MAT327 Lecture Notes

ARKY!! :3C

'24 Fall Semester

Contents

§1 Day 1: Open Sets and Continuity (Sep. 3, 2024)

> This class is *MAT327*; 3 meaning third year, 2 meaning the contents are on the fundamental side, and 7 meaning no mercy.
> – Dror Bar-Natan

Course administration matters first;

- The course link is given here (this will link straight to Quercus).
- The textbook is James Munkres' Topology (online PDF: ETH Zurich mirror); Prof Bar-Natan strongly recommends a paper copy, though (since people get distracted on the computer).

Today's reading in the textbook is on Ch. 1, sections 1 to 8, and Ch. 2, sections 12 to 13. Readings are supplementary to lecture material[1]. The goal of this course is to understand continuity in its most general form; in particular,

- In MAT157, we studied continuity in $f : \mathbb{R} \to \mathbb{R}$;
- In MAT257, we will study continuity in $f : \mathbb{R}^n \to \mathbb{R}^m$;

but in this class, we will study continuity in $F : X \to Y$, where X, Y are arbitrary spaces, such as (but not limited to) $\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{N}}$ (binary sequences), and so on. We start with some refreshers on previous coursework;

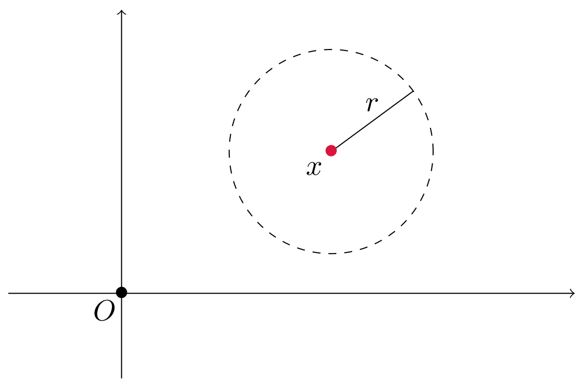
Definition 1.1 (Continuity in $\mathbb{R}^n \to \mathbb{R}^m$)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called "continuous" if it is continuous at all points in $\mathbb{R}^n$. Specifically, for all $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

With this, we state our main theorem for today (proof given later),

Theorem 1.2 (Continuity on $\mathbb{R}^n$ if and only pre-image of open subsets is open)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if all open subsets $U \in \mathbb{R}^m$ have $f^{-1}(U)$ open.

In order to build up to the above, we start by defining some terms;

Definition 1.3 (Open Ball)**.** Let $r > 0$, and $x \in \mathbb{R}^n$. An open ball of radius r about x is given formally by $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Visually, we have



Note that the border of the circle is not in $B_r(x)$, as the distance metric asks for a strict inequality.

[1] iirc it won't be tested unless specified. its still good to learn tho

In a similar fashion, we have

Definition 1.4 (Open Set)**.** A set $U \subset \mathbb{R}^n$ is called *open* if, for all $x \in U$, there exists an open ball about x contained in U. Specifically, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

Intuitively, we could say that the set U does not contain its edge; if it did, let x be on said edge; then we would not be able to fit an open ball about x in U. Here are some examples of open sets from lecture;

1. The whole set $U = \mathbb{R}^n$ is open.
2. The empty set $\emptyset$ is open. Since there does not exist any $x \in \emptyset$, no conditions on balls need to be satisfied; "every dog in the empty set of dogs is green".
3. $(0, 1)$ is open on $\mathbb{R}^1$ (and any open interval, for that matter).
4. $B_r(x) \in \mathbb{R}^n$ is open. To see this, observe that for any $y \in B_r(x)$, we may pick $\varepsilon < r - |x - y|$; by triangle inequality, all elements in $B_\varepsilon(y)$ must also be in $B_r(x)$.

