Project Report on

GRAPHS AND EIGEN VALUES

Submitted by:

ANEESH PANCHAL - 2K20/MC/21

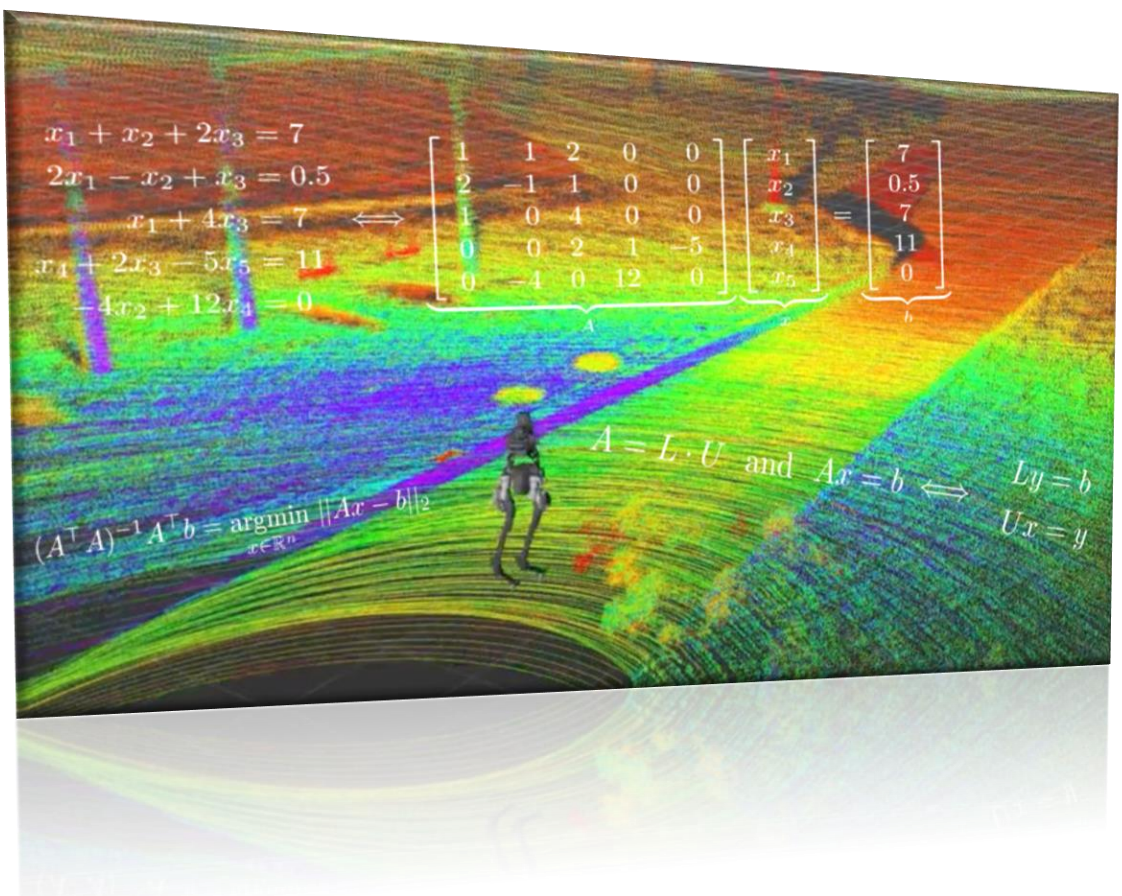
AYUSHI SAGAR - 2K20/MC/35

Submitted to:

DR. DINESH UDAR

Department of Applied Mathematics

Delhi Technological University



Certificate

I hereby certify that the project dissertation titled "Graphs and Eigen Values" which is submitted by Aneesh Panchal (2K20/MC/21) and Ayushi Sagar (2K20/MC/35) of Mathematics and Computing Department, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Bachelor of Technology, is a record of the project work carried out by the students. To the best of my knowledge this work has not been submitted in part or full for any Degree or Diploma to this university or elsewhere.

DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042

Acknowledgement

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We came to know about many new things, we are really thankful to him. We would also like to thank our classmates who have also helped whenever we were stuck at some point.

