MAT354 Lecture Notes

Arky!! :3c

'25 Fall Semester

Contents

§1 Day 1: Recap of Preliminaries (Sep. 2, 2025)

We start by discussing the complex plane and complex numbers. Given $z \in \mathbb{C}$, we say that $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of z respectively, i.e., $z = x + iy$. $\mathbb{C}$ is the set of all complex numbers. In this manner, we may identify $z = x + iy$ with $(x, y) \in \mathbb{R}^2$ using the standard complex plane.

(a) The complex *conjugate* of z is given by $\bar{z} = x - iy$, where we have that

$$\Re(z) = \frac{z + \bar{z}}{2}, \qquad \Im(z) = \frac{z - \bar{z}}{2i}.$$

(b) We now define addition and mlutiplication for the complex numbers. For all $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have that

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \end{aligned}$$

We have that $(\mathbb{C}, +, \times)$ is a field, with $(\mathbb{R}, +, \times)$ as a subfield. To verify this, we need to check that it indeed satisfies:

- Commutativity; for all $z_1, z_2 \in \mathbb{C}$, we have that $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$.
- Associativity: for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
- Distributivity: for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

(c) The absolute value of a complex number $z = x + iy$ is given by $|z| = \sqrt{x^2 + y^2}$. In particular, this yields the triangle inequality, where for any $z, w \in \mathbb{C}$, we have that $|z + w| \leq |z| + |w|$. The proof either comes visually or through explicit computation, both of which I will not write out here for brevity.[1]

As an extension of the inequality, we also automatically have that

$$|\Re z| \leq |z|, \qquad |\Im z| \leq |z|,$$

and that for all $z, w \in \mathbb{C}$, we have

$$||z| - |w|| \leq |z - w|.$$

Proof. Using the triangle inequality, we have that

$$\begin{aligned} |z| &= |(z - w) + w| \leq |z - w| + |w|, \\ |w| &= |(w - z) + z| \leq |z - w| + |z|, \end{aligned}$$

of which both imply that $|z| - |w| \leq |z - w|$ and $|w| - |z| \leq |z - w|$. □

For any $z \in \mathbb{C}$, we have that $|z|^2 = z \cdot \bar{z}$.

Proof. Write $z = x + iy$; then $|z|^2 = x^2 + y^2$, where we may note that $z \cdot \bar{z} = (x + iy)(x - iy)$ which yields the right hand side of the earlier equation through expansion. □

[1] no full credit if you draw a picture on the exam lmao

Finally, for $z, w \in \mathbb{C}$, we have that $|zw| = |z|\,|w|$. This is left as an exercise to the student.

(d) The polar form of a nonzero complex number $z \neq 0$ is given by $z = \gamma e^{i\theta}$, where $\gamma > 0$ and $\theta \in \mathbb{R}$. Let us assume the Euler formula; for all $\theta \in \mathbb{R}$, we have that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Let $r = |z|$; we have that $|z| = \left|re^{i\theta}\right| = |r|\left|e^{i\theta}\right| = r \cdot 1 = r$. θ is the angle between the positive real axis to the half-line starting from 0 and passing through z. In this manner, $z = re^{i\theta} = |z|(\cos\theta + i\sin\theta) = |z|\cos\theta + i\,|z|\sin\theta$, which means we have that

$$\Re z = |z|\cos\theta, \qquad \Im z = |z|\sin\theta.$$

As an example, let us find all the complex numbers z such that $z^4 = i$. Since $i = e^{i\frac{\pi}{2}}$, $z = \rho e^{i\theta}$ satisfying $z^4 = i$ becomes $\rho^4 e^{i\cdot 4\theta} = e^{i\frac{\pi}{2}}$, meaning

$$\begin{cases} \rho^4 = 1, \\ 4\theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

This means $\rho = 1$ and $\theta = \frac{\pi}{8} + \frac{k\pi}{2}$, where $k \in \mathbb{Z}$. Considering the cases $k = 0, 1, 2, 3$ and observing that there are only 4 equivalence classes modulo 4 to consider, we have that

$$z_0 = e^{i\frac{\pi}{8}}, \quad z_1 = e^{i\frac{5\pi}{8}}, \quad z_2 = e^{i\frac{9\pi}{8}}, \quad z_3 = e^{i\frac{13\pi}{8}}.$$

We now discuss convergence. We say that a set of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ converges to $w \in \mathbb{C}$ if $\lim_{n\to\infty} |z_n - w| = 0$. We write it as $\lim_{n\to\infty} z_n = w$. In the complex plane, the convergence can be in any direction.

Lemma 1.1. $\{z_n\}_{n\in\mathbb{N}}$ converges to w if and only if $\{\Re z_n\}_{n\in\mathbb{N}}$ converges to $\Re w$ and $\{\Im z_n\}_{n\in\mathbb{N}}$ converges to $\Im w$.

Proof. We have that

$$\begin{aligned} |z_n - w| &= |(\Re z_n - \Re w) + i(\Im z_n - \Im w)| \\ &\leq |\Re z_n - \Re w| + |\Im z_n - \Im w|, \end{aligned}$$

where as $n \to \infty$, we have that the right hand side is given by $0 + 0$. For the opposite direction, we have that $|z| \geq |\Re z|$ or $|\Im z|$, so we have that

$$|\Re z_n - \Re w| = |\Re(z_n - w)| \leq |z_n - w|,$$

which approaches 0 as $n \to \infty$. The same argument goes for the imaginary portion. □

A sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ is called *Cauchy* if $|z_n - z_m| \to 0$ as $n, m \to \infty$. In $\varepsilon - \delta$, this means that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|z_n - z_m| < \varepsilon$ for all $n, m > N$.

Theorem 1.2 (Bolzano–Weierstrass Theorem)**.** $\mathbb{R}$ is *complete*, i.e., every Cauchy sequence of real numbers convergesto a real number.

Theorem 1.3. $\mathbb{C}$ is complete.

Proof. Take any Cauchy sequence of complex numbers $\{z_n\}$. Using the inequalities $|\Re z| \leq |z|$ and $\{\Im z\} \leq |z|$, we have that $\{\Re z_n\}$ and $\{\Im z_n\}$ are Cauchy sequences of real numbers. By Bolzano-Weierstrass, we have that $\Re z_n \to x_0 \in \mathbb{R}$ and $\Im z_n \to y_0 \in \mathbb{R}$. By the previous lemma, we actually have $\lim_{n\to\infty} z_n = x_0 + iy_0$. □

We now move onto topology in the complex plane. Given $z_0 \in \mathbb{C}$ and $r > 0$, we can form an open or closed disc centered at z_0 of radius r. We write both of these as

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$
$$\bar{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\},$$

Given a set $\Omega \subseteq \mathbb{C}$, a point z_0 is an interior point if there exists $r > 0$ such that $D_r(z) \subseteq \Omega$. The interior of Ω is given by the set of all such interior points. In particular, the interior of $\bar{D}_r(i)$ is $D_r(i)$.

A set Ω is called *open* if every point in Ω is an interior point. Ω is called *closed* if the complement of Ω, $\Omega^c = \mathbb{C} \setminus \Omega$, is open. As an example, the open right half-plane $\{z \in \mathbb{C} \mid \Re z > 0\}$ is open.

Proof. For any $z \in \Omega$, let $z = x + iy$, and take $r = \frac{x}{2} = \frac{\Re z}{2}$. Then we claim that $D_r(z) \subseteq \Omega$. For all $w \in D_r(z)$, we clearly have that

$$\Re w = \Re z - (\Re z - \Re w) \geq \Re z - |z - w| \geq \frac{\Re z}{2} > 0,$$

and so all such $w \in \Omega$, and we are done. $\square$

A point $z \in \mathbb{C}$ is a *limit point* of Ω if there exists a sequence $\{z_n\} \subset \Omega$ with $z_n \neq z$ such that $z_n \to z$.

As an example, we define D to be the open unit disc centered at 0. 0 and 1 are both limit points of D, but 1 is not contained in D itself.[2] The *closure* of Ω, $\bar{\Omega}$, is given by Ω unioned with all its limit points. The *boundary* of a set Ω, wirtten $\partial\Omega$, is given by $\bar{\Omega} \setminus \operatorname{int}\Omega$. A set $\Omega \subseteq \mathbb{C}$ is said to be compact if it is closed and bounded, i.e., there exists $M > 0$ such that $|z| \leq M$ for all $z \in \Omega$.

Theorem 1.4. A set $\Omega \subseteq \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω.

Proposition 1.5. If $\Omega_1 \supset \Omega_2 \cdots \supset \Omega_n \supset \ldots$ is a sequence of nonempty compact sets in $\mathbb{C}$, where $\operatorname{diam}(\Omega_n) = \sup_{z,w\in\Omega_n} |z - w| \to 0$ as $n \to \infty$, then there exists a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for every $n \in \mathbb{N}$.

Proof. For each Ω_n, pick a point $z_n \in \Omega_n$. Then $\{z_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence because the diameter of Ω_n approaches 0. By the Bolzano-Weierstrass theorem for complex numbers, this means that $\{z_n\}_{n\in\mathbb{N}}$ indeed does converge to some $w \in \mathbb{C}$. In particular, we have w is the limit of the subsequence $\{z_m\}_{m\geq n} \subseteq \Omega_n$, where Ω_n is compact, meaning the limit w should be in Ω_n. This means there exists a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for every $n \in \mathbb{N}$.

To show the uniqueness of w, we argue by contradiction; assume $w' \neq w$ satisfies the property. Then $|w' - w| > 0$. Since $w, w' \in \Omega_n$ for all n, this contradicts that $\operatorname{diam}(\Omega_n) \to 0$. $\square$

An open set Ω is called *connected* if it is not possible to find two disjoint nonempty open sets Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. A connected open set in $\mathbb{C}$ is called a *region*.

[2] hell is it disc or disk YKW LET'S COMPROMISE it's spelled disque actually (paint nails)

§2 Day 2: Functions on the Complex Plane (Sep. 4, 2025)

Let $f : \Omega \to \mathbb{C}$, where Ω is an open subset of $\mathbb{C}$. We say that f is continuous if at $z_0 \in \Omega$ if, for all $\varepsilon > 0$, there exists an open disk $D_\gamma(z_0)$ such that $|f(z) - f(z_0)| < \varepsilon$ for all $z \in D_\gamma(z_0)$. In particular, f is said to be continuous on Ω if it is continuous at every point in Ω.

Example 2.1. Consider $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$. Show that f is continuous.

Solution. For all complex z, z_0, we have that $|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0| = |z - z_0|$. Thus, we have that for any $\varepsilon > 0$, we obtain[3]

$$f(D_\varepsilon(z_0)) = D_\varepsilon(\overline{z_0}). \qquad \square$$

We now discuss holomorphic functions (i.e., complex differentiable functions). We say that $f : \Omega \to \mathbb{C}$ is *holomorphic* at $z_0 \in \Omega$ if

$$\frac{f(z_0 + h) - f(z_0)}{h}, \qquad h \in \mathbb{C} \setminus \{0\},$$

converges as $h \to 0$. If the limit exists, we let

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

be the derivative.

Example 2.2. Consider the exact same function as in the previous example, $f(z) = \bar{z}$. Is f holomorphic?

