22.01.2022

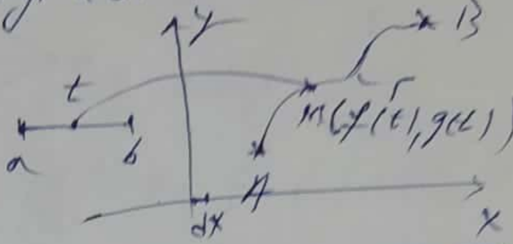
Pregătire examen (AM II)

① Integrale curbilinii de speța 1 și 2.

– extensii naturale ale integralei Riemann

în $\mathbb{R}^2$

$\Gamma: \begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad t \in [a, b].$



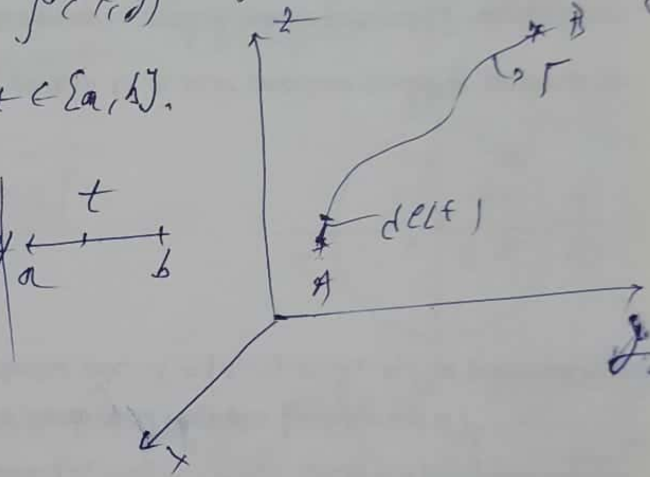
Integrala curbilinie de speța 1.

$\int_\Gamma f(x,y) \cdot dl(t)$ $\qquad dl = \sqrt{x'^2 + y'^2} \cdot dt = \sqrt{f'^2(t) + g'^2(t)}\, dt$

$\int_\Gamma dl(t) =$ lungimea curbei Γ.

$m(\Gamma) = \int_\Gamma \rho(x,y) \cdot dl(t)$; $\rho(x,y) =$ densitatea materială

$\Gamma \subset \mathbb{R}^3: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad t \in [a, b].$



$dl(t) = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\, dt$

$\int_\Gamma dl(t) = l(\Gamma)$

$m(\Gamma) = \int_\Gamma \rho(x,y,z) \cdot dl(t)$

Aplicații

① $\int_\Gamma y \cdot e^{-t} \cdot dl$; $\quad \Gamma: \begin{cases} x = \ln(1+t^2) \\ y = 2\,\mathrm{arctg}\, t - t + 1 \end{cases} \quad t \in [0, 1].$

$I = \int_\Gamma f(x,y) \cdot dl$; $\quad dl(t) = \sqrt{x'^2(t) + y'^2(t)} \cdot dt$

$x'(t) = \frac{2t}{1+t^2}$; $\quad y'(t) = \frac{2}{1+t^2} - 1$

$\sqrt{x'^2(t) + y'^2(t)} = \sqrt{\frac{4t^2}{(1+t^2)^2} + \frac{4}{(1+t^2)^2} - \frac{4}{1+t^2} + 1} =$

$= \sqrt{\frac{4t^2 + 4 - 4(1+t^2) + (1+t^2)^2}{(1+t^2)^2}} = \sqrt{\frac{(1+t^2)^2}{(1+t^2)^2}} = 1.$

$$I = \int_\Gamma y \cdot e^{-x} \cdot dl = \int_0^1 (2\,\mathrm{arctg}\,t - t + 1) \cdot e^{-\ln(1+t^2)} \cdot 1 \cdot dt =$$

$$= \int_0^1 (2\,\mathrm{arctg}\,t - t+1)\cdot \frac{1}{1+t^2}\, dt = 2\cdot\int_0^1 \frac{\mathrm{arctg}\,t}{1+t^2}\,dt -$$

