MAT351 Lecture Notes

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'25 Fall Semester

Contents

§1 Day 1: Course Administrative Details (Sep. 3, 2025)

[I'll put any of my own remarks in Square Brackets or in footnotes. I love making connections to other ideas, and it feels like killing my child to exclude them just because the prof didn't mention it explicitly.]

Put "MAT351" in subject if you're emailing about the course. The syllabus tab will have the list of topics covered each week. Attendance is strongly encouraged.[1]

- Problems are in *Partial Differential Equations: An Introduction* by Walter Strauss.
- Evans PDEs will cover the topics in more detail because it is a graduate textbook.
- PDEs in Action is more applied.

Midterm 2 will be $\geq$80% problems from the Problem Sets. Final exam is gonna be 2 hours rather than 3 because 3 hours is too long.

Topics:

- Intro
- Examples
- General Concepts
- Classify (There are many different kinds). By the end of the semester, when you see a PDE you should know what kind it is. Also, if you see some behavior in the wild, you can reach into your toolbox and think about which kind of PDE properly models the behavior.

We will go over all the main classes of PDEs

- First order PDEs
- Wave equation (Hyperbolic PDEs)
- Heat equation/Diffusion (Parabolic PDEs)
- Laplace/Harmonic Functions (Elliptic PDEs)
- Boundary/Initial Value Problems (Fourier Series)
- First Order Nonlinear PDEs (Shocks)
- Eigenvalues (Hydrogen Atom)
- Schrödinger and Fourier Transform (If there's time)
- Distributions and Weak Solutions[2] (If there's time)
- ...

PDE is a vast subject. It is one of the most connected fields of math, with connections to Analysis, Number Theory, Physics, Differential Geometry, Algebra, Numberical Analysis, Experiments, and so on. The first part of the class will be more computational, finding explicit solutions to our first four PDEs mentioned above, and the second part will be more theoretical.

[1] I'm looking at you, Isaac.
[2] This is my favourite topic in PDEs.

The rest of the lecture went over some examples of PDEs.

Example 1.1. A simple possible PDE would be as follows: Let $u = u(x, y)$ be a function of two variables. Then our PDE might be

$$u_x + u_y = 0.$$

Remark 1.2. $u_x = \frac{\partial}{\partial x}u$.

From here, a single solution would be $x - y$. We have a solution $u(x, y) = f(x - y)$ for any function f.

Remark 1.3. For ODEs, first order has one degree of freedom, second order has two degrees of freedom, etc. For PDEs, this first order PDE has infinitely many degrees of freedom.

We will also solve $u_x + c(x, y)u_y = 0$ next lecture (by method of characteristics).

Example 1.4 (Burgers' Equation)**.** We will return to first order PDEs at some point to look at nonlinear cases. One such example is

$$u_x + uu_y = 0.$$

This is a simple example used for fluid dynamics and part of why fluid dynamics is hard.

Example 1.5 (Wave Equation)**.** With the Wave Equation, there is one special coordinate, which is time. Take $u = u(t, x)$ with $t \in I \subseteq \mathbb{R}$, $x \in U \subseteq \mathbb{R}^m$, where I is an interval and U is an open set. Now, consider the $m = 1$ case. Then the wave equation is

$$u_{tt} - u_{xx} = 0.$$

For a general m, it's

$$u_{tt} - \Delta_x u = 0$$

Remark 1.6. If $f : \mathbb{R}^m \to \mathbb{R}$, then Δf is the Laplacian of f, $\frac{\partial^2}{\partial x_1^2}f + \cdots + \frac{\partial^2}{\partial x_m^2}f$. When f has two different variables ($f(t, x)$, then $\Delta_x f$ is the Laplacian with respect to x only. [An efficient way to write it is as the divergence of the gradient of a function, which will come up a bit later.]

We'll also look at

$$u_{tt} - \Delta u = f(x, t),$$

which is the non-homogeneous case.

Example 1.7 (Heat Equation)**.** The heat equation is

$$u_t - \Delta u = 0.$$

The difference between this and the Wave Equation is that this is the first derivative, whereas the Wave Equation is the second derivative. Makes a huge difference.

Example 1.8 (Laplace's Equation)**.** The Laplacian (Laplace's equation) is

$$-\Delta u = 0.$$

[Functions satisfying Laplace's Equation are sometimes called harmonic.] In this sense, Laplace's equation is the steady-state solution to the heat equation and the wave equation. Laplace's equation is very important for other fields as well. [Dirichlet's problem in the circle is just a special case of an initial value problem for Laplace's equation. We will almost certainly see this in 354. In this case, Complex Analysis turns out to be a very efficient way to solve this IVP.]

Remark 1.9. Why do we have the minus in front? One justification could be that Laplace's equation is inspired by physics. For example, in Electrostatics, if V is the electrical potential, then the electric field is given by $E = -\nabla V$. But from here, the electric charge is $\rho = \nabla \cdot E$ (up to a change in units), so it follows that $\rho = -\Delta V$.[3] It follows that in regions of space with no charge, the electrical potential should be a harmonic function.

Another more mathematical justification is that $-\Delta$ is a non-negative operator. The professor kinda trailed off while talking about this and never finished the proof, but here's what I imagine he was trying to say:

$$\begin{aligned}\langle -\Delta g, g\rangle &= \int_{\mathbb{R}^m} -\Delta g \ g \\ &= \int_{\mathbb{R}^m} \operatorname{div}(-\nabla g(x) g(x)) + \int_{\mathbb{R}^m} |\nabla g|^2 dx\end{aligned}$$

from here, assuming that $\nabla g \cdot g \to 0$ sufficiently quickly, it would follow (by the divergence theorem) that

$$\int_{\mathbb{R}^m} \operatorname{div}(-\nabla g(x) g(x)) = 0,$$

and therefore that this inner product would be non-negative. That being said, he was not very precise here.

Next time we start with the formal definition of a PDE and write out Navier-Stokes.

[3] I *told* you it would come up later.

§2 Day 2: Definitions, Examples, Solutions, Oh My! (Sep. 8, 2025)

Definition 2.1 (Partial Differential Equation)**.** A Partial Differential Equation is an equation for an unknown function u of more variables involving its partial derivatives, in general written as

$$F\left(D^k u, D^{k-1}u, \dots, Du, u, x\right) = 0. \tag{$*$}$$

The convention is that $D^k u$ is the kth partial derivatives of u.[4]

We'll generally look for $u : x \in U \subseteq \mathbb{R}^m \to X$, where X is typically $\mathbb{R}, \mathbb{C}, \mathbb{R}^m$. We also want $u \in C^k(U)$ (although, there are more general setups).[5] For the PDE $(*)$, we say the order is k (order will mainly be 1 or 2 in this class.) If $(*)$ can be written as

$$\mathcal{L}[u] = 0,$$

where $\mathcal{L}[u+v] = \mathcal{L}[u] + \mathcal{L}[v]$, then we call it linear. We also can include $\mathcal{L}[cu] = c\mathcal{L}[u]$, but this is usually redundant. Otherwise it's nonlinear. If $(*)$ has the form $\mathcal{L}[u] = f$ it's called (linear) non-homogeneous.

Example 2.2. The general second order pde linear: $u : (x, y) \in U \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$Q_{11}(x,y)u_{xx} + Q_{12}(x,y)u_{xy} + Q_{22}(x,y)u_{yy} + b_1(x,y)u_x + b_2(x,y)u_y + c(x,y)u = 0.$$

We say it's constant coefficient if Q_{ij}, b_k, c are constant.

Some examples from last class were constant coefficient

Example 2.3 (Wave Equation)**.**

$$u_{tt} - u_{xx} = 0$$

Example 2.4 (Heat Equation)**.**

$$u_t - u_{xx} = 0$$

Example 2.5 (Laplace's Equation)**.**

$$u_{xx} + u_{yy} = 0$$

Example 2.6.

$$u_x + Q(x,y)u_y = 0$$

Some more examples of linear PDEs:

Example 2.7 (Schrödinger Equation)**.**

$$u : (t, x) \subseteq I \times \mathbb{R}^m \to \mathbb{C}$$

where $I \subseteq \mathbb{R}$

$$iu_t - \Delta_x u = 0$$

We'll also look at $iu_t + (-\Delta + V(x))u = 0$.

[4] I imagine something like D^k is a k-tensor analogous to—in the case of $k = 2$—the Hessian matrix. Such formalism is pointless, however.

[5] Much of 457/8 is dedicated to things that make PDEs more general

On the other hand, here are some nonlinear PDEs:

Example 2.8 (Burgers' Type)**.**

$$u_t + Q(u)u_x = 0$$

Example 2.9.

$$u_t + u_x = u^2.$$

Since the right is in terms of u and nonlinear, it's a nonlinear PDE.

These are the only two types of nonlinear PDEs we'll solve. In some sense, the first one is "more" nonlinear because we're multiplying the partial derivative by something nonlinear, whereas the second is only multiplying u by something.

Example 2.10 (Nonlinear Wave Equation)**.**

$$u_{tt} - u_{xx} = u^p,$$

This might be a good model for general relativity, as general relativity is all about the nonlinear wave equation.

Example 2.11 (Maxwell's Equations)**.** Maxwell's Equations are an example of a linear system of PDEs.[6]

$$\begin{aligned} \frac{1}{c}\vec{E}_t &= \nabla \times \vec{B} \\ \frac{1}{c}\vec{B}_t &= -\nabla \times \vec{E}. \end{aligned}$$

Here, we take

$$\vec{E}, \vec{B} : (t, x) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3,$$

and c is the speed of light. We also have some constraints that

$$\begin{aligned} \nabla_x \cdot \vec{E} &= 0 \\ \nabla_x \cdot \vec{B} &= 0, \end{aligned}$$

and these are added for physical reasons and also to make the ODE solvable—in real life we indeed observe that the Electric field and the Magnetic field are conservative, so adding these requirements doesn't cause problems. If we don't have these, then there's too many degrees of freedom to solve. [Cf. Helmholtz Theorem]

Example 2.12 (Navier-Stokes' Equations)**.** These are a nonlinear system of PDEs.

$$\vec{u}_t + \vec{u} \cdot \nabla_x \vec{u} = -\nabla p + \nu \Delta_x \vec{u}.$$

Here, we have that $\vec{u} : (t, x) \in [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field, $\nu > 0$ is a constant which represents viscosity, we have that $p : (t, x) \to \mathbb{R}$ is pressure. We say also that

$$(\vec{u} \cdot \nabla \vec{u})_k = \sum_{j=1}^{3} \vec{u}_j \partial_j \vec{u}_k$$

and then we have

$$\nabla \cdot \vec{u} = 0$$

because otherwise there would be too many unknowns to solve it. [Another way to think about this term is that it represents consetvation of mass, so this boundary condition makes the fluid incompressible—so water, rather than air].

[6] "Emily gave a FANTASTIC example here. Kudos to her." – Isaac

§3 Day 3: I'm kind of getting sick of only doing examples (Sep. 10, 2025)

Definition 3.1. A PDE is fully nonlinear if it is nonlinear in the highest order derivative. This is, if the PDE is of the form

$$F(D^k u, D^{k-1}u, \dots, u, x) = 0,$$

where F is nonlinear in the $D^k u$ component.

Example 3.2.

$$u_t + uu_x = 0 \tag{1}$$
$$u_{tt} + u_x u_{xx} = 0 \tag{2}$$
$$u_t + u_x = u^2 \tag{3}$$
$$u_t^2 + u_x = 0 \tag{4}$$
$$u_t u_x + u = 0 \tag{5}$$

We see that (1) is not fully nonlinear, likewise with (2), and (3). On the other hand, (4) and (5) are fully nonlinear, because they are nonlinear in Du, and their order is 1. Finally, m

Example 3.3 (Mean Curvature)**.** This is from Differential Geometry.

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$

It is fully nonlinear as well.

We will now solve some equations "by hand".

Example 3.4. Let $u = u(x, y), (x, y) \in \mathbb{R}^2$, with

$$u_{xx} = 0.$$

We can just integrate twice to get the general solution here.

$$\begin{aligned} u_{xx} &= 0 \\ u_x &= f(y) \\ u &= xf(y) + g(y). \end{aligned}$$

This is clearly the general solution as we just integrated twice. Now, the question is how much information do we need to provide for there to be only one solution to the IVP? For this, if we specify

$$u(0, y) = H(y) \qquad \text{and} \qquad u_x(0, y) = L(y),$$

then this would give that

$$u(x, y) = L(y)x + H(y),$$

which is sufficient.

Example 3.5 (267 Exposure Therapy)**.** "I just want to test your ODE knowledge."

$$u_{xx} + u = 0.$$

Solutions are

$$u = a(y)\sin x + b(y)\cos x,$$

He then started talking about like factoring polynomials and exponentials to find the solution to the ODE but I fainted from the experience, so I'm not sure what he did.

Remark 3.6. When drawing, functions of t and x, make t the vertical axis, and x the horizontal axis. I don't know why. Also, he writes $u(t, x)$ but the book writes $u(x, t)$. I personally prefer $u(t, x)$, but I might end up using both accidentally.

Example 3.7. Let $u = u(t, x)$ with

$$u_{tt} - u_{xx} = 0. \tag{w}$$

Then we can write this as

$$(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0.$$

If we know $(\partial_t + \partial_x)u = 0$, then u is a solution of (w), but this can be solved by $F(t - x)$, for all function F. Similarly, if $(\partial_t - \partial_x)u = 0$, then u solves (w), which would yield solutions $G(t + x)$ of (w), for all functions G.

It turns out that

$$u(t, x) = F(t - x) + G(t + x)$$

is actually the general solution—we will prove this later.

We saw that solving a PDE gives infinitely many degrees of freedom. So how do we pose a PDE so that we can identify a specific solution? Answer: Give additional info in the form of (A) initial conditions, (B) boundary conditions. This motivates the following definition:

Definition 3.8 (Well-Posed Problem)**.** We say that a Well-Posed Problem is a PDE along with Boundary Conditions and Initial Conditions such that the solutions have

1. Existence of Solutions
2. Uniqueness of Solutions
3. Continuous dependence[7] on this additional data.

Example 3.9 (Poisson's Equation)**.** Let u be the unknown, and let our equation be

$$-\Delta u = f.$$

Then there's two ways to make it well-posed. If we say that $u = u(x)$, and $x \in D \subseteq \mathbb{R}^m$, and $g : \partial D \to \mathbb{R}$, then we can either do

1. Let $u = g$, $x \in \partial D$. These are the Dirichlet Boundary Conditions [a.k.a. The Dirichlet Circle Problem]

[7] I felt a visceral sense of dread typing this.

2. Let
$$\frac{\partial u}{\partial \hat{n}} = \nabla u \cdot \hat{n},$$
where $\hat{n}$ is the unit outward normal of ∂D. Then the Neumann boundary conditions are
$$\frac{\partial u}{\partial \hat{n}} = g,$$
where $x \in \partial D$

3. The Robin boundary conditions are
$$\frac{\partial u}{\partial \hat{n}} + Qu = g$$
where $x \in \partial D$.

For the Neumann boundary conditions we need some care to prove this is well-posed, which will appear in the homework. Giving both the Dirichlet and Neumann boundary data is not a good idea. One might remark that this seems strange—after all, it's a second-order PDE, so it might look like (as with a second-order ODE) we'd have to specify stuff about both the function and the derivatives. But we will see this is not the case.

Example 3.10 (Dirichlet's Boundary Conditions)**.** Consider Poisson's equation along with the Dirichlet boundary conditions:
$$\begin{aligned} -\Delta u &= f & (x \in D) \\ u &= g & (x \in \partial D) \end{aligned}$$
Are solutions of this problem unique? Assume there exist u_1, u_2 solutions, then look at $v = u_1 - u_2$. Then this yields
$$\begin{aligned} -\Delta v &= 0 & (x \in D) \\ v &= 0 & (x \in \partial D) \end{aligned}$$
The idea is to integrate our PDE against v. First, notice the identity that
$$\begin{aligned} \int_D \Delta v \cdot v \, dx &= \int_D \sum_{i=1}^m v \, \partial_{x_i}^2 v \, dx \\ &= \int_D \sum_{i=1}^m \partial_{x_i} \left(v \, \partial_{x_i} v \right) dx - \int_D \sum_{i=1}^m \partial_{x_i} v \cdot \partial_{x_i} v dx. \\ &= \int_D \nabla \cdot (v \, \nabla v) - \int_D \nabla v \cdot \nabla v \, dx. \end{aligned}$$
Using this, we can derive
$$\begin{aligned} 0 &= \int_D v \Delta v \, dx \\ &= \int_D \nabla \cdot (\nabla v) \, v \, dx \\ &= \int_D \nabla \cdot (\nabla v \, v) \, dx - \int_D \nabla v \cdot \nabla v \, dx \qquad (1) \\ &= \int_{\partial D} \Delta v \cdot v \cdot \hat{n} \, dx - \int_D |\nabla v|^2 dx \end{aligned}$$

Since $v \equiv 0$ on the boundary, $\Delta v \cdot v \cdot \hat{n} \equiv 0$ on the boundary, so it follows that $\int_D |\nabla v|^2 dx = 0$. It follows that $\nabla v \equiv 0$ on D, thus v is constant. But we know $v \equiv 0$ on ∂D so $v \equiv 0$ on the whole domain. This proves uniqueness.

