Temă seminar 02

E1 / 190

a) $f(x) = x^4,\ x \in \mathbb{R}$

$F(x) = \int f(x)\,dx = \int x^4\,dx = \frac{x^5}{5} + C$ $\qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

b) $f(x) = 8x^7,\ x \in \mathbb{R}$

$F(x) = \int f(x)\,dx = \int 8x^7\,dx = 8\int x^7\,dx = 8 \cdot \frac{x^8}{8} + C = x^8$ $\qquad \int a \cdot x^n\,dx = a \cdot \frac{x^{n+1}}{n+1}$

c) $f(x) = x^{\frac{4}{5}}$

$F(x) = \int f(x)\,dx = \int x^{\frac{4}{5}}\,dx = \frac{x^{\frac{4}{5}+1}}{\frac{4}{5}+1} = \frac{x^{\frac{9}{5}}}{\frac{9}{5}} = \frac{5}{9} \cdot x^{\frac{4}{5}} + C = \frac{5}{9} \cdot \sqrt[5]{x^9} + C$

$= \frac{5}{9} \cdot x \cdot \sqrt[5]{x^4} + C$

d) $f(x) = \sqrt[5]{x^3},\ x \in \mathbb{R}$

$F(x) = \int f(x)\,dx = \int x^{\frac{3}{5}}\,dx = \frac{x^{\frac{3}{5}+1}}{\frac{3}{5}+1} + C = \frac{x^{\frac{8}{5}}}{\frac{8}{5}} + C = \frac{5}{8} \cdot \sqrt[5]{x^8} + C$

$= \frac{5}{8} \cdot x\sqrt[5]{x^3} + C$

$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$

e) $f(x) = x^{-\frac{8}{3}},\ x > 0$

$F(x) = \int f(x)\,dx = \int x^{-\frac{8}{3}}\,dx = \frac{x^{-\frac{8}{3}+\frac{3}{3}}}{-\frac{8}{3}+\frac{3}{3}} = \frac{x^{-\frac{5}{3}}}{-\frac{5}{3}} + C = -\frac{3}{5} \cdot x^{-\frac{5}{3}} + C$

$F(x) = -\frac{3}{5} \cdot \sqrt[3]{x^{-5}} + C = -\frac{3}{5} \cdot \sqrt[3]{\frac{1}{x^5}} + C = -\frac{3}{5} \cdot \frac{1}{x} \sqrt[3]{\frac{1}{x^2}} + C$

f) $f(x) = 11 \cdot x \cdot \sqrt[4]{x^3},\ x > 0$ $\qquad \int x^n\,dx = \frac{x^{n+1}}{n+1}$

$f(x) = 11 \cdot x \cdot x^{\frac{3}{4}} = 11 \cdot x^{1+\frac{3}{4}} = 11 \cdot x^{\frac{4+3}{4}} = 11 \cdot x^{\frac{7}{4}}$

$F(x) = \int f(x)\,dx = \int 11\, x^{\frac{7}{4}}\,dx = 11\int x^{\frac{7}{4}}\,dx = 11 \cdot \frac{x^{\frac{7}{4}+\frac{4}{4}}}{\frac{7}{4}+\frac{4}{4}} + C$

$F(x) = 11 \cdot \frac{x^{\frac{11}{4}}}{\frac{11}{4}} + C = \cancel{11} \cdot \frac{4}{\cancel{11}} \cdot x^{\frac{11}{4}} = 4 \cdot x^{\frac{11}{4}} + C$

$F(x) = 4 \cdot \sqrt[4]{x^{11}} + C = 4 \cdot x^2 \cdot \sqrt[4]{x^3} + C$

g) $f(x) = \frac{1}{\sqrt[3]{x^2}},\ x > 0$ $\qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

$f(x) = \frac{1}{x^{\frac{2}{3}}} = x^{-\frac{2}{3}}$

$F(x) = \int f(x)\,dx = \int x^{-\frac{2}{3}}dx = \frac{x^{-\frac{2}{3}+\frac{3}{3}}}{-\frac{2}{3}+\frac{3}{3}} + C = \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + C = 3\sqrt[3]{x} + C$

h) $f(x) = e^x$, $x \in \mathbb{R}$

$\int e^x\,dx = e^x + C$

$\int f(x) = \int e^x\,dx = e^x + C$

$F(x) = e^x + C$

i) $f(x) = 2^x$, $x \in \mathbb{R}$

$\int a^x dx = \frac{a^x}{\ln a} + C$

$F(x) = \int f(x)\,dx = \int 2^x\,dx = \frac{2^x}{\ln 2} + C$

j) $f(x) = \frac{1}{x-1}$, $x > 1$

$(\ln u)' = \frac{1}{u} \cdot u'$

$(\ln(x-1))' = \frac{1}{x-1} \cdot (x-1)' = \frac{1}{x-1}$

$F(x) = \int \frac{1}{x-1}\,dx = \int (\ln(x-1))'dx$

$\int f'(x)\,dx = f(x)$

$F(x) = \ln(x-1) + C$

k) $f(x) = \frac{1}{x^2-9}$, $x \in (-3, 3)$

$\int \frac{1}{x^2-a^2} = \frac{1}{2a} \cdot \ln\left|\frac{x-a}{x+a}\right| + C$

$F(x) = \int \frac{1}{x^2-9}\,dx = \int \frac{1}{x^2-3^2}\,dx = \frac{1}{2 \cdot 3} \cdot \ln\left|\frac{x-3}{x+3}\right| + C$

$F(x) = \frac{1}{6} \cdot \ln\left|\frac{x-3}{x+3}\right| + C$

l) $f(x) = \frac{1}{16+x^2}$, $x \in \mathbb{R}$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\, \text{arctg}\, \frac{x}{a}$

$F(x) = \int f(x)\,dx = \int \frac{1}{4^2+x^2}\,dx = \frac{1}{4} \cdot \text{arctg}\, \frac{x}{4} + C$

m) $f(x) = \frac{1}{\sqrt{x^2-4}}$, $x \in (-\infty, -2)$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x - \sqrt{x^2-a^2}\right| + C$

$F(x) = \int \frac{1}{\sqrt{x^2-4}}\,dx = \ln\left|x - \sqrt{x^2-4}\right| + C$

n) $f(x) = \dfrac{1}{\sqrt{4-x^2}}$, $x \in (-2,2)$ $\qquad \int \dfrac{1}{\sqrt{a^2-x^2}}dx = \arcsin \dfrac{x}{a} + C$

$F(x) = \int f(x)\,dx = \int \dfrac{1}{\sqrt{2^2-x^2}}dx = \arcsin \dfrac{x}{2} + C$

$\int \dfrac{1}{\sqrt{a^2+x^2}}dx = \ln\left(x + \sqrt{x^2+a^2}\right) + C$

o) $f(x) = \dfrac{1}{\sqrt{x^2+25}}$, $x \in \mathbb{R}$

$F(x) = \int f(x)\,dx = \int \dfrac{1}{\sqrt{x^2+25}}dx = \ln\left(x + \sqrt{x^2+25}\right) + C$

p) $f(x) = \dfrac{1}{\sqrt{(6-x)(6+x)}} = \dfrac{1}{\sqrt{36-x^2}}$ $\qquad \int \dfrac{1}{\sqrt{a^2-x^2}} = \arcsin \dfrac{x}{a} + C$

