MAT367 Lecture Notes

ARKY!! :3c

'26 Winter Semester

Contents

§1 Day 1: Recap of Preliminaries (Jan. 6, 2026)

Today's class can be followed more precisely on §1.2 to §1.4 of our textbook by Gross and Meinrenken. The slogan of this class is that a manifold is something that locally looks like $\mathbb{R}^n$. Specifically, an n-manifold can be covered n-dimensional charts $(U \subset M) \to \mathbb{R}^n$, with our main motivating example being solutions sets to equations. Recall the implicit function theorem,

Theorem 1.1. Given a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, consider the solution set $f(x_1, \ldots, x_{n+1}) = 0$ and a point $p \in \mathbb{R}^n$ such that $\nabla f(p) \neq 0$; then, for $(x_1, \ldots, x_{n+1})$ in said solution set near p, we can represent solutions as $(x_1, \ldots, x_n, g(x_1, \ldots, x_n))$, where g is also a smooth function.

In particular, if 0 is a regular value[1] of f, then we can cover $\{x \mid f(x) = 0\}$ by graphs/charts. We present some examples;

(i) Let $f(x, y) = xy$; then $\ker f$ is precisely the x and y axes, which is not a manifold, because it does not look like $\mathbb{R}^n$ (for any n) near the origin.

(ii) Let $f(x, y) = y - x^{2/3}$; then $\ker f$ can be graphed in desmos as $y = x^{2/3}$, which is not a smooth manifold because of its behavior at 0.

(iii) The n-sphere $S^n = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$ can be regarded as the level set of the ℓ^2-norm, for which $S^0 = \{\pm 1\} \subset \mathbb{R}$, S^1 is a circle, S^2 is the usual sphere. Note that we may use the stereographic projection as seen in complex analysis, to view S^3 (and any of the previous or subsequence S^n) as $\mathbb{R}^3 \cup \{\infty\}$.

(iv) The 2-dimensional torus T^2 is the surface of revolution obtained from a circle of radius r and R about an axis of revolution. It can be regarded as a level set by writing
$$T^2 = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - R)^2 + z^2 = r\}.$$

(v) The Möbius strip can't be a part of a level set (at a regular value) because level sets are orientable (2-sided), while the strip is not.

(vi) The Klein bottle is also not orientable; it is closed (doesn't have a boundary), and doesn't embed into $\mathbb{R}^3$. It can be immersed into $\mathbb{R}^3$, i.e., locally embedded but not globally, as seen in the textbook.

Theorem 1.2 (Whitney Embedding Theorem)**.** Every n-manifold has an embedding in $\mathbb{R}^{2n}$.

In this class, we prefer to deal with intrinsic descriptions of manifolds rather than extrinsic ones; a good motivation is given on p.7 in the textbook with respect to our 2-torus.

Now, consider M to be the rotations of a ball. We call this a configuration spcae; i.e., its points are a way of configuring another object. How do we put coordinates on a piece of this space? To start, we wish to describe M; one possible way is by considering the 3×3 orthonormal matrices, i.e., SO(3); another way is to first designate a point on the unit sphere as the north pole N (of which there are two degrees of freedom in this choice), then choosing where the vector $(1, 0, 0)$ at N is mapped to (of which we have one degree of freedom).

In this manner, we may regard M as a 3-dimensional manifold and define coordinates for

[1] note to self: what's a regular value?

points near the identity.

A second example is given by considering linkages, which are collections of line segments and joints (p.8-10); suppose we have four segments given by $n_1, \ldots, n_4$, to be regarded as vectors in $\mathbb{R}^2$; without additional constraints imposed, we see that this can be regarded as 8-dimensional. In imposing the conditions on *making* $n_1, \ldots, n_4$ a linkage, we see that $\|n_i\|$ for each $i = 1, \ldots, 4$ is fixed (whence 4 less dimensions), $n_1 + n_2 + n_3 + n_4 = 0$ (whence one less), and n_1 is fixed (whence one less again), so such a linkage can be regarded as a 2 dimensional manifold with coordinates θ and φ.

