Project Report on

Multiple Integrals and Riemann Sum

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Introduction:

Riemann Sum is a certain kind of approximation of an integral by a finite sum. It is named after 19th century German mathematician Bernhard Riemann. Riemann Sum helps us to visually understand the meaning of areas and volumes.

In this Project I am going to find out the applications of multiple integrals and understanding the concept of Riemann Sum.

Applications of Multiple Integrals:

Double Integrals:

One of the major task to find out the double integral is to find out the limits of integration over the specified region because by changing the order of integration the limits of integration also changes.

Changing the order of integration means if we are first integrating with respect to x(say 1st variable) and then with respect to y (say 2nd variable). So, after changing the order of integration we will be first integrating with respect to y and then with respect to x.

1. Area (Double integrals):

We can find out the area A enclosed between curves y = f(x) and y = g(x) in the xy-plane where f(x) and g(x) are continues functions.

$$Area, A = \iint dA$$

2. Volume (Double integrals):

We can find out the volume V of the solid G enclosed between surface z=f(x,y) and a region R in the xy-plane where f(x,y) is continues and nonnegative on R.

$$Volume, V = \iint f(x,y)dA$$

3. Mass (Double integrals):

We can find the mass M of a lamina (a region R in the xy-plane) whose density (the mass per unit area) is a continues non negative function p(x,y)

$$Mass\ of\ lamina, M = \iint p(x,y)dA$$

4. Centre of Gravity (Double integrals):

We can find out the centre of gravity (x_o, y_o) of a body whose total mass is M and mass density is a continues non negative function p(x,y)

$$Total\ Mass, M = \iint p(x,y)dA$$

As centre of gravity have the coordinates (x_o, y_o) which are given by,

$$x_o = \frac{\iint xp(x,y)dA}{\iint p(x,y)dA} \qquad y_o = \frac{\iint yp(x,y)dA}{\iint p(x,y)dA}$$

Triple Integrals:

One of the major task to find out the triple integral is to find out the limits of integration over the specified region because by changing the order of integration the limits of integration also changes. (same as double integrals)

1. Volume (Triple integrals):

We can find out the volume V of the solid G enclosed in a region R in the xyz-plane.

$$Volume, V = \iiint dV$$

2. Mass (Triple integrals):

We can find the mass M of a lamina (a region R in the xyz-plane) whose density (the mass per unit area) is a continues non negative function p(x,y,z)

$$Mass\ of\ lamina, M = \iiint p(x,y,z)dV$$

3. Centre of Gravity (Triple integrals):

We can find out the centre of gravity (x_o, y_o, z_o) of a body whose total mass is M and mass density is a continues non negative function p(x,y,z)

$$Total\ Mass, M = \iiint p(x,y,z)dV$$

As centre of gravity have the coordinates (x_o, y_o, z_o) which are given by,

$$x_o = \frac{\iiint xp(x,y,z)dV}{\iiint p(x,y,z)dV}$$

$$y_o = \frac{\iiint yp(x,y,z)dV}{\iiint p(x,y,z)dV}$$

$$z_o = \frac{\iiint zp(x,y,z)dV}{\iiint p(x,y,z)dV}$$

Riemann Sums:

Let f: $[a, b] \rightarrow \mathrm{R}$ be a function defined on a closed interval [a,b] of real numbers R, and

$P = \{[x_o, x_1], [x_1, x_2], \ldots \ldots, [x_{n-1}, x_n]\}$

Be a partition of interval, I(say), where

$a = x_o < x_1 < x_2 < \cdots < x_n = b$

A Riemann Sum S of function f over I with partition P is defined as

$$S = \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

Where $\Delta x_i = x_i - x_{i-1}$ and $x_i^* \in [x_{i-1}, x_i]$.

If function is Riemann integrate-able then distance or width of the summand $\Delta x_i \rightarrow 0$.

Finding Areas and Volumes using Riemann Sum:

Let f be a continues function on a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$.

- Partition the interval [a,b] into m subintervals of equal length $\Delta x = \frac{b-a}{m}$

  Let $x_o, x_1, \ldots \ldots, x_m$ be the endpoints of these subintervals, where

  $$a = x_o < x_1 < x_2 < \cdots < x_m = b$$

- Partition the interval [c,d] into m subintervals of equal length $\Delta y = \frac{d-c}{n}$

  Let $y_o, y_1, \ldots \ldots, y_n$ be the endpoints of these subintervals, where

  $$c = y_o < y_1 < y_2 < \cdots < y_m = d$$

- These two partitions create a partition of the rectangle R into mn sub-rectangles R_{ij} with opposite vertices (x_{i-1}, y_{j-1}) *and* (x_i, y_j) for i between 1 and m and j between 1 and n.

