

# MAT354 Lecture Notes

ARKY!! :3C

'25 Fall Semester

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## §1 Day 1: Recap of Preliminaries (Sep. 2, 2025)

We start by discussing the complex plane and complex numbers. Given  $z \in \mathbb{C}$ , we say that  $\Re(z)$  and  $\Im(z)$  are the real and imaginary parts of  $z$  respectively, i.e.,  $z = x + iy$ .  $\mathbb{C}$  is the set of all complex numbers. In this manner, we may identify  $z = x + iy$  with  $(x, y) \in \mathbb{R}^2$  using the standard complex plane.

- (a) The complex *conjugate* of  $z$  is given by  $\bar{z} = x - iy$ , where we have that

$$\Re(z) = \frac{z + \bar{z}}{2}, \quad \Im(z) = \frac{z - \bar{z}}{2i}.$$

- (b) We now define addition and multiplication for the complex numbers. For all  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ , we have that

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \end{aligned}$$

We have that  $(\mathbb{C}, +, \times)$  is a field, with  $(\mathbb{R}, +, \times)$  as a subfield. To verify this, we need to check that it indeed satisfies:

- Commutativity: for all  $z_1, z_2 \in \mathbb{C}$ , we have that  $z_1 + z_2 = z_2 + z_1$  and  $z_1 z_2 = z_2 z_1$ .
  - Associativity: for all  $z_1, z_2, z_3 \in \mathbb{C}$ , we have that  $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$  and  $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ .
  - Distributivity: for all  $z_1, z_2, z_3 \in \mathbb{C}$ , we have that  $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ .
- (c) The absolute value of a complex number  $z = x + iy$  is given by  $|z| = \sqrt{x^2 + y^2}$ . In particular, this yields the triangle inequality, where for any  $z, w \in \mathbb{C}$ , we have that  $|z + w| \leq |z| + |w|$ . The proof either comes visually or through explicit computation, both of which I will not write out here for brevity.<sup>1</sup>

As an extension of the inequality, we also automatically have that

$$|\Re z| \leq |z|, \quad |\Im z| \leq |z|,$$

and that for all  $z, w \in \mathbb{C}$ , we have

$$||z| - |w|| \leq |z - w|.$$

*Proof.* Using the triangle inequality, we have that

$$\begin{aligned} |z| &= |(z - w) + w| \leq |z - w| + |w|, \\ |w| &= |(w - z) + z| \leq |z - w| + |z|, \end{aligned}$$

of which both imply that  $|z| - |w| \leq |z - w|$  and  $|w| - |z| \leq |z - w|$ . □

For any  $z \in \mathbb{C}$ , we have that  $|z|^2 = z \cdot \bar{z}$ .

*Proof.* Write  $z = x + iy$ ; then  $|z|^2 = x^2 + y^2$ , where we may note that  $z \cdot \bar{z} = (x + iy)(x - iy)$  which yields the right hand side of the earlier equation through expansion. □

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<sup>1</sup>no full credit if you draw a picture on the exam lmao

Finally, for  $z, w \in \mathbb{C}$ , we have that  $|zw| = |z||w|$ . This is left as an exercise to the student.

- (d) The polar form of a nonzero complex number  $z \neq 0$  is given by  $z = \gamma e^{i\theta}$ , where  $\gamma > 0$  and  $\theta \in \mathbb{R}$ . Let us assume the Euler formula; for all  $\theta \in \mathbb{R}$ , we have that

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Let  $r = |z|$ ; we have that  $|z| = |re^{i\theta}| = |r| |e^{i\theta}| = r \cdot 1 = r$ .  $\theta$  is the angle between the positive real axis to the half-line starting from 0 and passing through  $z$ . In this manner,  $z = re^{i\theta} = |z|(\cos \theta + i \sin \theta) = |z| \cos \theta + i |z| \sin \theta$ , which means we have that

$$\Re z = |z| \cos \theta, \quad \Im z = |z| \sin \theta.$$

As an example, let us find all the complex numbers  $z$  such that  $z^4 = i$ . Since  $i = e^{i\frac{\pi}{2}}$ ,  $z = \rho e^{i\theta}$  satisfying  $z^4 = i$  becomes  $\rho^4 e^{i4\theta} = e^{i\frac{\pi}{2}}$ , meaning

$$\begin{cases} \rho^4 = 1, \\ 4\theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