$F(x) = \int f(x)\,dx = \int \dfrac{1}{\sqrt{36-x^2}}dx = \arcsin \dfrac{x}{6} + C$

E2/190

a) $\int (5x^4 - 4x^3 + 3x^2 - 6x + 1)\,dx$ $\qquad \int x^n dx = \dfrac{x^{n+1}}{n+1}$

$= 5\int x^4 dx - 4\int x^3 dx + 3\int x^2 dx - 6\int x\,dx + \int dx$

$= \cancel{5} \cdot \dfrac{x^5}{\cancel{5}} + C - \cancel{4}\left[\dfrac{x^4}{\cancel{4}} + C\right] + 3 \cdot \dfrac{x^3}{3} + C - 6\,\dfrac{x^2}{2}\,C + x + C$

$= x^5 - x^4 + x^3 - 3x^2 + x + C$

! putem face abstracție de C până la final pt că este const.

b) $\int (x^2 - 2x)^3\,dx$ $\qquad (a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

$\int \left[(x^2)^3 - (2x)^3 - 3(x^2)^2 \cdot 2x + 3 \cdot x^2 (2x)^2\right] dx$ $\qquad \int x^n dx = \dfrac{x^{n+1}}{n+1} + C$

$\int \left[x^6 - 8x^3 - 3x^4 \cdot 2x + 3x^2 \cdot 4x^2\right] dx = \int \left[x^6 - 8x^3 - 6x^5 + 12 \cdot x^4\right] dx$

$\int x^6 dx - 8\int x^3 dx - 6\int x^5 dx + 12\int x^4 dx$

$\dfrac{x^7}{7} - 8\,\dfrac{x^4}{4} - 6\,\dfrac{x^6}{6} + 12\,\dfrac{x^5}{5} + C$

$\dfrac{1}{7}x^7 - 2x^4 - x^6 + \dfrac{12}{5}x^5 + C$

c) $\int\left(\frac{2}{x^3}-\frac{4}{x^5}-\frac{3}{x}\right)dx$

$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$

$\int \frac{1}{x}\,dx = \ln(x)+C$

$= 2\int \frac{1}{x^3}\,dx - 4\int\frac{1}{x^5}\,dx - 3\int\frac{1}{x}\,dx$

$= 2\cdot\int x^{-3}\,dx - 4\int x^{-5}\,dx - 3\cdot\ln(x) + C$

$= 2\cdot\frac{x^{-3+1}}{-3+1} - 4\cdot\frac{x^{-5+1}}{-5+1} - 3\ln(x)+C$

$= 2\cdot\frac{x^{-2}}{-2} - 4\cdot\frac{x^{-4}}{-4} - 3\ln(x)+C$

$= -x^{-2} + x^{-4} - 3\ln(x)+C$

$= -\frac{1}{x^2} + \frac{1}{x^4} - 3\ln(x)+C$

d) $\int\left(8x^2\sqrt{x} + 7x\sqrt[4]{x^3}\right)dx \quad , \quad x>0$

$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$

$x^a\cdot x^b = x^{a+b}$

$= \int 8\cdot x^2\cdot x^{\frac{1}{2}}\,dx + \int 7x\cdot x^{\frac{3}{4}}\,dx$

$= 8\int x^{2+\frac{1}{2}}\,dx + 7\int x^{1+\frac{3}{4}}\,dx$

$= 8\int x^{\frac{4+1}{2}}\,dx + 7\int x^{\frac{4+3}{4}}\,dx$

$= 8\int x^{\frac{5}{2}}\,dx + 7\int x^{\frac{7}{4}} = 8\cdot\frac{x^{\frac{5}{2}+\frac{2}{2}}}{\frac{5}{2}+\frac{2}{2}} + 7\cdot\frac{x^{\frac{7}{4}+\frac{4}{4}}}{\frac{7}{4}+\frac{4}{4}}$

$= 8\cdot\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + 7\cdot\frac{x^{\frac{11}{4}}}{\frac{11}{4}}$

$= 8\cdot\frac{2}{7}\cdot x^{\frac{7}{2}} + 7\cdot\frac{4}{11}\cdot x^{\frac{11}{4}}$

$= \frac{16}{7}\cdot\sqrt{x^7} + \frac{28}{11}\cdot\sqrt[4]{x^{11}} = \frac{16}{7}\cdot x^3\sqrt{x} + \frac{28}{11}\cdot x^2\sqrt[4]{x^3} + C$

$= \frac{16}{7}x^3\sqrt{x} + \frac{28}{11}\cdot x^2\sqrt[4]{x^3} + C$

e) $\int \left(\frac{x}{\sqrt[3]{x^7}} - 21 x^4 \cdot \sqrt[4]{x} \right) dx \quad , x>0$

$x^a \cdot x^b = x^{a+b}$

$\frac{x^a}{x^b} = x^{a-b}$

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$

$\sqrt[a]{x^b} = x^{\frac{b}{a}}$

$= \int \frac{x}{x^{\frac{7}{3}}} dx - 21 \cdot \int (x^4 \cdot x^{\frac{1}{4}}) dx$

$= \int x^{1-\frac{7}{3}} dx - 21 \int x^{4+\frac{1}{4}} dx$

$= \int x^{\frac{3-7}{3}} dx - 21 \int x^{\frac{16+1}{4}} dx$

$= \int x^{-\frac{4}{3}} dx - 21 \int x^{\frac{17}{4}} dx$

$= \frac{x^{-\frac{4}{3}+1}}{-\frac{4}{3}+1} - 21 \cdot \frac{x^{\frac{17}{4}+1}}{\frac{17}{4}+1} + C$

$= \frac{x^{\frac{-4+3}{3}}}{\frac{-4+3}{3}} - 21 \cdot \frac{x^{\frac{17+4}{4}}}{\frac{17+4}{4}} + C$

$= \frac{x^{\frac{-1}{3}}}{\frac{-1}{3}} - 21 \cdot \frac{x^{\frac{21}{4}}}{\frac{21}{4}} + C = -3 \cdot x^{\frac{-1}{3}} - \cancel{21} \cdot \frac{4}{\cancel{21}} \cdot x^{\frac{21}{4}} + C$

$= -3 \cdot \sqrt[3]{\frac{1}{x}} - 4 \cdot \sqrt[4]{x^{21}} + C = -3 \sqrt[3]{\frac{1}{x}} - 4 \cdot x^5 \cdot \sqrt[4]{x} + C$

f) $\int \frac{1}{4x^2-1} dx \quad , x > \frac{1}{2}$

$\int \frac{u'}{u^2-a^2} dx = \frac{1}{2a} \cdot \ln \left| \frac{u-a}{u+a} \right| + C$

$= \int \frac{1}{(2x)^2 - 1^2} dx =$

$u = 2x \rightarrow u' = 2$

$= \int \frac{1}{2} \cdot \frac{2}{(2x)^2 - 1^2} dx = \frac{1}{2} \int \frac{(2x)'}{(2x)^2 - 1^2} dx = \frac{1}{2} \cdot \underbrace{\frac{1}{2} \cdot \ln \left| \frac{2x-1}{2x+1} \right|}_{\frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right|} + C$

