

# MAT354 Lecture Notes

ARKY!! :3C

'25 Fall Semester

## Contents

1	Day 1: Recap of Preliminaries (Sep. 2, 2025)	2
2	Day 2: Functions on the Complex Plane (Sep. 4, 2025)	5
3	Day 3: Holomorphic Functions and Power Series (Sep. 9, 2025)	7
4	Day 4: Complex Power Series (Sep. 11, 2025)	10
5	Day 5: Curves in the Complex Plane (Sep. 16, 2025)	12
6	Day 6: Cauchy's Theorem on a Disc (Sep. 18, 2025)	15
7	Day 7: Cauchy's Integral Formula and Corollaries (Sep. 23, 2025)	17

## §1 Day 1: Recap of Preliminaries (Sep. 2, 2025)

We start by discussing the complex plane and complex numbers. Given  $z \in \mathbb{C}$ , we say that  $\Re(z)$  and  $\Im(z)$  are the real and imaginary parts of  $z$  respectively, i.e.,  $z = x + iy$ .  $\mathbb{C}$  is the set of all complex numbers. In this manner, we may identify  $z = x + iy$  with  $(x, y) \in \mathbb{R}^2$  using the standard complex plane.

- (a) The complex *conjugate* of  $z$  is given by  $\bar{z} = x - iy$ , where we have that

$$\Re(z) = \frac{z + \bar{z}}{2}, \quad \Im(z) = \frac{z - \bar{z}}{2i}.$$

- (b) We now define addition and multiplication for the complex numbers. For all  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ , we have that

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \end{aligned}$$

We have that  $(\mathbb{C}, +, \times)$  is a field, with  $(\mathbb{R}, +, \times)$  as a subfield. To verify this, we need to check that it indeed satisfies:

- Commutativity: for all  $z_1, z_2 \in \mathbb{C}$ , we have that  $z_1 + z_2 = z_2 + z_1$  and  $z_1 z_2 = z_2 z_1$ .
  - Associativity: for all  $z_1, z_2, z_3 \in \mathbb{C}$ , we have that  $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$  and  $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ .
  - Distributivity: for all  $z_1, z_2, z_3 \in \mathbb{C}$ , we have that  $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ .
- (c) The absolute value of a complex number  $z = x + iy$  is given by  $|z| = \sqrt{x^2 + y^2}$ . In particular, this yields the triangle inequality, where for any  $z, w \in \mathbb{C}$ , we have that  $|z + w| \leq |z| + |w|$ . The proof either comes visually or through explicit computation, both of which I will not write out here for brevity.<sup>1</sup>

As an extension of the inequality, we also automatically have that

$$|\Re z| \leq |z|, \quad |\Im z| \leq |z|,$$

and that for all  $z, w \in \mathbb{C}$ , we have

$$||z| - |w|| \leq |z - w|.$$

*Proof.* Using the triangle inequality, we have that

$$\begin{aligned} |z| &= |(z - w) + w| \leq |z - w| + |w|, \\ |w| &= |(w - z) + z| \leq |z - w| + |z|, \end{aligned}$$

of which both imply that  $|z| - |w| \leq |z - w|$  and  $|w| - |z| \leq |z - w|$ . □

For any  $z \in \mathbb{C}$ , we have that  $|z|^2 = z \cdot \bar{z}$ .

*Proof.* Write  $z = x + iy$ ; then  $|z|^2 = x^2 + y^2$ , where we may note that  $z \cdot \bar{z} = (x + iy)(x - iy)$  which yields the right hand side of the earlier equation through expansion. □

---

<sup>1</sup>no full credit if you draw a picture on the exam lmao

Finally, for  $z, w \in \mathbb{C}$ , we have that  $|zw| = |z||w|$ . This is left as an exercise to the student.

- (d) The polar form of a nonzero complex number  $z \neq 0$  is given by  $z = \gamma e^{i\theta}$ , where  $\gamma > 0$  and  $\theta \in \mathbb{R}$ . Let us assume the Euler formula; for all  $\theta \in \mathbb{R}$ , we have that

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Let  $r = |z|$ ; we have that  $|z| = |re^{i\theta}| = |r| |e^{i\theta}| = r \cdot 1 = r$ .  $\theta$  is the angle between the positive real axis to the half-line starting from 0 and passing through  $z$ . In this manner,  $z = re^{i\theta} = |z|(\cos \theta + i \sin \theta) = |z| \cos \theta + i |z| \sin \theta$ , which means we have that

$$\Re z = |z| \cos \theta, \quad \Im z = |z| \sin \theta.$$

As an example, let us find all the complex numbers  $z$  such that  $z^4 = i$ . Since  $i = e^{i\frac{\pi}{2}}$ ,  $z = \rho e^{i\theta}$  satisfying  $z^4 = i$  becomes  $\rho^4 e^{i4\theta} = e^{i\frac{\pi}{2}}$ , meaning

$$\begin{cases} \rho^4 = 1, \\ 4\theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

This means  $\rho = 1$  and  $\theta = \frac{\pi}{8} + \frac{k\pi}{2}$ , where  $k \in \mathbb{Z}$ . Considering the cases  $k = 0, 1, 2, 3$  and observing that there are only 4 equivalence classes modulo 4 to consider, we have that

$$z_0 = e^{i\frac{\pi}{8}}, \quad z_1 = e^{i\frac{5\pi}{8}}, \quad z_2 = e^{i\frac{9\pi}{8}}, \quad z_3 = e^{i\frac{13\pi}{8}}.$$

We now discuss convergence. We say that a set of complex numbers  $\{z_n\}_{n \in \mathbb{N}}$  converges to  $w \in \mathbb{C}$  if  $\lim_{n \rightarrow \infty} |z_n - w| = 0$ . We write it as  $\lim_{n \rightarrow \infty} z_n = w$ . In the complex plane, the convergence can be in any direction.

**Lemma 1.1.**  $\{z_n\}_{n \in \mathbb{N}}$  converges to  $w$  if and only if  $\{\Re z_n\}_{n \in \mathbb{N}}$  converges to  $\Re w$  and  $\{\Im z_n\}_{n \in \mathbb{N}}$  converges to  $\Im w$ .

*Proof.* We have that

$$\begin{aligned} |z_n - w| &= |(\Re z_n - \Re w) + i(\Im z_n - \Im w)| \\ &\leq |\Re z_n - \Re w| + |\Im z_n - \Im w|, \end{aligned}$$

where as  $n \rightarrow \infty$ , we have that the right hand side is given by  $0 + 0$ . For the opposite direction, we have that  $|z| \geq |\Re z|$  or  $|\Im z|$ , so we have that

$$|\Re z_n - \Re w| = |\Re(z_n - w)| \leq |z_n - w|,$$

which approaches 0 as  $n \rightarrow \infty$ . The same argument goes for the imaginary portion.  $\square$

A sequence of complex numbers  $\{z_n\}_{n \in \mathbb{N}}$  is called *Cauchy* if  $|z_n - z_m| \rightarrow 0$  as  $n, m \rightarrow \infty$ . In  $\varepsilon - \delta$ , this means that for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $|z_n - z_m| < \varepsilon$  for all  $n, m > N$ .

**Theorem 1.2** (Bolzano-Weierstrass Theorem).  $\mathbb{R}$  is *complete*, i.e., every Cauchy sequence of real numbers converges to a real number.

