

MAT363 Lecture Notes

ARKY!! :3C

'25 Winter Semester

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§1 Day 1: Introduction to the Class (Jan. 6, 2025)

Course administrative details! First day slides are given [here](#). This is a class in classical differential geometry; the following 12 weeks will be split up as follows,

- (a) Curves, for two weeks;
- (b) Surfaces, for three weeks;
- (c) Curvature of surfaces, for three weeks;
- (d) Geodesics, for three weeks;
- (e) Gauss-Bonnet theorem, for one week.

Grading will be done by 5% on PCEs, 15% on problem sets, 15% on quizzes, 25% on the term test, 30% on the final exam, and 10% weighted towards your best test.

To start, consider the following maps $\gamma : I = (-10, 10) \rightarrow \mathbb{R}^3$, given by

$$\begin{aligned}\gamma(t) &= (t, t, t); \\ \gamma(t) &= (|t|, |t|, |t|); \\ \gamma(t) &= (t, t^2, t^3); \\ \gamma(t) &= (t^3, t^3, t^3); \\ \gamma(t) &= (\cos t, \sin t, t); \\ \gamma(t) &= (t \cos t, t \sin t, t).\end{aligned}$$

In this class, we say that a curve is a *parameterized curve* if it is a smooth function $\gamma : I \rightarrow \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval. In particular, of the six examples given above, only $t \mapsto (|t|, |t|, |t|)$ is not smooth.

Definition 1.1 (Regular Curve). Let $\gamma : I \rightarrow \mathbb{R}^n$ be a curve; it is said to be *regular* if $|\gamma'(t)| \neq 0$ for all $t \in I$, i.e. the speed is always nonzero.

Note that $\gamma'(t)$ and $|\gamma'(t)|$ describe different qualities, with the former describing velocity and the latter describing speed (i.e., one describes speed as well, while the other is a scalar quantity). As an example, consider the curve $\gamma(t) = (\cos t, \sin t, t)$. To find the distance travelled from $t = 0$ to $t = 2\pi$, we may observe that

$$|\gamma'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Since the speed is constant, the total distance traveled is simply $2\pi\sqrt{2}$. □

Definition 1.2 (Closed Curve). Consider a curve $\gamma : [a, b] \rightarrow \mathbb{R}^n$. We say that γ is a *closed curve* if $\gamma(a) = \gamma(b)$ and $\gamma^{(n)}(a) = \gamma^{(n)}(b)$ for all naturals n .

Definition 1.3 (Simple Curve). We say that γ is a *simple curve* if it is injective on $[a, b]$.

Note that while in topology we do not care if there is a “sharp corner” at $\gamma(a) = \gamma(b)$, such things do matter, as per the condition that the n th derivative of γ must agree on a and b (for example, the velocity γ' at a, b must be equal).

In this class, we automatically take the inner product $\langle \cdot, \cdot \rangle$ as the Euclidean inner product,

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n.$$

For any subspace $V \subset \mathbb{R}^n$, we may decompose any vector $x \in \mathbb{R}^n$ uniquely as $x = x^\parallel + x^\perp$, where $x^\parallel \in V$ and $\langle x^\perp, v \rangle = 0$ for any vector $v \in V$. Now, consider any curve $\gamma : I \rightarrow \mathbb{R}^n$. We have the following proposition,

Proposition 1.4. If $|\gamma(t)|$ is constant, then $\langle \gamma(t), \gamma'(t) \rangle = 0$ for all $t \in I$.

To see this, let $|\gamma(t)|^2 = c$ be constant; then

$$\frac{d}{dt} |\gamma(t)|^2 = 0 \implies \frac{d}{dt} (\langle \gamma(t), \gamma(t) \rangle) = \langle \gamma'(t), \gamma(t) \rangle + \langle \gamma(t), \gamma'(t) \rangle = 0$$

i.e. $\langle \gamma(t), \gamma'(t) \rangle = 0$ as desired. □

Given a regular curve $\gamma : I \rightarrow \mathbb{R}^n$, we may compute the velocity and acceleration as $\gamma'(t), \gamma''(t)$, which are denoted $v(t), a(t)$ respectively. In particular, we may write

$$a(t) = a^{\parallel}(t) + a^{\perp}(t),$$

with $a^{\parallel}(t)$ being the tangential acceleration, and $a^{\perp}(t)$ being the normal acceleration. We may find these by projecting $a(t)$ into the subspace $\text{span}\{v\}$ (i.e., the span of the velocity vector).

§2 Day 2: Curvature of a Curve (Jan. 9, 2025)

Definition 2.1. Suppose that $\gamma : I \rightarrow \mathbb{R}^n$ is a regular curve. A *reparameterization* of γ is a function of the form $\tilde{\gamma} = \gamma \circ \phi : \tilde{I} \rightarrow \mathbb{R}^n$, where \tilde{I} is an interval, and $\phi : \tilde{I} \rightarrow I$ is a smooth bijection with nowhere vanishing derivative $\phi'(t) \neq 0$ for all $t \in \tilde{I}$.

