THE TACOMA NARROWS BRIDGE “GALLOPING GERTIE”

MC207

Engineering Analysis & Design

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LOCATION & HISTORY OF BRIDGE

The **Tacoma Narrows Bridge**, was a suspension bridge in the **U.S.** state of **Washington** that spanned between **Tacoma and the Kitsap Peninsula**.

The bridge was idea since 1889.
Construction of the bridge was started on **Sept,1938.**

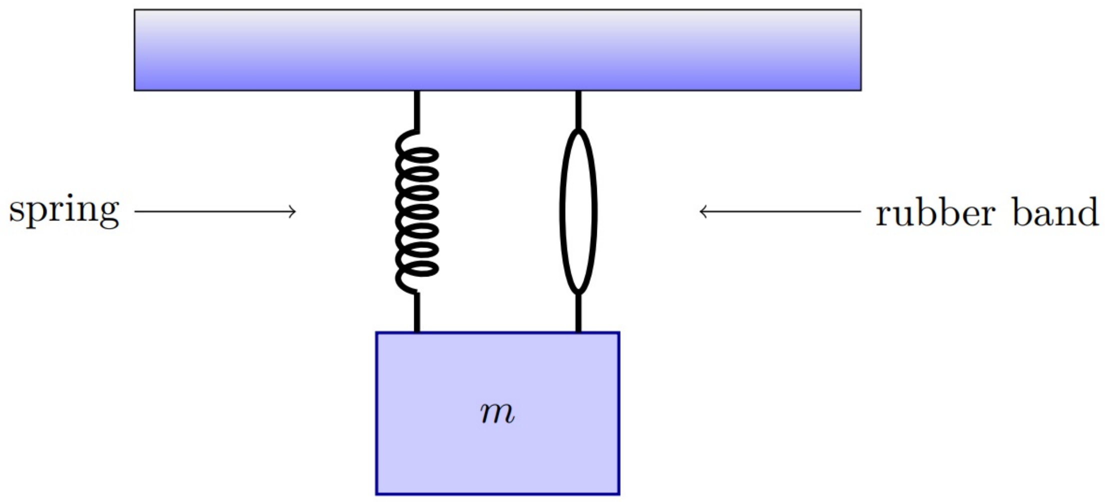
Bridge was designed by **Leon Moisseiff.**

On **July 1,1940**, the Tacoma Narrows Bridge was completed and opened to traffic.
Even on the first day of operation, people driving bridge experienced vertical oscillations of several feet.
On the **morning of November 7**, these **Vertical oscillations** were as large as **3 feet**. Later that morning, the bridge began to experience **Torsional oscillations** as great as **23 degrees**. The bridge **failed at 11:10 a.m.**

Possible Reason for Collapse of Bridge

If the collapse of the bridge was due to resonance, the forcing frequency of a forced harmonic oscillator must be close to its natural frequency, but the wind (the forcing) simply does not behave like this.

One explanation might be the fact that **cables are not true springs**. They act more like rubber bands. Imagine a mass suspended by both a spring and a rubber band. The rubber band acts like a spring when it is stretched, but there is no restoring force if the oscillator is in a compressed position.



A spring-rubber band system

Spring Rubber Band System Explaination

Suppose that y(t) is vertical displacement of the mass the spring-rubber band model, where the displacement is positive in the downward direction and negative in the upward direction. We will also assume that we have a damped system. If this is the case, we can model the motion of the mass with the differential equation

$$\boldsymbol{my'' = -by' - k_1 y - k_2 y^+ + F(t)}$$

where $y^+ = \max(y, 0)$.

Here, m is the mass,

b is the damping coefficient,

k_1 is the spring constant, and

F(t) is the forcing term.

The term $k_2 y^+$ corresponds to the rubber band.

