

# MAT363 Lecture Notes

ARKY!! :3C

'25 Winter Semester

## Contents

## §1 Day 1: Introduction to the Class (Jan. 6, 2025)

*Course administrative details!* First day slides are given [here](#). This is a class in classical differential geometry; the following 12 weeks will be split up as follows,

- (a) Curves, for two weeks;
- (b) Surfaces, for three weeks;
- (c) Curvature of surfaces, for three weeks;
- (d) Geodesics, for three weeks;
- (e) Gauss-Bonnet theorem, for one week.

Grading will be done by 5% on PCEs, 15% on problem sets, 15% on quizzes, 25% on the term test, 30% on the final exam, and 10% weighted towards your best test.

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To start, consider the following maps  $\gamma : I = (-10, 10) \rightarrow \mathbb{R}^3$ , given by

$$\begin{aligned}\gamma(t) &= (t, t, t); \\ \gamma(t) &= (|t|, |t|, |t|); \\ \gamma(t) &= (t, t^2, t^3); \\ \gamma(t) &= (t^3, t^3, t^3); \\ \gamma(t) &= (\cos t, \sin t, t); \\ \gamma(t) &= (t \cos t, t \sin t, t).\end{aligned}$$

In this class, we say that a curve is a *parameterized curve* if it is a smooth function  $\gamma : I \rightarrow \mathbb{R}^n$ , where  $I \subset \mathbb{R}$  is an interval. In particular, of the six examples given above, only  $t \mapsto (|t|, |t|, |t|)$  is not smooth.

**Definition 1.1** (Regular Curve). Let  $\gamma : I \rightarrow \mathbb{R}^n$  be a curve; it is said to be *regular* if  $|\gamma'(t)| \neq 0$  for all  $t \in I$ , i.e. the speed is always nonzero.

Note that  $\gamma'(t)$  and  $|\gamma'(t)|$  describe different qualities, with the former describing velocity and the latter describing speed (i.e., one describes speed as well, while the other is a scalar quantity). As an example, consider the curve  $\gamma(t) = (\cos t, \sin t, t)$ . To find the distance travelled from  $t = 0$  to  $t = 2\pi$ , we may observe that

$$|\gamma'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Since the speed is constant, the total distance traveled is simply  $2\pi\sqrt{2}$ . □

**Definition 1.2** (Closed Curve). Consider a curve  $\gamma : [a, b] \rightarrow \mathbb{R}^n$ . We say that  $\gamma$  is a *closed curve* if  $\gamma(a) = \gamma(b)$  and  $\gamma^{(n)}(a) = \gamma^{(n)}(b)$  for all naturals  $n$ .

**Definition 1.3** (Simple Curve). We say that  $\gamma$  is a *simple curve* if it is injective on  $[a, b]$ .

Note that while in topology we do not care if there is a “sharp corner” at  $\gamma(a) = \gamma(b)$ , such things do matter, as per the condition that the  $n$ th derivative of  $\gamma$  must agree on  $a$  and  $b$  (for example, the velocity  $\gamma'$  at  $a, b$  must be equal).

In this class, we automatically take the inner product  $\langle \cdot, \cdot \rangle$  as the Euclidean inner product,

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n.$$

For any subspace  $V \subset \mathbb{R}^n$ , we may decompose any vector  $x \in \mathbb{R}^n$  uniquely as  $x = x^\parallel + x^\perp$ , where  $x^\parallel \in V$  and  $\langle x^\perp, v \rangle = 0$  for any vector  $v \in V$ . Now, consider any curve  $\gamma : I \rightarrow \mathbb{R}^n$ . We have the following proposition,

**Proposition 1.4.** If  $|\gamma(t)|$  is constant, then  $\langle \gamma(t), \gamma'(t) \rangle = 0$  for all  $t \in I$ .

