# MAT327 Lecture Notes

ARKY!! :3C

'24 Fall Semester

## Contents

# §1 Day 1: Open Sets and Continuity (Sep. 3, 2024)

> This class is *MAT327*; 3 meaning third year, 2 meaning the contents are on the fundamental side, and 7 meaning no mercy.
> – Dror Bar-Natan

Course administration matters first;

- The course link is given here (this will link straight to Quercus).
- The textbook is James Munkres' Topology (online PDF: ETH Zurich mirror); Prof Bar-Natan strongly recommends a paper copy, though (since people get distracted on the computer).

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Today's reading in the textbook is on Ch. 1, sections 1 to 8, and Ch. 2, sections 12 to 13. Readings are supplementary to lecture material[1]. The goal of this course is to understand continuity in its most general form; in particular,

- In MAT157, we studied continuity in $f : \mathbb{R} \to \mathbb{R}$;
- In MAT257, we will study continuity in $f : \mathbb{R}^n \to \mathbb{R}^m$;

but in this class, we will study continuity in $F : X \to Y$, where $X, Y$ are arbitrary spaces, such as (but not limited to) $\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{N}}$ (binary sequences), and so on. We start with some refreshers on previous coursework;

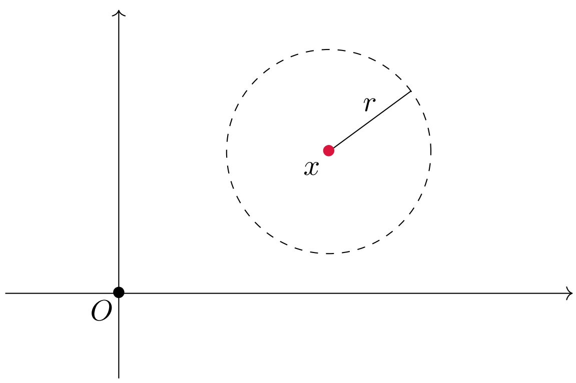
**Definition 1.1** (Continuity in $\mathbb{R}^n \to \mathbb{R}^m$)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called "continuous" if it is continuous at all points in $\mathbb{R}^n$. Specifically, for all $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

With this, we state our main theorem for today (proof given later),

**Theorem 1.2** (Continuity on $\mathbb{R}^n$ if and only pre-image of open subsets is open)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if all open subsets $U \in \mathbb{R}^m$ have $f^{-1}(U)$ open.

In order to build up to the above, we start by defining some terms;

**Definition 1.3** (Open Ball)**.** Let $r > 0$, and $x \in \mathbb{R}^n$. An open ball of radius $r$ about $x$ is given formally by $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Visually, we have




Note that the border of the circle is not in $B_r(x)$, as the distance metric asks for a strict inequality.

[1] iirc it won't be tested unless specified. its still good to learn tho

In a similar fashion, we have

**Definition 1.4** (Open Set)**.** A set $U \subset \mathbb{R}^n$ is called *open* if, for all $x \in U$, there exists an open ball about $x$ contained in $U$. Specifically, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

Intuitively, we could say that the set $U$ does not contain its edge; if it did, let $x$ be on said edge; then we would not be able to fit an open ball about $x$ in $U$. Here are some examples of open sets from lecture;

1. The whole set $U = \mathbb{R}^n$ is open.
2. The empty set $\emptyset$ is open. Since there does not exist any $x \in \emptyset$, no conditions on balls need to be satisfied; "every dog in the empty set of dogs is green".
3. $(0, 1)$ is open on $\mathbb{R}^1$ (and any open interval, for that matter).
4. $B_r(x) \in \mathbb{R}^n$ is open. To see this, observe that for any $y \in B_r(x)$, we may pick $\varepsilon < r - |x - y|$; by triangle inequality, all elements in $B_\varepsilon(y)$ must also be in $B_r(x)$.

We now define images and pre-images; let us have a function between sets $f : X \to Y$. For subsets $A \subset X$ and $B \subset Y$,

$$f(A) = \{f(a) \mid a \in A\},$$
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Since $f$ need not be injective, $f^{-1}$ does not necessarily exist. While images of a union of subsets is a union of the images, the same is not true for intersections; let $A_1, \dots, A_n$ be subsets of $X$; then

$$f\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f(A_i), \qquad f\left(\bigcap_{i=1}^n A_i\right) \subset \bigcap_{i=1}^n f(A_i).$$

On the other hand, pre-images preserve both union and intersection; let $B_1, \dots, B_n \subset Y$; then

$$f^{-1}\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f^{-1}(A_i), \qquad f^{-1}\left(\bigcap_{i=1}^n A_i\right) = \bigcap_{i=1}^n f^{-1}(A_i).$$

As for set complements, we have

$$f^{-1}(B^C) = f^{-1}(B)^C,$$

but the same cannot be said for images; $f(A)^C \subset f(A^C)$ for surjective $f$, and vice versa for injective $f$. Equality occurs only if $f$ is a bijection.

**Note:** for the proof of Theorem 1.2, I'll leave it out in case Prof. Bar-Natan proves it later on Thursday.

# §2 Day 2: Basic Definitions and Topological Spaces (Sep. 5, 2024)

We start by recapping the previous lecture; we introduced

- Open sets $U \subset \mathbb{R}^n$, where there exists an open ball of radius $\varepsilon > 0$ about any $x \in U$.
- Continuity in $f : \mathbb{R}^n \to \mathbb{R}^m$.

Recall the continuity property from last class, that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if the pre-image of an open set $U$ is open.

($\Rightarrow$) To start, pick any $x_0 \in f^{-1}(U)$, and let us have a small enough $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \in U$ (this is possible since $U$ is open). By continuity, there exists $\delta > 0$ such that any $x \in B_\delta(x_0)$ satisfies $f(x) \in B_\varepsilon(f(x_0)) \subset U$; this means $x \in f^{-1}(U)$ by definition of pre-image, and since $x$ was arbitrary, we see $B_\delta(x_0) \in f^{-1}(U)$. Moreover, since $x_0$ was also arbitrary, our construction shows that there always exists a $\delta$-ball about any point in the pre-image, and so $f^{-1}(U)$ is open. □

($\Leftarrow$) For the other direction, take any $x_0 \in \mathbb{R}^n$, and $\varepsilon > 0$. Since $B_\varepsilon(f(x_0))$ is open we have that $f^{-1}(B_\varepsilon(f(x_0)))$ is open as well. This means we may pick a small enough $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x_0)))$ (by definition of openness), and we immediately see

$$f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)).$$

This is a reconstruction of the epsilon-delta definition of continuity, and so we are done. □

Open sets in $\mathbb{R}^n$ have a number of properties;

(a) $\emptyset, \mathbb{R}^n$ are open sets.

(b) The union of open sets are open; specificay, let $S$ be a set of indices, and let $A_\alpha \subset \mathbb{R}^n$ for all $\alpha \in S$. Then

$$\bigcup_{\alpha \in S} A_\alpha$$

is an open set.

(c) The finite intersection of open sets are also open. Let $A_1, \dots, A_n \subset \mathbb{R}^n$ be open; then

$$\bigcap_{i=1}^{n} A_i$$

is open.

We now proceed with the proofs for these properties.

(a) This was proven last lecture.

(b) For any $x$ in the union, by definition, there exists $\alpha \in S$ such that $x \in A_\alpha$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A_\alpha$, and $A_\alpha \subset \bigcup_{\alpha \in S} A_\alpha$. This concludes that the union is open.

(c) For any $x$ in the finite intersection, then $x$ is an element of each of $A_1, \dots, A_n$. Let us have $\varepsilon_1, \dots, \varepsilon_n > 0$ such that $B_{\varepsilon_i}(x) \varepsilon A_i$, and take $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$ (note that minimum is defined only for finite lists), which means $B_\varepsilon(x) \subset \bigcap_{i=1}^{n} A_i$.

In the case of infinite intersection of open sets, the resulting set need not be open. For example, let us consider the infinite intersection of intervals

$$\bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, \frac{1}{j}\right) = \{0\},$$

which is not open.

**Definition 2.1** (Topology on a Set)**.** Let $X$ be a set equipped with topology $\mathcal{T}$ on $X$. $\mathcal{T}$ is a collection of subsets of $X$, i.e. $\mathcal{T} \subset \mathcal{P}(X)$, with properties

- $\varepsilon, X \in \mathcal{T}$.
- The union of subsets of $\mathcal{T}$ is also in $\mathcal{T}$.
- The finite intersection of subsets of $\mathcal{T}$ is also in $\mathcal{T}$.

We call $(X, \mathcal{T})$ a *topological space*, which may be abbreviated to $X$ is $\mathcal{T}$ is given or obvious. Moreover, we define another notion of openness, where $U \in \mathcal{T}$ is said to "be open relative to $\mathcal{T}$," or that "$U$ is open." Here are a few examples of such topological spaces.

(a) The standard topology on $\mathbb{R}^n$ is given by

$$\mathbb{R}_{std}\left(\mathbb{R}, \mathcal{T}_{std} = \{U \subset \mathbb{R}^n \mid U \text{ is open in the ``old sense''}\}\right),$$

i.e. collection of open intervals[2].

(b) The discrete topology, $X_{discrete} = (X, \mathcal{T} = \mathcal{P}(x))$ can be defined over any set $X$, equipped with $\mathcal{T}$ as the collection of all subsets.

(c) The trivial topology, $X_{trivial} = (X, \mathcal{T} = \{\emptyset, X\})$.

**Definition 2.2** (Continuity between Topological Spaces)**.** If $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are topological spaces, and $f : X \to Y$ is a function between said spaces, then we say $f$ is continuous if, for all $U \in \mathcal{T}_Y$, we have that $f^{-1}(U) \in \mathcal{T}_X$. This draws from our definition from the notion that pre-images of open sets are open.

Now for the examples from lecture;

(a) $f : X_{discrete} \to \mathbb{R}_{std}$ is always continuous.

(b) $f : X_{trivial} \to \mathbb{R}_{std}$ is continuous if and only if $f$ is constant.

(c) $f : \mathbb{R}_{std} \to X_{trivial}$ is always continuous.

(d) $f : \mathbb{R}_{std} \to X_{discrete}$ is almost never continuous, except when $X$ is empty or a singleton (in which case, $X_{discrete} = X_{trivial}$). If $x_0 \in X$ yet $X \setminus \{x_0\} \neq \emptyset$, then let us have

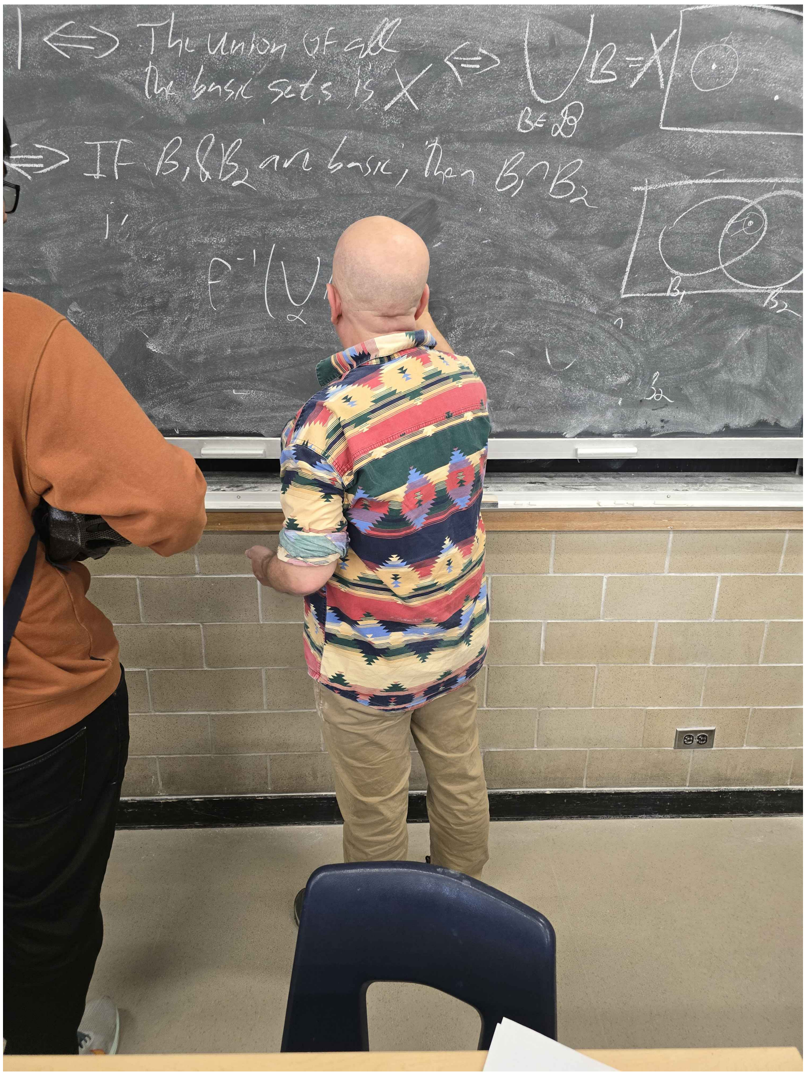
$$A := f^{-1}(\{x_0\}),$$
$$B := f^{-1}(X \setminus \{x_0\}).$$

While $A, B$ are both open, we see that $A \cup B = \mathbb{R}$, and $A \cap B = \emptyset$.[3]

[2] this is how i understood it, he might clarify next time?
[3] this one i don't completely get yet.

# §3 Day 3: Homeomorphisms and Bases (Sep. 10, 2024)

Outfit of the day (by popular request)




Course administrative details first;

- The reading for this week is on sections 12 to 14 (this week will cover these contents), and 15 to 16 as prereading.

Recap of last lecture:

- A topology $\mathcal{T} \subset \mathcal{P}(X)$ is a collection of subsets of $X$, of which we require $\{\emptyset, X\} \subset \mathcal{T}$. We also require $\mathcal{T}$ to be closed under arbitrary unions and finite intersections.

- We say a function $F : X \to Y$ is continuous if and only if for all $U \in \mathcal{T}_Y$, we have $f^{-1}(U) \in \mathcal{T}_X$.

Today we will cover homeomorphisms and bases. To start, recall the example topologies, such as $\mathcal{T}_{\text{std}}$ on $\mathbb{R}^n$ (where $\mathcal{T}_{\text{std}}$ consists of the open balls), $\mathcal{T}_{\text{triv}}$, and $\mathcal{T}_{\text{disc}}$. We also introduce a new example topology (where FC standards for finite complement),

$$\mathcal{T}_{\text{FC}} = \{U \subset X \mid X \setminus U \text{ is finite, or } U = \emptyset\}.$$

Note that Prof. Bar-Natan may interchange the notations $-$ or $\setminus$ to represent set difference.

**Theorem 3.1** (Composition of Continuous Functions is Continuous)**.** Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f : X \to Z$ is continuous (relative to the same topologies on $X$ and $Z$).

If $U \in \mathcal{T}_Z$, we have $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U)$, where by definition of continuity, we see that pre-images of open sets are open, and we have $g^{-1}(U)$ is open in $Y$, and similarly $f^{-1}(g^{-1}(U))$ is also open in $X$. $\square$

In tutorial, we equipped $X$ with topologies $\mathcal{T}_1, \mathcal{T}_2$ (i.e., $X$ is a topological space in two ways). We say $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if $T_1 \supset T_2$, and coarser for the opposite direction; the words bigger and stronger may be used interchangeably with finer, and smaller or weaker for coarser. For example, the identity map

$$\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$$

is continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$. To see this, let $U$ be an open set in $(X, \mathcal{T}_2)$; then $U$ must be open in $(X, \mathcal{T}_1)$ as well, which is true for any $U$ only if $\mathcal{T}_1 \supset \mathcal{T}_2$[4].

**Definition 3.2** (Homeomorphism)**.** A map $h : X \to Y$ is called a homeomorphism if $h$ is continuous, bijective, and $h^{-1}$ is continuous as well.[5]

Note that continuous bijective maps $h$ need not have continuous inverses; for example, let us have $\mathrm{id} : X_{\text{disc}} \to X_{\text{triv}}$. id is continuous as per our above example, while its inverse is not. Another example is to consider $[0, 2\pi) \to S^1$ (unit circle), where $x \mapsto (\cos 2x\pi, \sin 2x\pi)$; we see that the inverse is discontinuous at 0 and $2\pi$ radians, even if the map is continuous and bijective (also observe that $[0, 1)$ is not compact while $S^1$ is).
An example of a homeomorphism is as follows (as per tutorial); let us consider[6]

$$(-1, 1)_{\text{std}} \cong \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)_{\text{std}} \cong \mathbb{R}_{\text{std}},$$

where we may map the first to the second by $x \mapsto \frac{\pi}{2}x$, and the second to third by $x \mapsto \tan x$. Since the composition of continuous maps is continuous, we also see $(-1, 1)_{\text{std}}$ is homeomorphic to $\mathbb{R}_{\text{std}}$ (any open interval is homophobic to $\mathbb{R}$ for that matter).

Another example of a homemorphism is $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ if $\mathcal{T}_1 = \mathcal{T}_2$.

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A "basis" for a topology on $X$ is a collection $\mathcal{B} \subset \mathcal{P}(X)$ of subsets such that

1. For all $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$. We call $B$ a *basic set.*
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists a third basic set $B_3$ such that $x \in B_3 \subset B_1 \cap B_2$.

[4]includes the case $\mathcal{T}_1 = \mathcal{T}_2$; i'm following florian notation here with $\subset$ and $\subsetneq$ for explicit non equality
[5]smth smth coffee cup and donut "extra homework: go find a nice video on why this is true on youtube"
[6]dror was using $\sim$ for homeomorphism symbol today. if he keeps using that i'll adjust my notes, but for now i'll use $\cong$ cuz afaik its used more...?

In particular, the first condition is equivalent to $\bigcup_{B\in\mathcal{B}} B = X$ (i.e., the basis forms a covering of $X$), and the second condition is equivalent to the basic sets contained in $B_1 \cap B_2$ forming a cover of $B_1 \cap B_2$, i.e.

$$B_1 \cap B_2 = \bigcup_{\substack{B\in\mathcal{B} \\ B\subset B_1\cap B_2}} B.$$

Here are some examples;

(a) $\{B_r(x_0)\} \subset \mathcal{P}(\mathbb{R}^n)$, i.e. the open balls on $\mathbb{R}^n$ form a basis.

(b) The one-dimensional analogue of case (a) is $\{(a, b) \mid a < b\}$, and it forms a basis on $\mathbb{R}$.

(c) $\{[a, b) \mid a < b\}$ is called the lower limit topology, and it forms a basis on $\mathbb{R}$.

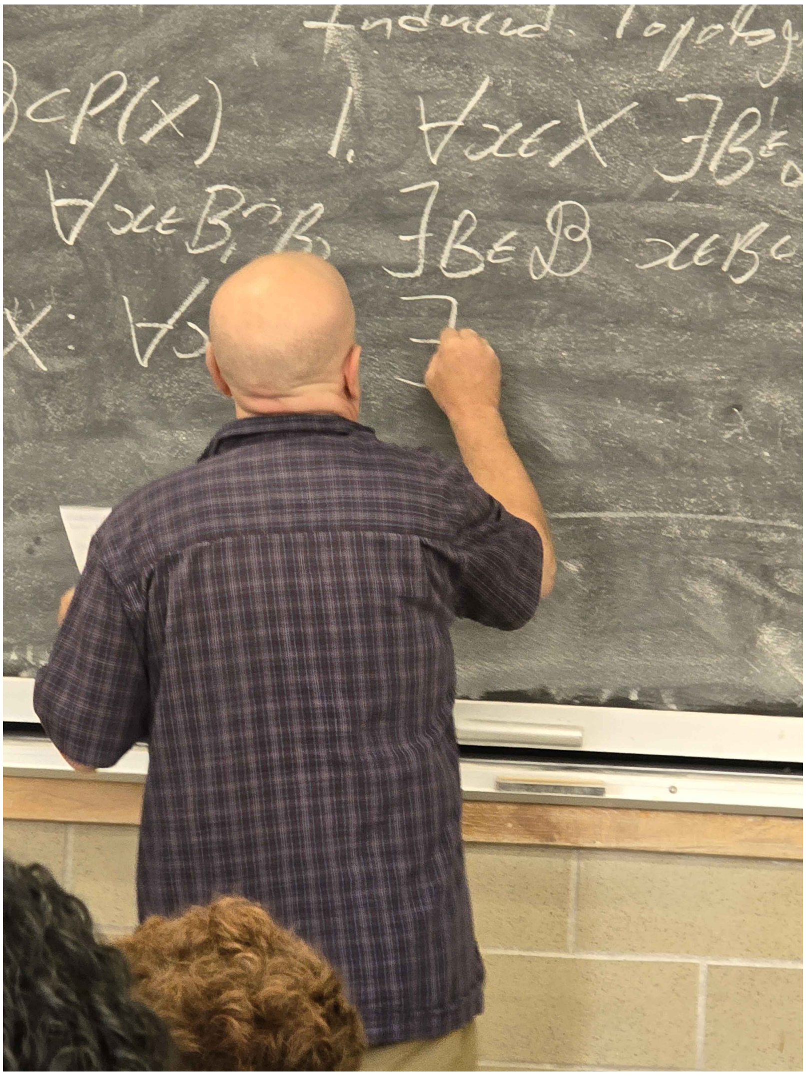
(d) $\{[a, b] \mid a < b\}$ implies $[a, b] \cap [b, c] = \{b\}$, which forces the basic set to include all singletons on $\mathbb{R}$. In that case, this is simply the discrete topology (?).

**Theorem 3.3.** $\mathcal{T}_{\mathcal{B}} = \{U \subset X \mid \forall x \in U \implies \exists B \in \mathcal{B}$ such that $x \in B \subset U\}$, i.e. the collection of all unique basic sets.

This will be expanded on next lecture.

# §4 Day 4: Basis; Orderings on Sets; Product Topology (Sep. 10, 2024)

Outfit of the day! Gives King Dice vibes tbh (from cuphead) i like the purple a lot :3



Recap of last lecture:

- We define the basis $\mathcal{B} \subset \mathcal{P}(X)$ of a topology to have the following properties (which we will refer to as the first and second axioms),
  1. For all $x \in X$, there exists a basic set $B \in \mathcal{B}$ such that $x \in B$.
  2. For any $x$ in the intersection of two basic sets (i.e. $x \in B_1 \cap B_2$), there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

With this, we may construct the topology generated by $\mathcal{B}$, i.e.

$$\mathcal{T}_{\mathcal{B}} := \{U \subset X \mid \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U\}.$$

**Theorem 4.1** (Basis Topology is a Topology on $X$)**.** We claim that $\mathcal{T}_{\mathcal{B}}$ is a topology on $X$.

We proceed by checking that $\mathcal{T}_{\mathcal{B}}$ satisfies the required properties.

1. Observe that $U = \emptyset, X$ are both in $\mathcal{T}_{\mathcal{B}}$; if $U = \emptyset$, then there are no elements $x \in U$ to bother about, of which the condition is vacuously true; if $U = X$, then the condition is true by the basis axioms.

2. Let us consider an indexing set $I$, and let us consider the arbitrary union of subsets $U_\alpha \subset X$
$$U := \bigcup_{\alpha \in I} U_\alpha.$$
Then any $x \in U$ belongs to $x \in U_\alpha$ for some index $\alpha \in I$; since $U_\alpha$ satisfies the condition, we see that the union $U$ also satisfies the condition as well.

3. Finally, for intersections, let us take $U_1, U_2 \in \mathcal{T}$ where[7]
$$U_1 = \bigcup_{\alpha_1 \in A_1} B_{1,\alpha_1},$$
$$U_2 = \bigcup_{\alpha_1 \in A_2} B_{2,\alpha_2},$$
where each $B_{i,\alpha_j}$ for $i, j \in \{1, 2\}$ above is a basic set in $\mathcal{B}$. Then
$$U_1 \cap U_2 = \left( \bigcup_{\alpha_1 \in A_1} B_{1,\alpha_1} \right) \cap \left( \bigcup_{\alpha_2 \in A_2} B_{2,\alpha_2} \right) = \bigcup_{\substack{\alpha_1 \in A_1 \\ \alpha_2 \in A_2}} \left( B_{1,\alpha_1} \cap B_{2,\alpha_2} \right).$$
Observing that $B_{1,\alpha_1} \cap B_{2,\alpha_2}$ is open by the second axiom of bases, we are done by quoting that the union of open sets is open. $\square$

**Theorem 4.2** ($\mathcal{T}_{\mathcal{B}}$ is the minimal topology containing $\mathcal{B}$)**.**

To start, we obviously have $\mathcal{T}_{\mathcal{B}} \supset \mathcal{B}$; now, let us have $\mathcal{T}'$ be another topology that contains $\mathcal{B}$. Since $\mathcal{T}_{\mathcal{B}}$ is the set of all unions of elements of $\mathcal{B}$, if $\mathcal{T}'$ contains $\mathcal{B}$, then it also contains $\mathcal{T}_{\mathcal{B}}$.

In fact, we may prove that the minimal topology containing $\mathcal{B}$ is unique.[8] Let $\mathcal{T}', ST''$ be minimal topologies that contain $\mathcal{B}$; then by the above argument, they both contain $\mathcal{T}_{\mathcal{B}}$; since they are minimal, we see $\mathcal{T}_{\mathcal{B}} = \mathcal{T}' = \mathcal{T}''$, and so $\mathcal{T}' = \mathcal{T}''$. $\square$

**Theorem 4.3** (Continuity on Basic Sets)**.** It is enough to test for continuity on basic sets. *(Originally left as exercise)*

Suppose $\mathcal{B}_Y$ is a basis of $\mathcal{T}_Y$ on $Y$, and suppose $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is the function we want to test for continuity. Then the topology of $\mathcal{T}_Y$ contains the basis topology $\mathcal{T}_{\mathcal{B}_Y}$, meaning that all open sets $\mathcal{T}_Y$ are unions and intersections of elements from $\mathcal{T}_{\mathcal{B}_Y}$. If the pre-image of the basic sets $B \in \mathcal{B}_Y$ are open in $X$, then so are the unions and intersections of such basic sets. $\square$

[7] note that dror used superscript to separate the two types of base sets, but i'm going to subscript them in a list to keep it clear.

[8] thought: measure theory gives that minimal $\sigma$-algebra generated by generator is unique; similar situation here?

