MAT425 Lecture Notes

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Contents

§1 Day 1: Manifolds or Whatever (Jan. 6, 2025)

The textbook is Differential Topology, by Guillemin & Pollack. This course is about smooth manifolds. But this is a very vast subject, so it's more accurate to say that it's all about a specific circle of questions which are related to something called transversality.

What is a manifold? Well a manifold is... a subset of $\mathbb{R}^n$ with some requirements on dimensionality. We don't loose any generality this way (due to Whitney's Embedding Theorem) therefore for our purposes this is sufficient.

Assume that $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ and $f : X \to Y$ is smooth. What does this mean? Well let's recall an easier case. When X and Y are open, then this is just our standard definition of differentiability (locally linear and such). On the other hand, if X and Y aren't open, then it can be much harder—if X and Y are some complicated fractal sets then how do you define it?

We say a map $f : X \to Y$ is smooth at a point $x \in X$ if there exists some open neighborhood U of x and some function $F : U \cup X \to \mathbb{R}^m$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X \cup U & \xrightarrow{\ F\ } & \mathbb{R}^m \\
\big\uparrow & & \big\uparrow \\
X & \xrightarrow[\ f\ ]{} & Y
\end{array}
$$

and additionally, F is smooth at x. We say that f is smooth if it is smooth everywhere in its domain.

Definition 1.1. Two sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are diffeomorphic if there is a map $f : X \to Y$ which is bijective, smooth, and f^{-1} is smooth as well.

Definition 1.2. X is a k-dimensional smooth manifold if for all $x \in X$ there exists some open $U \subseteq \mathbb{R}^n$ such that $x \in U$ and $U \cap X$ is diffeomorphic to an open $V \subseteq \mathbb{R}^m$.

Thus, there exists $\varphi : U \cap X \to V$ and $\psi : V \to U \cap X$ with $\varphi^{-1} = \psi$ and φ, ψ are smooth maps.

Definition 1.3. $X \subseteq \mathbb{R}^n$ is a k-dimensional smooth manifold if it is locally diffeomorphic to open sets of $\mathbb{R}^k$.

I have no clue why he wrote this as a separate definition. Assume that $X \subseteq \mathbb{R}^n$ is a k-dimensional manifold. Then there exists a local parameterization $\varphi : U \subseteq \mathbb{R}^k \to V \cap X$. The opposite direction $(V \to U)$ is called a coordinate system for our manifold.

Let us now define the tangent spaces. If X is a k-dimensional manifold, the tangent space is given by the local parameterizations. Suppose that $\varphi : U \to V \cap X$ is a parameterization of our manifold. Suppose that $u_0 \in U$ and $f(u_0) = x$. Then we say that the tangent space of X at x is

$$
T_x X = d\varphi(u_0)(\mathbb{R}^k) \subseteq \mathbb{R}^n.
$$

Notice that the tangent spae is independent of parameterization. He drew a photo of the tangent line to a curve. Also notice that the tangent space is centered at 0, but it's natural to imagine it centered at x (and infinitesimal) instead.

To define the derivative for a smooth manifold naturally, we need two things. First, it should agree with our usual Euclidean definition of derivative. Second, it should satisfy the chain rule. For it to make sense, $df(x)$ should map $T_x X$ to $T_{f(x)} Y$.

Let U and V be open neighborhoods, and $\phi.$ and $\psi.$ be parameterizations and charts of their manifolds. Define h such that the following diagram commutes.

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
\varphi_X \big\uparrow\big\downarrow \psi_X & \varphi_Y \big\uparrow\big\downarrow \psi_Y & \\
U & \xrightarrow{\;h\;} & V
\end{array}
$$

Namely, $h = \psi_Y \circ f \circ \varphi_X$. Suppose (for simplicity) that $0 \in U, V$, $\varphi_X(0) = x$, and $\psi_Y(f(x)) = 0$. Then we have that

$$df(x) = d\varphi_Y(0) \circ dh(0) \circ d\psi_X(x).$$

[This part is quite dense in the notation, so draw a picture to make sure you're clear with the notation—it helped me a ton to work through it].

