

# MAT417 Lecture Notes

ARKY!! :3C

'25 Fall Semester

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## §1 Day 1: Course Administrative Details and Preliminaries (Sep. 2, 2025)

Course materials will be free and available online; here is a list of reference materials:

- Serre's *Course in Arithmetics* up to Chapter 4,
- Lecture notes by Noam Elkies (which will be posted on Quercus).

Homework will be posted every Thursday and due the following Thursday, and is worth **20%** of the course grade.

The central question of number theory is about the structure of prime numbers, of which the main analytic tools used are the Riemann  $\zeta$ -functions and its relatives (the  $L$ -functions). We may discuss things like modular forms, Hecke operators and  $L$ -functions related to Galois representation later on.

Let us consider the following two questions;

- (a) How many primes are there? There are infinitely many of them.
- (b) Can you say something about how the primes are distributed?

Given  $x > 0$ , where  $x$  may be a natural or a real, let us define

$$\pi(x) = \#\{p \text{ is prime} \mid p \leq x\}.$$

Can we estimate how  $\pi(x)$  grows? The prime number theorem states that the growth of  $\pi(x)$  is proportional to  $\frac{x}{\log x}$ , i.e.,

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x / \log x} = 1, \quad \frac{\pi(x)}{x} \rightarrow 0 \text{ as } x \rightarrow \infty.$$

As an exercise, show that the prime number theorem informally says that the  $n$ th prime  $p_n$  is of the size  $n \log n$ .

**Theorem 1.1** (Dirichlet Theorem). Let  $a, d$  be coprime naturals where  $a < d$ . Consider all numbers of the form  $a + kd$ , where  $k$  is also a natural; infinitely many of these numbers are prime.

*Proof.* Done with  $L$ -functions. Check [here](#). □

**Theorem 1.2** (Fundamental Theorem of Arithmetic). Any natural number  $N$  can be written uniquely as  $p_1^{a_1} \dots p_n^{a_n}$ , where  $p_i$  are primes and  $a_i > 0$ .

**Proposition 1.3** (Euclid's Argument on the Infinitude of Primes). Assume that  $p_1 < p_2 < \dots < p_n$  constitute all the primes. Then it is clear that  $p_1 \dots p_n + 1$  is coprime to any  $p_i$ . By the fundamental theorem of arithmetic, this means that  $p_1 \dots p_n + 1$  is divisible by a prime less than  $p_1 \dots p_n + 1$  not given by some  $p_i$ , which is a contradiction.

Can we use this to get an estimate on  $\pi(x)$ ? We claim that  $\pi(x) > \log_2 \log_2 x$ . Let  $p_n$  be the  $n$ th prime. Then

$$p_{n+1} < 1 + \prod_{i=1}^n p_i < \prod_{i=1}^n p_n.$$

If equality always held then we would have  $p_n = 2^{2^{n-1}}$ . However, in actuality,  $p_n < 2^{2^{n-1}}$ , so we must have that  $\pi(x) > \log_2 \log_2 x$ .

The Riemann-Zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

**Claim 1.4.**  $\zeta$  is absolutely convergent for any  $s > 1$ .

*Proof.* Will be given next class. □

**Lemma 1.5.** For  $s > 1$ , we have that

$$\zeta(s) \leq \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}}.$$

*Proof.* This is given directly by geometric series, i.e.,

$$\frac{1}{1 - p^{-s}} = \sum_{i=0}^{\infty} p^{-is} = \sum_{\substack{p_1 < \dots < p_n \\ a_1, \dots, a_n > 0}} p_1^{a_1} \dots p_n^{a_n}. \quad \square$$

Moreover, if we had finitely many primes, we could apply this to  $s = 1$  and obtain that the sum of  $\frac{1}{n}$  is convergent, which is clearly false. This also implies that the sum of the reciprocals of primes is divergent, and you can't have  $\pi(x)$  be bounded from above by  $Cx^D$ , where  $C > 0, D < 1$ .

