MAT436 Lecture Notes

Arky!! :3c

'25 Fall Semester

Contents

§1 Day 1: Inaudible Lecture (Sep. 3, 2025)

Attendance won't be taken every lecture. This class is glorified linear algebra. Otherwise, incomprehensible lecture (professor speaks quieter than the air conditioner). This class is truly a test if you have tinnitus or not.

Throughout this class, we will use H to refer to a Hilbert space, U a unitary operator (where $U^* = U^{-1}$), and $T \in B(H)$ a bounded operator on H. In particular, we have that $(UTU^{-1})^* = UT^*U^{-1}$.

Exercise from class. Let $S \subseteq H$. If S is a closed linear subspace, show that $S^{\perp\perp} = S$ and that S admits an orthogonal complement. Is this a sufficient and necessary condition?

§2 Day 2: (Sep. 5, 2025)

Exercise 2.1. How many diagonals are there in a parallelepiped in an arbitrary vector space?

When we started talking about the projection theorem, which is one of the properties of Hilbert spaces, i.e., for every closed linear subspace, there is a complementary linear subspace where the intersection is 0 and the sum is the whole space.

Any Hilbert space has an orthonormal basis.

Exercise 2.2. Hilbert spaces can be vector spaces. Given a Hilbert space basis, if it is not finite dimensional, then this cannot be a vector space basis.

Let H be a hilbert space, and let K be a closed subspace of H. We call $K^\perp$ the *orthogonal complement*, consisting of the vectors perpendicular to all vectors in K. Is it true that $H = K + K^\perp$? Note that even though $K \cap K^\perp = \{0\}$, we do not use the direct sum here; we refer to $+$ as the internal direct sum. $\oplus$ is reserved for external direct sums, if we wish to create a third Hilbert space from two separate Hilbert spaces.

Let H be a Hilbert space, and let S be a closed and convex subset of H, and let $x \in H$. There exists a unique closest point of S to x. It can be proven using the parallelogram law. Similarly, given $H = K + K^\perp$ and $x \in H$, there exists a closest point to x in K, which we shall call y. Then we have that $x - y \in K^\perp$.

Try to find something related to the lectures for the next class. Specifically, index theory, etc, finding stuff on Wikipedia is fine.

§3 Day 3: (Sep. 8, 2025)

We return to listen to our dear preacher for another hour. Here is a list of professor priest's quotes from today:

Given a Banach space and a closed subspace, there exists a complement. This is topologically equivalent to a Hilbert space (to the Hilbert space norm).

Every Hilbert space has an orthonormal basis.

There is a paper by J. Hogan and S. Li.

Exercise 3.1. Linear transformations induce a new inner product.

Exercise 3.2. We know that a unit circle in $\mathbb{R}^2$ centered about the origin is induced by some norm, and there are infinitely many automorphisms on $\mathbb{R}^2$ sending D^1 to itself. An ellipse centered about the origin in $\mathbb{R}^2$ is likewise induced by some norm; can you find an example of a function sending the ellipse back to itself, and show that there are infinitely many such functions?

§4 Episode 4: Professor Priest Finds a Microphone (Sep. 10, 2025)

Actually nevermind even though he has a microphone he walks so far away from it every single time that his voice is magnified like only twenty percent of the time...

Something something, discussion that all bounded operators are adjoinable?

Tutorial notes! Josh won't email you feedback about the homework, but you can always ask and know what your grade is for the class so far. Tutorial class participation is like, recorded on a piece of paper so they know who comes. There aren't really any hard deadlines for the class. Homework is graded largely on how much effort you put into it, not one million percent correctness (quite literally, "you tried"... he won't check details for the homework. He will know if you bullshit it though).

There will be an email saying the final date to submit all the homework (like at the end of the class in December). You should not worry about the homework grade as long as you're submitting your homework.

There are two essays. George will have "rough deadlines" for these, and there will be like a list of topics sent out for them. Examples are very nice in essays, because they showed you thought about the concept. If it's just an overview, it's not really... that good of an idea.

Get George to learn who you are and your name by the end of your class.

George will talk about two major things: spectrum and index. This is quite literally the entire syllabus.

Definition 4.1. When it is defined for the operator P, we have that $\operatorname{ind}(P) = \dim\ker P - \dim\operatorname{coker} P \in \mathbb{Z}$. $\operatorname{ind} P$ is said to be the *index* of P.

What sucks is, per the rank-nullity theorem, the index of $P : V \to V$ is always zero if V is finite dimensional; this is the *index theorem* for finite-dimensional vector spaces. In a way, this means we only really care about the index when we're talking about infinite dimensional vector spaces V, i.e., if $\dim V = \infty$, then it can happen that $\operatorname{ind}(P) \neq 0$.

As an example, consider the space $V = \ell^2(\mathbb{N})$, and consider P to be the operator,

$$P : (x_0, x_1, z_2, \dots) \mapsto (x_1, x_2, x_3, \dots);$$

clearly, we have that $\operatorname{ind} P = 1 - 0 = 1$. In particular, this operator P is commonly written S^*, which is the adjoint of the unilateral shift operator,

$$S : (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, \dots),$$

where $\operatorname{ind} S = -1$.

