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Seminar 2

Sisteme de ecuații liniare

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$a_{ij}, b_i \in K; \ i = \overline{1,m}, \ j = \overline{1,n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$AX = B$

Sisteme
- compatibile
  - → Determinate {soluție unică}
  - → Nedeterminate {inf de soluții}
- necompatibile {nu au soluții}

Dacă $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ → sisteme omogenee

OBS: Sistemele omogenee au tot timpul soluția banală $x_1, x_2, \dots, x_n = \emptyset$

În cazul în care $A \in M_{n \times n}(K)$

Metoda lui Cramer

$x_i = \frac{dx_i}{d}$
- → $d = \det(A)$
- → $dx_i = \begin{vmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ \vdots & & \vdots & & \\ a_{m1} & & b_n & & a_{mn} \end{vmatrix}$

Ex. Să se determine soluția sistemului

$$\begin{cases} x+y-2z=-1 \\ 2x-y+4z=-4 \\ 4x+y+4z=-2 \end{cases} \rightarrow A=\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 4 \\ 4 & 1 & 4 \end{bmatrix} \quad x=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B=\begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix}$$

$\det(A)=12 \rightarrow$ Se poate aplica Cramer

$d_x=4 \quad d_y=8 \quad d_z=-12$

$x=\frac{1}{3} \quad y=\frac{2}{3} \quad z=-1$

Metoda eliminării a lui Gauss

Algoritmu

Fie $\bar{A}=(A|B)$

1. Se stabilește PIVOTUL $\rightarrow a_{11}\neq 0$
2. Elementele de pe linia PIVOTULUI se transcriu neschimbate
3. Elementele de pe coloana PIVOTULUI se transcriu cu 0
4. Celelalte elemente din $\bar{A}$ se calculează cu regula dreptunghiului

Exemplu

$$\begin{cases} x+y-2z=-1 \\ 2x-y+4z=-4 \\ 2x+y+4z=-2 \end{cases} \quad \bar{A}=\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 2 & -1 & 4 & -4 \\ 4 & 1 & 4 & -2 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & 0 & -2 \\ 0 & -3 & -4 & 2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} -3 & 0 & -6 & 5 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 12 & -12 \end{array}\right] :12 \rightarrow \left[\begin{array}{ccc|c} -3 & 0 & -6 & 5 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -3 & 0 & 0 & -1 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$= \begin{cases} -3x=-1 \\ -3y=-2 \\ z=-1 \end{cases} \rightarrow \begin{aligned} x &= 1/3 \\ y &= 2/3 \\ z &= -1 \end{aligned}$$

Exemplu 2

$$\begin{cases} 3x - 6y + 12z = 6 \\ -2x + 5y - 9z = -7 \\ -x + 3y - 5z = -4 \end{cases}$$

$$\begin{array}{c} :3 \\ \\ \\ \end{array} \overline{A} = \left[\begin{array}{ccc|c} 3 & -6 & 12 & 6 \\ -2 & 5 & -9 & -7 \\ -1 & 3 & -5 & -4 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ -2 & 5 & -9 & -7 \\ -1 & 3 & -5 & -4 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \begin{cases} x + 2z = -4 \\ y - z = -3 \\ 0 = 1 \end{cases} \Rightarrow \text{Incompatibil}$$

IMPOSIBIL (F)

Exemplu 3

$$\begin{cases} -x_3 + 4x_4 = 2 \\ x_1 - 2x_2 + 4x_3 + 3x_4 = 4 \\ 3x_1 - 6x_2 + 8x_3 + 5x_4 = 0 \end{cases}$$

$$\overline{A} = \begin{array}{c} \left[\begin{array}{cccc|c} 0 & 0 & -1 & 4 & 2 \\ 1 & -2 & 4 & 3 & 4 \\ 3 & -6 & 8 & 5 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 4 & 3 & 4 \\ 0 & 0 & -1 & 4 & 4 \\ 3 & -6 & 8 & 5 & 0 \end{array}\right]$$

$$\simeq \left[\begin{array}{cccc|c} 1 & -2 & 4 & 3 & 4 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & -4 & -4 & -12 \end{array}\right] \begin{array}{c} \\ \\ :-4 \end{array} \simeq \left[\begin{array}{cccc|c} 1 & -2 & 4 & 3 & 4 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

