Rezolvarea sistemelor de ecuatii liniare

24-Mar-20

Sistem de ecuatii liniare cu n ecuatii si n necunoscute

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

- Coeficientii se cunosc.

$$a_{ij}, i, j \in \{1,2,\ldots,n\}$$

- $$b_i, i \in \{1,2,\ldots,n\}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

- Sistemul de mai sus se poate scrie sub forma

$$Ax = b$$

Metode de rezolvare

- Directe
  - Regula lui Cramer
  - Metoda eliminarii a lui Gauss
  - Metoda Gauss- Jordan
  - Metoda descompunerii LU
- Iterative
  - Gauss- Jacobi
  - Gauss-Seidel

Metode iterative

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

Din fiecare ecuatie putem exprima, pe rand x1, x2, ..., xn, in functie de celelalte necunoscute.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n)}{a_{22}}$$

$$x_i = \frac{b_i - (a_{i1}x_1 + \ldots + a_{i,i-1}\, x_{i-1} + a_{i,i+1}x_{i+1} \ldots + a_{in}x_n)}{a_{ii}}$$

$$x_n = \frac{b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1})}{a_{nn}}$$

$$x_i = \frac{b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}} \quad pentru\, orice\, i = 1,\ldots n$$

- Vrem sa determinam $x=(x_i)_{i=1,\ldots,n}$ astfel incat $Ax=b$.
- Metoda iterativa de rezolvare presupune ca

  se pleaca de la o valoare initiala a lui x,

  notata $x^{(0)} \quad =(x_i^{\,0})_{i=1,\ldots,n}$

si se construieste un sir $(x^{(k)})_k$ ce converge la solutia sistemului.

- Sirul se defineste sub forma:
- $x^{(k+1)}=f\,(x^{(k)})$ pt $k>=1$

Metoda Jacobi

$$x_i^{(k+1)} = \frac{b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k)}}{a_{ii}} \quad pentru\, orice\, i = 1, \ldots n$$

- Algoritmul se opreste cand $x^{(k+1)}$ si $x^{(k)}$ sunt suficient de aproape adica

$$\left| x_i^{(k+1)} - x_i^{(k)} \right| \leq \in \; pentru\, orice\, i = 1, \ldots, n$$

Teorema

Daca A este o matrice diagonal dominanta atunci metoda Jacobi este convergenta.

Def:

A este o matrice diagonal dominanta daca

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \forall i = 1, \dots, n$$

Care din urmatoarele matrici sunt diagonal dominante?

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & 2 \\ -1 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}$$

A si C

$$|a_{11}| \geq |a_{12}| + |a_{13}| \qquad |+3| \geq |-2| + |+1|$$

$$|a_{22}| \geq |a_{21}| + |a_{23}| \qquad |-3| \geq |+1| + |+2|$$

$$|a_{33}| \geq |a_{31}| + |a_{32}| \qquad |+4| \geq |-1| + |+2|$$

La fel se arata ca C este matrice strict diagonal dominanta.

$$B = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

nu este diagonal dominanta
pt ca

$$|b_{11}| < |b_{12}| + |b_{13}| \qquad |-2| < |+2| + |+1|$$

Exemplu

$$\begin{bmatrix} -5 & -1 & 2 \\ 2 & 6 & -3 \\ 2 & 1 & 7 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 32 \end{bmatrix}$$

Matricea sistemului este diagonal dominanta?

DA

Rezolvarea sistemului

$$\begin{cases} -5x_1 - x_2 + 2x_3 = 1 \\ 2x_1 + 6x_2 - 3x_3 = 2 \\ 2x_1 + x_2 + 7x_3 = 32 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{1}{5}(1 + x_2 - 2x_3) \\ x_2 = \frac{1}{6}(2 - 2x_1 + 3x_3) \\ x_3 = \frac{1}{7}(32 - 2x_1 - x_2) \end{cases} \Rightarrow$$

Metoda Gauss-Jacobi

$$\begin{cases} x_1^{(k+1)} = -\frac{1}{5}(1 + x_2^{(k)} - 2x_3^{(k)}) \\ x_2^{(k+1)} = \frac{1}{6}(2 - 2x_1^{(k)} + 3x_3^{(k)}) \\ x_3^{(k+1)} = \frac{1}{7}(32 - 2x_1^{(k)} - x_2^{(k)}) \end{cases}$$

