MAT354 Lecture Notes

Arky!! :3c

'25 Fall Semester

Contents

§1 Day 1: Recap of Preliminaries (Sep. 2, 2025)

We start by discussing the complex plane and complex numbers. Given $z \in \mathbb{C}$, we say that $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of z respectively, i.e., $z = x + iy$. $\mathbb{C}$ is the set of all complex numbers. In this manner, we may identify $z = x + iy$ with $(x, y) \in \mathbb{R}^2$ using the standard complex plane.

(a) The complex *conjugate* of z is given by $\bar{z} = x - iy$, where we have that

$$\Re(z) = \frac{z + \bar{z}}{2}, \qquad \Im(z) = \frac{z - \bar{z}}{2i}.$$

(b) We now define addition and mlutiplication for the complex numbers. For all $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have that

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2). \end{aligned}$$

We have that $(\mathbb{C}, +, \times)$ is a field, with $(\mathbb{R}, +, \times)$ as a subfield. To verify this, we need to check that it indeed satisfies:

- Commutativity; for all $z_1, z_2 \in \mathbb{C}$, we have that $z_1 + z_2 = z_2 + z_1$ and $z_1z_2 = z_2z_1$.
- Associativity: for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1z_2)z_3 = z_1(z_2z_3)$.
- Distributivity: for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.

(c) The absolute value of a complex number $z = x + iy$ is given by $|z| = \sqrt{x^2 + y^2}$. In particular, this yields the triangle inequality, where for any $z, w \in \mathbb{C}$, we have that $|z + w| \le |z| + |w|$. The proof either comes visually or through explicit computation, both of which I will not write out here for brevity.[1]

As an extension of the inequality, we also automatically have that

$$|\Re z| \le |z|, \qquad |\Im z| \le |z|,$$

and that for all $z, w \in \mathbb{C}$, we have

$$||z| - |w|| \le |z - w|.$$

Proof. Using the triangle inequality, we have that

$$\begin{aligned} |z| &= |(z - w) + w| \le |z - w| + |w|, \\ |w| &= |(w - z) + z| \le |z - w| + |z|, \end{aligned}$$

of which both imply that $|z| - |w| \le |z - w|$ and $|w| - |z| \le |z - w|$. □

For any $z \in \mathbb{C}$, we have that $|z|^2 = z \cdot \bar{z}$.

Proof. Write $z = x + iy$; then $|z|^2 = x^2 + y^2$, where we may note that $z \cdot \bar{z} = (x + iy)(x - iy)$ which yields the right hand side of the earlier equation through expansion. □

[1] no full credit if you draw a picture on the exam lmao

Finally, for $z, w \in \mathbb{C}$, we have that $|zw| = |z|\,|w|$. This is left as an exercise to the student.

(d) The polar form of a nonzero complex number $z \neq 0$ is given by $z = \gamma e^{i\theta}$, where $\gamma > 0$ and $\theta \in \mathbb{R}$. Let us assume the Euler formula; for all $\theta \in \mathbb{R}$, we have that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Let $r = |z|$; we have that $|z| = \left|re^{i\theta}\right| = |r|\left|e^{i\theta}\right| = r \cdot 1 = r$. θ is the angle between the positive real axis to the half-line starting from 0 and passing through z. In this manner, $z = re^{i\theta} = |z|(\cos\theta + i\sin\theta) = |z|\cos\theta + i\,|z|\sin\theta$, which means we have that

$$\Re z = |z|\cos\theta, \qquad \Im z = |z|\sin\theta.$$

As an example, let us find all the complex numbers z such that $z^4 = i$. Since $i = e^{i\frac{\pi}{2}}$, $z = \rho e^{i\theta}$ satisfying $z^4 = i$ becomes $\rho^4 e^{i\cdot 4\theta} = e^{i\frac{\pi}{2}}$, meaning

$$\begin{cases} \rho^4 = 1, \\ 4\theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

