MAT327 Lecture Notes

Arky!! :3c

'24 Fall Semester

Contents

§1 Day 1: Open Sets and Continuity (Sep. 3, 2024)

> This class is *MAT327*; 3 meaning third year, 2 meaning the contents are on the fundamental side, and 7 meaning no mercy.
> – Dror Bar-Natan

Course administration matters first;

- The course link is given here (this will link straight to Quercus).
- The textbook is James Munkres' Topology (online PDF: ETH Zurich mirror); Prof Bar-Natan strongly recommends a paper copy, though (since people get distracted on the computer).

Today's reading in the textbook is on Ch. 1, sections 1 to 8, and Ch. 2, sections 12 to 13. Readings are supplementary to lecture material[1]. The goal of this course is to understand continuity in its most general form; in particular,

- In MAT157, we studied continuity in $f : \mathbb{R} \to \mathbb{R}$;
- In MAT257, we will study continuity in $f : \mathbb{R}^n \to \mathbb{R}^m$;

but in this class, we will study continuity in $F : X \to Y$, where X, Y are arbitrary spaces, such as (but not limited to) $\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{N}}$ (binary sequences), and so on. We start with some refreshers on previous coursework;

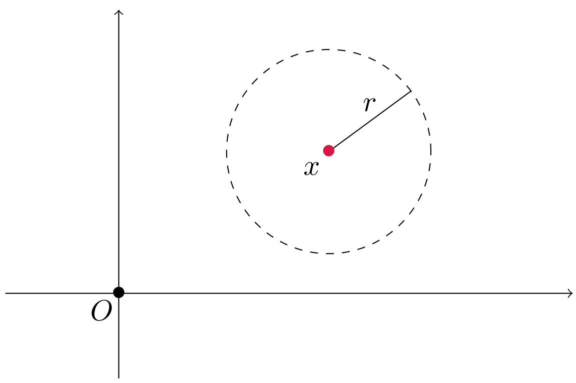
Definition 1.1 (Continuity in $\mathbb{R}^n \to \mathbb{R}^m$)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called "continuous" if it is continuous at all points in $\mathbb{R}^n$. Specifically, for all $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

With this, we state our main theorem for today (proof given later),

Theorem 1.2 (Continuity on $\mathbb{R}^n$ if and only pre-image of open subsets is open)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if all open subsets $U \in \mathbb{R}^m$ have $f^{-1}(U)$ open.

In order to build up to the above, we start by defining some terms;

Definition 1.3 (Open Ball)**.** Let $r > 0$, and $x \in \mathbb{R}^n$. An open ball of radius r about x is given formally by $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Visually, we have



Note that the border of the circle is not in $B_r(x)$, as the distance metric asks for a strict inequality.

[1] iirc it won't be tested unless specified. its still good to learn tho

In a similar fashion, we have

Definition 1.4 (Open Set)**.** A set $U \subset \mathbb{R}^n$ is called *open* if, for all $x \in U$, there exists an open ball about x contained in U. Specifically, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

Intuitively, we could say that the set U does not contain its edge; if it did, let x be on said edge; then we would not be able to fit an open ball about x in U. Here are some examples of open sets from lecture;

1. The whole set $U = \mathbb{R}^n$ is open.

2. The empty set $\emptyset$ is open. Since there does not exist any $x \in \emptyset$, no conditions on balls need to be satisfied; "every dog in the empty set of dogs is green".

3. $(0, 1)$ is open on $\mathbb{R}^1$ (and any open interval, for that matter).

4. $B_r(x) \in \mathbb{R}^n$ is open. To see this, observe that for any $y \in B_r(x)$, we may pick $\varepsilon < r - |x - y|$; by triangle inequality, all elements in $B_\varepsilon(y)$ must also be in $B_r(x)$.

