MAT351 Lecture Notes

EMILY

'25 Fall Semester

Contents

§1 Day 1: Course Administrative Details (Sep. 3, 2025)

[I'll put any of my own remarks in Square Brackets or in footnotes. I love making connections to other ideas, and it feels like killing my child to exclude them just because the prof didn't mention it explicitly.]

Put "MAT351" in subject if you're emailing about the course. The syllabus tab will have the list of topics covered each week. Attendance is strongly encouraged.[1]

- Problems are in *Partial Differential Equations: An Introduction* by Walter Strauss.
- Evans PDEs will cover the topics in more detail because it is a graduate textbook.
- PDEs in Action is more applied.

Midterm 2 will be ≥80% problems from the Problem Sets. Final exam is gonna be 2 hours rather than 3 because 3 hours is too long.

Topics:

- Intro
- Examples
- General Concepts
- Classify (There are many different kinds). By the end of the semester, when you see a PDE you should know what kind it is. Also, if you see some behavior in the wild, you can reach into your toolbox and think about which kind of PDE properly models the behavior.

We will go over all the main classes of PDEs

- First order PDEs
- Wave equation (Hyperbolic PDEs)
- Heat equation/Diffusion (Parabolic PDEs)
- Laplace/Harmonic Functions (Elliptic PDEs)
- Boundary/Initial Value Problems (Fourier Series)
- First Order Nonlinear PDEs (Shocks)
- Eigenvalues (Hydrogen Atom)
- Schrödinger and Fourier Transform (If there's time)
- Distributions and Weak Solutions[2] (If there's time)
- ...

PDE is a vast subject. It is one of the most connected fields of math, with connections to Analysis, Number Theory, Physics, Differential Geometry, Algebra, Numberical Analysis, Experiments, and so on. The first part of the class will be more computational, finding explicit solutions to our first four PDEs mentioned above, and the second part will be more theoretical.

[1] I'm looking at you, Isaac.
[2] This is my favourite topic in PDEs.

The rest of the lecture went over some examples of PDEs.

Example 1.1. A simple possible PDE would be as follows: Let $u = u(x, y)$ be a function of two variables. Then our PDE might be

$$u_x + u_y = 0.$$

Remark 1.2. $u_x = \frac{\partial}{\partial x}u$.

From here, a single solution would be $x - y$. We have a solution $u(x, y) = f(x - y)$ for any function f.

Remark 1.3. For ODEs, first order has one degree of freedom, second order has two degrees of freedom, etc. For PDEs, this first order PDE has infinitely many degrees of freedom.

We will also solve $u_x + c(x, y)u_y = 0$ next lecture (by method of characteristics).

Example 1.4 (Burgers' Equation)**.** We will return to first order PDEs at some point to look at nonlinear cases. One such example is

$$u_x + uu_y = 0.$$

This is a simple example used for fluid dynamics and part of why fluid dynamics is hard.

Example 1.5 (Wave Equation)**.** With the Wave Equation, there is one special coordinate, which is time. Take $u = u(t, x)$ with $t \in I \subseteq \mathbb{R}$, $x \in U \subseteq \mathbb{R}^m$, where I is an interval and U is an open set. Now, consider the $m = 1$ case. Then the wave equation is

$$u_{tt} - u_{xx} = 0.$$

For a general m, it's

$$u_{tt} - \Delta_x u = 0$$

Remark 1.6. If $f : \mathbb{R}^m \to \mathbb{R}$, then Δf is the Laplacian of f, $\frac{\partial^2}{\partial x_1^2}f + \cdots + \frac{\partial^2}{\partial x_m^2}f$. When f has two different variables ($f(t, x)$, then $\Delta_x f$ is the Laplacian with respect to x only. [An efficient way to write it is as the divergence of the gradient of a function, which will come up a bit later.]

We'll also look at

$$u_{tt} - \Delta u = f(x, t),$$

which is the non-homogeneous case.

Example 1.7 (Heat Equation)**.** The heat equation is

$$u_t - \Delta u = 0.$$

The difference between this and the Wave Equation is that this is the first derivative, whereas the Wave Equation is the second derivative. Makes a huge difference.

Example 1.8 (Laplace's Equation)**.** The Laplacian (Laplace's equation) is

$$-\Delta u = 0.$$

[Functions satisfying Laplace's Equation are sometimes called harmonic.] In this sense, Laplace's equation is the steady-state solution to the heat equation and the wave equation. Laplace's equation is very important for other fields as well. [Dirichlet's problem in the circle is just a special case of an initial value problem for Laplace's equation. We will almost certainly see this in 354. In this case, Complex Analysis turns out to be a very efficient way to solve this IVP.]

Remark 1.9. Why do we have the minus in front? One justification could be that Laplace's equation is inspired by physics. For example, in Electrostatics, if V is the electrical potential, then the electric field is given by $E = -\nabla V$. But from here, the electric charge is $\rho = \nabla \cdot E$ (up to a change in units), so it follows that $\rho = -\Delta V$.[3] It follows that in regions of space with no charge, the electrical potential should be a harmonic function.

Another more mathematical justification is that $-\Delta$ is a non-negative operator. The professor kinda trailed off while talking about this and never finished the proof, but here's what I imagine he was trying to say:

$$\begin{aligned}\langle -\Delta g, g\rangle &= \int_{\mathbb{R}^m} -\Delta g \; g \\ &= \int_{\mathbb{R}^m} \operatorname{div}\left(-\nabla g(x) g(x)\right) + \int_{\mathbb{R}^m} |\nabla g|^2 dx\end{aligned}$$

from here, assuming that $\nabla g \cdot g \to 0$ sufficiently quickly, it would follow (by the divergence theorem) that

$$\int_{\mathbb{R}^m} \operatorname{div}\left(-\nabla g(x) g(x)\right) = 0,$$

and therefore that this inner product would be non-negative. That being said, he was not very precise here.

Next time we start with the formal definition of a PDE and write out Navier-Stokes.

[3] I *told* you it would come up later.

§2 Day 2: Definitions, Examples, Solutions, Oh My! (Sep. 8, 2025)

Definition 2.1 (Partial Differential Equation)**.** A Partial Differential Equation is an equation for an unknown function u of more variables involving its partial derivatives, in general written as

$$F\left(D^k u, D^{k-1}u, \ldots, Du, u, x\right) = 0. \tag{$*$}$$

The convention is that $D^k u$ is the kth partial derivatives of u.[4]

We'll generally look for $u : x \in U \subseteq \mathbb{R}^m \to X$, where X is typically $\mathbb{R}, \mathbb{C}, \mathbb{R}^m$. We also want $u \in C^k(U)$ (although, there are more general setups).[5] For the PDE $(*)$, we say the order is k (order will mainly be 1 or 2 in this class.) If $(*)$ can be written as

$$\mathcal{L}[u] = 0,$$

where $\mathcal{L}[u+v] = \mathcal{L}[u] + \mathcal{L}[v]$, then we call it linear. We also can include $\mathcal{L}[cu] = c\mathcal{L}[u]$, but this is usually redundant. Otherwise it's nonlinear. If $(*)$ has the form $\mathcal{L}[u] = f$ it's called (linear) non-homogeneous.

