MAT351 Lecture Notes

Emily

'25 Fall Semester

Contents

§1 Day 1: Course Administrative Details (Sep. 3, 2025)

[I'll put any of my own remarks in Square Brackets or in footnotes. I love making connections to other ideas, and it feels like killing my child to exclude them just because the prof didn't mention it explicitly.]

Put "MAT351" in subject if you're emailing about the course. The syllabus tab will have the list of topics covered each week. Attendance is strongly encouraged.[1]

- Problems are in *Partial Differential Equations: An Introduction* by Walter Strauss.
- Evans PDEs will cover the topics in more detail because it is a graduate textbook.
- PDEs in Action is more applied.

Midterm 2 will be $\geq$80% problems from the Problem Sets. Final exam is gonna be 2 hours rather than 3 because 3 hours is too long.

Topics:

- Intro
- Examples
- General Concepts
- Classify (There are many different kinds). By the end of the semester, when you see a PDE you should know what kind it is. Also, if you see some behavior in the wild, you can reach into your toolbox and think about which kind of PDE properly models the behavior.

We will go over all the main classes of PDEs

- First order PDEs
- Wave equation (Hyperbolic PDEs)
- Heat equation/Diffusion (Parabolic PDEs)
- Laplace/Harmonic Functions (Elliptic PDEs)
- Boundary/Initial Value Problems (Fourier Series)
- First Order Nonlinear PDEs (Shocks)
- Eigenvalues (Hydrogen Atom)
- Schrödinger and Fourier Transform (If there's time)
- Distributions and Weak Solutions[2] (If there's time)
- ...

PDE is a vast subject. It is one of the most connected fields of math, with connections to Analysis, Number Theory, Physics, Differential Geometry, Algebra, Numberical Analysis, Experiments, and so on. The first part of the class will be more computational, finding explicit solutions to our first four PDEs mentioned above, and the second part will be more theoretical.

[1] I'm looking at you, Isaac.
[2] This is my favourite topic in PDEs.

The rest of the lecture went over some examples of PDEs.

Example 1.1. A simple possible PDE would be as follows: Let $u = u(x, y)$ be a function of two variables. Then our PDE might be

$$u_x + u_y = 0.$$

Remark 1.2. $u_x = \frac{\partial}{\partial x} u$.

From here, a single solution would be $x - y$. We have a solution $u(x, y) = f(x - y)$ for any function f.

Remark 1.3. For ODEs, first order has one degree of freedom, second order has two degrees of freedom, etc. For PDEs, this first order PDE has infinitely many degrees of freedom.

We will also solve $u_x + c(x, y)u_y = 0$ next lecture (by method of characteristics).

Example 1.4 (Burgers' Equation)**.** We will return to first order PDEs at some point to look at nonlinear cases. One such example is

$$u_x + uu_y = 0.$$

This is a simple example used for fluid dynamics and part of why fluid dynamics is hard.

Example 1.5 (Wave Equation)**.** With the Wave Equation, there is one special coordinate, which is time. Take $u = u(t, x)$ with $t \in I \subseteq \mathbb{R}$, $x \in U \subseteq \mathbb{R}^m$, where I is an interval and U is an open set. Now, consider the $m = 1$ case. Then the wave equation is

$$u_{tt} - u_{xx} = 0.$$

For a general m, it's

$$u_{tt} - \Delta_x u = 0$$

Remark 1.6. If $f : \mathbb{R}^m \to \mathbb{R}$, then Δf is the Laplacian of f, $\frac{\partial^2}{\partial x_1^2} f + \cdots + \frac{\partial^2}{\partial x_m^2} f$. When f has two different variables ($f(t, x)$, then $\Delta_x f$ is the Laplacian with respect to x only. [An efficient way to write it is as the divergence of the gradient of a function, which will come up a bit later.]

We'll also look at

$$u_{tt} - \Delta u = f(x, t),$$

which is the non-homogeneous case.

Example 1.7 (Heat Equation)**.** The heat equation is

$$u_t - \Delta u = 0.$$

The difference between this and the Wave Equation is that this is the first derivative, whereas the Wave Equation is the second derivative. Makes a huge difference.

