# MAT327 Lecture Notes

Arky!! :3c

'24 Fall Semester

## Contents

# §1 Day 1: Open Sets and Continuity (Sep. 3, 2024)

> This class is *MAT327*; 3 meaning third year, 2 meaning the contents are on the fundamental side, and 7 meaning no mercy.
> – Dror Bar-Natan

Course administration matters first;

- The course link is given here (this will link straight to Quercus).
- The textbook is James Munkres' Topology (online PDF: ETH Zurich mirror); Prof Bar-Natan strongly recommends a paper copy, though (since people get distracted on the computer).

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Today's reading in the textbook is on Ch. 1, sections 1 to 8, and Ch. 2, sections 12 to 13. Readings are supplementary to lecture material[1]. The goal of this course is to understand continuity in its most general form; in particular,

- In MAT157, we studied continuity in $f : \mathbb{R} \to \mathbb{R}$;
- In MAT257, we will study continuity in $f : \mathbb{R}^n \to \mathbb{R}^m$;

but in this class, we will study continuity in $F : X \to Y$, where $X, Y$ are arbitrary spaces, such as (but not limited to) $\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{N}}$ (binary sequences), and so on. We start with some refreshers on previous coursework;

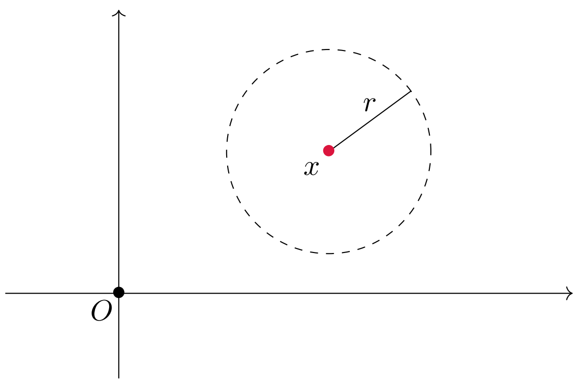
**Definition 1.1** (Continuity in $\mathbb{R}^n \to \mathbb{R}^m$)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called "continuous" if it is continuous at all points in $\mathbb{R}^n$. Specifically, for all $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

With this, we state our main theorem for today (proof given later),

**Theorem 1.2** (Continuity on $\mathbb{R}^n$ if and only pre-image of open subsets is open)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if all open subsets $U \in \mathbb{R}^m$ have $f^{-1}(U)$ open.

In order to build up to the above, we start by defining some terms;

**Definition 1.3** (Open Ball)**.** Let $r > 0$, and $x \in \mathbb{R}^n$. An open ball of radius $r$ about $x$ is given formally by $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Visually, we have




Note that the border of the circle is not in $B_r(x)$, as the distance metric asks for a strict inequality.

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[1] iirc it won't be tested unless specified. its still good to learn tho

In a similar fashion, we have

**Definition 1.4** (Open Set)**.** A set $U \subset \mathbb{R}^n$ is called *open* if, for all $x \in U$, there exists an open ball about $x$ contained in $U$. Specifically, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

Intuitively, we could say that the set $U$ does not contain its edge; if it did, let $x$ be on said edge; then we would not be able to fit an open ball about $x$ in $U$. Here are some examples of open sets from lecture;

1. The whole set $U = \mathbb{R}^n$ is open.
2. The empty set $\emptyset$ is open. Since there does not exist any $x \in \emptyset$, no conditions on balls need to be satisfied; "every dog in the empty set of dogs is green".
3. $(0, 1)$ is open on $\mathbb{R}^1$ (and any open interval, for that matter).
4. $B_r(x) \in \mathbb{R}^n$ is open. To see this, observe that for any $y \in B_r(x)$, we may pick $\varepsilon < r - |x - y|$; by triangle inequality, all elements in $B_\varepsilon(y)$ must also be in $B_r(x)$.

We now define images and pre-images; let us have a function between sets $f : X \to Y$. For subsets $A \subset X$ and $B \subset Y$,

$$f(A) = \{f(a) \mid a \in A\},$$
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Since $f$ need not be injective, $f^{-1}$ does not necessarily exist. While images of a union of subsets is a union of the images, the same is not true for intersections; let $A_1, \ldots, A_n$ be subsets of $X$; then

$$f\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f(A_i), \quad f\left(\bigcap_{i=1}^n A_i\right) \subset \bigcap_{i=1}^n f(A_i).$$

On the other hand, pre-images preserve both union and intersection; let $B_1, \ldots, B_n \subset Y$; then

$$f^{-1}\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f^{-1}(A_i), \quad f^{-1}\left(\bigcap_{i=1}^n A_i\right) = \bigcap_{i=1}^n f^{-1}(A_i).$$

As for set complements, we have

$$f^{-1}(B^C) = f^{-1}(B)^C,$$

but the same cannot be said for images; $f(A)^C \subset f(A^C)$ for surjective $f$, and vice versa for injective $f$. Equality occurs only if $f$ is a bijection.

**Note:** for the proof of Theorem 1.2, I'll leave it out in case Prof. Bar-Natan proves it later on Thursday.

# §2 Day 2: Basic Definitions and Topological Spaces (Sep. 5, 2024)

We start by recapping the previous lecture; we introduced

- Open sets $U \subset \mathbb{R}^n$, where there exists an open ball of radius $\varepsilon > 0$ about any $x \in U$.
- Continuity in $f : \mathbb{R}^n \to \mathbb{R}^m$.

Recall the continuity property from last class, that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if the pre-image of an open set $U$ is open.

($\Rightarrow$) To start, pick any $x_0 \in f^{-1}(U)$, and let us have a small enough $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \in U$ (this is possible since $U$ is open). By continuity, there exists $\delta > 0$ such that any $x \in B_\delta(x_0)$ satisfies $f(x) \in B_\varepsilon(f(x_0)) \subset U$; this means $x \in f^{-1}(U)$ by definition of pre-image, and since $x$ was arbitrary, we see $B_\delta(x_0) \in f^{-1}(U)$. Moreover, since $x_0$ was also arbitrary, our construction shows that there always exists a $\delta$-ball about any point in the pre-image, and so $f^{-1}(U)$ is open. $\square$

($\Leftarrow$) For the other direction, take any $x_0 \in \mathbb{R}^n$, and $\varepsilon > 0$. Since $B_\varepsilon(f(x_0))$ is open we have that $f^{-1}(B_\varepsilon(f(x_0)))$ is open as well. This means we may pick a small enough $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x_0)))$ (by definition of openness), and we immediately see

$$f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)).$$

This is a reconstruction of the epsilon-delta definition of continuity, and so we are done. $\square$

Open sets in $\mathbb{R}^n$ have a number of properties;

(a) $\emptyset, \mathbb{R}^n$ are open sets.

(b) The union of open sets are open; specificay, let $S$ be a set of indices, and let $A_\alpha \subset \mathbb{R}^n$ for all $\alpha \in S$. Then

$$\bigcup_{\alpha \in S} A_\alpha$$

is an open set.

(c) The finite intersection of open sets are also open. Let $A_1, \ldots, A_n \subset \mathbb{R}^n$ be open; then

$$\bigcap_{i=1}^{n} A_i$$

is open.

We now proceed with the proofs for these properties.

(a) This was proven last lecture.

(b) For any $x$ in the union, by definition, there exists $\alpha \in S$ such that $x \in A_\alpha$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A_\alpha$, and $A_\alpha \subset \bigcup_{\alpha \in S} A_\alpha$. This concludes that the union is open.

(c) For any $x$ in the finite intersection, then $x$ is an element of each of $A_1, \ldots, A_n$. Let us have $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $B_{\varepsilon_i}(x) \varepsilon A_i$, and take $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$ (note that minimum is defined only for finite lists), which means $B_\varepsilon(x) \subset \bigcap_{i=1}^n A_i$.

In the case of infinite intersection of open sets, the resulting set need not be open. For example, let us consider the infinite intersection of intervals

$$\bigcap_{j=1}^{\infty}\left(-\frac{1}{j},\frac{1}{j}\right)=\{0\},$$

which is not open.

**Definition 2.1** (Topology on a Set)**.** Let $X$ be a set equipped with topology $\mathcal{T}$ on $X$. $\mathcal{T}$ is a collection of subsets of $X$, i.e. $\mathcal{T}\subset\mathcal{P}(X)$, with properties

- $\varepsilon, X\in\mathcal{T}$.
- The union of subsets of $\mathcal{T}$ is also in $\mathcal{T}$.
- The finite intersection of subsets of $\mathcal{T}$ is also in $\mathcal{T}$.

We call $(X,\mathcal{T})$ a *topological space*, which may be abbreviated to $X$ is $\mathcal{T}$ is given or obvious. Moreover, we define another notion of openness, where $U\in\mathcal{T}$ is said to "be open relative to $\mathcal{T}$," or that "$U$ is open." Here are a few examples of such topological spaces.

(a) The standard topology on $\mathbb{R}^n$ is given by

$$\mathbb{R}_{std}\left(\mathbb{R},\mathcal{T}_{std}=\{U\subset\mathbb{R}^n\mid U\text{ is open in the ``old sense''}\}\right),$$

i.e. collection of open intervals[2].

(b) The discrete topology, $X_{discrete}=(X,\mathcal{T}=\mathcal{P}(x))$ can be defined over any set $X$, equipped with $\mathcal{T}$ as the collection of all subsets.

(c) The trivial topology, $X_{trivial}=(X,\mathcal{T}=\{\emptyset,X\})$.

**Definition 2.2** (Continuity between Topological Spaces)**.** If $(X,\mathcal{T}_X),(Y,\mathcal{T}_Y)$ are topological spaces, and $f:X\to Y$ is a function between said spaces, then we say $f$ is continuous if, for all $U\in\mathcal{T}_Y$, we have that $f^{-1}(U)\in\mathcal{T}_X$. This draws from our definition from the notion that pre-images of open sets are open.

Now for the examples from lecture;

(a) $f:X_{discrete}\to\mathbb{R}_{std}$ is always continuous.

(b) $f:X_{trivial}\to\mathbb{R}_{std}$ is continuous if and only if $f$ is constant.

(c) $f:\mathbb{R}_{std}\to X_{trivial}$ is always continuous.