## §2 Day 2: More accurate treatment of the Riemann-Zeta function (Sep. 4, 2025)

Note that I won't be here for the second hour of Thursday classes because I have complex analysis during that time. Isaac will be taking the full hour's worth of notes, though. *I lied I'm staying for this lecture*

Today's lesson agenda is as follows,

- (i) More accurate treatment of  $\zeta(s)$ ;
- (ii) Prove that  $\sum_{p \text{ is prime}} \frac{1}{p}$  is divergent (per Euler),
- (iii) Start doing preaportory material for the Dirichlet theorem, and introduce the Dirichlet  $L$ -functions.

**Lemma 2.1.** The Riemann-Zeta function is convergent for  $s \in \mathbb{R}$ ,  $s > 1$ ; it is absolutely convergent for  $s \in \mathbb{C}$ ,  $\Re s > 1$ .

We will later prove that for  $\Re s > 1$ ,  $\zeta(s)$  is a holomorphic function. Let's start by comparing  $\sum \frac{1}{n^s}$  to  $\int_1^\infty x^{-s} dx$ ; observe that

$$\int_1^a x^{-s} dx = \left. \frac{x^{1-s}}{1-s} \right|_1^a = \frac{a^{1-s}}{1-s} - \frac{1}{1-s},$$

of which  $a^{1-s}$  approaches 0 as  $a \rightarrow \infty$ . Thus, we have that

$$\int_1^\infty x^{-s} dx = \frac{1}{s-1}.$$

We also have that

$$\sum_{n=2}^\infty n^{-s} \leq \int_1^\infty x^{-s} dx = \frac{1}{s-1},$$

and

$$\sum_{n=2}^N n^{-s} \leq \int_1^N x^{-s} dx,$$

which yields convergence. Thus, we have that inequality that  $\zeta(s) \leq 1 + \frac{1}{s-1}$ .

**Exercise 2.2.** Run a very similar argument and prove that  $\zeta(s) > \frac{1}{s-1}$ . In particular,

$$\frac{1}{s-1} < \zeta(s) < 1 + \frac{1}{s-1}.$$

In particular, the Riemann-Zeta function can also be written in the *Euler product* form, given by

$$\zeta(s) = \prod_{p \text{ prime}} \left( \frac{1}{1-p^{-s}} \right).$$

Taking the log of both sides, we get that

$$\log \zeta(s) = - \sum_p \log(1-p^{-s}).$$

From here on, we simply write a subscript of  $p$  on summations or products to indicate that they're prime (unless stated otherwise). Clearly, the above is divergent for  $s = 1$ .

**Lemma 2.3.** (i) For all  $s_0 > 1$ , there exists some constant  $M > 0$  such that

$$\log \left| \sum_p p^{-s} - \log \frac{1}{s-1} \right| < M \text{ for all } 1 < s \leq s_0.$$

(ii) The sum of  $\frac{1}{p}$  over all primes diverge.

*Proof.* We may rewrite the equation in the first line as follows,

$$\sum_p p^{-s} = \log \frac{1}{s-1} + O(1) \text{ as } s \rightarrow 1,$$

where we may note  $O(1)$  is some bounded function. Recall the following,

**Definition 2.4.** Let  $f, g$  be functions on some space  $X$ , where  $g \geq 0$ . We say that  $f = O(g)$  if  $|f| \leq Mg$ , where  $M$  is some constant.

In this manner, saying  $f = O(1)$  is equivalent to saying that  $|f|$  is bounded. Now, let us take the log of the entire following inequality,

$$\begin{aligned} \frac{1}{s-1} &< \zeta(s) < 1 + \frac{1}{s-1} = \frac{s}{s-1}, \\ \log \left( \frac{1}{s-1} \right) &< - \sum_p \log(1 - p^{-s}) < \log \left( \frac{s}{s-1} \right), \\ 0 &< - \left( \log(s-1) + \sum_p \log(1 - p^{-s}) \right) < \log s \end{aligned} \quad (*)$$

where the Taylor expansion of  $|\log(1 - p^{-s}) - p^{-s}|$  is less than  $p^{-2s}$ .