Because our manifolds are smooth we can discuss general position. What does this mean? Well consider two lines. They might intersect at one point, no points, or infinitely many points. But if we slightly perturb our lines, they will only intersect at one point. Therefore, we say that they generally intersect once.

We can do the exact same thing on a torus. It's possible to find two lines which intersect infinitely many times, but we can find a general number of times they intersect. Using these ideas we can formulate all sorts of interesting topological invariants. This is all about transversality (he didn't define).

We can formulate these ideas in a general setting. Sard's theorem is most at home in Euclidean space (telling you that the critical points of a function form a measure zero set), but we can also do it for manifolds.

Definition 1.4. A critical point $x \in X$ of a smooth map $f : X \to Y$ between manifolds is a point where $df(x)$ does not have full rank. Otherwise, x is regular.

Definition 1.5. A critical value $y \in Y$ of a smooth map $f : X \to Y$ is a point which is the image of some critical point. Otherwise, y is regular.

Sard's theorem tells us that the set of critical values of our manifold has measure 0. This is important because we want to perturb into regular values.

Next class we will cover the implicit function theorem (i might have misheard, he might have said inverse), normal maps, Sard's theorem, degrees of maps, Transversality, topological invariants, then some bonus if there's time (Morse theory).

§2 Day 2: IFT, Immersions, Submersions (Jan. 8, 2025)

Review: $M \subseteq \mathbb{R}^n$ is a k-manifold if it is locally diffeomorphic to $\mathbb{R}^k$. Our manifolds are locally parameterized by k-dimensional space. This means that for every $x \in M$ there exists a φ such that $\varphi(0) = x$, and $\varphi : U \subseteq \mathbb{R}^k \to M$ is a local diffeomorphism.

We can think of φ as a collection of n functions from $\mathbb{R}^k$ to $\mathbb{R}$.

When we think about the differential, sometimes it's nice to imagine a curve through 0 $\alpha : \mathbb{R} \to \mathbb{R}^k$, and then we have

$$d(\varphi \circ \alpha) = d\varphi(0) \cdot \alpha'(0).$$

But otherwise, we can just do what we did last time.

We say that if $N^k \subseteq M^n \subseteq \mathbb{R}^N$, and M^n is a manifold in $\mathbb{R}^N$, then N^k is a submanifold of M^n.[1]

§2.1 Inverse and Implicit Function Theorem

Suppose that $f : M \to N$ is a smooth map, and that at $x_0 \in M$, we have that $df(x_0) : TM_{x_0} \to TN_{f(x_0)}$ is an isomorphism. Then we have that f is a local diffeomorphism near x_0. That is, there exists an open subset $U \subseteq M$, and $x_0 \in U$, and an open set $V \subseteq N$, with $f(x_0) \in V$ such that $f|_U$ is a diffeomorphism. Notice that because the differential is an isomorphism, M and N have the same dimension.

This version of the theorem can be derived by the regular inverse function theorem in Euclidean space. This is because given parameterizations $\varphi : U \to M^k$ and $\psi : V \to N^k$,

$$
\begin{array}{ccc}
M^k & \xrightarrow{\quad f \quad} & N^k \\
\varphi \uparrow & & \psi \uparrow \\
U & \xrightarrow[\psi^{-1} \circ f \circ \varphi]{} & V
\end{array}
$$

we can use the regular inverse function theorem on $\psi^{-1} \circ f \circ \varphi$, to get an inverse function $\varphi^{-1} \circ f^{-1} \circ \psi$ locally, and then just compose with ψ^{-1} and φ to get our local inverse function for f. Nabutovsky drew a picture, but it was not at all helpful so I will not include it.

Nabutovsky then went over the regular inverse function theorem. This is because nobody in the class could recall the proof of the inverse function theorem from memory. Nabutovsky gave an interesting proof. Notice that

$$f(\vec{x}) = f(\vec{x}_0) + Df(\vec{x}_0)(\vec{x} - \vec{x}_0) + O(||\vec{x} - \vec{x}_0||^2),$$

therefore it's possible to define a contraction mapping and then use this to get a unique inverse point. If we can show this point varies continuously on our output point, then chain rule shows it's differentiable and gives the formula. The contraction mapping was blocked by Nabutovsky's body, but if we say $f(x^*) = y$, and assume without loss of generality that $f(\vec{x}_0) = 0$ and $x_0 = 0$, then

$$\Gamma(x^*)(p) = df(0)^{-1}(f(p) - y) + x^*.$$

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[1] Nabutovsky seems to be using the notation M^n to refer to an n-dimensional manifold. It's not the product.