This means $\rho = 1$ and $\theta = \frac{\pi}{8} + \frac{k\pi}{2}$, where $k \in \mathbb{Z}$. Considering the cases $k = 0, 1, 2, 3$ and observing that there are only 4 equivalence classes modulo 4 to consider, we have that

$$z_0 = e^{i\frac{\pi}{8}}, \quad z_1 = e^{i\frac{5\pi}{8}}, \quad z_2 = e^{i\frac{9\pi}{8}}, \quad z_3 = e^{i\frac{13\pi}{8}}.$$

We now discuss convergence. We say that a set of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ converges to $w \in \mathbb{C}$ if $\lim_{n\to\infty} |z_n - w| = 0$. We write it as $\lim_{n\to\infty} z_n = w$. In the complex plane, the convergence can be in any direction.

Lemma 1.1. $\{z_n\}_{n\in\mathbb{N}}$ converges to w if and only if $\{\Re z_n\}_{n\in\mathbb{N}}$ converges to $\Re w$ and $\{\Im z_n\}_{n\in\mathbb{N}}$ converges to $\Im w$.

Proof. We have that

$$\begin{aligned} |z_n - w| &= |(\Re z_n - \Re w) + i(\Im z_n - \Im w)| \\ &\leq |\Re z_n - \Re w| + |\Im z_n - \Im w|, \end{aligned}$$

where as $n \to \infty$, we have that the right hand side is given by $0 + 0$. For the opposite direction, we have that $|z| \geq |\Re z|$ or $|\Im z|$, so we have that

$$|\Re z_n - \Re w| = |\Re(z_n - w)| \leq |z_n - w|,$$

which approaches 0 as $n \to \infty$. The same argument goes for the imaginary portion. □

A sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ is called *Cauchy* if $|z_n - z_m| \to 0$ as $n, m \to \infty$. In $\varepsilon - \delta$, this means that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|z_n - z_m| < \varepsilon$ for all $n, m > N$.

Theorem 1.2 (Bolzano-Weierstrass Theorem)**.** $\mathbb{R}$ is *complete*, i.e., every Cauchy sequence of real numbers convergesto a real number.

Theorem 1.3. $\mathbb{C}$ is complete.

Proof. Take any Cauchy sequence of complex numbers $\{z_n\}$. Using the inequalities $|\Re z| \leq |z|$ and $\{\Im z\} \leq |z|$, we have that $\{\Re z_n\}$ and $\{\Im z_n\}$ are Cauchy sequences of real numbers. By Bolzano-Weierstrass, we have that $\Re z_n \to x_0 \in \mathbb{R}$ and $\Im z_n \to y_0 \in \mathbb{R}$. By the previous lemma, we actually have $\lim_{n\to\infty} z_n = x_0 + iy_0$. □

We now move onto topology in the complex plane. Given $z_0 \in \mathbb{C}$ and $r > 0$, we can form an open or closed disc centered at z_0 of radius r. We write both of these as

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$
$$\bar{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\},$$

Given a set $\Omega \subseteq \mathbb{C}$, a point z_0 is an interior point if there exists $r > 0$ such that $D_r(z) \subseteq \Omega$. The interior of Ω is given by the set of all such interior points. In particular, the interior of $\bar{D}_r(i)$ is $D_r(i)$.

A set Ω is called *open* if every point in Ω is an interior point. Ω is called *closed* if the complement of Ω, $\Omega^c = \mathbb{C} \setminus \Omega$, is open. As an example, the open right half-plane $\{z \in \mathbb{C} \mid \Re z > 0\}$ is open.

Proof. For any $z \in \Omega$, let $z = x + iy$, and take $r = \frac{x}{2} = \frac{\Re z}{2}$. Then we claim that $D_r(z) \subseteq \Omega$. For all $w \in D_r(z)$, we clearly have that

$$\Re w = \Re z - (\Re z - \Re w) \geq \Re z - |z - w| \geq \frac{\Re z}{2} > 0,$$

and so all such $w \in \Omega$, and we are done. □

A point $z \in \mathbb{C}$ is a *limit point* of Ω if there exists a sequence $\{z_n\} \subset \Omega$ with $z_n \neq z$ such that $z_n \to z$.