**Exercise 2.5.** Check that  $|\log(1 - y) - y| < y^2$  for  $0 < y < 1$  for  $y \in \mathbb{R}$ . This is done by expanding  $\log(1 + x)$  around  $x = 0$ .

Specifically, summing over all  $p$  and applying the triangle inequality, the above tells us that

$$\left| \sum_p (p^{-s} + \log(1 - p^{-s})) \right| < \sum_p p^{-2s} < \zeta(2).$$

Using both inequalities together, we obtain

$$\begin{aligned} &\left| \sum_p p^{-s} - \log \frac{1}{s-1} \right| \\ &= \left| \left( \sum_p p^{-s} + \sum_p \log(1 - p^{-s}) \right) - \left( \log \frac{1}{s-1} + \sum_p \log(1 - p^{-s}) \right) \right| \\ &\leq \zeta(2) + \log s \leq \zeta(2) + s_0 - 1, \end{aligned}$$

if  $1 < s \leq s_0$ . Indeed, this shows that  $M = s_0 - 1 + \zeta(2)$  for (i). The second part of the lemma is also left as homework.  $\square$

We now discuss Dirichlet series and Dirichlet  $L$ -functions. Let  $m \in \mathbb{N}$ , and let  $(\mathbb{Z}/m\mathbb{Z})^*$  be the invertible elements in the ring  $\mathbb{Z}/m\mathbb{Z}$ . Specifically, these are the residues modulo  $m$  which are prime to  $m$ . This forms an abelian group under multiplication, of which its size is given by the totient  $\varphi(m)$ .

**Exercise 2.6.** If  $m$  is prime, then  $(\mathbb{Z}/m\mathbb{Z})^*$  is the cyclic group of order  $m - 1$ .

Fix a character  $\chi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$ , where  $\mathbb{C}^*$  are the nonzero complex numbers. Extend  $\chi$  as a map  $\mathbb{Z} \rightarrow \mathbb{C}$  such that  $\chi(n)\chi(m) = \chi(nm)$  as follows,

$$\chi(n) = \begin{cases} 0 & \text{if } \gcd(n, m) \neq 1, \\ \chi(n \bmod m) & \text{if } \gcd(n, m) = 1. \end{cases}$$

As an example, let  $m = 3$ , and consider  $(\mathbb{Z}/3\mathbb{Z})^* = \{\pm 1\}$ . Then

$$\chi(n) = \begin{cases} 0 & \text{if } 3 \mid n, \\ 1 & \text{if } n \equiv 1 \pmod{3}, \\ -1 & \text{if } n \equiv -1 \pmod{3}. \end{cases}$$

For all  $m$ , we have the trivial homomorphism  $(\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$ . Let  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$  be the function

$$\chi(n) = \begin{cases} 1 & \text{if } \gcd(n, m) = 1, \\ 0 & \text{if } \gcd(n, m) \neq 1. \end{cases}$$

Then we may define the  $L$ -function

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left( \frac{1}{1 - \frac{\chi(p)}{p^s}} \right).$$

**Claim 2.7.**  $L(\chi, x)$  is absolutely convergent for  $\Re s > 1$ .

**Theorem 2.8.** (i)  $L(\chi, s)$  is holomorphic for  $\Re s > 1$ . (ii) Assume the extension of  $\chi$  is not equal to 1. Then  $L(\chi, s)$  converges for  $\Re s > 0$  and defines a holomorphic function there. (iii) If the extension of  $\chi$  is not equal to 1, then  $L(\chi, 1) \neq 0$ .

Let  $G$  be a finite abelian group. Consider all characters  $\chi : G \rightarrow \mathbb{C}^*$ ; they form a group  $G^\vee$  under multiplication.

**Claim 2.9.** (i)  $G^\vee$  is (non-canonically) isomorphic to  $G$ , and  $\#G^\vee = \#G$ . (ii)  $(G^\vee)^\vee \cong G$  canonically.