This means  $\rho = 1$  and  $\theta = \frac{\pi}{8} + \frac{k\pi}{2}$ , where  $k \in \mathbb{Z}$ . Considering the cases  $k = 0, 1, 2, 3$  and observing that there are only 4 equivalence classes modulo 4 to consider, we have that

$$z_0 = e^{i\frac{\pi}{8}}, \quad z_1 = e^{i\frac{5\pi}{8}}, \quad z_2 = e^{i\frac{9\pi}{8}}, \quad z_3 = e^{i\frac{13\pi}{8}}.$$

We now discuss convergence. We say that a set of complex numbers  $\{z_n\}_{n \in \mathbb{N}}$  converges to  $w \in \mathbb{C}$  if  $\lim_{n \rightarrow \infty} |z_n - w| = 0$ . We write it as  $\lim_{n \rightarrow \infty} z_n = w$ . In the complex plane, the convergence can be in any direction.

**Lemma 1.1.**  $\{z_n\}_{n \in \mathbb{N}}$  converges to  $w$  if and only if  $\{\Re z_n\}_{n \in \mathbb{N}}$  converges to  $\Re w$  and  $\{\Im z_n\}_{n \in \mathbb{N}}$  converges to  $\Im w$ .

*Proof.* We have that

$$\begin{aligned} |z_n - w| &= |(\Re z_n - \Re w) + i(\Im z_n - \Im w)| \\ &\leq |\Re z_n - \Re w| + |\Im z_n - \Im w|, \end{aligned}$$

where as  $n \rightarrow \infty$ , we have that the right hand side is given by  $0 + 0$ . For the opposite direction, we have that  $|z| \geq |\Re z|$  or  $|\Im z|$ , so we have that

$$|\Re z_n - \Re w| = |\Re(z_n - w)| \leq |z_n - w|,$$

which approaches 0 as  $n \rightarrow \infty$ . The same argument goes for the imaginary portion.  $\square$

A sequence of complex numbers  $\{z_n\}_{n \in \mathbb{N}}$  is called *Cauchy* if  $|z_n - z_m| \rightarrow 0$  as  $n, m \rightarrow \infty$ . In  $\varepsilon - \delta$ , this means that for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $|z_n - z_m| < \varepsilon$  for all  $n, m > N$ .

**Theorem 1.2** (Bolzano-Weierstrass Theorem).  $\mathbb{R}$  is *complete*, i.e., every Cauchy sequence of real numbers converges to a real number.

**Theorem 1.3.**  $\mathbb{C}$  is complete.

*Proof.* Take any Cauchy sequence of complex numbers  $\{z_n\}$ . Using the inequalities  $|\Re z| \leq |z|$  and  $|\Im z| \leq |z|$ , we have that  $\{\Re z_n\}$  and  $\{\Im z_n\}$  are Cauchy sequences of real numbers. By Bolzano-Weierstrass, we have that  $\Re z_n \rightarrow x_0 \in \mathbb{R}$  and  $\Im z_n \rightarrow y_0 \in \mathbb{R}$ . By the previous lemma, we actually have  $\lim_{n \rightarrow \infty} z_n = x_0 + iy_0$ .  $\square$

We now move onto topology in the complex plane. Given  $z_0 \in \mathbb{C}$  and  $r > 0$ , we can form an open or closed disc centered at  $z_0$  of radius  $r$ . We write both of these as

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$

$$\bar{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\},$$

Given a set  $\Omega \subseteq \mathbb{C}$ , a point  $z_0$  is an interior point if there exists  $r > 0$  such that  $D_r(z_0) \subseteq \Omega$ . The interior of  $\Omega$  is given by the set of all such interior points. In particular, the interior of  $\bar{D}_r(i)$  is  $D_r(i)$ .

