$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \overset{\infty-\infty}{=} \lim_{n\to\infty} \frac{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$$

$$(a+1)(a-1) = a^2 - 1^2$$

$$= \lim_{n\to\infty} \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\infty} = 0$$

(2) $$\lim_{n\to\infty} (\sqrt{n^2+n} - \sqrt{n^2-n}) \overset{\infty-\infty}{=} \lim_{n\to\infty} \frac{(n^2+n)(n^2-n)}{\sqrt{n^2+n}+\sqrt{n^2-n}} =$$

$$= \lim_{n\to\infty} \frac{n^2+n-n^2-n}{\sqrt{n^2(1+\frac{1}{n})}+\sqrt{n^2(1-\frac{1}{n})}} = \lim_{n\to\infty} \frac{2n}{n\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}$$

$$\overset{\frac{\infty}{\infty}}{=} \frac{2}{1+1} = \frac{2}{2} = 1$$

se rezolvă prin simplificare

amplificăm cu conjugata

(3) $$\lim_{n\to\infty} (\sqrt{3n^2+n} - \sqrt{n^2+1}) \overset{\infty-\infty}{=} \lim_{n\to\infty} \frac{3n^2+n-n^2-1}{\sqrt{3n^2+n}+\sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} = \frac{2n^2+n-1}{n\sqrt{3+\frac{1}{n}}+n\sqrt{1+\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{n^2(2+\underbrace{\tfrac{1}{n}}_{0}-\underbrace{\tfrac{1}{n^2}}_{0})}{n\left(\sqrt{3+\underbrace{\tfrac{1}{n}}_{0}}+\sqrt{1+\underbrace{\tfrac{1}{n^2}}_{0}}\right)} = \infty$$

(4) $$\lim_{n\to\infty} n\sqrt{n}(\sqrt{n+1}+\sqrt{n-1}-2\sqrt{n}) \overset{\infty(\infty-\infty)}{=}$$

$$\lim_{n\to\infty} n\sqrt{n}\left[(\sqrt{n+1}-\sqrt{n})+(\sqrt{n-1}-\sqrt{n})\right] \overset{\infty(\infty-\infty)}{=}$$

$$\lim_{n\to\infty} n\sqrt{n}\left[\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}+\frac{\cancel{n}-1-\cancel{n}}{\sqrt{n-1}+\sqrt{n}}\right]=$$

$$=\lim n\sqrt{n}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}-\frac{1}{\sqrt{n-1}+\sqrt{n}}\right)\overset{\infty-0}{=}$$

$$\lim_{n\to\infty} n\sqrt{n}\cdot\frac{\sqrt{n-1}+\cancel{\sqrt{n}}-\sqrt{n+1}-\cancel{\sqrt{n}}}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n-1}+\sqrt{n})}=$$

$$\lim_{n\to\infty}\frac{n\sqrt{n}(\overset{\infty}{\sqrt{n-1}}\overset{-}{+}\overset{\infty}{\sqrt{n+1}})}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n-1}+\sqrt{n})}$$

$\Rightarrow$ amplificăm cu conjugata

$$\overset{\infty-\infty}{=}\lim_{n\to\infty}\frac{n\sqrt{n}(\sqrt{n-1}+\sqrt{n+1})(\sqrt{n-1}-\sqrt{n+1})}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n-1}+\sqrt{n})(\sqrt{n-1}+\sqrt{n+1})}$$

$$=\lim_{n\to\infty}\frac{\cancel{n\sqrt{n}}(\cancel{n}-1-\cancel{n}-1)}{\cancel{(\sqrt{n})^3}\left(\sqrt{1+\frac{1}{n}}+1\right)\left(\sqrt{1-\frac{1}{n}}+1\right)\left(\sqrt{1-\frac{1}{n}}+\sqrt{1+\frac{1}{n}}\right)}$$

$\sqrt{n}^3=n\sqrt{n}$

pt că sunt 3 paranteze

$$=\frac{-1}{2\cdot1\cdot2}=\frac{1}{4}$$

⑤

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^3}{n^2(n^2+1)}$$

$$S_1 = \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$$

$$S_2 = \sum_{k=1}^{n} k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$ (Gauss)

$$S_3 = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_4 = \sum_{k=1}^{n} k^4 = ?$$

