

# MAT327 Lecture Notes

ARKY!! :3C

'24 Fall Semester

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## §1 Day 1: Open Sets and Continuity (Sep. 3, 2024)

This class is *MAT327*; 3 meaning third year, 2 meaning the contents are on the fundamental side, and 7 meaning no mercy.

– Dror Bar-Natan

Course administration matters first;

- The course link is given [here](#) (this will link straight to Quercus).
- The textbook is [James Munkres' Topology](#) (online PDF: [ETH Zurich mirror](#)); Prof Bar-Natan strongly recommends a paper copy, though (since people get distracted on the computer).

Today's readings in the textbook is on Ch. 1, sections 1 to 8, and Ch. 2, sections 12 to 13. Readings are supplementary to lecture material<sup>1</sup>. The goal of this course is to understand continuity in its most general form; in particular,

- In MAT157, we studied continuity in  $f : \mathbb{R} \rightarrow \mathbb{R}$ ;
- In MAT257, we will study continuity in  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ;

but in this class, we will study continuity in  $F : X \rightarrow Y$ , where  $X, Y$  are arbitrary spaces, such as (but not limited to)  $\mathbb{R}^n$ ,  $\mathbb{R}^{\mathbb{N}}$ ,  $\{0, 1\}^{\mathbb{N}}$  (binary sequences), and so on. We start with some refreshers on previous coursework;

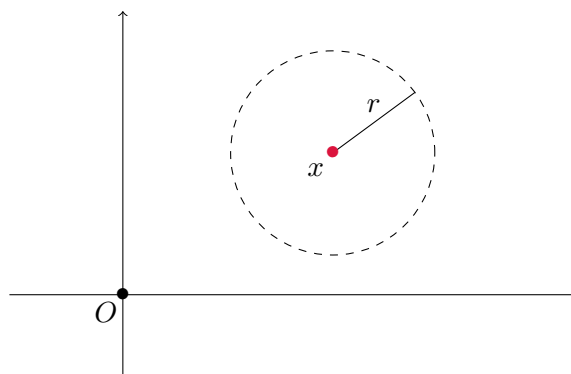
**Definition 1.1** (Continuity in  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ ). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called “continuous” if it is continuous at all points in  $\mathbb{R}^n$ . Specifically, for all  $x_0 \in \mathbb{R}^n$  and  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$ .

With this, we state our main theorem for today (proof given later),

**Theorem 1.2** (Continuity on  $\mathbb{R}^n$  if and only pre-image of open subsets is open). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous if and only if all open subsets  $U \in \mathbb{R}^m$  have  $f^{-1}(U)$  open.

In order to build up to the above, we start by defining some terms;

**Definition 1.3** (Open Ball). Let  $r > 0$ , and  $x \in \mathbb{R}^n$ . An open ball of radius  $r$  about  $x$  is given formally by  $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$ . Visually, we have



Note that the border of the circle is not in  $B_r(x)$ , as the distance metric asks for a strict inequality.

<sup>1</sup>irc it won't be tested unless specified. its still good to learn tho

In a similar fashion, we have

**Definition 1.4** (Open Set). A set  $U \subset \mathbb{R}^n$  is called *open* if, for all  $x \in U$ , there exists an open ball about  $x$  contained in  $U$ . Specifically, there is some  $\varepsilon > 0$  such that  $B_\varepsilon(x) \subset U$ .

Intuitively, we could say that the set  $U$  does not contain its edge; if it did, let  $x$  be on said edge; then we would not be able to fit an open ball about  $x$  in  $U$ . Here are some examples of open sets from lecture;

1. The complete set<sup>2</sup>  $U = \mathbb{R}^n$  is open.
2. The empty set  $\emptyset$  is open. Since there does not exist any  $x \in \emptyset$ , no conditions on balls need to be satisfied; “every dog in the empty set of dogs is green”.
3.  $(0, 1)$  is open on  $\mathbb{R}^1$  (and any open interval, for that matter).
4.  $B_r(x) \subset \mathbb{R}^n$  is open. To see this, observe that for any  $y \in B_r(x)$ , we may pick  $\varepsilon < r - |x - y|$ ; by triangle inequality, all elements in  $B_\varepsilon(y)$  must also be in  $B_r(x)$ .

We now define images and pre-images; let us have a function between sets  $f : X \rightarrow Y$ . For subsets  $A \subset X$  and  $B \subset Y$ ,

$$f(A) = \{f(a) \mid a \in A\},$$

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Since  $f$  need not be injective,  $f^{-1}$  does not necessarily exist. While images of a union of subsets is a union of the images, the same is not true for intersections; let  $A_1, \dots, A_n$  be subsets of  $X$ ; then

$$f\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f(A_i), \quad f\left(\bigcap_{i=1}^n A_i\right) \subset \bigcap_{i=1}^n f(A_i).$$

On the other hand, pre-images preserve both union and intersection; let  $B_1, \dots, B_n \subset Y$ ; then

$$f^{-1}\left(\bigcup_{i=1}^n B_i\right) = \bigcup_{i=1}^n f^{-1}(B_i), \quad f^{-1}\left(\bigcap_{i=1}^n B_i\right) = \bigcap_{i=1}^n f^{-1}(B_i).$$

As for set complements, we have

$$f^{-1}(B^C) = f^{-1}(B)^C,$$

but the same cannot be said for images; one side includes the other depending on injectivity and surjectivity<sup>3</sup>.

**Note:** for the proof of Theorem 1.2, I'll leave it out in case Prof. Bar-Natan proves it later on Thursday.

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<sup>2</sup>check wording?

<sup>3</sup>check if i'm right on this, i did a quick run thru in my head