Project Report on

Mandelbrot Set and Fractals

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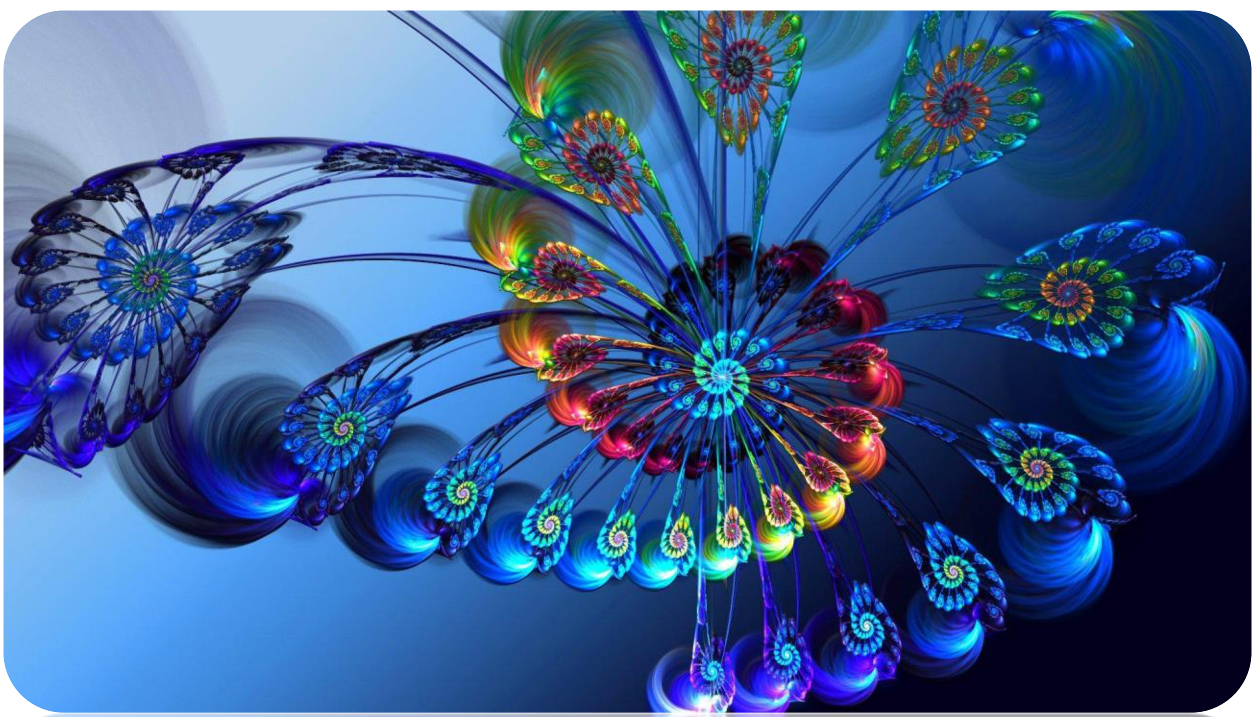

Certificate

I hereby certify that the project dissertation titled “**Mandelbrot Set and Fractals**” which is submitted by **Aneesh Panchal (2K20/MC/21)** and **Ayushi Sagar (2K20/MC/35)** of Mathematics and Computing Department, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Bachelor of Technology, is a record of the project work carried out by the students. To the best of my knowledge this work has not been submitted in part or full for any Degree or Diploma to this university or elsewhere.

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Acknowledgement

We, **Aneesh Panchal(2K20/MC/21)** and **Ayushi Sagar (2K20/MC/35)** would like to express our special thanks of gratitude to **Mr. Kamaljeet.S Gangania**, Mathematics and Computing Department, Delhi Technological University for his able guidance and support in completing our project report.

We came to know about many new things, we are really thankful to him. We would also like to thank our classmates who have also helped whenever we were stuck at some point.

Thanking You

Aneesh Panchal (2K20/MC/21)
Ayushi Sagar (2K20/MC/35)

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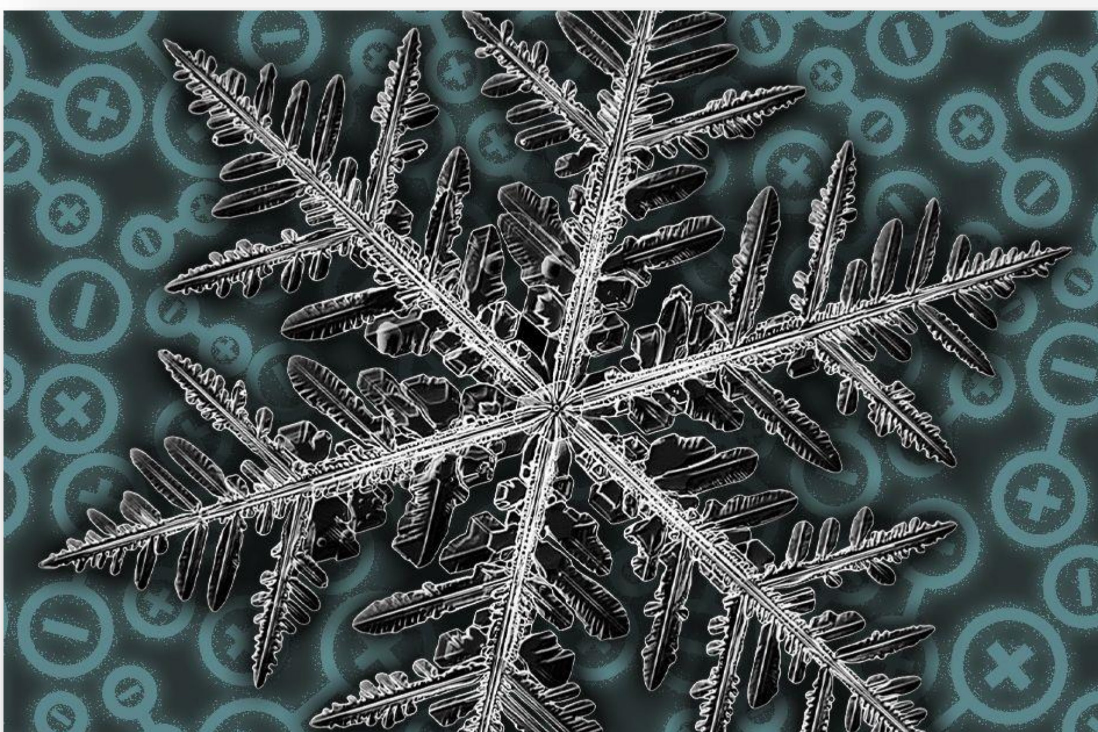


Introduction:

Basic Defination I:

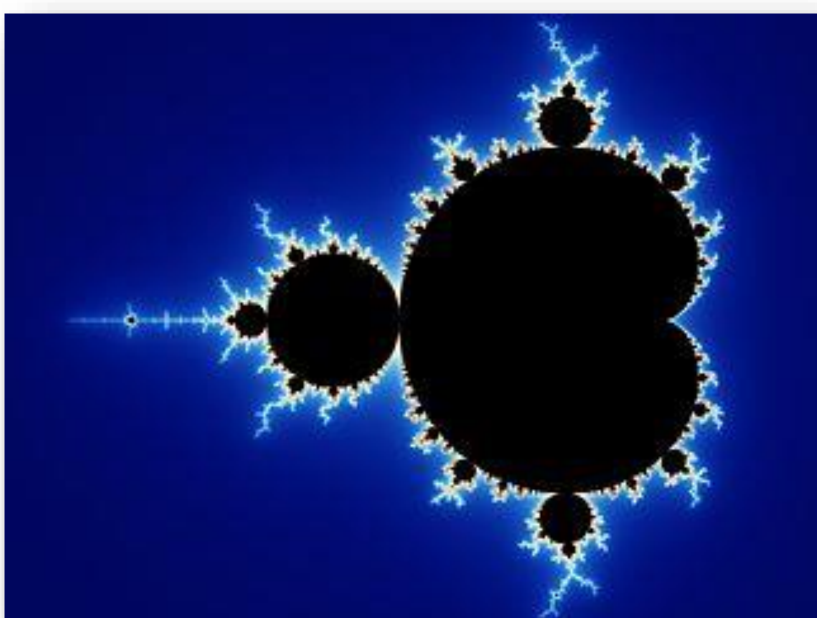
The Mandelbrot set is the set of complex numbers **c** for which the function $\boldsymbol{f_c(z) = z^2 + c}$ does not diverge to infinity when iterated from **z = 0**, i.e., for which the sequence $\boldsymbol{f_c(0)}$, $\boldsymbol{f_c(f_c(0))}$, etc., remains bounded in absolute value.

Its definition is credited to **Adrien Douady** who named it in tribute to the mathematician **Benoit Mandelbrot**, a pioneer of fractal geometry.

Basic Defination II:

The Mandelbrot set is a picture in the complex "c-plane" of the fate of the orbit of 0 under iteration of the function $\boldsymbol{x^2 + c}$. A c-value is in the Mandelbrot set if the orbit of 0 under iteration of $\boldsymbol{x^2 + c}$ for the particular value of c does not tend to infinity.

If the orbit of 0 tends to infinity, then that c-value is not in the Mandelbrot set.



Mandelbrot Set:

Formal Defination:

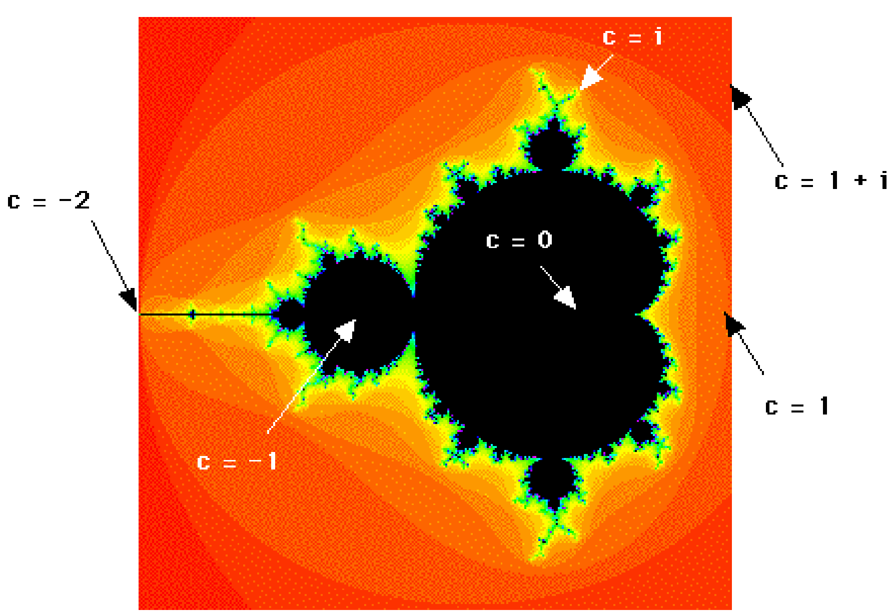
The Mandelbrot set is the set of values of c in the complex plane for which the orbit of the critical point z = 0 under iteration of the quadratic map

$$\boldsymbol{z_{n+1} = z_n{}^2 + c}$$

remains bounded. Thus, a complex number c is a member of the Mandelbrot set if, when starting with $z_0 = 0$ and applying the iteration repeatedly, the absolute value of $\boldsymbol{z_n}$ **remains bounded for all n>0**.

