

# MAT354 Lecture Notes

ARKY!! :3C

'25 Fall Semester

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## §1 Day 1: Recap of Preliminaries (Sep. 2, 2025)

We start by discussing the complex plane and complex numbers. Given  $z \in \mathbb{C}$ , we say that  $\Re(z)$  and  $\Im(z)$  are the real and imaginary parts of  $z$  respectively, i.e.,  $z = x + iy$ .  $\mathbb{C}$  is the set of all complex numbers. In this manner, we may identify  $z = x + iy$  with  $(x, y) \in \mathbb{R}^2$  using the standard complex plane.

- (a) The complex *conjugate* of  $z$  is given by  $\bar{z} = x - iy$ , where we have that

$$\Re(z) = \frac{z + \bar{z}}{2}, \quad \Im(z) = \frac{z - \bar{z}}{2i}.$$

- (b) We now define addition and multiplication for the complex numbers. For all  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ , we have that

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \end{aligned}$$

We have that  $(\mathbb{C}, +, \times)$  is a field, with  $(\mathbb{R}, +, \times)$  as a subfield. To verify this, we need to check that it indeed satisfies:

- Commutativity: for all  $z_1, z_2 \in \mathbb{C}$ , we have that  $z_1 + z_2 = z_2 + z_1$  and  $z_1 z_2 = z_2 z_1$ .
  - Associativity: for all  $z_1, z_2, z_3 \in \mathbb{C}$ , we have that  $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$  and  $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ .
  - Distributivity: for all  $z_1, z_2, z_3 \in \mathbb{C}$ , we have that  $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ .
- (c) The absolute value of a complex number  $z = x + iy$  is given by  $|z| = \sqrt{x^2 + y^2}$ . In particular, this yields the triangle inequality, where for any  $z, w \in \mathbb{C}$ , we have that  $|z + w| \leq |z| + |w|$ . The proof either comes visually or through explicit computation, both of which I will not write out here for brevity.<sup>1</sup>

As an extension of the inequality, we also automatically have that

$$|\Re z| \leq |z|, \quad |\Im z| \leq |z|,$$

and that for all  $z, w \in \mathbb{C}$ , we have

$$||z| - |w|| \leq |z - w|.$$

*Proof.* Using the triangle inequality, we have that

$$\begin{aligned} |z| &= |(z - w) + w| \leq |z - w| + |w|, \\ |w| &= |(w - z) + z| \leq |z - w| + |z|, \end{aligned}$$

of which both imply that  $|z| - |w| \leq |z - w|$  and  $|w| - |z| \leq |z - w|$ . □

For any  $z \in \mathbb{C}$ , we have that  $|z|^2 = z \cdot \bar{z}$ .

*Proof.* Write  $z = x + iy$ ; then  $|z|^2 = x^2 + y^2$ , where we may note that  $z \cdot \bar{z} = (x + iy)(x - iy)$  which yields the right hand side of the earlier equation through expansion. □

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<sup>1</sup>no full credit if you draw a picture on the exam lmao

Finally, for  $z, w \in \mathbb{C}$ , we have that  $|zw| = |z||w|$ . This is left as an exercise to the student.

- (d) The polar form of a nonzero complex number  $z \neq 0$  is given by  $z = \gamma e^{i\theta}$ , where  $\gamma > 0$  and  $\theta \in \mathbb{R}$ . Let us assume the Euler formula; for all  $\theta \in \mathbb{R}$ , we have that

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Let  $r = |z|$ ; we have that  $|z| = |re^{i\theta}| = |r| |e^{i\theta}| = r \cdot 1 = r$ .  $\theta$  is the angle between the positive real axis to the half-line starting from 0 and passing through  $z$ . In this manner,  $z = re^{i\theta} = |z|(\cos \theta + i \sin \theta) = |z| \cos \theta + i |z| \sin \theta$ , which means we have that

$$\Re z = |z| \cos \theta, \quad \Im z = |z| \sin \theta.$$

As an example, let us find all the complex numbers  $z$  such that  $z^4 = i$ . Since  $i = e^{i\frac{\pi}{2}}$ ,  $z = \rho e^{i\theta}$  satisfying  $z^4 = i$  becomes  $\rho^4 e^{i4\theta} = e^{i\frac{\pi}{2}}$ , meaning

$$\begin{cases} \rho^4 = 1, \\ 4\theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

