Exemple dezvoltări în serie Mac Laurine

ex 1

$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = e^x$

$f^{(n)}(x) = e^x \quad \forall\, n \in \mathbb{N}, x \in \mathbb{R} \qquad (e^x)' = e^x$

$\hookrightarrow f^{n}(0) = 1$

formula lui Mac Laurine în restul lui Lagrange

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} \cdot f''(0) + \dots + \frac{x^n}{n!} f^{n}(0) + \frac{x^{n+1}}{(n+1)!} f(\theta x)$$

$$e^x = e^0 + \frac{x}{1!} \cdot 1 + \frac{x^2}{2!} \cdot 1 + \dots + \frac{x^n}{n!} + \underbrace{\frac{x^{n+1}}{(n+1)!} \cdot e^{\theta x}}_{R_n = \frac{x^{n+1}}{(n+1)!} \cdot f(\theta x)}$$

$$\lim_{n \to \infty} \frac{x^{n+1}}{(n+1)!} e^{\theta x} = 0 \to e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

ex 2

$f(x) = \sin x$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f''''(x) = \sin x$

$f'(0) = 1$

$f''(0) = 0$

$f'''(0) = -1$

$f''''(1) = 0$

$$f^{n}(x) = \sin\left(x + \frac{n\pi}{2}\right)$$

formula lui Mac Laurine

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1) \frac{x^{2n-1}}{(2n-1)!} + R_n(x)$$

$$R_n(x) = (-1)^n \frac{x^{2n+1}}{(2n+1)!} \cdot \cos(\theta x)$$

$$R_n = \frac{x^{n+1}}{(n+1)!} \cdot f'(\theta x)$$

$$\lim_{n\to\infty} |R_n(x)| = 0 \Rightarrow$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \dots$$

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ex3

$f(x) = \cos x$

$f'(x) = -\sin(x)$

$f''(x) = -\cos(x)$

$f'''(x) = \sin(x)$

$f''''(x) = \cos(x)$

general $f^{(n)}(x) = \cos\left(x + n\frac{\pi}{2}\right)$

$f(0) = 1$

$f'(0) = -\sin 0 = 0$

$f''(0) = -\cos 0 = -1$

$f'''(0) = \sin 0 = 0$

$f''''(0) = \cos 0 = 1$

$$\cos x = f(0) - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \underbrace{(-1)^{n+1}\frac{x^{2n+2}}{(2n+2)!}\cdot\cos(\theta x)}_{R_n}$$

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \cos(\theta x) \right| = 0$$

factorial $\to \infty$ mai repede decat putere

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

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ex4 $f(x) = \ln(1+x)$

ex5 $f(x) = \ln(1-x)$

} ex. anterioare

$$e^{ix} = \cos x + i\sin x, \quad \forall x \in \mathbb{R},\ i^2 = -1$$

$\ln(e^{ix}) = \ln(\cos x + i\sin x) \Leftrightarrow \ln(\cos t + i\cdot\sin t) = it$

dacă derivatele sunt egale atunci cele două funcții diferă printr-o constantă

$(\ln u)' = \frac{u'}{u}$

$(x^n)' = n(x)^{n-1}$

$u(x) = \ln(\cos x + i\sin x)$

$v(x) = ix$

$u(x) = \frac{1}{\cos x + i\sin x} \cdot (\cos x + i\sin x)'$

$$u'(x) = \frac{(-\sin x + i\cos x)(\cos x - i\sin x)}{(\cos x + i\sin x)(\cos x - i\sin x)}$$

$$u'(x) = \frac{-\sin x\cos x + i\sin^2 x + i\cos^2 x - i^2 \sin x\cos x}{\cos^2 x + \sin^2 x}$$

$$u'(x) = \frac{i(\sin^2 x + \cos^2 x)}{\cos^2 + \sin^2 x} = i$$

$v'(x) = (ix)' = i \cdot (x)' = i$

$v'(x) = u'(x) = i$

$u(x) = v(x) + C$

$\ln(\cos x + i\sin x) = ix + C$

$\cos x + i\sin x = e^{ix+C} \quad \forall x$

pt $x = 0 \quad \cos 0 = 1 \quad \sin 0 = 1$

$1 = e^C \Rightarrow C = 0$

$\rightarrow u(x) = v(x)$

$\Leftrightarrow \ln(\cos x + i\sin x) = ix$

$\Leftrightarrow \cos x + i\sin x = e^{ix}$

cosinus hiperbolic $ch(x) = \frac{e^x + e^{-x}}{2}$

sinus hiperbolic $sh(x) = \frac{e^x - e^{-x}}{2}$

$$ch^2(x) - sh^2(x) = \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4}$$

$ch^2(x) - sh^2(x) = 1$ $\quad$ $\cos^2 x + \sin^2 x = 1$ fundamentala trig. circulară

