# Rezolvarea sistemelor de ecuatii liniare

24-May-8

# Sistem de ecuatii liniare cu n ecuatii si n necunoscute

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

- Sistemul de mai sus se poate scrie sub forma

$$Ax = b$$

# Metode de rezolvare

- Directe
  - Regula lui Cramer
  - Metoda eliminarii a lui Gauss
  - Metoda Gauss- Jordan
  - Metoda descompunerii LU
- Iterative
  - Gauss- Jacobi
  - Gauss-Seidel

# Metode iterative

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

Din fiecare ecuatie putem exprima, pe rand x1, x2, ..., xn, in functie de celelalte necunoscute.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n)}{a_{22}}$$

$$x_i = \frac{b_i - (a_{i1}x_1 + \ldots + a_{i,i-1}\,x_{i-1} + a_{i,i+1}x_{i+1} \ldots + a_{in}x_n)}{a_{ii}}$$

$$x_n = \frac{b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1})}{a_{nn}}$$

$$x_i = \frac{b_i - \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}} \quad pentru\ orice\ i = 1,...n$$

- Vrem sa determinam $x=(x_i)_{i=1,\ldots,n}$ astfel incat $Ax=b$.
- Metoda iterativa de rezolvare presupune ca

  se pleaca de la o valoare initiala a lui x,

  notata $x^{(0)} \quad =(x_i^{\,0})_{i=1,\ldots,n}$

si se construieste un sir $(x^{(k)})_k$ ce converge la solutia sistemului.

- Sirul se defineste sub forma:
- $x^{(k+1)}=f\,(x^{(k)})$ pt $k>=1$

# Metoda Jacobi

$$x_i^{(k+1)} = \frac{b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k)}}{a_{ii}} \quad pentru\ orice\ i = 1, \ldots n$$

- Algoritmul se opreste cand $x^{(k+1)}$ si $x^{(k)}$ sunt suficient de aproape adica

epsilon

$$\left| x_i^{(k+1)} - x_i^{(k)} \right| \leq \in \ pentru\ orice\ i = 1, \ldots, n$$

Teorema

Daca A este o matrice strict diagonal dominanta atunci metoda Jacobi este convergenta.

<u>Def:</u>

A este o <u>matrice</u> strict <u>diagonal dominanta</u> daca

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \forall i = 1, \ldots, n$$

# Care din urmatoarele matrici sunt strict diagonal dominante?

diagonal dominantă

A si C

$\geq$

La fel se arata ca C este matrice strict diagonal dominanta.

nu este diagonal dominanta
pt ca

# Exemplu

$$\begin{bmatrix} -5 & -1 & 2 \\ 2 & 6 & -3 \\ 2 & 1 & 7 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 32 \end{bmatrix}$$

Matricea sistemului este strict diagonal dominanta?

$|-5| > |-1| + |2|$

$|6| > |2| + |-3|$

$|7| > |2| + |1|.$

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# Rezolvarea sistemului

$$\begin{cases} -5x_1 - x_2 + 2x_3 = 1 \\ 2x_1 + 6x_2 - 3x_3 = 2 \\ 2x_1 + x_2 + 7x_3 = 32 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{1}{5}(1 + x_2 - 2x_3) \\ x_2 = \frac{1}{6}(2 - 2x_1 + 3x_3) \\ x_3 = \frac{1}{7}(32 - 2x_1 - x_2) \end{cases} \Rightarrow$$

Metoda Gauss-Jacobi

$$\begin{cases} x_1^{(k+1)} = -\frac{1}{5}(1 + x_2^{(k)} - 2x_3^{(k)}) \\ x_2^{(k+1)} = \frac{1}{6}(2 - 2x_1^{(k)} + 3x_3^{(k)}) \\ x_3^{(k+1)} = \frac{1}{7}(32 - 2x_1^{(k)} - x_2^{(k)}) \end{cases}$$

