MAT367 Lecture Notes

Arky!! :3c

'26 Winter Semester

Contents

1 Day 1: Recap of Preliminaries (Jan. 6, 2026) 2

2 Day 2: (Jan. 8, 2026) 4

§1 Day 1: Recap of Preliminaries (Jan. 6, 2026)

Today's class can be followed more precisely on §1.2 to §1.4 of our textbook by Gross and Meinrenken. The slogan of this class is that a manifold is something that locally looks like $\mathbb{R}^n$. Specifically, an n-manifold can be covered n-dimensional charts $(U \subset M) \to \mathbb{R}^n$, with our main motivating example being solutions sets to equations. Recall the implicit function theorem,

Theorem 1.1. Given a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, consider the solution set $f(x_1, \ldots, x_{n+1}) = 0$ and a point $p \in \mathbb{R}^n$ such that $\nabla f(p) \neq 0$; then, for $(x_1, \ldots, x_{n+1})$ in said solution set near p, we can represent solutions as $(x_1, \ldots, x_n, g(x_1, \ldots, x_n))$, where g is also a smooth function.

In particular, if 0 is a regular value[1] of f, then we can cover $\{x \mid f(x) = 0\}$ by graphs/charts. We present some examples;

(i) Let $f(x, y) = xy$; then $\ker f$ is precisely the x and y axes, which is not a manifold, because it does not look like $\mathbb{R}^n$ (for any n) near the origin.

(ii) Let $f(x, y) = y - x^{2/3}$; then $\ker f$ can be graphed in desmos as $y = x^{2/3}$, which is not a smooth manifold because of its behavior at 0.

(iii) The n-sphere $S^n = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$ can be regarded as the level set of the ℓ^2-norm, for which $S^0 = \{\pm 1\} \subset \mathbb{R}$, S^1 is a circle, S^2 is the usual sphere. Note that we may use the stereographic projection as seen in complex analysis, to view S^3 (and any of the previous or subsequence S^n) as $\mathbb{R}^3 \cup \{\infty\}$.

(iv) The 2-dimensional torus T^2 is the surface of revolution obtained from a circle of radius r and R about an axis of revolution. It can be regarded as a level set by writing
$$T^2 = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - R)^2 + z^2 = r\}.$$

(v) The Möbius strip can't be a part of a level set (at a regular value) because level sets are orientable (2-sided), while the strip is not.

(vi) The Klein bottle is also not orientable; it is closed (doesn't have a boundary), and doesn't embed into $\mathbb{R}^3$. It can be immersed into $\mathbb{R}^3$, i.e., locally embedded but not globally, as seen in the textbook.

Theorem 1.2 (Whitney Embedding Theorem)**.** Every n-manifold has an embedding in $\mathbb{R}^{2n}$.

In this class, we prefer to deal with intrinsic descriptions of manifolds rather than extrinsic ones; a good motivation is given on p.7 in the textbook with respect to our 2-torus.

Now, consider M to be the rotations of a ball. We call this a configuration spcae; i.e., its points are a way of configuring another object. How do we put coordinates on a piece of this space? To start, we wish to describe M; one possible way is by considering the 3×3 orthonormal matrices, i.e., SO(3); another way is to first designate a point on the unit sphere as the north pole N (of which there are two degrees of freedom in this choice), then choosing where the vector $(1, 0, 0)$ at N is mapped to (of which we have one degree of freedom).

In this manner, we may regard M as a 3-dimensional manifold and define coordinates for

[1] note to self: what's a regular value?

points near the identity.

A second example is given by considering linkages, which are collections of line segments and joints (p.8-10); suppose we have four segments given by $n_1, \ldots, n_4$, to be regarded as vectors in $\mathbb{R}^2$; without additional constraints imposed, we see that this can be regarded as 8-dimensional. In imposing the conditions on *making* $n_1, \ldots, n_4$ a linkage, we see that $\|n_i\|$ for each $i = 1, \ldots, 4$ is fixed (whence 4 less dimensions), $n_1 + n_2 + n_3 + n_4 = 0$ (whence one less), and n_1 is fixed (whence one less again), so such a linkage can be regarded as a 2 dimensional manifold with coordinates θ and φ.

Finally, denote $\mathbb{R}\mathrm{P}^2$ the real projective plane, given by lines passing through the origin in $\mathbb{R}^3$; equivalently, we may regard this as the pairs of antipodal $\{x, -x\}$ points in S^2.

Fact 1.3. "Closed" (equivalently, compact) surfaces are easy to enumerate.

Fedya then said something about orientability, but that's left in the textbook and easier to read there.