As an example, we define D to be the open unit disc centered at 0. 0 and 1 are both limit points of D, but 1 is not contained in D itself.[2] The *closure* of Ω, $\bar{\Omega}$, is given by Ω unioned with all its limit points. The *boundary* of a set Ω, wirtten $\partial\Omega$, is given by $\bar{\Omega} \setminus \operatorname{int} \Omega$. A set $\Omega \subseteq \mathbb{C}$ is said to be compact if it is closed and bounded, i.e., there exists $M > 0$ such that $|z| \leq M$ for all $z \in \Omega$.

Theorem 1.4. A set $\Omega \subseteq \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω.

Proposition 1.5. If $\Omega_1 \supset \Omega_2 \cdots \supset \Omega_n \supset \dots$ is a sequence of nonempty compact sets in $\mathbb{C}$, where $\operatorname{diam}(\Omega_n) = \sup_{z,w \in \Omega_n} |z - w| \to 0$ as $n \to \infty$, then there exists a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for every $n \in \mathbb{N}$.

Proof. For each Ω_n, pick a point $z_n \in \Omega_n$. Then $\{z_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence because the diameter of Ω_n approaches 0. By the Bolzano-Weierstrass theorem for complex numbers, this means that $\{z_n\}_{n \in \mathbb{N}}$ indeed does converge to some $w \in \mathbb{C}$. In particular, we have w is the limit of the subsequence $\{z_m\}_{m \geq n} \subseteq \Omega_n$, where Ω_n is compact, meaning the limit w should be in Ω_n. This means there exists a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for every $n \in \mathbb{N}$.

To show the uniqueness of w, we argue by contradiction; assume $w' \neq w$ satisfies the property. Then $|w' - w| > 0$. Since $w, w' \in \Omega_n$ for all n, this contradicts that $\operatorname{diam}(\Omega_n) \to 0$. □

An open set Ω is called *connected* if it is not possible to find two disjoint nonempty open sets Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. A connected open set in $\mathbb{C}$ is called a *region*.

[2] hell is it disc or disk YKW LET'S COMPROMISE it's spelled disque actually (paint nails)

§2 Day 2: Functions on the Complex Plane (Sep. 4, 2025)

Let $f : \Omega \to \mathbb{C}$, where Ω is an open subset of $\mathbb{C}$. We say that f is continuous if at $z_0 \in \Omega$ if, for all $\varepsilon > 0$, there exists an open disk $D_\gamma(z_0)$ such that $|f(z) - f(z_0)| < \varepsilon$ for all $z \in D_\gamma(z_0)$. In particular, f is said to be continuous on Ω if it is continuous at every point in Ω.

Example 2.1. Consider $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$. Show that f is continuous.

Solution. For all complex z, z_0, we have that $|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0| = |z - z_0|$. Thus, we have that for any $\varepsilon > 0$, we obtain[3]

$$f(D_\varepsilon(z_0)) = D_\varepsilon(\overline{z_0}). \qquad \square$$

We now discuss holomorphic functions (i.e., complex differentiable functions). We say that $f : \Omega \to \mathbb{C}$ is *holomorphic* at $z_0 \in \Omega$ if

$$\frac{f(z_0 + h) - f(z_0)}{h}, \qquad h \in \mathbb{C} \setminus \{0\},$$

converges as $h \to 0$. If the limit exists, we let

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

be the derivative.

Example 2.2. Consider the exact same function as in the previous example, $f(z) = \bar{z}$. Is f holomorphic?

Solution. For all $z_0 \in \mathbb{C}$ and $h \in \mathbb{C} \setminus \{0\}$, we have that

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{z_0 + h} - \overline{z_0}}{h} = \frac{\overline{h}}{h} = \frac{\rho e^{-i\theta}}{\rho e^{i\theta}} = e^{-2i\theta}.$$

If we take $h \to 0$ along the real line, we may let $h = \rho$, which means the fraction is equal to 1 as $h \to 0$. If we take $\rho \to 0$ along the complex axis, however, then we have that $h = \rho e^{i\pi/2}$, where we obtain the fraction is equal to -1 as $\rho \to 0$. Thus, f cannot be holomorphic. $\square$

Proposition 2.3. Let Ω be open in $\mathbb{C}$. If f, g are holomorphic on Ω, then

(i) $f + g$ is holomorphic on Ω, and $(f + g)' = f' + g'$.

