

MAT354 Lecture Notes

ARKY!! :3C

'25 Fall Semester

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§1 Day 1: Recap of Preliminaries (Sep. 2, 2025)

We start by discussing the complex plane and complex numbers. Given $z \in \mathbb{C}$, we say that $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of z respectively, i.e., $z = x + iy$. \mathbb{C} is the set of all complex numbers. In this manner, we may identify $z = x + iy$ with $(x, y) \in \mathbb{R}^2$ using the standard complex plane.

- (a) The complex *conjugate* of z is given by $\bar{z} = x - iy$, where we have that

$$\Re(z) = \frac{z + \bar{z}}{2}, \quad \Im(z) = \frac{z - \bar{z}}{2i}.$$

- (b) We now define addition and multiplication for the complex numbers. For all $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have that

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \end{aligned}$$

We have that $(\mathbb{C}, +, \times)$ is a field, with $(\mathbb{R}, +, \times)$ as a subfield. To verify this, we need to check that it indeed satisfies:

- Commutativity: for all $z_1, z_2 \in \mathbb{C}$, we have that $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$.
 - Associativity: for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
 - Distributivity: for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.
- (c) The absolute value of a complex number $z = x + iy$ is given by $|z| = \sqrt{x^2 + y^2}$. In particular, this yields the triangle inequality, where for any $z, w \in \mathbb{C}$, we have that $|z + w| \leq |z| + |w|$. The proof either comes visually or through explicit computation, both of which I will not write out here for brevity.¹

As an extension of the inequality, we also automatically have that

$$|\Re z| \leq |z|, \quad |\Im z| \leq |z|,$$

and that for all $z, w \in \mathbb{C}$, we have

$$||z| - |w|| \leq |z - w|.$$

Proof. Using the triangle inequality, we have that

$$\begin{aligned} |z| &= |(z - w) + w| \leq |z - w| + |w|, \\ |w| &= |(w - z) + z| \leq |z - w| + |z|, \end{aligned}$$

of which both imply that $|z| - |w| \leq |z - w|$ and $|w| - |z| \leq |z - w|$. □

For any $z \in \mathbb{C}$, we have that $|z|^2 = z \cdot \bar{z}$.

Proof. Write $z = x + iy$; then $|z|^2 = x^2 + y^2$, where we may note that $z \cdot \bar{z} = (x + iy)(x - iy)$ which yields the right hand side of the earlier equation through expansion. □

¹no full credit if you draw a picture on the exam lmao

Finally, for $z, w \in \mathbb{C}$, we have that $|zw| = |z||w|$. This is left as an exercise to the student.

- (d) The polar form of a nonzero complex number $z \neq 0$ is given by $z = \gamma e^{i\theta}$, where $\gamma > 0$ and $\theta \in \mathbb{R}$. Let us assume the Euler formula; for all $\theta \in \mathbb{R}$, we have that

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Let $r = |z|$; we have that $|z| = |re^{i\theta}| = |r| |e^{i\theta}| = r \cdot 1 = r$. θ is the angle between the positive real axis to the half-line starting from 0 and passing through z . In this manner, $z = re^{i\theta} = |z|(\cos \theta + i \sin \theta) = |z| \cos \theta + i |z| \sin \theta$, which means we have that

$$\Re z = |z| \cos \theta, \quad \Im z = |z| \sin \theta.$$

As an example, let us find all the complex numbers z such that $z^4 = i$. Since $i = e^{i\frac{\pi}{2}}$, $z = \rho e^{i\theta}$ satisfying $z^4 = i$ becomes $\rho^4 e^{i4\theta} = e^{i\frac{\pi}{2}}$, meaning

$$\begin{cases} \rho^4 = 1, \\ 4\theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

This means $\rho = 1$ and $\theta = \frac{\pi}{8} + \frac{k\pi}{2}$, where $k \in \mathbb{Z}$. Considering the cases $k = 0, 1, 2, 3$ and observing that there are only 4 equivalence classes modulo 4 to consider, we have that

$$z_0 = e^{i\frac{\pi}{8}}, \quad z_1 = e^{i\frac{5\pi}{8}}, \quad z_2 = e^{i\frac{9\pi}{8}}, \quad z_3 = e^{i\frac{13\pi}{8}}.$$

We now discuss convergence. We say that a set of complex numbers $\{z_n\}_{n \in \mathbb{N}}$ converges to $w \in \mathbb{C}$ if $\lim_{n \rightarrow \infty} |z_n - w| = 0$. We write it as $\lim_{n \rightarrow \infty} z_n = w$. In the complex plane, the convergence can be in any direction.

Lemma 1.1. $\{z_n\}_{n \in \mathbb{N}}$ converges to w if and only if $\{\Re z_n\}_{n \in \mathbb{N}}$ converges to $\Re w$ and $\{\Im z_n\}_{n \in \mathbb{N}}$ converges to $\Im w$.