Thanking You

Aneesh Panchal (2K20/MC/21)
Ayushi Sagar (2K20/MC/35)

Introduction:

A graph is a structure amounting to a set of objects in which some pairs of the objects are in some sense "related".

It is a mathematical structure consisting of two finite sets V and E. The elements of V are called vertices or nodes and the elements of E are called edges. Each edge is associated with a set consisting of either one or two vertices called its endpoints.

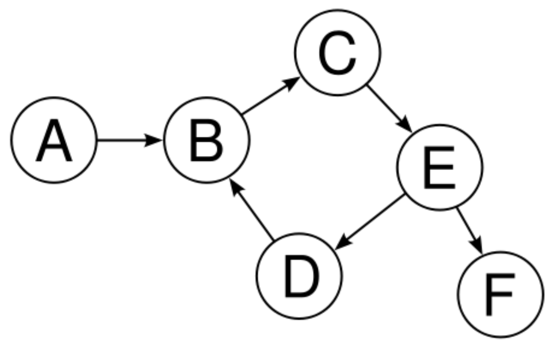
A graph (denoted as G=(V,E)) consists of a non-empty set of vertices or nodes V and a set of edges E.

Let G = (V, E) be a graph

- **Vertices** - The elements of V are called the vertices of G
- **Edges** - The elements of E are called the edges of G.

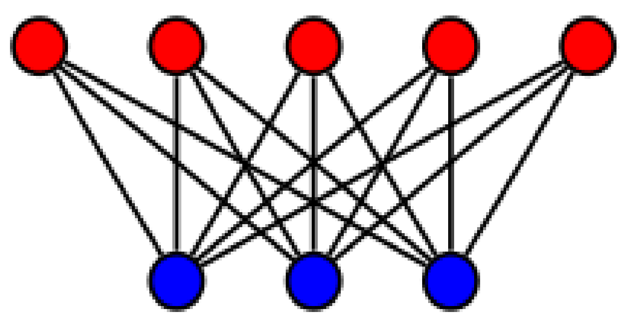
DIRECTED GRAPH (DIGRAPH):

Directed graph (digraph) is an ordered pair: G = (V, E), where: V is the vertex set E is the edge set (or arc set) each edge e = (v,w) in E is an ordered pair of vertices from V



BIPARTITE GRAPH (BIGRAPH):

In the Graph theory, a bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V. Vertex sets U and V are usually called the parts of the graph.



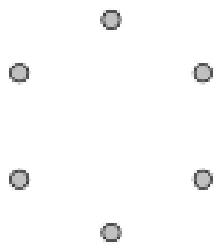
REGULAR GRAPH:

In graph theory, a regular graph is a graph where each vertex has the same number of neighbours i.e. every vertex has the same degree or valency.

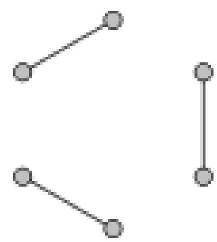
A regular graph contains an even number of vertices with odd degree.

EXAMPLES OF REGULAR GRAPHS:

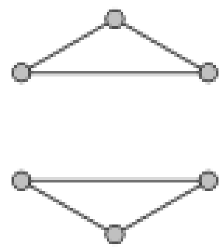
0 – Regular Graph



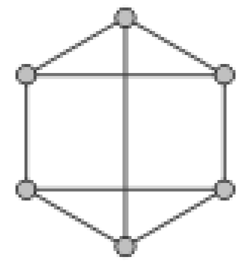
1 – Regular Graph



2 – Regular Graph



3 – Regular Graph



MATRIX REPRESENTATION OF GRAPH:

A graph can be represented inside a computer by using adjacency matrices.

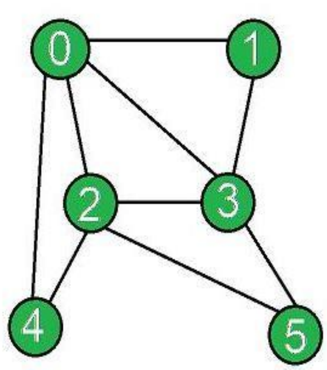
ADJACENCY MATRIX:

In graph theory, an adjacency matrix is a square matrix used to represent a finite graph. The elements of the matrix indicate whether pairs of vertices are adjacent or not in the graph.