This means  $\rho = 1$  and  $\theta = \frac{\pi}{8} + \frac{k\pi}{2}$ , where  $k \in \mathbb{Z}$ . Considering the cases  $k = 0, 1, 2, 3$  and observing that there are only 4 equivalence classes modulo 4 to consider, we have that

$$z_0 = e^{i\frac{\pi}{8}}, \quad z_1 = e^{i\frac{5\pi}{8}}, \quad z_2 = e^{i\frac{9\pi}{8}}, \quad z_3 = e^{i\frac{13\pi}{8}}.$$

We now discuss convergence. We say that a set of complex numbers  $\{z_n\}_{n \in \mathbb{N}}$  converges to  $w \in \mathbb{C}$  if  $\lim_{n \rightarrow \infty} |z_n - w| = 0$ . We write it as  $\lim_{n \rightarrow \infty} z_n = w$ . In the complex plane, the convergence can be in any direction.

**Lemma 1.1.**  $\{z_n\}_{n \in \mathbb{N}}$  converges to  $w$  if and only if  $\{\Re z_n\}_{n \in \mathbb{N}}$  converges to  $\Re w$  and  $\{\Im z_n\}_{n \in \mathbb{N}}$  converges to  $\Im w$ .

*Proof.* We have that

$$\begin{aligned} |z_n - w| &= |(\Re z_n - \Re w) + i(\Im z_n - \Im w)| \\ &\leq |\Re z_n - \Re w| + |\Im z_n - \Im w|, \end{aligned}$$

where as  $n \rightarrow \infty$ , we have that the right hand side is given by  $0 + 0$ . For the opposite direction, we have that  $|z| \geq |\Re z|$  or  $|\Im z|$ , so we have that

$$|\Re z_n - \Re w| = |\Re(z_n - w)| \leq |z_n - w|,$$

which approaches 0 as  $n \rightarrow \infty$ . The same argument goes for the imaginary portion.  $\square$

A sequence of complex numbers  $\{z_n\}_{n \in \mathbb{N}}$  is called *Cauchy* if  $|z_n - z_m| \rightarrow 0$  as  $n, m \rightarrow \infty$ . In  $\varepsilon - \delta$ , this means that for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $|z_n - z_m| < \varepsilon$  for all  $n, m > N$ .

**Theorem 1.2** (Bolzano-Weierstrass Theorem).  $\mathbb{R}$  is *complete*, i.e., every Cauchy sequence of real numbers converges to a real number.

**Theorem 1.3.**  $\mathbb{C}$  is complete.

*Proof.* Take any Cauchy sequence of complex numbers  $\{z_n\}$ . Using the inequalities  $|\Re z| \leq |z|$  and  $|\Im z| \leq |z|$ , we have that  $\{\Re z_n\}$  and  $\{\Im z_n\}$  are Cauchy sequences of real numbers. By Bolzano-Weierstrass, we have that  $\Re z_n \rightarrow x_0 \in \mathbb{R}$  and  $\Im z_n \rightarrow y_0 \in \mathbb{R}$ . By the previous lemma, we actually have  $\lim_{n \rightarrow \infty} z_n = x_0 + iy_0$ .  $\square$

We now move onto topology in the complex plane. Given  $z_0 \in \mathbb{C}$  and  $r > 0$ , we can form an open or closed disc centered at  $z_0$  of radius  $r$ . We write both of these as

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$

$$\bar{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\},$$

Given a set  $\Omega \subseteq \mathbb{C}$ , a point  $z_0$  is an interior point if there exists  $r > 0$  such that  $D_r(z_0) \subseteq \Omega$ . The interior of  $\Omega$  is given by the set of all such interior points. In particular, the interior of  $\bar{D}_r(i)$  is  $D_r(i)$ .

A set  $\Omega$  is called *open* if every point in  $\Omega$  is an interior point.  $\Omega$  is called *closed* if the complement of  $\Omega$ ,  $\Omega^c = \mathbb{C} \setminus \Omega$ , is open. As an example, the open right half-plane  $\{z \in \mathbb{C} \mid \Re z > 0\}$  is open.

*Proof.* For any  $z \in \Omega$ , let  $z = x + iy$ , and take  $r = \frac{x}{2} = \frac{\Re z}{2}$ . Then we claim that  $D_r(z) \subseteq \Omega$ . For all  $w \in D_r(z)$ , we clearly have that

$$\Re w = \Re z - (\Re z - \Re w) \geq \Re z - |z - w| \geq \frac{\Re z}{2} > 0,$$

and so all such  $w \in \Omega$ , and we are done.  $\square$

A point  $z \in \mathbb{C}$  is a *limit point* of  $\Omega$  if there exists a sequence  $\{z_n\} \subset \Omega$  with  $z_n \neq z$  such that  $z_n \rightarrow z$ .