*Proof.* The claim lets us say that if  $G$  is finite and abelian, then  $G$  is isomorphic to a product of finite cyclic groups

$$G \cong \prod_{i=1}^k (\mathbb{Z}/a_i\mathbb{Z}), \quad a_i > 1.$$

Using the fact that  $(G \times H)^\vee \cong G^\vee \times H^\vee$ , we see that specifying  $\chi : G \times H \rightarrow \mathbb{C}^\times$  is equivalent to specifying characters  $\chi_1, \chi_2$  on  $G$  and  $H$  respectively. Letting  $a > 1$ , we have that if  $\chi : \mathbb{Z}/a\mathbb{Z} \rightarrow \mathbb{C}^\times$  and  $g^a = 1$ , we have that  $\chi(g) \in \mathbb{C}^*$  and  $\chi(g)^a = 1$ . This means that  $\chi(g)$  must be an  $a$ th root of unity. All the roots of 1 of order  $a$  form a cyclic group of order  $a$ .

For the second part of the claim, in the direction of  $G \rightarrow (G^\vee)^\vee$ , we have that for each  $g \in G$ , we obtain a canonical map  $G^\vee \rightarrow \mathbb{C}^*$  where all  $x \in G^\vee \mapsto \chi(g)$ .  $\square$

**Lemma 2.10.** This map is an isomorphism.

**Lemma 2.11.** (i) All  $\chi \in G^\vee$  form a basis of  $\mathbb{C}(G)$ , the complex valued functions on  $G$ .  
(ii) This basis is orthonormal with respect to  $\langle f_1, f_2 \rangle = \frac{1}{\#G} \sum_g f_1(g) \bar{f}_2(g)$ .

*Proof.* We know that  $\dim \mathbb{C}(G) = \#G = \#G^\vee$ . Recall that we have

$$\langle \chi, \chi \rangle = \frac{1}{\#G} \sum_g \chi(g) \bar{\chi}(g) = \frac{1}{\#G} \sum_g \chi(g) \chi_g^{-1} = \frac{1}{\#G} \sum_g \chi(gg^{-1}) = 1,$$

since  $\chi(1) = 1$ . Now, let us evaluate  $\#G \langle \chi, 1 \rangle = \sum_g \chi(g)$ . We have that since  $\chi$  is not uniformly 1, there must exist some  $h \in G$  such that  $\chi(h) \neq 1$ ; and so

$$\chi(h) \sum_g \chi(g) = \sum_g \chi(hg) = \sum_g \chi(g),$$

meaning  $\sum_g \chi(g) = 0$ , as  $\chi(h)$  is nonzero as well. Thus, we obtain that

$$\#G \langle \chi_1, \chi_2 \rangle = \sum_g \chi_1(g) \bar{\chi}_2(g) = \sum_g \chi_1(g) \chi_2^{-1}(g),$$

meaning that  $\#G \langle \chi_1 \chi_2^{-1}, 1 \rangle$ . If  $\chi_1 \chi_2^{-1} \neq 1$  (i.e., if  $\chi_1 \neq \chi_2$ ), then this is 0.  $\square$

Let  $x_n$  be a sequence of elements of  $\mathbb{R}_{>0}$  such that  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ . The main example we will be looking at is  $\lambda_n = \log n$  (or  $\lambda_n = n$ ), and the Dirichlet series  $\sum_n a_n e^{-\lambda_n z}$  where  $a_n \in \mathbb{C}$ .

Next lecture, we will do some general analysis of convergence and analytic properties of such series. We will apply this to  $L(\chi, s)$ .

