MAT267 Lecture Notes

Arky!! :3c

'25 Winter Semester

Contents

§1 Day 1: Introduction to Class (Jan. 8, 2025)

Class administrative details! Some classes will be on Zoom (such as this Friday); class materials are Hirsch-Smale-Devaney's *Differential Equations, Dynamical Systems, and Linear Algebra*, Tenenbaum and Pollard's *Ordinary Differential Equations*, Perko's *Differential Equations and Dynamical Systems*, and Paul's online notes on ODEs.

Exam testing topics include statements of theorems (along with their proofs), harder homework questions (easier homework questions can show up on quizzes as well). Quizzes will be held in approximately the first ten minutes of class, every Wednesday, aside from weeks in which there are midterms.

We start with a few examples of ODEs. Newton's law states that $F = ma$, where F represents force, m represents mass, and a represents acceleration. Consider a moving object;

- $x(t)$ represents the displacement of an object;
- $x'(t) = v(t)$ represents the velocity of the object;
- $x''(t) = a(t)$ represents the acceleration of the object.

Hooke's Law states that $F(x) = -kx$, i.e. $mx''(t) = -kx(t)$. In a swinging pendulum system, where α is the angle of the pendulum from the vertical, we have that $m\alpha'' = k\sin\alpha$; $x''(t) = -\frac{k}{m}x(t)$, where a solution could be given as $x(t) = \cos(\omega t)$, with $\omega = \sqrt{\frac{k}{m}}$. More generally, $x(t) = A\cos(\sqrt{\frac{k}{m}}t) + B\sin(\sqrt{\frac{k}{m}}t)$, with A, B constants. These are all the possible solutions to the system.

Definition 1.1. An ODE is an equation $F(t, x(t), x'(t), \ldots, x^{(k)}(t)) = 0$, where x is a vector valued function on an open interval $I \subset \mathbb{R}$, which is k-times differentiable.

Note that this means

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}; \quad F = \begin{pmatrix} F^1 \\ \vdots \\ F^m \end{pmatrix},$$

where $m = n$ usually; if $m > 1$, this is a vector valued system of ODEs.

Definition 1.2. A *classical solution* of an ODE $F(t, x, x', \ldots, x^{(k)}) = 0$ is a function $\phi : I \to \mathbb{R}^n$ (where I is an open interval) which is $\mathcal{C}^k(I)$, such that

$$F(t, \phi(t), \phi'(t), \ldots, \phi^{(k)}(t)) = 0$$

for all $t \in I$.

A *non-example* of a function is $y = \sqrt{-(1 + x^2)}$, and is not a solution to an ODE. An example of an ODE is $x + yy' = 0$. As another example, $\left|\frac{dy}{dx}\right| + |y| + 1 = 0$ has no solutinos.

As an example of notation conventions, x' and $\dot{x}$ are examples of ways to write derivatives, with the former being more common in math and latter more common in physics. In this class, it is expected to be clear on which derivative is being taken.

Definition 1.3. The general solution for an ODE is a formula for *all* possible solutions.

For example, $mx'' + kx = 0$ has a general solution

$$\varphi(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right).$$

An ODE is said to be in *implicit form* if it is written as $F(t, x, \ldots, x^{(k)}) = 0$. It is said to be written in *explicit* or *standard form* if it is written as $x^{(k)} = G(t, x, \ldots, x^{(k-1)})$.

There is a standard trick we can perform with ODEs; we can turn a higher order ODE into a system of first order ODEs, the former being more useful to solve, and the latter being more useful for abstract theorems (existence, uniqueness, etc.). For example, let $mx'' = -kx - cx'$. Then let $x_1 = x$, $x_2 = x'$, and we may construct the system

$$\begin{aligned} x_1' &= x_2, \\ mx_2' &= -kx_1 - cx_2. \end{aligned}$$

From this, we get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{pmatrix}.$$

The philosophy of the course is that there may be problems from philosophy, economics, physics, etc. and we wish to order them into first order ODE systems (i.e. vector fields in $\mathbb{R}^n$), of which we will find the general solution, and turn into a group of $\mathcal{C}^1$ or $\mathcal{C}^k$ diffeomorphisms (with structure provided from composition).