(d) $f:\mathbb{R}_{std}\to X_{discrete}$ is almost never continuous, except when $X$ is empty or a singleton (in which case, $X_{discrete}=X_{trivial}$). If $x_0\in X$ yet $X\setminus\{x_0\}\neq\emptyset$, then let us have

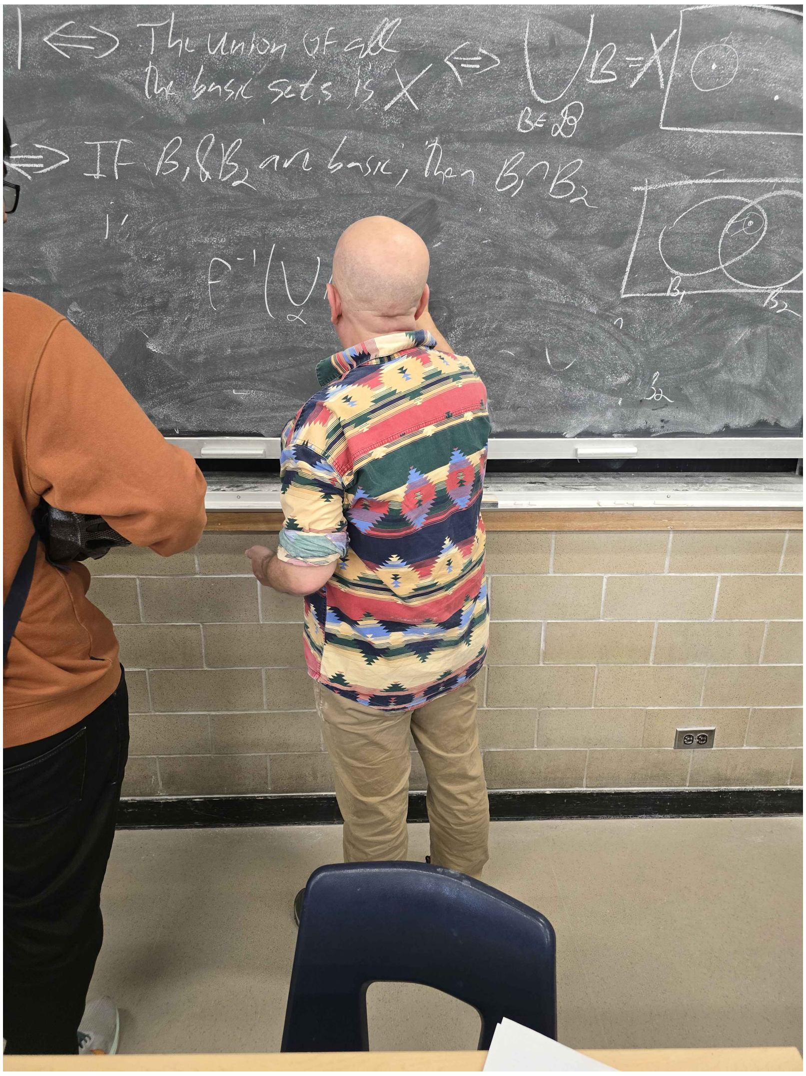
$$\begin{aligned}A&:=f^{-1}(\{x_0\}),\\B&:=f^{-1}(X\setminus\{x_0\}).\end{aligned}$$

While $A,B$ are both open, we see that $A\cup B=\mathbb{R}$, and $A\cap B=\emptyset$.[3]

[2] this is how i understood it, he might clarify next time?
[3] this one i don't completely get yet.

# §3 Day 3: Homeomorphisms and Bases (Sep. 10, 2024)

Outfit of the day (by popular request)




Course administrative details first;

- The reading for this week is on sections 12 to 14 (this week will cover these contents), and 15 to 16 as prereading.

Recap of last lecture:

- A topology $\mathcal{T} \subset \mathcal{P}(X)$ is a collection of subsets of $X$, of which we require $\{\emptyset, X\} \subset \mathcal{T}$. We also require $\mathcal{T}$ to be closed under arbitrary unions and finite intersections.

- We say a function $F : X \to Y$ is continuous if and only if for all $U \in \mathcal{T}_Y$, we have $f^{-1}(U) \in \mathcal{T}_X$.

Today we will cover homeomorphisms and bases. To start, recall the example topologies, such as $\mathcal{T}_{\text{std}}$ on $\mathbb{R}^n$ (where $\mathcal{T}_{\text{std}}$ consists of the open balls), $\mathcal{T}_{\text{triv}}$, and $\mathcal{T}_{\text{disc}}$. We also introduce a new example topology (where FC standards for finite complement),

$$\mathcal{T}_{\text{FC}} = \{U \subset X \mid X \setminus U \text{ is finite, or } U = \emptyset\}.$$

Note that Prof. Bar-Natan may interchange the notations $-$ or $\setminus$ to represent set difference.

**Theorem 3.1** (Composition of Continuous Functions is Continuous)**.** Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f : X \to Z$ is continuous (relative to the same topologies on $X$ and $Z$).

If $U \in \mathcal{T}_Z$, we have $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U)$, where by definition of continuity, we see that pre-images of open sets are open, and we have $g^{-1}(U)$ is open in $Y$, and similarly $f^{-1}(g^{-1}(U))$ is also open in $X$. □

In tutorial, we equipped $X$ with topologies $\mathcal{T}_1, \mathcal{T}_2$ (i.e., $X$ is a topological space in two ways). We say $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if $T_1 \supset T_2$, and coarser for the opposite direction; the words bigger and stronger may be used interchangeably with finer, and smaller or weaker for coarser. For example, the identity map

$$\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$$

is continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$. To see this, let $U$ be an open set in $(X, \mathcal{T}_2)$; then $U$ must be open in $(X, \mathcal{T}_1)$ as well, which is true for any $U$ only if $\mathcal{T}_1 \supset \mathcal{T}_2$[4].

**Definition 3.2** (Homeomorphism)**.** A map $h : X \to Y$ is called a homeomorphism if $h$ is continuous, bijective, and $h^{-1}$ is continuous as well.[5]

Note that continuous bijective maps $h$ need not have continuous inverses; for example, let us have $\mathrm{id} : X_{\text{disc}} \to X_{\text{triv}}$. id is continuous as per our above example, while its inverse is not. Another example is to consider $[0, 2\pi) \to S^1$ (unit circle), where $x \mapsto (\cos 2x\pi, \sin 2x\pi)$; we see that the inverse is discontinuous at 0 and $2\pi$ radians, even if the map is continuous and bijective (also observe that $[0, 1)$ is not compact while $S^1$ is).
An example of a homeomorphism is as follows (as per tutorial); let us consider[6]

$$(-1, 1)_{\text{std}} \cong \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)_{\text{std}} \cong \mathbb{R}_{\text{std}},$$

where we may map the first to the second by $x \mapsto \frac{\pi}{2}x$, and the second to third by $x \mapsto \tan x$. Since the composition of continuous maps is continuous, we also see $(-1, 1)_{\text{std}}$ is homeomorphic to $\mathbb{R}_{\text{std}}$ (any open interval is homophobic to $\mathbb{R}$ for that matter).

Another example of a homemorphism is $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ if $\mathcal{T}_1 = \mathcal{T}_2$.

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A "basis" for a topology on $X$ is a collection $\mathcal{B} \subset \mathcal{P}(X)$ of subsets such that

1. For all $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$. We call $B$ a *basic set.*

2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists a third basic set $B_3$ such that $x \in B_3 \subset B_1 \cap B_2$.

[4] includes the case $\mathcal{T}_1 = \mathcal{T}_2$; i'm following florian notation here with $\subset$ and $\subsetneq$ for explicit non equality
[5] smth smth coffee cup and donut "extra homework: go find a nice video on why this is true on youtube"
[6] dror was using $\sim$ for homeomorphism symbol today. if he keeps using that i'll adjust my notes, but for now i'll use $\cong$ cuz afaik its used more...?

In particular, the first condition is equivalent to $\bigcup_{B\in\mathcal{B}} B = X$ (i.e., the basis forms a covering of $X$), and the second condition is equivalent to the basic sets contained in $B_1 \cap B_2$ forming a cover of $B_1 \cap B_2$, i.e.

$$B_1 \cap B_2 = \bigcup_{\substack{B\in\mathcal{B} \\ B\subset B_1\cap B_2}} B.$$

Here are some examples;

(a) $\{B_r(x_0)\} \subset \mathcal{P}(\mathbb{R}^n)$, i.e. the open balls on $\mathbb{R}^n$ form a basis.

(b) The one-dimensional analogue of case (a) is $\{(a,b) \mid a < b\}$, and it forms a basis on $\mathbb{R}$.

(c) $\{[a,b) \mid a < b\}$ is called the lower limit topology, and it forms a basis on $\mathbb{R}$.

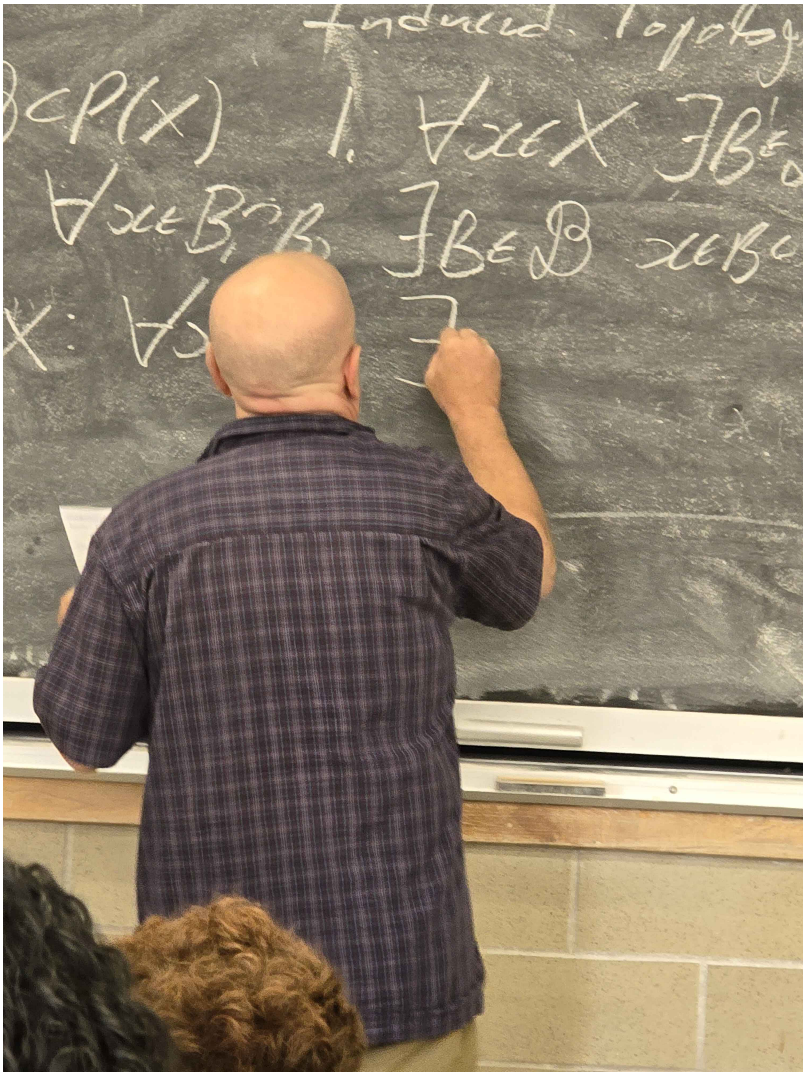
(d) $\{[a,b] \mid a < b\}$ implies $[a,b] \cap [b,c] = \{b\}$, which forces the basic set to include all singletons on $\mathbb{R}$. In that case, this is simply the discrete topology (?).