Adjacency matrix for undirected graph is always symmetric.

Adjacency Matrix is also used to represent weighted graphs.

If adj[i][j] = w, then there is an edge from vertex i to vertex j with weight w.



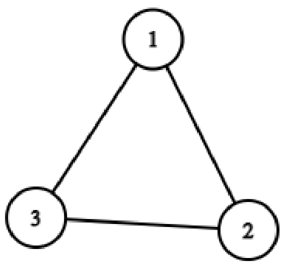
	0	1	2	3	4	5
0	0	1	1	1	1	0
1	1	0	0	1	0	0
2	1	0	0	1	1	1
3	1	1	1	0	0	1
4	1	0	1	0	0	0
5	0	0	1	1	0	0

SPECTRUM OF GRAPH:

The adjacency matrix of an undirected simple graph is symmetric, and therefore has a complete set of real eigenvalues and an orthogonal eigenvector basis.
The set of eigenvalues of a graph is the Spectrum of the graph.
It is common to denote the eigenvalues by,
$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

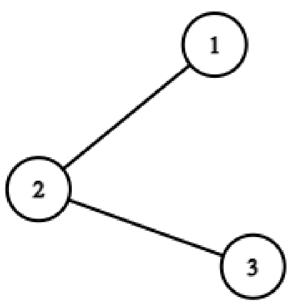
Property I:

The greatest eigenvalue λ_1 is bounded above by the maximum degree.



$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Maximum degree of graph is 2.
$\lambda_1, \lambda_2, \lambda_3 = 2, -1, -1$
$\lambda_1 = 2 \leq 2$



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

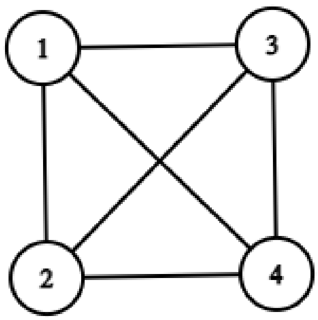
Maximum degree of graph is 2.
$\lambda_1, \lambda_2, \lambda_3 = \sqrt{2}, -\sqrt{2}, 0$
$\lambda_1 = \sqrt{2} \leq 2$

Property II:

For d-regular graphs, d is the first eigenvalue of A, λ_i. The multiplicity of this eigenvalue is the number of connected components of G.

d3 regular graph:

It is regular graph because all nodes have degree 3.



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Eigen Values:

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 = 3, -1, -1, -1$

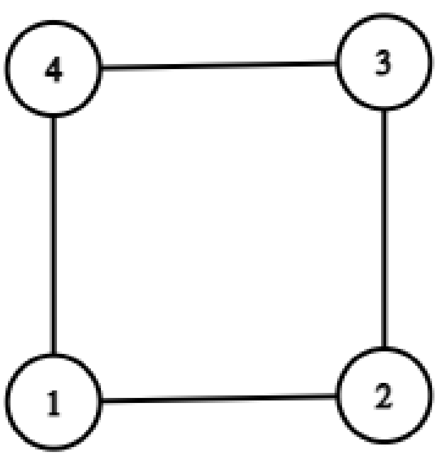
$\lambda_1 = 3$

Multiplicity, m = 1

As it is clearly seen that there is one connected component i.e. whole graph.

d2 regular graph:

It is regular graph because all nodes have degree 2.



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Eigen Values:

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 = 2, -2, 0, 0$

$\lambda_1 = 2$

Multiplicity, m = 1

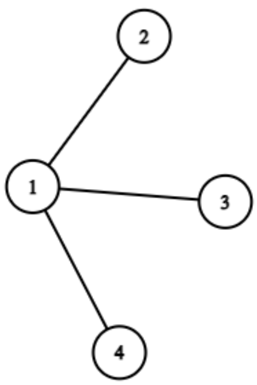
As it is clearly seen that there is one connected component i.e. whole graph.

Property III:

It can be shown that for each eigenvalue λ_i , its opposite $\lambda_i = -\lambda_{n+1-i}$ is also an eigenvalue of A if G is a bipartite graph.

