MAT354 Lecture Notes

Arky!! :3c

'25 Fall Semester

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§1 Day 1: Recap of Preliminaries (Sep. 2, 2025)

We start by discussing the complex plane and complex numbers. Given $z \in \mathbb{C}$, we say that $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of z respectively, i.e., $z = x + iy$. $\mathbb{C}$ is the set of all complex numbers. In this manner, we may identify $z = x + iy$ with $(x, y) \in \mathbb{R}^2$ using the standard complex plane.

(a) The complex *conjugate* of z is given by $\bar{z} = x - iy$, where we have that

$$\Re(z) = \frac{z + \bar{z}}{2}, \qquad \Im(z) = \frac{z - \bar{z}}{2i}.$$

(b) We now define addition and mlutiplication for the complex numbers. For all $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have that

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2 y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2). \end{aligned}$$

We have that $(\mathbb{C}, +, \times)$ is a field, with $(\mathbb{R}, +, \times)$ as a subfield. To verify this, we need to check that it indeed satisfies:

- Commutativity; for all $z_1, z_2 \in \mathbb{C}$, we have that $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$.
- Associativity: for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
- Distributivity: for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

(c) The absolute value of a complex number $z = x + iy$ is given by $|z| = \sqrt{x^2 + y^2}$. In particular, this yields the triangle inequality, where for any $z, w \in \mathbb{C}$, we have that $|z + w| \leq |z| + |w|$. The proof either comes visually or through explicit computation, both of which I will not write out here for brevity.[1]

As an extension of the inequality, we also automatically have that

$$|\Re z| \leq |z|, \qquad |\Im z| \leq |z|,$$

and that for all $z, w \in \mathbb{C}$, we have

$$||z| - |w|| \leq |z - w|.$$

Proof. Using the triangle inequality, we have that

$$\begin{aligned} |z| &= |(z - w) + w| \leq |z - w| + |w|, \\ |w| &= |(w - z) + z| \leq |z - w| + |z|, \end{aligned}$$

of which both imply that $|z| - |w| \leq |z - w|$ and $|w| - |z| \leq |z - w|$. $\square$

For any $z \in \mathbb{C}$, we have that $|z|^2 = z \cdot \bar{z}$.

Proof. Write $z = x + iy$; then $|z|^2 = x^2 + y^2$, where we may note that $z \cdot \bar{z} = (x + iy)(x - iy)$ which yields the right hand side of the earlier equation through expansion. $\square$

[1] no full credit if you draw a picture on the exam lmao

Finally, for $z, w \in \mathbb{C}$, we have that $|zw| = |z|\,|w|$. This is left as an exercise to the student.

(d) The polar form of a nonzero complex number $z \neq 0$ is given by $z = \gamma e^{i\theta}$, where $\gamma > 0$ and $\theta \in \mathbb{R}$. Let us assume the Euler formula; for all $\theta \in \mathbb{R}$, we have that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Let $r = |z|$; we have that $|z| = \left|re^{i\theta}\right| = |r|\left|e^{i\theta}\right| = r \cdot 1 = r$. θ is the angle between the positive real axis to the half-line starting from 0 and passing through z. In this manner, $z = re^{i\theta} = |z|(\cos\theta + i\sin\theta) = |z|\cos\theta + i\,|z|\sin\theta$, which means we have that

$$\Re z = |z|\cos\theta, \qquad \Im z = |z|\sin\theta.$$

As an example, let us find all the complex numbers z such that $z^4 = i$. Since $i = e^{i\frac{\pi}{2}}$, $z = \rho e^{i\theta}$ satisfying $z^4 = i$ becomes $\rho^4 e^{i\cdot 4\theta} = e^{i\frac{\pi}{2}}$, meaning

$$\begin{cases} \rho^4 = 1, \\ 4\theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

This means $\rho = 1$ and $\theta = \frac{\pi}{8} + \frac{k\pi}{2}$, where $k \in \mathbb{Z}$. Considering the cases $k = 0, 1, 2, 3$ and observing that there are only 4 equivalence classes modulo 4 to consider, we have that

$$z_0 = e^{i\frac{\pi}{8}}, \quad z_1 = e^{i\frac{5\pi}{8}}, \quad z_2 = e^{i\frac{9\pi}{8}}, \quad z_3 = e^{i\frac{13\pi}{8}}.$$

We now discuss convergence. We say that a set of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ converges to $w \in \mathbb{C}$ if $\lim_{n\to\infty} |z_n - w| = 0$. We write it as $\lim_{n\to\infty} z_n = w$. In the complex plane, the convergence can be in any direction.

Lemma 1.1. $\{z_n\}_{n\in\mathbb{N}}$ converges to w if and only if $\{\Re z_n\}_{n\in\mathbb{N}}$ converges to $\Re w$ and $\{\Im z_n\}_{n\in\mathbb{N}}$ converges to $\Im w$.

Proof. We have that

$$\begin{aligned} |z_n - w| &= |(\Re z_n - \Re w) + i(\Im z_n - \Im w)| \\ &\leq |\Re z_n - \Re w| + |\Im z_n - \Im w|, \end{aligned}$$

where as $n \to \infty$, we have that the right hand side is given by $0 + 0$. For the opposite direction, we have that $|z| \geq |\Re z|$ or $|\Im z|$, so we have that

$$|\Re z_n - \Re w| = |\Re(z_n - w)| \leq |z_n - w|,$$

which approaches 0 as $n \to \infty$. The same argument goes for the imaginary portion. □

A sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ is called *Cauchy* if $|z_n - z_m| \to 0$ as $n, m \to \infty$. In $\varepsilon - \delta$, this means that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|z_n - z_m| < \varepsilon$ for all $n, m > N$.

Theorem 1.2 (Bolzano-Weierstrass Theorem)**.** $\mathbb{R}$ is *complete*, i.e., every Cauchy sequence of real numbers convergesto a real number.

Theorem 1.3. $\mathbb{C}$ is complete.

Proof. Take any Cauchy sequence of complex numbers $\{z_n\}$. Using the inequalities $|\Re z| \leq |z|$ and $\{\Im z\} \leq |z|$, we have that $\{\Re z_n\}$ and $\{\Im z_n\}$ are Cauchy sequences of real numbers. By Bolzano-Weierstrass, we have that $\Re z_n \to x_0 \in \mathbb{R}$ and $\Im z_n \to y_0 \in \mathbb{R}$. By the previous lemma, we actually have $\lim_{n\to\infty} z_n = x_0 + iy_0$. □

We now move onto topology in the complex plane. Given $z_0 \in \mathbb{C}$ and $r > 0$, we can form an open or closed disc centered at z_0 of radius r. We write both of these as

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$
$$\bar{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\},$$

Given a set $\Omega \subseteq \mathbb{C}$, a point z_0 is an interior point if there exists $r > 0$ such that $D_r(z) \subseteq \Omega$. The interior of Ω is given by the set of all such interior points. In particular, the interior of $\bar{D}_r(i)$ is $D_r(i)$.

A set Ω is called *open* if every point in Ω is an interior point. Ω is called *closed* if the complement of Ω, $\Omega^c = \mathbb{C} \setminus \Omega$, is open. As an example, the open right half-plane $\{z \in \mathbb{C} \mid \Re z > 0\}$ is open.