**Theorem 1.3.**  $\mathbb{C}$  is complete.

*Proof.* Take any Cauchy sequence of complex numbers  $\{z_n\}$ . Using the inequalities  $|\Re z| \leq |z|$  and  $|\Im z| \leq |z|$ , we have that  $\{\Re z_n\}$  and  $\{\Im z_n\}$  are Cauchy sequences of real numbers. By Bolzano-Weierstrass, we have that  $\Re z_n \rightarrow x_0 \in \mathbb{R}$  and  $\Im z_n \rightarrow y_0 \in \mathbb{R}$ . By the previous lemma, we actually have  $\lim_{n \rightarrow \infty} z_n = x_0 + iy_0$ .  $\square$

We now move onto topology in the complex plane. Given  $z_0 \in \mathbb{C}$  and  $r > 0$ , we can form an open or closed disc centered at  $z_0$  of radius  $r$ . We write both of these as

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$

$$\bar{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\},$$

Given a set  $\Omega \subseteq \mathbb{C}$ , a point  $z_0$  is an interior point if there exists  $r > 0$  such that  $D_r(z_0) \subseteq \Omega$ . The interior of  $\Omega$  is given by the set of all such interior points. In particular, the interior of  $\bar{D}_r(i)$  is  $D_r(i)$ .

A set  $\Omega$  is called *open* if every point in  $\Omega$  is an interior point.  $\Omega$  is called *closed* if the complement of  $\Omega$ ,  $\Omega^c = \mathbb{C} \setminus \Omega$ , is open. As an example, the open right half-plane  $\{z \in \mathbb{C} \mid \Re z > 0\}$  is open.

*Proof.* For any  $z \in \Omega$ , let  $z = x + iy$ , and take  $r = \frac{x}{2} = \frac{\Re z}{2}$ . Then we claim that  $D_r(z) \subseteq \Omega$ . For all  $w \in D_r(z)$ , we clearly have that

$$\Re w = \Re z - (\Re z - \Re w) \geq \Re z - |z - w| \geq \frac{\Re z}{2} > 0,$$

and so all such  $w \in \Omega$ , and we are done.  $\square$

A point  $z \in \mathbb{C}$  is a *limit point* of  $\Omega$  if there exists a sequence  $\{z_n\} \subset \Omega$  with  $z_n \neq z$  such that  $z_n \rightarrow z$ .

As an example, we define  $D$  to be the open unit disc centered at 0. 0 and 1 are both limit points of  $D$ , but 1 is not contained in  $D$  itself.<sup>2</sup> The *closure* of  $\Omega$ ,  $\bar{\Omega}$ , is given by  $\Omega$  unioned with all its limit points. The *boundary* of a set  $\Omega$ , written  $\partial\Omega$ , is given by  $\bar{\Omega} \setminus \text{int } \Omega$ . A set  $\Omega \subseteq \mathbb{C}$  is said to be compact if it is closed and bounded, i.e., there exists  $M > 0$  such that  $|z| \leq M$  for all  $z \in \Omega$ .

**Theorem 1.4.** A set  $\Omega \subseteq \mathbb{C}$  is compact if and only if every sequence  $\{z_n\} \subset \Omega$  has a subsequence that converges to a point in  $\Omega$ .

**Proposition 1.5.** If  $\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_n \supset \dots$  is a sequence of nonempty compact sets in  $\mathbb{C}$ , where  $\text{diam}(\Omega_n) = \sup_{z, w \in \Omega_n} |z - w| \rightarrow 0$  as  $n \rightarrow \infty$ , then there exists a unique  $w \in \mathbb{C}$  such that  $w \in \Omega_n$  for every  $n \in \mathbb{N}$ .

*Proof.* For each  $\Omega_n$ , pick a point  $z_n \in \Omega_n$ . Then  $\{z_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence because the diameter of  $\Omega_n$  approaches 0. By the Bolzano-Weierstrass theorem for complex numbers, this means that  $\{z_n\}_{n \in \mathbb{N}}$  indeed does converge to some  $w \in \mathbb{C}$ . In particular, we have  $w$  is the limit of the subsequence  $\{z_m\}_{m \geq n} \subseteq \Omega_n$ , where  $\Omega_n$  is compact, meaning the limit  $w$  should be in  $\Omega_n$ . This means there exists a unique  $w \in \mathbb{C}$  such that  $w \in \Omega_n$  for every  $n \in \mathbb{N}$ .

To show the uniqueness of  $w$ , we argue by contradiction; assume  $w' \neq w$  satisfies the property. Then  $|w' - w| > 0$ . Since  $w, w' \in \Omega_n$  for all  $n$ , this contradicts that  $\text{diam}(\Omega_n) \rightarrow 0$ .  $\square$

An open set  $\Omega$  is called *connected* if it is not possible to find two disjoint nonempty open sets  $\Omega_1$  and  $\Omega_2$  such that  $\Omega = \Omega_1 \cup \Omega_2$ . A connected open set in  $\mathbb{C}$  is called a *region*.

<sup>2</sup>hell is it disc or disk YKW LET'S COMPROMISE it's spelled disque actually (paint nails)

## §2 Day 2: Functions on the Complex Plane (Sep. 4, 2025)

Let  $f : \Omega \rightarrow \mathbb{C}$ , where  $\Omega$  is an open subset of  $\mathbb{C}$ . We say that  $f$  is continuous if at  $z_0 \in \Omega$  if, for all  $\varepsilon > 0$ , there exists an open disk  $D_\gamma(z_0)$  such that  $|f(z) - f(z_0)| < \varepsilon$  for all  $z \in D_\gamma(z_0)$ . In particular,  $f$  is said to be continuous on  $\Omega$  if it is continuous at every point in  $\Omega$ .

**Example 2.1.** Consider  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by  $f(z) = \bar{z}$ . Show that  $f$  is continuous.

*Solution.* For all complex  $z, z_0$ , we have that  $|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0| = |z - z_0|$ . Thus, we have that for any  $\varepsilon > 0$ , we obtain<sup>3</sup>

$$f(D_\varepsilon(z_0)) = D_\varepsilon(\bar{z}_0). \quad \square$$

We now discuss holomorphic functions (i.e., complex differentiable functions). We say that  $f : \Omega \rightarrow \mathbb{C}$  is *holomorphic* at  $z_0 \in \Omega$  if

$$\frac{f(z_0 + h) - f(z_0)}{h}, \quad h \in \mathbb{C} \setminus \{0\},$$

converges as  $h \rightarrow 0$ . If the limit exists, we let

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

be the derivative.

**Example 2.2.** Consider the exact same function as in the previous example,  $f(z) = \bar{z}$ . Is  $f$  holomorphic?

*Solution.* For all  $z_0 \in \mathbb{C}$  and  $h \in \mathbb{C} \setminus \{0\}$ , we have that

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{z_0 + h} - \bar{z}_0}{h} = \frac{\bar{h}}{h} = \frac{\rho e^{-i\theta}}{\rho e^{i\theta}} = e^{-2i\theta}.$$

If we take  $h \rightarrow 0$  along the real line, we may let  $h = \rho$ , which means the fraction is equal to 1 as  $h \rightarrow 0$ . If we take  $\rho \rightarrow 0$  along the complex axis, however, then we have that  $h = \rho e^{i\pi/2}$ , where we obtain the fraction is equal to  $-1$  as  $\rho \rightarrow 0$ . Thus,  $f$  cannot be holomorphic.  $\square$

**Proposition 2.3.** Let  $\Omega$  be open in  $\mathbb{C}$ . If  $f, g$  are holomorphic on  $\Omega$ , then

- (i)  $f + g$  is holomorphic on  $\Omega$ , and  $(f + g)' = f' + g'$ .
- (ii)  $fg$  is holomorphic on  $\Omega$ , and  $(fg)' = f'g + fg'$ .
- (iii) If  $g(z_0) \neq 0$  where  $z_0 \in \Omega$ , then  $\frac{f}{g}$  is also holomorphic at  $z_0$ , where

$$\left(\frac{f}{g}\right)' = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

- (iv) If  $f : \Omega \rightarrow U$  and  $g : U \rightarrow \mathbb{C}$  are holomorphic, then  $g \circ f$  is also holomorphic, and we obtain the chain rule

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

<sup>3</sup>note to self: ol is better than bar for this stuff...