Then I believe Banach gives us that x^* is the unique fixed point, and we can use another argument to get continuity. Nabutovsky however said that there is an entirely elementary proof of the inverse function theorem, which we can prove by first proving the implicit function theorem, then showing their equivalence.

Let $\vec{x}_0 \in \Omega \subseteq \mathbb{R}^{n+m}$, and $f : \Omega \to \mathbb{R}^n$. Then $Df(\vec{x}_0)$ is an $n \times (n+m)$ matrix

$$\begin{pmatrix} \partial_1 f_1 & \cdots & \partial_{n+m} f_1 \\ \vdots & \ddots & \vdots \\ \partial_1 f_n & \cdots & \partial_{n+m} f_n \end{pmatrix}.$$

If we assume that $Df|_{(x_1,\ldots,x_n)}(\vec{x}_n)$ is invertible, then there exists some $\Omega_1 \subseteq \mathbb{R}^n$, $\Omega_2 \subseteq \mathbb{R}^m$ (both open), and a map $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{pmatrix}$ with $\varphi : \Omega_1 \to \Omega_2$ such that

$$f(x_1, \ldots, x_n, \varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)) = f(\vec{x}_0),$$

for $(x_1, \ldots, x_n) \in \Omega_1$.

We can actually prove the inverse function theorem from the implicit function theorem (the other implication is in Spivak, but he did do it explicitly in class), by defining

$$F(\vec{x}, \vec{b}) = f(\vec{x}) - \vec{b},$$

and then if $F(\vec{x}, \vec{b}) = 0$, we can find $\vec{x}$ as a function of $\vec{b}$. Therefore to prove all our theorems, we need only prove the implicit function theorem. But there is an elementary proof.

Proof (Implicit Function Theorem). Suppose that $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Then, suppose without loss of generality that $f(0,0) = 0$, and $Df_{(x_1,\ldots,x_n)}(0,0)$ is invertible.

For the purposes of illustration, let us first consider the case where $n = 1$. Then in this case we have that $\partial_1 f(0,0) \neq 0$. This implies that on some neighborhood U of 0, $\partial_1 f(0,0)$ all has the same sign. Let us assume without loss of generality that it is positive. This implies that along the x-axis our function is strictly increasing. But since $f(0,0) = 0$, this implies that for $(x,0) \in U$, $f(x,0) < 0$ when $x < 0$ and $f(x,0) > 0$, when $x > 0$.

Fix some $a > 0$ such that $(-a, a) \times \{0\} \subseteq U$. Now, find an $\varepsilon > 0$ such that for $(x,y) \in B_\varepsilon(-a, 0)$, $f(x,y) < 0$, and for $(x,y) \in B_\varepsilon(a, 0)$, $f(x,y) > 0$, and

$$[-a, a] \times (-\varepsilon, \varepsilon) \subseteq U.$$

It follows that for any $y_0 \in (-\varepsilon, \varepsilon)$,

$$f(-a, y_0) < 0, \qquad f(a, y_0) > 0,$$

and so by the intermediate value theorem, there exists $x_0 \in (-a, a)$ such that $f(x_0, y_0) = 0$.

This implies that each y_0 has a unique zero along it. We also know it depends continuously on y_0 by the continuity of f, and therefore we can handwave away any requirements of differentiability or invertibility (Nabutovsky said it's easy—so it probably isn't[2]).

The proof for $n \neq 1$ is the exact same except with parallelepipeds instead of rectangles. To prove it for $m \neq 1$, use induction on m to reduce it into showing it for $m-1$ dimensions and 1 dimension independently. This is fairly clear. $\square$

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[2]Many of the above details were filled in by me, he mostly just gave intuition and a proof sketch

x) So let's return to the inverse function theorem on manifolds. Given X, Y manifolds of the same dimension k, if $df(x_0) : T_{x_0}X^k \to T_{f(x_0)}Y^k(x_0)$ is an isomorphism, then by the inverse function theorem in $\mathbb{R}^n$, we've proven the inverse function theorem for manifolds.

