

# MAT327 Lecture Notes

ARKY!! :3C

'24 Fall Semester

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## §1 Day 1: Open Sets and Continuity (Sep. 3, 2024)

This class is *MAT327*; 3 meaning third year, 2 meaning the contents are on the fundamental side, and 7 meaning no mercy.

– Dror Bar-Natan

Course administration matters first;

- The course link is given [here](#) (this will link straight to Quercus).
- The textbook is [James Munkres' Topology](#) (online PDF: [ETH Zurich mirror](#)); Prof Bar-Natan strongly recommends a paper copy, though (since people get distracted on the computer).

Today's readings in the textbook is on Ch. 1, sections 1 to 8, and Ch. 2, sections 12 to 13. Readings are supplementary to lecture material<sup>1</sup>. The goal of this course is to understand continuity in its most general form; in particular,

- In MAT157, we studied continuity in  $f : \mathbb{R} \rightarrow \mathbb{R}$ ;
- In MAT257, we will study continuity in  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ;

but in this class, we will study continuity in  $F : X \rightarrow Y$ , where  $X, Y$  are arbitrary spaces, such as (but not limited to)  $\mathbb{R}^n$ ,  $\mathbb{R}^{\mathbb{N}}$ ,  $\{0, 1\}^{\mathbb{N}}$  (binary sequences), and so on. We start with some refreshers on previous coursework;

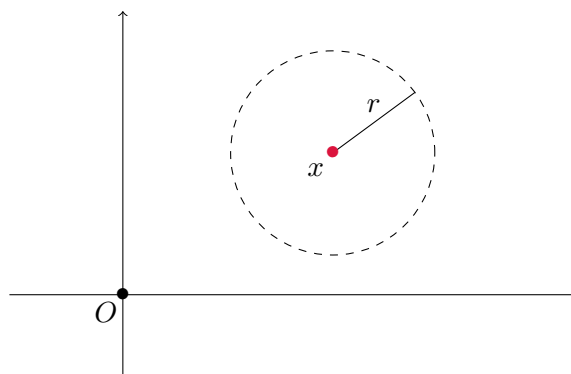
**Definition 1.1** (Continuity in  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ ). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called “continuous” if it is continuous at all points in  $\mathbb{R}^n$ . Specifically, for all  $x_0 \in \mathbb{R}^n$  and  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$ .

With this, we state our main theorem for today (proof given later),

**Theorem 1.2** (Continuity on  $\mathbb{R}^n$  if and only pre-image of open subsets is open). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous if and only if all open subsets  $U \in \mathbb{R}^m$  have  $f^{-1}(U)$  open.

In order to build up to the above, we start by defining some terms;

**Definition 1.3** (Open Ball). Let  $r > 0$ , and  $x \in \mathbb{R}^n$ . An open ball of radius  $r$  about  $x$  is given formally by  $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$ . Visually, we have



Note that the border of the circle is not in  $B_r(x)$ , as the distance metric asks for a strict inequality.

<sup>1</sup>irc it won't be tested unless specified. its still good to learn tho

In a similar fashion, we have

**Definition 1.4** (Open Set). A set  $U \subset \mathbb{R}^n$  is called *open* if, for all  $x \in U$ , there exists an open ball about  $x$  contained in  $U$ . Specifically, there is some  $\varepsilon > 0$  such that  $B_\varepsilon(x) \subset U$ .

Intuitively, we could say that the set  $U$  does not contain its edge; if it did, let  $x$  be on said edge; then we would not be able to fit an open ball about  $x$  in  $U$ . Here are some examples of open sets from lecture;

1. The whole set  $U = \mathbb{R}^n$  is open.
2. The empty set  $\emptyset$  is open. Since there does not exist any  $x \in \emptyset$ , no conditions on balls need to be satisfied; “every dog in the empty set of dogs is green”.
3.  $(0, 1)$  is open on  $\mathbb{R}^1$  (and any open interval, for that matter).
4.  $B_r(x) \subset \mathbb{R}^n$  is open. To see this, observe that for any  $y \in B_r(x)$ , we may pick  $\varepsilon < r - |x - y|$ ; by triangle inequality, all elements in  $B_\varepsilon(y)$  must also be in  $B_r(x)$ .

