MAT363 Lecture Notes

Arky!! :3c

'25 Winter Semester

Contents

§1 Day 1: Introduction to the Class (Jan. 6, 2025)

Course administrative details! First day slides are given here. This is a class in classical differential geometry; the following 12 weeks will be split up as follows,

(a) Curves, for two weeks;

(b) Surfaces, for three weeks;

(c) Curvature of surfaces, for three weeks;

(d) Geodesics, for three weeks;

(e) Gauss-Bonnet theorem, for one week.

Grading will be done by 5% on PCEs, 15% on problem sets, 15% on quizzes, 25% on the term test, 30% on the final exam, and 10% weighted towards your best test.

To start, consider the following maps $\gamma : I = (-10, 10) \to \mathbb{R}^3$, given by

$$\begin{aligned}
\gamma(t) &= (t, t, t);\\
\gamma(t) &= (|t|, |t|, |t|);\\
\gamma(t) &= (t, t^2, t^3);\\
\gamma(t) &= (t^3, t^3, t^3);\\
\gamma(t) &= (\cos t, \sin t, t);\\
\gamma(t) &= (t\cos t, t\sin t, t).
\end{aligned}$$

In this class, we say that a curve is a *parameterized curve* if it is a smooth function $\gamma : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval. In particular, of the six examples given above, only $t \mapsto (|t|, |t|, |t|)$ is not smooth.

Definition 1.1 (Regular Curve)**.** Let $\gamma : I \to \mathbb{R}^n$ be a curve; it is said to be *regular* if $|\gamma'(t)| \neq 0$ for all $t \in I$, i.e. the speed is always nonzero.

Note that $\gamma'(t)$ and $|\gamma'(t)|$ describe different qualities, with the former describing velocity and the latter describing speed (i.e., one describes speed as well, while the other is a scalar quantity). As an example, consider the curve $\gamma(t) = (\cos t, \sin t, t)$. To find the distance travelled from $t = 0$ to $t = 2\pi$, we may observe that

$$\left|\gamma'(t)\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Since the speed is constant, the total distance traveled is simply $2\pi\sqrt{2}$. $\square$

Definition 1.2 (Closed Curve)**.** Consider a curve $\gamma : [a, b] \to \mathbb{R}^n$. We say that γ is a *closed curve* if $\gamma(a) = \gamma(b)$ and $\gamma^{(n)}(a) = \gamma^{(n)}(b)$ for all naturals n.

Definition 1.3 (Simple Curve)**.** We say that γ is a simple curve if it is injective on $[a, b)$.

Note that while in topology we do not care if there is a "sharp corner" at $\gamma(a) = \gamma(b)$, such things do matter, as per the condition that the nth derivative of γ must agree on a and b (for example, the velocity γ' at a, b must be equal).

In this class, we automatically take the inner product $\langle , \rangle$ as the Euclidean inner product,

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n.$$

For any subspace $V \subset \mathbb{R}^n$, we may decompose any vector $x \in \mathbb{R}^n$ uniquely as $x = x^{\|} + x^{\perp}$, where $x^{\|} \in V$ and $\left\langle x^{\perp}, v \right\rangle = 0$ for any vector $v \in V$. Now, consider any curve $\gamma : I \to \mathbb{R}^n$. We have the following proposition,

Proposition 1.4. If $|\gamma(t)|$ is constant, then $\langle \gamma(t), \gamma'(t) \rangle = 0$ for all $t \in I$.

To see this, let $|\gamma(t)|^2 = c$ be constant; then

$$\frac{d}{dt} |\gamma(t)|^2 = 0 \implies \frac{d}{dt} (\langle \gamma(t), \gamma(t) \rangle) = \left\langle \gamma'(t), \gamma(t) \right\rangle + \left\langle \gamma(t), \gamma'(t) \right\rangle = 0m$$

i.e. $\langle \gamma(t), \gamma'(t) \rangle = 0$ as desired. $\square$

Given a regular curve $\gamma : I \to \mathbb{R}^n$, we may compute the velocity and acceleration as $\gamma'(t), \gamma''(t)$, which are denoted $v(t), a(t)$ respectively. In particular, we may write

$$a(t) = a^{\parallel}(t) + a^{\perp}(t),$$

with $a^{\parallel}(t)$ being the tangential acceleration, and $a^{\perp}(t)$ being the normal acceleration. We may find these by projecting $a(t)$ into the subspace $\operatorname{span}\{v\}$ (i.e., the span of the velocity vector).