Trebuie să cunoaștem: $\sum k$; $\sum k^2$; $\sum k^3$

Se pornește de la dezvoltarea $(k+1)^5$

$$(k+1)^5 = k^5 + C_5^1 \cdot k^4 + C_5^2 \cdot k^3 + C_5^3 \cdot k^2 + C_5^4 \cdot k^1 + C_5^5 \cdot 1$$

$$(k+1)^5 = k^5 + 5 \cdot k^4 + 10 k^3 + 5k^2 + 5k + 1$$

$k=1$: $2^5 = 1^5 + 5 \cdot 1^4 + 10 \cdot 1^3 + 10 \cdot 1^2 + 5 \cdot 1 + 1$

$k=2$: $3^5 = 2^5 + 5 \cdot 2^4 + 10 \cdot 2^3 + 10 \cdot 2^2 + 5 \cdot 2 + 1$

. — — — — — — — — — — — — — —

$k=n$: $(n+1)^5 = n^5 + 5 \cdot n^4 + 10 \cdot n^3 + 10 \cdot n^2 + 5n + 1$

Total: $\cancel{2^5+3^5+\ldots m^5}+(m+1)^5 = \cancel{1^5+2^5+\ldots+m^5}+$
$+5(1^4+2^4+\ldots+m^4)+$
$+10(1^3+2^3+\ldots+m^3)+$
$+10(1^2+2^2+\ldots+m^2)+$
$+5(1+2+\ldots+m)+m$

$$(m+1)^5 = 5\sum_{k=1}^{m} k^4 + 10\sum_{k=1}^{m} k^3 + 10\sum_{k=1}^{m} k^2 + 5\sum_{k=1}^{m} k + m$$

$$5\cdot S_4 = -(10\cdot S_3 + 10\cdot S_2 + 5\cdot S_1) - m - 1 + (m+1)^5$$

$$\Rightarrow 5\cdot S_4 = (m+1)^5 - (m+1) - 10S_3 - 10S_2 - 5\cdot S_1$$

$$\lim_{m\to\infty} \frac{\sum_{k=1}^{m} k^3}{m^2(m^2+1)} = \lim_{m\to\infty} \frac{\frac{\cancel{m^2}(m+1)^2}{4}}{\cancel{m^2}(m^2+1)} \overset{\frac{\infty}{\infty}}{=} \frac{1}{4}\lim \frac{m^2+2m+1}{m^2+1}$$

$$\overset{\frac{\infty}{\infty}}{=} \frac{1}{4}\lim_{m\to\infty} \frac{\cancel{m^2}\left(1+\frac{2}{m}+\frac{1}{m^2}\right)}{\cancel{m^2}\left(1+\frac{1}{m^2}\right)} = \frac{1}{4}$$

6) $$\lim_{m\to\infty} \frac{2^m+3^m}{3^m+4^m} \overset{\frac{\infty}{\infty}}{=} \lim_{m\to\infty} \frac{3^m\left(\left(\frac{2}{3}\right)^m+1\right)}{4^m\left(\left(\frac{3}{4}\right)^m+1\right)} = \lim_{m\to\infty} \left(\frac{3}{4}\right)^m + \frac{1+\left(\frac{2}{3}\right)^m}{1+\left(\frac{3}{4}\right)^3}$$

$$= 0\cdot\frac{1+0}{1+0} = 0\cdot 1 = 0$$

7) $$e \simeq 2{,}718\ldots = \lim_{m\to\infty} = \left(1+\frac{1}{m}\right)^m$$

$$\lim_{m\to\infty} \frac{\ln(1+e^{5m})}{\ln(1+e^{2m})} \overset{\frac{\infty}{\infty}}{=} \lim_{m\to\infty} \frac{\ln e^{5m}\left(\frac{1}{e^{5m}}+1\right)}{\ln e^{2m}\left(\frac{1}{e^{2m}}+1\right)}$$

$$= \lim_{n\to\infty} = \frac{\ln e^{5n} + \ln\left(1+\frac{1}{e^{5n}}\right)}{\ln e^{2n} + \ln\left(1+\frac{1}{e^{2n}}\right)} = \lim_{n\to\infty} \frac{5n \cdot \ln e + \ln\left(1+\frac{1}{e^{5n}}\right)}{2n \cdot \ln e = \ln\left(1+\frac{1}{e^{2n}}\right)}$$

$$\overset{\frac{\infty}{\infty}}{=}$$

$$\lim_{n\to\infty} \frac{\cancel{n}\left[5+\frac{1}{n}\cdot \ln\left(1+\frac{1}{e^{5n}}\right)\right]}{\cancel{n}\left[2+\frac{1}{n}\cdot \ln\left(1+\frac{1}{e^{2n}}\right)\right]} = \frac{5}{2}$$

$\ln(a\cdot b) = \ln a + \ln b$

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$\ln a^b = b \cdot \ln a$

$\ln 1 = 0$

$\ln e = 0$

(7) $$\lim_{n\to\infty} \frac{\ln(n^3+n-1)}{\ln(n^6+2n^3+n)} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{\ln n^3\left(1+\frac{1}{n^2}-\frac{1}{n^3}\right)}{\ln n^6\left(1+\frac{2}{n^3}+\frac{1}{n^5}\right)} =$$

$$\overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{\ln n^3 + \ln\left(1+\frac{1}{n^2}-\frac{1}{n^3}\right)}{\ln n^6 + \ln\left(1+\frac{2}{n^3}+\frac{1}{n^5}\right)} = \lim_{n\to\infty} \frac{3\ln n + \ln\left(1+\frac{1}{n^2}-\frac{1}{n^3}\right)}{6\ln n + \ln\left(1+\frac{2}{n^3}+\frac{1}{n^5}\right)}$$

$$\overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{\cancel{\ln n}\left[3+\frac{1}{\ln n}\cdot \ln\left(1+\frac{1}{n^2}-\frac{1}{n^3}\right)\right]}{\cancel{\ln n}\left[6+\frac{1}{\ln n}\cdot \ln\left(1+\frac{2}{n^3}+\frac{1}{n^5}\right)\right]} = \frac{3}{6}$$