We now define the notion of orders (corresponding Munkres section 14). A "complete" (otherwise called simple, line, or total) order on a set $X$ is a relation $<$ on $X \times X$ such that

- $<$ is transitive; i.e., if $x < y$ and $y < z$, then $x < z$.
- If $x, y \in X$, then exactly one of the following is true (recall trichotomy from 157):
  (a) $x < y$
  (b) $x > y$
  (c) $x = y$.

If $X$ is a set with a simple order relation, let $\mathcal{B}$ be the collection of all sets of the following types,

- All open intervals $(a, b)$ in $X$,
- All intervals of the form $[a_0, b)$ where $a_0$ is the smallest element of $X$ (if it exists),
- All intervals of the form $(a, b_0]$ where $b_0$ is the largest element of $X$ (if it exists).
- In the case that $X$ is a singleton set, said singleton is in $\mathcal{B}$.

The collection $\mathcal{B}$ is a basis for a topology on $X$; we call this the order topology. Now, we give some examples of orderings:

(a) $(\mathbb{R}, <_{\text{std}})$ and $(\mathbb{Q}, <_{\text{std}})$ are basic examples. Note that here, $<_{\text{std}}$ refers to the standard comparison.

(b) English words in lexicographiacl order, such as

$$\text{apple} < \text{ton} < \text{topo} < \text{topple} < \text{zebra}.$$

(c) $\{0, 1\} \times \mathbb{N}$ in dictionary order; we say $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$ if $\alpha_1 < \alpha_2$, or $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$.

(d) Alternatively, if we consider $\mathbb{N} \times \{0, 1\}$ (i.e. binary sequences), then it's the same idea as above just for infinite sequences of $\{\alpha_i\}_{i \in \mathbb{N}}, \{\beta_i\}_{i \in \mathbb{N}}$. Note that $\mathcal{T}_{\{0,1\} \times \mathbb{N}}$ and $\mathcal{T}_{\mathbb{N} \times \{0,1\}}$ are not homeomorphic. *(Originally left as exercise.)*

(e) $\mathbb{R} \times \mathbb{R}$ in dictionary order. Here, open sets can be thought of as $(a, b) = \{x \mid a < x < b\}$, where we may note $x \in \mathbb{R} \times \mathbb{R}$.

For another example, consider $X$ to be the set of finite strings over the usual alphabet; then we may observe that the open sets are dense in $X$; for example, the open set[9]

$$(\text{potato}, \text{tomato}) \supset \{\text{potatoo}, \text{potatooo}, \text{potatoooooooo}\}$$

contains as many strings as we want to fit into it. In particular, a consequence is that the order topology on $X$ is not equal to the discrete topology on $X$; since the discrete topology contains singleton sets (i.e., single lettersoups), and the unions and intersections of open sets of lettersoups is another open set, the two topologies are not equal.

In general, though, if $X$ is a finite set (i.e., similar to the above example, if $X$ is the set of *english words*) and we equip it with a strict total ordering, then the order topology is equal to the discrete topology.

[9] i'm done calling these things words. they are hereby known from now on as lettersoups

We now move onto product topologies. Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces; we want to construct a topology on $X \times Y$ such that $(X, Y)$ restricts to $X$ on $X$ and $Y$ on $Y$, i.e.

$$\pi_X : (X, Y) \to X, \quad \pi_Y : (X, Y) \to Y,$$

and we want it to come with the properties

1. $\pi_X$, $\pi_Y$ are continuous (as per above, they are the projections to $X$ and $Y$),
2. Suppose $f, g$ are two continuous maps $f : Z \to X$ and $g : Z \to Y$. Then $f \times g : Z \to X \times Y$, where $(f \times g)(z) = (f(z), g(z))$.

Note that the above condition is equivalent to taking $h : Z \to X \times Y$; if $\pi_X \circ h$ and $\pi_Y \circ h$ are continuous, then $h$ is continuous.

**Theorem 4.4** (Existence of Unique Product Topology)**.** There exists a unique topology on $X \times Y$ satisfying the above two conditions; we will call this the product topology on $X \times Y$.

To start, by definition of continuity, we need that for all $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$, we have that

$$\pi_X^{-1}(U) = \{(x, y) \mid x \in U\}, \quad \pi_Y^{-1}(V) = \{(x, y) \mid y \in V\}$$

must be open sets. Let us claim that the collection $\mathcal{B}_{X \times Y} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ forms a basis of $X \times Y$. Then let $\mathcal{T}_{X \times Y}$ be generated by $\mathcal{B}_{X \times Y}$; we claim that it satisfies the two properties outlined above.

1. $\pi_X^{-1}(U) = U \times Y \in \mathcal{T}_{U \times Y}$, and $\pi_Y^{-1}(V) = X \times V \in \mathcal{T}_{X \times V}$ show that continuity of $\pi_X, \pi_Y$ is satisfied on basic sets, and so they are satisfied in general.
2. Let $f : Z \to X$ and $g : Z \to Y$ be continuous. Consider $f \times g : Z \to X \times Y$, and let $U \subset \mathcal{T}_X$ and $V \subset \mathcal{T}_Y$. Then $U \times V$ is open, and we may write

$$\begin{aligned}(f \times g)^{-1}(U \times V) &= \{(x, y) \mid (f \times g)(x, y) \in U \times V\} \\ &= \{z \mid (f(z), g(z)) \in U \times V\} \\ &= \{z \mid f(z) \in U \text{ and } g(z) \in V\} \\ &= \{z \mid f(z) \in U\} \cap \{z \mid g(z) \in V\} \\ &= f^{-1}(U) \cap g^{-1}(V),\end{aligned}$$

which is the intersection of open sets, and so is open. Thus, our claim is complete. Now, to demonstrate uniqueness, suppose $\mathcal{T}', \mathcal{T}''$ are topologies in $X \times Y$ which satisfy our two properties from earlier. Then let us have the mapping

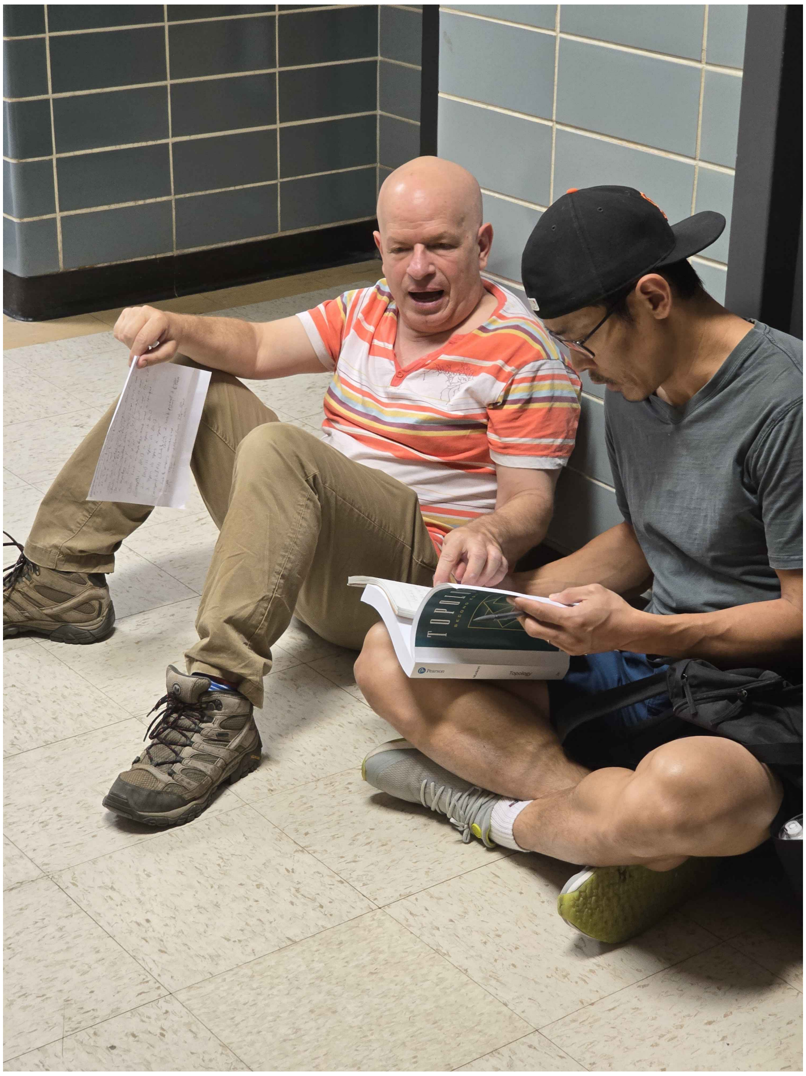
$$\begin{aligned}\mathrm{id} = \pi_X \times \pi_Y : (X \times Y)_{\mathcal{T}'} &\to (X \times Y)_{\mathcal{T}''} \\ (x, y) &\mapsto (\pi_X(x, y), \pi_Y(x, y)) = (x, y).\end{aligned}$$

In particular, $\pi_X : (X \times Y)_{\mathcal{T}'} \to X$ is continuous by property 1 of $\mathcal{T}'$ (same with $\pi_Y$). This means $\pi_X \times \pi_Y$ is continuous by property 2 of $\mathcal{T}''$[10]. We make the same argument for $\mathcal{T}''$; this means $\mathcal{T}' \subset \mathcal{T}''$ and vice versa, which means $\mathcal{T}' = \mathcal{T}''$ as desired. □

[10]phytor correction. ty!!

# §5 Day 5: Induced Topologies (Sep. 17, 2024)

Outfit of the day! Gives sushi shop california roll vibes :3



This week, we will cover sections 17 to 18 in Munkres, and the pre-reading for next week will be from section 19 to section 20. Now for recap;

Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, we define the product topology $\mathcal{T}_{X\times Y}$ on $X \times Y$ to be the unique topology with the properties

1. The projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.
2. For any function $h : Z \to X \times Y$, if $\pi_X \circ h, \pi_Y \circ h$ are continuous, then so is $h$.

Moreover, we have that $\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ is a basis for $\mathcal{T}_{X\times Y}$.

> **Remark 5.1** (Homeomorphism to Cartesian Product with Singleton)**.** Any topological space $X$ is homeomorphic to $X \times \{*\}$, where $\{*\}$ is some singleton. It is also homeomorphic to $\{*\} \times X$.

We now introduce the subspace topology (ref: section 16, Munkres). Given $(X, \mathcal{T}_X)$ and some subset $Y \subset X$, we wish to construct a topology on $Y$ such that

1. The inclusion map $\iota_Y : Y \hookrightarrow X$ is continuous.[11]
2. If $f : Z \to Y$ has that $\iota_Y \circ f$ is continuous, then $f$ is also continuous.

**Theorem 5.2** (Subspace Topology Exists and is Unique)**.** The topology on $Y$ satisfying the above two properties exists and is unique.

We start with existence; let us claim that $\mathcal{T}_Y := \{\iota_Y^{-1}(U) = Y \cap U \mid U \in \mathcal{T}_X\}$ satisfies the above two properties.

1. By construction of the set, $\iota_Y$ is obviously continuous.
2. If $f : Z \to Y$ and $\iota_Y \circ f$ is continuous, then let us take any open $V \in Y$, and consider $f^{-1}(Y)$. Pick $U \in \mathcal{T}_X$ such that $V = U \cap X$. Then we have

   $$f^{-1}(V) = f^{-1}(U \cap X) = f^{-1}(\iota_Y^{-1}(U)) = (\iota_Y \circ f)^{-1}(U).$$

   Since $\iota_Y \circ f$ is continuous, we have that $(\iota_Y \circ f)^{-1}(U)$ is open, and so we conclude $f$ is continuous. This concludes that $\mathcal{T}_Y$ satisfies the properties.

To prove uniqueness, suppose we have $\mathcal{T}'_Y$ and $\mathcal{T}''_Y$ satisfying the above two properties. Then observe the commutative triangle,

$$\begin{tikzcd}
 & X & \\
(Y, \mathcal{T}'_Y) \arrow[ur, hook, "\iota'_Y"] \arrow[rr, leftrightarrow, "Id"'] & & (Y, \mathcal{T}''_Y) \arrow[ul, hook', "\iota''_Y"']
\end{tikzcd}$$

We have that $\iota''_Y \circ \mathrm{Id} = \iota'_Y$; by proposition 1 of $\mathcal{T}'_Y$, we have that $\iota'_Y$ is continuous, meaning $\iota''_Y \circ \mathrm{Id}$ is as well. By proposition 2 of $\mathcal{T}''_Y$, this means the identity between $(Y, \mathcal{T}'_Y)$ and $(Y, \mathcal{T}''_Y)$ is continuous, concluding that $\mathcal{T}'_Y = \mathcal{T}''_Y$. □

We now give some examples of subset topologies.

(a) Consider $[0, 1] \subset \mathbb{R}$ (where we identify $Y$ with $[0, 1]$ and $\mathbb{R}$ with $X$).[12] Then the topology $\mathcal{T}_{[0,1],\text{std}}$ is given by $\{U \cap [0, 1] \mid U \in \mathbb{R}_{\text{std}}\}$. Note that we may consider a subset topology on $\mathbb{R}$ even if $Y$ is not a open set.

   If $Y \subset X$ is an open set, then open sets in $\mathcal{T}_Y$ are automatically open in $X$ as well.

(b) Now, suppose $Y' \subset Y$ and $X' \subset X$. Then $X' \times Y'$ has two topologies:

   (a) As a subset of the product topology $X \times Y$,

   (b) As a product of two subsets, one of $X$ and the other of $Y$.

   We claim that these topologies are the same. *(Originally left as exercise.)*

(c) Let us have $Z \subset Y \subset X$. Let $Y$ have the subspace topology induced by $X$, and let $Z$ have the subspace topology induced by $Y$. Then $Z$ also has subspace topology induced by $X$.

[11] dror uses $i_Y$ instead of iota

[12] Note that the convention is that if the topology on $\mathbb{R}$ is not specified, then it is automatically the standard topology $\mathbb{R}_{\text{std}}$.

We now give some examples of product topologies.

(a) Let us have the product topologies $X \times Y, Y \times X$. Is $X \times Y = Y \times X$? Not necessariy, but we may construct a homeomorphism between them (by swapping coordinates). Note that this is different; if we induce an order topology on $X \times Y$ and $Y \times X$, then this is almost never true.

(b) Let $X, Y, Z$ be topological spaces. Then the cartesian product $(X \times Y) \times Z \neq X \times (Y \times Z)$ is generally not associative, since the sets have structure $\{((x, y), z)\}$ and $\{(x, (y, z))\}$. However, it is common to identify both of them as $X \times Y \times Z$ as an abuse of notation, with elements $\{(x, y, z)\}$. Thus, the product of finitely many topological spaces makes sense from induction on above, i.e.

$$X_1 \times \cdots \times X_n.$$

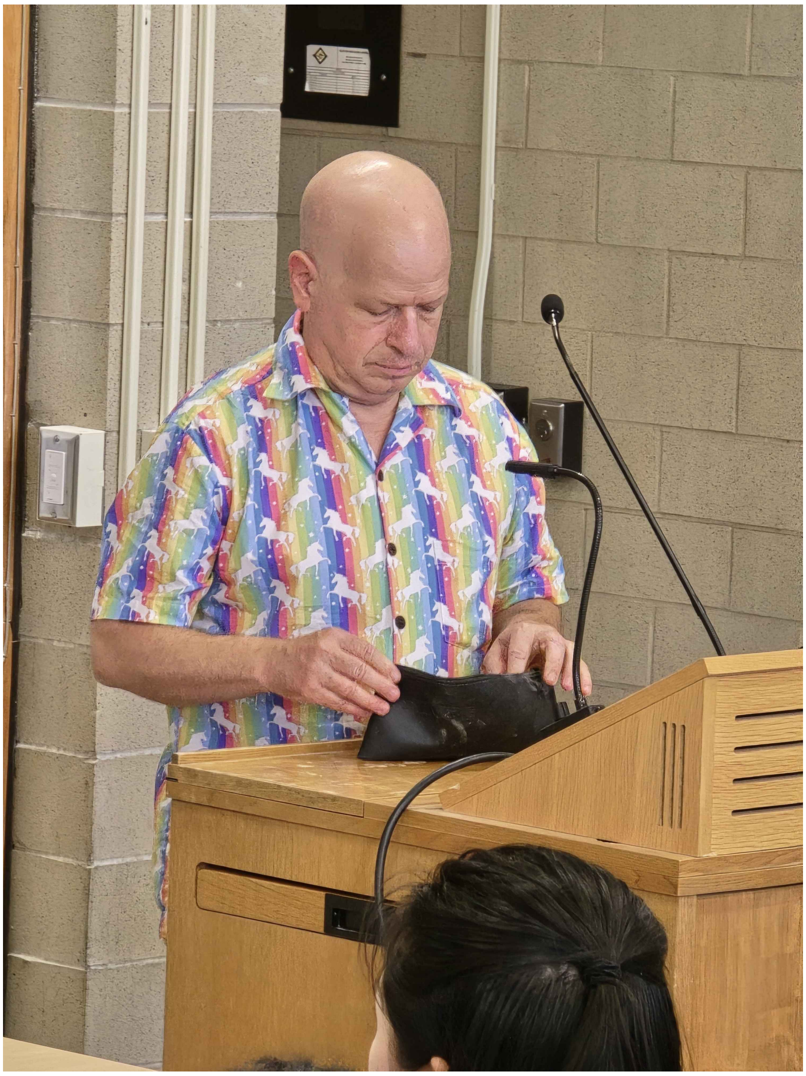
In general, induced topologies interact well with bases; for example, $\mathcal{T}_{X\times Y}$ is the topology generated by $\{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$. In terms of basis, we may write $\mathcal{B} = \{U \times V \mid U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$ with bases $\mathcal{B}_X, \mathcal{B}_Y$ from $X, Y$ respectively.

The subspace topology $Y \subset X$ also interacts well with bases, i.e. $\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}_X\}$. One of the only situations that the subspace topology does not work well with bases is with the order topology; let $X$ be ordered, and let $Y \subset X$. Then $Y$ can either inherits order or topology from $X$. Suppose these two are called by $Y_{\text{ord}}$ and $Y_{\text{subs}}$. Sometimes, $Y_{\text{ord}} \neq Y_{\text{subs}}$, but they are equal if $Y$ is convex. This example is gone over in Munkres.[13]

[13] yeah i'm confused here.

# §6 Day 6: Closed Sets, Cantor Construction, Closures and Interiors; Continuity (Sep. 19, 2024)

Outfit of the day![14] Rainbow unicorns owo!



Course administrative details;

- Office hours have been moved to Tuesdays, on 9:30 to 10:30AM.

If $X$ is a topological space, we say that a set $B \subset X$ is closed if its complement, $B^c = X \setminus B$ is open. *Sets are not doors; they can be open, closed, both, or neither.* For example, in the discrete topology, all sets are both open and closed (in general, $\emptyset$ and $X$ are the only sets that are always clopen).

Closed sets have the following properties,

(a) $\emptyset, X$ are both closed sets.

(b) The arbitary intersection of closed sets is closed.

(c) The finite union of closed sets is closed.

[14] livetexing this sum bitch- so i can't dror pic. sike

In particular, the last two properties are derived through considering the complements of the union and intersection properties of open sets. Specifically, if $A_\alpha$ is a collection of open sets, then

$$\left(\bigcup_\alpha A_\alpha\right)^c = \bigcap_\alpha A_\alpha^c, \qquad \left(\bigcap_\alpha A_\alpha\right)^c = \bigcup_\alpha A_\alpha^c.$$

**Theorem 6.1** (Continuity, defined by Pre-Images of Closed Sets)**.** Any function $F : X \to Y$ (with topological spaces $X, Y$) is continuous if and only if, for every closed $C \subset Y$, we have $f^{-1}(C)$ is closed in $X$.

To see this, we may write

$$f^{-1}(C) = f^{-1}((C^c)^c) = f^{-1}(C^c)^c.$$

Since $C^c$ is an open set, we have that $f^{-1}(C^c)$ is open in $X$, and so its complement is closed. This means $f^{-1}(C)$ is closed in $X$. □

> **Remark 6.2.** Having defined continuity, we note that the last two properties of closed sets could have also been used to define the notion of topologies instead of open sets.

Now, let us consider the following example: consider $\mathbb{R}_{\text{std}}$ containing $C_0 = [0, 1]$. Then $C_0^c = (-\infty, 0) \cup (1, \infty)$. Next, define $C_1 = [0, 1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, i.e. $C_0$ with its inner third removed. Let $C_2$ be $C_1$ with the inner thirds of each of its intervals removed, etc.;

Notice that each $C_n$ is a union of closed intervals on $\mathbb{R}_{\text{std}}$. Define

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right).$$

Since the arbitrary union of open intervals is open, we see that the complement of them is closed, meaning $\mathcal{C}$ is closed. Moreover, we claim that $\mathcal{C}$ is not empty. Start by writing $\mathcal{C}$ as the set $\{x \mid x_3 \text{ has no 1s}\}$, i.e. $x_3$ representing the ternary expansion of $x$. For any number whose ternary decimal representation consists of only 0s and 2s, we may write said number as an infinitely recurring decimal containing 1, since $0.1\overline{2}_3 = 0.2_3$, by rewriting the last 2 in the expansion of any such number.

Clearly, $\mathcal{C}$ is uncountable. In addition, it exhibits the properties,[15]

- $\mathcal{C}$ is uncountable, yet the length of $\mathcal{C}$ is 0.
- *(Originally left as exercise; Devil's Staircase Construction)* There exists a continuous $F : [0, 1] \to [0, 1]$ such that $F(0) = 0, F(1) = 1$ and for all $x \notin \mathcal{C}$, we have that $F'(X) = 0$.
- *(Originally left as exercise)* What is $\mathcal{C} + \mathcal{C} = \{x + y \mid x, y \in \mathcal{C}\}$? Alternatively, consider $f : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ where $f(x, y) = x + y$.

[15] read more here ig? linkylinkylink

When is a set closed in the product of two spaces? Suppose $A \subset X, B \subset Y$ are closed sets in their respective topological spaces. Then observe that we cannot just take the complement as follows,

$$(A \times B)^c \neq A^c \times B^c,$$

since $(A \times B)^c = (A^c \times Y) \cup (X \times B^c)$. Observing that both of these sets are open, we see that $(A \times B)^c$ is also open, meaning $A \times B$ is closed. □

**Claim 6.3.** If $Y \subset X$ are topological spaces with $Y$ having the subspace topology, then $C \subset Y$ is closed if and only if there exists some closed $B \subset X$ such that $C = B \cap Y$.

**Claim 6.4.** If $Y \subset X$ is closed, and $B \subset Y$ is closed in $Y$, then $B$ is closed in $X$.

We now define the interior and closure of sets.

- The interior of $A$ is denoted $\text{int}_X A = \mathring{A}$. It is the largest open set contained in $A$. By the arbitrary union property of open sets, this is given by the union of all open sets contained in $A$.
- The closure of $A$, $\text{cl}_X A = \overline{A}$, is the smallest closed set containing $A$, namely the intersection of all closet sets containing $A$.

For an example, we have that

$$[0,1)^o = (0,1),$$
$$\overline{[0,1)} = [0,1].$$

If $A$ is open, then $\mathring{A} = A$. If $A$ is closed, then $\overline{A} = A$. Here are some more properties on interiors and closures:

- The interior of the interior of $A$ is equal to the interior of $A$, i.e. $\mathring{\mathring{A}} = \mathring{A}$.
- The same holds for the closure, i.e. $\overline{\overline{A}} = \overline{A}$.
- The interior and closure of any clopen set is the clopen set itself. For example, $\mathring{\emptyset} = \overline{\emptyset} = \emptyset$, and the same holds for the whole set.
- In general, we don't know what $\mathring{\overline{A}}$ or $\overline{\mathring{A}}$ is.
- The complement of $\overline{A}$ is given by

$$\overline{A}^c = \left( \bigcap_{\substack{F \supset A \\ F \text{ closed}}} F \right)^c = \bigcup_{\substack{F \supset A \\ F \text{ closed}}} F = \bigcup_{\substack{U^c \supset A \\ U^c \text{ closed}}} U \underbrace{=}_{F^c = U} \bigcup_{\substack{U \subset A^c \\ U \text{ open}}} U = (\mathring{A^c}).$$

  Thus, the complement of the closure is the interior of the complement.
- Likewise, we have $(\mathring{A})^c = \overline{A^c}$.
- *(Challenge Exercise)* Prove that we can make 14 distinct sets from any general set $A$ using complement, closure, and interior.

**Theorem 6.5.** Let $X$ be a topological space, and let $A \subset X$. $x$ is in the closure of $A$ if and only if every neighborhood of $x$ intersects $A$. A neighborhood of $x$ is defined as an open set containing $x$.

Specifically, the condition above may be written as $\forall U \in \mathcal{T}_X$, we have $x \in U \implies U \cap A \neq \emptyset$.

- ($\Rightarrow$) Assume $x \in \overline{A}$ by contradiction. Assume that $U$ is open and $x \in U$, with $U \cap A = \emptyset$. We have $U^c \supset A$, but $U^C$ is closed. Since $x \notin U^c \supset \overline{A} \ni x$ is a contradiction, we are done.
- ($\Leftarrow$) Assume every neighborhood of $X$ intersects $A$. By contradiction, also assume that $x \notin \overline{A}$. Then $\overline{A}^c \ni x$ is a neighborhood of $x$, not intersecting $A$. This is a contradiction, so we are done. $\square$

In fact, we may check basic neighborhoods (i.e., open set in basis containing $x$) only instead of all neighborhoods.

**Claim 6.6** (Closure of $\mathbb{Q}$ is $\mathbb{R}$)**.** We have that $\overline{Q} = \mathbb{R}$.

For all $x \in \mathbb{R}$, we have $x \in \overline{\mathbb{Q}}$ if and only if $x \in (a, b)$ implies $(a, b) \cap \mathbb{Q} \neq \mathbb{Q}$. $\square$

**Definition 6.7.** Given $A \subset X$, $x \in X$ is a limit of $A$ if every neighborhood of $x$ contains a point of $A$ other than $x$ itself. This is equivalent to saying that $x \in \overline{A \setminus \{x\}}$.