Proof. For any $z \in \Omega$, let $z = x + iy$, and take $r = \frac{x}{2} = \frac{\Re z}{2}$. Then we claim that $D_r(z) \subseteq \Omega$. For all $w \in D_r(z)$, we clearly have that

$$\Re w = \Re z - (\Re z - \Re w) \geq \Re z - |z - w| \geq \frac{\Re z}{2} > 0,$$

and so all such $w \in \Omega$, and we are done. $\square$

A point $z \in \mathbb{C}$ is a *limit point* of Ω if there exists a sequence $\{z_n\} \subset \Omega$ with $z_n \neq z$ such that $z_n \to z$.

As an example, we define D to be the open unit disc centered at 0. 0 and 1 are both limit points of D, but 1 is not contained in D itself.[2] The *closure* of Ω, $\bar{\Omega}$, is given by Ω unioned with all its limit points. The *boundary* of a set Ω, wirtten $\partial\Omega$, is given by $\bar{\Omega} \setminus \operatorname{int}\Omega$. A set $\Omega \subseteq \mathbb{C}$ is said to be compact if it is closed and bounded, i.e., there exists $M > 0$ such that $|z| \leq M$ for all $z \in \Omega$.

Theorem 1.4. A set $\Omega \subseteq \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω.

Proposition 1.5. If $\Omega_1 \supset \Omega_2 \cdots \supset \Omega_n \supset \dots$ is a sequence of nonempty compact sets in $\mathbb{C}$, where $\operatorname{diam}(\Omega_n) = \sup_{z,w \in \Omega_n} |z - w| \to 0$ as $n \to \infty$, then there exists a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for every $n \in \mathbb{N}$.

Proof. For each Ω_n, pick a point $z_n \in \Omega_n$. Then $\{z_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence because the diameter of Ω_n approaches 0. By the Bolzano-Weierstrass theorem for complex numbers, this means that $\{z_n\}_{n\in\mathbb{N}}$ indeed does converge to some $w \in \mathbb{C}$. In particular, we have w is the limit of the subsequence $\{z_m\}_{m\geq n} \subseteq \Omega_n$, where Ω_n is compact, meaning the limit w should be in Ω_n. This means there exists a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for every $n \in \mathbb{N}$.

To show the uniqueness of w, we argue by contradiction; assume $w' \neq w$ satisfies the property. Then $|w' - w| > 0$. Since $w, w' \in \Omega_n$ for all n, this contradicts that $\operatorname{diam}(\Omega_n) \to 0$. $\square$

An open set Ω is called *connected* if it is not possible to find two disjoint nonempty open sets Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. A connected open set in $\mathbb{C}$ is called a *region*.

[2] hell is it disc or disk YKW LET'S COMPROMISE it's spelled disque actually (paint nails)

§2 Day 2: Functions on the Complex Plane (Sep. 4, 2025)

Let $f : \Omega \to \mathbb{C}$, where Ω is an open subset of $\mathbb{C}$. We say that f is continuous if at $z_0 \in \Omega$ if, for all $\varepsilon > 0$, there exists an open disk $D_\gamma(z_0)$ such that $|f(z) - f(z_0)| < \varepsilon$ for all $z \in D_\gamma(z_0)$. In particular, f is said to be continuous on Ω if it is continuous at every point in Ω.

Example 2.1. Consider $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$. Show that f is continuous.

Solution. For all complex z, z_0, we have that $|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0| = |z - z_0|$. Thus, we have that for any $\varepsilon > 0$, we obtain[3]

$$f(D_\varepsilon(z_0)) = D_\varepsilon(\overline{z_0}). \qquad \square$$

We now discuss holomorphic functions (i.e., complex differentiable functions). We say that $f : \Omega \to \mathbb{C}$ is *holomorphic* at $z_0 \in \Omega$ if

$$\frac{f(z_0 + h) - f(z_0)}{h}, \qquad h \in \mathbb{C} \setminus \{0\},$$

converges as $h \to 0$. If the limit exists, we let

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

be the derivative.

Example 2.2. Consider the exact same function as in the previous example, $f(z) = \bar{z}$. Is f holomorphic?

Solution. For all $z_0 \in \mathbb{C}$ and $h \in \mathbb{C} \setminus \{0\}$, we have that

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{z_0 + h} - \overline{z_0}}{h} = \frac{\overline{h}}{h} = \frac{\rho e^{-i\theta}}{\rho e^{i\theta}} = e^{-2i\theta}.$$

If we take $h \to 0$ along the real line, we may let $h = \rho$, which means the fraction is equal to 1 as $h \to 0$. If we take $\rho \to 0$ along the complex axis, however, then we have that $h = \rho e^{i\pi/2}$, where we obtain the fraction is equal to -1 as $\rho \to 0$. Thus, f cannot be holomorphic. $\square$

Proposition 2.3. Let Ω be open in $\mathbb{C}$. If f, g are holomorphic on Ω, then

(i) $f + g$ is holomorphic on Ω, and $(f + g)' = f' + g'$.

(ii) fg is holomorphic on Ω, and $(fg') = f'g + fg'$.

(iii) If $g(z_0) \neq 0$ where $z_0 \in \Omega$, then $\frac{f}{g}$ is also holomorphic at z_0, where

$$\left(\frac{f}{g}\right)' = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

(iv) If $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, then $g \circ f$ is also holomorphic, and we obtain the chain rule

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

[3] note to self: ol is better than bar for this stuff...

We now discuss complex differentiability versus real differentiability. A holomorphic function $f : \Omega \to \mathbb{C}$ can be identified with a function $F : \Omega \to \mathbb{R}^2$ given by $(x, y) \mapsto (u(x, y), v(x, y)) = (\Re f(x, y), \Im f(x, y))$. Consider the partial derviative of F at (x_0, y_0); these exist if there exists some linear transformation $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{\|F(P_0 + H) - F(P_0) - J(H)\|}{\|H\|} \to 0$$

as $H \to 0$. Or, we may define $\Psi(H)$ to take on the fraction above, and we see that F is indeed differentiable at $P_0 = (x_0, y_0)$ if $\Psi(H) \to 0$ as $H \to 0$. We now deal with complex differentiability. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic at $z_0 = x_0 + iy_0$. Then we have partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}.$$

Naturally,

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h - f(z_0)}{h}$$

along any path; in particular, we take $h \in \mathbb{R} \setminus \{0\}$ and observe that

$$\begin{aligned}
f'(z_0) &= \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{h\to 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\
&= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0),
\end{aligned}$$

and so both exist, and they are $\Re f'(z_0)$ and $\Im f'(z_0)$ respectively. Similarly, we may take $h = ik$ where $k \in \mathbb{R} \setminus \{0\}$ and obtain

$$\begin{aligned}
f'(z_0) &= \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{k\to 0} \frac{u(x_0, y_0 + k) + iv(x_0, y_0 + k) - u(x_0, y_0) - iv(x_0, y_0)}{ik} \\
&= \lim_{k\to 0} \frac{-i(u(x_0, y_0 + k) - u(x_0, y_0)) + v(x_0, y_0 + k) - v(x_0, y_0)}{k} \\
&= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0),
\end{aligned}$$

and so both partials also exist and they are $\Re f'(z_0)$ and $-\Im f'(z_0)$ respectively.