We now define images and pre-images; let us have a function between sets $f : X \to Y$. For subsets $A \subset X$ and $B \subset Y$,

$$f(A) = \{f(a) \mid a \in A\},$$
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Since f need not be injective, f^{-1} does not necessarily exist. While images of a union of subsets is a union of the images, the same is not true for intersections; let $A_1, \ldots, A_n$ be subsets of X; then

$$f\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f(A_i), \qquad f\left(\bigcap_{i=1}^n A_i\right) \subset \bigcap_{i=1}^n f(A_i).$$

On the other hand, pre-images preserve both union and intersection; let $B_1, \ldots, B_n \subset Y$; then

$$f^{-1}\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f^{-1}(A_i), \qquad f^{-1}\left(\bigcap_{i=1}^n A_i\right) = \bigcap_{i=1}^n f^{-1}(A_i).$$

As for set complements, we have

$$f^{-1}(B^C) = f^{-1}(B)^C,$$

but the same cannot be said for images; $f(A)^C \subset f(A^C)$ for surjective f, and vice versa for injective f. Equality occurs only if f is a bijection.

Note: for the proof of Theorem 1.2, I'll leave it out in case Prof. Bar-Natan proves it later on Thursday.

§2 Day 2: Basic Definitions and Topological Spaces (Sep. 5, 2024)

We start by recapping the previous lecture; we introduced

- Open sets $U \subset \mathbb{R}^n$, where there exists an open ball of radius $\varepsilon > 0$ about any $x \in U$.
- Continuity in $f : \mathbb{R}^n \to \mathbb{R}^m$.

Recall the continuity property from last class, that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if the pre-image of an open set U is open.

($\Rightarrow$) To start, pick any $x_0 \in f^{-1}(U)$, and let us have a small enough $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \in U$ (this is possible since U is open). By continuity, there exists $\delta > 0$ such that any $x \in B_\delta(x_0)$ satisfies $f(x) \in B_\varepsilon(f(x_0)) \subset U$; this means $x \in f^{-1}(U)$ by definition of pre-image, and since x was arbitrary, we see $B_\delta(x_0) \in f^{-1}(U)$. Moreover, since x_0 was also arbitrary, our construction shows that there always exists a δ-ball about any point in the pre-image, and so $f^{-1}(U)$ is open. □

($\Leftarrow$) For the other direction, take any $x_0 \in \mathbb{R}^n$, and $\varepsilon > 0$. Since $B_\varepsilon(f(x_0))$ is open we have that $f^{-1}(B_\varepsilon(f(x_0)))$ is open as well. This means we may pick a small enough $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x_0)))$ (by definition of openness), and we immediately see

$$f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)).$$

This is a reconstruction of the epsilon-delta definition of continuity, and so we are done. □

Open sets in $\mathbb{R}^n$ have a number of properties;

(a) $\emptyset, \mathbb{R}^n$ are open sets.

(b) The union of open sets are open; specificay, let S be a set of indices, and let $A_\alpha \subset \mathbb{R}^n$ for all $\alpha \in S$. Then

$$\bigcup_{\alpha \in S} A_\alpha$$

is an open set.

(c) The finite intersection of open sets are also open. Let $A_1, \ldots, A_n \subset \mathbb{R}^n$ be open; then

$$\bigcap_{i=1}^{n} A_i$$

is open.

We now proceed with the proofs for these properties.

(a) This was proven last lecture.

(b) For any x in the union, by definition, there exists $\alpha \in S$ such that $x \in A_\alpha$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A_\alpha$, and $A_\alpha \subset \bigcup_{\alpha \in S} A_\alpha$. This concludes that the union is open.

(c) For any x in the finite intersection, then x is an element of each of $A_1, \ldots, A_n$. Let us have $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $B_{\varepsilon_i}(x) \varepsilon A_i$, and take $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$ (note that minimum is defined only for finite lists), which means $B_\varepsilon(x) \subset \bigcap_{i=1}^{n} A_i$.

In the case of infinite intersection of open sets, the resulting set need not be open. For example, let us consider the infinite intersection of intervals

$$\bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, \frac{1}{j}\right) = \{0\},$$

which is not open.

Definition 2.1 (Topology on a Set)**.** Let X be a set equipped with topology $\mathcal{T}$ on X. $\mathcal{T}$ is a collection of subsets of X, i.e. $\mathcal{T} \subset \mathcal{P}(X)$, with properties

- $\varepsilon, X \in \mathcal{T}$.
- The union of subsets of $\mathcal{T}$ is also in $\mathcal{T}$.
- The finite intersection of subsets of $\mathcal{T}$ is also in $\mathcal{T}$.