Solution. For all $z_0 \in \mathbb{C}$ and $h \in \mathbb{C} \setminus \{0\}$, we have that

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{z_0 + h} - \overline{z_0}}{h} = \frac{\overline{h}}{h} = \frac{\rho e^{-i\theta}}{\rho e^{i\theta}} = e^{-2i\theta}.$$

If we take $h \to 0$ along the real line, we may let $h = \rho$, which means the fraction is equal to 1 as $h \to 0$. If we take $\rho \to 0$ along the complex axis, however, then we have that $h = \rho e^{i\pi/2}$, where we obtain the fraction is equal to -1 as $\rho \to 0$. Thus, f cannot be holomorphic. $\square$

Proposition 2.3. Let Ω be open in $\mathbb{C}$. If f, g are holomorphic on Ω, then

(i) $f + g$ is holomorphic on Ω, and $(f + g)' = f' + g'$.

(ii) fg is holomorphic on Ω, and $(fg') = f'g + fg'$.

(iii) If $g(z_0) \neq 0$ where $z_0 \in \Omega$, then $\frac{f}{g}$ is also holomorphic at z_0, where

$$\left(\frac{f}{g}\right)' = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

(iv) If $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, then $g \circ f$ is also holomorphic, and we obtain the chain rule

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

[3] note to self: ol is better than bar for this stuff...

We now discuss complex differentiability versus real differentiability. A holomorphic function $f : \Omega \to \mathbb{C}$ can be identified with a function $F : \Omega \to \mathbb{R}^2$ given by $(x, y) \mapsto (u(x, y), v(x, y)) = (\Re f(x, y), \Im f(x, y))$. Consider the partial derviative of F at (x_0, y_0); these exist if there exists some linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{\|F(P_0 + H) - F(P_0) - J(H)\|}{\|H\|} \to 0$$

as $H \to 0$. Or, we may define $\Psi(H)$ to take on the fraction above, and we see that F is indeed differentiable at $P_0 = (x_0, y_0)$ if $\Psi(H) \to 0$ as $H \to 0$. We now deal with complex differentiability. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic at $z_0 = x_0 + iy_0$. Then we have partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}.$$

Naturally,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h - f(z_0)}{h}$$

along any path; in particular, we take $h \in \mathbb{R} \setminus \{0\}$ and observe that

$$\begin{aligned}
f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{h \to 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\
&= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0),
\end{aligned}$$

and so both exist, and they are $\Re f'(z_0)$ and $\Im f'(z_0)$ respectively. Similarly, we may take $h = ik$ where $k \in \mathbb{R} \setminus \{0\}$ and obtain

$$\begin{aligned}
f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{k \to 0} \frac{u(x_0, y_0 + k) + iv(x_0, y_0 + k) - u(x_0, y_0) - iv(x_0, y_0)}{ik} \\
&= \lim_{k \to 0} \frac{-i(u(x_0, y_0 + k) - u(x_0, y_0)) + v(x_0, y_0 + k) - v(x_0, y_0)}{k} \\
&= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0),
\end{aligned}$$

and so both partials also exist and they are $\Re f'(z_0)$ and $-\Im f'(z_0)$ respectively.

§3 Day 3: Holomorphic Functions and Power Series (Sep. 9, 2025)

Let $f : \Omega \to \mathbb{C}$ (where Ω is an open set in $\mathbb{C}$). We say that f is holomorphic at z_0 if

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}, \quad h \in \mathbb{C}\setminus\{0\}$$

exists. Recall that $\mathbb{C}$ can be identified with $\mathbb{R}^2$ by considering any $z = x+iy \in \mathbb{C}$ as a tuple $(x,y) \in \mathbb{R}^2$. In this way, given a function $f : \Omega \to \mathbb{C}$, we can define $F : \Omega \to \mathbb{R}^2$, where $F : (x,y) \mapsto (u(x,y), v(x,y))$, given by $u = \Re f$ and $v = \Im f$.

Proposition 3.1. If $f = u+iv$ is holomorphic at $z_0 = x_0 + iy_0$, then we have that all four partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist and they satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \Re f(z_0), \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \Im f(z_0).$$

We also have that F is differentiable at $P_0 = (x_0, y_0)$.

Definition 3.2. We say that F is differentiable at P_0 if there exists a linear transforamtion (the derivative) $J = J_F(x_0,y_0) : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{H\to 0} \frac{\|F(P_0+H) - F(P_0) - J(H)\|}{\|H\|} = 0.$$

Before we discuss the complex definition, let us recall another property of real differentiability; if F is differentiable at $P_0 = (x_0, y_0)$, then all four partial derivatives exist, and

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is called the Jacobian matrix of F at (x_0, y_0). To see this, consider the association $P_0 = (x_0, y_0)$ with $z_0 = x_0 + iy_0$, and $H = (h_1, h_2)$ with $h = h_1 + ih_2$; then we have that

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 \\ \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 \end{pmatrix}.$$

This is a vector in $\mathbb{R}^2$, which we may associate with the complex number

$$\left(\frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2\right) + i\left(\frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2\right) = \left(\frac{\partial u}{\partial x} + i\frac{\partial y}{\partial x}\right) h_1 + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) h_2,$$

which, by the Cauchy–Riemann equations, we obtain

$$\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) h_1 + i\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\right) h_2 = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2),$$

which is precisely equal to $f(z_0+h) - f(z_0) - f(z_0)h$. In particular,

$$\lim_{h\to 0} \left| \frac{f(z_0+h) - f(z_0) - f(z_0)h}{h} \right| = \lim_{h\to 0} \left| \frac{f(z_0+h)-f(z_0)}{h} - f(z_0) \right| = 0.$$

Similarly, per the definition of the Jacobian, we must have

$$\lim_{H\to 0} \frac{\|F(P_0+H) - F(P_0) - J(H)\|}{\|H\|} = 0,$$

and this concludes the proof of proposition 3.1. $\square$

Theorem 3.3. Suppose $f = u + iv$ is a complex-valued function defined on an open set $\Omega \subset \mathbb{C}$. If $u, v : \Omega \to \mathbb{R}$, are continuously differentiable and satisfy the Cauchy–Riemann equations, then f is holomorphic on Ω and $f'(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)$.

Proof. Since u is continuously differentiable at the point $(x, y) \in \Omega$, there exists a linear transformation $J_u : \mathbb{R}^2 \to \mathbb{R}$ where

$$\frac{|u(x + h_1, y + h_2) - u(x, y) - J_0(h_1, h_2)|}{\|(h_1, h_2)\|} \to 0, \quad (h_1, h_2) \to 0.$$

In particular, $J_u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$. The above fraction is equivalent to

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + \|h\|\, \psi_1(h),$$

where $\psi_1 : U \to \mathbb{R}$, where U is some open neighborhood of $0 \in \mathbb{R}^2$, with $\psi_1(h) \to 0$ as $h \to 0$. Similarly, we have that

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + \|h\|\, \psi_2(h)$$

with $\psi_2(h) \to 0$ as $h \to 0$. We want to show that f is holomorphic at $z = x + iy$. We have that

$$\begin{aligned} f(z + h) - f(z) &= (u(x + h_1, y + h_2) - u(x, y)) + i(v(x + h_1, y + h_2) - v(x, y)) \\ &= \left(\frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial h_y}h_2\right) + \|h\|\, \psi_1(h) + i\left(\frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2\right) + i\,\|h\|\, \psi_2(h) \\ &= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + \|h\|\, \psi_1(h) + i\,\|h\|\, \psi_2(h) \end{aligned}$$

from Cauchy–Riemann. Thus, we have that[4]

$$\begin{aligned} \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} &= \lim_{h \to 0} \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + \frac{\|h\|}{h}(\psi_1(h) + i\psi_2(h)) \\ &= \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right). \end{aligned}$$ ◻

We now discuss complex power series.

Definition 3.4. A complex power series is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

with $a_n \in \mathbb{C}$ and z a complex variable. We say that $\sum_{n=0}^{\infty} a_n z^n$ converges at $z_0 \in \mathbb{C}$ if there exists some $w \in \mathbb{C}$ such that, for all $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $N \geq N_0$ satisfies

$$\left|\sum_{n=0}^{N} a_0 z^n - w\right| < \varepsilon.$$

The series converges *absolutely* at z_0 if there exists $w \in \mathbb{R}$ such that

$$\left|\sum_{n=0}^{N} |a_n|\, |z_0|^n - w\right| < \varepsilon.$$

[4] i swear wenyu has an invisible key wired into her back like nano from nichijou and it's permanently cranked on

Proposition 3.5. If $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely at $z_0 \in \mathbb{C}$, then $\sum_{n=0}^{\infty} a_n z^n$ converges at $z_1 \in \mathbb{C}$ with $|z_1| \leq |z_0|$.

Proof. For all $z_1 \in \mathbb{C}$ with $|z_1| \leq |z_0|$, consider the sequence of partial sums $\{S_m(z_1)\}_{m\in\mathbb{N}}$ given by

$$S_m(z_1) = \sum_{n=0}^{m} a_n z_1^n.$$

We want to show that such a sequence converges. Since $\mathbb{C}$ is complete, it suffices to show that said sequence is Cauchy. For all $m < k \in \mathbb{N}$, we have that

$$|S_k(z_1) - S_m(z_1)| = \left|\sum_{n=m+1}^{k} a_n z_1^n\right| \leq \sum_{n=m+1}^{k} |a_n| |z_1|^n \leq \sum_{n=m+1}^{k} |a_n| |z_0|^k . \qquad \square$$

We now provide a few examples.

(i) The complex exponential function for all $z \in \mathbb{C}$, given by

$$e^z := \sum_{n=0}^{\infty} = \frac{z^n}{n!}.$$

For all $z \in \mathbb{C}$, this sum converges because it converges absolutely (consider $e^{|z|}$).

(ii) The geometric series $\sum_{n=0}^{\infty} z^n$, where $|z| < 1$, converges; otherwise, is $|z| \geq 1$, it diverges. In particular, if $\sum_{n=0}^{\infty} z^n$ converges, then $|z^n| \to 0$ as $n \to \infty$.

Theorem 3.6 (Shakarchi, Thm. 2.5)**.** Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $R \in [0, \infty)$ such that (i) if $|z| < R$, the series converges, and (ii) if $|z| > R$, the series diverges. We call R the *radius of convergence* of $\sum_{n=0}^{\infty} a_n z^n$, and $\{z \in \mathbb{C} \mid |z| < R\}$ the disc[5] of convergence. Moreover, R is given by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} =: L,$$

where we use the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proof. For all $z \in \mathbb{C}$ with $|z| < r < R$, there exists some $\varepsilon > 0$ such that

$$(L + \varepsilon) |z| = r < 1.$$

By definition of L, we have $|a_n|^{1/n} \leq L + \varepsilon$ for all large n, meaning that

$$|a_n| |z|^n = \left(|a_n|^{1/n} |z|\right)^n \leq ((L + \varepsilon) |z|)^n = r^n, \quad r \in (0, 1),$$

whereby comparison with the geometric series $\sum r^n$, we see that $\sum |a_n| |z|^n$ converges. Similarly, if $|z| > R$, we have that

$$\left(\frac{1}{r} - \varepsilon\right) |z| > 1,$$

where, using the definition of R, there exists an infinite subsequence a_{n_k} such that $|a_{n_k}|^{1/n_k} \geq \frac{1}{R} - \varepsilon$. We have that

$$|a_{n_k} z^{n_k}| = \left(|a_{n_k}|^{1/n_k} |z|\right)^{n_k} \geq \left[\left(\frac{1}{R} - \varepsilon\right) |z|\right]^{n_k} > 1. \qquad \square$$

[5] disque. ok i'll stop

§4 Day 4: Complex Power Series (Sep. 11, 2025)

As per given in the previous lecture, recall that the complex power series is defined as an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, z \in \mathbb{C},$$

i.e., z as a complex variable.