We now define images and pre-images; let us have a function between sets $f : X \to Y$. For subsets $A \subset X$ and $B \subset Y$,

$$f(A) = \{f(a) \mid a \in A\},$$
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Since f need not be injective, f^{-1} does not necessarily exist. While images of a union of subsets is a union of the images, the same is not true for intersections; let $A_1, \dots, A_n$ be subsets of X; then

$$f\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f(A_i), \qquad f\left(\bigcap_{i=1}^n A_i\right) \subset \bigcap_{i=1}^n f(A_i).$$

On the other hand, pre-images preserve both union and intersection; let $B_1, \dots, B_n \subset Y$; then

$$f^{-1}\left(\bigcup_{i=1}^n A_i\right) = \bigcup_{i=1}^n f^{-1}(A_i), \qquad f^{-1}\left(\bigcap_{i=1}^n A_i\right) = \bigcap_{i=1}^n f^{-1}(A_i).$$

As for set complements, we have

$$f^{-1}(B^C) = f^{-1}(B)^C,$$

but the same cannot be said for images; $f(A)^C \subset f(A^C)$ for surjective f, and vice versa for injective f. Equality occurs only if f is a bijection.

Note: for the proof of Theorem 1.2, I'll leave it out in case Prof. Bar-Natan proves it later on Thursday.

§2 Day 2: Basic Definitions and Topological Spaces (Sep. 5, 2024)

We start by recapping the previous lecture; we introduced

- Open sets $U \subset \mathbb{R}^n$, where there exists an open ball of radius $\varepsilon > 0$ about any $x \in U$.
- Continuity in $f : \mathbb{R}^n \to \mathbb{R}^m$.

Recall the continuity property from last class, that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if the pre-image of an open set U is open.

($\Rightarrow$) To start, pick any $x_0 \in f^{-1}(U)$, and let us have a small enough $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \in U$ (this is possible since U is open). By continuity, there exists $\delta > 0$ such that any $x \in B_\delta(x_0)$ satisfies $f(x) \in B_\varepsilon(f(x_0)) \subset U$; this means $x \in f^{-1}(U)$ by definition of pre-image, and since x was arbitrary, we see $B_\delta(x_0) \in f^{-1}(U)$. Moreover, since x_0 was also arbitrary, our construction shows that there always exists a δ-ball about any point in the pre-image, and so $f^{-1}(U)$ is open. $\square$

($\Leftarrow$) For the other direction, take any $x_0 \in \mathbb{R}^n$, and $\varepsilon > 0$. Since $B_\varepsilon(f(x_0))$ is open we have that $f^{-1}(B_\varepsilon(f(x_0)))$ is open as well. This means we may pick a small enough $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x_0)))$ (by definition of openness), and we immediately see

$$f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)).$$

This is a reconstruction of the epsilon-delta definition of continuity, and so we are done. $\square$

Open sets in $\mathbb{R}^n$ have a number of properties;

(a) $\emptyset, \mathbb{R}^n$ are open sets.

(b) The union of open sets are open; specificay, let S be a set of indices, and let $A_\alpha \subset \mathbb{R}^n$ for all $\alpha \in S$. Then

$$\bigcup_{\alpha \in S} A_\alpha$$

is an open set.

(c) The finite intersection of open sets are also open. Let $A_1, \ldots, A_n \subset \mathbb{R}^n$ be open; then

$$\bigcap_{i=1}^{n} A_i$$

is open.

We now proceed with the proofs for these properties.

(a) This was proven last lecture.

(b) For any x in the union, by definition, there exists $\alpha \in S$ such that $x \in A_\alpha$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A_\alpha$, and $A_\alpha \subset \bigcup_{\alpha \in S} A_\alpha$. This concludes that the union is open.

(c) For any x in the finite intersection, then x is an element of each of $A_1, \ldots, A_n$. Let us have $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $B_{\varepsilon_i}(x)\varepsilon A_i$, and take $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$ (note that minimum is defined only for finite lists), which means $B_\varepsilon(x) \subset \bigcap_{i=1}^{n} A_i$.

In the case of infinite intersection of open sets, the resulting set need not be open. For example, let us consider the infinite intersection of intervals

$$\bigcap_{j=1}^{\infty}\left(-\frac{1}{j},\frac{1}{j}\right)=\{0\},$$

which is not open.

Definition 2.1 (Topology on a Set)**.** Let X be a set equipped with topology $\mathcal{T}$ on X. $\mathcal{T}$ is a collection of subsets of X, i.e. $\mathcal{T}\subset\mathcal{P}(X)$, with properties

- $\varepsilon, X\in\mathcal{T}$.
- The union of subsets of $\mathcal{T}$ is also in $\mathcal{T}$.
- The finite intersection of subsets of $\mathcal{T}$ is also in $\mathcal{T}$.