Example 1.8 (Laplace's Equation)**.** The Laplacian (Laplace's equation) is

$$-\Delta u = 0.$$

[Functions satisfying Laplace's Equation are sometimes called harmonic.] In this sense, Laplace's equation is the steady-state solution to the heat equation and the wave equation. Laplace's equation is very important for other fields as well. [Dirichlet's problem in the circle is just a special case of an initial value problem for Laplace's equation. We will almost certainly see this in 354. In this case, Complex Analysis turns out to be a very efficient way to solve this IVP.]

Remark 1.9. Why do we have the minus in front? One justification could be that Laplace's equation is inspired by physics. For example, in Electrostatics, if V is the electrical potential, then the electric field is given by $E = -\nabla V$. But from here, the electric charge is $\rho = \nabla \cdot E$ (up to a change in units), so it follows that $\rho = -\Delta V$.[3] It follows that in regions of space with no charge, the electrical potential should be a harmonic function.

Another more mathematical justification is that $-\Delta$ is a non-negative operator. The professor kinda trailed off while talking about this and never finished the proof, but here's what I imagine he was trying to say:

$$\begin{aligned}\langle -\Delta g, g\rangle &= \int_{\mathbb{R}^m} -\Delta g \; g \\ &= \int_{\mathbb{R}^m} \operatorname{div}\left(-\nabla g(x) g(x)\right) + \int_{\mathbb{R}^m} |\nabla g|^2 dx\end{aligned}$$

from here, assuming that $\nabla g \cdot g \to 0$ sufficiently quickly, it would follow (by the divergence theorem) that

$$\int_{\mathbb{R}^m} \operatorname{div}\left(-\nabla g(x) g(x)\right) = 0,$$

and therefore that this inner product would be non-negative. That being said, he was not very precise here.

Next time we start with the formal definition of a PDE and write out Navier-Stokes.

[3] I *told* you it would come up later.

§2 Day 2: Definitions, Examples, Solutions, Oh My! (Sep. 8, 2025)

Definition 2.1 (Partial Differential Equation)**.** A Partial Differential Equation is an equation for an unknown function u of more variables involving its partial derivatives, in general written as

$$F\left(D^k u, D^{k-1}u, \ldots, Du, u, x\right) = 0. \tag{$*$}$$

The convention is that $D^k u$ is the kth partial derivatives of u.[4]

We'll generally look for $u : x \in U \subseteq \mathbb{R}^m \to X$, where X is typically $\mathbb{R}, \mathbb{C}, \mathbb{R}^m$. We also want $u \in C^k(U)$ (although, there are more general setups).[5] For the PDE $(*)$, we say the order is k (order will mainly be 1 or 2 in this class.) If $(*)$ can be written as

$$\mathcal{L}[u] = 0,$$

where $\mathcal{L}[u+v] = \mathcal{L}[u] + \mathcal{L}[v]$, then we call it linear. We also can include $\mathcal{L}[cu] = c\mathcal{L}[u]$, but this is usually redundant. Otherwise it's nonlinear. If $(*)$ has the form $\mathcal{L}[u] = f$ it's called (linear) non-homogeneous.

Example 2.2. The general second order pde linear: $u : (x,y) \in U \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$Q_{11}(x,y)u_{xx} + Q_{12}(x,y)u_{xy} + Q_{22}(x,y)u_{yy} + b_1(x,y)u_x + b_2(x,y)u_y + c(x,y)u = 0.$$

We say it's constant coefficient if Q_{ij}, b_k, c are constant.

Some examples from last class were constant coefficient

Example 2.3 (Wave Equation)**.**

$$u_{tt} - u_{xx} = 0$$

Example 2.4 (Heat Equation)**.**

$$u_t - u_{xx} = 0$$

Example 2.5 (Laplace's Equation)**.**

$$u_{xx} + u_{yy} = 0$$

Example 2.6.

$$u_x + Q(x,y)u_y = 0$$

Some more examples of linear PDEs:

Example 2.7 (Schrödinger Equation)**.**

$$u : (t,x) \subseteq I \times \mathbb{R}^m \to \mathbb{C}$$

where $I \subseteq \mathbb{R}$

$$iu_t - \Delta_x u = 0$$

We'll also look at $iu_t + (-\Delta + V(x))u = 0$.

[4] I imagine something like D^k is a k-tensor analogous to—in the case of $k = 2$—the Hessian matrix. Such formalism is pointless, however.