Theorem 4.1 (Thm. 2.6, Shakarchi)**.** The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function on its disc of convergence. The derivative of f is given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Moreover, f' has the same radius of convergence as f.

Proof. Let g be the power series defining f', and let $R \geq 0$ be the radius of convergence of f. The radius of convergence of g is also R, per Hadamard's formula,

$$\limsup_{n\to\infty} |na_n|^{\frac{1}{n-1}} \overset{(*)}{=} \limsup_{n\to\infty} |a_n|^{\frac{1}{n}\cdot\frac{n}{n-1}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R},$$

since

$$n^{\frac{1}{n-1}} = e^{\frac{\log n}{n-1}} \xrightarrow{n\to\infty} e^0 = 1. \tag{$*$}$$

For all $z_0 \in \mathbb{C}$ with $|z_0| < r < R$ and $h \in \mathbb{C} \setminus \{0\}$ with $|z_0 + h| < r$, let us compute the following,

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right|;$$

to start,

$$f(z) = \underbrace{\sum_{n=0}^{N} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{E_N(z)},$$

where $N \in \mathbb{N}$ is to be determined; we have that

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right) + \left(S_N'(z_0) - g(z_0)\right) + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h} \right).$$

We compute each part individually.

$$\begin{aligned}
\left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| &= \left| \frac{\sum_{n=N+1}^{\infty} a_n (z_0+h)^n - \sum_{n=N+1}^{\infty} a_n z_0^n}{h} \right| \\
&\leq \sum_{n=N+1}^{\infty} \frac{|a_n|}{h} \left|(z_0+h)^n - z_0^n\right| \\
&\leq \sum_{n=N+1}^{\infty} |a_n| \left|(z_0+h)^{n-1} + (z_0+h)^{n-2} + \cdots + z_0^{n-1}\right| \\
&\leq \sum_{n=N+1}^{\infty} |a_n| \, \gamma^{n-1} \cdot n \xrightarrow{N\to\infty} 0,
\end{aligned}$$

as y has the radius of convergence of $R > r$. Next,

$$\left|S_N'(z_0) - g(z_0)\right| \xrightarrow{N\to\infty} 0,$$

since $S_N'(z_0) = \sum_{n=1}^N na_n z_0^{n-1}$ and $g(z_0) = \sum_{n=1}^\infty na_n z_0^{n-1}$. Given any $\varepsilon > 0$, we may choose a sufficiently large N such that

$$\left|S_N'(z_0) - g(z_0)\right| < \varepsilon, \quad \left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| < \varepsilon,$$

per our two computations above. Since $S_N(z)$ is a finite polynomial, $S_N'(z_0)$ is the derivative of $S_N(z)$ at z_0, and so there exists $\delta > 0$ such that, for all $0 \leq |h| < \delta$, we have

$$\left|\frac{S_N(z_0+h) - S_N(z_0)h}{h} - S_N'(z_0)\right| < \varepsilon,$$

which resolves all three parts of our expansion, and so we are done. □

Corollary 4.2. The power series $f(z) = \sum_{n=0}^\infty a_n z^n$ is infinitely complex differentiable on its disc of convergence. For $k \in \mathbb{N}$, its kth derivative $f^{(k)}$ is given by

$$f^{(k)}(z) = \sum_{n=0}^\infty (a_n z^n)^k.$$

Definition 4.3. A function $f : \Omega \to \mathbb{C}$ is said to be *analytic* at $z_0 \in \Omega$ if there exists a power series $\sum_{n=0}^\infty a_n(z-z_0)^n$ with positive radius of convergence such that

$$f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n$$

on a neighborhood of $z_0 \in \Omega$.

In particular, this means that if $f : \Omega \to \mathbb{C}$ is holomorphic, we have that f is holomorphic at $z_0 \in \Omega$, and so f is analytic at $z_0 \in \Omega$ as well. The implication that analytic implies holomorphic was given by our earlier theorem; the direction that holomorphic implies analytic is given by Cauchy's integral formula, but we need to first define integration along curves.

(i) A parameterized curve is a function $z : [a,b] \to \mathbb{C}$, where $t \mapsto z(t)$. This gives the orientation from $z(a)$ to $z(b)$.

(ii) (*Regularity conditions on curves*). We say that the parameterized curve is smooth if $z'(t)$ exists, is continuous on $[a,b]$, and $z'(t) \neq 0$ for $t \in [a,b]$. We say that the parameterized curve z is piecewise smooth if z is continuous on $[a,b]$ and there exists a partition of $[a,b]$ with $a = a_0 < \cdots < a_n = b$ such that $z(t)$ is smooth on each $[a_r, a_{r+1}]$.

§5 Day 5: Curves in the Complex Plane (Sep. 16, 2025)

We say that a parameterized curve is a function $z : [a, b] \to \mathbb{C}$ where $t \mapsto z(t)$; in particular, z gives the orientation from $z(a)$ to $z(b)$.We say that z is *smooth* if $z'(t)$ exists and is continuous on $[a, b]$, where $z'(t) \neq 0$ for $t \in [a, b]$. We say it's *piecewise smooth* if z is continuous on $[a, b]$ and we have a partition $a = a_0 < \cdots < a_n = b$ such that $z(t)$ is smooth on each $[a_k, a_{k+1}]$.

Example 5.1. Let $z : [0, 2\pi] \to \mathbb{C}$, where $t \mapsto z_0 + Re^{it}$, and $z_1 : [0, \frac{\pi}{2}] \to \mathbb{C}$, where $t_0 \mapsto z_0 = Re^{i4t}$.

We say that two smooth parameterizations, $z : [a, b] \to \mathbb{C}$ and $\widetilde{z} : [c, d] \to \mathbb{C}$, are *equivalent* if they have the same image and orientation; i.e., if there exists a continuously differentiable bijection $s \mapsto t(s)$ from $[c, d]$ to $[a, b]$ such that $t'(s) > 0$ (read: same orientation) and $\widetilde{z} = z \circ t$. In this way, all equivalent smooth parameterizations of $z : [a, b] \to \mathbb{C}$ can be written as a smooth curve γ with image $z([a, b])$ and orientation from $z(a)$ to $z(b)$. In addition, we denote γ^- as said smooth curve, but with reversed orientation.

A smooth or piecewise smooth curve given by $z : [a, b] \to \mathbb{C}$ is said to be *closed* if $z(a) = z(b)$, and *simple* if $z(t) \neq z(s)$ for all $t \neq s$ in the time interval (note that if the curve is closed, we allow $s = a$, $t = b$ to satisfy $z(s) = z(t)$). We now define integration along curves.

Definition 5.2. Let $f : \Omega \to \mathbb{C}$ be a continuous function, and let γ be a smooth curve in Ω parameterized by $z : [a, b] \to \mathbb{C}$. Then

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt,$$

where we may realize $f \circ z : [a, b] \to \mathbb{C}$. The length of γ is defined as $\text{length}(\gamma) = \int_a^b |z'(t)|\,dt$.

Example 5.3. Consider the function $f(z) = z^{-1}$ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.[6] Let C be a circle in $\mathbb{C}^*$ centered at z_0 with radius $R > 0$, equipped with an anticlockwise orientation. Compute $\int_C f(z)\,dz$.

While this example seems trivial, there is a lot of casework to work through, and we don't have the prerequisite knowledge for it yet.

Proposition 5.4. Integration of continuous functions along smooth (or piecewise smooth) curves satisfy the following properties,

(i) (*Linearity*) For all $\alpha, \beta \in \mathbb{C}$, we have that

$$\int_\gamma (\alpha f + \beta g)(z) = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz$$

(ii) If γ^- is γ with reversed orientation, then

$$\int_{\gamma^-} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

[6] 417 notation seeping into my 354 work

(iii) We have the following inequality,

$$\left| \int_\gamma f(z)\, dz \right| \leq \left(\sup_{z \in \gamma} |f(z)| \right) \cdot \text{length}(\gamma).$$

Exercise 5.5. Check that the definition of integration is well-defined.

We now prove the above proposition.

Proof. Assume γ is smooth and parameterized by $z : [a, b] \to \mathbb{C}$. Then

$$\left| \int_\gamma f(z)\, dz \right| = \left| \int_a^b f(z(t)) \cdot z'(t)\, dt \right| \leq \int_a^b \left| f(z(t)) \cdot z'(t) \right|\, dt,$$

which we note is true by considering

$$\left| \sum_i u(t_i) + iv(t_i) \Delta t \right| \leq \sum_i |u(t_i) + iv(t_i)|\, \Delta t,$$

so we indeed have that

$$\int_a^b \left| f(z(t)) \cdot z'(t) \right|\, dt \leq \left(\sup_{z \in [a,b]} |f(z)| \right) \cdot \int_a^b \left| z'(t) \right|\, dt = \left(\sup_{z \in [a,b]} |f(z)| \right) \cdot \text{length}(\gamma) \quad \square$$

Suppose $f : \Omega \to \mathbb{C}$. A *primitive* for f on Ω is a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \Omega$.

Theorem 5.6 (Complex Fundamental Theorem of Calculus)**.** If a continuous function f has a primitive F on Ω, and γ is a curve that begins at w_1 and ends at w_2, then

$$\int_\gamma f(z)\, dz = F(w_2) - F(w_1).$$

Proof. Suppose γ is smooth and parameterized by $z : [a, b] \to \mathbb{C}$ with $z(a) = w_1$ and $z(b) = w_2$. Then

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t)) z'(t)\, dt = \int_a^b (F(z(t)))'\, dt,$$

since we may note that $(F \circ z)' = (F' \circ z) \cdot z' = (f \circ z) \cdot z'$, whereby we note that the above integral evaluates to $F(z(b)) - F(z(a)) = F(w_2) - F(w_1)$. $\square$

Corollary 5.7. If f is holomorphic on a region Ω and $f' = 0$, then f is constant.

Recall that Ω is called a region if it is an open connected set. Alternatively, connectedness is equivalent to path connectedness here, since if Ω is path connected, it is connected (by Medusa), and if it is connected, then it is locally path connected, and through a partition, local path connectedness implies path connectedness.

Proof. Note that Ω is path connected per our earlier digression; fix $z_0 \in \Omega$. We will show that $f(z) = f(z_0)$ for all $z \in \Omega$; let z, z_0 be joined by a piecewise smooth curve γ. Then we have

$$0 = \int_\gamma f'(z)\, dz = f(z) - f(z_0),$$

and so f is constant on Ω. $\square$

Theorem 5.8 (Goursat's Theorem)**.** If Ω is an open set in $\mathbb{C}$ and $T \subset \Omega$ is a triangle whose interior is also in Ω, then for any holomorphic function f on Ω, we have $\int_T f(z)\,dz = 0$.