We now define images and pre-images; let us have a function between sets  $f : X \rightarrow Y$ . For subsets  $A \subset X$  and  $B \subset Y$ ,

$$f(A) = \{f(a) \mid a \in A\},$$

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Since  $f$  need not be injective,  $f^{-1}$  does not necessarily exist. While images of a union of subsets is a union of the images, the same is not true for intersections; let  $A_1, \dots, A_n$  be subsets of  $X$ ; then

$$f\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f(A_i), \quad f\left(\bigcap_{i=1}^n A_i\right) \subset \bigcap_{i=1}^n f(A_i).$$

On the other hand, pre-images preserve both union and intersection; let  $B_1, \dots, B_n \subset Y$ ; then

$$f^{-1}\left(\bigcup_{i=1}^n B_i\right) = \bigcup_{i=1}^n f^{-1}(B_i), \quad f^{-1}\left(\bigcap_{i=1}^n B_i\right) = \bigcap_{i=1}^n f^{-1}(B_i).$$

As for set complements, we have

$$f^{-1}(B^C) = f^{-1}(B)^C,$$

but the same cannot be said for images;  $f(A)^C \subset f(A^C)$  for surjective  $f$ , and vice versa for injective  $f$ . Equality occurs only if  $f$  is a bijection.

**Note:** for the proof of Theorem 1.2, I'll leave it out in case Prof. Bar-Natan proves it later on Thursday.

## §2 Day 2: Basic Definitions and Topological Spaces (Sep. 5, 2024)

We start by recapping the previous lecture; we introduced

- Open sets  $U \subset \mathbb{R}^n$ , where there exists an open ball of radius  $\varepsilon > 0$  about any  $x \in U$ .
- Continuity in  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

Recall the continuity property from last class, that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous if and only if the pre-image of an open set  $U$  is open.

( $\Rightarrow$ ) To start, pick any  $x_0 \in f^{-1}(U)$ , and let us have a small enough  $\varepsilon > 0$  such that  $B_\varepsilon(f(x_0)) \subset U$  (this is possible since  $U$  is open). By continuity, there exists  $\delta > 0$  such that any  $x \in B_\delta(x_0)$  satisfies  $f(x) \in B_\varepsilon(f(x_0)) \subset U$ ; this means  $x \in f^{-1}(U)$  by definition of pre-image, and since  $x$  was arbitrary, we see  $B_\delta(x_0) \subset f^{-1}(U)$ . Moreover, since  $x_0$  was also arbitrary, our construction shows that there always exists a  $\delta$ -ball about any point in the pre-image, and so  $f^{-1}(U)$  is open.  $\square$

( $\Leftarrow$ ) For the other direction, take any  $x_0 \in \mathbb{R}^n$ , and  $\varepsilon > 0$ . Since  $B_\varepsilon(f(x_0))$  is open we have that  $f^{-1}(B_\varepsilon(f(x_0)))$  is open as well. This means we may pick a small enough  $\delta > 0$  such that  $B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$  (by definition of openness), and we immediately see

$$f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)).$$

This is a reconstruction of the epsilon-delta definition of continuity, and so we are done.  $\square$

Open sets in  $\mathbb{R}^n$  have a number of properties;

- $\emptyset, \mathbb{R}^n$  are open sets.
- The union of open sets are open; specifically, let  $S$  be a set of indices, and let  $A_\alpha \subset \mathbb{R}^n$  for all  $\alpha \in S$ . Then

$$\bigcup_{\alpha \in S} A_\alpha$$

is an open set.

- The finite intersection of open sets are also open. Let  $A_1, \dots, A_n \subset \mathbb{R}^n$  be open; then

$$\bigcap_{i=1}^n A_i$$

is open.

We now proceed with the proofs for these properties.

- This was proven last lecture.
- For any  $x$  in the union, by definition, there exists  $\alpha \in S$  such that  $x \in A_\alpha$ . Then there exists  $\varepsilon > 0$  such that  $B_\varepsilon(x) \subset A_\alpha$ , and  $A_\alpha \subset \bigcup_{\alpha \in S} A_\alpha$ . This concludes that the union is open.
- For any  $x$  in the finite intersection, then  $x$  is an element of each of  $A_1, \dots, A_n$ . Let us have  $\varepsilon_1, \dots, \varepsilon_n > 0$  such that  $B_{\varepsilon_i}(x) \subset A_i$ , and take  $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$  (note that minimum is defined only for finite lists), which means  $B_\varepsilon(x) \subset \bigcap_{i=1}^n A_i$ .