$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} si \in= 0.005$

Construirea sirului

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1^{(1)} = -\frac{1}{5}(1 + x_2^{(0)} - 2x_3^{(0)}) = -0.2 \\ x_2^{(1)} = \frac{1}{6}(2 - 2x_1^{(0)} + 3x_3^{(0)}) = 0.3333 \\ x_3^{(1)} = \frac{1}{7}(32 - 2x_1^{(0)} - x_2^{(0)}) = 4.5714 \end{cases}$$

$$\left| x^{(1)} - x^{(0)} \right| = \begin{bmatrix} 0.2 \\ 0.3333 \\ 4.5714 \end{bmatrix}$$

$$\begin{cases} x_1^{(2)} = -\dfrac{1}{5}(1 + x_2^{(1)} - 2x_3^{(1)}) = 1.5619 \\ x_2^{(2)} = \dfrac{1}{6}(2 - 2x_1^{(1)} + 3x_3^{(1)}) = 2.6857 \\ x_3^{(2)} = \dfrac{1}{7}(32 - 2x_1^{(1)} - x_2^{(1)}) = 4.5810 \end{cases}$$

$$\left| x^{(2)} - x^{(1)} \right| = \begin{bmatrix} 1.7619 \\ 2.3524 \\ 0.0096 \end{bmatrix}$$

k	$x1^{(k)}$	$x2^{(k)}$	$x3^{(k)}$	$\|x1^{(k)}-x1^{(k-1)}\|$	$\|x2^{(k)}-x2^{(k-1)}\|$	$\|x3^{(k)}-x3^{(k-1)}\|$
0	0	0	0			
1	-0.2	0.3333	4.5714			
2	1.5619	2.6857	4.5810			
3	1.0952	2.1032	3.7415			
4	0.8760	1.8390	3.9580			
5	1.0154	2.0204	4.0584			
6	1.0193	2.0241	3.9927			
7	0.9923	1.9899	3.9910			
8	0.9984	1.9981	4.0037	0.0061	0.0082	0.0127
9	**1.0018**	**2.0023**	**4.0007**	0.0034	0.0042	0.003
10	0.9998	1.9997	3.9991	0.002	0.0026	0.0016

Metoda Gauss - Seidel

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n)}{a_{22}}$$

$$x_i = \frac{b_i - (a_{i1}x_1 + \ldots + a_{i,i-1}\, x_{i-1} + a_{i,i+1}x_{i+1} \ldots + a_{in}x_n)}{a_{ii}}$$

$$x_n = \frac{b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1})}{a_{nn}}$$

Metoda Gauss - Seidel

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}} \quad pentru\, orice\, i = 1,\ldots n$$

- Algoritmul se opreste cand $x^{(k+1)}$ si $x^{(k)}$ sunt suficient de aproape adica

$$\left| x_i^{(k+1)} - x_i^{(k)} \right| \leq \in \; pentru\, orice\, i = 1,\ldots, n$$

Teorema

Daca A este o matrice diagonal dominanta atunci metoda Gauss-Seidel este convergenta.

Exemplu

$$\begin{bmatrix} -5 & -1 & 2 \\ 2 & 6 & -3 \\ 2 & 1 & 7 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 32 \end{bmatrix}$$

Matricea sistemului este diagonal dominanta?

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Rezolvarea sistemului

$$\begin{cases} -5x_1 - x_2 + 2x_3 = 1 \\ 2x_1 + 6x_2 - 3x_3 = 2 \\ 2x_1 + x_2 + 7x_3 = 32 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{1}{5}(1 + x_2 - 2x_3) \\ x_2 = \frac{1}{6}(2 - 2x_1 + 3x_3) \\ x_3 = \frac{1}{7}(32 - 2x_1 - x_2) \end{cases} \Rightarrow$$

Metoda Gauss-Seidel

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} si \in = 0.005$$

$$\begin{cases} x_1^{(k+1)} = -\frac{1}{5}(1 + x_2^{(k)} - 2x_3^{(k)}) \\ x_2^{(k+1)} = \frac{1}{6}(2 - 2x_1^{(k+1)} + 3x_3^{(k)}) \\ x_3^{(k+1)} = \frac{1}{7}(32 - 2x_1^{(k+1)} - x_2^{(k+1)}) \end{cases}$$