We now discuss complex differentiability versus real differentiability. A holomorphic function  $f : \Omega \rightarrow \mathbb{C}$  can be identified with a function  $F : \Omega \rightarrow \mathbb{R}^2$  given by  $(x, y) \mapsto (u(x, y), v(x, y)) = (\Re f(x, y), \Im f(x, y))$ . Consider the partial derivative of  $F$  at  $(x_0, y_0)$ ; these exist if there exists some linear transformation  $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that

$$\frac{\|F(P_0 + H) - F(P_0) - J(H)\|}{\|H\|} \rightarrow 0$$

as  $H \rightarrow 0$ . Or, we may define  $\Psi(H)$  to take on the fraction above, and we see that  $F$  is indeed differentiable at  $P_0 = (x_0, y_0)$  if  $\Psi(H) \rightarrow 0$  as  $H \rightarrow 0$ . We now deal with complex differentiability. Suppose  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic at  $z_0 = x_0 + iy_0$ . Then we have partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}.$$

Naturally,

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

along any path; in particular, we take  $h \in \mathbb{R} \setminus \{0\}$  and observe that

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0), \end{aligned}$$

and so both exist, and they are  $\Re f'(z_0)$  and  $\Im f'(z_0)$  respectively. Similarly, we may take  $h = ik$  where  $k \in \mathbb{R} \setminus \{0\}$  and obtain

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{k \rightarrow 0} \frac{u(x_0, y_0 + k) + iv(x_0, y_0 + k) - u(x_0, y_0) - iv(x_0, y_0)}{ik} \\ &= \lim_{k \rightarrow 0} \frac{-i(u(x_0, y_0 + k) - u(x_0, y_0)) + v(x_0, y_0 + k) - v(x_0, y_0)}{k} \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0), \end{aligned}$$

and so both partials also exist and they are  $\Re f'(z_0)$  and  $-\Im f'(z_0)$  respectively.

### §3 Day 3: Holomorphic Functions and Power Series (Sep. 9, 2025)

Let  $f : \Omega \rightarrow \mathbb{C}$  (where  $\Omega$  is an open set in  $\mathbb{C}$ ). We say that  $f$  is holomorphic at  $z_0$  if

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}, \quad h \in \mathbb{C} \setminus \{0\}$$

exists. Recall that  $\mathbb{C}$  can be identified with  $\mathbb{R}^2$  by considering any  $z = x + iy \in \mathbb{C}$  as a tuple  $(x, y) \in \mathbb{R}^2$ . In this way, given a function  $f : \Omega \rightarrow \mathbb{C}$ , we can define  $F : \Omega \rightarrow \mathbb{R}^2$ , where  $F : (x, y) \mapsto (u(x, y), v(x, y))$ , given by  $u = \Re f$  and  $v = \Im f$ .

**Proposition 3.1.** If  $f = u + iv$  is holomorphic at  $z_0 = x_0 + iy_0$ , then we have that all four partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist and they satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \Re f(z_0), \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \Im f(z_0).$$

We also have that  $F$  is differentiable at  $P_0 = (x_0, y_0)$ .

**Definition 3.2.** We say that  $F$  is differentiable at  $P_0$  if there exists a linear transformation (the derivative)  $J = J_F(x_0, y_0) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that

$$\lim_{H \rightarrow 0} \frac{\|F(P_0 + H) - F(P_0) - J(H)\|}{\|H\|} = 0.$$

Before we discuss the complex definition, let us recall another property of real differentiability; if  $F$  is differentiable at  $P_0 = (x_0, y_0)$ , then all four partial derivatives exist, and

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is called the Jacobian matrix of  $F$  at  $(x_0, y_0)$ . To see this, consider the association  $P_0 = (x_0, y_0)$  with  $z_0 = x_0 + iy_0$ , and  $H = (h_1, h_2)$  with  $h = h_1 + ih_2$ ; then we have that

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 \\ \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 \end{pmatrix}.$$

This is a vector in  $\mathbb{R}^2$ , which we may associate with the complex number

$$\left( \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 \right) + i \left( \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 \right) = \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) h_1 + \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) h_2,$$

which, by the Cauchy-Riemann equations, we obtain

$$\left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) h_1 + i \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right) h_2 = \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h_1 + ih_2),$$

which is precisely equal to  $f(z_0 + h) - f(z_0) - f'(z_0)h$ . In particular,

$$\lim_{h \rightarrow 0} \left| \frac{f(z_0 + h) - f(z_0) - f'(z_0)h}{h} \right| = \lim_{h \rightarrow 0} \left| \frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0) \right| = 0.$$

Similarly, per the definition of the Jacobian, we must have

$$\lim_{H \rightarrow 0} \frac{\|F(P_0 + H) - F(P_0) - J(H)\|}{\|H\|} = 0,$$

and this concludes the proof of proposition 3.1. □

**Theorem 3.3.** Suppose  $f = u + iv$  is a complex-valued function defined on an open set  $\Omega \subset \mathbb{C}$ . If  $u, v : \Omega \rightarrow \mathbb{R}$ , are continuously differentiable and satisfy the Cauchy-Riemann equations, then  $f$  is holomorphic on  $\Omega$  and  $f'(z) = \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{1}{i} \frac{\partial f}{\partial y} \right)$ .

*Proof.* Since  $u$  is continuously differentiable at the point  $(x, y) \in \Omega$ , there exists a linear transformation  $J_u : \mathbb{R}^2 \rightarrow \mathbb{R}$  where

$$\frac{|u(x + h_1, y + h_2) - u(x, y) - J_u(h_1, h_2)|}{\|(h_1, h_2)\|} \rightarrow 0, \quad (h_1, h_2) \rightarrow 0.$$

In particular,  $J_u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$ . The above fraction is equivalent to

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + \|h\| \psi_1(h),$$

where  $\psi_1 : U \rightarrow \mathbb{R}$ , where  $U$  is some open neighborhood of  $0 \in \mathbb{R}^2$ , with  $\psi_1(h) \rightarrow 0$  as  $h \rightarrow 0$ . Similarly, we have that

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + \|h\| \psi_2(h)$$

with  $\psi_2(h) \rightarrow 0$  as  $h \rightarrow 0$ . We want to show that  $f$  is holomorphic at  $z = x + iy$ . We have that

$$\begin{aligned} f(z + h) - f(z) &= (u(x + h_1, y + h_2) - u(x, y)) + i(v(x + h_1, y + h_2) - v(x, y)) \\ &= \left( \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 \right) + \|h\| \psi_1(h) + i \left( \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 \right) + i \|h\| \psi_2(h) \\ &= \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h_1 + ih_2) + \|h\| \psi_1(h) + i \|h\| \psi_2(h) \end{aligned}$$

from Cauchy-Riemann. Thus, we have that<sup>4</sup>

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(z + h) - f(z)}{h} &= \lim_{h \rightarrow 0} \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} + \frac{\|h\|}{h} (\psi_1(h) + i \psi_2(h)) \\ &= \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{1}{i} \frac{\partial f}{\partial y} \right). \quad \square \end{aligned}$$

We now discuss complex power series.

**Definition 3.4.** A complex power series is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

with  $a_n \in \mathbb{C}$  and  $z$  a complex variable. We say that  $\sum_{n=0}^{\infty} a_n z^n$  converges at  $z_0 \in \mathbb{C}$  if there exists some  $w \in \mathbb{C}$  such that, for all  $\varepsilon > 0$ , there exists  $N_0 \in \mathbb{N}$  such that  $N \geq N_0$  satisfies

$$\left| \sum_{n=0}^N a_n z^n - w \right| < \varepsilon.$$

The series converges *absolutely* at  $z_0$  if there exists  $w \in \mathbb{R}$  such that

$$\left| \sum_{n=0}^N |a_n| |z_0|^n - w \right| < \varepsilon.$$

<sup>4</sup>i swear wenyu has an invisible key wired into her back like nano from nichijou and it's permanently cranked on



**Proposition 3.5.** If  $\sum_{n=0}^{\infty} a_n z^n$  converges absolutely at  $z_0 \in \mathbb{C}$ , then  $\sum_{n=0}^{\infty} a_n z^n$  converges at  $z_1 \in \mathbb{C}$  with  $|z_1| \leq |z_0|$ .

