Fall 2024 – 16-642 Manipulation, Estimation, and Control
Problem Set 1

Due: Monday September 23rd 2024

GUIDELINES:

- You must *neatly* write up your solutions and submit all required material electronically via Canvas by **11:59 pm of the posted due date.**
- Include any MATLAB scripts that were used in finding the solutions. Also include any plots generated with MATLAB as a part of the question.
- You are encouraged to work with other students in the class, however you must turn in your own *unique* solution.
- Late Policy: You are given **in total 72 hours** of grace period for **ALL** homework across the semester, which you can use to give yourself extra time without penalty. Late work handed in when you have run out of grace will not be accepted.

1. Consider the classic example of an ideal Spring-Mass-Damper system with mass m, damping constant μ, spring constant k. Linear position of the mass from rest is represented as q, and the input force to the system is u.

   (a) (2 points) Write out the second-order ODE, derived from applying Newton's Law, using m, μ, k, q, and u. (You are not required to draw the free body diagram, but it can be helpful).

   (b) (3 points) Represent the ODE as a linear state space model. Assume that $y = q$. Your answer should have two equations (one with A, B and the other with C).

   (c) (5 points) Under what conditions for the variables is this system stable? Under what conditions is it unstable? Note that in this model, m, μ, and k must have physically plausible values.

   (d) (5 points) Under what conditions for the variables is this system controllable?

   (e) (5 points) Let $m = 1, \mu = 0.5$, and $k = 5$. Let the initial state vector be

   $$x_0 = x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

   Using the MATLAB `expm` command, plot the output of the unforced system for $t \in [0, 5]$.

   (f) (5 points) Use the MATLAB `place` command to find the matrix K such that the matrix $A - BK$ contains eigenvalues $\{-1 + i, \ -1 - i\}$.

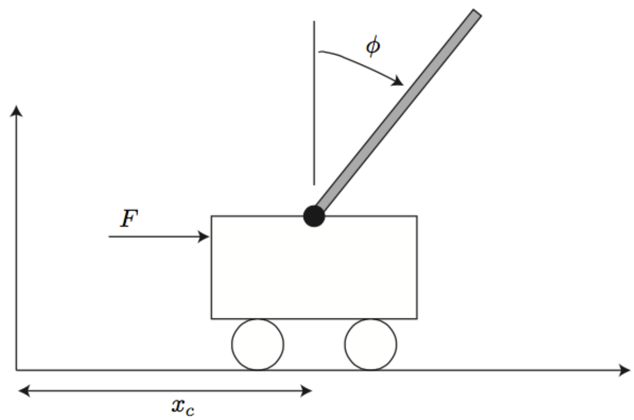
   Additionally, explain why we prefer $A - BK$ to have eigenvalues λ where $Re(\lambda) < 0$.

   (g) (5 points) Let the initial state vector be

   $$x_0 = x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   Use the MATLAB `expm` command and plot the output of the system under the feedback law $u(t) = -Kx(t)$ for $t \in [0, 10]$. Use the K found in the previous problem.

2. Consider the "pendulum on a cart" system:



The equations of motion for this system are

$$\gamma \ddot{x}_c - \beta \ddot{\phi} \cos\phi + \beta \dot{\phi}^2 \sin\phi + \mu \dot{x}_c = F$$

$$\alpha \ddot{\phi} - \beta \ddot{x}_c \cos\phi - D \sin\phi = 0,$$

where α, β, γ, D, and μ are physical constants determined by the masses of the cart and pendulum, pendulum length, and friction, and F is an externally applied force. All quantities use their respective meters/kilograms/seconds/radians standard units.

(a) (10 points) Define the state vector

$$x = \begin{bmatrix} x_c \\ \phi \\ \dot{x}_c \\ \dot{\phi} \end{bmatrix}$$

and the input $u = F$. Write the equations of motion as a nonlinear state space equation.

If you try doing this the "direct" way by hand (you should try it for 10 minutes), you will notice that it becomes complicated fast! Instead, we will use a helpful observation. We define

$$M = \begin{bmatrix} \gamma & -\beta\cos\phi \\ -\beta\cos\phi & \alpha \end{bmatrix}$$

which has the useful property that it is always invertible.