**Theorem 6.8.** $\overline{A} = A \cup A'$ where $A'$ is given by the set of limit points of $A$.

- ($\supset$) $\overline{A} \supset A$ is trivially true. If $x \in A'$, then $x \in \overline{A \setminus \{x\}} \subset \overline{A}$.
- ($\subset$) Take $x \in \overline{A}$. If $x \in A$, we're automatically done. Thus, let us assume that $x \notin A$; then $x \in \overline{A} = \overline{A \setminus \{x\}}$, implying that $x \in A'$. $\square$

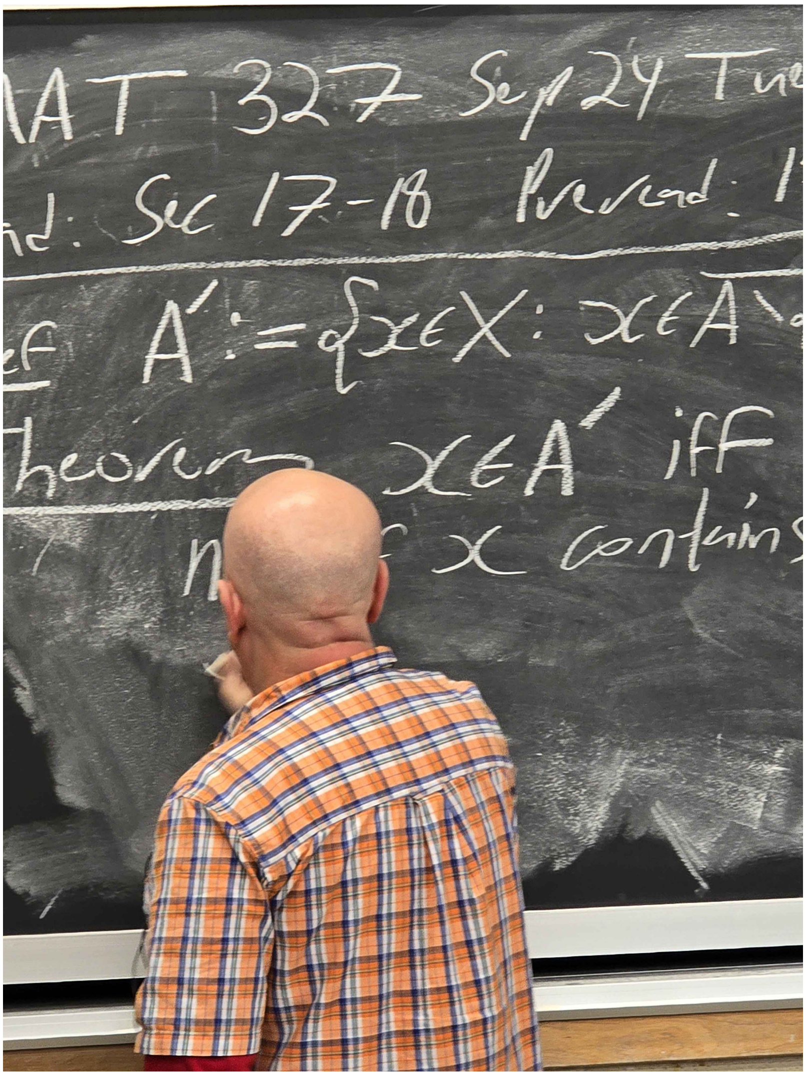
We now present a fheorem (false theorem). If $x$ is a limit point of some set $A$, then every neighborhood of $x$ contains infinitely many elements of $A$, i.e.

$$x \in U \in \mathcal{T}_X \implies |U \cap A| = \infty.$$

We now present a froof (false proof). For any $x$, we may take any neighborhood about $x$. Then we may remove a point $a_1$; since the neighborhood with $a_1$ removed is still an open set, we may remove $a_2$, $a_3$, and so on inductively. $\square$

# §7 Day 7: Separation Axioms and $T_2$/Hausdorff Spaces, More on Continuity (Sep. 24, 2024)

Outfit of the day! Plaid outfit!



Recap of last lecture:

- We define $A' := \{x \in X \mid x \in \overline{A \setminus \{x\}}$ to be the set of limit points of $A$ in $X$.
- We present a "fheorem" (read: false theorem): $x \in A'$ if and only if every neighborhood of $X$ contains infinitely many points of $A$, i.e.

  $$x \in U \subset X \implies |U \cap A| = \infty,$$

  where $U$ is open in $X$.

  We expand on our "froof" from last time: indeed, if $U$ is a neighborhood of $X$, then as $x \in \overline{A \setminus \{x\}}$, we have that $U \cap (A \setminus \{x\}) \neq \emptyset$. Now, let us pick $a_1 \in U \cap (A \setminus \{x\})$, and consider $U_1 = U \setminus \{a_1\}$.

  Since $U_1$ is a neighborhood of $x$, we have that $U_1 \cap \overline{A \setminus \{x\}} \neq \emptyset$, so we may pick $a_2 \in U_1 \cap \overline{A \cap \{x\}}$; then we consider $U_2$, etc... We argue by induction that there are infinitely many such points. Thus, there exists an infinite sequence of distinct points $\{a_i\}_{i\in\mathbb{N}}$ in $U \cap A \setminus \{x\}$. □

This proof is in fact wrong because $U_1$ is not necessarily a neighborhood of $x$ (i.e., we don't know if $X \setminus \{a_1\}$ is open or not). For a blatant example, consider the trivial topology. Then $X \setminus \{a_1\}$ is obviously not open if $X$ is not a singleton.

To expand on our recap of last lecture, we introduce the separation axioms.

$T_1$: A space $X$ is called $T_1$ if and only if, for all $x \in X$, $\{x\}$ is closed, and $X \setminus \{x\}$ is open, and for all $x, y \in X$ where $x \neq y$, we can find a neighborhood of $y$ that does not contain $x$, i.e. $\exists U$ open in $X$ such that $x \in U$ but $y \notin U$. These three conditions are equivalent to each other; though usually, it is useful to just identify it as singletons being closed, or their complements being open.

In particular, our "froof" from earlier holds if $X$ is $T_1$ (i.e., any singleton is closed, and so the union of singletons are closed).

$T_2$: We say a space $X$ is $T_2$ (aka Hausdorff or *separated*) if, for all $x, y \in X$ where $x \neq y$, then there exists open sets $U_1, U_2$ in $X$ such that $U_1 \cap U_2 = \emptyset$.

**Exercise 7.1.** Show that if $x_1, \ldots, x_n \in X$, then there exists open sets $U_1, \ldots, U_n$ such that $x_i \in U_i$ and $i \neq j \implies U_i \cap U_j = \emptyset$.

**Claim 7.2.** We claim that if $X$ is $T_2$, then it is also $T_1$.

This is clear by picking neighborhoods $U_1, U_2$ for $x, y \in X$ where $x \neq y$, such that $U_1 \cap U_2 = \emptyset$. Clearly, $U_1$ does not contain $y$, and $U_2$ does not contain $x$. □

**Claim 7.3.** In a $T_2$ space, any sequence has at most one limit.

In particular, we define sequence convergence as follows; if $\{a_n\}_{n\in\mathbb{N}} \in X$, then we say that $a_n$ converges to $a \in X$, i.e. $a_n \to a$ as $n \to \infty$, or $\lim_{n\to\infty} a_n = a$, if for every neighborhood $U$ of $a$, there exists $N$ such that $n > N$ implies $a_n \in U$.

We now prove claim 7.3. Assume $\{a_n\}_{n\in\mathbb{N}}$ is a sequence that converges to $a$ and $a'$, i.e. $a_n \to a$ and $a_n \to a'$. Then by $T_2$, there exists neighborhoods $U, U'$ of $a, a'$ respectively such that $U \cap U' = \emptyset$. Then, let us pick a large enough $N$ such that $n > N \implies a_n \in U$, and $N'$ large enough such that $n > N' \implies a_n \in U'$. Then just consider $n = \max\{N, N'\} + 1$, then $a_n \in U$ and $U'$ at once, while $U \cap U' = \emptyset$. This is contradictory, and so we are done. □

We now make the following claim.

**Claim 7.4.** We claim that a subspace of a $T_2$ space is $T_2$, and that the product of $T_2$ spaces is also $T_2$.

We start by proving the claim on products[16]. Suppose $X, Y$ are $T_2$, and let us have $a_1, a_2 \in X \times Y$ where $a_1 = (x_1, y_1) \neq (x_2, y_2) = a_2$. Then $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality, suppose $y_1 \neq y_2$. Then there exists neighborhoods $V_1, V_2$ in $Y$ of $y_1, y_2$ respectively, where $V_1 \cap V_2 = \emptyset$. Then we see that $X \times V_1$ and $X \times V_2$ are suitable choices of neighborhoods for $a_1, a_2$ to conclude that $X \times Y$ is $T_2$.

[16] im not including the froof from class because i think it's kind of immediate, sorry. also i didn't catch it entirely so oh well

We now move onto a different topic.

**Theorem 7.5.** Given $f : X \to Y$, we have that the following are equivalent:

(a) $f$ is continuous.

(b) $f(\overline{A}) \subset \overline{f(A)}$, i.e. $f$ maps the closure of $A$ into the closure of $f(A)$.

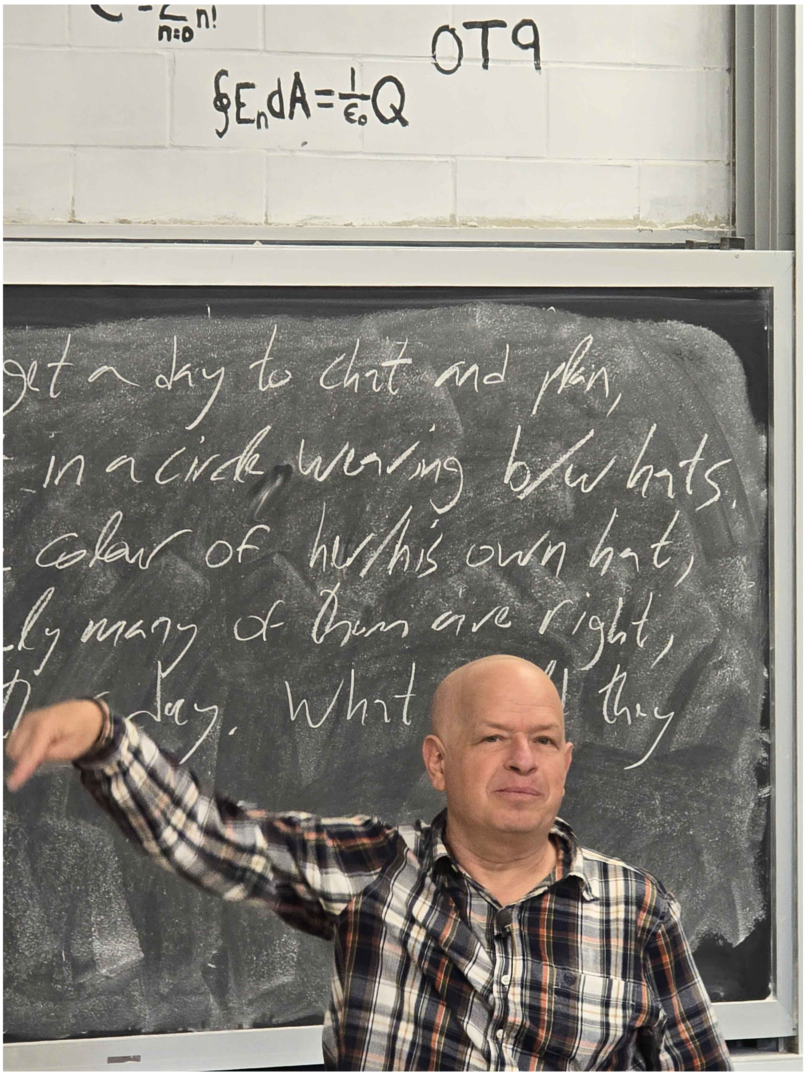
(c) If $B \subset Y$ is closed, then $f^{-1}(B)$ is closed.

(d) For all $x_0 \in X$, if $V$ is a neighborhood of $f(x_0)$, then there is a negihborhood $U$ of $x_0$ such that $f(U) \subset V$.

Note that we have already proven that (a) if and only if (c), (a) iff (d), and we will prove that (a) implies (b) and (b) implies (c) on Thursday.

# §8 Day 8: Continuous Functions and Infinite Products (Sep. 26, 2024)

Outfit of the day! Sniper monkey 0-2-5 outfit!!!! very demure, very mindful




Recap from last lecture: for any $f : X \to Y$, we have that the following are equivalent:

(a) $f$ is continuous.

(b) For all $A \subset X$, we have that $f(\overline{A}) \subset \overline{f(A)}$.

(c) The pre-image of a closed set is closed.

(d) For all $x \in X$ and for all negihborhoods $V$ of $f(x)$, there exists a neighborhood $U$ of $x$ such that $f(U) \subset V$.

Note that we have already established that $(a) \iff (c)$ and $(a) \iff (d)$. We now prove that $(a) \implies (b)$.

Take $x$ in the closure of $A$, and pick a neighborhood $U$ of $f(x)$. Then $f^{-1}(U)$ is open (by continuity), and it contains $x$. Thus, $f^{-1}(U)$ intersects $A$. Let us pick $y \in f^{-1}(U) \cap A$; then we have $f(y) \in U \cap f(A)$, and so every neighborhood of $f(x)$ intersects $f(A)$, and so $f(x) \in \overline{f(A)}$. $\square$

As an example, let us $X = \mathbb{R}^2$, and consider $A = \{(x, \frac{1}{x}) \mid x \neq 0\}$. Then if we let $f$ be the function projecting $A$ to the $x$-axis, i.e. $f(A) = \mathbb{R} \setminus \{0\}$, and we have that

$$f(\overline{A}) = f(A) = \mathbb{R} \setminus \{0\} \subsetneq \overline{f(A)} = \mathbb{R}.$$

We now continue on our previous claim, and show that $(b) \implies (c)$. Let $B \in Y$ be a closed set, and let $f^{-1}(B) =: A$. Then we may write,

$$f(\overline{A}) \overset{(b)}{\subset} \overline{f(A)} = \overline{f(f^{-1}(B))} \subset \overline{B} = B,$$

since the closure of a closed set is itself. Thus, we have that $\overline{A} \subset f^{-1}(B) = A$, implying $\overline{A} = A$, and so $A$ is closed, i.e. the pre-image of $B$ is closed. $\square$

We now move onto infinite products. Suppose $X_\alpha$ is a set for every $\alpha \in I$, and let $X = \prod X_\alpha = \{x : I \to \bigcup_\alpha X_\alpha \mid \forall \alpha \in I, x(\alpha) \in X_\alpha\}$[17]. Per the axiom of choice (which we will refer to as $AC$ from now on), if all $X_\alpha$s are not empty, then $X = \prod X_\alpha$ is non-empty.

Assuming AC, we have that $\prod_{\emptyset \neq A \subset \mathbb{R}} A$ is non-empty. Let us consider the function $c : \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\} \to \mathbb{R}$ such that $c(A) \in A$. There exists a function on the nonempty sets of reals that picks an element from each set of reals.

Let $S = \mathbb{R}^{\mathbb{N}} = \{(a_i) + i \in \mathbb{N} \mid a_i \in \mathbb{R}\}$. Then we claim that there exists a function $h : S \to S$ such that

(a) $h(a)$ depends only on the tail of $a$, meaning if $a \sim b$ (i.e. they have the same tail, meaning $\exists N \in \mathbb{N}$ such that for all $n > N$, $a_n = b_n$), then $h(a) = h(b)$.

(b) $h(a) \sim a$, i.e. they only differ in finitely many coordinates, and are equal past some index $N$.

**Claim 8.1.** If such a function exists, we can save infinitely many prisoners; recall that the prisoner problem is that every prisoner is given a white or black hat (they cannot see their own hat), and they have to call the color of their own hat out.

Assuming AC, we know that $h$ exists, and so by property (b) of $h$, only finitely many prisoners will die.

Let $I = S/\sim$ (read: equivalence classes as per $\sim$ defined earlier, in $S$), and consider $X = \prod_{\alpha \in I} \alpha \ni h'$. Let $h'$ be a function taking the tail of $\alpha$ that finds another sequence whose tail is the same as $\alpha$'s. Then $h(a) = h'(\text{tail of } \alpha)$; unfortunately, to find $h'$, we need to use AC. $\square$

Suppose $X_\alpha$ is a topological space, and we want a topology on $X = \prod X_\alpha$. We may approach this in two ways:

1. We may choose to generalize the construction,

$$\mathcal{B} = \left\{ \prod_{\alpha \in I} U_\alpha \mid \forall \alpha, U_\alpha \subset X_\alpha \right\},$$

   where each $U_\alpha$ is open. *(We note that this is a dead end.)*

2. Alternatively, we may generalize the requirements. We want a topology such that:
   (a) $\forall \alpha, \prod_\alpha : X \to X_\alpha$, i.e. $\prod_\alpha(x) = x(\alpha) = x_\alpha$
   (b) If $g : Z \to X$ has that $\forall \alpha, \pi_\alpha \circ g$ is continuous, then $g$ is continuous.

[17] alternatively, the function $x$ can be seen as just a "collection" of $x_\alpha$s.

**Claim 8.2.** There exists a unique topology on $X$ satisfying properties 1 and 2.

Observe that at least $\prod_\alpha^{-1}(U_\alpha)$ must be open for every $U_\alpha$, where $\alpha \in I$, to be open in $X$. Let's start by defining

$$\mathcal{B} = \{\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n}) \mid \alpha_1, \ldots, \alpha_n \in I;\ U_{\alpha_i} \text{ open in } X_i\};$$
$$\mathcal{B}_{\text{cyl}} = \left\{ \prod_{\alpha \in I} U_\alpha \mid \forall \alpha, U_\alpha \subset X_\alpha \text{ is open; } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha\text{'s.} \right\}.$$

This is exactly the previous basis $\mathcal{B}$ (note that we read cyl as cylinder here[18]). We claim that this works; observe that

(a) $\mathcal{B}_{\text{cyl}}$ is indeed a basis, so let $\mathcal{T}_{\text{cyl}}$ be the topology it generates, meaning that property 1 holds,

(b) We also have that property 2 holds; assume $g : Z \to X$ and $\pi_\alpha \circ g$ is continuous for all $\alpha$. Then let us write,

$$g^{-1}\left(\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})\right) = \bigcap_{i=1}^{n} g^{-1}(\pi_\alpha^{-1}(U_{\alpha_i})) = \bigcap_{i=1}^{n} (\pi_\alpha \circ g)^{-1}(U_{\alpha_i}).$$

Since each $\pi_\alpha \circ g$ is assumed to be continuous, we have that this is a finite intersection of open sets, and we conclude that the above is indeed open.

Now that we have proved existence, we claim that we also have uniqueness. If $\mathcal{T}'$ and $\mathcal{T}''$ are topologies on $X = \prod X_\alpha$ satisfying 1 and 2, then $\mathcal{T}' = \mathcal{T}''$; to start, consider the identity map $g = \text{id}$,

$$\left(\prod X_\alpha, \mathcal{T}'\right) \xrightarrow{\text{id}} \left(\prod X_\alpha, \mathcal{T}''\right).$$

Then $\pi_\alpha \circ g = \pi_\alpha \circ \text{id} = \pi_\alpha$ is continuous by property 1 of $\mathcal{T}'$; thus, by property 2 of $\mathcal{T}''$, we see that the identity map is continuous, and we proceed as per our previous uniqueness proofs to see that $\mathcal{T}' = \mathcal{T}''$. $\square$

For concreteness, we now present an example. Let $c : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$, where $c(t) = (t, t, \ldots)$, i.e. the constant sequence consisting of $t$s. Then $c : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}_{\text{cyl}}$ is continuous. $(\pi_k \circ c)(t) = t$ (since $c = \text{id}_{\mathbb{R}}$), and so $c$ is continuous.

Now, consider $c : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}_{\text{box}}$. Then

$$c^{-1}\left(\prod_{k \in \mathbb{N}} \left(-\frac{1}{k}, \frac{1}{k}\right)\right) = \{0\},$$

which is not open.

Note that $\mathcal{T}_{\text{box}} \neq \mathcal{T}_{\text{cyl}}$. In fact, the cylinder topology is contained in the box topology, and the inclusion is not an equality, $\mathcal{T}_{\text{cyl}} \subsetneq \mathcal{T}_{\text{box}}$. In both topologies,

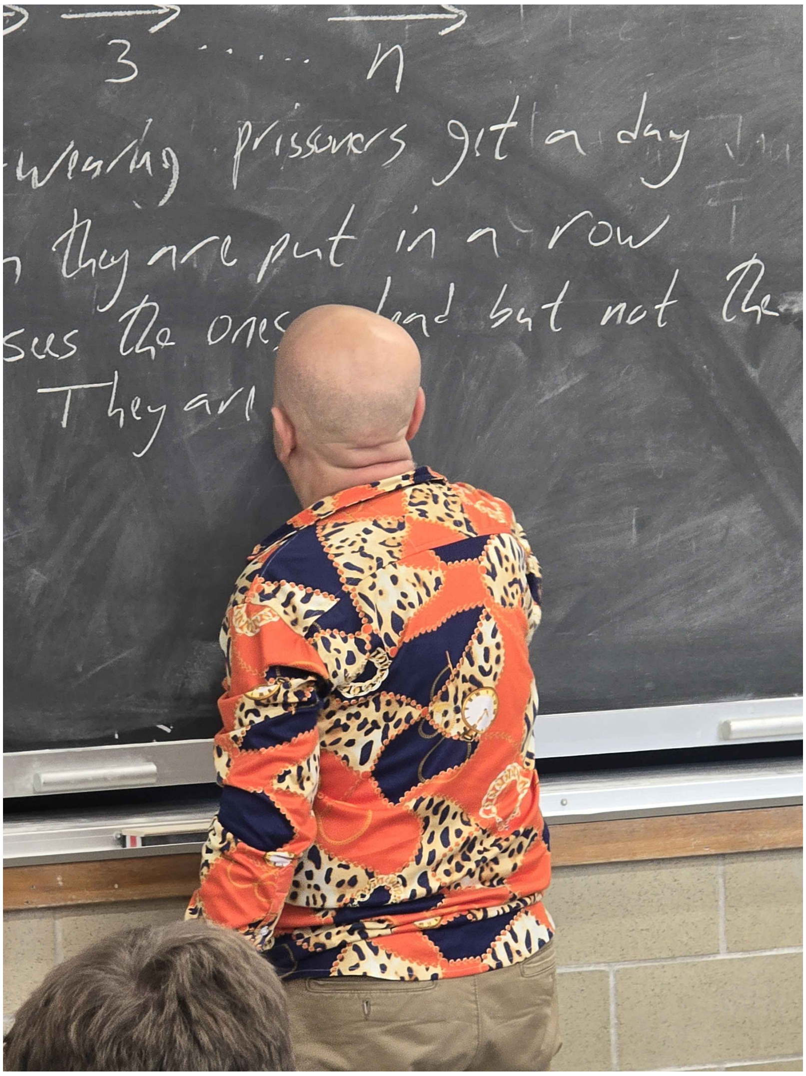
(a) If $A_\alpha \subset X_\alpha$, for every $\alpha$, $\prod A_\alpha$ as a subset of a product is the same as $\prod A_\alpha$ as a product of subsets. *(The proof is messy. We leave it alone for now.)*

(b) If, for all $\alpha$, $X_\alpha$ is $T_2$, then $(\prod_\alpha X_\alpha)_{\text{box}}, (\prod_\alpha X_\alpha)_{\text{cyl}}$ are $T_2$.

[18] and i have no idea why it's cylinder, but we play along for now

# §9 Day 9: Infinite Products, Metrics (Oct. 1, 2024)

Outfit of the day! smth smth butterflies !!! very demure, very mindful




Recap of last lecture; topologies on $\prod_\alpha X_\alpha$ generalize the following;

(a) The basis $\mathcal{B}_{\text{box}} = \{\prod U_\alpha \mid U_\alpha \subset X_\alpha \text{ open}\} \rightsquigarrow \mathcal{T}_{\text{box}}$;

(b) The requirements $\mathcal{B}_{\text{cyl}} = \{\prod U_\alpha \mid U_\alpha \subset X_\alpha \text{ open, almost always } U_\alpha = X_\alpha\} \rightsquigarrow \mathcal{T}_{\text{cyl}}$.

Observe that the function sending a constant $t \in \mathbb{R}$ to the constant sequence $(t, t, \dots) \in \mathbb{R}^{\mathbb{N}}$ is continuous in the cylinder topology, but not in the box topology (read: here). This means $\mathcal{T}_{\text{cyl}} \supsetneq \mathcal{T}_{\text{box}}$ (in general, they are not the same); yet in both, both subspace and the Hausdorff property behaves (i.e., products preserve Hausdorff-ness).

**Claim 9.1** (Theorem 19.5, Munkres)**.** Let $\{X_\alpha\}$ be an indexed family of spaces, and let $A_\alpha \subset X_\alpha$ for each $\alpha$. Then we have that $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$ if $\prod X_\alpha$ is given the box or cylinder topologies.

We prove this by double inclusion:

($\Leftarrow$) Let $x = (x_\alpha) \in \overline{\prod_\alpha A_\alpha}$. Recall that $x$ is in the closure of $\prod A_\alpha$ if and only if every basic neighborhood of $x$ intersects $\prod_\alpha A_\alpha$. This condition is equivalent to saying that for all open neighborhood $U_\alpha \subset X_\alpha$, $x \in \prod U_\alpha \implies \prod U_\alpha \cap \prod U_\alpha \neq \emptyset$, which is also equivalent to saying that for all $\alpha$ where $x_\alpha \in U_\alpha$, we have $U_\alpha \cap A_\alpha \neq \emptyset$. Thus, every neighborhood $U_\alpha$ about $x$ intersect $A_\alpha$. This means for all $\alpha, x_\alpha \in \overline{A_\alpha}$, and we conclude that $x \in \overline{\prod_\alpha A_\alpha}$.