Cazul 1^∞ $$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \overset{1^\infty}{=} e$$

Dacă $\lim_{n\to\infty} x_n = \infty \Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{x_n}\right)^{x_n} = e$

notăm $y_n = \frac{1}{x_n}$

Dacă $\lim_{n\to\infty} y_n = 0 \Rightarrow \lim_{n\to\infty}(1+y_n)^{\frac{1}{y_n}} = e$

1) $\lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^{3n} \overset{1^\infty}{=} \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n} = \lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^{3} = e^3$

$\lim\limits_{n\to\infty}(a_n)^{b_n} = \left(\lim\limits_{n\to\infty} a^n\right)^{\lim\limits_{n\to\infty} b_n} = a^b$

Dacă a^b are sens!
$(1^\infty, 0^0, \infty^0)$

0^∞ - nu este nedeterminare

2) $\lim\limits_{n\to\infty}\left(\frac{5n+1}{5n}\right)^{6n-4} \overset{1^\infty}{=} \lim\limits_{n\to\infty}\left(1+\frac{5n+1}{5n}-1\right)^{6n-4} = \lim\limits_{n\to\infty}\left(1+\frac{1}{5n}\right)^{6n-4}$

$= \underbrace{\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{5n}\right)^{5n}\right]^{\frac{6n-4}{5n}}}_{(a^b)^c = a^{b\cdot c}} = e^{\lim\limits_{n\to\infty}\frac{6n-4}{5n}} = e^{\frac{6}{5}}$

3) $\lim\limits_{n\to\infty}\left(\frac{2n+3}{2n}\right)^{n+1}\cdot\left(\frac{3n+1}{3n}\right)^{2n+1}$; $\lim(a_n\cdot b_n) = \lim a_n\cdot\lim b_n$

$\lim\left(\frac{2n+3}{2n}\right)^{n+1} = \lim\left(1+\frac{3}{2n}\right)^{n+1} = \lim\left[\left(1+\frac{3}{2n}\right)^{\frac{2n}{3}}\right]^{\frac{3}{2n}\cdot(n+1)}$

$= e^{\lim\limits_{n\to\infty}\frac{3(n+1)}{2n}} = e^{\frac{3}{4}}$

$\frac{3(n+1)}{2n} \;\|\; \frac{3}{2}$

$\lim\left(\frac{3n+1}{3n}\right)^{2n+1} = \lim\limits_{n\to\infty}\left(1+\frac{3n+1}{3n}-1\right)^{2n+1} = \lim\left(1+\frac{1}{3n}\right)^{2n+1}$

$= \lim\left[\left(1+\frac{1}{3n}\right)^{3n}\right]^{\frac{2n+1}{3n}} = e^{\lim\frac{2n+1}{3n}} = e^{\frac{2}{3}}$

(4) $\lim\limits_{n\to\infty}\left(\dfrac{a^{n}+b^{n}}{2}\right)^{n}$ 1^{∞} $\lim\limits_{n\to\infty} 1+\dfrac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}-1=\lim\limits_{n\to\infty}\left(1+\dfrac{a^{\frac{1}{n}}+b^{\frac{1}{n}}-2}{2}\right)^{n}$

când avem 1^{∞}
adăugăm și scădem 1

$$=\lim_{n\to\infty}\left[(1+x_n)^{\frac{1}{x_n}}\right]^{x_n\cdot n}=e^{\lim\limits_{n\to\infty} n\cdot x_n}$$

$$\lim_{n\to\infty} n\cdot x_n=\lim_{n\to\infty} n\cdot\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}-2}{2}\overset{\infty\cdot 0}{=}\lim_{n\to\infty}\frac{1}{2}\left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}-2}{\frac{1}{n}}\right)$$

$$=\frac{1}{2}\lim_{n\to\infty}\left(\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}+\frac{b^{\frac{1}{n}}-1}{\frac{1}{n}}\right)=\frac{1}{2}(\ln a+\ln b)=\frac{1}{2}\cdot\ln(ab)$$

$$=\ln(ab)^{\frac{1}{2}}=\ln\sqrt{ab}$$

$$\Rightarrow \lim\left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^{n}=e^{\lim n\cdot x_n}=e^{\ln\sqrt{ab}}=\sqrt{ab}$$