Example 2.2. The general second order pde linear: $u : (x,y) \in U \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$Q_{11}(x,y)u_{xx} + Q_{12}(x,y)u_{xy} + Q_{22}(x,y)u_{yy} + b_1(x,y)u_x + b_2(x,y)u_y + c(x,y)u = 0.$$

We say it's constant coefficient if Q_{ij}, b_k, c are constant.

Some examples from last class were constant coefficient

Example 2.3 (Wave Equation)**.**

$$u_{tt} - u_{xx} = 0$$

Example 2.4 (Heat Equation)**.**

$$u_t - u_{xx} = 0$$

Example 2.5 (Laplace's Equation)**.**

$$u_{xx} + u_{yy} = 0$$

Example 2.6.

$$u_x + Q(x,y)u_y = 0$$

Some more examples of linear PDEs:

Example 2.7 (Schrödinger Equation)**.**

$$u : (t,x) \subseteq I \times \mathbb{R}^m \to \mathbb{C}$$

where $I \subseteq \mathbb{R}$

$$iu_t - \Delta_x u = 0$$

We'll also look at $iu_t + (-\Delta + V(x))u = 0$.

[4] I imagine something like D^k is a k-tensor analogous to—in the case of $k = 2$—the Hessian matrix. Such formalism is pointless, however.

[5] Much of 457/8 is dedicated to things that make PDEs more general

On the other hand, here are some nonlinear PDEs:

Example 2.8 (Burgers' Type)**.**

$$u_t + Q(u)u_x = 0$$

Example 2.9.

$$u_t + u_x = u^2.$$

Since the right is in terms of u and nonlinear, it's a nonlinear PDE.

These are the only two types of nonlinear PDEs we'll solve. In some sense, the first one is "more" nonlinear because we're multiplying the partial derivative by something nonlinear, whereas the second is only multiplying u by something.

Example 2.10 (Nonlinear Wave Equation)**.**

$$u_{tt} - u_{xx} = u^p,$$

This might be a good model for general relativity, as general relativity is all about the nonlinear wave equation.

Example 2.11 (Maxwell's Equations)**.** Maxwell's Equations are an example of a linear system of PDEs.[6]

$$\frac{1}{c}\vec{E}_t = \nabla \times \vec{B}$$
$$\frac{1}{c}\vec{B}_t = -\nabla \times \vec{E}.$$

Here, we take

$$\vec{E}, \vec{B} : (t, x) \in \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3,$$

and c is the speed of light. We also have some constraints that

$$\nabla_x \cdot \vec{E} = 0$$
$$\nabla_x \cdot \vec{B} = 0,$$

and these are added for physical reasons and also to make the ODE solvable—in real life we indeed observe that the Electric field and the Magnetic field are conservative, so adding these requirements doesn't cause problems. If we don't have these, then there's too many degrees of freedom to solve. [Cf. Helmholtz Theorem]

Example 2.12 (Navier-Stokes' Equations)**.** These are a nonlinear system of PDEs.

$$\vec{u}_t + \vec{u} \cdot \nabla_x \vec{u} = -\nabla p + \nu \Delta_x \vec{u}.$$

Here, we have that $\vec{u} : (t, x) \in [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field, $\nu > 0$ is a constant which represents viscosity, we have that $p : (t, x) \to \mathbb{R}$ is pressure. We say also that

$$(\vec{u} \cdot \nabla \vec{u})_k = \sum_{j=1}^{3} \vec{u}_j \partial_j \vec{u}_k$$

and then we have

$$\nabla \cdot \vec{u} = 0$$

because otherwise there would be too many unknowns to solve it. [Another way to think about this term is that it represents consetvation of mass, so this boundary condition makes the fluid incompressible—so water, rather than air].

[6] "Emily gave a FANTASTIC example here. Kudos to her." – Isaac

§3 Day 3: I'm kind of getting sick of only doing examples (Sep. 10, 2025)

Definition 3.1. A PDE is fully nonlinear if it is nonlinear in the highest order derivative. This is, if the PDE is of the form

$$F(D^k u, D^{k-1}u, \ldots, u, x) = 0,$$

where F is nonlinear in the $D^k u$ component.

Example 3.2.

$$u_t + uu_x = 0 \tag{1}$$
$$u_{tt} + u_x u_{xx} = 0 \tag{2}$$
$$u_t + u_x = u^2 \tag{3}$$
$$u_t^2 + u_x = 0 \tag{4}$$
$$u_t u_x + u = 0 \tag{5}$$

We see that (1) is not fully nonlinear, likewise with (2), and (3). On the other hand, (4) and (5) are fully nonlinear, because they are nonlinear in Du, and their order is 1. Finally, m

Example 3.3 (Mean Curvature)**.** This is from Differential Geometry.

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$

It is fully nonlinear as well.

We will now solve some equations "by hand".

Example 3.4. Let $u = u(x, y), (x, y) \in \mathbb{R}^2$, with

$$u_{xx} = 0.$$

We can just integrate twice to get the general solution here.

$$\begin{aligned} u_{xx} &= 0 \\ u_x &= f(y) \\ u &= xf(y) + g(y). \end{aligned}$$

This is clearly the general solution as we just integrated twice. Now, the question is how much information do we need to provide for there to be only one solution to the IVP? For this, if we specify

$$u(0, y) = H(y) \qquad \text{and} \qquad u_x(0, y) = L(y),$$

then this would give that

$$u(x, y) = L(y)x + H(y),$$

which is sufficient.

Example 3.5 (267 Exposure Therapy)**.** "I just want to test your ODE knowledge."

$$u_{xx} + u = 0.$$

Solutions are

$$u = a(y)\sin x + b(y)\cos x,$$

He then started talking about like factoring polynomials and exponentials to find the solution to the ODE but I fainted from the experience, so I'm not sure what he did.

Remark 3.6. When drawing, functions of t and x, make t the vertical axis, and x the horizontal axis. I don't know why. Also, he writes $u(t,x)$ but the book writes $u(x,t)$. I personally prefer $u(t,x)$, but I might end up using both accidentally.

Example 3.7. Let $u = u(t,x)$ with

$$u_{tt} - u_{xx} = 0. \tag{w}$$

Then we can write this as

$$(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0.$$

If we know $(\partial_t + \partial_x)u = 0$, then u is a solution of (w), but this can be solved by $F(t-x)$, for all function F. Similarly, if $(\partial_t - \partial_x)u = 0$, then u solves (w), which would yield solutions $G(t+x)$ of (w), for all functions G.

It turns out that

$$u(t,x) = F(t-x) + G(t+x)$$

is actually the general solution—we will prove this later.

We saw that solving a PDE gives infinitely many degrees of freedom. So how do we pose a PDE so that we can identify a specific solution? Answer: Give additional info in the form of (A) initial conditions, (B) boundary conditions. This motivates the following definition:

Definition 3.8 (Well-Posed Problem)**.** We say that a Well-Posed Problem is a PDE along with Boundary Conditions and Initial Conditions such that the solutions have

1. Existence of Solutions
2. Uniqueness of Solutions
3. Continuous dependence[7] on this additional data.

Example 3.9 (Poisson's Equation)**.** Let u be the unknown, and let our equation be

$$-\Delta u = f.$$

Then there's two ways to make it well-posed. If we say that $u = u(x)$, and $x \in D \subseteq \mathbb{R}^m$, and $g : \partial D \to \mathbb{R}$, then we can either do

1. Let $u = g$, $x \in \partial D$. These are the Dirichlet Boundary Conditions [a.k.a. The Dirichlet Circle Problem]

[7] I felt a visceral sense of dread typing this.