(ii) fg is holomorphic on Ω, and $(fg') = f'g + fg'$.

(iii) If $g(z_0) \neq 0$ where $z_0 \in \Omega$, then $\frac{f}{g}$ is also holomorphic at z_0, where

$$\left(\frac{f}{g}\right)' = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

(iv) If $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, then $g \circ f$ is also holomorphic, and we obtain the chain rule

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

[3] note to self: ol is better than bar for this stuff...

We now discuss complex differentiability versus real differentiability. A holomorphic function $f : \Omega \to \mathbb{C}$ can be identified with a function $F : \Omega \to \mathbb{R}^2$ given by $(x, y) \mapsto (u(x, y), v(x, y)) = (\Re f(x, y), \Im f(x, y))$. Consider the partial derviative of F at (x_0, y_0); these exist if there exists some linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{\|F(P_0 + H) - F(P_0) - J(H)\|}{\|H\|} \to 0$$

as $H \to 0$. Or, we may define $\Psi(H)$ to take on the fraction above, and we see that F is indeed differentiable at $P_0 = (x_0, y_0)$ if $\Psi(H) \to 0$ as $H \to 0$. We now deal with complex differentiability. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic at $z_0 = x_0 + iy_0$. Then we have partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}.$$

Naturally,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h - f(z_0)}{h}$$

along any path; in particular, we take $h \in \mathbb{R} \setminus \{0\}$ and observe that

$$\begin{aligned}
f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{h \to 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\
&= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0),
\end{aligned}$$

and so both exist, and they are $\Re f'(z_0)$ and $\Im f'(z_0)$ respectively. Similarly, we may take $h = ik$ where $k \in \mathbb{R} \setminus \{0\}$ and obtain

$$\begin{aligned}
f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{k \to 0} \frac{u(x_0, y_0 + k) + iv(x_0, y_0 + k) - u(x_0, y_0) - iv(x_0, y_0)}{ik} \\
&= \lim_{k \to 0} \frac{-i(u(x_0, y_0 + k) - u(x_0, y_0)) + v(x_0, y_0 + k) - v(x_0, y_0)}{k} \\
&= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0),
\end{aligned}$$

and so both partials also exist and they are $\Re f'(z_0)$ and $-\Im f'(z_0)$ respectively.

§3 Day 3: Holomorphic Functions and Power Series (Sep. 9, 2025)

Let $f : \Omega \to \mathbb{C}$ (where Ω is an open set in $\mathbb{C}$). We say that f is holomorphic at z_0 if

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}, \quad h \in \mathbb{C}\setminus\{0\}$$

exists. Recall that $\mathbb{C}$ can be identified with $\mathbb{R}^2$ by considering any $z = x+iy \in \mathbb{C}$ as a tuple $(x, y) \in \mathbb{R}^2$. In this way, given a function $f : \Omega \to \mathbb{C}$, we can define $F : \Omega \to \mathbb{R}^2$, where $F : (x, y) \mapsto (u(x, y), v(x, y))$, given by $u = \Re f$ and $v = \Im f$.

Proposition 3.1. If $f = u + iv$ is holomorphic at $z_0 = x_0 + iy_0$, then we have that all four partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist and they satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \Re f(z_0), \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \Im f(z_0).$$

We also have that F is differentiable at $P_0 = (x_0, y_0)$.