[5] Much of 457/8 is dedicated to things that make PDEs more general

On the other hand, here are some nonlinear PDEs:

Example 2.8 (Burgers' Type)**.**

$$u_t + Q(u)u_x = 0$$

Example 2.9.

$$u_t + u_x = u^2.$$

Since the right is in terms of u and nonlinear, it's a nonlinear PDE.

These are the only two types of nonlinear PDEs we'll solve. In some sense, the first one is "more" nonlinear because we're multiplying the partial derivative by something nonlinear, whereas the second is only multiplying u by something.

Example 2.10 (Nonlinear Wave Equation)**.**

$$u_{tt} - u_{xx} = u^p,$$

This might be a good model for general relativity, as general relativity is all about the nonlinear wave equation.

Example 2.11 (Maxwell's Equations)**.** Maxwell's Equations are an example of a linear system of PDEs.[6]

$$\frac{1}{c}\vec{E}_t = \nabla \times \vec{B}$$
$$\frac{1}{c}\vec{B}_t = -\nabla \times \vec{E}.$$

Here, we take

$$\vec{E}, \vec{B} : (t, x) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3,$$

and c is the speed of light. We also have some constraints that

$$\nabla_x \cdot \vec{E} = 0$$
$$\nabla_x \cdot \vec{B} = 0,$$

and these are added for physical reasons and also to make the ODE solvable—in real life we indeed observe that the Electric field and the Magnetic field are conservative, so adding these requirements doesn't cause problems. If we don't have these, then there's too many degrees of freedom to solve. [Cf. Helmholtz Theorem]

Example 2.12 (Navier-Stokes' Equations)**.** These are a nonlinear system of PDEs.

$$\vec{u}_t + \vec{u} \cdot \nabla_x \vec{u} = -\nabla p + \nu \Delta_x \vec{u}.$$

Here, we have that $\vec{u} : (t, x) \in [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field, $\nu > 0$ is a constant which represents viscosity, we have that $p : (t, x) \to \mathbb{R}$ is pressure. We say also that

$$(\vec{u} \cdot \nabla \vec{u})_k = \sum_{j=1}^{3} \vec{u}_j \partial_j \vec{u}_k$$

and then we have

$$\nabla \cdot \vec{u} = 0$$

because otherwise there would be too many unknowns to solve it. [Another way to think about this term is that it represents consetvation of mass, so this boundary condition makes the fluid incompressible—so water, rather than air].

[6] "Emily gave a FANTASTIC example here. Kudos to her." – Isaac

§3 Day 3: I'm kind of getting sick of only doing examples (Sep. 10, 2025)

Definition 3.1. A PDE is fully nonlinear if it is nonlinear in the highest order derivative. This is, if the PDE is of the form

$$F(D^k u, D^{k-1}u, \dots, u, x) = 0,$$

where F is nonlinear in the $D^k u$ component.

Example 3.2.

$$u_t + uu_x = 0 \tag{1}$$
$$u_{tt} + u_x u_{xx} = 0 \tag{2}$$
$$u_t + u_x = u^2 \tag{3}$$
$$u_t^2 + u_x = 0 \tag{4}$$
$$u_t u_x + u = 0 \tag{5}$$

We see that (1) is not fully nonlinear, likewise with (2), and (3). On the other hand, (4) and (5) are fully nonlinear, because they are nonlinear in Du, and their order is 1. Finally, m

Example 3.3 (Mean Curvature)**.** This is from Differential Geometry.

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$

It is fully nonlinear as well.

We will now solve some equations "by hand".

Example 3.4. Let $u = u(x, y), (x, y) \in \mathbb{R}^2$, with

$$u_{xx} = 0.$$

We can just integrate twice to get the general solution here.

$$\begin{aligned} u_{xx} &= 0 \\ u_x &= f(y) \\ u &= xf(y) + g(y). \end{aligned}$$

This is clearly the general solution as we just integrated twice. Now, the question is how much information do we need to provide for there to be only one solution to the IVP? For this, if we specify

$$u(0, y) = H(y) \qquad \text{and} \qquad u_x(0, y) = L(y),$$

then this would give that

$$u(x, y) = L(y)x + H(y),$$

which is sufficient.