The Tacoma Bridge Model
General Model

$$my'' + ay' - by - cy^{+} = -gm + F(t)$$

where

- y is the height of the road bed at the center of the bridge.
- m is the mass of the bridge.
- αy′ is the damping term. We assume that α is small since suspension bridges are relatively flexible structures.
- by is the force provided by the material of the bridge pulling the bridge back to y = 0.
- cy^{+} is the force provided by the pull of the cables.
- −gm is the force provided by gravity.
- F(t) is the periodic force provided by the wind. Yes, wind does provide a periodic force.

We can think of how a flag flutters in a wind.

DERIVATION OF EQUATION

If a spring with spring constant k is extended by a distance y, the potential energy is $ky^2/2$. Let θ be the angle of the rod from the horizontal. If a rod of mass m and length 2L rotates about its center of gravity with angular velocity dθ/dt,

then its kinetic energy is given by $\frac{1}{6}ml^2\left(\frac{d\theta}{dt}\right)^2$

The potential due to gravity is −mgy.

The extension of one spring is given by (y - $l\sin(\theta))^+$ and $(y + l\sin(\theta))^+$ in the other, where

$y^+ = \max(y, 0)$.

The total potential energy is given by

$$V = \left(\frac{k}{2}\right)\left(((y - l\sin\theta)^+)^2 + (y + l\sin\theta)^+\right)^2 - mgy,$$

the total kinetic energy is given by

$$T = \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{6}ml^2\left(\frac{d\theta}{dt}\right)^2.$$

Derivation of Equation (cont.)

As we know that total energy of the system remains constant that means

Total Energy(E) = T + V = c(constant)

If we differentiate the equation of E with respect to θ and y partially then we get 2 differential equations,

$$\frac{1}{3}ml^2\theta'' = kl\cos\theta[(y - l\sin\theta)^+ - (y + l\sin\theta)^+]$$

$$y'' = -k[(y - l\sin\theta)^+ + (y + l\sin\theta)^+] + mg$$

Adding forcing and damping terms, and assume that cable never lose tension, that means:

$(y - \mathrm{l}\sin(\theta))^+ = y - \mathrm{l}\sin(\theta)$, and $(y + \mathrm{l}\sin(\theta))^+ = y + \mathrm{l}\sin(\theta)$

We get resultant differential equation as,

$$\theta'' = -\delta\theta' - \frac{6k}{m}\cos\theta\sin\theta + F(t)$$

$$y'' = -\delta y' - \frac{2k}{m}y + g.$$

Final Model for Tacoma Bridge Analysis

We have to take damping coefficient very small as bridges are flexible structures.

We take wind (forcing term) as a sinusoidal function.

As bridge collapse due to torsional oscillations which were as large as 23 degrees.

Let us make Linear and Non Linear models for the bridge by assuming constants (constants taken from research paper):

Linear Model: (we assume $\cos(\theta) = 1$ and $\sin(\theta) = \theta$ when we assume torsional oscillations very small)

$$\theta'' = -0.01\theta' - (2.4)\theta + (1.2)\sin(0.06t)$$

Non Linear Model: (actual model)

$$\theta'' = -0.01\theta' - (2.4)\sin(\theta)\cos(\theta) + (1.2)\sin(0.06t)$$

NON LINEAR DIFFERENTIAL HAMILTONIAN SYSTEM

Theorem *Let*

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y),$$

where f and g are continuously differentiable.

If the system is Hamiltonian, then

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}.$$

Suppose that (x_0, y_0) is an equilibrium solution for the system

$$\frac{dx}{dt} = f(x,y) = \frac{\partial H}{\partial y}(x,y)$$
$$\frac{dy}{dt} = g(x,y) = -\frac{\partial H}{\partial x}(x,y).$$

In order to determine the nature of the equilibrium solution, we will compute the Jacobian matrix of the system at (x_0, y_0),

$$J = \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} = \begin{pmatrix} H_{yx}(x_0,y_0) & H_{yy}(x_0,y_0) \\ -H_{xx}(x_0,y_0) & -H_{xy}(x_0,y_0) \end{pmatrix}.$$

If we let

$$\alpha = H_{xy}(x_0,y_0) = H_{yx}(x_0,y_0)$$
$$\beta = H_{yy}(x_0,y_0)$$

$$\gamma = -H_{xx}(x_0,y_0),$$

then J becomes

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\lambda^2 - \alpha^2 - \beta\gamma.$$

Therefore, our matrix has eigenvalues

$$\lambda = \pm\sqrt{\alpha^2 + \beta\gamma},$$

and we have the following possibilities.

