MAT351 Lecture Notes

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Contents

§1 Day 1: Course Administrative Details (Sep. 3, 2025)

[I'll put any of my own remarks in Square Brackets or in footnotes. I love making connections to other ideas, and it feels like killing my child to exclude them just because the prof didn't mention it explicitly.]

Put "MAT351" in subject if you're emailing about the course. The syllabus tab will have the list of topics covered each week. Attendance is strongly encouraged.[1]

- Problems are in *Partial Differential Equations: An Introduction* by Walter Strauss.

- Evans PDEs will cover the topics in more detail because it is a graduate textbook.

- PDEs in Action is more applied.

Midterm 2 will be $\geq 80\%$ problems from the Problem Sets. Final exam is gonna be 2 hours rather than 3 because 3 hours is too long.

Topics:

- Intro

- Examples

- General Concepts

- Classify (There are many different kinds). By the end of the semester, when you see a PDE you should know what kind it is. Also, if you see some behavior in the wild, you can reach into your toolbox and think about which kind of PDE properly models the behavior.

We will go over all the main classes of PDEs

- First order PDEs

- Wave equation (Hyperbolic PDEs)

- Heat equation/Diffusion (Parabolic PDEs)

- Laplace/Harmonic Functions (Elliptic PDEs)

- Boundary/Initial Value Problems (Fourier Series)

- First Order Nonlinear PDEs (Shocks)

- Eigenvalues (Hydrogen Atom)

- Schrödinger and Fourier Transform (If there's time)

- Distributions and Weak Solutions[2] (If there's time)

- . . .

PDE is a vast subject. It is one of the most connected fields of math, with connections to Analysis, Number Theory, Physics, Differential Geometry, Algebra, Numberical Analysis, Experiments, and so on. The first part of the class will be more computational, finding explicit solutions to our first four PDEs mentioned above, and the second part will be more theoretical.

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[1] I'm looking at you, Isaac.
[2] This is my favourite topic in PDEs.

The rest of the lecture went over some examples of PDEs.

Example 1.1. A simple possible PDE would be as follows: Let $u = u(x, y)$ be a function of two variables. Then our PDE might be

$$u_x + u_y = 0.$$

Remark 1.2. $u_x = \frac{\partial}{\partial x} u$.

From here, a single solution would be $x - y$. We have a solution $u(x, y) = f(x - y)$ for any function f.

Remark 1.3. For ODEs, first order has one degree of freedom, second order has two degrees of freedom, etc. For PDEs, this first order PDE has infinitely many degrees of freedom.

We will also solve $u_x + c(x, y)u_y = 0$ next lecture (by method of characteristics).

Example 1.4 (Burgers' Equation)**.** We will return to first order PDEs at some point to look at nonlinear cases. One such example is

$$u_x + uu_y = 0.$$

This is a simple example used for fluid dynamics and part of why fluid dynamics is hard.

Example 1.5 (Wave Equation)**.** With the Wave Equation, there is one special coordinate, which is time. Take $u = u(t, x)$ with $t \in I \subseteq \mathbb{R}$, $x \in U \subseteq \mathbb{R}^m$, where I is an interval and U is an open set. Now, consider the $m = 1$ case. Then the wave equation is

$$u_{tt} - u_{xx} = 0.$$

For a general m, it's

$$u_{tt} - \Delta_x u = 0$$

Remark 1.6. If $f : \mathbb{R}^m \to \mathbb{R}$, then Δf is the Laplacian of f, $\frac{\partial^2}{\partial x_1^2} f + \cdots + \frac{\partial^2}{\partial x_m^2} f$. When f has two different variables ($f(t, x)$, then $\Delta_x f$ is the Laplacian with respect to x only. [An efficient way to write it is as the divergence of the gradient of a function, which will come up a bit later.]

We'll also look at

$$u_{tt} - \Delta u = f(x, t),$$

which is the non-homogeneous case.