$= \frac{1}{4} \cdot \ln \left| \frac{2x-1}{2x+1} \right| + C$

g) $\displaystyle\int \frac{30}{9x^2-25}\,dx$ $\qquad \displaystyle\int \frac{1\cdot u'}{u^2-a^2}\,dx = \frac{1}{2a}\cdot \ln\left|\frac{u-a}{u+a}\right| + C$

$u = 3x$; $u' = 3$

$$\int \frac{3\cdot 10}{(3x)^2-5^2}\,dx = 10\int \frac{3}{(3x)^2-5^2}\,dx = 10\cdot\int \frac{(3x)'}{(3x)^2-5^2}\,dx$$

$$\cancel{10}\,\frac{1}{\cancel{2\cdot 5}}\cdot \ln\left|\frac{3x-5}{3x+5}\right| + C = \ln\left|\frac{3x-5}{3x+5}\right| + C$$

h) $\displaystyle\int \frac{8}{4x^2+1}\,dx = \int \frac{2\cdot 4}{4x^2+1}\,dx$ $\qquad \displaystyle\int \frac{u'}{u^2+a^2}\,dx = \frac{1}{a}\cdot \operatorname{arctg}\frac{u}{a} + C$

$$= \int \frac{(2x)'\cdot 4}{(2x)^2+1^2}\,dx = 4\int \frac{(2x)'}{(2x)^2+1^2}\,dx = 4\cdot\frac{1}{1}\cdot \operatorname{arctg}\frac{2x}{1} + C$$

$$= 4\cdot \operatorname{arctg} 2x + C$$

h) $\displaystyle\int \frac{8}{4x^2+1}\,dx = \int \frac{(2x)'\cdot 4}{(2x)^2+1^2}\,dx = 4\int \frac{(2x)'}{(2x)^2+1^2}\,dx$

$$= 4\cdot\frac{1}{1}\cdot \operatorname{arctg}\frac{2x}{1} + C \qquad \int \frac{u'}{u^2+a^2}\,dx = \frac{1}{a}\cdot \operatorname{arctg}\frac{u}{a} + C$$

$$= 4\cdot \operatorname{arctg} 2x + C$$

i) $\displaystyle\int \frac{18}{3x^2+27}\,dx = \int \frac{\sqrt{3}\cdot(\sqrt{3}\cdot 6)}{(\sqrt{3}x)^2+(3\sqrt{3})^2}\,dx = 6\sqrt{3}\int \frac{(\sqrt{3}x)'}{(\sqrt{3}x)^2+(3\sqrt{3})^2}\,dx$

$$= 6\sqrt{3}\cdot\frac{1}{3\sqrt{3}}\cdot \operatorname{arctg}\frac{\sqrt{3}x}{3\sqrt{3}} + C \qquad \int \frac{u'}{u^2+a^2} = \frac{1}{a}\cdot \operatorname{arctg}\frac{u}{a} + C$$

$$= 2\cdot \operatorname{arctg}\frac{x}{3} + C$$

j) $\displaystyle\int (5^x\cdot \ln 5 - 4^x\cdot \ln 16)\,dx$ $\qquad \displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$

$$= \ln 5\cdot\int 5^x\,dx - \ln 16\cdot\int 4^x\,dx \qquad \ln a^2 = 2\cdot \ln a$$

$$= \cancel{\ln 5}\cdot\frac{5^x}{\cancel{\ln 5}} - \ln 4^2\cdot\frac{4^x}{\ln 4} + C$$

$$= 5^x - 2\cdot\cancel{\ln 4}\cdot\frac{4^x}{\cancel{\ln 4}} + C = 5^x - 2\cdot 4^x + C$$

k) $\int \frac{1}{\sqrt{6x^2+24}}\,dx$

$\int \frac{1 \cdot u'}{\sqrt{u^2+a^2}}\,du = \ln\left[u + \sqrt{u^2+a^2}\right] + C$

$= \int \frac{1}{(\sqrt{6}x)^2 + (\sqrt{24})^2}\,dx \quad \int \frac{(\sqrt{6}x)'}{(\sqrt{6}x)^2 + (2\sqrt{6})^2} \cdot \frac{1}{\sqrt{6}}\,dx = \frac{1}{\sqrt{6}} \cdot \ln\left[\sqrt{6}x + \sqrt{6x^2+24}\right] + C$

l) $\int \frac{1}{\sqrt{2x^2-18}}\,dx = \int \frac{(\sqrt{2}x)'}{\sqrt{(\sqrt{2}x)^2 - (3\sqrt{2})^2}} \cdot \frac{1}{\sqrt{2}}\,dx$

$= \frac{1}{\sqrt{2}} \cdot \ln\left|\sqrt{2}x + \sqrt{2x^2-18}\right| + C$

$\int \frac{1}{\sqrt{u^2-a^2}}\,du = \ln\left|u + \sqrt{u^2-a^2}\right| + C$

m) $\int \frac{\sqrt{3}}{\sqrt{48-3x^2}}\,dx$

$\int \frac{u'}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C$

$\int \frac{(\sqrt{3}x)'}{\sqrt{(4\sqrt{3})^2 - (\sqrt{3}x)^2}}\,dx = \arcsin\frac{\sqrt{3}x}{4\sqrt{3}} + C = \arcsin\frac{x}{4} + C$

Es/190

a) $\int (3\sin x + 4\cos x)\,dx$

$\int \sin x\,dx = -\cos x$

$\int \cos x\,dx = \sin x$

$= 3\int \sin x\,dx + 4\int \cos x\,dx$

$= 3(-\cos x) + 4\sin x + C = 4\sin x - 3\cos x + C$

b) $\int (2\sin^2 x - \sqrt[3]{-8}\cos^2 x)\,dx$

$\sin^2 x + \cos^2 x = 1$

$\int dx = x$

$= \int (2\sin^2 x - (-2)\cos^2 x)\,dx$

$= \int (2\sin^2 x + 2\cos^2 x)\,dx = 2\int (\sin^2 x + \cos^2 x)\,dx = 2x + C$

c) $\int 2\sin\frac{x}{2}\cos\frac{x}{2}\,dx$

$(\sin x)' = \cos x$

$(\sin u)' = \sin u \cdot u'$

$= \int 2 \cdot \sin\frac{x}{2} \cdot \left(\cos\frac{x}{2} \cdot \frac{1}{2}\right) \cdot 2\,dx$

$(\sin\frac{x}{2})' = \cos\frac{x}{2} \cdot \frac{1}{2}$

$(\frac{x}{2})'$

$= 4\int \sin\frac{x}{2} \cdot (\sin\frac{x}{2})' = \cancel{4}^{2}\,\frac{\sin(\frac{x}{2})^2}{\cancel{2}} + C$

$\int f \cdot f'\,dx = \frac{f^2}{2} + C$

d) $\int 2\cos^2\frac{x}{2}\,dx$ $\qquad$ $\int \cos u \cdot u' = \sin u + C$

$\int 2\cdot\cos^2\frac{x}{2}\cdot\left(\frac{x}{2}\right)'\cdot 2\,dx$ $\qquad$ $\left(\frac{x}{2}\right)' = \frac{1}{2}$

$(\cos u)' = -\sin u \cdot u'$

$2\cdot 2\int \cos^2\frac{x}{2}\cdot\left(\frac{x}{2}\right)' dx$ $\qquad$ $\int \sin u \cdot u' = -\cos u + C$

