Seminar 3

1. Metoda integrării directe (continuare)

2) $\int \frac{dx}{9x^2-4} = \frac{1}{9}\int \frac{dx}{x^2-\left(\frac{2}{3}\right)^2} = \frac{1}{2\cdot\frac{2}{\cancel{3}}\cdot\cancel{9}\,3} \ln\left|\frac{x-\frac{2}{3}}{x+\frac{2}{3}}\right| + c = \frac{1}{12}\ln\left|\frac{3x-2}{3x+2}\right| + C$

3) $\int \frac{(2x^2+3)\,dx}{(x^2-1)(x^2+4)} = \int \frac{x^2-1+x^2+4}{(x^2-1)(x^2+4)}\,dx = \int \frac{\cancel{x^2-1}}{(\cancel{x^2-1})(x^2+4)}\,dx + \int \frac{\cancel{x^2+4}}{(x^2-1)(\cancel{x^2+4})}\,dx =$

$= \int \frac{dx}{x^2+2^2} + \int \frac{dx}{x^2-1} = \frac{1}{2}\operatorname{arctg}\frac{x}{2} + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

4) $\int \frac{\sqrt{2x^2-8}+1}{2x^2-8}\,dx = \int \frac{\sqrt{2x^2-8}+1}{2x^2-8}\,dx + \int \frac{dx}{2x^2-8} = \int \frac{dx}{\sqrt{2x^2-8}} + \frac{1}{2}\int \frac{dx}{\cancel{2}x^2-4} =$

$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2-4}} + \frac{1}{8}\ln\left|\frac{x-2}{x+2}\right| = \frac{\sqrt{2}}{2}\ln\left|x+\sqrt{x^2-4}\right| + \frac{1}{8}\ln\left|\frac{x-2}{x+2}\right| + C$

2. Metoda integrării prin părți:

Această metodă, ca și metodele de schimbare de variabilă, se utilizează atunci când expresia de sub integrală nu se regăsește în tabelele primitivelor imediate sau este un produs de 2 funcții.

$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\,g(x)\,dx$

Alegerea funcțiilor f și g' din integrala de calculat, se face, urmărind ca integrala din membrul dx, care se obține aplicând formula, să fie mai ușor de calculat decât cea inițială;
Uneori metoda se aplică succesiv {De mai multe ori} până când se ajunge la rezultatul dorit.

2. Ex.)

1) $I = \int x^3 \ln^2 x\, dx$

$f(x) = \ln^2 x \Rightarrow f'(x) = \frac{2}{x}\ln x$

$g'(x) = x^3 \Rightarrow g(x) = \frac{x^4}{4}$

$\Big\}\ I = \frac{x^4}{4}\cdot\ln^2 x - \int \frac{2}{\cancel{x}}\cdot\frac{x^{\cancel{4}}}{\cancel{4}}\cdot\ln x\,dx =$

$= \frac{x^4}{4}\ln^2 x - \frac{1}{2}\underbrace{\int x^3\ln x\,dx}_{J}$

$J = \int x^3 \ln x\,dx$

$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

$g'(x) = x^3 \Rightarrow g(x) = \frac{x^4}{4}$

$\Big\}\ = \frac{x^4}{4}\cdot\ln x - \int \frac{x^3}{4}\,dx = \frac{x^4}{4}\ln x - \frac{1}{4}\cdot\frac{x^4}{4} + C$

$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$

$$I = \frac{x^4}{4} \cdot \ln^2 x - \frac{1}{2}\left(\frac{x^4}{4}\ln x - \frac{x^4}{16}\right) + C$$

$$= \frac{x^4}{4}\left(\ln^2 x - \frac{\ln x}{2} + \frac{1}{8}\right) + C = \frac{x^4}{32}\left(8\ln^2 x - 4\ln x + 1\right) + C$$

2.) $I = \int e^{\alpha x} \cdot \sin \beta x \, dx$; $\quad J = \int e^{\alpha x} \cdot \cos \beta x \, dx$

I: $f(x) = e^{\alpha x} \Rightarrow f'(x) = \alpha \cdot e^{\alpha x}$

$g'(x) = \sin \beta x \quad g(x) = \int \sin \beta x \, dx = -\frac{1}{\beta}\int (-\beta) \cdot \sin \beta x \, dx = -\frac{1}{\beta}\int (\cos \beta x)' \, dx = -\frac{\cos \beta x}{\beta}$

$$I = -\frac{e^{\alpha x}}{\beta} \cdot \cos \beta x - \int \alpha \cdot e^{\alpha x} \cdot \left(-\frac{\cos \beta x}{\beta}\right) dx = -\frac{e^{\alpha x}}{\beta}\cos \beta x + \frac{\alpha}{\beta}\underbrace{\int e^{\alpha x}\cos \beta x \, dx}_{J}$$

$$I = -\frac{e^{\alpha x}}{\beta}\cos \beta x + \frac{\alpha}{\beta} J \Rightarrow \boxed{\frac{\alpha}{\beta} J - I = \frac{e^{\alpha x}}{\beta}\cos \beta x}$$