2. Let
$$\frac{\partial u}{\partial \hat{n}} = \nabla u \cdot \hat{n},$$
where $\hat{n}$ is the unit outward normal of ∂D. Then the Neumann boundary conditions are
$$\frac{\partial u}{\partial \hat{n}} = g,$$
where $x \in \partial D$

3. The Robin boundary conditions are
$$\frac{\partial u}{\partial \hat{n}} + Qu = g$$
where $x \in \partial D$.

For the Neumann boundary conditions we need some care to prove this is well-posed, which will appear in the homework. Giving both the Dirichlet and Neumann boundary data is not a good idea. One might remark that this seems strange—after all, it's a second-order PDE, so it might look like (as with a second-order ODE) we'd have to specify stuff about both the function and the derivatives. But we will see this is not the case.

Example 3.10 (Dirichlet's Boundary Conditions)**.** Consider Poisson's equation along with the Dirichlet boundary conditions:

$$\begin{aligned} -\Delta u &= f && (x \in D) \\ u &= g && (x \in \partial D) \end{aligned}$$

Are solutions of this problem unique? Assume there exist u_1, u_2 solutions, then look at $v = u_1 - u_2$. Then this yields

$$\begin{aligned} -\Delta v &= 0 && (x \in D) \\ v &= 0 && (x \in \partial D) \end{aligned}$$

The idea is to integrate our PDE against v. First, notice the identity that

$$\begin{aligned} \int_D \Delta v \cdot v \, dx &= \int_D \sum_{i=1}^m v \, \partial_{x_i}^2 v \, dx \\ &= \int_D \sum_{i=1}^m \partial_{x_i} (v \, \partial_{x_i} v) \, dx - \int_D \sum_{i=1}^m \partial_{x_i} v \cdot \partial_{x_i} v dx. \\ &= \int_D \nabla \cdot (v \, \nabla v) - \int_D \nabla v \cdot \nabla v \, dx. \end{aligned}$$

Using this, we can derive

$$\begin{aligned} 0 &= \int_D v \Delta v \, dx \\ &= \int_D \nabla \cdot (\nabla v) \, v \, dx \\ &= \int_D \nabla \cdot (\nabla v \, v) \, dx - \int_D \nabla v \cdot \nabla v \, dx \qquad (1) \\ &= \int_{\partial D} \Delta v \cdot v \cdot \hat{n} \, dx - \int_D |\nabla v|^2 dx \end{aligned}$$

Since $v \equiv 0$ on the boundary, $\Delta v \cdot v \cdot \hat{n} \equiv 0$ on the boundary, so it follows that $\int_D |\nabla v|^2 dx = 0$. It follows that $\nabla v \equiv 0$ on D, thus v is constant. But we know $v \equiv 0$ on ∂D so $v \equiv 0$ on the whole domain. This proves uniqueness.

Example 3.11 (Vibrating String)**.** Let $D = [0, \infty) \times [0, L]$. A vibrating string can be modelled by the laws

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(t, 0) &= w_1(t) \\ u(t, L) &= w_2(t) \end{aligned}$$

where $u : D \to \mathbb{R}$ [a longer derivation of these laws can be found in Stein and Shakarchi Book 1], where $w_1, w_2 : [0, \infty) \to \mathbb{R}$ are how the end-points of the string move. It turns out this is not enough to get a unique solution. We also need

$$u(0, x) = f(x), \qquad u_t(0, x) = g(x).$$

He did not show this, but he showed show in the case of Laplace's equation, these boundary conditions don't yield a well-posed problem.

Example 3.12 (Laplace's Equation)**.** Suppose we have $u : \mathbb{H}^n \to \mathbb{R}$. Then impose the initial conditions

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x, 0) &= 0 \\ u_y(x, 0) &= \frac{1}{m} \sin(mx). \end{aligned}$$

Notice that this is analogous to providing both the Dirichlet and Neumann boundary conditions for Poisson's equations above (but it's not, because it's not bounded). This is a sequence of PDEs, which we'll call $u^{(m)}$ the solution.

As $m \to \infty$ the boundary conditions go to 0, so they roughly go to the solution $u \equiv 0$. If we solve for m large, then our solution should be close to 0 to get continuous dependence. But is this the case?

We can solve this by ODEs like we did earlier (rewriting as $(\partial_t - i\partial_x)(\partial_t + i\partial_x)u = 0$ then solving each root individually—he left actually doing this as an exercise). This gives solutions of

$$u^{(m)} = \frac{e^{-\sqrt{m}}}{m^2} \sin(mx) \sinh(my).$$

The question is whether this goes to 0 as $m \to \infty$. But notice that sinh is basically exponential, so this is going to go to infinity everywhere. Thus, in this case, continuous dependence on boundary conditions doesn't hold, so this isn't a well-posed problem.

§4 Day 4: Solving some linear PDEs (Sep. 15, 2025)

Well-Posed Problems have three parts

- Existence, which we discussed in the last class, and non-existence, which we saw in the homework
- Uniqueness, which we saw examples and a proof of last class
- Continuous dependence on boundary data, which we saw a counterexample of last week for the Laplace problem.

We will mostly study well-posed problems almost all the time. We'll have to prove that we have well-posedness for a given problem (but we might have to leave this for another class if it's too hard). [Also, one can ask what's called a regularity question—this won't come up in this class, but it's about when we can prove a weak solution is continuous or differentiable or smooth or etc. This can be defined precisely using methods of 436/457/458.]

First order PDEs (we are doing $\mathbb{R}^2$ for now) can be classified in a couple different ways.

1. Linear homogeneous, with constant coefficients, which is of the form
$$au_x + bu_y + cu = 0,$$
2. Linear homogeneous with non-constant coefficients, which is of the form
$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0$$
3. Linear non-homogeneous, which is of the form
$$a(x,y)u_x + b(x,y)u_y = f(x,y)$$
4. Semilinear, of the form
$$au_x + bu_y = f(u),$$
where f is nonlinear.
5. Quasilinear, which we will study rigorously later. An example is
$$u_x + uu_y = 0.$$
[This is where the regularity problem I discussed above can pop up.]

We will now solve these.

1. Previously, we mentioned that the solutions to $au_x + bu_y = 0$ are of the form
$$u(x,y) = F(bx - ay).$$
However, we will do this more rigorously now. We will add additional data (boundary/initial conditions) later—working backwards in some sense. There are three methods we can employ to solve this problem.

a) The geometric approach. This PDE says $\nabla u \cdot (a,b) = 0$. In other words, $\nabla u \perp (a,b)$. It follows that starting at any point, our function must be constant in the direction of (a,b). As such, to make the problem well posed, we just need to provide initial conditions along any (smooth) curve which is never tangent to the level sets. These level sets are called coordinate curves.