($\Rightarrow$) *(Not covered in class)* Let $x = (x_\alpha)$ lie in the closure of $\prod A_\alpha$. Then for any index $\beta$, we have $x_\beta \in \overline{A_\beta}$. Let $V_\beta$ be an arbitrary open set of $X_\beta$ containing $x_\beta$. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology, it contains $y = (y_\alpha) \in \prod A_\alpha$. Then $y_\beta$ belongs to $V_\beta \cap A_\beta$, which means $x_\beta \in \overline{A_\beta}$. $\square$

We now move onto metric spaces. We say a metric on a set $X$ is a function $d : X \times X \to (\mathbb{R} \geq 0)$ such that:

(a) $d(x,y) \geq 0$, with $d(x,y) = 0 \iff x = y$ (non-negativity);

(b) $d(x,y) = d(y,x)$ (symmetry);

(c) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

We also define that $B_r(x_0) = \{x \mid d(x_0, x) < r\}$. Note that balls in non-euclidean metrics may look different from a sphere; we now give some examples of metrics:

(a) The Euclidean metric ($L^2$) on $\mathbb{R}^n$, $d(x,y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

(b) The Manhattan distance ($L^1$) on $\mathbb{R}^n$, $d(x,y) = \sum_{i=1}^n |x_i - y_i|$. Note that here, a ball looks more like a fucked-up rhombus.

(c) For any set $X$, let us define $d(x,y) = 1$ for $x \neq y$, and $d(x,y) = 0$ if $x = y$.

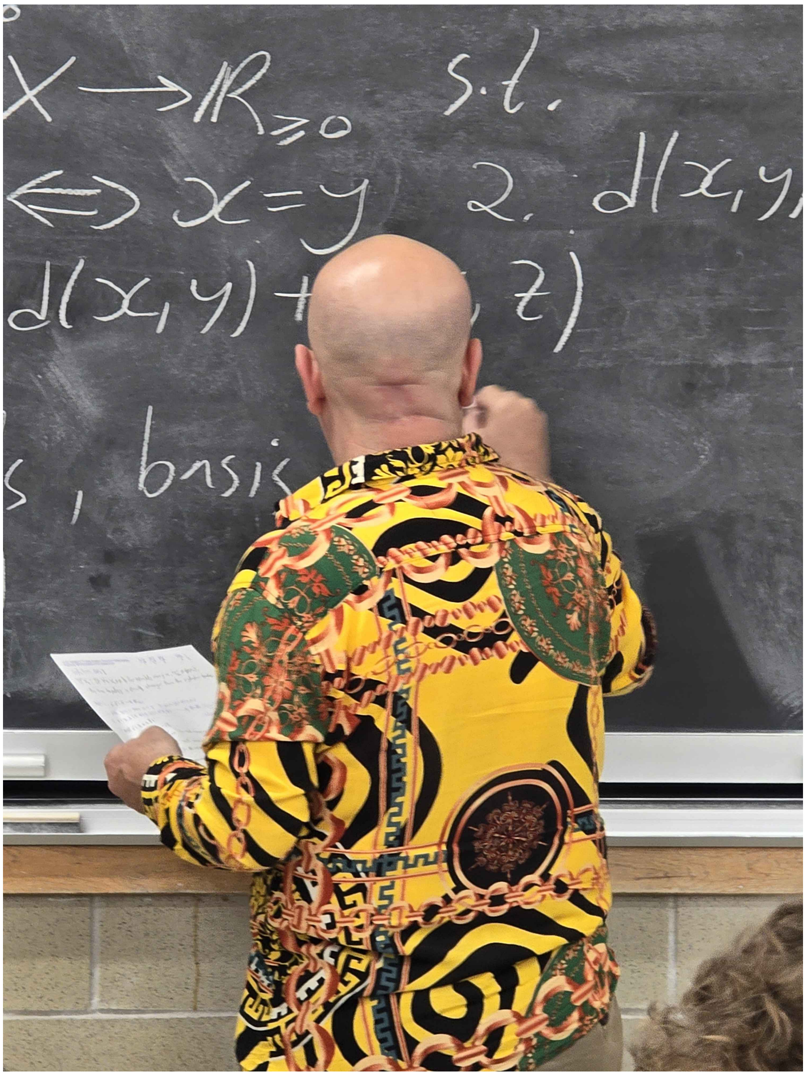
(d) For bounded sequences $(a_i) \in \mathbb{R}^{\mathbb{N}}$, we may define $d(a,b) = \sup |a_i - b_i|$ (aka $L^\infty$).

A set with a metric on it is called a *metric space*; on metric spaces, we set $\mathcal{B}_{\mathrm{d}} = \{B_r(x_0)\}$, i.e. the set of all open balls, and we claim that this is indeed a basis *(Left as exercise, but it's obvious)*. In particular, every metric space has a topology; namely, it is a topological space; we claim that the metrics introduced above induce the same topology, i.e. the discrete topology on $\mathbb{R}^n$, except (d).

All metric spaces are Hausdorff by considering the triangle inequality.

# §10 Day 10: Metric Spaces (Oct. 3, 2024)

Outfit of the day! jungle story / africa circles shirt




Recap: let $d : X \times X \to \mathbb{R}_{\geq 0}$ be a metric. Then $d$ is defined to have the properties,

(a) $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, z) \leq d(x, y) + d(y, z)$.

In particular, the balls $B_r(x)$ forming a basis on the metric topologoy means that such spaces are always $T_2$.

Let us consider $\prod X_\alpha$, and consider $H_\alpha \subset X_\alpha$ for all indices $\alpha$ to be open, nonempty, and non-full (not $X_\alpha$). Then $\prod H_\alpha$ is open in the box topology, but it contains not even one cylinder. So the box topology is strictly stronger than the cylinder topology, except if there exists $\alpha$ such that some $X_\alpha = \emptyset$ and not too many of them have the trivial topology.[19]

[19] wtf? time for me to read more munkres

A topological space $X$ is called "metrizable" if there is a metric $d$ on $X$ that induces its topology. Note that $d$ is far from unique. We now consider some examples:

(a) $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is not metrizable.

(b) $\mathbb{R}^{\mathbb{R}}_{\text{cyl}}$ is also not metrizable (read: set of functions $f : \mathbb{R} \to \mathbb{R}$).

Recall that in a topological space, $x_n$ converges to $x$ (i.e., $x_n \to x$) if every neighborhood of $x$ contains almost all of the $x_n$'s (read: an infinite amount or except for finitely many $x_n$'s). A similar definition holds in metric spaces; however, here, it is enough to look at the balls $B_\varepsilon(x)$ for all $\varepsilon > 0$. We may write this as

$$\forall \varepsilon > 0, \exists N \, s.t. \; n > N \implies (x_n \in B_\varepsilon(x) \iff d(x_n, x) < \varepsilon) \, .$$

Thus, $d(x_n, x) \xrightarrow{n \to \infty} 0$.

Given a topological space $X$ with $A \subset X$, then the sequential closure of $A$ in $X$, i.e. $\text{scl}_X(A) = \{x \in X \mid \exists x_1, x_2, \ldots \; s.t. \; x_n \to x\}$.[20]

**Claim 10.1** (Sequential Closure is in Closure)**.** We claim that $\text{scl}\,(A) \subset \text{cl}\,(A)$.

For any $x$ in the sequential closure of $A$ such that $x_n \to x$, every neighborhood of $x$ contains almost all of the $x_n$'s, so it intersects $A$. Thus, $x \in \text{cl}\,(A)$. $\square$

**Claim 10.2** (Equality of Sequential Closure and Closure in Metrizable $X$)**.** If $X$ is metrizable, and $A \subset X$, then $\text{cl}(A) = \text{scl}(A)$.

Pick any metric $d$ that induces the topology of $X$. For any $n \geq 1$, $B_{d,\frac{1}{n}}(x)$ (new notation! read: the ball of radius $\frac{1}{n}$ about $x$ w.r.t. the metric $d$) is an open neighborhood of $x$, and so $B_{d,\frac{1}{n}}(x) \cap A \neq \emptyset$. So choose $x_n \in B_{d,\frac{1}{n}}(x) \cap A$, meaning $x_n \to x$. Indeed, if $U$ is a neighborhood of $x$, so $U$ contains some ball of radius $\varepsilon$ which contains $x$. Namely, there exists $y \in X$ and there exists $\varepsilon > 0$ such that $x \in B_\varepsilon(y) \subset U$.

Take $\delta = \varepsilon - d(x, y)$, and now $x \in B_\delta(x) \subset B_\varepsilon(y) \subset U$. If $n > \frac{1}{\delta}$, then $\frac{1}{n} < \delta$, so $x_n \in B_{\frac{1}{n}}(x) \subset B_\delta(x) \subset U$. Thus, the sequence $x_n$ indeed converges to the point $x$. $\square$

We now return to prove examples (a) and (b).

(a) We will prove this by considering both directions. Let $X = \mathbb{R}^{\mathbb{N}}_{\text{box}}$, $A = \mathbb{R}^{\mathbb{N}}_{>0}$, i.e. $A$ is the set of sequences of positive numbers. Let $x = \overline{0} = (0, 0, 0, \ldots)$. Clearly, $\overline{0}$ is in the closure of $A$. Indeed, if $\prod(a_i, b_i)$ is a basic neighborhood of $\overline{0}$, i.e. $a_i < 0 < b_i$, then the sequence $\left(\frac{b_i}{2}\right)_{i=1}^{\infty} \in U \cap A$, and so $U \cap A \neq \emptyset$.

For the harder side, we claim that $\overline{0} \notin \text{scl}\,(A)$. For contradictoin, suppose that $\overline{0}$ is in the sequential closure, i.e. $x_i \in A$ such that $x_i \to \overline{0}$ (read: each $x_i$ itself is a sequence). Namely, $x_i = (x_{ij})_{j \in \mathbb{N}}$, where $x_{ij} > 0$ for all $i, j$. Let $U = \prod(a_i, b_i)$. We may take $a_i = -327$ (random negative number). We may then take $b_1 = \frac{x_{11}}{2}$, $b_2 = \frac{x_{22}}{2}$, etc... with $b_i = \frac{x_{ii}}{2}$. Thus, we have

$$U = \prod_{i=1}^{\infty} \left(-327, \frac{x_{ii}}{2}\right).$$

Then $\overline{0} \in U$, yet for all $i$, $x_i \notin U$ as $x_{ii} \geq \frac{x_{ii}}{2}$. So it is not true that almost all of the $x_i$'s belong to $U$. $\square$

[20] note that in class, dror uses seq-cl instead of scl for sequential closure. i just can't get it to work on latex :¡

(b) Let $X = \mathbb{R}^{\mathbb{R}}_{\text{cyl}}$. Let

$$\begin{aligned} B &:= \{\text{beds of nails}\} = \{f : \mathbb{R} \to \mathbb{R} \mid f = 0 \text{ except on finitely many } x\} \\ &= \{f \mid \exists \text{ finite } A_f \subset \mathbb{R}\, s.t.\ f(x) = 0 \text{ if } x \notin A_f\}. \end{aligned}$$

Then $\text{cl}\,(B) = X$. The sequential closure $\text{scl}\,(B)$ is given as an exercise. Indeed, for example, the function $e^x \in \text{cl}\,(B)$, even if $e^x$ is clearly not a bed of nails. Suppose $U$ is a neighborhood of $e^x$. $U$ is defined by constraining finitely many coordinates $x \in \mathbb{R}$ in open intervals $(a_x, b_x)$ (read: on the $y$-axis) for each given $x$. Then there is a bed of nails satisfying these constraints.

**Claim 10.3.** We claim that the constant function, $\overline{1}(x) = 1$, does not belong in the sequential closure of beds of nails.

Suppose $f_n \in B$ (a sequence of $f_1, f_2, \ldots$ where we sequentially add more and more nails with each $f_i$), and $f_n \xrightarrow{\text{cyl}} \overline{1}$. Then let $A_n = \sup(f_n) = \{x \mid f_n(x) \neq 0\}$ be a finite set. $\bigcup A_n$ is countable, hence $\bigcup A_n \subsetneq \mathbb{R}$. Thus, we may pick $y \notin \bigcup A_n$, meaning $f_n(y) = 0$ for every $n$.

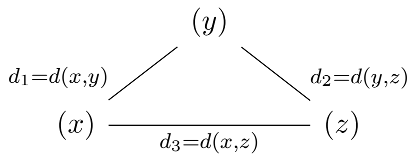
Pick a neighborhood of $\overline{1}$, $U = \{f \mid f(y) > 0\}$. Then $U$ is open in the cylinder topology because we've only constrained 1 coordinate to be in the open set. Thus, $\overline{1} \in U$ and for all $n$, $f_n \in U$, and so $f_n \not\to \overline{1}$.

**Theorem 10.4** (Countable Product of Metrizable Spaces)**.** The countable product of metrizable spaces is also metrizable in the cylinder topology, i.e., if $X_n$ is metrizable for all $n$, then $(\prod_{n=1}^{\infty} X_n)_{\text{cyl}}$ is metrizable.

Given $(X_n, d_n)$, we need a metric on $\prod X_n = \{(x_n) \mid x_n \in X_n\}$. Then $d\left((x_n), (y_n)\right) = \sup\left(\frac{\overline{d_n}(x_n, y_n)}{n}\right)$.

**Lemma 10.5.** If $d$ is a metric, then so is $\overline{d}(x, y) = \min\{1, d(x, y)\}$, and it induces the same topology as $d$.

We start by noting that $\overline{d}$ is a metric.




If $d_1 + d_2 \leq 1$, then $d_3 \leq 1$ and now $\overline{d_i} = d_i$. If $d_1 + d_2 > 1$, then $\overline{d_1} + \overline{d_2} \geq 1 \geq \overline{d_3}$. Now, let $\mathcal{B} = \{B_r(x)\}$ be replaced with the set $\{B_r(x) \mid r < \frac{1}{7}\}$ or some arbitrary constant less than 1 (for now, we pick $\frac{1}{7}$). Then $d, \overline{d}$ induce the same topology because their small balls are the same. $\square$

We now claim that if $(X_n, d_n)$ are metric spaces, then defining $d((x_n), (y_n)) = \sup_n \overline{d_n}(x_n, y_n)$ is a metric on $X = \prod X_n$ called "the uniform metric on $X$". It defines neither the box nor the cylinder topology.

We will continue the proof next lecture.

# §11 Day 11: Metric Spaces and Products; Quotient Spaces (Oct. 8, 2024)

Outfit of the day! RAINBOW HAT!!!


Recap of last lecture; suppose $n \in \mathbb{N}$ and $X_n \neq \emptyset$ is metrizable. Then $X = \prod_n X_n$ is metrizable. Also, given $(X_n, d_n)$ with $d_n$ bounded by 1 (Lemma 10.5), set $d((x),(y)) = \sup_n \frac{1}{n} d(x_n, y_n)$. Then we claim that

(a) $d((x),(y))$ as given above is a metric.

(b) Said metric induces the cylinder topology.

We start by proving (a); i.e., we will check that $\sup_n d_n(x_n, y_n)$ is a metric (i.e., the maximum across all $(X_n, d_n)$). The first two axioms are obvious, so we only check that $d$

satisfies the triangle inequality. Let us have $(x), (y), (z) \in X$, and write

$$\begin{aligned} d((x),(y)) + d((y),(z)) &\geq \sup\underbrace{(d_n(x_n,y_n))}_{a_n} + \sup\underbrace{(d_n(y_n,z_n))}_{b_n} \\ &= \sup(a_n) + \sup(b_n) \\ &= \sup\left(d_n(x_n,y_n) + d_n(y_n,z_n)\right) \qquad \text{(Triangle Ineq. in } X_n) \\ &\geq \sup d_n(x_n,z_n) = d((x),(z)). \quad \square \end{aligned}$$

We now check that (b) holds as well; the proof in class was unclear, so we will simply refer to Theorem 20.5 in Munkres.

We now move onto quotient spaces. To start, we define the notion of equivalence relations. An *equivalence relation* on a set $X$ is a relation $\sim: X \times X \to \{0,1\}$ (read: true or false) such that it satisfies the following properties,

(a) $x \sim x$ (reflexive property);

(b) $x \sim y \implies y \sim x$ (symmetric property);

(c) $x \sim y, y \sim z \implies x \sim z$ (transitive property).

We now go over a few examples,

(a) "Examples" $\sim$ "examples" under the relation where uppercase and lowercase are conisdered equivalent.

(b) '=' is an equivalence relation.

(c) Let $f : X \to Y$. Define $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Without loss of generality, we may let $f$ be surjective.

**Theorem 11.1.** All equivalence relations come in this way, i.e. equivalence relations on $X$ are equivalent to surjections with domain $X$.

We start by providing a few more definitions.

**Definition 11.2.** Given $X$ be equipped with an equivalence relation $\sim$, i.e. $(X, \sim)$, and let $x_0 \in X$; we have that $X \supset [x_0]$, i.e. $[x_0]$, the equivalence class of $x_0$, which is given by $\{x \in X \mid x_0 \sim x\}$.

**Lemma 11.3.** Let us have equivalence classes $[x_0], [x_1]$. Such equivalence classes are either equal or disjoint.

Suppose $[x_0] \cap [x_1] \ni z$. Then by definition, $x_0 \sim z \sim x_1$, meaning $x_0 \sim x_1$. Then every element in $[x_0]$ and $[x_1]$ are equivalent to each other by transitivity. Thus, $[x_0] = [x_1]$; if no such $z$ exists, clearly $[x_0]$ and $[x_1]$ are disjoint. $\square$

**Lemma 11.4.** The union of $[x_0]$ over all $x_0 \in X$ is equal to the whole of $X$.

**Definition 11.5.** The set $X/\sim$ is called "$X$ mod $\sim$". It is the set of all qeuivalence classes in $X$, $\{[x_0] \mid x_0 \in X\}$. Then let $\pi : X \to X/\sim$, and $\pi : x_0 \mapsto [x_0]$.

We now prove the theorem. Given $\sim$ on $X$, $Y := X/\sim$, and $f = \pi$. Then we claim that $\sim_\pi = \sim$. The rest of the proof is skipped.

Suppose $X$ is a topological space, and $\sim$ is an equivalence relation on $X$. We seek a topology on $X/\sim$ such that

(a) $\pi : X \to X/\sim$ is continuous.

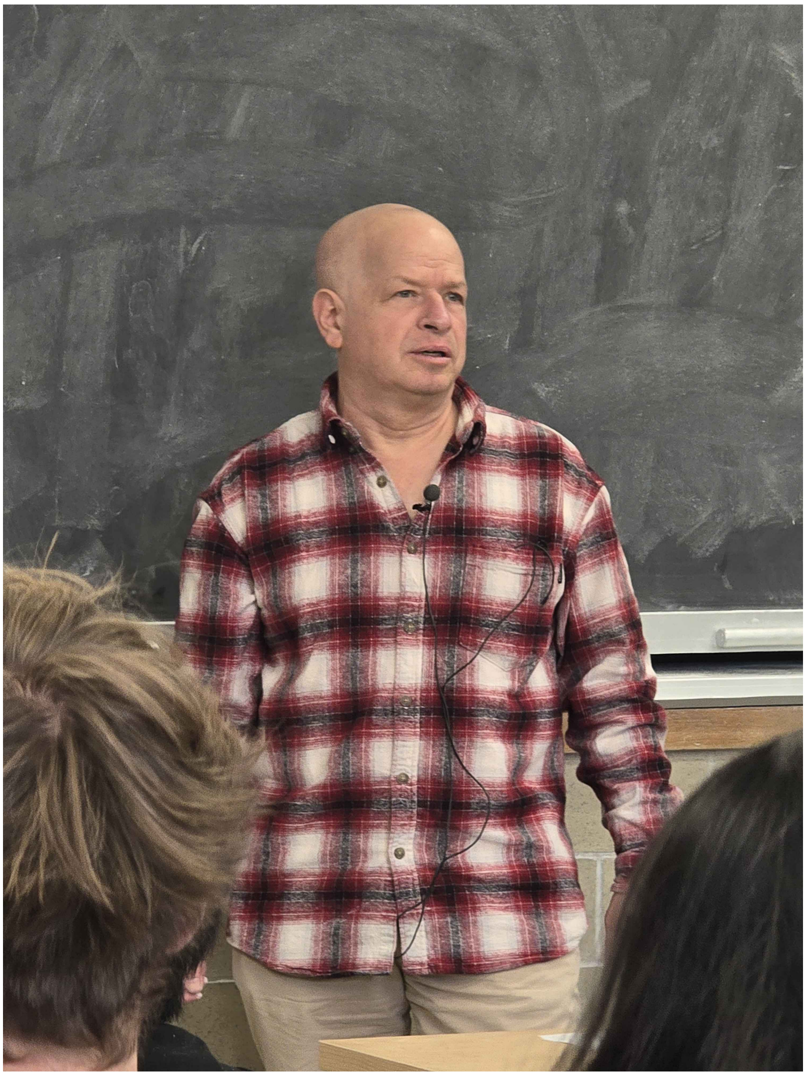
(b) Given $g : X/\sim \to Z$, if $g \circ \pi$ is continuous, then so is $g$.

**Theorem 11.6.** Such a topology satisfying the above conditions exists, and is unique.

The proof is left to next class / as an exercise or something.

# §12 Day 12: Quotient Topology Examples; Connectivity (Oct. 10, 2024)

Outfit of the day! no rainbow hat :c



Recap! Given a surjection $\pi : X \to Y$ or a quotient map $\pi : X \to X/{\sim}$, seek a topology on $X/{\sim}$ such that

(a) $\pi : X \to X/{\sim}$ is continuous

(b) If $f : X/{\sim} \to Z$ is such that $f \circ \pi$ is continuous, then so is $f$.

**Theorem 12.1.** Such a topology satisfying the above conditions exists, and is unique.

We start with uniqueness. Observe the following diagram,

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ (X/{\sim})_1 \arrow[rr, "\mathrm{id}"'] & & (X/{\sim})_2 \end{tikzcd}$$

Then the proof is similar to every other proof we have done before now; a continuous identity means they're the same. To check existence, let us have $\mathcal{T}_{X/\sim} := \{V \subset X/{\sim} \mid \pi^{-1}(v) \in \mathcal{T}_X\}$.

**Claim 12.2.** $\mathcal{T}_{X/\sim}$ is indeed a topology on $X/\sim$ that satisfies (2).

*(Proof is left as an exercise)* read this link mayb?

We now go over some examples.

(a) Consider $S^1 = {}^{[0,1]}/_{0 \sim 1} = \{x \sim x \mid x \in [0,1]\} \cup \{0 \sim 1, 1 \sim 0\}$. Then this is equivalent to $(0,1) \cup N$, where $N$ is a new point next to 0 and 1 (read: $N$ is a point connecting the endpoints of the interval $[0,1]$ to "join" it into the circle $S^1$).

(b) $X = {}^{[0,1] \times [0,1]}\Big/_{(0,y) \sim (1,y)}$ for all $y \in [0,1]$. Here, imagine a unit square; we are identifying its vertical edges as identical; this is similar to (a), except we have a square instead of a unit interval. Thus, we see $X$ is basically a cylinder.

(c) $X = {}^{[0,1] \times [0,1]}\Big/_{(0,y) \sim (1,1-y)}$ is a Möbius band.

(d) $X = {}^{[0,1] \times [0,1]}\Big/_{\substack{(0,y) \sim (1,y) \\ (x,0) \sim (x,1)}}$ is a torus.

(e) $X = {}^{[0,1] \times [0,1]}\Big/_{\substack{(0,y) \sim (1,y) \\ (x,0) \sim (1-x,1)}}$ is a Klein bottle.

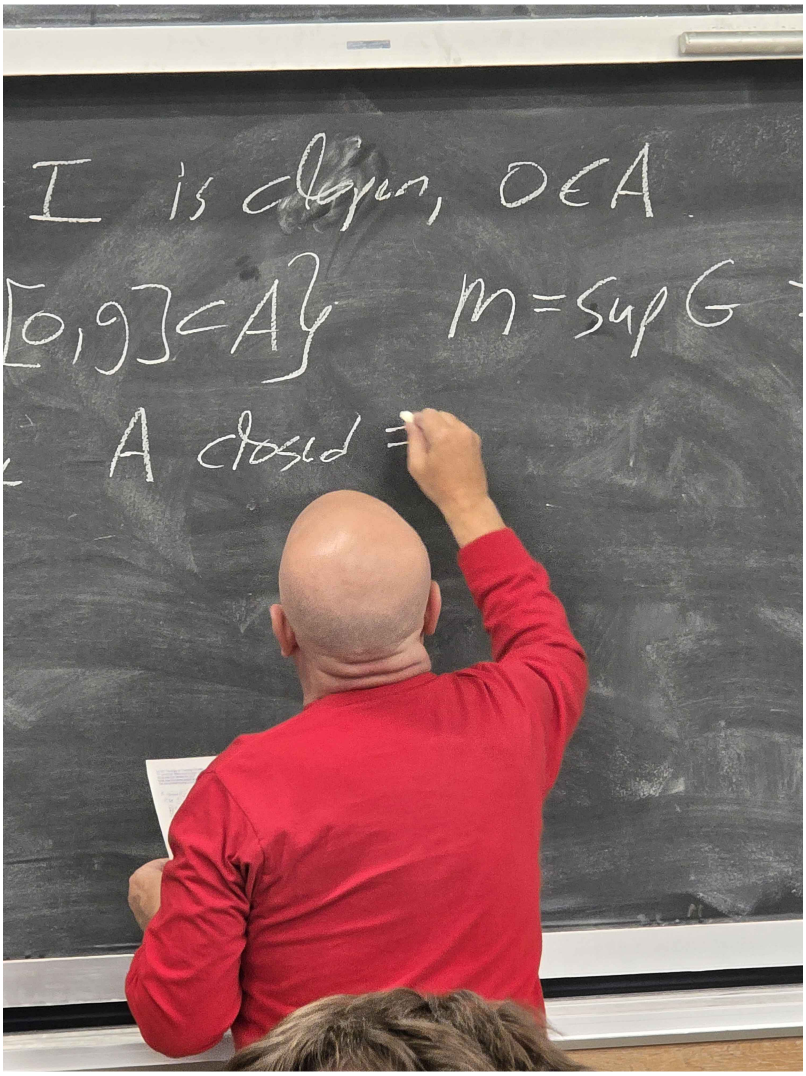
(f) $\mathbb{RP}^2$ or $\mathbb{P}_2(\mathbb{R})$ (the real projective space, here), given by ${}^{[0,1] \times [0,1]}\Big/_{\substack{(0,y) \sim (1,1-y) \\ (x,0) \sim (1-x,1)}}$ cannot be embedded in $\mathbb{R}^3$;

(g) octagon type shit

(h) Let us consider vectors $v$ as having magnitude and direction; we may write them as $|v| \in \mathbb{R}$, and the direction belongs in $\mathrm{dir}(v) = {}^{v}\Big/_{\substack{v_1 \sim \alpha v_2 \\ \alpha > 0}}$.