**Theorem 3.3.** $\mathcal{T}_{\mathcal{B}} = \{U \subset X \mid \forall x \in U \implies \exists B \in \mathcal{B}$ such that $x \in B \subset U\}$, i.e. the collection of all unique basic sets.

This will be expanded on next lecture.

# §4 Day 4: Basis; Orderings on Sets; Product Topology (Sep. 10, 2024)

Outfit of the day! Gives King Dice vibes tbh (from cuphead) i like the purple a lot :3



Recap of last lecture:

- We define the basis $\mathcal{B} \subset \mathcal{P}(X)$ of a topology to have the following properties (which we will refer to as the first and second axioms),
  1. For all $x \in X$, there exists a basic set $B \in \mathcal{B}$ such that $x \in B$.
  2. For any $x$ in the intersection of two basic sets (i.e. $x \in B_1 \cap B_2$), there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

With this, we may construct the topology generated by $\mathcal{B}$, i.e.

$$\mathcal{T}_{\mathcal{B}} := \{U \subset X \mid \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U\}.$$

**Theorem 4.1** (Basis Topology is a Topology on $X$)**.** We claim that $\mathcal{T}_\mathcal{B}$ is a topology on $X$.

We proceed by checking that $\mathcal{T}_\mathcal{B}$ satisfies the required properties.

1. Observe that $U = \emptyset, X$ are both in $\mathcal{T}_\mathcal{B}$; if $U = \emptyset$, then there are no elements $x \in U$ to bother about, of which the condition is vacuously true; if $U = X$, then the condition is true by the basis axioms.

2. Let us consider an indexing set $I$, and let us consider the arbitrary union of subsets $U_\alpha \subset X$
$$U := \bigcup_{\alpha \in I} U_\alpha.$$
Then any $x \in U$ belongs to $x \in U_\alpha$ for some index $\alpha \in I$; since $U_\alpha$ satisfies the condition, we see that the union $U$ also satisfies the condition as well.

3. Finally, for intersections, let us take $U_1, U_2 \in \mathcal{T}$ where[7]
$$U_1 = \bigcup_{\alpha_1 \in A_1} B_{1,\alpha_1},$$
$$U_2 = \bigcup_{\alpha_1 \in A_2} B_{2,\alpha_2},$$
where each $B_{i,\alpha_j}$ for $i, j \in \{1, 2\}$ above is a basic set in $\mathcal{B}$. Then
$$U_1 \cap U_2 = \left( \bigcup_{\alpha_1 \in A_1} B_{1,\alpha_1} \right) \cap \left( \bigcup_{\alpha_2 \in A_2} B_{2,\alpha_2} \right) = \bigcup_{\substack{\alpha_1 \in A_1 \\ \alpha_2 \in A_2}} \left( B_{1,\alpha_1} \cap B_{2,\alpha_2} \right).$$
Observing that $B_{1,\alpha_1} \cap B_{2,\alpha_2}$ is open by the second axiom of bases, we are done by quoting that the union of open sets is open. $\square$

**Theorem 4.2** ($\mathcal{T}_\mathcal{B}$ is the minimal topology containing $\mathcal{B}$)**.**

To start, we obviously have $\mathcal{T}_\mathcal{B} \supset \mathcal{B}$; now, let us have $\mathcal{T}'$ be another topology that contains $\mathcal{B}$. Since $\mathcal{T}_\mathcal{B}$ is the set of all unions of elements of $\mathcal{B}$, if $\mathcal{T}'$ contains $\mathcal{B}$, then it also contains $\mathcal{T}_\mathcal{B}$.

In fact, we may prove that the minimal topology containing $\mathcal{B}$ is unique.[8] Let $\mathcal{T}', ST''$ be minimal topologies that contain $\mathcal{B}$; then by the above argument, they both contain $\mathcal{T}_\mathcal{B}$; since they are minimal, we see $\mathcal{T}_\mathcal{B} = \mathcal{T}' = \mathcal{T}''$, and so $\mathcal{T}' = \mathcal{T}''$. $\square$

**Theorem 4.3** (Continuity on Basic Sets)**.** It is enough to test for continuity on basic sets. *(Originally left as exercise)*

Suppose $\mathcal{B}_Y$ is a basis of $\mathcal{T}_Y$ on $Y$, and suppose $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is the function we want to test for continuity. Then the topology of $\mathcal{T}_Y$ contains the basis topology $\mathcal{T}_{\mathcal{B}_Y}$, meaning that all open sets $\mathcal{T}_Y$ are unions and intersections of elements from $\mathcal{T}_{\mathcal{B}_Y}$. If the pre-image of the basic sets $B \in \mathcal{B}_Y$ are open in $X$, then so are the unions and intersections of such basic sets. $\square$

[7] note that dror used superscript to separate the two types of base sets, but i'm going to subscript them in a list to keep it clear.

[8] thought: measure theory gives that minimal $\sigma$-algebra generated by generator is unique; similar situation here?

We now define the notion of orders (corresponding Munkres section 14). A "complete" (otherwise called simple, line, or total) order on a set $X$ is a relation $<$ on $X \times X$ such that

- $<$ is transitive; i.e., if $x < y$ and $y < z$, then $x < z$.
- If $x, y \in X$, then exactly one of the following is true (recall trichotomy from 157):
  (a) $x < y$
  (b) $x > y$
  (c) $x = y$.

If $X$ is a set with a simple order relation, let $\mathcal{B}$ be the collection of all sets of the following types,

- All open intervals $(a, b)$ in $X$,
- All intervals of the form $[a_0, b)$ where $a_0$ is the smallest element of $X$ (if it exists),
- All intervals of the form $(a, b_0]$ where $b_0$ is the largest element of $X$ (if it exists).
- In the case that $X$ is a singleton set, said singleton is in $\mathcal{B}$.

The collection $\mathcal{B}$ is a basis for a topology on $X$; we call this the order topology. Now, we give some examples of orderings:

(a) $(\mathbb{R}, <_{\text{std}})$ and $(\mathbb{Q}, <_{\text{std}})$ are basic examples. Note that here, $<_{\text{std}}$ refers to the standard comparison.

(b) English words in lexicographiacl order, such as

$$\text{apple} < \text{ton} < \text{topo} < \text{topple} < \text{zebra}.$$

(c) $\{0, 1\} \times \mathbb{N}$ in dictionary order; we say $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$ if $\alpha_1 < \alpha_2$, or $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$.

(d) Alternatively, if we consider $\mathbb{N} \times \{0, 1\}$ (i.e. binary sequences), then it's the same idea as above just for infinite sequences of $\{\alpha_i\}_{i\in\mathbb{N}}, \{\beta_i\}_{i\in\mathbb{N}}$. Note that $\mathcal{T}_{\{0,1\}\times\mathbb{N}}$ and $\mathcal{T}_{\mathbb{N}\times\{0,1\}}$ are not homeomorphic. *(Originally left as exercise.)*

(e) $\mathbb{R} \times \mathbb{R}$ in dictionary order. Here, open sets can be thought of as $(a, b) = \{x \mid a < x < b\}$, where we may note $x \in \mathbb{R} \times \mathbb{R}$.

For another example, consider $X$ to be the set of finite strings over the usual alphabet; then we may observe that the open sets are dense in $X$; for example, the open set[9]

$$(\text{potato}, \text{tomato}) \supset \{\text{potatoo}, \text{potatooo}, \text{potatoooooooo}\}$$

contains as many strings as we want to fit into it. In particular, a consequence is that the order topology on $X$ is not equal to the discrete topology on $X$; since the discrete topology contains singleton sets (i.e., single lettersoups), and the unions and intersections of open sets of lettersoups is another open set, the two topologies are not equal.

In general, though, if $X$ is a finite set (i.e., similar to the above example, if $X$ is the set of *english words*) and we equip it with a strict total ordering, then the order topology is equal to the discrete topology.

[9] i'm done calling these things words. they are hereby known from now on as lettersoups

We now move onto product topologies. Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces; we want to construct a topology on $X \times Y$ such that $(X, Y)$ restricts to $X$ on $X$ and $Y$ on $Y$, i.e.

$$\pi_X : (X, Y) \to X, \qquad \pi_Y : (X, Y) \to Y,$$

and we want it to come with the properties

1. $\pi_X$, $\pi_Y$ are continuous (as per above, they are the projections to $X$ and $Y$),
2. Suppose $f, g$ are two continuous maps $f : Z \to X$ and $g : Z \to Y$. Then $f \times g : Z \to X \times Y$, where $(f \times g)(z) = (f(z), g(z))$.

Note that the above condition is equivalent to taking $h : Z \to X \times Y$; if $\pi_X \circ h$ and $\pi_Y \circ h$ are continuous, then $h$ is continuous.