   One can deduce that $u(x,y) = F(bx - ay)$ is the general solution from this reasoning, but he did not do this explicitly.

   Example 4.1.

$$\begin{cases} au_x + bu_y = 0 \\ u(x,0) = g(x) \end{cases}$$

   We know $u(x,y) = F(bx - ay)$, so

$$g(x) = u(x,0) = F(bx),$$

   thus we have $F(x) = g\left(\frac{x}{b}\right)$, so

$$u(x,y) = g\left(x - \frac{a}{b}y\right).$$

b) Change coordinates—this method is very important in any PDE, as you might be able to reduce the dimensionality of your problem if you are clever. We will do this next time.

c) The general approach is the method of characteristics, which we will use later for semilinear, quasilinear, etc. We will also do this next time.

§5 Day 5: More on First-Order Equations (Sep. 17, 2025)

Problem set 2 due next Thursday, as the TAs are going to have office hours on Wednesday.

(1) We solved $au_x + bu_y = 0$ for $u = u(x, y)$, a, b constants by the geometric method last time.

(2) Let's use change of variables. Set $u = u(x, y)$ and set $v(x', y') = u(x, y)$, where x', y' represent a change of coordinates from x, y. Differentiating $u(x, y)$ yields

$$u_x = v_{x'} \frac{dx'}{dx} + v_{y'} \frac{dy'}{dx}$$
$$u_y = v_{x'} \frac{dx'}{dy} + v_{y'} \frac{dy'}{dy}.$$

From here, set

$$\begin{cases} x' = ax + by \\ y' = bx - ay \end{cases}$$

Then we have that

$$\begin{cases} u_x = av_{x'} + bv_{y'} \\ u_y = bv_{x'} - av_{y'} \end{cases}$$

so plugging into our PDE yields

$$a(av_{x'} + bv_{y'}) + b(bv_{x'} - av_{y'}) = 0,$$

and then simplifying,

$$a^2 v_{x'} + b^2 v_{x'} = 0.$$

But if $a^2 + b^2 = 0$, then our PDE is quite boring, so assuming it is positive, this implies $v_{x'} = 0$. From here, we get that v is constant with respect to x', so

$$v = f(y') = f(bx - ay) = u(x, y),$$

which solves the PDE.

(3) The book *mentions* this method, but not really systematically, only briefly.[8] It's a more general method. Let's look at curves in the plane $\gamma(s) = (x(s), y(s))$ in parametric form.[9] Look at u along this curve. We define

$$z(s) = u(x(s), y(s)),$$

then differentiate along this curve

$$\dot{z}(s) = \dot{x}(s) u_x(x(s), y(s)) + \dot{y}(s) u_y(x(y), y(s)).$$

Using the original ODE, we can get $\dot{z} \equiv 0$ if $\dot{x} \equiv a$ and $\dot{y} \equiv b$. If we can merely deduce what all the curves satisfying the system

$$\begin{cases} \dot{x} \equiv a \\ \dot{y} \equiv b \end{cases},$$

[8] See Evans 3.2 if you want the systematic development. Fair warning, Evans is not for the faint of heart.

[9] It gets a bit hard to follow, because he switches between x and y being functions, and being independent variables at times.

then we will have curves along which $\dot{z} \equiv 0$, so u is constant along these curves. We call these curves the characteristic curves of our PDE, and these equations are the characteristic ODEs. The solutions to these ODEs are

$$(x(s), y(s)) = (as + x(0), bs + y(0)),$$

which implies that the solution can be determined by values along all curves of this form. This is equivalent to the general solution provided by the other two methods.

Now, let's suppose that we have data on the x-axis. Suppose $y(0) = 0$. Then we have that

$$u(as + x(0), bs) = u(x(0), 0), \quad s \in \mathbb{R},$$

so it follows that $u(x, y) = u(x - \frac{a}{b}y, 0)$. This gives us a unique solution for any [sufficiently regular] data on the x-axis.

Example 5.1. Suppose we are given

$$\begin{cases} u_x + yu_y = 0 \\ u(0, y) = y^2 \end{cases}$$

Of the three methods, try to do the first two as an exercise. But we will use the method of characteristics once again. Set $z(s) = u(x(s), y(s))$. Then once again we have $\dot{z} = u_x\dot{x} + u_y\dot{y}$, so set

$$\begin{cases} \dot{x}(s) = 1 \\ \dot{y}(s) = y(s) \end{cases}$$

to get $\dot{z}(s) = 0$. These yield curves of the form

$$x(s) = s + x_0, \quad y(s) = y_0e^s.$$

Rearranging, we can get that

$$y(x) = y_0e^{x - x_0},$$

which makes things slightly nicer. From here, to solve the ODE, notice that $u(x, Ce^x) = u(0, C)$ and so $u(x, y) = f(ye^{-x})$. To solve f given our constraints, notice that

$$f(ye^0) = u(0, y) = y^2,$$

so our solutions are

$$u(x, y) = \left(ye^{-x}\right)^2 = y^2e^{-2x}.$$

This is the outline of the general method. Find some way to make the function nice along curves z [they need not be constant, just solvable—if we had $\dot{z}(s) = s^2$ we could solve something like this], then determine what data is required to specify the characteristic curves (in the above, y_0 uniquely determined the curve), then find the curve that intersects (x, y) in terms of this data, and this will yield the value of u at this point.

Example 5.2.

$$u_x + 2xy^2u_y = 0,$$

then we have that

$$\dot{x} = 1, \dot{y} = 2xy^2,$$

so we can get that

$$\frac{dy}{dx} = 2xy^2.$$

From here, we need to use our ODE knowledge to solve this (it's separable).

$$\begin{aligned}\int \frac{1}{y^2}\,dy &= \int 2x\,dx\\ \frac{-1}{y} &= x^2 + C\\ y &= \left(-x^2 + C\right)^{-1},\end{aligned}$$

and thus these are our characteristic curves, so u is constant along them. He then graphed these functions as C varied. You can do this on desmos if you want. Notably, its asymptotes are at $\pm\sqrt{C}$ and its y-intercept is $\frac{1}{C}$. It follows that $u(0, \frac{1}{C}) = f(C) = f_0(\frac{1}{y} + x^2)$

Remark 5.3. Let $u = u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^m$. In general, we can solve first order equations like

$$u_t + a(t,x)u_x + b(t,x)u = f(t,x). \tag{$*$}$$

By the same method as before, we want to reduce this into a system of ODEs. Set $z(s) = u(t(s), x(s))$. Then we have that

$$\begin{aligned} z'(s) &= t'(s)u_t(t(s),x(s)) + x'(s)u_x(t(s),x(s))\\ &= f(t(s),x(s)) - b(t(s),x(s))u(t(s)x(s)) && \text{(plugging into ($*$))}\\ &= f(t(s),x(s)) - b(t(s),x(s))z(s).\end{aligned}$$

Thus for $(*)$, we say that the system of characteristics is

$$\begin{cases} \dot{t} = 1\\ \dot{x} = a(t,x)\\ \dot{z} = f(t,x) - b(t,x)z\end{cases}$$

We set $\dot{t} = 1$ because the parameterization of the path doesn't matter. This is a nonlinear system of ODEs, but it works.