As an example, we define  $D$  to be the open unit disc centered at 0. 0 and 1 are both limit points of  $D$ , but 1 is not contained in  $D$  itself.<sup>2</sup> The *closure* of  $\Omega$ ,  $\bar{\Omega}$ , is given by  $\Omega$  unioned with all its limit points. The *boundary* of a set  $\Omega$ , written  $\partial\Omega$ , is given by  $\bar{\Omega} \setminus \text{int } \Omega$ . A set  $\Omega \subseteq \mathbb{C}$  is said to be compact if it is closed and bounded, i.e., there exists  $M > 0$  such that  $|z| \leq M$  for all  $z \in \Omega$ .

**Theorem 1.4.** A set  $\Omega \subseteq \mathbb{C}$  is compact if and only if every sequence  $\{z_n\} \subset \Omega$  has a subsequence that converges to a point in  $\Omega$ .

**Proposition 1.5.** If  $\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_n \supset \dots$  is a sequence of nonempty compact sets in  $\mathbb{C}$ , where  $\text{diam}(\Omega_n) = \sup_{z, w \in \Omega_n} |z - w| \rightarrow 0$  as  $n \rightarrow \infty$ , then there exists a unique  $w \in \mathbb{C}$  such that  $w \in \Omega_n$  for every  $n \in \mathbb{N}$ .

*Proof.* For each  $\Omega_n$ , pick a point  $z_n \in \Omega_n$ . Then  $\{z_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence because the diameter of  $\Omega_n$  approaches 0. By the Bolzano-Weierstrass theorem for complex numbers, this means that  $\{z_n\}_{n \in \mathbb{N}}$  indeed does converge to some  $w \in \mathbb{C}$ . In particular, we have  $w$  is the limit of the subsequence  $\{z_m\}_{m \geq n} \subseteq \Omega_n$ , where  $\Omega_n$  is compact, meaning the limit  $w$  should be in  $\Omega_n$ . This means there exists a unique  $w \in \mathbb{C}$  such that  $w \in \Omega_n$  for every  $n \in \mathbb{N}$ .

To show the uniqueness of  $w$ , we argue by contradiction; assume  $w' \neq w$  satisfies the property. Then  $|w' - w| > 0$ . Since  $w, w' \in \Omega_n$  for all  $n$ , this contradicts that  $\text{diam}(\Omega_n) \rightarrow 0$ .  $\square$

An open set  $\Omega$  is called *connected* if it is not possible to find two disjoint nonempty open sets  $\Omega_1$  and  $\Omega_2$  such that  $\Omega = \Omega_1 \cup \Omega_2$ . A connected open set in  $\mathbb{C}$  is called a *region*.

<sup>2</sup>hell is it disc or disk YKW LET'S COMPROMISE it's spelled disque actually (paint nails)

## §2 Day 2: Functions on the Complex Plane (Sep. 4, 2025)

Let  $f : \Omega \rightarrow \mathbb{C}$ , where  $\Omega$  is an open subset of  $\mathbb{C}$ . We say that  $f$  is continuous if at  $z_0 \in \Omega$  if, for all  $\varepsilon > 0$ , there exists an open disk  $D_\gamma(z_0)$  such that  $|f(z) - f(z_0)| < \varepsilon$  for all  $z \in D_\gamma(z_0)$ . In particular,  $f$  is said to be continuous on  $\Omega$  if it is continuous at every point in  $\Omega$ .

**Example 2.1.** Consider  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by  $f(z) = \bar{z}$ . Show that  $f$  is continuous.

*Solution.* For all complex  $z, z_0$ , we have that  $|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0| = |z - z_0|$ . Thus, we have that for any  $\varepsilon > 0$ , we obtain<sup>3</sup>

$$f(D_\varepsilon(z_0)) = D_\varepsilon(\bar{z}_0). \quad \square$$

We now discuss holomorphic functions (i.e., complex differentiable functions). We say that  $f : \Omega \rightarrow \mathbb{C}$  is *holomorphic* at  $z_0 \in \Omega$  if

$$\frac{f(z_0 + h) - f(z_0)}{h}, \quad h \in \mathbb{C} \setminus \{0\},$$

converges as  $h \rightarrow 0$ . If the limit exists, we let

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

be the derivative.