Example 3.11 (Vibrating String)**.** Let $D = [0, \infty) \times [0, L]$. A vibrating string can be modelled by the laws

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(t, 0) &= w_1(t) \\ u(t, L) &= w_2(t) \end{aligned}$$

where $u : D \to \mathbb{R}$ [a longer derivation of these laws can be found in Stein and Shakarchi Book 1], where $w_1, w_2 : [0, \infty) \to \mathbb{R}$ are how the end-points of the string move. It turns out this is not enough to get a unique solution. We also need

$$u(0, x) = f(x), \qquad u_t(0, x) = g(x).$$

He did not show this, but he showed show in the case of Laplace's equation, these boundary conditions don't yield a well-posed problem.

Example 3.12 (Laplace's Equation)**.** Suppose we have $u : \mathbb{H}^n \to \mathbb{R}$. Then impose the initial conditions

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x, 0) &= 0 \\ u_y(x, 0) &= \frac{1}{m} \sin(mx). \end{aligned}$$

Notice that this is analogous to providing both the Dirichlet and Neumann boundary conditions for Poisson's equations above (but it's not, because it's not bounded). This is a sequence of PDEs, which we'll call $u^{(m)}$ the solution.

As $m \to \infty$ the boundary conditions go to 0, so they roughly go to the solution $u \equiv 0$. If we solve for m large, then our solution should be close to 0 to get continuous dependence. But is this the case?

We can solve this by ODEs like we did earlier (rewriting as $(\partial_t - i\partial_x)(\partial_t + i\partial_x)u = 0$ then solving each root individually—he left actually doing this as an exercise). This gives solutions of

$$u^{(m)} = \frac{e^{-\sqrt{m}}}{m^2} \sin(mx) \sinh(my).$$

The question is whether this goes to 0 as $m \to \infty$. But notice that sinh is basically exponential, so this is going to go to infinity everywhere. Thus, in this case, continuous dependence on boundary conditions doesn't hold, so this isn't a well-posed problem.

§4 Day 4: Solving some linear PDEs (Sep. 15, 2025)

Well-Posed Problems have three parts

- Existence, which we discussed in the last class, and non-existence, which we saw in the homework
- Uniqueness, which we saw examples and a proof of last class
- Continuous dependence on boundary data, which we saw a counterexample of last week for the Laplace problem.

We will mostly study well-posed problems almost all the time. We'll have to prove that we have well-posedness for a given problem (but we might have to leave this for another class if it's too hard). [Also, one can ask what's called a regularity question—this won't come up in this class, but it's about when we can prove a weak solution is continuous or differentiable or smooth or etc. This can be defined precisely using methods of 436/457/458.]

First order PDEs (we are doing $\mathbb{R}^2$ for now) can be classified in a couple different ways.

1. Linear homogeneous, with constant coefficients, which is of the form
$$au_x + bu_y + cu = 0,$$
2. Linear homogeneous with non-constant coefficients, which is of the form
$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0$$
3. Linear non-homogeneous, which is of the form
$$a(x,y)u_x + b(x,y)u_y = f(x,y)$$
4. Semilinear, of the form
$$au_x + bu_y = f(u),$$
where f is nonlinear.
5. Quasilinear, which we will study rigorously later. An example is
$$u_x + uu_y = 0.$$
[This is where the regularity problem I discussed above can pop up.]

We will now solve these.

1. Previously, we mentioned that the solutions to $au_x + bu_y = 0$ are of the form
$$u(x,y) = F(bx - ay).$$
However, we will do this more rigorously now. We will add additional data (boundary/initial conditions) later—working backwards in some sense. There are three methods we can employ to solve this problem.

a) The geometric approach. This PDE says $\nabla u \cdot (a, b) = 0$. In other words, $\nabla u \perp (a, b)$. It follows that starting at any point, our function must be constant in the direction of (a, b). As such, to make the problem well posed, we just need to provide initial conditions along any (smooth) curve which is never tangent to the level sets. These level sets are called coordinate curves.

   One can deduce that $u(x, y) = F(bx - ay)$ is the general solution from this reasoning, but he did not do this explicitly.

   Example 4.1.
$$\begin{cases} au_x + bu_y = 0 \\ u(x, 0) = g(x) \end{cases}$$
   We know $u(x, y) = F(bx - ay)$, so
$$g(x) = u(x, 0) = F(bx),$$
   thus we have $F(x) = g\left(\frac{x}{b}\right)$, so
$$u(x, y) = g\left(x - \frac{a}{b}y\right).$$

b) Change coordinates—this method is very important in any PDE, as you might be able to reduce the dimensionality of your problem if you are clever. We will do this next time.

c) The general approach is the method of characteristics, which we will use later for semilinear, quasilinear, etc. We will also do this next time.

§5 Day 5: More on First-Order Equations (Sep. 17, 2025)

Problem set 2 due next Thursday, as the TAs are going to have office hours on Wednesday.

(1) We solved $au_x + bu_y = 0$ for $u = u(x, y)$, a, b constants by the geometric method last time.

(2) Let's use change of variables. Set $u = u(x, y)$ and set $v(x', y') = u(x, y)$, where x', y' represent a change of coordinates from x, y. Differentiating $u(x, y)$ yields

$$\begin{aligned} u_x &= v_{x'}\frac{dx'}{dx} + v_{y'}\frac{dy'}{dx} \\ u_y &= v_{x'}\frac{dx'}{dy} + v_{y'}\frac{dy'}{dy}. \end{aligned}$$

From here, set

$$\begin{cases} x' = ax + by \\ y' = bx - ay \end{cases}$$

Then we have that

$$\begin{cases} u_x = av_{x'} + bv_{y'} \\ u_y = bv_{x'} - av_{y'} \end{cases}$$

so plugging into our PDE yields

$$a(av_{x'} + bv_{y'}) + b(bv_{x'} - av_{y'}) = 0,$$

and then simplifying,

$$a^2 v_{x'} + b^2 v_{x'} = 0.$$

But if $a^2 + b^2 = 0$, then our PDE is quite boring, so assuming it is positive, this implies $v_{x'} = 0$. From here, we get that v is constant with respect to x', so

$$v = f(y') = f(bx - ay) = u(x, y),$$

which solves the PDE.

(3) The book *mentions* this method, but not really systematically, only briefly.[8] It's a more general method. Let's look at curves in the plane $\gamma(s) = (x(s), y(s))$ in parametric form.[9] Look at u along this curve. We define

$$z(s) = u(x(s), y(s)),$$

then differentiate along this curve

$$\dot{z}(s) = \dot{x}(s)u_x(x(s), y(s)) + \dot{y}(s)u_y(x(y), y(s)).$$

Using the original ODE, we can get $\dot{z} \equiv 0$ if $\dot{x} \equiv a$ and $\dot{y} \equiv b$. If we can merely deduce what all the curves satisfying the system

$$\begin{cases} \dot{x} \equiv a \\ \dot{y} \equiv b \end{cases},$$

[8] See Evans 3.2 if you want the systematic development. Fair warning, Evans is not for the faint of heart.

[9] It gets a bit hard to follow, because he switches between x and y being functions, and being independent variables at times.

then we will have curves along which $\dot{z} \equiv 0$, so u is constant along these curves. We call these curves the characteristic curves of our PDE, and these equations are the characteristic ODEs. The solutions to these ODEs are

$$(x(s), y(s)) = (as + x(0), bs + y(0)),$$

which implies that the solution can be determined by values along all curves of this form. This is equivalent to the general solution provided by the other two methods.

Now, let's suppose that we have data on the x-axis. Suppose $y(0) = 0$. Then we have that

$$u(as + x(0), bs) = u(x(0), 0), \quad s \in \mathbb{R},$$

so it follows that $u(x, y) = u(x - \frac{a}{b}y, 0)$. This gives us a unique solution for any [sufficiently regular] data on the x-axis.

Example 5.1. Suppose we are given

$$\begin{cases} u_x + yu_y = 0 \\ u(0, y) = y^2 \end{cases}$$

Of the three methods, try to do the first two as an exercise. But we will use the method of characteristics once again. Set $z(s) = u(x(s), y(s))$. Then once again we have $\dot{z} = u_x\dot{x} + u_y\dot{y}$, so set

$$\begin{cases} \dot{x}(s) = 1 \\ \dot{y}(s) = y(s) \end{cases}$$

to get $\dot{z}(s) = 0$. These yield curves of the form

$$x(s) = s + x_0, \quad y(s) = y_0 e^s.$$

Rearranging, we can get that

$$y(x) = y_0 e^{x - x_0},$$

which makes things slightly nicer. From here, to solve the ODE, notice that $u(x, Ce^x) = u(0, C)$ and so $u(x, y) = f(ye^{-x})$. To solve f given our constraints, notice that

$$f(ye^0) = u(0, y) = y^2,$$

so our solutions are

$$u(x, y) = \left(ye^{-x}\right)^2 = y^2 e^{-2x}.$$

This is the outline of the general method. Find some way to make the function nice along curves z [they need not be constant, just solvable—if we had $\dot{z}(s) = s^2$ we could solve something like this], then determine what data is required to specify the characteristic curves (in the above, y_0 uniquely determined the curve), then find the curve that intersects (x, y) in terms of this data, and this will yield the value of u at this point.

Example 5.2.

$$u_x + 2xy^2 u_y = 0,$$

then we have that

$$\dot{x} = 1, \dot{y} = 2xy^2,$$

so we can get that

$$\frac{dy}{dx} = 2xy^2.$$

From here, we need to use our ODE knowledge to solve this (it's separable).

$$\begin{aligned}\int \frac{1}{y^2}\,dy &= \int 2x\,dx \\ \frac{-1}{y} &= x^2 + C \\ y &= \left(-x^2 + C\right)^{-1},\end{aligned}$$

and thus these are our characteristic curves, so u is constant along them. He then graphed these functions as C varied. You can do this on desmos if you want. Notably, its asymptotes are at $\pm\sqrt{C}$ and its y-intercept is $\frac{1}{C}$. It follows that $u(0, \frac{1}{C}) = f(C) = f_0(\frac{1}{y} + x^2)$

Remark 5.3. Let $u = u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^m$. In general, we can solve first order equations like

$$u_t + a(t,x)u_x + b(t,x)u = f(t,x). \tag{$*$}$$

By the same method as before, we want to reduce this into a system of ODEs. Set $z(s) = u(t(s), x(s))$. Then we have that

$$\begin{aligned}z'(s) &= t'(s)u_t(t(s),x(s)) + x'(s)u_x(t(s),x(s)) \\ &= f(t(s),x(s)) - b(t(s),x(s))u(t(s)x(s)) && \text{(plugging into $(*)$)} \\ &= f(t(s),x(s)) - b(t(s),x(s))z(s).\end{aligned}$$

Thus for $(*)$, we say that the system of characteristics is

$$\begin{cases} \dot{t} = 1 \\ \dot{x} = a(t,x) \\ \dot{z} = f(t,x) - b(t,x)z \end{cases}$$

We set $\dot{t} = 1$ because the parameterization of the path doesn't matter. This is a nonlinear system of ODEs, but it works.

Example 5.4 (Semilinear)**.** Find characteristics for

$$\begin{cases} u_t + 2tu_x = u^2 \\ u(0,x) = g(x) \end{cases}$$

To do this, we have that the system of ODEs is

$$\begin{cases} \dot{t} = 1 \\ \dot{x} = 2t \\ \dot{z} = z^2 \end{cases},$$

so if we let our initial data be $t(0) = 0$, $x(0) = x_0$, $z(0) = g(x_0)$, then solving yields

$$\begin{cases} t = s \\ x = t^2 + x_0 \\ z = \frac{g(x_0)}{1 - sg(x_0)} \end{cases}.$$

Rearranging, we have that

$$\begin{cases} s = t \\ x_0 = x - t^2 \\ u(t(s),x(s)) = \frac{g(x_0)}{1 - sg(x_0)} \end{cases},$$

so therefore
$$u(t,x) = \frac{g(x-t^2)}{1-tg(x-t^2)}.$$
This solution could blow up depending on the values of g, so just like nonlinear ODEs ($x' = 1 + x^2$), we have less assurances of nice properties when we don't have linearity.

§6 Day 6: Method of Characteristics (Sep. 22, 2025)

Theorem 6.1 (Non-Homogeneous 1st Order)**.** Non-homogeneous 1st-order PDEs are of the form

$$u_t + b\nabla u = f(t, x),$$

where $b \in \mathbb{R}^m$ (constant), $u : (t, x) \in \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, and $f : (t, x) \in \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$. Given the homogeneous case

$$v_t + b\nabla v = 0,$$

we have that the general solution is $F(x - bt)$.

We'll solve the transport equation with forcing. First, we will apply the method of characteristics. Let $(t(s), x(s), z(s))$ be the characteristic curves. Then we have that our system of ODEs is

$$\begin{cases} \dot{t}(s) = 1 & t(0) = 0 \\ \dot{x}(s) = b & x(0) = x_0 \\ \dot{z}(s) = f(t, x) & z(0) = g(x_0) \end{cases}$$

where the right side gives some additional data with which we can hopefully have a unique solution. It follows that $t(s) = s$ and $x(s) = x_0 + bs$. From here,

$$\dot{z}(s) = \frac{d}{ds} u(s, x_0 + bs) = f(x_0 + bs),$$

so it follows that

$$z(t) = \int_0^t f(s, x_0 + bs)\, ds = \mu(t, x_0 + bt) - \mu(0, x_0).$$

Rearranging gives

$$\mu(t, x_0 + bt) = \mu(0, x_0) + \int_0^t f(s, x_0 + bs)\, ds$$

thus we have that $(x = x_0 + bt)$

$$\mu(t, x) = g(x - bt) + \int_0^t f(s, x - b(t - s))ds.$$

Remark 6.2. Solution of non-homogeneous is the superposition of solution of homogeneous problem and the solution of the non-homogeneous problem without any data. Namely, the $g(x - bt)$ is the homogeneous solution, and the integral is the solution of the non-homogeneous without any data.

Remark 6.3. This is an example of Duhamel's principle.

$$u(t, x) = u_0 + \int_0^t v,$$

where u_0 solves the homogeneous problem, and v solves

$$\begin{cases} v_t + b \cdot \nabla v = 0 \\ v(t) = v(s) = f(s, x) \end{cases}.$$

Example 6.4. We will tackle

$$\begin{cases} u_t + uu_x = 0 \\ u(0,x) = x^2 \end{cases}$$

If we have constant coefficients, (constant c instead of u) then c would be the speed of transport. As such, this is a model for shocks/explosions. Compressible fluids and the like.

We will try the method of characteristics. Our system with data is

$$\begin{cases} \dot{t} = 1 & t(0) = 0 \\ \dot{x} = z & x(0) = x_0 \\ \dot{z} = 0 & z(0) = x_0^2 \end{cases}$$

so it has turned this Quasilinear first order PDE into a system of ODEs. This yields the solution

$$\begin{cases} t = s \\ x = sx_0^2 + x_0 \\ z = x_0^2 \end{cases}$$

From here we have to invert the relation. The above tells us that

$$u(s, sx_0^2 + x_0) = z(s) = x_0^2,$$

so we just need to find x_0 as a function of t and x. On Wednesday he is going to ask us for the answer, so be prepared (jk I know most of you will not be there).

§7 Day 7: Lots of Calculations (Sep. 22, 2025)

Remark 7.1 (Method of Characteristics)**.** For the PDE

$$P(t,x,u)u_t + Q(t,x,y)u_x = f(t,x,u)$$

the characteristic curves $(t(s), x(s))$ are given as solutions of

$$\begin{cases} \dot{t} = P(t,x,z) \\ \dot{x} = Q(t,x,z) \\ \dot{z} = f(t,x,z). \end{cases}$$

These are not necessarily level curves.

We will do second-order PDEs now. We start with the 2D case, for linear 2nd order. The general case is

$$\begin{aligned} a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} \\ + a_1(x,y)u_x + a_2(x,y)u_y + c(x,y)u = 0. \end{aligned} \qquad \text{(lower order terms)}$$

The PDE can have very different behavior depending on the coefficients a_{11}, a_{12}, and a_{22}. So first we will start with the constant coefficient case.