For example, for c = 1, the sequence is 0, 1, 2, 5, 26, ..., which tends to infinity, so **1 is not an element of the Mandelbrot set**.
On the other hand, for c = -1, the sequence is 0, −1, 0, −1, 0, ..., which is bounded, so **−1 does belong to the Mandelbrot set**.

The Mandelbrot set can also be defined as the **connectedness locus of a family of polynomials**



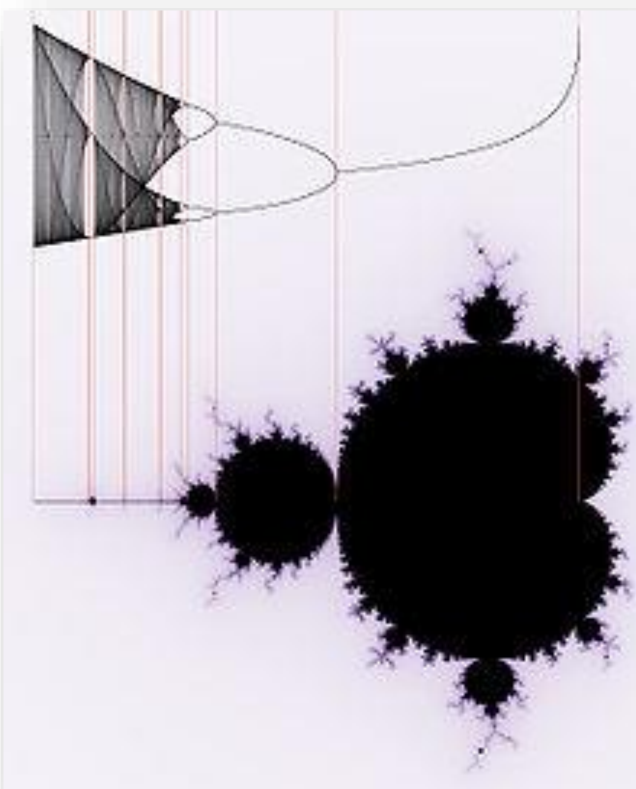
The Mandelbrot set is a compact set, since it is closed and contained in the closed disk of radius 2 around the origin. More specifically, a point **c** belongs to the Mandelbrot set if and only if $|z_n| \leq 2$ for all **n >= 0**.

In other words, the absolute value of z_n must remain at or below 2 for **c** to be in the Mandelbrot set, M, as if that absolute value exceeds 2, the sequence will escape to infinity.

The intersection of M with the real axis is precisely the interval **[-2, 1/4]**. The parameters along this interval can be put in one-to-one correspondence with those of the real logistic family,

$x_{n+1} = rx_n(1 - x_n)$, **r belongs to [1,4]**

$z = r\left(\frac{1}{2} - x\right)$, $c = \frac{r}{2}\left(1 - \frac{r}{2}\right)$



Period Bulbs:

For every rational number p/q, with p and q coprime, there is such a bulb that is tangent at the parameter

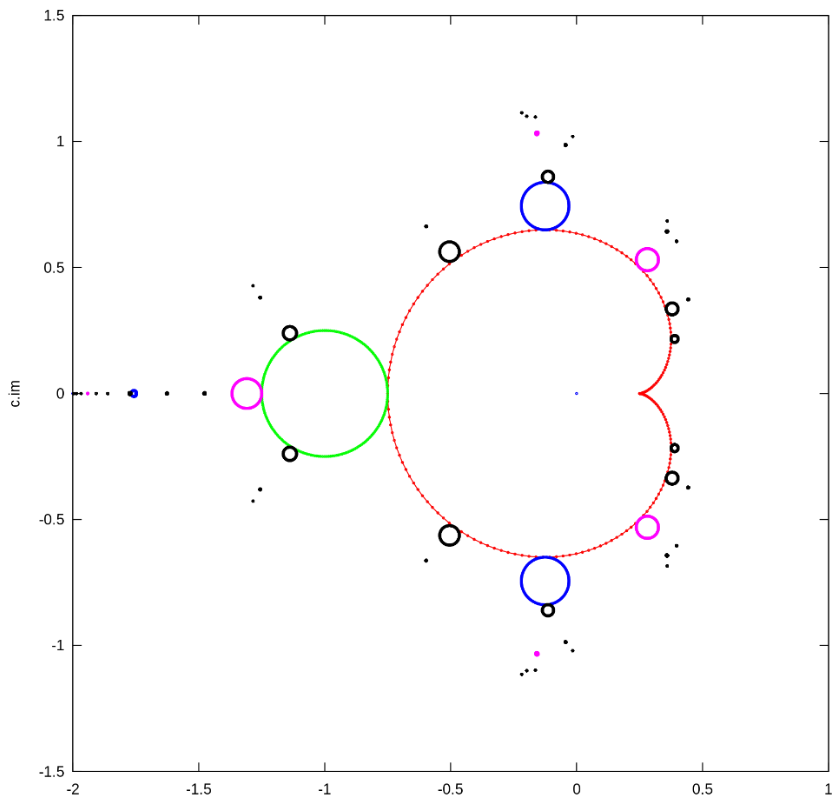
$$c_{\frac{p}{q}} = \frac{e^{2\pi i \frac{p}{q}}}{2}\left(1 - \frac{e^{2\pi i \frac{p}{q}}}{2}\right)$$

This bulb is called the **p/q bulb** of the Mandelbrot Set.

It consists of parameters that have an attracting **Cycle of Period q** and **Combinatorial Rotation Number p/q**.

Hyperbolic Components:

All the bulbs which are interior components of the Mandelbrot set and maps f_c having an attracting periodic cycle are known as the **hyperbolic components**.



Geometry of Mandelbrot Set:

A point **c** belongs to the Mandelbrot set if and only if $|\boldsymbol{r_{max}}| = \mathbf{2}$.

The kidney bean shaped portion of the Mandelbrot set turns out to be bordered by a cardioid with equations:

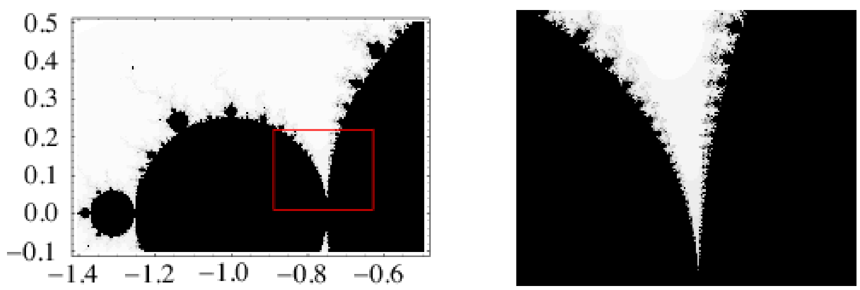
$$\mathbf{4x = 2\cos(t) - \cos(2t)}$$

$$\mathbf{4y = 2\sin(t) - \sin(2t)}$$

Sea Horse Valley:

The region of the Mandelbrot set centered around **-0.75 + 0.1i** is sometimes known as the sea horse valley because the spiral shapes appearing in it resemble sea horse tails (Giffin, Munafo).

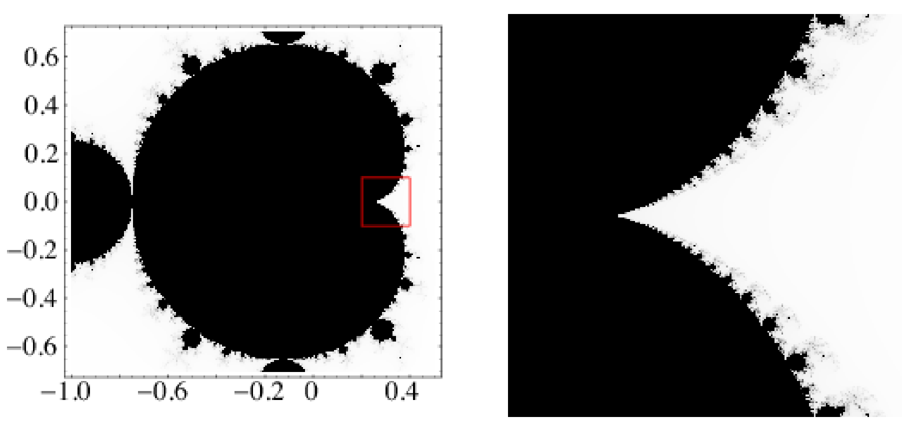
sea horse valley



Elephant Valley:

The portion of the Mandelbrot set centered around **0.3 + 0i** with size approximately **0.1 + 0.1i** is known as elephant valley.

elephant valley



Fractals:

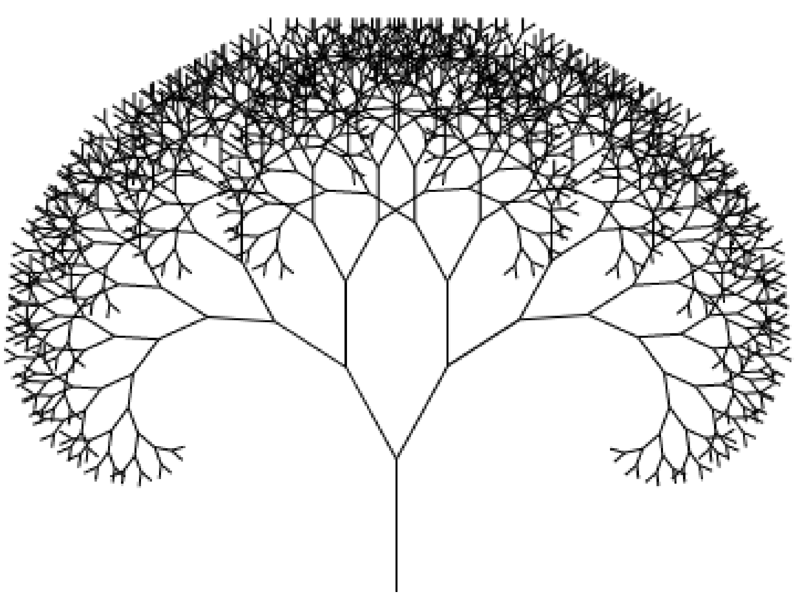
A fractal is a curve or geometrical figure, each part of which has the same statistical character as the whole. They are **useful in modelling structures** (such as snowflakes) in which similar patterns recur at progressively smaller scales, and in describing partly random or chaotic phenomena such as crystal growth and galaxy formation.