*Proof.* For all  $z_1 \in \mathbb{C}$  with  $|z_1| \leq |z_0|$ , consider the sequence of partial sums  $\{S_m(z_1)\}_{m \in \mathbb{N}}$  given by

$$S_m(z_1) = \sum_{n=0}^m a_n z_1^n.$$

We want to show that such a sequence converges. Since  $\mathbb{C}$  is complete, it suffices to show that said sequence is Cauchy. For all  $m < k \in \mathbb{N}$ , we have that

$$|S_k(z_1) - S_m(z_1)| = \left| \sum_{n=m+1}^k a_n z_1^n \right| \leq \sum_{n=m+1}^k |a_n| |z_1|^n \leq \sum_{n=m+1}^k |a_n| |z_0|^k. \quad \square$$

We now provide a few examples.

- (i) The complex exponential function for all  $z \in \mathbb{C}$ , given by

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

For all  $z \in \mathbb{C}$ , this sum converges because it converges absolutely (consider  $e^{|z|}$ ).

- (ii) The geometric series  $\sum_{n=0}^{\infty} z^n$ , where  $|z| < 1$ , converges; otherwise, is  $|z| \geq 1$ , it diverges. In particular, if  $\sum_{n=0}^{\infty} z^n$  converges, then  $|z^n| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Theorem 3.6** (Shakarchi, Thm. 2.5). Given a power series  $\sum_{n=0}^{\infty} a_n z^n$ , there exists  $R \in [0, \infty)$  such that (i) if  $|z| < R$ , the series converges, and (ii) if  $|z| > R$ , the series diverges. We call  $R$  the *radius of convergence* of  $\sum_{n=0}^{\infty} a_n z^n$ , and  $\{z \in \mathbb{C} \mid |z| < R\}$  the disc<sup>5</sup> of convergence. Moreover,  $R$  is given by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n} =: L,$$

where we use the convention that  $\frac{1}{0} = \infty$  and  $\frac{1}{\infty} = 0$ .

*Proof.* For all  $z \in \mathbb{C}$  with  $|z| < r < R$ , there exists some  $\varepsilon > 0$  such that

$$(L + \varepsilon) |z| = r < 1.$$

By definition of  $L$ , we have  $|a_n|^{1/n} \leq L + \varepsilon$  for all large  $n$ , meaning that

$$|a_n| |z|^n = \left( |a_n|^{1/n} |z| \right)^n \leq ((L + \varepsilon) |z|)^n = r^n, \quad r \in (0, 1),$$

whereby comparison with the geometric series  $\sum r^n$ , we see that  $\sum |a_n| |z|^n$  converges. Similarly, if  $|z| > R$ , we have that

$$\left( \frac{1}{r} - \varepsilon \right) |z| > 1,$$

where, using the definition of  $R$ , there exists an infinite subsequence  $a_{n_k}$  such that  $|a_{n_k}|^{1/n_k} \geq \frac{1}{R} - \varepsilon$ . We have that

$$|a_{n_k} z^{n_k}| - \left( |a_{n_k}|^{1/n_k} |z| \right)^{n_k} \geq \left[ \left( \frac{1}{R} - \varepsilon \right) |z| \right]^{n_k} > 1. \quad \square$$

---

<sup>5</sup>disque. ok i'll stop

## §4 Day 4: Complex Power Series (Sep. 11, 2025)

As per given in the previous lecture, recall that the complex power series is defined as an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, z \in \mathbb{C},$$

i.e.,  $z$  as a complex variable.

**Theorem 4.1.** The power series  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  defines a holomorphic function on its disc of convergence. The derivative of  $f$  is given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Moreover,  $f'$  has the same radius of convergence as  $f$ .

*Proof.* Let  $g$  be the power series defining  $f'$ , and let  $R \geq 0$  be the radius of convergence of  $f$ . The radius of convergence of  $g$  is also  $R$ , per Hadamard's formula,

$$\limsup_{n \rightarrow \infty} |n a_n|^{\frac{1}{n-1}} \stackrel{(*)}{=} \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n} \cdot \frac{n}{n-1}} = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \frac{1}{R},$$

since

$$n^{\frac{1}{n-1}} = e^{\frac{\log n}{n-1}} \xrightarrow{n \rightarrow \infty} e^0 = 1. \quad (*)$$

For all  $z_0 \in \mathbb{C}$  with  $|z_0| < r < R$  and  $h \in \mathbb{C} \setminus \{0\}$  with  $|z_0 + h| < r$ , let us compute the following,

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right|;$$

to start,

$$f(z) = \underbrace{\sum_{n=0}^N a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{E_N(z)},$$

where  $N \in \mathbb{N}$  is to be determined; we have that

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) &= \left( \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right) + \\ &\quad (S'_N(z_0) - g(z_0)) + \left( \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right). \end{aligned}$$

We compute each part individually.

$$\begin{aligned} \left| \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right| &= \left| \frac{\sum_{n=N+1}^{\infty} a_n (z_0 + h)^n - \sum_{n=N+1}^{\infty} a_n z_0^n}{h} \right| \\ &\leq \sum_{n=N+1}^{\infty} \frac{|a_n|}{h} |(z_0 + h)^n - z_0^n| \\ &\leq \sum_{n=N+1}^{\infty} |a_n| |(z_0 + h)^{n-1} + (z_0 + h)^{n-2} + \dots + z_0^{n-1}| \\ &\leq \sum_{n=N+1}^{\infty} |a_n| \gamma^{n-1} \cdot n \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

as  $y$  has the radius of convergence of  $R > r$ . Next,

$$|S'_N(z_0) - g(z_0)| \xrightarrow{N \rightarrow \infty} 0,$$

since  $S'_N(z_0) = \sum_{n=1}^N n a_n z_0^{n-1}$  and  $g(z_0) = \sum_{n=1}^{\infty} n a_n z_0^{n-1}$ . Given any  $\varepsilon > 0$ , we may choose a sufficiently large  $N$  such that

$$|S'_N(z_0) - g(z_0)| < \varepsilon, \quad \left| \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right| < \varepsilon,$$

per our two computations above. Since  $S_N(z)$  is a finite polynomial,  $S'_N(z_0)$  is the derivative of  $S_N(z)$  at  $z_0$ , and so there exists  $\delta > 0$  such that, for all  $0 \leq |h| < \delta$ , we have

$$\left| \frac{S_N(z_0 + h) - S_N(z_0)h}{h} - S'_N(z_0) \right| < \varepsilon,$$

which resolves all three parts of our expansion, and so we are done.  $\square$

**Corollary 4.2.** The power series  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  is infinitely complex differentiable on its disc of convergence. For  $k \in \mathbb{N}$ , its  $k$ th derivative  $f^{(k)}$  is given by

$$f^{(k)}(z) = \sum_{n=0}^{\infty} (a_n z^n)^k.$$

**Definition 4.3.** A function  $f : \Omega \rightarrow \mathbb{C}$  is said to be *analytic* at  $z_0 \in \Omega$  if there exists a power series  $\sum_{n=0}^{\infty} a_n (z - z_0)^n$  with positive radius of convergence such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

on a neighborhood of  $z_0 \in \Omega$ .

In particular, this means that if  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic, we have that  $f$  is holomorphic at  $z_0 \in \Omega$ , and so  $f$  is analytic at  $z_0 \in \Omega$  as well. The implication that analytic implies holomorphic was given by our earlier theorem; the direction that holomorphic implies analytic is given by Cauchy's integral formula, but we need to first define integration along curves.