(1) Now find a matrix N involving the equations of motion such that $M \begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} = N$

(2) Now try writing the nonlinear state space equation. You should be able to use M, N and the fact that the inverse of a 2D matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Although annoying, please do this entire section by hand.

(b) (5 points) Verify your answer for the previous answer by using Matlab's symbolic equation toolbox. Specifically, directly solve the nonlinear equation symbolically and check that if it matches your solution. You can just show your Matlab solution for $\ddot{x}_c, \ddot{\phi}$ and we note that our Matlab solution was only 4 lines.

(c) (10 points) Describe the set of equilibrium points for the system. Describe them mathematically (i.e., with equations) and in English (i.e., what do they physically mean).

(d) (5 points) Let $\gamma = 2$, $\alpha = 1$, $\beta = 1$, $D = 1$, and $\mu = 3$. The linearized system about the equilibrium point at $x = 0$ is

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u(t).$$

Compute the eigenvalues of A, and use them to say whatever you can about the stability of the equilibrium point at $x = 0$ for both the linearized system and original nonlinear system.

(e) (15 points) Assume that the entire state can be measured directly (i.e., there is no need for an observer). Letting $Q_u = 10$ and

$$Q_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

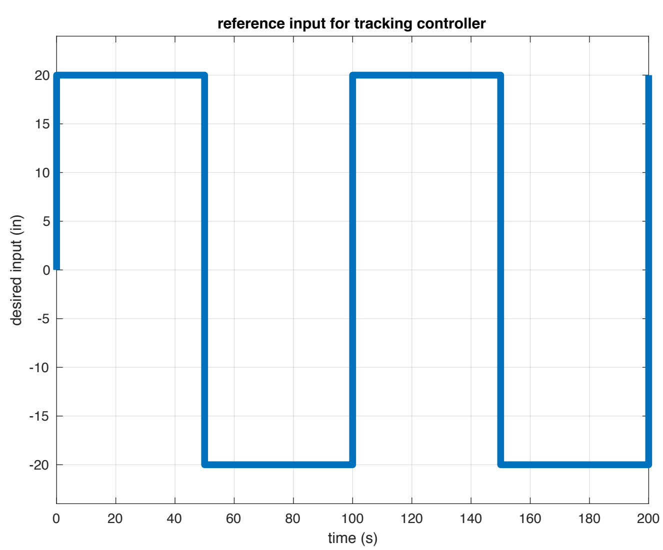
use the MATLAB lqr command to find the corresponding optimal feedback control $u(t) = -K_c x(t)$.

Using a timestep of $T = 0.01$ seconds and a final time of $t_f = 30$ seconds and an initial state of $x_0 = [\,0,\ 0.1,\ 0,\ 0\,]^T$, calculate and plot the state of the *linearized system* under the feedback control law above. You may either write your own 4th order Runge-Kutta routine to solve the state equation, or use the MATLAB function `ode45`. Repeat for $x_0 = [\,0,\ 0.5,\ 0,\ 0\,]^T$, $x_0 = [\,0\,1.0886\,0\,0\,]^T$, and $x_0 = [\,0,\ 1.1,\ 0,\ 0\,]^T$.

(f) (5 points) Repeat part 2e using the full nonlinear state equations in the simulation (as opposed to the linearized state equations). Explain any differences you see in the results.

(g) (5 points) Assume you have an output y given by a sensor that measures the cart position *in inches*. Find the matrix C so that $y = Cx$.

(h) (10 points) Using the LQR controller designed above, create a tracking controller that allows you to specify a desired cart position trajectory. Test this tracking controller for a desired output y_d that is a square wave as shown in the picture below. Simulate the full nonlinear dynamics using $T = 0.01$ seconds, $t_f = 200$ seconds, and $x_0 = [\,0,\ 0,\ 0,\ 0\,]^T$. Plot the state vs. time. On a separate graph, make a plot that overlays the desired and actual outputs. Explain what is going on.



(i) (10 points) Choose new Q_x and Q_u for the LQR controller to make the tracking controller better. Explain what you mean by "better", and describe your reasoning for how you changed Q_x and Q_u.

Simulate the system using the same parameters as above, and demonstrate your improvement by plotting the desired and actual outputs.