  These rectangles all have equal areas $\Delta A = \Delta x . \Delta y$

- Area of curve y=f(x) over the region $S = \{x : a \leq x \leq b\}$ is given by

  $$Area, A = \sum_{i=1}^{m} f(x_i) . \Delta x$$

- Volume of curve z=f(x,y) which defined over region R is given by

$$Volume, V = \sum_{j=1}^{n}\sum_{i=1}^{m} f\left(x_{ij}, y_{ij}\right).\Delta A$$

Area of Rectangle:

Let a Rectangle have vertices (0,0), (a,0), (a,b), (0,b)

- Using Riemann Sum:

  Here region is $R = \{(x, y): 0 \le x \le a, 0 \le y \le b\}$.

  Let the interval [0,a] divided into n subintervals of equal length $\Delta x = \frac{a}{n}$

  Curve of rectangle is f(x) = b and x lies in [0,a]

  $f(x_1) = f(x_2) = \cdots = f(x_i) = \cdots = f(x_n) = b$

  Area of the Rectangle is given by

$$A = \sum_{i=1}^{m} f(x_i).\Delta x = b\Delta x + b\Delta x + \cdots upto\ n\ terms$$

$$A = nb\Delta x = nb.\frac{a}{n} = ab\ sq.units$$

- Using Double Integrals:

  Here region is $R = \{(x, y): 0 \le x \le a, 0 \le y \le b\}$.

$$A = \int_0^b \int_0^a dx.dy = \int_0^b ady = ab\ sq.units$$

Volume of Cuboid:

Let a Cuboid of region $R = \{(x, y, z): 0 \le x \le a, 0 \le y \le b, 0 \le z \le c\}$.

- Using Riemann Sum:

  Let the interval [0,a] divided into m subintervals of equal length $\Delta x = \frac{a}{m}$

  And let the interval [0,b] divided into n subintervals of equal length $\Delta y = \frac{b}{n}$

Curve of rectangle is f(x,y) = c

$f(x_{ij}, y_{ij}) = c$

Volume of Cuboid is given by

$$V = \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}, y_{ij}).\Delta A = \sum_{j=1}^{n} mc\Delta A = mnc\Delta A$$

$$V = mnc\Delta x.\Delta y = mnc\frac{a}{m}\frac{b}{n} = abc\ cubic\ units$$

- **Using Double Integrals:**

$$V = \int_0^b \int_0^a f(x,y)dx.dy = \int_0^b \int_0^a c.dx.dy = \int_0^b ac.dy = abc\ cubic\ units$$

- **Using Triple Integrals:**

$$V = \int_0^c \int_0^b \int_0^a dx.dy.dz = \int_0^c \int_0^b a.dy.dz = \int_0^c ab.dz = abc\ cubic\ units$$

Area of Circle:

Let a circle be $x^2 + y^2 = a^2$ that is region $R = \{(x,y): -a \le x \le a, -a \le y \le a\}$

- **Using Riemann Sum:**

  Let us calculate the region of circle in 1st quadrant and then makes it 4 times to get the area of the circle.

  Let the interval [0,a] divided into n subintervals of equal length $\Delta x = \frac{a}{n}$