In simpler words, a fractal is a **never-ending pattern**. Fractals are infinitely complex patterns that are **self-similar** across different scales. They are created by repeating a simple process over and over in an ongoing feedback loop. Driven by recursion, fractals are images of dynamic systems – the pictures of Chaos.

Geometrically, they exist in between our familiar dimensions. Fractal patterns are extremely familiar since nature is full of fractals.

For instance: trees, rivers, coastlines, mountains, clouds, seashells, hurricanes, etc.

Abstract fractals – such as the Mandelbrot Set – can be generated by a computer calculating a simple equation over and over.



Mandelbrot Set and Julia Set:

- **Julia Set:**

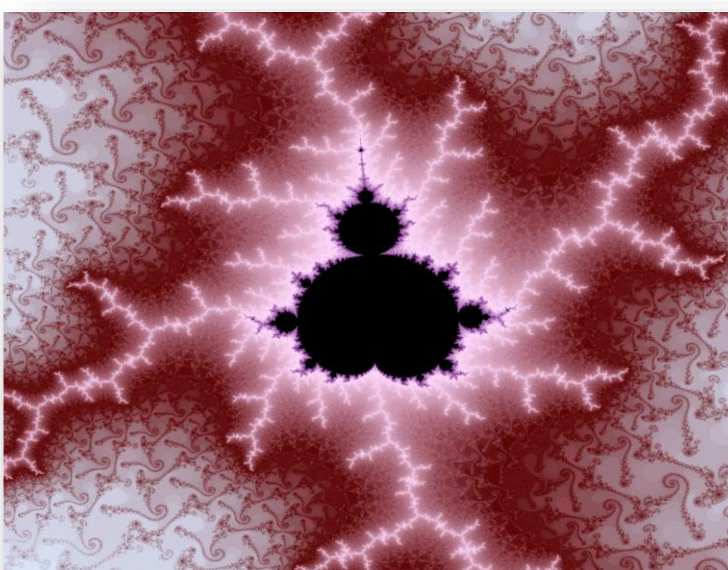
  For a complex number c, the filled-in Julia set of c is the set of all z for which the iteration **z →** z^2**+ c** does not diverge to infinity. The Julia set is the boundary of the filled-in Julia set. For almost all c, these sets are fractals.

- **Mandelbrot Set:**

  The Mandelbrot set is the set of all c for which the iteration **z →** z^2**+ c**, starting from z = 0, does not diverge to infinity.

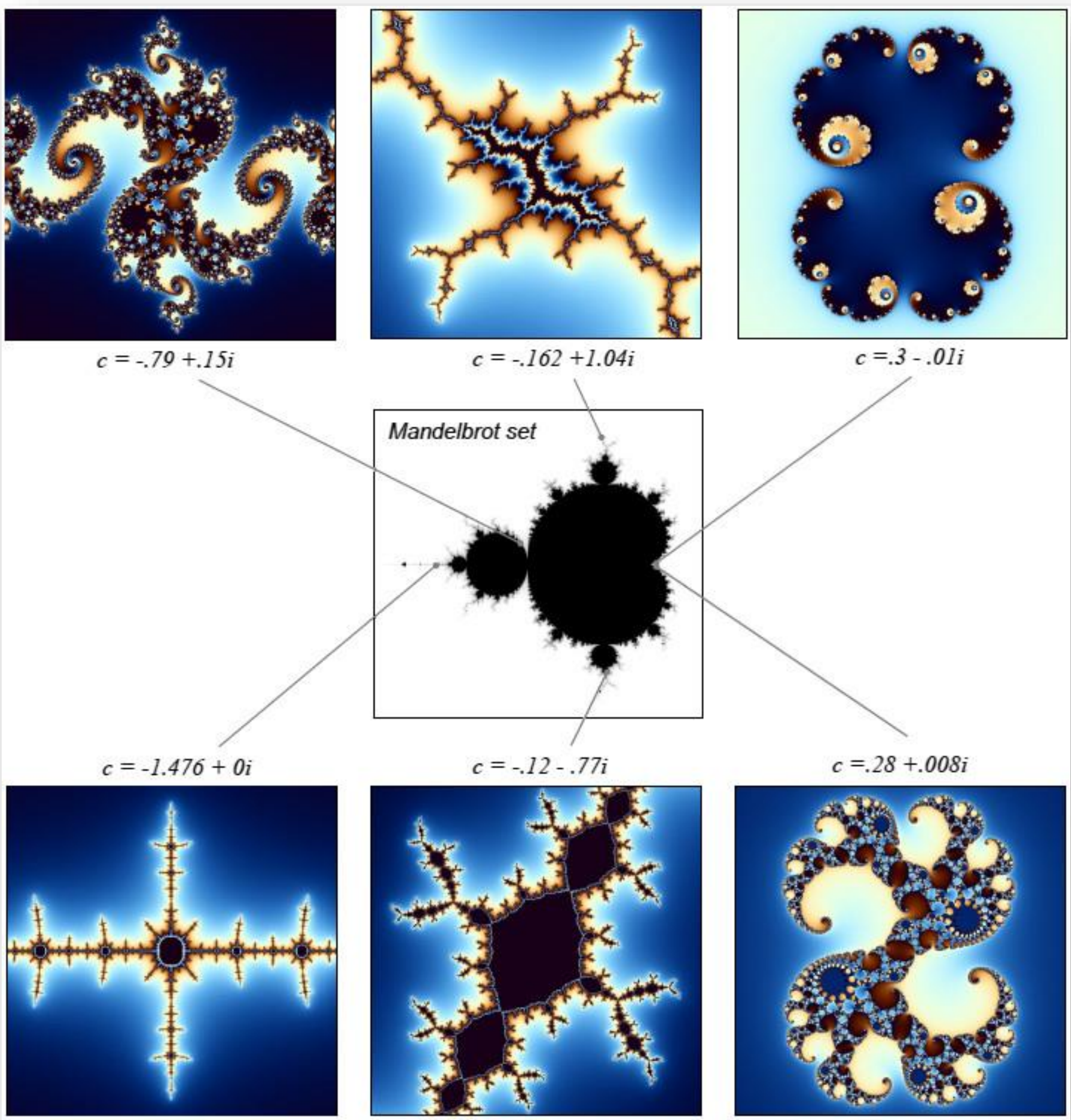
  Julia sets are either connected (one piece) or a dust of infinitely many points. The Mandelbrot set is those c for which the Julia set is connected.

A specific Julia set can be defined by a point in the Mandelbrot set matching its constant c value, and the look of an entire Julia set is usually similar in style to the Mandelbrot set at that corresponding location. Points near the edges of the Mandelbrot set typically give the most interesting Julia sets.



The Mandelbrot set is a dictionary of all Julia sets:

Six Julia Sets and their corresponding locations in the Mandelbrot Set:



Fractal Creation:

Fractals are mathematical sets, usually obtained through recursion, that exhibit interesting dimensional properties. We'll explore what that sentence means through the rest of the chapter. For now, we can begin with the idea of self-similarity, a characteristic of most fractals.

A shape is self-similar when it looks essentially the same from a distance as it does closer up.

Iterated Fractals:

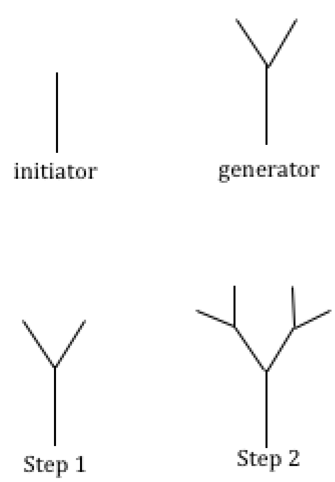
This self-similar behaviour can be replicated through recursion: repeating a process over and over.

An **initiator** is a starting shape.

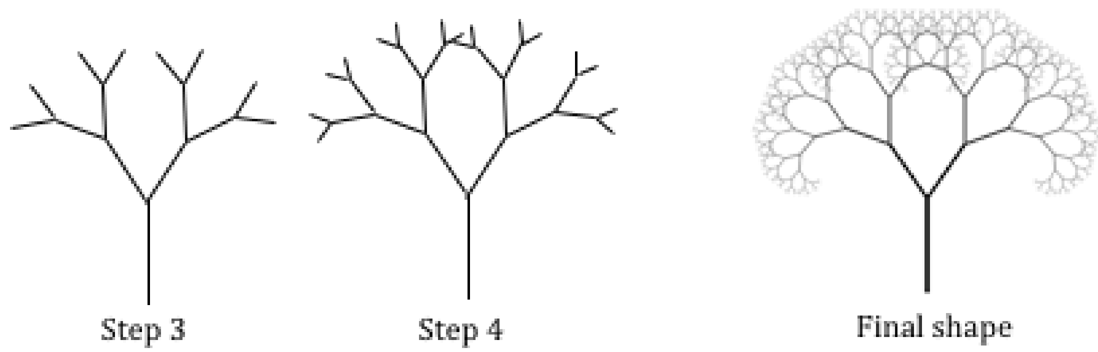
A **generator** is an arranged collection of scaled copies of the initiator.

