can be evaluated by the fundamental theorem of calculus or approximated by numerical integration.

Riemann Integral:

Let f be a non-negative real-valued function on the interval [a, b], and let S be the region of the plane under the graph of the function f and above the interval [a, b]. See the figure on the top right. This region can be expressed in set-builder notation as

$$S = \{(x, y) : a \leq x \leq b, 0 < y < f(x)\}.$$

We are interested in measuring the area of S. Once we have measured it, we will denote the area in the usual way by

$$\int_a^b f(x)dx.$$

The basic idea of the Riemann integral is to use very simple approximations for the area of S. By taking better and better approximations, we can say that "in the limit" we get exactly the area of S under the curve.

When f(x) can take negative values, the integral equals the signed area between the graph of f and the x-axis: that is, the area above the x-axis minus the area below the x-axis.

Change of Variable

It is often convenient to rewrite the usual formula for the change of variable

$$\int_{G(a)}^{G(b)} f = \int_a^b f \circ G \cdot G'$$

in the form,

$$\int_{G(a)}^{G(b)} f = \int_a^b f \circ G \cdot g, \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (1)}$$

with $G(t) = G(a) + \int_a^t g$

Throughout this project we shall assume that g is Riemann integrable on [a,b] and G its Riemann primitive.

According to Hobson Lebesgue as far back as in 1909 proved (1) for a monotone G and R-integrable f.

In this project we show that (1) holds as long the integrand on either side of (1) is Riemann integrable.

Our aim is also to give proofs within the framework of Riemann theory and accessible to undergraduates.

First, we prove (1) for a monotone G.

Monotone Substitution

We need a weak variant of Henstock's lemma for the Riemann integral which is an immediate consequence of the definition based on upper and lower sums. Given ε > 0 there is a partition

$D \equiv \mathrm{a} = t_o < t_1 < \cdots < t_n = b$ ________(2)

such that,

$S(g, D) = \sum_{i=1}^{k} f(\tau_i)(\mathrm{t}_i - \mathrm{t}_{i-1})$ ____________(3)

for any set of τ_i with $\mathrm{t}_{i-1}-1 \le \tau_i \le \mathrm{t}_i$.

Moreover (3) persists for any refinement of D.

Theorem 1

If f is bounded on the range of G and g ≥ 0 on [a,b] then

$$\overline{\int_{G(a)}^{G(b)}} f = \overline{\int_{a}^{b}} f \circ G \cdot g, \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (4)}$$

$$\underline{\int_{G(a)}^{G(b)}} f = \underline{\int_{a}^{b}} f \circ G \cdot g. \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (5)}$$

If either g ≥ 0 or g ≤ 0 then if one side of (1) exists as a Riemann integral so does the other and equality holds.

Proof:

It suffices to prove (4), the relation for the lower integrals follows by using (4) on −f. The rest of the theorem follows from (4–5).

Let |f| ≤ M^f, |g| ≤ M^g, denote by M_i and $M_i^{\sim}$ the least upper bound of f ∘ G and f ∘ G · g, respectively,

on the interval $[t_{i-1}, t_i]$. We choose a partition (2) such that

$$\sum_1^n \tilde{M}_i(t_i - t_{i-1}) \le \overline{\int_a^b} f \circ G \cdot g + \varepsilon$$

and (3) hold simultaneously.

On every $[t_{i-1}, t_i]$ we find τ_i such that $M_i - \varepsilon < f(G(\tau_i))$Mi − ε < f(G(τi)),

denote $x_i = G(t_i)^1$ and have

$$\begin{aligned}\overline{\int_{G(a)}^{G(b)}} f &\le \sum_1^n M_i(x_i - x_{i-1}) = \sum_1^n M_i[G(t_i) - G(t_{i-1})] \\ &\le \sum_1^n M_i g(\tau_i)(t_i - t_{i-1}) + \sum_1^n M_i\left[G(t_i) - G(t_{i-1}) - g(\tau_i)(t_i - t_{i-1})\right] \\ &\le \sum_1^n [f(G(\tau_i)) + \varepsilon] g(\tau_i)(t_i - t_{i-1}) + M^f \varepsilon\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{1}^{n}[f(G(\tau_i))]g(\tau_i)(t_i - t_{i-1}) + \left(M^f + M^g(b-a)\right)\varepsilon \\
&\leq \sum_{1}^{n} \tilde{M}_i(t_i - t_{i-1}) + \left(M^f + M^g(b-a)\right)\varepsilon \\
&\leq \int_a^{\overline{b}} f \circ G \cdot g + \left(M^f + M^g(b-a) + 1\right)\varepsilon.
\end{aligned}
$$

Consequently, we have (4) with = replaced by ≤. For proving the reversed inequality we find a partition (2) such that

$$
\sum_{1}^{n} M_i[G(t_i) - G(t_{i-1})] \leq \int_{G(a)}^{\overline{G(b)}} f + \varepsilon
$$

and (3) hold simultaneously.

