Temă curs 02

Rezolvați ecuațiile:

a) $\left(\frac{1}{y} - \frac{y}{x^2}\right)dx + \left(\frac{1}{x} - \frac{x}{y^2}\right)dy = 0$

$P(x,y) = \frac{1}{y} - \frac{y}{x^2}$

$Q(x,y) = \frac{1}{x} - \frac{x}{y^2}$

$\frac{\partial P}{\partial y} = \left(\frac{1}{y}\right)' - \frac{1}{x^2}\cdot(y)' = -\frac{1}{y^2} - \frac{1}{x^2}$

$\frac{\partial Q}{\partial x} = \left(\frac{1}{x}\right)' - \frac{1}{y^2}\cdot x' = -\frac{1}{x^2} - \frac{1}{y^2}$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \int_{x_0}^{x} P(t,y_0)dt - \int_{y_0}^{y} Q(x,t)dt =$

$= \int_{x_0}^{x}\left(\frac{1}{y_0} - \frac{y_0}{t^2}\right)dt + \int_{y_0}^{y}\left(\frac{1}{x} - \frac{x}{t^2}\right)dt =$

$= \int_{x_0}^{x}\frac{1}{y_0}dt - \int_{x_0}^{x}\frac{y_0}{t^2}dt + \int_{y_0}^{y}\left(\frac{1}{x}\right)dt - \int_{y_0}^{y}\frac{x}{t^2}dt =$

$= t\cdot\frac{1}{y_0}\Big|_{x_0}^{x} - y_0\cdot\frac{t^{-1}}{-1}\Big|_{x_0}^{x} + \frac{1}{x}\cdot t\Big|_{y_0}^{y} - x\frac{(-1)}{t}\Big|_{y_0}^{y} =$

$= \cancel{\frac{x}{y_0}} - \frac{x_0}{y_0} + \cancel{\frac{y_0}{x}} - \frac{y_0}{x_0} + \frac{y}{x} - \cancel{\frac{y_0}{x}} + \frac{x}{y} - \cancel{\frac{x}{y_0}} =$

$= \left(-\frac{x_0}{y_0} - \frac{y_0}{x_0}\right) + \left(\frac{y}{x} + \frac{x}{y}\right) = \underbrace{\left(\frac{x}{y} + \frac{y}{x}\right)}_{F(x,y)=C} - \left(\frac{x_0}{y_0} + \frac{y_0}{x_0}\right)$

$\hookrightarrow \frac{x}{y} + \frac{y}{x} = C$ ✓

$\hookrightarrow$ forma implicită

b) $\left(-\frac{1}{x} + y + \frac{y}{x^2+y^2}\right)dx + \left(\frac{1}{y} + x - \frac{x}{x^2+y^2}\right)dy = 0$

$P(x,y) = -\frac{1}{x} + y + \frac{y}{x^2+y^2}$

$Q(x,y) = \frac{1}{y} + x - \frac{x}{x^2+y^2}$

$\frac{\partial P}{\partial y} = \left(-\frac{1}{x}\right)' + y' + \left(\frac{y}{x^2+y^2}\right)' = 1 + \frac{y'\cdot(x^2+y^2) - (x^2+y^2)'\cdot y}{(x^2+y^2)^2}$

$\frac{\partial P}{\partial y} = 1 + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = 1 + \frac{x^2-y^2}{(x^2+y^2)^2}$ ✓

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$$\frac{\partial Q}{\partial x} = \left(\frac{1}{y}\right)'_x + x' - \left(\frac{x}{x^2+y^2}\right)'_x = 1 - \frac{x'(x^2+y^2) - (x^2+y^2)'\cdot x}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = 1 - \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = 1 + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\hookrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \checkmark \Rightarrow$$

$$\Rightarrow \int_{x_0}^{x} P(t,y_0)\,dt + \int_{y_0}^{y} Q(x,t)\,dt =$$

$$= \int_{x_0}^{x}\left(-\frac{1}{t} + y_0 + \frac{y_0}{t^2+y_0^2}\right)dt + \int_{y_0}^{y}\left(\frac{1}{t} + x - \frac{x}{x^2+t^2}\right)dt =$$

$$= -\int_{x_0}^{x}\frac{1}{t}\,dt + y_0\int_{x_0}^{x}dt + y_0\int_{x_0}^{x}\frac{1}{t^2+y_0^2}\,dt + \int_{y_0}^{y}\frac{1}{t}\,dt + x\int_{y_0}^{y}dt - x\int_{y_0}^{y}\frac{1}{x^2+t^2}\,dt =$$

$$= -\ln(t)\Big|_{x_0}^{x} + y_0\cdot t\Big|_{x_0}^{x} + \cancel{y_0}\cdot\frac{1}{\cancel{t}\,y_0}\cdot\operatorname{arctg}\frac{t}{y_0}\Big|_{x_0}^{x} + \ln(t)\Big|_{y_0}^{y} + x\cdot t\Big|_{y_0}^{y} - \cancel{x}\cdot\frac{1}{\cancel{t}\,x}\cdot\operatorname{arctg}\frac{t}{x}\Big|_{y_0}^{y}$$