In the case of infinite intersection of open sets, the resulting set need not be open. For example, let us consider the infinite intersection of intervals

$$\bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, \frac{1}{j}\right) = \{0\},$$

which is not open.

**Definition 2.1** (Topology on a Set). Let  $X$  be a set equipped with topology  $\mathcal{T}$  on  $X$ .  $\mathcal{T}$  is a collection of subsets of  $X$ , i.e.  $\mathcal{T} \subset \mathcal{P}(X)$ , with properties

- $\emptyset, X \in \mathcal{T}$ .
- The union of subsets of  $\mathcal{T}$  is also in  $\mathcal{T}$ .
- The finite intersection of subsets of  $\mathcal{T}$  is also in  $\mathcal{T}$ .

We call  $(X, \mathcal{T})$  a *topological space*, which may be abbreviated to  $X$  if  $\mathcal{T}$  is given or obvious. Moreover, we define another notion of openness, where  $U \in \mathcal{T}$  is said to “be open relative to  $\mathcal{T}$ ,” or that “ $U$  is open.” Here are a few examples of such topological spaces.

- (a) The standard topology on  $\mathbb{R}^n$  is given by

$$\mathbb{R}_{std}(\mathbb{R}, \mathcal{T}_{std} = \{U \subset \mathbb{R}^n \mid U \text{ is open in the “old sense”}\}),$$

i.e. collection of open intervals<sup>2</sup>.

- (b) The discrete topology,  $X_{discrete} = (X, \mathcal{T} = \mathcal{P}(X))$  can be defined over any set  $X$ , equipped with  $\mathcal{T}$  as the collection of all subsets.
- (c) The trivial topology,  $X_{trivial} = (X, \mathcal{T} = \{\emptyset, X\})$ .

**Definition 2.2** (Continuity between Topological Spaces). If  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  are topological spaces, and  $f : X \rightarrow Y$  is a function between said spaces, then we say  $f$  is continuous if, for all  $U \in \mathcal{T}_Y$ , we have that  $f^{-1}(U) \in \mathcal{T}_X$ . This draws from our definition from the notion that pre-images of open sets are open.

Now for the examples from lecture;

- (a)  $f : X_{discrete} \rightarrow \mathbb{R}_{std}$  is always continuous.
- (b)  $f : X_{trivial} \rightarrow \mathbb{R}_{std}$  is continuous if and only if  $f$  is constant.
- (c)  $f : \mathbb{R}_{std} \rightarrow X_{trivial}$  is always continuous.
- (d)  $f : \mathbb{R}_{std} \rightarrow X_{discrete}$  is almost never continuous, except when  $X$  is empty or a singleton (in which case,  $X_{discrete} = X_{trivial}$ ). If  $x_0 \in X$  yet  $X \setminus \{x_0\} \neq \emptyset$ , then let us have

$$\begin{aligned} A &:= f^{-1}(\{x_0\}), \\ B &:= f^{-1}(X \setminus \{x_0\}). \end{aligned}$$