Let our a all be real values, and likewise u. Then our constant coefficient case is given by

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + \text{lower order} = 0. \qquad (*)$$

We will classify it based on the discriminant of the polynomial $a_{11}\lambda^2 + 2a_{12}\lambda + a_{22}$.

Theorem 7.2. By a linear change of independent variables, the equation $(*)$ can be writen in one of the 3 forms below:

1. If $a_{12}^2 - a_{11}a_{22} < 0$, then it's
$$u_{xx} + u_{yy} + \text{lower order} = 0.$$
2. If $a_{12}^2 - a_{11}a_{22} = 0$ then it's,
$$u_{xx} + \text{lower order} = 0.$$
3. If $a_{12}^2 - a_{11}a_{22} > 0$ then it's,
$$u_{xx} - u_{yy} + \text{lower order} = 0.$$

Zander pointed out that this is naturally motivated by algebraic properties of quadratic forms. He then said the word "sheaf" and I ran away to hide.

Proof. We will automatically disregard any lower order terms. We can rescale $a_{11} = 1$ without loss of generality (unless $a_{11} = a_{22} = 0$). Then we have

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0.$$

We can therefore rewrite this as

$$(\partial_x + a_{12}\partial_y)^2 u + (a_{22} - a_{12}^2)u_{yy} = 0.$$

From here, we're going to use a change of variables

$$\begin{cases} x = \xi \\ y = a_{12}\xi + \eta \end{cases}$$

which yields

$$\begin{cases} \partial_\xi = \partial_x + a_{12}\partial_y \\ \partial_\eta = \partial_y. \end{cases},$$

so we get that

$$\partial_\xi^2 u + b^2 \partial_\eta u = 0,$$

where $b^2 = (a_{22} - a_{12}^2)$. This gives us 1 and 2. For 3, set $b^2 = a_{12}^2 - a_{22} > 0$. PDE is

$$(\partial_x + a_{12}\partial_y)^2 u - b^2 u_{yy} = 0.$$

□

In HW 3 we will see problems on classification of PDEs and the relation to the characteristic polynomial $a_{11}\lambda^2 + 2a_{12}\lambda + a_{22}$.

For the general case of dimensions higher than 2, we look at the linear PDE

$$\sum_{i,j=1}^{m} a_{ij}(x)\partial^2_{x_i x_j} u + b(x)\nabla u + c(x)u = 0,$$

where $x \in \mathbb{R}^m$. From here we look at the Eigenvalues and all that stuff.

Theorem 7.3. We can classify according to the signature of the matrix $A = (a_{i,j})_{i,j=1}^m$. Notice that it is real symmetric, so[10] it's diagonalizable.

1. If all the eigenvalues have the same sign (nonzero), then the PDE is elliptic.
2. If all the eigenvalues have the same sign except one is zero, it's parabolic.
3. If all the eigenvalues have one sign, except one with the opposite sign, it's hyperbolic.

We don't classify all the possible cases, but these are the main things we see anyways. If we restrict it to 2 dimensions.

We'll start with the wave equation. Surely this won't be so hard. This is just the equation

$$u_{tt} - c^2 u_{xx} = 0.$$

We will

1. Find the general solution rigorously ($f(x+ct) + g(x-ct)$),
2. Find the solution to the IVP (D'Alembert's Equation)
3. Look at physical consequences (e.g. finite speed)
4. Look at a derivation of equation

Proposition 7.4. The general solution to the 1d wave equation is $u(x,t) = f(x+ct) + g(x-ct)$.

[10]by spectral theorem :-)

Proof. We have shown that this is a collection of possible solutions, but we haven't shown it's the general solution. To do this, there' a bunch of possible ways to do it, but we'll do it rigorously. There's two possible ways to go about this,

1. Using the method of change of variables, set
$$\begin{cases} \phi = x - ct \\ \psi = x + ct. \end{cases}$$
which yields
$$\begin{cases} \partial_\varphi = \frac{1}{2}\partial_x - \frac{1}{2c}\partial_t \\ \partial_\psi = \frac{1}{2}\partial_x + \frac{1}{2c}\partial_t. \end{cases}$$
"Why did I choose this? Good question. Anyways, [...]" - Fabio. We can rewrite this as
$$\begin{cases} 2c\partial_\psi = c\partial_x - \partial_t \\ 2c\partial_\psi = c\partial_x + \partial_t. \end{cases}$$
From here, it follows that our PDE is just
$$\partial_\phi\partial_\psi v = 0,$$
so it turns out that
$$\partial_\psi v = F(\psi),$$
and thus
$$v = f(\psi) + g(\phi),$$
which is exactly what we want.

2. For the method of characteristics, we are going to use the following trick:
$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ \partial_t^2 u - c^2 \partial_x^2 u &= 0 \\ \frac{\partial_t^2}{\partial_x^2} u - c^2 u &= 0 && \text{(Dividing by } \partial_x^2) \\ \lambda^2 u - c^2 u &= 0 && (\lambda = \tfrac{\partial_t}{\partial_x}) \\ \lambda^2 = c^2 \implies \lambda &= \pm c. && \text{(Dividing by } u. \text{ Wait, huh?)} \end{aligned}$$
This *does* work formally, although not for the (tongue-in-cheek) reasons I wrote. Fabio didn't formalize it himself, but my best guess is that the third line is using the chain rule, then the fifth line just uses the fact that if λ^2 and c^2 agree on every function, then they must be the same operator.

□

Theorem 7.5 (D'Alembert's Formula)**.** Consider the IVP for the wave equation.
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0, x) = \phi(x) \\ u_t(0, x) = \psi(x). \end{cases}$$
Then
$$u(t, x) = \frac{1}{2}\left(\phi(x + ct) + \phi(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$
[Fabio didn't write the formula until the end of the proof. I only wrote it earlier than he did for completeness.]

Proof. We know $u(t,x) = F(x+ct) + G(x-ct)$. From here, we will impose the initial conditions.

$$\begin{cases} u(0,x) = F(x) + G(x) = \phi(x) \\ u_t(0,x) = cF'(x) - cG'(x) = \psi(x), \end{cases}$$

so we have the equation

$$\begin{cases} F' + G' = \phi \\ F' - G' = \frac{1}{c}\psi \end{cases}$$

thus solving this yields

$$2F' = \phi' + \frac{1}{c}\psi \qquad 2G' = \phi' - \frac{1}{c}\psi.$$

From here, we can determine that

$$\begin{cases} F = \frac{1}{2}\phi + \frac{1}{2c}\int_0^x \psi(s)ds + A \\ G = \frac{1}{2}\phi - \frac{1}{2c}\int_0^x \psi(s)ds + B. \end{cases}$$

Since we need $F + G = \phi$, it follows that $A + B = 0$. We can therefore do

$$u(t,x) = \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(s)ds + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi(s)ds,$$

which yields a final formula of

$$u(t,x) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Notice that this proof simultaneously shows uniqueness, because at each step of the proof, our hand was forced. There was no step along the way at which we could've deviated and gotten a different final solution, so it must be the unique solution. We can show uniqueness independently as well, of course, this is just a quick way to verify it. □

We can deduce many properties of the solution of the wave equation from this formula. We can also deduce some abstractly.

Example 7.6. Consider the system

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,x) = e^{-x^2} \\ u_t(0,x) = 0. \end{cases}.$$

Then this is just solved by

$$u(t,x) = \frac{e^{-(x+ct)^2} + e^{-(x-ct)^2}}{2}.$$

Example 7.7. Consider the system

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,x) = \chi_{[-1,1]}(x) \\ u_t(0,x) = 0 \end{cases}.$$

By D'Alembert, this is just

$$u(t,x) = \frac{1}{2}\chi_{[-1,1]}(x+ct) + \frac{1}{2}\chi_{[-1,1]}(x-ct).$$

He then graphed out how this would look as time evolved.

This models the propagation of signals, or spacetime. If we look at the supports of a function subject to the wave equation over time, it will resemble this graph, which is why we can use it to model the above. A plot of it reveals that after a finite time, the signal will leave where it originated ($\frac{1}{c}$ seconds). We see that the edges of where the signal have propagated to form a sort of "light cone," which represents all the points in spacetime that the signal could have reached after the initial transmission. This is because there's a finite speed of propagation. Given the initial conditions, nothing can move faster than c, so we can call c the speed of light, or something like that. It's interesting stuff, but probably not required for the class lol.

§8 Day 8: Wave Equation (Sep. 29, 2025)

The wave equation is time-reversible.

Theorem 8.1 (D'Alembert's Equation)**.** The unique solution to the system

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

is given by

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.$$

Remark 8.2. We assumed some regularity for the data ϕ, ψ (for example, $\phi \in C^1$. Nevertheless, the formula makes sense for rougher ϕ and ψ.

Remark 8.3 (Trust me on this)**.**

1. When deriving identities or inequalities, we are allowed to think our functions are very nice. This is because reasonable functions can approximate most functions we care about, and then we can take the limit. Usually this works. [The Stone-Weierstrass approximation theorem tells us that actually almost any collection of functions you can think of that's closed under pointwise addition and multiplication will be dense in this sense.]
2. If $\phi \in C^2$ and $\psi \in C^1$, then $u \in C^2$, which makes it a perfectly valid solution of our function. We call this a classical solution. But, there are ways to say that rougher functions solve the PDE. Not in the classical sense, in some other sense. Most things in the world aren't necessarily smooth. So for the Burgers' equation and such we will see solutions in some other sense.

Remark 8.4 (Finite Speed)**.** If the data ϕ, ψ vanishes in $|x| > R$ (for some $R \in \mathbb{R}$), then $u(x,t)$ vanishes on the set

$$S = \{(t,x) : |x| > R + |ct|\}.$$

We can imagine this as drawing cones over every point in our initial data whose slope is $\frac{1}{c}$ along the boundary.

Let's choose a point (x_0, t_0). Which points in the plane "care" about the value of u at this point? We call this the domain of influence of (x_0, t_0). We also say this is the light cone of this region. It consists precisely of a (two-sided) cone centered around our point, with slope $\frac{1}{c}$. The one in the positive t direction is the forward light cone, and the one in the negative t direction is the backwards light cone.

We call the points on our initial data within the light cone of our point the domain of dependence, because outside of this region, the data would not have enough time to reach our point, and so the value of our point depends exclusively on the value of our function at this point.

Claim 8.5. The finite speed can be seen from the PDE alone without the explicit formula for the solution using "energy".[11]

Definition 8.6. The energy density is given by

$$\sigma(x,t) = \frac{1}{2}(u_t^2 + c^2|\nabla u|^2(x,t),$$

[11] Isaac losing his mind rn

so we have that the total energy of u at a time t is just integrating σ over $\mathbb{R}^d$. This gives

$$E(t) = \int_{\mathbb{R}^d} \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\, dx,$$

where we have that our wave is in $\mathbb{R} \times \mathbb{R}^d$. We say that u_t^2 is the kinetic energy, and $c^2|\nabla u|^2$ is the potential energy.

Claim 8.7. $\dot{E}(t) \equiv 0$.

Proof.

$$\frac{d}{dt}E(t) = \int_{\mathbb{R}^d} \frac{d}{dt}\frac{1}{2}\left(u_t^2 + c^2|\nabla u|^2\right) dx$$

Notice that the interchange of the limit and derivative imposes some requirements. Both because we need some sort of smoothness of the integrand, but also because of problems with our integral diverging. To solve this, we can assume that the data of our PDE has compact support—call it P.

$$\begin{aligned}
&= \int_{\mathbb{R}^d} u_t u_{tt} + c^2 \nabla u \nabla u_t dx \\
&= \int_{\mathbb{R}^d} u_t u_{tt} - c^2 \Delta u u_t dx && \text{(Gauss)} \\
&= \int_{\mathbb{R}^d} u_t (u_{tt} - c^2 \Delta u) dx \\
&= 0. && \text{(Plug in Wave Equation)}
\end{aligned}$$

Recall that the multi-variable wave equation is

$$u_{tt} - c^2 \Delta u = 0,$$

which is where this last line derives. For the application of Gauss to get this last part, we need some domain for which our functions are identically 0 over the boundary. Fortunately, since we assumed our support is compact[12] we can find a region for which this works. He remarked that we will need this type of argument a lot, especially for Elliptic PDEs, so we should make sure we are very comfortable with it. □

[12] Fabio called it finite support, which I figure just means bounded. Of course this means its closure is compact, so it still works the same.

§9 Day 9: More on Wave Equation, Heat Equation (Sep. 29, 2025)

Let's look at the wave equation in $d+1$ dimensions. This is

$$\begin{cases} u_{tt} - c^2\Delta_x u = 0 & t \in \mathbb{R}, x \in \mathbb{R}^d \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

We defined the energy density $\sigma(x,t) = \frac{1}{2}|u_t|^2 + \frac{1}{2}c^2|\nabla u|^2$, and the total energy at the time t as

$$E(t) = \int_{\mathbb{R}^d} \sigma(x,t)dx.$$

We proved for sufficiently nice solutions that $\dot{E} \equiv 0$.

Theorem 9.1 (Finite Speed of Propagation)**.** If ϕ, ψ vanish for $|x| > R$ then $u(x,t) = 0$ for $|x| > R + ct$ where $t > 0$. If $d = 1$ this is just D'Alembert.

Proof. The past cone (the domain of dependence of (x_0, t_0)) is

$$|x - x_0| = c(t_0 - t). \qquad (0 \leq t \leq t_0)$$

Taking the gradient of the function

$$F(x,t) = |x - x_0| - c(t_0 - t)$$

yields the unit outward normal to our cone C. It is precisely

$$\left(x - x_0, -c\frac{t - t_0}{|t - t_0|}\right)\frac{1}{\sqrt{1 + c^2}}.$$

Define $R = x - x_0$ and write

$$\hat{m} = \left(\frac{R}{|R|}, c\right)\frac{1}{\sqrt{1 + c^2}}$$

on C. Then we have that

$$\partial_t\sigma(x,t) - \nabla \cdot p(x,t) = 0,$$

where

$$\sigma(x,t) = \frac{1}{2}|u_t|^2 + \frac{1}{2}c^2|\nabla u|^2, \qquad p(x,t) = c^2 u_t \nabla u.$$

This can be found just by plugging these in to the above and cancelling out. Call D the solid region of the cone between 0 and t. For the rest of this proof refer to this desmos graph for the sets. Then

$$\int_D \partial_t\sigma - \nabla \cdot p dV = 0,$$

but this is just the divergence (in both x and t) of the vector field $(-p, \sigma)$, so applying Gauss, this is

$$\int_{\partial D} (-p, \sigma) \cdot \hat{m}\, da,$$

where $\partial D = T \cup B \cup K$, where T is the top of our slice of the cone, B is the bottom slice, and K is the boundary connecting them (again, refer to our graph).

$$\begin{aligned}
\int_T (-p,\sigma)\cdot(0,\dots,0,1)dx &= \int_T \sigma dx \\
&= \frac{1}{2}\int_T u_t^2 + c^2|\nabla u(t)|^2 dx \\
\int_B (-p,\sigma)\cdot(0,\dots,-1)dx &= \frac{-1}{2}\int_B \psi^2 + c^2|\nabla\phi|^2 dx \qquad (*) \\
\int_K (-p,\sigma)\cdot\hat{m} &= \int_K -p\hat{m}_x + \sigma\hat{m}_t \\
&= \frac{1}{\sqrt{1+c^2}}\int_K -c^2 u_t \nabla u\cdot\frac{R}{|R|} + c\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right) da \\
\int_T \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2 &\le \int_B \frac{1}{2}\psi^2 + \frac{1}{2}c^2|\nabla\phi|^2 - \int_K \cdots.
\end{aligned}$$

We will show that our integral over K is positive to show that we can drop this term in the inequality. To do this, set $u_r = \nabla u\cdot\frac{R}{|R|}$, to represent the radial derivative of u. Then

$$\begin{aligned}
\frac{1}{c}I &= -cu_t u_r + \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2 \\
&= \frac{1}{2}(u_t - cu_r)^2 + \frac{c^2}{2}(|\nabla u|^2 - u_r^2) \\
&= \frac{1}{2}(u_t - cu_r)^2 + \frac{c^2}{2}\left|\nabla u - u_r\cdot\frac{R}{|R|}\right|^2 \\
&\ge 0.
\end{aligned}$$

It follows that
$$\int_T \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2 \le \int_B \frac{1}{2}\psi^2 + \frac{c^2}{2}|\nabla\phi|^2,$$
so if $\phi,\psi = 0$ on B, then $\int_T u_t^2 + |\nabla u|^2 dx = 0$. This implies $u \equiv 0$ on T so $u \equiv 0$ on the whole cone. Also, $u(x_0,t_0)$ is zero if data is zero on the base of the past cone of (x_0,t_0) (i.e. the domain of dependence). □

What we showed also implied finite speed property for $d+1$ dimensional wave equation. "No more cones for a while. We may have ellipses and spheres, but no cones. Well, maybe I'll say one thing..." Think about how the energy inequality implies uniqueness of our solution. Also, think about how $\dot{E}(t) = 0$ implies uniqueness.