**Theorem 4.4** (Existence of Unique Product Topology)**.** There exists a unique topology on $X \times Y$ satisfying the above two conditions; we will call this the product topology on $X \times Y$.

To start, by definition of continuity, we need that for all $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$, we have that

$$\pi_X^{-1}(U) = \{(x, y) \mid x \in U\}, \qquad \pi_Y^{-1}(V) = \{(x, y) \mid y \in V\}$$

must be open sets. Let us claim that the collection $\mathcal{B}_{X\times Y} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ forms a basis of $X \times Y$. Then let $\mathcal{T}_{X\times Y}$ be generated by $\mathcal{B}_{X\times Y}$; we claim that it satisfies the two properties outlined above.

1. $\pi_X^{-1}(U) = U \times Y \in \mathcal{T}_{U\times Y}$, and $\pi_Y^{-1}(V) = X \times V \in \mathcal{T}_{X\times V}$ show that continuity of $\pi_X, \pi_Y$ is satisfied on basic sets, and so they are satisfied in general.
2. Let $f : Z \to X$ and $g : Z \to Y$ be continuous. Consider $f \times g : Z \to X \times Y$, and let $U \subset \mathcal{T}_X$ and $V \subset \mathcal{T}_Y$. Then $U \times V$ is open, and we may write

$$\begin{aligned}(f \times g)^{-1}(U \times V) &= \{(x, y) \mid (f \times g)(x, y) \in U \times V\} \\ &= \{z \mid (f(z), g(z)) \in U \times V\} \\ &= \{z \mid f(z) \in U \text{ and } g(z) \in V\} \\ &= \{z \mid f(z) \in U\} \cap \{z \mid g(z) \in V\} \\ &= f^{-1}(U) \cap g^{-1}(V),\end{aligned}$$

which is the intersection of open sets, and so is open. Thus, our claim is complete. Now, to demonstrate uniqueness, suppose $\mathcal{T}', \mathcal{T}''$ are topologies in $X \times Y$ which satisfy our two properties from earlier. Then let us have the mapping

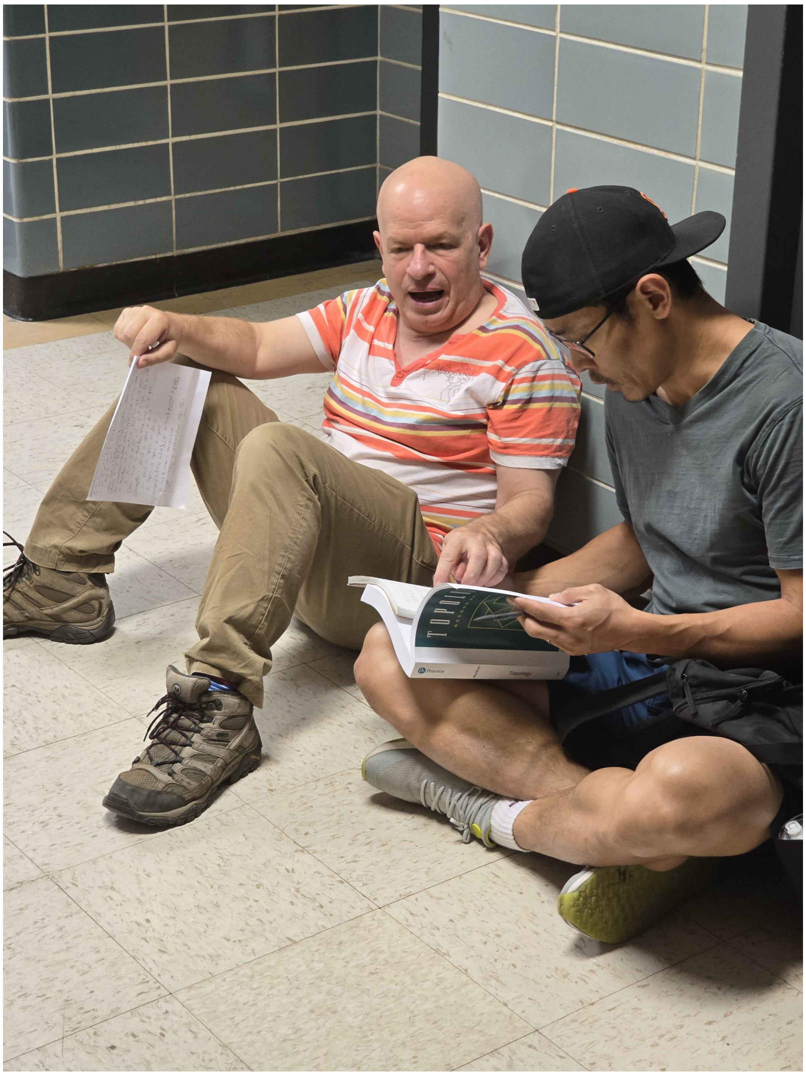
$$\begin{aligned}\mathrm{id} = \pi_X \times \pi_Y : (X \times Y)_{\mathcal{T}'} &\to (X \times Y)_{\mathcal{T}''} \\ (x, y) &\mapsto (\pi_X(x, y), \pi_Y(x, y)) = (x, y).\end{aligned}$$

In particular, $\pi_X : (X \times Y)_{\mathcal{T}'} \to X$ is continuous by property 1 of $\mathcal{T}'$ (same with $\pi_Y$). This means $\pi_X \times \pi_Y$ is continuous by property 2 of $\mathcal{T}''$[10]. We make the same argument for $\mathcal{T}''$; this means $\mathcal{T}' \subset \mathcal{T}''$ and vice versa, which means $\mathcal{T}' = \mathcal{T}''$ as desired. □

[10] phytor correction. ty!!

# §5 Day 5: Induced Topologies (Sep. 17, 2024)

Outfit of the day! Gives sushi shop california roll vibes :3



This week, we will cover sections 17 to 18 in Munkres, and the pre-reading for next week will be from section 19 to section 20. Now for recap;

Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, we define the product topology $\mathcal{T}_{X\times Y}$ on $X \times Y$ to be the unique topology with the properties

1. The projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.
2. For any function $h : Z \to X \times Y$, if $\pi_X \circ h, \pi_Y \circ h$ are continuous, then so is $h$.

Moreover, we have that $\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ is a basis for $\mathcal{T}_{X\times Y}$.

> **Remark 5.1** (Homeomorphism to Cartesian Product with Singleton)**.** Any topological space $X$ is homeomorphic to $X \times \{*\}$, where $\{*\}$ is some singleton. It is also homeomorphic to $\{*\} \times X$.

We now introduce the subspace topology (ref: section 16, Munkres). Given $(X, \mathcal{T}_X)$ and some subset $Y \subset X$, we wish to construct a topology on $Y$ such that

1. The inclusion map $\iota_Y : Y \hookrightarrow X$ is continuous.[11]
2. If $f : Z \to Y$ has that $\iota_Y \circ f$ is continuous, then $f$ is also continuous.

**Theorem 5.2** (Subspace Topology Exists and is Unique)**.** The topology on $Y$ satisfying the above two properties exists and is unique.

We start with existence; let us claim that $\mathcal{T}_Y := \{\iota_Y^{-1}(U) = Y \cap U \mid U \in \mathcal{T}_X\}$ satisfies the above two properties.

1. By construction of the set, $\iota_Y$ is obviously continuous.
2. If $f : Z \to Y$ and $\iota_Y \circ f$ is continuous, then let us take any open $V \in Y$, and consider $f^{-1}(Y)$. Pick $U \in \mathcal{T}_X$ such that $V = U \cap X$. Then we have
$$f^{-1}(V) = f^{-1}(U \cap X) = f^{-1}(\iota_Y^{-1}(U)) = (\iota_Y \circ f)^{-1}(U).$$
Since $\iota_Y \circ f$ is continuous, we have that $(\iota_Y \circ f)^{-1}(U)$ is open, and so we conclude $f$ is continuous. This concludes that $\mathcal{T}_Y$ satisfies the properties.

To prove uniqueness, suppose we have $\mathcal{T}'_Y$ and $\mathcal{T}''_Y$ satisfying the above two properties. Then observe the commutative triangle,

$$\begin{tikzcd}
& X & \\
(Y, \mathcal{T}'_Y) \arrow[ur, hook, "\iota'_Y"] \arrow[rr, leftrightarrow, "Id"'] & & (Y, \mathcal{T}''_Y) \arrow[ul, hook', "\iota''_Y"']
\end{tikzcd}$$

We have that $\iota''_Y \circ \mathrm{Id} = \iota'_Y$; by proposition 1 of $\mathcal{T}'_Y$, we have that $\iota'_Y$ is continuous, meaning $\iota''_Y \circ \mathrm{Id}$ is as well. By proposition 2 of $\mathcal{T}''_Y$, this means the identity between $(Y, \mathcal{T}'_Y)$ and $(Y, \mathcal{T}''_Y)$ is continuous, concluding that $\mathcal{T}'_Y = \mathcal{T}''_Y$. □

We now give some examples of subset topologies.

(a) Consider $[0, 1] \subset \mathbb{R}$ (where we identify $Y$ with $[0, 1]$ and $\mathbb{R}$ with $X$).[12] Then the topology $\mathcal{T}_{[0,1],\text{std}}$ is given by $\{U \cap [0, 1] \mid U \in \mathbb{R}_{\text{std}}\}$. Note that we may consider a subset topology on $\mathbb{R}$ even if $Y$ is not a open set.

If $Y \subset X$ is an open set, then open sets in $\mathcal{T}_Y$ are automatically open in $X$ as well.

(b) Now, suppose $Y' \subset Y$ and $X' \subset X$. Then $X' \times Y'$ has two topologies:

  (a) As a subset of the product topology $X \times Y$,

  (b) As a product of two subsets, one of $X$ and the other of $Y$.

We claim that these topologies are the same. *(Originally left as exercise.)*

(c) Let us have $Z \subset Y \subset X$. Let $Y$ have the subspace topology induced by $X$, and let $Z$ have the subspace topology induced by $Y$. Then $Z$ also has subspace topology induced by $X$.

[11]dror uses $i_Y$ instead of iota

[12]Note that the convention is that if the topology on $\mathbb{R}$ is not specified, then it is automatically the standard topology $\mathbb{R}_{\text{std}}$.

We now give some examples of product topologies.

