Temă de control pentru V.I. nr. 1

1. Ecuații diferențiale cu variabile separate

1. $$\frac{y\,dy}{\sqrt{1+y^2}} + \frac{x\,dx}{\sqrt{1+x^2}} = 0$$

$$\int \frac{y\,dy}{\sqrt{1+y^2}} + \int \frac{x\,dx}{\sqrt{1+x^2}} = c \Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} = c$$

$$\Rightarrow \sqrt{1+y^2} = c - \sqrt{1+x^2} \Rightarrow 1+y^2 = \left(c-\sqrt{1+x^2}\right)^2 \Rightarrow$$

$$\Rightarrow y^2 = \left(c-\sqrt{1+x^2}\right)^2 - 1 \Rightarrow y^2 = c^2 - 2c\sqrt{1+x^2} \cancel{+1} + x^2 \cancel{-1}$$

$$\Rightarrow y = \pm\sqrt{c^2+x^2-2c\sqrt{1+x^2}}$$

2. $$\frac{dx}{1-x} = \frac{y\,dy}{1-y^2}, \quad x_0 = -1, \quad y_0 = 3$$

$$\int \frac{dx}{1-x} = \int \frac{y}{1-y^2}\,dy \Rightarrow -\ln(1-x) \cancel{+C} = \sqrt{1-y^2} \cancel{+C}$$

$$\cancel{\Rightarrow \left[-\ln(1-x)\right]^2 = \left[\sqrt{1-y^2}\right]^2 \Rightarrow y^2 = 1 - \ln^2(1-x)}$$

Fie $c = \ln e^k \Rightarrow \ln e^k - \ln(1-x) = \sqrt{1-y^2}$

$$\left(\ln\frac{e^k}{1-x}\right)^2 = \left(\sqrt{1-y^2}\right)^2 \Rightarrow \ln^2\frac{e^k}{1-x} = 1-y^2$$ forma explicită

2. Ecuații diferențiale cu variabile separate

1. $$y' = \overset{x(y^2+2)}{1} + \overset{y^2+2}{\frac{1}{x}} - \overset{x}{\frac{1}{y^2+2}} - \frac{1}{x(y^2+2)}$$

$$y' = \frac{xy^2+2x+y^2+2-x-1}{x(y^2+2)} \Rightarrow y' = \frac{y^2(x+1)+x+1}{x(y^2+2)}$$

$$\Rightarrow y' = \frac{(y^2+1)(x+1)}{x(y^2+2)} \Rightarrow \frac{y^2+2}{y^2+1}\,dy = \frac{x+1}{x}\,dx$$

$\left(1+\frac{1}{y^2+1}\right)dy=\left(1+\frac{1}{x}\right)dx$ – ecuație diferențială cu variabile separate

$\int\left(1+\frac{1}{y^2+1}\right)dy=\int\left(1+\frac{1}{x}\right)dx+C \Rightarrow y+\operatorname{arctg} y = x+\ln|x|+C$

↳ forma implicită

2\. $\frac{dy}{1-y}=dx-\frac{dx}{1+x}$

$\frac{dy}{1-y}=\left(1-\frac{1}{1+x}\right)dx \Rightarrow \int\frac{dy}{1-y}=\int\left(1-\frac{1}{1+x}\right)dx+C$

$\Rightarrow -\ln(1-y)=x-\ln(x+1)+C \Rightarrow \ln(x+1)-\ln(1-y)=x+C$

$\Rightarrow \ln\frac{x+1}{1-y}=\ln e^{x+C} \Leftrightarrow \frac{x+1}{1-y}=e^{x+C} \Rightarrow y=\frac{x+1}{e^{x+C}}-1$

↳ forma ~~implicită~~ explicită

3\. Ecuații diferențiale omogene

1\. $(x+y)\,dy+(y-x)\,dx=0$

$(x+y)dy=(x-y)dx \Rightarrow dy=\frac{x-y}{x+y}dx \Rightarrow dy=\frac{1-\frac{y}{x}}{1+\frac{y}{x}}dx \Bigg\} \Rightarrow$

Fie $y=z\cdot x \Rightarrow y'=dy=z'\cdot x+z$

sau $z=y/x$

$\Rightarrow z'\cdot x+z=\frac{1-z}{1+z}$ ~~$dx \Rightarrow dz\cdot x$~~ $\Rightarrow z'\cdot x=\frac{1-z}{1+z}-z \Rightarrow$

$\Rightarrow z'\cdot x=\frac{1-2z-z^2}{1+z} \Rightarrow \frac{dz}{dx}\cdot x=\frac{1-2z-z^2}{1+z} \Rightarrow \frac{1+z}{1-2z-z^2}dz=\frac{dx}{x}$

