Capitol 3

3.2.2

$T(x) = (x_1 - 2x_3,\ x_1 + 3x_2 + 2x_3,\ -x_1 + x_2 + x_3)$

a) 1) $T(x+y) = T(x) + T(y) \rightarrow$

$\rightarrow T(x+y) = T((x_1, x_2, x_3) + (y_1, y_2, y_3)) =$

$= T((x_1+y_1,\ x_2+y_2,\ x_3+y_3)) =$

$= ((x_1+y_1 - 2(x_3+y_3),\ (x_2+y_2) + 3(x_2+y_2) + 2(x_3+y_3),$

$-(x_1+y_1) + (x_2+y_2) + (x_3+y_3))$

$= (x_1+y_1 - 2x_3 - 2y_3,\ x_1+y_1 + 3x_2 + 3y_2 + 2x_3 + 2y_3,$

$-x_1 - y_1 + x_2 + y_2 + x_3 + y_3) =$

$= (x_1 - 2x_3,\ x_1 + 3x_2 + 2x_3,\ -x_1 + x_2 + x_3) +$

$(y_1 - 2y_3,\ y_1 + 3y_2 + 2y_3,\ -y_1 + y_2 + y_3) =$

$= T(x) + T(y)$ ✓

2) $T(\alpha x) = \alpha T(x) \rightarrow$

$\rightarrow T(\alpha x) = T(\alpha(x_1, x_2, x_3)) = T(\alpha x_1, \alpha x_2, \alpha x_3) =$

$= (\alpha x_1 - 2\alpha x_3,\ \alpha x_1 + 3\alpha x_2 + 2\alpha x_3,\ -\alpha x_1 + \alpha x_2 + \alpha x_3) =$

$= (\alpha(x_1 - 2x_3),\ \alpha(x_1 + 3x_2 + 2x_3),\ \alpha(-x_1 + x_2 + x_3)) =$

$= \alpha \cdot ((x_1 - 2x_3),\ (x_1 + 3x_2 + 2x_3),\ (-x_1 + x_2 + x_3)) =$

$= \alpha \cdot T(x)$ ✓

b)

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

Ex 3.2.4

$f(x_1 x_2 x_3) = (-x_1 + x_2 + 2x_3, 3x_1 + 3x_2 + 4x_3, 2x_1 + x_2 + 2x_3)$

1. Checking for bijection

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 3 & 3 & 4 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & -6 & -10 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & +3 & 5 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -3 & 0 & 1 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to$$

$$= \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$x_1 = x_2 = x_3 = 0$

f: bijective

2/ Checking inverse f^{-1}

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & y_1 \\ 3 & 3 & 4 & y_2 \\ 2 & 1 & 2 & y_3 \end{array}\right) = \left(\begin{array}{ccc|c} -1 & 1 & 2 & y_1 \\ 0 & -6 & -10 & -3y_1-y_2 \\ 0 & -3 & -6 & -2y_1-y_3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} +6 & 0 & -2 & -3y_1+y_2 \\ 0 & -6 & -10 & -3y_1-y_2 \\ 0 & 0 & -6 & 3y_1-3y_2+6y_3 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|c} 6 & 0 & -2 & -3y_1+y_2 \\ 0 & -6 & -10 & -3y_1-y_2 \\ 0 & 0 & 2 & y_1-y_2+2y_3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 12 & 0 & 0 & -4y_1+4y_3 \\ 0 & -12 & 0 & 4y_1-12y_2+20y_3 \\ 0 & 0 & 2 & y_1-y_2+2y_3 \end{array}\right)$$

$$y_1 = \frac{-y_1+y_3}{3}$$

$$\rightarrow y_2 = \frac{-y_1+3y_2-5y_3}{3}$$

$$y_3 = \frac{y_1-y_2+2y_3}{2}$$

$f^{-1}(0,0,0) = (0,0,0)$

$f^{-1}(0,1,2) = \left(\frac{2}{3}, -\frac{7}{3}, \frac{3}{2}\right)$

$f^{-1}(1,5,2) = \left(\frac{1}{3}, \frac{4}{3}, 0\right)$

Ex 3.2.12

$T(x) = (\mathcal{E}_1 - \mathcal{E}_2 + \mathcal{E}_3, \mathcal{E}_2 + \mathcal{E}_3, \mathcal{E}_1 - \mathcal{E}_2)$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) \to A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

baza $\{f_1, f_2, f_3\} \Rightarrow f_1 = (1,1,1)$ $f_2 = (0,1,1)$ $f_3 = (0,0,1)$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -2 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Ex 3.3.4