§3 Day 3: Holomorphic Functions and Power Series (Sep. 9, 2025)

Let $f : \Omega \to \mathbb{C}$ (where Ω is an open set in $\mathbb{C}$). We say that f is holomorphic at z_0 if

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}, \quad h \in \mathbb{C} \setminus \{0\}$$

exists. Recall that $\mathbb{C}$ can be identified with $\mathbb{R}^2$ by considering any $z = x + iy \in \mathbb{C}$ as a tuple $(x, y) \in \mathbb{R}^2$. In this way, given a function $f : \Omega \to \mathbb{C}$, we can define $F : \Omega \to \mathbb{R}^2$, where $F : (x, y) \mapsto (u(x, y), v(x, y))$, given by $u = \Re f$ and $v = \Im f$.

Proposition 3.1. If $f = u + iv$ is holomorphic at $z_0 = x_0 + iy_0$, then we have that all four partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist and they satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \Re f(z_0), \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \Im f(z_0).$$

We also have that F is differentiable at $P_0 = (x_0, y_0)$.

Definition 3.2. We say that F is differentiable at P_0 if there exists a linear transforamtion (the derivative) $J = J_F(x_0, y_0) : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{H\to 0} \frac{\|F(P_0+H) - F(P_0) - J(H)\|}{\|H\|} = 0.$$

Before we discuss the complex definition, let us recall another property of real differentiability; if F is differentiable at $P_0 = (x_0, y_0)$, then all four partial derivatives exist, and

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is called the Jacobian matrix of F at (x_0, y_0). To see this, consider the association $P_0 = (x_0, y_0)$ with $z_0 = x_0 + iy_0$, and $H = (h_1, h_2)$ with $h = h_1 + ih_2$; then we have that

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 \\ \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 \end{pmatrix}.$$

This is a vector in $\mathbb{R}^2$, which we may associate with the complex number

$$\left(\frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2\right) + i\left(\frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2\right) = \left(\frac{\partial u}{\partial x} + i\frac{\partial y}{\partial x}\right) h_1 + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) h_2,$$

which, by the Cauchy-Riemann equations, we obtain

$$\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) h_1 + i\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\right) h_2 = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2),$$

which is precisely equal to $f(z_0 + h) - f(z_0) - f(z_0)h$. In particular,

$$\lim_{h\to 0} \left| \frac{f(z_0+h) - f(z_0) - f(z_0)h}{h} \right| = \lim_{h\to 0} \left| \frac{f(z_0+h) - f(z_0)}{h} - f(z_0) \right| = 0.$$

Similarly, per the definition of the Jacobian, we must have

$$\lim_{H\to 0} \frac{\|F(P_0+H) - F(P_0) - J(H)\|}{\|H\|} = 0,$$

and this concludes the proof of proposition 3.1. □

Theorem 3.3. Suppose $f = u + iv$ is a complex-valued function defined on an open set $\Omega \subset \mathbb{C}$. If $u, v : \Omega \to \mathbb{R}$, are continuously differentiable and satisfy the Cauchy-Riemann equations, then f is holomorphic on Ω and $f'(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)$.

Proof. Since u is continuously differentiable at the point $(x, y) \in \Omega$, there exists a linear transformation $J_u : \mathbb{R}^2 \to \mathbb{R}$ where

$$\frac{|u(x + h_1, y + h_2) - u(x, y) - J_0(h_1, h_2)|}{\|(h_1, h_2)\|} \to 0, \quad (h_1, h_2) \to 0.$$

In particular, $J_u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$. The above fraction is equivalent to

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + \|h\| \psi_1(h),$$

where $\psi_1 : U \to \mathbb{R}$, where U is some open neighborhood of $0 \in \mathbb{R}^2$, with $\psi_1(h) \to 0$ as $h \to 0$. Similarly, we have that

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + \|h\| \psi_2(h)$$

with $\psi_2(h) \to 0$ as $h \to 0$. We want to show that f is holomorphic at $z = x + iy$. We have that

$$\begin{aligned} f(z + h) - f(z) &= (u(x + h_1, y + h_2) - u(x, y)) + i(v(x + h_1, y + h_2) - v(x, y)) \\ &= \left(\frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial h_y} h_2\right) + \|h\| \psi_1(h) + i\left(\frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2\right) + i \|h\| \psi_2(h) \\ &= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + \|h\| \psi_1(h) + i \|h\| \psi_2(h) \end{aligned}$$

from Cauchy-Riemann. Thus, we have that[4]

$$\begin{aligned} \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} &= \lim_{h \to 0} \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + \frac{\|h\|}{h}(\psi_1(h) + i\psi_2(h)) \\ &= \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right). \end{aligned}$$ □

We now discuss complex power series.

Definition 3.4. A complex power series is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

with $a_n \in \mathbb{C}$ and z a complex variable. We say that $\sum_{n=0}^{\infty} a_n z^n$ converges at $z_0 \in \mathbb{C}$ if there exists some $w \in \mathbb{C}$ such that, for all $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $N \geq N_0$ satisfies

$$\left|\sum_{n=0}^{N} a_0 z^n - w\right| < \varepsilon.$$

The series converges *absolutely* at z_0 if there exists $w \in \mathbb{R}$ such that

$$\left|\sum_{n=0}^{N} |a_n| |z_0|^n - w\right| < \varepsilon.$$

[4] i swear wenyu has an invisible key wired into her back like nano from nichijou and it's permanently cranked on

Proposition 3.5. If $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely at $z_0 \in \mathbb{C}$, then $\sum_{n=0}^{\infty} a_n z^n$ converges at $z_1 \in \mathbb{C}$ with $|z_1| \leq |z_0|$.

Proof. For all $z_1 \in \mathbb{C}$ with $|z_1| \leq |z_0|$, consider the sequence of partial sums $\{S_m(z_1)\}_{m\in\mathbb{N}}$ given by

$$S_m(z_1) = \sum_{n=0}^{m} a_n z_1^n.$$

We want to show that such a sequence converges. Since $\mathbb{C}$ is complete, it suffices to show that said sequence is Cauchy. For all $m < k \in \mathbb{N}$, we have that

$$|S_k(z_1) - S_m(z_1)| = \left|\sum_{n=m+1}^{k} a_n z_1^n\right| \leq \sum_{n=m+1}^{k} |a_n| \, |z_1|^n \leq \sum_{n=m+1}^{k} |a_n| \, |z_0|^k. \qquad \square$$

We now provide a few examples.

(i) The complex exponential function for all $z \in \mathbb{C}$, given by

$$e^z := \sum_{n=0}^{\infty} = \frac{z^n}{n!}.$$

For all $z \in \mathbb{C}$, this sum converges because it converges absolutely (consider $e^{|z|}$).

(ii) The geometric series $\sum_{n=0}^{\infty} z^n$, where $|z| < 1$, converges; otherwise, is $|z| \geq 1$, it diverges. In particular, if $\sum_{n=0}^{\infty} z^n$ converges, then $|z^n| \to 0$ as $n \to \infty$.