We now give some ODE examples.

(i) Consider $x' = 0$ on $\mathbb{R}$. Then the general solution is given by $x(t) = c$; this is true by the mean value theorem, since $x(t_1) - x(t_2) = (t_1 - t_2)x'(c)$ for some $c \in (t_1, t_2)$; since $x'(c) = 0$ as per the ODE, we have that $x(t_1) = x(t_2)$, and we have that x is constant.

(ii) Let $x' = f(t)$. By FTC, the general solution is $x(t) = x_0 + \int_0^t f(s)\,ds$. We may let $x_0 = x(0)$; this is called the parameter.

(iii) Let $x' = ax$. Then $a \in \mathbb{R}$ fixes $x(t) = Ae^{at}$ as the general solution. Let $y(t) = e^{-at}x(t)$. Assume $x(t)$ is a solution. Then

$$y'(t) = \frac{d}{dt}(e^{-at}x(t)) = -ae^{-at}x + e^{-at}\underbrace{x'}_{=ax} = 0.$$

By MVT, we have that $y(t) = A$ yields $x(t) = Ae^{at}$ as desired. $\square$

(iv) Something something on unstable and stable stationary points, the idea that if you deviate a little on an unstable stationary point you will "leave" it, but if you deviate a little on a stable one you will go back.

List of readings to do before next class; chapter 1 of HSD, and 1-5 of TP.

Consider the Logistic EQ (*note: page 4 in HSD, section 1.2*),

$$x' = ax\left(1 - \frac{x}{N}\right),$$

where $x \in \mathbb{R}$, and a, N being fixed constants. Without loss of generality, let $N = 1$. Then the ODE reduces to $x' = f_a(x) = ax(1 - x)$. This is an example of a first order, autonomous, and nonlinear ODE (definitions in the book); we now solve the ODE through separation of variables.

In a nutshell, the idea is separating all the x's to one side, and all the t's to the other side, then integrating. Recall that

$$\frac{dx}{dt} = f(t)g(x),$$

with f, g continuous. If $g(x_0) = 0$, then $x(t) = x_0$ is a solution. We may directly write as follows,

$$\frac{dx}{g(x)} = f(t)\,dt \implies \int \frac{dx}{g(x)} = \int f(t)\,dt \implies G(x) = F(x) + C,$$

where C is a constant, $G'(x) = \frac{1}{g(x)}$ and $F'(t) = f(t)$. Then we claim that $x = G^{-1}(F(t) + C)$ on (a, b) if G^{-1} exists. As justification, through a change of variables, we have

$$\frac{x'(t)}{g(x(t))} = f(t) \iff \int \frac{x'(t)\,dt}{g(x(t))} = \int f(t)\,dt \implies G(x(t)) = F(t) + C,$$

where our change of variables is given by $\frac{d}{dt}G(x(t)) = \frac{x'(t)}{g(x(t))} = f(t)$. Note that this process works as long as $g(x) \neq 0$. $\square$

IUT proceeded to integrate both sides as page 5 in the textbook which I did not bother to write out lmao. He also drew the slope field for the ODE (figure 1.3).

§2 Day 2: Existence and Uniqueness Theorem (Jan. 10, 2025)

Theorem 2.1 (Existence and Uniqueness Theorem)**.** Let $x' = F(t, x)$, $x(t_0) = x_0$ for $x \in U \subset \mathbb{R}^n$ where U is open, and $t_0 \in \mathbb{R}$. If F is $\mathcal{C}^1(I \times U)$, where I represents time and U is the domain, then for $t_0 \in I$, for all $x_0 \in U$, there exists a unique solution $x(t)$ with $x(t_0) = x_0$. $x(t)$ is defined on a time interval J depending on (t_0, x_0).[1]