Definition 2.2. We say that $\gamma, \tilde{\gamma}$ have the same orientation (i.e., the parameterization is *orientation-preserving*) if $\phi' > 0$, and orientation-reversing if $\phi' < 0$.

Note that since ϕ is smooth, it is impossible for ϕ' to have places on which it is greater than 0 and less than 0, since IVT holds on \tilde{I} and $\phi'(t) \neq 0$.

We now present an example;

$$\begin{aligned}\gamma : [0, 1] &\rightarrow \mathbb{R}^3 \text{ with } \gamma(t) = (t, t), \\ \beta : [0, \pi/2] &\rightarrow \mathbb{R}^3 \text{ with } \beta(t) = (\sin t, \sin t).\end{aligned}$$

While these both parameterize the same curve, we prefer γ greatly because $|\gamma'(t)| = \sqrt{2}$ while $|\beta'(t)| = \cos t$; specifically, constant speed parameterizations are much nicer to deal with. Even better,

Definition 2.3. A curve γ is said to be parameterized by arclength if $|\gamma'(t)| = 1$.

In particular, we may turn our above $\gamma : t \mapsto (t, t)$ into an arclength parameterization by considering it to be $t \mapsto (\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}})$ instead, since we would then have

$$|\phi'(t)| = \sqrt{2 \cdot \left(\frac{1}{\sqrt{2}}\right)^2} = 1. \quad \square$$

We now introduce the idea of curvatures of curves. A few ideas first:

- A smaller circle has *larger* curvature (naturally, it is more curved than a large circle).
- The curvature of a curve is given by a function $\kappa : I \rightarrow [0, \infty)$; i.e., it cannot be negative. Specifically, it is given by

$$\kappa(t) = \frac{|a^\perp(t)|}{|v|^2}.$$

Curvature enjoys two main properties; κ is independent of the parameterization of the curve, and that $\kappa = \frac{1}{r}$, where r is the radius of the circle that approximates the curve at the particular point. If γ is parameterized by arclength, then $\kappa(t) = a(t)$. To see this, observe that

$$\kappa(t) = \frac{|a^\perp(t)|}{|v|^2} = \left|a^\perp(t)\right| = |a(t)|.$$

In particular, since $|v(t)|$ is constant, so is $|v(t)|^2$. Then

$$\frac{d}{dt} \langle v(t), v(t) \rangle = 0 \implies \langle v'(t), v(t) \rangle = 0 \implies \langle a(t), v(t) \rangle = 0.$$

From this, along with $a(t) = a^\perp(t) + a^\parallel(t)$, we have that $a^\parallel(t) = 0$.¹ \square

¹for more detail, check proposition 1.18 in the textbook; γ' and γ'' are perpendicular if γ is a curve with constant speed.

Definition 2.4. Let $\gamma : I \rightarrow \mathbb{R}^n$ be a regular curve. We define the unit tangent vector and unit normal vector as

$$T(t) = \frac{v(t)}{|v(t)|}; \quad n(t) = \frac{a^\perp(t)}{|a^\perp(t)|},$$

respectively.

Note that the textbook uses \mathbf{t}, \mathbf{n} respectively, but these are hard to write on paper.

Definition 2.5 (Osculating Plane). At a fixed point on a regular curve with $\kappa \neq 0$, we define the *osculating plane* by

$$\text{span}\{T, n\},$$

and we define the *osculating circle* to be the unique circle with these properties:

- (a) Radius $\frac{1}{\kappa}$,
- (b) It is in the osculating plane,
- (c) It is centered at 0.

§3 Day 3: Angle Function and Rotation Index (Jan. 13, 2025)

We begin with a few questions.

- (a) Why is $\kappa_s(t)$ only defined for planar curves and not in \mathbb{R}^n for $n > 2$? In \mathbb{R}^2 , the orthogonal vector is 1-dimensional; in higher dimensions, it is $n - 1$ -dimensional, which does not make sense to apply a 90-degree rotation to; i.e., the notions of “clockwise” and “counter-clockwise” rotation are not that well defined in \mathbb{R}^n .
- (b) How can the angle function $\theta(t)$ increase by more than 2π along a curve? This is possible by having multiple loops; check figure 1.23, on page 37 in the textbook.
- (c) What is the geometric meaning behind the equation $\theta'(t) = \kappa_s(t)$? Intuitively, θ' measures the change in the angle measuring how much the curve deviates from a straight line.
- (d) Why is the rotation index of a unit-speed closed plane curve always an integer? Since the curve is closed and smooth, θ has an equal evaluation at the beginning and end of the interval the curve is parameterized on, with a difference of a multiple of 2π per definition; thus, we have that

$$\frac{\theta(b) - \theta(a)}{2\pi}$$

evaluates out to an integer always.