Now let's start on the Heat/Diffusion equation in 1d. Recall this is

$$u_t - ku_{xx} = 0. \qquad (k > 0)$$

We may assign initial data $u(x,0) = \phi(x)$. We may also look at boundary conditions. Here, only take $t \ge 0$. For $t < 0$ the problem is generally ill-posed. Notice that if you change t to $-t$ this is not the same equation, because we are requiring $k > 0$.

We will solve it explicitly later. First, let's look at some properties that can be obtained only from the equation.

Theorem 9.2 (Weak Maximum Principle)**.** Consider the rectangle $0 \le x \le \ell$ and $0 \le t \le T$. Suppose we have a solution here. Namely, we have that $u_t = ku_{xx}$ in $(x,t) \in [0,\ell]\times[0,T] = R$. Then the max value for u in R is obtained on either $x = 0$, $x = \ell$ or $t = 0$.

Remark 9.3. The strong maximum principle says the max can only be on the boundaries mentioned (unless the function is constant). The weak maximum principle is provable easily, the strong maximum principle is very hard.

We will save the proof for next time. It's a basic trick in analysis. It's different in flavour from what we've done so far—no integration by parts or integrals at all.

We also have the min principle. Just apply the max principle to $-u$. We will also have this same idea for the elliptic equation.[13]

[13] There was something else at the end but I really had to pee so I wasn't paying attention at the very end. It was something about uniqueness in the heat equation and boundary conditions, but I figure he'll go over it again if it's important.

§10 Day 10: Heat/Diffusion Equation (Oct. 6th, 2025)

Theorem 10.1. Let u solve $u_t = ku_{xx}$, for $k > 0$ in the rectangle $R = [0, \ell] \times [0, T]$ with initial/boundary conditions

$$\begin{cases} u(x,0) = \phi(x) & 0 \leq x \leq \ell \\ u(0,t) = g(t) & 0 \leq t \leq T \\ u(\ell,t) = h(t) & 0 \leq t \leq T. \end{cases}$$

Notice here that the $u(0,t)$ and $u(\ell,t)$ terms represent some sort of "forcing," which represents either pumping in or removing heat or something of that sort.

Then we have that u attains its max somewhere along $B = \partial R \setminus \{(x,T) : 0 < x < \ell\}$.

Proof. If u has a max at a point p in the interior of R, then $u_t|_p = 0$ and $u_{xx}|_p \leq 0$. This is almost violating the PDE. The idea is to create a big of room.

Let $v(x,t) = u(x,t) + \varepsilon x^2$. Then v solves

$$v_t - kv_{xx} = u_t - ku_{xx} - 2\varepsilon k < 0,$$

if $\varepsilon > 0$. But then $v = \mu + \varepsilon x^2$, $\varepsilon > 0$ arbitrary cannot have an interior max Q, as otherwise, $v_t|_Q = 0$, $v_{xx}|_Q \leq 0$, which is a contradiction. Thus, v attains its max on the boundary.

Assume that v attains its max along the top. Specifically, suppose it attains a max at (x_0, T). It follows that

$$\lim_{\delta \to 0} \frac{v(x_0,T) - v(x_0, T-\delta)}{\delta} \geq 0.$$

Then we still get a contradiction to $v_t - v_{xx} = 0$.

To show u attains its max along the boundary, let $M = \max_R u$. We have therefore that $\max_R v = \max_B v$. This implies

$$\max_R u + \varepsilon x^2 = \max_B u + \varepsilon x^2$$

and thus

$$\max_R u \leq \max_B u + \varepsilon \ell^2,$$

for ε arbitrary. This implies that $\max_R u \leq \max_B u$. □

Remark 10.2. We also have a minimum principle, as discussed last time. The PDE is linear, so just negate our solution and apply the above.

From here, uniqueness follows and also continuous dependence on data.

Say that $g = h = 0$. Then u_j solves with $u_j(x,0) = \phi_j(x)$, $j = 1, 2$. Because $u_1 - u_2$ and $u_2 - u_1$ are solutions, we get

$$\max_R |u_1 - u_2| \leq \max_{0 \leq x \leq \ell} |\phi_1(x) - \phi_2(x)|.$$

This gives stability/continuous dependence in the sup norm.

Proposition 10.3. We can find the uniqueness/continuous dependence using the energy method. Define

$$E(t) = \frac{1}{2} \int_0^\ell u^2(x,t) dx.$$

Then we have

$$\begin{aligned}\dot{E}(t) &= \int_0^\ell u(x,t)u_t(x,t)dx \\ &= \int_0^\ell uku_{xx}dx \\ &= -k\int_0^\ell (u_x)^2(x,t)dx \leq 0.\end{aligned}$$

Then this gives uniqueness and continuous dependence. Uniqueness because given u_1, u_2 solving, $v = u_1 - u_2$ satisfies

$$\frac{1}{2}\int_0^\ell v^2(x,t)dx \leq \frac{1}{2}\int_0^\ell v^2(x,0)dx = 0,$$

so thus $u_1 = u_2$. Continuous dependence because given ϕ_1, ϕ_2 that solve, $v = u_1 - u_2$. Then we have by the same argument above that solutions depend continuously wrt the L^2 norm. Here, we have that the L^2 norm of ϕ is

$$||\phi||_{L^2} = \left(\int_0^\ell \phi^2(x)dx.\right)^{\frac{1}{2}}.$$

Remark 10.4. Continuous dependence in ODEs is nice and simple, because the initial conditions are just points, so just take the standard Euclidean norm. Furthermore, since most norms on $\mathbb{R}^n$ are equivalent, we don't have to worry too much about which norm the dependence is continuous with respect to. For PDEs, it's not quite the same. The L^∞ norm (sup norm) and the L^2 norm are indeed not equivalent, so when we talk about continuous dependence, we have to specify which norm it depends continuously upon. That is, even if solutions are continuously dependent on the data (which is a function) with respect to the L^2 norm, it might not depend continuously on the data with respect to the L^∞ norm.

§11 Day 11: Heat/Diffusion Again (Oct. 8, 2025)

We will consider the following equation where $x \in \mathbb{R}$ and $t \geq 0$. We have a fixed parameter $k > 0$.

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = \phi(x) \end{cases}$$

The goal is to find an explicit formula for the solution fo the IVP above. We will eventually state the full theorem.

The idea is to use invariance properties and symmetries of our equation to reduce it to ODEs. Here are some nice invariances of our equation. These are only invariances of the PDE and not the initial data.

1. If $u(x,t)$ is a solution, then this implies that $u(x-y,t)$ is a solution, for all $y \in \mathbb{R}$.
2. If u is a solution, then $\partial_x u$, $\partial_t u$, $\partial_{xx} u$ are also solutions.
3. Linear combinations of solutions are solutions (superposition principle).
4. If $S = S(x,t)$ is a solution, then
$$\int_{\mathbb{R}} S(x-y,t)\psi(y)dy$$
is also a solution, for any nice enough ψ.
Remark 11.1. The general solutions of many linear PDEs have solutions in the form
$$\int_{\mathbb{R}} K(x-y,t)\phi(y)dy,$$
for some K called the fundamental solution. This is called the convolution of $K(\cdot,t)$ and ϕ.
5. If u is a solution, then $u(\sqrt{a}x, at)$ is a solution for all $a > 0$.

Remark 11.2 (By me, not Fabio)**.** We are going to solve our PDE with initial conditions $u(x,0) = \chi_{\{x \geq 0\}}$. As usual, I use χ for the indicator function.[14] This is the distribution function for the Kronecker delta function. For those unaware, the Kronecker delta function is a generalized function δ such that

$$\int_a^b \delta(x)dx = \begin{cases} 1 & 0 \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

Obviously this function doesn't actually exist because it would have to be infinite height at 0, but using measure theory this can be made perfectly formal. In some sense as well, we have that $\partial_x \phi = \delta$. Not in the classical sense of course, because ϕ is discontinuous, but we *would* have that an analog of the fundamental theorem of calculus holds:

$$\int_a^b \delta(x)dx = \phi(b) - \phi(a).$$

This is idea behind the Radon-Nikodym derivative, for those curious.

But why do we care about all this fancy stuff? Suppose that we have some function $\phi : \mathbb{R} \to \mathbb{R}$. Then it would follow that

$$\phi(x) = \int_{\mathbb{R}} \delta(x-y)\phi(y)dy.$$

[14]If you object, please DM me how to write `\mathbb{1}` for future.

So if we can set $K(x,t)$ to have $K(x,0) = \delta(x)$ (whatever this means) and K solves our ODE (whatever *this* means), then by symmetry 4, we should have that the function

$$u(x,t) = \int_{\mathbb{R}} K(x-y,t)\phi(y)dy$$

solves our ODE, and also

$$u(x,0) = \int_{\mathbb{R}} K(x-y,0)\phi(y)dy = \int_{\mathbb{R}} \delta(x-y)\phi(y)dy = \phi(x).$$

In other words, u would satisfy the initial conditions $u(x,0) = \phi(x)$.

So we will start by trying to solve the PDE for the (mostly normal) function $u(x,0) = \chi_{\{x\geq 0\}}(x)$, then applying symmetry 2, we should get a solution for

$$v(x,0) = u_x(x,0) = \delta(x).$$

From here, we should have a formula to solve our PDE with any initial conditions. If we have done this, then we can retroactively verify it works, instead of having to formalize these middle steps. Alright, back to what Fabio said.

We will find the fundamental solution of $u_t - u_{xx} = 0$. Let's start by solving with initial conditions

$$u(x,0) = \chi_{\{x\geq 0\}}.$$

By symmetry 5, we look for $Q(x,t) = g(p)$ with $p = \frac{x}{\sqrt{4t}}$. Calculating the PDE for Q yields

$$\partial_t Q = g'(p)\frac{x}{\sqrt{4t}}\left(\frac{-1}{2t}\right) = \frac{-1}{2t}pg'(p).$$

Also,

$$\begin{cases} \partial_x Q = g'(p)\frac{1}{\sqrt{4t}} \\ \partial_{xx} Q = g''(p)\frac{1}{4t} \end{cases}$$

Which means Q solves if

$$\begin{aligned} -\frac{1}{2t}pg'(p) - \frac{1}{4t}g''(p) &= 0 \\ v' + 2pv &= 0 && (v = g') \\ v(p) &= e^{-p^2}C_1. \end{aligned}$$

So it follows that for some $C_1, C_2 \in \mathbb{R}$,

$$g(p) = C_1 \int_{-\infty}^{p} e^{-q^2}dq + C_2,$$

and therefore we can solve the PDE with

$$Q(x,t) = C_1 \int_{-\infty}^{\frac{x}{\sqrt{4t}}} e^{-p^2}dp + C_2.$$

Now, we want to find C_1, C_2 for the given data ϕ. Formally, take $t \searrow 0$, because we want $\lim_{t\searrow 0} Q(x,t) = \phi(x)$. Then

$$\begin{aligned} \lim_{t\searrow 0} Q(x,t) &= C_1\frac{\sqrt{\pi}}{2} + C_2 && (x > 0) \\ \lim_{t\searrow 0} Q(x,t) &= -\frac{C_1\sqrt{\pi}}{2} + C_2\left[\frac{\sqrt{\pi}}{2}\int_0^\infty e^{-p^2}dp\right] && (x < 0) \end{aligned}$$

$$\begin{cases} C_1\frac{\sqrt{\pi}}{2} + C_2 = 1 \\ -C_1\frac{\sqrt{\pi}}{2} + C_2 = 0 \end{cases}$$

and thus $C_2 = \frac{1}{2}$ and $C_1 = \frac{1}{\sqrt{\pi}}$. It follows that

$$Q(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4t}}} e^{-p^2} dp + \frac{1}{2}$$

is a solution. Therefore $\partial_x Q = S(x,t)$ is a solution, and

$$S(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

This is just the Gaussian. From here, for any data ϕ,

$$\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \phi(y) dy$$

is a solution.

Claim 11.3. This is $u(x,t)$ solving

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = \phi(x) \end{cases}$$

for all given ϕ ($t \geq 0$).

Proof. The first part is satisfied (well, we will be more rigorous later) by our properties above. The second part we have also verified in some sense, but we will be more rigorous here. Our question is whether

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \phi(y) dy \to \phi(x) \tag{$*$}$$

as $t \searrow 0$. To show it does, notice that

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \partial_x Q(x-y) \phi(y) dy \\
&= \int_{-\infty}^{\infty} -\partial_y Q(x-y) \phi(y) dy \\
&= [-Q(x-y)\phi(y)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} Q(x-y)\phi'(y) dy \\
&= \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y) dy
\end{aligned}$$

We assume that $\phi(\infty) = \phi(-\infty) = 0$ to make the boundary conditions cancel out. You should know ($*$) by heart.

Notice that ($*$) makes sense for all $t > 0$, when ϕ is not growing too fast at $y = \pm\infty$. This is because the first term (our Gaussian) becomes very very small, so it won't end up affecting the integral too much. Now, let's take

$$\begin{aligned}
\lim_{t \searrow 0} u(x,t) &= \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y) dy \\
&= \int_{-\infty}^{x} \phi'(y) dy = \phi(x)
\end{aligned}$$

□

As such we say that $S(x,t)$ is the fundamental solution of the heat equation on $\mathbb{R}$. Actually, $S(x,t)$ is singular as $t \searrow 0$ if $x = 0$ [as you should expect by my remark]. But it's smooth in x and if we integrate over $\mathbb{R}$, then for all $t > 0$,

$$\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} dx = 1.$$

and S solves $S_t - S_{xx} = 0$, when $t > 0$.

Remark 11.4. For the wave equation, singularities would be propagated, but for the heat equation, they are immediately smoothed out. Also, we have that the heat equation in some sense has infinite speed of propagation. To prove this, just look at the formula.

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \phi(y) dy.$$

If ϕ has compact support (also, suppose it's positive), then for any $x \in \mathbb{R}$, we can choose some $y \in \mathbb{R}$ such that $\phi(y)$ is nonzero, and although the integrand will be multiplied by $e^{-\frac{(x-y)^2}{4t}}$, this is strictly positive (albeit very small). Accordingly, every point on $\mathbb{R}$ will immediately be nonzero.

Theorem 11.5. Let ϕ be continuous and bounded. Then $(*)$ is a solution to

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = \phi \end{cases}$$

and moreover,

1. $u \in C^\infty(\mathbb{R} \times (0,\infty))$
2. $\lim_{t \searrow 0} u(x,t) = \phi(x)$ for all x.

Proof.

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}} \phi(x-z) dz && (z = x-y) \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{p^2}{4}} \phi(x - \sqrt{t}\, p) dp && (p = \tfrac{z}{\sqrt{t}}) \end{aligned}$$

We write convolutions as $f * g(x) = \int f(x-y)g(y)dy$. Now, let's argue that we can bring partial derivatives inside the integral.

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \phi(y) dy \\ |u(x,t)| &\le \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} |\phi(y)| dy \\ &\le \sup_y |\phi(y)|, \end{aligned}$$

so we want to show that

$$\partial_x u(x,t) = \int_{-\infty}^{\infty} \partial_x S(x-y,t) \phi(y) dy.$$

To do this, think of this as a limit

$$\begin{aligned} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} F(x,y) dy &\cong \lim_{m \to \infty} \int_{-\infty}^{\infty} F_m(x,y) dy \\ &= \int \lim_{m \to \infty} F_m(x,y) dy \end{aligned}$$

provided $F_m(y)$ converges as $m \to \infty$ uniformly and this limit is absolutely integrable. $\square$

§12 Day 12: Heat/Diffusion Part 3 (Oct. 15, 2025)

Theorem 12.1. Let ϕ be bounded and continuous, $\phi = \phi(x)$, $x \in \mathbb{R}$. Then the formula $u(x,t) = \int_{\mathbb{R}} S(x-y,t)\phi(y)dy$, where

$$S(z,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}},$$

is such that

1. $u \in C^\infty(\mathbb{R} \times (0,\infty))$,
2. u solves $u_t - u_{xx} = 0$.
3. $\lim_{t \searrow 0} u(x,t) = \phi(x)$ for all x.

Notably, even if ϕ is not smooth, u is.

Proof. We can write

$$u(x,t) = \int_{\mathbb{R}} S(z,t)\phi(x-z)dz.$$

This is just the convolution $S(\cdot,t) * \phi$. Now, if we set $z = \sqrt{t}p$, then this yields

$$u(x,t) \int_{\mathbb{R}} e^{-\frac{p^2}{x}} \phi(x - \sqrt{t}p)dp.$$

Notice that $|u(x,t)| \le \sup_y |\phi(y)|$.