$$\simeq \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 3 & 4 \\ 0 & -1 & 0 & 4 & 2 \\ 0 & 1 & 0 & 1 & 3 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_3 & x_2 & x_4 & b \end{array} \end{array} \simeq \left[\begin{array}{cccc|c} -1 & 0 & 2 & -19 & -12 \\ 0 & -1 & 0 & 4 & 2 \\ 0 & 0 & 0 & -5 & -5 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} -1 & 0 & 2 & -19 & -12 \\ 0 & -1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \sim \begin{array}{c} \left[\begin{array}{cccc|c} -1 & 0 & -19 & 2 & -12 \\ 0 & -1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_3 & x_4 & x_2 & b \end{array} \end{array}$$

$$\sim \begin{array}{c} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 2 & 7 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_3 & x_4 & x_2 & b \end{array} \end{array} \rightarrow \begin{cases} -x_1 + 2x_2 = 7 \\ -x_3 = -2 \\ x_4 = 1 \end{cases}$$

$x_1 = 2\alpha - 7$

$x_3 = 2$

$x_4 = 1$

$x_2 = \alpha$

$$S = \left\{ (2\alpha - 7), (\alpha), (2), (1) \right\}, \ \alpha \in \mathbb{R}$$

OBS | Kronicker-Capelli:

$Ax = B$ den m ecuații și n necunoscute $\Leftrightarrow$ este compatibil dacă $rang(A) = rang(A|B)$.

Rangul unei Matrici

Algoritm

1. Se alege PIVOTUL $a_{11} \neq 0$. PIVOTUL se înlocuiește cu 1
2. Elementele de pe linia și coloana pivotului se înlocuiesc cu 0
3. Celelalte elemente -> regula dreptunghiului

Exemplu 1 Să se calculeze rangul

$$A = \begin{bmatrix} 2 & 3 & 1 & 5 & 4 & 6 \\ 1 & 3 & 0 & 4 & 3 & 4 \\ 3 & -1 & 2 & 2 & 1 & 4 \\ -2 & 4 & -2 & 2 & 2 & 0 \\ 4 & 2 & -1 & 6 & 3 & 7 \end{bmatrix} \in \mathcal{M}_{5,6}(\mathbb{R})$$

$rang \leq 5$

$$\simeq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 3 & 2 & 2 \\ 0 & -11 & 1 & -11 & -10 & -10 \\ 0 & 14 & -2 & 14 & 12 & 12 \\ 0 & -8 & -2 & -8 & -10 & -10 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 3 & 2 & 2 \\ 0 & -11 & 1 & -11 & -10 & -10 \\ 0 & +7 & -1 & 7 & 6 & 6 \\ 0 & +4 & 1 & +4 & +5 & +5 \end{bmatrix} \simeq$$

$$\simeq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & -8 & -8 \\ 0 & 0 & 4 & 0 & 4 & 4 \\ 0 & 0 & 7 & 0 & 7 & 7 \end{bmatrix} \begin{array}{l} \\ \\ :-8 \\ :4 \\ :7 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\simeq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rang $(A) = 3$

Calculul inversei unei Matrici

$A \in \mathcal{M}_{n \times n}(K)$

$(A_n \mid I_n) \to (A_n^{(1)} \mid I_n^{(1)}) \to \dots \to (I_n \mid A^{-1})$

Exemplu 1 $\quad A^{-1} = ?$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 7 & -7 & -2 & 1 & 0 \\ 0 & 7 & -7 & -3 & 0 & 1 \end{array}\right] \simeq \left[\begin{array}{ccc|ccc} 7 & 0 & 7 & 3 & 2 & 0 \\ 0 & 7 & -7 & -2 & 1 & 0 \\ 0 & 0 & 42 & 11 & 5 & 7 \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|ccc} 294 & 0 & 0 & 49 & 49 & -49 \\ 0 & 294 & 0 & -7 & 77 & 49 \\ 0 & 0 & 42 & 11 & 5 & 7 \end{array}\right] \simeq \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & -\frac{1}{42} & \frac{11}{42} & \frac{1}{6} \\ 0 & 0 & 1 & \frac{11}{42} & \frac{5}{42} & \frac{1}{6} \end{array}\right]$$

$$\underbrace{\begin{matrix} \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{42} & \frac{11}{42} & \frac{1}{6} \\ \frac{11}{42} & \frac{5}{42} & \frac{1}{6} \end{matrix}}_{A^{-1}}$$