Finally, denote $\mathbb{R}\mathrm{P}^2$ the real projective plane, given by lines passing through the origin in $\mathbb{R}^3$; equivalently, we may regard this as the pairs of antipodal $\{x, -x\}$ points in S^2.

Fact 1.3. "Closed" (equivalently, compact) surfaces are easy to enumerate.

Fedya then said something about orientability, but that's left in the textbook and easier to read there.

We now wish to define manifolds. For a first attempt, we want to say that a smooth manifold is a set M covered by a set of smooth charts $\varphi_i : (U_i \subset M) \to \mathbb{R}^n$, such that each $p \in M$ is covered by some U_i. However, we run into a problem; what does "smooth" mean? $f : U \to \mathbb{R}^n$ recall that a function is said to be smooth if it has partial derivatives of all orders. A function $f : U \to V$ is a diffeomorphism if it's a smooth bijectino and its inverse is also smooth.

Definition 1.4. A coordinate chart is an injective map $\varphi : U \to \mathbb{R}^n$ with open image for some $U \subset M$.

In particular, we say that two charts $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ are compatible if the transition $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism. Clearly, if $U \cap V = \varnothing$, then said maps are compatible; note, however, that compatibility is not an equivalence relation, since, while it is symmetric and reflexive, it is not necessarily transitive.

§2 Day 2: Atlases (Jan. 8, 2026)

We correct an error from last class. Let M be a set, and consider $\varphi : U \to \mathbb{R}^n$, where $U \subset M$. We call φ a (coordinate) chart if $\varphi(U)$ is open and φ is injective; indeed, we identify U with an open subset of $\mathbb{R}^n$. Given two charts $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$, we say they are *compatible* if

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a diffeomorphism of open subsets.[2] We now present an idea; indeed, we may regard φ as a coordinate system on U, i.e., $\varphi(p) = (x^1, \ldots, x^n)$ are *coordinates* of p (with respect to φ), for which our transition maps are realized as coordinate changes.

Definition 2.1. A set of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ that covers M (i.e., every $p \in M$ is in some U_α) is called an *atlas*.

From this we may draft the following definition,

Definition. A *manifold* is a set M with an atlas $\mathcal{A}$ of charts.

Observe the following examples,

(i) Consider the n-sphere S^n, for which we have the stereographic projection φ_N, projecting N through a point on the sphere onto $\mathbb{R}^n$, which we may define as follows,
$$\varphi_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n).$$
However, it is evident that φ_N does not admit N in its domain; thus, we may similarly define the stereographic projection from the south pole by
$$\varphi_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n)$$
in order to cover S^n. Do these two maps form an atlas for S^n? It suffices to check that they are compatible; directly write as follows,[3]
$$\varphi_S \circ \varphi_N^{-1}(y_1, \ldots, y_n) = \frac{1}{|\vec{y}|^2}(y_1, \ldots, y_n),$$
where $\vec{y} = (y_1, \ldots, y_n)$.[4]

(ii) Let M be the set of straight lines in $\mathbb{R}^2$, i.e., of the form "$ax + by = c$". Consider the charts φ mapping $mx + b$ to $(m, b) \in \mathbb{R}^2$ and ψ mapping $x = ny + c$ to $(n, c) \in \mathbb{R}^2$ (in this manner, we account for both horizontal and vertical lines). Then the transition map between them is realized as
$$\psi \circ \varphi^{-1}(m, b) = \left(\frac{1}{m}, -\frac{b}{m}\right)$$
whenever $m \neq 0$ (when the line is neither horizontal nor vertical).

Given two atlases, how do we test that they define the same manifold?

[2] defn 2.4 in gross meinrenken

[3] see: p.22-23 for the S^1 case

[4] fedya u seem cool n all but we cannot be friends if u use this notation (that is, until i change my mind and start appreciating $\vec{\bullet}$)

Definition 2.2 (§2.8)**.** A chart $\varphi : U \to \mathbb{R}^n$ is *compatible* with an atlas if $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ if φ is compatible with every φ_α.