# §13 Day 13: Connectivity (Oct. 15, 2024)

Outfit of the day! red shirt!!!




Term test information is given here. Recap! Consider the function space $\mathbb{R}^{\mathbb{R}} = \mathrm{SL}(\mathbb{R}, \mathbb{R})$, aka all functions $f : \mathbb{R} \to \mathbb{R}$. We claim that $\mathbb{R}^{\mathbb{R}}$ is *not* metrizable. Consider the subset $A \subset \mathbb{R}^{\mathbb{R}}$ where $A$ is given by

$$A = \{f : \mathbb{R} \to \mathbb{R} \mid |\mathrm{supp} f| < \infty\}.$$

We make the following claims:

(a) The function $\overline{1}(x) = 1$ is in the closure of $A$, but

(b) $\overline{1} \notin \mathrm{scl}(A)$. This means that $\mathbb{R}^{\mathbb{R}}_{\mathrm{cyl}}$ is not metrizable.

We prove the first claim. $A$ intersects every open set containing non-empty open set, i.e.

$$U = \prod_{\alpha \in \mathbb{R}} U_\alpha,$$

where we see that $U_\alpha = \mathbb{R}$ for all but finitely many indices; let these indices be given by $\alpha_1, \ldots, \alpha_n$. Pick $y_i \in U_{\alpha_i}$, and let $f(\alpha_i) = y_i$, and $f(\alpha) = 0$ if $\alpha \neq \alpha_i$ for some $i$. Then supp$f \subset \{\alpha_i\}$, so $f \in A$, and also $f \in U$ as $f(\alpha) \in U_\alpha$ for all $\alpha$. Indeed, this holds over the indices $\alpha_1, \ldots, \alpha_n$, and for the rest there is nothing to check.

For the second claim, suppose $f_n \to \overline{1}$, and $f_n \in A$. Then the support of $f_n$ is finite, and so the union of the supports of all $f_i$ is also countable. Then choose $\alpha \notin \bigcup_{n=1}^{\infty} \operatorname{supp}(f_n)$, meaning $f_n(\alpha) = 0$ for all $\alpha$. Then $U = \prod U_\beta$, where $U_\beta$ can be picked as $(\frac{1}{2}, \frac{3}{2})$ if $\alpha = \beta$, and $\mathbb{R}$ otherwise. Then $U$ is open, $\overline{1} \in U$, none of $f_n$ are in $U$, as $f_n(\alpha) = 0 \notin U_\alpha$. $\square$

We now return to connectedness. If $X$ is connected, then $X$ is not empty, and has no clopen subsets aside from $\emptyset$ and $X$.

**Theorem 13.1.** Let $X$ be connected, and consider a continuous function $f : X \to \mathbb{R}$, where $f(x_0) < 0, f(x_1) > 0$ implies that there exists $x$ such that $f(x) = 0$.

Suppose that there does not exist $x$ such that $f(x) = 0$; then we have that $X = f^{-1}((-\infty, 0)) \cup f^{-1}((0, \infty))$. Each of the pre-images are open, since they are the pre-images of open sets; they are also nonempty, since $x_0$ belongs to the former and $x_1$ belongs to the latter. Since there does not exist $x$ such that $f(x) = 0$, their intersection is empty, and so have a non-trivial separation of $X$, contradicting the connectiveness of $X$. $\square$

**Theorem 13.2.** $I = [0, 1]$ is connected.

Assume $A \subset I$ is clopen, and $0 \in A$. Let $G := \{g \in I \mid [0, g] \subset A\}$, and let $m = \sup G > 0$. Then $m = 1$; if not, we would have $m < 1$: using that $A$ is closed, we have $m \in A$. Moreover, since $A$ is open, there exists $\varepsilon > 0$ such that $(m - \varepsilon, m + \varepsilon) \subset A$, and then $m + \frac{\varepsilon}{2} \in G$, meaning $m \geq m + \frac{\varepsilon}{2}$, which is obviously untrue. It remains to show that $1 \in G$. Indeed, $A$ is closed, and so $G \subset A$, meaning $1 = m = \sup G \in \overline{G} \subset \overline{A} = A$. But $A$ is open, and so for some $\varepsilon > 0$, $(1 - \varepsilon, 1] \in A$. Pick some $g \in G$ such that $g > 1 - \varepsilon$. Then $[0, 1] = [0, g] \cup (1 - \varepsilon, 1] \subset A$. The former is in $A$, since $g \in G$, and the latter is in $A$ by choice. $\square$

**Claim 13.3.** A continuous image of a connected set is connected; e.g., $[a, b] = f([0, 1])$.

Suppose $A$ is a non-trivial clopen in $Y$; then $f^{-1}(A)$ is a non-trivial clopen in $X$.

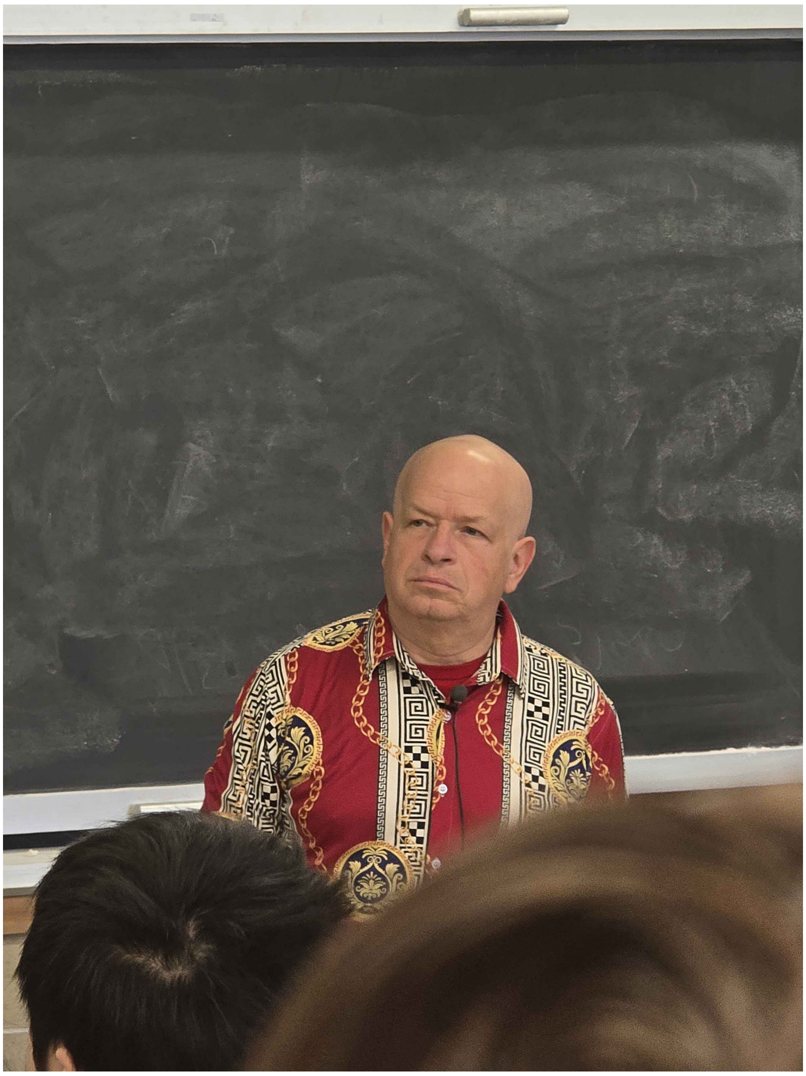
**Claim 13.4.** If, for all indices $\alpha$, the subsets $A_\alpha \subset X$ is connected, and $\bigcap A_\alpha \neq \emptyset$, then $\bigcup A_\alpha$ is connected.

Suppose $B \subset \bigcup A_\alpha$ is clopen and non-empty. Then, for some $\alpha_0$, $B \cap A_{\alpha_0} \neq \emptyset$. But $A_{\alpha_0}$ is connected, so $B \supset A_{\alpha 0}$, meaning $B \supset \bigcap A_| alpha \neq \emptyset$, and so for all indices $\alpha_i$, $B \cap A_{\alpha_i} \neq \emptyset$, but $A_{\alpha_1}$ is connected, and so $B \cap A_{\alpha_1} = A_{\alpha_1}$, so for all $\alpha$, $B \supset A_\alpha$, so $B \supset \bigcup A_\alpha$. $\square$

As a corollary, $[a, b), (a, \infty), (-\infty, a)$ are connected. To prove this, write them as a union of closed intervals, etc...

# §14 Day 14: Connectedness of Products. Path Connectedness (Oct. 17, 2024)

Outfit of the day. not sure what to call this one "i'm terribly sorry you have to see me for 3 days in a row, but the good news is in a week and a half, you won't have to see me for a week".



Recap: $[0,1]$ is connected, continuous image of connected is connected, and a union of connected sets with a nonempty intersection is connected. This means we have the following theorem,

**Theorem 14.1.** A subset of $\mathbb{R}$ is connected if and only if it is an interval, i.e.the interval can be open, closed, half-open, etc... whatever, or if it is a ray, such as $(-\infty, a)$, $(-\infty, a]$, etc..., or the whole of $\mathbb{R}$.

The theorem is equivalent to saying that if a subset $A \subset \mathbb{R}$ is convex, if $a, b \in A$ with $a < b$, then $[a, b] \subset A$.

($\Leftarrow$) All of the 9 cases above are easy. For example,

$$(9, \infty) = \bigcup_{n=1}^{\infty} \left[9 + \frac{1}{n}, n + 20\right].$$

($\Rightarrow$) If $A$ is not convex, then there exists $a, b, c$ such that $a < c < b$ with $a, b \in A$ but $c \notin A$. Then $A = (A \cap (-\infty, c)) \cup (A \cup (c, \infty))$. Clearly, both sets in the union are open, so we see this is a separation of $A$. □

**Theorem 14.2.** $X, Y \neq \emptyset$ is connected if and only if $X \times Y$ is connected.

($\Leftarrow$) Indeed, $X = \pi_X(X \times Y)$ is a continuous image of connected sets, so it is also connected. We also have $Y = \pi_Y(X \times Y)$, which is also connected. Note that this theorem *isn't* in Munkres because Munkres does not assume $X, Y$ are nonempty.

($\Rightarrow$) Start by observing that $\{x_0\} \times Y$, $X \times \{y_0\}$ is connected. We may pick any arbitrary $x_0, y_0$ in $X, Y$ respectively, so write (we decided to vary on $y_0$ during class)

$$X \times Y = \bigcup_{y_0 \in Y} (\{x_0\} \times Y) \cup (X \times \{y_0\}),$$

which is a union of connected sets with non-empty intersection, i.e. the intersection is $\{x_0\} \times Y$. Note that if either of $X$ or $Y$ are empty, we can simply write the union in terms of the other set. □

We now look at some examples. Consider $\mathbb{R}^{\mathbb{N}}_{\text{box}}$, which is the union of bounded and non-bounded sequences (as seen on the 2010 and 2018 midterms). We know that $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is not connected as a result. To check this, observe that all bounded sets $x = \{x_n\}_{n \in \mathbb{N}}$ are bounded above by $\sup |x_n| \leq M$; then

$$U_x = \prod_{n=1}^{\infty} (x_n - 1, x_n + 1)$$

is a neighborhood of $x$, while all sequences in $U_x$ are bounded by $M + 1$. By construction, we see that there exists an open neighborhood of bounded sequences about any bounded sequence, so the set is open. Likewise, the set of non-bounded sequences is bounded: we may take $U_x$ as the same construction to see that all sequences in there are unbounded as well. Since the sets of bounded and non-bounded sequences are disjoint, we see that $\mathbb{R}^{\mathbb{N}}$ is not connected in the box topology.

**Theorem 14.3.** For all indices $\alpha$, $X_\alpha$ is connected if and only if $\prod X_\alpha$ is connected in the product topology.[21]

($\Leftarrow$) Same argument as earlier (14.2) literally.

($\Rightarrow$) Intuition: let us fix a point in each $X_\alpha$; then every product of a finite number of spaces and fixed points in the rest of the spaces is connected; taking the union of all these products, we have that its closure is the whole space.

Pick $x_\alpha \in X_\alpha$ for each $\alpha$; for a finite set $F \subset A$, consider

$$B_F = \left\{ y : A \to \bigcup X_\alpha \mid y_\alpha \begin{cases} = x_\alpha & \text{if } \alpha \notin F \\ \in X_\alpha & \text{otherwise} \end{cases} \right\}.$$

For each $F$, $B_F \simeq \prod_{\alpha \in F} X_\alpha$ is a finite product of connected space, hence it is connected. Also, $x = (x_\alpha)$ clearly belongs to $B_F$ for every $F$, so the intersection

[21] never saying cylinders again fuck me

of $B_F$ over all $F$ is clearly nonempty. Thus, the union of all $B_F$ over finite sets $F \subset A$ is connected; for convenience, denote the union as $B$. Then we may write

$$B = \bigcup B_F = \left\{ y \in \prod X_\alpha \mid y_\alpha = x_\alpha \text{ for all but finitely many } \alpha \right\}.$$

We now claim that the closure of $B$ is the full set; $B$ is dense in $\prod X_\alpha$, i.e. $\overline{B} = \prod X_\alpha$, which is equivalent to saying that every non-empty open set in $\prod X_\alpha$ intersects $B$. Indeed, let $U = \prod U_\alpha$ with $U_\alpha = X_\alpha$ almost always be a basic open set in the product $\prod X_\alpha$. Then for $y = (y_\alpha)$, we have that $y_\alpha$ is some element of $U_\alpha$ if $U_\alpha \neq X_\alpha$, and $x_\alpha$ if $U_\alpha$ is the full set. By construction, we see that $y \in B \cap U$. $\square$

**Lemma 14.4.** Let $B \subset X$ be connected, and $B \subset A \subset \overline{B}$. Then $A$ is connected.

Clearly, the lemma completes the *if* part of the theorem, because $B$ is connected in our case and hence also $\overline{B}$. We now check the lemma. If $C$ is clopen in $A$, then without loss of generality, $C \cap B \neq \emptyset$. Then $C \cap B$ is clopen in $B$, but $B$ is connected, so $C \supset B$. Note that we have $C = \mathrm{cl}_A C$, and we want to compare the closure of $C$ inside $A$ (which is itself) and the closure of $C$ inside $X$. Observe that $\mathrm{cl}_A C = A \cap \mathrm{cl}_X C$, and we have that

$$A \cap \mathrm{cl}_X C \supset A \cap \mathrm{cl}_X B = A,$$

so $C$ is either trivial, meaning that $A$ has no separation.

We used the fact that $C \subset A \subset X$ implies $\mathrm{cl}_A C = A \cap \mathrm{cl}_X C$ earlier; we prove it now. If $x \in \mathrm{cl}_A C$, then $x \in A$ and every neighborhood of $A$ containing $x$ intersects $C$. This is equivalent to saying that every neighborhood of $x$ in $X$ intersects $C$, which is also equivalent to saying that $x \in \mathrm{cl}_X C$, and we have that $x \in A \cap \mathrm{cl}_X C$. $\square$

We now move onto a lighter topic. We say that a set $X$ is *path-connected* if there exists a "path" connecting any two points. Formally, for every $a, b \in X$, there exists a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = a, \gamma(1) = b$.

**Theorem 14.5.** If $X$ is path-connected, it is also connected.

Pick a point $m \in X$, and for every $b \in X$, pick a path, i.e., a continuous function $\gamma_b : [0,1] \to X$ such that $g_b(0) = m$ and $g_b(1) = b$. Then $S_b = \gamma_b([0,1])$ is connected with $b \in S_b$ and $m \in S_b$. Now,

$$\bigcup_{b \in X} S_b = X, \qquad \bigcap S_b \supset \{m\},$$

meaning this union of $S_b$ is a construction to show that $X$ is connected. Note that the converse is a harder topic we do not talk about yet. $\square$

We now talk about a few properties.

(a) We give an example of a connected, but not path-connected set. The *topologist's since curve* is given by the closure of $S = (-\infty, 0] \times \{0\} \cup \{(x, \sin \frac{1}{x}) \mid x > 0\}$, i.e. $\overline{S} = \{0\} \times [-1,1] \cup S$.

(b) IVT holds for p.c. spaces.

(c) $[0,1]$ is p.c.

(d) The continuous image of p.c. is p.c.. Suppose $X$ is p.c., and $f : X \to Y$ is continuous. Then $f(X)$ is also p.c. because the composition of continuous functions is continuous, i.e. $f \circ \gamma_b$ for any $b \in X$ yields a path for $f(m)$ to $f(b)$.

(e) The product, $X = \prod X_\alpha$ is p.c., if and only if for all indices $\alpha$, $X_\alpha$ is also p.c.

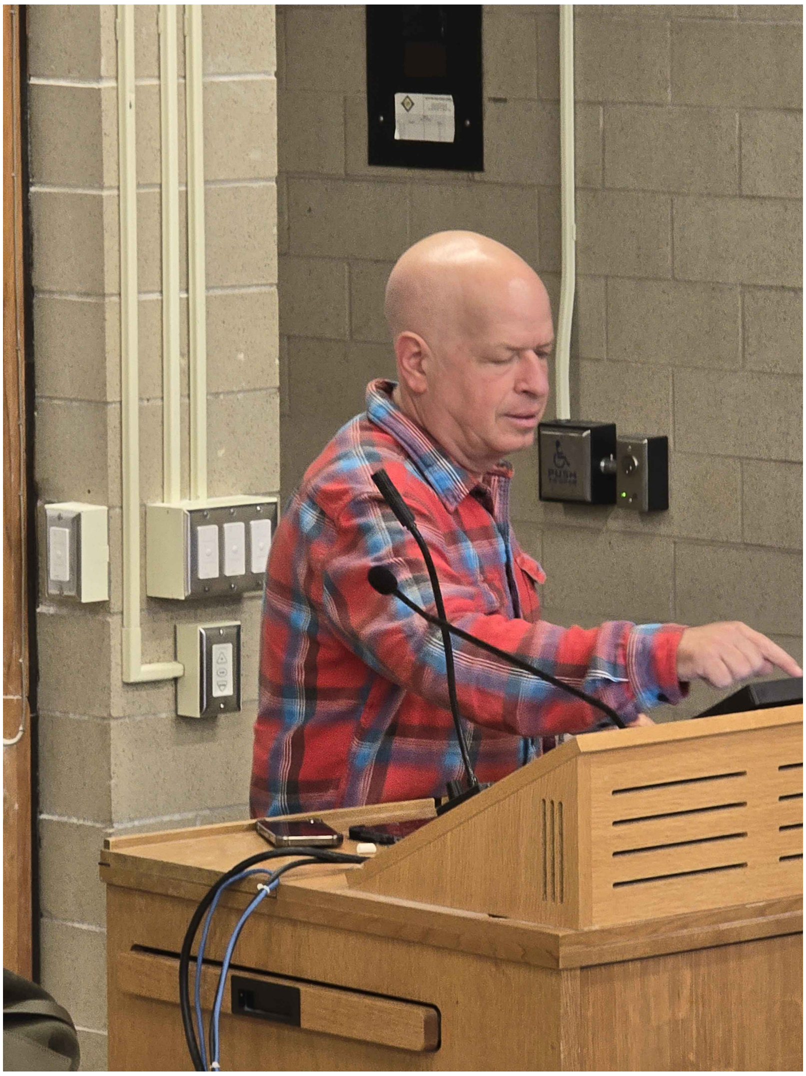
($\Rightarrow$) Writing $X_\alpha = \pi_\alpha(\prod X_\alpha)$, we see that the projection function is continuous, so any path function on $X$ composed with $\pi_\alpha$ yields a path on $X_\alpha$.

($\Leftarrow$) If $X_\alpha$ is p.c., then suppose $x = (x_\alpha)$, $y = (y_\alpha)$ are in $X$. We use the p.c. of each $X_\alpha$ to find a continuous $\gamma_\alpha : [0,1] \to X_\alpha$ such that $\gamma_\alpha(0) = x_\alpha$ and $\gamma_\alpha(1) = y_\alpha$. Now, consider the combined path $\gamma : [0,1] \to X$ by $\gamma(t)_\alpha = \gamma_\alpha(t)$ for $t \in [0,1]$. $\gamma$ is continuous because the second axiom of product spaces states that if a function is continuous in its coordinates (i.e., $\gamma_\alpha$), then the function $\gamma$ itself is continuous. $\gamma(0)$ is clearly $x$, $\gamma(1)$ is clearly $y$, and we are done; we conclude that $X$ is p.c.. $\square$

We end class with a riddle: do there exist disjoint connected paths $A, B$ on $[0,1] \times [0,1]$ such that $A$ goes from $(0,0)$ to $(1,1)$ and $B$ goes from $(1,0)$ to $(0,1)$ in the standard topology?

# §15 Day 15: Compactness (Oct. 22, 2024)

Outfit of the day: balatro theme



Our goal for today is to imitate the proof that, on a compact set, every continuous function $F : X \to \mathbb{R}$ is bounded. Start by observing that every continuous function is "locally bounded" (read: $\varepsilon - \delta$), where *locally* means that every point has a neighborhood in which $f$ is bounded[22].

To prove this, given a point $x \in X$, we have that $U = f^{-1}((f(x) - 1, f(x) + 1))$ be a neighborhood of $x$ on which $f$ is bounded. Compactness is a property of spaces that intuitively means a local property is true globally as well. A topological space $X$ is called *compact* if it has a finite cover by open sets.

**Definition 15.1.** A *cover* of a set $X$ is a collection $A$ of subsets of $X$ such that $\bigcup_{A \in U} A = X$. If all the sets in $A$ are open, we say that $U$ is an open cover.

We say that a topological space $X$ is called compact if every open cover of $X$ has a finite subcovering.

[22] In general, *locally* [topo property] means that said property holds in some open neighborhood about every point. Something on basic sets maybe, check lance

**Theorem 15.2.** A continuous function on a compact space is bounded. Namely, if $X$ is compact and $f : X \to \mathbb{R}$ is continuous, then there exists $M > 0$ such that, for all $x \in X$, $|f(x)| < M$.

Let $\mathcal{O} = \{U_x = f^{-1}((f(x) - 1, f(x) + 1)) \mid x \in X\}$ is an open cover of $X$, as each $U_x$ is open, and all $x \in X$ are in $U_x$, we see that $\mathcal{O}$ covers $X$; by compactness, there exists a finite subcover $U_{x_1}, \ldots, U_{x_n}$; then we may take a maximum of the bounds on each $U_{x_i}$, i.e. $M = \max\{|f(x_1)| + 1, \ldots, |f(x_n)| + 1\}$ to see that $f$ is bounded above by $M$. □

**Theorem 15.3.** $[0, 1]$ is compact.

Let $\mathcal{O}$ be an open cover of $[0, 1]$, and set $G = \{g \in [0, 1] \mid [0, g]$ covered by finitely many $U \in \mathcal{O}\}$. Start by observing that $G$ is not empty because $0 \in G$, and $G \in [0, 1]$ so it is bounded. Let $m = \sup G$; we want to prove that $1 \in G$. To start, $m > 0$; indeed, pick $U \in \mathcal{O}$ such that $0 \in U$, which is possible because the cover covers the entire of $[0, 1]$. $U$ is open, so for some $\varepsilon > 0$, $[0, \varepsilon) \in U$, so $U$ covers $[0, \frac{\varepsilon}{2}]$, meaning $\frac{\varepsilon}{2} \in G$ and so $m \geq \frac{\varepsilon}{2} > 0$.

We now show that $m = 1$. Assume not, i.e. $0 < m < 1$. Let $U \in \mathcal{O}$ be a neighborhood of $m$; then $(m - \varepsilon, m + \varepsilon) \in U$ for $\varepsilon > 0$, meaning $[m - \frac{\varepsilon}{2}, m + \frac{\varepsilon}{2}] \in U$; taking a finite subcover of $[0, m']$ (for some $m - \frac{\varepsilon}{2} < m' < m$, which is necessarily in $G$, otherwise $m$ is not the supremum) and appending $[m - \frac{\varepsilon}{2}, m + \frac{\varepsilon}{2}]$, we see that $[0, m + \frac{\varepsilon}{2}] \in G$, meaning $\sup G \neq m$, which is a contradiction. Thus, $m = 1$.

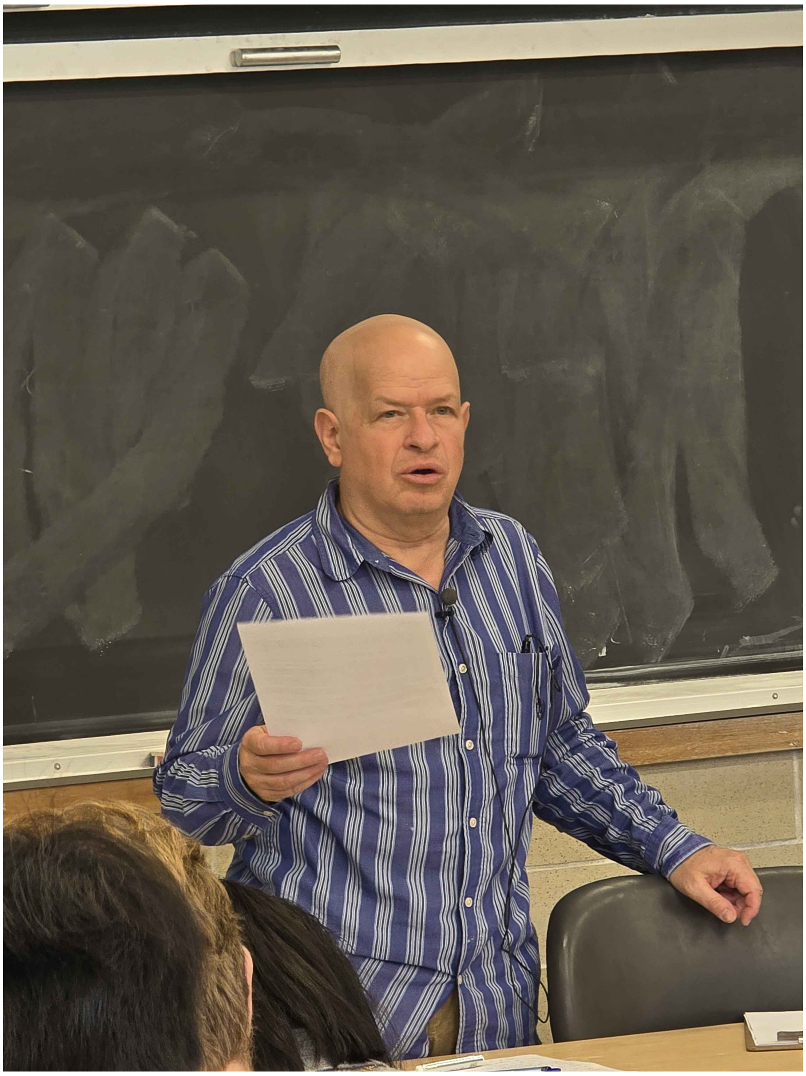
Indeed, find $U \in \mathcal{O}$ such that $1 \in U$; let $\varepsilon > 0$ such that $(1 - \varepsilon, 1] \in U$. Find $g \in G$ such that $1 - \varepsilon < g \leq 1$. Then $[0, g]$ has a finite cover, and $[0, g] \cup U = [0, g] \cup (1 - \varepsilon, 1] = [0, 1]$, which has a finite cover as well. □

**Theorem 15.4.** A closed subset $C$ of a compact set $X$ is compact.

Picture proof was given, but just use Heine-Borel... (closed and bounded).