$= 2\int \cos\frac{x}{2}\cdot\cos\frac{x}{2}\,dx$ $\qquad$ $\sin u' = \cos u \cdot u'$

$\int \cos\frac{x}{2}\cdot\cos\frac{x}{2}\,dx = 2\int\left(\sin\frac{x}{2}\right)'\cdot\cos\left(\frac{x}{2}\right)dx$

$\underbrace{\left(\frac{1}{2}\cos\frac{x}{2}\right)}_{\left(\sin\frac{x}{2}\right)'}\cdot\cos\frac{x}{2}\cdot 2$ $\qquad$ $= 2\left[\sin\frac{x}{2}\cdot\cos\frac{x}{2} - \int \sin\frac{x}{2}\cdot\left[\left(\cos\frac{x}{2}\right)'\right]dx\right]$

$= 2\left[\sin\frac{x}{2}\cos\frac{x}{2} - \int \sin\frac{x}{2}\left(-\sin\frac{x}{2}\right)\cdot\frac{1}{2}\,dx\right.$

$= 2\left[\sin\frac{x}{2}\cos\frac{x}{2} + \frac{1}{2}\int \sin^2\frac{x}{2}\,dx\right]$

$\int \cos^2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2} + \int 1 - \cos^2\frac{x}{2}\,dx$

$\int \cos^2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2} + x - \int \frac{\cos^2 x}{2}\,dx$

$2\int \cos^2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2} + x + C = x + \sin x + C$

$\int 2\cos^2\frac{x}{2}\,dx = 2\sin\frac{x}{2}\cos\frac{x}{2} + C$

$(\cos u)' = -(\sin u)\cdot u'$

e) $\int 2\sin^2\frac{x}{2}\,dx$

$\int 2\cdot\sin\frac{x}{2}\cdot\sin\frac{x}{2}\,dx$

$\cancel{2\cdot\left[2\cdot\left(-1\cdot\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}\right)\cdot(-1)\cdot\sin\frac{x}{2}\right] = 2\int\left(\cos\frac{x}{2}\right)'\cdot\sin\left(\frac{x}{2}\right)\cdot(-1)}$

$\cancel{(\cos u)'}$ $\qquad$ $\cancel{= 2\left[-\cos\frac{x}{2}\cdot\sin\frac{x}{2} + \int\right.}$

$\int 2\sin^2\frac{x}{2}\,dx = 2\int\frac{1}{2}(1-\cos x)\,dx = \int(1-\cos x)\,dx$

$= x - \int \cos x\,dx = x - \sin x + C$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

a) $\int \frac{3x^5+x^2+x-1}{x^3}\,dx \;,\; x>0$

$= \int \left[\frac{3x^5}{x^3} + \frac{x^2}{x^3} + \frac{x}{x^3} - \frac{1}{x^3}\right] dx = \int \frac{3x^5}{x^3}dx + \int \frac{x^2}{x^3}dx + \int \frac{x}{x^3}dx - \int \frac{1}{x^3}dx$

$= 3\int x^{5-3}dx + \int x^{2-3}dx + \int x^{1-3}dx - \int x^{-3}dx$

$= 3\int x^2 dx + \int \frac{1}{x}dx + \int x^{-2}dx - \int x^{-3}dx$

$= \cancel{3}\cdot\frac{x^3}{\cancel{3}} + \ln(x) + \frac{x^{-1}}{-1} - \frac{x^{-2}}{-2} + C$

$= x^3 + \ln(x) - \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2} + C$

$= \ln(x) - \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2} + x^3 + C$

b) $\int \frac{2x^3-x^4}{\sqrt{x}}\,dx = \int 2\frac{x^3}{x^{\frac{1}{2}}}dx - \int \frac{x^4}{x^{\frac{1}{2}}}dx$

$= 2\int x^{3-\frac{1}{2}}dx - \int x^{4-\frac{1}{2}}dx = 2\cdot\int x^{\frac{5}{2}}dx - \int x^{\frac{7}{2}}dx$

$= \frac{2\cdot x^{\frac{5+2}{2}}}{\frac{5+2}{2}} - \frac{x^{\frac{7+2}{2}}}{\frac{7+2}{2}} + C$

$= 2\cdot\frac{x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{x^{\frac{9}{2}}}{\frac{9}{2}} + C = 2\cdot\frac{2}{7}\cdot x^{\frac{7}{2}} - \frac{2}{9}\cdot x^{\frac{9}{2}} + C$

$= \frac{4}{7}\cdot\sqrt[2]{x^7} - \frac{2}{9}\sqrt[2]{x^9} + C$

$= \frac{4}{7}\cdot x^3\sqrt{x} - \frac{2}{9}\cdot x^4\cdot\sqrt{x} + C$

$= x^3\sqrt{x}\left(\frac{4}{7} - \frac{2}{9}x\right) + C$

c) $\int (x\sqrt[5]{x} - \sqrt[3]{x^2} - \ln 3 \cdot 9^x)\,dx$

$\int x^r\,dx = \frac{x^{r+1}}{r+1}$

$\int a^x\,dx = \frac{a^x}{\ln a}$

$= \int x \cdot x^{\frac{1}{5}}\,dx - \int x^{\frac{2}{3}}\,dx - \ln 3 \cdot \int 9^x\,dx$

$= \int x^{\frac{6}{5}}\,dx - \int x^{\frac{2}{3}}\,dx - \ln 3 \cdot \frac{9^x}{\ln 3 + \ln 3} + C$

$= \frac{x^{\frac{6}{5}+1}}{\frac{6}{5}+1} - \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} - \cancel{\ln 3} \cdot \frac{9^x}{2\cancel{\ln 3}} + C$

$= x^{\frac{11}{5}} \cdot \frac{5}{11} - x^{\frac{5}{3}} \cdot \frac{3}{5} - \frac{9^x}{2} + C$

$= \frac{5}{11} \cdot \sqrt[5]{x^{11}} - \frac{3}{5} \cdot \sqrt[3]{x^5} - \frac{1}{2} \cdot 9^x + C$

$= \frac{5}{11} \cdot x\sqrt[5]{x^2} - \frac{3}{5} \cdot x\sqrt[3]{x^2} - \frac{1}{2} 9^x + C$

d) $\int \frac{x\sqrt[3]{x} + 2x^2\sqrt[4]{x^2}}{\sqrt{x}}\,dx$

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$

$= \int \frac{x\sqrt[3]{x}}{\sqrt{x}}\,dx + \int \frac{2x^2\sqrt[4]{x^2}}{\sqrt{x}}\,dx$

$= \int \frac{x \cdot x^{\frac{1}{3}}}{x^{\frac{1}{2}}}\,dx + 2 \cdot \int \frac{x^2 \cdot x^{\frac{2}{4}}}{x^{\frac{1}{2}}}\,dx$

$= \int x^{1+\frac{1}{3}-\frac{1}{2}}\,dx + 2\int x^{2+\frac{2}{4}-\frac{1}{2}}\,dx = \int x^{\frac{6+2-3}{6}}\,dx + 2\int x^{\frac{8+2-2}{4}}\,dx$