Theorem 3.6 (Shakarchi, Thm. 2.5)**.** Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $R \in [0, \infty)$ such that (i) if $|z| < R$, the series converges, and (ii) if $|z| > R$, the series diverges. We call R the *radius of convergence* of $\sum_{n=0}^{\infty} a_n z^n$, and $\{z \in \mathbb{C} \mid |z| < R\}$ the disc[5] of convergence. Moreover, R is given by Hadamard's formula,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} =: L,$$

where we use the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proof. For all $z \in \mathbb{C}$ with $|z| < r < R$, there exists some $\varepsilon > 0$ such that

$$(L + \varepsilon)\,|z| = r < 1.$$

By definition of L, we have $|a_n|^{1/n} \leq L + \varepsilon$ for all large n, meaning that

$$|a_n|\,|z|^n = \left(|a_n|^{1/n}\,|z|\right)^n \leq ((L+\varepsilon)\,|z|)^n = r^n, \quad r \in (0,1),$$

whereby comparison with the geometric series $\sum r^n$, we see that $\sum |a_n|\,|z|^n$ converges. Similarly, if $|z| > R$, we have that

$$\left(\frac{1}{r} - \varepsilon\right)|z| > 1,$$

where, using the definition of R, there exists an infinite subsequence a_{n_k} such that $|a_{n_k}|^{1/n_k} \geq \frac{1}{R} - \varepsilon$. We have that

$$|a_{n_k} z^{n_k}| - \left(|a_{n_k}|^{1/n_k}\,|z|\right)^{n_k} \geq \left[\left(\frac{1}{R} - \varepsilon\right)|z|\right]^{n_k} > 1. \qquad \square$$

[5] disque. ok i'll stop

§4 Day 4: Complex Power Series (Sep. 11, 2025)

As per given in the previous lecture, recall that the complex power series is defined as an infinite sum of the form

$$\sum +n = 0^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, z \in \mathbb{C},$$

i.e., z as a complex variable.

Theorem 4.1. The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function on its disc of convergence. The derivative of f is given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Moreover, f' has the same radius of convergence as f.

Proof. Let g be the power series defining f', and let $R \geq 0$ be the radius of convergence of f. The radius of convergence of g is also R, per Hadamard's formula,

$$\limsup_{n\to\infty} |na_n|^{\frac{1}{n-1}} \overset{(*)}{=} \limsup_{n\to\infty} |a_n|^{\frac{1}{n}\cdot\frac{n}{n-1}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R},$$

since

$$n^{\frac{1}{n-1}} = e^{\frac{\log n}{n-1}} \xrightarrow{n\to\infty} e^0 = 1. \tag{$*$}$$

For all $z_0 \in \mathbb{C}$ with $|z_0| < r < R$ and $h \in \mathbb{C} \setminus \{0\}$ with $|z_0 + h| < r$, let us compute the following,

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right|;$$

to start,

$$f(z) = \underbrace{\sum_{n=0}^{N} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{E_N(z)},$$

where $N \in \mathbb{N}$ is to be determined; we have that

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right) + \\ \left(S_N'(z_0) - g(z_0)\right) + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h} \right).$$

We compute each part individually.

$$\begin{aligned} \left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| &= \left| \frac{\sum_{n=N+1}^{\infty} a_n (z_0+h)^n - \sum_{n=N+1}^{\infty} a_n z_0^n}{h} \right| \\ &\leq \sum_{n=N+1}^{\infty} \frac{|a_n|}{h} \left|(z_0+h)^n - z_0^n\right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n| \left|(z_0+h)^{n-1} + (z_0+h)^{n-2} + \cdots + z_0^{n-1}\right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n| \gamma^{n-1} \cdot n \xrightarrow{n\to\infty} 0, \end{aligned}$$

as y has the radius of convergence of $R > r$. Next,

$$\left|S_N'(z_0) - g(z_0)\right| \xrightarrow{N\to\infty} 0,$$

since $S_N'(z_0) = \sum_{n=1}^{N} na_n z_0^{n-1}$ and $g(z_0) = \sum_{n=1}^{\infty} na_n z_0^{n-1}$. Given any $\varepsilon > 0$, we may choose a sufficiently large N such that

$$\left|S_N'(z_0) - g(z_0)\right| < \varepsilon, \quad \left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| < \varepsilon,$$

per our two computations above. Since $S_N(z)$ is a finite polynomial, $S_N'(z_0)$ is the derivative of $S_N(z)$ at z_0, and so there exists $\delta > 0$ such that, for all $0 \leq |h| < \delta$, we have

$$\left|\frac{S_N(z_0+h) - S_N(z_0)h}{h} - S_N'(z_0)\right| < \varepsilon,$$

which resolves all three parts of our expansion, and so we are done. □

Corollary 4.2. The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is infinitely complex differentiable on its disc of convergence. For $k \in \mathbb{N}$, its kth derivative $f^{(k)}$ is given by

$$f^{(k)}(z) = \sum_{n=0}^{\infty} (a_n z^n)^k.$$

Definition 4.3. A function $f : \Omega \to \mathbb{C}$ is said to be *analytic* at $z_0 \in \Omega$ if there exists a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ with positive radius of convergence such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

on a neighborhood of $z_0 \in \Omega$.

In particular, this means that if $f : \Omega \to \mathbb{C}$ is holomorphic, we have that f is holomorphic at $z_0 \in \Omega$, and so f is analytic at $z_0 \in \Omega$ as well. The implication that analytic implies holomorphic was given by our earlier theorem; the direction that holomorphic implies analytic is given by Cauchy's integral formula, but we need to first define integration along curves.

(i) A parameterized curve is a function $z : [a,b] \to \mathbb{C}$, where $t \mapsto z(t)$. This gives the orientation from $z(a)$ to $z(b)$.

(ii) (*Regularity conditions on curves*). We say that the parameterized curve is smooth if $z'(t)$ exists, is continuous on $[a,b]$, and $z'(t) \neq 0$ for $t \in [a,b]$. We say that the parameterized curve z is piecewise smooth if z is continuous on $[a,b]$ and there exists a partition of $[a,b]$ with $a = a_0 < \cdots < a_n = b$ such that $z(t)$ is smooth on each $[a_r, a_{r+1}]$.

§5 Day 5: Curves in the Complex Plane (Sep. 16, 2025)

We say that a parameterized curve is a function $z : [a, b] \to \mathbb{C}$ where $t \mapsto z(t)$; in particular, z gives the orientation from $z(a)$ to $z(b)$.We say that z is *smooth* if $z'(t)$ exists and is continuous on $[a, b]$, where $z'(t) \neq 0$ for $t \in [a, b]$. We say it's *piecewise smooth* if z is continuous on $[a, b]$ and we have a partition $a = a_0 < \cdots < a_n = b$ such that $z(t)$ is smooth on each $[a_k, a_{k+1}]$.

Example 5.1. Let $z : [0, 2\pi] \to \mathbb{C}$, where $t \mapsto z_0 + Re^{it}$, and $z_1 : [0, \frac{\pi}{2}] \to \mathbb{C}$, where $t_0 \mapsto z_0 = Re^{i4t}$.

We say that two smooth parameterizations, $z : [a, b] \to \mathbb{C}$ and $\widetilde{z} : [c, d] \to \mathbb{C}$, are *equivalent* if they have the same image and orientation; i.e., if there exists a continuously differentiable bijection $s \mapsto t(s)$ from $[c, d]$ to $[a, b]$ such that $t'(s) > 0$ (read: same orientation) and $\widetilde{z} = z \circ t$. In this way, all equivalent smooth parameterizations of $z : [a, b] \to \mathbb{C}$ can be written as a smooth curve γ with image $z([a, b])$ and orientation from $z(a)$ to $z(b)$. In addition, we denote γ^- as said smooth curve, but with reversed orientation.

A smooth or piecewise smooth curve given by $z : [a, b] \to \mathbb{C}$ is said to be *closed* if $z(a) = z(b)$, and *simple* if $z(t) \neq z(s)$ for all $t \neq s$ in the time interval (note that if the curve is closed, we allow $s = a$, $t = b$ to satisfy $z(s) = z(t)$). We now define integration along curves.