Bipartite graph:

It is a bipartite graph because every edge connects the node of X = {1} with the node of Y = {2,3,4}



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Eigen Values:

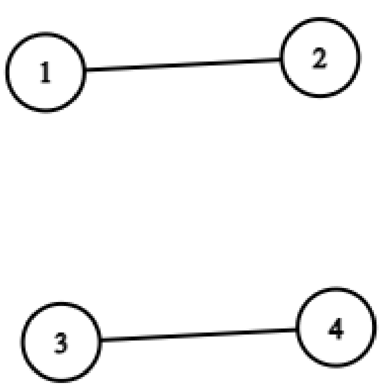
$\lambda_1, \lambda_2, \lambda_3, \lambda_4 = \sqrt{3}, 0, 0, -\sqrt{3}$

$\lambda_1 = \sqrt{3} = -(-\sqrt{3}) = \ -\lambda_4 = -\lambda_{4-1+1}$
$\lambda_2 = 0 = -(0) = \ -\lambda_3 = -\lambda_{4-2+1}$
Hence, $\lambda_i = -\lambda_{n+1-i}$

Regular Bipartite Graph:

d1 regular graph:

It is regular graph because all nodes have degree 1.



$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Bipartite graph:

It is a bipartite graph because every edge connects the node of X = {1,3} with the node of Y = {2,4}

Eigen Values:

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 = 1, 1, -1, -1$

$\lambda_1 = 1$

Multiplicity, m = 2

As it is clearly seen that there are 2 connected components i.e. (1,2) and (3,4).

$\lambda_1 = 1 = -(-1) = \ -\lambda_4 = -\lambda_{4-1+1}$

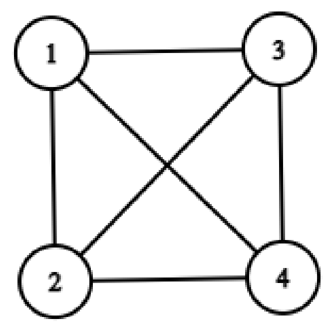
$\lambda_2 = 1 = -(-1) = \ -\lambda_3 = -\lambda_{4-2+1}$

Hence, $\lambda_i = -\lambda_{n+1-i}$

Matrix powers:

Theorem:

Let G be a graph with adjacency matrix A and k be a positive integer. Then the matrix power A^k gives the matrix where Aij counts the number of paths of length k between vertices vi and vj



For graph G with degree 3 i.e. for n = 3

$$\textbf{A*A*A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix}$$

$$\textbf{A*A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

Matrix Powers are used to find out the **Geometry of the graph** without making it.

Applications of Matrix Powers:

- **Find out number of lines in a graph:**

  To find out number of lines in a graph first we have to find out number of walk length for which i != j in matrix of order 2:

  Walks, w = (2+2+2+2+2+2)*2 = 24

  Number of lines, n =$\frac{w}{2*2} = \frac{24}{4} = 6$ lines

  As we can see that total number of lines are 6 only,

  Lines = {(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)}

- **Find out number of triangles in a graph:**

  To find out number of triangles in a graph first we have to find out number of walk length for which i = j in matrix of order 3:

  Walks, w = (6+6+6+6) = 24

  Number of lines, n =$\frac{w}{3*2} = \frac{24}{6} = 4$ triangles

  As we can see that total number of triangles are 4 only,

  Triangles = {(1,2,4), (1,3,4), (2,4,3), (1,2,3)}

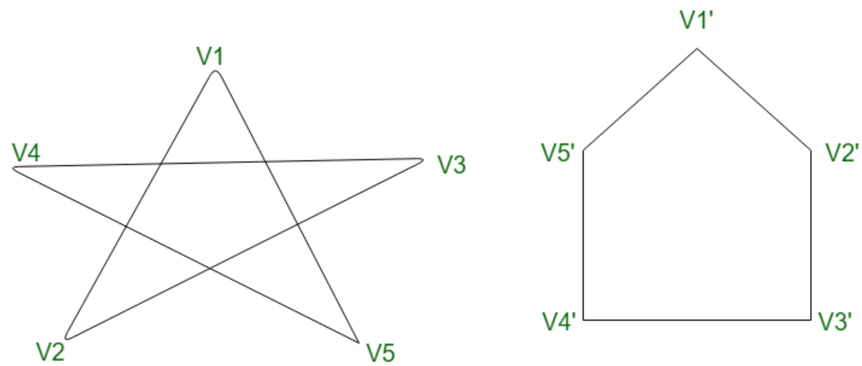
Conjugate Matrices:

If G1 and G2 are a pair of isomorphic graphs with adjacency matrices A1, A2, then A1 and A2 are conjugate via a permutation matrix P:

$$\boldsymbol{A2 = P(A1)P^{-1}}$$

Notes:

1. G1 and G2 are isomorphic if
   - Same number of vertices and edges
   - Same degree of vertices
   - Same length cycles
2. P is permutation matrix if it contains only 0's and 1's and each row/column contain only single 1
3. A1 and A2 have same set of eigen values



G G'

Here we have 2 different Graphs G and G’
Both have 5 edges and 5 vertices
Both have vertices of same degree i.e. 2
Both have only 1 cycle
So, G and G’ are Isomorphic

As here we get,

A1 = A2 = $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

So we get the Permutation matrix here as Identity(I)
As we know inverse of I is again I
So,
$A2 = A1$
$A2 = I(A1)I$
$A2 = I(A1)I^{-1}$
Also, A1 = A2. So, both have same set of eigen values.

Chromatic Number:

The chromatic number of a graph is the minimal number of colours needed to colour the vertices in such a way that no two adjacent vertices have the same colour.

Theorem:

Let G be a finite, connected, undirected graph, without loops or multiple edges. Then,

k < 1 + λ

where, k is chromatic number of G

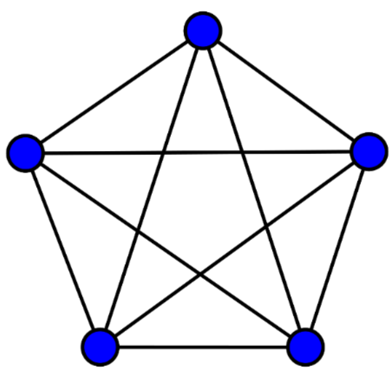
$\lambda \leq max_{v \epsilon G}(degree\ of\ vetrex\ v)$ **(Perron Frobenius Theorem)**

with equality iff G is a complete graph or an odd circuit graph.

Graph G have E edges and n vertices, Then

$$k \leq 1 + \left\{2\left(1 - \frac{1}{n}\right)E\right\}^{\frac{1}{2}}$$

with equality iff G is complete graph



Here we have K_5 complete graph,

Edges, E = 10

Vertices, n = 5

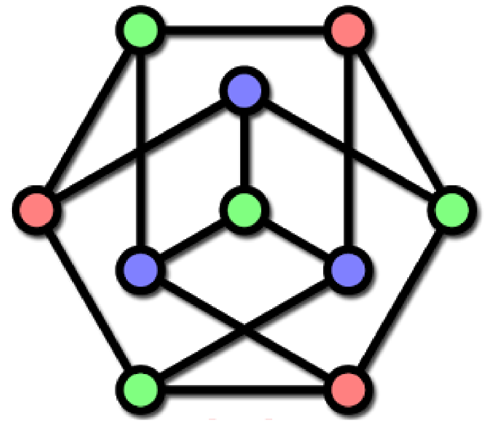
As here we have complete graph, so equality must hold

We can clearly see that we require 5 as chromatic number (k=5)

$$2\left(1 - \frac{1}{n}\right)E = 2\left(1 - \frac{1}{5}\right)10 = 2\left(\frac{4}{5}\right)10 = 16$$

$$k = 1 + \left\{2\left(1 - \frac{1}{n}\right)E\right\}^{\frac{1}{2}} = 1 + \{16\}^{\frac{1}{2}} = 1 + 4 = 5$$

Here we have simple graph with
Edges, E = 15
Vertices, n = 10
It is not a complete graph,
Here we need 3 as chromatic number (k=3)



$$2\left(1-\frac{1}{n}\right)E = 2\left(1-\frac{1}{10}\right)15 = 2\left(\frac{9}{10}\right)15 = 27$$

$$1+\left\{2\left(1-\frac{1}{n}\right)E\right\}^{\frac{1}{2}} = 1+\{27\}^{\frac{1}{2}} = 1+5.2 = 6.2$$