§2 Day 2: Curvature of a Curve (Jan. 9, 2025)

Definition 2.1. Suppose that $\gamma : I \to \mathbb{R}^n$ is a regular curve. A *reparameterization* of γ is a function of the form $\tilde{\gamma} = \gamma \circ \phi : \tilde{I} \to \mathbb{R}^n$, where $\tilde{I}$ is an interval, and $\phi : \tilde{I} \to I$ is a smooth bijection with nowhere vanishing derivative $\phi'(t) \neq 0$ for all $t \in \tilde{I}$.

Definition 2.2. We say that $\gamma, \tilde{\gamma}$ have the same orientation (i.e., the parameterization is *orientation-preserving*) if $\phi' > 0$, and orientation-reversing if $\phi' < 0$.

Note that since ϕ is smooth, it is impossible for ϕ' to have places on which it is greater than 0 and less than 0, since IVT holds on $\tilde{I}$ and $\phi'(t) \neq 0$.

We now present an example;

$$\begin{aligned} &\gamma : [0,1] \to \mathbb{R}^3 \text{ with } \gamma(t) = (t,t), \\ &\beta : [0,\pi/2] \to \mathbb{R}^3 \text{ with } \beta(t) = (\sin t, \sin t). \end{aligned}$$

While these both parameterize the same curve, we prefer γ greatly because $|\gamma'(t)| = \sqrt{2}$ while $|\beta'(t)| = \cos t$; specifically, constant speed parameterizations are much nicer to deal with. Even better,

Definition 2.3. A curve γ is said to be parameterized by arclength if $|\gamma'(t)| = 1$.

In particular, we may turn our above $\gamma : t \mapsto (t,t)$ into an arclength parameterization by considering it to be $t \mapsto (\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}})$ instead, since we would then have

$$\left|\phi'(t)\right| = \sqrt{2 \cdot \left(\frac{1}{\sqrt{2}}\right)^2} = 1. \quad \square$$

We now introduce the idea of curvatures of curves. A few ideas first:

- A smaller circle has *larger* curvature (naturally, it is more curved than a large circle).
- The curvature of a curve is given by a function $\kappa : I \to [0,\infty)$; i.e., it cannot be negative. Specifically, it is given by

$$\kappa(t) = \frac{\left|a^\perp(t)\right|}{|v|^2}.$$

Curvature enjoys two main properties; κ is independent of the parameterization of the curve, and that $\kappa = \frac{1}{r}$, where r is the radius of the circle that approximates the curve at the particular point. If γ is parameterized by arclength, then $\kappa(t) = a(t)$. To see this, observe that

$$\kappa(t) = \frac{\left|a^\perp(t)\right|}{|v|^2} = \left|a^\perp(t)\right| = |a(t)|.$$

In particular, since $|v(t)|$ is constant, so is $|v(t)|^2$. Then

$$\frac{d}{dt}\langle v(t), v(t)\rangle = 0 \implies \langle v;(t), v(t)\rangle = 0 \implies \langle a(t), v(t)\rangle = 0.$$

From this, along with $a(t) = a^\perp(t) + a^\parallel(t)$, we have that $a^\parallel(t) = 0$.[1] $\square$

[1] for more detail, check proposition 1.18 in the textbook; γ' and γ'' are perpendicular if γ is a curve with constant speed.

Definition 2.4. Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. We define the unit tangent vector and unit normal vector as

$$T(t) = \frac{v(t)}{|v(t)|}; \qquad n(t) = \frac{a^\perp(t)}{|a^\perp(t)|},$$

respectively.

Note that the textbook uses $\mathfrak{t}, \mathfrak{n}$ respectively, but these are hard to write on paper.

Definition 2.5 (Osculating Plane)**.** At a fixed point on a regular curve with $\kappa \neq 0$, we define the *osculating plane* by

$$\operatorname{span}\{T, n\},$$

and we define the *osculating circle* to be the unique circle with these properties:

(a) Radius $\frac{1}{\kappa}$,

(b) It is in the osculating plane,

(c) It is centered at 0.

§3 Day 3: Angle Function and Rotation Index (Jan. 13, 2025)

We begin with a few questions.

(a) Why is $\kappa_s(t)$ only defined for planar curves and not in $\mathbb{R}^n$ for $n > 2$? In $\mathbb{R}^2$, the orthogonal vector is 1-dimensional; in higher dimensions, it is $n-1$-dimensional, which does not make sense to apply a 90-degree rotation to; i.e., the notions of "clockwise" and "counter-clockwise" rotation are not that well defined in $\mathbb{R}^n$.

(b) How can the angle function $\theta(t)$ increase by more than 2π along a curve? This is possible by having multiple loops; check figure 1.23, on page 37 in the textbook.

(c) What is the geometric meaning behind the equation $\theta'(t) = \kappa_s(t)$? Intuitively, θ' measures the change in the angle measuring how much the curve deviates from a straight line.