Example 3.5 (267 Exposure Therapy)**.** "I just want to test your ODE knowledge."

$$u_{xx} + u = 0.$$

Solutions are

$$u = a(y)\sin x + b(y)\cos x,$$

He then started talking about like factoring polynomials and exponentials to find the solution to the ODE but I fainted from the experience, so I'm not sure what he did.

Remark 3.6. When drawing, functions of t and x, make t the vertical axis, and x the horizontal axis. I don't know why. Also, he writes $u(t, x)$ but the book writes $u(x, t)$. I personally prefer $u(t, x)$, but I might end up using both accidentally.

Example 3.7. Let $u = u(t, x)$ with

$$u_{tt} - u_{xx} = 0. \tag{w}$$

Then we can write this as

$$(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0.$$

If we know $(\partial_t + \partial_x)u = 0$, then u is a solution of (w), but this can be solved by $F(t-x)$, for all function F. Similarly, if $(\partial_t - \partial_x)u = 0$, then u solves (w), which would yield solutions $G(t + x)$ of (w), for all functions G.

It turns out that

$$u(t, x) = F(t - x) + G(t + x)$$

is actually the general solution—we will prove this later.

We saw that solving a PDE gives infinitely many degrees of freedom. So how do we pose a PDE so that we can identify a specific solution? Answer: Give additional info in the form of (A) initial conditions, (B) boundary conditions. This motivates the following definition:

Definition 3.8 (Well-Posed Problem)**.** We say that a Well-Posed Problem is a PDE along with Boundary Conditions and Initial Conditions such that the solutions have

1. Existence of Solutions
2. Uniqueness of Solutions
3. Continuous dependence[7] on this additional data.

Example 3.9 (Poisson's Equation)**.** Let u be the unknown, and let our equation be

$$-\Delta u = f.$$

Then there's two ways to make it well-posed. If we say that $u = u(x)$, and $x \in D \subseteq \mathbb{R}^m$, and $g : \partial D \to \mathbb{R}$, then we can either do

1. Let $u = g$, $x \in \partial D$. These are the Dirichlet Boundary Conditions [a.k.a. The Dirichlet Circle Problem]

[7] I felt a visceral sense of dread typing this.

2. Let
$$\frac{\partial u}{\partial \hat{n}} = \nabla u \cdot \hat{n},$$
where $\hat{n}$ is the unit outward normal of ∂D. Then the Neumann boundary conditions are
$$\frac{\partial u}{\partial \hat{n}} = g,$$
where $x \in \partial D$

3. The Robin boundary conditions are
$$\frac{\partial u}{\partial \hat{n}} + Qu = g$$
where $x \in \partial D$.

For the Neumann boundary conditions we need some care to prove this is well-posed, which will appear in the homework. Giving both the Dirichlet and Neumann boundary data is not a good idea. One might remark that this seems strange—after all, it's a second-order PDE, so it might look like (as with a second-order ODE) we'd have to specify stuff about both the function and the derivatives. But we will see this is not the case.

Example 3.10 (Dirichlet's Boundary Conditions)**.** Consider Poisson's equation along with the Dirichlet boundary conditions:
$$\begin{aligned} -\Delta u &= f && (x \in D) \\ u &= g && (x \in \partial D) \end{aligned}$$

Are solutions of this problem unique? Assume there exist u_1, u_2 solutions, then look at $v = u_1 - u_2$. Then this yields
$$\begin{aligned} -\Delta v &= 0 && (x \in D) \\ v &= 0 && (x \in \partial D) \end{aligned}$$

The idea is to integrate our PDE against v. First, notice the identity that
$$\begin{aligned} \int_D \Delta v \cdot v \, dx &= \int_D \sum_{i=1}^m v \, \partial_{x_i}^2 v \, dx \\ &= \int_D \sum_{i=1}^m \partial_{x_i} (v \, \partial_{x_i} v) \, dx - \int_D \sum_{i=1}^m \partial_{x_i} v \cdot \partial_{x_i} v dx. \\ &= \int_D \nabla \cdot (v \, \nabla v) - \int_D \nabla v \cdot \nabla v \, dx. \end{aligned}$$