# §16 Day 16: Compactness, mostly in $\mathbb{R}^2$ (Oct. 24, 2024)

Outfit of the day: stripes of pinstripes



We define compactness as the property that every open cover of a set $C$ admits a finite subcover.

**Theorem 16.1.** A continuous function on a compact space is bounded.

**Theorem 16.2.** $[0,1]$ is compact.

**Theorem 16.3.** A closed subset of a compact set is compact.

The proofs for all three were given in last lecture.

**Theorem 16.4.** A compact subset $C$ of a Hausdorff space $X$ is closed.

Let $x \in X \setminus C$. By Hausdorffness, for any $y \in C$, there exists open neighborhoods $U_y \ni x, V_y \ni y$ and $U_y \cap V_y = \emptyset$. The colllection $\{V_y \mid y \in C\}$ is an open cover of $C$, so by compactness, it has a finite subcover $\{V_{y_1}, \dots, V_{y_n}\}$. Consider $U = \bigcap_{i=1}^n U_{y_i}$, in which $x \in U$ and $U$ is open. Then

$$U \cap \bigcup_{i=1}^n V_{y_i} = \bigcup_{i=1}^n (U \cap V_{y_i}) \subset \bigcup_{i=1}^n (U_{y_i} \cap V_{y_i}) \subset \emptyset,$$

so $U$ is disjoint from $C$. □

**Definition 16.5** (Regular / $T_3$)**.** A space $X$ is called $T_3$ or *regular* if you can separate points from closed sets: for any $x \in X$ and closed subset $A \subset X$, then there exists open $U, V$ such that $U \ni x$, $A \subset V$, and $U \cap V = \emptyset$.

**Theorem 16.6.** A compact $T_2$ space is $T_3$.

Given $x, A$ as per the definition of a regular space, $A$ is a closed subset of a compact space, so it is compact. Now, following the previous proof that a compact subset of a Hausdorff space is closed, let

$$U = \bigcap U_{y_i}, \quad V = \bigcup V_{y_i}.$$

This is the desired separation. □

As a corollary, a subset $C$ of $\mathbb{R}$ is compact if and only if it is closed and bounded (Heine-Borel).

($\Leftarrow$) $C \subset [-M, M]$ for some $M$ by boundedness; since $[-M, M]$ is compact, and $C$ is a closed subset, $C$ is compact.

($\Rightarrow$) By a previous theorem and the fact that $R$ is Hausdorff, $C$ is closed. It is bounded because the union of $(-1, 1), (-2, 2), \ldots$ covers $\mathbb{R}$, and so we may take a finite subcover to contain $C$ to see that it is bounded. The inclusion $i : C \hookrightarrow \mathbb{R}$ is continuous on a compact set, so it is bounded. □

**Theorem 16.7.** A continuous image of a compact set is compact, i.e. if $f : A \to Y$ is continuous and $A$ is compact, then $f(A)$ is compact.

We start with a corollary, aka EVT; if $f : X \to \mathbb{R}$ is continuous and $X \neq \emptyset$ is compact, then $f$ attains its minimum and maximum, namely, $\exists a, b \in X$ such that for all $x \in X$, $f(a) \leq f(x) \leq f(b)$. We now prove this statement.

$f(X)$ is compact, hence it is bounded and closed. Let $m = \inf f(X)$, $M = \sup f(X)$; since $f(X)$ is closed, $m, F \in f(X)$, implying that there exists $a, b$ such that $m = f(a), M = f(b)$. □

We now prove the original theorem. Let $\mathcal{O}$ be an open cover of $f(A)$, and consider $\mathcal{O}' = \{f^{-1}(U) \mid U \in \mathcal{O}\}$, where we see that $\mathcal{O}'$ is an open cover of $A$. Since $A$ is compact, there exists a finite subcover of $\mathcal{O}'$ given by $f^{-1}(U_1), \ldots, f^{-1}(U_n)$ for $U_1, \ldots, U_n \in \mathcal{O}$; clearly, $U_1, \ldots, U_n$ cover $f(A)$. □

**Theorem 16.8.** If $X, Y$ is compact, then $X \times Y$ is compact.

We consider the reverse direction first;

($\Leftarrow$) This direction is more of a remark if anything; note that the converse of the theorem statement is true only if $X, Y$ are nonempty, since in this case, $\pi_X(X \times Y)$ and $\pi_Y(X \times Y)$ are surjective onto $X, Y$ respectively, and both spaces are continuous images of a compact sets, and so are compact themselves.

($\Rightarrow$) We start by introducing a lemma.

**Lemma 16.9** (Tube Lemma)**.** If $\mathcal{O}$ is an open cover of $X \times Y$ (with $Y$ compact), and $x \in X$, then there is some open neighborhood $W_x$ of $x$ such that $W_x \times Y$ can be covered with finitely many members of $\mathcal{O}$.

We now prove this lemma. Consider $\mathcal{O}'$ to be the set of $A \times B$, where $A \subset X$, $B \subset Y$, and $A \times B \subset U$ for some open set $U \in \mathcal{O}$. Clearly, $\mathcal{O}'$ covers $X \times Y$, so it covers $\{x\} \times Y$. Let us take a finite subcover of $\mathcal{O}'$, $A_i \times B_i \subset \mathcal{O}'$ for $i = 1, \dots, n$; without loss of generality, $x \in A_i$ for all $i$ (simple get rid of $A_i \times B_i$ from the finite subcover we're considering if $x \notin A_i$). Then let $W = \bigcap_{i=1}^n A_i \ni x$, so $W$ is a neighborhood of $x$ as required.

For each $i$, $A_i \times B_i \in \mathcal{O}'$, so we may find $U_i \in \mathcal{O}$ such that $A_i \times B_i \subset U_i$; now,

$$\bigcup_{i=1}^n U_i \supset \bigcup_{i=1}^n A_i \times B_i \supset \bigcup_{i=1}^n W \times B_i \supset W \times Y. \quad \square$$

We now prove the theorem. By the lemma, let $\mathcal{O}$ be an open cover of $X \times Y$, and by the lemma, for each $x \in X$, find a neighborhood $W_x$ and $U_{x,1}, \dots, U_{x,n_x} \in \mathcal{O}$ where $n_x \in \mathbb{N}$ such that $W_x \times Y \subset \bigcup_{i=1}^{n_x} U_{x,i}$. Since $\{W_x \mid x \in X\}$ is an open cover of $X$, it has a finite subcover $W_{x_1}, \dots, W_{x_m}$. The set

$$\{U_{x_1,1}, \dots, U_{x_1,n_{x_1}}, U_{x_2,1}, \dots, U_{x_2,n_{x_2}}, \dots, U_{x_m,1}, \dots, U_{x_m,n_{x_m}}\}$$

can be viewed more conveniently by considering each "cluster" as $W_{x_1} \times Y \cup \dots W_{x_m} \times Y$ instead, which is equal to $(W_{x_1} \cup \dots \cup W_{x_m}) \times Y = X \times Y$. Thus, $\mathcal{O}$ has a finite subcover by construction.

We now have two corollaries;

(a) The product of closed intervals, $\prod_{i=1}^n [a_i, b_i]$, in $\mathbb{R}^n$ is compact.

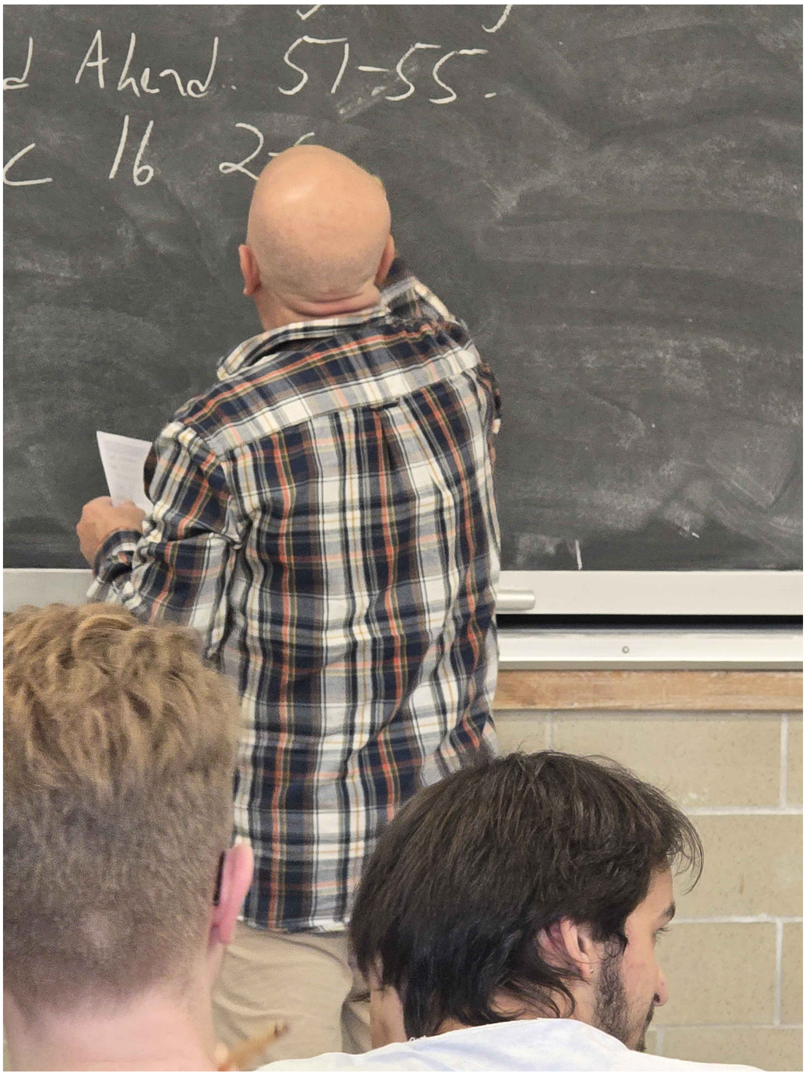
(b) $C \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

- ($\Leftarrow$) $C$ is closed and bounded, meaning we may take $C \subset \prod_{i=1}^n [-M, M]$ by boundedness; since the product of $[-M, M]$ is compact, $C$ is compact by being a closed subset of a compact set.
- ($\Rightarrow$) The argument is same as before; we see that $C$ is bounded by considering $C \subset \bigcup_{r>0} B_r(0)$ and considering a finite subcover of $\mathcal{O} = \{B_r(0) \mid r > 0\}$; if we consider $n : \mathbb{R}^n \to \mathbb{R}$ to be the norm function, i.e. $n(x) = \|x\|$, we see that it is continuous, hence we have boundedness. $\square$

We also talked a bit on Riemann integrals to introduce uniform continuity on arbitrary metric spaces. Define $f : X \to Y$ to be *uniformly continuous* where $X, Y$ are metric spaces, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \varepsilon$.

# §17 Day 17: Uniform Continuity... and with great pain, Intro to Algebraic Topology (Nov. 5, 2024)

Outfit of the day: looks like a valorant skin i forgot which one




Read sections 26-27, and read ahead on sections 51-55 in Chapter 9. Recall that compactness is the property that every open cover admits a finite subcover. Recall that the definition of uniform continuity is given as follows; if $f : X \to Y$ is uniformly continuous, where $X, Y$ are both metric spaces; then $d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$.

**Theorem 17.1.** If $X$ is compact, then any continuous function from $X$ is uniformly continuous as well.

To prove this, we start with a lemma.

**Lemma 17.2** (Lebesgue Number Lemma)**.** If $X$ is a compact metric space and $\mathcal{O}$ is an open cover of $X$, then there exists some $\varepsilon > 0$ such that for every $x \in X$, there exists $U \in \mathcal{O}$ such that $B_\varepsilon(x) \in U$.

Define the following function,

$$\Delta(x) = \sup\{\delta \leq 1 \mid \exists U \in \mathcal{O} \text{ s.t. } B_\delta(x) \subset U\},$$

of which we note that the set we are taking the supremum of is necessarily nonempty since $x$ is in some open set in $\mathcal{O}$, and since the set is bounded above by 1, the supremum exists. For $y$ satisfying $d(x, y) < \varepsilon$, we have $\Delta(y) \geq \Delta(x) - \varepsilon$. By symmetry, we also have that $\Delta(x) \geq \Delta(y) - \varepsilon$. This means $|\Delta(x) - \Delta(y)| < \varepsilon$, yielding that $\Delta$ is a continuous function.

Let $\delta_0 = \min_{x \in X} \Delta(x) > 0$. For $x \in X$, we have $\Delta(x) > \delta_0$, and so there exists $U \in \mathcal{O}$ such that we may pick $\delta = \frac{\delta_0}{2}$ to see $B_\delta(x) \subset U$, and we are done. We call $\delta$ the *Lebesgue number* of $\mathcal{O}$. [23] $\square$

We now prove the theorem. Let $\varepsilon > 0$ be given; by continuity, for every $x \in X$, find $\delta_x$ such that if $y \in B_{\delta_x}(x)$, then $d_Y(f(x), f(y)) < \frac{\varepsilon}{2}$. The collection $\{B_{\delta_x}(x) \mid x \in X\}$ is an open cover of $X$, so it has a Lebesgue number $\delta > 0$ as per our lemma. Now, if $d_X(x_1, x_2) < \delta$,then $x_2 \in B_\delta(x_1)$, so there exists some $x \in X$ such that $x_1, x_2 \in B_\delta(x_1) \subset B_{\delta_x}(x)$, so

$$d_Y(f(x_1), f(x_2)) \leq d_Y(f(x_1), f(x)) + d_Y(f(x), f(x_2)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

**Theorem 17.3** (Tychonoff's Theorem)**.** An arbitrary product of compact spaces is compact.

This won't be proven in class, but the Axiom of Choice is used ~~so it's a fuckin' lie~~. A whole other slew of stuff also won't be covered in class pwp... here's the list!

We now go onto algebraic topology. Notice that in MAT327, we study topological spaces and continuous maps; in MAT240, we studied vector spaces and linear maps; set theory deals with sets and functions, and MAT347 deals with groups and homomorphisms. On Thursday, we will construct a functor $\pi_1$ from topological spaces to groups and continuous maps to homomorphisms.

[23] i think the wikipedia proof is better... here

# §18 Day 18: The Fundamental Group (Nov. 7, 2024)

Outfit of the day: walter white

Turns out, we are bad at telling spaces apart (in terms of homeomorphisms). Here are some examples.

- Closed intervals and open intervals are not homeomorphic because one is compact, and the other isn't.
- Open intervals and half-open intervals are not homeomorphic either, since if we remove a point in an open interval, it would cause a disconnection, while we may remove the point at the closed end of the half-open interval and retain connectedness.
- Consider $D_0, D_1, D_2, \ldots$, i.e. $D_i$ is an open disc with $i$ disjoint open discs removed from it (read: punctured disc).
- Surfaces of different genus are not homeomorphic, but we don't have the tools to prove it yet.
- The circle and the trefoil knot are homeomorphic, since we may parameterize the curves (read: Jordan curve theorem; simple closed curve is homeomorphic to $S^1$.)

We start with some fundamental definitions in group theory.

**Definition 18.1.** A group is a set $G$ along with the three following operations:

- $m : G \times G \to G$, i.e., multiplication. Note that in practice, we will never write $m$ as multiplication, we're just writing it as a function for now, where $(a, b) \mapsto m(a, b) = a \cdot b = ab$.
- $\iota : G \to G$, $a \mapsto \iota(a) = a^{-1}$, i.e. the unary operation, or the inverse.
- $e = 1 \in G$, called the nullary operation; it specifies "the unit" in the group $G$.

Groups satisfy the axioms,

(a) Associativity; for $a, b, c \in G$, we have $(ab)c = a(bc)$.

(b) Identity; for $a \in G$, $ea = ae = a$.

(c) Inverse; $aa^{-1} = a^{-1}a = 1 = e$.

We now give some examples.

(a) If $V$ is a vector space, we have that it is also a group with $m = +$, $\iota = (v \mapsto -v)$, and $e = 0$, i.e. $1_G = 0_V$.

(b) The integers $\mathbb{Z}$ equipped with the addition operation is a group, i.e. $(\mathbb{Z}, +)$.

(c) The integres modulo 2, $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, equipped with multiplication is a group, i.e. $(\mathbb{Z}/2\mathbb{Z}, \times)$.

(d) The real numbers equipped with $+$ is a group, i.e. $(\mathbb{R}, +)$.

(e) The real numbers without $\{0\}$ equipped with $\times$ is also a group, i.e. $(\mathbb{R} \setminus \{0\}, \times)$.

(f) $(\mathbb{R}_{>0}, \times)$ is a group.

(g) $(\mathbb{Q} \setminus \{0\}, \times)$ and $(\mathbb{C} \setminus \{0\}, \times)$ are also groups.

It happens that each of these five examples are such that their elements commute under their respective operations. It is not required for elements to commute for them to form a group; when they do, we call them Abelian.

(h) The set of all invertible $n \times n$ matrices with entries in $\mathbb{C}$, equipped with multiplication, i.e. $\mathrm{GL}_n(\mathbb{C})$, is not commutative. Specifically, we have $m : (A, B) \mapsto AB$ (matrix multiplication), $e = I_n$, and $\iota$ being the matrix inversion operation.

**Claim 18.2.** In any group, $(ab)^{-1} = b^{-1}a^{-1}$.

We simply have $(ab)(b^{-1}a^{-1}) = aea^{-1} = aa^{-1} = e$. $\square$

(i) Let $[n] = \{1, \ldots, n\}$. $S_n = \{\sigma : [n] \to [n] \mid \sigma \text{ is invertible } \iff \sigma \text{ inj. and cont.}\}$. Then the set of permutations forms a a group under function composition, i.e. $\mathfrak{S}_n = (S_n, \circ)$.

(j) Given any set $X$,

$$S(X) = \{\sigma : X \to X \mid \sigma \text{ is invertible } \}$$

is a group under function composition, $(S(X), \circ)$.

(k) *Relations of a cube:* There are 24 different rotations of a cube; let these moves be given by $S_C$. We note that these moves are non-commutative under composition (i.e., swapping the order of moves).

(l) Let $G$ be the set of possible combinations we can get on a Rubik's cube with legal moves. Then we have

$$|G| = \frac{8! \cdot 3^8 \cdot 12! \cdot 2^12}{12} = 43,252,003,274,489,856,000.$$

**Definition 18.3.** If $G_1, G_2$ are groups, then a homomorphism $\varphi : G_1 \to G_2$ is a function such that $\varphi(ab) = \varphi(a)\varphi(b)$, $\varphi(a^{-1} = \varphi(a)^{-1}$, and $\varphi(e_{G_1}) = e_{G_2}$.

We now give some examples of group homomorphisms.

(a) The inclusion map yields the following embeddings,

$$(\mathbb{Z}, +) \hookrightarrow (\mathbb{Q}, +) \hookrightarrow (\mathbb{R}, +) \hookrightarrow (\mathbb{C}, +).$$

(b) $\pi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ is another example of a morphism, where $\pi(k) = 0$ if $k$ is even, and 1 if $k$ is odd.

(c) $\det : \mathrm{GL}_n(\mathbb{C}) \to (\mathbb{C} \setminus \{0\}, \times)$, where $\det(AB) = \det(A)\det B)$.

(d) $\alpha : \mathfrak{S}_n \to \mathfrak{S}_{n+1}$, where $\alpha$ is sort of an "inclusion" map in the sense that we permute the first $n$ elements as $\sigma \in S_n$ does, and we fix the $n+1$th element to yield a permutation in $S_{n+1}$.

(e) $\beta : \mathfrak{S}_4 \to \mathfrak{S}_3$. $S_4$ is the set of rotations of a tetrahedron. Permutations $\sigma \in S_4$ permute the positions of the vertices; then we may partition the edges of the tetrahedron (of which there are 6) into 3 pairs of non-adjacent sides; since there are only 3 possible positions for the pairs of edges to go to, we see $\sigma \in S_4$ induces a permutation of pairs of edges, i.e. on $S_3$ (reference: here). We can check that this is a homomorphism because it preserves multiplicative structure.

We now move back to fundamental groups. There was some discussion in class on why the 1-torus and 2-torus are not homeomorphic by the fundamental group, but I don't really understand it.

**Definition 18.4.** A path in a topological space $X$ is a continuous function $\gamma : [0, 1] \to X$.

**Definition 18.5.** Paths $\gamma_0, \gamma_1 \in X$ are called *path-homotopic* if $\gamma_0 \sim_p \gamma_1$, i.e. there exists a continuous function $H = H(s, t) : [0, 1]_s \times [0, 1]_t \to X$ such that for all $s, t \in [0, 1]$, $H(s, 0) = \gamma_0(s)$, $H(s, 1) = \gamma_1(s)$, $H(0, t) = \gamma_0(0) = \gamma_1(0)$, and $H(1, t) = \gamma_0(1) = \gamma_1(1)$.

Intuitively, the idea is that we can deform $\gamma_0$ to $\gamma_1$ while fixing the endpoints. Think of $s$ as the "where you are on the curve" parameter and $t$ as the time parameter.

**Claim 18.6.** $\sim_p$ is an equivalence relation.

We check $\sim_p$ satisfies reflexivity, symmetry, and transitivity below respectively.

(a) By the homotopy, we have $\gamma \sim_p \gamma$, i.e. $H(s, t) = \gamma(s)$.

(b) Suppose $\gamma_0$ is path-homotopic to $\gamma_1$. This means there exists $H$ s.t. $H(s,0) = \gamma_0(s)$ and $H(s,1) = \gamma_1(s)$. Define $H'(s,t) = H(s,1-t)$; then

$$H'(s,0) = H(s,t) = \gamma_1(s),$$
$$H'(s,1) = H(s,0) = \gamma_0(s).$$

So $\gamma_1 \sim_p \gamma_0$.

(c) Let $\gamma_0 \sim_p \gamma_1$ by $H'$, and $\gamma_1 \sim_p \gamma_2$ by $H''$. Define

$$H(s,t) = \begin{cases} H'(s,2t) & t \leq \frac{1}{2}, \\ H''(s,2t-1) & t \geq \frac{1}{2}. \end{cases}$$

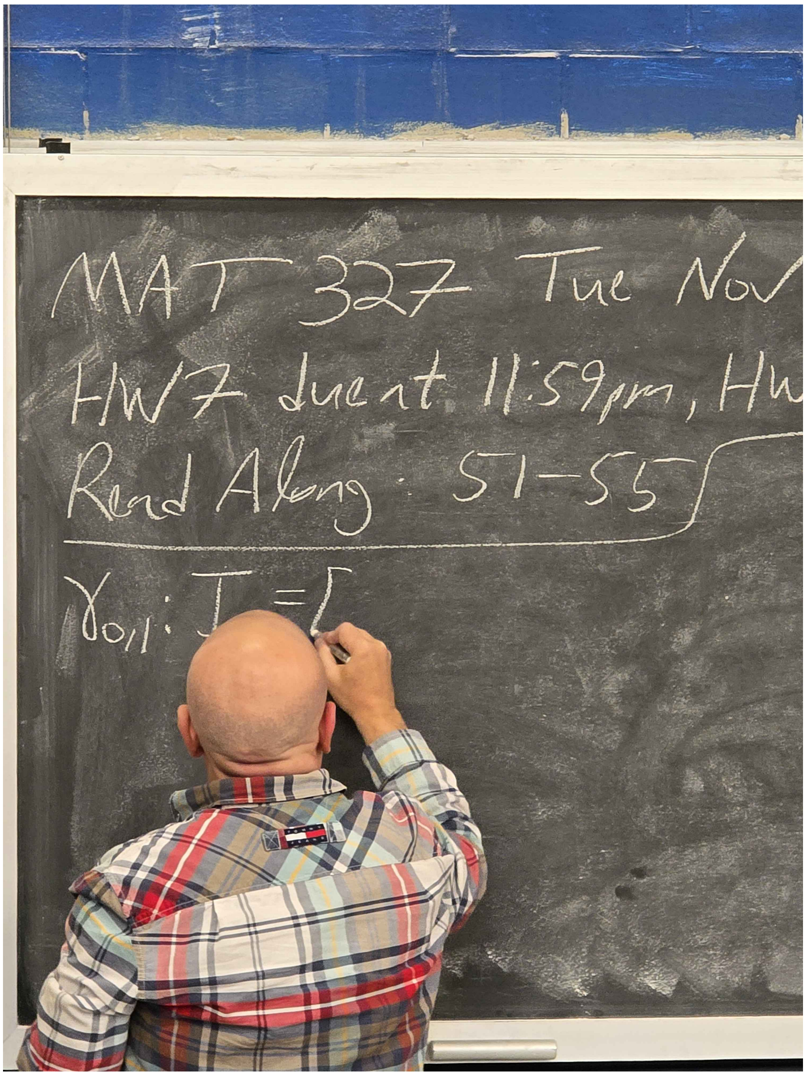
This gives $\gamma_0 \sim_p \gamma_2$ by $H$.