Proof. Let $T^{(0)}$ be the original triangle. Let $d^{(0)}, p^{(0)}$ be the diameter and perimeter of $T^{(0)}$ respectively. Take the midpoints of each side of $T^{(0)}$, and form 4 smaller triangles with orientation consistent to the orientation of $T^{(0)}$; we will call these triangles $T_1^{(1)}, \dots, T_4^{(1)}$. Clearly,

$$\int_{T^{(0)}} f(z)\,dz = \sum_{k=1}^{4} \int_{T_k^{(1)}} f(z)\,dz,$$

along with

$$\left|\int_{T^{(0)}} f(z)\,dz\right| = \sum_{k=1}^{4} \left|\int_{T_k^{(1)}} f(z)\,dz\right|.$$

Let $T_j^{(1)}$ be chosen to be such that $\left|\int_{T_k^{(1)}} f(z)\,dz\right|$ is maximal among $k \in \{1, \dots, 4\}$; we will write $T^{(1)} = T_j^{(1)}$, and iterate this process to obtain a sequence of triangles $\{T_0, T_1, \dots\}$, where

$$\left|\int_{T^{(0)}} f(z)\,dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)\,dz\right|.$$

$d^{(k)}, p^{(k)}$ are defined analogously, where

$$d^{(k)} = \frac{1}{2^k} d^{(0)}, \quad p^{(k)} = \frac{1}{2^k} p^{(0)}.$$

Let $\mathcal{T}^{(n)}$ be the solid triangle enclosed by $T^{(n)}$. Clearly, $\mathcal{T}^{(0)} \supset \mathcal{T}^{(1)} \supset \cdots \supset \mathcal{T}^{(n)}$, and there exists a unique $z_0 \in \mathbb{C}$ such that $z_0 \in \mathcal{T}^{(n)}$ for every n; since f is holomorphic at z_0, we have that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

with $\psi(z) \to 0$ as $z \to z_0$. We may write,

$$\int_{T^{(n)}} f(z)\,dz = \int_{T^{(n)}} f(z_0)\,dz + \int_{T^{(n)}} f'(z_0)(z - z_0)\,dz + \int_{T^{(n)}} \psi(z)(z - z_0)\,dz.$$

The first two terms vanish, since $f(z)$, $f'(z_0)(z-z_0)$ have primitives $f(z_0)z$ and $\frac{1}{2}f(z_0)(z-z_0)^2$ respectively. It remains to compute the last term; we have that

$$\left|\int_{T^{(n)}} \psi(z)(z - z_0)\,dz\right| \leq \left(\sup_{z \in T^{(n)}} |\psi(z)|\right)\left(\sup_{z \in T^{(n)}} |z - z_0|\right) \operatorname{length} T^{(n)}$$

where we note the first term approaches 0 as $n \to \infty$, the second term is bounded above by $2^{-n} d^{(0)}$, and the third term is bounded above by $2^{-n} p^{(n)}$. We may combine everything to obtain

$$\left|\int_{T^{(0)}} f(z)\,dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)\,dz\right| \leq d^{(0)} p^{(0)} \left(\sup_{z \in T^{(n)}} |\psi(z)|\right) \xrightarrow{n \to \infty} 0. \qquad \square$$

§6 Day 6: Cauchy's Theorem on a Disc (Sep. 18, 2025)

Recall Goursat's theorem from last class, where if $\Omega \subset \mathbb{C}$ is open and $T \subset \Omega$ is a triangle whose interior is contained in Ω, then for any holomorphic function f on Ω, we have that

$$\int_T f(z)\,dz = 0.$$

We introduce a follow-up to this thoerem.

Theorem 6.1. If f is holomorphic on a disc, then $\int_\gamma f(z)\,dz = 0$ for any closed curve γ in that disc.

To prove this, we start by using Goursat's theorem to show f has a primitive, and then we complete the proof using the complex FTC, i.e., if f is holomorphic on a disc, then f has a primitive on that disc.

Proof. After a translation, we may assume that the center of the disc is 0. Define $F : D \to \mathbb{C}$, given by $z \mapsto \int_{\gamma_z} f(u)\,du$. To show that F is holomorphic and $F'(z) = f(z)$, fix $z \in D$, and observe that for any $h \in \mathbb{C} \setminus \{0\}$ with $z + h \in D$, we have that

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(u)\,du - \int_{\gamma_z} f(u)\,du.$$

Regard this as the path from z to 0 to $z+h$. Let us add to the expression the integrals over two paths, going both directions so that we do not change the value of $F(z+h) - F(z)$, one between z and $\Re(z+h) + i\Im(z)$, and one between z and $z+h$ directly. In this manner, we've created a rectangular region and a triangular region on which we have path integrals over, and per Goursat's theorem, they all vanish, and we are left with the integral on the path η from z to $z+h$. This means all that remains is to compute[7]

$$F(z+h) - F(z) = \int_\eta f(w)\,dw$$

Since f is continuous at z, we may write $f(w) = f(z) + \psi(w)$, where $\psi(w) \to 0$ as $w \to z$. This means we may write

$$\int_\eta f(w)\,dw = \int_\eta f(z)\,dw + \int_\eta \psi(w)\,dw = f(z)(z+h-z) + \int_\eta \psi(w)\,dw,$$

upon which we may rearrange and rewrite the above RHS to obtain

$$\begin{aligned}\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &= \left|\frac{1}{h}\int_\eta \psi(w)\,dw\right| \\ &\le \frac{1}{|h|}\sup_{w\in\eta}|\psi(w)|\underbrace{\text{length}(\eta)}_{=|h|} = \sup_{w\in\eta}|\psi(w)| \xrightarrow{h\to 0} 0.\end{aligned}$$

This concludes the hard part of the proof in showing that f has a primitive; by complex FTC, we immediately see that $\int_\gamma f(z)\,dz = 0$, since γ is a closed curve and its endpoints are equal to each other. □

[7] GOD KNOWS if this is a w or an ω, i'm just going to use w for now. forensic analysis on yalls handwriting holy shit

We now give an example.

Problem 6.2. For all $\xi \in \mathbb{R}$, let $\mathcal{F}$ denote the Fourier transform, and let

$$(\mathcal{F}f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\, dx.$$

Show that if $f(x) = e^{-\pi x^2}$, we have that $(\mathcal{F}f)(\xi) = f(\xi) = e^{-\pi\xi^2}$.

Solution. In the $\xi = 0$ case, we immediately have that

$$(\mathcal{F}f)(0) = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} e^{-\pi x^2} = 1,$$

from computation through the Gaussian integral (polar coordinate transform). If $\xi > 0$ (we note that $\xi < 0$ follows analogously), let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = e^{-\pi z^2}$. Then, for $R > 0$, let us integrate on the rectangle from $-R$ to R, R to $R+i\xi$, $R+i\xi$ to $-R+i\xi$, and $-R+i\xi$ to $-R$, where the latter three paths are denoted I_1, I_3, I_2 respectively (we intentionally number this way because the two opposing sides I_1, I_2 can be tackled together at once). We have that

$$0 = \int_{\gamma_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{I_1} f(z)\, dz + \int_{I_2} f(z)\, dz + \int_{I_3} f(z)\, dz.$$

Let us consider the integral $\left|\int_{I_1} f(z)\, dz\right|$, with parameter $I_1 : [0, \xi] \to \mathbb{C}$, given by $t \mapsto R + it$; we have that

$$\left|\int_{I_1} f(z)\, dz\right| = \left|\int_0^{\xi} f(R+it) i\, dt\right| = \left|\int_0^{\xi} e^{-\pi(R+it)^2} i\, dt\right|,$$

for which we observe that the integrand

$$\left|e^{-\pi(R+it)^2} i\right| = \left|e^{-\pi(R^2-t^2)}\right| |i| \left|e^{i\pi 2Rt}\right| \leq \left|e^{-\pi(R^2-\xi^2)}\right| \to 0, \quad R \to +\infty,$$

so the integrals on $I_1, I_2 \to 0$ for large enough R (we note that the same conclusion held for I_2 because the computation follows analogously). For the last part, consider that

$$\int_{I_3} f(z)\, dz = \int_{-R}^{R} f(t+i\xi)\, dt = \int_{-R}^{R} e^{-\pi(t+i\xi)^2}\, dt = e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi t^2} e^{-2\pi i \xi t}\, dt,$$

upon which we obtain $e^{\pi\xi^2}(\mathcal{F}f)(\xi)$ as $R \to \infty$. This means we have that $-e^{\pi\xi^2}(\mathcal{F}f)(\xi)$ vanishes, where the minus sign is from the orientation of I_3, and so we may conclude that $(\mathcal{F}f)(\xi) = e^{-\pi\xi^2} = f(\xi)$. $\square$

§7 Day 7: Cauchy's Integral Formula and Corollaries (Sep. 23, 2025)

We start with an example.

Example 7.1 (Fresnel integrals; Shakarchi Ex. §2.1))**.** Prove that

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

Solution. To do this, we proceed by Cauchy's theorem, i.e., using functions of complex variables. Let e^{ix^2}, which, per Euler's formula, is equal to $\cos(x^2) + i\sin(x^2)$ for $x \in \mathbb{R}$. Let us reframe the question by integrating e^{z^2}, where $z \in \mathbb{C}$, over the contour (closed curve) given by a $\frac{\pi}{4}$ radian sector of the circle of radius $R > 0$ centered at 0; specifically, the contour is given by $0 \to R$, $R \to Re^{i\pi/4}$ along the arc, and $Re^{i\pi/4} \to 0$. In this manner, let $z = \rho e^{i\pi/4}$, where $\rho \in (0, R)$, we have that

$$e^{-(\rho e^{i\pi/4})^2} = e^{-\rho^2\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^2} = e^{-\frac{\rho^2}{2}(1+i)^2} = e^{-\rho^2 i} = \cos(\rho^2) + i\sin(\rho^2).$$

Let the three paths in the contour (which we will call γ_R) be given by I_1, I_2, I_3 in order; we have that, by Cauchy's theorem,

$$0 = \int_{\gamma_R} f(z)\,dz = \int_{I_1} f(z)\,dz + \int_{I_2} f(z)\,dz + \int_{I_3} f(z)\,dz.$$

Directly compute as follows, where $f(z) = e^{-z^2}$,

$$\int_{I_1} f(z)\,dz = \int_0^R e^{-x^2}\,dx \xrightarrow{R\to\infty} \frac{\sqrt{\pi}}{2},$$

Let $-I_3 : [0, R] \to \mathbb{C}$ be given by $t \mapsto te^{i\pi/4}$; we have,

$$\begin{aligned}
\int_{-I_3} f(z)\,dz &= \int_0^R f\left(te^{i\pi/4}\right) e^{i\pi/4}\,dt \\
&= e^{i\pi/4} \int_0^R e^{-(te^{i\pi/4})^2}\,dt \\
&= e^{i\pi/4} \left[\int_0^R \cos t^2\,dt - i\int_0^R \sin(t^2)\,dt\right],
\end{aligned}$$

and finally, for the integral on I_2 (where $I_2 : [0, \frac{\pi}{4}]^2 \to \mathbb{C}$ and $t \mapsto Re^{it}$), we have that

$$\int_{I_2} f(z)\,dz = \int_0^{\pi/4} e^{-(Re^{it})^2} iRe^{it}\,dt,$$

for which we may bound the integrand as follows,

$$\left|e^{-(Re^{it})^2} iRe^{it}\right| \le R\left|e^{-(Re^{it})^2}\right| = R\left|e^{-R^2(\cos(2t)+i\sin(2t))}\right| = Re^{-R^2(\cos 2t)}.$$

This means we may write

$$\left|\int_{I_2} f(z)\,dz\right| \le \int_0^{\pi/4} \left|e^{-(Re^{it})^2} iRe^{it}\right|\,dt = \int_0^{\pi/4} Re^{-R^2(\cos 2t)}\,dt.$$

Let us compute $\cos(2t)$; we have that $2t \in [0, \frac{\pi}{2}]$, so $\cos(2t) = \sin(\frac{\pi}{2} - 2t)$, and $\sin(\theta) \geq \frac{2}{\pi}\theta$ by appealing to geometric intuition; this means

$$\begin{aligned}
\int_0^{\pi/4} Re^{-R^2(\cos 2t)}\,dt &\leq \int_0^{\pi/4} Re^{-R^2\frac{2}{\pi}\left(\frac{\pi}{2}-2t\right)}\,dt \qquad \text{(Let } s = \tfrac{\pi}{2} - 2t\text{)} \\
&= \frac{1}{2}\int_0^{\pi/2} Re^{-R^2\frac{2}{\pi}s}\,ds \\
&= \frac{1}{2}\int_0^{\pi/2} Rd\left(\frac{e^{-R^2\frac{2}{\pi}s}}{-R^2\frac{2}{\pi}}\right) \\
&= \frac{1}{2}\cdot\frac{1}{R\frac{2}{\pi}}\left(e^{-R^2} - 1\right) \xrightarrow{R\to\infty} 0.
\end{aligned}$$

Having established computations for I_1, I_2, I_3, we may now write

$$0 = \frac{\sqrt{\pi}}{2} - e^{i\pi/4}\left[\int_0^\infty \cos(x^2)\,dx - i\int_0^\infty \sin(x^2)\,dx\right]$$

This means we have

$$\begin{aligned}
\int_0^\infty \cos(x^2)\,dx - i\int_0^\infty \sin(x^2)\,dx &= e^{-i\pi/4}\frac{\sqrt{\pi}}{2} \\
&= \frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{\pi}}{2} - i\frac{\sqrt{\pi}}{2}\right) \\
&= \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}.
\end{aligned}$$

□

We now discuss Cauchy's integral formula. As another example, let D be a disc centered at z, and let f be a holomorphic function; we may express $f(z)$ using the values of f on ∂D.