Fractal Generation Rule:

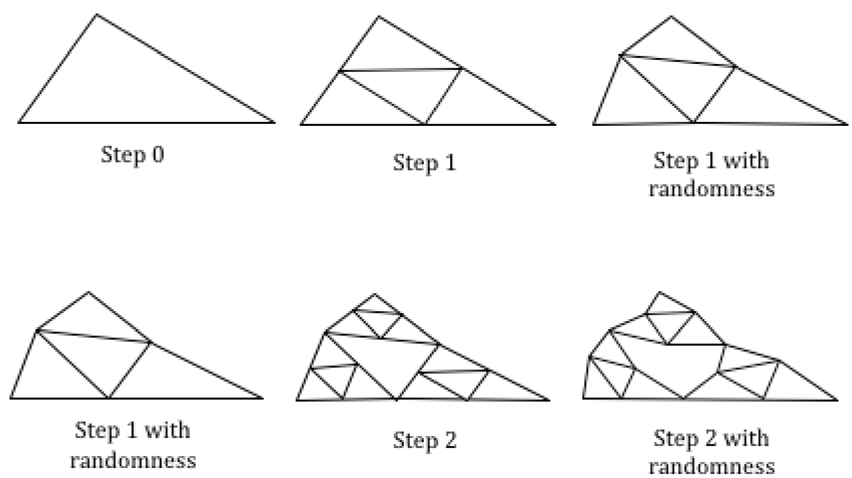
At each step, replace every copy of the initiator with a scaled copy of the generator, rotating as necessary.



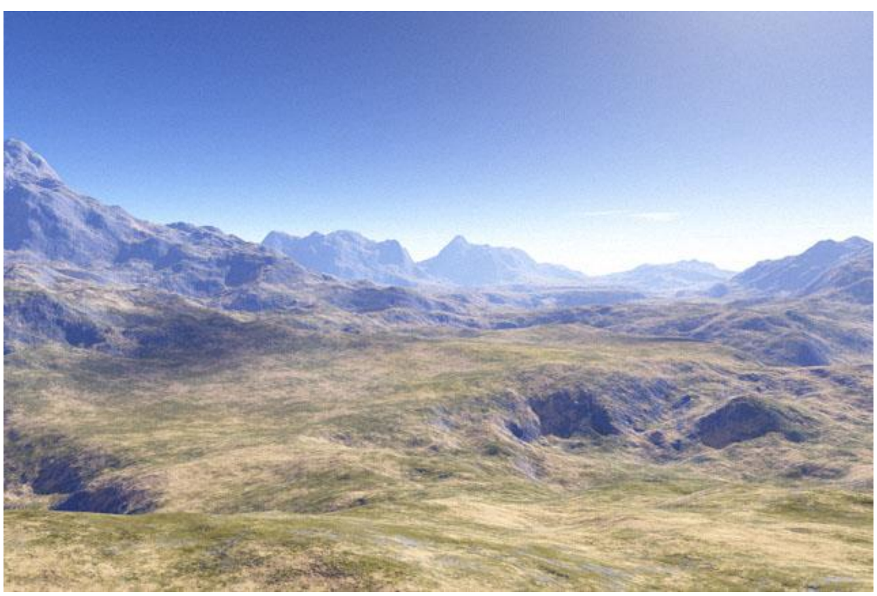
Step 1, the generator. Step 2, one iteration of the generator.



Example 2:

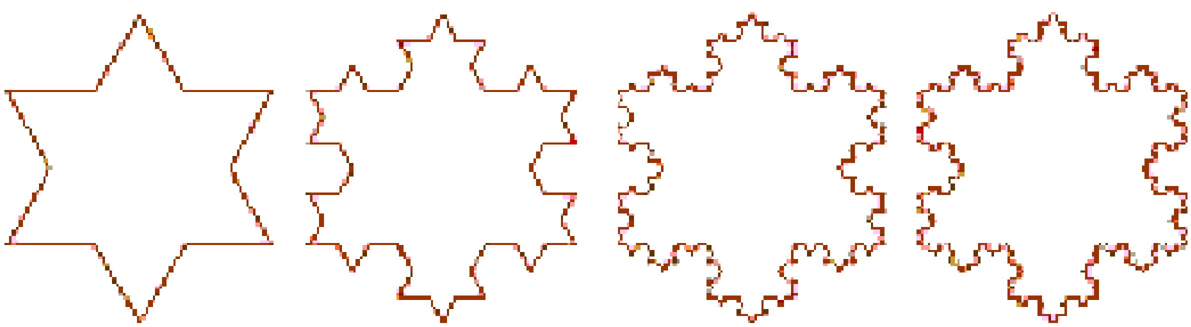


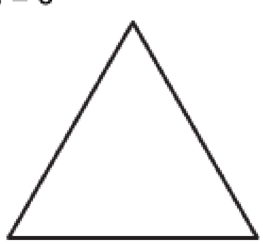
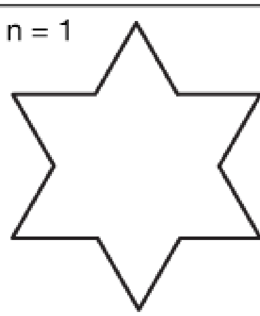
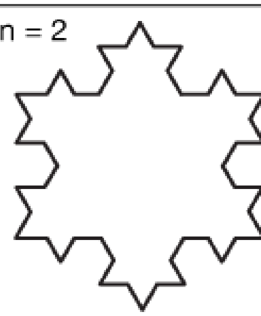
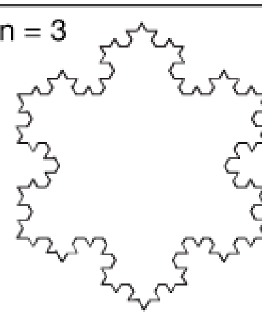
Continuing this process can create **mountain-like structures**. This landscape was created using fractals, then coloured and textured.



Koch Snowflake:

To construct the Koch Snowflake,
we have to **begin** with an **equilateral triangle** with sides of length,
In the middle of each side, we will add a new triangle one-third the size;
and repeat this process for an **infinite number of iterations**.
The length of the boundary is infinity.
However, the area remains less than the area of a circle drawn around the original triangle.
That means that an infinitely long line surrounds a finite area.
The end construction of a Koch Snowflake resembles the coastline of a shore.



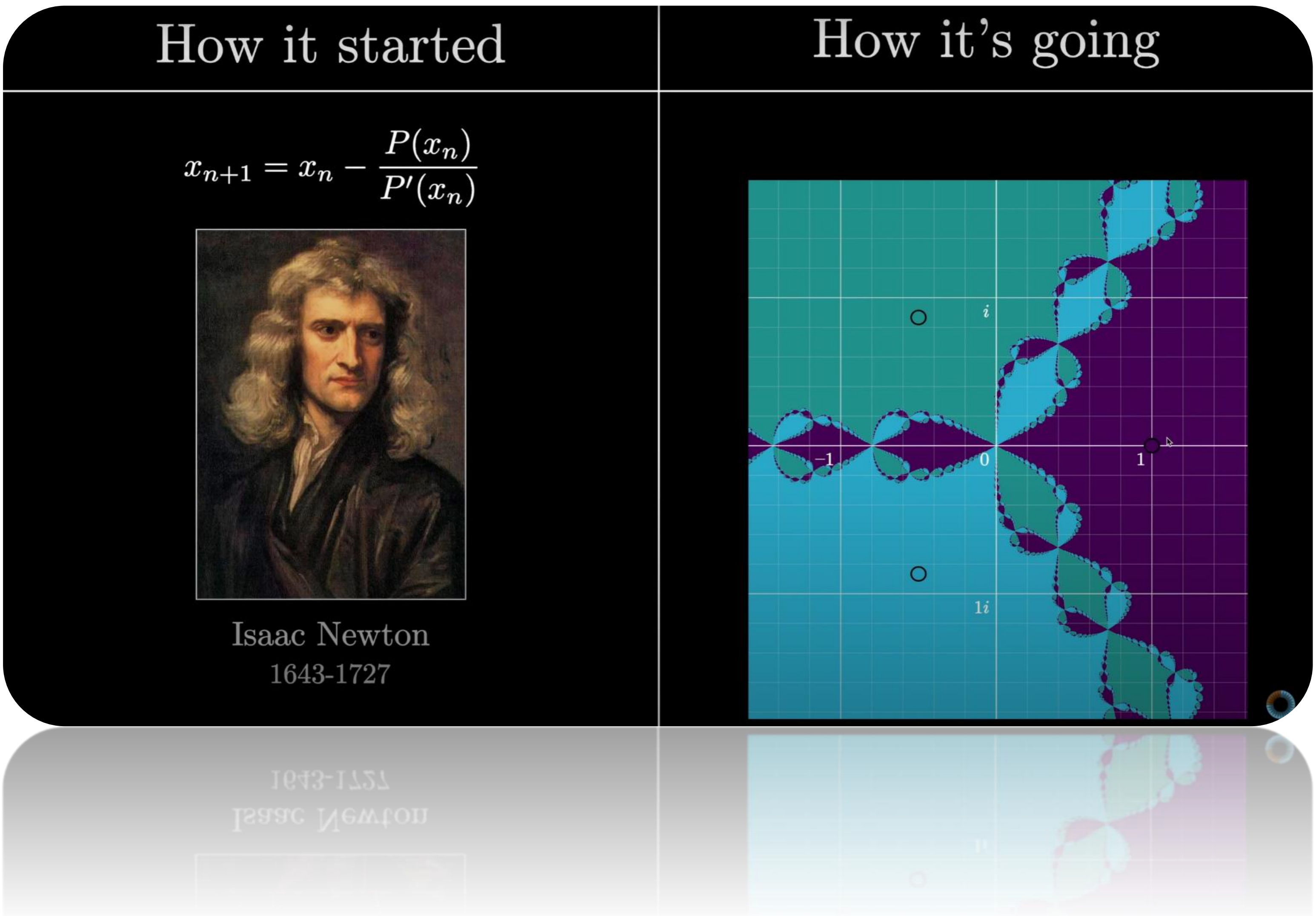
	n = 0	n = 1	n = 2	n = 3
				
Number of sides (N)	3	12	48	192
Side length (S)	1	1/3	1/9	1/27
Perimeter length (P)	3	4	5.33	9.11

Newton’s Theory and Fractals:

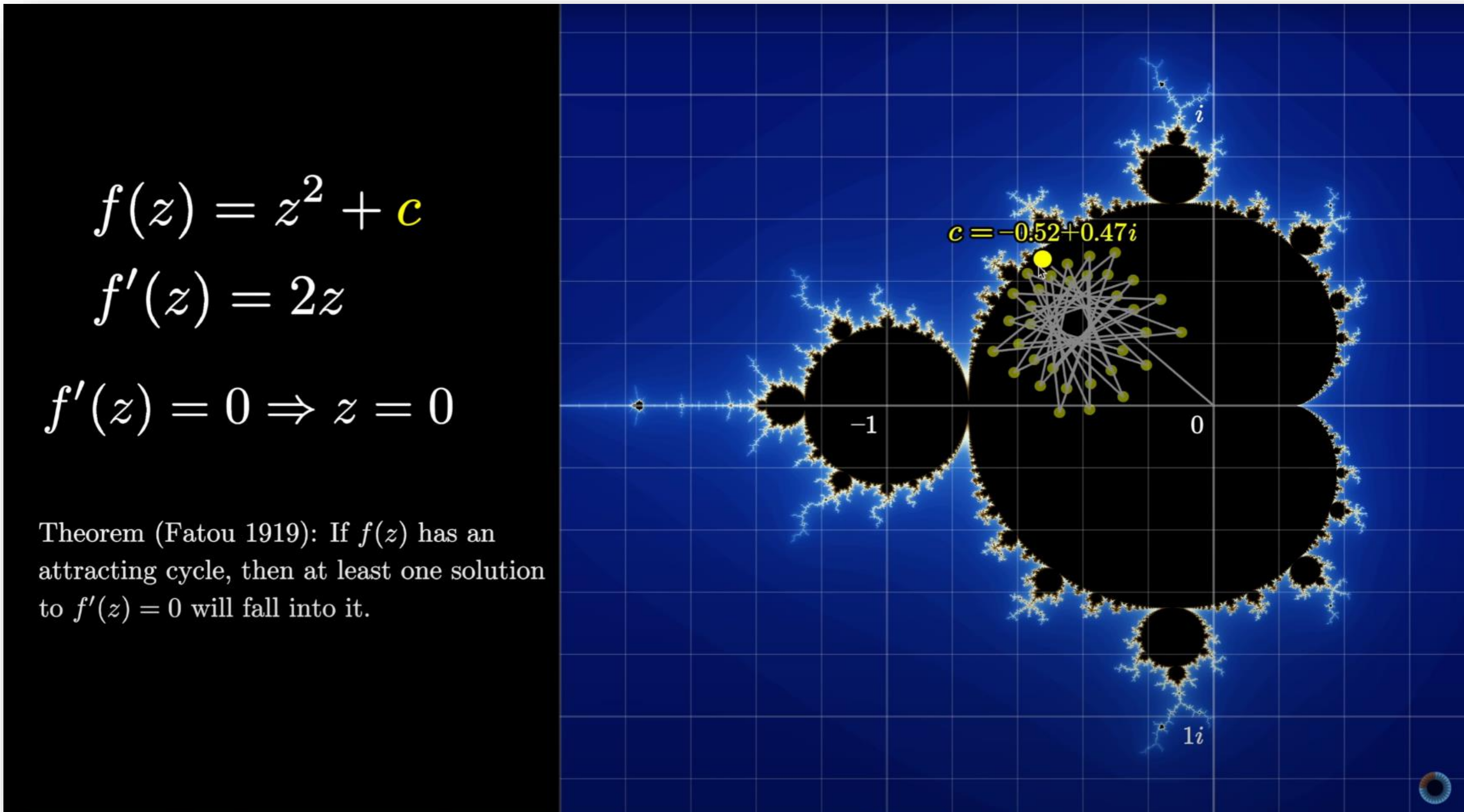
Issac Newton proposed a method to **solve a polynomial** of any degree using graphs and coordinate geometry by using the formula:

$$\boldsymbol{x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}}$$

This formula give rise to Fractal graph as follows which are nowadays known as Newton Fractals:



Newton Fractals and Mandelbrot Set:



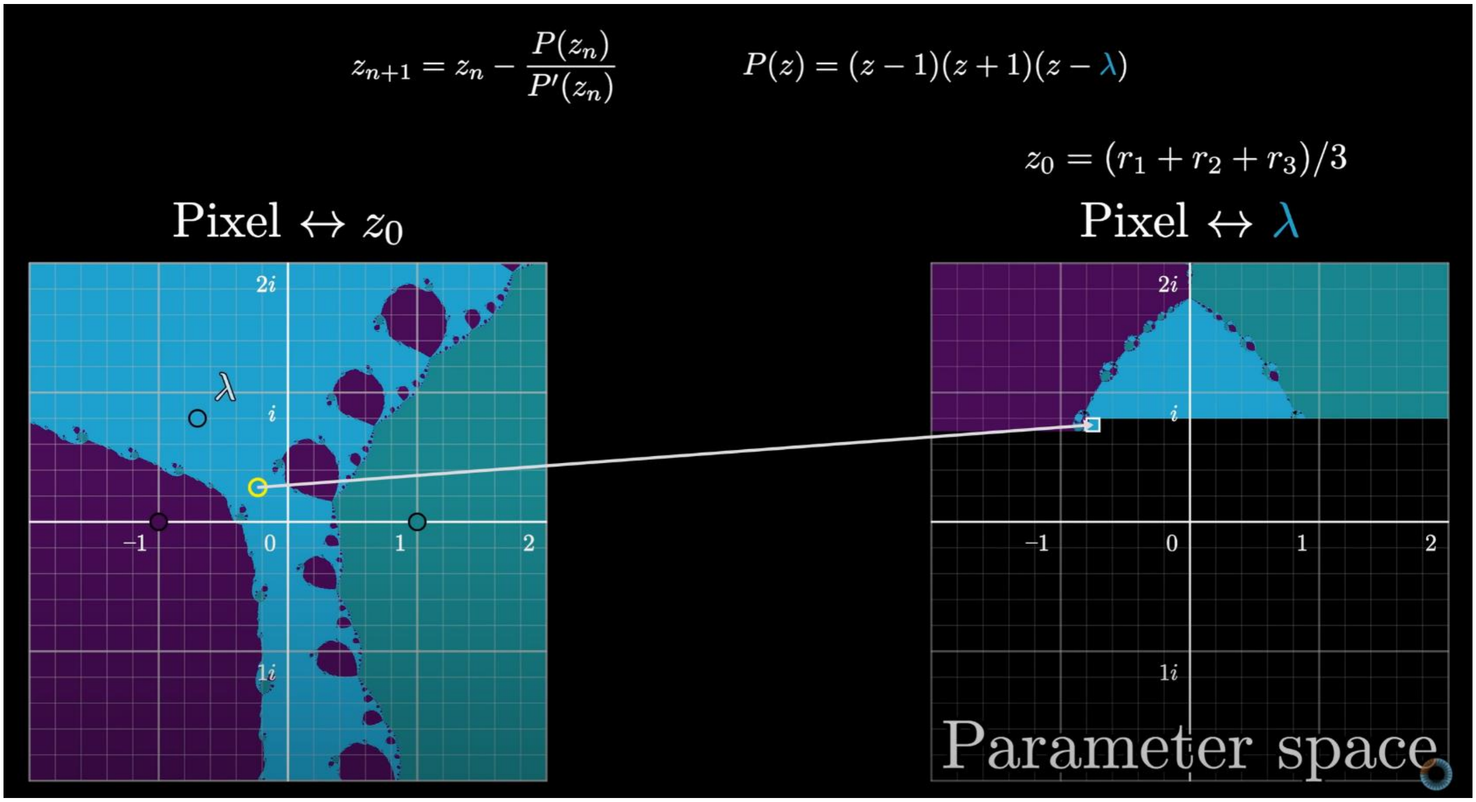
We take Newton's formula:

$$z_{n+1} = z_n - \frac{P(z_n)}{P\prime(z_n)}$$

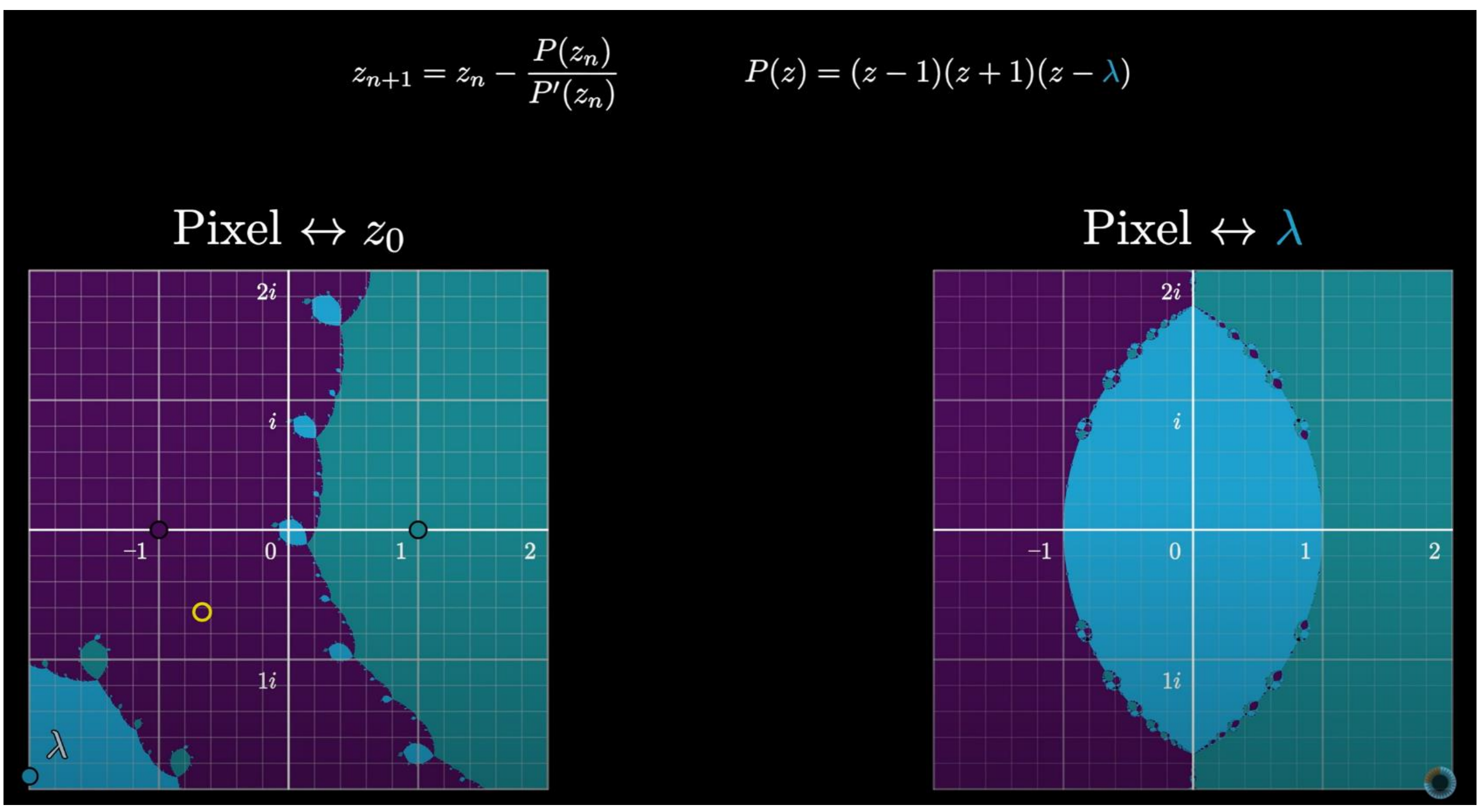
Let we fix 2 points **z = 1**, **z = -1** and take an arbitrary point **z = λ** which lies in **|z| <= 2** which is required condition for Mandelbrot set and map the point $z = \frac{-1+1+\lambda}{3}$ onto a parametric space