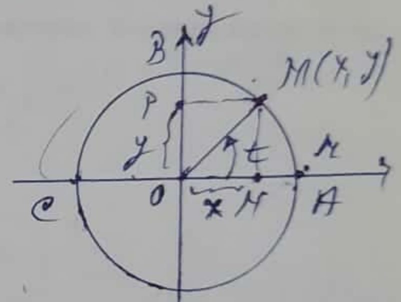
$$- \int_0^1 \frac{t}{1+t^2}\,dt + \int_0^1 \frac{1}{1+t^2}\,dt = 2\cdot\frac{1}{2}\cdot \mathrm{arctg}^2 t\Big|_0^1 -$$

$$- \frac12 \cdot \int_0^1 \frac{2t}{1+t^2}\,dt + \mathrm{arctg}\,t\Big|_0^1 = \left(\frac{\pi}{4}\right)^2 - 0 - \frac12\cdot\ln(1+t^2)\Big|_0^1 +$$

$$\int_a^b f(x)\cdot dx = F(x)\Big|_a^b = F(b)-F(a) \quad \text{(Leibniz-Newton)}$$

$$+\frac{\pi}{4} - 0 = \frac{\pi^2}{16} + \frac{\pi}{4} - \frac12(\ln 2 - \ln 1) = \frac{\pi^2}{16}+\frac{\pi}{4}-\frac12\ln 2$$

2° Să se calculeze lungimea cercului de rază r



$$d(O,M) = \sqrt{(x-0)^2+(y-0)^2} = r$$

$$\sqrt{x^2+y^2} = r \iff \boxed{x^2+y^2=r^2}$$

$$\int_{ABC} dl(t) = l(ABC)$$

$t = \measuredangle(Ox, OM)$, măsurat în sens direct trigonometric. În $\triangle ONM$: $\sin t = \frac{NM}{OM} = \frac{y}{r} \Rightarrow \boxed{y = r\cdot \sin t}$

$\cos t = \frac{ON}{OM} = \frac{x}{r} \Rightarrow \boxed{x = r\cos t}$

$$\Gamma = \overset{\frown}{ABC}: \begin{cases} x = r\cos t\\ y = r\sin t\end{cases} \quad t\in[0,\pi]$$

$$\begin{cases} \begin{cases}\cos t = \frac{x}{r}\\ \sin t = \frac{y}{r}\end{cases}\\ \sin^2 t + \cos^2 t = 1\\ \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1\\ x^2+y^2 = r^2\end{cases}$$

$$dl(t) = \sqrt{x'^2(t)+y'^2(t)}\cdot dt =$$

$$= \sqrt{(-r\sin t)^2 + (r\cos t)^2}\cdot dt =$$

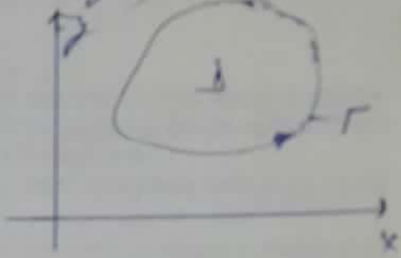
$$= \sqrt{r^2(\sin^2 t+\cos^2 t)}\cdot dt = r\cdot dt$$

$$l(\overset{\frown}{ABC}) = \int_{ABC} dl(t) = \int_0^\pi r\cdot dt = r\cdot\int_0^\pi dt = r\cdot t\Big|_0^\pi = r(\pi - 0) = r\pi$$

$$l(\overset{\frown}{ABCA}) = 2r\pi$$

Formula lui Green – leagă integrala curbilinie de tipul doi, pe o curbă închisă din plan, parcursă în sens direct, care închide în interiorul său un domeniu simplu convex, D, și integrala dublă pe domeniul respectiv, conform relației:

$$\oint_\Gamma P(x,y)\cdot dx + Q(x,y)\cdot dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$$\oint_\Gamma x\cdot dy = \iint_D 1\cdot dx\,dy = \text{aria}(D)$$