- If $\alpha^2 + \beta\gamma > 0$, we have two real eigenvalues of opposite signs. Therefore, our equilibrium solution is a saddle.
- If $\alpha^2 + \beta\gamma = 0$, the only eigenvalue is zero.
- If $\alpha^2 + \beta\gamma < 0$, we have two purely imaginary eigenvalues.

ANALYSIS OF SOLUTION

$\theta'' = -0.01\theta' - (2.4)\theta + (1.2)\sin(0.06t)$

$\theta'' + 0.01\theta' + 2.4\theta = (1.2)\sin(0.06t)$

for CF: $(D^2 + 0.01D + 2.4)\theta = 0$

$m^2 + 0.01m + 2.4 = 0$

$m = -0.005 \pm 1.549i$

which gives finally,,

CF: $[K_1 \sin(1.549t) + K_2 \cos(1.549t)]e^{-0.005t}$

after solving for PI we get,,

PI: $\frac{7988000}{15952037}\sin\left[\frac{3}{50}t\right] - \frac{2000}{15952037}\cos\left[\frac{3}{50}t\right]$

$\therefore$ final sol^n for linear model is

$\theta = CF + PI$

Non Linear Model

$\theta'' = -0.01\theta' - 2.4\sin\theta\cos\theta + 1.2\sin(0.06t)$

$\frac{d\theta}{dt} = \theta' = \omega = f(\theta,\omega)$

$\frac{d\omega}{dt} = -0.01\omega - 1.2\sin(2\theta) + 1.2\sin(0.06t) = g(\theta,\omega)$

$\frac{\partial f}{\partial \theta} = 0$ and $\frac{\partial g}{\partial \omega} = -0.01$

as we can clearly see that $\frac{\partial f}{\partial \theta} \neq \frac{-\partial g}{\partial \omega}$

$\therefore$ The given system is not Hamiltonian

That means we can't analyse system directly

We also want to find out equillibrium sol^n,,

$\frac{d\theta}{dt} = \omega = 0$ (at equillibrium position)

$\omega' + 0.01\omega + 2.4\theta = (1.2)\sin(0.06t)$

$\frac{d\omega}{dt} = -0.01\omega - 2.4\theta + (1.2)\sin(0.06t) = 0$

$\omega = \frac{1.2\sin(0.06t) - 2.4\theta}{0.01} = 0$ (for equi. sol^n)