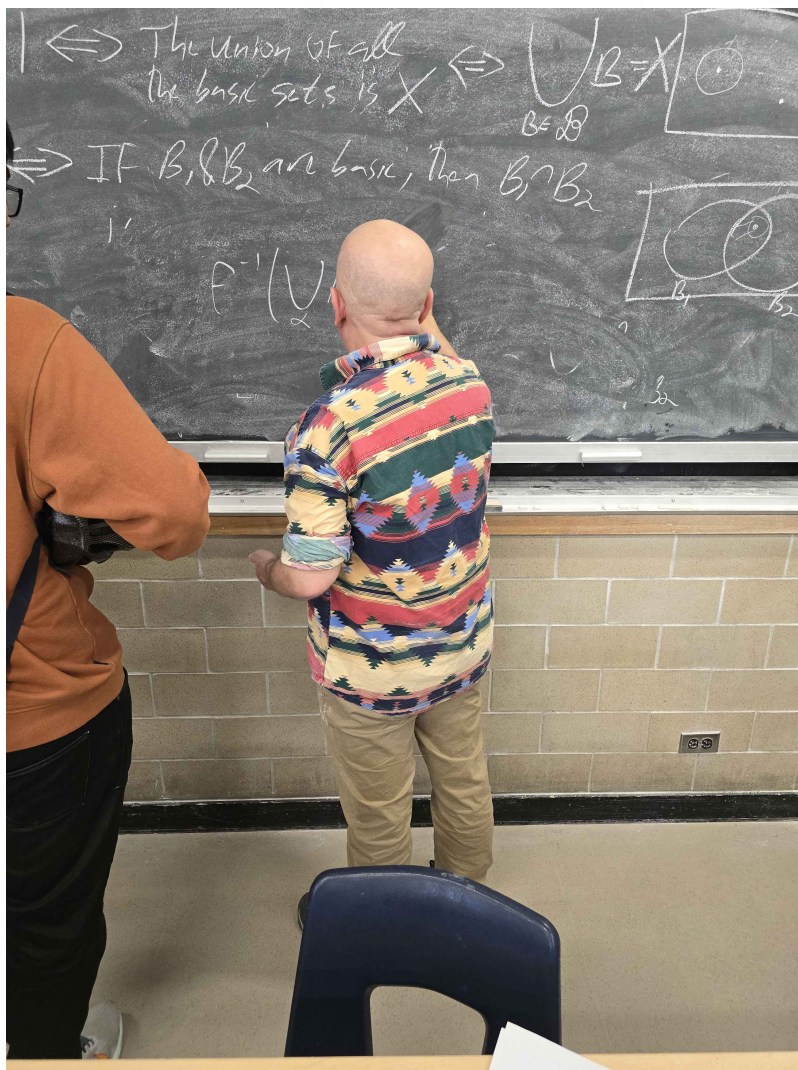
While  $A, B$  are both open, we see that  $A \cup B = \mathbb{R}$ , and  $A \cap B = \emptyset$ .<sup>3</sup>

<sup>2</sup>this is how i understood it, he might clarify next time?

<sup>3</sup>this one i don't completely get yet.

### §3 Day 3: Homeomorphisms and Bases (Sep. 10, 2024)

Outfit of the day (by popular request)



Course administrative details first;

- The reading for this week is on sections 12 to 14 (this week will cover these contents), and 15 to 16 as prereading.

Recap of last lecture:

- A topology  $\mathcal{T} \subset \mathcal{P}(X)$  is a collection of subsets of  $X$ , of which we require  $\{\emptyset, X\} \subset \mathcal{T}$ . We also require  $\mathcal{T}$  to be closed under arbitrary unions and finite intersections.
- We say a function  $F : X \rightarrow Y$  is continuous if and only if for all  $U \in \mathcal{T}_Y$ , we have  $f^{-1}(U) \in \mathcal{T}_X$ .

Today we will cover homeomorphisms and bases. To start, recall the example topologies, such as  $\mathcal{T}_{\text{std}}$  on  $\mathbb{R}^n$  (where  $\mathcal{T}_{\text{std}}$  consists of the open balls),  $\mathcal{T}_{\text{triv}}$ , and  $\mathcal{T}_{\text{disc}}$ . We also introduce a new example topology (where FC stands for finite complement),

$$\mathcal{T}_{\text{FC}} = \{U \subset X \mid X \setminus U \text{ is finite, or } U = \emptyset\}.$$

Note that Prof. Bar-Natan may interchange the notations  $-$  or  $\setminus$  to represent set difference.

**Theorem 3.1** (Composition of Continuous Functions is Continuous). Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous functions. Then  $g \circ f : X \rightarrow Z$  is continuous (relative to the same topologies on  $X$  and  $Z$ ).

If  $U \in \mathcal{T}_Z$ , we have  $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U)$ , where by definition of continuity, we see that pre-images of open sets are open, and we have  $g^{-1}(U)$  is open in  $Y$ , and similarly  $f^{-1}(g^{-1}(U))$  is also open in  $X$ .  $\square$

In tutorial, we equipped  $X$  with topologies  $\mathcal{T}_1, \mathcal{T}_2$  (i.e.,  $X$  is a topological space in two ways). We say  $\mathcal{T}_1$  is finer than  $\mathcal{T}_2$  if  $\mathcal{T}_1 \supset \mathcal{T}_2$ , and coarser for the opposite direction; the words bigger and stronger may be used interchangeably with finer, and smaller or weaker for coarser. For example, the identity map

$$\text{id} : (X, \mathcal{T}_1) \rightarrow (X, \mathcal{T}_2)$$

is continuous if and only if  $\mathcal{T}_1$  is finer than  $\mathcal{T}_2$ . To see this, let  $U$  be an open set in  $(X, \mathcal{T}_2)$ ; then  $U$  must be open in  $(X, \mathcal{T}_1)$  as well, which is true for any  $U$  only if  $\mathcal{T}_1 \supset \mathcal{T}_2$ <sup>4</sup>.