$x = Q;\ P = 0$

$$\oint_\Gamma (-y)\,dx = \iint_D +1\,dx\,dy = \text{aria}(D)$$

$-y = P;\ Q = 0$

$$\text{aria}(D) = \oint_\Gamma x\cdot dy = \int_\Gamma (-y)\cdot dx = \frac{1}{2}\oint_\Gamma -y\cdot dx + x\cdot dy$$

Aria cercului de rază r

$x^2 + y^2 = r^2 \iff \begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \quad t \in [0, 2\pi].$

$$\begin{cases} dx = -r\sin t\cdot dt \\ dy = r\cos t\cdot dt \end{cases}$$

$$A(D) = \frac{1}{2}\cdot\int_0^{2\pi}\left[(-r\sin t)\cdot(-r\sin t) + r\cos t\cdot r\cos t\right]\cdot dt$$

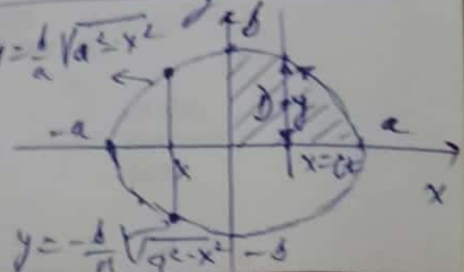
$$= \frac{1}{2}\int_0^{2\pi}(r^2\sin^2 t + r^2\cos^2 t)\,dt = \frac{1}{2}\cdot\int_0^{2\pi} r^2\cdot dt =$$

$$= \frac{1}{2}\cdot r^2\cdot\int_0^{2\pi} dt = \frac{r^2}{2}\cdot t\Big|_0^{2\pi} = \frac{r^2}{2}(2\pi - 0) = \pi r^2.$$

Să se calculeze aria elipsei de semiaxe a și b

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (E)\begin{cases} \dfrac{x}{a} = \cos t \\ \dfrac{y}{b} = \sin t \\ t \in [0, 2\pi], \end{cases}$$

$$A = \int_E x\cdot dy$$



$x = a\cos t$

$y = b\cdot\sin t \Rightarrow dy = y'(t)\cdot dt = b\cos t\cdot dt$

$\mathcal{A}(\mathcal{E}) = \oint_{\mathcal{E}} x\cdot dy = \int_0^{2\pi} a\cos t\cdot b\cos t\, dt = ab\cdot\int_0^{2\pi}\cos^2 t\, dt =$

$= ab\cdot\int_0^{2\pi}\frac{1+\cos 2t}{2}\cdot dt = \frac{ab}{2}\cdot\int_0^{2\pi}(1+\cos 2t)\cdot dt =$

$= \frac{ab}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_0^{2\pi} = \frac{ab}{2}\left[(2\pi - 0) + \frac{1}{2}(\sin 4\pi - \sin 0)\right] =$

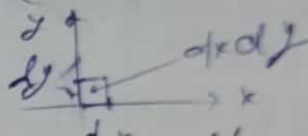
$= \frac{ab}{2}\cdot 2\pi = \pi ab$ $\quad\boxed{\Rightarrow \mathcal{A}(\mathcal{E}) = \pi ab}$

(1°) Calculul ariei $\mathcal{E}$ cu ajutorul integralei duble

Fie D = domeniul „sfertul elipsei din cadranul 1.

$\mathcal{A}(\mathcal{E}) = 4\cdot \text{aria}(D)$. $\quad dxdy$ = elementul infinitesimal de arie în plan

$\text{Aria}(D) = \iint_D dx\,dy$;



$\int_a^b dx = (b-a)$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$; $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$

$x \in [-a, a]$

dreapta $x = ct$ intersectează frontiera lui D în exact 2 puncte:

$D : \begin{cases} 0 \le x \le a \\ 0 \le y \le \frac{b}{a}\sqrt{a^2 - x^2} \end{cases}$ $\quad D$ = simplu în raport cu axa Oy.