On $[t_{i-1}, t_i]$ we choose τ_i such that M˜ < f ∘ G · g(τ) + ε. Then we have,

$$
\begin{aligned}
\int_a^{\overline{b}} f \circ G \cdot g &\leq \sum_{1}^{n} \tilde{M}_i(t_i - t_{i-1}) \leq \sum_{1}^{n} f(G(\tau_i))g(\tau_i)(t_i - t_{i-1}) + \varepsilon(b-a) \\
&\leq \sum_{1}^{n} M_i\left[G(t_i) - G(t_{i-1})\right] + \varepsilon(b-a) \\
&\qquad + M^f \sum_{1}^{n} |G(t_i) - G(t_{i-1}) - g(\tau_i)(t_i - t_{i-1})| \\
&\leq \int_{G(a)}^{\overline{G(b)}} f + \varepsilon(b - a + 1 + M^f).
\end{aligned}
$$

Substitution Merely an Integral

In this situation the existence of the integral $\int_{G(a)}^{G(b)} f$ places no restriction on the behaviour of f outside [G(a),G(b)] therefore it is natural to replace this assumption by integrability of f on the range of G.

Theorem 2

The change of variables formula (1) is valid if either

1. f is Riemann integrable on the range of G, or
2. f is bounded on the range of G and the Riemann integral on the right hand side of (1) exists.

For the proof we need the following,

Lemma 3

If z is a Lipschitz function on [a, b] and $z^0 \le 0$ almost everywhere then z(b) ≤ z(a).
If z is Lipschitz and $z^0 = 0$ almost everywhere then z(b) = z(a).

Proof

It suffices to prove the first part of the lemma.
Let K > 0 be a Lipschitz constant for z.
Given ε > 0, there exists a countable system of open disjoint intervals J_n, n = 1,2,... covering the set where z^0 either does not exist or is positive and such that

$$\sum_1^\infty |J_n| < \frac{\varepsilon}{2K}$$

Since,

$$z' < \frac{\varepsilon}{2(b-a)}$$

on

$$[a,\, b] \setminus \bigcup_1^\infty J_n$$

it is easily seen that the least upper bound of all x ∈ [a, b] such that

$$z(x) - z(a) \leq \frac{\varepsilon(x-a)}{2(b-a)} + \sum_{1}^{\infty} K\,|J_n \cap (a,x)|$$

is the number b, and consequently z(b) − z(a) ≤ ε.

Proof of Theorem 2

Let E = {x ∈ (a, b); g continuous at x},
E_0 = {x ∈ E; g(x) = 0},
E_- = {x ∈ E; g(x) < 0},
E_+ = {x ∈ E; g(x) > 0}.

Assuming (i),
let

$$\begin{aligned} F(t) &= \int_{G(a)}^{G(t)} f, \\ L(t) &= \int_{\underline{a}}^{t} f \circ G g. \end{aligned}$$

On E_0 we have F^0(t) = L^0(t) = 0 because f is bounded, G^0(t) = 0 and g continuous at t with g(t) = 0.
Let t ∈ E_+, there is a positive α such that g > 0 on [t − α, t + α]. If |h| < α then
$F(t+h) - F(t) = L(t+h) - L(t)$

By Theorem 1. It follows that
$(F-L)^0(t) = 0$ ___________ (6)
for t ∈ E_+.

Similarly for t ∈ E_{-}. Consequently (6) holds on E and hence a.e. on [a, b].
By the Lemma with z = F − L we obtain

$$\int_{G(a)}^{G(b)} f = \int_{\underline{a}}^{b} f \circ Gg. \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_ (7)}$$

Similarly,

$$\int_{G(a)}^{G(b)} f = \int_{a}^{\overline{b}} f \circ Gg. \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_ (8)}$$

Consequently,

$$\int_{\underline{a}}^{b} f \circ Gg = \int_{a}^{\overline{b}} f \circ Gg = \int_{G(a)}^{G(b)} f.$$

Assuming (ii),
let

$$\mathcal{L}(t) \quad = \quad \underline{\int_{G(a)}^{G(t)}} f,$$

$$\mathcal{F}(t) \quad = \quad \int_{a}^{t} f \circ Gg.$$

The first part of the proof applies mutas mutandis with F, L replaced by F, L, respectively.
The assumption that f is bounded in (ii) seems undesirable but, in fact, is essential. If f(1/n) = n and g(1/n) = 0 for n = 1,2,..., and f(x) = 0, g(x) = 1 otherwise then the integral on the right hand side of (1) exists but the one on the left hand side does not when a = 0 and b = 1.

Conclusion:

Theorem 1 remains valid if Riemann integrability is replaced by Lebesgue or Perron integrability. On the other hand the counterpart (i) of Theorem 2 for L-integral is false. This is because a composition of two AC functions need not be AC. The analog of Theorem 2 is not valid for the Perron integral either. If both integrals in (1) exist as Lebesgue or Perron integral then equation (1) holds. For the Lebesgue integral this was established by de la Val´ee Poussin already in 1915, the proof (of a more general theorem) for the Perron integral is in Goodman. Theorem 2 is perhaps one of the rare examples when a theorem naturally and generally formulated within the framework of a theory is valid for the Riemann but not for the Lebesgue integral.

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