MAT257 Lecture Notes

ARKY!! :3C

'24 Fall & '25 Winter Semester

Contents

§1 Day 1: Course Administrative Details and Equivalent Norms (Sep. 4, 2024)

Course administrative details!

- The prerequisites to this class are calculus and linear algebra; specifically, this class is about calculus on manifolds.
- The main three theorems this class will cover are the inclusive function theorem, change of variable in integral, and Stokes' theorem (generalization of FTC).
- The main techniques we will study this class are manifolds, theory of integration, and differential forms.

To give an example of how linear algebra can be used in defining differentiability, we start by recalling that for a function $f : \mathbb{R} \to \mathbb{R}$ to be differentiable in 1 variable, we need

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} =: c = f'(a)$$

to exist for any $a \in \mathbb{R}$. Specifically, this may be rewritten as

$$\lim_{h\to 0} \frac{f(a+h) - (f(a) + ch)}{h} = 0,$$

where we have $f(a) + f'(a)h$ as the best local linear approximation of $f(a+h)$ at $h = 0$, and $h \mapsto ch$ is a linear transformation $\mathbb{R} \to \mathbb{R}$. For the higher dimensional analogue of the above, let us take $f : \mathbb{R}^n \to \mathbb{R}^m$, and have

$$\begin{aligned} a &= (a_1, \ldots, a_n), \\ x &= (x_1, \ldots, x_n), \\ h &= (h_1, \ldots, h_n). \end{aligned}$$

If we let $y = f(x)$, then $y \in \mathbb{R}^m$, and we can write it as a column vector with components $y_1, \ldots, y_m$;

$$f(x) = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix}.$$

In this way, each y_i for $1 \le i \le n$ can be viewed as a function in n variables. We say f is differentiable at $a \in \mathbb{R}^n$ if there exists a linear transformation $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that[1]

$$\lim_{h\to 0} \frac{|f(a+h) - (f(a) + \lambda h)|}{|h|} = 0$$

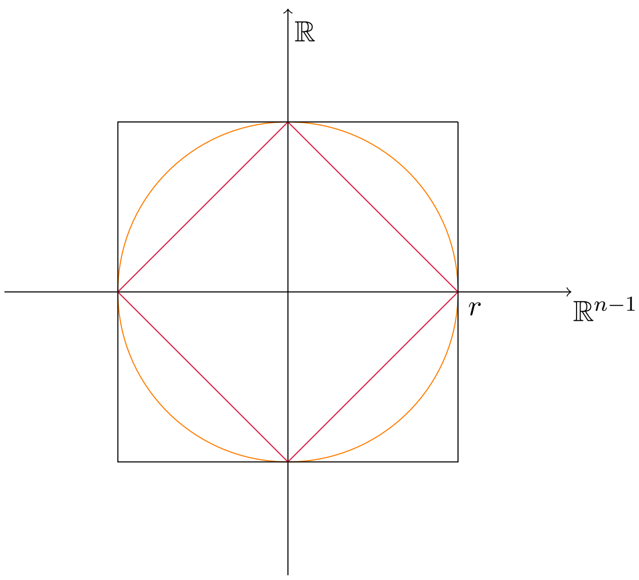
In this way, we have that $f(a) + \lambda h$ is the best linear approximation of $f(a+h)$ at $h = 0$. Notice that compared to the $\mathbb{R} \to \mathbb{R}$ differentiability condition, we impose a norm on $f(a+h) - (f(a) + \lambda h)$ and h, since we cannot divide by $\mathbb{R}^n, \mathbb{R}^m$ vectors.

[1] i think we need a norm in the numerator for the limit below; bierstone didn't do it in class though, so... idk. also lambda is jacobian?

This also brings the topic of what sort of norms we could be looking at; for example, we have[2]

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2},$$
$$\|x\| = \max\{|x_1|, \ldots, |x_n|\},$$
$$|||x||| = |x_1| + \cdots + |x_n|.$$

We say that two norms p, q are equivalent to each other if we may find a constant C where $p(x) \leq Cq(x)$ (and vice versa). In fact, the norms $|x|, \|x\|, |||x|||$ are all equivalent to each other; here is a visual proof,



where each figure is the level sets of their respective norms. In particular, $|x| \leq \sqrt{n}\,\|x\|$, and $|||x||| \leq \sqrt{n}\,|x|$. To prove the latter analytically, we may write the norm as an inner product, $|||x||| = \langle x, u(x)\rangle < |x|\,|u(x)| \leq \sqrt{n}r$, where u_i is the sign of each component of x.[3]

[2] these are (in order) euclidean, maximum, and taxicab norms; or $\ell^2, \ell^\infty, \ell^1$

[3] i kinda get where this proof is going, but at the same time i dont really know how he defined u in the first place, so yeah. alternate proof would be to give Cauchy-Schwarz

§2 Day 2: Review of MAT247, Metric Spaces (Sep. 6, 2024)

Course administrative details!