If solutions cross at $\overline{x_0}$, then at $\overline{x_0}$ we would have more than one solution, which is a contradiction; and so, solutions don't cross. If a solution set covers $\mathbb{R}^2$, then we have the general solution. As a consequence, $x \equiv 1, x \equiv 0$ are solutions if $x_0 \in (0, 1) \implies x(t) \in (0, 1)$. As an example, let

$$\frac{dy}{dx} = \sqrt{|y|}.$$

Then $y \equiv 0$ is a solution. To see this, let us separate the variables as follows,

$$\begin{aligned} \frac{dy}{\sqrt{|y|}} = dx &\implies 2\sqrt{|y|} = x - c \\ &\implies \begin{cases} y &= \frac{1}{4}(x-c)^2, \\ y &= -\frac{1}{4}(x-c)^2. \end{cases}, \end{aligned}$$

where $x \in (c, \infty)$ is a solution for all $c \in \mathbb{R}$, and the latter if $x \in (-\infty, c)$ is a solution. In particular,

$$y_{ab}(x) = \begin{cases} -\frac{(x-a)^2}{4} & , x < a \\ 0 & , a \leq x \leq b \\ \frac{(x-b)^2}{4} & , x > b, \end{cases}$$

for all $x \in \mathbb{R}$ and $a < b$. Recall that the initial value problem $y' = \sqrt{|y|}$, $y(0) = 0$ has infinitely many solutions, including $y \equiv 0$. Our problem at hand is that if $y(x_0) = x_0 < 0$, then the solution reaches 0 in finite time, which allows the solution to be continued by 0. If $y(x_0) = x_0 > 0$, the same happens by going backwards in time. Consider

$$x' = ax(1-x) = \int a(x).$$

Near 0, $x > 0$ implies $\int a(x) > 0$, with slopes greater than 0, and so the solution is increasing. If $x < 0$, the opposite holds. Thus, we are moving away from 0, i.e., 0 is the source. Similarly, $x = 1$ is a sink. Analytically,

$$\int a'(x) = a - 2ax$$

Evaluating at $x = 0$, we have that $\int a'(0) = a > 0$, i.e., the slopes are increasing through 0 as x passes through 0. Specifically, the slopes are < 0 for $x < 0$, and > 0 for $x > 0$. Thus, the solutions are moving through 0.

We now enter chapter 2 material: linear systems of ODEs. Let $X' = A(t)x + f(t)$. A is an $n \times n$ matrix of coefficients, $f : I \to \mathbb{R}^n$, and $f(t)$ is called the *inhomogeneity*. Let

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix};$$

[1] 7.2 in HSD.. picard lindelöf? page 144.

if $f(t) \equiv 0$, then $X' = A(t)x + f(t)$ is said to be homogeneous. If $A(t) = A$ constantly, then the system is said to be of constant coefficient ODEs. Morally, $x' - Ax = f$ implies $Bx = C$, and we have $x = B^{-1}C$. We now examine cases.

(a) Suppose $A(t) = A$ constantly. Then $x' = Ax$. A is a $\mathbb{R}^{n \times n}$ fixed matrix; if $n = 1$, we would have $x' = ax \implies x = Ae^{at}$. If $x \equiv 0$, then we have an equilibrium point. We *guess* for now that $x(t) = x_0 e^{At}$, where $x \in \mathbb{R}^n$. For example,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We *hope* that

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

First, does the series above converge? Second, is it differentiable w.r.t. t? Ideally, we want $\frac{d}{dt} e^{tA} = Ae^{tA}$. Note that at $t = e$, $e^{0A} = \mathrm{id}$. To start, $x(t) = x_0 e^{tA}$ solves $x' = Ax$ for all $x_0 \in \mathbb{R}^n$, and is a general solution. In particular,

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \mathrm{id} + tA + \frac{t^2}{2} A^2 + \frac{t^3}{6} A^3 + \dots.$$

The idea is that we wish to seek solutions $x' = Ax$ of the form

$$x(t) = e^{\lambda t} \sigma,$$

where λ is the unknown parameter, and σ is to be determined as well. Let us solve it as follows; [2]

$$\begin{aligned} x'(t) &= \lambda e^{\lambda t} \sigma \\ \implies x(t) &= e^{\lambda t} v \\ \implies e^{\lambda t} \lambda \sigma &= e^{\lambda t} A v. \end{aligned}$$

Another example was presented in class but the handwriting was illegible.

[2] the hell

§3 Day 3: Linear ODE Systems (Jan. 15, 2025)

Our quiz today was on exercise (*§1.13*) from the textbook, and to state the simple version of the existence and uniqueness theorem.