Example 5.4 (Semilinear)**.** Find characteristics for

$$\begin{cases} u_t + 2tu_x = u^2\\ u(0,x) = g(x)\end{cases}$$

To do this, we have that the system of ODEs is

$$\begin{cases} \dot{t} = 1\\ \dot{x} = 2t\\ \dot{z} = z^2\end{cases},$$

so if we let our initial data be $t(0) = 0$, $x(0) = x_0$, $z(0) = g(x_0)$, then solving yields

$$\begin{cases} t = s\\ x = t^2 + x_0\\ z = \frac{g(x_0)}{1 - sg(x_0)}\end{cases}.$$

Rearranging, we have that

$$\begin{cases} s = t\\ x_0 = x - t^2\\ u(t(s),x(s)) = \frac{g(x_0)}{1 - sg(x_0)}\end{cases},$$

so therefore
$$u(t,x) = \frac{g(x-t^2)}{1-tg(x-t^2)}.$$
This solution could blow up depending on the values of g, so just like nonlinear ODEs ($x' = 1 + x^2$), we have less assurances of nice properties when we don't have linearity.

§6 Day 6: Method of Characteristics (Sep. 22, 2025)

Theorem 6.1 (Non-Homogeneous 1st Order)**.** Non-homogeneous 1st-order PDEs are of the form

$$u_t + b\nabla u = f(t, x),$$

where $b \in \mathbb{R}^m$ (constant), $u : (t, x) \in \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, and $f : (t, x) \in \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$. Given the homogeneous case

$$v_t + b\nabla v = 0,$$

we have that the general solution is $F(x - bt)$.

We'll solve the transport equation with forcing. First, we will apply the method of characteristics. Let $(t(s), x(s), z(s))$ be the characteristic curves. Then we have that our system of ODEs is

$$\begin{cases} \dot{t}(s) = 1 & t(0) = 0 \\ \dot{x}(s) = b & x(0) = x_0 \\ \dot{z}(s) = f(t, x) & z(0) = g(x_0) \end{cases}$$

where the right side gives some additional data with which we can hopefully have a unique solution. It follows that $t(s) = s$ and $x(s) = x_0 + bs$. From here,

$$\dot{z}(s) = \frac{d}{ds} u(s, x_0 + bs) = f(x_0 + bs),$$

so it follows that

$$z(t) = \int_0^t f(s, x_0 + bs)\, ds = \mu(t, x_0 + bt) - \mu(0, x_0).$$

Rearranging gives

$$\mu(t, x_0 + bt) = \mu(0, x_0) + \int_0^t f(s, x_0 + bs)\, ds$$

thus we have that $(x = x_0 + bt)$

$$\mu(t, x) = g(x - bt) + \int_0^t f(s, x - b(t - s)) ds.$$

Remark 6.2. Solution of non-homogeneous is the superposition of solution of homogeneous problem and the solution of the non-homogeneous problem without any data. Namely, the $g(x - bt)$ is the homogeneous solution, and the integral is the solution of the non-homogeneous without any data.

Remark 6.3. This is an example of Duhamel's principle.

$$u(t, x) = u_0 + \int_0^t v,$$

where u_0 solves the homogeneous problem, and v solves

$$\begin{cases} v_t + b \cdot \nabla v = 0 \\ v(t) = v(s) = f(s, x) \end{cases}.$$

Example 6.4. We will tackle

$$\begin{cases} u_t + uu_x = 0 \\ u(0,x) = x^2 \end{cases}$$

If we have constant coefficients, (constant c instead of u) then c would be the speed of transport. As such, this is a model for shocks/explosions. Compressible fluids and the like.

We will try the method of characteristics. Our system with data is

$$\begin{cases} \dot{t} = 1 & t(0) = 0 \\ \dot{x} = z & x(0) = x_0 \\ \dot{z} = 0 & z(0) = x_0^2 \end{cases}$$

so it has turned this Quasilinear first order PDE into a system of ODEs. This yields the solution

$$\begin{cases} t = s \\ x = sx_0^2 + x_0 \\ z = x_0^2 \end{cases}$$

From here we have to invert the relation. The above tells us that

$$u(s, sx_0^2 + x_0) = z(s) = x_0^2,$$

so we just need to find x_0 as a function of t and x. On Wednesday he is going to ask us for the answer, so be prepared (jk I know most of you will not be there).

§7 Day 7: Lots of Calculations (Sep. 22, 2025)

Remark 7.1 (Method of Characteristics)**.** For the PDE

$$P(t,x,u)u_t + Q(t,x,y)u_x = f(t,x,u)$$

the characteristic curves $(t(s), x(s))$ are given as solutions of

$$\begin{cases} \dot{t} = P(t,x,z) \\ \dot{x} = Q(t,x,z) \\ \dot{z} = f(t,x,z). \end{cases}$$

These are not necessarily level curves.

We will do second-order PDEs now. We start with the 2D case, for linear 2nd order. The general case is

$$\begin{aligned} &a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} \\ &\quad + a_1(x,y)u_x + a_2(x,y)u_y + c(x,y)u = 0. \end{aligned} \qquad \text{(lower order terms)}$$

The PDE can have very different behavior depending on the coefficients a_{11}, a_{12}, and a_{22}. So first we will start with the constant coefficient case.

Let our a all be real values, and likewise u. Then our constant coefficient case is given by

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + \text{lower order} = 0. \tag{$*$}$$

We will classify it based on the discriminant of the polynomial $a_{11}\lambda^2 + 2a_{12}\lambda + a_{22}$.

Theorem 7.2. By a linear change of independent variables, the equation $(*)$ can be writen in one of the 3 forms below:

1. If $a_{12}^2 - a_{11}a_{22} < 0$, then it's
$$u_{xx} + u_{yy} + \text{lower order} = 0.$$
2. If $a_{12}^2 - a_{11}a_{22} = 0$ then it's,
$$u_{xx} + \text{lower order} = 0.$$
3. If $a_{12}^2 - a_{11}a_{22} > 0$ then it's,
$$u_{xx} - u_{yy} + \text{lower order} = 0.$$

Zander pointed out that this is naturally motivated by algebraic properties of quadratic forms. He then said the word "sheaf" and I ran away to hide.

Proof. We will automatically disregard any lower order terms. We can rescale $a_{11} = 1$ without loss of generality (unless $a_{11} = a_{22} = 0$). Then we have

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0.$$

We can therefore rewrite this as

$$(\partial_x + a_{12}\partial_y)^2 + (a_{22} - a_{12}^2)u_{yy} = 0.$$

From here, we're going to use a change of variables

$$\begin{cases} x = \xi \\ y = a_{12}\xi + \eta \end{cases}$$

which yields

$$\begin{cases} \partial_\xi = \partial_x + a_{12}\partial_y \\ \partial_\eta = \partial_y. \end{cases},$$

so we get that

$$\partial_\xi^2 u + b^2 \partial_\eta u = 0,$$

where $b^2 = (a_{22} - a_{12}^2)$. This gives us 1 and 2. For 3, set $b^2 = a_{12}^2 - a_{22} > 0$. PDE is

$$(\partial_x + a_{12}\partial_y)^2 u - b^2 u_{yy} = 0.$$

□

In HW 3 we will see problems on classification of PDEs and the relation to the characteristic polynomial $a_{11}\lambda^2 + 2a_{12}\lambda + a_{22}$.