Lemma 2.3 (§2.10)**.** If $\varphi : U \to \mathbb{R}^n$, $\psi : V \to \mathbb{R}^n$ are compatible with the same atlas $\mathcal{A}$, then they're compatible with each other.

Proof. It is straightforward to check the properties, so just read the book. □

Theorem 2.4 (§2.11)**.** Given an atlas $\mathcal{A}$ on M, there's a unique *maximal* atlas $\widetilde{\mathcal{A}}$ which consists of all charts compatible to $\mathcal{A}$. Every chart compatible with $\mathcal{A}$ is already in $\widetilde{\mathcal{A}}$.

Proof. In one direction, if a chart is compatible with $\tilde{\mathcal{A}}$, then it is compatible with $\mathcal{A}$, and is therefore in $\widetilde{\mathcal{A}}$; in the other direction, $\widetilde{\mathcal{A}}$ is an atlas, so it covers because it contains $\mathcal{A}$ and is pairwise compatible by the lemma, so we are done. □

With this, we give a second attempt and defining a manifold.

Definition. A manifold is a set M with a maximal atlas $\mathcal{A}$.

§3 Day 3: Real and Complex Projective Plane; Real and Complex Grassmannians (Jan. 13, 2026)

Last class, we finished with our "second draft" of a definition of a manifold, i.e., a set M with a maximal atlas $\mathcal{A}$ (if a chart is compatible with $\mathcal{A}$, then it is already in $\mathcal{A}$). In practice, it is sufficient to take a "reasonably large" atlas, from which it can be uniquely completed to a maximal atlas. Two atlases generate the same maximal atlas if and only if each chart from one atlas is compatible with every chart in the other.

There are still some problems with our definition, though. For example, if M is "too big", such as $M = \mathbb{R}$ and we consider the atlas $\varphi_r : U_r \to \mathbb{R}^0 = \{0\}$, where each $U_r = \{r\} \in \mathbb{R}$, then M is just "uncountable dust", and doesn't fit in every $\mathbb{R}^N$.[5]

Definition 3.1. A manifold $(M, \mathcal{A})$ is *second countable* if it can be covered by countably many charts.

As another example, consider the line with two origins, $\mathbb{R} \cup \{0'\}$; we may give this two charts, $\mathrm{id} : \mathbb{R} \to \mathbb{R}$, and $\mathbb{R} \setminus \{0\} \cup \{0'\} \to \mathbb{R}$ given by the identity everywhere except $0' \mapsto 0$.

Definition 3.2 (§2.16b)**.** A manifold $(M, \mathcal{A})$ is *Hausdorff* if every pair of points lie in disjoint charts, i.e., for all $p_1, p_2 \in M$, there are

$$\varphi_1 : U_1 \to \mathbb{R}^n, \quad \varphi_2 : U_2 \to \mathbb{R}^n,$$

such that $U_1 \cap U_2 = \varnothing$, $p_1 \in U_1$, and $p_2 \in U_2$. Note that in a maximal atlas, given a chart $\varphi : U \to \mathbb{R}^n$, the restriction

$$\varphi\big|_V : V \to \mathbb{R}^n$$

is also a chart as long as $\varphi(V)$ is open.

In particular, this means that if p_1, p_2 are in the same chart, they can't obstruct the Hausdorff property. With this, we can define the manifold properly,

Definition 3.3. A manifold is a set M with a maximal atlas $\mathcal{A}$ that is second countable and Hausdorff.

We now present some examples.

(a) (§2.18) Let S^n be equipped with the maximal atlas containing the stereographic projections φ_N and φ_S. M is second countable because it is covered by two charts, and Hausdorff because the only interesting case to look at is when $p_1 = N$ and $p_2 = S$ (otherwise, we can cover p_1, p_2 with the same chart), but we can then restrict φ_S and φ_N on the open hemispheres.

