MAT357 Lecture Notes

ARKY!! :3C

'26 Winter Semester

Contents

§1 Day 1: Recap of Preliminaries (Jan. 6, 2026)

We give a review of the necessary definitions.

Theorem 1.1. $\mathbb{R}$ has the least upper bound property.

Specifically, for all subsets $S \subset \mathbb{R}$, we say M is an upper bound for S if, for all $x \in S$, we have $x \leq M$. Any non-empty $S \subset \mathbb{R}$ that is bounded above has a least upper bound; i.e., there exists M such that $x \leq M$ for all $x \in S$, and if $\hat{M}$ is another upper bound for S, then $M \leq \hat{M}$. Note that M is unique.

Theorem 1.2 (Archimedean Property)**.** If $a, b \in \mathbb{R}$ such that $a < b$, then there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Proof. Assume $0 < a < b$; let $M \in \mathbb{N}$ such that $M > \frac{1}{b-a}$, i.e., $M(b-a) > 1$. Let N be the largest natural number such that $N \leq Ma$; then $q = \frac{N+1}{M}$ satisfies the above, i.e.,

$$N + 1 > Ma \implies a < \frac{N+1}{M}, \quad Mb > Ma + 1 \geq N + 1 \implies b > \frac{N+1}{M}. \qquad \square$$

We now discuss sequences. We say (a_n) converges to $a \in \mathbb{R}$ if, for all $\varepsilon > 0$, there exists N such that for all $n \geq N$, we have $|a - a_n| < \varepsilon$. In this case, we write $\lim_{n\to\infty} a_n = a$, or $a_n \to a$ as $n \to \infty$.

Definition 1.3. (a_n) is said to be *Cauchy* if, for all $\varepsilon > 0$, there exists $N > 0$ such that, for all indices $n, m \geq N$, we have $|a_n - a_m| < \varepsilon$.

Theorem 1.4. A sequence converges to some $a \in \mathbb{R}$ if and only if it is Cauchy.

Proof. The proof for convergent implies Cauchy is immediate, so it is left as an exercise. For the other direction, suppose (a_n) is Cauchy; we start by claiming that the sequence is bounded. Indeed, let $\varepsilon = 1$; then there exists some N such that $|a_n - a_m| \leq 1$ for all $n, m \geq N$, and so $|a_n| \leq |a_N| + 1$ for all $n \geq N$, whence the tail is bounded.

Now, consider the set B to be the set of all $x \in \mathbb{R}$ for which there are infinitely many n such that $a_n \geq x$. Then B is nonempty and bounded above; let $a = \sup B$. Take $\varepsilon > 0$ and N large enough such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$, and that for all $n \geq N$, we have $a_n \leq a + \varepsilon$. In this manner, we must have $a - \varepsilon \in B$, so we can choose $M \geq N$ such that $a_m \geq a - \varepsilon$. If $n \geq m$, then $|a - a_n| \leq |a_n - a_m| + |a_m - a| \leq 2\varepsilon$. $\square$

Corollary 1.5. $\mathbb{R}$ is complete.

Definition 1.6. A metric space is a set M equipped with a function $d : M \times M \to \mathbb{R}$ (which we will call a metric) such that

(i) (*Non-negativity*) $d(x, y) \geq 0$; we have equality, i.e., $d(x, y) = 0$ if and only if $x = y$.

(ii) (*Symmetry*) $d(x, y) = d(y, x)$ for all $x, y \in M$.

(iii) (*Triangle inequality*) $d(x, z) \leq d(x, y) + d(y, z)$.

For some examples, consider

(i) $M = \mathbb{R}$, $d(x, y) = |x - y|$

(ii) If $M = \mathbb{R}^n$, then $d_1(x,y) = \sum_{i=1}^n |x_i - y_i|$ is the taxicab norm; we also have

$$d_2(x,y) = \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{1/2}$$

is the standard L^2-norm.

(iii) $\ell^1 := \{(a_n) \mid \sum_{n=1}^\infty |a_n| < \infty\}$; then $d_1(x,y) = \sum_{i=1}^\infty |x_i - y_i|$ is a normed vector space with $\|a\| = \sum_{n=1}^\infty |a_n|$.

(iv) $C([0,1])$ is the space of continuous functions $f : [0,1] \to \mathbb{R}$ with metric

$$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

It is a normed vector space with norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$.

Definition 1.7. A normed vector space is a vector space V equipped with a norm $\|\cdot\| : V \to \mathbb{R}$ such that

(i) $\|v\| \geq 0$ for all $v \in V$. $\|v\| = 0$ if and only if $v = 0$.

(ii) $\|\lambda v\| = |\lambda| \|v\|$

(iii) $\|v + w\| \leq \|v\| + \|w\|$.

If $(V, \|\cdot\|)$ is a normed vector space, then (V, d) is a metric space with $d(v,w) = \|v - w\|$.