1. To prove that $u \in C^\infty(\mathbb{R} \times (0,\infty))$, first we will state a nice general lemma.

   Lemma 12.2.
   $$\lim_{m\to\infty} \int_{\mathbb{R}} F_m(y)dy = \int_{\mathbb{R}} \lim_{m\to\infty} F_m(y)dy,$$
   provided that $\lim F_m(y)$ exists for all $y \in \mathbb{R}$, this limit is absolutely integrable, and $F_m \rightrightarrows F$.

   From here, notice that if $F_m(x) = \frac{u(x+h_m,t)-u(x,t)}{h_m}$, then applying this to suitable sequences h_m yields that

   $$\frac{\partial}{\partial x} u(x,t) = \int_{\mathbb{R}} \frac{\partial}{\partial x} S(x-y,t)\phi(y)\, dy$$

   To do this, let us check our conditions. For absolute integrability,

   $$\begin{aligned}
   \int_{\mathbb{R}} \frac{\partial}{\partial x} S(x-y,t)\phi(y)\, dy &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} -\frac{x-y}{2t} e^{-\frac{(x-y)^2}{4t}} \phi(y)dy \\
   &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \frac{1}{2\sqrt{t}} e^{-p^2} p\phi(x - p\sqrt{t})dp. \qquad \left(\tfrac{x-y}{\sqrt{t}} = p\right) \\
   \implies \left| \int_{\mathbb{R}} \frac{\partial}{\partial x} S(x-y,t)\phi(y)dy \right| &\le \frac{C}{t} \sup|\phi|.
   \end{aligned}$$

   This proves that this middle part is absolutely integrable. Also, by Taylor's formula,

   $$\begin{aligned}
   \left| \frac{S(z+h_m,t) - S(z,t)}{h_m} - \frac{\partial}{\partial z} S(z,t) \right| &= \left| \frac{\partial}{\partial z} S(z+\xi,t) - \frac{\partial}{\partial z} S(z,t) \right| \\
   &\qquad \text{(For some } 0 \le |\xi| \le |h_m|) \\
   &= |\xi| \cdot |S''(z+\xi_0,t)| \qquad (0 \le |\xi_0| \le |\xi|) \\
   &\le |h_m| \sup |S''(z,t)|
   \end{aligned}$$

and thus we have that our convergence is uniform. It follows that we can interchange the integral and limit. It follows that the first derivative of u exists. We can do similar proofs to show it is C^k for any k, so it is C^∞.

2. In the proof of (i) we showed that

$$\frac{\partial^2}{\partial x^2}u(x,t) = \int_{\mathbb{R}} \frac{\partial^2}{\partial x^2}S(x-y,t)\phi(y)dy.$$

We also showed that

$$\frac{\partial}{\partial t}u(x,t) = \int_{\mathbb{R}} \frac{\partial}{\partial t}S(x-y,t)\phi(y)dy$$

thus, since S solves the heat equation for $t > 0$,

$$u_t - u_{xx} = 0.$$

3. We want to show that $\lim_{t\searrow 0} u(x,t) = \phi(x)$.

$$u(x,t) - \phi(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi}}e^{-\frac{p^2}{4}}\phi(x-\sqrt{t}p) - \frac{1}{\sqrt{4\pi}}e^{-\frac{p^2}{2}}\phi(x)\,dp,$$

so we need to show that

$$\lim_{t\searrow 0}\int_{\mathbb{R}} e^{-\frac{p^2}{4}}\left[\phi(x-\sqrt{t}p) - \phi(x)\right]dp = 0.$$

To do this, we can just use the continuity of ϕ. It $\sqrt{t}p \ll 1$, this should be roughly 0 close to x, and $e^{-\frac{p^2}{4}}$ makes things small elsewhere. Specifically,

$$\int_{\mathbb{R}} e^{-\frac{p^2}{4}}\left[\phi(x-\sqrt{t}p) - \phi(x)\right]dp = I_1 + I_2,$$

where we have that

$$\begin{aligned} I_1 &= \int_{|p\sqrt{t}|<\delta} e^{-\frac{p^2}{4}}\left[\phi(x-\sqrt{t}p) - \phi(x)\right]dp \\ I_2 &= \int_{|p\sqrt{t}|\geq\delta} e^{-\frac{p^2}{4}}\left[\phi(x-\sqrt{t}p) - \phi(x)\right]dp. \end{aligned}$$

Fix $\varepsilon > 0$ arbitrary. We want to show that there exists $t_0 > 0$ such that for all $t \leq t_0$ we have that

$$|I_1| < \frac{\varepsilon}{2}, \quad |I_2| < \frac{\varepsilon}{2}.$$

To do this, let δ be such that

$$|\phi(x-y) - \phi(x)| < \frac{\varepsilon}{2}, \qquad |y| < \delta.$$

a) To estimate I_1, notice that

$$\begin{aligned} |I_1| &\leq \int_{|\sqrt{t}p|<\delta} e^{-\frac{p^2}{4}}|\phi(x-\sqrt{t}p) - \phi(x)|\,dp \\ &\leq \int_{\mathbb{R}} e^{-\frac{p^2}{4}}\frac{\varepsilon}{2\sqrt{4\pi}}\,dp \leq \frac{\varepsilon}{2}. \end{aligned}$$

b) To estimate I_2,

$$\begin{aligned}|I_2| &\le \left(2\sup_{y\in\mathbb{R}}\phi(y)\right)\int_{|p\sqrt{t}|\ge\delta} e^{-\frac{p^2}{4}}\,dp \\ &= \left(2\sup_{y\in\mathbb{R}}\phi(y)\right)\int_{|p|<\frac{\delta}{\sqrt{t}}} e^{-\frac{p^2}{4}}\,dp \\ &= 4\sup|\phi|\int_{\frac{\delta}{\sqrt{t}}}^{\infty} e^{-\frac{p^2}{4}}\,dp.\end{aligned}$$

Given ε and δ, pick t_0 small enough such that

$$\int_{\frac{\delta}{\sqrt{t_0}}}^{\infty} e^{-\frac{p^2}{4}}\,dp < \frac{\varepsilon}{8\sup|\phi|},$$

so we are done.

□

Remark 12.3. We always said that $u(x,t) = S(\cdot,t) * \phi$. This is **a** solution to the IVP

$$\begin{cases} u_t = u_{xx} & t>0 \quad x\in\mathbb{R} \\ u=\phi & t=0 \quad x\in\mathbb{R}. \end{cases}$$

We said this because it's not unique. In particular, there exists u solving

$$\begin{cases} u_t = u_{xx} & t>0 \\ u(x,0)=0, \end{cases}$$

but with $u \not\equiv 0$. At the same time, under some conditions (see homework) we can get uniqueness.

Remark 12.4. $u_t = u_{xx}$ is ill-posed for $t<0$. To see this, consider the data

$$\frac{1}{m}\sin(mx) = \phi_m(x).$$

Then the solution is

$$u_m(x,t) = \frac{1}{m}\sin(mx)e^{-m^2t}.$$

If $t=-\varepsilon$, then $u_m(x,-\varepsilon) = \frac{1}{m}\sin(mx)e^{\varepsilon m^2}$, so there's no continuous dependence on data.

Example 12.5 (Diffusion with a source)**.** This is the non-homogeneous case.

$$u_t - u_{xx} = f(x,t).$$

We'll say that $t>0$, and that f has to be nice enough such that the following stuff still works. We want to find a solution using Duhamel's formula/operator method. To do this, we want to "think of the pde as an ode." To do this, recall how Duhamel works.

$$\begin{cases} u_t + Au = f \\ u(0) = \phi, \end{cases}$$

where A is a matrix (linear operator). Use integrating factors.

$$v = e^{At}u \implies v_t = e^{At}f.$$

This yields that

$$v(t) = \underbrace{v(0)}_{\phi} + \int_0^t e^{As} f(s)\, ds$$

$$u(t) = e^{-At}\phi + \int_0^t e^{-A(t-s)} f(s)\, ds.$$

In the ODE case, $e^{-At}\phi$ is the solution of the homogeneous problem. In the PDE case, this is given by

$$e^{-At\phi} \leftrightarrow S_t * \phi.$$

We can also write this as $e^{t\partial_x^2}\phi$ [I do hate this, sorry Fabio]. Thus, writing our Duhamel's formuls for PDEs yields

$$u(x,t) = S_t * \phi + \int_0^t S_{t-s}(\cdot) * f(\cdot, s) ds.$$

We will work with this more next time.

§13 Day 13: Heat/Diffusion Part 4 (Oct. 15, 2025)

Midterm: everything we convered in class up to and including section 3.5.

I was a bit late (why have you forsaken me TTC) so I missed the start. Notes were harder to infer because I missed the critical initial explanation. I've done my best to fill it in.

He had rewrote the formula for solutions of IVP. I presume before I arrived he discussed that we would apply Duhamel's to the case of ODEs, and then show that it extends naturally to the case of PDEs. I believe the correspondence between the PDEs and ODEs comes from essentially just ignoring the x derivatives. So the heat equation $u_t - u_{xx} = f$ gives $u_t - u = f$. Likewise, the wave equation $u_{tt} - u_{xx} = f$ gives $u_{tt} - u = f$. It is kind of intuitive that this should work, but I did not catch Fabio's justification of it.

With that aside, we started with the heat equation. Use Duhamel's on

$$u'(t) = u(t) + f(t)$$

to get $u(t) = e^t\phi + \int_0^t e^{t-s} f(s)ds$.

Translated to the PDE, in the above, the $e^t\phi$ gives the solution of homogeneous problem at time t. We solved the homogeneous heat equation as

$$\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * \phi,$$

where $*$ is convolution, so we would hope that in the Duhamel's formula, this substitutes for $e^t\phi$. But notice that actually

$$\begin{cases} u_t = u_{xx} + f(x,t) \\ u(x,0) = \phi(x) \end{cases}$$

is solved by

$$u(t) = S(t)\phi + \int_0^t S(t-s)ds,$$

where

$$S(t)g = \frac{e^{\frac{-x^2}{4t}}}{\sqrt{4\pi t}} * g,$$

so indeed we see that it is the case.

This is what the book called using the "operator method."

Let's do the same thing for the wave equation. I strongly recommend you check out page 77 and 78 of Strauss. I couldn't follow what Fabio was doing.

$$\begin{cases} u_{tt} - u_{xx} = f(x,t) \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

The ODE analogue is

$$\begin{cases} \ddot{u} + Bu = f(t) \\ u(0) = g \\ \dot{u}(0) = h \end{cases}$$

(where g and h are constants—this is the notation he used). Solving this requires we have some information about B. So let's assume $B > 0$. Then, we can solve this as roughly looking like sin and cos. Specifically, the solution of homogeneous problem is

$$\cos(Bt)g + \frac{\sin(Bt)}{B}h.$$

For the non-homogeneous problem, it's

$$u(t) = \cos(Bt)g + \frac{\sin(Bt)}{B}h + \int_0^t \frac{\sin(B(t-s))}{B} f(s)ds.$$

Notice that this is Duhamels, where $S(t) = \frac{\sin(Bt)}{B}h$. Specifically

$$u(t) = S'(t)g + S(t)\psi + \int_0^t S(t-s)f(s)ds.$$

Comparing with D'Alembert's, the homogeneous solution $\frac{\sin(Bt)}{B}h$ looks unrecognizable.

We will see that[15]

$$\cos(Bt)g \cong \frac{1}{2}(g(x+t) + g(x-t))$$

and

$$\frac{\sin(Bt)}{B}h \cong \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$$

Later we'll see why $\cos(i\partial_x t)g = \frac{1}{2}(g(x+t)+g(x-t))$. [He mentioned pset 5, but I forget what it was regarding.]

From here, Duhamel's gives

$$\frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)dyds = \frac{1}{2}\iint_{(x,t)} fdS.$$

[We have to prove this formula of course, but he didn't do this in class.]

Odd even extension to solve heat/wave on half-line. This is assigned reading.

Fabio then asked if we want him to manually write out the solution for heat with source on the whole line. The class said yes. Here it is (assuming he wrote it right and I copied it right).

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}} \cdot \frac{1}{\sqrt{4\pi(t-s)}} f(y,s)dyds.$$

Now, we just have to check that this is correct. A posteriori we have to show it makes sense. Of course we got here from theory so we know it is correct, but we still have to prove it.

[15] He did not write $\cong$, he just wrote $=$. This pains me, so I did not.

§14 Day 14: Conserved Quantities and Fourier Series (Oct. 22, 2025)

We talked about conserved quantities/energy. For example, energy in the wave equation and momentum in the wave equation. But we found both of these based on physical intuition. How do we find conserved quantities?

One way is that invariances of the equation (technically the Lagrangian) and conserved quantities have a correspondence (Noether's Theorem).

Connection back to ODEs is that we use the principle of least action [this is from classical mechanics and the calculus of variations], and then if the action is invariant under symmetries, then applying these symmetries to our solution will preserve their satisfying the principle of least action, so they will also be solutions.

Let's use a different method from what we previously had to prove that energy is conserved by the wave equation.

Example 14.1. Recall that the wave equation is $u_{tt} - u_{xx} = 0$. From here, we're going to basically guess a factor that should make things invariant.

$$\int_{-\infty}^{\infty} (au + bu_x + cu_t)(u_{tt} - u_{xx})dx = 0$$

But we're not actually guessing at *random* what this factor is. We can cancel out some factors based on the symmetries of the equations.

$$\begin{aligned} &\int_{-\infty}^{\infty} u_t u_{tt} - u_t u_{xx} dx \\ &= \int_{-\infty}^{\infty} \partial_t \frac{u_t^2}{2} + \partial_x u_t u_x dx \qquad \text{(IBP)} \\ &= \frac{d}{dt} \int_{-\infty}^{\infty} \frac{u_t^2}{2} + \frac{u_x^2}{2} dx \end{aligned}$$

Specifically, if $u(x,t) = u(x,t+h)$, then taking the derivative with respect to h yields

$$\left.\frac{\partial}{\partial h}\right|_{h=0} u(x,t+h) = u_t.$$

So using our Lagrangian, this motivates the choice of u_t.

From here, we will start something completely different. We haven't yet solved our wave and heat equations on a finite interval. Rather, we've only done it on the whole line. The goal is to solve it on some interval $x \in [0,\ell]$, and $t \in \mathbb{R}$ with the conditions

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ \text{Boundary Conditions at } x = 0, \ell \end{cases}$$

and a similar initial value problem for $u_t - u_{xx}$ with boundary conditions on an interval. [This method is going to be similar in theory to how we solved the heat equation on the whole line, where we find one particular solution, then combine it with itself in ways such that they approximate our initial conditions well.]

To do this, we will look for a solution of the wave equation in separated form

$$u(x,t) = X(x)T(t),$$

and we will impose that it solves the equation (only the equation, not the boundary condition). But this uses quite a clever trick [in my opinion—Fabio said it's boring].

$$X(x)T''(t) = c^2 X''(x)T(t)$$
$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)}$$

thus since the left side is a function of x and the right side is a function of t, they must both be constant. We will assume (and justify later) that the constant is negative. This gives

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\beta^2.$$

For later we will say $\beta^2 = \lambda$. But then if $X'' = -\beta^2 X$ and $T'' = -c^2\beta^2 T$, then we have that

$$\begin{cases} X(x) = C\cos(\beta x) + D\sin(\beta x) \\ T(t) = A\cos(c\beta t) + B\sin(c\beta t). \end{cases}$$

Now, we'll impose the Dirichlet boundary conditions (later we will use Neumann, Mixed and Robin boundary conditions). That is, $u(0,t) = 0 = u(\ell, t)$. It follows that $X(0)T(t) = 0 = X(\ell)T(t)$ so $X(0) = 0 = X(\ell)$, so

$$\begin{cases} X(0) = C = 0 \\ X(\ell) = D\sin(\beta\ell) = 0 \end{cases}$$

But we don't want to set $D = 0$ or else our solution sucks [that's the technical term] so we instead use this to find $\beta = \frac{\pi n}{\ell}$.

Remark 14.2. We found that $\beta = \frac{\pi n}{\ell}$. But $-\beta^2$ is an eigenvalue for the problem

$$\begin{cases} X'' = (-\beta^2)X \\ X(0) = 0 = X(\ell). \end{cases}$$

and so what we've done here is find the eigenfunction $D_n \sin\left(\frac{\pi n}{\ell}x\right)$.

Therefore we found that

$$u_n(x,t) = \sin\left(\frac{\pi n}{\ell}x\right)\left(A_n\cos\left(c\frac{\pi n}{\ell}t\right) + B_n\sin\left(c\frac{\pi n}{\ell}t\right)\right)$$

In particular, we have that linear combinations of these solutions will be a solution. Further, even an infinite series will be a solution, assuming the convergence is nice and yields a nice enough function.