$f: \mathbb{R}^3 \to \mathbb{R}^3$: $f(x) = (2x_1 + x_2 - x_3, x_1 - x_2 + 2x_3, 3x_1 + x_3)$

Este aplicație liniară? Da

$f(x+y) = f(x) + f(y)$ și $f(\alpha x) = \alpha f(x)$

$$f(x+y) = f((x_1, x_2, x_3) + (y_1, y_2, y_3)) =$$
$$= f(x_1 + y_1, x_2 + y_2, x_3 + y_3) =$$
$$= (2\cdot(x_1+y_1) + (x_2+y_2) - (x_3+y_3), (x_1+y_1) +$$
$$- (x_2+y_2) + 2\cdot(x_3+y_3), 3(x_1+y_1) + (x_3+y_3))$$
$$= (2x_1 + 2y_1 + x_2 + y_2 - x_3 - y_3, x_1 + y_1 - x_2 - y_2 +$$
$$+ 2x_3 + 2y_3, 3x_1 + 3y_1 + x_3 + y_3) =$$
$$= ((2x_1 + x_2 - x_3, x_1 - x_2 + 2x_3, 3x_1 + x_3) +$$
$$(2y_1 + y_2 - y_3, y_1 - y_2 + 2y_3, 3y_1 + y_3)) =$$
$$= (f(x) + f(y))$$ ✓

$$f(\alpha x) = f(\alpha(x_1, x_2, x_3)) = f(\alpha x_1, \alpha x_2, \alpha x_3) =$$
$$= (2\alpha x_1 + \alpha x_2 - \alpha x_3, \alpha x_1 - \alpha x_2 + 2\alpha x_3, 3\alpha x_1 + \alpha x_3) =$$
$$= (\alpha(2x_1 + x_2 - x_3), \alpha(x_1 - x_2 + 2x_3), \alpha(3x_1 + x_3))$$
$$= \alpha(2x_1 + x_2 - x_3, x_1 - x_2 + 2x_3, 3x_1 + x_3) =$$
$$= \alpha f(x)$$ ✓

Ker f - Im f - TR

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -3 & 5 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \to$$ L.i -> Dim Ker = 0

↳ rank 3 Dim Im f = 3

Teorema rangului

Dim $\mathbb{R}^3$ = Dim Ker + Dim Im

3 0 3 ✓

& 3.3.6

$f: \mathbb{R}^3 \to \mathbb{R}^3,\ f(x) = (x_1 - x_2,\ x_2 + x_3,\ -2x_1 + 3x_2 + x_3)$

Este Aplicație Liniară? Da

$$f(x+y) = f(x) + f(y)$$

$$= f((x_1, x_2, x_3) + (y_1\ y_2\ y_3)) =$$

$$= f((x_1+y_1,\ x_2+y_2,\ x_3+y_3)) =$$

$$= ((x_1+y_1) - (x_2+y_2),\ (x_2+y_2) + (x_3+y_3),$$

$$-2\cdot(x_1+y_1) + 3(x_2+y_2) + (x_3+y_3)) =$$

$$= ((x_1+y_1) + (-x_2-y_2),\ x_2+y_2+x_3+y_3,$$

$$-2x_1 - 2y_1 + 3x_2 + 3y_2 + x_3 + y_3) =$$

$$= ((x_1 - x_2,\ x_2 + x_3,\ -2x_1 + 3x_2 + x_3) +$$

$$(y_1 - y_2,\ y_2 + y_3,\ -2y_1 + 3y_2 + y_3)) =$$

$$= f(x) + f(y) \checkmark$$

$$f(\alpha x) = f(\alpha(x_1, x_2, x_3)) =$$

$$= f(\alpha x_1, \alpha x_2, \alpha x_3) =$$

$$= (\alpha x_1 - \alpha x_2,\ \alpha x_2 + \alpha x_3,\ -2\alpha x_1 + 3\alpha x_2 + \alpha x_3)$$

$$= (\alpha(x_1 - x_2),\ \alpha(x_2 + x_3),\ \alpha(-2x_1 + 3x_2 + x_3))$$

$$= \alpha \cdot f(x) \checkmark$$

Ker f - Im f - TR

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -2 & 3 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to$$

rank = 2

↳ Dim Ker f = 1

Dim Im f = 2

$$\underset{3}{\text{Dim } \mathbb{R}^3} = \underset{1}{\text{Dim Ker } f} + \underset{2}{\text{Dim Im } f}$$

✓