$\Rightarrow$ ecuație cu variabile separate

$$\int \frac{1+z}{1-2z-z^2}\,dz = \int \frac{1}{x}\,dx$$

Fie $1-2z-z^2 \stackrel{not}{=} u$, $(\ln u)' = \frac{u'}{u} = \frac{(1-2z-z^2)'}{1-2z-z^2} = \frac{-2(1+z)}{1-2z-z^2}$

$$\Rightarrow -\frac{1}{2}\ln(1-2z-z^2) = \ln|x| - \frac{1}{2}\ln c \Rightarrow \ln(1-2z-z^2)^{-\frac{1}{2}} + \ln c^{\frac{1}{2}} = \ln|x|$$

$$\Rightarrow \ln\frac{\sqrt{c}}{\sqrt{(1-2z-z^2)}} = \ln|x| \Rightarrow (\sqrt{c})^2 = (|x|\sqrt{1-2z-z^2})^2 \Rightarrow$$

$$\Rightarrow c^2 = x^2(1-2z-z^2)$$

$z = \frac{y}{x}$

$$\Rightarrow c^2 = x^2\left(1 - 2\frac{y}{x} - \frac{y^2}{z^2}\right) \Rightarrow$$

$$\Rightarrow \underline{c^2 = x^2 - 2xy - y^2}$$

3. $(3x^2 - y^2)\,dy = 2xy\,dx$, $x_0 = 0$, $y_0 = 1$

$$(3x^2 - y^2)\,dy = 2xy\,dx \;\Big|\; : x^2 \Rightarrow \left[3 - \left(\frac{y}{x}\right)^2\right]dy = 2\frac{y}{x}\,dx$$

Fie $\frac{y}{x} = u \Rightarrow y = u\cdot x \Rightarrow y' = dy = du\cdot x + x \;|\; : dx \Rightarrow$

$$\frac{dy}{dx} = \frac{du}{dx}\cdot x + \frac{x}{dx} = u$$

$$\Rightarrow (3-u^2)\,dy = 2u\,dx \Rightarrow \frac{dy}{dx} = \frac{2u}{3-u^2}$$

$$\Rightarrow \cancel{\frac{dy}{dx}} \;\; \frac{du}{dx}\cdot x + u = \frac{2u}{3-u^2} \Rightarrow$$

$$\Rightarrow 3x\,du - u x\,du + 3u\,dx - u^3\,dx = 2u\,dx$$

$$u\,dx - u^3\,dx = ux\,du - 3x\,du$$

$$(u-u^3)\,dx = x(u-3)\,du \Rightarrow \frac{u-3}{u-u^3}\,du = \frac{1}{x}\,dx =$$

= ecuație cu variabile separate

$$\Rightarrow \int \frac{u-3}{u-u^3}\,du = \int \frac{1}{x}\,dx$$

$$\int \frac{1}{x} = \ln|x| + \ln c = \ln c|x|, \quad c>0$$

$$\int \frac{u-3}{u-u^3}\,du = \int \frac{u}{u-u^3}\,du + 3\int \frac{1}{u^3-u}\,du =$$

$$= -\int \frac{1}{u^2-1}\,du + 3\int \frac{1}{u(u-1)(u+1)}\,du$$

$$\frac{1}{u(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1} = \frac{A(u^2-1) + B(u^2+u) + C(u^2-u)}{u(u-1)(u+1)}$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ -A=1 \\ B-C=0 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=C=\frac{1}{2} \end{cases}$$

$$\Rightarrow -\int \frac{1}{u^2-1}\,du + 3\left[-\int \frac{1}{u}\,du + \frac{1}{2}\int \frac{1\,du}{u-1} + \frac{1}{2}\int \frac{1}{u+1}\,du\right] =$$

$$= -\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + 3\left(-\ln|u| + \frac{1}{2}\ln|u-1| + \frac{1}{2}\ln|u+1|\right) =$$

$$= 3\left(\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| - \ln|u|\right) - \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| =$$

$$= \frac{3}{2}\ln\left|\frac{u-1}{u+1}\right| - 3\ln|u| - \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| =$$

$$= \ln\left|\frac{u-1}{u+1}\right| - 3\ln|u| = \ln\left|\frac{u-1}{(u+1)u^3}\right|$$

$$\Rightarrow \ln\left|\frac{u-1}{(u+1)u^3}\right| = \ln C|x| \Rightarrow C|x| = \left|\frac{u-1}{u^3(u+1)}\right|$$

$$C|x| = \left|\frac{\frac{y}{x} - \frac{x}{1}}{\frac{y^3}{x^3}\left(\frac{y}{x} + \frac{x}{1}\right)}\right| \Rightarrow C|x| = \left|\frac{\frac{y-x}{\cancel{x}}}{\frac{y^3}{x^3}\cdot\frac{y+x}{\cancel{x}}}\right| \Rightarrow$$

$$\Rightarrow C|x| = \left|\frac{x^3}{y^3}\cdot\frac{y-x}{y+x}\right| \Rightarrow C = \left|\frac{x^2(y-x)}{y^3(y+x)}\right|,\ C>0 \Rightarrow$$

$$\Rightarrow x^2y - x^3 = C(y^4+xy).\quad \text{Pt } x_0=0,\ y_0=1 \Rightarrow C=0$$

$$\Rightarrow \boxed{x^2y - x^3 = 0}$$

2. $y\,dx + (2\sqrt{xy} - x)\,dy = 0 \quad |:x$

$\Rightarrow \frac{y}{x}dx + \left(2\sqrt{\frac{y}{x}} - 1\right)dy = 0 \Rightarrow \frac{y}{x}dx = \left(1-2\sqrt{\frac{y}{x}}\right)dy \Rightarrow$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}}{1-2\sqrt{\frac{y}{x}}}$ (1)