To see this, let  $|\gamma(t)|^2 = c$  be constant; then

$$\frac{d}{dt} |\gamma(t)|^2 = 0 \implies \frac{d}{dt} (\langle \gamma(t), \gamma(t) \rangle) = \langle \gamma'(t), \gamma(t) \rangle + \langle \gamma(t), \gamma'(t) \rangle = 0$$

i.e.  $\langle \gamma(t), \gamma'(t) \rangle = 0$  as desired. □

Given a regular curve  $\gamma : I \rightarrow \mathbb{R}^n$ , we may compute the velocity and acceleration as  $\gamma'(t), \gamma''(t)$ , which are denoted  $v(t), a(t)$  respectively. In particular, we may write

$$a(t) = a^{\parallel}(t) + a^{\perp}(t),$$

with  $a^{\parallel}(t)$  being the tangential acceleration, and  $a^{\perp}(t)$  being the normal acceleration. We may find these by projecting  $a(t)$  into the subspace  $\text{span}\{v\}$  (i.e., the span of the velocity vector).

## §2 Day 2: Curvature of a Curve (Jan. 9, 2025)

**Definition 2.1.** Suppose that  $\gamma : I \rightarrow \mathbb{R}^n$  is a regular curve. A *reparameterization* of  $\gamma$  is a function of the form  $\tilde{\gamma} = \gamma \circ \phi : \tilde{I} \rightarrow \mathbb{R}^n$ , where  $\tilde{I}$  is an interval, and  $\phi : \tilde{I} \rightarrow I$  is a smooth bijection with nowhere vanishing derivative  $\phi'(t) \neq 0$  for all  $t \in \tilde{I}$ .

**Definition 2.2.** We say that  $\gamma, \tilde{\gamma}$  have the same orientation (i.e., the parameterization is *orientation-preserving*) if  $\phi' > 0$ , and orientation-reversing if  $\phi' < 0$ .

Note that since  $\phi$  is smooth, it is impossible for  $\phi'$  to have places on which it is greater than 0 and less than 0, since IVT holds on  $\tilde{I}$  and  $\phi'(t) \neq 0$ .

We now present an example;

$$\begin{aligned}\gamma : [0, 1] &\rightarrow \mathbb{R}^3 \text{ with } \gamma(t) = (t, t), \\ \beta : [0, \pi/2] &\rightarrow \mathbb{R}^3 \text{ with } \beta(t) = (\sin t, \sin t).\end{aligned}$$

While these both parameterize the same curve, we prefer  $\gamma$  greatly because  $|\gamma'(t)| = \sqrt{2}$  while  $|\beta'(t)| = \cos t$ ; specifically, constant speed parameterizations are much nicer to deal with. Even better,

**Definition 2.3.** A curve  $\gamma$  is said to be parameterized by arclength if  $|\gamma'(t)| = 1$ .

In particular, we may turn our above  $\gamma : t \mapsto (t, t)$  into an arclength parameterization by considering it to be  $t \mapsto (\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}})$  instead, since we would then have

$$|\phi'(t)| = \sqrt{2 \cdot \left(\frac{1}{\sqrt{2}}\right)^2} = 1. \quad \square$$

We now introduce the idea of curvatures of curves. A few ideas first:

- A smaller circle has *larger* curvature (naturally, it is more curved than a large circle).
- The curvature of a curve is given by a function  $\kappa : I \rightarrow [0, \infty)$ ; i.e., it cannot be negative. Specifically, it is given by

$$\kappa(t) = \frac{|a^\perp(t)|}{|v|^2}.$$

Curvature enjoys two main properties;  $\kappa$  is independent of the parameterization of the curve, and that  $\kappa = \frac{1}{r}$ , where  $r$  is the radius of the circle that approximates the curve at the particular point. If  $\gamma$  is parameterized by arclength, then  $\kappa(t) = a(t)$ . To see this, observe that

$$\kappa(t) = \frac{|a^\perp(t)|}{|v|^2} = |a^\perp(t)| = |a(t)|.$$

In particular, since  $|v(t)|$  is constant, so is  $|v(t)|^2$ . Then

$$\frac{d}{dt} \langle v(t), v(t) \rangle = 0 \implies \langle v'(t), v(t) \rangle = 0 \implies \langle a(t), v(t) \rangle = 0.$$

From this, along with  $a(t) = a^\perp(t) + a^\parallel(t)$ , we have that  $a^\parallel(t) = 0$ .<sup>1</sup>  $\square$

<sup>1</sup>for more detail, check proposition 1.18 in the textbook;  $\gamma'$  and  $\gamma''$  are perpendicular if  $\gamma$  is a curve with constant speed.