We call $(X,\mathcal{T})$ a *topological space*, which may be abbreviated to X is $\mathcal{T}$ is given or obvious. Moreover, we define another notion of openness, where $U\in\mathcal{T}$ is said to "be open relative to $\mathcal{T}$," or that "U is open." Here are a few examples of such topological spaces.

(a) The standard topology on $\mathbb{R}^n$ is given by

$$\mathbb{R}_{std}\left(\mathbb{R},\mathcal{T}_{std}=\{U\subset\mathbb{R}^n\mid U\text{ is open in the “old sense”}\}\right),$$

i.e. collection of open intervals[2].

(b) The discrete topology, $X_{discrete}=(X,\mathcal{T}=\mathcal{P}(x))$ can be defined over any set X, equipped with $\mathcal{T}$ as the collection of all subsets.

(c) The trivial topology, $X_{trivial}=(X,\mathcal{T}=\{\emptyset,X\})$.

Definition 2.2 (Continuity between Topological Spaces)**.** If $(X,\mathcal{T}_X),(Y,\mathcal{T}_Y)$ are topological spaces, and $f:X\to Y$ is a function between said spaces, then we say f is continuous if, for all $U\in\mathcal{T}_Y$, we have that $f^{-1}(U)\in\mathcal{T}_X$. This draws from our definition from the notion that pre-images of open sets are open.

Now for the examples from lecture;

(a) $f:X_{discrete}\to\mathbb{R}_{std}$ is always continuous.

(b) $f:X_{trivial}\to\mathbb{R}_{std}$ is continuous if and only if f is constant.

(c) $f:\mathbb{R}_{std}\to X_{trivial}$ is always continuous.

(d) $f:\mathbb{R}_{std}\to X_{discrete}$ is almost never continuous, except when X is empty or a singleton (in which case, $X_{discrete}=X_{trivial}$). If $x_0\in X$ yet $X\setminus\{x_0\}\neq\emptyset$, then let us have

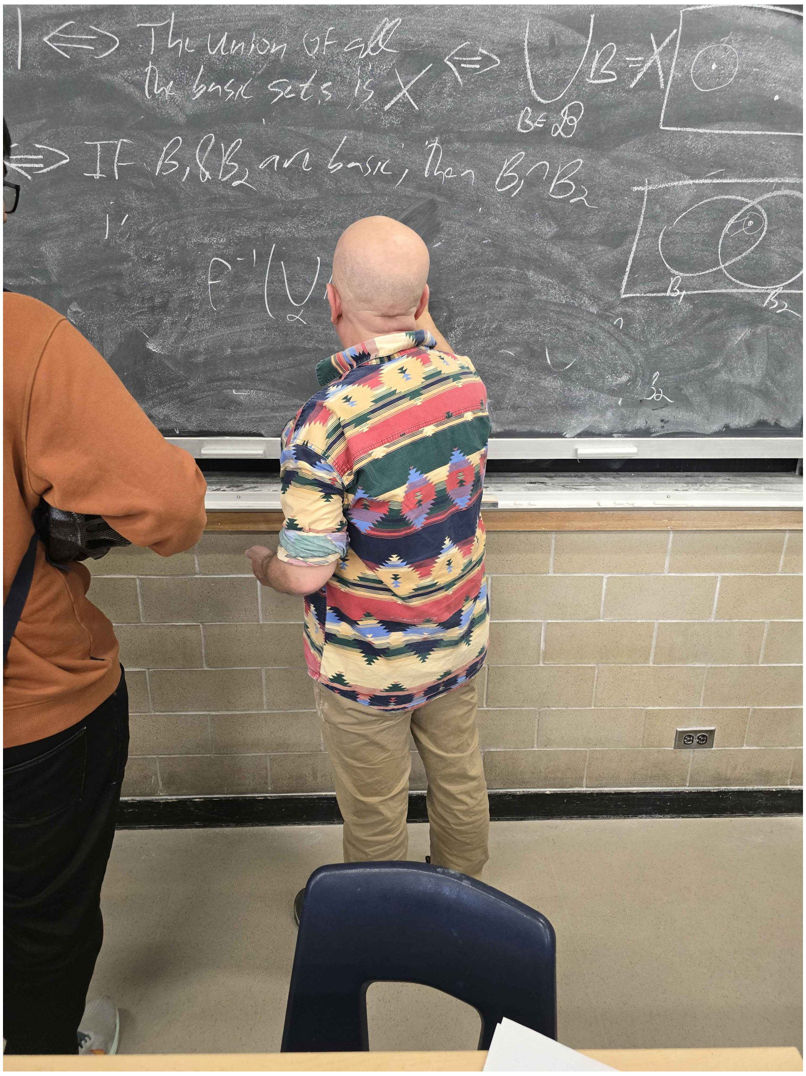
$$A:=f^{-1}(\{x_0\}),$$
$$B:=f^{-1}(X\setminus\{x_0\}).$$