We say that a function $f : X^n \to Y^m$ is an immersion if for all $x \in X^n$, $df(x)$ is injective. If $n > m$ then it is clear that f will never be an immersion. Thus we have the counterpart of a submersion, which is a map such that $df(x)$ is surjective, for all $x \in X^n$.

What the inverse function theorem for manifolds proves is that if f is an immersion and a submersion, then f is a *local* diffeomorphism. There is a related notion. We say that $x \in X^n$ is a regular point of X if $df(x)(T_xX^n) = T_{f(x)}Y^m$. Therefore, f is a submersion if and only if every point of X^n is regular.

We say that x is critical if it is not regular. If $f : X^n \to Y^m$ and $n < m$, then every point of X^n is a critical point.

If x is a critical point, then we call $f(x)$ a critical value. A regular value $y \in Y^m$ is a value which is not critical—for all critical point $x \in X^n$, $f(x) \neq y$. It follows that if $n < m$, then $f(X^n)$ are the critical values of Y^m. Notice that $Y^m \setminus f(X^n)$ are all regular values.[3]

Now how do we actually apply the implicit and inverse function theorem? Well consider $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and f is a submersion, with $f(\vec{x}_0) = \vec{0} \in \mathbb{R}^n$. Suppose that

$$f(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = \vec{a}.$$

Then a priori we know that $\{\vec{a} : f(\vec{x}) = \vec{a}\}$ is nonempty, but not much else. On the other hand, with the implicit function theorem, we know that this set is an m-dimensional manifold.

If $df(\vec{x}) : T_{\vec{x}}\mathbb{R}^{n+m} \to T_{\vec{a}}\mathbb{R}^n$ is a surjection. Then it has full rank when we consider it as a matrix, so we can use the implicit function theorem to get φ which parameterizes the image locally.

Let us consider a generalization: If $f : X^n \to Y^m$ is a submersion (we know therefore that $n \geq m$), then for all $y \in f(X^n) \subseteq Y^m$, then $f^{-1}(y)$ is an $(m-n)$-manifold. Even further, if y is regular, then $f^{-1}(y)$ is an $(m-n)$-manifold. This is the preimage theorem.

Think about our domain as being X^n and our range being $\mathbb{R}^n$, with $f(x) = x$ for simplicity. Then we can just use our local parameterization to get our theorem, because the implicit function theorem is a local statement.

Let's look more at immersions and submersions. If f is an immersion, then the canonical immersion is from $\mathbb{R}^n \to \mathbb{R}^m$ where

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \overbrace{0, \ldots, 0}^{n-m\ 0\text{'s}})$$

If $f : X^n \to Y^m$ is an immersion, then for all $x \in X^n$, there exists some open sets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$, where $x \in U$, $f(x) \in V$, with paramaterizations $\varphi : U \to X^n$, $\psi : V \to Y^m$, such that

$$
\begin{array}{ccc}
X^n & \xrightarrow{\ f\ } & Y^m \\
\varphi \uparrow & & \psi \uparrow \\
U & \xrightarrow[\text{canon}]{} & V
\end{array}
\quad .
$$

Analogously, the canonical submersion maps $(x_1, \ldots, x_m, \ldots, x_n)$ to $(x_1, \ldots, x_m)$. This is just the orthogonal projection. The submersion theorem shows us that the exact same commutative diagram commutes, except it's the canonical surjection at the bottom this time. This implies the preimage theorem when f is a submersion immediately.

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[3]cf. Sard's Theorem

§3 Day 3: Submersion, Immersion, Preimage Theorems (Jan. 13, 2025)

As a notation (which Nabutovsky did not use) I will say that π is our canonical submersion and ι is our canonical immersion. This is because Nabutovsky wrote it out in plain text every time he referred to it, which is harder to fit into latex. Use this notation at your own risk.