Definition 5.2. Let $f : \Omega \to \mathbb{C}$ be a continuous function, and let γ be a smooth curve in Ω parameterized by $z : [a, b] \to \mathbb{C}$. Then

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t)) z'(t)\, dt,$$

where we may realize $f \circ z : [a, b] \to \mathbb{C}$. The length of γ is defined as $\text{length}(\gamma) = \int_a^b |z'(t)|\, dt$.

Example 5.3. Consider the function $f(z) = z^{-1}$ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.[6] Let C be a circle in $\mathbb{C}^*$ centered at z_0 with radius $R > 0$, equipped with an anticlockwise orientation. Compute $\int_C f(z)\, dz$.

While this example seems trivial, there is a lot of casework to work through, and we don't have the prerequisite knowledge for it yet.

Proposition 5.4. Integration of continuous functions along smooth (or piecewise smooth) curves satisfy the following properties,

(i) (*Linearity*) For all $\alpha, \beta \in \mathbb{C}$, we have that

$$\int_\gamma (\alpha f + \beta g)(z) = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz$$

(ii) If γ^- is γ with reversed orientation, then

$$\int_{\gamma^-} f(z)\, dz = -\int_\gamma f(z)\, dz.$$

[6] 417 notation seeping into my 354 work

(iii) We have the following inequality,

$$\left|\int_\gamma f(z)\,dz\right| \leq \left(\sup_{z\in\gamma}|f(z)|\right)\cdot \text{length}(\gamma).$$

Exercise 5.5. Check that the definition of integration is well-defined.

We now prove the above proposition.

Proof. Assume γ is smooth and parameterized by $z : [a,b] \to \mathbb{C}$. Then

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))\cdot z'(t)\,dt\right| \leq \int_a^b \left|f(z(t))\cdot z'(t)\right|\,dt,$$

which we note is true by considering

$$\left|\sum_i u(t_i) + iv(t_i)\Delta t\right| \leq \sum_i |u(t_i)+iv(t_i)|\,\Delta t,$$

so we indeed have that

$$\int_a^b \left|f(z(t))\cdot z'(t)\right|\,dt \leq \left(\sup_{z\in[a,b]}|f(z)|\right)\cdot\int_a^b |z'(t)|\,dt = \left(\sup_{z\in[a,b]}|f(z)|\right)\cdot \text{length}(\gamma) \quad \square$$

Suppose $f : \Omega \to \mathbb{C}$. A *primitive* for f on Ω is a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \Omega$.

Theorem 5.6 (Complex Fundamental Theorem of Calculus)**.** If a continuous function f has a primitive F on Ω, and γ is a curve that begins at w_1 and ends at w_2, then

$$\int_\gamma f(z)\,dz = F(w_2) - F(w_1).$$

Proof. Suppose γ is smooth and parameterized by $z : [a,b] \to \mathbb{C}$ with $z(a) = w_1$ and $z(b) = w_2$. Then

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt = \int_a^b (F(z(t)))'\,dt,$$

since we may note that $(F\circ z)' = (F'\circ z)\cdot z' = (f\circ z)\cdot z'$, whereby we note that the above integral evaluates to $F(z(b)) - F(z(a)) = F(w_2) - F(w_1)$. $\square$

Corollary 5.7. If f is holomorphic on a region Ω and $f' = 0$, then f is constant.

Recall that Ω is called a region if it is an open connected set. Alternatively, connectedness is equivalent to path connectedness here, since if Ω is path connected, it is connected (by Medusa), and if it is connected, then it is locally path connected, and through a partition, local path connectedness implies path connectedness.

Proof. Note that Ω is path connected per our earlier digerssion; fix $z_0 \in \Omega$. We will show that $f(z) = f(z_0)$ for all $z \in \Omega$; let z, z_0 be joined by a piecewise smooth curve γ. Then we have

$$0 = \int_\gamma f'(z)\,dz = f(z) - f(z_0),$$

and so f is constant on Ω. $\square$

Theorem 5.8 (Goursat's Theorem)**.** If Ω is an open set in $\mathbb{C}$ and $T \subset \Omega$ is a triangle whose interior is also in Ω, then for any holomorphic function f on Ω, we have $\int_T f(z)\,dz = 0$.

Proof. Let $T^{(0)}$ be the original triangle. Let $d^{(0)}, p^{(0)}$ be the diameter and perimeter of $T^{(0)}$ respectively. Take the midpoints of each side of $T^{(0)}$, and form 4 smaller triangles with orientation consistent to the orientation of $T^{(0)}$; we will call these triangles $T_1^{(1)}, \dots, T_4^{(1)}$. Clearly,

$$\int_{T^{(0)}} f(z)\,dz = \sum_{k=1}^{4} \int_{T_k^(1)} f(z)\,dz,$$

along with

$$\left|\int_{T^{(0)}} f(z)\,dz\right| = \sum_{k=1}^{4} \left|\int_{T_k^(1)} f(z)\,dz\right|.$$

Let $T_j^{(1)}$ be chosen to be such that $\left|\int_{T_k^{(1)}} f(z)\,dz\right|$ is maximal among $k \in \{1, \dots, 4\}$; we will write $T^{(1)} = T_j^{(1)}$, and iterate this process to obtain a sequence of triangles $\{T_0, T_1, \dots\}$, where

$$\left|\int_{T^{(0)}} f(z)\,dz\right| \le 4^n \left|\int_{T^{(n)}} f(z)\,dz\right|.$$

$d^{(k)}, p^{(k)}$ are defined analogously, where

$$d^{(k)} = \frac{1}{2^k} d^{(0)}, \quad p^{(k)} = \frac{1}{2^k} p^{(0)}.$$

Let $\mathcal{T}^{(n)}$ be the solid triangle enclosed by $T^{(n)}$. Clearly, $\mathcal{T}^{(0)} \supset \mathcal{T}^{(1)} \supset \cdots \supset \mathcal{T}^{(n)}$, and there exists a unique $z_0 \in \mathbb{C}$ such that $z_0 \in \mathcal{T}^{(n)}$ for every n; since f is holomorphic at z_0, we have that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

with $\psi(z) \to 0$ as $z \to z_0$. We may write,

$$\int_{T^{(n)}} f(z)\,dz = \int_{T^{(0)}} f(z_0)\,dz + \int_{T^{(n)}} f'(z_0)(z - z_0)\,dz + \int_{T^{(0)}} \psi(z)(z - z_0)\,dz.$$

The first two terms vanish, since $f(z)$, $f'(z_0)(z-z_0)$ have primitives $f(z_0)z$ and $\frac{1}{2}f(z_0)(z-z_0)^2$ respectively. It remains to compute the last term; we have that

$$\left|\int_{T^{(n)}} \psi(z)(z - z_0)\,dz\right| \le \left(\sup_{z \in T^{(n)}} |\psi(z)|\right)\left(\sup_{z \in T^{(n)}} |z - z_0|\right) \operatorname{length} T^{(n)}$$

where we note the first term approaches 0 as $n \to 0$, the second term is bounded above by $2^{-n} d^{(0)}$, and the third term is bounded above by $2^{-n} p^{(n)}$. We may combine everything to obtain

$$\left|\int_{T^{(0)}} f(z)\,dz\right| \le 4^n \left|\int_{T^{(n)}} f(z)\,dz\right| \le d^{(0)} p^{(0)} \left(\sup_{z \in T^{(n)}} |\psi(z)|\right) \xrightarrow{n \to \infty} 0. \qquad \square$$

§6 Day 6: Cauchy's Theorem on a Disc (Sep. 18, 2025)

Recall Goursat's theorem from last class, where if $\Omega \subset \mathbb{C}$ is open and $T \subset \Omega$ is a triangle whose interior is contained in Ω, then for any holomorphic function f on Ω, we have that

$$\int_T f(z)\, dz = 0.$$

We introduce a follow-up to this thoerem.