What is left is that we have the make our function satisfy the initial conditions ϕ and ψ. So given a solution of the form

$$\sum_{m\geq 1} \sin\left(\frac{\pi n x}{\ell}\right)\left(A_n\cos\left(\frac{c\pi n}{\ell}t\right) + B_n\sin\left(\frac{c\pi n}{\ell}t\right)\right) \tag{$*$}$$

If we impose

1. $u(x,0) = \phi(x)$, then we have that

$$\phi(x) = \sum_{n\geq 1} A_n \sin\left(\frac{\pi n x}{\ell}\right) = \phi(x)$$

so if ϕ has a Fourier series representation as above for a specific A_m, we have found our A_m.

2. $u_t(x,0) = \psi$, then we have that

$$\psi(x) = \sum_{n\geq 1} \sin\left(\frac{\pi n x}{\ell}\right) \cdot \frac{c\pi n}{\ell} B_n.$$

These both follow just by plugging in $(*)$ into these equations in the place of u. In summary, if everything works out nicely, then we have solved initial + dirichlet boundary value problem for the wave, provided that these sums work nicely.

The only consideration remaining is to show that the constant from separating variables is negative. So recall that in the separation of variables, we got the eigenvalue problem

$$\begin{cases} X'' = kX \\ X(0) = 0 = X(\ell). \end{cases}$$

Let k be a constant. We said that our constant k should be negative. Why? Well if our constant is 0, then our solution is linear, and therefore to satisfy the boundary conditions, it must be identically 0. If the constant is positive, then we end up with hyperbolic sine and cosine.

$$X(x) = ae^{\gamma x} + be^{-\gamma x}.$$

But if $X(0) = a + b = 0$ and $X(\ell) = ae^{\gamma\ell} + be^{-\gamma\ell} = 0$, then we have that

$$ae^{2\gamma\ell} + b = 0,$$

so $e^{2\gamma\ell} = 1$, which constradicts $\gamma > 0$. Thus, our only possible solutions are $\gamma < 0$ and the trivial case of $\gamma = 0$.

Next we'll solve Neumann boundary conditions. The book covers Robin and Mixed conditions in great detail. These are $u_x(0,t) = 0$ and $u_x(\ell,t) = 0$. This boils down to basically the same thing, just with a different eigenvalue problem.

§15 Day 15: Boundary and IVPs and Fourier Series (Nov. 3, 2025)

Happy birthday to my Dad (not Fabio's). Midterm in class on Wednesday (Nov. 5 i think?) if you weren't aware. Usual room and all that. Also a review session TODAY (Nov. 3) from 5 to 7. We get to meet the one and only Matt.

Last time we used "formal" calculations to write down solutions for the problems

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \ell, t \in \mathbb{R} \\ u(0,t) = 0 = u(\ell,t) & t \in \mathbb{R} \\ u(x,0) = \phi(x) & u_t(x,0) = \psi(x) \quad 0 < x < \ell \end{cases} \tag{W}$$

which is the wave on an interval with Dirichlet boundary conditions (the second line), and

$$\begin{cases} u_t = k u_{xx} \\ u(0,t) = 0 = u(\ell,t) \\ u(x,0) = \phi(x) \end{cases} \tag{H}$$

which is the heat equation on an interval. We can consider boundary conditions other than the Dirichlet boundary conditions as well.

To solve the Dirichlet boundary conditions, using separation of variables and solving the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < \ell \\ X(0) = 0 = X(\ell) \end{cases}$$

yields an eigenfunction X for the eigenvalue λ of the system.

Via this method, we arrived at the solution for the Dirichlet boundary conditions

$$u(x,t) = \sum_{n \in \mathbb{Z}} \left[A_n \cos\left(\frac{cn\pi t}{\ell}\right) + B_n \sin\left(\frac{cn\pi t}{\ell}\right) \right] \underbrace{\sin\left(\frac{n\pi x}{\ell}\right)}_{X_n(x)}.$$

Using the same method as last time, we can also solve the heat equation with

$$u(x,t) = \sum_{n \in \mathbb{Z}} A_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \underbrace{\sin\left(\frac{n\pi x}{\ell}\right)}_{X_n(x)}.$$

Notice that in these two solutions, our X_n is the same. This is because in the first step, look at $u(x,t) = X(x)T(t)$ to get that when we plug into $u_t = k u_{xx}$,

$$XT' = kX''T \implies \frac{X''}{X} = \frac{T'}{kT} = -\lambda,$$

where $\lambda \in \mathbb{R}_{>0}$ is a constant. We justified why this constant must be negative last time. Since the $\frac{X''}{X} = -\lambda$ is the same in both, this is why our X_n is the same between question.

But what can λ possibly be? Well we need X to satisfy

$$\begin{cases} -X'' = \lambda X \\ X(0) = 0 = X(\ell) \end{cases}$$

for u to satisfy our boundary value problem. Thus, $\lambda = \left(\frac{\pi n}{\ell}\right)^2$. This implies $X(x) = \sin\left(\frac{\pi n}{\ell} x\right) = \sin\left(\sqrt{\lambda_n x}\right)$. We also have to solve for $T(t)$, but

$$T' = -k\lambda T \implies T(t) = Ae^{-k\lambda t} = Ae^{-k\left(\frac{\pi n}{\ell}\right)^2 t}.$$

This yields our solution of (H) above.

Now, let's solve the Schrödinger equation for fun. Recall that this is $u_t = iu_{xx}$. "It's like the heat, with $k \to i$." He then shouted loudly "k STILL HAS TO BE REAL!!" which scared me. Nevertheless, if we just naively plug in to the formula for the heat solution, we get a solution to the Schrödinger equation

$$u(x,t) = \sum_{n\in\mathbb{Z}} A_n e^{-i\left(\frac{n\pi}{\ell}\right)^2 t} \sin\left(\frac{n\pi x}{\ell}\right).$$

This solution is found by literally replacing every k with an i in our proof of the solution of the heat equation.

We can get uniqueness using the energy for the wave and heat equation on an interval. Of course for the Schrödinger equation we have to be more careful because it's complex valued. The Schrödinger equation has conserved mass

$$M(t) = \frac{1}{2}\int_0^\ell |u(x,t)|^2 dx.$$

Next week we will need to find the coefficients A_m and B_m, which will lead us to Fourier series.

§16 Day 16: Boundary Value Problems with a view towards Fourier Series (Nov. 10, 2025)

We did separation of variables to find solutions to initial and boundary value problems for the heat, wave, and Schrödinger equation. Our boundary conditions can be

1. Dirichlet, where $u(x,0) = u(\ell, 0) = 0$
2. Neumann, where $u_x(0,t) = u_x(\ell, t) = 0$
3. Robin (we will skip—but he might give up problems on it).

We also have initial conditions of $u(x,0) = \phi(x)$, where $0 \leq x \leq \ell$. We did Dirichlet boundary conditions last time, so let's do Neumann boundary conditions. In this case, we will consider the wave equation $u_{tt} = c^2 u_{xx}$.

Example 16.1. Suppose $u_x(0,t) = u_x(\ell,t) = 0$. Now, let's look for a separable solution. $u(x,t) = X(x)T(t)$. Plugging into our PDE yields

$$\frac{X''(x)}{X(x)} = -\lambda, \qquad \frac{T''(t)}{c^2 T(t)} = -\lambda,$$

where λ is a positive contant. Thus,

$$\begin{cases} -X'' = \lambda X & 0 < x < \ell \\ X'(0) = 0 = X'(\ell) \end{cases} \tag{EP}$$

If $\lambda = \beta^2$, then the solution to this ODE (without the boundary conditions) is

$$X(x) = A\cos(\beta x) + B\sin(\beta x),$$

for some constants A, B. From here, differentiating yields

$$\begin{cases} X'(0) = \beta B\cos(0) - A\beta\sin(\beta 0) = 0 \\ X'(\ell) = \beta B\cos(\beta\ell) - A\beta\sin(\beta\ell) = 0, \end{cases}$$

so therefore $B = 0$, so as long as $A \neq 0$, $\beta\ell = n\pi$ for some $n \in \mathbb{N}$. Thus, $\beta = \frac{n\pi}{\ell}$. As usual, always think of X as an eigenvector, and λ as an eigenvalue.

It follows that the eigenvalue problem (EP) has solutions

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \qquad X_n(x) = \cos\left(\frac{n\pi}{\ell}x\right).$$

We will solve for T now. The eigenvalue problem for T is

$$-T'' = \lambda c^2 T,$$

and so solving gives

$$u(x,t) = \sum_{n\geq 1} \cos\left(\frac{n\pi}{\ell}x\right)\left[A_n\cos\left(\frac{n\pi c}{\ell}t\right) B_n\sin\left(\frac{n\pi c}{\ell}t\right)\right].$$

But something is missing here. This is correct, but not fully complete. This is because we only considered $\lambda > 0$. This was justified for the Dirichlet boundary conditions, because the other cases of λ yielded no solutions. But let's ask if the same is the case here. When $\lambda < 0$, then we have $X'' = aX$ for $a > 0$, but then it's impossible to satisfy our boundary conditions $X'(0) = X'(\ell) = 0$ (if X is nontrivial). On the other hand, if $\lambda = 0$, then a linear function satisfies $X'' = 0$. To make it satisfy our boundary conditions $X'(0) = X'(\ell) = 0$, we have to have X be constant. This yields an ode $T'' = 0$, which would give $T(t)$ is linear. Thus, our general solution would be

$$u(x,t) = \left(\sum_{n \geq 1} \cos\left(\frac{n\pi}{\ell} x\right) \left[A_n \cos\left(\frac{n\pi c}{\ell} t\right) B_n \sin\left(\frac{n\pi c}{\ell} t\right) \right] \right) + A_0 + B_0 t.$$

Example 16.2 (Mixed Boundary Conditions)**.** The mixed boundary conditions are

$$X(0) = 0 = X'(\ell).$$

These are distinct from Robin boundary conditions, which are $au(0,t) + bu_x(0,t) = 0$. For Schrödinger, Heat, and Wave, you always end up with the same ODE for X, so you should know the general solution.

$$\begin{cases} -X''(x) = \lambda X(x) \\ X(0) = 0 = X'(\ell). \end{cases}$$

But then this is solved by (assuming $\lambda = \beta^2$)

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

Now, this implies that $A = 0$, and so if $X'(\ell) = 0$, this implies

$$0 = X'(\ell) = B\beta\cos(\beta\ell),$$

so assuming $B \neq 0$, we have $\beta\ell = n + \frac{\pi}{2}$ for some $n \in \mathbb{N}$. Thus,

$$X(x) = B\sin\left(\left(n + \frac{1}{2}\right)\frac{\pi}{\ell} x\right).$$

Robin is a bit harder he claims.

This will have to be done on the midterm, in the problems, etc.

We saw that we can solve Heat/Wave/Schrödinger equations by separating variables and solving the Eigenvalue problem, and we arrive at the solution u as a series of $\sin(\beta_n x)$ and/or $\cos(\beta_n x)$ times T_n. But we still haven't solved the IVP part ($u(x,0) = \phi(x)$). We will do this next time.

§17 Day 17: Fourier Series (Nov. 12, 2025)

Before we've solved our 1+1 dimensional wave, heat, and Schrödinger boundary value problems by writing our solutions as a series of sin, cos, and so on. For example,

$$\begin{cases} u_t = ku_{xx} \\ \text{boundary conditions at } x = 0 \text{ and } x = \ell \end{cases}$$

is solved by

$$u(x,t) = \sum_{n \geq 0} e^{-\lambda_m kt} \left[A_n \cos\left(\sqrt{\lambda_n} x\right) + B_n \sin\left(\sqrt{\lambda_n x}\right)\right],$$

where λ_n is such that $-X'' = \lambda_n X$ has nontrivial solutions which satisfy our boundary conditions.[16]

Now, we will focus on the initial value problem. Let's forget about the PDE for now, and just consider how to solve the initial value problem

$$u(x,0) = \sum_{n \geq 0} \left[A_n \cos\left(\sqrt{\lambda_n x}\right) + B_n \sin\left(\sqrt{\lambda_n x}\right)\right],$$

for a given sequence of λ_n. We will look at a theoretical/general approach to the series using real and also functional analysis.

The sine seies (dirichlet boundary conditions gave us this) is

$$\sum_{n \geq 1} A_n \sin\left(\frac{\pi n}{\ell} x\right) = \phi(x).$$

If this is true, what should our A_n be? Well multiply both sides by $\sin\left(\frac{\pi m}{\ell} x\right)$ and integrate from 0 to ℓ.

$$\int_0^\ell \phi(x) \sin\left(\frac{m\pi}{\ell} x\right) dx = \sum_{n \geq 1} A_n \int_0^\ell \sin\left(\frac{n\pi}{\ell} x\right) \sin\left(\frac{m\pi}{\ell} x\right).$$

Our claim is that

$$\int_0^\ell \sin\left(\frac{n\pi}{\ell} x\right) \sin\left(\frac{m\pi}{\ell} x\right) dx = \frac{\ell}{2} \delta_{nm}.$$

This can be proven by showing that

$$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y),$$

so integrating this will yield 0 since it is periodic and even. You can prove this using the polarization identity for sin and cos. Since

$$\begin{aligned} \int_0^\ell \phi(x) \sin\left(\frac{m\pi}{\ell} x\right) dx &= \sum_{n \geq 1} A_n \int_0^\ell \sin\left(\frac{n\pi}{\ell} x\right) \sin\left(\frac{m\pi}{\ell} x\right) dx \\ &= \sum_{n \geq 1} A_n \delta_{n,m} \cdot \frac{\ell}{2} \\ &= A_m \frac{\ell}{2}. \end{aligned}$$

[16] Depending on our boundary conditions, we might be able to solve $X'' = \lambda_n X$ with $\lambda_n > 0$.

It follows that

$$A_m = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{m\pi}{\ell}x\right)dx,$$

so we have found our coefficients A_n. We can do the same this with cosine using a similar product identity. If

$$\phi(x) = \sum_{n\geq 0} B_m \cos\left(\frac{n\pi}{\ell}x\right),$$

then by almost the exact same argument,

$$B_m = \frac{2}{\ell}\int_0^\ell \cos\left(\frac{m\pi}{\ell}x\right)\phi(x)dx.$$

Example 17.1. Suppose that $\phi(x) \equiv 1$. Then

$$A_n = \frac{2}{\ell}\sin\left(\frac{n\pi}{\ell}xdx\right) = \frac{2}{\ell}\left[-\cos\left(\frac{n\pi}{\ell}x\right)\frac{\ell}{n\pi}\right]_0^\ell = \frac{2}{n\pi}\left(1-\cos(n\pi)\right).$$

Thus,

$$1 = \sum_{n \text{ odd}} \frac{4}{\pi n}\sin\left(\frac{\pi n}{\ell}x\right) = \sum_{n\geq 0}\frac{4}{\pi(2n+1)}\sin\left(\frac{\pi(2n+1)}{\ell}x\right).$$

In the problem set we use Fourier series methods to solve some interesting questions in number theory, such as the Basel problem.

Example 17.2. Set $\phi(x) = x$. Then we want to find the sine series for ϕ. This is just

$$\begin{aligned}
A_n &= \frac{2}{\ell}\int_0^\ell \sin\left(\frac{n\pi}{\ell}x\right)xdx \\
&= \frac{2}{n\pi}\left[-\cos\left(\frac{n\pi}{\ell}x\right)\cdot x\right]_0^\ell + \frac{2}{n\pi}\int_0^\ell \cos\left(\frac{n\pi}{dx}x\right)dx \\
&= \frac{2}{n\pi}\left[-\cos\left(n\pi\right)\ell\right] + \frac{2}{n\pi}\left[\sin\left(n\pi\right)\cdot\frac{\ell}{n\pi}\right] \\
&= \frac{-2\ell}{n\pi}\cos(n\pi) + \frac{2}{n\pi}\cdot\frac{\ell}{n\pi}\sin(n\pi) \\
&= \begin{cases} \frac{-2\ell}{n\pi} & n \text{ even} \\ \frac{2\ell}{n\pi} & n \text{ odd.} \end{cases} \\
&= (-1)^{n+1}\frac{2\ell}{n\pi}
\end{aligned}$$

So the sine series is $A_n = \frac{(-1)^{n+1}2\ell}{n\pi}$, and the cosine series is $\frac{-4\ell}{n^2\pi^2}$, when n is odd, and 0 otherwise. This second one you have to verify for yourself, but it's a similar calculation.