We call $(X, \mathcal{T})$ a *topological space*, which may be abbreviated to X is $\mathcal{T}$ is given or obvious. Moreover, we define another notion of openness, where $U \in \mathcal{T}$ is said to "be open relative to $\mathcal{T}$," or that "U is open." Here are a few examples of such topological spaces.

(a) The standard topology on $\mathbb{R}^n$ is given by

$$\mathbb{R}_{std}\left(\mathbb{R}, \mathcal{T}_{std} = \{U \subset \mathbb{R}^n \mid U \text{ is open in the “old sense”}\}\right),$$

i.e. collection of open intervals[2].

(b) The discrete topology, $X_{discrete} = (X, \mathcal{T} = \mathcal{P}(x))$ can be defined over any set X, equipped with $\mathcal{T}$ as the collection of all subsets.

(c) The trivial topology, $X_{trivial} = (X, \mathcal{T} = \{\emptyset, X\})$.

Definition 2.2 (Continuity between Topological Spaces)**.** If $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are topological spaces, and $f : X \to Y$ is a function between said spaces, then we say f is continuous if, for all $U \in \mathcal{T}_Y$, we have that $f^{-1}(U) \in \mathcal{T}_X$. This draws from our definition from the notion that pre-images of open sets are open.

Now for the examples from lecture;

(a) $f : X_{discrete} \to \mathbb{R}_{std}$ is always continuous.

(b) $f : X_{trivial} \to \mathbb{R}_{std}$ is continuous if and only if f is constant.

(c) $f : \mathbb{R}_{std} \to X_{trivial}$ is always continuous.

(d) $f : \mathbb{R}_{std} \to X_{discrete}$ is almost never continuous, except when X is empty or a singleton (in which case, $X_{discrete} = X_{trivial}$). If $x_0 \in X$ yet $X \setminus \{x_0\} \neq \emptyset$, then let us have

$$\begin{aligned} A &:= f^{-1}(\{x_0\}), \\ B &:= f^{-1}(X \setminus \{x_0\}). \end{aligned}$$

While A, B are both open, we see that $A \cup B = \mathbb{R}$, and $A \cap B = \emptyset$.[3]

[2] this is how i understood it, he might clarify next time?
[3] this one i don't completely get yet.

§3 Day 3: Homeomorphisms and Bases (Sep. 10, 2024)

Course administrative details first;

- The reading for this week is on sections 12 to 14 (this week will cover these contents), and 15 to 16 as prereading.

Recap of last lecture:

- A topology $\mathcal{T} \subset \mathcal{P}(X)$ is a collection of subsets of X, of which we require $\{\emptyset, X\} \subset \mathcal{T}$. We also require $\mathcal{T}$ to be closed under arbitrary unions and finite intersections.
- We say a function $F : X \to Y$ is continuous if and only if for all $U \in \mathcal{T}_Y$, we have $f^{-1}(U) \in \mathcal{T}_X$.

Today we will cover homeomorphisms and bases. To start, recall the example topologies, such as $\mathcal{T}_{\text{std}}$ on $\mathbb{R}^n$ (where $\mathcal{T}_{\text{std}}$ consists of the open balls), $\mathcal{T}_{\text{triv}}$, and $\mathcal{T}_{\text{disc}}$. We also introduce a new example topology (where FC standards for finite complement),

$$\mathcal{T}_{\text{FC}} = \{U \subset X \mid X \setminus U \text{ is finite, or } U = \emptyset\}.$$

Note that Prof. Bar-Natan may interchange the notations $-$ or $\setminus$ to represent set difference.

Theorem 3.1 (Composition of Continuous Functions is Continuous)**.** Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f : X \to Z$ is continuous (relative to the same topologies on X and Z).