Proof. We have that

$$\begin{aligned} |z_n - w| &= |(\Re z_n - \Re w) + i(\Im z_n - \Im w)| \\ &\leq |\Re z_n - \Re w| + |\Im z_n - \Im w|, \end{aligned}$$

where as $n \rightarrow \infty$, we have that the right hand side is given by $0 + 0$. For the opposite direction, we have that $|z| \geq |\Re z|$ or $|\Im z|$, so we have that

$$|\Re z_n - \Re w| = |\Re(z_n - w)| \leq |z_n - w|,$$

which approaches 0 as $n \rightarrow \infty$. The same argument goes for the imaginary portion. \square

A sequence of complex numbers $\{z_n\}_{n \in \mathbb{N}}$ is called *Cauchy* if $|z_n - z_m| \rightarrow 0$ as $n, m \rightarrow \infty$. In $\varepsilon - \delta$, this means that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|z_n - z_m| < \varepsilon$ for all $n, m > N$.

Theorem 1.2 (Bolzano-Weierstrass Theorem). \mathbb{R} is *complete*, i.e., every Cauchy sequence of real numbers converges to a real number.

Theorem 1.3. \mathbb{C} is complete.

Proof. Take any Cauchy sequence of complex numbers $\{z_n\}$. Using the inequalities $|\Re z| \leq |z|$ and $|\Im z| \leq |z|$, we have that $\{\Re z_n\}$ and $\{\Im z_n\}$ are Cauchy sequences of real numbers. By Bolzano-Weierstrass, we have that $\Re z_n \rightarrow x_0 \in \mathbb{R}$ and $\Im z_n \rightarrow y_0 \in \mathbb{R}$. By the previous lemma, we actually have $\lim_{n \rightarrow \infty} z_n = x_0 + iy_0$. \square

We now move onto topology in the complex plane. Given $z_0 \in \mathbb{C}$ and $r > 0$, we can form an open or closed disc centered at z_0 of radius r . We write both of these as

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$

$$\bar{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\},$$

Given a set $\Omega \subseteq \mathbb{C}$, a point z_0 is an interior point if there exists $r > 0$ such that $D_r(z_0) \subseteq \Omega$. The interior of Ω is given by the set of all such interior points. In particular, the interior of $\bar{D}_r(i)$ is $D_r(i)$.

A set Ω is called *open* if every point in Ω is an interior point. Ω is called *closed* if the complement of Ω , $\Omega^c = \mathbb{C} \setminus \Omega$, is open. As an example, the open right half-plane $\{z \in \mathbb{C} \mid \Re z > 0\}$ is open.

Proof. For any $z \in \Omega$, let $z = x + iy$, and take $r = \frac{x}{2} = \frac{\Re z}{2}$. Then we claim that $D_r(z) \subseteq \Omega$. For all $w \in D_r(z)$, we clearly have that

$$\Re w = \Re z - (\Re z - \Re w) \geq \Re z - |z - w| \geq \frac{\Re z}{2} > 0,$$

and so all such $w \in \Omega$, and we are done. \square

A point $z \in \mathbb{C}$ is a *limit point* of Ω if there exists a sequence $\{z_n\} \subset \Omega$ with $z_n \neq z$ such that $z_n \rightarrow z$.

As an example, we define D to be the open unit disc centered at 0. 0 and 1 are both limit points of D , but 1 is not contained in D itself.² The *closure* of Ω , $\bar{\Omega}$, is given by Ω unioned with all its limit points. The *boundary* of a set Ω , written $\partial\Omega$, is given by $\bar{\Omega} \setminus \text{int } \Omega$. A set $\Omega \subseteq \mathbb{C}$ is said to be compact if it is closed and bounded, i.e., there exists $M > 0$ such that $|z| \leq M$ for all $z \in \Omega$.

Theorem 1.4. A set $\Omega \subseteq \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω .

Proposition 1.5. If $\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_n \supset \dots$ is a sequence of nonempty compact sets in \mathbb{C} , where $\text{diam}(\Omega_n) = \sup_{z, w \in \Omega_n} |z - w| \rightarrow 0$ as $n \rightarrow \infty$, then there exists a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for every $n \in \mathbb{N}$.

Proof. For each Ω_n , pick a point $z_n \in \Omega_n$. Then $\{z_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence because the diameter of Ω_n approaches 0. By the Bolzano-Weierstrass theorem for complex numbers, this means that $\{z_n\}_{n \in \mathbb{N}}$ indeed does converge to some $w \in \mathbb{C}$. In particular, we have w is the limit of the subsequence $\{z_m\}_{m \geq n} \subseteq \Omega_n$, where Ω_n is compact, meaning the limit w should be in Ω_n . This means there exists a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for every $n \in \mathbb{N}$.

To show the uniqueness of w , we argue by contradiction; assume $w' \neq w$ satisfies the property. Then $|w' - w| > 0$. Since $w, w' \in \Omega_n$ for all n , this contradicts that $\text{diam}(\Omega_n) \rightarrow 0$. \square

An open set Ω is called *connected* if it is not possible to find two disjoint nonempty open sets Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. A connected open set in \mathbb{C} is called a *region*.