For the general case of dimensions higher than 2, we look at the linear PDE

$$\sum_{i,j=1}^{m} a_{ij}(x)\partial_{x_i x_j}^2 u + b(x)\nabla u + c(x)u = 0,$$

where $x \in \mathbb{R}^m$. From here we look at the Eigenvalues and all that stuff.

Theorem 7.3. We can classify according to the signature of the matrix $A = (a_{i,j})_{i,j=1}^m$. Notice that it is real symmetric, so[10] it's diagonalizable.

1. If all the eigenvalues have the same sign (nonzero), then the PDE is elliptic.
2. If all the eigenvalues have the same sign except one is zero, it's parabolic.
3. If all the eigenvalues have one sign, except one with the opposite sign, it's hyperbolic.

We don't classify all the possible cases, but these are the main things we see anyways. If we restrict it to 2 dimensions.

We'll start with the wave equation. Surely this won't be so hard. This is just the equation

$$u_{tt} - c^2 u_{xx} = 0.$$

We will

1. Find the general solution rigorously ($f(x + ct) + g(x - ct)$),
2. Find the solution to the IVP (D'Alembert's Equation)
3. Look at physical consequences (e.g. finite speed)
4. Look at a derivation of equation

Proposition 7.4. The general solution to the 1d wave equation is $u(x, t) = f(x + ct) + g(x - ct)$.

[10] by spectral theorem :-)

Proof. We have shown that this is a collection of possible solutions, but we haven't shown it's the general solution. To do this, there' a bunch of possible ways to do it, but we'll do it rigorously. There's two possible ways to go about this,

1. Using the method of change of variables, set
$$\begin{cases} \phi = x - ct \\ \psi = x + ct. \end{cases}$$
which yields
$$\begin{cases} \partial_\varphi = \frac{1}{2}\partial_x - \frac{1}{2c}\partial_t \\ \partial_\psi = \frac{1}{2}\partial_x + \frac{1}{2c}\partial_t. \end{cases}$$
"Why did I choose this? Good question. Anyways, [...]" - Fabio. We can rewrite this as
$$\begin{cases} 2c\partial_\psi = c\partial_x - \partial_t \\ 2c\partial_\psi = c\partial_x + \partial_t. \end{cases}$$
From here, it follows that our PDE is just
$$\partial_\phi \partial_\psi v = 0,$$
so it turns out that
$$\partial_\psi v = F(\psi),$$
and thus
$$v = f(\psi) + g(\phi),$$
which is exactly what we want.

2. For the method of characteristics, we are going to use the following trick:
$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ \partial_t^2 u - c^2 \partial_x^2 u &= 0 \\ \frac{\partial_t^2}{\partial_x^2} u - c^2 u &= 0 && \text{(Dividing by } \partial_x^2\text{)} \\ \lambda^2 u - c^2 u &= 0 && (\lambda = \tfrac{\partial_t}{\partial_x}) \\ \lambda^2 = c^2 \implies \lambda &= \pm c. && \text{(Dividing by } u\text{. Wait, huh?)} \end{aligned}$$
This *does* work formally, although not for the (tongue-in-cheek) reasons I wrote. Fabio didn't formalize it himself, but my best guess is that the third line is using the chain rule, then the fifth line just uses the fact that if λ^2 and c^2 agree on every function, then they must be the same operator.

□

Theorem 7.5 (D'Alembert's Formula)**.** Consider the IVP for the wave equation.
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0, x) = \phi(x) \\ u_t(0, x) = \psi(x). \end{cases}$$
Then
$$u(t, x) = \frac{1}{2}\left(\phi(x + ct) + \phi(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s) ds.$$
[Fabio didn't write the formula until the end of the proof. I only wrote it earlier than he did for completeness.]

Proof. We know $u(t,x) = F(x+ct) + G(x-ct)$. From here, we will impose the initial conditions.

$$\begin{cases} u(0,x) = F(x) + G(x) = \phi(x) \\ u_t(0,x) = cF'(x) - cG'(x) = \psi(x), \end{cases}$$

so we have the equation

$$\begin{cases} F' + G' = \phi \\ F' - G' = \frac{1}{c}\psi \end{cases}$$

thus solving this yields

$$2F' = \phi' + \frac{1}{c}\psi \qquad 2G' = \phi' - \frac{1}{c}\psi.$$

From here, we can determine that

$$\begin{cases} F = \frac{1}{2}\phi + \frac{1}{2c}\int_0^x \psi(s)ds + A \\ G = \frac{1}{2}\phi - \frac{1}{2c}\int_0^x \psi(s)ds + B. \end{cases}$$

Since we need $F + G = \phi$, it follows that $A + B = 0$. We can therefore do

$$u(t,x) = \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(s)ds + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi(s)ds,$$

which yields a final formula of

$$u(t,x) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Notice that this proof simultaneously shows uniqueness, because at each step of the proof, our hand was forced. There was no step along the way at which we could've deviated and gotten a different final solution, so it must be the unique solution. We can show uniqueness independently as well, of course, this is just a quick way to verify it. □

We can deduce many properties of the solution of the wave equation from this formula. We can also deduce some abstractly.

Example 7.6. Consider the system

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,x) = e^{-x^2} \\ u_t(0,x) = 0. \end{cases}.$$

Then this is just solved by

$$u(t,x) = \frac{e^{-(x+ct)^2} + e^{-(x-ct)^2}}{2}.$$

Example 7.7. Consider the system

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,x) = \chi_{[-1,1]}(x) \\ u_t(0,x) = 0 \end{cases}.$$

By D'Alembert, this is just

$$u(t,x) = \frac{1}{2}\chi_{[-1,1]}(x+ct) + \frac{1}{2}\chi_{[-1,1]}(x-ct).$$

He then graphed out how this would look as time evolved.

This models the propagation of signals, or spacetime. If we look at the supports of a function subject to the wave equation over time, it will resemble this graph, which is why we can use it to model the above. A plot of it reveals that after a finite time, the signal will leave where it originated ($\frac{1}{c}$ seconds). We see that the edges of where the signal have propagated to form a sort of "light cone," which represents all the points in spacetime that the signal could have reached after the initial transmission. This is because there's a finite speed of propagation. Given the initial conditions, nothing can move faster than c, so we can call c the speed of light, or something like that. It's interesting stuff, but probably not required for the class lol.

§8 Day 8: Wave Equation (Sep. 29, 2025)

The wave equation is time-reversible.

Theorem 8.1 (D'Alembert's Equation)**.** The unique solution to the system

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

is given by

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.$$

Remark 8.2. We assumed some regularity for the data ϕ, ψ (for example, $\phi \in C^1$. Nevertheless, the formula makes sense for rougher ϕ and ψ.

Remark 8.3 (Trust me on this)**.**

1. When deriving identities or inequalities, we are allowed to think our functions are very nice. This is because reasonable functions can approximate most functions we care about, and then we can take the limit. Usually this works. [The Stone-Weierstrass approximation theorem tells us that actually almost any collection of functions you can think of that's closed under pointwise addition and multiplication will be dense in this sense.]