(a) Let us have the product topologies $X \times Y, Y \times X$. Is $X \times Y = Y \times X$? Not necessariy, but we may construct a homeomorphism between them (by swapping coordinates). Note that this is different; if we induce an order topology on $X \times Y$ and $Y \times X$, then this is almost never true.

(b) Let $X, Y, Z$ be topological spaces. Then the cartesian product $(X \times Y) \times Z \neq X \times (Y \times Z)$ is generally not associative, since the sets have structure $\{((x, y), z)\}$ and $\{(x, (y, z))\}$. However, it is common to identify both of them as $X \times Y \times Z$ as an abuse of notation, with elements $\{(x, y, z)\}$. Thus, the product of finitely many topological spaces makes sense from induction on above, i.e.

$$X_1 \times \cdots \times X_n.$$

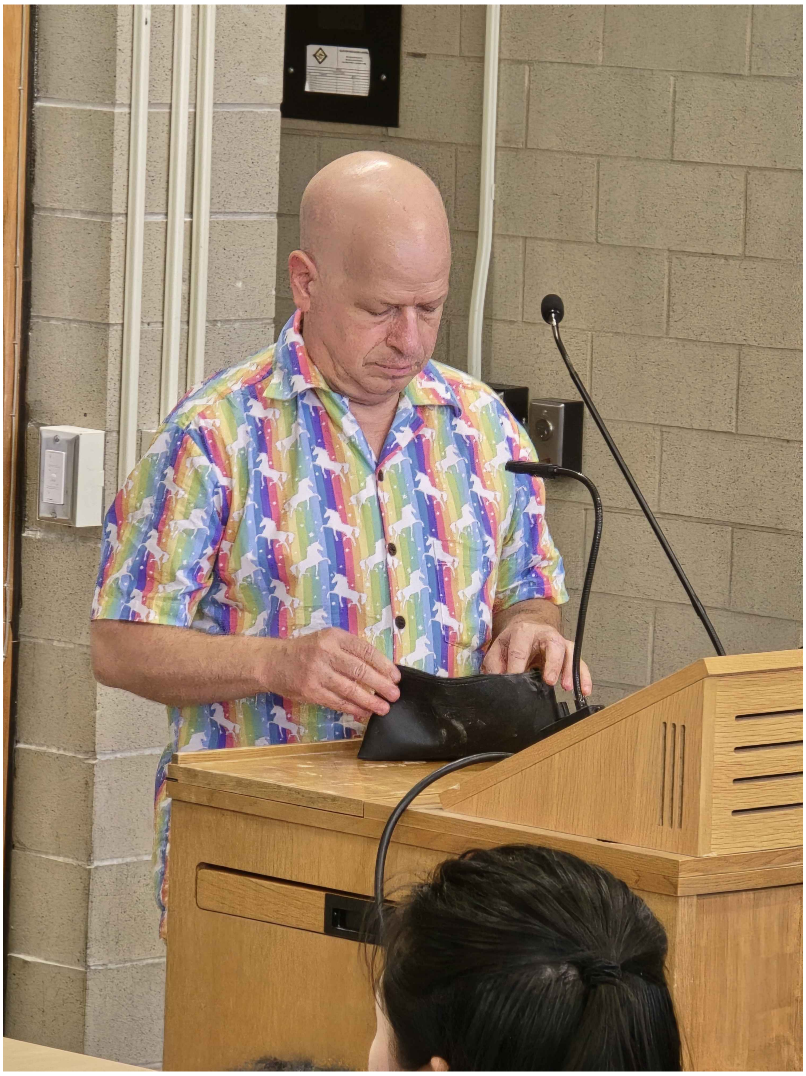
In general, induced topologies interact well with bases; for example, $\mathcal{T}_{X\times Y}$ is the topology generated by $\{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$. In terms of basis, we may write $\mathcal{B} = \{U \times V \mid U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$ with bases $\mathcal{B}_X, \mathcal{B}_Y$ from $X, Y$ respectively.

The subspace topology $Y \subset X$ also interacts well with bases, i.e. $\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}_X\}$. One of the only situations that the subspace topology does not work well with bases is with the order topology; let $X$ be ordered, and let $Y \subset X$. Then $Y$ can either inherits order or topology from $X$. Suppose these two are called by $Y_{\text{ord}}$ and $Y_{\text{subs}}$. Sometimes, $Y_{\text{ord}} \neq Y_{\text{subs}}$, but they are equal if $Y$ is convex. This example is gone over in Munkres.[13]

[13] yeah i'm confused here.

# §6 Day 6: Closed Sets, Cantor Construction, Closures and Interiors; Continuity (Sep. 19, 2024)

Outfit of the day![14] Rainbow unicorns owo!



Course administrative details;

- Office hours have been moved to Tuesdays, on 9:30 to 10:30AM.

If $X$ is a topological space, we say that a set $B \subset X$ is closed if its complement, $B^c = X \setminus B$ is open. *Sets are not doors; they can be open, closed, both, or neither.* For example, in the discrete topology, all sets are both open and closed (in general, $\emptyset$ and $X$ are the only sets that are always clopen).

Closed sets have the following properties,

(a) $\emptyset, X$ are both closed sets.

(b) The arbitary intersection of closed sets is closed.

(c) The finite union of closed sets is closed.

[14] livetexing this sum bitch- so i can't dror pic. sike

In particular, the last two properties are derived through considering the complements of the union and intersection properties of open sets. Specifically, if $A_\alpha$ is a collection of open sets, then

$$\left(\bigcup_\alpha A_\alpha\right)^c = \bigcap_\alpha A_\alpha^c, \quad \left(\bigcap_\alpha A_\alpha\right)^c = \bigcup_\alpha A_\alpha^c.$$

**Theorem 6.1** (Continuity, defined by Pre-Images of Closed Sets)**.** Any function $F : X \to Y$ (with topological spaces $X, Y$) is continuous if and only if, for every closed $C \subset Y$, we have $f^{-1}(C)$ is closed in $X$.

To see this, we may write

$$f^{-1}(C) = f^{-1}((C^c)^c) = f^{-1}(C^c)^c.$$

Since $C^c$ is an open set, we have that $f^{-1}(C^c)$ is open in $X$, and so its complement is closed. This means $f^{-1}(C)$ is closed in $X$. $\square$

> **Remark 6.2.** Having defined continuity, we note that the last two properties of closed sets could have also been used to define the notion of topologies instead of open sets.

Now, let us consider the following example: consider $\mathbb{R}_{\text{std}}$ containing $C_0 = [0,1]$. Then $C_0^c = (-\infty, 0) \cup (1, \infty)$. Next, define $C_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, i.e. $C_0$ with its inner third removed. Let $C_2$ be $C_1$ with the inner thirds of each of its intervals removed, etc.;

Notice that each $C_n$ is a union of closed intervals on $\mathbb{R}_{\text{std}}$. Define

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n = [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right).$$

Since the arbitrary union of open intervals is open, we see that the complement of them is closed, meaning $\mathcal{C}$ is closed. Moreover, we claim that $\mathcal{C}$ is not empty. Start by writing $\mathcal{C}$ as the set $\{x \mid x_3 \text{ has no 1s}\}$, i.e. $x_3$ representing the ternary expansion of $x$. For any number whose ternary decimal representation consists of only 0s and 2s, we may write said number as an infinitely recurring decimal containing 1, since $0.1\overline{2}_3 = 0.2_3$, by rewriting the last 2 in the expansion of any such number.

Clearly, $\mathcal{C}$ is uncountable. In addition, it exhibits the properties,[15]

- $\mathcal{C}$ is uncountable, yet the length of $\mathcal{C}$ is 0.
- *(Originally left as exercise; Devil's Staircase Construction)* There exists a continuous $F : [0,1] \to [0,1]$ such that $F(0) = 0, F(1) = 1$ and for all $x \notin \mathcal{C}$, we have that $F'(X) = 0$.
- *(Originally left as exercise)* What is $\mathcal{C} + \mathcal{C} = \{x + y \mid x, y \in \mathcal{C}\}$? Alternatively, consider $f : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ where $f(x,y) = x + y$.

[15] read more here ig? linkylinkylink

When is a set closed in the product of two spaces? Suppose $A \subset X, B \subset Y$ are closed sets in their respective topological spaces. Then observe that we cannot just take the complement as follows,

$$(A \times B)^c \neq A^c \times B^c,$$

since $(A \times B)^c = (A^c \times Y) \cup (X \times B^c)$. Observing that both of these sets are open, we see that $(A \times B)^c$ is also open, meaning $A \times B$ is closed. $\square$

**Claim 6.3.** If $Y \subset X$ are topological spaces with $Y$ having the subspace topology, then $C \subset Y$ is closed if and only if there exists some closed $B \subset X$ such that $C = B \cap Y$.

**Claim 6.4.** If $Y \subset X$ is closed, and $B \subset Y$ is closed in $Y$, then $B$ is closed in $X$.

We now define the interior and closure of sets.

- The interior of $A$ is denoted $\text{int}_X A = \mathring{A}$. It is the largest open set contained in $A$. By the arbitrary union property of open sets, this is given by the union of all open sets contained in $A$.
- The closure of $A$, $\text{cl}_X A = \overline{A}$, is the smallest closed set containing $A$, namely the intersection of all closet sets containing $A$.

For an example, we have that

$$[0,1)^o = (0,1),$$
$$\overline{[0,1)} = [0,1].$$

If $A$ is open, then $\mathring{A} = A$. If $A$ is closed, then $\overline{A} = A$. Here are some more properties on interiors and closures:

- The interior of the interior of $A$ is equal to the interior of $A$, i.e. $\mathring{\mathring{A}} = \mathring{A}$.
- The same holds for the closure, i.e. $\overline{\overline{A}} = \overline{A}$.
- The interior and closure of any clopen set is the clopen set itself. For example, $\mathring{\emptyset} = \overline{\emptyset} = \emptyset$, and the same holds for the whole set.
- In general, we don't know what $\mathring{\overline{A}}$ or $\overline{\mathring{A}}$ is.
- The complement of $\overline{A}$ is given by

$$\overline{A}^c = \left( \bigcap_{\substack{F \supset A \\ F \text{ closed}}} F \right)^c = \bigcup_{\substack{F \supset A \\ F \text{ closed}}} F = \bigcup_{\substack{U^c \supset A \\ U^c \text{ closed}}} U \underbrace{=}_{F^c = U} \bigcup_{\substack{U \subset A^c \\ U \text{ open}}} U = (\mathring{A^c}).$$

  Thus, the complement of the closure is the interior of the complement.