Theorem 6.1. If f is holomorphic on a disc, then $\int_\gamma f(z)\, dz = 0$ for any closed curve γ in that disc.

To prove this, we start by using Goursat's theorem to show f has a primitive, and then we complete the proof using the complex FTC, i.e., if f is holomorphic on a disc, then f has a primitive on that disc.

Proof. After a translation, we may assume that the center of the disc is 0. Define $F : D \to \mathbb{C}$, given by $z \mapsto \int_{\gamma_z} f(u)\, du$. To show that F is holomorphic and $F'(z) = f(z)$, fix $z \in D$, and observe that for any $h \in \mathbb{C} \setminus \{0\}$ with $z + h \in D$, we have that

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(u+h)\, du - \int_{\gamma_z} f(u)\, du.$$

Regard this as the path from z to 0 to $z+h$. Let us add to the expression the integrals over two paths, going both directions so that we do not change the value of $F(z+h) - F(z)$, one between z and $\Re(z+h) + i\Im(z)$, and one between z and $z+h$ directly. In this manner, we've created a rectangular region and a triangular region on which we have path integrals over, and per Goursat's theorem, they all vanish, and we are left with the integral on the path η from z to $z+h$. This means all that remains is to compute[7]

$$F(z+h) - F(z) = \int_\eta f(w)\, dw$$

Since f is continuous at z, we may write $f(w) = f(z) + \psi(w)$, where $\psi(w) \to 0$ as $w \to z$. This means we may write

$$\int_\eta f(w)\, dw = \int_\eta f(z)\, dw + \int_\eta \psi(w)\, dw = f(z)(z+h-z) + \int_\eta \psi(w)\, dw,$$

upon which we may rearrange and rewrite the above RHS to obtain

$$\begin{aligned} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \left| \frac{1}{h} \int_\eta \psi(w)\, dw \right| \\ &\leq \frac{1}{|h|} \sup_{w \in \eta} |\psi(w)| \underbrace{\text{length}(\eta)}_{=|h|} = \sup_{w \in \eta} |\psi(w)| \xrightarrow{h \to 0} 0. \end{aligned}$$

This concludes the hard part of the proof in showing that f has a primitive; by complex FTC, we immediately see that $\int_\gamma f(z)\, dz = 0$, since γ is a closed curve and its endpoints are equal to each other. ∎

[7] GOD KNOWS if this is a w or an ω, i'm just going to use w for now. forensic analysis on yalls handwriting holy shit

We now give an example.

Problem 6.2. For all $\xi \in \mathbb{R}$, let $\mathcal{F}$ denote the Fourier transform, and let

$$(\mathcal{F}f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx.$$

Show that if $f(x) = e^{-\pi x^2}$, we have that $(\mathcal{F}f)(\xi) = f(\xi) = e^{-\pi\xi^2}$.

Solution. In the $\xi = 0$ case, we immediately have that

$$(\mathcal{F}f)(0) = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} e^{-\pi x^2} = 1,$$

from computation through the Gaussian integral (polar coordinate transform). If $\xi > 0$ (we note that $\xi < 0$ follows analogously), let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = e^{-\pi z^2}$. Then, for $R > 0$, let us integrate on the rectangle from $-R$ to R, R to $R+i\xi$, $R+i\xi$ to $-R+i\xi$, and $-R+i\xi$ to $-R$, where the latter three paths are denoted I_1, I_3, I_2 respectively (we intentionally number this way because the two opposing sides I_1, I_2 can be tackled together at once). We have that

$$0 = \int_{\gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{I_1} f(z)\,dz + \int_{I_2} f(z)\,dz + \int_{I_3} f(z)\,dz.$$

Let us consider the integral $\left|\int_{I_1} f(z)\,dz\right|$, with parameter $I_1 : [0,\xi] \to \mathbb{C}$, given by $t \mapsto R + it$; we have that

$$\left|\int_{I_1} f(z)\,dz\right| = \left|\int_0^{\xi} f(R+it)i\,dt\right| = \left|\int_0^{\xi} e^{-\pi(R+it)^2} i\,dt\right|,$$

for which we observe that the integrand

$$e^{-\pi(R+it)^2} i = e^{-\pi(R^2-t^2)} e^{i\pi 2Rt} \le e^{-\pi(R^2-\xi^2)} \to 0, \quad R \to +\infty,$$

so the integrals on $I_1, I_2 \to 0$ for large enough R (we note that the same conclusion held for I_2 because the computation follows analogously). For the last part, consider that

$$\int_{I_3} f(z)\,dz = \int_{-R}^{R} f(t+i\xi)\,dt = \int_{-R}^{R} e^{-\pi(t+i\xi)^2}\,dt = e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi t^2} e^{-2\pi i\xi t}\,dt,$$

upon which we obtain $e^{\pi\xi^2}(\mathcal{F}f)(\xi)$ as $R \to \infty$. This means we have that $0 = -e^{\pi\xi^2}(\mathcal{F}f)(\xi)$, where the minus sign is from the orientation of I_3. This means we may conclude that $(\mathcal{F}f)(\xi) = e^{-\pi\xi^2} = f(\xi)$. $\square$

§7 Day 7: Cauchy's Integral Formula and Corollaries (Sep. 23, 2025)

We start with an example.

Example 7.1 (Fresnel Integrals)**.** Prove that

$$\int_0^\infty (\sin x^2)\, dx = \int_0^\infty \cos(x^2)\, dx = \frac{\sqrt{2\pi}}{4}.$$

Solution. To do this, we proceed by Cauchy's theorem, i.e., using functions of complex variables. Let e^{ix^2}, which, per Euler's formula, is equal to $\cos(x^2) + i\sin(x^2)$ for $x \in \mathbb{R}$. Let us reframe the question by integrating e^{z^2}, where $z \in \mathbb{C}$, over the contour (closed curve) given by a $\frac{\pi}{4}$ radian sector of the circle of radius $R > 0$ centered at 0; specifically, the contour is given by $0 \to R$, $R \to Re^{i\pi/4}$ along the arc, and $Re^{i\pi/4} \to 0$. In this manner, let $z = \rho e^{i\pi/4}$, where $\rho \in (0, R)$, we have that

$$e^{-(\rho e^{i\pi/4})^2} = e^{-\rho^2\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^2} = e^{-\frac{\rho^2}{2}(1+i)^2} = e^{-\rho^2 i} = \cos(\rho^2) + i\sin(\rho^2).$$

Let the three paths in the contour (which we will call γ_R) be given by I_1, I_2, I_3 in order; we have that, by Cauchy's theorem,

$$0 = \int_{\gamma_R} f(z)\, dz = \int_{I_1} f(z)\, dz + \int_{I_2} f(z)\, dz + \int_{I_3} f(z)\, dz.$$

Directly compute as follows, where $f(z) = e^{-z^2}$,

$$\int_{I_1} f(z)\, dz = \int_0^R e^{-x^2}\, dx \xrightarrow{R\to\infty} \frac{\sqrt{\pi}}{2},$$