- (i) A parameterized curve is a function  $z : [a, b] \rightarrow \mathbb{C}$ , where  $t \mapsto z(t)$ . This gives the orientation from  $z(a)$  to  $z(b)$ .
- (ii) (*Regularity conditions on curves*). We say that the parameterized curve is smooth if  $z'(t)$  exists, is continuous on  $[a, b]$ , and  $z'(t) \neq 0$  for  $t \in [a, b]$ . We say that the parameterized curve  $z$  is piecewise smooth if  $z$  is continuous on  $[a, b]$  and there exists a partition of  $[a, b]$  with  $a = a_0 < \dots < a_n = b$  such that  $z(t)$  is smooth on each  $[a_r, a_{r+1}]$ .

## §5 Day 5: Curves in the Complex Plane (Sep. 16, 2025)

We say that a parameterized curve is a function  $z : [a, b] \rightarrow \mathbb{C}$  where  $t \mapsto z(t)$ ; in particular,  $z$  gives the orientation from  $z(a)$  to  $z(b)$ . We say that  $z$  is *smooth* if  $z'(t)$  exists and is continuous on  $[a, b]$ , where  $z'(t) \neq 0$  for  $t \in [a, b]$ . We say it's *piecewise smooth* if  $z$  is continuous on  $[a, b]$  and we have a partition  $a = a_0 < \cdots < a_n = b$  such that  $z(t)$  is smooth on each  $[a_k, a_{k+1}]$ .

**Example 5.1.** Let  $z : [0, 2\pi] \rightarrow \mathbb{C}$ , where  $t \mapsto z_0 + Re^{it}$ , and  $z_1 : [0, \frac{\pi}{2}] \rightarrow \mathbb{C}$ , where  $t_0 \mapsto z_0 = Re^{i4t}$ .

We say that two smooth parameterizations,  $z : [a, b] \rightarrow \mathbb{C}$  and  $\tilde{z} : [c, d] \rightarrow \mathbb{C}$ , are *equivalent* if they have the same image and orientation; i.e., if there exists a continuously differentiable bijection  $s \mapsto t(s)$  from  $[c, d]$  to  $[a, b]$  such that  $t'(s) > 0$  (read: same orientation) and  $\tilde{z} = z \circ t$ . In this way, all equivalent smooth parameterizations of  $z : [a, b] \rightarrow \mathbb{C}$  can be written as a smooth curve  $\gamma$  with image  $z([a, b])$  and orientation from  $z(a)$  to  $z(b)$ . In addition, we denote  $\gamma^-$  as said smooth curve, but with reversed orientation.

A smooth or piecewise smooth curve given by  $z : [a, b] \rightarrow \mathbb{C}$  is said to be *closed* if  $z(a) = z(b)$ , and *simple* if  $z(t) \neq z(s)$  for all  $t \neq s$  in the time interval (note that if the curve is closed, we allow  $s = a, t = b$  to satisfy  $z(s) = z(t)$ ). We now define integration along curves.

**Definition 5.2.** Let  $f : \Omega \rightarrow \mathbb{C}$  be a continuous function, and let  $\gamma$  be a smooth curve in  $\Omega$  parameterized by  $z : [a, b] \rightarrow \mathbb{C}$ . Then

$$\int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt,$$

where we may realize  $f \circ z : [a, b] \rightarrow \mathbb{C}$ . The length of  $\gamma$  is defined as  $\text{length}(\gamma) = \int_a^b |z'(t)| dt$ .

**Example 5.3.** Consider the function  $f(z) = z^{-1}$  on  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ .<sup>6</sup> Let  $C$  be a circle in  $\mathbb{C}^*$  centered at  $z_0$  with radius  $R > 0$ , equipped with an anticlockwise orientation. Compute  $\int_C f(z) dz$ .

While this example seems trivial, there is a lot of casework to work through, and we don't have the prerequisite knowledge for it yet.

**Proposition 5.4.** Integration of continuous functions along smooth (or piecewise smooth) curves satisfy the following properties,

(i) (*Linearity*) For all  $\alpha, \beta \in \mathbb{C}$ , we have that

$$\int_{\gamma} (\alpha f + \beta g)(z) = \alpha \int_{\gamma} f(z) dz + \beta \int_{\gamma} g(z) dz$$

(ii) If  $\gamma^-$  is  $\gamma$  with reversed orientation, then

$$\int_{\gamma^-} f(z) dz = - \int_{\gamma} f(z) dz.$$

<sup>6</sup>417 notation seeping into my 354 work

(iii) We have the following inequality,

$$\left| \int_{\gamma} f(z) dz \right| \leq \left( \sup_{z \in \gamma} |f(z)| \right) \cdot \text{length}(\gamma).$$

**Exercise 5.5.** Check that the definition of integration is well-defined.

We now prove the above proposition.

*Proof.* Assume  $\gamma$  is smooth and parameterized by  $z : [a, b] \rightarrow \mathbb{C}$ . Then

$$\left| \int_{\gamma} f(z) dz \right| = \left| \int_a^b f(z(t)) \cdot z'(t) dt \right| \leq \int_a^b |f(z(t)) \cdot z'(t)| dt,$$

which we note is true by considering

$$\left| \sum_i u(t_i) + iv(t_i) \Delta t \right| \leq \sum_i |u(t_i) + iv(t_i)| \Delta t,$$

so we indeed have that

$$\int_a^b |f(z(t)) \cdot z'(t)| dt \leq \left( \sup_{z \in [a, b]} |f(z)| \right) \cdot \int_a^b |z'(t)| dt = \left( \sup_{z \in [a, b]} |f(z)| \right) \cdot \text{length}(\gamma) \quad \square$$

Suppose  $f : \Omega \rightarrow \mathbb{C}$ . A *primitive* for  $f$  on  $\Omega$  is a holomorphic function  $F : \Omega \rightarrow \mathbb{C}$  such that  $F'(z) = f(z)$  for all  $z \in \Omega$ .

**Theorem 5.6** (Complex Fundamental Theorem of Calculus). If a continuous function  $f$  has a primitive  $F$  on  $\Omega$ , and  $\gamma$  is a curve that begins at  $w_1$  and ends at  $w_2$ , then

$$\int_{\gamma} f(z) dz = F(w_2) - F(w_1).$$

*Proof.* Suppose  $\gamma$  is smooth and parameterized by  $z : [a, b] \rightarrow \mathbb{C}$  with  $z(a) = w_1$  and  $z(b) = w_2$ . Then

$$\int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt = \int_a^b (F(z(t)))' dt,$$

since we may note that  $(F \circ z)' = (F' \circ z) \cdot z' = (f \circ z) \cdot z'$ , whereby we note that the above integral evaluates to  $F(z(b)) - F(z(a)) = F(w_2) - F(w_1)$ .  $\square$

**Corollary 5.7.** If  $f$  is holomorphic on a region  $\Omega$  and  $f' = 0$ , then  $f$  is constant.

Recall that  $\Omega$  is called a region if it is an open connected set. Alternatively, connectedness is equivalent to path connectedness here, since if  $\Omega$  is path connected, it is connected (by Medusa), and if it is connected, then it is locally path connected, and through a partition, local path connectedness implies path connectedness.

*Proof.* Note that  $\Omega$  is path connected per our earlier digression; fix  $z_0 \in \Omega$ . We will show that  $f(z) = f(z_0)$  for all  $z \in \Omega$ ; let  $z, z_0$  be joined by a piecewise smooth curve  $\gamma$ . Then we have

$$0 = \int_{\gamma} f'(z) dz = f(z) - f(z_0),$$

and so  $f$  is constant on  $\Omega$ .  $\square$

**Theorem 5.8** (Goursat's Theorem). If  $\Omega$  is an open set in  $\mathbb{C}$  and  $T \subset \Omega$  is a triangle whose interior is also in  $\Omega$ , then for any holomorphic function  $f$  on  $\Omega$ , we have  $\int_T f(z) dz = 0$ .