- Likewise, we have $(\mathring{A})^c = \overline{A^c}$.
- *(Challenge Exercise)* Prove that we can make 14 distinct sets from any general set $A$ using complement, closure, and interior.

**Theorem 6.5.** Let $X$ be a topological space, and let $A \subset X$. $x$ is in the closure of $A$ if and only if every neighborhood of $x$ intersects $A$. A neighborhood of $x$ is defined as an open set containing $x$.

Specifically, the condition above may be written as $\forall U \in \mathcal{T}_X$, we have $x \in U \implies U \cap A \neq \emptyset$.

($\Rightarrow$) Assume $x \in \overline{A}$ by contradiction. Assume that $U$ is open and $x \in U$, with $U \cap A = \emptyset$. We have $U^c \supset A$, but $U^C$ is closed. Since $x \notin U^c \supset \overline{A} \ni x$ is a contradiction, we are done.

($\Leftarrow$) Assume every neighborhood of $X$ intersects $A$. By contradiction, also assume that $x \notin \overline{A}$. Then $\overline{A}^c \ni x$ is a neighborhood of $x$, not intersecting $A$. This is a contradiction, so we are done. □

In fact, we may check basic neighborhoods (i.e., open set in basis containing $x$) only instead of all neighborhoods.

**Claim 6.6** (Closure of $\mathbb{Q}$ is $\mathbb{R}$)**.** We have that $\overline{Q} = \mathbb{R}$.

For all $x \in \mathbb{R}$, we have $x \in \overline{\mathbb{Q}}$ if and only if $x \in (a, b)$ implies $(a, b) \cap \mathbb{Q} \neq \mathbb{Q}$. □

**Definition 6.7.** Given $A \subset X$, $x \in X$ is a limit of $A$ if every neighborhood of $x$ contains a point of $A$ other than $x$ itself. This is equivalent to saying that $x \in \overline{A \setminus \{x\}}$.

**Theorem 6.8.** $\overline{A} = A \cup A'$ where $A'$ is given by the set of limit points of $A$.

($\supset$) $\overline{A} \supset A$ is trivially true. If $x \in A'$, then $x \in \overline{A \setminus \{x\}} \subset \overline{A}$.

($\subset$) Take $x \in \overline{A}$. If $x \in A$, we're automatically done. Thus, let us assume that $x \notin A$; then $x \in \overline{A} = \overline{A \setminus \{x\}}$, implying that $x \in A'$. □

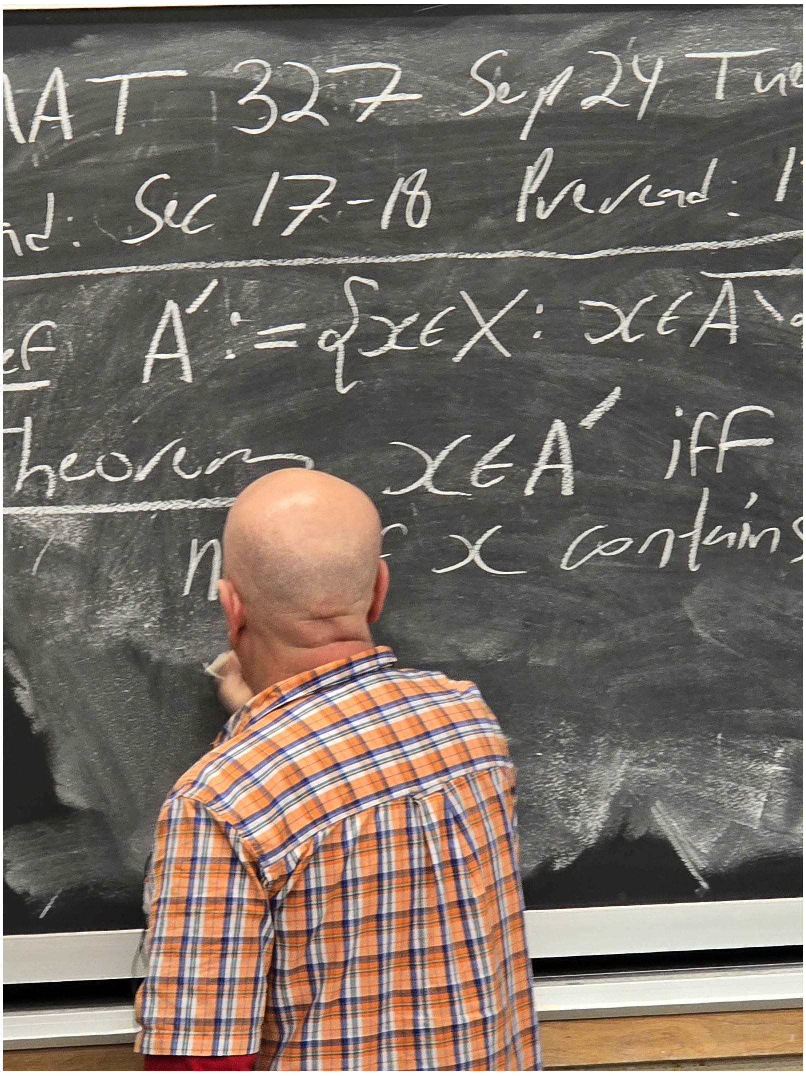
We now present a fheorem (false theorem). If $x$ is a limit point of some set $A$, then every neighborhood of $x$ contains infinitely many elements of $A$, i.e.

$$x \in U \in \mathcal{T}_X \implies |U \cap A| = \infty.$$

We now present a froof (false proof). For any $x$, we may take any neighborhood about $x$. Then we may remove a point $a_1$; since the neighborhood with $a_1$ removed is still an open set, we may remove $a_2$, $a_3$, and so on inductively. □

# §7 Day 7: Separation Axioms and $T_2$/Hausdorff Spaces, More on Continuity (Sep. 24, 2024)

Outfit of the day! Plaid outfit!




Recap of last lecture:

- We define $A' := \{x \in X \mid x \in \overline{A \setminus \{x\}}$ to be the set of limit points of $A$ in $X$.
- We present a "fheorem" (read: false theorem): $x \in A'$ if and only if every neighborhood of $X$ contains infinitely many points of $A$, i.e.

  $$x \in U \subset X \implies |U \cap A| = \infty,$$

  where $U$ is open in $X$.

  We expand on our "froof" from last time: indeed, if $U$ is a neighborhood of $X$, then as $x \in \overline{A \setminus \{x\}}$, we have that $U \cap (A \setminus \{x\}) \neq \emptyset$. Now, let us pick $a_1 \in U \cap (A \setminus \{x\})$, and consider $U_1 = U \setminus \{a_1\}$.

  Since $U_1$ is a neighborhood of $x$, we have that $U_1 \cap \overline{A \setminus \{x\}} \neq \emptyset$, so we may pick $a_2 \in U_1 \cap \overline{A \cap \{x\}}$; then we consider $U_2$, etc... We argue by induction that there are infinitely many such points. Thus, there exists an infinite sequence of distinct points $\{a_i\}_{i\in\mathbb{N}}$ in $U \cap A \setminus \{x\}$. □

This proof is in fact wrong because $U_1$ is not necessarily a neighborhood of $x$ (i.e., we don't know if $X \setminus \{a_1\}$ is open or not). For a blatant example, consider the trivial topology. Then $X \setminus \{a_1\}$ is obviously not open if $X$ is not a singleton.

To expand on our recap of last lecture, we introduce the separation axioms.

$T_1$: A space $X$ is called $T_1$ if and only if, for all $x \in X$, $\{x\}$ is closed, and $X \setminus \{x\}$ is open, and for all $x, y \in X$ where $x \neq y$, we can find a neighborhood of $y$ that does not contain $x$, i.e. $\exists U$ open in $X$ such that $x \in U$ but $y \notin U$. These three conditions are equivalent to each other; though usually, it is useful to just identify it as singletons being closed, or their complements being open.

In particular, our "froof" from earlier holds if $X$ is $T_1$ (i.e., any singleton is closed, and so the union of singletons are closed).

$T_2$: We say a space $X$ is $T_2$ (aka Hausdorff or *separated*) if, for all $x, y \in X$ where $x \neq y$, then there exists open sets $U_1, U_2$ in $X$ such that $U_1 \cap U_2 = \emptyset$.

**Exercise 7.1.** Show that if $x_1, \ldots, x_n \in X$, then there exists open sets $U_1, \ldots, U_n$ such that $x_i \in U_i$ and $i \neq j \implies U_i \cap U_j = \emptyset$.

**Claim 7.2.** We claim that if $X$ is $T_2$, then it is also $T_1$.

This is clear by picking neighborhoods $U_1, U_2$ for $x, y \in X$ where $x \neq y$, such that $U_1 \cap U_2 = \emptyset$. Clearly, $U_1$ does not contain $y$, and $U_2$ does not contain $x$. □

**Claim 7.3.** In a $T_2$ space, any sequence has at most one limit.

In particular, we define sequence convergence as follows; if $\{a_n\}_{n\in\mathbb{N}} \in X$, then we say that $a_n$ converges to $a \in X$, i.e. $a_n \to a$ as $n \to \infty$, or $\lim_{n\to\infty} a_n = a$, if for every neighborhood $U$ of $a$, there exists $N$ such that $n > N$ implies $a_n \in U$.

We now prove claim 7.3. Assume $\{a_n\}_{n\in\mathbb{N}}$ is a sequence that converges to $a$ and $a'$, i.e. $a_n \to a$ and $a_n \to a'$. Then by $T_2$, there exists neighborhoods $U, U'$ of $a, a'$ respectively such that $U \cap U' = \emptyset$. Then, let us pick a large enough $N$ such that $n > N \implies a_n \in U$, and $N'$ large enough such that $n > N' \implies a_n \in U'$. Then just consider $n = \max\{N, N'\} + 1$, then $a_n \in U$ and $U'$ at once, while $U \cap U' = \emptyset$. This is contradictory, and so we are done. □

We now make the following claim.

