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Matricea de Trecere de la o bază la alta. Formula de schimbare a coordonatelor unui vector la schimbarea bazei

Fie V un spațiu vectorial și 2 baze ale sale, din V = n vectori

$$B_1 = \{e_1, e_2, \dots, e_n\}$$

$$B_2 = \{f_1, f_2, \dots, f_n\}$$

Matricea de trecere de la baza B_1 la B_2 este o matrice $T = (t_{ij})_{i,j}$, unde

$$f_i = \sum_{j=1}^{n} t_{ij} e_j \quad , \forall i = \overline{1,n}$$

Ex. Considerăm bazele din $\mathbb{R}^2$,

$$B_1 = \{v_1, v_2\} \quad , v_1 = (1,2) \quad , v_2 = (1,-1)$$

$$B_2 = \{u_1, u_2\} \quad , u_1 = (2,1) \quad , u = (0,1)$$

Elementele matricii T, $B_1 \xrightarrow{T} B_2$, se găsesc rezolvând

$$u_i = \sum_{j=1}^{2} t_{ji} v_j \quad , i = \overline{1,2}$$

$i=1 \Rightarrow u_1 = \sum_{j=1}^{2} t_{j1} v_j = t_{11} v_1 + t_{21} v_2$

$(2,1) = t_{11} \cdot (1,2) + t_{21} \cdot (1,-1)$

$(2,1) = (t_{11}, 2t_{11}) + (t_{21}, -t_{21})$

$(2,1) = (t_{11} + t_{21}, 2t_{11} - t_{21})$

$$\begin{cases} t_{11} + t_{21} = 2 & \rightarrow t_{11} = 1 \\ 2t_{11} - t_{21} = 1 & \rightarrow t_{21} = 1 \end{cases}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \quad x = \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$i=2 \rightarrow u_2 = \sum_{j=1}^{2} t_{j2} v_j = t_{12} v_1 + t_{21} v_2$

$(0,1) = t_{12} \cdot (1,2) + t_{22} \cdot (1,-1)$

$(0,1) = (t_{12}, 2t_{12}) + (t_{22}, -t_{22})$

$(0,1) = (t_{12} + t_{22}, 2t_{12} - t_{22})$

$$\begin{cases} t_{12} + t_{22} = 0 & t_{12} = \frac{1}{3} \\ 2t_{12} - t_{22} = 1 & t_{22} = -\frac{1}{3} \end{cases}$$

$$B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \quad x = \begin{bmatrix} t_{12} \\ t_{22} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 1/3 \\ 1 & -1/3 \end{bmatrix}$$

Varianta 2

$$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} -3 & 0 & -3 & -1 \\ 0 & -3 & -3 & 1 \end{array}\right]$$

$$\begin{array}{cccc} v_1 & v_2 & u_1 & u_2 \end{array}$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1/3 \\ 0 & 1 & 1 & -1/3 \end{array}\right]$$

T

OBS: $B_1 \xrightarrow{T} B_2$

$B_2 \xrightarrow{S} B_1$

$S = T^{-1}$

Gasim inversa

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{3} & 1 & 0 \\ 1 & -\frac{1}{3} & 0 & 1 \end{array}\right] \simeq \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 3/2 & -3/2 \end{array}\right] \qquad T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix}$$

S

Varianta 2

$B_2 \xrightarrow{S} B_1$

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 1 & 1 & 2 & -1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 3/2 & -3/2 \end{array}\right]$$

S

Exemplu 2 $\quad V = \mathbb{R}^3$, $B = \{e_1, e_2, e_3\}$, $B' = \{f_1, f_2, f_3\}$

$e_1 = (1,0,1)$ $\quad e_2 = (2,1,3)$ $\quad e_3 = (1,1,1)$

$f_1 = (1,-1,2)$ $\quad f_2 = (2,0,1)$ $\quad f_3 = (1,1,0)$

Să se scrie matricea de trecere T de $B \xrightarrow{T} B'$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 1 & 3 & 1 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 3 & 1 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{matrix}}_{T}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ +1 & -1 & -1 & 0 & 1 & 0 \\ -2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{5}{2} & 1 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{7}{2} & 2 \end{array}\right]$$

$$\underbrace{\begin{matrix} \frac{1}{2} & \frac{5}{2} & 1 \\ 0 & -2 & -1 \\ \frac{1}{2} & \frac{7}{2} & 2 \end{matrix}}_{S=T^{-1}}$$

Fie $x \in V$ si 2 Baze $B_1 = \{e_1, \dots, e_n\}$, $B_2 = \{f_1, \dots, f_n\}$ ale V

Fie $x = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n$

$x = \beta_1 f_1 + \dots + \beta_n f_n$

T matricea de trecere de la baza B_1 la B_2.
Relatiile de schimbare a coordonatelor sunt:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = T^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Exemplu În $\mathbb{R}^3$ consideram Baza $\{v_1, v_2, v_3\} = B$

$v_1 = (2, 1, 1)$ $v_2 = (-1, 4, -2)$ $v_3 = (-1, -2, 2)$

Aflati coordonatele vectorului $u = (2, 10, 10)$ în baza B

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 1 & 4 & -2 & 10 \\ 1 & -2 & 2 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$\alpha_1 = 6$

$\alpha_2 = 4$

$\alpha_3 = 6$

$u = 6v_1 + 4v_2 + 6v_3$

$\mathcal{B}_c = \{e_1, e_2, e_3\}$ $\qquad$ $\mathcal{B} = \{v_1, v_2, v_3\}$

$\uparrow$

Canonică

$$u = (2,10,10) = 2 \cdot \overset{e_1}{(1,0,0)} + 10 \cdot \overset{e_2}{(0,1,0)} + 10 \cdot \overset{e_3}{(0,0,1)}$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = T^{-1} \cdot \begin{pmatrix} 2 \\ 10 \\ 10 \end{pmatrix} \qquad \mathcal{B}_c \xrightarrow{T^{-1}} \mathcal{B}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 4 & -2 \\ 0 & 0 & 1 & 1 & -2 & 2 \end{array}\right) \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 2 & -1 & -1 \\ 1 & 4 & -2 \\ 1 & -2 & 2 \end{matrix}}_{T} \\ 0 & 0 & 1 & \end{array}\right]$$

$$T^{-1} = \begin{bmatrix} 2/9 & 2/9 & 1/3 \\ -2/9 & 5/18 & 1/6 \\ -1/3 & 1/6 & 1/2 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 2/9 & 2/9 & 1/3 \\ -2/9 & 5/18 & 1/6 \\ -1/3 & 1/6 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 10 \\ 10 \end{pmatrix} = \begin{pmatrix} \frac{4}{9} + \frac{20}{9} + \frac{10}{3} \\ -\frac{4}{9} + \frac{50}{18} + \frac{10}{6} \\ -\frac{2}{3} + \frac{10}{6} + \frac{10}{2} \end{pmatrix} = \begin{matrix} \beta_1 = 6 \\ \beta_2 = 4 \\ \beta_3 = 6 \end{matrix}$$