

MAT327 Lecture Notes

ARKY!! :3C

'24 Fall Semester

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§1 Day 1: Open Sets and Continuity (Sep. 3, 2024)

This class is *MAT327*; 3 meaning third year, 2 meaning the contents are on the fundamental side, and 7 meaning no mercy.

– Dror Bar-Natan

Course administration matters first;

- The course link is given [here](#) (this will link straight to Quercus).
- The textbook is [James Munkres' Topology](#) (online PDF: [ETH Zurich mirror](#)); Prof Bar-Natan strongly recommends a paper copy, though (since people get distracted on the computer).

Today's readings in the textbook is on Ch. 1, sections 1 to 8, and Ch. 2, sections 12 to 13. Readings are supplementary to lecture material¹. The goal of this course is to understand continuity in its most general form; in particular,

- In MAT157, we studied continuity in $f : \mathbb{R} \rightarrow \mathbb{R}$;
- In MAT257, we will study continuity in $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$;

but in this class, we will study continuity in $F : X \rightarrow Y$, where X, Y are arbitrary spaces, such as (but not limited to) \mathbb{R}^n , $\mathbb{R}^{\mathbb{N}}$, $\{0, 1\}^{\mathbb{N}}$ (binary sequences), and so on. We start with some refreshers on previous coursework;

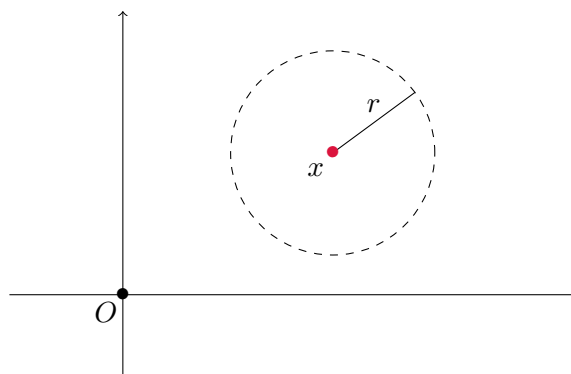
Definition 1.1 (Continuity in $\mathbb{R}^n \rightarrow \mathbb{R}^m$). A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called “continuous” if it is continuous at all points in \mathbb{R}^n . Specifically, for all $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

With this, we state our main theorem for today (proof given later),

Theorem 1.2 (Continuity on \mathbb{R}^n if and only pre-image of open subsets is open). A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous if and only if all open subsets $U \in \mathbb{R}^m$ have $f^{-1}(U)$ open.

In order to build up to the above, we start by defining some terms;

Definition 1.3 (Open Ball). Let $r > 0$, and $x \in \mathbb{R}^n$. An open ball of radius r about x is given formally by $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Visually, we have



Note that the border of the circle is not in $B_r(x)$, as the distance metric asks for a strict inequality.

¹irc it won't be tested unless specified. its still good to learn tho

In a similar fashion, we have

Definition 1.4 (Open Set). A set $U \subset \mathbb{R}^n$ is called *open* if, for all $x \in U$, there exists an open ball about x contained in U . Specifically, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

Intuitively, we could say that the set U does not contain its edge; if it did, let x be on said edge; then we would not be able to fit an open ball about x in U . Here are some examples of open sets from lecture;

1. The whole set $U = \mathbb{R}^n$ is open.
2. The empty set \emptyset is open. Since there does not exist any $x \in \emptyset$, no conditions on balls need to be satisfied; “every dog in the empty set of dogs is green”.
3. $(0, 1)$ is open on \mathbb{R}^1 (and any open interval, for that matter).
4. $B_r(x) \subset \mathbb{R}^n$ is open. To see this, observe that for any $y \in B_r(x)$, we may pick $\varepsilon < r - |x - y|$; by triangle inequality, all elements in $B_\varepsilon(y)$ must also be in $B_r(x)$.

We now define images and pre-images; let us have a function between sets $f : X \rightarrow Y$. For subsets $A \subset X$ and $B \subset Y$,

$$f(A) = \{f(a) \mid a \in A\},$$

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Since f need not be injective, f^{-1} does not necessarily exist. While images of a union of subsets is a union of the images, the same is not true for intersections; let A_1, \dots, A_n be subsets of X ; then

$$f\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f(A_i), \quad f\left(\bigcap_{i=1}^n A_i\right) \subset \bigcap_{i=1}^n f(A_i).$$

On the other hand, pre-images preserve both union and intersection; let $B_1, \dots, B_n \subset Y$; then

$$f^{-1}\left(\bigcup_{i=1}^n B_i\right) = \bigcup_{i=1}^n f^{-1}(B_i), \quad f^{-1}\left(\bigcap_{i=1}^n B_i\right) = \bigcap_{i=1}^n f^{-1}(B_i).$$

As for set complements, we have

$$f^{-1}(B^C) = f^{-1}(B)^C,$$

but the same cannot be said for images; $f(A)^C \subset f(A^C)$ for surjective f , and vice versa for injective f . Equality occurs only if f is a bijection.

Note: for the proof of Theorem 1.2, I'll leave it out in case Prof. Bar-Natan proves it later on Thursday.