Suppose that we are given a $\phi(x)$ on $[-\ell, \ell]$. Then we will express it as the full series

$$\phi(x) = \frac{A_0}{2} + \sum_{n\geq 1} A_n\cos\left(\frac{\pi n}{\ell}x\right) + B_n\sin\left(\frac{\pi n}{\ell}x\right).$$

To find A_m and B_m, we apply the same trick, integrating against $\cos\left(\frac{m\pi}{\ell}x\right)$ and $\sin\left(\frac{m\pi}{\ell}x\right)$ on $-\ell < x < \ell$. With the previous claims plus

$$\int_{-\ell}^{\ell}\sin\left(\frac{\pi a x}{\ell}\right)\cos\left(\frac{\pi b x}{\ell}\right)dx = 0,$$

for all $a, b \in \mathbb{Z}$. This leads us to a general formula for our coefficients that

$$\phi(x) = \frac{A_0}{2} + \sum_{n\geq 1} A_n \cos\left(\frac{\pi n x}{\ell}\right) + B_n \sin\left(\frac{\pi n x}{\ell}\right),$$

where

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\phi(x)dx,$$
$$B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)\phi(x)dx.$$

Example 17.3. Consider $\phi(x) = x$ on the interval $[-\ell, \ell]$ rather than $[0, \ell]$. Then we need the full Fourier series. Then

$$x = \frac{A_0}{2} + \sum_{n\geq 1} A_n \cos\left(\frac{\pi n}{\ell}x\right) + \sum_{n\geq 1} B_n \sin\left(\frac{\pi n}{\ell}x\right).$$

But ϕ is odd, so all of the A_n cancel out, including $\frac{A_0}{2}$. Now, since ϕ is odd and sin is odd, we have that their product is even, so

$$\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)\phi(x)dx = 2\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\phi(x)\right)dx.$$

But since our formula for B_n in the full Fourier series differs by a factor of 2 from our B_n in the Sine series, we have that the coefficients of $\phi(x) = x$ on $[0, \ell]$ is the same as the coefficients of $\phi(x) = x$ on $[-\ell, \ell]$. In other words,

$$B_n = (-1)^{n+1}\frac{2\ell}{n\pi}.$$

Another way to think about it is graphically, where since ϕ is odd and our sine series is odd, keeping our coefficients the same will yield ϕ on our full interval.

I just remembered that the subsection command exists, so expect the notes to henceforth be a bit more organized.

He talked a bit about extending ϕ using even, odd, and periodic extensions.

§18 Day 18: Fourier Series Part 2. (Nov. 12, 2025)

Solving PDEs on bounded domains (intervals for now) with boundary conditions led to Fourier Series.

Example 18.1. We would like to represent a function $f(x) = \sum_{n\in\mathbb{Z}} c_n X_n(x)$ where X_m solves the Eigenvalue Problem

$$\begin{cases} -X_n'' = \lambda_n X_n \\ \text{Boundary Conditions} \end{cases}.$$

Remark 18.2. If we want to write $f(x) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi}{\ell}x\right)$,, then we can do this by integrating against $\sin\left(\frac{m\pi}{\ell}x\right)$ so long as we can switch the integral with our sum. This will yield

$$c_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{\pi n}{\ell}x\right)dx.$$

Furthermore, if this works, then it's the unique coefficient. This used the orthogonality of our sine functions.

If we are given f which is a function on $[0, \ell]$ and define

$$c_n = \int_0^\ell f(x)X_n(x)dx,$$

where X_n satisfies our eigenvalue problem. Is it the case that

$$f(x) = \sum_n c_n X_n(x)?$$

Well there's a limit here in our sum, so we have to answer with respect to which function space are we considering the convergence? From here we can ask which class of functions we can pick f from to make it work. Notice that it's also possible that our sum converges, but not to f. Let's start with a version that is a bit abstract.

§18.1 Orthogonality and Symmetric Boundary Conditions

Fiven two functions f, g on $[a, b]$, define an inner product for functions

$$(f, g) = \int_a^b f(x)g(x)dx.$$

[He did indeed use the round parentheses here. Come on Fabio.] If we are using Complex Valued functions, we should use the conjugate on the second function to make it a Hermitian inner product. Refer back to 247. We say $f \perp g$ if $(f, g) = 0$. Refer back to 247.

Question 18.3. What kind of functions are f and g? If f and g are continuous, then this will be an inner product, but if we just require integrable, then we might have that there exists $f \neq 0$ but $(f, f) = 0$, which would contradict our inner product axioms. [This means that as it stands, (f, f) is a pre-inner product (which is defined as you'd guess)].

An example of such a function is the indicator function on a singleton. But for the sake of our theory, instead of worrying about this, we will think of functions as being parts of equivalence classes, where

$$f \sim g \iff (f - g, f - g) = 0.$$

This turns out to be equivalent to $||f - g||_{L^2} = 0$, which is fun. After modding out by the self-orthogonal vectors, we have that $(\cdot, \cdot)$ is an inner product. Fabio defined it in class as f and g differing only on a set with measure zero, but I like this definition more because it is more direct in my humble opinion.

Definition 18.4. We say that a Hilbert space H is a vector space with an inner product $\langle \cdot, \cdot \rangle$such that H is complete with respect to the norm $||f|| = \langle f, f \rangle^{\frac{1}{2}}$.[17] Completeness here is defined such that Cauchy sequences with respect to $|| \cdot ||$ converge in H.

Hilbert spaces have lots of geometric properties. They are very similar to finite dimensional space. They are "closest to finite dimensional". The problem is that in infinite dimensions we lose compactness of the unit closed ball. In this general context, define $v \perp w$ if $\langle v, w \rangle = 0$. Let us go over properties of Hilbert spaces.

Lemma 18.5.

1. Cauchy-Schwarz holds for Hilbert Spaces.

$$|\langle u, v \rangle| \leq ||u|| \, ||v||$$

2. The Parallelogram Identity holds for Hilbert Spaces.

$$||u + v||^2 + ||u - v||^2 = 2 \left(||u||^2 + ||v||^2 \right).$$

3. Pythagorean Theorem holds for Hilbert Spaces.

$$u \perp v \implies ||u + v||^2 = ||u||^2 + ||v||^2.$$

Proposition 18.6. If $(w_n)_{n \geq 1}$ is an orthonormal sequence—namely, $\langle w_n, w_m \rangle = \delta_{n,m}$, then for all v,

$$\sum \langle v, w_n \rangle^2 \leq ||v||^2.$$

Corollary 18.7. If we look at the series $\sum \langle v, w_n \rangle w_n$, then this converges in H.

Proof. This is because our sum is absolutely convergent. □

[17] Isaac wants to point out that "erm technically[...]" I will not repeat any such remarks here. You, dear reader, had the choice to take 436 if you wanted to.

§19 Day 19: Fourier Series Part 3. (Nov. 12, 2025)

Intended midterm average was 60-65. In the past his averages have been 55-60. Our average was around 51-52. He will curve depending on the results of the other midterms.

Last time we talked about a bit of functional analysis. Hilbert spaces, inner products, and some basic properties.

Example 19.1. If $X \subseteq \mathbb{R}^n$, then $L^2(X)$ space is the functions

$$L^2(X) = \left\{ f : \int_X |f|^2 dx < \infty \right\}.$$

The norm is induced by the inner product from last time

$$(f, g) = \int_X f \cdot \bar{g}\, dx,$$

where the $\bar{\cdot}$ is the complex conjugate. We take the conjugate when our codomain is the complex numbers (or something like it where integrals make sense), to make this a Hermetian inner product. If our codomain is a real vector space, we can exclude it.

To show that L^2 is a Hilbert space, we need to show completeness. Of course this requires that we either take the Lebesgue measure, or we do a hack by taking closure of compactly supported functions or something. Ultimately these things aren't of large importance to this class.

Example 19.2.

$$\ell^2 = \left\{ (a_n)_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty \right\},$$

where the norm is given by

$$||(a_n)||_{\ell^2} = \left(\sum_{n\in\mathbb{Z}} |a_n|^2 \right)^{\frac{1}{2}}.$$

This norm is induced by the inner product

$$((a_n), (b_n)) = \sum_{n\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} a_n \cdot \bar{b}_n.$$

Our remarks about complex vector spaces are the same as before.

Example 19.3. $\mathbb{R}^n$ with the dot product and usual norm.

Example 19.4. Consider continuous functions on $[0, 1]$ with the norm

$$||f|| = \left(\int_0^1 |f(x)|^2\, dx \right)^{\frac{1}{2}}.$$

We claim that this is not a Hilbert space, because it's not complete. For exmple,

$$d(x^n, x^m) = \left(\int_0^1 |x^n - x^m|^2 dx \right)^{\frac{1}{2}} \to 0,$$

as $n, m \to \infty$. But these converge to a discontinuous function (it's 0 everywhere except for at 1, where it's 1). On the other hand, with the uniform norm (L^∞, or sup norm), they are not Cauchy, so they do not converge.

One should remark that if we take this sequence in the L^2 space, then these do converge, and they converge to 0. This is because the indicator function on $\{1\}$ is in the same equivalence class as the 0 function in the L^2 space.

Example 19.5. Consider continuous functions on $[0,1]$ with the L^∞ norm. We call this space $C^0([0,1])$. Then this space *is* complete, because the uniform limit of continuous functions is continuous.

Example 19.6. For L^p spaces, they are not Hilbert spaces (when $p \neq 2$), but they are complete normed vector spaces, so they are Banach spaces. Of course,

$$||f||_{L^p} = \left(\int |f(x)|^p dx \right)^{\frac{1}{p}}.$$

To show this, we can show that the parallelogram identity does not hold for the L^p norm ($p \neq 2$).

§19.1 Orthogonality

We are trying to write $f(x) = \sum_{n \geq 0} a_n X_n(x)$, where X_n) are eigenfunctions of $-\partial_x^2$.

Proposition 19.7. Two eigenfunctions of $-\partial_x^2$, $-X_i'' = \lambda_i X_i$, $i = 1,2$ with $\lambda_1 \neq \lambda_2$, then we claim that $X_1 \perp X_2$ with respect to $(\cdot,\cdot)$.

Proof.

$$\begin{aligned}
(\lambda_1 - \lambda_2) \int_a^b X_1 X_2 &= \int_a^b \lambda_1 X_1 X_2 - \int_a^b X_1 (\lambda_2 X_2) \\
&= \int_a^b -X_1'' X_2 + \int_a^b X_1 X_2'' \\
&= \left[-X_1' X_2\right]_a^b + \int_a^b X_1' X_2' \\
&+ \left[X_1 X_2'\right]_a^b - \int_a^b X_1' X_2' \\
&= 0,
\end{aligned}$$

so long as our boundary conditions ensure that

$$\left[X_1 X_2' - X_1' X_2\right]_a^b = 0.$$

□

Definition 19.8. Boundary conditions (for the eigenvalue problem $-X'' = \lambda X$) for which

$$\left[X_1 X_2' - X_1' X_2\right]_a^b = 0$$

are called symmetric boundary conditions.

Example 19.9.

1. Dirichlet $X(a) = 0 = X(b)$
2. Neumann $X'(a) = 0 = X'(b)$
3. Robin [Check in the Homework]
4. Periodic $X(a) = X(b)$, $X'(a) = X'(b)$

Remark 19.10. We may have 2 or more eigenfunctions with the same eigenvalue. For example, $-X'' = \lambda X$ on $[0, 2\pi]$,

$$X(0) = X(2\pi), X'(0) = X'(2\pi)$$

have eigenfunctions of $\sin(nx)$ and $\cos(nx)$ with eigenvalue n^2 for all n. Also, any linear combination. That being said, fortunately $\sin(nx)$ and $\cos(nx)$ are already orthogonal. In general, by Gram-Schmidt you can make these eigenfunctions orthogonal. We know that the eigenspace is finite dimensional because it's a homogeneous second order ODE, so the general solution is a 2D vector space.

Corollary 19.11.

1. With symmetric boundary conditions, eigenfunctions of $-X'' = \lambda X$ with different eigenvalues are orthogonal.
2. The eigenvalues of $-X'' = \lambda X$ are real.
3. We can make eigenfunctions with the same value orthogonal.

Proof. We already proved the first and third part. The second part follows doing the same proof as the other part, but on

$$(\lambda - \bar{\lambda}) \int X\bar{X}.$$

□

As a consequence, with symmetric boundary conditions, if

$$f(x) = \sum_{n \geq 0} a_n X_n(x)$$

(which we will call a generalized Fourier series) then it follows that

$$a_n = \frac{(f, X_n)}{(X_n, X_n)},$$

where our X_n are an orthogonal set (which we are sure we can find).

Theorem 19.12. For $-X'' = \lambda X$ with symmetric boundary conditions, there exists infinitely many eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$, with $\lambda_n \nearrow \infty$.

Question 19.13. When we write $f(x) = \lim_{N\to\infty} \sum_{n=1}^{N} a_n X_n(x)$ where

$$a_n = \frac{(f, X_n)}{(X_n, X_n)},$$

with respect to which norm does this limit converge? We are interested in the different modes of convergence of the series of functions

$$\sum_{n=1}^{N} a_n X_n(x)$$

and the convergence to what. That is, it may not converge to f even if it does converge.

§19.2 Notions of Convergence

Definition 19.14. We say that $\sum f_n(x)$ converges pointwise to f in (a, b) if

$$\lim_{N\to\infty}\left|\sum_{n=1}^{N} f_n(x) - f(x)\right| = 0.$$

Definition 19.15. We say $\sum f_n(x)$ converges uniformly to f on $[a, b]$ if

$$\lim_{N\to\infty}\sup_{[a,b]}\left|\sum_{n=0}^{N} F_n(x) - f(x)\right| = 0.$$

Definition 19.16. We say $\sum f_n$ converges to f in $L^2([a, b])$ if

$$\lim_{N\to\infty}\left|\left|\sum_{n=0}^{N} f_n(x) - f(x)\right|\right|_{L^2([a,b])} = 0.$$

Remark 19.17. Uniform Convergence implies Pointwise convergence and L^2 convergence.

Remark 19.18. Uniform convergence is strictly stronger than Pointwise convergence, as x^n converges pointwise but not uniformly.

Remark 19.19. Pointwise convergence does not imply L^2 convergence, which we can see by taking functions like $g_n = \chi_{(\frac{n-1}{n},1)} \cdot \sqrt{n}$. [This is my own example. If you round the edges you can make it continuous or smooth if you want. Fabio used triangles, which was basically the same idea.]

Remark 19.20. L^2 convergence also does not imply pointwise convergence. Consider

$$g_n = \chi_{(\sin(n-\frac{1}{n}),\sin(n+\frac{1}{n}))},$$

where as usual, χ is the indicator function. Then by Diophantine approximation, we have that this sequence will be nonzero at every point infinitely many times (indeed, even if we set it to be $(\sin(n - \frac{1}{n^2}), \sin(n + \frac{1}{n^2}))$,[18] it would diverge on a comeagre set—a set whose compliment has measure 0), but its L^2 norm also clearly goes to 0, as sin is a contraction mapping, so we have that the diameter of our interval goes to 0, and therefore the L^2 norm of our sequence goes to 0. As such, we are done.

Once again, this was my example. But in this case, Fabio did not give an example (or at least he didn't write one down. He may have said one out loud.)

[18] On the other hand, if instead of squaring it we take an exponent of α with $\alpha > 2$, then it has measure 0 and Hausdorff (fractal) dimension less than 2. Wenyu proved this fact in class in MAT335.

§20 Day 20: Convergence of Fourier Series (Nov. 24, 2025)

We did some formal non-rigorous calculations with fourier series already to solve boundary and initial value problems. We also talked about the theory behind it with orthogonality and that stuff. We also discussed in the more general context of Hilbert spaces. So now we will consider questions of convergence.

Our plan is to state and prove three theorems about convergence. One about uniform convergence, one about L^2 convergence, and one about pointwise convegence. This is going to be the most theoretical part so far.

Theorem 20.1. Assume that f on $[a, b]$ is such that f, f', and f'' are all continuous. Then the Fourier series

$$\sum_{n=1}^{N} A_n X_n(x) \rightrightarrows f$$

on $[a, b]$.

Remark 20.2. Here, as in the previous lecture, X_n is defined as solutions to the ODE $-X'' = \lambda X$ with some symmetric boundary conditions, and

$$A_n = \frac{(f, X_n)}{(X_n, X_n)}.$$

We can get the same conclusion with different assumptions, but for that we need to do something slightly different.

Proof. Use Theorem 1.4 and the homework. □

Theorem 20.3. Let $f \in L^2([a, b])$. Namely, we want f to have L^2 norm less than infinity. Then

$$\sum_{n=1}^{N} A_n X_n(x) \to f$$

in $L^2([a, b])$.