### §3 Day 3: Characters (Sep. 9, 2025)

Recall that given  $m \in \mathbb{Z}_{\geq n}$ , we have  $\chi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$  and  $\tilde{\chi} : \mathbb{Z} \rightarrow \mathbb{C}$  satisfies

$$\tilde{\chi}(n) = \begin{cases} 0 & n \text{ is not prime to } m, \\ \chi(n, \text{mod } m) & \text{if } \gcd(n, m) = 1. \end{cases}$$

Also, we ask that  $|\chi(n)| \leq 1$  for all  $n$  (so the magnetude does not spiral off to infinity). Recall that the  $L$ -function is defined as

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

which converges absolutely for  $\Re s > 1$ . Then we have the following theorem,

**Theorem 3.1.**  $L(\chi, s)$  is holomorphic in  $s$  for  $\Re s \geq 1$ , and it extends meromorphically to  $\Re s > 0$ . If  $\chi \neq 1$ , then  $L(\chi, s)$  is holomorphic for  $\Re s > 0$  and the series  $\sum \frac{\chi(n)}{n^s}$  is convergent for  $\Re s > 0$ . Moreover, if  $\chi = 1$ , then  $L(\chi, s)$  has a simple pole at  $s = 1$  and has no other poles.

In fact,  $L(\chi, s)$  is meromorphic for all  $s \in \mathbb{C}$ .

**Theorem 3.2.** If  $\chi \neq 1$ , then  $L(\chi, 1) \neq 0$ .

We plan to prove theorem 3.1, then, assuming theorem 3.2, we will deduce the Dirichlet theorem about primes in an arithmetic progression. We will follow Serre's book [here](#) (section 2.2, Dirichlet series).

Let  $x_n$  be a sequecne of positive real numbers tending to infinity, i.e.,  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ . A *Dirichlet series* is a series, where, given  $\{a_n\}$  a sequence of complex numbers, we write

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad (a_n \in \mathbb{C}, z \in \mathbb{C}).$$

Two such examples of Dirichlet series are given by setting  $\lambda_n = \log n$  (the ordinary Dirichlet series), where such a series is written  $\sum \frac{a_n}{n^s}$ , and  $\lambda_n = n$  where by setting  $t = e^{-z}$ , the series turns into a power series in  $t$  as follows,

$$\sum_{n=1}^{\infty} a_n e^{-nz} = \sum_{n=0}^{\infty} a_n t^n.$$

**Theorem 3.3.** Assume that  $f(z) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n z}$  is convergent for  $z = z_0$ . Then it is convergent uniformly on every set of the form  $\Re(z - z_0) \geq 0$ , where  $\arg(z - z_0) \leq \alpha$  with  $\alpha < \frac{\pi}{2}$ .

**Exercise 3.4.** Analyze what this means for  $\lambda_n = n$  and realize that you know this statement.

**Lemma 3.5.** Suppose  $\{f_n(z)\}$  is a sequence of holomorphic functions on some domain  $U \subset \mathbb{C}$ . Assume there exists  $f(z) = \lim_{n \rightarrow \infty} f_n(z)$  for all  $z \in U$  such that the convergence is uniform on every compact subset of  $U$ . Then  $f(z)$  is holomorphic, and moreover,  $f'(z) = \lim_{n \rightarrow \infty} f'_n(z)$ .

In particular, if we let  $U = \{z \mid \Re(z) > \Re(z_0)\}$ , then every compact set can be covered by finitely many sectors, meaning there exists a uniform convergence no every compact set.



**Corollary 3.6.** Let  $L(\chi, s)$  be holomorphic for  $\Re s > 1$ .

The following lemma is necessary to study series with summands of the form  $a_n b_n$ .