$I = \iint_D dxdy = \int_0^a\left(\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\right)\cdot dx = \int_0^a\left(y\Big|_0^{\frac{b}{a}\sqrt{a^2-x^2}}\right)\cdot dx$

$= \int_0^a\left(\frac{b}{a}\sqrt{a^2-x^2} - 0\right)dx = \frac{b}{a}\cdot\int_0^a\sqrt{a^2-x^2}\cdot dx$

$x = a\cdot\sin t \Rightarrow dx = a\cos t\cdot dt$; $\sin t = \frac{x}{a}$

$t = \arcsin\frac{x}{a}$

x	0	a
t	0	$\frac{\pi}{2}$

$x = a \Rightarrow t = \arcsin 1 = \frac{\pi}{2}$

$$I = \frac{b}{a}\cdot\int_0^a \sqrt{a^2-x^2}\cdot dx = \frac{b}{a}\cdot\int_0^{\frac{\pi}{2}} \sqrt{a^2-a^2\sin^2 t}\cdot(a\cos t)\cdot dt$$

$$= \frac{b}{a}\cdot a^2\cdot\int_0^{\frac{\pi}{2}}\sqrt{\cos^2 t}\cdot\cos t\cdot dt = b\,a\int_0^{\frac{\pi}{2}}\cos^2 t\, dt =$$

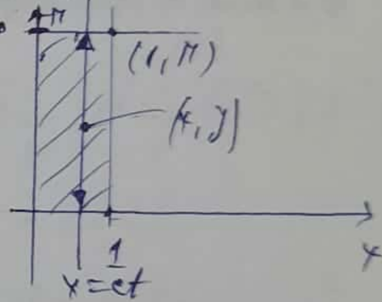
$$= ab\cdot\int_0^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\cdot dt = \frac{ab}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2t)\,dt =$$

$$= \frac{ab}{2}\cdot\left(t\Big|_0^{\frac{\pi}{2}} + \frac{\sin 2t}{2}\Big|_0^{\frac{\pi}{2}}\right) = \frac{ab}{2}\left(\frac{\pi}{2}+0\right) = \frac{\pi ab}{4}.$$

$$\text{Aria}(D) = \frac{\pi ab}{4} \Rightarrow \mathcal{A}(\mathcal{E}) = 4\cdot\frac{\pi ab}{4} = \pi ab$$

(2°) Să se calculeze integrala dublă:

$$I = \iint_D x\cdot\sin(x\cdot y)\,dx\,dy, \qquad D = [0,1]\times[0,\pi].$$



$$I = \iint_D f(x,y)\,dx\,dy = \int_0^1\left(\int_0^\pi x\cdot\sin(xy)\,dy\right)\cdot dx$$

$$= \int_0^1 x\cdot\left(\int_0^\pi(\sin xy)\,dy\right)\cdot dx$$

$$\int_0^\pi \sin(xy)\cdot dy = -\frac{\cos(xy)}{x}\Big|_{y=0}^{y=\pi} = -\frac{1}{x}\left(\cos(\pi x)-\cos 0\right) =$$

$$= -\frac{1}{x}\cdot(\cos(\pi\cdot x)-1) = -\frac{1}{x}\cdot\cos(\pi x)+\frac{1}{x}$$

$$\Rightarrow I = \int_0^1 x\cdot\left(\frac{1}{x}-\frac{1}{x}\cdot\cos\pi x\right)\cdot dx = \int_0^1(1-\cos\pi x)\,dx =$$

$$= x\Big|_0^1 - \frac{\sin\pi x}{\pi}\Big|_0^1 = 1-\frac{1}{\pi}\left(\underbrace{\sin\pi - \sin 0}_{=0}\right) = 1$$

$u(x,y) = \ln(x^3+y^2)$

$(2y+1)\cdot\frac{\partial u}{\partial x} - 3x^2\cdot\frac{\partial u}{\partial y} - \frac{3x^2}{x^3+y^2} = 0$

$(\ln u(x))' = \frac{u'(x)}{u(x)}$

$\frac{\partial u}{\partial x} = u'_x = \frac{3x^2}{x^3+y^2}$

$\frac{\partial u}{\partial y} = u'_y = \frac{2y}{x^3+y^2}$

$(2y+1)\cdot\frac{3x^2}{x^3+y^2} - 3x^2\cdot\frac{2y}{x^3+y^2} - \frac{3x^2}{x^3+y^2} = \frac{6x^2y+3x^2-6x^2y-3x^2}{x^3+y^2} = 0$

$F(x,y,z)$; $F: A\subset\mathbb{R}^3\to\mathbb{R}$.

