MAT367 Lecture Notes

Arky!! :3c

'26 Winter Semester

Contents

§1 Day 1: Recap of Preliminaries (Jan. 6, 2026)

Today's class can be followed more precisely on §1.2 to §1.4 of our textbook by Gross and Meinrenken. The slogan of this class is that a manifold is something that locally looks like $\mathbb{R}^n$. Specifically, an n-manifold can be covered n-dimensional charts $(U \subset M) \to \mathbb{R}^n$, with our main motivating example being solutions sets to equations. Recall the implicit function theorem,

Theorem 1.1. Given a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, consider the solution set $f(x_1, \ldots, x_{n+1}) = 0$ and a point $p \in \mathbb{R}^n$ such that $\nabla f(p) \neq 0$; then, for $(x_1, \ldots, x_{n+1})$ in said solution set near p, we can represent solutions as $(x_1, \ldots, x_n, g(x_1, \ldots, x_n))$, where g is also a smooth function.

In particular, if 0 is a regular value[1] of f, then we can cover $\{x \mid f(x) = 0\}$ by graphs/charts. We present some examples;

(i) Let $f(x, y) = xy$; then $\ker f$ is precisely the x and y axes, which is not a manifold, because it does not look like $\mathbb{R}^n$ (for any n) near the origin.

(ii) Let $f(x, y) = y - x^{2/3}$; then $\ker f$ can be graphed in desmos as $y = x^{2/3}$, which is not a smooth manifold because of its behavior at 0.

(iii) The n-sphere $S^n = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$ can be regarded as the level set of the ℓ^2-norm, for which $S^0 = \{\pm 1\} \subset \mathbb{R}$, S^1 is a circle, S^2 is the usual sphere. Note that we may use the stereographic projection as seen in complex analysis, to view S^3 (and any of the previous or subsequence S^n) as $\mathbb{R}^3 \cup \{\infty\}$.

(iv) The 2-dimensional torus T^2 is the surface of revolution obtained from a circle of radius r and R about an axis of revolution. It can be regarded as a level set by writing
$$T^2 = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - R)^2 + z^2 = r\}.$$

(v) The Möbius strip can't be a part of a level set (at a regular value) because level sets are orientable (2-sided), while the strip is not.

(vi) The Klein bottle is also not orientable; it is closed (doesn't have a boundary), and doesn't embed into $\mathbb{R}^3$. It can be immersed into $\mathbb{R}^3$, i.e., locally embedded but not globally, as seen in the textbook.

Theorem 1.2 (Whitney Embedding Theorem)**.** Every n-manifold has an embedding in $\mathbb{R}^{2n}$.

In this class, we prefer to deal with intrinsic descriptions of manifolds rather than extrinsic ones; a good motivation is given on p.7 in the textbook with respect to our 2-torus.

[1] note to self: what's a regular value?