While A,B are both open, we see that $A\cup B=\mathbb{R}$, and $A\cap B=\emptyset$.[3]

[2] this is how i understood it, he might clarify next time?
[3] this one i don't completely get yet.

§3 Day 3: Homeomorphisms and Bases (Sep. 10, 2024)

Outfit of the day (by popular request)



Course administrative details first;

- The reading for this week is on sections 12 to 14 (this week will cover these contents), and 15 to 16 as prereading.

Recap of last lecture:

- A topology $\mathcal{T} \subset \mathcal{P}(X)$ is a collection of subsets of X, of which we require $\{\emptyset, X\} \subset \mathcal{T}$. We also require $\mathcal{T}$ to be closed under arbitrary unions and finite intersections.
- We say a function $F : X \to Y$ is continuous if and only if for all $U \in \mathcal{T}_Y$, we have $f^{-1}(U) \in \mathcal{T}_X$.

Today we will cover homeomorphisms and bases. To start, recall the example topologies, such as $\mathcal{T}_{\text{std}}$ on $\mathbb{R}^n$ (where $\mathcal{T}_{\text{std}}$ consists of the open balls), $\mathcal{T}_{\text{triv}}$, and $\mathcal{T}_{\text{disc}}$. We also introduce a new example topology (where FC standards for finite complement),

$$\mathcal{T}_{\text{FC}} = \{U \subset X \mid X \setminus U \text{ is finite, or } U = \emptyset\}.$$

Note that Prof. Bar-Natan may interchange the notations $-$ or $\setminus$ to represent set difference.

Theorem 3.1 (Composition of Continuous Functions is Continuous)**.** Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f : X \to Z$ is continuous (relative to the same topologies on X and Z).

If $U \in \mathcal{T}_Z$, we have $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U)$, where by definition of continuity, we see that pre-images of open sets are open, and we have $g^{-1}(U)$ is open in Y, and similarly $f^{-1}(g^{-1}(U))$ is also open in X. $\square$

In tutorial, we equipped X with topologies $\mathcal{T}_1, \mathcal{T}_2$ (i.e., X is a topological space in two ways). We say $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if $T_1 \supset T_2$, and coarser for the opposite direction; the words bigger and stronger may be used interchangeably with finer, and smaller or weaker for coarser. For example, the identity map

$$\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$$

is continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$. To see this, let U be an open set in $(X, \mathcal{T}_2)$; then U must be open in $(X, \mathcal{T}_1)$ as well, which is true for any U only if $\mathcal{T}_1 \supset \mathcal{T}_2$[4].

Definition 3.2 (Homeomorphism)**.** A map $h : X \to Y$ is called a homeomorphism if h is continuous, bijective, and h^{-1} is continuous as well.[5]

Note that continuous bijective maps h need not have continuous inverses; for example, let us have $\mathrm{id} : X_{\text{disc}} \to X_{\text{triv}}$. id is continuous as per our above example, while its inverse is not. Another example is to consider $[0, 2\pi) \to S^1$ (unit circle), where $x \mapsto (\cos 2x\pi, \sin 2x\pi)$; we see that the inverse is discontinuous at 0 and 2π radians, even if the map is continuous and bijective (also observe that $[0, 1)$ is not compact while S^1 is).
An example of a homeomorphism is as follows (as per tutorial); let us consider[6]

$$(-1, 1)_{\text{std}} \cong \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)_{\text{std}} \cong \mathbb{R}_{\text{std}},$$

where we may map the first to the second by $x \mapsto \frac{\pi}{2}x$, and the second to third by $x \mapsto \tan x$. Since the composition of continuous maps is continuous, we also see $(-1, 1)_{\text{std}}$ is homeomorphic to $\mathbb{R}_{\text{std}}$ (any open interval is homophobic to $\mathbb{R}$ for that matter).