$dF(x,y,z) = ?$

$d^2F(x,y,z) = ?$

$dF(x,y,z) = \frac{\partial F}{\partial x}\cdot dx + \frac{\partial F}{\partial y}\cdot dy + \frac{\partial F}{\partial z}\cdot dz$

$d^2F(x,y,z) = \left(\frac{\partial}{\partial x}\cdot dx + \frac{\partial}{\partial y}\cdot dy + \frac{\partial}{\partial z}\cdot dz\right)^{(2)} F(x,y,z)$

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$

$d^2F(x,y,z) = \frac{\partial^2 F}{\partial x^2}\cdot(dx)^2 + \frac{\partial^2 F}{\partial y^2}\cdot(dy)^2 + \frac{\partial^2 F}{\partial z^2}\cdot(dz)^2 + 2\frac{\partial^2 F}{\partial x\partial y}\cdot dx\,dy + 2\frac{\partial^2 F}{\partial x\partial z}\cdot dx\,dz + 2\frac{\partial^2 F}{\partial y\partial z}\,dy\,dz$

Să se calculeze diferențialele de ordinul 1 și 2 ale funcției $f(x,y) = x^2 - xy + 2y^2 + 3x - 5y + 7$ în punctul $(1,2)$

$df(x,y) = \left(\frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy\right)(1,2)$

$\begin{cases} \frac{\partial f}{\partial x} = 2x - y + 3 \\ \frac{\partial f}{\partial y} = -x + 4y - 5 \end{cases}$

$\frac{\partial f}{\partial x}(1,2) = 2-2+3 = 3$

$\frac{\partial f}{\partial y}(1,2) = -1+8-5 = 2$

$\Rightarrow df(1,2) = 3\cdot dx + 2\cdot dy$

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$$d^2f(x,y) = \left(\frac{\partial}{\partial x}\cdot dx + \frac{\partial}{\partial y}\cdot dy\right)^{(2)}(f(x,y)) = \frac{\partial^2 f}{\partial x^2}\cdot(dx)^2 + 2\frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial y^2}\cdot(dy)^2.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x-y+3) = 2$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x-y+3) = -1$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-x+4y-5) = 4$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-x+4y-5) = -1$$

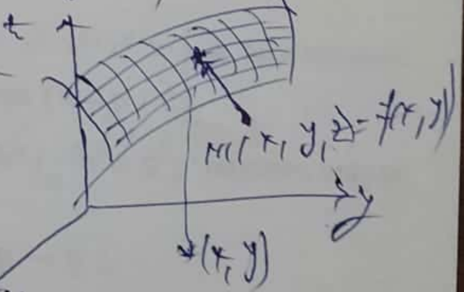
Obs. că $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$ (Schwarz)

$$d^2f(1,2) = 2\cdot dx^2 + 2\cdot(-1)\cdot dxdy + 4\cdot\frac{\partial^2 f}{\partial y^2}(dy)^2$$

Să se calculeze punctele de extrem local ale funcției $f(x,y) = x^2 - xy + 2y^2 + 3x - 5y + 7$, $f:\mathbb{R}^2 \to \mathbb{R}$

condiții necesare de extrem

Etape.



① Se determină punctele staționare ($\bar{x}$ – punctele critice; $f'(x)=0$)

$$\begin{cases}\frac{\partial f}{\partial x}=0\\ \frac{\partial f}{\partial y}=0\end{cases} \Leftrightarrow \begin{cases}2x-y=-3\\ -x+4y=5 \,/\cdot 2\end{cases} \quad \begin{cases}2x-y=-3\\ -2x+8y=10\end{cases}$$

$$/\; 7y = 7 \Rightarrow \boxed{y=1}$$

$$\begin{cases}x = 4y-5 \Rightarrow \boxed{x=-1}\\ y=1\end{cases}$$

$\boxed{M(-1,1)}$ = punctul staționar.