An example graph of $\theta(t)$ for curve γ was given in class; we record the answers here, but not the figure.

- (a) The rotation index is $\frac{1}{4}$.
- (b) The curve is not closed, since the rotation index is not an integer.
- (c) Clockwise means negative θ , and counterclockwise means positive θ .
- (d) The maximum curvature, $\max |\theta'(t)|$, is approximately 6π (which is a guess).

Let $\gamma : [a, b] \rightarrow \mathbb{R}^2$ be a simple closed curve. Let $C = \gamma([a, b])$ denote its trace.

Theorem 3.1 (Hopf’s Umlaufsatz). The rotation index of γ is either -1 or 1 .

Theorem 3.2 (Jordan Curve Theorem). $\mathbb{R}^2 \setminus C = \{p \in \mathbb{R}^2 \mid p \notin C\}$ has exactly two path connected components. Their common boundary is C . One component (which we will call the interior) is bounded, while the other (which we call the exterior) is unbounded.

Definition 3.3. In the above context, we say that γ is positively oriented if the rotation index of γ is 1 , and negatively oriented if the index is -1 .

Definition 3.4. A *piecewise regular curve* in \mathbb{R}^n is a continuous function $\gamma : [a, b] \rightarrow \mathbb{R}^n$ with partition $a = t_0 < \dots < t_n = b$ such that each $\gamma|_{[t_i, t_{i+1}]}$ is a regular curve. We call the points $\gamma(t_i)$ for $i = 1, \dots, n - 1$ the “corners” of γ .

Definition 3.5. In the above context, the *signed angle* at $\gamma(t_i)$ denoted by $\alpha_i \in [-\pi, \pi]$ is given by the angle between $v^-(t_i)$ and $v^+(t_i)$ with $\alpha_i > 0$ for a counterclockwise and $\alpha_i < 0$ for a clockwise turn.

Theorem 3.6 (Generalized Hopf's Umlaufsatz). Let $\gamma : [a, b] \rightarrow \mathbb{R}^n$ be a unit-speed positively oriented piecewise regular simple closed plane curve. Let κ_s denote its signed curvature function, and let $(\alpha_i)_i$ be the list of signed angles at its corners. Then

$$\int_a^b \kappa_s(t) dt + \sum \alpha_i = 2\pi.$$

We now move onto space curves. Note that \mathbb{R}^2 may be embedded into \mathbb{R}^3 by considering $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

Definition 3.7. Consider a regular space curve $\gamma : I \rightarrow \mathbb{R}^3$ and $t \in I$ such that $\kappa(t) \neq 0$. The *Frenet frame* at t is the orthonormal basis $\{T, n, B\}$ of \mathbb{R}^3 defined by

$$\begin{aligned} T(t) &= \frac{v(t)}{|v(t)|}, \\ n(t) &= \frac{a^\perp(t)}{|a^\perp(t)|}, \\ B(t) &= T(t) \times n(t). \end{aligned}$$

In particular, $|B'|$ is a natural choice to measure the changing tilt of the osculating plane because B is constant length, and that it is orthogonal to both B and T . Moreover, $\langle B', n \rangle$ and $|B'|$ are related as follows,

$$\langle B', n \rangle = |B'| |n| \cos \varphi = \pm |B'|,$$

where φ is the angle between B and n (note that this is either 0 or π , since $B \parallel n$).

Definition 3.8. Consider a regular space curve $\gamma : I \rightarrow \mathbb{R}^3$ and $t \in I$ such that $\kappa(t) \neq 0$. The torsion of γ at t is given by

$$\tau(t) = \frac{-\langle B'(t), n(t) \rangle}{|v(t)|}.$$

The torsion is independent of parameterization.

As an example, if $\gamma : I \rightarrow \mathbb{R}^3$ is such that for all $t \in I$, $\kappa(t) \neq 0$ and $\tau(t) = 0$, we have that $|B'| = 0$, B is constant, and so $\text{span}\{T, u\} = P$. In particular, this means that γ is planar; i.e., for all $t \in I$, $\gamma(t) \in \gamma(t_0) + P$.

Theorem 3.9 (Frenet Equations). For a regular curve $\gamma : I \rightarrow \mathbb{R}^3$, the following equations hold whenever $\kappa(t) \neq 0$:

$$\begin{aligned} T' &= |v| \kappa n \\ n' &= |v| [-\kappa T + \tau B] \\ B' &= -|v| \tau n. \end{aligned}$$

In matrix form, this is written as

$$\begin{pmatrix} T \\ n \\ B \end{pmatrix}' = |v| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ B \end{pmatrix}.$$

Next class, we will prove the above; in particular, there exists a unique solution by the Picard-Lindelöf theorem. We did not cover Taylor approximations because we ran out of time.