²hell is it disc or disk YKW LET'S COMPROMISE it's spelled disque actually (paint nails)

§2 Day 2: Functions on the Complex Plane (Sep. 4, 2025)

Let $f : \Omega \rightarrow \mathbb{C}$, where Ω is an open subset of \mathbb{C} . We say that f is continuous if at $z_0 \in \Omega$ if, for all $\varepsilon > 0$, there exists an open disk $D_\gamma(z_0)$ such that $|f(z) - f(z_0)| < \varepsilon$ for all $z \in D_\gamma(z_0)$. In particular, f is said to be continuous on Ω if it is continuous at every point in Ω .

Example 2.1. Consider $f : \mathbb{C} \rightarrow \mathbb{C}$ given by $f(z) = \bar{z}$. Show that f is continuous.

Solution. For all complex z, z_0 , we have that $|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0| = |z - z_0|$. Thus, we have that for any $\varepsilon > 0$, we obtain³

$$f(D_\varepsilon(z_0)) = D_\varepsilon(\bar{z}_0). \quad \square$$

We now discuss holomorphic functions (i.e., complex differentiable functions). We say that $f : \Omega \rightarrow \mathbb{C}$ is *holomorphic* at $z_0 \in \Omega$ if

$$\frac{f(z_0 + h) - f(z_0)}{h}, \quad h \in \mathbb{C} \setminus \{0\},$$

converges as $h \rightarrow 0$. If the limit exists, we let

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

be the derivative.

Example 2.2. Consider the exact same function as in the previous example, $f(z) = \bar{z}$. Is f holomorphic?

Solution. For all $z_0 \in \mathbb{C}$ and $h \in \mathbb{C} \setminus \{0\}$, we have that

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{z_0 + h} - \bar{z}_0}{h} = \frac{\bar{h}}{h} = \frac{\rho e^{-i\theta}}{\rho e^{i\theta}} = e^{-2i\theta}.$$

If we take $h \rightarrow 0$ along the real line, we may let $h = \rho$, which means the fraction is equal to 1 as $h \rightarrow 0$. If we take $\rho \rightarrow 0$ along the complex axis, however, then we have that $h = \rho e^{i\pi/2}$, where we obtain the fraction is equal to -1 as $\rho \rightarrow 0$. Thus, f cannot be holomorphic. \square

Proposition 2.3. Let Ω be open in \mathbb{C} . If f, g are holomorphic on Ω , then

- (i) $f + g$ is holomorphic on Ω , and $(f + g)' = f' + g'$.
- (ii) fg is holomorphic on Ω , and $(fg)' = f'g + fg'$.
- (iii) If $g(z_0) \neq 0$ where $z_0 \in \Omega$, then $\frac{f}{g}$ is also holomorphic at z_0 , where

$$\left(\frac{f}{g}\right)' = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

- (iv) If $f : \Omega \rightarrow U$ and $g : U \rightarrow \mathbb{C}$ are holomorphic, then $g \circ f$ is also holomorphic, and we obtain the chain rule

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

³note to self: ol is better than bar for this stuff...

We now discuss complex differentiability versus real differentiability. A holomorphic function $f : \Omega \rightarrow \mathbb{C}$ can be identified with a function $F : \Omega \rightarrow \mathbb{R}^2$ given by $(x, y) \mapsto (u(x, y), v(x, y)) = (\Re f(x, y), \Im f(x, y))$. Consider the partial derivative of F at (x_0, y_0) ; these exist if there exists some linear transformation $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\frac{\|F(P_0 + H) - F(P_0) - J(H)\|}{\|H\|} \rightarrow 0$$

as $H \rightarrow 0$. Or, we may define $\Psi(H)$ to take on the fraction above, and we see that F is indeed differentiable at $P_0 = (x_0, y_0)$ if $\Psi(H) \rightarrow 0$ as $H \rightarrow 0$. We now deal with complex differentiability. Suppose $f : \Omega \rightarrow \mathbb{C}$ is holomorphic at $z_0 = x_0 + iy_0$. Then we have partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}.$$

Naturally,

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

along any path; in particular, we take $h \in \mathbb{R} \setminus \{0\}$ and observe that

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0), \end{aligned}$$

and so both exist, and they are $\Re f'(z_0)$ and $\Im f'(z_0)$ respectively. Similarly, we may take $h = ik$ where $k \in \mathbb{R} \setminus \{0\}$ and obtain

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{k \rightarrow 0} \frac{u(x_0, y_0 + k) + iv(x_0, y_0 + k) - u(x_0, y_0) - iv(x_0, y_0)}{ik} \\ &= \lim_{k \rightarrow 0} \frac{-i(u(x_0, y_0 + k) - u(x_0, y_0)) + v(x_0, y_0 + k) - v(x_0, y_0)}{k} \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0), \end{aligned}$$

and so both partials also exist and they are $\Re f'(z_0)$ and $-\Im f'(z_0)$ respectively.