(d) Why is the rotation index of a unit-speed closed plane curve always an integer? Since the curve is closed and smooth, θ has an equal evaluation at the beginning and end of the interval the curve is parameterized on, with a difference of a multiple of 2π per definition; thus, we have that

$$\frac{\theta(b) - \theta(a)}{2\pi}$$

evaluates out to an integer always.

An example graph of $\theta(t)$ for curve γ was given in class; we record the answers here, but not the figure.

(a) The rotation index is $\frac{1}{4}$.

(b) The curve is not closed, since the rotation index is not an integer.

(c) Clockwise means negative θ, and counterclockwise means positive θ.

(d) The maximum curvature, $\max |\theta'(t)|$, is approximately 6π (which is a guess).

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a simple closed curve. Let $C = \gamma([a, b])$ denote its trace.

Theorem 3.1 (Hopf's Umlaufsatz)**.** The rotation index of γ is either -1 or 1.

Theorem 3.2 (Jordan Curve Theorem)**.** $\mathbb{R}^2 \setminus C = \{p \in \mathbb{R}^2 \mid p \notin C\}$ has exactly two path connected components. Their common boundary is C. One component (which we will call the interior) is bounded, while the other (which we call the exterior) is unbounded.

Definition 3.3. In the above context, we say that γ is positively oriented if the rotation index of γ is 1, and negatively oriented if the index is -1.

Definition 3.4. A *piecewise regular curve* in $\mathbb{R}^n$ is a continuous function $\gamma : [a, b] \to \mathbb{R}^n$ with partition $a = t_0 < \cdots < t_n = b$ such that each $\gamma|_{[t_i, t_{i+1}]}$ is a regular curve. We call the points $\gamma(t_i)$ for $i = 1, \ldots, n-1$ the "corners" of γ.

Definition 3.5. In the above context, the *signed angle* at $\gamma(t_i)$ denoted by $\alpha_i \in [-\pi, \pi]$ is given by the angle between $v^-(t_i)$ and $v^+(t_i)$ with $\alpha_i > 0$ for a counterclockwise and $\alpha_i < 0$ for a clockwise turn.

Theorem 3.6 (Generalized Hopf's Umlaufsatz)**.** Let $\gamma : [a, b] \to \mathbb{R}^n$ be a unit-speed positively oriented piecewise regular simple closed plane curve. Let κ_s denote its signed curvature function, and let $(\alpha_i)_i$ be the list of signed angles at its corners. Then

$$\int_a^b \kappa_s(t)\, dt + \sum \alpha_i = 2\pi.$$

We now move onto space curves. Note that $\mathbb{R}^2$ may be embedded into $\mathbb{R}^3$ by considering $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

Definition 3.7. Consider a regular space curve $\gamma : I \to \mathbb{R}^3$ and $t \in I$ such that $\kappa(t) \neq 0$. The *Frenet frame* at t is the orthonormal basis $\{T, n, B\}$ of $\mathbb{R}^3$ defined by

$$\begin{aligned} T(t) &= \frac{v(t)}{|v(t)|}, \\ n(t) &= \frac{a^\perp(t)}{|a^\perp(t)|}, \\ B(t) &= T(t) \times n(t). \end{aligned}$$

In particular, $|B'|$ is a natural choice to measure the changing tilt of the osculating plane because B is constant length, and that it is orthogonal to both B and T. Moreover, $\langle B', n\rangle$ and $|B'|$ are related as follows,

$$\left\langle B', n \right\rangle = \left|B'\right| |n| \cos\varphi = \pm \left|B'\right|,$$

where φ is the angle between B and n (note that this is either 0 or π, since $B \parallel n$).

Definition 3.8. Consider a regular space curve $\gamma : I \to \mathbb{R}^3$ and $t \in I$ such that $\kappa(t) \neq 0$. The torsion of γ at t is given by

$$\tau(t) = \frac{-\langle B'(t), n(t)\rangle}{|v(t)|}.$$

The torsion is independent of parameterization.

As an example, if $\gamma : I \to \mathbb{R}^3$ is such that for all $t \in I$, $\kappa(t) \neq 0$ and $\tau(t) = 0$, we have that $|B'| = 0$, B is constant, and so $\operatorname{span}\{T, u\} = P$. In particular, this means that γ is planar; i.e., for all $t \in I$, $\gamma(t) \in \gamma(t_0) + P$.