?

$$= -\ln x + \ln x_0 + \cancel{y_0 x} - y_0x_0 + \cancel{y_0\cdot\frac{1}{y_0}\cdot\operatorname{arctg}\frac{x}{y_0}} - \cancel{y_0}\cdot\frac{1}{\cancel{x_0}}\operatorname{arctg}\frac{x_0}{y_0} + \ln y - \ln y_0$$

$$+ x\cdot y - \cancel{x\cdot y_0} - x\cdot\frac{1}{\cancel{x}}\cdot\operatorname{arctg}\frac{y}{x} + \cancel{x\cdot\frac{1}{y_0}}\cdot\operatorname{arctg}\frac{y_0}{x}$$

$$= -\ln x + \ln x_0 - y_0x_0 - \underbrace{y_0\cdot\frac{1}{x_0}\cdot\operatorname{arctg}\frac{x_0}{y_0}} + \ln y - \ln y_0 + xy - \frac{x}{y}\cdot\operatorname{arctg}\frac{y}{x}$$

$$= \left(-\ln(x) + \ln y + xy - \frac{x}{y}\operatorname{arctg}\frac{y}{x}\right) - \left(-\ln x_0 + \ln y_0 + y_0x_0 + \frac{y_0}{x_0}\operatorname{arctg}\frac{x_0}{x_0}\right)$$

$$\boxed{\operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} = \frac{\pi}{2}} \Rightarrow$$

$F(x,y) = C$

$$\hookrightarrow -\ln x + \ln y + xy - \frac{x}{y}\cdot\operatorname{arctg}\frac{y}{x} = C$$

$$= -\ln x + \ln x_0 + \cancel{y_0x} - y_0x_0 + \operatorname{arctg}\frac{x}{y_0} - \operatorname{arctg}\frac{x_0}{y_0} +$$

$$+ \ln y - \ln y_0 + xy - \cancel{xy_0} - \operatorname{arctg}\frac{y}{x} + \operatorname{arctg}\frac{y_0}{x} =$$

$$= -\ln x + \ln y + xy - \operatorname{arctg}\frac{y}{x} - \left(-\ln x_0 + \ln y_0 + x_0y_0 + \operatorname{arctg}\frac{x_0}{y_0}\right) \underbrace{+ \operatorname{arctg}\frac{x}{y_0} + \operatorname{arctg}\frac{y_0}{x}}_{=\frac{\pi}{2}} \quad (*)$$

$$\boxed{\operatorname{arctg}\frac{x_0}{y_0} = \frac{\pi}{2} - \operatorname{arctg}\frac{y_0}{x_0}}$$

$$x\cdot(y^2+1)\,dx + \left(x^2y + \frac{1}{\sqrt{1-y^2}}\right)dy = 0$$

$$P(x,y) = x(y^2+1)$$

$$Q(x,y) = x^2y + \frac{1}{\sqrt{1-y^2}}$$

$$\left.\begin{aligned}\frac{\partial P}{\partial y} &= (xy^2+x)'_y = x\cdot 2y = 2xy \checkmark\\ \frac{\partial Q}{\partial x} &= (x^2y)'_x + \frac{1}{\sqrt{1-y^2}} = 2xy \checkmark\end{aligned}\right\}\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\int_{x_0}^{x} P(t,y_0)\,dt + \int_{y_0}^{y} Q(x,t)\,dt =$$

$$= \int_{x_0}^{x} t\,(y_0^2+1)\,dt + \int_{y_0}^{y}\left((x^2t) + \frac{1}{\sqrt{1-t^2}}\right)dt =$$

$$= (y_0^2+1)\int_{x_0}^{x} t\,dt + x^2\int_{y_0}^{y} t\,dt + \int_{y_0}^{y}\frac{1}{\sqrt{1-t^2}}\,dt =$$

$$= (y_0^2+1)\,\frac{t^2}{2}\Big|_{x_0}^{x} + x^2\cdot\frac{t^2}{2}\Big|_{y_0}^{y} + \arcsin\frac{t}{1}\Big|_{y_0}^{y} =$$

$$= (y_0^2+1)\frac{x^2}{2} - (y_0^2+1)\cdot\frac{x_0^2}{2} + x^2\cdot\frac{y^2}{2} - x^2\cdot\frac{y_0^2}{2} + \arcsin y - \arcsin y_0 =$$