Let $f : X^k \to Y^\ell$ be a function (where $k \leq \ell$). Suppose that for some $x_0 \in X^k$, $df(x_0) : TX^k \to TY^\ell$ is injective. We want to show that there are open sets $U \subseteq \mathbb{R}^k$, $V \subseteq \mathbb{R}^\ell$, with $x_0 \in U$, and parameterizations $\varphi : U \to X^k$ and $\psi : V \to Y^\ell$ such that the following diagram commutes.

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\varphi \uparrow & & \uparrow \psi \\
U & \xrightarrow{\ \iota\ } & V
\end{array}
$$

Recall that our canonical immersion $\iota : \mathbb{R}^k \to \mathbb{R}^\ell$ is just

$$
\iota(x_1, \ldots, x_k) = (x_1, \ldots, x_k, \underbrace{0, \ldots, 0}_{\ell - k \text{ zeroes}}).
$$

Despite Nabutovsky's incomprehensible summary, the idea behind this is actually quite simple. Choose arbitrary parameterizations $\varphi : U \to X^k$ and $\psi : U' \to Y^\ell$ such that $x_0 \in \operatorname{im} \varphi$, and the following diagram commutes for some $g : U \to U'$

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\varphi \uparrow & & \uparrow \psi \\
U & \xrightarrow{\ g\ } & U'
\end{array}
$$

Without loss of generality, we can assume that the top $\ell \times \ell$ minor of Dg (that is, the first ℓ rows of Dg) is invertible. Define $G : \mathbb{R}^k \times \mathbb{R}^{\ell - k} \to \mathbb{R}^\ell$ by

$$
G(x, z) = g(x) + z.
$$

Then we have that

$$
DG = \left(\ Dg \ \middle| \ \begin{array}{c} 0 \\ \hline I_{k-\ell} \end{array} \right),
$$

so G is a local diffeomorphism. Why do we care? Let $V = \iota(U)$. Then we have that the following diagram commutes:

$$
\begin{array}{ccccc}
V & \xleftarrow{\ \iota\ } & U & \xrightarrow{\ \varphi\ } & X \\
 & {}_{G}\searrow & \downarrow g & & \downarrow f \\
 & & U' & \xrightarrow{\ \psi\ } & Y
\end{array}
$$

Therefore removing the extraneous pieces, the following diagram commutes

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\varphi \uparrow & & \uparrow {\psi \circ G} \\
U & \xrightarrow{\ \iota\ } & V
\end{array}
$$

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Since G is a local diffeomorphism, by shrinking our sets U and V sufficiently (but ensuring that that $x_0 \in U$), we can get that $G|_U$ is a global diffeomorphism and therefore $\psi \circ G|_U$ is a parameterization, proving the theorem.

For the submersion theorem, we do almost the exact same thing. We want that if $f : X^k \to Y^\ell$ and $x_0 \in X^k$ is such that $df(x_0)$ is surjective, then there exists parameterizations $\varphi : U \to X^k$, $\psi : V \to Y^\ell$ such that $x_0 \in \operatorname{im} \varphi$, andthe following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \varphi \uparrow & & \uparrow \psi \\ U & \xrightarrow{\ \pi\ } & V \end{array} \ .$$

For this purpose, choose parameterizations $\varphi : U \to X^k$ and $\psi : U' \to Y^\ell$ such that the following commutes.

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \varphi \uparrow & & \uparrow \psi \\ U & \xrightarrow{\ g\ } & U' \end{array} \ .$$

If we set $G : U \to U' \times \mathbb{R}^{k-\ell}$ and

$$G(x_1, \ldots, x_k) = (g(x_1, \ldots, x_k), x_{\ell+1}, \ldots, x_k) ,$$

$$DG = \left(\frac{Dg}{0 \ | \ I_{k-\ell}} \right) ,$$

therefore (once again, up to a change of basis) DG is invertible, and so G is a local diffeomorphism. If we set $V = G(U)$, then the following diagram commutes

$$\begin{array}{ccccc} V & \xleftarrow{\ G\ } & U & \xrightarrow{\ \varphi\ } & X \\ & {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle g} & & \downarrow {\scriptstyle f} \\ & & U' & \xrightarrow{\ \psi\ } & Y \end{array}$$

which implies that so too does

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \varphi \circ G^{-1}} \uparrow & & \uparrow {\scriptstyle \psi} \\ U & \xrightarrow{\ \pi\ } & V \end{array}$$

which once more yields our desired result by shrinking U and V.

