MAT363 Lecture Notes

Arky!! :3c

'25 Winter Semester

Contents

§1 Day 1: Introduction to the Class (Jan. 6, 2025)

Course administrative details! First day slides are given here. This is a class in classical differential geometry; the following 12 weeks will be split up as follows,

(a) Curves, for two weeks;

(b) Surfaces, for three weeks;

(c) Curvature of surfaces, for three weeks;

(d) Geodesics, for three weeks;

(e) Gauss-Bonnet theorem, for one week.

Grading will be done by 5% on PCEs, 15% on problem sets, 15% on quizzes, 25% on the term test, 30% on the final exam, and 10% weighted towards your best test.

To start, consider the following maps $\gamma : I = (-10, 10) \to \mathbb{R}^3$, given by

$$\begin{aligned}
\gamma(t) &= (t, t, t); \\
\gamma(t) &= (|t|, |t|, |t|); \\
\gamma(t) &= (t, t^2, t^3); \\
\gamma(t) &= (t^3, t^3, t^3); \\
\gamma(t) &= (\cos t, \sin t, t); \\
\gamma(t) &= (t \cos t, t \sin t, t).
\end{aligned}$$

In this class, we say that a curve is a *parameterized curve* if it is a smooth function $\gamma : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval. In particular, of the six examples given above, only $t \mapsto (|t|, |t|, |t|)$ is not smooth.

Definition 1.1 (Regular Curve)**.** Let $\gamma : I \to \mathbb{R}^n$ be a curve; it is said to be *regular* if $|\gamma'(t)| \neq 0$ for all $t \in I$, i.e. the speed is always nonzero.

Note that $\gamma'(t)$ and $|\gamma'(t)|$ describe different qualities, with the former describing velocity and the latter describing speed (i.e., one describes speed as well, while the other is a scalar quantity). As an example, consider the curve $\gamma(t) = (\cos t, \sin t, t)$. To find the distance travelled from $t = 0$ to $t = 2\pi$, we may observe that

$$\left|\gamma'(t)\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Since the speed is constant, the total distance traveled is simply $2\pi\sqrt{2}$. □

Definition 1.2 (Closed Curve)**.** Consider a curve $\gamma : [a, b] \to \mathbb{R}^n$. We say that γ is a *closed curve* if $\gamma(a) = \gamma(b)$ and $\gamma^{(n)}(a) = \gamma^{(n)}(b)$ for all naturals n.

Definition 1.3 (Simple Curve)**.** We say that γ is a simple curve if it is injective on $[a, b)$.

Note that while in topology we do not care if there is a "sharp corner" at $\gamma(a) = \gamma(b)$, such things do matter, as per the condition that the nth derivative of γ must agree on a and b (for example, the velocity γ' at a, b must be equal).

In this class, we automatically take the inner product $\langle , \rangle$ as the Euclidean inner product,

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n.$$

For any subspace $V \subset \mathbb{R}^n$, we may decompose any vector $x \in \mathbb{R}^n$ uniquely as $x = x^{\parallel} + x^{\perp}$, where $x^{\parallel} \in V$ and $\left\langle x^{\perp}, v \right\rangle = 0$ for any vector $v \in V$. Now, consider any curve $\gamma : I \to \mathbb{R}^n$. We have the following proposition,

Proposition 1.4. If $|\gamma(t)|$ is constant, then $\langle \gamma(t), \gamma'(t) \rangle = 0$ for all $t \in I$.

To see this, let $|\gamma(t)|^2 = c$ be constant; then

$$\frac{d}{dt} |\gamma(t)|^2 = 0 \implies \frac{d}{dt} (\langle \gamma(t), \gamma(t) \rangle) = \left\langle \gamma'(t), \gamma(t) \right\rangle + \left\langle \gamma(t), \gamma'(t) \right\rangle = 0m$$

i.e. $\langle \gamma(t), \gamma'(t) \rangle = 0$ as desired. $\square$

Given a regular curve $\gamma : I \to \mathbb{R}^n$, we may compute the velocity and acceleration as $\gamma'(t), \gamma''(t)$, which are denoted $v(t), a(t)$ respectively. In particular, we may write

$$a(t) = a^{\|}(t) + a^{\perp}(t),$$

with $a^{\|}(t)$ being the tangential acceleration, and $a^{\perp}(t)$ being the normal acceleration. We may find these by projecting $a(t)$ into the subspace $\text{span}\{v\}$ (i.e., the span of the velocity vector).