**Lemma 3.7** (Abel's lemma). Let  $A_{m,p} = \sum_{n=m}^p a_n$  and let  $B_{m,m'} = \sum_{n=m}^{m'} a_n b_n$ . Then we have

$$S_{m,m'} = \sum_{n=m}^{m'-1} A_{m,n}(b_n - b_{n+1}) + A_{m,m'} b_{m'}.$$

**Lemma 3.8.** Let  $\alpha, \beta \in \mathbb{R}$ , and let  $0 < \alpha < \beta$ . Then  $z = x + iy$  with  $x > 0$ ; then

$$\left| e^{-\alpha z} - e^{-\beta z} \right| \leq \left| \frac{z}{x} \right| (e^{-\alpha x} - e^{-\beta x}).$$

For  $z = z_0$ ,  $f(z_0)$  converges and  $\sum a_n$  converges, meaning that for all  $\varepsilon$ , there exists  $N$  such that for all  $m, m' \geq N$ , we have that  $|A_{m,m'}| < \varepsilon$ . Applying the lemma with  $b_n = e^{-\lambda_n z}$ , we have that

$$S_{m,m'} = \sum_{n=m}^{m'-1} A_{m,n}(e^{-\lambda_n z} - e^{-\lambda_{n+1} z}) + A_{m,m'} e^{-\lambda_{m'} z},$$

and putting  $z = x + iy$  and applying lemma 3.8, we have that

$$|S_{m,m'}| \leq \varepsilon \left( 1 + \frac{|z|}{x} \sum_{n=m}^{m'-1} (e^{-\lambda_n x} - e^{-\lambda_{n+1} x}) \right) \leq \varepsilon (1 + k(e^{-\lambda_m x} - e^{-\lambda_{m'} x})) \leq \varepsilon (1 + k),$$

and so uniform convergence is clear. Note that I am not entirely confident about this argument, so re-check the proof of proposition 6 in Serre's book if confused.

## §4 Day 4: (Sep. 11, 2025)

Last time, we proved that  $L(\chi, s)$  are holomorphic for  $\Re s > 1$ , up to some lemma; next, we are going to show that all  $L(s, \chi)$  are in fact, meromorphic, for  $\Re s > 0$ .

1. (Page 71, Prop. 11) If  $\chi = 1$ , then  $\zeta(s)$  is meromorphic for  $\Re s > 0$  and has a unique simple pole for  $s = 1$ .
2. (Prop. 12) If  $\chi \neq 1$ , then  $L(s, \chi)$  is holomorphic for  $\Re s > 0$ .

Later today, we will show that (prop. 13)

$$\zeta_m(s) = \prod_{\chi} L(s, \chi) \iff \forall x, x \neq 1, L(1, \chi) \neq 0.$$

We note that  $\zeta_m$  has a simple pole at  $s = 1$ . We also have the unproved lemma from last time, where if  $0 < \alpha < \beta$ , then for  $z \in \mathbb{C}$  with  $\Re z > 0$ , written  $z = x + iy$ , we have that

$$\left| e^{-\alpha z} - e^{-\beta z} \right| \leq \frac{|z|}{x} \left( e^{-\alpha x} - e^{-\beta x} \right).$$

This is true by writing

$$z \int_{\alpha}^{\beta} e^{-tz} dt = e^{-\alpha z} - e^{-\beta z} \implies \left| e^{-\alpha z} - e^{-\beta z} \right| \leq |z| \int_{\alpha}^{\beta} e^{-tx} dt = \frac{|z|}{x} \left( e^{-\alpha x} - e^{-\beta x} \right).$$

We now discuss proposition 10. In the case  $\chi = 1$ , we claim the following,

**Claim 4.1** (Prop. 10).  $\zeta(s) = \frac{1}{s-1} + \varphi(s)$ , where  $\varphi(s)$  is holomorphic in  $\Re s > 0$ .

*Proof.* We have that

$$\frac{1}{s-1} = \int_1^{\infty} t^{-s} dt = \sum_{n=1}^{\infty} \int_n^{n+1} t^{-s} dt,$$

meaning we may write

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - t^{-s}) dt.$$

With this, we may construct a sequence of  $\varphi_n$ ,

$$\varphi_n(s) = \int_n^{n+1} (n^{-s} - t^{-s}) dt, \quad \varphi(s) = \sum_{n=1}^{\infty} \varphi_n(s),$$

where each  $\varphi_n(s)$  is holomorphic for  $\Re s > 0$ . Since each  $\varphi_n(s)$  holds this property, it suffices to check that the series converges normally, of which we have that  $\sum_{n=1}^{\infty} \|\varphi_n\|$  converges, where  $\|\varphi_n\| = \sup_{s \in S} |\varphi_n(s)|$ . We claim that normal convergence implies uniform absolute convergence, i.e., for all  $\varepsilon > 0$ , the series of  $\varphi_n(s)$  is normally convergent in  $\Re s \geq \varepsilon$ .