We give some examples. In $\mathbb{R}^n$, if $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$, then $\gamma_0 \sim_p \gamma_1$, since we may construct

$$H(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s).$$

# §19 Day 19: Homotopies (Nov. 12, 2024)

Outfit of the day: scary uwu




Recall from last lecture that $\gamma_{0,1} : I_s \to [0,1]_s \to X$ and $\gamma_0 \sim_p \gamma_1$ means that path homotopy is an equivalence relation; i.e., there exists $H : I^2_{s,t} \to X$ such that

$$\begin{aligned}
H(0,t) &= \gamma_0(0) = \gamma_1(0), \\
H(1,t) &= \gamma_0(1) = \gamma_1(1), \\
H(s,0) &= \gamma_0(s), \\
H(s,1) &= \gamma_1(s).
\end{aligned}$$

Let $\gamma : I \to X$ be a path $[\gamma] = \{\gamma' \mid \gamma' \sim \gamma\}$. $[\gamma]$ is called a homotopy class of paths.

**Definition 19.1** (Path Composition / Product)**.** If $\gamma_1, \gamma_2$ are paths and $\gamma_1(1) = \gamma_2(0)$, then $(\gamma_1 * \gamma_2)(s) = \gamma_1(2s)$ when $s \leq \frac{1}{2}$, and $\gamma_2(2s-1)$ when $s \geq \frac{1}{2}$.

**Claim 19.2.** Path composition descends to homotopy classes of paths; i.e.,

$$\begin{tikzcd}
\{\text{paths}\} \arrow[d] & \times & \{\text{paths}\} \arrow[r, "*"] \arrow[d] & \{\text{paths}\} \arrow[d] \\
\{\text{h.c.o.p.}\} & \times & \{\text{h.c.o.p.}\} \arrow[r, "*"] & \{\text{h.c.o.p.}\}
\end{tikzcd}$$

where we note that "h.c.o.p." is shorthand for homotopy class of paths. Namely, $[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2]$ is well-defined. If $\gamma_1 \sim_p \gamma_1'$ and $\gamma_2 \sim_p \gamma_2'$, then we claim that

$$\gamma_1 * \gamma_2 \sim_p \gamma_1' * \gamma_2'.$$

We now prove the claim. Assume $\gamma_1 \sim_p^{H_1} \gamma_1'$ and $\gamma_2 \sim_p^{H_1} \gamma_2'$ (read: they are equivalent by the functions $H_1, H_2$ respectively). Then $\gamma_1 * \gamma_2 \sim_p^{H} \gamma_1' * \gamma_2'$. If $H_1, H_2$ are the path homotopies of $[\gamma_1], [\gamma_2]$, we may define

$$H(s,t) = \begin{cases} H_1(2s,t) & s \leq \frac{1}{2}, \\ H_2(2s-1,t) & s \geq \frac{1}{2}, \end{cases}$$

since $H_1(1,t) = x_1 = H_2(0,t)$ for all $t$. This means $H$ is well-defined, and is continuous by the pasting lemma (Sec. 51, Page 326 in Munkres).

**Theorem 19.3** (Theorem 51.2 in Munkres; Groupoid Structure)**.** The $*$ operation forms a groupoid on the homotopy class of paths, i.e.

(a) $[\gamma_1] * ([\gamma_2] * [\gamma_3]) = ([\gamma_1] * [\gamma_2]) * [\gamma_3]$ (associativity),

(b) $[\gamma] * [e] = [\gamma] = [e] * [\gamma]$ (identity),

(c) there exists $\overline{\gamma}$ for all $\gamma$ such that $[\gamma] * [\overline{\gamma}] = [e]$.

We check that all the properties hold.

(a) For associativity, we have

$$[\gamma_1] * ([\gamma_2] * [\gamma_3]) = [\gamma_1 * (\gamma_2 * \gamma_3)]; \qquad ([\gamma_1] * [\gamma_2]) * [\gamma_3] = [(\gamma_1 * \gamma_2) * \gamma_3].$$

Note that $[\gamma_1 * (\gamma_2 * \gamma_3) \neq (\gamma_1 * \gamma_2) * \gamma_3$ in general, though, since $*$ divides the time parameter in half for each of its inputs, i.e. on the left hand side, we are "running on" $\gamma_1$ for half the time and $\gamma_2, \gamma_3$ for a quarter of the time, versus $\gamma_1, \gamma_2$ for a quarter and $\gamma_3$ for half on the RHS.

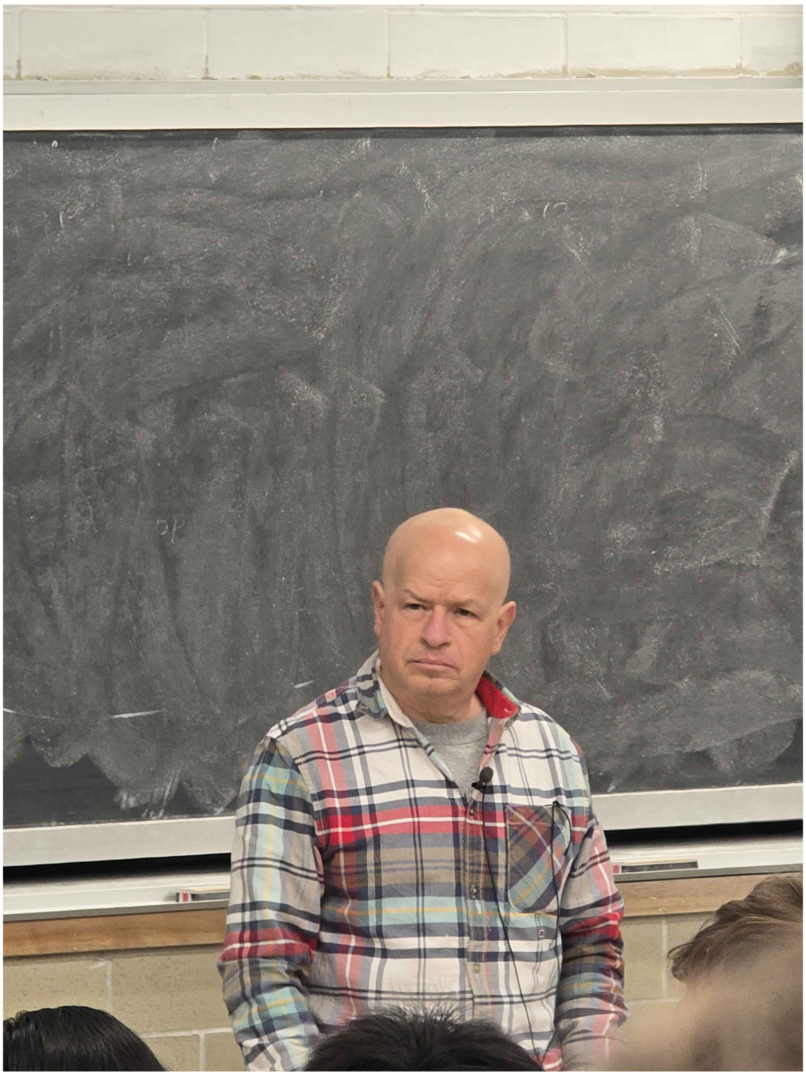
There was some diagram that was drawn that concluded the proof. Weird... i'll check textbook later.

(b) Given $x \in X$, let us have $e_x(s) = x$; if $\gamma(0) = x$ and $\gamma(1) = y$, then $[e_x * \gamma] = [\gamma * e_y] = [\gamma]$.

(c) We now check for inverses. Let $\overline{\gamma}(s) := \gamma(1-s)$. We claim that $[\gamma * \overline{\gamma}] = e_x$. More diagram proof. Check Munkres i guess.

# §20 Day 20: Homotopy Constructions (Nov. 14, 2024)

Outfit of the day: more plaid!!



Recall that we have $[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2]$ assuming $\gamma_1(1) = \gamma_2(0)$; the operation $*$ is a composition of paths, i.e. concatenation (read: product of paths in Munkres). As shown last time, $*$ is associative, has identities $[e_x]$, and has inverses $[\gamma] \mapsto [\overline{\gamma}]$.

Let $X$ be a space, and let $x_0$ be a point in $X$. A path in $X$ that begins and ends at $x_0$ is called a loop based at $x_0$, and the set of path homotopy classes of loops based at $x_0$, equipped with $*$, is called the fundamental group of $X$ relative to base point $x_0$, denoted by $\pi_1(X, x_0)$.

A base space is defined as $(X, x_0)$, where $x_0 \in X$ is called the base point as per above. The convention is that we write $f : (X, x_0) \to (Y, y_0)$ meaning $f : X \to Y$ and $f(x_0) = y_0$, where $f$ is continuous.

**Definition 20.1.** Given $(X, x_0)$, $\pi_1(X, x_0) = \{[\gamma] \mid \gamma : [0,1] \to X; \gamma(0) = \gamma(1) = x_0\}$. We see that $\pi_1(X, x_0)$ indeed satisfy the axioms for a group; we write $e = e_{x_0}$, $e_{x_0}(s) = x_0$ for identity.

We now give a few examples.

(a) $\pi_1(\mathbb{R}^n, 0) = \{[e_0]\} = \{e\}$ is the trivial group, since if $f$ is a loop in $\mathbb{R}^n$ based at 0, the straight-line homotopy is a path homotopy that sends $f$ to the constant path at 0. Notice that this argument works to show that $\pi_1(X, x_0)$ where $X$ is a convex subset of $\mathbb{R}^n$ and $x_0 \in X$ is also a trivial group, as these all consist of nothing but $\{[x_0]\}$.

**Theorem 20.2.** If $X$ is path connected and $x_{0,1} \in X$ then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$, meaning there exists $\varphi : \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $\psi : \pi_1(X, x_1) \to \pi_1(X, x_0)$, where $\varphi, \psi$ are group homomorphisms and inverses of each other. In particular, we say that these are *isomorphisms.*

Pick a path $\lambda$ such that $\lambda(0) = x_0$, $\lambda(1) = x_1$, and $\lambda(i) = x_i$. Define $\psi : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $[\gamma] \mapsto [\lambda] * [\gamma] * [\overline{\lambda}] = [\lambda * \gamma * \overline{\lambda}]$, where $[\gamma]$ is an element of $\pi_1(X, x_1)$, where we recall that it is the equivalence class of loops $\gamma(0) = \gamma(1) = x_1$. Indeed, this is a group homomorphism. (note that for ease of reading, we omit $*$ and just concatenate per convention) We do this by checking that $\psi([\gamma_1][\gamma_2]) = [\lambda\gamma_1\gamma_2\overline{\lambda}]$ is equivalent to $\psi([\gamma_1])\psi([\gamma_2])$, which we may see from the following,

$$\begin{aligned}
\psi([\gamma_1])\psi([\gamma_2]) &= [\lambda\gamma_1\overline{\lambda}][\lambda\gamma_2\overline{\lambda}] \\
&= [\lambda\gamma_1\overline{\lambda}\lambda\gamma_2\overline{\lambda}] \\
&= [\lambda\gamma_1][\overline{\lambda}\lambda][\gamma_2\overline{\lambda}] \\
&= [\lambda\gamma_1][e_{x_1}][\gamma_2\overline{\lambda}] \\
&= [\lambda\gamma_1][\gamma_2\overline{\lambda}] \\
&= [\lambda\gamma_1\gamma_2\overline{\lambda}],
\end{aligned}$$

which is the left hand side as desired. We may also check that $\varphi : [\gamma] \mapsto [\overline{\lambda}\gamma\lambda]$ is also a group homomorphism.

**Claim 20.3.** $(\psi \circ \varphi)([\gamma]) = \psi(\overline{\lambda}\gamma\lambda) = [\lambda\overline{\lambda}\gamma\lambda\overline{\lambda}] = [\gamma]$. Thus, $\psi \circ \varphi = \mathrm{id}_{\pi_1(X, x_0)}$.

Obviously, this holds in the other direction as well. □

**Definition 20.4.** We say that $X$ is *simply connected* if it is path connected and $\pi_1(X, x_0) = \{e\}$ for some $x_0$, i.e. $\pi_1(X, x_0)$ is the trivial (one element) group.

Notice that the definition may be modified to say that *all* $\pi_1(X, x_0)$ for all $x_0 \in X$ are trivial, by our previous isomorphism construction. In particular, $\mathbb{R}^n$ is a simply connected space.

**Theorem 20.5.** Recall that $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\} = \{z \in \mathbb{C}, |z| = 1\}$. We have that $\pi_1(S^1, 1) \cong \mathbb{Z}$.

The intuition is that $\pi_1(S^1, 1)$ consists of equivalence classes of paths that stay at 1, go around the circle once, go around the circle twice, etc... we may then map the number of times that they go around the circle to the integers as desired, with the sign being related to whether the path is clockwise or counterclockwise. However, this is not a proper proof.

Quick digression; in the following diagram,

$$\begin{tikzcd}
 & X \arrow[d, "p"] \\
Z \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

$\tilde{f}$ is called a *lift* of $f$. Now, let us consider the following lift of $\gamma$,

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & S^1 \end{tikzcd}$$

where $I$ is an interval on $\mathbb{R}$, $p : t \mapsto e^{2\pi it}$, $\gamma$ is a path where $\gamma(0) = \gamma(1) = 1$, and $\tilde{\gamma}$ is a path to a "slinky". To rigorously prove ;

(a) $p : E \to B$, where $E$ is a covering and $B$ is a base space,

(b) Path lifting for $p : E \to B$,

(c) Homotopy lifting for $p : E \to B$.

To start, we say that $p : E \to B$ is a covering map; informally speaking, locally, $E$ is the product of $B$ with a discrete space. Formally defined, if every $x \in B$ has a neighborhood $U$, and a set $D$ with homeomorphism $\varphi : U \times D \to p^{-1}(U)$, where

$$\begin{tikzcd} U \times D \arrow[r, "\varphi"'] \arrow[d, "\pi_1"] & p^{-1}(U) \arrow[d, "p"'] \\ U \arrow[r, "\mathrm{id}"] & U \end{tikzcd}$$

is a commutative diagram illustrating the relation. We give a few eaxmples.

(a) Let $X$ be any space; then $\mathrm{id} : X \to X$ is the identity map; it is also a trivial covering map. If we let $E = X \times D$, then $p : E \to X$ with $p(x, i) = x$ for all $i \in D$ is also a covering map.

(b) $p : R \to S^1$ by $t \mapsto e^{2\pi it}$ as per earlier,

(c) Let $E = S^1$, and $p : S^1 \to S^1$, given by $z \mapsto z^3$. This is also a covering map.

(d) Those parking garages that when you loop around you go up or down one floor; this is also a covering map onto the footprint of the building.

(e) Mexican cross onto the chinese lucky space (figure eight) is a covering map.[24]

(f) Long quipu to abaab...????????

(g) Olympic rings to the figure eight.

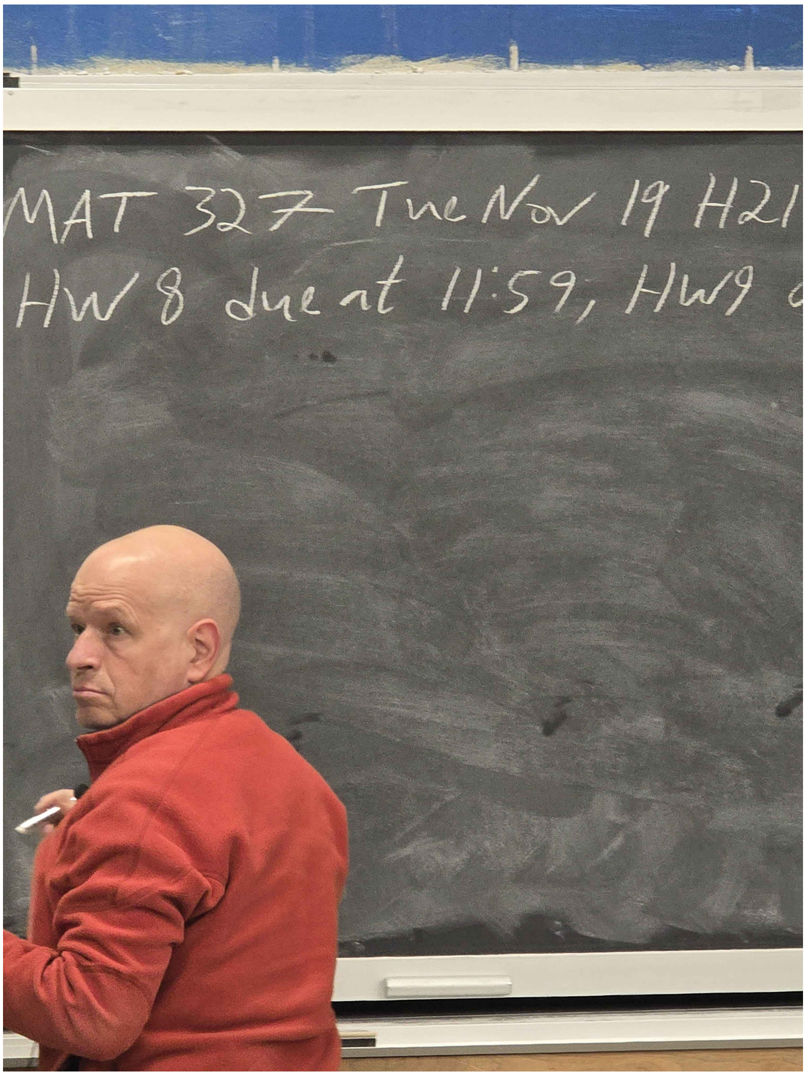
**Claim 20.6.** If $B$ is connected and $P : E \to B$ is a covering that is finite-to-one, then $f(x) = \left|p^{-1}(\{x\})\right|$ is consatnt.

This is true because $f(x)$ is locally constant.

[24] what the hell?

# §21 Day 21: Coverings (Nov. 19, 2024)

Outfit of the day: the look speaks for itself




Recall that the fundamental group, where $(X, x_0)$ is a based space, is given by

$$\pi_1(X, x_0) = \{[\gamma] \mid \gamma : [0,1] \to X; \gamma(0) = \gamma(1) = x_0\}.$$

In particular, $\pi_1(X, x_0)$, as the name suggests is a group; for example, $\pi_1(S^1, 1) \cong \mathbb{Z}$. The notation $f : (X, x_0) \to (Y, y_0)$ represents a function $f : X \to Y$ with $f(x_0) = y_0$. If $p : E \to B$ is a covering, it means by $B$ can be covered with open $U_\alpha$ such that, for all $\alpha$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets such that $p$ is a homeomorphism of each of them with $U_\alpha$.

An example of a covering space is the map $\mathbb{R} \to S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, where $x \mapsto e^{ix}$.

**Lemma 21.1.** If $(X, x_0)$ is a connected based space, $p : (E, e_0) \to (B, b_0)$ is a covering, and $\psi : (X, x_0) \to (B, b_0)$ is "small", i.e. $\mathrm{Im}(\psi) \subset U_\alpha$, where $U_\alpha$ is as in the definition of a covering, then it has a unique lift $\tilde{\psi} : (X, x_0) \to (E, e_0)$ such that $p \circ \tilde{\psi} = \psi$.

$p^{-1}(U_\alpha) = \bigcup_\beta U_{\alpha_\beta}$, where $U_{\alpha_\beta}$ are open in $E$, and $p : U_{\alpha_\beta} \to U_\alpha$ is a homeomorphism for each $\beta$. Only one of $U_{\alpha_\beta}$, say, $U_{\alpha_{\beta_0}}$, contains $e_0$, and we lift $\psi$ to $\tilde{\psi} : (X, x_0) \to (U_{\alpha_{\beta_0}}, e_0)$, using the fact that $U_{\alpha_{\beta_0}}$ is homeomorphic to $U_\alpha$. □

**Theorem 21.2** (The Path Lifting Property / Lemma 54.1; Munkres)**.** Let $p : (E, e_0) \to (B, b_0)$ a covering, $\gamma : (I, 0) \to (B, b_0)$ a path be given; then there exists a unique lift $\tilde{\gamma} : (I, 0) \to (E, e_0)$ such that $p \circ \tilde{\gamma} = \gamma$.

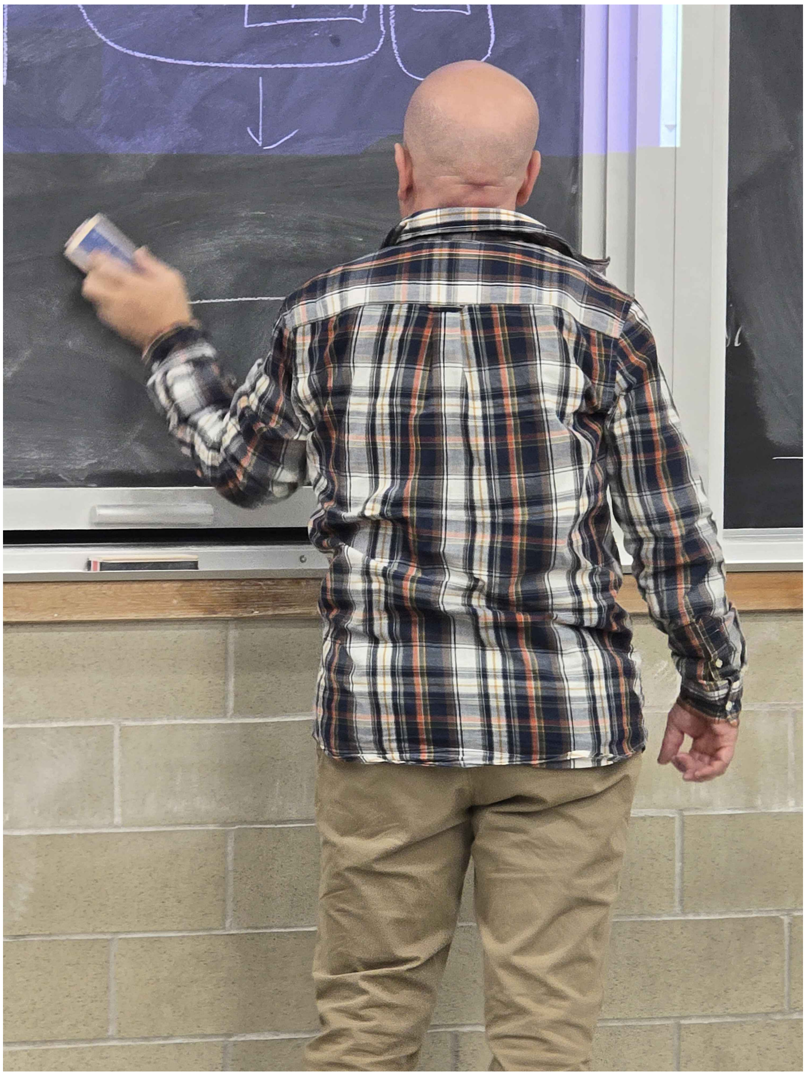
Let $\{U_\alpha\}$ be an open cover of $B$ such that for all $\alpha$, $p^{-1}(U_\alpha)$ is etc etc etc. then $\{\gamma^{-1}(U_\alpha)\}$ is an open cover of $I$. By the lebesgue covering lemma (since $I$ is compact), there exists $\delta$ such that every subinterval of $I$ of length less than $\delta$ is contained in one of the $\gamma^{-1}(U_\alpha)$'s. Pick $n > \frac{1}{\delta}$ and consider $I = \bigcup I_k$ where $I_k = [\frac{k-1}{n}, \frac{k}{n}]$ for $k = 1, \dots, n$. $\gamma|_{I_k}$ is small in the sense of the previous lemma, and so we can lift $\gamma$ uniquely to $\bigcup_{k=1}^{m} I_k$ by induction on $m$. $\square$

**Theorem 21.3** (The Homotopy Lifting Property / Lemma 54.2; Munkres)**.** Let $p : (E, e_0) \to (B, b_0)$ be a covering, $H : (I^2, 0) \to (B, b_0)$ be given, then there exists a unique homotopy $\tilde{H} : (I^2, 0) \to (E, e_0)$ such that $p \circ \tilde{H} = H$.

The proof is the exact same; except in the end, instead of constructing intervals, we have a square instead. Check Munkres p.344 for the diagram. $\square$

# §22 Day 22: Lifting, Fundamental Group of Circle; Categories (Nov. 21, 2024)

Outfit of the day: homeland defender sniper monkey palette



Recall the path/homotopy lifting property: Given a covering $p : (E, e_0) \to (B, b_0)$ and $\gamma : (I, 0) \to B, b_0)$ or $H : (I^2, 0) \to (b, b_0)$, there exists a unique lift $\tilde{\gamma} : (I, 0) \to (E, e_0)$ or $\tilde{H} : (I^2, 0) \to (E, e_0)$ such that $p \circ \tilde{\gamma} = \gamma$ and $p \circ \tilde{H} = H$.

**Theorem 22.1.** $\pi_1(s^1, 1) \cong \mathbb{Z}$.

For the purpose of the proof, fix $(E, e_0) = (\mathbb{R}, 0)$, and $(B, b_0) = (S^1, 1)$, and $p : (E, e_0) \to (B, b_0)$ is just the complex exponential map, $p(x) = e^{2\pi ix}$.

For $n \in \mathbb{Z}$, $s \in I$, let $\varphi(n)(s)$ be a homotopy class of a path, given by

$$\varphi(n)(s) = p(ns) = e^{2\pi ins}.$$

Here, $\gamma : [0, 1] \to S^1$, $\gamma(0) = \gamma(1) = 1$, and we have that $\psi([\gamma]) = \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the lift of $\gamma$ as in the proof above. We also have that $\tilde{\gamma}(0) = 0$. Note that $p(\tilde{\gamma}(1)) = \gamma(1) = 1$, since $[\gamma] \in \pi_1(S^1, 1)$. Thus, $\tilde{\gamma}(1) \in p^{-1}(1) = \mathbb{Z}$. We need to show:

(a) $\psi$ is well-defined.

(b) $\psi \circ \varphi = \mathrm{id}_{\mathbb{Z}}$.

(c) $\varphi \circ \psi : \mathrm{id}_{\pi_1}$.

(d) $\varphi$, $\psi$ are group homomorphisms if and only if $[\varphi(n+m)] = [\varphi(n)] * [\varphi(m)]$.

We prove the claims now.

(a) If $\gamma_1 \sim_p \gamma_2$ and $\gamma_1(0) = \gamma_2(0) = \gamma_1(1) = \gamma_2(1) = 1$, then we need to show that $\tilde{\gamma_1}(1) = \psi([\gamma_1]) = \psi([\gamma_2]) = \tilde{\gamma_2}(1)$. We know that there exists $H : I^2 \to (B, b_0)$ such that $H(s,0) = \gamma_1(s), H(s,1) = \gamma_2(s)$, and $H(0,t) = H(1,t) = 1$. Let $\tilde{H}$ be the lift of $H$.