$$= \cancel{y_0^2\frac{x^2}{2}} + \frac{x^2}{2} - \frac{y_0^2x_0^2}{2} - \frac{x_0^2}{2} + \frac{x^2y^2}{2} - \cancel{\frac{x^2y_0^2}{2}} + \arcsin y - \arcsin y_0 =$$

$$= \underbrace{\left(\frac{x^2}{2} + \frac{x^2y^2}{2} + \arcsin y\right)}_{F(x,y)} - \underbrace{\left(\frac{x_0^2}{2} + \frac{x_0^2y_0^2}{2} + \arcsin y_0\right)}_{F(x_0,y_0)} =$$

$$F(x,y) = C$$

$$\frac{x^2}{2} + \frac{x^2y^2}{2} + \arcsin y = C$$

$$\frac{1}{2}\left(x^2 + x^2y^2\right) + \arcsin y = C$$

$$\frac{1}{2}\cdot x^2(1+y^2) + \arcsin y = C \checkmark$$

(*) $-\ln x + \ln y + xy - \operatorname{arctg}\frac{y}{x} - \left(-\ln x_0 + \ln y_0 + x_0y_0 + \frac{\pi}{2} - \operatorname{arctg}\frac{y_0}{x_0}\right) + \frac{\pi}{2} =$

$= F(x,y) - F(x_0,y_0) - \frac{\pi}{2} + \frac{\pi}{2}$

$= F(x,y) - F(x_0,y_0)$

unde

$\underbrace{-\ln x + \ln y + xy - \operatorname{arctg}\frac{y}{x}}_{F(x,y)} = C$

$F(x,y) = C$

$\operatorname{arctg}\frac{y}{x}\,dx + \frac{1}{2}\ln(x^2+y^2)\,dy = 0$

$P(x,y) = \operatorname{arctg}\frac{y}{x}$

$Q(x,y) = \frac{1}{2}\ln(x^2+y^2)$

$$\frac{\partial P(x,y)}{\partial y} = \left(\operatorname{arctg}\frac{y}{x}\right)'_y = \frac{1}{\left(\frac{y}{x}\right)^2+1}\cdot\left(\frac{1}{x}\cdot y\right)'_y = \frac{\frac{1}{x}}{\frac{y^2+x^2}{x^2}} = \frac{x^{\cancel{2}}}{\cancel{x}(y^2+x^2)}$$

$$\frac{\partial Q}{\partial x} = \frac{1}{2}\left(\ln(x^2+y^2)\right)'_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot(x^2+y^2)'_x = \frac{\cancel{2}x}{\cancel{2}(x^2+y^2)}$$

$$\hookrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \checkmark$$

$$\int_{x_0}^{x} P(t,y_0)\,dt + \int_{y_0}^{y} Q(x,t)\,dt =$$

$$= \int_{x_0}^{x}\operatorname{arctg}\frac{y_0}{t}\,dt + \int_{y_0}^{y}\frac{1}{2}\ln(x^2+t^2)\,dt =$$

$$= \int_{x_0}^{x}\operatorname{arctg}\frac{y_0}{t}\cdot 1\,dt + \int_{y_0}^{y}\frac{1}{2}\ln(x^2+t^2)\,dt$$????

$$x\cdot\operatorname{arctg}\left(\frac{y_0}{x}\right) + \frac{1}{2}\cdot\left[y_0\cdot\ln(y_0^2+x^2) - y_0\cdot\ln(y^2+x_0^2)\right] - x_0\cdot\operatorname{arctg}\frac{y}{x_0}$$

$$+\frac{1}{2}\cdot\left[y\cdot\ln(x^2+y^2) + 2\cdot\operatorname{arctg}\frac{y}{x} - 2y - y_0\ln(x^2+y_0^2) - 2x\cdot\operatorname{arctg}\frac{y_0}{x} + 2y_0\right]$$

$$= x\cdot\operatorname{arctg}\frac{y_0}{x} + \cancel{\frac{1}{2}y_0\cdot\ln(y_0^2+x^2)} - \frac{1}{2}y_0\ln(y^2+x_0^2) - x_0\cdot\operatorname{arctg}\frac{y}{x_0}$$

$$\boxed{+\frac{1}{2}y\cdot\ln(x^2+y^2)}\ \boxed{+\operatorname{arctg}\frac{y}{x}}\ \boxed{-y} - \cancel{\frac{1}{2}y_0\ln(x^2+y_0^2)} - x\operatorname{arctg}\frac{y_0}{x} + y_0$$

$$\cancel{\left(\frac{1}{2}y\ln(x^2+y^2) + \operatorname{arctg}\frac{y}{x} - y\right) - \left(\frac{1}{2}y_0\ln(x^2+\right.}$$

Trebuie refăcută!