Definition 3.2. We say that F is differentiable at P_0 if there exists a linear transforamtion (the derivative) $J = J_F(x_0, y_0) : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{H\to 0} \frac{\|F(P_0+H)-F(P_0)-J(H)\|}{\|H\|} = 0.$$

Before we discuss the complex definition, let us recall another property of real differentiability; if F is differentiable at $P_0 = (x_0, y_0)$, then all four partial derivatives exist, and

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is called the Jacobian matrix of F at (x_0, y_0). To see this, consider the association $P_0 = (x_0, y_0)$ with $z_0 = x_0 + iy_0$, and $H = (h_1, h_2)$ with $h = h_1 + ih_2$; then we have that

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 \\ \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 \end{pmatrix}.$$

This is a vector in $\mathbb{R}^2$, which we may associate with the complex number

$$\left(\frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2\right) + i\left(\frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2\right) = \left(\frac{\partial u}{\partial x} + i\frac{\partial y}{\partial x}\right)h_1 + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)h_2,$$

which, by the Cauchy-Riemann equations, we obtain

$$\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)h_1 + i\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\right)h_2 = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2),$$

which is precisely equal to $f(z_0 + h) - f(z_0) - f(z_0)h$. In particular,

$$\lim_{h\to 0}\left|\frac{f(z_0+h)-f(z_0)-f(z_0)h}{h}\right| = \lim_{h\to 0}\left|\frac{f(z_0+h)-f(z_0)}{h} - f(z_0)\right| = 0.$$

Similarly, per the definition of the Jacobian, we must have

$$\lim_{H\to 0} \frac{\|F(P_0+H)-F(P_0)-J(H)\|}{\|H\|} = 0,$$

and this concludes the proof of proposition 3.1. $\square$

Theorem 3.3. Suppose $f = u + iv$ is a complex-valued function defined on an open set $\Omega \subset \mathbb{C}$. If $u, v : \Omega \to \mathbb{R}$, are continuously differentiable and satisfy the Cauchy-Riemann equations, then f is holomorphic on Ω and $f'(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)$.

Proof. Since u is continuously differentiable at the point $(x, y) \in \Omega$, there exists a linear transformation $J_u : \mathbb{R}^2 \to \mathbb{R}$ where

$$\frac{|u(x + h_1, y + h_2) - u(x, y) - J_0(h_1, h_2)|}{\|(h_1, h_2)\|} \to 0, \quad (h_1, h_2) \to 0.$$

In particular, $J_u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$. The above fraction is equivalent to

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + \|h\| \psi_1(h),$$

where $\psi_1 : U \to \mathbb{R}$, where U is some open neighborhood of $0 \in \mathbb{R}^2$, with $\psi_1(h) \to 0$ as $h \to 0$. Similarly, we have that

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + \|h\| \psi_2(h)$$

with $\psi_2(h) \to 0$ as $h \to 0$. We want to show that f is holomorphic at $z = x + iy$. We have that

$$\begin{aligned} f(z + h) - f(z) &= (u(x + h_1, y + h_2) - u(x, y)) + i(v(x + h_1, y + h_2) - v(x, y)) \\ &= \left(\frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial h_y}h_2\right) + \|h\| \psi_1(h) + i\left(\frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2\right) + i\|h\| \psi_2(h) \\ &= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + \|h\| \psi_1(h) + i\|h\| \psi_2(h) \end{aligned}$$

from Cauchy-Riemann. Thus, we have that[4]

$$\begin{aligned} \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} &= \lim_{h \to 0} \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + \frac{\|h\|}{h}(\psi_1(h) + i\psi_2(h)) \\ &= \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right). \end{aligned}$$

□

We now discuss complex power series.

Definition 3.4. A complex power series is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

with $a_n \in \mathbb{C}$ and z a complex variable. We say that $\sum_{n=0}^{\infty} a_n z^n$ converges at $z_0 \in \mathbb{C}$ if there exists some $w \in \mathbb{C}$ such that, for all $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $N \geq N_0$ satisfies

$$\left|\sum_{n=0}^{N} a_0 z^n - w\right| < \varepsilon.$$

The series converges *absolutely* at z_0 if there exists $w \in \mathbb{R}$ such that

$$\left|\sum_{n=0}^{N} |a_n| \, |z_0|^n - w\right| < \varepsilon.$$

[4] i swear wenyu has an invisible key wired into her back like nano from nichijou and it's permanently cranked on

Proposition 3.5. If $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely at $z_0 \in \mathbb{C}$, then $\sum_{n=0}^{\infty} a_n z^n$ converges at $z_1 \in \mathbb{C}$ with $|z_1| \leq |z_0|$.