② condiții suficiente pt. existența extremului

- se calculează d.p. de ord. 2, în fiecare punct staționar. $M(-1,1)$

$$\frac{\partial^2 f}{\partial x^2}(-1,1) = 2\;;\quad \frac{\partial^2 f}{\partial x\partial y}(-1,1) = -1\;;\quad \frac{\partial^2 f}{\partial y^2}(-1,1) = 4.$$

$$d^2f(-1,1) = 2\cdot(dx)^2 - 2dxdy + 4\cdot(dy)^2$$

Se calculează matricea Hessiană a funcției f în punctul staționar:

$$H(-1,1) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(-1,1) & \frac{\partial^2 f}{\partial x \partial y}(-1,1) \\ \frac{\partial^2 f}{\partial y \partial x}(-1,1) & \frac{\partial^2 f}{\partial y^2}(-1,1) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$$

Se calculează determinanții lui Jacobi (Sylvester)

$\Delta_1 = \frac{\partial^2 f}{\partial x^2}(-1,1) = 2$; $\Delta_2 = \det(H) = 8 - 1 = 7$

1° Dacă $\Delta_1 < 0$ și $\Delta_2 > 0 \Rightarrow M(-1,1)$ este punct maxim local.

2° Dacă $\Delta_1 > 0$ și $\Delta_2 > 0$, $\Rightarrow M(-1,1)$ este punct de minim local.

$\Delta_1 = 2 > 0$; $\Delta_2 = 7 > 0$. $\Rightarrow M(-1,1)$ este punct de minim local valoarea minimului $= f(-1,1) =$

Pentru o funcție de 3 variabile

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & f''_{y^2} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{z^2} \end{pmatrix}$$ (în punctul staționar).

$\Delta_1 = f''(x^2)$

$\Delta_2 = f''_{x^2} \cdot f''_{y^2} - (f''_{xy})^2$

$\Delta_3 = \det(H)$

Dacă

1° $\Delta_1 < 0$, $\Delta_2 > 0$; $\Delta_3 < 0$ $\Rightarrow M(x_0, y_0, z_0) =$ maxim

2° $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0 \Rightarrow$ $M(x_0, y_0, z_0) =$ minim

Șiruri și serii de numere reale

(1°) Să se calculeze limita șirului:

$$x_n = \frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}}; \quad p \in \mathbb{N}^* \ (p \geq 1.)$$

Cesaro-Stolz; $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = ?$

Dacă a) v_n este strict crescător și nemărginit.

b) $\exists \lim\limits_{n\to\infty} \frac{u_{n+1}-u_n}{v_{n+1}-v_n} = \ell$.

Atunci există și $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \ell$.

a) $v_n = n^{p+1} \nearrow +\infty$

b) $\lim\limits_{n\to\infty} \frac{u_{n+1}-u_n}{v_{n+1}-v_n} = \lim\limits_{\infty} \frac{\cancel{1^p + 2^p + \dots + n^p} + (n+1)^p \cancel{- 1^p - 2^p - \dots - n^p}}{(n+1)^{p+1} - n^{p+1}}$

$= \lim\limits_{\infty} \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}} = \lim\limits_{n\to\infty} \frac{(n(1+\frac{1}{n}))^p}{\cancel{C_{p+1}^0 \cdot n^{p+1}} + C_{p+1}^1 \cdot n^p + \dots + 1 \cancel{- n^{p+1}}}$

$(C_{p+1}^0 = 1)$

$\left(\frac{\infty}{\infty}\right) = \lim\limits_{n\to\infty} \frac{\cancel{n^p} (1+\frac{1}{n})^p}{\cancel{n^p}(C_{p+1}^1 + C_{p+1}^2 \cdot \frac{1}{n} + \dots + \frac{1}{n^p})} = \frac{1}{C_{p+1}^1} = \frac{1}{p+1}$

$x_n \leq y_n \leq z_n$

$\Downarrow$

ℓ

(2°) **criteriul cleștelui**

$\lim\limits_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \dots + \frac{n}{n^2+n}\right) \overset{\infty \cdot 0}{=}$