$$\boxed{\operatorname{arctg}\alpha + \operatorname{arctg}\frac{1}{\alpha} = \frac{\pi}{2}}$$

$$\frac{1}{2}\int_{y_0}^{y} t'\ln(x^2+t^2)\,dt = \frac{1}{2}\left[t\ln(x^2+t^2)\Big|_{y_0}^{y} - \int_{y_0}^{y} t\cdot\frac{2t}{t^2+x^2}\,dt\right] =$$

$$= \frac{1}{2}\left[y\ln(x^2+y^2) - y_0\ln(x^2+y_0^2) - \int_{y_0}^{y}\frac{2t^2+2x^2-2x^2}{t^2+x^2}\,dt\right] =$$

$$= \frac{1}{2}\left[y\ln(x^2+y^2) - y_0\ln(x^2+y_0^2) - 2\int_{y_0}^{y}dt + 2x^2\int_{y_0}^{y}\frac{dt}{t^2+x^2}\right] =$$

$= \frac{y}{2} \ln(x^2+y^2) - \frac{y_0}{2} \ln(x^2+y_0^2) - y + y_0 + x \operatorname{arctg} \frac{y}{x} - x \operatorname{arctg} \frac{y_0}{x}$

$$(1 + x\sqrt{x^2+y^2})\,dx + (\sqrt{x^2+y^2} - 1)\,y\,dy = 0$$

$$P(x,y) = 1 + x\sqrt{x^2+y^2}$$

$$Q(x,y) = (\sqrt{x^2+y^2} - 1)\,y$$

$$\frac{dP}{dy} = \underbrace{1' + x' \cdot \sqrt{x^2+y^2}}_{0} + x\sqrt{x^2+y^2}\,' = x \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{dQ}{dx} = (\sqrt{x^2+y^2} - 1)' \cdot y + \underbrace{y' \cdot (\sqrt{x^2+y^2} - 1)}_{0}$$

$$\frac{dQ}{dx} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x \cdot y = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\to \frac{dP}{dy} = \frac{dQ}{dx}$$

$$\int_{x_0}^{x} P(t, y_0)\,dt + \int_{y_0}^{y} Q(x,t)\,dt =$$

$$= \int_{x_0}^{x} (1 + t\sqrt{t^2+y_0^2})\,dt + \int_{y_0}^{y} (\sqrt{x^2+t^2} - 1)\,t\,dt =$$

$$= \int_{x_0}^{x} dt + \int_{x_0}^{x} \frac{1}{2}(t^2+y_0^2)' \cdot (t^2+y_0^2)^{\frac{1}{2}}\,dt + \int_{y_0}^{y} (x^2+t^2)' \cdot \frac{1}{2}\sqrt{x^2+t^2}\,dt - \int_{y_0}^{y} t\,dt =$$

$$= t\Big|_{x_0}^{x} + \frac{1}{2} \cdot \frac{(t^2+y_0^2)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_{x_0}^{x} + \frac{1}{2}\,\frac{(x^2+t^2)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_{y_0}^{y} - \frac{t^2}{2}\Big|_{y_0}^{y} =$$

$$= x - x_0 + \frac{1}{\cancel{2}}(t^2+y_0^2)^{\frac{3}{2}} \cdot \frac{\cancel{2}}{3}\Big|_{x_0}^{x} + \frac{1}{\cancel{2}} \cdot (x^2+t^2)^{\frac{3}{2}} \cdot \frac{\cancel{2}}{3}\Big|_{y_0}^{y} - \frac{y^2}{2} + \frac{y_0^2}{2} =$$

$$= x - x_0 + \frac{1}{3}(t^2+y_0^2)^{\frac{3}{2}}\Big|_{x_0}^{x} + \frac{1}{3} \cdot (x^2+t^2)^{\frac{3}{2}}\Big|_{y_0}^{y} - \frac{y^2}{2} + \frac{y_0^2}{2} =$$

$$= x - x_0 + \cancel{\frac{1}{3}(x^2+y_0^2)^{\frac{3}{2}}} - \frac{1}{3}(x_0^2+y_0^2)^{\frac{3}{2}} + \frac{1}{3}(x^2+y^2)^{\frac{3}{2}} - \cancel{\frac{1}{3}(x^2+y_0^2)^{\frac{3}{2}}} - \frac{y^2}{2} + \frac{y_0^2}{2}$$

$$= \underbrace{\left[x - \frac{y^2}{2} + \frac{1}{3}(x^2+y^2)^{\frac{3}{2}}\right]}_{F(x,y)} - \underbrace{\left[x_0 - \frac{y_0^2}{2} + \frac{1}{3}(x_0^2+y_0^2)^{\frac{3}{2}}\right]}_{F(x_0,y_0)} \to F(x,y) = C$$

$$x - \frac{y^2}{2} + \frac{1}{3}(x^2+y^2)^{\frac{3}{2}} = C$$

Perfect! Bravo!

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