Example 7.2 (Steady-State Heat Equation)**.** Let $g(x, y)$ be continuous on $\mathbb{R}^2$. Find $u(x, y)$ satisfying

$$\begin{cases} \Delta u = 0 & \text{on } D, \\ u = g & \text{on } \partial D, \end{cases}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator. The solution is given by considering $(x, y) = (r\cos\theta, r\sin\theta)$, where

$$u(r, \theta) = \int P_r(\theta, \varphi)g(\cos\varphi, \sin\varphi)\,d\varphi, \quad P_r(\theta, \varphi) = \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2}$$

where P_r is called the *Poisson kernel*.

Theorem 7.3 (Cauchy's Integral Formula)**.** Suppose f is holomorphic in an open set Ω that contains the closure of a disc D. Let $C = \partial D$ equipped with the anticlockwise orientation. Then for any $z \in D$,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof. We start by constructing a "keyhole contour" on D, where δ is the width of the corridor, and ε is the radius of the circle centered at z. The contour can be thought of as picking a point in C and connecting it to the ε-circle about z with a δ-wide corridor.

Let the contour be called $\Gamma_{\delta,\varepsilon}$. Let $F(\zeta) = \frac{f(\zeta)}{\zeta - z}$; clearly, it is holomorphic on $\Omega \setminus \{z\}$. By Cauchy's theorem,

$$0 = \int_{\Gamma_{\delta,\varepsilon}} F(\zeta)\, d\zeta = \int_{I_1} F(\zeta)\, d\zeta + \int_{I_2} F(\zeta)\, d\zeta + \int_{I_3} F(\zeta)\, d\zeta + \int_{I_4} F(\zeta)\, d\zeta,$$

where I_1, I_3 represent the paths on C and the ε-circle about z respectively, and I_2, I_4 the "walls of the corridor". We start with some basic observations;

(i) If we let $\delta \to 0^+$, then
$$\int_{I_1} F(\zeta)\, d\zeta = \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

(ii) Again, if we let $\delta \to 0^+$, we have that
$$\int_{I_2} F(\zeta)\, d\zeta = -\int_{I_4} F(\zeta)\, d\zeta,$$
since they are simply two path integrals of the opposite orientation.

(iii) For I_3, we may first write
$$\int_{I_3} F(\zeta)\, d\zeta = \int_{I_3} \frac{f(\zeta)}{\zeta - z}\, d\zeta;$$
if we let $\varepsilon \to 0^+$, we see that this is problematic, since we have a singularity at z. However, we notice that the integrand resembles the definition of the derivative, i.e., we may write
$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z},$$
so we obtain
$$\int_{I_3} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{I_3} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta + f(z) \int_{I_3} \frac{1}{\zeta - z}\, d\zeta,$$
where the latter term is equal to $-2\pi i f(z)$, per (p.47 in Shakarchi)
$$\int_{I_3} \frac{f(z)}{\zeta - z}\, d\zeta = f(z) \int_{I_3} \frac{d\zeta}{\zeta - z} = -f(z) \int_0^{2\pi} \frac{\varepsilon i e^{-it}}{\varepsilon e^{-it}}\, dt = -f(z) 2\pi i.$$
For the former term, there exists $\varepsilon_0 > 0$ such that for all $\zeta \in D_{\varepsilon_0}(z)$, we have that
$$\left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \le \left| f'(z) \right| + 2.$$
We obtain
$$\left| \int_{I_3} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \right| \le \left(\left| f'(z) \right| + 2 \right) \cdot 2\pi\varepsilon \xrightarrow{\varepsilon \to 0} 0.$$

Combining all these observations, we obtain

$$0 = \int_{\Gamma_{\delta,\varepsilon}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \xrightarrow{\delta,\varepsilon \to 0^+} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta - 2\pi i f(z),$$

from which we conclude Cauchy's integral formula.[8] □

[8] reference: p.45-47 Shakarchi

Theorem 7.4 (Cor. 4.2, Shakarchi)**.** "A holomorphic function is infinitely complex differentiable."[9] Suppose f is holomorphic in an open set Ω. Then f has infinitely many complex derivatives in Ω. Moreover, for any $z \in \Omega$ and $n \in \mathbb{Z}_{\geq 0}$, we have that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Proof. We proceed by induction on n. The base case $n = 0$ is immediately given by Cauchy's integral formula; assuming that the statement is true for $n-1$, for any $h \in \mathbb{C}\backslash\{0\}$ such that $z + h \in D$, we have that

$$\begin{aligned} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} &= \frac{(n-1)!}{2\pi i h} \int_C \left[\frac{f(\zeta)}{(\zeta - z - h)^n} - \frac{f(\zeta)}{(\zeta - z)^n} \right] d\zeta \\ &= \frac{(n-1)!}{2\pi i h} \int_C f(\zeta) \left[\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right] d\zeta. \end{aligned}$$

By binomial expansion, we have that

$$\begin{aligned} &\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \\ &= \frac{1}{(\zeta - z - h)^n (\zeta - z)^n} \left[(\zeta - z)^n - (\zeta - z - h)^n \right] \\ &= \frac{h}{(\zeta - z - h)^n (\zeta - z)^n} \left[(\zeta - z)^{n-1} + (\zeta - z)^{n-2}(\zeta - z - h) + \cdots + (\zeta - z - h)^{n-1} \right]. \end{aligned}$$

By taking h sufficiently small, we obtain

$$\frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{2n}} n(\zeta - z)^{n-1}\, d\zeta = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta. \qquad \square$$

Theorem 7.5 (Thm. 4.4, Shakarchi)**.** "A holomorphic function is locally a power series". Suppose f is holomorphic in an open set Ω. If D is a disc centered at z_0 whose closure is contained in Ω, then f has a power series expansion at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in D$, and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \geq 0.$$

Proof. Fix any $z \in D$; by Cauchy's integral formula, we have that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Note that per our previous corollary, the idea is to write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 + z_0 - z} = \frac{1}{(\zeta - z_0)} \frac{1}{\left(1 - \frac{z - z_0}{\zeta - z_0}\right)},$$

[9] hell, i need to run a marathon with 20mg of thc in my system. props wenyu

where we observe that since $z \in D$ is fixed and $\zeta \in C$, we know that there exists some $r \in (0, 1)$ such that

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < r,$$

so we may regard the geometric series representation

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n,$$

for which the series converges uniformly for any $\zeta \in C$. This means we may interchange the integral and the sum to obtain

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right) (z - z_0)^n. \qquad \square$$

Corollary 7.6 (Liouville's Theorem: Thm. 4.5, Shakarchi)**.** If f is entire and bounded, then f is constant. We say that f is *entire* if it is holomorphic on the whole of $\mathbb{C}$.

Proof. We will prove this later on. Though, it is done by observing that $\mathbb{C}$ is connected (hence a region, i.e., open connected set), then checking $f' = 0$, and so f is constant. $\square$

Corollary 7.7 (Cauchy's Inequality)**.** If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 with radius R, then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! \, \|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z \in C} |f(z)|$ (and C is the boundary of D.)

Proof. We have that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta;$$

if we let $C : [0, 2\pi] \to \mathbb{C}$ be given by $t \mapsto z_0 + Re^{it}$, then the above is equal to

$$\frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^{n+1} e^{i(n+1)t}} iRe^{it} \, dt,$$

for which we may write

$$\left| \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^{n+1} e^{i(n+1)t}} iRe^{it} \, dt \right| \leq \frac{n!}{2\pi} \cdot \frac{\|f\|_C}{R^n} \cdot 2\pi = \frac{n! \, \|f\|_C}{R^n},$$

which finishes the proof. $\square$

§8 Day 8: Morera's Theorem and Distribution of Zeros of Holomorphic Functions (Sep. 26, 2025)

Recall Cauchy's integral formula, where if f is holomorphic on an open set Ω containing the closure of disc D, then let $C = \partial D$; we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$$
$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta, \quad n \in \mathbb{N},$$

i.e., f is infinitely complex differentiable. Recall that we also have that if γ is a closed curve with interior in Ω, then $0 = \int_\gamma f$.

Theorem 8.1 (Morera's Theorem)**.** Suppose f is continuous on an open disc D such that for any triangle T contained in D, we have $\int_T f(z)\,dz = 0$. Then f is holomorphic.

Proof. Recall our earlier proof of Cauchy's theorem on a disc, where we first used Goursat's theorem, then f has a primitive on D. In the second step, we only used that $\int_T f(z)\,dz = 0$, so f has a primitive on the disc, and we may apply the proof to our new f to find F with $F' = f$. Since F is holomorphic, it is infinitely complex differentiable, so we conclude that f is holomorphic as desired. □

Theorem 8.2 (Distribution of zeros of holomorphic functions)**.** Suppose f is holomorphic in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω itself. Then $f = 0$ on Ω (i.e., the zeros are isolated).

Proof. We start by showing that $f = 0$ on a neighborhood of the limit point z_0. Let D be a disc centered at z_0 in Ω; we have that f coincides with a power series on D,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

(where we assume $f \neq 0$). Then there exists some non-negative a_n, per our assumption. Let m be the smallest index such that $a_m \neq 0$, and write

$$f(z) = a_m (z - z_0)^m \left[1 + \frac{1}{a_m}\sum_{n>m} a_n (z - z_0)^{n-m}\right],$$

where we let $g(z)$ be given by $f(z) = a_m(z - z_0)^m(1 + g(z))$. Clearly, $g(z)$ converges on D, since

$$|a_n|^{\frac{1}{n-m}} = |a_n|^{\frac{1}{n}\frac{n}{n-m}} \xrightarrow{n\to\infty} |a_n|^{\frac{1}{n}},$$

so by Hadamard's formula, $g(z)$ has some radius of convergence, as $f(z), g(z) \to 0$ with $z \to z_0$.

Set $z = w_k \neq z_0$ in D, where w_k is some element of the sequence of distinct points. Then we have

$$0 = f(w_k) = a_m (w_k - z_0)^m (1 + g(w_k)),$$

for which all three terms are nonzero (the third can be made to be nonzero by picking k large enough such that $|g(w_k)| < 1$). This means that for a sufficiently large k, we get a contradiction, and so $a_m = 0$ and $f = 0$ on D. This establishes that f vanishes on a local disc about z_0.