**Definition 2.4.** Let  $\gamma : I \rightarrow \mathbb{R}^n$  be a regular curve. We define the unit tangent vector and unit normal vector as

$$T(t) = \frac{v(t)}{|v(t)|}; \quad n(t) = \frac{a^\perp(t)}{|a^\perp(t)|},$$

respectively.

Note that the textbook uses  $\mathbf{t}, \mathbf{n}$  respectively, but these are hard to write on paper.

**Definition 2.5** (Osculating Plane). At a fixed point on a regular curve with  $\kappa \neq 0$ , we define the *osculating plane* by

$$\text{span}\{T, n\},$$

and we define the *osculating circle* to be the unique circle with these properties:

- (a) Radius  $\frac{1}{\kappa}$ ,
- (b) It is in the osculating plane,
- (c) It is centered at 0.

### §3 Day 3: Angle Function and Rotation Index (Jan. 13, 2025)

We begin with a few questions.

- (a) Why is  $\kappa_s(t)$  only defined for planar curves and not in  $\mathbb{R}^n$  for  $n > 2$ ? In  $\mathbb{R}^2$ , the orthogonal vector is 1-dimensional; in higher dimensions, it is  $n - 1$ -dimensional, which does not make sense to apply a 90-degree rotation to; i.e., the notions of “clockwise” and “counter-clockwise” rotation are not that well defined in  $\mathbb{R}^n$ .
- (b) How can the angle function  $\theta(t)$  increase by more than  $2\pi$  along a curve? This is possible by having multiple loops; check figure 1.23, on page 37 in the textbook.
- (c) What is the geometric meaning behind the equation  $\theta'(t) = \kappa_s(t)$ ? Intuitively,  $\theta'$  measures the change in the angle measuring how much the curve deviates from a straight line.
- (d) Why is the rotation index of a unit-speed closed plane curve always an integer? Since the curve is closed and smooth,  $\theta$  has an equal evaluation at the beginning and end of the interval the curve is parameterized on, with a difference of a multiple of  $2\pi$  per definition; thus, we have that

$$\frac{\theta(b) - \theta(a)}{2\pi}$$

evaluates out to an integer always.

An example graph of  $\theta(t)$  for curve  $\gamma$  was given in class; we record the answers here, but not the figure.

- (a) The rotation index is  $\frac{1}{4}$ .
- (b) The curve is not closed, since the rotation index is not an integer.
- (c) Clockwise means negative  $\theta$ , and counterclockwise means positive  $\theta$ .
- (d) The maximum curvature,  $\max |\theta'(t)|$ , is approximately  $6\pi$  (which is a guess).

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  be a simple closed curve. Let  $C = \gamma([a, b])$  denote its trace.

**Theorem 3.1** (Hopf’s Umlaufsatz). The rotation index of  $\gamma$  is either  $-1$  or  $1$ .

**Theorem 3.2** (Jordan Curve Theorem).  $\mathbb{R}^2 \setminus C = \{p \in \mathbb{R}^2 \mid p \notin C\}$  has exactly two path connected components. Their common boundary is  $C$ . One component (which we will call the interior) is bounded, while the other (which we call the exterior) is unbounded.

**Definition 3.3.** In the above context, we say that  $\gamma$  is positively oriented if the rotation index of  $\gamma$  is  $1$ , and negatively oriented if the index is  $-1$ .

**Definition 3.4.** A *piecewise regular curve* in  $\mathbb{R}^n$  is a continuous function  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  with partition  $a = t_0 < \dots < t_n = b$  such that each  $\gamma|_{[t_i, t_{i+1}]}$  is a regular curve. We call the points  $\gamma(t_i)$  for  $i = 1, \dots, n - 1$  the “corners” of  $\gamma$ .