*Proof.* Let  $T^{(0)}$  be the original triangle. Let  $d^{(0)}, p^{(0)}$  be the diameter and perimeter of  $T^{(0)}$  respectively. Take the midpoints of each side of  $T^{(0)}$ , and form 4 smaller triangles with orientation consistent to the orientation of  $T^{(0)}$ ; we will call these triangles  $T_1^{(1)}, \dots, T_4^{(1)}$ . Clearly,

$$\int_{T^{(0)}} f(z) dz = \sum_{k=1}^4 \int_{T_k^{(1)}} f(z) dz,$$

along with

$$\left| \int_{T^{(0)}} f(z) dz \right| = \sum_{k=1}^4 \left| \int_{T_k^{(1)}} f(z) dz \right|.$$

Let  $T_j^{(1)}$  be chosen to be such that  $\left| \int_{T_k^{(1)}} f(z) dz \right|$  is maximal among  $k \in \{1, \dots, 4\}$ ; we will write  $T^{(1)} = T_j^{(1)}$ , and iterate this process to obtain a sequence of triangles  $\{T_0, T_1, \dots\}$ , where

$$\left| \int_{T^{(0)}} f(z) dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z) dz \right|.$$

$d^{(k)}, p^{(k)}$  are defined analogously, where

$$d^{(k)} = \frac{1}{2^k} d^{(0)}, \quad p^{(k)} = \frac{1}{2^k} p^{(0)}.$$

Let  $\mathcal{T}^{(n)}$  be the solid triangle enclosed by  $T^{(n)}$ . Clearly,  $\mathcal{T}^{(0)} \supset \mathcal{T}^{(1)} \supset \dots \supset \mathcal{T}^{(n)}$ , and there exists a unique  $z_0 \in \mathbb{C}$  such that  $z_0 \in \mathcal{T}^{(n)}$  for every  $n$ ; since  $f$  is holomorphic at  $z_0$ , we have that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

with  $\psi(z) \rightarrow 0$  as  $z \rightarrow z_0$ . We may write,

$$\int_{T^{(n)}} f(z) dz = \int_{T^{(0)}} f(z_0) dz + \int_{T^{(n)}} f'(z_0)(z - z_0) dz + \int_{T^{(0)}} \psi(z)(z - z_0) dz.$$

The first two terms vanish, since  $f(z), f'(z_0)(z - z_0)$  have primitives  $f(z_0)z$  and  $\frac{1}{2}f'(z_0)(z - z_0)^2$  respectively. It remains to compute the last term; we have that

$$\left| \int_{T^{(n)}} \psi(z)(z - z_0) dz \right| \leq \left( \sup_{z \in T^{(n)}} |\psi(z)| \right) \left( \sup_{z \in T^{(n)}} |z - z_0| \right) \text{length } T^{(n)}$$

where we note the first term approaches 0 as  $n \rightarrow \infty$ , the second term is bounded above by  $2^{-n}d^{(0)}$ , and the third term is bounded above by  $2^{-n}p^{(0)}$ . We may combine everything to obtain

$$\left| \int_{T^{(0)}} f(z) dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z) dz \right| \leq d^{(0)}p^{(0)} \left( \sup_{z \in T^{(n)}} |\psi(z)| \right) \xrightarrow{n \rightarrow \infty} 0. \quad \square$$

## §6 Day 6: Cauchy's Theorem on a Disc (Sep. 18, 2025)

Recall Goursat's theorem from last class, where if  $\Omega \subset \mathbb{C}$  is open and  $T \subset \Omega$  is a triangle whose interior is contained in  $\Omega$ , then for any holomorphic function  $f$  on  $\Omega$ , we have that

$$\int_T f(z) dz = 0.$$

We introduce a follow-up to this theorem.

**Theorem 6.1.** If  $f$  is holomorphic on a disc, then  $\int_\gamma f(z) dz = 0$  for any closed curve  $\gamma$  in that disc.

To prove this, we start by using Goursat's theorem to show  $f$  has a primitive, and then we complete the proof using the complex FTC, i.e., if  $f$  is holomorphic on a disc, then  $f$  has a primitive on that disc.

*Proof.* After a translation, we may assume that the center of the disc is 0. Define  $F : D \rightarrow \mathbb{C}$ , given by  $z \mapsto \int_{\gamma_z} f(u) du$ . To show that  $F$  is holomorphic and  $F'(z) = f(z)$ , fix  $z \in D$ , and observe that for any  $h \in \mathbb{C} \setminus \{0\}$  with  $z + h \in D$ , we have that

$$F(z + h) - F(z) = \int_{\gamma_{z+h}} f(u) du - \int_{\gamma_z} f(u) du.$$

Regard this as the path from  $z$  to 0 to  $z + h$ . Let us add to the expression the integrals over two paths, going both directions so that we do not change the value of  $F(z + h) - F(z)$ , one between  $z$  and  $\Re(z + h) + i\Im(z)$ , and one between  $z$  and  $z + h$  directly. In this manner, we've created a rectangular region and a triangular region on which we have path integrals over, and per Goursat's theorem, they all vanish, and we are left with the integral on the path  $\eta$  from  $z$  to  $z + h$ . This means all that remains is to compute<sup>7</sup>

$$F(z + h) - F(z) = \int_\eta f(w) dw$$

Since  $f$  is continuous at  $z$ , we may write  $f(w) = f(z) + \psi(w)$ , where  $\psi(w) \rightarrow 0$  as  $w \rightarrow z$ . This means we may write

$$\int_\eta f(w) dw = \int_\eta f(z) dw + \int_\eta \psi(w) dw = f(z)(z + h - z) + \int_\eta \psi(w) dw,$$

upon which we may rearrange and rewrite the above RHS to obtain

$$\begin{aligned} \left| \frac{F(z + h) - F(z)}{h} - f(z) \right| &= \left| \frac{1}{h} \int_\eta \psi(w) dw \right| \\ &\leq \frac{1}{|h|} \sup_{w \in \eta} |\psi(w)| \underbrace{\text{length}(\eta)}_{=|h|} = \sup_{w \in \eta} |\psi(w)| \xrightarrow{h \rightarrow 0} 0. \end{aligned}$$

This concludes the hard part of the proof in showing that  $f$  has a primitive; by complex FTC, we immediately see that  $\int_\gamma f(z) dz = 0$ , since  $\gamma$  is a closed curve and its endpoints are equal to each other.  $\square$

<sup>7</sup>GOD KNOWS if this is a  $w$  or an  $\omega$ , i'm just going to use  $w$  for now. forensic analysis on yalls handwriting holy shit

We now give an example.

**Problem 6.2.** For all  $\xi \in \mathbb{R}$ , let  $\mathcal{F}$  denote the Fourier transform, and let

$$(\mathcal{F}f)(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

Show that if  $f(x) = e^{-\pi x^2}$ , we have that  $(\mathcal{F}f)(\xi) = f(\xi) = e^{-\pi \xi^2}$ .

*Solution.* In the  $\xi = 0$  case, we immediately have that

$$(\mathcal{F}f)(0) = \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} e^{-\pi x^2} = 1,$$

from computation through the Gaussian integral (polar coordinate transform). If  $\xi > 0$  (we note that  $\xi < 0$  follows analogously), let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be given by  $f(z) = e^{-\pi z^2}$ . Then, for  $R > 0$ , let us integrate on the rectangle from  $-R$  to  $R$ ,  $R$  to  $R + i\xi$ ,  $R + i\xi$  to  $-R + i\xi$ , and  $-R + i\xi$  to  $-R$ , where the latter three paths are denoted  $I_1, I_3, I_2$  respectively (we intentionally number this way because the two opposing sides  $I_1, I_2$  can be tackled together at once). We have that

$$0 = \int_{\gamma_R} f(z) dz = \int_{-R}^R f(x) dx + \int_{I_1} f(z) dz + \int_{I_2} f(z) dz + \int_{I_3} f(z) dz.$$