We now check that $f = 0$ on the entire of Ω by using the connectedness of Ω. Let U be the interior of $\{z \in \Omega \mid f(z) = 0\}$, and observe that $U \neq \varnothing$ as $D \subset U$ and U is open. It suffices to check that U is closed; let $\{z_n\} \subset U$ be any sequence such that $z_n \to z$ for some $z \in \Omega$. Since f is continuous, we have that $f(z) = 0$. By our previous argument, f is zero on an open neighborhood of z, and so $z \in U$, meaning U contains all its limit points, and is therefore closed. We conclude that U is clopen in Ω, so $U = \Omega$ as desired. $\square$

Corollary 8.3. Suppose f, g are holomorphic in a region Ω, and $f(z) = g(z)$ on a nonempty open subset of Ω. Then $f(z) = g(z)$ on all of Ω.

Remark 8.4. Given f, F analytic in regions Ω, Ω' respectively with $\Omega \subset \Omega'$, if f and F agree on Ω we say that F is an analytic continuation of f into Ω'. Such analytic continuations are always unique.

§9 Day 9: Applications of Cauchy's Integral Formula (Sep. 30, 2025)

Recall Liouville's theorem that if f is an entire (holomorphic on the whole complex plane) bounded function, then f is constant.

Corollary 9.1 (Fundamental Theorem of Algebra)**.** Every nonconstant polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ with complex coefficients has a root in $\mathbb{C}$.

Proof. Proceed by contradiction by means of Liouville's theorem. Suppose $P(z)$ is nonconstant and admits no roots in $\mathbb{C}$; then $P(z)^{-1}$ is entire, and it remains to check that it is bounded. It is enough to get a lower bound for $a_n z^n$, since the dominating term of $P(z)$ is $a_n z^n$; supposing $a_n \neq 0$, we have that

$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_n}{z^n}\right),$$

of which we know is defined on $\mathbb{C} \setminus \{0\}$; taking $|z| \to +\infty$, we have that $\frac{P(z)}{z^n} \to a_n$, so there exists $R > 0$ such that

$$|P(z)| \geq \frac{|a_n|}{2} |z|^n$$

for all $|z| > R$. This means

$$\frac{1}{|P(z)|} \leq \frac{1}{\frac{|a_n|}{2} |z|^n} \leq \frac{1}{\frac{|a_n|}{2} R^n}, \quad \text{for } |z| > R.$$

For any $z \in \overline{D_R(0)}$, we have that $P(z) \neq 0$. Since P is continuous, there exists an open neighborhood D_z of z and $c_z > 0$ such that $|P(z')| \geq c_z > 0$ for any $z' \in D_z$. Since $\overline{D_R(0)}$ is compact, there exists finitely many $D_{z_1}, \ldots, D_{z_k}$ such that $\overline{D}_R(0) \subset \bigcup_{i=1}^k D_{z_i}$. Then $|P(z)| \geq \min\{C_{z_1}, \ldots, C_{z_k}\} > 0$ on $\overline{D_R(0)}$. Since we have a lower bound for $P(z)$ on the compact set $\overline{D_R(0)}$ and outside of it, we see that $P(z)^{-1}$ is bounded on $\mathbb{C}$, and so per Liouville's theorem, $P(z)^{-1}$ is constant, yielding that $P(z)$ is constant, contradicting the assumption. □

Corollary 9.2. Every polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$.

Proof. Left as an exercise. □

We now discuss the applications of Cauchy's integral formula. Let $\{f_n\}_{n=1}^\infty$ be a sequence of holomorphic functions.

Theorem 9.3 (Thm. 5.2, Shakarchi)**.** Let Ω be an open subset of $\mathbb{C}$, and let $\{f_n\}_{n=1}^\infty$ be a sequence of holomorphic functions that converge uniformly to a function f on every compact subest of Ω. Then (i) f is holomorphic on Ω, (ii) $\{f_n'\}_{n=1}^\infty$ converges uniformly to f' on every compact subset of Ω.

We give some examples of such sequences.

(i) Let $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ on $\mathbb{R}$; we have that each f_n is differentiable and $f_n(x) \to f(x) = |x|$ as $n \to \infty$ on compact intervals, but $f(x)$ itself is not differentiable at 0.

(ii) (*Weierstrass approximation theorem*) Every continuous function on a closed bounded interval $[a, b]$ can be uniformly approximate by a polynomial. Specifically, for every $\varepsilon > 0$, there exists a polynomial $P(x)$ such that $\sup_{x\in[a,b]} |f(x) - P(x)| < \varepsilon$.

An additional side remark; $C([a, b])$, i.e., the set of all continuous functions on $[a, b]$, equipped with the uniform norm, has the set of polynomials dense in itself.

Proof. We now prove the theorem. Let us start by showing that $\{f_n\}$ converges uniformly to f on every compact subset of Ω and show that f is holomorphic. The proof idea here is to use Morera's theorem to show that f is continuous on a disc D, and so for every triangle $T \subset D$, we have that $\int_T f = 0$, meaning that f is holomorphic. We may use Cauchy's theorem to see that each $\int_T f_n$ is equal to 0, so $\int_T f = 0$, since

$$\left| \int_T f_n - \int_T f \right| = \left| \int_T f_n - f \right| \leq \sup_T |f_n - f| \operatorname{length}(T) \xrightarrow{n\to\infty} 0.$$

For the second part, we wish to show that $\{f_n'\}$ converges uniformly to f' on every compact subset of Ω. For $\delta > 0$, define $\Omega_\delta = \{z \in \Omega \mid \overline{D_\delta(z)} \subset \Omega\}$. Any compact subset of Ω is contained in some Ω_δ, so it suffces to show that $\{f_n'\}$ converges uniformly to f' on Ω_δ for each $\delta > 0$. We claim that if F is holomorphic on Ω, then

$$\sup_{z\in\Omega_\delta} \left|F'\right| \leq \frac{1}{\delta} \sup_{z\in\Omega} |F| .$$

Applying the claim to $f_n - f$, we see that

$$\sup_{\Omega_\delta} \left|f_n' - f'\right| \leq \frac{1}{\delta} \sup_{\Omega} |f_n - f| \xrightarrow{n\to\infty} 0,$$

so it remains to prove the claim itself. For all $z \in \Omega_\delta$, by Cauchy's integral formula for the derivative, we have that

$$F'(z) = \frac{1}{2\pi i} \int_{C_\delta(z)} \frac{F(\zeta)}{(\zeta - z)^2} d\zeta,$$

where $C_\delta(z) = \{w \mid |w - z| = \delta\}$, so for all $z \in \Omega_\delta$ we have,

$$\left|F'(z)\right| \leq \frac{1}{2\pi} \sup_{\zeta\in C_\delta(z)} \left| \frac{F(\zeta)}{(\zeta - z)^2} \right| \cdot 2\pi\delta = \frac{1}{\delta} \sup_{\zeta\in C_\delta(z)} |F(\zeta)| ,$$

meaning we make take the supremum over Ω_δ to get[10]

$$\begin{aligned} \sup_{z\in\Omega_\delta} \left|F'(z)\right| &\leq \sup_{z\in\Omega_\delta} \left(\frac{1}{2\pi} \sup_{\zeta\in C_\delta(z)} \left| \frac{F(\zeta)}{(\zeta - z)^2} \right| 2\pi\delta \right) \\ &\leq \sup_{z\in\Omega_\delta} \left(\frac{1}{\delta} \sup_{\zeta\in C_\delta(z)} |F(\zeta)| \right) \\ &\leq \frac{1}{\delta} \sup_{z\in\Omega} |F(z)| . \end{aligned}$$ □

The term test will be next Tuesday in class. It will be three problems; the first is to prove a theorem discussed in class, the second is a variation of a homework problem, and the third is a choice between another variation of a homework problem or a problem not in the homework, of which has higher marks (what?).

[10] page 54-55, shakarchi

We now discuss another application of Cauchy's integral formula; specifically, the Schwartz' reflection principle (Theorem 5.6 in Shakarchi, p. 60), which extends a holomorphic function analytically to a larger set. We start by presenting a counterexample.

Theorem 9.4 (Fabry (Gap) Theorem)**.** Consider a power series $f(z) = \sum_{k=0}^{\infty} a_{n_k} z^{n_k}$, where $\{n_k\}$ is a strictly increasing sequence of positive integers. Reference here.

Suppose that $\frac{n_k}{k} \to \infty$ as $n \to \infty$, and the radius of convergence of the power series is 1. Then f cannot be analytically extended beyond any point of the unit circle. Let $z \in \partial\mathbb{D}$; we want to show that we cannot find a holomorphic function $\tilde{f}$ defined on an open subset U of z such that

$$\tilde{f}\Big|_{U\cap\mathbb{D}} = f|_{U\cap\mathbb{D}}.$$

As an example, pick

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n^2}}{n^2},$$

for which $\frac{n^2}{n} = n \to \infty$ as $n \to \infty$. By Hadamard's formula, we have that the radius of convergence is indeed 1. The computation showing that the analytic extension does not extend beyond the unit circle is left as an exercise.

Theorem 9.5 (Schwartz reflection principle (Shakarchi 5.6))**.** Let Ω be an open subset of $\mathbb{C}$ that is symmetric with respect to the real line, i.e., $z \in \Omega$ if and only if $\overline{z} \in \Omega$. Define Ω^+, Ω^- to be subsets of Ω with positive and negative imaginary part respectively, and let $I = \Omega \cap \mathbb{R}$. Let f be a holomorphic function on Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists F holomorphic on all of Ω with $F|_{\Omega^+ \sqcup I} = f$.

To do this, we start by defining a holomorphic function F on Ω^-, then we prove that F is holomorphic on Ω (and F is holomorphic on I).

Theorem 9.6 (Symmetric principle)**.** Let f^+, f^- be holomorphic functions on Ω^+, Ω^- respectively that extend continuously on I such that they agree on I. Then the function f on Ω defined by

$$f(z) = \begin{cases} f^+(z) & z \in \Omega^+, \\ f^+(z) = f^-(z) & z \in I, \\ f^-(z) & z \in \Omega^- \end{cases}$$

is holomorphic on Ω.

To see that f is holomorphic on I, we may use Morera's theorem; pick any open disc D centered at a point $z \in I$ which is entirely contained in Ω. We will show that f is holomorphic on D. Observe that any $T \subset D$ is of four types; either it (i) does not intersect I, (ii) aligns with I with one of its sides, (iii) intersects with I at exactly one vertex, (iv) or intersects with I at two points.

For case (i), we have that Cauchy's theorem immediately shows that $\int_T f(z)\, dz = 0$. For cases (ii) and (iii), we may let T_ε (i.e., moved upwards or downwards by ε so it is of case (i)) be an affine shift of T; then

$$\int_{T_\varepsilon} f(z)\, dz \xrightarrow{\varepsilon\to 0} \int_T f(z)\, dz = 0.$$

For case (iv), we can partition the triangle into subtriangles satisfying case (ii) or (iii), and so we immediately have that the integral vanishes too.

With this, we may now prove the Schwartz reflection principle.