*Subproof.* To start, let us make the naive bound

$$\|\varphi_n(s)\| \leq \sup_{n \leq t \leq n+1} |n^{-s} - t^{-s}| \leq \sup_{n \leq t \leq n+1} \left| \frac{dt^{-s}}{dt} \right|,$$

which we have from the lemma that if  $f$  is a continuously differentiable function, we have that

$$|f(a) - f(b)| \leq \sup_{a \leq x \leq b} |f'(x)| (b - a).$$

In this manner, we also have that

$$\sup_{n \leq t \leq n+1} \left| \frac{dt^{-s}}{dt} \right| = \sup_{n \leq t \leq n+1} \left| \frac{s}{t^{s+1}} \right| = \frac{|s|}{n^{s+1}},$$

where we have that on  $\Re s \geq \varepsilon$ ,  $\sum_n \frac{|s|}{n^{s+1}}$  is convergent. ■

**Claim 4.2.**  $L(s, \chi)$  converges for  $\Re s > 0$ .

By what we did last time, this implies that  $L(s, \chi)$  is holomorphic in  $\Re s > 0$ . □

**Conjecture 4.3** (Riemann Hypothesis). For  $\Re s > 0$ , the only zeros of  $\zeta(s)$  have  $\Re = \frac{1}{2}$ .

We will discuss the motivations and applications for this later. We start by considering the section post-proposition 12,

**Lemma 4.4** (Proposition 9). Suppose we have a series  $\sum a_n n^{-s}$ . Assume that all partial sums of  $\{a_n\}$  are bounded; if all  $A_{m,m'}$ , given by

$$A_{k,k'} = \sum_{n=k}^{k'} a_n,$$

are bounded, then  $\sum a_n n^{-s}$  is convergent for  $\Re s > 0$ .

Consider the function,

$$\tilde{\chi}(n) = \begin{cases} 0 & \gcd(n, m) \neq 1, \\ \chi(n \bmod m) & \gcd(n, m) = 1; \end{cases}$$

if we let  $a_n = \tilde{\chi}_n$ , then for all  $k$ , we have that

$$\sum_{n=k}^{k+m-1} \tilde{\chi}(n) = 0.$$

*Proof.* Assume all  $|A_{k,k'}| \leq K$ ; by applying Abel's lemma, we have that

$$|S_{k,k'}| = \left| \sum_{n=k}^{k'} a_n \underbrace{n^{-s}}_{b_n} \right| \leq K \left( \sum_{n=k}^{k'} \left| \frac{1}{n^s} - \frac{1}{(n+1)^s} \right| + \left| \frac{1}{(k')^s} \right| \right).$$

If  $\Re s > 0$ , then the right hand side is simply equal to  $\frac{K}{k^s}$ , and for all  $\varepsilon > 0$ , there exists  $N$  such that if  $k \geq N$ , then the  $\frac{K}{k^s} \leq \varepsilon$ . □

So far, we've proven that

- (i) For all  $\chi$ ,  $L(s, \chi)$  is meromorphic for  $\Re s > 0$ .

(ii) If  $x = 1$ , there is a unique simple pole at  $s = 1$ .

(iii) If  $x \neq 1$ , there are no poles.