$$P(z_n) = (z - 1)(z + 1)(z - \lambda)$$
$$P'(z_n) = (z + 1)(z - \lambda) + (z - 1)(z - \lambda) + (z - 1)(z + 1)$$

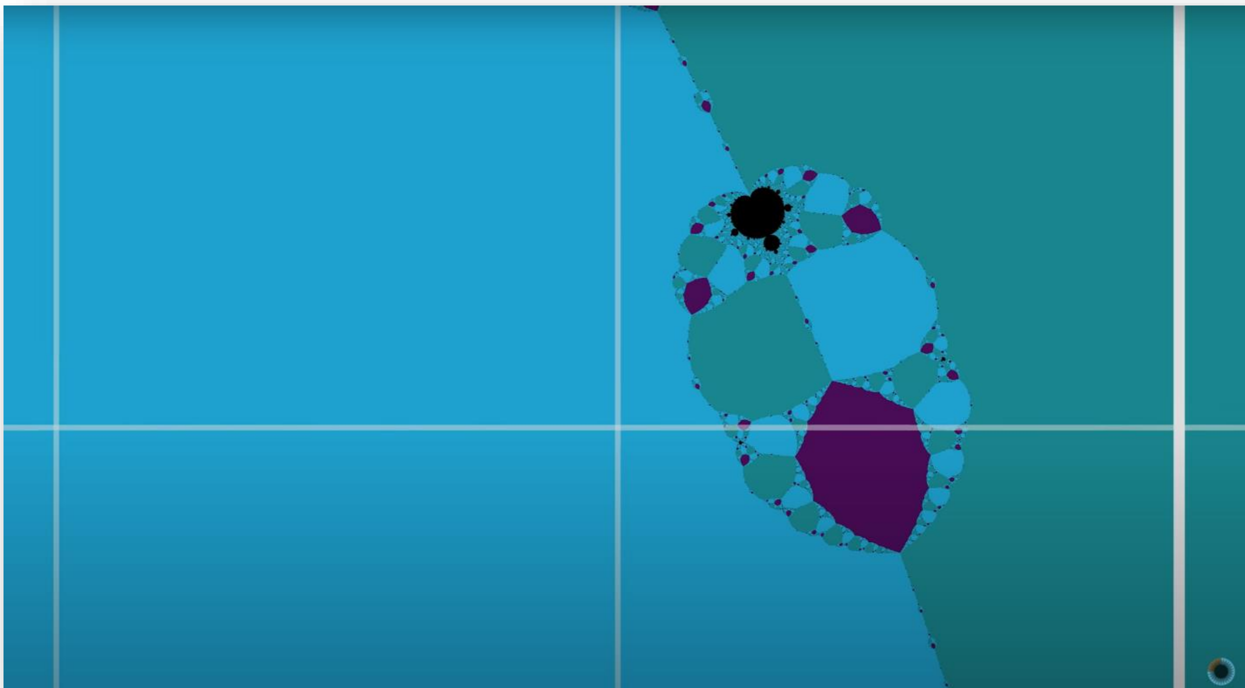
So first we have to map the point onto parametric space.

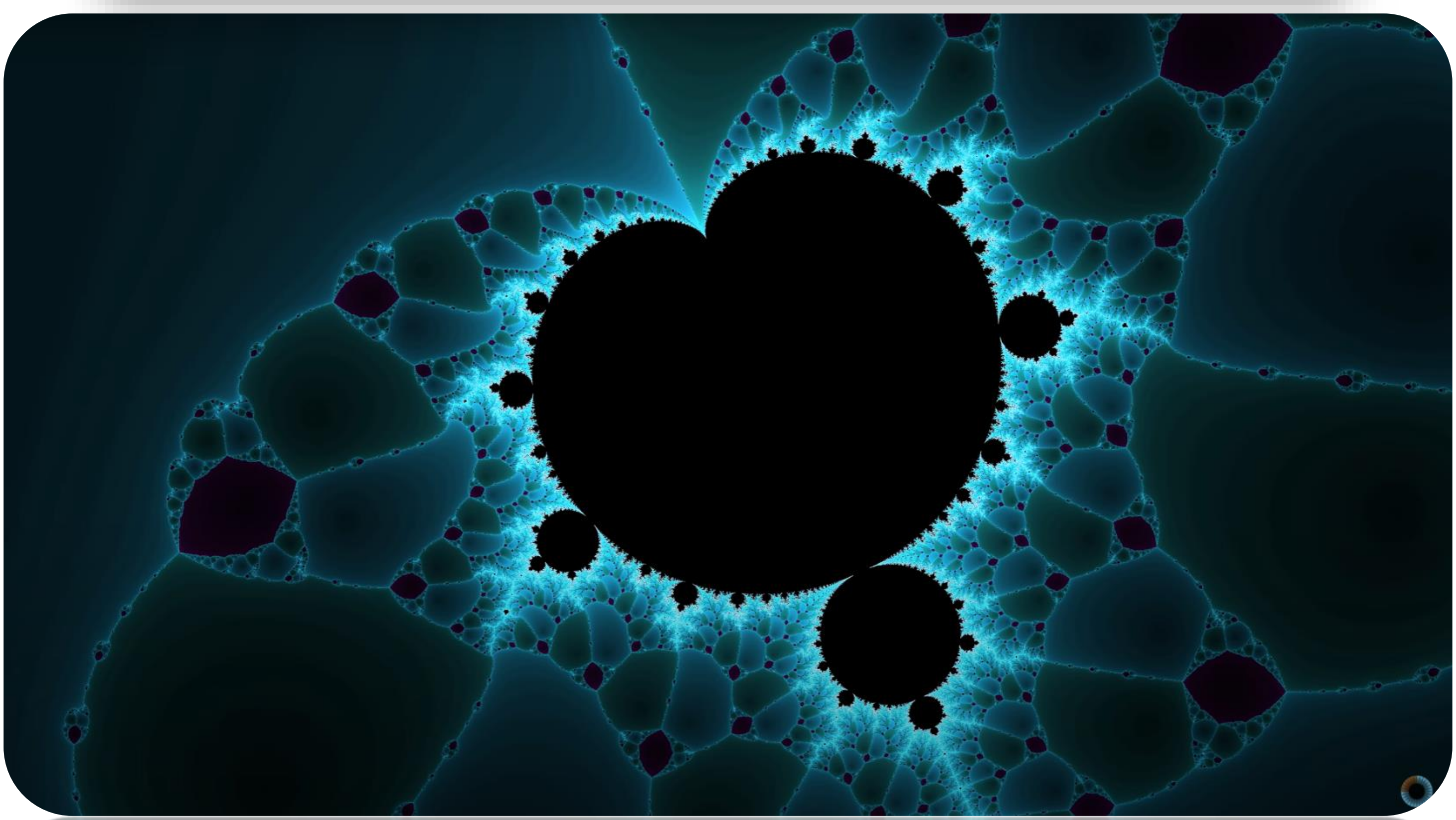


Finally we get this mapping,



Now if we closely observe this we get Mandelbrot figure,





Final Thoughts:

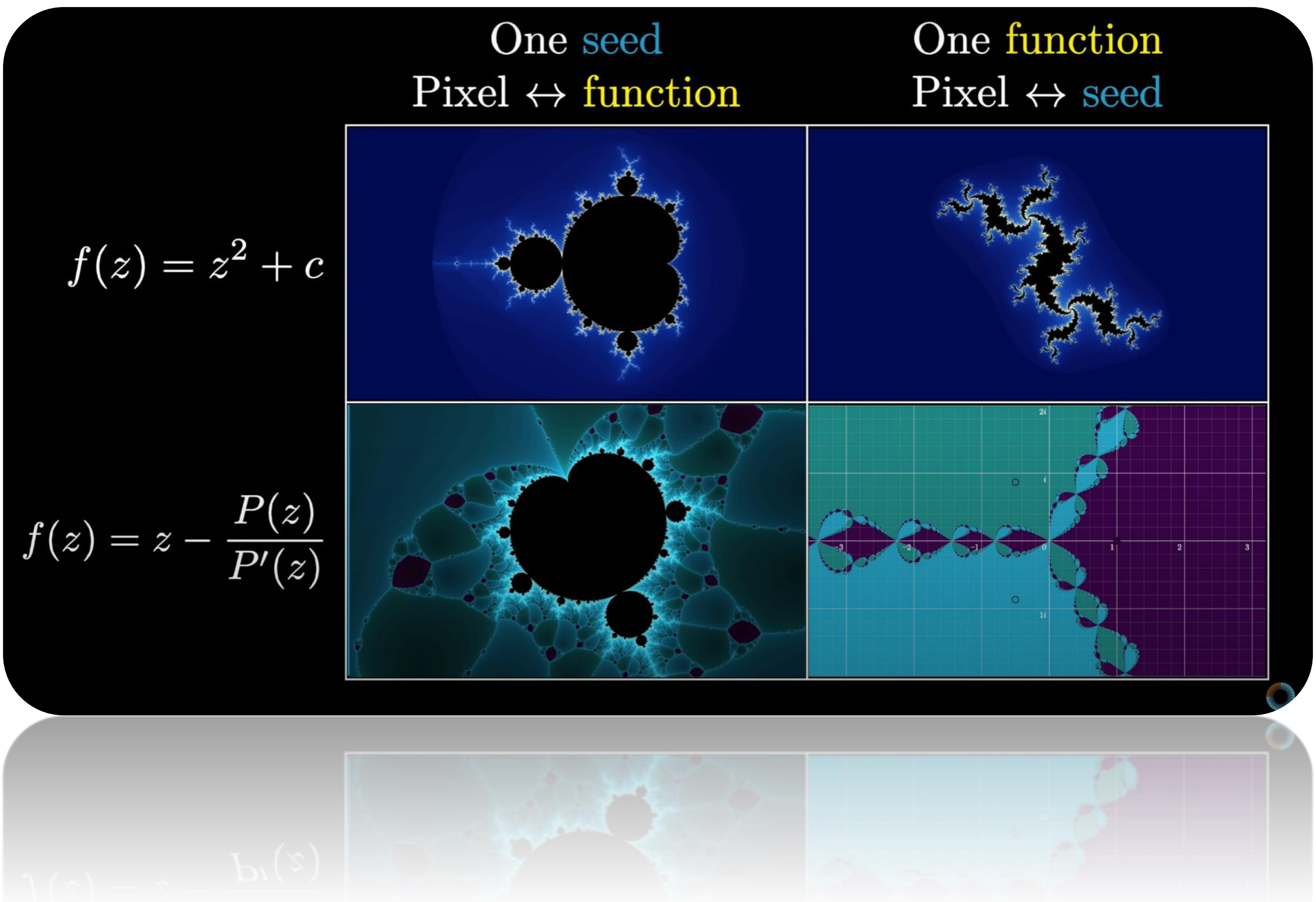
As we can clearly see that,

Newton Fractal can be considered as the universal fractal because we can derive Mandelbrot set from Newton Fractals.

Or we can say that any fractal can't be consider as the universal because there will be a limit to every fractal.

So, Mandelbrot is a special case of Newton Fractal when,

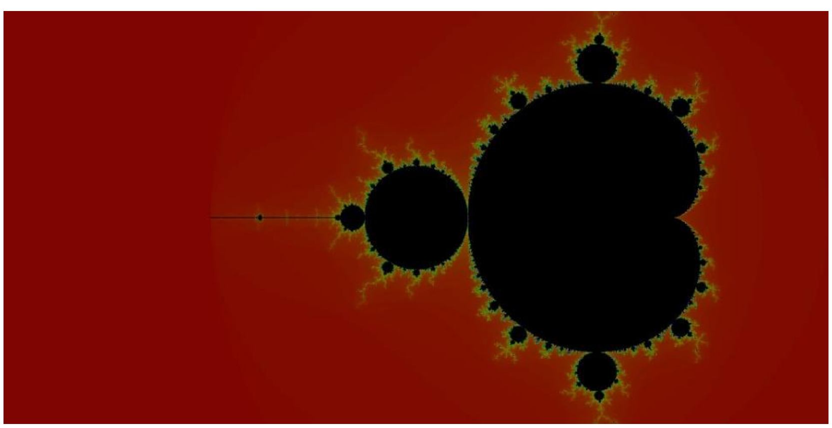
$$\boldsymbol{z_n - \frac{P(z_n)}{P'(z_n)} = z_n{}^2 + c}$$



Codes:

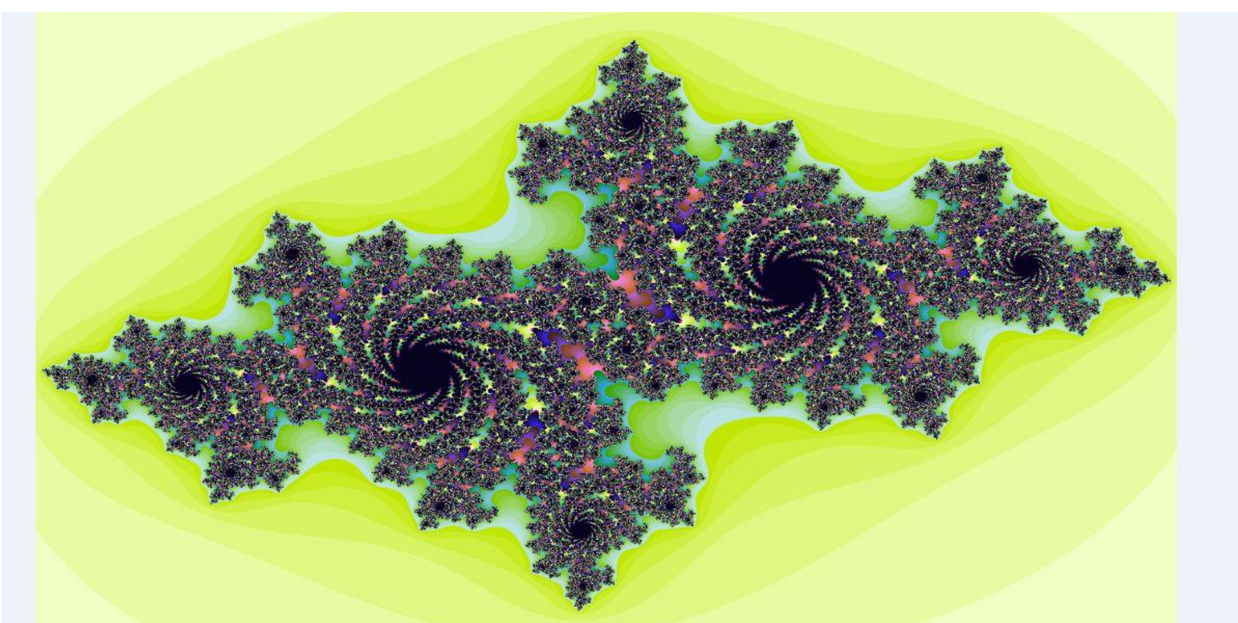
Mandelbrot Fractal:

```
// Aneesh Panchal 2K20/MC/21
// Ayushi Sagar 2K20/MC/35

from PIL import Image
from numpy import complex, array
import colorsys

WIDTH = 1024
def rgb_conv(i):
    color = 255 * array(colorsys.hsv_to_rgb(i / 255.0, 1.0, 0.5))
    return tuple(color.astype(int))

def mandelbrot(x, y):
    c0 = complex(x, y)
    c = 0
    for i in range(1, 1000):
        if abs(c) > 2:
            return rgb_conv(i)
        c = c * c + c0
    return (0, 0, 0)

img = Image.new('RGB', (WIDTH, int(WIDTH / 2)))
pixels = img.load()

for x in range(img.size[0]):
    for y in range(img.size[1]):
        pixels[x, y] = mandelbrot((x - (0.75 * WIDTH)) / (WIDTH / 4),
                                  (y - (WIDTH / 4)) / (WIDTH / 4))
img.show()
```



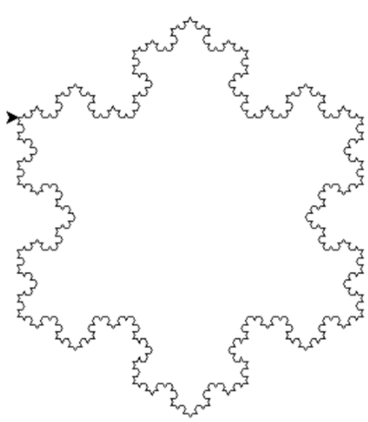
Julia Fractal:

```
// Aneesh Panchal 2K20/MC/21
// Ayushi Sagar 2K20/MC/35

from PIL import Image
if __name__ == "__main__":
    w, h, zoom = 1920,1080,1

    bitmap = Image.new("RGB", (w, h), "white")
    pix = bitmap.load()

    cX, cY = -0.7, 0.27015
    moveX, moveY = 0.0, 0.0
    maxIter = 255

    for x in range(w):
        for y in range(h):
            zx = 1.5*(x - w/2)/(0.5*zoom*w) + moveX
            zy = 1.0*(y - h/2)/(0.5*zoom*h) + moveY
            i = maxIter
            while zx*zx + zy*zy < 4 and i > 1:
                tmp = zx*zx - zy*zy + cX
                zy,zx = 2.0*zx*zy + cY, tmp
                i -= 1
            pix[x,y] = (i << 21) + (i << 10) + i*8
    bitmap.show()
```



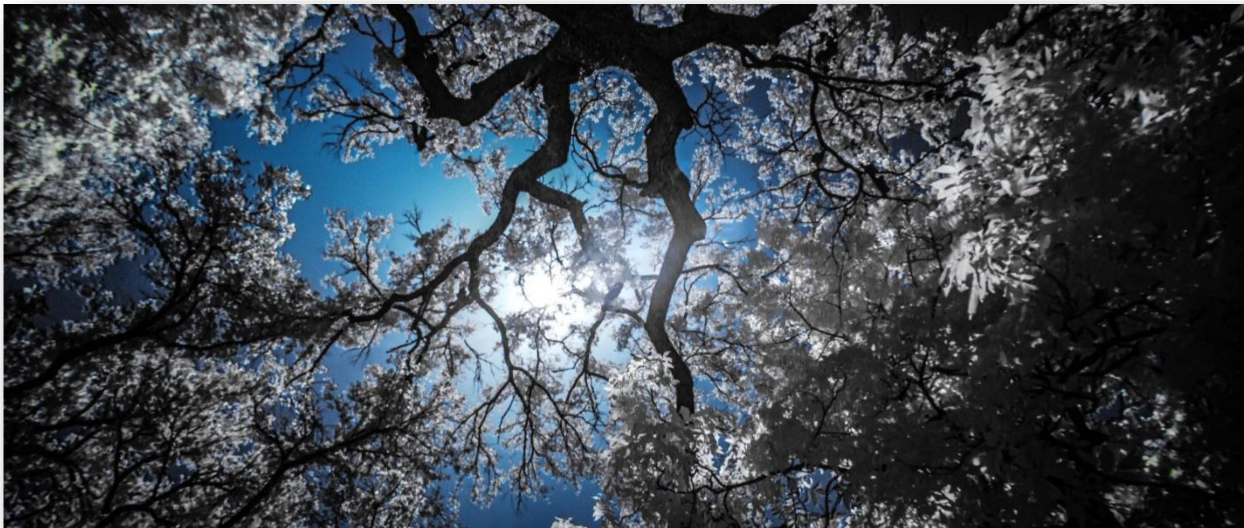
Koch Snowflake Curve:

```
// Aneesh Panchal 2K20/MC/21
// Ayushi Sagar 2K20/MC/35

from turtle import *
def snowflake(lengthSide, levels):
    if levels == 0:
        forward(lengthSide)
        return
    lengthSide /= 3.0
    snowflake(lengthSide, levels-1)
    left(60)
    snowflake(lengthSide, levels-1)
    right(120)
    snowflake(lengthSide, levels-1)
    left(60)
    snowflake(lengthSide, levels-1)

if __name__ == "__main__":
    speed(0)
    length = 300.0
    penup()

    backward(length/2.0)
    pendown()
    for i in range(3):
        snowflake(length, 4)
        right(120)

    mainloop()
```