Proof. Let f be holomorphic on Ω^+ and let it extend continuously to I such that it is real-valued on I. We claim that there exists F on Ω such that the restrction of F onto Ω^+ is equal to f on Ω^+. We may construct such F by having $f^-(z) = \overline{f(z)}$ for $z \in \Omega^-$. It suffices to check that f^- is holomorphic on Ω^-; for all $z, z_0 \in \Omega^-$, we have that $\overline{z}, \overline{z_0} \in \Omega^+$. Since f is holomorphic at $\overline{z_0}$, f admits a power series

$$f(\overline{z_0}) = \sum_{n=0}^{\infty} a_n(\overline{z} - \overline{z_0}),$$

which converges on some $D_r(\overline{z_0})$ with $r > 0$. In particular,

$$f^-(z) = \overline{f(\overline{z})} = \sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n,$$

which converges on $D_r(z_0)$, which by Hadamard's formula, admits the same radius of convergence as the power series about $f(\overline{z_0})$, i.e., the power series for $f^-(z)$ converges on $D_r(z_0)$. Hence, f^- is holomorphic at z_0, and since f extends continuous to I and is real valued on I, we have that $\overline{f(\overline{x})} = f(x)$ for all $x \in I$, and so f^- can be extended continuously to I such that $f^- = f^+$ on I. In this manner, we may apply the symmetric principle from earlier to obtain F satisfying the Schwartz reflection principle.[11] $\square$

[11] ref: p.60 shakarchi

§10 Day 10: Third Application of Cauchy's Formula; Analytic Extension of Gamma Function (Oct. 2, 2025)

We now discuss a third application of the Cauchy integral formula. Let f be given by

$$f(z) = \int_a^b F(z,s)\,ds,$$

where $F(z,s)$ is holomorphic for $z \in \mathbb{C}$ and continuous in $s \in \mathbb{R}$. Is f holomorphic in z?

Example 10.1. Consider the gamma function $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt$, which can be thought of as

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}t^{s-t}\,dt.$$

For $s > 0$, we have convergence, as for $t \to 0$, we have that $e^{-t}t^{s-1} \approx t^{s-1}$, which is integrable; for large t, the convergence is essentially "guaranteed" by the exponential decay of the integrand, i.e., $\left|e^{-t}t^{s-1}\right| \le Ce^{-\frac{t}{2}}$, which is integrable over ε^{-1} to infinity. We ask, is $\Gamma(s)$ convergent or analytic when $\Re s > 0$?

Theorem 10.2 (Thm. 5.4, Shakarchi)**.** Let $F(z,s)$ be defined for $(z,s) \in \Omega \times [0,1]$, where Ω is open in $\mathbb{C}$. Suppose F satisfies the following,

(i) $F(z,s)$ is holomorphic in z for all $s \in [0,1]$,

(ii) F is continuous on $\Omega \times [0,1]$.

Then $f(z) = \int_0^1 F(z,s)\,ds$ is holomorphic on Ω.

Proof. Observe that we may write $\int_0^1 F(z,s)\,ds$ in terms of Riemann sums. Define $\{f_n\}_{n\in\mathbb{N}}$ on Ω by

$$f_n(z) = \frac{1}{n}\sum_{k=1}^{n} F(z,k/n),$$

for which we may note that each f_n is holomorphic in z by property (i). We claim that $\{f_n\}$ converges uniformly to some f, i.e., as $n \to \infty$, we have that

$$\frac{1}{n}\sum_{k=1}^{n} F(z,k/n) \to \int_0^1 F(z,s)\,ds = f(z).$$

For any open disc D with closure $\overline{D} \subset \Omega$, we want to show that $\{f_n\} \to f$ uniformly on D. Since F is continuous on $\Omega \times [0,1] \supset$, it is also continuous on $\overline{D} \times [0,1]$, which we observe is compact, so we indeed have uniform continuity. In particular, this means that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup_{z\in D}|F(z,s_1) - F(z,s_2)| < \varepsilon$ whenever $|s_1 - s_2| < \delta$.

In this manner, we may take that given $n > \delta^{-1}$ and $z \in D$, we have that

$$\begin{aligned}
|f_n(z) - f(z)| &= \left|\sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} F(z,k/n) - F(z,s)\,ds\right| \\
&\le \sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} |F(z,k/n) - F(z,s)|\,ds && \text{(Triangle ineq.)} \\
&< \sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} \varepsilon\,ds = \sum_{k=1}^{n}\frac{\varepsilon}{n} = \varepsilon.
\end{aligned}$$

Thus, we establish that $f_n \to f$ on D, and by the first consequence of Cauchy's integral formula (equivalently, Theorem 5.2 in Shakarchi and 9.3 in these notes), we have that f is holomorphic in z on $\Omega \times [0, 1]$. $\square$

We now return to the Gamma function, which will follow Ch. 6 in Shakarchi.

Proposition 10.3 (Prop. 1.1, Shakarchi; p. 160)**.** $\Gamma(s)$ extends analytically on $\Re s > 0$, and is given by the same formula.

Proof. It suffices to consider that this holds for every strip $S_{\delta,m} = \{s \in \mathbb{C} \mid \delta < \Re s < M\}$, where δ, M are positive reals satisfying $\delta < M$. For all $s \in S_{\delta,M}$, denote $\Re s$ by σ. Then

$$\left|\int_0^\infty e^{-t}t^{s-1}\,dt\right| \le \int_0^\infty \left|e^{-t}t^{s-1}\right|\,dt = \int_0^\infty e^{-t}t^{\sigma-1}\,dt,$$

per Euler's formula, of which we may observe the latter integral converges by real convergence (as discussed earlier). Now, let $f(s,t) = e^{-t}t^{s-t}$ be holomorphic in s and continuous in (s,t). For $\varepsilon > 0$, define

$$F_\varepsilon(s) = \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}t^{s-1}\,dt;$$

per our previous result, we have that $F_\varepsilon(s)$ is holomorphic in s on $S_{\delta,M}$. Also, per the first result of Cauchy's formula, it suffices to check that F_ε converges uniformly to Γ on $S_{\delta,M}$. Directly write as follows,

$$|\Gamma(s) - F_\varepsilon(s)| \le \int_0^\varepsilon e^{-t}t^{\sigma-1}\,dt + \int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{\sigma-1}\,dt,$$

for which we may observe that

$$\int_0^\varepsilon e^{-t}t^{\sigma-1}\,dt = \frac{\varepsilon^\delta}{\delta} = \frac{\varepsilon^\delta}{\delta} \xrightarrow{\varepsilon\to 0} 0, \qquad \text{(on } 0 < \varepsilon < 1\text{)}$$

$$\left|\int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{\sigma-1}\,dt\right| \le \int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{M-1}\,dt \le C\int_{\frac{1}{\varepsilon}}^\infty e^{-\frac{t}{2}}\,dt \xrightarrow{\varepsilon\to 0} 0.$$

Thus, we obtain uniform convergence of $F_\varepsilon \to \Gamma$ independent of s, and we indeed have an analytic continuation of Γ to the half-plane $\Re s > 0$. $\square$

Let $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ be the Riemann ζ function, defined on $\Re s > 1$ (clearly, otherwise the series does not converge). ζ is analytic in the half-plane $\Re s > 1$, and we may show that it has an analytic extension using Γ. We know Γ can be analytically extended to a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$ using the relation $\Gamma(s+1) = s\Gamma(s)$ and the fact that $\Gamma(s)^{-1}$ is entire. Then we have that

$$\eta(s) = \pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s).$$

§11 Day 11: Singularities of Holomorphic Functions (Oct. 8, 2025)

Let f be a holomorphic function define in an open set Ω, except at $z_0 \in \Omega$. We call z_0 an isolated singularity of f. For an example, let $\Omega = \mathbb{D}$, $z_0 = 0$;

(i) Let $f_1(z) = z$ and $f_2(z) = z^2 + 1$. We may extend f_1, f_2 holomorphically to $z_0 = 0$ by plugging in $z = 0$ to the definitions of each to get $f_1(0) = 0$ and $f_2(0) = 1$. Clearly, f_1, f_2 in this manner are holomorphic on $\mathbb{D}$.

(ii) Let $f_1(z) = \frac{1}{z}$ and $f_2(z) = \frac{1}{z^2}$. In both cases, $\lim_{z\to 0} |f_i(z)| \to +\infty$.

(iii) Let $f(z) = e^{1/z}$. We can show that the limit as $z \to 0$ of $f(z)$ does not exist. For all $z \in \mathbb{C} \setminus \{0\}$, write $z = \rho e^{i\theta}$; we have that $f(z) = f(\rho e^{i\theta}) = \exp(\frac{1}{\rho} e^{-i\theta})$. Along $\theta = 0$, we have that $\exp(1/p) \to \infty$, while along $\theta = \pi$, we have that $\exp(-1/p) \to 0$, so the limit clearly does not exist.

Let f be a holomorphic function defined on an open set Ω, except at $z_0 \in \Omega$.

(i) If we can define f at z_0 in such a way that f becomes holomorphic on Ω, we say that z_0 is a removable singularity.

(ii) We say that z_0 is a pole of f if $1/f$ is holomorphic on $U \setminus \{z_0\}$ with U being some open neighborhood of z_0 in Ω, and setting $(1/f)(z_0) = 0$ makes $1/f$ holomorphic on U.

(iii) If z_0 is neither removable nor a pole, then we say that it is an essential singularity of f.

Theorem 11.1 (§3.3.1: Riemann's theorem on removable singularities)**.** Suppose f is holomorphic on $\Omega \setminus \{z_0\}$ for some $z_0 \in \Omega$; if f is bounded on $\Omega \setminus \{z_0\}$, then z_0 is a removable singularity.[12]

Proof. Assume $\Omega = D_r(z_0)$, where $r > 0$ and $C = \partial D$ with the counterclockwise orientation. It suffices to show that for all $z \in D_r(z_0) \setminus \{z_0\}$, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

i.e., Cauchy's integral formula holds. This is not immediate as $\operatorname{int} C \not\subset \Omega \setminus \{z_0\}$. If we indeed have Cauchy's integral formula, we can show that f is holomorphic at z_0 per the following,

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{z_0 + re^{i\theta}}{z_0 + re^{i\theta} - z} ie^{i\theta}\, d\theta,$$

where the integrand can be denoted $F(z, \theta)$ on $D_r(z_0) \times [0, 2\pi]$. By the third application of Cauchy's integral formula, we have that f is indeed holomorphic. Having established this, we now show the formula holds for all $z \in D$. Consider the double keyhole contour on C and about z, z_0. By sending the width of the corridors between C and z, z_0 to zero, we have that

$$0 = \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{C_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{C_\varepsilon(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

[12] i'm going to start using this to label shakarchi sections.

where we impose a radius of ε for the keyholes about z and z_0. By parameterizing the respective contours, we have

$$\int_{-C_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = -\int_0^{2\pi} \frac{f(z+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta}\, d\theta = -i\int_0^{2\pi} f(z+i\varepsilon i^{\theta})\, d\theta.$$

As $\varepsilon \to 0$, this is just $-i2\pi f(z)$. We may show that the contour integral about $-C_\varepsilon(z_0)$ also vanishes; directly write as follows,

$$\begin{aligned}\left|\int_{C_\varepsilon(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\right| &= \operatorname{length}(C_\varepsilon(z_0)) \sup_{C_\varepsilon(z_0)} \left|\frac{f(\zeta)}{\zeta - z}\right| \\ &\leq \operatorname{length}(C_\varepsilon(z_0)) \cdot \sup_{C_\varepsilon(z_0)} (f(\zeta)) \cdot \frac{1}{\frac{|z-z_0|}{2}} \cdot 2\pi\varepsilon,\end{aligned}$$

which converges to 0 as $\varepsilon \to 0$ using the fact that f is bounded. From here, it is immediate that

$$2\pi i f(z) = \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta. \qquad \square$$

Corollary 11.2 (§3.3.2)**.** Let f be holomorphically defined on an open Ω except at $z_0 \in \Omega$. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.

