MAT363 Lecture Notes

ARKY!! :3C

'25 Winter Semester

Contents

§1 Day 1: Introduction to the Class (Jan. 6, 2025)

Course administrative details! First day slides are given here. This is a class in classical differential geometry; the following 12 weeks will be split up as follows,

(a) Curves, for two weeks;

(b) Surfaces, for three weeks;

(c) Curvature of surfaces, for three weeks;

(d) Geodesics, for three weeks;

(e) Gauss-Bonnet theorem, for one week.

Grading will be done by 5% on PCEs, 15% on problem sets, 15% on quizzes, 25% on the term test, 30% on the final exam, and 10% weighted towards your best test.

To start, consider the following maps $\gamma : I = (-10, 10) \to \mathbb{R}^3$, given by

$$\begin{aligned}
\gamma(t) &= (t, t, t); \\
\gamma(t) &= (|t|, |t|, |t|); \\
\gamma(t) &= (t, t^2, t^3); \\
\gamma(t) &= (t^3, t^3, t^3); \\
\gamma(t) &= (\cos t, \sin t, t); \\
\gamma(t) &= (t\cos t, t\sin t, t).
\end{aligned}$$

In this class, we say that a curve is a *parameterized curve* if it is a smooth function $\gamma : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval. In particular, of the six examples given above, only $t \mapsto (|t|, |t|, |t|)$ is not smooth.

Definition 1.1 (Regular Curve)**.** Let $\gamma : I \to \mathbb{R}^n$ be a curve; it is said to be *regular* if $|\gamma'(t)| \neq 0$ for all $t \in I$, i.e. the speed is always nonzero.

Note that $\gamma'(t)$ and $|\gamma'(t)|$ describe different qualities, with the former describing velocity and the latter describing speed (i.e., one describes speed as well, while the other is a scalar quantity). As an example, consider the curve $\gamma(t) = (\cos t, \sin t, t)$. To find the distance travelled from $t = 0$ to $t = 2\pi$, we may observe that

$$\left|\gamma'(t)\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Since the speed is constant, the total distance traveled is simply $2\pi\sqrt{2}$. □

Definition 1.2 (Closed Curve)**.** Consider a curve $\gamma : [a, b] \to \mathbb{R}^n$. We say that γ is a *closed curve* if $\gamma(a) = \gamma(b)$ and $\gamma^{(n)}(a) = \gamma^{(n)}(b)$ for all naturals n.

Definition 1.3 (Simple Curve)**.** We say that γ is a simple curve if it is injective on $[a, b)$.

Note that while in topology we do not care if there is a "sharp corner" at $\gamma(a) = \gamma(b)$, such things do matter, as per the condition that the nth derivative of γ must agree on a and b (for example, the velocity γ' at a, b must be equal).

In this class, we automatically take the inner product $\langle , \rangle$ as the Euclidean inner product,

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n.$$

For any subspace $V \subset \mathbb{R}^n$, we may decompose any vector $x \in \mathbb{R}^n$ uniquely as $x = x^{\parallel} + x^{\perp}$, where $x^{\parallel} \in V$ and $\left\langle x^{\perp}, v \right\rangle = 0$ for any vector $v \in V$. Now, consider any curve $\gamma : I \to \mathbb{R}^n$. We have the following proposition,

Proposition 1.4. If $|\gamma(t)|$ is constant, then $\langle \gamma(t), \gamma'(t) \rangle = 0$ for all $t \in I$.

To see this, let $|\gamma(t)|^2 = c$ be constant; then

$$\frac{d}{dt} |\gamma(t)|^2 = 0 \implies \frac{d}{dt} (\langle \gamma(t), \gamma(t) \rangle) = \left\langle \gamma'(t), \gamma(t) \right\rangle + \left\langle \gamma(t), \gamma'(t) \right\rangle = 0m$$

i.e. $\langle \gamma(t), \gamma'(t) \rangle = 0$ as desired. $\square$

Given a regular curve $\gamma : I \to \mathbb{R}^n$, we may compute the velocity and acceleration as $\gamma'(t), \gamma''(t)$, which are denoted $v(t), a(t)$ respectively. In particular, we may write

$$a(t) = a^{\|}(t) + a^{\perp}(t),$$

with $a^{\|}(t)$ being the tangential acceleration, and $a^{\perp}(t)$ being the normal acceleration. We may find these by projecting $a(t)$ into the subspace $\text{span}\{v\}$ (i.e., the span of the velocity vector).