Theorem 3.9 (Frenet Equations)**.** For a regular curve $\gamma : I \to \mathbb{R}^3$, the following equations hold whenever $\kappa(t) \neq 0$:

$$\begin{aligned} T' &= |v|\, \kappa n \\ n' &= |v|\, |-\kappa T + \tau B| \\ B' &= -|v|\, \tau n. \end{aligned}$$

In matrix form, this is written as

$$\begin{pmatrix} T \\ n \\ B \end{pmatrix}' = |v| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ n \\ B \end{pmatrix}.$$

Next class, we will prove the above; in particular, there exists a unique solution by the Picard-Lindelöf theorem. We did not cover Taylor approximations because we ran out of time.

§4 Day 4: Rigid Motions (Jan. 16, 2025)

We start with a quick refresher of linear algebra properties;

Definition 4.1. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *orthogonal* if any of the following equivalent conditions are fulfilled,

(a) For all $p \in \mathbb{R}^n$, $|Ap| = |p|$, i.e., A is norm-preserving.

(b) For all $p, q \in \mathbb{R}^n$, $\langle Ap, Aq \rangle = \langle p, q \rangle$, i.e., inner product preserving.

(c) A sends an orthonormal basis of $\mathbb{R}^n$ to an orthonormal basis of $\mathbb{R}^n$.

(d) The columns of A are an orthonomal basis of $\mathbb{R}^n$.

(e) $A^\top A = I$.

In particular, if A is orthogonal, then $\det A = \pm 1$, and its inverse is also orthogonal; the product of two orthogonal matrices is also orthogonal.

Definition 4.2. A *rigid motion* of $\mathbb{R}^n$ means a function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$|f(p) - f(q)| = |p - q|$$

for all $p, q \in \mathbb{R}^n$.

Definition 4.3. For $A \in \mathbb{R}^{n \times n}$, we formally denote the associated linear map $L_A : \mathbb{R}^n \to \mathbb{R}^n$. For $q \in \mathbb{R}^n$, we define the translation $T_q : \mathbb{R}^n \to \mathbb{R}^n$ with $T_q(p) = p + q$. Note that translations are rigid motions.

Theorem 4.4. If f is a rigid motion of $\mathbb{R}^n$, then $f = T_q \circ L_A$ for a unique choice of q and A.

To prove this, observe that if we let $q := f(0)$ and define $g := (T_q)^{-1} \circ f$, then we may note that g is a rigid motion with $g(0) = 0$ by our choice of q, and that for any $v \in \mathbb{R}^n$, $|g(v)| = |g(v) - g(0)| = |v - 0| = |v|$ as desired. Thus, g preserves all norms, and so $g = L_A$ for some orthogonal matrix A, and we have that $f = T_q \circ L_A$ as desired. $\square$

We leave the proof of uniqueness as a quick exercise.

Definition 4.5. A rigid motion $f = T_q \circ L_A$ is said to be proper if $\det A = 1$ and improper if $\det A = -1$.

We are now able to answer one of the recurring questions in this class: if we have an object G, with geometric property P, and a map f of type T, does $f(G)$ still have the same property P, or can we at least predict some properties of it?

Theorem 4.6. A *proper* rigid motion preserves the curvature, torsion, and signed curvature of a curve, space curve, and plane curve respectively. An impropert rigid motion preserves curvature, but multiplies torsion and signed curvature by -1.

For simplicity, we prove that this holds for curvatures. Let $\gamma : I \to \mathbb{R}^n$ be smooth, and let $A \in \mathbb{R}^{n \times n}$ and its associated linear map L_A. Then

$$(L_A \circ \gamma)' = (dL_A \circ \gamma) \cdot \gamma' = A\gamma',$$

since the differential of L_A is just itself. Without loss of generality, let $|\gamma'| = 1$, and let $\overline{\gamma} = f \circ \gamma$. Then

$$\overline{\kappa} = |\overline{a}| = \left|(\overline{v})'\right| = \left|((f \circ \gamma)')'\right| = \left|((T_q \circ L_A \circ \gamma)')'\right| = \left|((L_A \circ \gamma)')'\right| = \left|A \circ \gamma''\right|,$$

and so $|\gamma''| = |a| + \kappa$. $\square$

Theorem 4.7. If $I \subset \mathbb{R}$ is an interval and $\kappa_s : I \to \mathbb{R}$ is smooth, then there exists a unit-speed plane curve $\gamma : I \to \mathbb{R}^2$ with signed curvature κ_s. γ is unique up to proper rigid motion.

Fix $t_0 \in I$. Define $\theta(t) := \int_{t_0}^t \kappa_s(u)\,du$, and define $v(t) = (\cos\theta(t), \sin\theta(t))$. Then we may define $\gamma(t) := \int_{t_0}^t v(u)\,du$. Then $\gamma(t)$ has signed curvature κ_s. We will check uniqueness next class. Also, something something about Frenet equations.