Let $x' = Ax$; then a solution to the system is given by $x(t) = ve^{\lambda t}$, where v is a vector in $\mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$. We know that $x(t)$ is a solution if $Av = \lambda v$; i.e., v is an eigenvector.

(a) Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$; specifically, the system can be written as

$$\begin{aligned} x' &= 2x + 3y, \\ y' &= x. \end{aligned}$$

To derive the general solutoin of this system, we may solve the characteristic polynomial $\det(A - \lambda I) = 0$ to obtain the eigenvalues (in our case, $\lambda_1 = 3, \lambda_2 = -1$), and then find vectors in the kernel of $A - \lambda_i I$. In this case, we have

$$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}; \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In this way, we have two solutions,

$$\begin{aligned} x_1(t) &= e^{3t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \\ x_2(t) &= e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Together, these two provide equilibrium solutions in the xy-plane in the form of two lines passing through the origin. Do note that, if $\tilde{\lambda} = 0$, then $x_i(t) = e^{\tilde{\lambda} t}v = v$, as expected.

Theorem 3.1 (Superposition Principle)**.** Suppose $x_1(t)$ solves $x' = A(t)x + f_1(t)$, and $x_2(t)$ solves $x' = A(t)x + f_2(t)$. If $a_1, a_2 \in \mathbb{R}$, then $x(t) = a_1x_1(t) + a_2x_2(t)$ solves $x' = A_t(x) + a_1f_1(t) + a_2f_2(t)$. In general, solutions to $x' = A(t)x$, i.e., a homogeneous system, are a vector space, and the general solution of $x' = A(t)x + f(t)$ is given by $\hat{x}(t) + y(t)$, where $\hat{x}(t)$ is the general solution of $x' = A(t)x$ (the homogeneous equation), and $y(t)$ is *one* particular solution of $x' = A(t)x + f(t)$.

Let $A(t) \in \mathbb{R}^{n\times n}$ be a vector space of $\dim n$. We start with a special case.

Claim 3.2. Let $v_1, \ldots, v_n$ be linearly independent eigenvectors of A, i.e. $Av_i = \lambda_i v_i$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then $x_i(t) = e^{\lambda_i t}v_i$ solves $x' = A_x$, $x(0) = v_i$.

Claim 3.3. We have that $x(t) = a_1x_1(t) + \cdots + a_nx_n(t)$, where $a_1, \ldots, a_n \in \mathbb{R}$ is the general solution.

The first statement is proven by the existence and uniqueness theorem. For the second claim, fix $a_1, \ldots, a_n \in \mathbb{R}$. Set $y(t) = a_1e^{\lambda_1 t}v_1 + \cdots + a_ne^{\lambda_n t}v_n$. By superposition, we have that $y' = Ay$ and $y(0) = a_1v_1 + \cdots + a_nv_n$.

Claim 3.4. $y(t)$ is the only solution of $x' = Ax$ with $x(0) = a_1v_1 + \cdots + a_nv_n = x_0$.

Assume $z(t)$ is another solution of the above; since $\{v_1, \ldots, v_n\}$ forms a basis of $\mathbb{R}^n$, we have that $z(t) = b_1(t)v_1 + \cdots + b_n(t)v_n$, and so $b_i(0) = a_i$, because $z(0) = x_0 + y(0)$. Now, write

$$\begin{aligned} z(t) &= b_1(t)v_1 + \cdots + b_n(t)v_n, \\ z'(t) &= b_1'(t)v_1 + \cdots + b_n'(t)v_n, \end{aligned}$$

and so

$$\begin{aligned} Az(t) &= A(b_1(t)v_1) + \cdots + b_n(t)v_n) \\ &= b_1(t)\lambda_1 v_1 + \cdots + b_n(t)\lambda_n v_n, \end{aligned}$$

i.e. $\{v_1, \ldots, v_n\}$ is a basis and so we have $b_i'(t) = \lambda_i b_(t)$ with $b_i(0) = a_i$ as an ODE on $\mathbb{R}$. Thus, the solution is provided by $b_i(t) = e^{\lambda_i t}a_i$, implying $z = y$. □

To expand on the one-line proof with existence and uniqueness for claim 3.1, consider the initial value problem $x' = A(t)x$ with $x(0) = v$. Then if S is the set of solutions to the IVP, with $x \in S$, then the operator $T : x \mapsto x(0) = v \in \mathbb{R}^n$. Since T is linear, onto, and injective, we have that it is a linear isomorphism, so S is a vector space of $\dim n$ as desired. □

Definition 3.5. The system $x' = Ax$ has a saddle point at 0 if A has eigenvalues $\lambda_2 < 0 < \lambda_1$.