**Claim 7.4.** We claim that a subspace of a $T_2$ space is $T_2$, and that the product of $T_2$ spaces is also $T_2$.

We start by proving the claim on products[16]. Suppose $X, Y$ are $T_2$, and let us have $a_1, a_2 \in X \times Y$ where $a_1 = (x_1, y_1) \neq (x_2, y_2) = a_2$. Then $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality, suppose $y_1 \neq y_2$. Then there exists neighborhoods $V_1, V_2$ in $Y$ of $y_1, y_2$ respectively, where $V_1 \cap V_2 = \emptyset$. Then we see that $X \times V_1$ and $X \times V_2$ are suitable choices of neighborhoods for $a_1, a_2$ to conclude that $X \times Y$ is $T_2$.

[16] im not including the froof from class because i think it's kind of immediate, sorry. also i didn't catch it entirely so oh well

We now move onto a different topic.

**Theorem 7.5.** Given $f : X \to Y$, we have that the following are equivalent:

(a) $f$ is continuous.

(b) $f(\overline{A}) \subset \overline{f(A)}$, i.e. $f$ maps the closure of $A$ into the closure of $f(A)$.

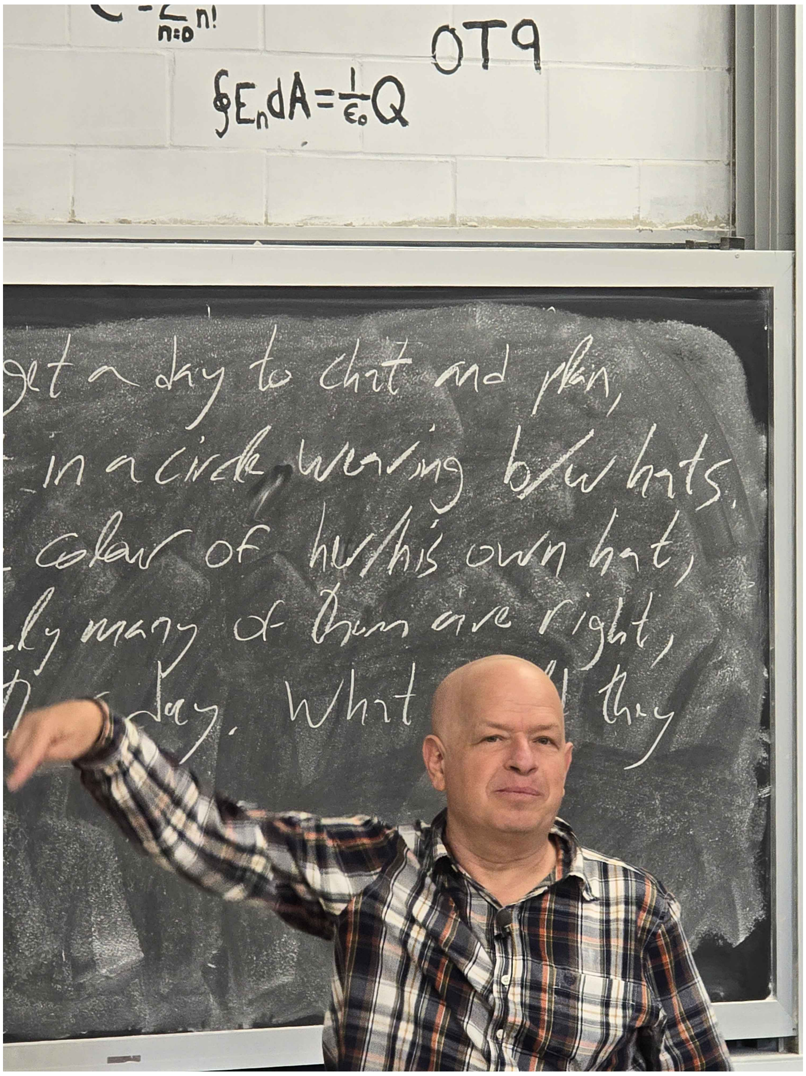
(c) If $B \subset Y$ is closed, then $f^{-1}(B)$ is closed.

(d) For all $x_0 \in X$, if $V$ is a neighborhood of $f(x_0)$, then there is a negihborhood $U$ of $x_0$ such that $f(U) \subset V$.

Note that we have already proven that (a) if and only if (c), (a) iff (d), and we will prove that (a) implies (b) and (b) implies (c) on Thursday.

# §8 Day 8: Continuous Functions and Infinite Products (Sep. 26, 2024)

Outfit of the day! Sniper monkey 0-2-5 outfit!!!! very demure, very mindful




Recap from last lecture: for any $f : X \to Y$, we have that the following are equivalent:

(a) $f$ is continuous.

(b) For all $A \subset X$, we have that $f(\overline{A}) \subset \overline{f(A)}$.

(c) The pre-image of a closed set is closed.

(d) For all $x \in X$ and for all negihborhoods $V$ of $f(x)$, there exists a neighborhood $U$ of $x$ such that $f(U) \subset V$.

Note that we have already established that $(a) \iff (c)$ and $(a) \iff (d)$. We now prove that $(a) \implies (b)$.

Take $x$ in the closure of $A$, and pick a neighborhood $U$ of $f(x)$. Then $f^{-1}(U)$ is open (by continuity), and it contains $x$. Thus, $f^{-1}(U)$ intersects $A$. Let us pick $y \in f^{-1}(U) \cap A$; then we have $f(y) \in U \cap f(A)$, and so every neighborhood of $f(x)$ intersects $f(A)$, and so $f(x) \in \overline{f(A)}$. $\square$

As an example, let us $X = \mathbb{R}^2$, and consider $A = \{(x, \frac{1}{x}) \mid x \neq 0\}$. Then if we let $f$ be the function projecting $A$ to the $x$-axis, i.e. $f(A) = \mathbb{R} \setminus \{0\}$, and we have that

$$f(\overline{A}) = f(A) = \mathbb{R} \setminus \{0\} \subsetneq \overline{f(A)} = \mathbb{R}.$$

We now continue on our previous claim, and show that $(b) \implies (c)$. Let $B \in Y$ be a closed set, and let $f^{-1}(B) =: A$. Then we may write,

$$f(\overline{A}) \overset{(b)}{\subset} \overline{f(A)} = \overline{f(f^{-1}(B))} \subset \overline{B} = B,$$

since the closure of a closed set is itself. Thus, we have that $\overline{A} \subset f^{-1}(B) = A$, implying $\overline{A} = A$, and so $A$ is closed, i.e. the pre-image of $B$ is closed. $\square$

We now move onto infinite products. Suppose $X_\alpha$ is a set for every $\alpha \in I$, and let $X = \prod X_\alpha = \{x : I \to \bigcup_\alpha X_\alpha \mid \forall \alpha \in I, x(\alpha) \in X_\alpha\}$[17]. Per the axiom of choice (which we will refer to as $AC$ from now on), if all $X_\alpha$s are not empty, then $X = \prod X_\alpha$ is non-empty.

Assuming AC, we have that $\prod_{\emptyset \neq A \subset \mathbb{R}} A$ is non-empty. Let us consider the function $c : \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\} \to \mathbb{R}$ such that $c(A) \in A$. There exists a function on the nonempty sets of reals that picks an element from each set of reals.

Let $S = \mathbb{R}^{\mathbb{N}} = \{(a_i) + i \in \mathbb{N} \mid a_i \in \mathbb{R}\}$. Then we claim that there exists a function $h : S \to S$ such that

(a) $h(a)$ depends only on the tail of $a$, meaning if $a \sim b$ (i.e. they have the same tail, meaning $\exists N \in \mathbb{N}$ such that for all $n > N$, $a_n = b_n$), then $h(a) = h(b)$.

(b) $h(a) \sim a$, i.e. they only differ in finitely many coordinates, and are equal past some index $N$.

**Claim 8.1.** If such a function exists, we can save infinitely many prisoners; recall that the prisoner problem is that every prisoner is given a white or black hat (they cannot see their own hat), and they have to call the color of their own hat out.

Assuming AC, we know that $h$ exists, and so by property (b) of $h$, only finitely many prisoners will die.

Let $I = S/\sim$ (read: equivalence classes as per $\sim$ defined earlier, in $S$), and consider $X = \prod_{\alpha \in I} \alpha \ni h'$. Let $h'$ be a function taking the tail of $\alpha$ that finds another sequence whose tail is the same as $\alpha$'s. Then $h(a) = h'(\text{tail of } \alpha)$; unfortunately, to find $h'$, we need to use AC. $\square$

Suppose $X_\alpha$ is a topological space, and we want a topology on $X = \prod X_\alpha$. We may approach this in two ways:

1. We may choose to generalize the construction,

$$\mathcal{B} = \left\{ \prod_{\alpha \in I} U_\alpha \mid \forall \alpha, U_\alpha \subset X_\alpha \right\},$$

   where each $U_\alpha$ is open. *(We note that this is a dead end.)*

2. Alternatively, we may generalize the requirements. We want a topology such that:
   (a) $\forall \alpha, \prod_\alpha : X \to X_\alpha$, i.e. $\prod_\alpha(x) = x(\alpha) = x_\alpha$
   (b) If $g : Z \to X$ has that $\forall \alpha, \pi_\alpha \circ g$ is continuous, then $g$ is continuous.

[17] alternatively, the function $x$ can be seen as just a "collection" of $x_\alpha$s.

**Claim 8.2.** There exists a unique topology on $X$ satisfying properties 1 and 2.

Observe that at least $\prod_\alpha^{-1}(U_\alpha)$ must be open for every $U_\alpha$, where $\alpha \in I$, to be open in $X$. Let's start by defining

$$\mathcal{B} = \{\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n}) \mid \alpha_1, \dots, \alpha_n \in I;\ U_{\alpha_i} \text{ open in } X_i\};$$
$$\mathcal{B}_{\text{cyl}} = \left\{ \prod_{\alpha \in I} U_\alpha \mid \forall \alpha, U_\alpha \subset X_\alpha \text{ is open; } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha\text{'s.} \right\}.$$

This is exactly the previous basis $\mathcal{B}$ (note that we read cyl as cylinder here[18]). We claim that this works; observe that

(a) $\mathcal{B}_{\text{cyl}}$ is indeed a basis, so let $\mathcal{T}_{\text{cyl}}$ be the topology it generates, meaning that property 1 holds,

(b) We also have that property 2 holds; assume $g : Z \to X$ and $\pi_\alpha \circ g$ is continuous for all $\alpha$. Then let us write,

$$g^{-1}\left(\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})\right) = \bigcap_{i=1}^{n} g^{-1}(\pi_\alpha^{-1}(U_{\alpha_i})) = \bigcap_{i=1}^{n} (\pi_\alpha \circ g)^{-1}(U_{\alpha_i}).$$

Since each $\pi_\alpha \circ g$ is assumed to be continuous, we have that this is a finite intersection of open sets, and we conclude that the above is indeed open.