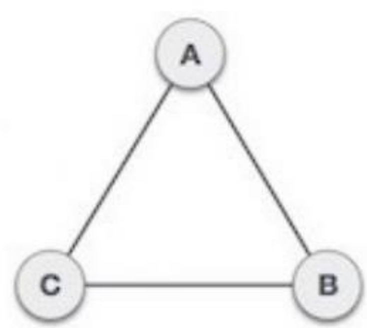
Clearly,
$k(=3) \leq 6.2$

Matrix Tree Theorem:

Let G be a graph of order n. Then the cofactor of any element of L(G) equals the number of spanning trees of G.

Laplacian Matrix:

L(G) = Dt(G) – A(G)
where, L(G) is Laplacian Matrix of graph G
A(G) is adjacency matrix of graph G
Dt(G) is diagonal matrix whose entries are the degrees of each vertex



As we know,
L(G) = Dt(G) – A(G)

$$L(G) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} * \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Barnette's conjecture

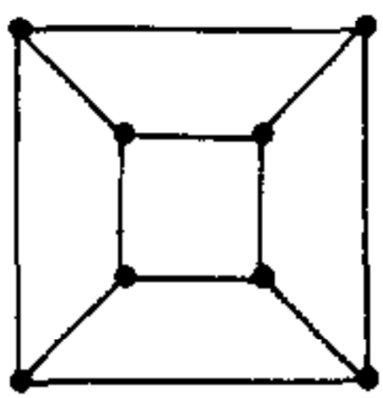
It states that every cubic bipartite three-connected planar graph has a Hamiltonian cycle.

Or,

It states that every bipartite polyhedral graph with three edges per vertex has a Hamiltonian cycle.

Hamiltonian Cycle,

A Hamiltonian cycle is a closed loop on a graph where every node (vertex) is visited exactly once.



It is a bipartite graph with 3 edges per vertex.

A =

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
1	0	1	0	1	1	0	0	0
2	1	0	1	0	0	1	0	0
3	0	1	0	1	0	0	1	0
4	1	0	1	0	0	0	0	1
5	1	0	0	0	0	1	0	1
6	0	1	0	0	1	0	1	0
7	0	0	1	0	0	1	0	1
8	0	0	0	1	1	0	1	0

A*A =

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
1	3	0	2	0	0	2	0	2
2	0	3	0	2	2	0	2	0
3	2	0	3	0	0	2	0	2
4	0	2	0	3	2	0	2	0
5	0	2	0	2	3	0	2	0
6	2	0	2	0	0	3	0	2
7	0	2	0	2	2	0	3	0
8	2	0	2	0	0	2	0	3

As we can clearly see here that diagonal elements of A*A have magnitude 3.
That is, we have 3 paths starting from each node and ending on the same node of length 2.
For node 1:
1 -> 2 -> 1
1 -> 4 -> 1
1 -> 5 -> 1

$A^8 =$

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
1	1641	0	1640	0	0	1640	0	1640
2	0	1641	0	1640	1640	0	1640	0
3	1640	0	1641	0	0	1640	0	1640
4	0	1640	0	1641	1640	0	1640	0
5	0	1640	0	1640	1641	0	1640	0
6	1640	0	1640	0	0	1641	0	1640
7	0	1640	0	1640	1640	0	1641	0
8	1640	0	1640	0	0	1640	0	1641

That is, we have total 1641 paths of length 8 starting and ending on the same node which consists of paths with unique and repeated edges.
So, there exists atleast 1 hamiltonian cycle (as there are 3 edges per vertex in the graph) in this graph.