Another example of a homemorphism is $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ if $\mathcal{T}_1 = \mathcal{T}_2$.

A "basis" for a topology on X is a collection $\mathcal{B} \subset \mathcal{P}(X)$ of subsets such that

1. For all $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$. We call B a *basic set.*

2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists a third basic set B_3 such that $x \in B_3 \subset B_1 \cap B_2$.

[4] includes the case $\mathcal{T}_1 = \mathcal{T}_2$; i'm following florian notation here with $\subset$ and $\subsetneq$ for explicit non equality
[5] smth smth coffee cup and donut "extra homework: go find a nice video on why this is true on youtube"
[6] dror was using $\sim$ for homeomorphism symbol today. if he keeps using that i'll adjust my notes, but for now i'll use $\cong$ cuz afaik its used more...?

In particular, the first condition is equivalent to $\bigcup_{B\in\mathcal{B}} B = X$ (i.e., the basis forms a covering of X), and the second condition is equivalent to the basic sets contained in $B_1 \cap B_2$ forming a cover of $B_1 \cap B_2$, i.e.

$$B_1 \cap B_2 = \bigcup_{\substack{B\in\mathcal{B} \\ B\subset B_1\cap B_2}} B.$$

Here are some examples;

(a) $\{B_r(x_0)\} \subset \mathcal{P}(\mathbb{R}^n)$, i.e. the open balls on $\mathbb{R}^n$ form a basis.

(b) The one-dimensional analogue of case (a) is $\{(a, b) \mid a < b\}$, and it forms a basis on $\mathbb{R}$.

(c) $\{[a, b) \mid a < b\}$ is called the lower limit topology, and it forms a basis on $\mathbb{R}$.

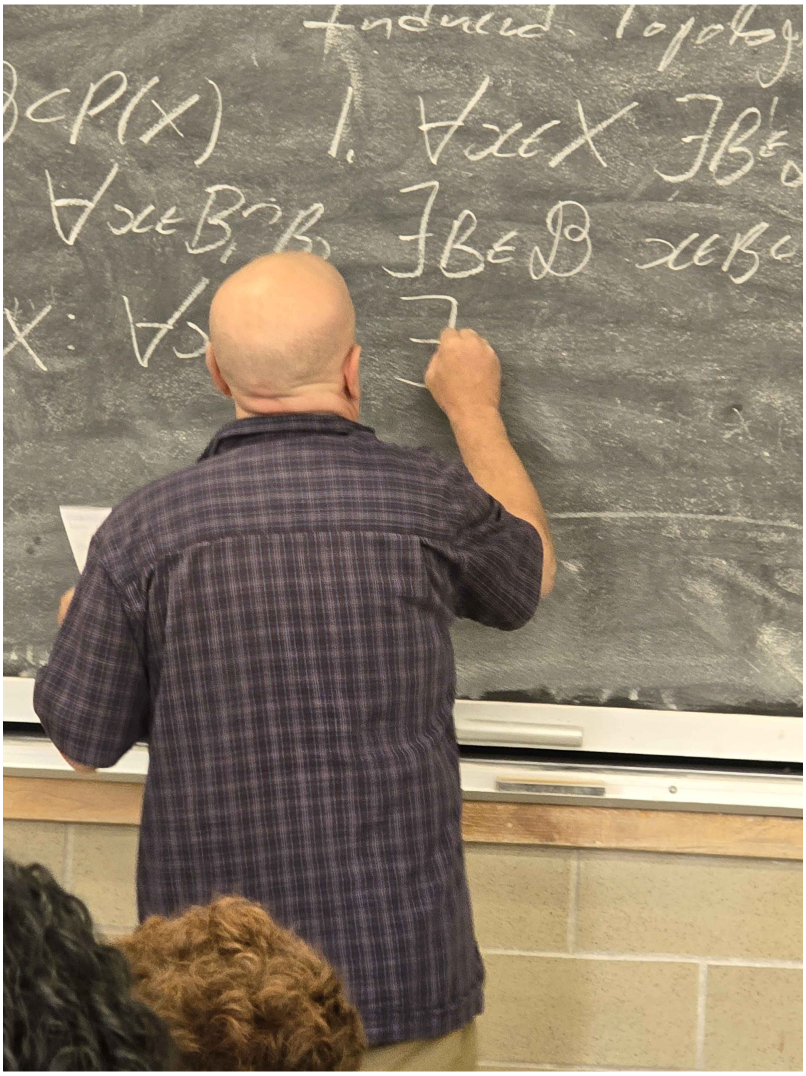
(d) $\{[a, b] \mid a < b\}$ implies $[a, b] \cap [b, c] = \{b\}$, which forces the basic set to include all singletons on $\mathbb{R}$. In that case, this is simply the discrete topology (?).