**Example 2.2.** Consider the exact same function as in the previous example,  $f(z) = \bar{z}$ . Is  $f$  holomorphic?

*Solution.* For all  $z_0 \in \mathbb{C}$  and  $h \in \mathbb{C} \setminus \{0\}$ , we have that

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{z_0 + h} - \bar{z}_0}{h} = \frac{\bar{h}}{h} = \frac{\rho e^{-i\theta}}{\rho e^{i\theta}} = e^{-2i\theta}.$$

If we take  $h \rightarrow 0$  along the real line, we may let  $h = \rho$ , which means the fraction is equal to 1 as  $h \rightarrow 0$ . If we take  $\rho \rightarrow 0$  along the complex axis, however, then we have that  $h = \rho e^{i\pi/2}$ , where we obtain the fraction is equal to  $-1$  as  $\rho \rightarrow 0$ . Thus,  $f$  cannot be holomorphic.  $\square$

**Proposition 2.3.** Let  $\Omega$  be open in  $\mathbb{C}$ . If  $f, g$  are holomorphic on  $\Omega$ , then

- (i)  $f + g$  is holomorphic on  $\Omega$ , and  $(f + g)' = f' + g'$ .
- (ii)  $fg$  is holomorphic on  $\Omega$ , and  $(fg)' = f'g + fg'$ .
- (iii) If  $g(z_0) \neq 0$  where  $z_0 \in \Omega$ , then  $\frac{f}{g}$  is also holomorphic at  $z_0$ , where

$$\left(\frac{f}{g}\right)' = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

- (iv) If  $f : \Omega \rightarrow U$  and  $g : U \rightarrow \mathbb{C}$  are holomorphic, then  $g \circ f$  is also holomorphic, and we obtain the chain rule

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

<sup>3</sup>note to self: ol is better than bar for this stuff...

We now discuss complex differentiability versus real differentiability. A holomorphic function  $f : \Omega \rightarrow \mathbb{C}$  can be identified with a function  $F : \Omega \rightarrow \mathbb{R}^2$  given by  $(x, y) \mapsto (u(x, y), v(x, y)) = (\Re f(x, y), \Im f(x, y))$ . Consider the partial derivative of  $F$  at  $(x_0, y_0)$ ; these exist if there exists some linear transformation  $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that

$$\frac{\|F(P_0 + H) - F(P_0) - J(H)\|}{\|H\|} \rightarrow 0$$

as  $H \rightarrow 0$ . Or, we may define  $\Psi(H)$  to take on the fraction above, and we see that  $F$  is indeed differentiable at  $P_0 = (x_0, y_0)$  if  $\Psi(H) \rightarrow 0$  as  $H \rightarrow 0$ . We now deal with complex differentiability. Suppose  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic at  $z_0 = x_0 + iy_0$ . Then we have partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}.$$

Naturally,

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

along any path; in particular, we take  $h \in \mathbb{R} \setminus \{0\}$  and observe that

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0), \end{aligned}$$

and so both exist, and they are  $\Re f'(z_0)$  and  $\Im f'(z_0)$  respectively. Similarly, we may take  $h = ik$  where  $k \in \mathbb{R} \setminus \{0\}$  and obtain

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{k \rightarrow 0} \frac{u(x_0, y_0 + k) + iv(x_0, y_0 + k) - u(x_0, y_0) - iv(x_0, y_0)}{ik} \\ &= \lim_{k \rightarrow 0} \frac{-i(u(x_0, y_0 + k) - u(x_0, y_0)) + v(x_0, y_0 + k) - v(x_0, y_0)}{k} \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0), \end{aligned}$$

and so both partials also exist and they are  $\Re f'(z_0)$  and  $-\Im f'(z_0)$  respectively.

### §3 Day 3: Holomorphic Functions and Power Series (Sep. 9, 2025)

Let  $f : \Omega \rightarrow \mathbb{C}$  (where  $\Omega$  is an open set in  $\mathbb{C}$ ). We say that  $f$  is holomorphic at  $z_0$  if

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}, \quad h \in \mathbb{C} \setminus \{0\}$$

exists. Recall that  $\mathbb{C}$  can be identified with  $\mathbb{R}^2$  by considering any  $z = x + iy \in \mathbb{C}$  as a tuple  $(x, y) \in \mathbb{R}^2$ . In this way, given a function  $f : \Omega \rightarrow \mathbb{C}$ , we can define  $F : \Omega \rightarrow \mathbb{R}^2$ , where  $F : (x, y) \mapsto (u(x, y), v(x, y))$ , given by  $u = \Re f$  and  $v = \Im f$ .