So we are done the immersion and submersion theorems. Let us look at the preimage theorem. If $y \in f(X)$ is regular for $f : X^k \to Y^\ell$, then $f^{-1}(y)$ is locally diffeomorphic to $\mathbb{R}^{k-\ell}$.

This is fairly intuitive in light of the submersion theorem. This is because a direct application submersion theorem yields our parameterization at any given point of X^k. Given $x_0 \in f^{-1}(y)$ (such that f is a submersion at x_0), we know that there exists φ, ψ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \varphi \uparrow & & \uparrow \psi \\ U & \xrightarrow{\ \pi\ } & V \end{array}$$

where $\varphi^{-1}(x_0) \in U$ and $\psi^{-1}(y) \in V$. But then $\pi^{-1}(\psi^{-1}(y))$ gives a subspace of U which parameterizes $f^{-1}(y)$ via the map φ.

Why do we care? All of this is kind of nebulous, so it might be nice to check out a concrete example. $M_{n \times n}$, the $n \times n$ matrices, can be thought of as basically $\mathbb{R}^{n^2}$. If we set $\mathrm{Sym}_{n \times n}$ to be the $n \times n$ symmetric matrices, then we have a subspace (it can be identified with $\mathbb{R}^{\frac{n(n+1)}{2}}$ by only controlling the entries of the lower diagonal). Now, if we consider the map $f : M_{n \times n} \to \mathrm{Sym}_{n \times n}$

$$f(A) = A^\top A,$$

then $f^{-1}(I)$ will be all the orthogonal matrices $O_{n \times n}$. To show this is a manifold, we can consider df, and show that $df(A)$ is surjective whenever A is orthogonal.

$$
\begin{aligned}
\lim_{h \to 0} \frac{f(A + hB) - f(A)}{h} &= \lim_{h \to 0} \frac{A^\top A + hB^\top A + hA^\top B + h^2 B^\top B - A^\top A}{h} \\
&= \lim_{h \to 0} B^\top A + hA^\top B + hB^\top B \\
&= B^\top A + A^\top B.
\end{aligned}
$$

Fix some symmetric matrix C. Then we want to construct B such that $B^\top A + A^\top B = C$. But we have that $C = C^\top$ by its orthogonality, so

$$
\begin{aligned}
B^\top A + A^\top B &= C \\
B^\top A + A^\top B &= \frac{1}{2}\left(C^\top + C\right) \\
A^\top B &= \frac{C}{2} \\
B &= \left(A^\top\right)^{-1} \frac{C}{2} \\
B &= \frac{AC}{2}
\end{aligned}
$$

therefore for all orthogonal A and symmetric C there exists some matrix B such that $df(A)(B) = C$, which implies that $df(A)$ is a surjection at every orthogonal A, and therefore I is a regular value of f. This implies that $f^{-1}(I) = O_{n \times n}$ is a manifold. Furthermore, its dimension will be $\dim M_{n \times n} - \dim S_{n \times n} = \frac{n(n-1)}{2}$.

Next up we ask when immersions yield manifolds. If $f : X^k \to Y^\ell$ is an immersion, is $f(X^k)$ a manifold? Well not necessarily, consider something like $X = \mathbb{R}$ and $f(x) = (\sin t, \sin t \cos t)$. This will be a lemniscate (looks a bit like ∞) which is not smooth at 0. What if we require f is injective? Then is it a manifold? Still not necessarily. Consider something like $f(x) = (x, \alpha x)$ for some irrational α, then consider the map from $\mathbb{R}^2 \to S^1 \times S^1$ given by

$$f(s, t) = (\cos s, \sin s, \cos t, \sin t).$$

The intuition is that this mapping will be dense in $S^1 \times S^1$ (by a Diophantine approximation argument), but its image should only be one dimensional because f is smooth, which is a contradiction.

Next time we will cover one way to ensure our image is a manifold.