Example 1.7 (Heat Equation)**.** The heat equation is

$$u_t - \Delta u = 0.$$

The difference between this and the Wave Equation is that this is the first derivative, whereas the Wave Equation is the second derivative. Makes a huge difference.

Example 1.8 (Laplace's Equation)**.** The Laplacian (Laplace's equation) is

$$-\Delta u = 0.$$

[Functions satisfying Laplace's Equation are sometimes called harmonic.] In this sense, Laplace's equation is the steady-state solution to the heat equation and the wave equation. Laplace's equation is very important for other fields as well. [Dirichlet's problem in the circle is just a special case of an initial value problem for Laplace's equation. We will almost certainly see this in 354. In this case, Complex Analysis turns out to be a very efficient way to solve this IVP.]

Remark 1.9. Why do we have the minus in front? One justification could be that Laplace's equation is inspired by physics. For example, in Electrostatics, if V is the electrical potential, then the electric field is given by $E = -\nabla V$. But from here, the electric charge is $\rho = \nabla \cdot E$ (up to a change in units), so it follows that $\rho = -\Delta V$.[3] It follows that in regions of space with no charge, the electrical potential should be a harmonic function.

Another more mathematical justification is that $-\Delta$ is a non-negative operator. The professor kinda trailed off while talking about this and never finished the proof, but here's what I imagine he was trying to say:

$$\langle -\Delta g, g \rangle = \int_{\mathbb{R}^m} -\Delta g \; g$$
$$= \int_{\mathbb{R}^m} \text{div}\left(-\nabla g(x)g(x)\right) + \int_{\mathbb{R}^m} |\nabla g|^2 dx$$

from here, assuming that $\nabla g \cdot g \to 0$ sufficiently quickly, it would follow (by the divergence theorem) that

$$\int_{\mathbb{R}^m} \text{div}\left(-\nabla g(x)g(x)\right) = 0,$$

and therefore that this inner product would be non-negative. That being said, he was not very precise here.

Next time we start with the formal definition of a PDE and write out Navier-Stokes.

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[3]I *told* you it would come up later.

§2 Day 2: Definitions, Examples, Solutions, Oh My! (Sep. 8, 2025)

Definition 2.1 (Partial Differential Equation). A Partial Differential Equation is an equation for an unknown function u of more variables involving its partial derivatives, in general written as

$$F\left(D^k u, D^{k-1}u, \ldots, Du, u, x\right) = 0. \tag{$*$}$$

The convention is that $D^k u$ is the kth partial derivatives of u.[4]

We'll generally look for $u : x \in U \subseteq \mathbb{R}^m \to X$, where X is typically $\mathbb{R}, \mathbb{C}, \mathbb{R}^m$. We also want $u \in C^k(U)$ (although, there are more general setups).[5] For the PDE $(*)$, we say the order is k (order will mainly be 1 or 2 in this class.) If $(*)$ can be written as

$$\mathcal{L}[u] = 0,$$

where $\mathcal{L}[u+v] = \mathcal{L}[u] + \mathcal{L}[v]$, then we call it linear. We also can include $\mathcal{L}[cu] = c\mathcal{L}[u]$, but this is usually redundant. Otherwise it's nonlinear. If $(*)$ has the form $\mathcal{L}[u] = f$ it's called (linear) non-homogeneous.

Example 2.2. The general second order pde linear: $u : (x, y) \in U \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$Q_{11}(x,y)u_{xx} + Q_{12}(x,y)u_{xy} + Q_{22}(x,y)u_{yy} + b_1(x,y)u_x + b_2(x,y)u_y + c(x,y)u = 0.$$

We say it's constant coefficient if Q_{ij}, b_k, c are constant.