Theorem 3.3. $\mathcal{T}_{\mathcal{B}} = \{U \subset X \mid \forall x \in U \implies \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}$, i.e. the collection of all unique basic sets.

This will be expanded on next lecture.

§4 Day 4: Basis; Orderings on Sets; Product Topology (Sep. 10, 2024)

Outfit of the day! Gives King Dice vibes tbh (from cuphead) i like the purple a lot :3



Recap of last lecture:

- We define the basis $\mathcal{B} \subset \mathcal{P}(X)$ of a topology to have the following properties (which we will refer to as the first and second axioms),
  1. For all $x \in X$, there exists a basic set $B \in \mathcal{B}$ such that $x \in B$.
  2. For any x in the intersection of two basic sets (i.e. $x \in B_1 \cap B_2$), there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

With this, we may construct the topology generated by $\mathcal{B}$, i.e.

$$\mathcal{T}_{\mathcal{B}} := \{U \subset X \mid \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U\}.$$

Theorem 4.1 (Basis Topology is a Topology on X)**.** We claim that $\mathcal{T}_\mathcal{B}$ is a topology on X.

We proceed by checking that $\mathcal{T}_\mathcal{B}$ satisfies the required properties.

1. Observe that $U = \emptyset, X$ are both in $\mathcal{T}_\mathcal{B}$; if $U = \emptyset$, then there are no elements $x \in U$ to bother about, of which the condition is vacuously true; if $U = X$, then the condition is true by the basis axioms.

2. Let us consider an indexing set I, and let us consider the arbitrary union of subsets $U_\alpha \subset X$
$$U := \bigcup_{\alpha \in I} U_\alpha.$$
Then any $x \in U$ belongs to $x \in U_\alpha$ for some index $\alpha \in I$; since U_α satisfies the condition, we see that the union U also satisfies the condition as well.

3. Finally, for intersections, let us take $U_1, U_2 \in \mathcal{T}$ where[7]
$$U_1 = \bigcup_{\alpha_1 \in A_1} B_{1,\alpha_1},$$
$$U_2 = \bigcup_{\alpha_1 \in A_2} B_{2,\alpha_2},$$
where each B_{i,α_j} for $i, j \in \{1, 2\}$ above is a basic set in $\mathcal{B}$. Then
$$U_1 \cap U_2 = \left(\bigcup_{\alpha_1 \in A_1} B_{1,\alpha_1} \right) \cap \left(\bigcup_{\alpha_2 \in A_2} B_{2,\alpha_2} \right) = \bigcup_{\substack{\alpha_1 \in A_1 \\ \alpha_2 \in A_2}} \left(B_{1,\alpha_1} \cap B_{2,\alpha_2} \right).$$
Observing that $B_{1,\alpha_1} \cap B_{2,\alpha_2}$ is open by the second axiom of bases, we are done by quoting that the union of open sets is open. $\square$

Theorem 4.2 ($\mathcal{T}_\mathcal{B}$ is the minimal topology containing $\mathcal{B}$)**.**

To start, we obviously have $\mathcal{T}_\mathcal{B} \supset \mathcal{B}$; now, let us have $\mathcal{T}'$ be another topology that contains $\mathcal{B}$. Since $\mathcal{T}_\mathcal{B}$ is the set of all unions of elements of $\mathcal{B}$, if $\mathcal{T}'$ contains $\mathcal{B}$, then it also contains $\mathcal{T}_\mathcal{B}$.