Let us consider the integral  $\left| \int_{I_1} f(z) dz \right|$ , with parameter  $I_1 : [0, \xi] \rightarrow \mathbb{C}$ , given by  $t \mapsto R + it$ ; we have that

$$\left| \int_{I_1} f(z) dz \right| = \left| \int_0^\xi f(R + it) i dt \right| = \left| \int_0^\xi e^{-\pi(R+it)^2} i dt \right|,$$

for which we observe that the integrand

$$e^{-\pi(R+it)^2} i = e^{-\pi(R^2 - t^2)} e^{i\pi 2Rt} \leq e^{-\pi(R^2 - \xi^2)} \rightarrow 0, \quad R \rightarrow +\infty,$$

so the integrals on  $I_1, I_2 \rightarrow 0$  for large enough  $R$  (we note that the same conclusion held for  $I_2$  because the computation follows analogously). For the last part, consider that

$$\int_{I_3} f(z) dz = \int_{-R}^R f(t + i\xi) dt = \int_{-R}^R e^{-\pi(t+i\xi)^2} dt = e^{\pi\xi^2} \int_{-R}^R e^{-\pi t^2} e^{-2\pi i \xi t} dt,$$

upon which we obtain  $e^{\pi\xi^2}(\mathcal{F}f)(\xi)$  as  $R \rightarrow \infty$ . This means we have that  $0 = -e^{\pi\xi^2}(\mathcal{F}f)(\xi)$ , where the minus sign is from the orientation of  $I_3$ . This means we may conclude that  $(\mathcal{F}f)(\xi) = e^{-\pi\xi^2} = f(\xi)$ .  $\square$



## §7 Day 7: Cauchy's Integral Formula and Corollaries (Sep. 23, 2025)

We start with an example.

**Example 7.1** (Fresnel Integrals). Prove that

$$\int_0^\infty (\sin x^2) dx = \int_0^\infty \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}.$$

*Solution.* To do this, we proceed by Cauchy's theorem, i.e., using functions of complex variables. Let  $e^{iz^2}$ , which, per Euler's formula, is equal to  $\cos(x^2) + i\sin(x^2)$  for  $x \in \mathbb{R}$ . Let us reframe the question by integrating  $e^{z^2}$ , where  $z \in \mathbb{C}$ , over the contour (closed curve) given by a  $\frac{\pi}{4}$  radian sector of the circle of radius  $R > 0$  centered at 0; specifically, the contour is given by  $0 \rightarrow R$ ,  $R \rightarrow Re^{i\pi/4}$  along the arc, and  $Re^{i\pi/4} \rightarrow 0$ . In this manner, let  $z = \rho e^{i\pi/4}$ , where  $\rho \in (0, R)$ , we have that

$$e^{-(\rho e^{i\pi/4})^2} = e^{-\rho^2 \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)^2} = e^{-\frac{\rho^2}{2}(1+i)^2} = e^{-\rho^2 i} = \cos(\rho^2) + i \sin(\rho^2).$$

Let the three paths in the contour (which we will call  $\gamma_R$ ) be given by  $I_1, I_2, I_3$  in order; we have that, by Cauchy's theorem,

$$0 = \int_{\gamma_R} f(z) dz = \int_{I_1} f(z) dz + \int_{I_2} f(z) dz + \int_{I_3} f(z) dz.$$

Directly compute as follows, where  $f(z) = e^{z^2}$ ,

$$\int_{I_1} f(z) dz = \int_0^R e^{-x^2} dx \xrightarrow{R \rightarrow \infty} \frac{\sqrt{\pi}}{2},$$

Let  $-I_3 : [0, R] \rightarrow \mathbb{C}$  be given by  $t \mapsto te^{i\pi/4}$ ; we have,

$$\begin{aligned} \int_{-I_3} f(z) dz &= \int_0^R f(te^{i\pi/4}) e^{i\pi/4} dt \\ &= e^{i\pi/4} \int_0^R e^{-(te^{i\pi/4})^2} dt \\ &= e^{i\pi/4} \left[ \int_0^R \cos t^2 dt - i \int_0^R \sin(t^2) dt \right], \end{aligned}$$

and finally, for the integral on  $I_2$  (where  $I_2 : [0, \frac{\pi}{4}]^2 \rightarrow \mathbb{C}$  and  $t \mapsto Re^{it}$ ), we have that

$$\int_{I_2} f(z) dz = \int_0^{\pi/4} e^{-(Re^{it})^2} iRe^{it} dt,$$

for which we may bound the integrand as follows,

$$\left| e^{-(Re^{it})^2} iRe^{it} \right| \leq R \left| e^{-(Re^{it})^2} \right| = R \left| e^{-R^2(\cos(2t) + i\sin(2t))} \right| = Re^{-R^2(\cos 2t)}.$$

This means we may write

$$\left| \int_{I_2} f(z) dz \right| \leq \int_0^{\pi/4} \left| e^{-(Re^{it})^2} iRe^{it} \right| dt = \int_0^{\pi/4} Re^{-R^2(\cos 2t)} dt.$$

Let us compute  $\cos(2t)$ ; we have that  $2t \in [0, \frac{\pi}{2}]$ , so  $\cos(2t) = \sin(\frac{\pi}{2} - 2t)$ , and  $\sin(\theta) \geq \frac{2}{\pi}\theta$  by appealing to geometric intuition; this means

$$\begin{aligned} \int_0^{\pi/4} R e^{-R^2(\cos 2t)} dt &\leq \int_0^{\pi/4} R e^{-R^2 \frac{2}{\pi}(\frac{\pi}{2} - 2t)} dt && (\text{Let } s = \frac{\pi}{2} - 2t) \\ &= \frac{1}{2} \int_0^{\pi/2} R e^{-R^2 \frac{2}{\pi} s} ds \\ &= \frac{1}{2} \int_0^{\pi/2} R d \left( \frac{e^{-R^2 \frac{2}{\pi} s}}{-R^2 \frac{2}{\pi}} \right) \\ &= \frac{1}{2} \cdot \frac{1}{R^2 \frac{2}{\pi}} (e^{-R^2} - 1) \xrightarrow{R \rightarrow \infty} 0. \end{aligned}$$

Having established computations for  $I_1, I_2, I_3$ , we may now write

$$0 = \frac{\sqrt{\pi}}{2} - e^{i\pi/4} \left[ \int_0^\infty \cos(x^2) dx - i \int_0^\infty \sin(x^2) dx \right]$$

This means we have

$$\begin{aligned} \int_0^\infty \cos(x^2) dx - i \int_0^\infty \sin(x^2) dx &= e^{-i\pi/4} \frac{\sqrt{\pi}}{2} \\ &= \frac{\sqrt{\pi}}{2} \left( \frac{\sqrt{\pi}}{2} - i \frac{\sqrt{\pi}}{2} \right) \\ &= \frac{\sqrt{2\pi}}{4} - i \frac{\sqrt{2\pi}}{4}. \end{aligned} \quad \square$$

We now discuss Cauchy's integral formula. As another example, let  $D$  be a disc centered at  $z$ , and let  $f$  be a holomorphic function; we may express  $f(z)$  using the values of  $f$  on  $\partial D$ .

**Example 7.2** (Steady-State Heat Equation). Let  $g(x, y)$  be continuous on  $\mathbb{R}^2$ . Find  $u(x, y)$  satisfying

$$\begin{cases} \Delta u = 0 & \text{on } D, \\ u = g & \text{on } \partial D, \end{cases}$$

where  $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  is the Laplacian operator. The solution is given by considering  $(x, y) = (r \cos \theta, r \sin \theta)$ , where

$$u(r, \theta) = \int P_r(\theta, \varphi) g(\cos \varphi, \sin \varphi) d\varphi, \quad P_r(\theta, \varphi) = \frac{1 - r^2}{1 - 2r \cos(\theta - \varphi) + r^2}$$

where  $P_r$  is called the *Poisson kernel*.

**Theorem 7.3** (Cauchy's Integral Formula). Suppose  $f$  is holomorphic in an open set  $\Omega$  that contains the closure of a disc  $D$ . Let  $C = \partial D$  equipped with the anticlockwise orientation. Then for any  $z \in D$ ,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

*Proof.* We start by constructing a “keyhole contour” on  $D$ , where  $\delta$  is the width of the corridor, and  $\varepsilon$  is the radius of the circle centered at  $z$ . The contour can be thought of as picking a point in  $C$  and connecting it to the  $\varepsilon$ -circle about  $z$  with a  $\delta$ -wide corridor.