(b) (Section §2.3.2) Consider the real projective n-space $\mathbb{R}\mathrm{P}^n$, which can be regarded as the set of lines through 0 in $\mathbb{R}^{n+1}$, or

$$(\mathbb{R}^{n+1} \setminus \{0\}) / \big\{v \sim cv \mid c \in \mathbb{R} \setminus \{0\}\big\},$$

or $S^n / v \sim -v$ (read: equivalence classes), or $\overline{B^n} / \{v \sim -v \mid v \in \partial \overline{B_n}\}$, where $B^n = \{x \in \mathbb{R}^n \mid \|x\|^2 \leq 1\}$. Note that all of these ways to express $\mathbb{R}\mathrm{P}^n$ are equivalent to each other.

[5]this is example §2.14 in the textbook, p.26

Definition 3.4. Since any nonzero vector in $\mathbb{R}^{n-1}$ determines a line (through the origin), we write $(x^0 : x^1 : \cdots : x^n)$ to be the *homogeneous coordinates* in the equivalence class of $(x^0, \ldots, x_n) \in \mathbb{R}^{n-1} \setminus \{0\}$ with equivalence if two points agree up to a nonzero scalar multiple.

The projective space $\mathbb{RP}^n$ can be given a chart, i.e., a projection taking the open northern hemisphere to $\mathbb{R}^n$ by

$$\varphi : (x^0 : x^1 : \cdots : x^n) \mapsto \left(\frac{x^1}{x^0}, \ldots, \frac{x^n}{x^0}\right),$$

for which φ_0 admits the domain $U_0 := \{(x^0 : x^1 : \cdots : x^n) \mid x^0 \neq 0\}$. We may, in this manner, construct $\varphi_1, \ldots, \varphi_n$ similarly covering $\mathbb{RP}^n$ (where we omit the jth coordinate for φ_j)[6], since every point in $\mathbb{RP}^n$ has some nonzero homogeneous coordinate. In this manner, we may view U_j to consists of the lines intersecting the affine hyperplane $H_j = \{x \in \mathbb{R}^{n+1} \mid x^j = 1\}$.

Exercise 3.5. Compute the transition maps for $\mathbb{RP}^n$ and verify that they're smooth. What do we need in order to show that this is a manifold? We need to show that (i) the charts cover $\mathbb{RP}^n$, (ii) the transition maps are smooth (which induces some maximal atlas), (iii) is second countable, (iv) and is Hausdorff. Note that the first two conditions are shown because we already demonstrated earlier that the $\varphi_0, \ldots, \varphi_n$ form a chart, and that second countability comes from (i). We will show (iv) later.

(c) (Section §2.3.3) We now discuss the complex projective space $\mathbb{CP}^n$, which we take to be $\mathbb{C}^{n+1} \setminus \{0\}$ quotientied out by constant multiples, or S^{2n+1} quotiented out by the relation $v \sim e^{i\theta} v$, where $\theta \in [0, 2\pi)$ (each equivalence class is a circle). Here, we write the homogeneous coordinates to be $(z^0 : z^1 : \cdots : z^n)$, and we have a similar construction for charts,

$$\varphi_i(z^0 : z^1 : \cdots : z^n) = \left(\frac{z^0}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^n}{z^i}\right),$$

where $\varphi_i : U_i \to \mathbb{R}^{2n} \cong \mathbb{C}^n$. As an example, consider $\mathbb{RP}^1 \cong \mathbb{R} \cup \{(1:0)\} \cong S^1$, and $\mathbb{CP}^1 \cong \mathbb{C} \cup \{(1:0)\} \cong S^2$, where we view $\mathbb{C} = \{(z:1) \mid z \in \mathbb{C}\}$. Note that this is not a rigorous visualization.

Remark 3.6. The transition maps here are not just diffeomorphic, but also holomorphic. $\mathbb{CP}^n$ can be regarded as a complex manifold, and in general, you can define manifolds with some "extra structure" if their transition maps have said extra structure.

(d) (Section §2.3.4) The *real Grassmannian* is denoted $\mathrm{Gr}(k, n)$ and defined as the k-dimensional vector subspaces of $\mathbb{R}^n$. As an example,

$$\mathbb{RP}^n = \mathrm{Gr}(1, n+1) \cong \mathrm{Gr}(n, n+1).$$

How do we build charts in this manifold? As an idea, we can think of the subspaces as graphs of linear functions; one such chart might be

$$\mathrm{span}\begin{pmatrix} 1 & 0 & a \\ 0 & a & b \end{pmatrix} \to (a, b) \in \mathbb{R}^2;$$

in general, there are $\binom{n}{k}$ total charts[7].

