Calcul diferential si integral
An I ZI si ID, Exercitii pregatitoare pt LC si examen

1) Sa se arate ca sirul cu termenul general $\left(a_n\right)_{n\in\mathbb{N}}$ este şir Cauchy si sa se calculeze limita sa:

$$a_n = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots \frac{1}{(3n-2)(3n+1)};$$

$$a_n = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{3.7} + \cdots \frac{1}{(2n-1)(2n+1)};$$

$$a_n = \frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \cdots \frac{1}{n(n+2)};$$

$$a_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n};$$

2) Sa se arate ca sirul cu termenul general $\left(a_n\right)_{n\in\mathbb{N}}$:

$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$ nu este şir Cauchy;

$a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots \frac{1}{\sqrt{n}}$ nu este convergent;

$a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$ este convergent

3) a) Să se enunţe teorema lui Cesaro-Stolz şi să se calculeze limita şirului cu termenul general $\left(u_n\right)_{n\in\mathbb{N}}$ utilizând această teoremă:

$$u_n = \frac{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}}{\sqrt{n}};$$

$$u_n = \frac{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}{\ln n};$$

$$u_n = \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3};$$

$$u_n = \frac{1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}}{n\sqrt{n}};$$

$$u_n = \frac{1 + 2\sqrt{2} + 3\sqrt{3} + \cdots + n\sqrt{n}}{n^2\sqrt{n}};$$

$$u_n = \frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}}, p \in \mathbb{N}^*.$$

b) Să se calculeze limita şirului cu termenul general $(u_n)_{n\in\mathbb{N}}$ utilizând criteriul clestelui:

$$a_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \frac{n}{n^2+3} + \cdots \frac{n}{n^2+n};$$

$$a_n = \frac{1^2+1}{n^3+1} + \frac{2^2+2}{n^3+2} + \cdots + \frac{k^2+k}{n^3+k} + \cdots \frac{n^2+n}{n^3+n};$$

$$u_n = \frac{[x] + [2^2 x] + [3^2 x] + \ldots + [n^2 x]}{n^3};$$

$$u_n = \frac{1! + 2! + 3! + \cdots + n!}{n!};$$

$$u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)};$$

$$a_n = \frac{C_n^1}{2^n+1} + \frac{C_n^2}{2^n+2} + \frac{C_n^3}{2^n+3} + \cdots \frac{C_n^n}{2^n+n}.$$

4) Să se calculeze limita şirului cu termenul general $(u_n)_{n\in\mathbb{N}}$ utilizand criteriul raportului sau al lui Cauchy-d'Alembert:

$$u_n = \frac{\sqrt[n]{n!}}{n};\quad u_n = \sqrt[n]{\frac{(n!)^2}{(2n)!\cdot 8^n}};\quad u_n = \sqrt[n]{\frac{(2n)!}{(n!)^2}};\quad u_n = \frac{\sqrt[n]{(n+1)(n+2)(n+3)\ldots(n+n)}}{n};$$

$$u_n = \frac{\sqrt[n]{(a+1)(a+2)(a+3)\ldots(a+n)}}{n}, a>0;\ u_n = \sqrt[n]{\frac{n!\cdot(2n)!}{(3n)!}}.$$

5) Aratati convergenta sirurilor urmatoare utilizand, eventual, criteriul lui Weierstrass:

$$u_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2};$$

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - \ln n;$$

$$u_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots \frac{1}{n+n};$$

$$u_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n\cdot(n+1)};$$

$$u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n!};$$

$$u_n = \sum_{k=2}^{n} \ln(1 - \frac{1}{k^2})$$

Fie seria $\sum_{n=1}^{\infty} u_n$, $u_n > 0$, convergenta. Aratati ca seriile: $\sum_{n=1}^{\infty} \sin u_n$, $\sum_{n=1}^{\infty} \arcsin u_n$, $\sum_{n=1}^{\infty} arctgu_n$,

$\sum_{n=1}^{\infty}\ln(1+u_n)$, $\sum_{n=1}^{\infty}(a^{u_n}-1)$, sunt, de asemenea, convergente.

6) Stabiliti natura urmatoarelor serii cu termeni pozitivi si in caz de convergenta calculati si sumele lor:

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)};\quad \sum_{n=1}^{\infty}\frac{1}{n(n+1)};\quad \sum_{n=1}^{\infty}\frac{1}{(4n-3)(4n+1)};\quad \sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n+1)};$$

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+3)};\quad \sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)};\quad \sum_{n=1}^{\infty}\frac{1}{(n+a)(n+a+1)},\ a>0;\quad \sum_{n=0}^{\infty}\frac{1}{a^n},\ a>1;$$

$$\sum_{n=1}^{\infty}\frac{\sqrt{2n+1}-\sqrt{2n-1}}{\sqrt{4n^2-1}};\quad \sum_{n=1}^{\infty}\frac{\sqrt{n^2+2}-\sqrt{n^2-2}}{\sqrt{n^4-4}};\quad \sum_{n=2}^{\infty}\frac{\ln(n+1)-\ln n}{\ln(n+1)\cdot\ln n};\quad \sum_{n=0}^{\infty}\frac{1}{n};\quad \sum_{n=2}^{\infty}\frac{1}{\ln n};$$

$$\sum_{n=1}^{\infty}\frac{\sin\left(\frac{1}{n(n+1)}\right)}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}};\quad \sum_{n=0}^{\infty}\frac{1}{2^n};\quad \sum_{n=1}^{\infty}\frac{1.3.5...(2n-1)}{2.4.6...(2n)};\quad \sum_{n=1}^{\infty}\frac{1.3.5...(2n-1)}{2.4.6...(2n)}\cdot\frac{1}{2n+1};$$

$$\sum_{n=1}^{\infty}\frac{a(a+1)(a+2)\cdots(a+n-1)}{b(b+1)(b+2)\cdots(b+n-1)},\ a,b>0;\qquad \sum_{n=1}^{\infty}a^n\cdot\left(\frac{n^2+n+1}{n^2}\right)^n,\ a>0;$$

$$\sum_{n=1}^{\infty}\frac{a^n\ln n}{n},a>0\quad \sum_{n=1}^{\infty}a^n\cdot\left(\frac{n+1}{n}\right)^n,\ a>0;\quad \sum_{n=1}^{\infty}\frac{a^n\ln n}{n},a>0;\quad \sum_{n=1}^{\infty}a^{\ln n},\ a>0;$$

$\sum_{n=1}^{\infty}n^{\ln a},\ a>0$, discutie dupa valorile parametrilor $a,b>0$.

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7) Stabiliti daca seriile urmatoare sunt absolut convergente sau semiconvergente:

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt[p]{n}},p\geq 2,p\in\mathbb{N}$; $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\alpha}}$, discutie dupa valorile parametrului $\alpha\in\mathbb{R}_+^*$;

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n};\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\cdot 3^n};\quad \sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n};\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1.3.5...(2n-1)}{2.4.6...(2n)};\quad \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n^2}.$$