$\theta = \frac{\sin(0.06t)}{2}$

$\therefore$ here θ changes as t changes sinusoidally

That means equillibrium θ varies from $[-28.648, 28.648]$

as we know, at equillibrium position $\omega = 0$

$\frac{d\omega}{dt} = -0.01\omega - 1.2\sin(2\theta) + 1.2\sin(0.06t) = 0$

$\omega = 120[\sin(0.06t) - \sin 2\theta]$

$\omega = 240\sin(0.03t - \theta)\cos(0.03t + \theta) = 0$

That give rise to 2 cases,,

(i) $0.03t - \theta = n\pi$

(ii) $0.03t + \theta = \frac{(2n+1)\pi}{2}$

In both of the cases θ varies with t linearly

That means equillibrium value of θ can reach upto $180°$

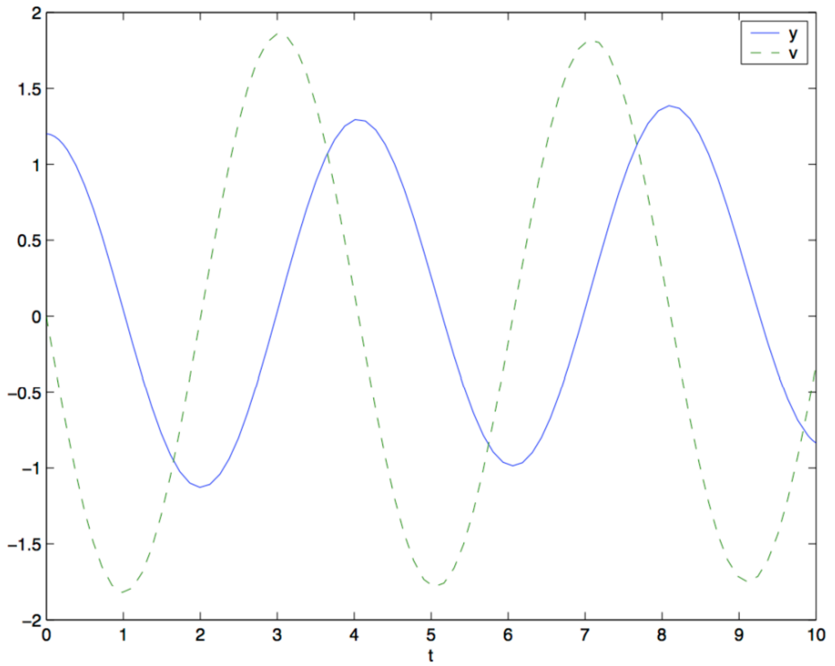
for particular value of n and t

FINAL GRAPHS FOR MODELS OF TORSION

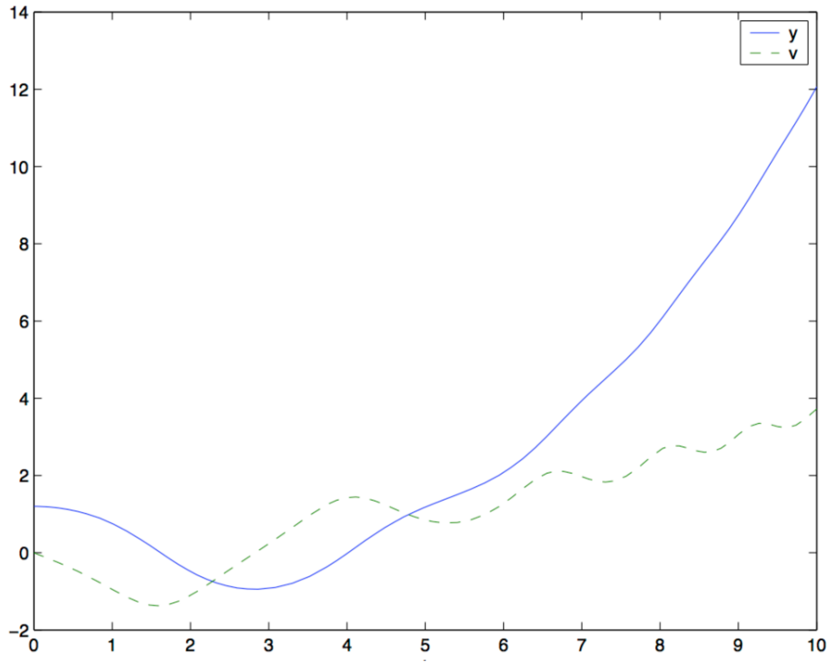
Finally if we take some initial conditions for graph we can create the graph for the 2 models as:

θ(0) = 1.2, θ'(0) = 0

Solution of Linear Model of Torsion



Solution of Non Linear Model of Torsion



FINAL CONCLUSION (PERSONAL VIEWS) & (PUBLISHED CONCLUSIONS)

- The walls of the bridge were made of steel causing no wind flow.
- If we think that there was no problem in the structure then cables of the bridge were not true spring causing them behaving differently.
- If we think that the structure is good enough then the engineers must have considered Linear Model which was not ready for the worst case.
- Best possible solution may be that the construction is made according to Spring Rubber Non Linear Model of Torsion.
- Outcome of this incident was study of Construction Aerodynamics increased drastically around the globe.