Some examples from last class were constant coefficient

Example 2.3 (Wave Equation).

$$u_{tt} - u_{xx} = 0$$

Example 2.4 (Heat Equation).

$$u_t - u_{xx} = 0$$

Example 2.5 (Laplace's Equation).

$$u_{xx} + u_{yy} = 0$$

Example 2.6.

$$u_x + Q(x,y)u_y = 0$$

Some more examples of linear PDEs:

Example 2.7 (Schrödinger Equation).

$$u : (t, x) \subseteq I \times \mathbb{R}^m \to \mathbb{C}$$

where $I \subseteq \mathbb{R}$

$$iu_t - \Delta_x u = 0$$

We'll also look at $iu_t + (-\Delta + V(x))u = 0$.

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[4] I imagine something like D^k is a k-tensor analogous to—in the case of $k = 2$—the Hessian matrix. Such formalism is pointless, however.

[5] Much of 457/8 is dedicated to things that make PDEs more general

On the other hand, here are some nonlinear PDEs:

Example 2.8 (Burgers' Type)**.**

$$u_t + Q(u)u_x = 0$$

Example 2.9.

$$u_t + u_x = u^2.$$

Since the right is in terms of u and nonlinear, it's a nonlinear PDE.

These are the only two types of nonlinear PDEs we'll solve. In some sense, the first one is "more" nonlinear because we're multiplying the partial derivative by something nonlinear, whereas the second is only multiplying u by something.

Example 2.10 (Nonlinear Wave Equation)**.**

$$u_{tt} - u_{xx} = u^p,$$

This might be a good model for general relativity, as general relativity is all about the nonlinear wave equation.

Example 2.11 (Maxwell's Equations)**.** Maxwell's Equations are an example of a linear system of PDEs.[6]

$$\frac{1}{c}\vec{E}_t = \nabla \times \vec{B}$$

$$\frac{1}{c}\vec{B}_t = -\nabla \times \vec{E}.$$

Here, we take

$$\vec{E}, \vec{B} : (t, x) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3,$$

and c is the speed of light. We also have some constraints that

$$\nabla_x \cdot \vec{E} = 0$$

$$\nabla_x \cdot \vec{B} = 0,$$

and these are added for physical reasons and also to make the ODE solvable—in real life we indeed observe that the Electric field and the Magnetic field are conservative, so adding these requirements doesn't cause problems. If we don't have these, then there's too many degrees of freedom to solve. [Cf. Helmholtz Theorem]

Example 2.12 (Navier-Stokes' Equations)**.** These are a nonlinear system of PDEs.

$$\vec{u}_t + \vec{u} \cdot \nabla_x \vec{u} = -\nabla p + \nu \Delta_x \vec{u}.$$

Here, we have that $\vec{u} : (t, x) \in [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field, $\nu > 0$ is a constant which represents viscosity, we have that $p : (t, x) \to \mathbb{R}$ is pressure. We say also that

$$(\vec{u} \cdot \nabla \vec{u})_k = \sum_{j=1}^{3} \vec{u}_j \partial_j \vec{u}_k$$

and then we have

$$\nabla \cdot \vec{u} = 0$$

because otherwise there would be too many unknowns to solve it. [Another way to think about this term is that it represents consetvation of mass, so this boundary condition makes the fluid incompressible—so water, rather than air].

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[6] "Emily gave a FANTASTIC example here. Kudos to her." – Isaac

§3 Day 3: I'm kind of getting sick of only doing examples (Sep. 10, 2025)

Definition 3.1. A PDE is fully nonlinear if it is nonlinear in the highest order derivative. This is, if the PDE is of the form

$$F(D^k u, D^{k-1} u, \ldots, u, x) = 0,$$

where F is nonlinear in the $D^k u$ component.

Example 3.2.

$$u_t + u u_x = 0 \tag{1}$$
$$u_{tt} + u_x u_{xx} = 0 \tag{2}$$
$$u_t + u_x = u^2 \tag{3}$$
$$u_t^2 + u_x = 0 \tag{4}$$
$$u_t u_x + u = 0 \tag{5}$$

We see that (1) is not fully nonlinear, likewise with (2), and (3). On the other hand, (4) and (5) are fully nonlinear, because they are nonlinear in Du, and their order is 1. Finally, m

Example 3.3 (Mean Curvature)**.** This is from Differential Geometry.