Using this, we can derive
$$\begin{aligned} 0 &= \int_D v \Delta v \, dx \\ &= \int_D \nabla \cdot (\nabla v) \, v \, dx \\ &= \int_D \nabla \cdot (\nabla v \, v) \, dx - \int_D \nabla v \cdot \nabla v \, dx \qquad (1) \\ &= \int_{\partial D} \Delta v \cdot v \cdot \hat{n} \, dx - \int_D |\nabla v|^2 dx \end{aligned}$$

Since $v \equiv 0$ on the boundary, $\Delta v \cdot v \cdot \hat{n} \equiv 0$ on the boundary, so it follows that $\int_D |\nabla v|^2 dx = 0$. It follows that $\nabla v \equiv 0$ on D, thus v is constant. But we know $v \equiv 0$ on ∂D so $v \equiv 0$ on the whole domain. This proves uniqueness.

Example 3.11 (Vibrating String)**.** Let $D = [0, \infty) \times [0, L]$. A vibrating string can be modelled by the laws

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(t,0) &= w_1(t) \\ u(t,L) &= w_2(t) \end{aligned}$$

where $u : D \to \mathbb{R}$ [a longer derivation of these laws can be found in Stein and Shakarchi Book 1], where $w_1, w_2 : [0, \infty) \to \mathbb{R}$ are how the end-points of the string move. It turns out this is not enough to get a unique solution. We also need

$$u(0,x) = f(x), \qquad u_t(0,x) = g(x).$$

He did not show this, but he showed show in the case of Laplace's equation, these boundary conditions don't yield a well-posed problem.

Example 3.12 (Laplace's Equation)**.** Suppose we have $u : \mathbb{H}^n \to \mathbb{R}$. Then impose the initial conditions

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x,0) &= 0 \\ u_y(x,0) &= \frac{1}{m}\sin(mx). \end{aligned}$$

Notice that this is analogous to providing both the Dirichlet and Neumann boundary conditions for Poisson's equations above (but it's not, because it's not bounded). This is a sequence of PDEs, which we'll call $u^{(m)}$ the solution.

As $m \to \infty$ the boundary conditions go to 0, so they roughly go to the solution $u \equiv 0$. If we solve for m large, then our solution should be close to 0 to get continuous dependence. But is this the case?

We can solve this by ODEs like we did earlier (rewriting as $(\partial_t - i\partial_x)(\partial_t + i\partial_x)u = 0$ then solving each root individually—he left actually doing this as an exercise). This gives solutions of

$$u^{(m)} = \frac{e^{-\sqrt{m}}}{m^2}\sin(mx)\sinh(my).$$

The question is whether this goes to 0 as $m \to \infty$. But notice that sinh is basically exponential, so this is going to go to infinity everywhere. Thus, in this case, continuous dependence on boundary conditions doesn't hold, so this isn't a well-posed problem.

§4 Day 4: Solving some linear PDEs (Sep. 15, 2025)

Well-Posed Problems have three parts

- Existence, which we discussed in the last class, and non-existence, which we saw in the homework
- Uniqueness, which we saw examples and a proof of last class
- Continuous dependence on boundary data, which we saw a counterexample of last week for the Laplace problem.

We will mostly study well-posed problems almost all the time. We'll have to prove that we have well-posedness for a given problem (but we might have to leave this for another class if it's too hard). [Also, one can ask what's called a regularity question—this won't come up in this class, but it's about when we can prove a weak solution is continuous or differentiable or smooth or etc. This can be defined precisely using methods of 436/457/458.]

First order PDEs (we are doing $\mathbb{R}^2$ for now) can be classified in a couple different ways.

1. Linear homogeneous, with constant coefficients, which is of the form
$$au_x + bu_y + cu = 0,$$
2. Linear homogeneous with non-constant coefficients, which is of the form
$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0$$
3. Linear non-homogeneous, which is of the form
$$a(x,y)u_x + b(x,y)u_y = f(x,y)$$
4. Semilinear, of the form
$$au_x + bu_y = f(u),$$
where f is nonlinear.
5. Quasilinear, which we will study rigorously later. An example is
$$u_x + uu_y = 0.$$
[This is where the regularity problem I discussed above can pop up.]

We will now solve these.

