MAT327 Lecture Notes

Arky!! :3c

'24 Fall Semester

Contents

§1 Day 1: Open Sets and Continuity (Sep. 3, 2024)

> This class is *MAT327*; 3 meaning third year, 2 meaning the contents are on the fundamental side, and 7 meaning no mercy.
> – Dror Bar-Natan

Course administration matters first;

- The course link is given here (this will link straight to Quercus).
- The textbook is James Munkres' Topology (online PDF: ETH Zurich mirror); Prof Bar-Natan strongly recommends a paper copy, though (since people get distracted on the computer).

Today's reading in the textbook is on Ch. 1, sections 1 to 8, and Ch. 2, sections 12 to 13. Readings are supplementary to lecture material[1]. The goal of this course is to understand continuity in its most general form; in particular,

- In MAT157, we studied continuity in $f : \mathbb{R} \to \mathbb{R}$;
- In MAT257, we will study continuity in $f : \mathbb{R}^n \to \mathbb{R}^m$;

but in this class, we will study continuity in $F : X \to Y$, where X, Y are arbitrary spaces, such as (but not limited to) $\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{N}}$ (binary sequences), and so on. We start with some refreshers on previous coursework;

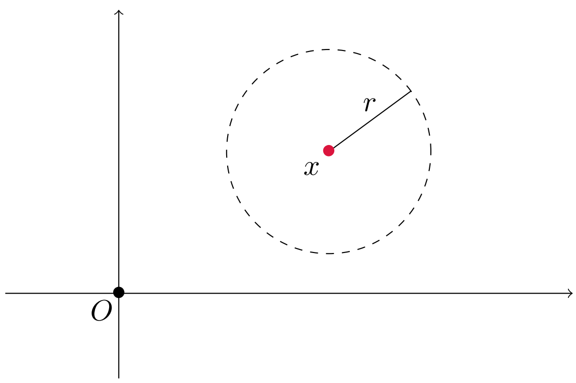
Definition 1.1 (Continuity in $\mathbb{R}^n \to \mathbb{R}^m$)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called "continuous" if it is continuous at all points in $\mathbb{R}^n$. Specifically, for all $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

With this, we state our main theorem for today (proof given later),

Theorem 1.2 (Continuity on $\mathbb{R}^n$ if and only pre-image of open subsets is open)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if all open subsets $U \in \mathbb{R}^m$ have $f^{-1}(U)$ open.

In order to build up to the above, we start by defining some terms;

Definition 1.3 (Open Ball)**.** Let $r > 0$, and $x \in \mathbb{R}^n$. An open ball of radius r about x is given formally by $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Visually, we have



Note that the border of the circle is not in $B_r(x)$, as the distance metric asks for a strict inequality.

[1] iirc it won't be tested unless specified. its still good to learn tho

In a similar fashion, we have

Definition 1.4 (Open Set)**.** A set $U \subset \mathbb{R}^n$ is called *open* if, for all $x \in U$, there exists an open ball about x contained in U. Specifically, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

Intuitively, we could say that the set U does not contain its edge; if it did, let x be on said edge; then we would not be able to fit an open ball about x in U. Here are some examples of open sets from lecture;

1. The whole set $U = \mathbb{R}^n$ is open.

2. The empty set $\emptyset$ is open. Since there does not exist any $x \in \emptyset$, no conditions on balls need to be satisfied; "every dog in the empty set of dogs is green".

3. $(0, 1)$ is open on $\mathbb{R}^1$ (and any open interval, for that matter).

4. $B_r(x) \in \mathbb{R}^n$ is open. To see this, observe that for any $y \in B_r(x)$, we may pick $\varepsilon < r - |x - y|$; by triangle inequality, all elements in $B_\varepsilon(y)$ must also be in $B_r(x)$.

We now define images and pre-images; let us have a function between sets $f : X \to Y$. For subsets $A \subset X$ and $B \subset Y$,

$$f(A) = \{f(a) \mid a \in A\},$$
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Since f need not be injective, f^{-1} does not necessarily exist. While images of a union of subsets is a union of the images, the same is not true for intersections; let $A_1, \ldots, A_n$ be subsets of X; then

$$f\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f(A_i), \qquad f\left(\bigcap_{i=1}^n A_i\right) \subset \bigcap_{i=1}^n f(A_i).$$

On the other hand, pre-images preserve both union and intersection; let $B_1, \ldots, B_n \subset Y$; then

$$f^{-1}\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f^{-1}(A_i), \qquad f^{-1}\left(\bigcap_{i=1}^n A_i\right) = \bigcap_{i=1}^n f^{-1}(A_i).$$

As for set complements, we have

$$f^{-1}(B^C) = f^{-1}(B)^C,$$

but the same cannot be said for images; $f(A)^C \subset f(A^C)$ for surjective f, and vice versa for injective f. Equality occurs only if f is a bijection.

Note: for the proof of Theorem 1.2, I'll leave it out in case Prof. Bar-Natan proves it later on Thursday.