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$

It is fully nonlinear as well.

We will now solve some equations "by hand".

Example 3.4. Let $u = u(x, y), (x, y) \in \mathbb{R}^2$, with

$$u_{xx} = 0.$$

We can just integrate twice to get the general solution here.

$$u_{xx} = 0$$
$$u_x = f(y)$$
$$u = x f(y) + g(y).$$

This is clearly the general solution as we just integrated twice. Now, the question is how much information do we need to provide for there to be only one solution to the IVP? For this, if we specify

$$u(0, y) = H(y) \qquad \text{and} \qquad u_x(0, y) = L(y),$$

then this would give that

$$u(x, y) = L(y)x + H(y),$$

which is sufficient.

Example 3.5 (267 Exposure Therapy)**.** "I just want to test your ODE knowledge."

$$u_{xx} + u = 0.$$

Solutions are

$$u = a(y)\sin x + b(y)\cos x,$$

He then started talking about like factoring polynomials and exponentials to find the solution to the ODE but I fainted from the experience, so I'm not sure what he did.

Remark 3.6. When drawing, functions of t and x, make t the vertical axis, and x the horizontal axis. I don't know why. Also, he writes $u(t, x)$ but the book writes $u(x, t)$. I personally prefer $u(t, x)$, but I might end up using both accidentally.

Example 3.7. Let $u = u(t, x)$ with

$$u_{tt} - u_{xx} = 0. \tag{w}$$

Then we can write this as

$$(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0.$$

If we know $(\partial_t + \partial_x)u = 0$, then u is a solution of (w), but this can be solved by $F(t - x)$, for all function F. Similarly, if $(\partial_t - \partial_x)u = 0$, then u solves (w), which would yield solutions $G(t + x)$ of (w), for all functions G.

It turns out that

$$u(t, x) = F(t - x) + G(t + x)$$

is actually the general solution—we will prove this later.

We saw that solving a PDE gives infinitely many degrees of freedom. So how do we pose a PDE so that we can identify a specific solution? Answer: Give additional info in the form of (A) initial conditions, (B) boundary conditions. This motivates the following definition:

Definition 3.8 (Well-Posed Problem)**.** We say that a Well-Posed Problem is a PDE along with Boundary Conditions and Initial Conditions such that the solutions have

1. Existence of Solutions

2. Uniqueness of Solutions

3. Continuous dependence[7] on this additional data.

Example 3.9 (Poisson's Equation)**.** Let u be the unknown, and let our equation be

$$-\Delta u = f.$$

Then there's two ways to make it well-posed. If we say that $u = u(x)$, and $x \in D \subseteq \mathbb{R}^m$, and $g : \partial D \to \mathbb{R}$, then we can either do

1. Let $u = g$, $x \in \partial D$. These are the Dirichlet Boundary Conditions [a.k.a. The Dirichlet Circle Problem]

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[7]I felt a visceral sense of dread typing this.

2. Let
$$\frac{\partial u}{\partial \hat{n}} = \nabla u \cdot \hat{n},$$
where $\hat{n}$ is the unit outward normal of ∂D. Then the Neumann boundary conditions are
$$\frac{\partial u}{\partial \hat{n}} = g,$$
where $x \in \partial D$

3. The Robin boundary conditions are
$$\frac{\partial u}{\partial \hat{n}} + Qu = g$$
where $x \in \partial D$.

For the Neumann boundary conditions we need some care to prove this is well-posed, which will appear in the homework. Giving both the Dirichlet and Neumann boundary data is not a good idea. One might remark that this seems strange—after all, it's a second-order PDE, so it might look like (as with a second-order ODE) we'd have to specify stuff about both the function and the derivatives. But we will see this is not the case.