**Definition 3.5.** In the above context, the *signed angle* at  $\gamma(t_i)$  denoted by  $\alpha_i \in [-\pi, \pi]$  is given by the angle between  $v^-(t_i)$  and  $v^+(t_i)$  with  $\alpha_i > 0$  for a counterclockwise and  $\alpha_i < 0$  for a clockwise turn.

**Theorem 3.6** (Generalized Hopf's Umlaufsatz). Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be a unit-speed positively oriented piecewise regular simple closed plane curve. Let  $\kappa_s$  denote its signed curvature function, and let  $(\alpha_i)_i$  be the list of signed angles at its corners. Then

$$\int_a^b \kappa_s(t) dt + \sum \alpha_i = 2\pi.$$

We now move onto space curves. Note that  $\mathbb{R}^2$  may be embedded into  $\mathbb{R}^3$  by considering  $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ .

**Definition 3.7.** Consider a regular space curve  $\gamma : I \rightarrow \mathbb{R}^3$  and  $t \in I$  such that  $\kappa(t) \neq 0$ . The *Frenet frame* at  $t$  is the orthonormal basis  $\{T, n, B\}$  of  $\mathbb{R}^3$  defined by

$$\begin{aligned} T(t) &= \frac{v(t)}{|v(t)|}, \\ n(t) &= \frac{a^\perp(t)}{|a^\perp(t)|}, \\ B(t) &= T(t) \times n(t). \end{aligned}$$

In particular,  $|B'|$  is a natural choice to measure the changing tilt of the osculating plane because  $B$  is constant length, and that it is orthogonal to both  $B$  and  $T$ . Moreover,  $\langle B', n \rangle$  and  $|B'|$  are related as follows,

$$\langle B', n \rangle = |B'| |n| \cos \varphi = \pm |B'|,$$

where  $\varphi$  is the angle between  $B$  and  $n$  (note that this is either 0 or  $\pi$ , since  $B \parallel n$ ).

**Definition 3.8.** Consider a regular space curve  $\gamma : I \rightarrow \mathbb{R}^3$  and  $t \in I$  such that  $\kappa(t) \neq 0$ . The torsion of  $\gamma$  at  $t$  is given by

$$\tau(t) = \frac{-\langle B'(t), n(t) \rangle}{|v(t)|}.$$

The torsion is independent of parameterization.

As an example, if  $\gamma : I \rightarrow \mathbb{R}^3$  is such that for all  $t \in I$ ,  $\kappa(t) \neq 0$  and  $\tau(t) = 0$ , we have that  $|B'| = 0$ ,  $B$  is constant, and so  $\text{span}\{T, u\} = P$ . In particular, this means that  $\gamma$  is planar; i.e., for all  $t \in I$ ,  $\gamma(t) \in \gamma(t_0) + P$ .

**Theorem 3.9** (Frenet Equations). For a regular curve  $\gamma : I \rightarrow \mathbb{R}^3$ , the following equations hold whenever  $\kappa(t) \neq 0$ :

$$\begin{aligned} T' &= |v| \kappa n \\ n' &= |v| [-\kappa T + \tau B] \\ B' &= -|v| \tau n. \end{aligned}$$

In matrix form, this is written as

$$\begin{pmatrix} T \\ n \\ B \end{pmatrix}' = |v| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ B \end{pmatrix}.$$

Next class, we will prove the above; in particular, there exists a unique solution by the Picard-Lindelöf theorem. We did not cover Taylor approximations because we ran out of time.

## §4 Day 4: Rigid Motions (Jan. 16, 2025)

We start with a quick refresher of linear algebra properties;

**Definition 4.1.** A matrix  $A \in \mathbb{R}^{n \times n}$  is said to be *orthogonal* if any of the following equivalent conditions are fulfilled,

- (a) For all  $p \in \mathbb{R}^n$ ,  $|Ap| = |p|$ , i.e.,  $A$  is norm-preserving.
- (b) For all  $p, q \in \mathbb{R}^n$ ,  $\langle Ap, Aq \rangle = \langle p, q \rangle$ , i.e., inner product preserving.
- (c)  $A$  sends an orthonormal basis of  $\mathbb{R}^n$  to an orthonormal basis of  $\mathbb{R}^n$ .
- (d) The columns of  $A$  are an orthonormal basis of  $\mathbb{R}^n$ .
- (e)  $A^\top A = I$ .