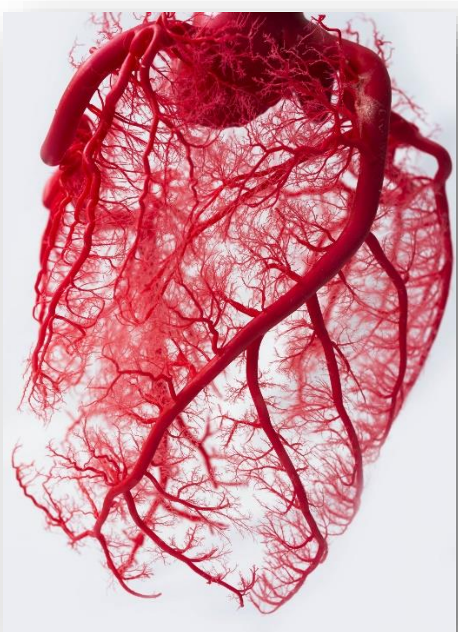


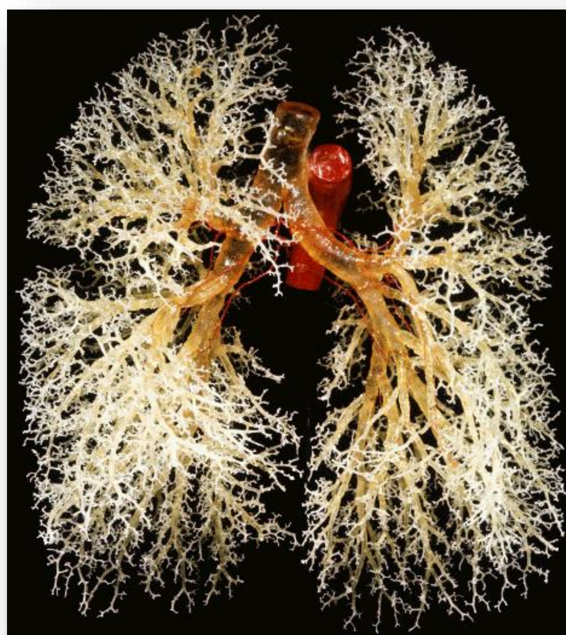
Fractal Examples:

➢ **Fractal Trees**



➢ **Fractals in Animal Bodies**



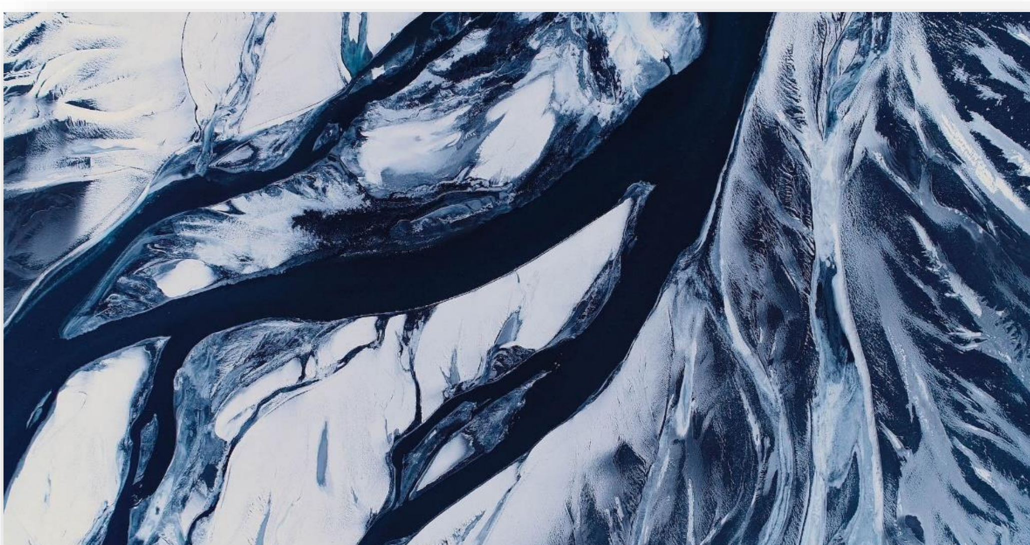


- **Fractal Snowflakes**

- **Fractal Lightning and Electricity**

➢ Fractals in Plants and Leaves

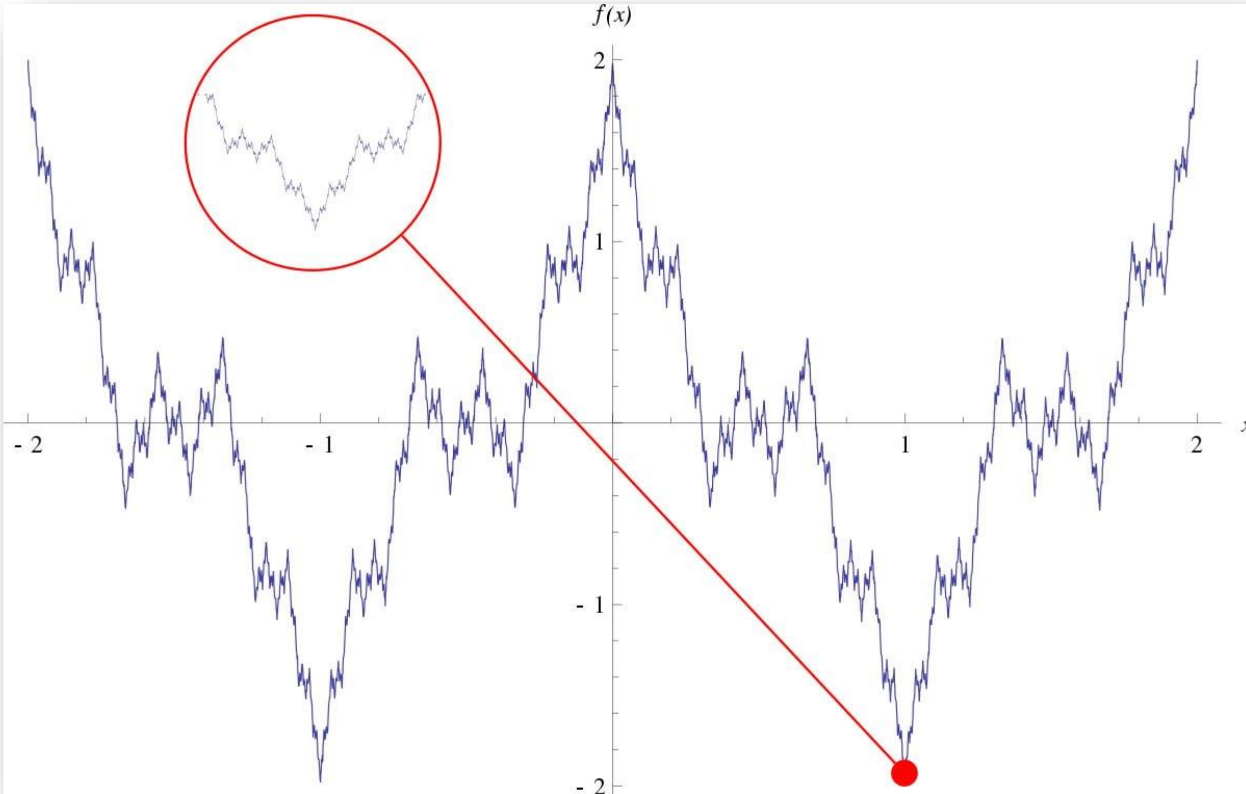
➢ Fractals in Geography, Rivers, and Terrain



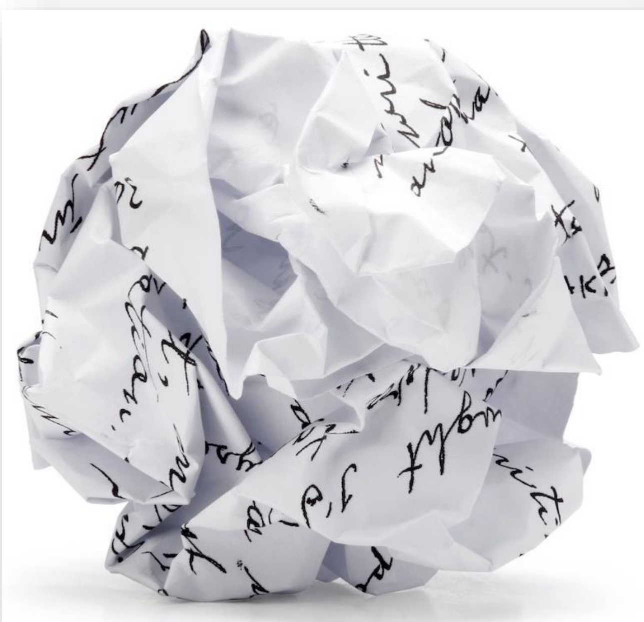
➢ Fractals in Clouds

➢ Fractals in Crystals

➢ Fractals in 2D Modelling



➢ General day to day Fractal



Thoughts of Mandelbrot:

Mandelbrot famously wrote:

"**Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.**"

The chaos and irregularity of the world - Mandelbrot referred to it as "**roughness**" - is **something to be celebrated**.

It would be a shame if clouds really were spheres, and mountains cones.

References:

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