Example 3.10 (Dirichlet's Boundary Conditions). Consider Poisson's equation along with the Dirichlet boundary conditions:

$$-\Delta u = f \qquad\qquad\qquad (x \in D)$$
$$u = g \qquad\qquad\qquad (x \in \partial D)$$

Are solutions of this problem unique? Assume there exist u_1, u_2 solutions, then look at $v = u_1 - u_2$. Then this yields

$$-\Delta v = 0 \qquad\qquad\qquad (x \in D)$$
$$v = 0 \qquad\qquad\qquad (x \in \partial D)$$

The idea is to integrate our PDE against v. First, notice that

$$\int_D \Delta v \cdot v \, dx = \int_D \sum_{i=1}^{m} v \, \partial_{x_i}^2 v \, dx$$
$$= \int_D \sum_{i=1}^{m} \partial_{x_i} \left(v \, \partial_{x_i} v \right) dx - \int_D \sum_{i=1}^{m} \partial_{x_i} v \cdot \partial_{x_i} v dx.$$

$$
\begin{aligned}
0 &= \int_D v \Delta v \, dx \\
&= \int_D \nabla \cdot (\nabla v) \, v \, dx \\
&= \int_D \nabla \cdot (\nabla v \, v) \, dx - \int_D \nabla v \cdot \nabla v \, dx \\
&= \int_{\partial D} \Delta v \cdot v \cdot \hat{n} \, dx - \int_D |\partial v|^2 dx
\end{aligned}
\qquad (1)
$$

Since $v \equiv 0$ on the boundary, $\Delta v \cdot v \cdot \hat{n} \equiv 0$ on the boundary, so it follows that $\int_D |\nabla v|^2 dx = 0$. It follows that $\nabla v \equiv 0$ on D, thus v is constant. But we know $v \equiv 0$ on ∂D so $v \equiv 0$ on the whole domain. This proves uniqueness.

Example 3.11 (Vibrating String). Let $D = [0, \infty) \times [0, L]$. A vibrating string can be modelled by the laws

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(t, 0) = w_1(t)$$
$$u(t, L) = w_2(t)$$

where $u : D \to \mathbb{R}$ [a longer derivation of these laws can be found in Stein and Shakarchi Book 1], where $w_1, w_2 : [0, \infty) \to \mathbb{R}$ are how the end-points of the string move. It turns out this is not enough to get a unique solution. We also need

$$u(0, x) = f(x), \qquad u_t(0, x) = g(x).$$

He did not show this, but he showed show in the case of Laplace's equation, these boundary conditions don't yield a well-posed problem.

Example 3.12 (Laplace's Equation). Suppose we have $u : \mathbb{H}^n \to \mathbb{R}$. Then impose the initial conditions

$$u_{xx} + u_{yy} = 0$$
$$u(x, 0) = 0$$
$$u_y(x, 0) = \frac{1}{m} \sin(mx).$$

Notice that this is analogous to providing both the Dirichlet and Neumann boundary conditions for Poisson's equations above (but it's not, because it's not bounded). This is a sequence of PDEs, which we'll call $u^{(m)}$ the solution.

As $m \to \infty$ the boundary conditions go to 0, so they roughly go to the solution $u \equiv 0$. If we solve for m large, then our solution should be close to 0 to get continuous dependence. But is this the case?

We can solve this by ODEs like we did earlier (rewriting as $(\partial_t - i\partial_x)(\partial_t + i\partial_x)u = 0$ then solving each root individually—he left actually doing this as an exercise). This gives solutions of

$$u^{(m)} = \frac{e^{-\sqrt{m}}}{m^2} \sin(mx) \sinh(my).$$

The question is whether this goes to 0 as $m \to \infty$. But notice that sinh is basically exponential, so this is going to go to infinity everywhere. Thus, in this case, continuous dependence on boundary conditions doesn't hold, so this isn't a well-posed problem.