We now wish to define manifolds. For a first attempt, we want to say that a smooth manifold is a set M covered by a set of smooth charts $\varphi_i : (U_i \subset M) \to \mathbb{R}^n$, such that each $p \in M$ is covered by some U_i. However, we run into a problem; what does "smooth" mean? $f : U \to \mathbb{R}^n$ recall that a function is said to be smooth if it has partial derivatives of all orders. A function $f : U \to V$ is a diffeomorphism if it's a smooth bijectino and its inverse is also smooth.

Definition 1.4. A coordinate chart is an injective map $\varphi : U \to \mathbb{R}^n$ with open image for some $U \subset M$.

In particular, we say that two charts $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ are compatible if the transition $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism. Clearly, if $U \cap V = \varnothing$, then said maps are compatible; note, however, that compatibility is not an equivalence relation, since, while it is symmetric and reflexive, it is not necessarily transitive.

§2 Day 2: (Jan. 8, 2026)

We correct an error from last class. Let M be a set, and consider $\varphi : U \to \mathbb{R}^n$, where $U \subset M$. We call φ a (coordinate) chart if $\varphi(U)$ is open and φ is injective; indeed, we identify U with an open subset of $\mathbb{R}^n$. Given two charts $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$, we say they are *compatible* if

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a diffeomorphism of open subsets.[2] We now present an idea; indeed, we may regard φ as a coordinate system on U, i.e., $\varphi(p) = (x^1, \ldots, x^n)$ are *coordinates* of p (with respect to φ), for which our transition maps are realized as coordinate changes.

Definition 2.1. A set of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ that covers M (i.e., every $p \in M$ is in some U_α) is called an *atlas*.

From this we may draft the following definition,

Definition. A *manifold* is a set M with an atlas $\mathcal{A}$ of charts.

Observe the following examples,

(i) Consider the n-sphere S^n, for which we have the stereographic projection φ_N, projecting N through a point on the sphere onto $\mathbb{R}^n$, which we may define as follows,

$$\varphi_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n).$$

However, it is evident that φ_N does not admit N in its domain; thus, we may similarly define the stereographic projection from the south pole by

$$\varphi_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n)$$

in order to cover S^n. Do these two maps form an atlas for S^n? It suffices to check that they are compatible; directly write as follows,[3]

$$\varphi_S \circ \varphi_N^{-1}(y_1, \ldots, y_n) = \frac{1}{|\vec{y}|^2}(y_1, \ldots, y_n),$$

where $\vec{y} = (y_1, \ldots, y_n)$.[4]

(ii) Let M be the set of straight lines in $\mathbb{R}^2$, i.e., of the form "$ax + by = c$". Consider the charts φ mapping $mx + b$ to $(m, b) \in \mathbb{R}^2$ and ψ mapping $x = ny + c$ to $(n, c) \in \mathbb{R}^2$ (in this manner, we account for both horizontal and vertical lines). Then the transition map between them is realized as

$$\psi \circ \varphi^{-1}(m, b) = \left(\frac{1}{m}, -\frac{b}{m}\right)$$

whenever $m \neq 0$ (when the line is neither horizontal nor vertical).

Given two atlases, how do we test that they define the same manifold?

[2] defn 2.4 in gross meinrenken

[3] see: p.22-23 for the S^1 case

[4] fedya u seem cool n all but we cannot be friends if u use this notation (that is, until i change my mind and start appreciating $\vec{\bullet}$)

Definition 2.2 (§2.8)**.** A chart $\varphi : U \to \mathbb{R}^n$ is *compatible* with an atlas if $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ if φ is compatible with every φ_α.

Lemma 2.3 (§2.10)**.** If $\varphi : U \to \mathbb{R}^n$, $\psi : V \to \mathbb{R}^n$ are compatible with the same atlas $\mathcal{A}$, then they're compatible with each other.

Proof. It is straightforward to check the properties, so just read the book. □

Theorem 2.4 (§2.11)**.** Given an atlas $\mathcal{A}$ on M, there's a unique *maximal* atlas $\widetilde{\mathcal{A}}$ which consists of all charts compatible to $\mathcal{A}$. Every chart compatible with $\mathcal{A}$ is already in $\widetilde{\mathcal{A}}$.

Proof. In one direction, if a chart is compatible with $\tilde{\mathcal{A}}$, then it is compatible with $\mathcal{A}$, and is therefore in $\widetilde{\mathcal{A}}$; in the other direction, $\widetilde{\mathcal{A}}$ is an atlas, so it covers because it contains $\mathcal{A}$ and is pairwise compatible by the lemma, so we are done. □

With this, we give a second attempt and defining a manifold.

Definition. A manifold is a set M with a maximal atlas $\mathcal{A}$.