

# MAT367 Lecture Notes

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'26 Winter Semester

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## §1 Day 1: Recap of Preliminaries (Jan. 6, 2026)

Today's class can be followed more precisely on §1.2 to §1.4 of our textbook by **Gross and Meinrenken**. The slogan of this class is that a manifold is something that locally looks like  $\mathbb{R}^n$ . Specifically, an  $n$ -manifold can be covered  $n$ -dimensional charts  $(U \subset M) \rightarrow \mathbb{R}^n$ , with our main motivating example being solutions sets to equations. Recall the implicit function theorem,

**Theorem 1.1.** Given a smooth function  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , consider the solution set  $f(x_1, \dots, x_{n+1}) = 0$  and a point  $p \in \mathbb{R}^n$  such that  $\nabla f(p) \neq 0$ ; then, for  $(x_1, \dots, x_{n+1})$  in said solution set near  $p$ , we can represent solutions as  $(x_1, \dots, x_n, g(x_1, \dots, x_n))$ , where  $g$  is also a smooth function.

In particular, if 0 is a regular value<sup>1</sup> of  $f$ , then we can cover  $\{x \mid f(x) = 0\}$  by graphs/charts. We present some examples;

- (i) Let  $f(x, y) = xy$ ; then  $\ker f$  is precisely the  $x$  and  $y$  axes, which is not a manifold, because it does not look like  $\mathbb{R}^n$  (for any  $n$ ) near the origin.
- (ii) Let  $f(x, y) = y - x^{2/3}$ ; then  $\ker f$  can be graphed in desmos as  $y = x^{2/3}$ , which is not a smooth manifold because of its behavior at 0.
- (iii) The  $n$ -sphere  $S^n = \{x \in \mathbb{R}^n \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$  can be regarded as the level set of the  $\ell^2$ -norm, for which  $S^0 = \{\pm 1\} \subset \mathbb{R}$ ,  $S^1$  is a circle,  $S^2$  is the usual sphere. Note that we may use the stereographic projection as seen in complex analysis, to view  $S^3$  (and any of the previous or subsequence  $S^n$ ) as  $\mathbb{R}^3 \cup \{\infty\}$ .
- (iv) The 2-dimensional torus  $T^2$  is the surface of revolution obtained from a circle of radius  $r$  and  $R$  about an axis of revolution. It can be regarded as a level set by writing
 
$$T^2 = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\}.$$
- (v) The Möbius strip can't be a part of a level set (at a regular value) because level sets are orientable (2-sided), while the strip is not.
- (vi) The Klein bottle is also not orientable; it is closed (doesn't have a boundary), and doesn't embed into  $\mathbb{R}^3$ . It can be immersed into  $\mathbb{R}^3$ , i.e., locally embedded but not globally, as seen in the textbook.

**Theorem 1.2** (Whitney Embedding Theorem). Every  $n$ -manifold has an embedding in  $\mathbb{R}^{2n}$ .

In this class, we prefer to deal with intrinsic descriptions of manifolds rather than extrinsic ones; a good motivation is given on p.7 in the textbook with respect to our 2-torus.

Now, consider  $M$  to be the rotations of a ball. We call this a configuration space; i.e., its points are a way of configuring another object. How do we put coordinates on a piece of this space? To start, we wish to describe  $M$ ; one possible way is by considering the  $3 \times 3$  orthonormal matrices, i.e.,  $\text{SO}(3)$ ; another way is to first designate a point on the unit sphere as the north pole  $N$  (of which there are two degrees of freedom in this choice), then choosing where the vector  $(1, 0, 0)$  at  $N$  is mapped to (of which we have one degree of freedom).

In this manner, we may regard  $M$  as a 3-dimensional manifold and define coordinates for

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<sup>1</sup>note to self: what's a regular value?

points near the identity.

A second example is given by considering linkages, which are collections of line segments and joints (p.8-10); suppose we have four segments given by  $n_1, \dots, n_4$ , to be regarded as vectors in  $\mathbb{R}^2$ ; without additional constraints imposed, we see that this can be regarded as 8-dimensional. In imposing the conditions on *making*  $n_1, \dots, n_4$  a linkage, we see that  $\|n_i\|$  for each  $i = 1, \dots, 4$  is fixed (whence 4 less dimensions),  $n_1 + n_2 + n_3 + n_4 = 0$  (whence one less), and  $n_1$  is fixed (whence one less again), so such a linkage can be regarded as a 2 dimensional manifold with coordinates  $\theta$  and  $\varphi$ .

Finally, denote  $\mathbb{RP}^2$  the real projective plane, given by lines passing through the origin in  $\mathbb{R}^3$ ; equivalently, we may regard this as the pairs of antipodal  $\{x, -x\}$  points in  $S^2$ .

**Fact 1.3.** “Closed” (equivalently, compact) surfaces are easy to enumerate.

Fedya then said something about orientability, but that’s left in the textbook and easier to read there.

We now wish to define manifolds. For a first attempt, we want to say that a smooth manifold is a set  $M$  covered by a set of smooth charts  $\varphi_i : (U_i \subset M) \rightarrow \mathbb{R}^n$ , such that each  $p \in M$  is covered by some  $U_i$ . However, we run into a problem; what does “smooth” mean?  $f : U \rightarrow \mathbb{R}^n$  recall that a function is said to be smooth if it has partial derivatives of all orders. A function  $f : U \rightarrow V$  is a diffeomorphism if it’s a smooth bijection and its inverse is also smooth.

**Definition 1.4.** A coordinate chart is an injective map  $\varphi : U \rightarrow \mathbb{R}^n$  with open image for some  $U \subset M$ .

In particular, we say that two charts  $\varphi : U \rightarrow \mathbb{R}^n$  and  $\psi : V \rightarrow \mathbb{R}^n$  are compatible if the transition  $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$  is a diffeomorphism. Clearly, if  $U \cap V = \emptyset$ , then said maps are compatible; note, however, that compatibility is not an equivalence relation, since, while it is symmetric and reflexive, it is not necessarily transitive.