Fie $\frac{y}{x} = z \Rightarrow y = z\cdot x \Rightarrow dy = y' = d(z\cdot x) = \underbrace{\overset{z'}{dz}\cdot x + \overset{1}{dx}\cdot z}_{}\ \Big|:dx$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx}\cdot x + z$ (2)

(1), (2) $\Rightarrow \frac{x\,dz}{dx} + z = \frac{z}{1-2\sqrt{z}} \Rightarrow x\,dz - 2x\sqrt{z}\,dz + \cancel{z\,dx} - 2z\sqrt{z}\,dx = \cancel{z\,dx} \Rightarrow x(1-2\sqrt{z})dz = 2z\sqrt{z}\,dx \Rightarrow$

$\Rightarrow \frac{dx}{x} = \frac{(1-2\sqrt{z})}{2z\sqrt{z}}dz$ = ecuație cu variabile separate

$\Rightarrow \int\frac{dx}{x} = \int\frac{1-2\sqrt{z}}{2z\sqrt{z}}dz \Rightarrow \int\frac{1}{x}dx = \int\frac{1}{2z\sqrt{z}}dz - \int\frac{1}{z}dz$

$\Rightarrow \int\frac{1}{x}dx = \frac{1}{2}\int\frac{1}{z^{\frac{3}{2}}}dz - \int\frac{1}{z}dz$

$\Rightarrow \int\frac{1}{x}dx = \frac{1}{2}\int z^{-\frac{3}{2}}dz - \int\frac{1}{z}dz$

$\Rightarrow \ln|x| + \ln c = \frac{1}{2}\cdot\frac{(-2)}{1}z^{-\frac{1}{2}} - \ln|z| \Rightarrow \ln c|x| = -z^{-\frac{1}{2}} - \ln|z|$

$\Rightarrow \ln c|x| = -\left(\frac{y}{x}\right)^{-\frac{1}{2}} - \ln\left|\frac{y}{x}\right| \Rightarrow \underbrace{\ln c|y| = -\sqrt{\frac{x}{y}}}_{}$.

4. Ecuații diferențiale reductibile la ecuații omogene:

2. $(3x+3y-1)dx + (x+y+1)dy = 0$

$$dy = y' = \frac{3x+3y-1}{-x-y-1}\,dx$$

Ecuațiile sunt de forma $ax+by+c=0$ și $a'x+b'y+c'=0$, dar $ab'-a'b=0 \Rightarrow$ nu se pot face substituții de tipul $u = x - x_1$ și $v = y - y_1$, unde (x_1, y_1) soluție a sistemului format de cele două ecuații.

Fie $z = ax+by \Rightarrow z' = dy = 3+3y' \Rightarrow dy = y' = \frac{z'-3}{3}$

$$a'x+b'y+c' \overset{ab'=a'b}{=} \frac{ab'}{b}x + b'y + c' = \frac{b'}{b}(ax+by)+c' =$$
$$= \frac{b'}{b}z + c'$$

$$\cancel{\Rightarrow \frac{z'-3}{3}\,dz = \frac{(z-1)dx}{-\frac{1}{3}z-1} \Rightarrow \frac{z'-3}{3}\,dz = \frac{(z-1)dx}{\frac{-z-3}{3}} \Rightarrow}$$

$$\cancel{\Rightarrow 3+3dy = \frac{(-3z+3)dz}{z+3} \Rightarrow 3+3dy - 3\left(\frac{1-z}{z+3}\right)dz = 0}$$

~~= ecuație cu variabile separate.~~

$\cancel{\int 3+dy}$

$$\Rightarrow \frac{z'-3}{3} = \frac{z-1}{-\frac{1}{3}z-1} \Rightarrow \frac{z'-3}{3} = \frac{z-1}{\frac{-z-3}{3}} \Rightarrow \frac{z'-3}{3} = \frac{3(z-1)}{-z-3}$$

$$\Rightarrow 9z-9 = -zz'-3z'+3z+9 \Rightarrow 6z+z\cdot z'+3z'-18=0$$

$$\Rightarrow (z+3)z' = 18-6z \Rightarrow \frac{z+3}{6(3-z)}\cdot z' = 1 \Rightarrow -\frac{1}{6}\int\frac{z+3}{z-3}\,dz = \int dx$$

$$\Rightarrow -\frac{1}{6}\left[\int \cancel{\frac{z-3}{z-3}}\,dz + 6\int\frac{1}{z-3}\,dz\right] = \int dx \Rightarrow$$

$$\Rightarrow -\frac{1}{6}z + 6\ln|z-3| = \ln|x| + c$$

$$\Rightarrow -\frac{1}{6}(3x+3y) + 6\ln|3x+3y-3| = \ln|x|+c$$

$$\Rightarrow -\frac{1}{2}x - \frac{1}{2}y + 6\ln 3 + 6\ln(x+y-1) = \ln|x| + c$$