A set  $\Omega$  is called *open* if every point in  $\Omega$  is an interior point.  $\Omega$  is called *closed* if the complement of  $\Omega$ ,  $\Omega^c = \mathbb{C} \setminus \Omega$ , is open. As an example, the open right half-plane  $\{z \in \mathbb{C} \mid \Re z > 0\}$  is open.

*Proof.* For any  $z \in \Omega$ , let  $z = x + iy$ , and take  $r = \frac{x}{2} = \frac{\Re z}{2}$ . Then we claim that  $D_r(z) \subseteq \Omega$ . For all  $w \in D_r(z)$ , we clearly have that

$$\Re w = \Re z - (\Re z - \Re w) \geq \Re z - |z - w| \geq \frac{\Re z}{2} > 0,$$

and so all such  $w \in \Omega$ , and we are done.  $\square$

A point  $z \in \mathbb{C}$  is a *limit point* of  $\Omega$  if there exists a sequence  $\{z_n\} \subset \Omega$  with  $z_n \neq z$  such that  $z_n \rightarrow z$ .

As an example, we define  $D$  to be the open unit disc centered at 0. 0 and 1 are both limit points of  $D$ , but 1 is not contained in  $D$  itself.<sup>2</sup> The *closure* of  $\Omega$ ,  $\bar{\Omega}$ , is given by  $\Omega$  unioned with all its limit points. The *boundary* of a set  $\Omega$ , written  $\partial\Omega$ , is given by  $\bar{\Omega} \setminus \text{int } \Omega$ . A set  $\Omega \subseteq \mathbb{C}$  is said to be compact if it is closed and bounded, i.e., there exists  $M > 0$  such that  $|z| \leq M$  for all  $z \in \Omega$ .

**Theorem 1.4.** A set  $\Omega \subseteq \mathbb{C}$  is compact if and only if every sequence  $\{z_n\} \subset \Omega$  has a subsequence that converges to a point in  $\Omega$ .

**Proposition 1.5.** If  $\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_n \supset \dots$  is a sequence of nonempty compact sets in  $\mathbb{C}$ , where  $\text{diam}(\Omega_n) = \sup_{z, w \in \Omega_n} |z - w| \rightarrow 0$  as  $n \rightarrow \infty$ , then there exists a unique  $w \in \mathbb{C}$  such that  $w \in \Omega_n$  for every  $n \in \mathbb{N}$ .

*Proof.* For each  $\Omega_n$ , pick a point  $z_n \in \Omega_n$ . Then  $\{z_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence because the diameter of  $\Omega_n$  approaches 0. By the Bolzano-Weierstrass theorem for complex numbers, this means that  $\{z_n\}_{n \in \mathbb{N}}$  indeed does converge to some  $w \in \mathbb{C}$ . In particular, we have  $w$  is the limit of the subsequence  $\{z_m\}_{m \geq n} \subseteq \Omega_n$ , where  $\Omega_n$  is compact, meaning the limit  $w$  should be in  $\Omega_n$ . This means there exists a unique  $w \in \mathbb{C}$  such that  $w \in \Omega_n$  for every  $n \in \mathbb{N}$ .

To show the uniqueness of  $w$ , we argue by contradiction; assume  $w' \neq w$  satisfies the property. Then  $|w' - w| > 0$ . Since  $w, w' \in \Omega_n$  for all  $n$ , this contradicts that  $\text{diam}(\Omega_n) \rightarrow 0$ .  $\square$

An open set  $\Omega$  is called *connected* if it is not possible to find two disjoint nonempty open sets  $\Omega_1$  and  $\Omega_2$  such that  $\Omega = \Omega_1 \cup \Omega_2$ . A connected open set in  $\mathbb{C}$  is called a *region*.

<sup>2</sup>hell is it disc or disk YKW LET'S COMPROMISE it's spelled disque actually (paint nails)

**§2 Day 2: Functions on the Complex Plane (Sep. 4, 2025)**

Let  $f : \Omega \rightarrow \mathbb{C}$ , where  $\Omega$  is an open subset of  $\mathbb{C}$ . We say that  $f$  is continuous if