If $U \in \mathcal{T}_Z$, we have $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U)$, where by definition of continuity, we see that pre-images of open sets are open, and we have $g^{-1}(U)$ is open in Y, and similarly $f^{-1}(g^{-1}(U))$ is also open in X. □

In tutorial, we equipped X with topologies $\mathcal{T}_1, \mathcal{T}_2$ (i.e., X is a topological space in two ways). We say $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if $T_1 \supset T_2$, and coarser for the opposite direction; the words bigger and stronger may be used interchangeably with finer, and smaller or weaker for coarser. For example, the identity map

$$\text{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$$

is continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$. To see this, let U be an open set in $(X, \mathcal{T}_2)$; then U must be open in $(X, \mathcal{T}_1)$ as well, which is true for any U only if $\mathcal{T}_1 \supset \mathcal{T}_2$[4].

Definition 3.2 (Homeomorphism)**.** A map $h : X \to Y$ is called a homeomorphism if h is continuous, bijective, and h^{-1} is continuous as well.[5]

Note that continuous bijective maps h need not have continuous inverses; for example, let us have $\text{id} : X_{\text{disc}} \to X_{\text{triv}}$. id is continuous as per our above example, while its inverse is not. Another example is to consider $[0, 2\pi) \to S^1$ (unit circle), where $x \mapsto (\cos 2x\pi, \sin 2x\pi)$; we see that the inverse is discontinuous at 0 and 2π radians, even if the map is continuous and bijective (also observe that $[0, 1)$ is not compact while S^1 is).

[4] includes the case $\mathcal{T}_1 = \mathcal{T}_2$; i'm following florian notation here with $\subset$ and $\subsetneq$ for explicit non equality

[5] smth smth coffee cup and donut "extra homework: go find a nice video on why this is true on youtube"

An example of a homeomorphism is as follows (as per tutorial); let us consider[6]

$$(-1,1)_{\text{std}} \cong \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)_{\text{std}} \cong \mathbb{R}_{\text{std}},$$

where we may map the first to the second by $x \mapsto \frac{\pi}{2}x$, and the second to third by $x \mapsto \tan x$. Since the composition of continuous maps is continuous, we also see $(-1,1)_{\text{std}}$ is homeomorphic to $\mathbb{R}_{\text{std}}$ (any open interval is homophobic to $\mathbb{R}$ for that matter).

Another example of a homemorphism is $\text{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ if $\mathcal{T}_1 = \mathcal{T}_2$.

A "basis" for a topology on X is a collection $\mathcal{B} \subset \mathcal{P}(X)$ of subsets such that

1. For all $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$. We call B a *basic set*.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists a third basic set B_3 such that $x \in B_3 \subset B_1 \cap B_2$.

In particular, the first condition is equivalent to $\bigcup_{B \in \mathcal{B}} B = X$ (i.e., the basis forms a covering of X), and the second condition is equivalent to the basic sets contained in $B_1 \cap B_2$ forming a cover of $B_1 \cap B_2$, i.e.

$$B_1 \cap B_2 = \bigcup_{\substack{B \in \mathcal{B} \\ B \subset B_1 \cap B_2}} B.$$

Here are some examples;

(a) $\{B_r(x_0)\} \subset \mathcal{P}(\mathbb{R}^n)$, i.e. the open balls on $\mathbb{R}^n$ form a basis.

(b) The one-dimensional analogue of case (a) is $\{(a,b) \mid a < b\}$, and it forms a basis on $\mathbb{R}$.

(c) $\{[a,b) \mid a < b\}$ is called the lower limit topology, and it forms a basis on $\mathbb{R}$.

(d) $\{[a,b] \mid a < b\}$ implies $[a,b] \cap [b,c] = \{b\}$, which forces the basic set to include all singletons on $\mathbb{R}$. In that case, this is simply the discrete topology (?).

Theorem 3.3. $\mathcal{T}_{\mathcal{B}} = \{U \subset X \mid \forall x \in U \implies \exists B \in \mathcal{B}$ such that $x \in B \subset U\}$, i.e. the collection of all unique basic sets.

This will be expanded on next lecture.

[6] dror was using $\sim$ for homeomorphism symbol today. if he keeps using that i'll adjust my notes, but for now i'll use $\cong$ cuz afaik its used more...?