[6] see p.31 for a more careful treatment

[7] i will leave some of the exposition to the textbook, i don't like what fedya is doing

Claim 3.7. These charts cover $\mathrm{Gr}(k, n)$, i.e., in each k-dimensional subspace E, we can find $e_{i_1}, \ldots, e_{i_k}$, such that E is the graph of a function sending $(x^{i_1}, \ldots, x^{i_k})$ to the other $n - k$ coordinates in $\mathbb{R}^n$.

Since I've basically lost what Fedya is saying because he writes in cursive and his jumping around everywhere in the textbook treatment for real Grassmannians, I direct you to read pages 34-36 for the same treatment of this subject that has already been written down.

§4 Day 4: Topology on Manifolds (Jan. 20, 2026)

Given a manifold M with a maximal atlas $\{(U_\alpha, \varphi_\alpha)\}$, we define a subset $U \subset M$ to be open if, for all charts $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$, $\varphi_\alpha(U \cap U_\alpha)$ is open in $\mathbb{R}^n$.

Proposition 4.1. U is open if and only if $\varphi_\beta(U \cap U_\beta)$ is open in (U_β, φ_β) covering U (e.g., a non-maximal atlas).

For example, $U \subset S^n$ is open if and only if $\varphi_N(U \setminus \{N\})$ and $\varphi_S(U \setminus \{S\})$ is open.

Proof. The forward direction is clear, so we will check the converse only. Let $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$ be another chart in $\mathcal{A}$; then $U \cap U_\alpha = \bigcup_\beta U \cap U_\alpha \cap U_\beta$, whence

$$\begin{aligned}\varphi_\alpha(U \cap U_\alpha \cap U_\beta) &= (\varphi_\alpha \circ \varphi_\beta^{-1}) \circ \varphi_\beta(U \cap U_\alpha \cap U_\beta) \\ &= (\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\beta(U \cap U_\beta) \cap \varphi_\beta(U_\alpha \cap U_\beta)).\end{aligned}$$

Indeed, each $\varphi_\beta(U \cap U_\beta)$ is open by assumption and each $\varphi_\beta(U_\alpha \cap U_\beta)$ is open because φ_α and φ_β are compatible charts (as they're in the same atlas). Thus, $\varphi_\beta(U \cap U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$, and so $\varphi_\alpha(U \cap U_\alpha \cap U_\beta)$ is open, so $\varphi_\alpha(U \cap U_\alpha)$ is a union of open sets and is therefore itself open. □

Fact 4.2. If $\mathcal{A}$ is an atlas on M and $U \subset M$ is open, then $A_U = \{(U \cap U_\alpha, \varphi_\alpha|_{U \cap U_\alpha})\}$ is an atlas on U. The proof of this fact is left as an exericse.

Moreover, observe that if $\mathcal{A}_U$ inherits maximality, second countability, and Hausdorff-ness from $\mathcal{A}$, meaning that if $(M, \mathcal{A})$ is a manifold, then $(U, \mathcal{A}_U)$ is one as well.

Proposition 4.3. Using our definition from earlier, the collection of open subsets of M forms a topology on M. Specifically, $\varnothing, M$ are open, the finite intersection of open sets are open, and the arbitrary union of open sets is also open.

The proof sketch is that we will use the fact that the above is indeed true in $\mathbb{R}^n$, so we will adapt that idea down here. We start by defining more things in topology, though. We call $N \subset M$ a neighborhood of p if there exists open U such that $p \in U \subset N$. A set A is *closed* if its complement is open. M is *disocnnected* if it can be written as $U \sqcup V$, where U, V are open and nonempty (equivalently, this means there are no nonempty proper subsets of M that are clopen).

For example, $S^0 = \{-1, 1\} \subset \mathbb{R}$ has $\{-1\}$ and $\{1\}$ as open sets, so they are both disconnected. We say M is connected if it is not disconnected. We say M is Hausdorff if any two points have disjoint neighborhoods.