In particular, if  $A$  is orthogonal, then  $\det A = \pm 1$ , and its inverse is also orthogonal; the product of two orthogonal matrices is also orthogonal.

**Definition 4.2.** A *rigid motion* of  $\mathbb{R}^n$  means a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$|f(p) - f(q)| = |p - q|$$

for all  $p, q \in \mathbb{R}^n$ .

**Definition 4.3.** For  $A \in \mathbb{R}^{n \times n}$ , we formally denote the associated linear map  $L_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . For  $q \in \mathbb{R}^n$ , we define the translation  $T_q : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $T_q(p) = p + q$ . Note that translations are rigid motions.

**Theorem 4.4.** If  $f$  is a rigid motion of  $\mathbb{R}^n$ , then  $f = T_q \circ L_A$  for a unique choice of  $q$  and  $A$ .

To prove this, observe that if we let  $q := f(0)$  and define  $g := (T_q)^{-1} \circ f$ , then we may note that  $g$  is a rigid motion with  $g(0) = 0$  by our choice of  $q$ , and that for any  $v \in \mathbb{R}^n$ ,  $|g(v)| = |g(v) - g(0)| = |v - 0| = |v|$  as desired. Thus,  $g$  preserves all norms, and so  $g = L_A$  for some orthogonal matrix  $A$ , and we have that  $f = T_q \circ L_A$  as desired.  $\square$

We leave the proof of uniqueness as a quick exercise.

**Definition 4.5.** A rigid motion  $f = T_q \circ L_A$  is said to be *proper* if  $\det A = 1$  and *improper* if  $\det A = -1$ .

We are now able to answer one of the recurring questions in this class: if we have an object  $G$ , with geometric property  $P$ , and a map  $f$  of type  $T$ , does  $f(G)$  still have the same property  $P$ , or can we at least predict some properties of it?

**Theorem 4.6.** A *proper* rigid motion preserves the curvature, torsion, and signed curvature of a curve, space curve, and plane curve respectively. An *improper* rigid motion preserves curvature, but multiplies torsion and signed curvature by  $-1$ .

For simplicity, we prove that this holds for curvatures. Let  $\gamma : I \rightarrow \mathbb{R}^n$  be smooth, and let  $A \in \mathbb{R}^{n \times n}$  and its associated linear map  $L_A$ . Then

$$(L_A \circ \gamma)' = (dL_A \circ \gamma) \cdot \gamma' = A\gamma',$$

since the differential of  $L_A$  is just itself. Without loss of generality, let  $|\gamma'| = 1$ , and let  $\bar{\gamma} = f \circ \gamma$ . Then

$$\bar{\kappa} = |\bar{a}| = |(\bar{v})'| = |((f \circ \gamma)')'| = |((T_q \circ L_A \circ \gamma)')'| = |((L_A \circ \gamma)')'| = |A \circ \gamma''|,$$

and so  $|\gamma''| = |a| + \kappa$ .  $\square$



**Theorem 4.7.** If  $I \subset \mathbb{R}$  is an interval and  $\kappa_s : I \rightarrow \mathbb{R}$  is smooth, then there exists a unit-speed plane curve  $\gamma : I \rightarrow \mathbb{R}^2$  with signed curvature  $\kappa_s$ .  $\gamma$  is unique up to proper rigid motion.

Fix  $t_0 \in I$ . Define  $\theta(t) := \int_{t_0}^t \kappa_s(u) du$ , and define  $v(t) = (\cos \theta(t), \sin \theta(t))$ . Then we may define  $\gamma(t) := \int_{t_0}^t v(u) du$ . Then  $\gamma(t)$  has signed curvature  $\kappa_s$ . We will check uniqueness next class. Also, something something about Frenet equations.