For the same graph,
Laplacian Matrix is,

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
1	3	0	0	0	0	0	0	0
2	0	3	0	0	0	0	0	0
3	0	0	3	0	0	0	0	0
4	0	0	0	3	0	0	0	0
5	0	0	0	0	3	0	0	0
6	0	0	0	0	0	3	0	0
7	0	0	0	0	0	0	3	0
8	0	0	0	0	0	0	0	3

—

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
1	0	1	0	1	1	0	0	0
2	1	0	1	0	0	1	0	0
3	0	1	0	1	0	0	1	0
4	1	0	1	0	0	0	0	1
5	1	0	0	0	0	1	0	1
6	0	1	0	0	1	0	1	0
7	0	0	1	0	0	1	0	1
8	0	0	0	1	1	0	1	0

Laplacian Matrix, L(G) =

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
1	3	-1	0	-1	-1	0	0	0
2	-1	3	-1	0	0	-1	0	0
3	0	-1	3	-1	0	0	-1	0
4	-1	0	-1	3	0	0	0	-1
5	-1	0	0	0	3	-1	0	-1
6	0	-1	0	0	-1	3	-1	0
7	0	0	-1	0	0	-1	3	-1
8	0	0	0	-1	-1	0	-1	3

According to Matrix Tree Theorem,
Spanning Trees of matrix A is cofactor of any element
Let us take 1st diagonal element

Sign		A_1	A_2	A_3	A_4	A_5	A_6	A_7
+	1	3	-1	0	0	-1	0	0
	2	-1	3	-1	0	0	-1	0
	3	0	-1	3	0	0	0	-1
	4	0	0	0	3	-1	0	-1
	5	-1	0	0	-1	3	-1	0
	6	0	-1	0	0	-1	3	-1
	7	0	0	-1	-1	0	-1	3

i.e. determinant of this matrix which is equal to,
Number of Spanning Trees, NST = 384
As there are 3 edges per vertex and number of spanning trees is 384 and total number of path are 1641
So there must exist atleast 1 Hamiltonian Cycle.

Erdős–Gyárfás conjecture

It states that every graph with minimum degree 3 contains a simple cycle whose length is a power of two.

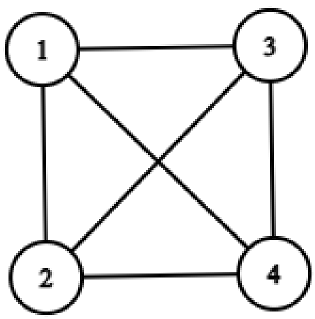
For a graph with number of vertices greater than or equal to 4.
As we know that adjacency matrix of any graph consists of diagonal elements as zero and atleast 3 elements in 1st row as 1's

As adjacency matrix of graph is symmetric.
So, 1st diagonal element of A*A can't be zero.
That is, we can say that there exist atleast 3 loops whose length is 2.

As, 1st diagonal element of A*A is +ve integer.
So, 1st diagonal element of A*A*A*A is again +ve
because all elements of A*A is +ve and sum of 2 +ve numbers are again +ve.
So, we can say that there exist atleast 9 loops whose length is 4.
As we know 2 and 4 are powers of 2.

So, Erdős–Gyárfás conjecture is true universally.

Let us take an example for clarification,
Complete graph G with 4 vertices,



A = $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ **A*A =** $\begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$

A*A*A = $\begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix}$ **A*A*A*A =** $\begin{bmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{bmatrix}$

Diagonal elements of A*A have magnitude 3
That is there are 3 paths of length 2 starting and ending on same node (cycle)
2 is a power of 2.

Diagonal elements of A*A*A*A have magnitude 21
That is there are 21 paths of length 4 starting and ending on same node (cycle)
4 is a power of 2.
Path consist of unique nodes (as minimum degree is 3) or repeated nodes.
Hence is Erdős–Gyárfás conjecture.

Conclusion:

Through this project we learned about different properties of Graphs & Adjacency Matrix, Relation between Graphs & Eigen Values of the Adjacency Matrix. We learned about Matrix powers & their applications in finding geometrical outcomes from the adjacency matrix. We also studied about Conjugate Matrices, Chromatic Number and Matrix Tree Theorem.
Through the whole project, our main focus is to study about Barnette's conjecture and Erdős–Gyárfás conjecture. We studied these conjuctures and tried to find out whether these conjuctures holds true or not by using examples for each.
We successfully studied and found these conjuctures true for the example we take, we generalized these conjuctures using simple language and found to be true. We tried and tried but we are unable to give a solid proof to these conjuctures.

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