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Pregătire lucrare de control. (Anul 1 ...)

Problema 1. Fie operatorul $f: \mathbb{R}^3 \to \mathbb{R}^3$, prin:

$$f(x_1, x_2, x_3) = (-x_1 + 3x_2 - x_3,\ -3x_1 + 5x_2 - x_3,\ -3x_1 + 3x_2 + x_3)$$

Să se determine:

a) Polinomul caracteristic $P_A(\lambda)$

b) valorile proprii

c) vectorii proprii corespunzători

d) matricea diagonalizatoare și forma diagonală

e) A^n; $f(X) = A \cdot X$.

Rezolvare: Fie $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$. $f(X) = A \cdot X \Rightarrow$

$$f(X) = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (A - \lambda I_3) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

a) Polinomul caracteristic:

$$P_A(\lambda) = \det(A - \lambda I_3) = \det \begin{pmatrix} -1-\lambda & 3 & -1 \\ -3 & 5-\lambda & -1 \\ -3 & 3 & 1-\lambda \end{pmatrix} \ \underline{\underline{C_1 = C_1 + C_2}}$$

$$= \begin{vmatrix} 2-\lambda & 3 & -1 \\ 2-\lambda & 5-\lambda & -1 \\ 0 & 3 & 1-\lambda \end{vmatrix} = (2-\lambda) \cdot \begin{vmatrix} 1 & 3 & -1 \\ 1 & 5-\lambda & -1 \\ 0 & 3 & 1-\lambda \end{vmatrix} \ \underline{\underline{L_2 = L_2 - L_1}}$$

$$= (2-\lambda) \cdot \begin{vmatrix} 1 & 3 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 3 & 1-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda)(-1)^{2+2} \cdot \begin{vmatrix} 1 & -1 \\ 0 & 1-\lambda \end{vmatrix} =$$

$$= (2-\lambda)^2 (1-\lambda)$$

b) valorile proprii: $P_A(\lambda) = 0$; $(2-\lambda)^2(1-\lambda) = 0 \Rightarrow$

$\lambda_1 = \lambda_2 = 2$; $\lambda_3 = 1$.

c) Vectorii proprii.

(c.1) $\lambda_3 = 1$

$$\begin{pmatrix} -2 & 3 & -1 \\ -3 & 4 & -1 \\ -3 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 3 & -1 & \vdots & 0 \\ -3 & 4 & -1 & \vdots & 0 \\ -3 & 3 & 0 & \vdots & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & 3 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 3 & -3 & \vdots & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 & 2 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

$x_1 \quad x_2 \quad x_3$

rang A = rang $\bar{A}$ = 2 → sistem compatibil nedeterminat (simplu): nec. pr: x_1, x_2; nec. sec. $x_3 = \alpha$; $\alpha \in \mathbb{R}$. → $\begin{cases} -2x_1 + 2\alpha = 0 \\ x_2 - \alpha = 0 \end{cases}$

$\begin{cases} x_1 = \alpha \\ x_2 = \alpha \\ x_3 = \alpha \end{cases}$ $\alpha \in \mathbb{R}$; $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \alpha$

vectorul propriu corespunzător lui $\boxed{\lambda_3 = 1 : V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

(C.2) $\lambda_1 = \lambda_2 = 2$ → valoare proprie având multiplicitatea algebrică egală cu 2 (valoare proprie dublă)

$$\begin{pmatrix} -3 & 3 & -1 & \vdots & 0 \\ -3 & 3 & -1 & \vdots & 0 \\ -3 & 3 & -1 & \vdots & 0 \end{pmatrix} \sim \begin{pmatrix} -3 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rang A = rang $\bar{A}$ = 1 ⇒ sistem compatibil dublu nedeterminat. Nec. pr: x_1. Nec. secundare:

$\begin{cases} x_2 = \alpha \\ x_3 = \beta \end{cases}$ $\alpha, \beta \in \mathbb{R}$ $\quad -3x_1 + 3x_2 - x_3 = 0$

$3x_1 = 3x_2 - x_3$

$3x_1 = 3\alpha - \beta$; $\begin{cases} x_1 = \alpha - \frac{1}{3}\beta \\ x_2 = \alpha \\ x_3 = \beta \end{cases}$; $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \alpha + \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \cdot \beta$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \alpha + \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \cdot \frac{\beta}{3}$ ⇒ $\boxed{V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \; V_2 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}}$

din $Sp[V_2, V_3] = 2$ = multiplicitatea geometrică multiplicitatea algebrică = multiplicitatea geometrică ⇒ matricea A este diagonalizabilă. Matricea diagonalizatoare este matricea ale cărei coloane sunt cei trei vectori proprii V_1, V_2, V_3 → sunt liniar indep.; rang$(V_1, V_2, V_3) = 3$

cei trei vectori formează o bază în $\mathbb{R}^3$ și matricea $T = (v_1, v_2, v_3)$ = matricea de trecere de la baza canonică: $\mathcal{B} = \{e_1, e_2, e_3\}$ la baza diagonalizatoare.