Proof. For all $z_1 \in \mathbb{C}$ with $|z_1| \leq |z_0|$, consider the sequence of partial sums $\{S_m(z_1)\}_{m\in\mathbb{N}}$ given by

$$S_m(z_1) = \sum_{n=0}^{m} a_n z_1^n.$$

We want to show that such a sequence converges. Since $\mathbb{C}$ is complete, it suffices to show that said sequence is Cauchy. For all $m < k \in \mathbb{N}$, we have that

$$|S_k(z_1) - S_m(z_1)| = \left| \sum_{n=m+1}^{k} a_n z_1^n \right| \leq \sum_{n=m+1}^{k} |a_n| \, |z_1|^n \leq \sum_{n=m+1}^{k} |a_n| \, |z_0|^k. \qquad \square$$

We now provide a few examples.

(i) The complex exponential function for all $z \in \mathbb{C}$, given by

$$e^z := \sum_{n=0}^{\infty} = \frac{z^n}{n!}.$$

For all $z \in \mathbb{C}$, this sum converges because it converges absolutely (consider $e^{|z|}$).

(ii) The geometric series $\sum_{n=0}^{\infty} z^n$, where $|z| < 1$, converges; otherwise, is $|z| \geq 1$, it diverges. In particular, if $\sum_{n=0}^{\infty} z^n$ converges, then $|z^n| \to 0$ as $n \to \infty$.

Theorem 3.6 (Shakarchi, Thm. 2.5)**.** Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $R \in [0, \infty)$ such that (i) if $|z| < R$, the series converges, and (ii) if $|z| > R$, the series diverges. We call R the *radius of convergence* of $\sum_{n=0}^{\infty} a_n z^n$, and $\{z \in \mathbb{C} \mid |z| < R\}$ the disc[5] of convergence. Moreover, R is given by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} =: L,$$

where we use the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proof. For all $z \in \mathbb{C}$ with $|z| < r < R$, there exists some $\varepsilon > 0$ such that

$$(L + \varepsilon)\, |z| = r < 1.$$

By definition of L, we have $|a_n|^{1/n} \leq L + \varepsilon$ for all large n, meaning that

$$|a_n| \, |z|^n = \left(|a_n|^{1/n} \, |z| \right)^n \leq ((L + \varepsilon)\, |z|)^n = r^n, \quad r \in (0, 1),$$

whereby comparison with the geometric series $\sum r^n$, we see that $\sum |a_n| \, |z|^n$ converges. Similarly, if $|z| > R$, we have that

$$\left(\frac{1}{r} - \varepsilon \right) |z| > 1,$$

where, using the definition of R, there exists an infinite subsequence a_{n_k} such that $|a_{n_k}|^{1/n_k} \geq \frac{1}{R} - \varepsilon$. We have that

$$|a_{n_k} z^{n_k}| - \left(|a_{n_k}|^{1/n_k} \, |z| \right)^{n_k} \geq \left[\left(\frac{1}{R} - \varepsilon \right) |z| \right]^{n_k} > 1. \qquad \square$$

[5]disque. ok i'll stop

§4 Day 4: Complex Power Series (Sep. 11, 2025)

As per given in the previous lecture, recall that the complex power series is defined as an infinite sum of the form

$$\sum +n = 0^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, z \in \mathbb{C},$$

i.e., z as a complex variable.

Theorem 4.1. The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function on its disc of convergence. The derivative of f is given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Moreover, f' has the same radius of convergence as f.