2. If $\phi \in C^2$ and $\psi \in C^1$, then $u \in C^2$, which makes it a perfectly valid solution of our function. We call this a classical solution. But, there are ways to say that rougher functions solve the PDE. Not in the classical sense, in some other sense. Most things in the world aren't necessarily smooth. So for the Burgers' equation and such we will see solutions in some other sense.

Remark 8.4 (Finite Speed)**.** If the data ϕ, ψ vanishes in $|x| > R$ (for some $R \in \mathbb{R}$), then $u(x,t)$ vanishes on the set

$$S = \{(t,x) : |x| > R + |ct|\}.$$

We can imagine this as drawing cones over every point in our initial data whose slope is $\frac{1}{c}$ along the boundary.

Let's choose a point (x_0, t_0). Which points in the plane "care" about the value of u at this point? We call this the domain of influence of (x_0, t_0). We also say this is the light cone of this region. It consists precisely of a (two-sided) cone centered around our point, with slope $\frac{1}{c}$. The one in the positive t direction is the forward light cone, and the one in the negative t direction is the backwards light cone.

We call the points on our initial data within the light cone of our point the domain of dependence, because outside of this region, the data would not have enough time to reach our point, and so the value of our point depends exclusively on the value of our function at this point.

Claim 8.5. The finite speed can be seen from the PDE alone without the explicit formula for the solution using "energy".[11]

Definition 8.6. The energy density is given by

$$\sigma(x,t) = \frac{1}{2}(u_t^2 + c^2|\nabla u|^2(x,t),$$

[11] Isaac losing his mind rn

so we have that the total energy of u at a time t is just integrating σ over $\mathbb{R}^d$. This gives

$$E(t) = \int_{\mathbb{R}^d} \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\,dx,$$

where we have that our wave is in $\mathbb{R} \times \mathbb{R}^d$. We say that u_t^2 is the kinetic energy, and $c^2|\nabla u|^2$ is the potential energy.

Claim 8.7. $\dot{E}(t) \equiv 0$.

Proof.

$$\frac{d}{dt}E(t) = \int_{\mathbb{R}^d} \frac{d}{dt}\frac{1}{2}\left(u_t^2 + c^2|\nabla u|^2\right)dx$$

Notice that the interchange of the limit and derivative imposes some requirements. Both because we need some sort of smoothness of the integrand, but also because of problems with our integral diverging. To solve this, we can assume that the data of our PDE has compact support—call it P.

$$\begin{aligned}
&= \int_{\mathbb{R}^d} u_t u_{tt} + c^2 \nabla u \nabla u_t dx \\
&= \int_{\mathbb{R}^d} u_t u_{tt} - c^2 \Delta u u_t dx && \text{(Gauss)} \\
&= \int_{\mathbb{R}^d} u_t (u_{tt} - c^2 \Delta u) dx \\
&= 0. && \text{(Plug in Wave Equation)}
\end{aligned}$$

Recall that the multi-variable wave equation is

$$u_{tt} - c^2 \Delta u = 0,$$

which is where this last line derives. For the application of Gauss to get this last part, we need some domain for which our functions are identically 0 over the boundary. Fortunately, since we assumed our support is compact[12] we can find a region for which this works. He remarked that we will need this type of argument a lot, especially for Elliptic PDEs, so we should make sure we are very comfortable with it. □

[12] Fabio called it finite support, which I figure just means bounded. Of course this means its closure is compact, so it still works the same.

§9 Day 9: More on Wave Equation, Heat Equation (Sep. 29, 2025)

Let's look at the wave equation in $d+1$ dimensions. This is

$$\begin{cases} u_{tt} - c^2\Delta_x u = 0 & t \in \mathbb{R}, x \in \mathbb{R}^d \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

We defined the energy density $\sigma(x,t) = \frac{1}{2}|u_t|^2 + \frac{1}{2}c^2|\nabla u|^2$, and the total energy at the time t as

$$E(t) = \int_{\mathbb{R}^d} \sigma(x,t)dx.$$

We proved for sufficiently nice solutions that $\dot{E} \equiv 0$.

Theorem 9.1 (Finite Speed of Propagation)**.** If ϕ, ψ vanish for $|x| > R$ then $u(x,t) = 0$ for $|x| > R + ct$ where $t > 0$. If $d = 1$ this is just D'Alembert.

Proof. The past cone (the domain of dependence of (x_0, t_0)) is

$$|x - x_0| = c(t_0 - t). \qquad (0 \le t \le t_0)$$

Taking the gradient of the function

$$F(x,t) = |x - x_0| - c(t_0 - t)$$

yields the unit outward normal to our cone C. It is precisely

$$\left(x - x_0, -c\frac{t - t_0}{|t - t_0|}\right)\frac{1}{\sqrt{1+c^2}}.$$

Define $R = x - x_0$ and write

$$\hat{m} = \left(\frac{R}{|R|}, c\right)\frac{1}{\sqrt{1+c^2}}$$

on C. Then we have that

$$\partial_t \sigma(x,t) - \nabla \cdot p(x,t) = 0,$$

where

$$\sigma(x,t) = \frac{1}{2}|u_t|^2 + \frac{1}{2}c^2|\nabla u|^2, \qquad p(x,t) = c^2 u_t \nabla u.$$

This can be found just by plugging these in to the above and cancelling out. Call D the solid region of the cone between 0 and t. For the rest of this proof refer to this desmos graph for the sets. Then

$$\int_D \partial_t \sigma - \nabla \cdot p dV = 0,$$

but this is just the divergence (in both x and t) of the vector field $(-p, \sigma)$, so applying Gauss, this is

$$\int_{\partial D} (-p, \sigma) \cdot \hat{m}\, da,$$

where $\partial D = T \cup B \cup K$, where T is the top of our slice of the cone, B is the bottom slice, and K is the boundary connecting them (again, refer to our graph).

$$\begin{aligned}
\int_T (-p,\sigma)\cdot(0,\dots,0,1)dx &= \int_T \sigma dx \\
&= \frac{1}{2}\int_T u_t^2 + c^2|\nabla u(t)|^2 dx \\
\int_B (-p,\sigma)\cdot(0,\dots,-1)dx &= \frac{-1}{2}\int_B \psi^2 + c^2|\nabla\phi|^2 dx \qquad (*) \\
\int_K (-p,\sigma)\cdot\hat{m} &= \int_K -p\hat{m}_x + \sigma\hat{m}_t \\
&= \frac{1}{\sqrt{1+c^2}}\int_K -c^2 u_t \nabla u\cdot\frac{R}{|R|} + c\left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right) da \\
\int_T \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2 &\le \int_B \frac{1}{2}\psi^2 + \frac{1}{2}c^2|\nabla\phi|^2 - \int_K \cdots .
\end{aligned}$$