Theorem 20.4. Let $f \in C([a, b])$ with f piecewise continuous. Then the classical fourier series (with sine and cosine) converges pointwise to f. And if f is piecewise continuous, then at any jump x_0, the Fourier series converges to

$$\frac{\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-} f(x)}{2}.$$

§20.1 The L^2-Theory

Theorem 20.5 (Least Square)**.** Let X_n be an orthogonal set of eigenfunctions and $||f||_{L^2} < \infty$. Then for N fixed,

$$\left|\left| f - \sum_{n=1}^{N} c_n X_n(x) \right|\right|_{L^2}$$

is going to be minimized by the Fourier coefficients

$$c_i = A_i = \frac{(f, X_n)}{(X_n, X_n)}.$$

Proof. This will give us some implications. Set $E_N = \left|\left|f - \sum_{n=1}^{N} c_n X_n\right|\right|_{L^2}^2$. Expanding this yields

$$E_N = ||f||_{L^2}^2 - 2\sum_{n=1}^{N} c_n \left(f, X_n\right) + \left|\left|\sum_{n=1}^{N} c_n^2 \left(X_n, X_n\right)\right|\right|_{L^2},$$

where we can bring the squares inside the norm by the pairwise orthogonality of our X_n. Now, we want to complete the squares.

$$0 \leq E_N = ||f||_{L^2}^2 + \underbrace{\sum_{n=1}^{N} \left(X_n, X_n\right)\left[c_n - \frac{\left(f, X_n\right)}{\left(X_n, X_n\right)}\right]^2} - \sum_{n=1}^{N} \frac{\left(f, X_n\right)^2}{\left(X_n, X_n\right)}.$$

where the term with the underbrace is the only term which varies with respect to our c_n. As such, by minimizing each of the individual terms of the sum, we minimize the series, which is accomplished by our hypotheses—when

$$c_n = \frac{\left(f, X_n\right)}{\left(X_n, X_n\right)}.$$

□

Corollary 20.6. With $c_n = A_n$, we get

$$\sum A_n^2 \left(X_n, X_n\right) \leq ||f||_{L^2}.$$

Proof. Rearrange the final line of our previous proof to get this. $\left(X_n, X_n\right)$ being multiplied and not divided is *not* a typo. □

We will continue on Wednesday.

§21 Day 21: Convergence of Fourier Series cont. (Nov. 24, 2025)

Last time we proved that our A_n really are the best coefficients. This is in the sense that

$$\sum_{n\geq 1} |A_n|^2 (X_n, X_n) \leq ||f||_{L^2}^2.$$

If $(X_n, X_n) = 1$, then

$$\sum_{n\geq \mathbb{N}} |A_n|^2 \leq ||f||_{L^2}^2, \quad A_n = (f, X_n).$$

So, we will keep this in mind. We also got Bessel's inequality

$$\begin{aligned} 0 \leq E_N &= \left|\left| f(x) - \sum_{n=1}^{N} A_n X_n(x) \right|\right|_{L^2}^2 \\ &= ||f||_{L^2}^2 - \sum_{n=1}^{N} \frac{|(f, X_n)|^2}{(X_n, X_n)}. \end{aligned}$$

But this is just the error between our function and the partial fourier sum (measured in L^2). This will be zero if our right hand sum actually converges to the norm squared (rather than just having our inequality above). Namely,

$$\sum_{n=1}^{\infty} |A_n|^2 (X_n, X_n) = ||f||_{L^2}^2.$$

This is called Parseval's identity, and when it holds, we have L^2 convergence of our Fourier series to f.

Theorem 21.1. Let w_n be an orthonormal sequence in a Hilbert space H. The following are equivalent:

1. Finite linear combinations
$$\sum_{n=1}^{N} b_n w_n$$
are dense ($b_n \in \mathbb{C}$) in the Hilbert space.
2. If $\langle u, w_n \rangle = 0$ for all n then $u = 0$.
3. Parseval's Identity.
$$||u||_H^2 = \sum_{n=1}^{\infty} |\langle u, w_n \rangle|^2.$$
4. (w_n) is an orthonormal basis for H.

Fabio didn't actually go through the proofs explicitly. It'll be easy to find this proof though.

It follows that to complete the proof of L^2 convergence in our setting, you can verify any of the above properties. A shortcut to prove property 1 is to use the Stone-Weierstrass theorem (from 357) to show that the trig polynomials are dense in the continuous functions with respect to the L^∞ norm. From here, the continuous functions are dense in L^2 with respect to the L^2 norm, and convergence in the L^∞ norm of a sequence our trig polynomials implies convergence (to the same limit) in the L^2 norm, therefore the trig polynomials are dense in the L^2 norm.

Theorem 21.2. Let f be a C^1 functions on $\mathbb{R}$ of period 2π. Define

$$(S_n f)(x) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos(nx) + \sum_{n=1}^{N} B_n \sin(nx).$$

Then $S_N f(x) \to f(x)$ pointwise.

Proof. Step 1 is to take S_N and write it as a convolution of a kernel with f.

$$\begin{aligned}
(S_N f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy \\
&+ \sum_{n=1}^{N}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)dy\right)\cos(nx) \\
&+ \sum_{n=1}^{N}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)dy\right)\sin(nx) \\
(S_N f)(x) = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(y)dy\right. \\
&+ \sum_{n=1}^{N}\left(\int_{-\pi}^{\pi} f(y)\cos(ny)dy\right)2\cos(nx) \\
&\left.+ \sum_{n=1}^{N}\left(\int_{-\pi}^{\pi} f(y)\sin(ny)dy\right)2\sin(nx)\right) \\
(S_N f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left(1 + \sum_{n=1}^{N} 2\cos(ny)\cos(nx) + 2\sin(ny)\sin(nx)\right)dy \\
(S_N f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left(1 + 2\sum_{n=1}^{N}\cos\left(n(x-y)\right)\right)dy
\end{aligned}$$

We notice that this kinda looks like a convolution, so we are going to set

$$K_N(x-y) = 1 + 2\sum_{n=1}^{N}\cos\left(n(x-y)\right).$$

We claim

$$K_N(\theta) = \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

This can be proven via residues! According to Fabio at least. He seems to have opted to do it directly.

$$\begin{aligned}
K_N(\theta) &= 1 + 2\sum_{n=1}^{N} \cos(n\theta) \\
&= 1 + 2\,\mathrm{Re}\left(\sum_{n=1}^{N} e^{in\theta}\right) \\
&= \sum_{n=-N}^{N} e^{in\theta} \\
&= e^{-iN\theta}\sum_{n=0}^{2N} e^{in\theta} \\
&= e^{-iN\theta}\frac{\left(1 - e^{i(2N+1)\theta}\right)}{1 - e^{i\theta}} && \text{(Geometric Series)} \\
&= \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} \cdot \frac{e^{-i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}} \\
&= \frac{e^{-i(N+\frac{1}{2})\theta} - e^{i(N+\frac{1}{2})\theta}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}} \\
&= \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}
\end{aligned}$$

This function will be nice everywhere except possibly at $\theta = 0$. But the limit as we approach 0 is just

$$\lim_{\theta\to 0} K_N(\theta) = \lim_{\theta\to 0}\frac{\sin\left((N+\frac{1}{2})\theta\right)}{\sin\left(\frac{\theta}{2}\right)} = \lim_{\theta\to 0}\frac{(N+\frac{1}{2})\theta + o(\theta)}{\frac{\theta}{2} + o(\theta)} = 2N+1,$$

so the limit exists, and equals our original sum.

Now, we want to calculate $S_N f(x) - f(x) \to 0$ for a fixed x. But notice that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)d\theta = 1, \tag{*}$$

by remarking that

$$K_N(\theta) = \sum_{n=-N}^{N} e^{in\theta},$$

thus the integral cancels out all the terms except $n = 0$, which will integrate to 2π. "This kernel is nice, but not *too* nice." Rewrite

$$S_N f(x) = \int_{-\pi}^{\pi} K_N(\theta) f(x+\theta)\frac{d\theta}{2\pi}.$$

This is by the periodicity of f. From here,

$$S_N f(x) - f(x) = \int_{-\pi}^{\pi} K_N(\theta)\left[f(x+\theta) - f(x)\right]\frac{d\theta}{2\pi}$$

which is because of the integral on line $(*)$.

$$= \int_{-\pi}^{\pi} \sin\left((N + \frac{1}{2}\theta\right)\frac{f(x+\theta) - f(x)}{\sin\left(\frac{\theta}{2}\right)}\frac{d\theta}{2\pi}$$

If we set

$$g_x(\theta) = \frac{f(x+\theta) - f(x)}{\sin\left(\frac{\theta}{2}\right)},$$

(x is fixed) we want to show that

$$\int_{-\pi}^{\pi} \sin\left(\left(N + \frac{1}{2}\right)\theta\right) g_x(\theta) d\theta \to 0$$

as $N \to \infty$. But we know that g is continuous in θ, and its limit as $\theta \to 0$, of g_x is just $2f'(x)$, so we have that it can be extended to continuous everywhere. Thus, we will try to use some recent knowledge. Recall Bessel's inequality. If h has $||h||_{L^2}^2 < \infty$, and X_n is an orthogonal set.

$$\sum \frac{(h, X_n)^2}{(X_n, X_n)} \leq ||h||_{L^2}^2.$$

If $|\,(X_n, X_n)\,| \leq C$ for some uniform bound C (independent of n), then we have that (h, X_n) is square summable, thus $(h, X_n) \to 0$ as $n \to \infty$.

Applying this, if we show g_x is L^2, and that our $\sin\left(\left(N + \frac{1}{2}\right)\theta\right)$ have bounded inner product, then this would prove the theorem, because

$$\int_{-\pi}^{\pi} \sin\left(\left(N + \frac{1}{2}\right)\theta\right) g_x(\theta) \frac{d\theta}{2\pi} = \left(\sin\left(\left(N + \frac{1}{2}\right)\theta\right), g_x(\theta)\right) \cdot \frac{1}{2\pi}.$$

Thus we just have to verify these.

1. We know that $||g_x||_{L^2} < \infty$ because $g(\theta) = \frac{f(x+\theta)-f(x)}{\sin\left(\frac{\theta}{2}\right)}$ is continuous on $[-\pi, \pi]$, and therefore it is bounded and integrable, so it is L^2.

2. $$\left|\int_{-\pi}^{\pi} \sin^2\left(N + \frac{1}{2}\theta\right) d\theta\right| \leq 2\pi \left|\sup_{\theta \in [-\pi,\pi]} \sin(\theta)\right| = 2\pi,$$

   so our functions are uniformly bounded.

3. For orthogonality, notice that our $\sin\left(\left(N + \frac{1}{2}\right)\theta\right)$ are the eigenfunctions of the eigenvalue problem

   $$\begin{cases} X'' = -\lambda X \\ X(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

   and so since the different N correspond to different eigenfunctions, they are orthogonal.

Thus, we are done. We have shown pointwise convergence. $\square$

§22 Day 22: Laplace Equation (Dec. 1, 2025)

Fabio has gone to New York. Slacker. We will be looking at the Laplace equation with the amazing TA Oliver. This is $\Delta u = 0$.

Where does this arise? It gives us stationary solutions to the heat and wave equation. This is of course because if $u(x,t) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is harmonic in x and constant in t, then

$$\partial_t^k u + c\Delta u = 0.$$

(for *all* $c \in \mathbb{R}$). Both the heat and wave eqution are of this form. Also the Schrödinger equation.

Example 22.1. Let's look at the Laplace equation in $[0,\pi]^2$.

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(\pi,y) = 0 \\ u(x,0) = 0, u(x,\pi) = g(x) \end{cases}.$$

Then as usual we'll look for a separable solution.

$$u(x,y) = X(x)Y(y) \implies \frac{X''}{X} = -\frac{Y''}{Y} = \lambda,$$

so we do what we've always done, and write the eigenvalue problem

$$\begin{cases} X'' = \lambda X \\ X(0) = 0 = X(\pi) \end{cases}.$$

Using some (very difficult and hard to prove) facts about trig functions, $\lambda = n^2$ for some $n \in \mathbb{Z}$, so

$$X(x) = a_n \cos(nx) + b_n \sin(nx),$$

and since we know all the solutions for second order constant coefficient ODEs, we can also deduce from our vast repertoire of ODE knowledge that

$$Y(y) = c_n \sinh(ny) + d_n \cosh(ny)$$

Plugging in the boundary conditions of $X(0) = 0 = X(\pi)$ and $Y(0) = 0$, we can deduce $a_n = 0$ and $d_n = 0$, so

$$X(x) = b_n \sin(nx), \qquad Y(y) = c_n \sinh(ny).$$

We can then write

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(nx)\sinh(ny).$$

Our final boundary condition (and orthogonality of eigenfunctions with different eigenvalues) gives us that

$$A_n \sinh(n\pi) = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)dx,$$

so since $\sinh(n\pi) \neq 0$, we are done.

Example 22.2. We're gonna do the problem on a disc.[19]

$$\begin{cases} \Delta u = 0 & B(0,1) \subseteq \mathbb{R}^2 \\ u = g & \partial B(0,1). \end{cases}$$

The natural way to do this is using polar coordinates. In polar coordinates,

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2.$$

If we write

$$u = R(r)\Theta(\theta),$$

then

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Then

$$-\frac{\Theta''}{\Theta} = \frac{r^2R'' + rR'}{R} = \Lambda.$$

What sign should λ be? Well we'll use the Θ term to exploit that since

$$\Theta'' = -\lambda\Theta$$

and Θ is 2π-periodic, then we either have that Θ is constant and $-\lambda = 0$, or $\Theta(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$ and $-\lambda$ is negative. We'll look at the latter case.

Notice that

$$r^2R'' + rR' - n^2R = 0,$$

so this is a Cauchy-Euler differential equation, and therefore setting as ansatz $R = r^\alpha$,

$$r^\alpha \left[\alpha(\alpha-1) + \alpha - n^2\right] = 0$$

thus $\alpha^2 - n^2 = 0$, so we have that $\alpha = \pm n$. Thus, this ODE will be solved by

$$R(r) = c_n r^n + d_n r^{-n}.$$

On the other hand, since we want this to be continuous on the disc, we need $d_n = 0$, as r^{-n} isn't defined at $r = 0$. It follows that

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

Since $g(\theta) = A_0 + \sum A_n \cos(n\theta) + B_n \sin(n\theta)$, we just need A_n and B_n to be the Fourier coefficients of g.

Example 22.3. Let's look at the Neumann boundary conditions.

$$\begin{cases} \Delta u = 0 & B(0,1) \\ \frac{\partial u}{\partial n} = h & \partial B(0,1). \end{cases}$$

Our deductions are the same to get up to the general solution

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right),$$

[19] "I wanna do something that's not on a *rectangle*." Oliver, your prejudices are showing.

so since we know (hello 257) $\frac{\partial u}{\partial n} = \nabla u \cdot n$ on the circle (and the normal vector at a point x on a circle of radius 1 is just x itself). Then we can deduce that

$$\frac{\partial u}{\partial n} = \sum_{n=1}^{\infty} n\left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] = h(\theta).$$

Integrating yields our Fourier coefficients as usual. We can't determined A_0 by this formula which is confirmation of our non-uniqueness. Also, $\hat{h}(0) = 0$ (the 0th Fourier coefficient for h is 0). It follows that

$$0 = \frac{1}{\pi}\int_0^{2\pi} h(\theta)d\theta.$$

Example 22.4. Let's do another wacky and fun domain.

$$\begin{cases} \Delta u = 0 & 1 < r < 2 \quad 0 < \theta < \pi \\ u(r, 0) = 0 = u(r, \pi) \\ u(1, \theta) = \sin\theta \\ u(2, \theta) = 0. \end{cases}$$

This will look like a rainbow (waow). $\Theta(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$ and $R(r) = c_n r^n + d_n r^{-n}$. In this case, we don't have any reason to cancel out the r^n and r^{-n}.

Now, $u(r, 0) = 0 = r(r, \pi)$ implies $\Theta(0) = 0 = \Theta(\pi)$, so $a_n = 0$. Thus,

$$u(r, \theta) = \sum_{n=1}^{\infty} \left(C_n r^n + D_n r^{-n}\right)\sin(n\theta).$$

There won't be a constant. When we set $r = 1$, we get that

$$\sum_{n=1}^{\infty} (C_n + D_n)\sin(n\theta) = \sin(\theta),$$

therefore

$$\begin{cases} C_1 + D_1 = 1 \\ C_n + D_n = 0 \quad n > 1. \end{cases}$$

When $r = 2$, then

$$\sum_{n=1}^{\infty} \left(C_n 2^n + D_n 2^{-n}\right)\sin(n\theta) = 0$$

and therefore

$$C_n 2^n + D_n 2^{-n} = 0, \qquad n \geq 1.$$

For each n, this is a linear system of two equations in two unknowns, so we can solve it. I won't write out that process, taking on faith that you have taken 240.

$$C_1 = \frac{-1}{3}, D_1 = \frac{4}{3}, \qquad C_n = 0, D_n = 0, \qquad n > 1.$$

Therefore

$$u(r, \theta) = \left(\frac{-r}{3} + \frac{4}{3r}\right)\sin(\theta).$$

Happy Holidays everyone. It's Dynamical Systems December for me.