§2 Day 2: Curvature of a Curve (Jan. 9, 2025)

Definition 2.1. Suppose that $\gamma : I \to \mathbb{R}^n$ is a regular curve. A *reparameterization* of γ is a function of the form $\tilde{\gamma} = \gamma \circ \phi : \tilde{I} \to \mathbb{R}^n$, where $\tilde{I}$ is an interval, and $\phi : \tilde{I} \to I$ is a smooth bijection with nowhere vanishing derivative $\phi'(t) \neq 0$ for all $t \in \tilde{I}$.

Definition 2.2. We say that $\gamma, \tilde{\gamma}$ have the same orientation (i.e., the parameterization is *orientation-preserving*) if $\phi' > 0$, and orientation-reversing if $\phi' < 0$.

Note that since ϕ is smooth, it is impossible for ϕ' to have places on which it is greater than 0 and less than 0, since IVT holds on $\tilde{I}$ and $\phi'(t) \neq 0$.

We now present an example;

$$\gamma : [0, 1] \to \mathbb{R}^3 \text{ with } \gamma(t) = (t, t),$$
$$\beta : [0, \pi/2] \to \mathbb{R}^3 \text{ with } \beta(t) = (\sin t, \sin t).$$

While these both parameterize the same curve, we prefer γ greatly because $|\gamma'(t)| = \sqrt{2}$ while $|\beta'(t)| = \cos t$; specifically, constant speed parameterizations are much nicer to deal with. Even better,

Definition 2.3. A curve γ is said to be parameterized by arclength if $|\gamma'(t)| = 1$.

In particular, we may turn our above $\gamma : t \mapsto (t, t)$ into an arclength parameterization by considering it to be $t \mapsto (\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}})$ instead, since we would then have

$$\left|\phi'(t)\right| = \sqrt{2 \cdot \left(\frac{1}{\sqrt{2}}\right)^2} = 1. \quad \square$$

We now introduce the idea of curvatures of curves. A few ideas first:

- A smaller circle has *larger* curvature (naturally, it is more curved than a large circle).
- The curvature of a curve is given by a function $\kappa : I \to [0, \infty)$; i.e., it cannot be negative. Specifically, it is given by

$$\kappa(t) = \frac{\left|a^{\perp}(t)\right|}{|v|^2}.$$

Curvature enjoys two main properties; κ is independent of the parameterization of the curve, and that $\kappa = \frac{1}{r}$, where r is the radius of the circle that approximates the curve at the particular point. If γ is parameterized by arclength, then $\kappa(t) = a(t)$. To see this, observe that

$$\kappa(t) = \frac{\left|a^{\perp}(t)\right|}{|v|^2} = \left|a^{\perp}(t)\right| = |a(t)|\,.$$

In particular, since $|v(t)|$ is constant, so is $|v(t)|^2$. Then

$$\frac{d}{dt}\langle v(t), v(t)\rangle = 0 \implies \langle v;(t), v(t)\rangle = 0 \implies \langle a(t), v(t)\rangle = 0.$$

From this, along with $a(t) = a^{\perp}(t) + a^{\parallel}(t)$, we have that $a^{\parallel}(t) = 0$.[1] $\square$

[1] for more detail, check proposition 1.18 in the textbook; γ' and γ'' are perpendicular if γ is a curve with constant speed.

Definition 2.4. Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. We define the unit tangent vector and unit normal vector as

$$T(t) = \frac{v(t)}{|v(t)|}; \qquad n(t) = \frac{a^{\perp}(t)}{|a^{\perp}(t)|},$$

respectively.

Note that the textbook uses $\mathfrak{t}, \mathfrak{n}$ respectively, but these are hard to write on paper.