  So the i^{th} x element will be $x_i = \frac{ai}{n} - 0 = \frac{ai}{n}$

  Hence, $f(x_i) = \sqrt{a^2 - \frac{a^2 i^2}{n^2}} = a\sqrt{1 - \frac{i^2}{n^2}}$

  Let us take only few terms in the expansion of $f(x_i)$ we get,

  $f(x_i)[approx] = a\left(1 - \frac{i^2}{2n^2}\right)$

Area of the Circle in 1st quadrant is given by

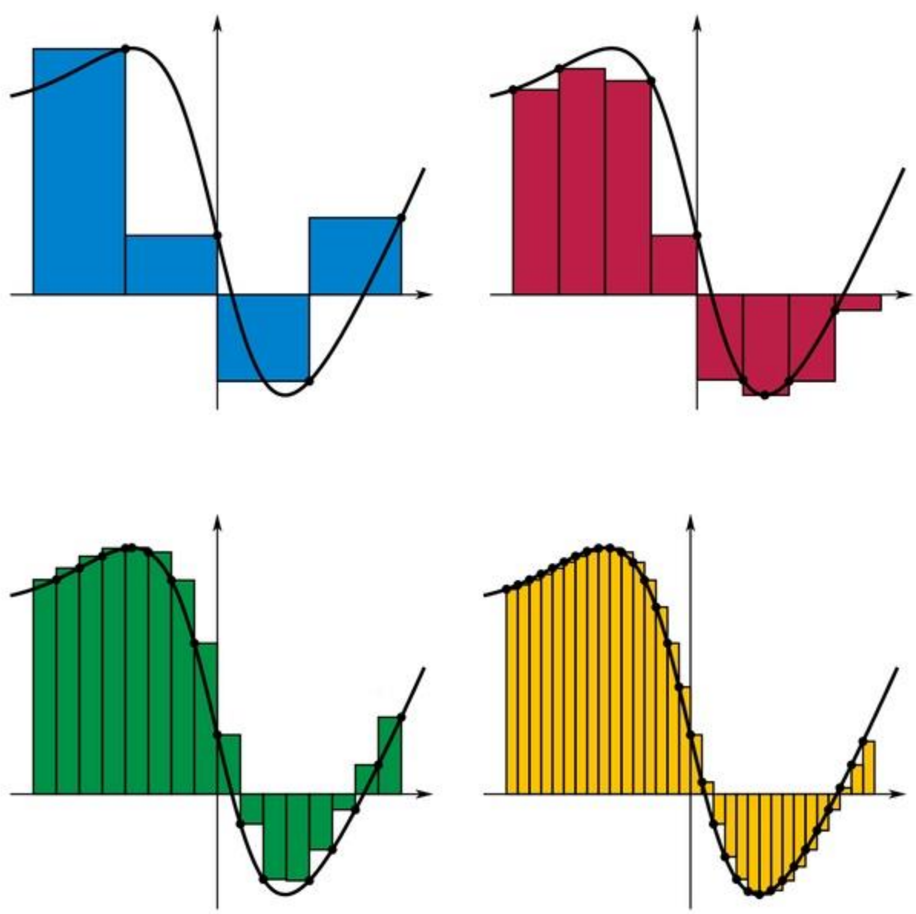
$$A_{\frac{1}{4}} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i).\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{a^2}{n}\left(1-\frac{i^2}{2n^2}\right)$$

$$= \lim_{n\to\infty}\frac{a^2}{n}\left(n-\frac{n(n+1)(2n+1)}{12n^2}\right)$$

$$A_{\frac{1}{4}} = \frac{5}{6}a^2\ sq.units$$

Total area of circle is

$$A = 4A_{\frac{1}{4}} = \frac{20}{6}a^2 = 3.334a^2\ sq.units$$



As area of circle is just near to actual value which is $3.14a^2$ and we have taken the approximation. So Riemann Sum gives the approximation of the area of the circle.

➢ Using Double Integrals:

$$A = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy.dx = \pi a^2 = 3.14a^2\ sq.units$$

Volume of Sphere:

Let a Sphere be $x^2 + y^2 + z^2 = a^2$ that is region $R = \{(x, y, z): -a \leq x \leq a, -a \leq y \leq a, -a \leq z \leq a\}$.

➢ Using Riemann Sum:

Let us calculate region of sphere in 1st octant and then makes it 8 times to get the full volume of the sphere.

Let the interval [0,a] divided into m subintervals of equal length $\Delta x = \frac{a}{m}$

Let the interval [0,a] divided into n subintervals of equal length $\Delta y = \frac{a}{n}$

So the i^{th} x element will be $x_i = \frac{ai}{m} - 0 = \frac{ai}{m}$

So the j^{th} y element will be $y_j = \frac{aj}{n} - 0 = \frac{aj}{n}$

Hence, $f(x_i, y_j) = \sqrt{a^2 - \frac{a^2i^2}{m^2} - \frac{a^2j^2}{n^2}} = a\sqrt{1 - \frac{i^2}{m^2} - \frac{j^2}{n^2}}$