§2 Day 2: Basic Definitions and Topological Spaces (Sep. 5, 2024)

We start by recapping the previous lecture; we introduced

- Open sets $U \subset \mathbb{R}^n$, where there exists an open ball of radius $\varepsilon > 0$ about any $x \in U$.
- Continuity in $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Recall the continuity property from last class, that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous if and only if the pre-image of an open set U is open.

(\Rightarrow) To start, pick any $x_0 \in f^{-1}(U)$, and let us have a small enough $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset U$ (this is possible since U is open). By continuity, there exists $\delta > 0$ such that any $x \in B_\delta(x_0)$ satisfies $f(x) \in B_\varepsilon(f(x_0)) \subset U$; this means $x \in f^{-1}(U)$ by definition of pre-image, and since x was arbitrary, we see $B_\delta(x_0) \subset f^{-1}(U)$. Moreover, since x_0 was also arbitrary, our construction shows that there always exists a δ -ball about any point in the pre-image, and so $f^{-1}(U)$ is open. \square

(\Leftarrow) For the other direction, take any $x_0 \in \mathbb{R}^n$, and $\varepsilon > 0$. Since $B_\varepsilon(f(x_0))$ is open we have that $f^{-1}(B_\varepsilon(f(x_0)))$ is open as well. This means we may pick a small enough $\delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$ (by definition of openness), and we immediately see

$$f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)).$$

This is a reconstruction of the epsilon-delta definition of continuity, and so we are done. \square

Open sets in \mathbb{R}^n have a number of properties;

- \emptyset, \mathbb{R}^n are open sets.
- The union of open sets are open; specifically, let S be a set of indices, and let $A_\alpha \subset \mathbb{R}^n$ for all $\alpha \in S$. Then

$$\bigcup_{\alpha \in S} A_\alpha$$

is an open set.

- The finite intersection of open sets are also open. Let $A_1, \dots, A_n \subset \mathbb{R}^n$ be open; then

$$\bigcap_{i=1}^n A_i$$

is open.

We now proceed with the proofs for these properties.

- This was proven last lecture.
- For any x in the union, by definition, there exists $\alpha \in S$ such that $x \in A_\alpha$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A_\alpha$, and $A_\alpha \subset \bigcup_{\alpha \in S} A_\alpha$. This concludes that the union is open.
- For any x in the finite intersection, then x is an element of each of A_1, \dots, A_n . Let us have $\varepsilon_1, \dots, \varepsilon_n > 0$ such that $B_{\varepsilon_i}(x) \subset A_i$, and take $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$ (note that minimum is defined only for finite lists), which means $B_\varepsilon(x) \subset \bigcap_{i=1}^n A_i$.

In the case of infinite intersection of open sets, the resulting set need not be open. For example, let us consider the infinite intersection of intervals

$$\bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, \frac{1}{j}\right) = \{0\},$$

which is not open.

Definition 2.1 (Topology on a Set). Let X be a set equipped with topology \mathcal{T} on X . \mathcal{T} is a collection of subsets of X , i.e. $\mathcal{T} \subset \mathcal{P}(X)$, with properties

- $\emptyset, X \in \mathcal{T}$.
- The union of subsets of \mathcal{T} is also in \mathcal{T} .
- The finite intersection of subsets of \mathcal{T} is also in \mathcal{T} .

We call (X, \mathcal{T}) a *topological space*, which may be abbreviated to X if \mathcal{T} is given or obvious. Moreover, we define another notion of openness, where $U \in \mathcal{T}$ is said to “be open relative to \mathcal{T} ,” or that “ U is open.” Here are a few examples of such topological spaces.

- (a) The standard topology on \mathbb{R}^n is given by

$$\mathbb{R}_{std}(\mathbb{R}, \mathcal{T}_{std} = \{U \subset \mathbb{R}^n \mid U \text{ is open in the “old sense”}\},$$

i.e. collection of open intervals².

- (b) The discrete topology, $X_{discrete} = (X, \mathcal{T} = \mathcal{P}(X))$ can be defined over any set X , equipped with \mathcal{T} as the collection of all subsets.
- (c) The trivial topology, $X_{trivial} = (X, \mathcal{T} = \{\emptyset, X\})$.

Definition 2.2 (Continuity between Topological Spaces). If $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are topological spaces, and $f : X \rightarrow Y$ is a function between said spaces, then we say f is continuous if, for all $U \in \mathcal{T}_Y$, we have that $f^{-1}(U) \in \mathcal{T}_X$. This draws from our definition from the notion that pre-images of open sets are open.

Now for the examples from lecture;

- (a) $f : X_{discrete} \rightarrow \mathbb{R}_{std}$ is always continuous.
- (b) $f : X_{trivial} \rightarrow \mathbb{R}_{std}$ is continuous if and only if f is constant.
- (c) $f : \mathbb{R}_{std} \rightarrow X_{trivial}$ is always continuous.
- (d) $f : \mathbb{R}_{std} \rightarrow X_{discrete}$ is almost never continuous, except when X is empty or a singleton (in which case, $X_{discrete} = X_{trivial}$). If $x_0 \in X$ yet $X \setminus \{x_0\} \neq \emptyset$, then let us have

$$\begin{aligned} A &:= f^{-1}(\{x_0\}), \\ B &:= f^{-1}(X \setminus \{x_0\}). \end{aligned}$$

While A, B are both open, we see that $A \cup B = \mathbb{R}$, and $A \cap B = \emptyset$.

²this is how i understood it, he might clarify next time?