MAT363 Lecture Notes

Arky!! :3c

'25 Winter Semester

Contents

§1 Day 1: Course Administrative Details and Equivalent Norms (Sep. 4, 2024)

Course administrative details!

- The prerequisites to this class are calculus and linear algebra; specifically, this class is about calculus on manifolds.

- The main three theorems this class will cover are the inclusive function theorem, change of variable in integral, and Stokes' theorem (generalization of FTC).

- The main techniques we will study this class are manifolds, theory of integration, and differential forms.

To give an example of how linear algebra can be used in defining differentiability, we start by recalling that for a function $f : \mathbb{R} \to \mathbb{R}$ to be differentiable in 1 variable, we need

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} =: c = f'(a)$$

to exist for any $a \in \mathbb{R}$. Specifically, this may be rewritten as

$$\lim_{h \to 0} \frac{f(a + h) - (f(a) + ch)}{h} = 0,$$

where we have $f(a) + f'(a)h$ as the best local linear approximation of $f(a + h)$ at $h = 0$, and $h \mapsto ch$ is a linear transformation $\mathbb{R} \to \mathbb{R}$. For the higher dimensional analogue of the above, let us take $f : \mathbb{R}^n \to \mathbb{R}^m$, and have

$$\begin{aligned} a &= (a_1, \ldots, a_n), \\ x &= (x_1, \ldots, x_n), \\ h &= (h_1, \ldots, h_n). \end{aligned}$$

If we let $y = f(x)$, then $y \in \mathbb{R}^m$, and we can write it as a column vector with components $y_1, \ldots, y_m$;

$$f(x) = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix}.$$

In this way, each y_i for $1 \leq i \leq n$ can be viewed as a function in n variables. We say f is differentiable at $a \in \mathbb{R}^n$ if there exists a linear transformation $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that[1]

$$\lim_{h \to 0} \frac{|f(a + h) - (f(a) + \lambda h)|}{|h|} = 0$$

In this way, we have that $f(a) + \lambda h$ is the best linear approximation of $f(a + h)$ at $h = 0$. Notice that compared to the $\mathbb{R} \to \mathbb{R}$ differentiability condition, we impose a norm on $f(a + h) - (f(a) + \lambda h)$ and h, since we cannot divide by $\mathbb{R}^n, \mathbb{R}^m$ vectors.

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[1] i think we need a norm in the numerator for the limit below; bierstone didn't do it in class though, so... idk. also lambda is jacobian?

This also brings the topic of what sort of norms we could be looking at; for example, we have[2]

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2},$$
$$\|x\| = \max\{|x_1|, \ldots, |x_n|\},$$
$$x = |x_1| + \cdots + |x_n|.$$

We say that two norms p, q are equivalent to each other if we may find a constant C where $p(x) \le Cq(x)$ (and vice versa). In fact, the norms $|x|, \|x\|, x$ are all equivalent to each other; here is a visual proof,
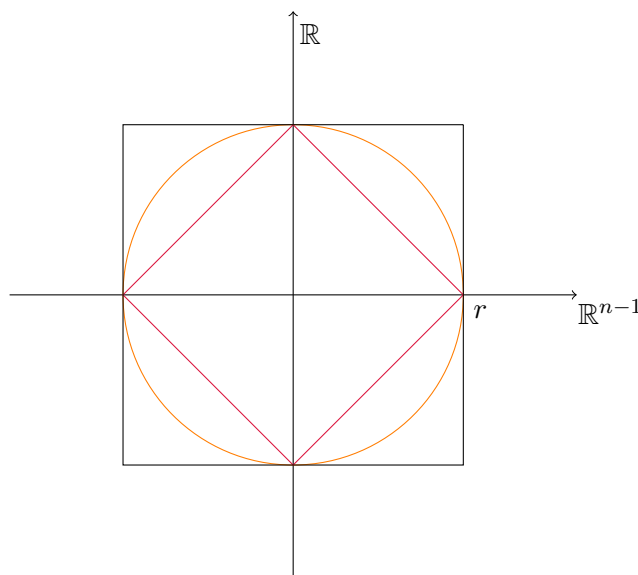
where each figure is the level sets of their respective norms. In particular, $|x| \le \sqrt{n} \|x\|$, and $x \le \sqrt{n} |x|$. To prove the latter analytically, we may write the norm as an inner product, $x = \langle x, u(x) \rangle < |x| |u(x)| \le \sqrt{n}r$, where u_i is the sign of each component of x.[3]

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[2]these are (in order) euclidean, maximum, and taxicab norms; or $\ell^2, \ell^\infty, \ell^1$

[3]i kinda get where this proof is going, but at the same time i dont really know how he defined u in the first place, so yeah. alternate proof would be to give Cauchy-Schwarz