$th(x) = \frac{sh\,x}{ch\,x}$ $\quad ctgh(x) = \frac{ch\,x}{sh\,x}$ $\quad tg\,x = \frac{\sin x}{\cos x}$

$$\begin{cases} \cos x + i\sin x = e^{ix} \\ \cos x - i\sin x = e^{-ix} \end{cases} \oplus$$

inlocuim x cu $-x$

$\oplus$ $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

$\Leftrightarrow$ $\cos x = ch(ix)$

$\sin x = sh(ix)$

$\ominus$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \dots$$

$$e^{-ix} = 1 - \frac{ix}{1!} - \frac{x^2}{2!} + \frac{ix^3}{3!} + \dots \quad \oplus$$

$$e^{ix} + e^{-ix} = 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n\frac{x^{2n}}{(2n)!}\right) \Big| :2$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!}$$

analog se poate calcula si dezvoltarea lui $\sin x$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

exemplu 9

$sh(x) = \frac{e^x - e^{-x}}{2}$ $\qquad ch(x) = \frac{e^x + e^{-x}}{2}$

aplicatie 10

Fie dezvoltarea $\sum_{1}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)}$ să se calculeze ρ, I, A

raza interval mult

si $S(x)$ suma seriei pe interval de convergenta

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$S(x) = -\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \dots + (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$

$$\rho = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$$

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{2n+1}}{\frac{(-1)^{n+1}}{2n+3}} \right| = \lim_{n\to\infty} \frac{2n+3}{2n+1} = 1$$

$J = (-\rho, \rho)$

$J = (-1, 1)$

convergența în capetele lui J $(-1, 1)$

pt $x=1$ seria devine $\sum_1^\infty (-1)^n \frac{1}{2n+1} = -\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + (-1)^n \frac{1}{2n+1} \dots$

serie alternată cu $a_n = \left(\frac{1}{2n+1}\right) \geq 0$ descrescător,

convergent la zero $\Rightarrow$ seria convergentă

seria modulelor $\sum_1^\infty \frac{1}{2n+1}$ divergentă, comparabilă $\sum \frac{1}{n}$

$\sum |a_n|$

armonica divergentă

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

$\Rightarrow$ cele 2 serii au aceeași natură dar $\sum \frac{1}{n}$ divergentă $\Rightarrow$

$\sum \frac{1}{2n+1}$ divergentă

criteriul lui Raabe $\lim_{n\to\infty} n \left(\frac{u_n}{u_{n+1}} - 1\right)$

$x=-1$ seria devine $\sum_1^\infty (-1)^n \cdot \frac{(-1)^{2n+1}}{2n+1} = \sum_1^\infty \frac{(-1)^n}{2n+1}$

serie alternată

→ seria anterioară cu semn schimbat

convergentă.

pt $x = \pm 1$ → serie semiconvergentă

$S(x) = -\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \cdot \frac{x^{2n+1}}{2n+1} + \dots$

$S'(x) = -x^2 + x^4 - x^6 + \dots + (-1)^n \cdot x^{2n} + \dots$

$S'(x) = -x^2 (1 - x^2 + x^4 + \dots + (-x^2)^n + \dots$

$S'(x) = (-x^2) \dfrac{1}{1-(-x^2)} = (-x^2) \dfrac{1}{1+x^2}$

$S'(x) = \dfrac{(-x^2) - 1 + 1}{1+x^2} = -1 + \dfrac{1}{1+x^2}$

$\int \dfrac{1}{a^2+x^2} = \operatorname{arctg} \dfrac{x}{a}$

$\int dx = x$

$S(x) = \int -1 + \dfrac{1}{1+x^2} = -x + \operatorname{arctg} x + C = -\dfrac{x^3}{3} + \dfrac{x^5}{5} + \dots$

pentru $x = 0$ $\boxed{C = 0} \Rightarrow S(x) = -x + \operatorname{arctg} x$

exemplu 11

seria geometrica $r = -x^2$

$1 + r + r^2 + \dots + r^{n-1} + \dots = \dfrac{1}{1-r}$

$1 - x^2 + x^4 - x^6 + \dots + (-1)^{n-1} x$

$\rho = \lim\limits_{n \to \infty} \left| \dfrac{a_n}{a_{n+1}} \right| = \lim\limits_{n \to \infty} \left| \dfrac{(-1)^n}{(-1)^{n+1}} \right| = 1 \qquad x \in (-1, 1)$

suma: $S(x) = \lim\limits_{n \to \infty} \dfrac{1 - (-x^2)^n}{1-(-x^2)} = \dfrac{1}{1+x^2}$

$\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \dots$

$x \in (-1, 1)$

$\Rightarrow \int \dfrac{1}{1+x^2} dx = \operatorname{arctg} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dots + (-1)^n \dfrac{x^{2n+1}}{2n+1} + \dots$

pt $x \pm 1$ seria e convergenta ca serie alternată

pt $x = 1 \Rightarrow \operatorname{arctg} 1 = \dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} + \dots + (-1)^n \cdot \dfrac{1}{2n+1} + \dots$