$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} si \in = 0.005$

# Construirea sirului

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1^{(1)} = -\frac{1}{5}(1 + x_2^{(0)} - 2x_3^{(0)}) = -0.2 \\ x_2^{(1)} = \frac{1}{6}(2 - 2x_1^{(0)} + 3x_3^{(0)}) = 0.3333 \\ x_3^{(1)} = \frac{1}{7}(32 - 2x_1^{(0)} - x_2^{(0)}) = 4.5714 \end{cases}$$

$$\left| x^{(1)} - x^{(0)} \right| = \begin{bmatrix} 0.2 \\ 0.3333 \\ 4.5714 \end{bmatrix}$$

$$\begin{cases} x_1^{(2)} = -\dfrac{1}{5}(1 + x_2^{(1)} - 2x_3^{(1)}) = 1.5619 \\ x_2^{(2)} = \dfrac{1}{6}(2 - 2x_1^{(1)} + 3x_3^{(1)}) = 2.6857 \\ x_3^{(2)} = \dfrac{1}{7}(32 - 2x_1^{(1)} - x_2^{(1)}) = 4.5810 \end{cases}$$

$$\left| x^{(2)} - x^{(1)} \right| = \begin{bmatrix} 1.7619 \\ 2.3524 \\ 0.0096 \end{bmatrix}$$

| k | x1 $^{(k)}$ | x2 $^{(k)}$ | x3 $^{(k)}$ | $\lvert x1^{(k)}-x1^{(k-1)}\rvert$ | $\lvert x2^{(k)}-x2^{(k-1)}\rvert$ | $\lvert x3^{(k)}-x3^{(k-1)}\rvert$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | | | |
| $x^{(1)}$ 1 | -0.2 | 0.3333 | 4.5714 | | | |
| $x^{(2)}$ 2 | 1.5619 | 2.6857 | 4.5810 | | | |
| 3 | 1.0952 | 2.1032 | 3.7415 | | | |
| 4 | 0.8760 | 1.8390 | 3.9580 | | | |
| 5 | 1.0154 | 2.0204 | 4.0584 | | | |
| 6 | 1.0193 | 2.0241 | 3.9927 | | | |
| 7 | 0.9923 | 1.9899 | 3.9910 | | | |
| 8 | 0.9984 | 1.9981 | 4.0037 | 0.0061 $> \varepsilon = 0.005$ | 0.0082 | 0.0127 |
| 9 | **1.0018** | **2.0023** | **4.0007** | 0.0034 $< \varepsilon$ | 0.0042 $< \varepsilon$ | 0.003 $< \varepsilon$ |
| 10 | 0.9998 | 1.9997 | 3.9991 | 0.002 | 0.0026 | 0.0016 |

Soluția exactă este $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

# Metoda Gauss - Seidel

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n)}{a_{22}}$$

$$x_i = \frac{b_i - (a_{i1}x_1 + \ldots + a_{i,i-1}\,x_{i-1} + a_{i,i+1}x_{i+1} \ldots + a_{in}x_n)}{a_{ii}} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} \cdot x_j - \sum_{j=i+1}^{n} a_{ij} \cdot x_j}{a_{ii}}$$

$$x_n = \frac{b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1})}{a_{nn}}$$

# Metoda Gauss - Seidel

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}} \; pentru \; orice \; i = 1,\ldots n$$

- Algoritmul se opreste cand $x^{(k+1)}$ si $x^{(k)}$ sunt suficient de aproape adica

$$\left| x_i^{(k+1)} - x_i^{(k)} \right| \leq \in \; pentru \; orice \; i = 1,\ldots, n$$

Teorema

Daca A este o matrice strict diagonal dominanta atunci metoda Gauss-Seidel este convergenta.

# Exemplu

$$\begin{bmatrix} -5 & -1 & 2 \\ 2 & 6 & -3 \\ 2 & 1 & 7 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 32 \end{bmatrix}$$

Matricea sistemului este diagonal dominanta?