Let $-I_3 : [0, R] \to \mathbb{C}$ be given by $t \mapsto te^{i\pi/4}$; we have,

$$\begin{aligned}
\int_{-I_3} f(z)\, dz &= \int_0^R f\left(te^{i\pi/4}\right) e^{i\pi/4}\, dt \\
&= e^{i\pi/4} \int_0^R e^{-(te^{i\pi/4})^2}\, dt \\
&= e^{i\pi/4} \left[\int_0^R \cos t^2\, dt - i\int_0^R \sin(t^2)\, dt\right],
\end{aligned}$$

and finally, for the integral on I_2 (where $I_2 : [0, \frac{\pi}{4}]^2 \to \mathbb{C}$ and $t \mapsto Re^{it}$), we have that

$$\int_{I_2} f(z)\, dz = \int_0^{\pi/4} e^{-(Re^{it})^2} iRe^{it}\, dt,$$

for which we may bound the integrand as follows,

$$\left|e^{-(Re^{it})^2} iRe^{it}\right| \le R\left|e^{-(Re^{it})^2}\right| = R\left|e^{-R^2(\cos(2t)+i\sin(2t))}\right| = Re^{-R^2(\cos 2t)}.$$

This means we may write

$$\left|\int_{I_2} f(z)\, dz\right| \le \int_0^{\pi/4} \left|e^{-(Re^{it})^2} iRe^{it}\right|\, dt = \int_0^{\pi/4} Re^{-R^2(\cos 2t)}\, dt.$$

Let us compute $\cos(2t)$; we have that $2t \in [0, \frac{\pi}{2}]$, so $\cos(2t) = \sin(\frac{\pi}{2} - 2t)$, and $\sin(\theta) \geq \frac{2}{\pi}\theta$ by appealing to geometric intuition; this means

$$\begin{aligned}
\int_0^{\pi/4} Re^{-R^2(\cos 2t)}\,dt &\leq \int_0^{\pi/4} Re^{-R^2\frac{2}{\pi}\left(\frac{pi}{2}-2t\right)}\,dt && \left(\text{Let } s = \tfrac{\pi}{2} - 2t\right) \\
&= \frac{1}{2}\int_0^{\pi/2} Re^{-R^2\frac{2}{\pi}s}\,ds \\
&= \frac{1}{2}\int_0^{\pi/2} Rd\left(\frac{e^{-R^2\frac{2}{\pi}s}}{-R^2\frac{2}{\pi}}\right) \\
&= \frac{1}{2}\cdot\frac{1}{R\frac{2}{\pi}}\left(e^{-R^2}-1\right) \xrightarrow{R\to\infty} 0.
\end{aligned}$$

Having established computations for I_1, I_2, I_3, we may now write

$$0 = \frac{\sqrt{\pi}}{2} - e^{i\pi/4}\left[\int_0^\infty \cos(x^2)\,dx - i\int_0^\infty \sin(x^2)\,dx\right]$$

This means we have

$$\begin{aligned}
\int_0^\infty \cos(x^2)\,dx - i\int_0^\infty \sin(x^2)\,dx &= e^{-i\pi/4}\frac{\sqrt{\pi}}{2} \\
&= \frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{\pi}}{2} - i\frac{\sqrt{\pi}}{2}\right) \\
&= \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}.
\end{aligned}$$

□

We now discuss Cauchy's integral formula. As another example, let D be a disc centered at z, and let f be a holomorphic function; we may express $f(z)$ using the values of f on ∂D.

Example 7.2 (Steady-State Heat Equation)**.** Let $g(x, y)$ be continuous on $\mathbb{R}^2$. Find $u(x, y)$ satisfying

$$\begin{cases} \Delta u = 0 & \text{on } D, \\ u = g & \text{on } \partial D, \end{cases}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator. The solution is given by considering $(x, y) = (r\cos\theta, r\sin\theta)$, where

$$u(r,\theta) = \int P_r(\theta,\varphi)g(\cos\varphi, \sin\varphi)\,d\varphi, \quad P_r(\theta,\varphi) = \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}$$

where P_r is called the *Poisson kernel.*

Theorem 7.3 (Cauchy's Integral Formula)**.** Suppose f is holomorphic in an open set Ω that contains the closure of a disc D. Let $C = \partial D$ equipped with the anticlockwise orientation. Then for any $z \in D$,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof. We start by constructing a "keyhole contour" on D, where δ is the width of the corridor, and ε is the radius of the circle centered at z. The contour can be thought of as picking a point in C and connecting it to the ε-circle about z with a δ-wide corridor.

Let the contour be called $\Gamma_{\delta,\varepsilon}$. Let $F(\zeta) = \frac{f(\zeta)}{\zeta - z}$; clearly, it is holomorphic on $\Omega \setminus \{z\}$. By Cauchy's theorem,

$$0 = \int_{\Gamma_{\delta,\varepsilon}} F(\zeta)\, d\zeta = \int_{I_1} F(\zeta)\, d\zeta + \int_{I_2} F(\zeta)\, d\zeta + \int_{I_3} F(\zeta)\, d\zeta + \int_{I_4} F(\zeta)\, d\zeta,$$

where I_1, I_3 represent the paths on C and the ε-circle about z respectively, and I_2, I_4 the "walls of the corridor". We start with some basic observations;

(i) If we let $\delta \to 0^+$, then

$$\int_{I_1} F(\zeta)\, d\zeta = \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

(ii) Again, if we let $\delta \to 0^+$, we have that

$$\int_{I_2} F(\zeta)\, d\zeta = -\int_{I_4} F(\zeta)\, d\zeta,$$

since they are simply two path integrals of the opposite orientation.

(iii) For I_3, we may first write

$$\int_{I_3} F(\zeta)\, d\zeta = \int_{I_3} \frac{f(\zeta)}{\zeta - z}\, d\zeta;$$

if we let $\varepsilon \to 0^+$, we see that this is problematic, since we have a singularity at z. However, we notice that the integrand resembles the definition of the derivative, i.e., we may write

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z},$$

so we obtain

$$\int_{I_3} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{I_3} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta + f(z) \int_{I_3} \frac{1}{\zeta - z}\, d\zeta,$$

where the latter term is equal to $-2\pi i f(z)$, per (p.47 in Shakarchi)

$$\int_{I_3} \frac{f(z)}{\zeta - z}\, d\zeta = f(z) \int_{I_3} \frac{d\zeta}{\zeta - z} = -f(z) \int_0^{2\pi} \frac{\varepsilon i e^{-it}}{\varepsilon e^{-it}}\, dt = -f(z) 2\pi i.$$

For the former term, there exists $e_0 > 0$ such that for all $\zeta \in D_{\varepsilon_0}(z)$, we have that

$$\left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \leq \left| f'(z) \right| + 2.$$

We obtain

$$\left| \int_{I_3} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \right| \leq (\left| f'(z) \right| + 2) \cdot 2\pi\varepsilon \xrightarrow{\varepsilon \to 0} 0.$$

Combining all these observations, we obtain

$$0 = \int_{\Gamma_{\delta,\varepsilon}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \xrightarrow{\delta,\varepsilon \to 0^+} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta - 2\pi i f(z),$$

from which we conclude Cauchy's integral formula.[8] □

[8] reference: p.45-47 Shakarchi

Theorem 7.4 (Cor. 4.2, Shakarchi)**.** "A holomorphic function is infinitely complex differentiable."[9] Suppose f is holomorphic in an open set Ω. Then f has infinitely many complex derivatives in Ω. Moreover, for any $z \in \Omega$ and $n \in \mathbb{Z}_{\geq 0}$, we have that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$