$= \int x^{\frac{5}{6}}\,dx + 2\int x^2\,dx = \frac{x^{\frac{5+6}{6}}}{\frac{5+6}{6}} + 2 \cdot \frac{x^3}{3} + C$

$= x^{\frac{11}{6}} \cdot \frac{6}{11} + \frac{2}{3}x^3 + C = \frac{6}{11}\sqrt[6]{x^{11}} + \frac{2}{3}x^3 + C$

$= \frac{6}{11}\,x\sqrt[6]{x^5} + \frac{2}{3}x^3 + C$

e) $\int [(2^x \cdot \ln \sqrt[3]{4}) - \ln 3 \cdot 9^x] dx$

$\int a^x dx = \frac{a^x}{\ln a}$

$= \ln^3 \sqrt{4} \int 2^x dx - \ln 3 \cdot \int 9^x dx$

$= \ln \sqrt[3]{4} \cdot \frac{2^x}{\ln 2} - \cancel{\ln 3} \cdot \frac{9^x}{2\cancel{\ln 3}} + C$

$= \ln \sqrt[3]{4} \cdot \frac{2^x}{\ln 2} - \frac{1}{2} \cdot 9^x + C$

f) $\int \left(\frac{1}{3+x^2} - \frac{1}{\sqrt{3+x^2}} \right) dx$

$\int \frac{1}{a^2+x^2} = \frac{1}{2} \cdot \operatorname{arctg} \frac{x}{a} + C$

$\int \frac{1}{\sqrt{a^2+x^2}} = \ln(x + \sqrt{x^2+a^2}) + C$

$= \int \frac{1}{(\sqrt{3})^2 + x^2} dx - \int \frac{1}{\sqrt{(\sqrt{3})^2 + x^2}} dx$

$= \frac{1}{2} \cdot \operatorname{arctg} \frac{x}{\sqrt{3}} - \ln(x + \sqrt{x^2+3}) + C$

g) $\int \frac{(x-1)^4}{x^2} dx = \int \frac{(x-1)^2 (x-1)^2}{x^2} dx = \int \frac{(x^2-2x+1)(x^2-2x+1)}{x^2}$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

$= \int \frac{x^4 - 2x^3 + x^2 - 2x^3 + 4x^2 - 2x + x^2 - 2x + 1}{x^2} dx$

$= \int \frac{x^4 - 4x^3 + 6x^2 - 4x + 1}{x^2} dx = \int \frac{x^4}{x^2} dx - \int 4 \frac{x^3}{x^2} dx + 6 \int \frac{x^2}{x^2} dx - 4 \int \frac{x}{x^2}$

$+ \int \frac{1}{x^2} dx = \int x^2 dx - \int 4x dx + 6 \int dx - 4 \int \frac{1}{x} dx + \int x^{-2} dx$

$= \frac{x^3}{3} - 4 \frac{x^2}{2} + 6x - 4 \ln(x) + \frac{x^{-1}}{-1} + C$

$= \frac{1}{3} x^3 - 2x^2 + 6x - 4 \ln(x) - \frac{1}{x} + C$

h) $\int \frac{\sqrt{x^2+4} - 1}{x^2+4} dx = \int \frac{\sqrt{x^2+4}}{x^2+4} dx - \int \frac{1}{x^2+4}$

$\int \frac{1}{\sqrt{x^2+4}} = \ln(x + \sqrt{x^2+a^2}) + C$

$\int \frac{1}{x^2+4} dx = \frac{1}{2} \cdot \operatorname{arctg} \frac{x}{a}$

$$= \int \frac{1}{\sqrt{x^2+4}} dx - \int \frac{1}{x^2+2^2}$$

$$= \ln\left(x+\sqrt{x^2+a^2}\right) - \frac{1}{2} \cdot \operatorname{arctg} \frac{x}{2} + C$$

i) $$\int \frac{\sqrt{x^2-4}+4}{x^2-4} dx = \int \left(\frac{\sqrt{x^2-4}}{x^2-4} + \frac{4}{x^2-4} \right) dx$$

$$\int \frac{1}{\sqrt{x^2-4}} = \ln\left|x-\sqrt{x^2-a^2}\right| + C$$

$$\int \frac{1}{x^2-a^2} = \frac{1}{2a} \cdot \ln\left|\frac{x-a}{x+a}\right| + C$$

$$= \int \frac{1}{\sqrt{x^2-4}} dx + 4\int \frac{1}{x^2-4} dx = \ln\left|x-\sqrt{x^2-4}\right| + \frac{4}{4} \cdot \ln\left|\frac{x-2}{x+2}\right| + C$$

$$= \ln\left|x-\sqrt{x^2-4}\right| + \ln\left|\frac{x-2}{x+2}\right| + C$$

j) $$\int \frac{\sqrt{2-x^2}+\sqrt{x^2+2}}{\sqrt{4-x^4}} dx = \int \frac{\sqrt{2-x^2}}{\sqrt{4-x^4}} dx + \int \frac{\sqrt{x^2+2}}{\sqrt{4-x^4}} dx$$

$$= \int \sqrt{\frac{2-x^2}{4-x^4}} dx + \int \sqrt{\frac{x^2+2}{4-x^4}} dx = \int \sqrt{\frac{\cancel{(2-x^2)}}{\cancel{(2-x^2)}(2+x^2)}} dx + \int \sqrt{\frac{\cancel{(x^2+2)}}{(2-x^2)\cancel{(2+x^2)}}}$$

$$= \int \frac{1}{\sqrt{x^2+2}} dx + \int \frac{1}{\sqrt{2-x^2}} dx$$

$$\int \frac{1}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C$$

$$= \ln\left(x+\sqrt{x^2+2}\right) + \arcsin \frac{x}{\sqrt{2}} + C$$

k) $$\int \frac{2x+1}{\sqrt{x^2-16}} dx = 2\int \frac{x}{\sqrt{x^2-16}} dx + \int \frac{1}{\sqrt{x^2-16}} dx$$

$$\int \frac{x}{\sqrt{x^2-a^2}} dx = \sqrt{x^2-a^2} + C \qquad \int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left|x-\sqrt{x^2-a^2}\right| + C$$

$= 2\cdot\sqrt{x^2-16} + \ln|x-\sqrt{x^2-16}| + C$

A2/191

a) $\int \frac{1}{\sin^2 x\cdot\cos^2 x}\,dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cdot\cos^2 x}\,dx$

$= \int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}\cdot\cos^2 x}\,dx + \int \frac{\cancel{\cos^2 x}}{\sin^2 x\cdot\cancel{\cos^2 x}}\,dx$

$= \int \frac{1}{\cos^2 x}\,dx + \int \frac{1}{\sin^2 x}\,dx$

$= \int tg'(x)\,dx + \int -(ctg\,x)'\,dx = tg\,x - ctg\,x + C$

$\cos 2x = \cos^2 x - \sin^2 x$

b) $\int \frac{\cos 2x}{\cos^2 x\cdot\sin^2 x}\,dx$

$= \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x\cdot\sin^2 x}\,dx = \int \frac{\cancel{\cos^2 x}}{\cancel{\cos^2 x}\cdot\sin^2 x}\,dx - \int \frac{\cancel{\sin^2 x}}{\cos^2 x\cdot\cancel{\sin^2 x}}\,dx$