S-a demonstrat că relația dintre matricea aplicației liniare f în baza canonică, A, și matricea aplicației f în baza formată din vectorii proprii este următoarea:

$$B = T^{-1}\cdot A\cdot T = \begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}$$

$$T = \begin{pmatrix}1&1&-1\\1&1&0\\1&0&3\end{pmatrix}; \quad (T, I_3) \simeq \dots \simeq (I_3, T^{-1})$$

$$\left(\underbrace{\begin{matrix}1&1&-1\\1&1&0\\1&0&3\end{matrix}}_{T}\ \middle|\ \underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I_3}\right) \simeq \left(\begin{matrix}1&1&-1\\0&0&1\\0&-1&4\end{matrix}\ \middle|\ \begin{matrix}1&0&0\\-1&1&0\\-1&0&1\end{matrix}\right) \simeq$$

$$\simeq \left(\begin{matrix}1&1&-1\\0&-1&4\\0&0&1\end{matrix}\ \middle|\ \begin{matrix}1&0&0\\-1&0&1\\-1&1&0\end{matrix}\right) \simeq \left(\begin{matrix}-1&0&-3\\0&-1&4\\0&0&-1\end{matrix}\ \middle|\ \begin{matrix}0&0&-1\\-1&0&1\\1&-1&0\end{matrix}\right) \simeq$$

$$\simeq \left(\begin{matrix}1&0&0\\0&1&0\\0&0&-1\end{matrix}\ \middle|\ \begin{matrix}3&-3&1\\-3&4&-1\\1&-1&0\end{matrix}\right)\Big|\cdot(-1) \simeq \left(\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I_3}\ \middle|\ \underbrace{\begin{matrix}3&-3&1\\-3&4&-1\\-1&1&0\end{matrix}}_{T^{-1}}\right)$$

calculăm $T^{-1}\cdot T = \begin{pmatrix}3&-3&1\\-3&4&-1\\-1&1&0\end{pmatrix}\cdot\begin{pmatrix}1&1&-1\\1&1&0\\1&0&3\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = I_3$

Forma diagonală: $B = T^{-1} A\, T \Leftrightarrow$

$$\underbrace{\begin{pmatrix}3&-3&1\\-3&4&-1\\-1&1&0\end{pmatrix}}_{T^{-1}}\cdot\underbrace{\begin{pmatrix}-1&3&-1\\-3&5&-1\\-3&3&1\end{pmatrix}}_{A}\cdot\underbrace{\begin{pmatrix}1&1&-1\\1&1&0\\1&0&3\end{pmatrix}}_{T} = \begin{pmatrix}3&-3&1\\-6&8&-2\\-2&2&0\end{pmatrix}\cdot\begin{pmatrix}1&1&-1\\1&1&0\\1&0&3\end{pmatrix} =$$

$$= \begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix} = \begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_2\end{pmatrix}$$

(e) Calculul matricei A^n

$B = T^{-1} A T \Rightarrow T\cdot B\cdot T^{-1} = \underbrace{(T\cdot T^{-1})}_{I_3} A\cdot \underbrace{(T T^{-1})}_{I_3} = A$

$A = T\cdot B\cdot T^{-1} \Rightarrow A^2 = T\cdot B\cdot \underbrace{(T^{-1}\cdot T)}_{I_3} B\cdot T^{-1} = T\cdot B^2\cdot T^{-1}$

$A^3 = A^2 A = T B^2 (T^{-1}\cdot T) B\cdot T^{-1} = T\cdot B^3\cdot T^{-1}$

Inducție. PP. $A^n = T\cdot B^n\cdot T^{-1}$ Dem că

$A^{n+1} = A^n\cdot A = T\cdot B^{n+1}\cdot T^{-1}$ q.e.d.

$A^{n+1} = T\cdot B^{n+1}\cdot T^{-1} \Rightarrow$

$$B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 2^2 \end{pmatrix} \dots$$

$$\Rightarrow B^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}$$

$$A^n = T\cdot B^n\cdot T^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}\cdot\begin{pmatrix} 3 & -3 & 1 \\ -3 & 4 & -1 \\ -1 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 2^n & -2^n \\ 1 & 2^n & 0 \\ 1 & 0 & 3\cdot 2^n \end{pmatrix}\cdot\begin{pmatrix} 3 & -3 & 1 \\ -3 & 4 & -1 \\ -1 & 1 & 0 \end{pmatrix} \simeq$$

$$= \begin{pmatrix} 3 - 3\cdot 2^n + 2^n & -3 + 4\cdot 2^n - 2^n & 1 - 2^n \\ 3 - 3\cdot 2^n & -3 + 4\cdot 2^n & 1 - 2^n \\ 3 - 3\cdot 2^n & -3 + 3\cdot 2^n & 1 \end{pmatrix} \quad (\forall)\, n \in \mathbb{N}^*.$$

Problema 2 Fie aplicația liniară $T: \mathbb{R}^3 \to \mathbb{R}^3$.

$T(X) = (x_1 + x_2 - x_3,\ x_1 - x_2 + x_3\ \ 5x_1 - x_2 + x_3)$ - matricea asociată, nucleul, imaginea, verificarea teoremei dimensiunii

$T(X) = A\cdot X;\ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 5 & -1 & 1 \end{pmatrix} = A.$

$\operatorname{Ker} T = \{X \in \mathbb{R}^3 \mid T(X) = 0_3\}.$

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ 5x_1 - x_2 + x_3 = 0 \end{cases};\quad \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 5 & -1 & 1 & 0 \end{array}\right) \simeq \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right) \begin{array}{l} \\ |:-2 \\ |:-6 \end{array}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ x_1 & x_2 & x_3 \end{array}$$

rang A = rang $\bar{A}$ = 2

NPR: x_1, x_2; NS: $x_3 = \alpha$

$$\begin{cases} x_1 = 0 \\ x_2 - x_3 = 0 \\ x_3 = \alpha \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = \alpha \\ x_3 = \alpha \end{cases} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \alpha$$

$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ = o bază pentru Ker T. $\boxed{\dim \operatorname{Ker} T = 1}$

$\operatorname{Im} T = S_p [T(e_1), T(e_2), T(e_3)]$

rang $A = 2 \Rightarrow \{T(e_1), T(e_2)\}$ = bază pt Im f

$\Rightarrow \boxed{\dim \operatorname{Im} f = 2}$

$\dim V = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$ ✓

$3 = 1 + 2$