Proof. We proceed in both directions.

($\Leftarrow$) $|f(z)| \to \infty$ as $z \to z_0$, so $(1/f)(z_0) = 0$. This means that $1/f$ is holomorphic on some $U \setminus \{z_0\}$ where U is an open neighborhood of z_0 in Ω; however, this means $1/f$ is bounded on $U \setminus \{z_0\}$, so Riemann's theorem on removable singularities shows that z_0 is a removable singularity, so z_0 has to be a pole of f.

($\Rightarrow$) Let $\tilde{f}$ denote the holomorphic extension onto U of $1/f$. In particular, we have that $1/f$ is continuous at z_0, so $(1/f)(z_0) = \lim_{z\to z_0}(1/f)(z) = 0$. $\square$

§12 Day 12: (Oct. 14, 2025)

Last time, we talked about singularities of holomorphic functions. Let f be holomorphic on an open set Ω except at $z_0 \in \Omega$; then z_0 is called an isolated singularity of f.

(i) z_0 is a removable singularity if we can define $f(z_0)$ such that f is holomorphic on Ω.

(ii) z_0 is a pole if $(1/f)(z_0) = 0$ makes $1/f$ a holomorphic function in a neighborhood of z_0.

(iii) z_0 is an essential singularity if it is neither removable nor a pole. One such example is given by $\exp(\frac{1}{z-z_0})$.

Theorem 12.1 (Casorati–Weierstrass, §3.3.3)**.** Suppose f is holomorphic in the punctured disc $D_r(z_0) \setminus \{z_0\}$ and has an essential singularity at z_0. Then the image of $(D_r(z_0) \setminus \{z_0\})$ under f is dense.

Proof. We will prove by contradiction. Suppose $f(D_r(z_0) \setminus \{z_0\})$ is not dense; then there exists $w \in \mathbb{C}$ and $\delta > 0$ such that $D_\delta(w) \cap f(D_r(z_0) - \{z_0\}) = \varnothing$, i.e., $|f(z) - w| \geq \delta$ for all $z \in D_r(z_0) \setminus \{z_0\}$. Consider $g : D_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ defined by

$$g(z) = \frac{1}{f(z) - w},$$

for which we note that $|g(z)| \leq \delta^{-1}$. We have that g is holomorphic on $D_r(z_0) \setminus \{z_0\}$, and it is bounded. By Riemann's theorem on removable singularities, we conclude that z_0 is a removable singularity of g, so g can be analytically extended to z_0. We usually denote the analytic extension by the same symbol. We have two cases to consider;

(i) If $g(z_0) = 0$, then $f(z) \to \infty$ as $z \to z_0$, so z_0 is a pole of f, which is a contradiction.

(ii) If $g(z_0) \neq 0$, then $f(z) - w$ can be analytically extended to z_0, which also contradicts the assumption. □

We now describe the local behaviors of singularities. Consider the zeroes of a holomorphic function.

Theorem 12.2 (§3.1.1)**.** Suppose f is holomorphic in a connected open set Ω, has a zero at $z_0 \in \Omega$, and does not vanish identically on Ω. Then there exists an open neighborhood $U \subset \Omega$ of z_0 and a non-vanishing holomorphic g on U with a unique $n \in \mathbb{N}$ such that

$$f(z) = (z - z_0)^n g(z), \quad \forall z \in U.$$

Proof. Since Ω is a connected open set and f does not vanish identically on Ω, we have that the zeroes of f are isolated. There exists an open disc $D_r(z_0)$ such that z_0 is the only zero of f on $D_r(z_0)$; then on $D_r(z_0)$, f has the power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = (z - z_0)^n \left[a_n + a_{n+1}(z - z_0) + \dots\right],$$

where we take n to be minimal and for which we denote the latter $g(z)$ as per our previous proofs. By Hadamard's formula, $g(z)$ is a convergent power series on $D_r(z_0)$, and hence holomorphic. Moreover, $g(z_0) = a_n \neq 0$, and so we may prove the uniqueness of n. Suppose that we can also write

$$(z - z_0)^n g(z) = (z - z_0)^m h(z)$$

with $m > n$. Then $g(z) = (z - z_0)^{m-n} h(z)$, where we take $z \to z_0$ to see $g(z_0) = 0$, which is contradictory. Thus, $m = n$ and $h = g$, so we are done here. □

Theorem 12.3 (§3.1.2)**.** If f has a pole at $z_0 \in \Omega$, then there exists an open neighborhood $U \subset \Omega$ of z_0, a non-vanishing holomorphic function h on U, and a unique $n \in \mathbb{N}$ such that

$$f(z) = (z - z_0)^{-n} h(z), \quad \forall z \in U \setminus \{z_0\}.$$

Proof. By the previous theorem, we have $1/f(z) = (z - z_0)^n g(z)$, where g is holomorphic and non-vanishing in a neighborhood of z_0, so the result is given by taking $h(z) = 1/g(z)$. □

Theorem 12.4 (§3.1.3)**.** If f has a pole of order n at z_0, we may write

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + G(z),$$

where $G(z) = \sum_{n \geq 0} b_n (z - z_0)^n$ is holomorphic on U.[13]

Proof. Since z_0 is a pole of f, we consider $F(z) = (1/f)(z)$, which admits z_0 as a zero. By the previous theorem, $F(z) = (z - z_0)^n g(z)$. □

Note that specifically, in the context above, we call

$$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0}$$

the *principal part* of f at z_0, and a_{-1} the residue of f at z_0.

Definition 12.5. A function f on an open set Ω if there exists a sequence of points $\{z_0, z_1, \ldots,\}$ that has no limit points in Ω, and such that (i) f is holomorphic on $\Omega \setminus \{z_0, z_1, \ldots\}$, and (ii) f has poles at $\{z_0, z_1, \ldots\}$.

We define the extended complex plane $\hat{\mathbb{C}}$ as $\mathbb{C} \cup \{\infty\}$, and we equip $\hat{\mathbb{C}}$ with the following topology; we say that U is open in $\hat{\mathbb{C}}$ if U is open in $\mathbb{C}$, or $\hat{\mathbb{C}} \setminus U$ is closed and bounded in $\mathbb{C}$. As an example, $U = \mathbb{C} \setminus \overline{D_r(z_0)} \cup \{\infty\}$ is open.

Proposition 12.6. $\hat{\mathbb{C}}$ is homeomorphic to a sphere S in $\mathbb{R}^3$.

Proof. Identify the complex plane with the xy-plane in $\mathbb{R}^3$, and suppose that the sphere is centered at $(0, 0, \frac{1}{2})$ with radius $\frac{1}{2}$. Let $\Phi : S \setminus \{N\} \to \mathbb{C}$, where $N = (0, 0, 1)$, be the stereographic projection of $W = (X, Y, Z) \mapsto w = (x, y, 0)$. We have that

$$\frac{X}{x} = \frac{Y}{y} = \frac{1 - z}{1} \implies x = \frac{x}{1 - z}, y = \frac{y}{1 - z}.$$

We may check that the stereographic projection Φ is a bijective homeomorphism by writing $\Phi^{-1} : \mathbb{C} \to S \setminus \{N\}$ is given by

$$\Phi^{-1}(x, y) = \left(\frac{x}{x^2 + y^2 + 1}, \frac{y}{x^2 + y^2 + 1}, \frac{x^2 + y^2}{x^2 + y^2 + 1} \right).$$

In fact, we may extend to $\Phi : S \to \hat{\mathbb{C}}$ by observing that $\Phi(W) \to \infty$ as $W \to N$, so we just set $\Phi(N) = \infty$. □

[13] this was bundled together with the proof of the previous theorem in class, but i'm separating because of shakarchi''

We say that a meromorphic function f in $\mathbb{C}$ is meromorphic in $\hat{\mathbb{C}}$ if (i) ∞ is an isolated singularity of f, i.e., f is holomorphic on $\mathbb{C} \setminus \overline{D_r(z_0)}$ for some $z_0 \in \mathbb{C}$ and $r > 0$, and (ii) f is either holomorphic at ∞ or has a pole at ∞, i.e., if $F(z) = f(1/z)$, then 0 is an unisolated singularity of F. f is holomorphic at ∞ if F is holomorphic at 0, and f has a pole at ∞ if F has a pole at 0.

Theorem 12.7 (§3.3.4)**.** The meromorphic functions on $\hat{\mathbb{C}}$ (the extended complex plane) are rational functions $P(z)/Q(z)$ (where $P(z), Q(z)$ are polynomials).

Proof. Let f be any meromorphic function on $\hat{\mathbb{C}}$. This means ∞ is an isolated singularity, which means f is holomorphic in $\mathbb{C} \setminus \overline{D_r(0)}$. Then f can only have finitely many poles in $\mathbb{C}$, say, $z_1, \ldots, z_n$. For each pole z_k, there exists an open neighborhood U_k of z_k, and a non-vanishing holomorphic function g_k on U_k such that

$$f(z) = \frac{a_{-n_k}}{(z - z_k)^{n_k}} + \cdots + \frac{a_{-1}}{z - z_k} + g_k(z)$$

on U_k, and define $f_k(z)$ to be the principal part of f at z_0, for which f_k is a poylnomial in $\frac{1}{z-z_k}$. Similarly, we can consider $F(z) = f(1/z)$, where 0 is an isolated singularity of F. If 0 is a pole of F, then there exists an open disc $D_r(0)$ and a holomorphic function g_0 such that

$$F(z) = \tilde{f}_0(z) + \tilde{g}_0(z)$$

on $D_r(0) \setminus \{0\}$; note that $\tilde{f}_0(z)$ denotes the principal part of F at 0 (and similarly for $\tilde{g}$), which is a polynomial in $1/z$. We may perform a final change of coordinates

$$f(z) = F(1/z) = \tilde{f}_0(1/z) + \tilde{g}_0(1/z),$$

where $\tilde{f}_0(1/z) = f_0(z)$ which is a polynomial in z. Finally, write

$$H(z) = f(z) - f_0(z) + \left[\sum_{k=1}^{n} f_k(z)\right],$$

for which we note that each term of the summation is a polynomial in $\frac{1}{z-z_k}$ and f_0 is a polynomial in z. It suffices to check that H is entire and bounded so we may apply Liouville's theorem to conclude. To see that H is entire, it suffices to show that each z_k is a removable singularity of H. Recall that f_k is the principal part of f at z_k; there exists an open neighborhood U_k of z_k and a holomorphic g_k on U_k such that $f(z) - f_k(z) = g_k(z)$, where g_k gives a holomorphic extension of $f - f_k$ to z_k. This means $\sum_{j \neq k} f_j(z) + f_0(z)$ is holomorphic on U_k, and so z_k is a removable singularity of H.

For boundedness, it suffices to check that H is bounded on $\mathbb{C} \setminus \overline{D_R(0)}$ for some $R > 0$ (per compactness implying boundedness immediately). Recall that f_0 comes from the principal part of $F(z) = f(1/z)$ at 0. We have that $F(z) = \tilde{f}_0(z) + g_0(z)$, which is holomorphic on $D_r(0)$, and g_0 is bounded on $\overline{D_{r/2}(0)}$, so $f(z) - \tilde{f}_0(z) = g_0(1/z)$ is bounded on $\mathbb{C} \setminus \overline{D_{2/r}(0)}$. $\square$