Remark 4.4. U is open if and only if it's a union of U_α, where $(U_\alpha, \varphi_\alpha)$ are charts in a maximal atlas. (A maximal atlas gives a basis for the manifold topology).

Proof. If $(U_\alpha, \varphi_\alpha)$ is a chart and $\varphi(U \cap U_\alpha)$ is open, then $(U \cap U_\alpha, \varphi_\alpha|_{U \cap U_\alpha})$ is also a chart in the maximal atlas. □

We now discuss compactness. $K \subset \mathbb{R}^n$ is compact if it is closed and bounded (for example, a closed ball).

Definition 4.5. $K \subset M$ is compact if $K = K_1 \cup \cdots \cup K_n$, where $K_i \subset U_i$ and $\varphi_i(K_i) \subset \mathbb{R}^n$ is compact.

In particular, in $\mathbb{R}^n$, finite unions of compact sets are compact. Alternatively, we can define K to be compact if $K \subset U_i$ and $\varphi_i(K)$ is compact in $\mathbb{R}^n$. We give some examples.

(i) S^n is compact. Moreover, if we take $\{x \in S^n \mid x_{n+1} \geq 0\}$ and $\{x \in S^n \mid x_{n+1} \leq 0\}$ to be the upper and lower hemispheres of S^n, then stereographic projection from the opposite poles will demonstrate that their images are compact in $\mathbb{R}^n$.

(ii) $\mathbb{RP}^n$ is compact, since we may regard it as $\{[x_0 : x_1 : \cdots : x_n]\}$ with an atlas $U_i = \{[x_0 : x_1 : \cdots : x_n] \mid x_i \neq 0\}$ and its associated map φ_i; by noticing that if we manually construct $x_0^2 + \cdots + x_n^2 = 1$, there is some element $x_i \geq \sqrt{n}^{-1}$, so $\|\varphi_i([x_0 : x_1 : \cdots : x_n])\| \leq \sqrt{n}$, i.e.,

$$\mathbb{RP}^n = \bigcup_i \varphi_i^{-1}\big(\overline{B_{\sqrt{n}}(0)}\big).$$

In topology, we have the following definition for compactness.

Definition 4.6. K is compact if, for all collections $\{U_\alpha\}$ of open sets such that $\bigcup_\alpha U_\alpha \supset K$ (i.e., the U_α cover K), there is a finite subcollection that also covers K.

Proposition 4.7. If K is topologically compact in a manifold M, then K is closed. If $C \subset M$ is closed, then $K \cap C$ is compact.

Proof. Suppose K is not compact; then $M \setminus K$ is not open, so there exists a point $y \notin K$ such that every open neighborhood of y intersects K. By Hausdorffness, for all x, there exists $U_x \ni x$ and $V_x \ni y$ an open disjoint set such that $\{U_x \mid x \in K\}$ covers K. This cover has a finite subcover, for which $K \subset U_{x_1} \cup \cdots \cup U_{x_n}$ and $K \cap (V_{x_1} \cap \cdots \cap V_{x_n}) = \varnothing$, which contradicts our assumption on y.

For the second part of the proposition, take $\{U_\alpha\}$ an open cover of $K \cap C$; then $\{U_\alpha\} \cup \{M \setminus C\}$ is an open cover of K. By compactness, this has a finite subcover for K; taking out $M \setminus C$, we get a finite subcover of $\{U_\alpha\}$ for $K \cap C$. □

Proposition 4.8. The two definitions for compactness are equivalent. Note that we assume this fact for $\mathbb{R}^n$ (per the Heine–Borel theorem).

Proof. Suppose K is charts-compact; then $K = K_1 \cup \cdots \cup K_n$, where $k_i \subset U_i$ for some charts (U_i, φ_i). Let $\{V_n\}$ be an open cover of K; by Heine–Borel, each V_i is covered by finitely many $\varphi_i(V_\alpha \cap U_i)$, so K admits a finite subcover of $\{V_n\}$.