Proof. Let g be the power series defining f', and let $R \geq 0$ be the radius of convergence of f. The radius of convergence of g is also R, per Hadamard's formula,

$$\limsup_{n\to\infty} |na_n|^{\frac{1}{n-1}} \overset{(*)}{=} \limsup_{n\to\infty} |a_n|^{\frac{1}{n}\cdot\frac{n}{n-1}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R},$$

since

$$n^{\frac{1}{n-1}} = e^{\frac{\log n}{n-1}} \xrightarrow{n\to\infty} e^0 = 1. \tag{$*$}$$

For all $z_0 \in \mathbb{C}$ with $|z_0| < r < R$ and $h \in \mathbb{C} \setminus \{0\}$ with $|z_0 + h| < r$, let us compute the following,

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right|;$$

to start,

$$f(z) = \underbrace{\sum_{n=0}^{N} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{E_N(z)},$$

where $N \in \mathbb{N}$ is to be determined; we have that

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right) + \\ \left(S_N'(z_0) - g(z_0) \right) + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h} \right).$$

We compute each part individually.

$$\begin{aligned}
\left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| &= \left| \frac{\sum_{n=N+1}^{\infty} a_n (z_0+h)^n - \sum_{n=N+1}^{\infty} a_n z_0^n}{h} \right| \\
&\leq \sum_{n=N+1}^{\infty} \frac{|a_n|}{h} \left| (z_0+h)^n - z_0^n \right| \\
&\leq \sum_{n=N+1}^{\infty} |a_n| \left| (z_0+h)^{n-1} + (z_0+h)^{n-2} + \cdots + z_0^{n-1} \right| \\
&\leq \sum_{n=N+1}^{\infty} |a_n| \gamma^{n-1} \cdot n \xrightarrow{n\to\infty} 0,
\end{aligned}$$

as y has the radius of convergence of $R > r$. Next,

$$\left|S_N'(z_0) - g(z_0)\right| \xrightarrow{N\to\infty} 0,$$

since $S_N'(z_0) = \sum_{n=1}^N na_n z_0^{n-1}$ and $g(z_0) = \sum_{n=1}^\infty na_n z_0^{n-1}$. Given any $\varepsilon > 0$, we may choose a sufficiently large N such that

$$\left|S_N'(z_0) - g(z_0)\right| < \varepsilon, \quad \left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| < \varepsilon,$$

per our two computations above. Since $S_N(z)$ is a finite polynomial, $S_N'(z_0)$ is the derivative of $S_N(z)$ at z_0, and so there exists $\delta > 0$ such that, for all $0 \leq |h| < \delta$, we have

$$\left|\frac{S_N(z_0+h) - S_N(z_0)h}{h} - S_N'(z_0)\right| < \varepsilon,$$

which resolves all three parts of our expansion, and so we are done. $\square$

Corollary 4.2. The power series $f(z) = \sum_{n=0}^\infty a_n z^n$ is infinitely complex differentiable on its disc of convergence. For $k \in \mathbb{N}$, its kth derivative $f^{(k)}$ is given by

$$f^{(k)}(z) = \sum_{n=0}^\infty (a_n z^n)^k.$$

Definition 4.3. A function $f : \Omega \to \mathbb{C}$ is said to be *analytic* at $z_0 \in \Omega$ if there exists a power series $\sum_{n=0}^\infty a_n (z - z_0)^n$ with positive radius of convergence such that

$$f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n$$

on a neighborhood of $z_0 \in \Omega$.

In particular, this means that if $f : \Omega \to \mathbb{C}$ is holomorphic, we have that f is holomorphic at $z_0 \in \Omega$, and so f is analytic at $z_0 \in \Omega$ as well. The implication that analytic implies holomorphic was given by our earlier theorem; the direction that holomorphic implies analytic is given by Cauchy's integral formula, but we need to first define integration along curves.

(i) A parameterized curve is a function $z : [a, b] \to \mathbb{C}$, where $t \mapsto z(t)$. This gives the orientation from $z(a)$ to $z(b)$.

(ii) (*Regularity conditions on curves*). We say that the parameterized curve is smooth if $z'(t)$ exists, is continuous on $[a, b]$, and $z'(t) \neq 0$ for $t \in [a, b]$. We say that the parameterized curve z is piecewise smooth if z is continuous on $[a, b]$ and there exists a partition of $[a, b]$ with $a = a_0 < \cdots < a_n = b$ such that $z(t)$ is smooth on each $[a_r, a_{r+1}]$.