$= \int \frac{1}{\sin^2 x}\,dx - \int \frac{1}{\cos^2 x}\,dx$

$= \int -(ctg\,x)'\,dx - \int tg(x)'\,dx$

$= -ctg(x) - tg(x) + C$

$\sin 2x = 2\sin x\cdot\cos x$

c) $\int \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2 dx = \int \left(\sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cdot\cos\frac{x}{2} + \cos^2\frac{x}{2}\right)dx$

$= \int 1 - 2\sin\frac{x}{2}\cos\frac{x}{2} = \int 1 - 2\sin 2\cdot\frac{x}{2} = \int(1-\sin x)\,dx$

$= \int dx - \int \sin x\,dx = x + \cos x + C$

d) $\int \frac{\sin^3 x - 8}{1-\cos^2 x}\,dx = \int \frac{\sin^3 x - 8}{\sin^2 x}\,dx = \int \frac{\sin^3 x}{\sin^2 x}\,dx - \int \frac{8}{\sin^2 x}\,dx$

$\int \sin x\,dx + 8\int -\frac{1}{\sin^2 x}\,dx = -\cos x + 8\cdot ctg\,x$

$(ctg\,x)' = \frac{-1}{\sin x^2}$

e) $\int \frac{3\cos 2x+1}{\sin^2 2x}\,dx$

$\cos 2x = \cos^2 a - \sin^2 a$

$\sin 2a = 2\cos a \sin a$

$\sin^2 2a = 4\cos^2 a\cdot\sin^2 a$

$= \int \frac{3(\cos^2 a - \sin^2 a)+1}{4\cos^2 a\cdot\sin^2 a}\,dx$

$= \int \frac{3\cos^2 x - 3\sin^2 a + 1}{4\cos^2 a\cdot\sin^2 a}\,dx = \frac{3}{4}\int\frac{\cos^2 a}{\cos^2 a\cdot\sin^2 a}\,dx - \frac{3}{4}\int\frac{\sin^2 a}{\cos^2 a\cdot\sin}$

$+\int\frac{1}{4\cos^2\cdot\sin^2 a}$

$= \frac{3}{4}\int\frac{1}{\sin^2 a}\,dx - \frac{3}{4}\int\frac{1}{\cos^2 a}\,dx + \frac{1}{4}\int\frac{1}{\cos^2 x\cdot\sin^2 x}\,dx$

$= \frac{3}{4}\int(-1)\cdot ctg(x)'\,dx - \frac{3}{4}\int tgx'\,dx + \frac{1}{2}\int\frac{1}{2\cos x\cdot\sin x}\cdot\frac{1}{\cos x\cdot\sin x}$

$= \frac{-3}{4}ctg x - \frac{3}{4}tg x + \int\frac{1}{\sin^2 2x}\,dx$

$= -\frac{3}{4}ctg x - \frac{3}{4}tg x - \frac{1}{2}ctg 2x + C$

f) $\int(1+tg^2 x)dx = \int dx + \int tg x\cdot tg x\,dx = x + \int tg x\cdot tg x\,dx$

$1+tg^2 x = \sec^2 x$

$1+ctg^2 x = cossec^2$

$\int tg x\, tg x = -x + tg x + C$

$\int(1+tg^2 x)\,dx = \cancel{x}\cancel{-x} + tg x + C = tg x + C$

g) $\int(1+ctg^2 x)\,dx = \int cosec^2 x\,dx = -ctg(x) + C$

$\boxed{A3/191}$

$I_1 = \int\frac{x^4+x^2+1}{x^2-x+1}\,dx = \frac{x(x^4+x^2+1)}{x^2-x+1} - \frac{2x^3}{3} - \frac{x^2}{2} + C$

$(x^2-x+1)' = 2x-1$

$(x^4+x^2+1)' = 4x^3+2x$

$I_1 = \int\frac{(x^2-x+1)+x+x^4}{x^2-x+1}\,dx = \int\frac{x^2-x+1}{x^2-x+1}\,dx + \int\frac{x+x^4}{x^2-x+1}\,dx$

$$\int 1\,dx + \int \frac{x^4+x}{x^2-x+1}\,dx = x + \int \frac{(x^2-x+1)' + 1 - x + x^4}{x^2-x+1}$$

$$= x + \int \frac{(x^2-x+1)'}{(x^2-x+1)}\,dx + \int \frac{x^4-x}{x^2-x+1}\,dx$$

$$= x + \ln(x^2-x+1) + \int \frac{x^4}{x^2-x+1} - \int \frac{x}{x^2-x+1}$$

$$\int \left(\frac{x^4+x^2+1}{x^2-x+1} + 1 - 1\right) dx = \int \left(\frac{x^4 \cancel{+x^2} \cancel{+1} \cancel{-x^2} + x \cancel{-1}}{x^2-x+1} + 1\right) dx$$

$$\int \left(\frac{x^4+x}{x^2-x+1}\right) dx + \int dx \;\ldots\ldots$$

A4/191

a) $\int 6x(3x^2+1)^7 dx$ $\qquad$ $\int u^n \cdot u'\,du = \frac{u^{n+1}}{n+1} + C$

$= \int (3x^2+1)' \cdot (3x^2+1)^7 dx = \frac{(3x^2+1)^8}{8} + C$ $\qquad$ $(x^n)' = n \cdot x^{n-1}$

b) $\int x^4 (1-x^5)^5 dx$

$(1-x^5)' = -5x^4$ $\Big\}$ $\int \frac{1}{-5} \cdot (1-x^5)' \cdot (1-x^5)^5 dx$

$= \frac{-1}{5} \cdot \frac{(1-x^5)^6}{6} + C = \frac{-1}{30}(1-x^5)^6 + C$ $\qquad$ $\int u^n u'\,du = \frac{u^{n+1}}{n+1}$

$(x^n)' = n \cdot x^{n-1}$

c) $\int x^4 \cdot \sqrt[3]{x^5+1}\,dx = \int x^4 \cdot (x^5+1)^{\frac{1}{3}} dx = \int \frac{1}{5} \cdot (x^5+1)' \cdot (x^5+1)^{\frac{1}{3}} dx$

$= \frac{1}{5} \cdot \frac{(x^5+1)^{\frac{1}{3}+\frac{3}{3}}}{\frac{4}{3}} + C = \frac{1}{5} \cdot \frac{3}{4} \sqrt[3]{(x^5+1)^4} + C = \frac{3}{20}(x^5+1)\sqrt[3]{(x^5+1)} + C$

d) $\int \frac{3x^2}{\sqrt{x^3+1}}\,dx = \int \frac{(x^3+1)'}{(x^3+1)^{\frac{1}{2}}}\,dx = \int (x^3+1)' \cdot (x^3+1)^{-\frac{1}{2}} dx$

$= \frac{(x^3+1)^{-\frac{1}{2}+\frac{2}{2}}}{-\frac{1}{2}+\frac{2}{2}} + C = \frac{\sqrt[2]{x^3+1}}{\frac{1}{2}} + C = 2\sqrt{x^3+1} + C$

e) $\int \frac{1}{x} \ln^4 x \, dx = \int (\ln x)' \cdot \ln^4 x \, dx$

$= \frac{\ln^5(x)}{5} + C$

$\int u' \cdot u^n \, du = \frac{u^{n+1}}{n+1} + C$

$\int \frac{f'}{f} df = \ln|f| + C$

f) $\int \frac{2x-5}{x^2-5x+7} dx = \int \frac{(x^2-5x+7)'}{x^2-5x+7} dx = \ln|x^2-5x+7| + C$

g) $\int \frac{x-1}{3x^2-6x+11} dx = \int \frac{1}{6} \frac{(3x^2-6x+11)'}{3x^2-6x+11} dx = \frac{1}{6} \ln|3x^2-6x+11| + C$