- Fall Office Hours will be held on Mondays from 10:30 to 11:30AM (likely in Bahen 6114).

We start with a review on functions and continuity. Given $\mathbb{R}^n$ with elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we define the following addition and scalar multiplication,

$$x + y = (x_1 + y_1, \ldots, x_n + y_n),$$
$$\lambda x = (\lambda x_1, \ldots, \lambda x_n), \lambda \in \mathbb{R}.$$

We may also equip it with the Euclidean inner product and norm,

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n,$$
$$|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} = \sqrt{\langle x, x \rangle},$$

of which it has the properties

(a) Non-negativity; $|x| \geq 0$, and is equal to 0 if and only if $x = 0$.

(b) Absolute Homogeneity; $|\lambda x| = |\lambda| \, |x|$ for any scalar $\lambda \in \mathbb{R}$.

(c) Triangle Inequality; $|x + y| \leq |x| + |y|$ for any vectors $x, y \in \mathbb{R}^n$.

We may check that the properties above hold for the Euclidean norm;

(a) Trivial, since the square of each component is non-negative.

(b) Not discussed in class, but we may directly expand the norm to get the equality,

$$|\lambda x| = \left(\sum_{i=1}^{n} (\lambda x_i)^2 \right)^{\frac{1}{2}} = \left(\lambda^2 \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = |\lambda| \, |x| \, .$$

(c) Start by observing that from inner product properties, we have $|\langle x, y \rangle| \leq |x| \, |y|$, with equality if and only if x, y are linearly dependent. If x, y are LI, observe $\lambda y - x \neq 0$ for all scalars $\lambda \in \mathbb{R}$, and we may write

$$\begin{aligned} 0 < |\lambda y - x|^2 &= \langle \lambda y - x, \lambda y - x \rangle \\ &= \lambda^2 \langle y, y \rangle - 2\lambda \langle x, y \rangle + \langle x, x \rangle \\ &= |y|^2 \lambda^2 - 2 \langle x, y \rangle \lambda + |x|^2 \, . \end{aligned}$$

Reading the above as a polynomial in λ, we observe that it must not have any real roots, and so the discriminant is necessarily negative, i.e. $4 \langle x, y \rangle^2 - 4 |x|^2 |y|^2 < 0$, which implies $|\langle x, y \rangle| < |x| \, |y|$. Returning to the triangle inequality, let us start by squaring both sides to obtain

$$|x + y|^2 = \langle x, x \rangle + 2 \langle x, y \rangle + \langle y, y \rangle \leq |x|^2 + 2 |x| \, |y| + |y|^2 \, .$$

This leaves us with $2 \langle x, y \rangle \leq 2 |x| \, |y|$, which, from the above, we find equality if x, y are linearly dependent, and "$<$" if LI.

Consider the linear transformation, $T : \mathbb{R}^n \to \mathbb{R}^m$, equipped with standard bases $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, and $\{f_1, \ldots, f_m\}$ of $\mathbb{R}^m$; let the matrix representation of T be given by the below matrix A,

$$y = T(x) \implies \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

In terms of basis vectors, we may visualize T by

$$T(e_i) = \sum_{j=1}^{m} a_{ji} f_j.$$

Now, suppose $S : \mathbb{R}^m \to \mathbb{R}^\ell$ be a linear transformation, with corresponding matrix representation $B \in M_{\ell\times m}(\mathbb{R})$. Then the composition $S \circ T : \mathbb{R}^n \to \mathbb{R}^\ell$ has matrix $B \cdot A \in M_{\ell \times n}(\mathbb{R})$.

Here is an example from class; if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $|Tx| \leq M\,|x|$ for some $M > 0$. To see this, let us start by writing

$$Tx = T\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i T(e_i).$$

This gives us

$$\begin{aligned} |Tx| \leq \sum_{i=1}^{n} |x_i T(e_i)| &= \sum_{i=1}^{n} |x_i|\,|T(e_i)| \\ &= \left(\sum_{i=1}^{n} a_{ji}^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}} =: |A|\,. \end{aligned}$$

Then we have

$$|Tx| \leq \left(\sum_{i=1}^{n} |x_i|\right) |A| \leq \sqrt{n}\,|A|\,|x|\,.$$

We simply pick $M \geq \sqrt{n}\,|A|$ and we are done. □

§2.1 Topology of $\mathbb{R}^n$

Let us equip $\mathbb{R}^n$ with the Euclidean norm; this is an example of a metric space (i.e., a set equipped with a distance function $d(x, y)$). The function $d : X \times X \to \mathbb{R}$ has the following properties,

(a) Symmetry, $d(x, y) = d(y, x)$.

(b) Non-negativity, $d(x, y) \geq 0$, with $d(x, y) = 0$ if $x = y$.

(c) Triangle inequality, $d(x, y) \leq d(x, z) + d(z, y)$.