Now that we have proved existence, we claim that we also have uniqueness. If $\mathcal{T}'$ and $\mathcal{T}''$ are topologies on $X = \prod X_\alpha$ satisfying 1 and 2, then $\mathcal{T}' = \mathcal{T}''$; to start, consider the identity map $g = \text{id}$,

$$\left(\prod X_\alpha, \mathcal{T}'\right) \xrightarrow{\text{id}} \left(\prod X_\alpha, \mathcal{T}''\right).$$

Then $\pi_\alpha \circ g = \pi_\alpha \circ \text{id} = \pi_\alpha$ is continuous by property 1 of $\mathcal{T}'$; thus, by property 2 of $\mathcal{T}''$, we see that the identity map is continuous, and we proceed as per our previous uniqueness proofs to see that $\mathcal{T}' = \mathcal{T}''$. $\square$

For concreteness, we now present an example. Let $c : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$, where $c(t) = (t, t, \dots)$, i.e. the constant sequence consisting of $t$s. Then $c : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}_{\text{cyl}}$ is continuous. $(\pi_k \circ c)(t) = t$ (since $c = \text{id}_{\mathbb{R}}$), and so $c$ is continuous.

Now, consider $c : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}_{\text{box}}$. Then

$$c^{-1}\left(\prod_{k \in \mathbb{N}} \left(-\frac{1}{k}, \frac{1}{k}\right)\right) = \{0\},$$

which is not open.

Note that $\mathcal{T}_{\text{box}} \neq \mathcal{T}_{\text{cyl}}$. In fact, the cylinder topology is contained in the box topology, and the inclusion is not an equality, $\mathcal{T}_{\text{cyl}} \subsetneq \mathcal{T}_{\text{box}}$. In both topologies,

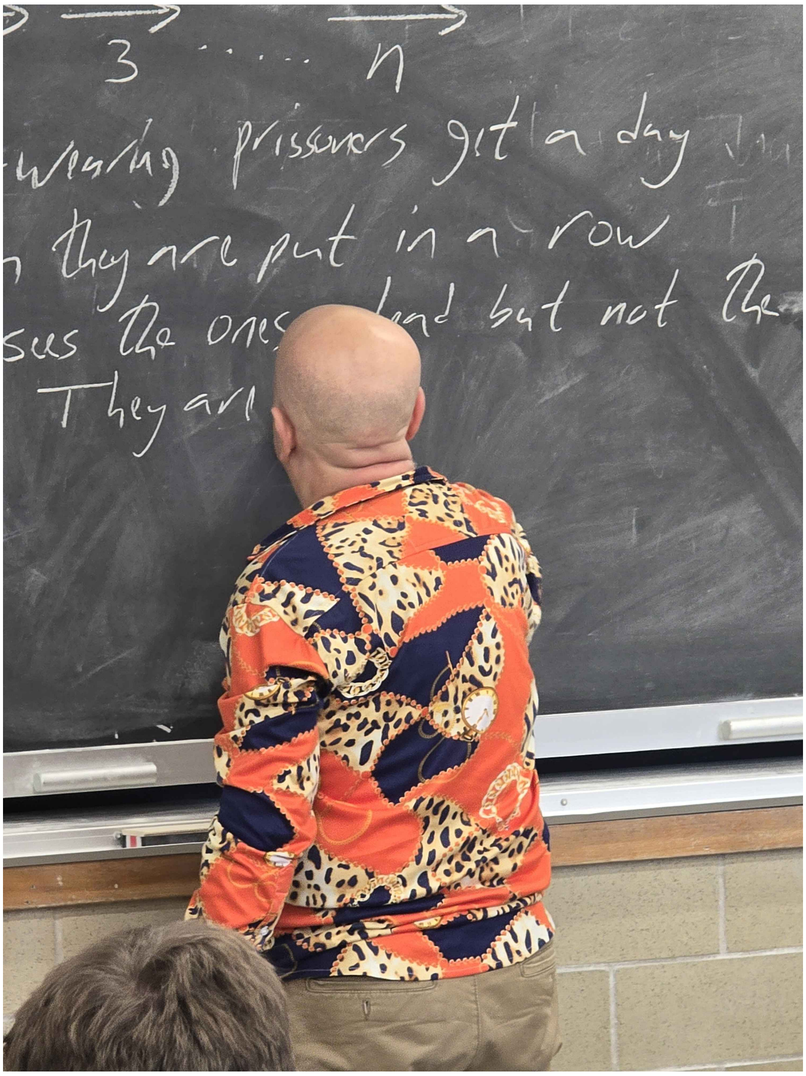
(a) If $A_\alpha \subset X_\alpha$, for every $\alpha$, $\prod A_\alpha$ as a subset of a product is the same as $\prod A_\alpha$ as a product of subsets. *(The proof is messy. We leave it alone for now.)*

(b) If, for all $\alpha$, $X_\alpha$ is $T_2$, then $(\prod_\alpha X_\alpha)_{\text{box}}, (\prod_\alpha X_\alpha)_{\text{cyl}}$ are $T_2$.

[18] and i have no idea why it's cylinder, but we play along for now

## §9 Day 9: Infinite Products, Metrics (Oct. 1, 2024)

Outfit of the day! smth smth butterflies !!! very demure, very mindful




Recap of last lecture; topologies on $\prod_\alpha X_\alpha$ generalize the following;

(a) The basis $\mathcal{B}_{\text{box}} = \{\prod U_\alpha \mid U_\alpha \subset X_\alpha \text{ open}\} \rightsquigarrow \mathcal{T}_{\text{box}}$;

(b) The requirements $\mathcal{B}_{\text{cyl}} = \{\prod U_\alpha \mid U_\alpha \subset X_\alpha \text{ open, almost always } U_\alpha = X_\alpha\} \rightsquigarrow \mathcal{T}_{\text{cyl}}$.

Observe that the function sending a constant $t \in \mathbb{R}$ to the constant sequence $(t, t, \dots) \in \mathbb{R}^{\mathbb{N}}$ is continuous in the cylinder topology, but not in the box topology (read: here). This means $\mathcal{T}_{\text{cyl}} \supsetneq \mathcal{T}_{\text{box}}$ (in general, they are not the same); yet in both, both subspace and the Hausdorff property behaves (i.e., products preserve Hausdorff-ness).

**Claim 9.1** (Theorem 19.5, Munkres)**.** Let $\{X_\alpha\}$ be an indexed family of spaces, and let $A_\alpha \subset X_\alpha$ for each $\alpha$. Then we have that $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$ if $\prod X_\alpha$ is given the box or cylinder topologies.

We prove this by double inclusion:

($\Leftarrow$) Let $x = (x_\alpha) \in \overline{\prod_\alpha A_\alpha}$. Recall that $x$ is in the closure of $\prod A_\alpha$ if and only if every basic neighborhood of $x$ intersects $\prod_\alpha A_\alpha$. This condition is equivalent to saying that for all open neighborhood $U_\alpha \subset X_\alpha$, $x \in \prod U_\alpha \implies \prod U_\alpha \cap \prod U_\alpha \neq \emptyset$, which is also equivalent to saying that for all $\alpha$ where $x_\alpha \in U_\alpha$, we have $U_\alpha \cap A_\alpha \neq \emptyset$. Thus, every neighborhood $U_\alpha$ about $x$ intersect $A_\alpha$. This means for all $\alpha, x_\alpha \in \overline{A_\alpha}$, and we conclude that $x \in \overline{\prod_\alpha A_\alpha}$.

($\Rightarrow$) *(Not covered in class)* Let $x = (x_\alpha)$ lie in the closure of $\prod A_\alpha$. Then for any index $\beta$, we have $x_\beta \in \overline{A_\beta}$. Let $V_\beta$ be an arbitrary open set of $X_\beta$ containing $x_\beta$. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology, it contains $y = (y_\alpha) \in \prod A_\alpha$. Then $y_\beta$ belongs to $V_\beta \cap A_\beta$, which means $x_\beta \in \overline{A_\beta}$. $\square$

We now move onto metric spaces. We say a metric on a set $X$ is a function $d : X \times X \to (\mathbb{R} \geq 0)$ such that:

(a) $d(x, y) \geq 0$, with $d(x, y) = 0 \iff x = y$ (non-negativity);

(b) $d(x, y) = d(y, x)$ (symmetry);

(c) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

We also define that $B_r(x_0) = \{x \mid d(x_0, x) < r\}$. Note that balls in non-euclidean metrics may look different from a sphere; we now give some examples of metrics:

(a) The Euclidean metric ($L^2$) on $\mathbb{R}^n$, $d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

(b) The Manhattan distance ($L^1$) on $\mathbb{R}^n$, $d(x, y) = \sum_{i=1}^n |x_i - y_i|$. Note that here, a ball looks more like a fucked-up rhombus.

(c) For any set $X$, let us define $d(x, y) = 1$ for $x \neq y$, and $d(x, y) = 0$ if $x = y$.

(d) For bounded sequences $(a_i) \in \mathbb{R}^{\mathbb{N}}$, we may define $d(a, b) = \sup |a_i - b_i|$ (aka $L^\infty$).

A set with a metric on it is called a *metric space*; on metric spaces, we set $\mathcal{B}_{\mathrm{d}} = \{B_r(x_0)\}$, i.e. the set of all open balls, and we claim that this is indeed a basis *(Left as exercise, but it's obvious)*. In particular, every metric space has a topology; namely, it is a topological space; we claim that the metrics introduced above induce the same topology, i.e. the discrete topology on $\mathbb{R}^n$, except (d).

All metric spaces are Hausdorff by considering the triangle inequality.