**Proposition 3.1.** If  $f = u + iv$  is holomorphic at  $z_0 = x_0 + iy_0$ , then we have that all four partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist and they satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \Re f(z_0), \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \Im f(z_0).$$

We also have that  $F$  is differentiable at  $P_0 = (x_0, y_0)$ .

**Definition 3.2.** We say that  $F$  is differentiable at  $P_0$  if there exists a linear transformation (the derivative)  $J = J_F(x_0, y_0) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that

$$\lim_{H \rightarrow 0} \frac{\|F(P_0 + H) - F(P_0) - J(H)\|}{\|H\|} = 0.$$

Before we discuss the complex definition, let us recall another property of real differentiability; if  $F$  is differentiable at  $P_0 = (x_0, y_0)$ , then all four partial derivatives exist, and

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is called the Jacobian matrix of  $F$  at  $(x_0, y_0)$ . To see this, consider the association  $P_0 = (x_0, y_0)$  with  $z_0 = x_0 + iy_0$ , and  $H = (h_1, h_2)$  with  $h = h_1 + ih_2$ ; then we have that

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 \\ \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 \end{pmatrix}.$$

This is a vector in  $\mathbb{R}^2$ , which we may associate with the complex number

$$\left( \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 \right) + i \left( \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 \right) = \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) h_1 + \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) h_2,$$

which, by the Cauchy-Riemann equations, we obtain

$$\left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) h_1 + i \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right) h_2 = \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h_1 + ih_2),$$

which is precisely equal to  $f(z_0 + h) - f(z_0) - f'(z_0)h$ . In particular,

$$\lim_{h \rightarrow 0} \left| \frac{f(z_0 + h) - f(z_0) - f'(z_0)h}{h} \right| = \lim_{h \rightarrow 0} \left| \frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0) \right| = 0.$$

Similarly, per the definition of the Jacobian, we must have

$$\lim_{H \rightarrow 0} \frac{\|F(P_0 + H) - F(P_0) - J(H)\|}{\|H\|} = 0,$$

and this concludes the proof of proposition 3.1. □

**Theorem 3.3.** Suppose  $f = u + iv$  is a complex-valued function defined on an open set  $\Omega \subset \mathbb{C}$ . If  $u, v : \Omega \rightarrow \mathbb{R}$ , are continuously differentiable and satisfy the Cauchy-Riemann equations, then  $f$  is holomorphic on  $\Omega$  and  $f'(z) = \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{1}{i} \frac{\partial f}{\partial y} \right)$ .

*Proof.* Since  $u$  is continuously differentiable at the point  $(x, y) \in \Omega$ , there exists a linear transformation  $J_u : \mathbb{R}^2 \rightarrow \mathbb{R}$  where

$$\frac{|u(x + h_1, y + h_2) - u(x, y) - J_u(h_1, h_2)|}{\|(h_1, h_2)\|} \rightarrow 0, \quad (h_1, h_2) \rightarrow 0.$$

In particular,  $J_u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$ . The above fraction is equivalent to

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + \|h\| \psi_1(h),$$

where  $\psi_1 : U \rightarrow \mathbb{R}$ , where  $U$  is some open neighborhood of  $0 \in \mathbb{R}^2$ , with  $\psi_1(h) \rightarrow 0$  as  $h \rightarrow 0$ . Similarly, we have that

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + \|h\| \psi_2(h)$$

with  $\psi_2(h) \rightarrow 0$  as  $h \rightarrow 0$ . We want to show that  $f$  is holomorphic at  $z = x + iy$ . We have that

$$\begin{aligned} f(z + h) - f(z) &= (u(x + h_1, y + h_2) - u(x, y)) + i(v(x + h_1, y + h_2) - v(x, y)) \\ &= \left( \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 \right) + \|h\| \psi_1(h) + i \left( \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 \right) + i \|h\| \psi_2(h) \\ &= \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h_1 + ih_2) + \|h\| \psi_1(h) + i \|h\| \psi_2(h) \end{aligned}$$

from Cauchy-Riemann. Thus, we have that<sup>4</sup>

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(z + h) - f(z)}{h} &= \lim_{h \rightarrow 0} \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} + \frac{\|h\|}{h} (\psi_1(h) + i \psi_2(h)) \\ &= \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{1}{i} \frac{\partial f}{\partial y} \right). \quad \square \end{aligned}$$

We now discuss complex power series.