$(3x^2-6x+11)' = 6x-6 = 6(x-1)$

h) $\int \frac{2x}{x^4-1} dx = \int \frac{2x}{(x^2-1)(x^2+1)} = 2\int \frac{x}{(x^2)^2-1^2} dx$

$(x^4-1)'' = (4x^3)' = 12x^2$

$(x^2-1) = 2x$

$= 2\int \frac{1}{2(x^4-1)} = 2 \cdot \frac{1}{2} \int \frac{1}{x^4-1} = \frac{2}{2} \int \frac{1}{(-x^4+1)} = \frac{2}{2}\left(-\int \frac{1}{1-x^4}\right)$

$\frac{2}{2}\left(-\int \frac{1}{1-x^4} dx\right) = 2 \cdot \frac{1}{2}\left(-\left(\frac{\ln|x^2+1|}{2} - \frac{\ln|x^2-1|}{2}\right)\right)$

$= -\frac{1}{2} \ln|x^2+1| + \frac{1}{2} \ln|x^2-1| + C$

i) $\cancel{\int \frac{x^2}{16-x^6} dx = \int \frac{x^2}{4^2-(x^3)^3} dx = \int -\frac{1}{3} \cdot \frac{(4^2-(x^3)^3)'}{4^2-(x^3)^3}}$

$\cancel{u = x^2 \quad u' = 2x}$

$\cancel{(4^2-u^3)' = -3u^2}$

i)? $\int \frac{x^2}{16-x^6} dx$

$\int \frac{u'}{u} = f|u| + C$

j) $\int \frac{x}{x^2+9} dx = \int \frac{(x^2+9)'}{x^2+9} \cdot \frac{1}{2} dx = \frac{1}{2} \ln|x^2+9| + C$

A5/192

a) $\int \frac{\text{arctg}^6 x}{1+x^2} dx = \int \text{arctg}^6 x \cdot \text{arctg}' x \, dx = \frac{\text{arctg}^7 x}{7} + C$

b) $\int \frac{\cos x}{\sin^2 x - 4} dx = -\frac{1}{4}\left(\ln\left|\frac{\sin x}{2}+1\right| - \ln\left|\frac{\sin x}{2}-1\right|\right) + C$

c) $\int \frac{\sin x}{9-\cos^2 x}\,dx = -\frac{1}{6}\left(\ln\left|\frac{\cos x}{3}+1\right| - \ln\left|\frac{\cos x}{3}-1\right|\right)+C$

d) $\int \frac{\cos x}{\sin^2 x+4}\,dx = \frac{1}{2}\cdot \operatorname{arctg}\left(\frac{\sin x}{2}\right)+C$

e) $\int 2x \sin(x^2+1)\cos(x^2+1)\,dx$

$(\sin u)' = \cos u \cdot u'$

$= \int \sin(x^2+1) \cdot \underbrace{\cos(x^2+1)\cdot(x^2+1)'}_{\sin(x^2+1)'}\,dx = \frac{1}{2}\sin^2(x^2+1)+C$

f) $\int 4x \sin 2(x^2+1)\,dx$

$\sin 2u = 2\sin u \cdot \cos u$

$(x^2+1)' = 2x$

$= \int 4\cdot x \cdot 2\cdot \sin(x^2+1)\cdot\cos(x^2+1)\,dx$

$= 4\int 2x\cdot\sin(x^2+1)\cdot\cos(x^2+1)\,dx = 4\int \sin(x^2+1)\cdot \underbrace{2x\cdot\cos(x^2+1)}_{(\sin u)'}\,dx$

$= 4\cdot \frac{\sin^2(x^2+1)}{2}+C = 2\cdot\sin^2(x^2+1)+C$

$tg'x = \frac{1}{\cos^2 x}$

g) $\int (tg^3 x + tg x)\,dx$

$= -\ln|\sec(x)| + \frac{\sec^2(x)}{2} - \ln|\cos x| + C$

$\int tg^3(x)\,dx = -\ln|\sec(x)| + \frac{\sec^2 x}{2}$

$\int tg(x) = -\ln|\cos x| + C$

$1+tg^2 x = \sec^2 x$

$(\sin x \sin x)' = \sin x' \sin x + \sin x \cdot \sin x'$

$= 2\sin x \cdot \sin x'$

$= 2\sin x \cos x$

$\sin 2x = 2\sin x \cos x$

A5 h, i

j) $\int \sin^3 x \cdot \cos^2 x\,dx$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

Integrarea prin parti $\int f \cdot g' dx = f \cdot g - \int f' \cdot g \, dx$

A6 / 192

a) $\int x^2 \cdot \ln x \, dx = \ln x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$

$f = \ln x$

$g' = x^2 \Rightarrow g = \int x^2 = \frac{x^3}{3}$

$= \ln x \cdot \frac{x^3}{3} - \frac{1}{3} \int x^2 dx = \frac{1}{3} x^3 \cdot \ln x - \frac{1}{3} \frac{x^3}{3} + C$

$= \frac{1}{3} x^3 \ln(x) - \frac{1}{9} \cdot x^3 + C$

b) $\int x \cdot e^{-x} dx = x \cdot (-e^{-x}) - \int 1 \cdot (-e^{-x}) dx$

$= -x \cdot e^{-x} + \int e^{-x}$

$(e^{-x})' = -e^{-x}$

$x' = 1$

$= -xe^{-x} - e^{-x} + C$

$\int x = \frac{x^2}{2} + C$

$= e^{-x}(-x-1) + C$

$\int e^{-x} = -e^{-x} + C$

$f = x \rightarrow f' = 1$

$g' = e^{-x} \rightarrow g = -e^{-x}$

c) $\int \sin^2 x \, dx = \int 1 - \cos^2 x \, dx = \int \frac{1 - \cos(2x)}{2} dx = \frac{1}{2} \int (1 - \cos(2x)) dx$

$= \int \sin^2 x \cdot x' dx = \sin^2 x \cdot x - \int (\sin^2 x)' \cdot x \, dx$

$(\sin x \cdot \sin x)' = \sin x \cdot \cos x + \sin x \cos x = 2 \sin x \cdot \cos x = \sin 2x$

$(\cos x \cdot \cos x)' = \cos x (-\sin x) + \cos x \cdot (-\sin x) = -2 \sin x \cos x$

$\cos 2a = \cos^2 a - \sin^2 a$

$= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x\right) + C$

f) $\int x \sqrt{x^2 - 9} \, dx = \int \frac{1}{2} (x^2-9)' \cdot (x^2-9)^{\frac{1}{2}} dx = \frac{1}{2} \cdot \frac{(x^2-9)^{\frac{1}{2}+1}}{\frac{1}{2}+1}$