We go back to giving examples.

(b) Let $B = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} = A + 2I$. We have that $\lambda_1 = 5, \lambda_2 = 1$, and $v_1 = \binom{3}{1}$, $v_2 = \binom{1}{-1}$, and so we have two solutions

$$\begin{aligned} x_1(t) &= e^{5t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \\ x_2(t) &= e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Since $0 < \lambda_2 < \lambda_1$, we have that the "speed" that we move along at on x_1 is called the "fast direction". Once again, we say that a node (in our case, the origin), is stable if it is a sink, unstable if it is a source, and semistable otherwise. Since the eigenvalues are both positive, it is indeed a source. In general, eigenvectors determine the direction in which the equilibrium lines are / the solutions, and eigenvalues determine the dynamics (i.e., speed). [3]

(c) Let $C = A - 5I$. Now, we have $\lambda_1 = -2$ and $\lambda_2 = -6$, with corresponding eigenvectors $\binom{3}{1}$ and $\binom{1}{-1}$. In this case, the phase portrait would have a sink at the origin (the node is at the origin since $\det C \neq 0$).

We now discuss complex eigenvectors.[4]

Lemma 3.6. Let $A \in \mathbb{R}^{n\times n}$, and choose an eigenvector v with eigenvalue λ. If $\lambda \notin \mathbb{R}$ (and we write $\lambda = \alpha + \beta i$, with $\beta \neq 0$), then:

(i) v is not real; in particular, $\Re(v), \Im(v)$ are linearly independent.

[3] intuitively, this is just because of how we put $e^\lambda t$ so the speed is exponential nyoooommmm

[4] iut quote: uhhh everybody fine with kindergarten complex numbers?

(ii) $\overline{\lambda}$ is also an eigenvalue with eigenvector $\overline{v}$.

Directly write, $Av = \lambda v \iff \overline{\lambda}\overline{v} = \overline{Av} = A\overline{v}$, since A is real. This resolves (ii); we now check (i). $\lambda = \alpha + \beta i$, where $\beta \neq 0$. Then $Av = \lambda v$, and $v = u + iw$, where $u, w \in \mathbb{R}^n$.

Claim 3.7. $\{\Re(v), \Im(v)\}, \{u, w\}$ are linearly independent over $\mathbb{R}$.

Proceed by contradiction; suppose they are LD. Then we may write $u = sv_0$, $w = tv_0$, where $s, t \in \mathbb{R}$ and $v_0 \in \mathbb{R}^n$. Then $v = u + iw = (s + it)v_0$; thus, we have that v_0 is also an eigenvector with eigenvalue λ, i.e. $Av_0 = \lambda v_0 = (\alpha + \beta i)v_0$. However, each of the components in this equation are real, aside from $\alpha + \beta i$, and so this is a contradiction. Thus, they are indeed linearly independent.

(d) Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, and consider $x' = Ax$. Then the characteristic polynomial is given by $\lambda^2 - 4\lambda + 5 = 0$, and the eigenvalues are $2 + i$ and $2 - i$, with corresponding eigenvectors $\binom{i}{1}$ and $\binom{-i}{1}$.

The general complex solution is given by

$$z(t) = c_1 e^{\lambda_1 t} v + c_2 e^{\overline{\lambda_1} t} + \overline{v},$$

where $c_1, c_2 \in \mathbb{C}$. The real solution is given by

$$x(t) = \Re(z(t)) = \frac{1}{2}(z + \overline{z}).$$

Given the complex solution, $z = e^{\lambda t} v$. We will construct the real solution explicitly next time.