**Definition 3.2** (Homeomorphism). A map  $h : X \rightarrow Y$  is called a homeomorphism if  $h$  is continuous, bijective, and  $h^{-1}$  is continuous as well.<sup>5</sup>

Note that continuous bijective maps  $h$  need not have continuous inverses; for example, let us have  $\text{id} : X_{\text{disc}} \rightarrow X_{\text{triv}}$ .  $\text{id}$  is continuous as per our above example, while its inverse is not. Another example is to consider  $[0, 2\pi) \rightarrow S^1$  (unit circle), where  $x \mapsto (\cos 2x\pi, \sin 2x\pi)$ ; we see that the inverse is discontinuous at 0 and  $2\pi$  radians, even if the map is continuous and bijective (also observe that  $[0, 1)$  is not compact while  $S^1$  is).

An example of a homeomorphism is as follows (as per tutorial); let us consider<sup>6</sup>

$$(-1, 1)_{\text{std}} \cong \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)_{\text{std}} \cong \mathbb{R}_{\text{std}},$$

where we may map the first to the second by  $x \mapsto \frac{\pi}{2}x$ , and the second to third by  $x \mapsto \tan x$ . Since the composition of continuous maps is continuous, we also see  $(-1, 1)_{\text{std}}$  is homeomorphic to  $\mathbb{R}_{\text{std}}$  (any open interval is homophobic to  $\mathbb{R}$  for that matter).

Another example of a homemorphism is  $\text{id} : (X, \mathcal{T}_1) \rightarrow (X, \mathcal{T}_2)$  if  $\mathcal{T}_1 = \mathcal{T}_2$ .

A “basis” for a topology on  $X$  is a collection  $\mathcal{B} \subset \mathcal{P}(X)$  of subsets such that

1. For all  $x \in X$ , there exists some  $B \in \mathcal{B}$  such that  $x \in B$ . We call  $B$  a *basic set*.
2. For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists a third basic set  $B_3$  such that  $x \in B_3 \subset B_1 \cap B_2$ .

<sup>4</sup>includes the case  $\mathcal{T}_1 = \mathcal{T}_2$ ; i'm following florian notation here with  $\subset$  and  $\subsetneq$  for explicit non equality

<sup>5</sup>smth smth coffee cup and donut “extra homework: go find a nice video on why this is true on youtube”

<sup>6</sup>dror was using  $\sim$  for homeomorphism symbol today. if he keeps using that i'll adjust my notes, but for now i'll use  $\cong$  cuz afaik its used more...?

In particular, the first condition is equivalent to  $\bigcup_{B \in \mathcal{B}} B = X$  (i.e., the basis forms a covering of  $X$ ), and the second condition is equivalent to the basic sets contained in  $B_1 \cap B_2$  forming a cover of  $B_1 \cap B_2$ , i.e.

$$B_1 \cap B_2 = \bigcup_{\substack{B \in \mathcal{B} \\ B \subset B_1 \cap B_2}} B.$$

Here are some examples;

- (a)  $\{B_r(x_0)\} \subset \mathcal{P}(\mathbb{R}^n)$ , i.e. the open balls on  $\mathbb{R}^n$  form a basis.
- (b) The one-dimensional analogue of case (a) is  $\{(a, b) \mid a < b\}$ , and it forms a basis on  $\mathbb{R}$ .
- (c)  $\{[a, b) \mid a < b\}$  is called the lower limit topology, and it forms a basis on  $\mathbb{R}$ .
- (d)  $\{[a, b] \mid a < b\}$  implies  $[a, b] \cap [b, c] = \{b\}$ , which forces the basic set to include all singletons on  $\mathbb{R}$ . In that case, this is simply the discrete topology (?).

**Theorem 3.3.**  $\mathcal{T}_{\mathcal{B}} = \{U \subset X \mid \forall x \in U \implies \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}$ , i.e. the collection of all unique basic sets.

This will be expanded on next lecture.