MAT417 Lecture Notes

ARKY!! :3C

'25 Fall Semester

Contents

§1 Day 1: Course Administrative Details (Sep. 2, 2025)

Course materials will be free and available online; here is a list of reference materials:

- Serre's *Course in Arithmetics* up to Chapter 4,
- Lecture notes by Noam Elkies (which will be posted on Quercus).

Homework will be posted every Thursday and due the following Thursday, and is worth **20%** of the course grade.

The central question of number theory is about the structure of prime numbers, of which the main analytic tools used are the Riemann ζ-functions and its relatives (the L-functions). We may discuss things like modular forms, Hecke operators and L-functions related to Galois representation later on.

Let us consider the following two questions;

(a) How many primes are there? There are infinitely many of them.

(b) Can you say something about how the primes are distributed?

Given $x > 0$, where x may be a natural or a real, let us define

$$\pi(x) = \#\{p \text{ is prime} \mid p \leq x\}.$$

Can we estimate how $\pi(x)$ grows? The prime number theorem states that the growth of $\pi(x)$ is proportional to $\frac{x}{\log x}$, i.e.,

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1, \qquad \frac{\pi(x)}{x} \to 0 \text{ as } x \to \infty.$$

As an exercise, show that the prime number theorem informally says that the nth prime p_n is of the size $n \log n$.

Theorem 1.1 (Dirichlet Theorem)**.** Let a, d be coprime naturals where $a < d$. Consider all numbers of the form $a + kd$, where k is also a natural; infinitely many of these numbers are prime.

Proof. Done with L-functions. Check here. □

Theorem 1.2 (Fundamental Theorem of Arithmetic)**.** Any nautral number N can be written uniquely as $p_1^{a_1} \dots p_n^{a_n}$, where p_i are primes and $a_i > 0$.

Proposition 1.3 (Euclid's Argument on the Infinitude of Primes)**.** Assume that $p_1 < p_2 < \cdots < p_n$ constitute all the primes. Then it is clear that $p_1 \dots p_n + 1$ is coprime to any p_i. By the fundamental theorem of arithmetic, this means that $p_1 \dots p_n + 1$ is divisible by a prime less than $p_1 \dots p_n + 1$ not given by some p_i, which is a contradiction.

Can we use this to get an estimate on $\pi(x)$? We claim that $\pi(x) > \log_2 \log_2 x$. Let p_n be the nth prime. Then

$$p_{n+1} < 1 + \prod_{i=1}^{n} p_i < \prod_{i=1}^{n} p_n.$$

If equality always held then we would have $p_n = 2^{2^{n-1}}$. However, in actuality, $p_n < 2^{2^{n-1}}$, so we must have that $\pi(x) > \log_2 \log_2 x$.

The Riemann-Zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Claim 1.4. ζ is absolutely convergent for any $s > 1$.

Proof. Will be given next class. $\square$

Lemma 1.5. For $s > 1$, we have that

$$\zeta(s) \leq \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}}.$$

Proof. This is given directly by geometric series, i.e.,

$$\frac{1}{1 - p^{-s}} = \sum_{i=0}^{\infty} p^{-is} = \sum_{\substack{p_1 < \cdots < p_n \\ a_1, \ldots, a_n > 0}} p_1^{a_1} \ldots p_n^{a_n}.$$

$\square$

Moreover, if we had finitely many primes, we could apply this to $s = 1$ and obtain that the sum of $\frac{1}{n}$ is convergent, which is clearly false. This also implies that the sum of the reciprocals of primes is divergent, and you can't have $\pi(x)$ be bounded from above by Cx^D, where $C > 0, D < 1$.