1. Previously, we mentioned that the solutions to $au_x + bu_y = 0$ are of the form
$$u(x,y) = F(bx - ay).$$
However, we will do this more rigorously now. We will add additional data (boundary/initial conditions) later—working backwards in some sense. There are three methods we can employ to solve this problem.

a) The geometric approach. This PDE says $\nabla u \cdot (a, b) = 0$. In other words, $\nabla u \perp (a, b)$. It follows that starting at any point, our function must be constant in the direction of (a, b). As such, to make the problem well posed, we just need to provide initial conditions along any (smooth) curve which is never tangent to the level sets. These level sets are called coordinate curves.

   One can deduce that $u(x, y) = F(bx - ay)$ is the general solution from this reasoning, but he did not do this explicitly.

   Example 4.1.
$$\begin{cases} au_x + bu_y = 0 \\ u(x, 0) = g(x) \end{cases}$$

   We know $u(x, y) = F(bx - ay)$, so
$$g(x) = u(x, 0) = F(bx),$$

   thus we have $F(x) = g\left(\frac{x}{b}\right)$, so
$$u(x, y) = g\left(x - \frac{a}{b}y\right).$$

b) Change coordinates—this method is very important in any PDE, as you might be able to reduce the dimensionality of your problem if you are clever. We will do this next time.

c) The general approach is the method of characteristics, which we will use later for semilinear, quasilinear, etc. We will also do this next time.

§5 Day 5: More on First-Order Equations (Sep. 17, 2025)

Problem set 2 due next Thursday, as the TAs are going to have office hours on Wednesday.

(1) We solved $au_x + bu_y = 0$ for $u = u(x, y)$, a, b constants by the geometric method last time.

(2) Let's use change of variables. Set $u = u(x, y)$ and set $v(x', y') = u(x, y)$, where x', y' represent a change of coordinates from x, y. Differentiating $u(x, y)$ yields

$$\begin{aligned} u_x &= v_{x'}\frac{dx'}{dx} + v_{y'}\frac{dy'}{dx} \\ u_y &= v_{x'}\frac{dx'}{dy} + v_{y'}\frac{dy'}{dy}. \end{aligned}$$

From here, set

$$\begin{cases} x' = ax + by \\ y' = bx - ay \end{cases}$$

Then we have that

$$\begin{cases} u_x = av_{x'} + bv_{y'} \\ u_y = bv_{x'} - av_{y'} \end{cases}$$

so plugging into our PDE yields

$$a(av_{x'} + bv_{y'}) + b(bv_{x'} - av_{y'}) = 0,$$

and then simplifying,

$$a^2 v_{x'} + b^2 v_{x'} = 0.$$

But if $a^2 + b^2 = 0$, then our PDE is quite boring, so assuming it is positive, this implies $v_{x'} = 0$. From here, we get that v is constant with respect to x', so

$$v = f(y') = f(bx - ay) = u(x, y),$$

which solves the PDE.

(3) The book *mentions* this method, but not really systematically, only briefly.[8] It's a more general method. Let's look at curves in the plane $\gamma(s) = (x(s), y(s))$ in parametric form.[9] Look at u along this curve. We define

$$z(s) = u(x(s), y(s)),$$

then differentiate along this curve

$$\dot{z}(s) = \dot{x}(s)u_x(x(s), y(s)) + \dot{y}(s)u_y(x(y), y(s)).$$

Using the original ODE, we can get $\dot{z} \equiv 0$ if $\dot{x} \equiv a$ and $\dot{y} \equiv b$. If we can merely deduce what all the curves satisfying the system

$$\begin{cases} \dot{x} \equiv a \\ \dot{y} \equiv b \end{cases},$$

[8] See Evans 3.2 if you want the systematic development. Fair warning, Evans is not for the faint of heart.

[9] It gets a bit hard to follow, because he switches between x and y being functions, and being independent variables at times.

then we will have curves along which $\dot{z} \equiv 0$, so u is constant along these curves. We call these curves the characteristic curves of our PDE, and these equations are the characteristic ODEs. The solutions to these ODEs are

$$(x(s), y(s)) = (as + x(0), bs + y(0)),$$

which implies that the solution can be determined by values along all curves of this form. This is equivalent to the general solution provided by the other two methods.