Finally, we need that  $L(1, \chi) \neq 0$  if  $\chi \neq 1$  (p.73, thm. 1). Define

$$\zeta_m(s) = \prod_x L(s, \chi),$$

which we already know to be meromorphic for  $\Re s > 0$ . We want to show that  $\zeta_m(s)$  has a unique simple pole at  $s = 1$ . As a quick digression, consider  $\mathbb{Q} \subset K \subset \mathbb{C}$ , where  $K$  is a finite extension of  $\mathbb{Q}$  (equivalently,  $\dim_{\mathbb{Q}}(K) < \infty$ ). There exists a notion that  $\zeta_K(s)$ , which is a  $\zeta$  function of a number field  $K$ . All of those have analytic properties similar to  $\zeta(s)$ . We have that  $\zeta(s) = \zeta_{\mathbb{Q}}(s)$  has a unique simple pole at  $s = 1$ ; if we fix  $m \geq 1$ , then the cyclotomic field of order  $m$ ,  $K_m$ , is given by  $K_m = \mathbb{Q}(\mu_m) = K(e^{2\pi i \frac{1}{m}})$ , where  $\mu_m$  are the roots of 1 of order  $m$ . Secretly, we have that  $\zeta_m(s) = \zeta_{K_m}(s)$ .

We write out the explicit Dirichlet series for  $\zeta_m(s)$ . Let  $p$  be a prime that does not divide into  $m$ , i.e.,  $\bar{p} = (\mathbb{Z}/m\mathbb{Z})^* = G(m)$ . Let  $f(p)$  be the order of  $\bar{p}$  in  $G(m)$ , and let  $g(p) = \frac{f(m)}{f(p)}$ , which is the order of  $G(m)$  quotiented by the subgroup generated by  $\bar{p}$ .

**Claim 4.5** (Proposition 13). We have that

$$\zeta_m(s) = \prod_{p \nmid m} \left( \frac{1}{1 - p^{-f(p)s}} \right)^{g(p)}.$$

*Proof.* Let  $T$  be a variable. Fix  $p$  where  $p \nmid m$ ; then we have

$$\prod_{\chi} (1 - \chi(\bar{p})T) = (1 - T^{f(p)})^{g(p)},$$

which follows from

$$\prod_w (1 - wT) = 1 - T^{f(p)},$$

and the product is taken over all  $w$  where  $w^{f(p)} = 1$ , i.e., the  $f(p)$ -th roots of unity (we note that  $f(p)$  can be any element of  $\mathbb{N}$ ). For all such  $w$ , there exist  $g(p)$  characters  $\chi$  such that  $\chi(\bar{p}) = w$ , which implies our result. To see why this is true, let  $A$  be a finite abelian group,  $B \subset A$  a subgroup, and let  $\chi_B : B \rightarrow \mathbb{C}^*$ . Then there exists exactly  $\#(A/B)$  extensions of  $\chi_B$  to  $A$ .

In our case, let  $A = (\mathbb{Z}/m\mathbb{Z})^*$ ,  $B$  be the subgroup generated by  $\bar{p}$ , and fix  $w$  such that  $w^{f(p)} = 1$ . There exists a unique character  $\chi_B$  of  $B$  such that  $\chi_B(\bar{p}) = w$ . An extension to  $A$  is a character  $\chi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$  such that  $\chi(\bar{p}) = w$ , and so

$$g(p) = \# \frac{(\mathbb{Z}/m\mathbb{Z})^*}{B},$$

meaning that for all  $w$  with  $w^{f(p)} = 1$ , there exist  $g(p)$  characters  $\chi$  such that  $\chi(\bar{p}) = w$ . Consider the chain

$$0 \rightarrow B \rightarrow A \rightarrow A/B \rightarrow 0.$$

If we let  $\widehat{\phantom{x}}$  denote the dual groups,

$$0 \rightarrow \widehat{A/B} \xrightarrow{\alpha} \widehat{A} \xrightarrow{\beta} \widehat{B} \rightarrow 0,$$

then we claim that  $\widehat{A/B} \rightarrow \widehat{A}$  is injective, and  $\ker \beta = \text{im } \alpha$ , which is obvious; since  $\#A = \#\widehat{A}$ , we have that  $\widehat{A} \xrightarrow{\beta} \widehat{B}$  is onto, and we are done.  $\square$