J: $f(x) = e^{\alpha x} \Rightarrow f'(x) = \alpha \cdot e^{\alpha x}$

$g'(x) = \cos \beta x \Rightarrow g(x) = \int \cos \beta x \, dx = \frac{\sin \beta x}{\beta}$

$$J = e^{\alpha x} \cdot \frac{\sin \beta x}{\beta} - \int \alpha \cdot e^{\alpha x} \cdot \frac{\sin \beta x}{\beta} dx = e^{\alpha x} \cdot \frac{\sin \beta x}{\beta} - \frac{\alpha}{\beta}\underbrace{\int e^{\alpha x}\sin \beta x \, dx}_{I}$$

$$J = \frac{e^{\alpha x}}{\beta}\sin \beta x - \frac{\alpha}{\beta} I \Rightarrow \boxed{\frac{\alpha}{\beta} I + J = \frac{e^{\alpha x}}{\beta}\sin \beta x}$$

$$\begin{cases} I - \frac{\alpha}{\beta} J = -\frac{e^{\alpha x}}{\beta}\cos \beta x & \Big| \cdot \left(-\frac{\alpha}{\beta}\right) \\ \frac{\alpha}{\beta} I + J = \frac{e^{\alpha x}}{\beta}\sin \beta x & \Big| \cdot \left(\frac{\alpha}{\beta}\right) \end{cases} \rightarrow \left(1 + \frac{\alpha^2}{\beta^2}\right) I = \frac{e^{\alpha x}}{\beta}\left(-\cos \beta x + \frac{\alpha}{\beta}\sin \beta x\right) \Rightarrow I$$

$$\Rightarrow \begin{cases} -\frac{\alpha}{\beta} I + \frac{\alpha^2}{\beta^2} J = \frac{\alpha e^{\alpha x}}{\beta}\cos \beta x \\ \frac{\alpha}{\beta} I + J = \frac{e^{\alpha x}}{\beta}\sin \beta x \quad (+) \end{cases}$$

$$/ \quad \left(1 + \frac{\alpha^2}{\beta^2}\right) J = \frac{e^{\alpha x}}{\beta}\left(\frac{\alpha}{\beta}\cos \beta x + \sin \beta x\right) \Rightarrow J$$

3. Metoda relațiilor de recurență pentru calculul primitivelor:

1) Să se stabilească formula de recurență pentru:

$$I_n = \int \ln^n x \, dx \,, \quad n \in \mathbb{N}$$

$$I_n = \int \ln^n x \cdot 1 \, dx$$

$$\left.\begin{aligned} f(x) &= \ln^n x \Rightarrow f'(x) = n \cdot \ln^{n-1} x \cdot \frac{1}{x} \\ g'(x) &= 1 \Rightarrow g(x) = x \end{aligned}\right\} \quad \begin{aligned} I_n &= x \cdot \ln^n x - \int n \cdot \ln^{n-1} x \, dx \\ I_n &= x \ln^n x - n \underbrace{\int \ln^{n-1} x \, dx}_{I_{n-1}} \end{aligned}$$

$$\boxed{I_n = x \ln^n x - n \cdot I_{n-1}}$$

Tema: $I_5 = ?$

2) $$I_n = \int \frac{x^n}{\sqrt{x^2+a^2}} \, dx = \int x^{n-1} \cdot \frac{2x}{2\sqrt{x^2+a^2}} \, dx = \frac{1}{2} \int x^{n-1} \cdot \left(\sqrt{x^2+a^2}\right)' dx =$$

$$= x^{n-1} \sqrt{x^2+a^2} - \int (n-1) x^{n-2} \cdot \sqrt{x^2+a^2} \, dx =$$

$$= x^{n-1} \cdot \sqrt{x^2+a^2} - (n-1) \int x^{n-2} \cdot \frac{x^2+a^2}{\sqrt{x^2+a^2}} \, dx =$$

$$= x^{n-1} \cdot \sqrt{x^2+a^2} - (n-1) \int \frac{x^n}{\sqrt{x^2+a^2}} \, dx - (n-1) \int \frac{a^2 \cdot x^{n-2}}{\sqrt{x^2+a^2}} \, dx$$

$$I_n = x^{n-1} \sqrt{x^2+a^2} - (n-1) I_n - a^2 (n-1) I_{n-2}$$

$$n I_n = x^{n-1} \sqrt{x^2+a^2} + a^2 (n-1) I_{n-2}$$

$$\boxed{I_n = \frac{x^{n-1}}{n} \sqrt{x^2+a^2} + \frac{n-1}{n} \cdot a^2 I_{n-2}}$$

Temă:

1) $J_n = \int \frac{x^n dx}{\sqrt{x^2-a^2}} \, dx$ 2) $K_n = \int \frac{x^n}{\sqrt{a^2-x^2}} \, dx$