Let the contour be called  $\Gamma_{\delta,\varepsilon}$ . Let  $F(\zeta) = \frac{f(\zeta)}{\zeta-z}$ ; clearly, it is holomorphic on  $\Omega \setminus \{z\}$ . By Cauchy's theorem,

$$0 = \int_{\Gamma_{\delta,\varepsilon}} F(\zeta) d\zeta = \int_{I_1} F(\zeta) d\zeta + \int_{I_2} F(\zeta) d\zeta + \int_{I_3} F(\zeta) d\zeta + \int_{I_4} F(\zeta) d\zeta,$$

where  $I_1, I_3$  represent the paths on  $C$  and the  $\varepsilon$ -circle about  $z$  respectively, and  $I_2, I_4$  the “walls of the corridor”. We start with some basic observations;

(i) If we let  $\delta \rightarrow 0^+$ , then

$$\int_{I_1} F(\zeta) d\zeta = \int_C \frac{f(\zeta)}{\zeta-z} d\zeta.$$

(ii) Again, if we let  $\delta \rightarrow 0^+$ , we have that

$$\int_{I_2} F(\zeta) d\zeta = - \int_{I_4} F(\zeta) d\zeta,$$

since they are simply two path integrals of the opposite orientation.

(iii) For  $I_3$ , we may first write

$$\int_{I_3} F(\zeta) d\zeta = \int_{I_3} \frac{f(\zeta)}{\zeta-z} d\zeta;$$

if we let  $\varepsilon \rightarrow 0^+$ , we see that this is problematic, since we have a singularity at  $z$ . However, we notice that the integrand resembles the definition of the derivative, i.e., we may write

$$\frac{f(\zeta)}{\zeta-z} = \frac{f(\zeta) - f(z)}{\zeta-z} + \frac{f(z)}{\zeta-z},$$

so we obtain

$$\int_{I_3} \frac{f(\zeta)}{\zeta-z} d\zeta = \int_{I_3} \frac{f(\zeta) - f(z)}{\zeta-z} d\zeta + f(z) \int_{I_3} \frac{1}{\zeta-z} d\zeta,$$

where the latter term is equal to  $-2\pi i f(z)$ , per (p.47 in Shakarchi)

$$\int_{I_3} \frac{f(z)}{\zeta-z} d\zeta = f(z) \int_{I_3} \frac{d\zeta}{\zeta-z} = -f(z) \int_0^{2\pi} \frac{\varepsilon i e^{-it}}{\varepsilon e^{-it}} dt = -f(z) 2\pi i.$$

For the former term, there exists  $\varepsilon_0 > 0$  such that for all  $\zeta \in D_{\varepsilon_0}(z)$ , we have that

$$\left| \frac{f(\zeta) - f(z)}{\zeta-z} \right| \leq |f'(z)| + 2.$$

We obtain

$$\left| \int_{I_3} \frac{f(\zeta) - f(z)}{\zeta-z} d\zeta \right| \leq (|f'(z)| + 2) \cdot 2\pi\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Combining all these observations, we obtain

$$0 = \int_{\Gamma_{\delta,\varepsilon}} \frac{f(\zeta)}{\zeta-z} d\zeta \xrightarrow{\delta,\varepsilon \rightarrow 0^+} \int_C \frac{f(\zeta)}{\zeta-z} d\zeta - 2\pi i f(z),$$

from which we conclude Cauchy's integral formula.<sup>8</sup> □

---

<sup>8</sup>reference: p.45-47 Shakarchi

**Theorem 7.4** (Cor. 4.2, Shakarchi). “A holomorphic function is infinitely complex differentiable.”<sup>9</sup> Suppose  $f$  is holomorphic in an open set  $\Omega$ . Then  $f$  has infinitely many complex derivatives in  $\Omega$ . Moreover, for any  $z \in \Omega$  and  $n \in \mathbb{Z}_{\geq 0}$ , we have that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

*Proof.* We proceed by induction on  $n$ . The base case  $n = 0$  is immediately given by Cauchy’s integral formula; assuming that the statement is true for  $n-1$ , for any  $h \in \mathbb{C} \setminus \{0\}$  such that  $z + h \in D$ , we have that

$$\begin{aligned} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} &= \frac{(n-1)!}{2\pi i h} \int_C \left[ \frac{f(\zeta)}{(\zeta - z - h)^n} - \frac{f(\zeta)}{(\zeta - z)^n} \right] d\zeta \\ &= \frac{(n-1)!}{2\pi i h} \int_C f(\zeta) \left[ \frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right] d\zeta. \end{aligned}$$

By binomial expansion, we have that

$$\begin{aligned} &\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \\ &= \frac{1}{(\zeta - z - h)^n (\zeta - z)^n} [(\zeta - z)^n - (\zeta - z - h)^n] \\ &= \frac{h}{(\zeta - z - h)^n (\zeta - z)^n} [(\zeta - z)^{n-1} + (\zeta - z)^{n-2}(\zeta - z - h) + \cdots + (\zeta - z - h)^{n-1}]. \end{aligned}$$

By taking  $h$  sufficiently small, we obtain

$$\frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{2n}} n(\zeta - z)^{n-1} d\zeta = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta. \quad \square$$

**Theorem 7.5** (Thm. 4.4, Shakarchi). “A holomorphic function is locally a power series”. Suppose  $f$  is holomorphic in an open set  $\Omega$ . If  $D$  is a disc centered at  $z_0$  whose closure is contained in  $\Omega$ , then  $f$  has a power series expansion at  $z_0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for  $z \in D$ , and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \geq 0.$$

*Proof.* Fix any  $z \in D$ ; by Cauchy’s integral formula, we have that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Note that per our previous corollary. The idea is to write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 + z_0 - z} = \frac{1}{(\zeta - z_0)} \frac{1}{\left(1 - \frac{z - z_0}{\zeta - z_0}\right)},$$

<sup>9</sup>hell, i need to run a marathon with 20mg of thc in my system. props wenyu

where we observe that since  $z \in D$  is fixed and  $\zeta \in C$ , we know that there exists some  $r \in (0, 1)$  such that

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < r,$$

so we may regard the geometric series representation

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^n,$$

for which the series converges uniformly for any  $\zeta \in C$ . This means we may interchange the integral and the sum to obtain

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0^n). \quad \square$$

**Corollary 7.6** (Liouville's Theorem: Thm. 4.5, Shakarchi). If  $f$  is entire and bounded, then  $f$  is constant. We say that  $f$  is *entire* if it is holomorphic on the whole of  $\mathbb{C}$ .

*Proof.* We will prove this later on. Though, it is done by observing that  $\mathbb{C}$  is connected (hence a region, i.e., open connected set), then checking  $f' = 0$ , and so  $f$  is constant.  $\square$

**Corollary 7.7** (Cauchy's Inequality). If  $f$  is holomorphic in an open set that contains the closure of a disc  $D$  centered at  $z_0$  with radius  $r$ , then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! \|f\|_C}{R^n},$$

where  $\|f\|_C = \sup_{z \in C} |f(z)|$  (and  $C$  is the boundary of  $D$ .)

*Proof.* We have that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta;$$

if we let  $C : [0, 2\pi] \rightarrow \mathbb{C}$  be given by  $t \mapsto z_0 + Re^{it}$ , then the above is equal to

$$\frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^{n+1} e^{i(n+1)t}} iRe^{it} dt,$$

for which we may write

$$\left| \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^{n+1} e^{i(n+1)t}} iRe^{it} dt \right| \leq \frac{n!}{2\pi} \cdot \frac{\|f\|_C}{R^n} \cdot 2\pi = \frac{n! \|f\|_C}{R^n},$$

which finishes the proof.  $\square$