We know $\tilde{H}(0,0) = 0$, and $p(\tilde{H}(s,0)) = H(s,0) = \gamma_1(s)$, and so $\tilde{H}(s,0)$ is a lift of $\gamma_1$. In particular, $\tilde{H}(s,0) = \tilde{\gamma_1}(s)$ by uniqueness in the theorem above. Similarly, $\tilde{H}(0,t) = 0$. In particular, $\tilde{H}(0,1) = 0$; this means we have that $\tilde{H}(s,1) = \tilde{\gamma_2}(s)$. Now, use the path lifting theorem but with $p : (\mathbb{R}, \tilde{\gamma}(1)) \to (S^1, 1)$. By uniqueness of path lifting, $\tilde{H}(1,t)$ is the constant $\tilde{\gamma_1}(1)$, and so $\tilde{\gamma_2}(t) = \tilde{H}(1,1) = \tilde{\gamma_1}(1)$. □

(b) Given $n$, $\varphi(n) = [\gamma_n]$, where $\gamma_n(s) = p(ns)$. Let $\tilde{\gamma_n}$ be the lift of $\gamma_n$; then $\tilde{\gamma_n}(s) = ns$. So $\psi(\varphi(n)) = \tilde{\gamma_n}(1) = n \cdot 1 = n$. □

(c) Let $[\gamma] \in \pi_1(S^1, 1)$. Let $\tilde{\gamma}$ be a lift of $\gamma$, $\tilde{\gamma} : [0,1] \to \mathbb{R}$, such that $p(\tilde{\gamma}(s)) = \gamma(s)$. Let $n = \tilde{\gamma}(1)$, and then $\varphi(n) = \gamma_n$. We need to show that $\gamma \sim_p \gamma_n$. Note that $\tilde{\gamma}(0) = 0$, $\tilde{(\gamma_n)}(0) = 0$; $\tilde{\gamma}(1) = n$, $\tilde{\gamma_n}(1) = n$, $\tilde{\gamma_n}(s) = ns$. So there exists a homotopy $\overline{H}$ in $\mathbb{R}$ from $\tilde{\gamma}$ to $\tilde{\gamma_n}$ with endpoints 0 to $n$. Let $H = p \circ \overline{H}$; then we have a homotopy $\gamma \sim_p \gamma_n$.

(d) We only sketch the proof for this one. $\varphi(n+m) = p(s \mapsto (n+m)s)$. Think of the expression as projecting the straight line given by $s \mapsto (n+m)s$. Then $\phi(n) * \phi(m)$ is intuitively a concatenation of $p$ on $s \mapsto ns$ and $s \mapsto ms$.

We now introduce categories. We start by giving a few examples for intuition. Colloquially, categories consist of objects, morphisms, and morphism composition.

(a) Maps between vector spaces as objects, i.e. the morphisms are linear transformations and compositions, are a category. Topological spaces equipped with continuous functions and composition is also a category. Groups, homomorphisms, and compositions is also a category. Sets, functions, and compositions are also a category. Respectively, these are from MAT240, 327, 347, and 407.

**Definition 22.2.** A *category* $\mathcal{C}$ is a quadruple $(\mathrm{obj}_{\mathcal{C}}, \mathrm{mor}_{\mathcal{C}}, \circ, \mathrm{id}_{\mathcal{C}})$, where:

- objects are a collection, i.e. "possibly a huge set",
- for any two objects $A, B \in \mathrm{obj}_{\mathcal{C}}$, we have a set $\mathrm{mor}_{\mathcal{C}}(A, B)$.
- for any three objects $A, B, C \in \mathrm{obj}_{\mathcal{C}}$, we have a function $\circ_{\mathcal{C},A,B,C} = \circ : \mathrm{mor}(A,B) \times \mathrm{mor}(B,C) \to \mathrm{mor}(A,C)$, where $(f,g) \mapsto g \circ f = \circ_{\mathcal{C},A,B,C}(f,g)$, for all $f \in \mathrm{mor}(A,B)$ and $g \in \mathrm{mor}(B,C)$.
- For each object, $A \in \mathrm{obj}$, a morphism $\mathrm{id}_A \in \mathrm{mor}(A,A)$, such that
  (i) Associativity; for all $A, B, C, D \in \mathrm{obj}$ and $f \in \mathrm{mor}(A,B)$, $g \in \mathrm{mor}(B,C)$, $h \in \mathrm{mor}(C,D)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) Given $A, B \in \text{obj}$, $f \in \text{mor}(A, B)$, we have $f \circ \text{id} = \text{id} \circ f = f$.

We now continue our examples.

(b) Based topological spaces, where the objects are given by $\{(X, x_0)\}$, morphisms given by functions $(X, x_0)$ to $(Y, y_0)$, and path composition $*$.

(c) Let $\text{obj} = X$, $\text{mor}(x, y)$ be the set of equivalence classes of paths starting at $x$ and ending at $y$ where $x, y \in \text{obj} = X$, composition given by $*$, and id given by the constant path $e_x$.

A category in which all morphisms are invertible, i.e.

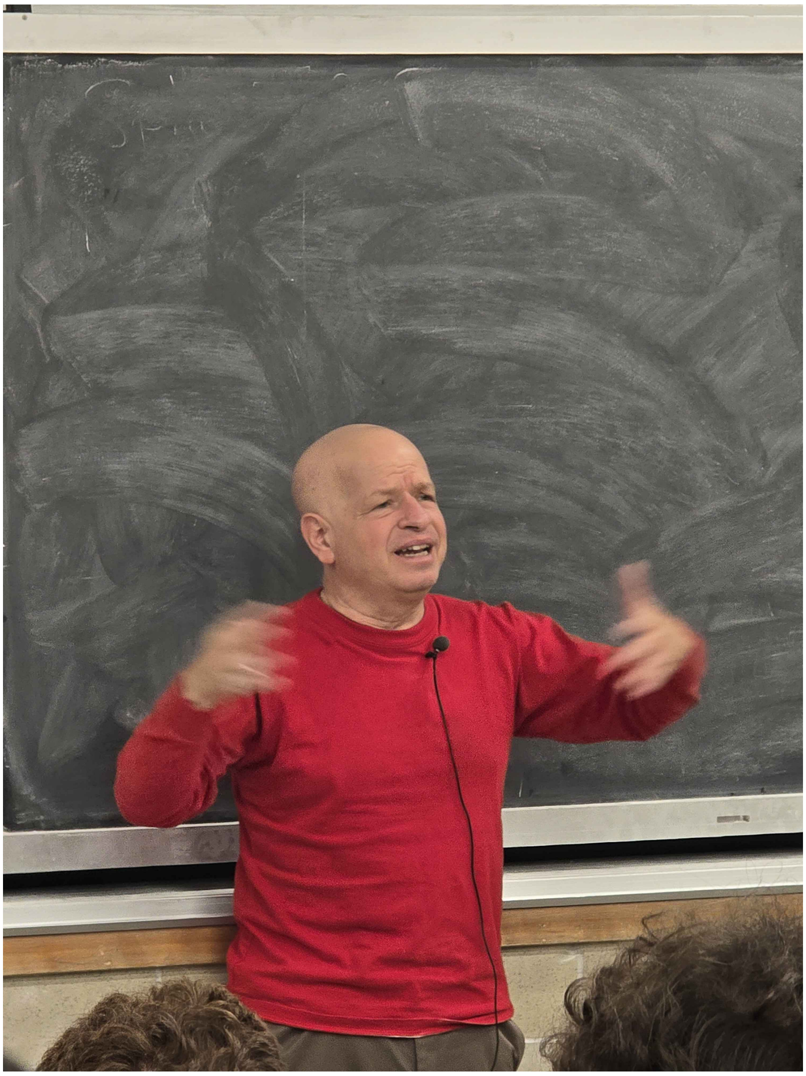
$$f \in \text{mor}(A, B) \implies \exists f^{-1} \in \text{mor}(B, A)$$

is called a groupoid. In particular, (c) is an example of a groupoid.

(d) The fifteen puzzle! The objects are given by the set of sixteen elements, of which the elements are where the empty square is placed. The morphisms between two fifteen puzzle positions are given by the ways to shuffle the tiles around; in particular, there are 15! shuffles (? proof?).

# §23 Day 23: Functors; Retracts, Brouwer Fixed Point (Nov. 26, 2024)

Outfit of the day: monocolor sweater. that's a first



We start by defining functors. A functor $F : \mathcal{C} \to \mathcal{D}$ is a mapping between categories $\mathcal{C} \to \mathcal{D}$; it associates each $A \in \mathrm{obj}_{\mathcal{C}}$ with an object $F(A) \in \mathrm{obj}_{\mathcal{D}}$; in particular, we may draw the following diagram,

$$\begin{tikzcd}
A \arrow[r, rightsquigarrow] \arrow[d, "f"] \arrow[dd, bend right, "g \circ f"'] & FA \arrow[d, "Ff"] \arrow[dd, bend left, "Fg \circ Ff"] \\
B \arrow[r, rightsquigarrow] \arrow[d, "g"] & FB \arrow[d, "Fg"] \\
C \arrow[r, rightsquigarrow] & FC
\end{tikzcd}$$

where $F : \mathrm{mor}_{\mathcal{C}}(A, B) \to \mathrm{mor}_{\mathcal{D}}(FA, FB)$ for any $A, B \in \mathrm{obj}_{\mathcal{C}}$ such that whenever $f \in \mathrm{mor}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{mor}_{\mathcal{C}}(B, C)$, we have that $F(g \circ f) = (Fg) \circ (Ff)$, and for any $A \in \mathrm{obj}_{\mathcal{C}}$, $F(\mathrm{id}_A) = \mathrm{id}_{FA}$. We start with some lousy examples.

(a) The forgetful functor, Forget : Top $\to$ Set, where Top is the category of topological spaces, and Set is the category of sets, is a functor that "forgets" the structures it is applied on. i.e., we have $(X, \mathcal{T}) \rightsquigarrow X$, a continuous function $f : X \to Y$ is mapped to a function $f : X \to Y$ (continuity not needed). Some other examples include

$$\begin{aligned} F_2 &: \text{Grp} \to \text{Set}, \\ F_3 &: \text{Vect} \to \text{Set}, \\ F_4 &: \text{Top}_0 \to \text{Top}, \end{aligned}$$

where $\text{Top}_0$ is the category of base spaces, and $F_4$ forgets the base point.

(b) Let us consider a functor $\hat{\underline{3}} : \text{Set} \to \text{Set}$, $A \rightsquigarrow A \times \underline{3}$, where $\underline{3} = \{1, 2, 3\}$. This particular functor applied to functions gives

$$\hat{\underline{3}}(F) : \hat{\underline{A}} \to \hat{\underline{B}},$$

i.e. $A \times \underline{3} \to B \times \underline{3}$, where $(\hat{\underline{3}} f)(a, \nu) = (F(a), \nu)$.

(c) Sophisticated example! Let $* : \text{Vect} \to \text{Vect}$ be a functor, mapping objects as follows, $V \to *V_i := V^*$. Given a linear transformation $L : V \to W \rightsquigarrow *L := L^* : W^* \to V^*$. Consider the contravariant functor $\mathcal{C} \rightsquigarrow \mathcal{C}^{\text{op}}$, where $\text{obj}\mathcal{C}^{\text{op}} = \text{obj}\mathcal{C}$, and $\text{mor}_{\mathcal{C}^{\text{op}}}(A, B) = \text{mor}_{\mathcal{C}}(B, A)$; intuitively, this is just "reversing the arrows".

(d) Our main example: let $\pi_1 : \text{Top}_0 \to \text{Grp}$, i.e. $\pi_1$ is a functor from based topological spaces into groups, where $\pi_1$ assigns $(X, x_0)$ to $\pi_1(X, x_0)$. If we have a continuous map $f : (X, x_0) \to (Y, y_0)$, then there is a morphism $\pi_1 f = f_* : \pi_1(X, x) \to \pi_1(Y, y_0)$. In particular, $f_*[\gamma] := [f \circ \gamma]$, where $\gamma : I \to X$ and $f \circ \gamma : I \to Y$. The diagram is as follows;

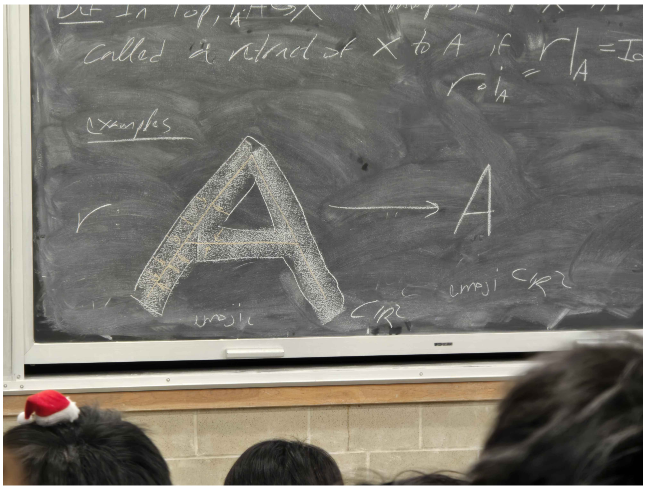
$$\begin{tikzcd} (X, x_0) \arrow[r, "\pi_1"] \arrow[d, "f"'] & \pi_1(X, x_0) \arrow[d, "\pi_1 f = f_*"] \\ (Y, y_0) \arrow[r, "\pi_1"] & \pi_1(Y, y_0) \end{tikzcd}$$

We now move onto retracts.

**Definition 23.1.** Let $A \subset X$; a morphism $r : X \to A$ is called a *retract* of $X$ to $A$ if $r|_A = \text{id}_A$. Specifically, $r \circ \iota_A = r|_A$.[25]

For example, a bigass fat $A$ mapping to a regular $A$ is an example of a retract.

[25] what




**Claim 23.2.** There does not exact a retract $r : D^1 \to S^0$.

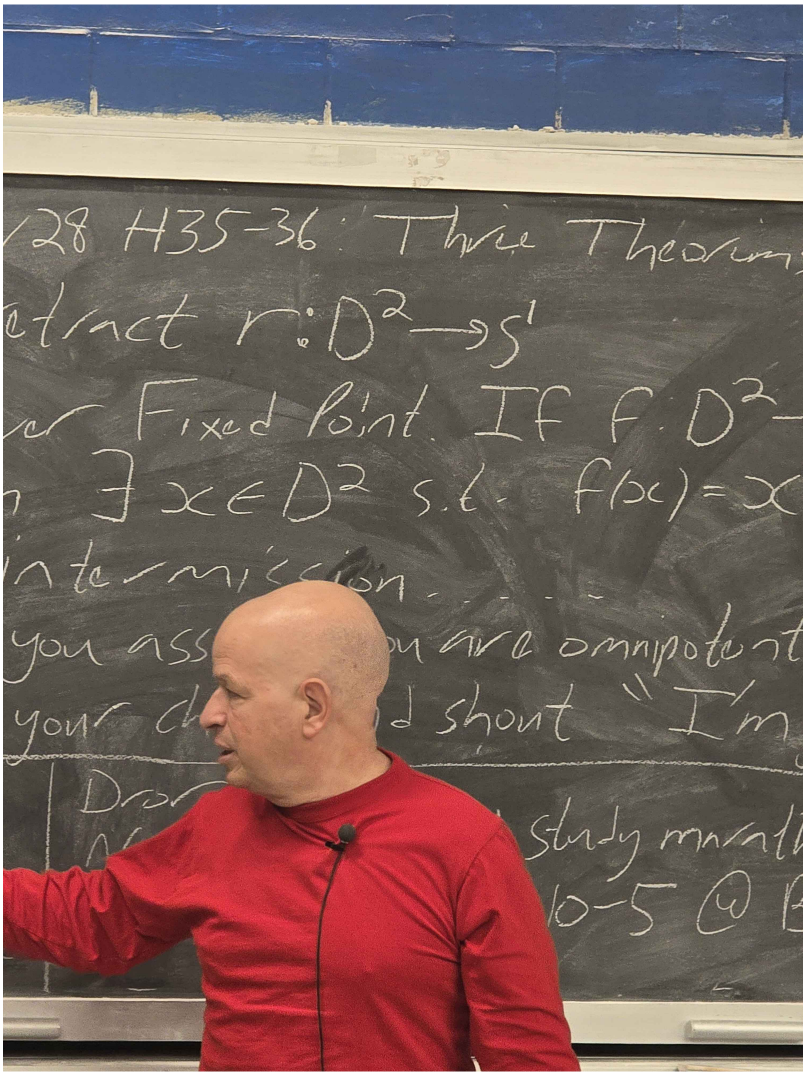
Recall that $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, and $\mathrm{d}S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$. Informally, there is no way to do this without tearing $S^0$; formally, it is because of connectivity.

For next lecture, we will prove the following claim:

**Claim 23.3.** There does not exist a retract $D^2 \to S^1$.

# §24 Day 23: No Retraction, Brouwer Fixed Point Theorem. Stone-Cech Compactification, Rigorous Definition of Limit (Nov. 26, 2024)

Outfit of the day: monocolor sweater. that's a second




**Office hours information!**

- Brinda will host office hours on Friday on Dec. 13, from 10am to 1pm in the grad lounge.
- Dror will not host office hours on Dec. 3, but will host them on Dec. 10, and will be available for a "study marathon" on Sunday, Dec. 15, from 10am to 5pm on Bahen's 6th Floor. He will roam around looking for topology students.

We will prove three major theorems today.

(i) There does not exist a retract $r : D^2 \to S^1$.

(ii) (Brouwer Fixed Point Theorem) If $f : D^2 \to D^2$ is continuous, then there exists $x \in D^2$ such that $f(x) = x$.

(iii) If you assume you are omnipotent, you get the pound on your chest and shout "I'm great!"

Recall that we have the following tools for today's lecture:

$$\pi_1(\mathbb{R}) = 0; \qquad \pi_1(S^1) = \mathbb{Z},$$

and that $\pi_1$ is a functor, i.e.

$$\begin{tikzcd} X \arrow[loop left, "\mathrm{id}_X"] \arrow[r, squiggly, "\pi_1"] \arrow[d, "F"'] & \pi_1(X) \arrow[d, "\pi_1 f = f_*"] \\ Y \arrow[loop left, "\mathrm{id}_Y"] \arrow[r, squiggly, "\pi_1"'] & \pi_2(Y) \end{tikzcd}$$

We now prove the aforementioned theorems.

**Theorem 24.1** (No Retraction)**.** There does not exist a retract $r : D^2 \to S^1$.

Suppose $r$ exists. Then we have the following diagram,

$$\begin{tikzcd} S^1 \arrow[r, hook, "\iota"] \arrow[dr, "I"'] & D^2 \arrow[d, "r"] \arrow[r, phantom, "\overset{\pi_1}{\rightsquigarrow}"] & \mathbb{Z} \arrow[r, "\pi_1 \iota"] \arrow[dr, "I"'] & \{e\} \arrow[d, "\pi_1 r"] \\ & S^1 & & \mathbb{Z} \end{tikzcd}$$

More generally, there does not exist a retraction $D^n \to S^{n-1}$ for $n > 2$, but this result will not be tested on the final. The proof is identical as the above, except we need a different functor.

**Theorem 24.2** (Brouwer Fixed Point)**.** If $f : D^2 \to D^2$ is continuous, then there exists $x \in D^2$ such that $f(x) = x$.

By contradiction, assume $f : D^2 \to D^2$ is continuous and fro all $x$, $f(x) \neq x$. Let $r(x)$ be the point where the straight ray from $f(x)$ to $x$ is continuous, and hits the circle. Then $r(x)$ is continuous; if $x \in S^1$, then $r(x) = x$. So $r : D^2 \to S^1$ is a retract, but retracts don't exist. $\square$

*Everything from this point onwards is not going to be tested on the final.*

**Theorem 24.3** (Formality of Limit)**.** Assuming the axiom of choice, then let Lim be a function be from bounded sequences to $\mathbb{R}$ such that

(i) Lim is linear, i.e. $\operatorname{Lim}(a_n + b_n) = \operatorname{Lim} a_n + \operatorname{Lim} b_n$, and $\operatorname{Lim}(ca_n) = c\operatorname{Lim} a_n$,

(ii) $\operatorname{Lim}(a_n b_n) = (\operatorname{Lim} a_n)(\operatorname{Lim} b_n)$

(iii) $\operatorname{Lim} a \in \bigcap_n \overline{\{a_k \mid k \geq n\}}$. If $a$ is convergent, then $\operatorname{Lim} a = \lim a$.

Note that lim sup doesn't really possess any "nice" linearity properties. We also cannot ask that Lim have the property

(iv) $\operatorname{Lim} a_{n+1} = \operatorname{Lim} a_n$, since we can just take the sequences $a_n = 0$ if $n$ is odd, and 1 if even.

Reminder that the axiom of choice implies Tychonoff's theorem; if $X_\alpha$ is compact, then for all $\alpha$, we have that $\prod_\alpha X_\alpha$ is compact. We instead assume the version that $I^W$ is compact even if $W$ is huge.

We now prove (iii). Let $W$ be the set of all bounded real sequences. For $a \in W$, let $I_a = [\inf a, \sup a] \ni a_k$. Let $X = \prod_{a \in W} I_a$, which is compact. Let $\alpha : \mathbb{N} \to X$ be given by $\alpha(n)_a = a_n$. Then $\alpha$ is an embedding, meaning that $\alpha$ is a homeomorphism into its image. This means that it is continuous, $\alpha$ is injective, and that $\alpha(\mathbb{N})$ is discrete.

Continuity is automatically given; the second property is given by observing that if $n \neq m$, then we may pick $a$ such that $a_n \neq a_m$, and then we have $\alpha(n)_a = a_n \neq a_m = \alpha(m)_a$. Given $n$, let $a_k^n = \delta_{nk}$, where $\delta$ is the Kronecker delta. Then $a^n = (0, \ldots, 0, 1, 0, \ldots, 0)$. Let $U_n = \prod_{a^n}^{-1}\left(\frac{1}{2}, \frac{3}{2}\right)$. Then $\alpha(m) \in U_n$ if and only if $\pi_{a^n}(\alpha(m)) \in \left(\frac{1}{2}, \frac{3}{2}\right)$, i.e. $a_m^n \in \left(\frac{1}{2}, \frac{3}{2}\right)$, which is true if and only if $m = n$ by the Kronecker delta.

Now, let $\beta\mathbb{N} = \overline{\alpha(\mathbb{N})}$, i.e. the "Stone-Cech compactification of $\mathbb{N}$" Clearly, this is compact, so $\alpha(\mathbb{N}) \neq \beta\mathbb{N}$ (more specifically, the former is included, but not equal to).

**Claim 24.4.** Every bounded sequence $a$ extends uniquely to a continuous $\tilde{a} : \beta\mathbb{N} \to \mathbb{R}$.

Note that the diagram is given by

$$\begin{tikzcd}
\mathbb{N} \arrow[rr, "\alpha"] \arrow[dr, "a"'] & & \beta\mathbb{N} \arrow[dl, "\tilde{a}"] \\
& \mathbb{R} &
\end{tikzcd}$$

Given $a$ and $\mu \in \beta\mathbb{N}$, set $\tilde{a}(\mu) = \mu_a$, i.e. $\tilde{a} = \pi_a|_{\beta\mathbb{N}}$. It is continuous, and $\tilde{\alpha}(\alpha(n)) = \alpha(n)_a = a_n$. It is also unique since, for $b, c : \beta\mathbb{N} \to \mathbb{R}$ such that

$$\begin{tikzcd}
\mathbb{N} \arrow[rr, "\alpha"] \arrow[dr, "a"'] & & \beta\mathbb{N} \arrow[dl, shift right, "c"'] \arrow[dl, shift left, "b"] \\
& \mathbb{R} &
\end{tikzcd}$$

then $b|_{\mathrm{Im}\alpha} = c|_{\mathrm{Im}\alpha}$, but $\mathrm{Im}\alpha$ is dense in $\beta\mathbb{N}$, so by the term test, we must have $b = c$. Now, pick any point $\mu \in \beta\mathbb{N} \setminus \alpha(\mathbb{N})$, which exists because of axiom of choice. Define $\operatorname{Lim} a = \operatorname{Lim}_\mu a = \tilde{a}(\mu) = \mu_a = \pi_a(\mu)$.

**Claim 24.5.** If $F : \mathbb{R}^2 \to \mathbb{R}$, then $\operatorname{Lim} F(a_n, b_n) = F(\operatorname{Lim} a_n, \operatorname{Lim} b_n)$.

Let $F(\tilde{a}, \tilde{b})$, $F(a, b)$ be functions $\beta\mathbb{N} \times \beta\mathbb{N} \to \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where the composition is given by $\tilde{a} \circ F$, $\tilde{b} \circ F$ respectively. Note that $F(\tilde{a}, \tilde{b}) = \widetilde{F(a, b)}$ since the extension of $F(a, b)$ to $\beta\mathbb{N}$ is unique. Evaluating on $\mu$, we get

$$F(\operatorname{Lim} a, \operatorname{Lim} b) = F(\tilde{a}(\mu), \tilde{b}(\mu)) = \operatorname{Lim} F(a, b). \quad \square$$

**Claim 24.6.** For all $n$, $\operatorname{Lim} a \in \overline{\{a_k \mid g \geq n\}}$.

Let $\mu \in \overline{\alpha(\mathbb{N}_{\geq n})}$. Let $\mu \in \overline{\alpha(\mathbb{N})} = \alpha(\mathbb{N})$, so $\mu \in \alpha(\mathbb{N})'$. So every neighborhood of $\mu$ contains infinitely many of the $\alpha(\mathbb{N})$'s as desired. Applying $\pi_a$, we have that

$$\operatorname{Lim} a = \pi_a(\mu) \in \pi_a(\overline{\alpha(\mathbb{N}_{\geq n})}) \subset \overline{\pi_a\left(\alpha(\mathbb{N}_{\geq n})\right)}.$$

This is the first time we're using that if $f$ is continuous, then $f(\overline{A}) \subset \overline{f(A)}$. In particular, we have

$$\overline{\pi_a\left(\alpha(\mathbb{N}_{\geq n})\right)} = \overline{\{\pi_a(\alpha(k)) \mid k \geq n\}} = \overline{\{\alpha(k)_a \mid k \geq n\}} = \overline{\{a_k \mid k \geq n\}}. \quad \square$$