**Definition 3.4.** A complex power series is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

with  $a_n \in \mathbb{C}$  and  $z$  a complex variable. We say that  $\sum_{n=0}^{\infty} a_n z^n$  converges at  $z_0 \in \mathbb{C}$  if there exists some  $w \in \mathbb{C}$  such that, for all  $\varepsilon > 0$ , there exists  $N_0 \in \mathbb{N}$  such that  $N \geq N_0$  satisfies

$$\left| \sum_{n=0}^N a_n z^n - w \right| < \varepsilon.$$

The series converges *absolutely* at  $z_0$  if there exists  $w \in \mathbb{R}$  such that

$$\left| \sum_{n=0}^N |a_n| |z_0|^n - w \right| < \varepsilon.$$

<sup>4</sup>i swear wenyu has an invisible key wired into her back like nano from nichijou and it's permanently cranked on



**Proposition 3.5.** If  $\sum_{n=0}^{\infty} a_n z^n$  converges absolutely at  $z_0 \in \mathbb{C}$ , then  $\sum_{n=0}^{\infty} a_n z^n$  converges at  $z_1 \in \mathbb{C}$  with  $|z_1| \leq |z_0|$ .

*Proof.* For all  $z_1 \in \mathbb{C}$  with  $|z_1| \leq |z_0|$ , consider the sequence of partial sums  $\{S_m(z_1)\}_{m \in \mathbb{N}}$  given by

$$S_m(z_1) = \sum_{n=0}^m a_n z_1^n.$$

We want to show that such a sequence converges. Since  $\mathbb{C}$  is complete, it suffices to show that said sequence is Cauchy. For all  $m < k \in \mathbb{N}$ , we have that

$$|S_k(z_1) - S_m(z_1)| = \left| \sum_{n=m+1}^k a_n z_1^n \right| \leq \sum_{n=m+1}^k |a_n| |z_1|^n \leq \sum_{n=m+1}^k |a_n| |z_0|^k. \quad \square$$

We now provide a few examples.

- (i) The complex exponential function for all  $z \in \mathbb{C}$ , given by

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

For all  $z \in \mathbb{C}$ , this sum converges because it converges absolutely (consider  $e^{|z|}$ ).

- (ii) The geometric series  $\sum_{n=0}^{\infty} z^n$ , where  $|z| < 1$ , converges; otherwise, is  $|z| \geq 1$ , it diverges. In particular, if  $\sum_{n=0}^{\infty} z^n$  converges, then  $|z^n| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Theorem 3.6** (Shakarchi, Thm. 2.5). Given a power series  $\sum_{n=0}^{\infty} a_n z^n$ , there exists  $R \in [0, \infty)$  such that (i) if  $|z| < R$ , the series converges, and (ii) if  $|z| > R$ , the series diverges. We call  $R$  the *radius of convergence* of  $\sum_{n=0}^{\infty} a_n z^n$ , and  $\{z \in \mathbb{C} \mid |z| < R\}$  the disc<sup>5</sup> of convergence. Moreover,  $R$  is given by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n} =: L,$$

where we use the convention that  $\frac{1}{0} = \infty$  and  $\frac{1}{\infty} = 0$ .

*Proof.* For all  $z \in \mathbb{C}$  with  $|z| < r < R$ , there exists some  $\varepsilon > 0$  such that

$$(L + \varepsilon) |z| = r < 1.$$

By definition of  $L$ , we have  $|a_n|^{1/n} \leq L + \varepsilon$  for all large  $n$ , meaning that

$$|a_n| |z|^n = \left( |a_n|^{1/n} |z| \right)^n \leq ((L + \varepsilon) |z|)^n = r^n, \quad r \in (0, 1),$$

whereby comparison with the geometric series  $\sum r^n$ , we see that  $\sum |a_n| |z|^n$  converges. Similarly, if  $|z| > R$ , we have that

$$\left( \frac{1}{r} - \varepsilon \right) |z| > 1,$$

where, using the definition of  $R$ , there exists an infinite subsequence  $a_{n_k}$  such that  $|a_{n_k}|^{1/n_k} \geq \frac{1}{R} - \varepsilon$ . We have that

$$|a_{n_k} z^{n_k}| - \left( |a_{n_k}|^{1/n_k} |z| \right)^{n_k} \geq \left[ \left( \frac{1}{R} - \varepsilon \right) |z| \right]^{n_k} > 1. \quad \square$$

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<sup>5</sup>disque. ok i'll stop