Construirea sirului

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1^{(1)} = -\frac{1}{5}(1 + x_2^{(0)} - 2x_3^{(0)}) = -0.2 \\ x_2^{(1)} = \frac{1}{6}(2 - 2x_1^{(1)} + 3x_3^{(0)}) = 0.4 \\ x_3^{(1)} = \frac{1}{7}(32 - 2x_1^{(1)} - x_2^{(1)}) = 4.5714 \end{cases}$$

$$\left| x^{(1)} - x^{(0)} \right| = \begin{bmatrix} 0.2 \\ 0.4 \\ 4.5714 \end{bmatrix}$$

$$
\begin{cases}
x_1{}^{(2)} = -\dfrac{1}{5}(1 + x_2{}^{(1)} - 2x_3{}^{(1)}) = 1.5486 \\
x_2{}^{(2)} = \dfrac{1}{6}(2 - 2x_1{}^{(2)} + 3x_3{}^{(1)}) = 2.1029 \\
x_3{}^{(2)} = \dfrac{1}{7}(32 - 2x_1{}^{(2)} - x_2{}^{(2)}) = 3.8286
\end{cases}
$$

$$
\left| x^{(2)} - x^{(1)} \right| = \begin{bmatrix} 1.7486 \\ 1.7029 \\ 0.7428 \end{bmatrix}
$$

k	$x1^{(k)}$	$x2^{(k)}$	$x3^{(k)}$	$\lvert x1^{(k)}-x1^{(k-1)} \rvert$	$\lvert x2^{(k)}-x2^{(k-1)} \rvert$	$\lvert x3^{(k)}-x3^{(k-1)} \rvert$
0	0	0	0			
1	-0.2	0.4	4.5714			
2	1.5486	2.1029	3.8286			
3	0.9109	1.9440	4.0335			
4	1.0246	2.0085	3.9918			
5	0.9950	1.9975	4.0018			
6	1.0012	2.0005	3.9996	0.0062		
7	**0.9997**	**1.9975**	**4.0001**	0.0015	0.0006	0.0005

Observatii

- Conditia ca A sa fie diagonal dominanta nu este obligatorie dar ea asigura convergenta metodelor GJ si GS. Exista A nediagonal dominanta pentru care metodele converg
- Daca valorile initiale sunt apropiate de solutie atunci convergenta este mai rapida.

Observatii

- Daca sistemul initial ar fi fost

$$\begin{cases} -5x_1 - x_2 + 2x_3 = 1 \\ 2x_1 + x_2 + 7x_3 = 32 \\ 2x_1 + 6x_2 - 3x_3 = 2 \end{cases}$$

Este matricea sistemului diagonal dominanta?

NU dar prin rearanjarea ecuatiilor ea poate deveni diagonal dominanta.

Tema 2

Algoritm Jacobi

```
 x=xinitial
iter=0
d=0
 while   d<n and iter < MAX
 d=0
             for i=1,n
                    S=0
                     for j=1,n
                            if i ≠j then S=S+a[i][j]*x[j]
                     endfor
                    xnew[i]=(b[i]-S)/a[i][i]
             endfor
             for i=1,n
                    diff[i]=|xnew[i]-x[i]|

                     if diff[i]<=epsilon    then d++
                     endif
            endfor
            x=xnew
            iter++
  endwhile
   if d<n then write 'nr iteratii depasit'
   endif
    for i=1,n
          Write x[i]
    endfor
```

Algoritm Gauss-Seidel

```
For i=1,n
      read x[i]
Endfor
Iter=0
Do
              d=0
               for i=1,n
                       xold[i]=x[i]
               endfor
               for i=1,n
                       S=0
                         for j=1,n
                                 if i ≠j then S=S+a[i][j]*x[j]
                        endfor
                       x[i]=(b[i]-S)/a[i][i]
                endfor
                for i=1,n
                       diff[i]=|x[i]-xold[i]|
                        if diff[i]<=epsilon    then d++
                        endif
              endfor
              iter++
  while   d<n and iter < MAX
   if d<n then write 'nr iteratii depasit'
   endif
    for i=1,n
           Write x[i]
    endfor
```