Proof. We proceed by induction on n. The base case $n = 0$ is immediately given by Cauchy's integral formula; assuming that the statement is true for $n-1$, for any $h \in \mathbb{C}\setminus\{0\}$ such that $z + h \in D$, we have that

$$\begin{aligned}\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} &= \frac{(n-1)!}{2\pi i h} \int_C \left[\frac{f(\zeta)}{(\zeta - z - h)^n} - \frac{f(\zeta)}{(\zeta - z)^n}\right] d\zeta \\ &= \frac{(n-1)!}{2\pi i h} \int_C f(\zeta) \left[\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n}\right] d\zeta.\end{aligned}$$

By binomial expansion, we have that

$$\begin{aligned}&\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \\ &= \frac{1}{(\zeta - z - h)^n (\zeta - z)^n} \left[(\zeta - z)^n - (\zeta - z - h)^n\right] \\ &= \frac{h}{(\zeta - z - h)^n (\zeta - z)^n} \left[(\zeta - z)^{n-1} + (\zeta - z)^{n-2}(\zeta - z - h) + \cdots + (\zeta - z - h)^{n-1}\right].\end{aligned}$$

By taking h sufficiently small, we obtain

$$\frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{2n}} n(\zeta - z)^{n-1} \, d\zeta = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta. \qquad \square$$

Theorem 7.5 (Thm. 4.4, Shakarchi)**.** "A holomorphic functoin is locally a power series". Suppose f is holomorphic in an open set Ω. If D is a disc centered at z_0 whose closure is contained in Ω, then f has a power series expansion at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in D$, and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \geq 0.$$

Proof. Fix any $z \in D$; by Cauchy's integral formula, we have that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Note that per our previous corollary. The idea is to write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 + z_0 - z} = \frac{1}{(\zeta - z_0)} \frac{1}{\left(1 - \frac{z - z_0}{\zeta - z_0}\right)},$$

[9] hell, i need to run a marathon with 20mg of thc in my system. props wenyu

where we observe that since $z \in D$ is fixed and $\zeta \in C$, we know that there exists some $r \in (0, 1)$ such that

$$\left|\frac{z - z_0}{\zeta - z_0}\right| < r,$$

so we may regard the geometric series representation

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n,$$

for which the series converges uniformly for any $\zeta \in C$. This means we may interchange the integral and the sum to obtain

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta\right) (z - z_0^n). \qquad \square$$

Corollary 7.6 (Liouville's Theorem: Thm. 4.5, Shakarchi)**.** If f is entire and bounded, then f is constant. We say that f is *entire* if it is holomorphic on the whole of $\mathbb{C}$.

Proof. We will prove this later on. Though, it is done by observing that $\mathbb{C}$ is connected (hence a region, i.e., open connected set), then checking $f' = 0$, and so f is constant. $\square$

Corollary 7.7 (Cauchy's Inequality)**.** If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 with radius r, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! \, \|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z \in C} |f(z)|$ (and C is the boundary of D.)

Proof. We have that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta;$$

if we let $C : [0, 2\pi] \to \mathbb{C}$ be given by $t \mapsto z_0 + Re^{it}$, then the above is equal to

$$\frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^{n+1} e^{i(n+1)t}} iRe^{it} \, dt,$$

for which we may write

$$\left|\frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^{n+1} e^{i(n+1)t}} iRe^{it} \, dt\right| \leq \frac{n!}{2\pi} \cdot \frac{\|f\|_C}{R^n} \cdot 2\pi = \frac{n! \, \|f\|_C}{R^n},$$

which finishes the proof. $\square$

§8 Day 8: Morera's Theorem and Distribution of Zeros of Holomorphic Functions (Sep. 26, 2025)

Recall Cauchy's integral formula, where if f is holomorphic on an open set Ω containing the closure of disc D, then let $C = \partial D$; we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta \\ f^{(n)}(z) &= \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta, \quad n \in \mathbb{N}, \end{aligned}$$

i.e., f is infinitely complex differentiable. Recall that we also have that if γ is a closed curve with interior in Ω, then $0 = \int_\gamma f$.

Theorem 8.1 (Morera's Theorem)**.** Suppose f is continuous on an open disc D such that for any triangle T contained in D, we have $\int_T f(z)\,dz = 0$. Then f is holomorphic.

Proof. Recall our earlier proof of Cauchy's theorem on a disc, where we first used Goursat's theorem, then f has a primitive on D. In the second step, we only used that $\int_T f(z)\,dz = 0$, so f has a primitive on the disc, and we may apply the proof to our new f to find F with $F' = f$. Since F is holomorphic, it is infinitely complex differentiable, so we conclude that f is holomorphic as desired. □

Theorem 8.2 (Distribution of zeros of holomorphic functions)**.** Suppose f is holomorphic in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω itself. Then $f = 0$ on Ω (i.e., the zeros are isolated).

Proof. We start by showing that $f = 0$ on a neighborhood of the limit point z_0. Let D be a disc centered at z_0 in Ω; we have that f coincides with a power series on D,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

(where we assume $f \neq 0$). Then there exists some non-negative a_n, per our assumption. Let m be the smallest index such that $a_m \neq 0$, and write

$$f(z) = a_m (z - z_0)^m \left[1 + \frac{1}{a_m}\sum_{n>m} a_n (z - z_0)^{n-m}\right],$$

where we let $g(z)$ be given by $f(z) = a_m(z - z_0)^m(1 + g(z))$. Clearly, $g(z)$ converges on D, since

$$|a_n|^{\frac{1}{n-m}} = |a_n|^{\frac{1}{n}\frac{n}{n-m}} \xrightarrow{n\to\infty} |a_n|^{\frac{1}{n}},$$

so by Hadamard's formula, $g(z)$ has some radius of convergence, as $f(z), g(z) \to 0$ with $z \to z_0$.

Set $z = w_k \neq z_0$ in D, where w_k is some element of the sequence of distinct points. Then we have

$$0 = f(w_k) = a_m (w_k - z_0)^m (1 + g(w_k)),$$

for which all three terms are nonzero (the third can be made to be nonzero by picking k large enough such that $|g(w_k)| < 1$). This means that for a sufficiently large k, we get a contradiction, and so $a_m = 0$ and $f = 0$ on D. This establishes that f vanishes on a local disc about z_0.

We now check that $f = 0$ on the entire of Ω by using the connectedness of Ω. Let U be the interior of $\{z \in \Omega \mid f(z) = 0\}$, and observe that $U \neq \emptyset$ as $D \subset U$ and U is open. It suffices to check that U is closed; let $\{z_n\} \subset U$ be any sequence such that $z_n \to z$ for some $z \in \Omega$. Since f is continuous, we have that $f(z) = 0$. By our previous argument, f is zero on an open neighborhood of z, and so $z \in U$, meaning U contains all its limit points, and is therefore closed. We conclude that U is clopen in Ω, so $U = \Omega$ as desired. $\square$

Corollary 8.3. Suppose f, g are holomorphic in a region Ω, and $f(z) = g(z)$ on a nonempty open subset of Ω. Then $f(z) = g(z)$ on all of Ω.

Remark 8.4. Given f, F analytic in regions Ω, Ω' respectively with $\Omega \subset \Omega'$, if f and F agree on Ω we say that F is an analytic continuation of f into Ω'. Such analytic continuations are always unique.