Now, let's suppose that we have data on the x-axis. Suppose $y(0) = 0$. Then we have that

$$u(as + x(0), bs) = u(x(0), 0), \quad s \in \mathbb{R},$$

so it follows that $u(x, y) = u(x - \frac{b}{a}y, 0)$. This gives us a unique solution for any [sufficiently regular] data on the x-axis.

Example 5.1. Suppose we are given

$$\begin{cases} u_x + yu_y = 0 \\ u(0, y) = y^2 \end{cases}$$

Of the three methods, try to do the first two as an exercise. But we will use the method of characteristics once again. Set $z(s) = u(x(s), y(s))$. Then once again we have $\dot{z} = u_x\dot{x} + u_y\dot{y}$, so set

$$\begin{cases} \dot{x}(s) = 1 \\ \dot{y}(s) = y(s) \end{cases}$$

to get $\dot{z}(s) = 0$. These yield curves of the form

$$x(s) = s + x_0, \quad y(s) = y_0 e^s.$$

Rearranging, we can get that

$$y(x) = y_0 e^{x - x_0},$$

which makes things slightly nicer. From here, to solve the ODE, notice that $u(x, Ce^x) = u(0, C)$ and so $u(x, y) = f(ye^{-x})$. To solve f given our constraints, notice that

$$f(ye^0) = u(0, y) = y^2,$$

so our solutions are

$$u(x, y) = \left(ye^{-x}\right)^2 = y^2 e^{-2x}.$$

This is the outline of the general method. Find some way to make the function nice along curves z [they need not be constant, just solvable—if we had $\dot{z}(s) = s^2$ we could solve something like this], then determine what data is required to specify the characteristic curves (in the above, y_0 uniquely determined the curve), then find the curve that intersects (x, y) in terms of this data, and this will yield the value of u at this point.

Example 5.2.

$$u_x + 2xy^2 u_y = 0,$$

then we have that

$$\dot{x} = 1, \dot{y} = 2xy^2,$$

so we can get that

$$\frac{dy}{dx} = 2xy^2.$$

From here, we need to use our ODE knowledge to solve this (it's separable).

$$\begin{aligned}\int \frac{1}{y^2}\,dy &= \int 2x\,dx\\ \frac{-1}{y} &= x^2 + C\\ y &= \left(-x^2 + C\right)^{-1},\end{aligned}$$

and thus these are our characteristic curves, so u is constant along them. He then graphed these functions as C varied. You can do this on desmos if you want. Notably, its asymptotes are at $\pm\sqrt{C}$ and its y-intercept is $\frac{1}{C}$. It follows that $u(0, \frac{1}{C}) = f(C) = f_0(\frac{1}{y} + x^2)$

Remark 5.3. Let $u = u(t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^m$. In general, we can solve first order equations like

$$u_t + a(t,x)u_x + b(t,x)u_y = f(t,x). \tag{$*$}$$

By the same method as before, we want to reduce this into a system of ODEs. Set $z(s) = u(t(s), x(s))$. For $(*)$, the system of characteristics is

$$\begin{cases} \dot{t} = 1\\ \dot{x} = a(t,x)\\ \dot{z} = f(t,x) - b(t,x)z \end{cases}$$

This is a nonlinear system of ODEs, but it works.

Example 5.4 (Semilinear)**.** Find characteristics for

$$\begin{cases} u_t + 2tu_x = u^2\\ u(0,x) = g(x) \end{cases}$$

To do this, we have that the system of ODEs is

$$\begin{cases} \dot{t} = 1\\ \dot{x} = 2t\\ \dot{z} = z^2 \end{cases},$$

so if we let our initial data be $t(0) = 0$, $x(0) = x_0$, $z(0) = g(x_0)$, then solving yields

$$\begin{cases} t = s\\ x = t^2 + x_0\\ z = \frac{g(x_0)}{1 - sg(x_0)} \end{cases}.$$

Rearranging, we have that

$$\begin{cases} s = t\\ x_0 = x - t^2\\ u(t(s), x(s)) = \frac{g(x_0)}{1 - sg(x_0)} \end{cases},$$

so therefore

$$u(t,x) = \frac{g(x - t^2)}{1 - tg(x - t^2)}.$$

This solution could blow up depending on the values of g, so just like nonlinear ODEs ($x' = 1 + x^2$), we have less assurances of nice properties when we don't have linearity.