In fact, we may prove that the minimal topology containing $\mathcal{B}$ is unique.[8] Let $\mathcal{T}', ST''$ be minimal topologies that contain $\mathcal{B}$; then by the above argument, they both contain $\mathcal{T}_\mathcal{B}$; since they are minimal, we see $\mathcal{T}_\mathcal{B} = \mathcal{T}' = \mathcal{T}''$, and so $\mathcal{T}' = \mathcal{T}''$. $\square$

Theorem 4.3 (Continuity on Basic Sets)**.** It is enough to test for continuity on basic sets. *(Originally left as exercise)*

Suppose $\mathcal{B}_Y$ is a basis of $\mathcal{T}_Y$ on Y, and suppose $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is the function we want to test for continuity. Then the topology of $\mathcal{T}_Y$ contains the basis topology $\mathcal{T}_{\mathcal{B}_Y}$, meaning that all open sets $\mathcal{T}_Y$ are unions and intersections of elements from $\mathcal{T}_{\mathcal{B}_Y}$. If the pre-image of the basic sets $B \in \mathcal{B}_Y$ are open in X, then so are the unions and intersections of such basic sets. $\square$

[7] note that dror used superscript to separate the two types of base sets, but i'm going to subscript them in a list to keep it clear.

[8] thought: measure theory gives that minimal σ-algebra generated by generator is unique; similar situation here?

We now define the notion of orders (corresponding Munkres section 14). A "complete" (otherwise called simple, line, or total) order on a set X is a relation $<$ on $X \times X$ such that

- $<$ is transitive; i.e., if $x < y$ and $y < z$, then $x < z$.
- If $x, y \in X$, then exactly one of the following is true (recall trichotomy from 157):
  (a) $x < y$
  (b) $x > y$
  (c) $x = y$.

If X is a set with a simple order relation, let $\mathcal{B}$ be the collection of all sets of the following types,

- All open intervals (a, b) in X,
- All intervals of the form $[a_0, b)$ where a_0 is the smallest element of X (if it exists),
- All intervals of the form $(a, b_0]$ where b_0 is the largest element of X (if it exists).
- In the case that X is a singleton set, said singleton is in $\mathcal{B}$.

The collection $\mathcal{B}$ is a basis for a topology on X; we call this the order topology. Now, we give some examples of orderings:

(a) $(\mathbb{R}, <_{\text{std}})$ and $(\mathbb{Q}, <_{\text{std}})$ are basic examples. Note that here, $<_{\text{std}}$ refers to the standard comparison.

(b) English words in lexicographiacl order, such as

$$\text{apple} < \text{ton} < \text{topo} < \text{topple} < \text{zebra}.$$

(c) $\{0, 1\} \times \mathbb{N}$ in dictionary order; we say $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$ if $\alpha_1 < \alpha_2$, or $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$.

(d) Alternatively, if we consider $\mathbb{N} \times \{0, 1\}$ (i.e. binary sequences), then it's the same idea as above just for infinite sequences of $\{\alpha_i\}_{i \in \mathbb{N}}, \{\beta_i\}_{i \in \mathbb{N}}$. Note that $\mathcal{T}_{\{0,1\} \times \mathbb{N}}$ and $\mathcal{T}_{\mathbb{N} \times \{0,1\}}$ are not homeomorphic. *(Originally left as exercise.)*

(e) $\mathbb{R} \times \mathbb{R}$ in dictionary order. Here, open sets can be thought of as $(a, b) = \{x \mid a < x < b\}$, where we may note $x \in \mathbb{R} \times \mathbb{R}$.