Let us take only few terms in the expansion of $f(x_i, y_j)$ we get,

$f(x_i, y_j)[approx] = a\left(1 - \frac{i^2}{2m^2} - \frac{j^2}{2n^2} - \frac{i^2j^2}{4m^2n^2}\right)$

Volume of sphere in 1st octant is given by

$V_{\frac{1}{8}} = \sum_{i=1}^{m}\sum_{j=1}^{n} a\left(1 - \frac{i^2}{2m^2} - \frac{j^2}{2n^2} - \frac{i^2j^2}{4m^2n^2}\right)\frac{a}{m}\frac{a}{n}$

$= \sum_{i=1}^{m} \frac{a^3}{mn}\left(n - \frac{ni^2}{2m^2} - \frac{n(n+1)(2n+1)}{12n^2} - \frac{i^2n(n+1)(2n+1)}{24m^2n^2}\right)$

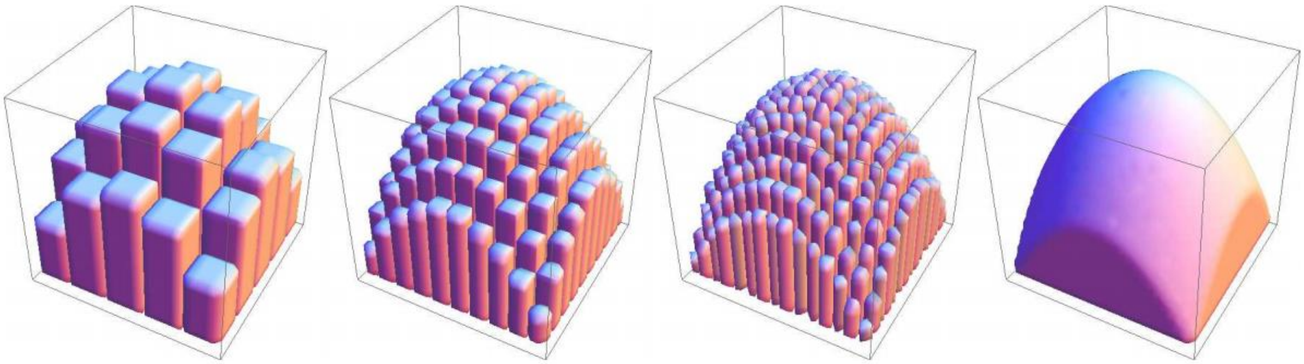
$= \frac{a^3}{mn}(mn - \frac{nm(m+1)(2m+1)}{12m^2} - \frac{mn(n+1)(2n+1)}{12n^2} - \frac{m(m+1)(2m+1)n(n+1)(2n+1)}{144m^2n^2})$

As we take infinitely many parts, so we get,

$Volume\ of\ sphere\ in\ 1st\ octant,\ V_{\frac{1}{8}} = \lim_{m,n\to\infty} \frac{a^3}{mn}(mn - \frac{nm(m+1)(2m+1)}{12m^2} - \frac{n(n+1)(2n+1)}{12n^2} - \frac{m(m+1)(2m+1)n(n+1)(2n+1)}{144m^2n^2})$

$V_{\frac{1}{8}} = 0.6388889a^3\ cubic\ units$

So total volume of the sphere is
V = 8 x 0.6388889a^3 = $5.11112a^3\ cubic\ units$



The volume of sphere is $4.1888a^2$ and we have taken the approximation. We have got the error due to small no. of terms we have consider if we consider large no. of terms then we will go near to the actual value.
As we can clearly see that volume unit always contain huge error as compare to area unit for the same figure. So Riemann Sum gives the approximation of the volume of the sphere.

- Using Double Integrals:

$$V = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2-y^2}\, dy.dx = \frac{4}{3}\pi a^3\ cubic\ units$$

- Using Triple Integrals:

$$V = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz.dy.dx = \frac{4}{3}\pi a^3\ cubic\ units$$

Conclusion:

Riemann Sum gives the approximation with an error of the integral when we consider only a limited no. of terms. And when we consider very large no. of terms that is no. of terms tends to infinity and $\Delta x \; and \; \Delta y$ be tending to zero then we get the more accurate and precise result from the Riemann Sum.

That is simply,

$$\int\int f(x,y)dA = \lim_{\substack{\Delta x \to 0,\ \Delta y \to 0 \\ n \to \infty,\ m \to \infty}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j$$

References:

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