For the other direction, suppose K is topologocailly compact. Take the open cover $\{V_{U,\varphi,x,\varepsilon}\}$, where $V_{U,\varphi,x,\varepsilon} = \varphi^{-1}(B_\varepsilon(x))$ where U, φ is a chart, $x \in \varphi(U)$, and $\overline{B_\varepsilon(x)} \subset \varphi(U)$. Take a finite subcover $V_{U_i,\varphi_i,x_i,\varepsilon_i}$. Then

$$K \subset \bigcup_{i=1}^{n} \varphi_i^{-1}(B_{\varepsilon_i}(x_i)) \implies K = \bigcup_{i=1}^{n} \varphi_i^{-1}(\overline{B_{\varepsilon_i}(x_i)} \cap K).$$

By the earlier proposition, this set is compact, so we are done. □

§5 Day 5: Orientability (Jan. 22, 2026)

Observe that the Möbius strip (following our construction in this class) only has "one side". We can associate the right hand rule with $\det(v_1 \mid v_2 \mid v_3) > 0$.

Definition 5.1. A linear transformation $T\colon V \to V$ is *orientation preserving* if $\det T > 0$; a transition map $\tau\colon V_1 \to V_2$ between two charts is *orentation preserving* if $\det(D\tau) > 0$ everywhere.

Exercise 5.2. If $\det(D\tau) > 0$ in one half of the Möbius strip, then it has $\det(D\tau) < 0$ in the other half, and vice versa.

Definition 5.3. We say φ_1 and φ_2 are *orientation compatible* if their transition map is orientation preserving.

Definition 5.4. An *oriented atlas* is one in which all charts are orientation compatible. A *maximal oriented atlas* contains all charts orientation compatible with a given oriented manifold $(M, \mathcal{A})$.

Note that the definition of maximal oriented atlas only makes senes if, in any oriented atlas, any two charts that are oriented compatibly to said atlas are orientation compatible to each other. This is left as an exercise. We now give examples.

(i) The sphere is orientable. We claim that the usual atlas $\{\varphi_N, \varphi_S\}$ is not oriented compatibly, but $\{\varphi_N, (-x_1, x_2, \ldots, x_n) \circ \varphi_S\}$ is. Observe that for $n \geq 2$, the intersection $U_N \cap U_S = S^n \setminus \{N, S\}$ is connected, meaning that the sign of $\det(D\tau) \neq 0$ on the whole set (by the intermediate value theorem).

Fact 5.5. Let X be a connected space, and let $f\colon X \to A$ (where A is taken to be some set) be locally constant, i.e., for every $p \in X$, there exists an open neighborhood $U \ni x$ such that $f|_U$ is constant. Then f is constant.

Proof. Observe that the preimage of fibers $f^{-1}(a) = \bigcup_{p \in f^{-1}(a)} U_p$ is open, so X can be realized as the disjoint union $\bigsqcup_{a \in A} f^{-1}(a)$; per connectedness of X, it must be that there is only one such a. □

(ii) $\mathbb{RP}^2$ is not orientable. We leave this as an exercise.

Definition 5.6. Let $(M, \mathcal{A})$ be an oriented manifold. Then there is an opposite orientation $\tilde{\mathcal{A}} = \{F(U), F \circ \varphi\}$ for each $(U, \varphi) \in \mathcal{A}$, where $F\colon \mathbb{R}^n \to \mathbb{R}^n$ and $x \mapsto (-x_1, x_2, \ldots, x_n)$.

Proposition 5.7. Let $(M, \mathcal{A})$ be an oriented manifold. Every connected chart (U, φ) is oriented compatible with $\mathcal{A}$ or $\tilde{\mathcal{A}}$.

Proof. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$, and define $\Sigma_p := \operatorname{sign}(\det D(\varphi_\alpha \circ \varphi^{-1})\big|_{\varphi(p)})$ is independent of α. We may note that $\Sigma_p\colon U \to \{+, -\}$ is a locally constant function; but U is connected, so it must be constant. □

We leave it as an exercise that if M is a connected manifold, then there is exactly one maximal oriented atlas.