Exercise 4.2. Show that, if P is an operator, then $\dim\operatorname{coker} P = \dim\ker P^*$.

We now move on; let P be a bounded operator on a Hilbert space. We say that P is nilpotent if the spectrum of P is $\{0\}$, unitary if $P^*P = PP^* = \mathrm{id}$, isometry if $P^*P = \mathrm{id}$ (co-isometry if $PP^* = \mathrm{id}$), Hermitian (self-adjoint) if $P = P^*$, normal if $PP^* = P^*P$, and finally, a projection if $PP = P = P^*$.

Exercise 4.3. Unitary operators are isometries, and vice versa, in finite-dimensional vector spaces.

Hermitian and normal operators are particularly relevant to spectral theory. As an example, the operators S, S^* from earlier satisfy $S^*S = \mathrm{id}$, and $SS^* : (x_0, x_1, \dots) \mapsto (0, x_1, \dots)$, i.e., unilateral shift is an isometry.

Lemma 4.4. The parallelogram law states that operators that preserve the norm also preserve inner products, and vice versa.

Lemma 4.5 (Wold Decomposition)**.** here, i don't even know what to be saying.

If $\operatorname{ind} P$ exists, at least one of the dimensions of the kernel or the cokernel is finite. Such an operator is called *Fredholm* (relevant to Calkin algebra).

Theorem 4.6. Let $F_t : [0, 1] \to \mathrm{Fred}(H)$ be a path in the subset of Fredholm operators in the space of bounded operators on H, $B(H)$, we have that $\mathrm{ind}(F_0) = \mathrm{ind}(F_1)$.

This means the index is a topological invariant (???), but the dimension of the kernel and the cokernel themselves *aren't* topological invariants; it's just that the amount that they change by is the same. The map of fredholm operators $\mathrm{Fred}(H) \xrightarrow{\mathrm{ind}} \mathbb{Z}$ forms a stratification of

$$\bigsqcup_{n\in\mathbb{Z}} \mathrm{ind}^{-1}(n).$$

Let $(H, \langle \cdot, \cdot \rangle)$ be a vector space over $(\mathbb{C}, \tau)$ (the real structure on $\mathbb{C}$), where $\tau z \mapsto \bar{z}$. The "real structure" is $\mathrm{Fix}(\tau) = \mathbb{R}$; in particular, $\langle a, b\rangle = \overline{\langle b, a\rangle} = \tau \langle b, a\rangle$, and we have

$$\langle Pv, w\rangle = \overline{\langle w, Pv\rangle} = \tau \langle w, Pv\rangle .$$

τ induces a real structured on $B(H)$, i.e., taking $P \mapsto P^*$. We have that $\mathrm{Fix}(*)$ are the self-adjoint operators.

The spectrum of an operator in infinite dimensions should be thought of the set $\{\lambda \in \mathbb{C} \mid P - \lambda \text{ is not invertible}\}$. Since the set of invertible operators is open, this tells us that the spectrum is a closet set; in particular, it is compact.

Theorem 4.7. The spectrum of any bounded operator is always nonempty.

This theorem is hard to prove.

§5 Episode 5: (Sep. 12, 2025)

Professor mumble rapper enters the room at 2:02pm. It appears that his hearing aids are out, which is why I am able to hear him today.

Anyone who has studied linear algebra will recognize the phrase row reduction. There are three numbers in relation to rank-nullity; d represents the dimension of the whole space, while n and r refer to the dimensions of the nullity and range respectively.

Exercise 5.1. Find a normed vector space in 2 dimension such that, for each 1 dimensional subspace, there's a projection of norm 1 onto it: an idempotent mapping taking the ball into itself (i.e., doesn't cast a shadow).

§6 Episode 6: (Sep. 15, 2025)

so there was more stuff that was discussed in class today but i have no idea what he's talking about. he's learned to use chalk though so... there's that!

§7 Episode 7: More Winding Numbers (Sep. 17, 2025)

I didn't go tutorial today.

§8 Episode 8: (Sep. 19, 2025)

§9 Episode 9: 50 minutes of attendance (Sep. 22, 2025)

§10 Episode 10: Power Outlet Tripping Hazard (Sep. 24, 2025)

A compact operator sends the unit ball to a compact set (or precompact, I guess). In particular, a compact operator sends an orthonormal basis to a sequence converging to 0 in norm.

Tutorial stuff. If X, Y are Banach spaces, then an operator $T : X \to Y$ is compact if $T(\overline{B_1^*})$ is compact in Y.

Theorem 10.1. If X and Y are Hilbert spaces, then T is compct if and only if there exists a sequence $T_n : X \to Y$ with $\operatorname{rank} T_n < \infty$ and $T_n \to T$ in $\|\cdot\|_{\mathrm{op}}$.

Fact 10.2. If S is compact, then ST and TS are compact for any other operator T, i.e., the compact operator K forms a two-sided ideal in $B(X)$.

Fact 10.3. The compact operators K are closed.

Definition 10.4. We say $B(H)/K(H) = Q(H)$ is a Calkin algebra.