$= \frac{1}{2} \frac{(x^2-9)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{1}{2} \cdot \frac{2}{3} (x^2-9)^{\frac{3}{2}} = \frac{1}{3} \sqrt[2]{(x^2-9)^3} = \frac{(x^2-9)}{3} \sqrt{x^2-9}$

h) $\int x \cdot \operatorname{arctg} x \, dx = \frac{x^2}{2} \operatorname{arctg} x - \int \frac{x^2}{2} \cdot \frac{1}{x^2+1} dx$

$= \frac{x^2}{2} \cdot \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2}{x^2+1} dx$

$= \frac{x^2}{2} \cdot \operatorname{arctg} x - \frac{1}{2} \left[\int \frac{x^2+1}{x^2+1} dx - \int \frac{1}{x^2+1} \right] dx$

$= \frac{x^2}{2} \cdot \operatorname{arctg} x - \frac{1}{2} \left(x - \operatorname{arctg} x \right) + C$

$= \frac{x^2}{2} \operatorname{arctg} x - \frac{x}{2} + \frac{1}{2} \operatorname{arctg} x + C$

Teste de evaluare

Testul 1
pagina 192

1/192

$f(x) = e^x \cdot \sin x$

$g(x) = e^x \cos x$

$\sin x' = \cos x$

$\cos x' = -\sin x$

a) $f + g = e^x (\sin x + \cos x)$

$f'(x) = e^{x'} \cdot \sin x + e^x \cdot (\sin x)'$

$f'(x) = e^x \sin x + e^x \cdot \cos x = f + g \Rightarrow f$ primitivă a lui $f+g$

b) $g - f = e^x \cos x - e^x \cdot \sin x$

$= e^x (\cos x - \sin x)$

$g'(x) = e^x \cdot \cos x + e^x (\cos x)' = e^x \cos x - e^x \sin(x)$

$= g - f \Rightarrow$ primitiva a lui $g - f$

2/192 $f(x) = (x^2 - 1) \ln x$

$F(x) = x(ax^2 - 1) \cdot \ln x - x\left(\frac{x^2}{9} - b\right)$

F primitiva $\Leftrightarrow F'(x) = f(x)$

$F(x) = (ax^3 - x) \cdot \ln x - \left(\frac{x^3}{9} - xb\right)$

$F'(x) = (ax^3 - x)' \cdot \ln x + (ax^3 - x) \cdot \ln x' - \left(\frac{1}{9} x^{3\,'} - (xb)'\right)$

$F'(x) = (3ax^2 - 1) \ln x + \frac{ax^3 - x}{x} - \left(\frac{3}{9} x^2 - b\right)$

$F'(x) = (3ax^2 - 1) \ln x + \left[\frac{ax^3 - x}{x} - \frac{1}{3} x^2 + b\right]$

$\int f(x)\,dx = \int \underbrace{(x^2-1)}_{g'} \cdot \underbrace{\ln x}_{f}\, dx = \ln x\left(\frac{x^3}{3}-x\right) - \int\left(\frac{x^3}{3}-x\right)\cdot\frac{1}{x}\,dx$

$\int x^2-1 = \frac{x^3}{3}-x$

$= \ln x\left(\frac{x^3-3x}{3}\right) - \int\left(\frac{x^3-3x}{3}\right)\cdot\frac{1}{x}\,dx$

$= \ln x\left(\frac{x^3-3x}{3}\right) - \int \frac{\cancel{x}(x^2-3)}{3}\cdot\frac{1}{\cancel{x}}\,dx$

$= \ln x\left(\frac{x^3-3x}{3}\right) - \frac{1}{3}\int (x^2-3)\,dx$

$= \ln x\left(\frac{x^3-3x}{3}\right) - \frac{1}{3}\left[\frac{x^3}{3} - 3x\right] + C$

$= \ln x\left(\frac{x^3-3x}{3}\right) - \frac{1}{3}\,\frac{x^3}{3} + \frac{1}{3}\,3x$

$= \ln x\left(\frac{x^3-3x}{3}\right) - \frac{x^3}{9} + x$

$F(x) = \underbrace{(ax^3-x)\cdot\ln x}_{} - \underbrace{\left(\frac{x^3}{9} - xb\right)}_{}$

$\underbrace{\ln x\left(\frac{x^3-3x}{3}\right)}_{} - \underbrace{\left(\frac{x^3}{9}+x\right)}_{}$

$\ln(x)\cdot\left(\frac{x^3}{3} - x\right) - \left(\frac{x^3}{9} - (-1)\cdot x\right)$

$ax^3 - x = \frac{x^3}{3} - x \;/ + x$

$ax^3 = \frac{x^3}{3} \rightarrow a = \frac{1}{3}$

$\frac{x^3}{9} - (-1)x = \frac{x^3}{9} - xb \;\Big|\; -\frac{x^3}{9}$

$-(-1)x = -b\cdot x \Rightarrow b = -1$

$\Bigg\}\; a = \frac{1}{3} \quad b = -1$

x	-3	0
$2x+6$	$- - - - - - - - 0 + + + + + +$	

$\sqrt{x^2+6x+9}$ pentru $x=-3$ $\}$ $\sqrt{9-18+9}=0$

$\hookrightarrow$ în $x=-3$ nu se poate $\dfrac{2x+6}{\sqrt{x^2+6x+9}}$

2/193

$f:(1,\infty)\to\mathbb{R},\ f(x)=\ln(1+\ln x)$

$f'(x)=\ln(1+\ln x)'=\dfrac{1}{1+\ln x}\cdot(1+\ln x)'=\dfrac{1}{1+\ln x}\cdot\dfrac{1}{x}$

$f'(x)=\dfrac{1}{x(1+\ln(x))}=g(x)\Rightarrow f$ este primitiva lui g

3/193

a) $\displaystyle\int\frac{x+2}{\sqrt{x^2+1}}dx=\int\frac{x}{\sqrt{x^2+1}}dx+2\int\frac{1}{\sqrt{x^2+1}}dx$

$\displaystyle=\frac{1}{2}\cdot\int(x^2+1)'\cdot(x^2+1)^{-\frac{1}{2}}dx+2\cdot\ln(x+\sqrt{x^2+1})+C$

$\displaystyle=\frac{1}{2}\cdot\frac{(x^2+1)^{-\frac{1}{2}+\frac{2}{2}}}{-\frac{1}{2}+\frac{2}{2}}+2\ln(x+\sqrt{x^2+1})+C$

$\displaystyle=\frac{1}{2}\cdot\frac{\sqrt[2]{x^2+1}}{\frac{1}{2}}+2\ln(x+\sqrt{x^2+1})+C$

$=\frac{1}{\cancel{2}}\cdot\cancel{2}\sqrt[2]{x^2+1}+2\ln(x+\sqrt{x^2+1})+C$

$=\sqrt[2]{x^2+1}+2\ln(x+\sqrt{x^2+1})+\mathcal{C}$

b) $\displaystyle\int(x+2)e^x\,dx=(x+2)\cdot e^x-\int1\cdot e^x\,dx=(x+2)e^x-e^x$

$\displaystyle\int(x+2)e^x\,dx=e^x(x+1)+C$

c) $\displaystyle\int\sin x\cdot\cos x\,dx=\int\frac{1}{2}\cdot\sin 2x=\int\sin x\cdot(\sin x)'=\frac{\sin^2 x}{2}+C$