For another example, consider X to be the set of finite strings over the usual alphabet; then we may observe that the open sets are dense in X; for example, the open set[9]

$$(\text{potato}, \text{tomato}) \supset \{\text{potatoo}, \text{potatooo}, \text{potatoooooooo}\}$$

contains as many strings as we want to fit into it. In particular, a consequence is that the order topology on X is not equal to the discrete topology on X; since the discrete topology contains singleton sets (i.e., single lettersoups), and the unions and intersections of open sets of lettersoups is another open set, the two topologies are not equal.

In general, though, if X is a finite set (i.e., similar to the above example, if X is the set of *english words*) and we equip it with a strict total ordering, then the order topology is equal to the discrete topology.

[9] i'm done calling these things words. they are hereby known from now on as lettersoups

We now move onto product topologies. Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces; we want to construct a topology on $X \times Y$ such that (X, Y) restricts to X on X and Y on Y, i.e.

$$\pi_X : (X, Y) \to X, \qquad \pi_Y : (X, Y) \to Y,$$

and we want it to come with the properties

1. π_X, π_Y are continuous (as per above, they are the projections to X and Y),

2. Suppose f, g are two continuous maps $f : Z \to X$ and $g : Z \to Y$. Then $f \times g : Z \to X \times Y$, where $(f \times g)(z) = (f(z), g(z))$.

Note that the above condition is equivalent to taking $h : Z \to X \times Y$; if $\pi_X \circ h$ and $\pi_Y \circ h$ are continuous, then h is continuous.

Theorem 4.4 (Existence of Unique Product Topology)**.** There exists a unique topology on $X \times Y$ satisfying the above two conditions; we will call this the product topology on $X \times Y$.

To start, by definition of continuity, we need that for all $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$, we have that

$$\pi_X^{-1}(U) = \{(x, y) \mid x \in U\}, \qquad \pi_Y^{-1}(V) = \{(x, y) \mid y \in V\}$$

must be open sets. Let us claim that the collection $\mathcal{B}_{X\times Y} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ forms a basis of $X \times Y$. Then let $\mathcal{T}_{X\times Y}$ be generated by $\mathcal{B}_{X\times Y}$; we claim that it satisfies the two properties outlined above.

1. $\pi_X^{-1}(U) = U \times Y \in \mathcal{T}_{U\times Y}$, and $\pi_Y^{-1}(V) = X \times V \in \mathcal{T}_{X\times V}$ show that continuity of π_X, π_Y is satisfied on basic sets, and so they are satisfied in general.

2. Let $f : Z \to X$ and $g : Z \to Y$ be continuous. Consider $f \times g : Z \to X \times Y$, and let $U \subset \mathcal{T}_X$ and $V \subset \mathcal{T}_Y$. Then $U \times V$ is open, and we may write

$$\begin{aligned}
(f \times g)^{-1}(U \times V) &= \{(x, y) \mid (f \times g)(x, y) \in U \times V\} \\
&= \{z \mid (f(z), g(z)) \in U \times V\} \\
&= \{z \mid f(z) \in U \text{ and } g(z) \in V\} \\
&= \{z \mid f(z) \in U\} \cap \{z \mid g(z) \in V\} \\
&= f^{-1}(U) \cap g^{-1}(V),
\end{aligned}$$

which is the intersection of open sets, and so is open. Thus, our claim is complete. Now, to demonstrate uniqueness, suppose $\mathcal{T}', \mathcal{T}''$ are topologies in $X \times Y$ which satisfy our two properties from earlier. Then let us have the mapping

$$\begin{aligned}
\mathrm{id} = \pi_X \times \pi_Y : (X \times Y)_{\mathcal{T}'} &\to (X \times Y)_{\mathcal{T}''} \\
(x, y) &\mapsto (\pi_X(x, y), \pi_Y(x, y)) = (x, y).
\end{aligned}$$

In particular, $\pi_X : (X \times Y)_{\mathcal{T}'} \to X$ is continuous by property 1 of $\mathcal{T}'$ (same with π_Y). This means $\pi_X \times \pi_Y$ is continuous by property 2 of $\mathcal{T}'$. We make the same argument for $\mathcal{T}''$; this means $\mathcal{T}' \subset \mathcal{T}''$ and vice versa, which means $\mathcal{T}' = \mathcal{T}''$ as desired. $\square$