Theorem 10.5 (Atkinson)**.** $\mathrm{Fred}(H) = \mathrm{GL}(Q(H))$, where $\mathrm{Fred}(H)$ denotes the operators with index on H.

In particular, the Calkin short exact sequence is given by

$$0 \to K \to B(H) \to Q(H) \to 0$$

of C^* algebras. The unilateral shift $S \in B(\ell^2(\mathbb{N}))$ is a C^* algebra generated by S, and $C^*(S)$ is defined as the smallest C^* algebra of $B(\ell^2(\mathbb{N}))$ containing S. In particular, $C^*(S)$ contains all $P(S, S^*)$ for $p \in \mathbb{C}[z, \overline{z}]$; for example, S, S^*, $1 - S^*S$, and 1 dimensional projections $e_{ij} = (S^*)^j(1 - SS^*)S^i$. We call $C^*(S)$ a *Toeplitz* algebra. The Toeplitz short exact sequence is given by

$$0 \to K \to C^*(S) \to \frac{C^*(S)}{K} \to 0.$$

We may write $\ell^2(\mathbb{N}) = \{(a_n) \in \ell^2(\mathbb{Z}) \mid a_n = 0 \text{ if } n \geq 0\}$, and consider

$$\ell^2(\mathbb{N}) \subset \ell^2(\mathbb{Z}) \to L^2(S^1)$$

by the Fourier transform by $(e_n^j = \delta_{nj}) \mapsto z^j$, where we are sending $(\dots, 0, 1, 0, \dots)$ with a 1 in the jth position to z^j.

Definition 10.6. $H^2 = \{f : \overline{\mathbb{D}} \to \mathbb{C} \mid f \text{ is holomorphic on } \mathrm{int}(\mathbb{D})\}$ is called a Hardy space, where $\overline{\mathbb{D}}$ is the closed unit disc in $\mathbb{C}$.

In this manner, any element of H^2 may be considered to map

$$\sum_{j=0}^{\infty} a_j z^j \mapsto \sum_{y=0}^{\infty} a_j z^{j+1} = z \sum_{j=0}^{\infty} a_j z^j$$

under S, and $C^*(S)$ embeds into $B(H^2)$ by $S \mapsto M_z$, $S^j \mapsto M_{z^j}$, $p(S) \mapsto M_{p(z)}$, where $M_f(g) = fg$ and $\|M_f\| = \|f\|_\infty$. This means that the image of $C^*(S) \hookrightarrow B(H^2)$ contains M_p, $p \in C(S^1)$.

§11 Episode 11: (Sep. 26, 2025)

Exercise 11.1. The algebra of bounded operators on a separable Hilbert space is not separable.

Andre Weil's autobiography is very worth reading. His nasty review in AMS is also nice to read.

Exercise 11.2. It was either "there is a one-to-one correspondence between unitary self-adjoint operators and projections" or "find all unitary self-adjoint operators". reference here and here.

§12 Episode 12: (Sep. 29, 2025)

§13 Episode 13: (Oct. 1, 2025)

I'm just going to type up the notes from tutorial. Let $A : H \to H$ be a compact operator; then A is the limit of A_n with $n \to \infty$, where each A_n is a fintie rank operator. If $A^* = A_n$, then $A_n \in M_n\mathbb{C}$ is a unitarily diagonalizable matrix with real eigenvalues as seen from linear algebra, i.e., $A_n = U_n D_n U_n^*$ where U_n is unitary and D_n is diagonal. In this manner, we have the following sketch for a proof,

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} (U_n D_n U_n^*) \stackrel{?}{=} U(\lim D_n)U^*,$$

where we *suspect* that the eigenvalues of A (henceforth denoted the spectrum $\operatorname{spec} A$) is given by the limit of the eigenvalues of A_n. Note that for any operator $T : H \to H$, we have that $T = A + iB$, where $A = \frac{1}{2}(T + T^*)$ and $B = \frac{1}{2i}(T - T^*)$; then $A \in M_n\mathbb{C}$, where $\det(A - I\lambda)$ is its characteristic polynomial, and we may define

$$\operatorname{spec}(A) = \{\lambda \in \mathbb{C} \mid A - I\lambda \text{ is not invertible}\},$$

for $A \in B(H)$. If A is compact, then we may follow the process,

(i) Find an eigenvector for the eigenvalue $\|A\|$ (from compactness).

(ii) dot dot dot idk

Theorem 13.1. $\operatorname{spec}(A) \subset B^C_{\|A\|}$.

i stopped paying attention after this

§14 Episode 14: (Oct. 3, 2025)

Exercise 14.1. Show that a Hilbert space with an uncountable orthonormal basis does not admit a countable dense subset (i.e., not separable).

Wrap an ε-ball about each element of the basis, and observe that any countable set cannot intersect each of these balls per uncountability. Thus, countable sets cannot be dense in such a Hilbert space.

Exercise 14.2. Take two invertible positive operators, which commute, (you may take one of them to be the identity, and the other self adjoint and positive and invertible). Look at the weighted average (perhaps start with the normal average); show that it is invertible.