We will show that our integral over K is positive to show that we can drop this term in the inequality. To do this, set $u_r = \nabla u \cdot \frac{R}{|R|}$, to represent the radial derivative of u. Then

$$\begin{aligned}
\frac{1}{c}I &= -cu_t u_r + \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2 \\
&= \frac{1}{2}\left(u_t - cu_r\right)^2 + \frac{c^2}{2}(|\nabla u|^2 - u_r^2) \\
&= \frac{1}{2}(u_t - cu_r)^2 + \frac{c^2}{2}\left|\nabla u - u_r\cdot\frac{R}{|R|}\right|^2 \\
&\ge 0.
\end{aligned}$$

It follows that

$$\int_T \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2 \le \int_B \frac{1}{2}\psi^2 + \frac{c^2}{2}|\nabla\phi|^2,$$

so if $\phi, \psi = 0$ on B, then $\int_T u_t^2 + |\nabla u|^2 dx = 0$. This implies $u \equiv 0$ on T so $u \equiv 0$ on the whole cone. Also, $u(x_0, t_0)$ is zero if data is zero on the base of the past cone of (x_0, t_0) (i.e. the domain of dependence). □

What we showed also implied finite speed property for $d+1$ dimensional wave equation. "No more cones for a while. We may have ellipses and spheres, but no cones. Well, maybe I'll say one thing..." Think about how the energy inequality implies uniqueness of our solution. Also, think about how $\dot{E}(t) = 0$ implies uniqueness.

Now let's start on the Heat/Diffusion equation in 1d. Recall this is

$$u_t - ku_{xx} = 0. \qquad (k > 0)$$

We may assign initial data $u(x, 0) = \phi(x)$. We may also look at boundary conditions. Here, only take $t \ge 0$. For $t < 0$ the problem is generally ill-posed. Notice that if you change t to $-t$ this is not the same equation, because we are requiring $k > 0$.

We will solve it explicitly later. First, let's look at some properties that can be obtained only from the equation.

Theorem 9.2 (Weak Maximum Principle)**.** Consider the rectangle $0 \le x \le \ell$ and $0 \le t \le T$. Suppose we have a solution here. Namely, we have that $u_t = ku_{xx}$ in $(x,t) \in [0,\ell]\times[0,T] = R$. Then the max value for u in R is obtained on either $x = 0$, $x = \ell$ or $t = 0$.

Remark 9.3. The strong maximum principle says the max can only be on the boundaries mentioned (unless the function is constant). The weak maximum principle is provable easily, the strong maximum principle is very hard.

We will save the proof for next time. It's a basic trick in analysis. It's different in flavour from what we've done so far—no integration by parts or integrals at all.

We also have the min principle. Just apply the max principle to $-u$. We will also have this same idea for the elliptic equation.[13]

[13] There was something else at the end but I really had to pee so I wasn't paying attention at the very end. It was something about uniqueness in the heat equation and boundary conditions, but I figure he'll go over it again if it's important.

§10 Day 10: Heat/Diffusion Equation (Oct. 6th, 2025)

Theorem 10.1. Let u solve $u_t = ku_{xx}$, for $k > 0$ in the rectangle $R = [0, \ell] \times [0, T]$ with initial/boundary conditions

$$\begin{cases} u(x,0) = \phi(x) & 0 \leq x \leq \ell \\ u(0,t) = g(t) & 0 \leq t \leq T \\ u(\ell,t) = h(t) & 0 \leq t \leq T. \end{cases}$$

Notice here that the $u(0,t)$ and $u(\ell,t)$ terms represent some sort of "forcing," which represents either pumping in or removing heat or something of that sort.

Then we have that u attains its max somewhere along $B = \partial R \setminus \{(x,T) : 0 < x < \ell\}$.

Proof. If u has a max at a point p in the interior of R, then $u_t|_p = 0$ and $u_{xx}|_p \leq 0$. This is almost violating the PDE. The idea is to create a big of room.

Let $v(x,t) = u(x,t) + \varepsilon x^2$. Then v solves

$$v_t - kv_{xx} = u_t - ku_{xx} - 2\varepsilon k < 0,$$

if $\varepsilon > 0$. But then $v = \mu + \varepsilon x^2$, $\varepsilon > 0$ arbitrary cannot have an interior max Q, as otherwise, $v_t|_Q = 0$, $v_{xx}|_Q \leq 0$, which is a contradiction. Thus, v attains its max on the boundary.

Assume that v attains its max along the top. Specifically, suppose it attains a max at (x_0, T). It follows that

$$\lim_{\delta \to 0} \frac{v(x_0,T) - v(x_0, T-\delta)}{\delta} \geq 0.$$

Then we still get a contradiction to $v_t - v_{xx} = 0$.

To show u attains its max along the boundary, let $M = \max_R u$. We have therefore that $\max_R v = \max_B v$. This implies

$$\max_R u + \varepsilon x^2 = \max_B u + \varepsilon x^2$$

and thus

$$\max_R u \leq \max_B u + \varepsilon \ell^2,$$

for ε arbitrary. This implies that $\max_R u \leq \max_B u$. □

Remark 10.2. We also have a minimum principle, as discussed last time. The PDE is linear, so just negate our solution and apply the above.

From here, uniqueness follows and also continuous dependence on data.

Say that $g = h = 0$. Then u_j solves with $u_j(x,0) = \phi_j(x)$, $j = 1, 2$. Because $u_1 - u_2$ and $u_2 - u_1$ are solutions, we get

$$\max_R |u_1 - u_2| \leq \max_{0 \leq x \leq \ell} |\phi_1(x) - \phi_2(x)|.$$

This gives stability/continuous dependence in the sup norm.

Proposition 10.3. We can find the uniqueness/continuous dependence using the energy method. Define

$$E(t) = \frac{1}{2} \int_0^\ell u^2(x,t) dx.$$

Then we have

$$\begin{aligned}\dot{E}(t) &= \int_0^\ell u(x,t)u_t(x,t)dx \\ &= \int_0^\ell uku_{xx}dx \\ &= -k\int_0^\ell (u_x)^2(x,t)dx \leq 0.\end{aligned}$$

Then this gives uniqueness and continuous dependence. Uniqueness because given u_1, u_2 solving, $v = u_1 - u_2$ satisfies

$$\frac{1}{2}\int_0^\ell v^2(x,t)dx \leq \frac{1}{2}\int_0^\ell v^2(x,0)dx = 0,$$

so thus $u_1 = u_2$. Continuous dependence because given ϕ_1, ϕ_2 that solve, $v = u_1 - u_2$. Then we have by the same argument above that solutions depend continuously wrt the L^2 norm. Here, we have that the L^2 norm of ϕ is

$$||\phi||_{L^2} = \left(\int_0^\ell \phi^2(x)dx.\right)^{\frac{1}{2}}.$$

Remark 10.4. Continuous dependence in ODEs is nice and simple, because the initial conditions are just points, so just take the standard Euclidean norm. Furthermore, since most norms on $\mathbb{R}^n$ are equivalent, we don't have to worry too much about which norm the dependence is continuous with respect to. For PDEs, it's not quite the same. The L^∞ norm (sup norm) and the L^2 norm are indeed not equivalent, so when we talk about continuous dependence, we have to specify which norm it depends continuously upon. That is, even if solutions are continuously dependent on the data (which is a function) with respect to the L^2 norm, it might not depend continuously on the data with respect to the L^∞ norm.