DA

# Rezolvarea sistemului

$$\begin{cases} -5x_1 - x_2 + 2x_3 = 1 \\ 2x_1 + 6x_2 - 3x_3 = 2 \\ 2x_1 + x_2 + 7x_3 = 32 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{1}{5}(1 + x_2 - 2x_3) \\ x_2 = \frac{1}{6}(2 - 2x_1 + 3x_3) \\ x_3 = \frac{1}{7}(32 - 2x_1 - x_2) \end{cases} \Rightarrow$$

Metoda Gauss-Seidel

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} si \in = 0.005$$

$$\begin{cases} x_1{}^{(k+1)} = -\frac{1}{5}(1 + x_2{}^{(k)} - 2x_3{}^{(k)}) \\ x_2{}^{(k+1)} = \frac{1}{6}(2 - 2x_1{}^{(k+1)} + 3x_3{}^{(k)}) \\ x_3{}^{(k+1)} = \frac{1}{7}(32 - 2x_1{}^{(k+1)} - x_2{}^{(k+1)}) \end{cases}$$

# Construirea sirului

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1^{(1)} = -\frac{1}{5}(1 + x_2^{(0)} - 2x_3^{(0)}) = -0.2 \\ x_2^{(1)} = \frac{1}{6}(2 - 2x_1^{(1)} + 3x_3^{(0)}) = 0.4 \\ x_3^{(1)} = \frac{1}{7}(32 - 2x_1^{(1)} - x_2^{(1)}) = 4.5714 \end{cases}$$

$$\left| x^{(1)} - x^{(0)} \right| = \begin{bmatrix} 0.2 \\ 0.4 \\ 4.5714 \end{bmatrix}$$

$$\begin{cases} x_1^{(2)} = -\frac{1}{5}(1 + x_2^{(1)} - 2x_3^{(1)}) = 1.5486 \\ x_2^{(2)} = \frac{1}{6}(2 - 2x_1^{(2)} + 3x_3^{(1)}) = 2.1029 \\ x_3^{(2)} = \frac{1}{7}(32 - 2x_1^{(2)} - x_2^{(2)}) = 3.8286 \end{cases}$$

$$\left| x^{(2)} - x^{(1)} \right| = \begin{bmatrix} 1.7486 \\ 1.7029 \\ 0.7428 \end{bmatrix}$$

| k | x1 $^{(k)}$ | x2 $^{(k)}$ | x3 $^{(k)}$ | $\|x1^{(k)}-x1^{(k-1)}\|$ | $\|x2^{(k)}-x2^{(k-1)}\|$ | $\|x3^{(k)}-x3^{(k-1)}\|$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | | | |
| 1 | -0.2 | 0.4 | 4.5714 | | | |
| 2 | 1.5486 | 2.1029 | 3.8286 | | | |
| 3 | 0.9109 | 1.9440 | 4.0335 | | | |
| 4 | 1.0246 | 2.0085 | 3.9918 | | | |
| 5 | 0.9950 | 1.9975 | 4.0018 | | | |
| 6 | 1.0012 | 2.0005 | 3.9996 | 0.0062 | | |
| 7 | **0.9997** | **1.9975** | **4.0001** | 0.0015 | 0.0006 | 0.0005 |

# Observatii

- Conditia ca A sa fie strict diagonal dominanta nu este obligatorie dar ea asigura convergenta metodelor GJ si GS. Exista A nediagonal dominanta pentru care metodele converg
- Daca valorile initiale sunt apropiate de solutie atunci convergenta este mai rapida.

# Observatii

- Daca sistemul initial ar fi fost

$$\begin{cases} -5x_1 - x_2 + 2x_3 = 1 \\ 2x_1 + x_2 + 7x_3 = 32 \\ 2x_1 + 6x_2 - 3x_3 = 2 \end{cases}$$

$$2x_1 + \boxed{6}x_2 - 3x_3 = 2$$
$$2x_1 + x_2 + \boxed{7}x_3 = 32.$$

Este matricea sistemului strict diagonal dominanta?

NU dar prin rearanjarea ecuatiilor ea poate devenii strict diagonal dominanta.