$\frac{n}{n^2+n} < \frac{n}{n^2+p} < \frac{n}{n^2+1}, \ (\forall)\ p = 1, 2, \dots, n$

$p=1: \quad \frac{n}{n^2+n} < \frac{n}{n^2+1} \leq \frac{n}{n^2+1}$

$p=2: \quad \frac{n}{n^2+n} < \frac{n}{n^2+2} < \frac{n}{n^2+1}$

$- \quad - \quad -$

$p=n: \quad \frac{n}{n^2+n} = \frac{n}{n^2+n} < \frac{n}{n^2+1}$

(+)

$n \cdot \frac{n}{n^2+n} < x_n < n \cdot \frac{n}{n^2+1}$

$\frac{n^2}{n^2+n} < x_n < \frac{n^2}{n^2+1}$

$\Downarrow$

1

$\Rightarrow \lim\limits_{n\to\infty} x_n = 1$

Criteriul raportului (d'Alembert)

Dacă $x_n > 0$, $(\forall) n$ și $(\exists) \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell$ și $\ell \in [0, 1)$ $\Rightarrow$ $\boxed{(\exists) \lim_{n\to\infty} x_n = 0}$

Ex $\log_a n << n << n^k << a^n << n! << n^n$
$a>1$ $\quad (k>1 \text{ finit})$ $\quad a>1$

Spunem că $x_n << y_n$ dacă $\lim_{n\to\infty} \frac{x_n}{y_n} = 0$.

$\lim_{n\to\infty} \frac{a^n}{n!} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{(a>1)}{=}}$; Notăm $x_n = \frac{a^n}{n!} > 0$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{\infty} \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \lim_{\infty} \frac{a}{n+1} = 0 < 1$

$\lim_{n\to\infty} \underbrace{\frac{n!}{n^n}}_{x_n > 0}$; $\lim_{\infty} \frac{x_{n+1}}{x_n} = \lim \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} =$

$= \lim \frac{\cancel{(n+1)} \cdot n^n}{\cancel{(n+1)}(n+1)^n} = \lim_{\infty} \frac{n^n}{(n+1)^n} = \lim \frac{1}{\left(\frac{n+1}{n}\right)^n} =$

$= \lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1 \Rightarrow \lim \frac{n!}{n^n} = 0 \Leftrightarrow \boxed{n! << n^n}$

Criteriul Cauchy – d'Alembert

Dacă $x_n > 0$ și $(\exists) \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell$, atunci există și $\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{\infty} \frac{x_{n+1}}{x_n} = \ell$,

Ex. $\lim_{n\to\infty} \sqrt[n]{\frac{n! \cdot (2n)!}{(3n)!}} = ?$

Fie $x_n = \frac{n! \cdot (2n)!}{(3n)!} > 0.$

Calculăm $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim \frac{(n+1)!\,(2n+2)!}{(3n+3)!} \cdot \frac{(3n)!}{n! \cdot (2n)!} =$

$= \lim \frac{\cancel{n!}(n+1)\cancel{(2n)!}(2n+1)(2n+2)}{\cancel{(3n)!}(3n+1)(3n+2)(3n+3)} \cdot \frac{\cancel{(3n)!}}{\cancel{n! \cdot (2n)!}} =$

$= \lim_{\infty} \frac{(n+1)(2n+1)(2n+2)}{(3n+1)(3n+2)(3n+3)} \overset{\left(\frac{\infty}{\infty}\right)}{=} \frac{4}{27} < 1$

$\Rightarrow (\exists) \lim \sqrt[n]{\frac{n!\,(2n)!}{(3n)!}} = \frac{4}{27}$

Obs. Deoarece $\lim_{\infty} \frac{x_{n+1}}{x_n} = \frac{4}{27} < 1 \Rightarrow$

$\Rightarrow \lim_{n\to\infty} x_n = 0$

$\Rightarrow$ Seria $\sum_{n=0}^{\infty} \frac{n! \cdot (2n)!}{(3n)!}$ este convergentă