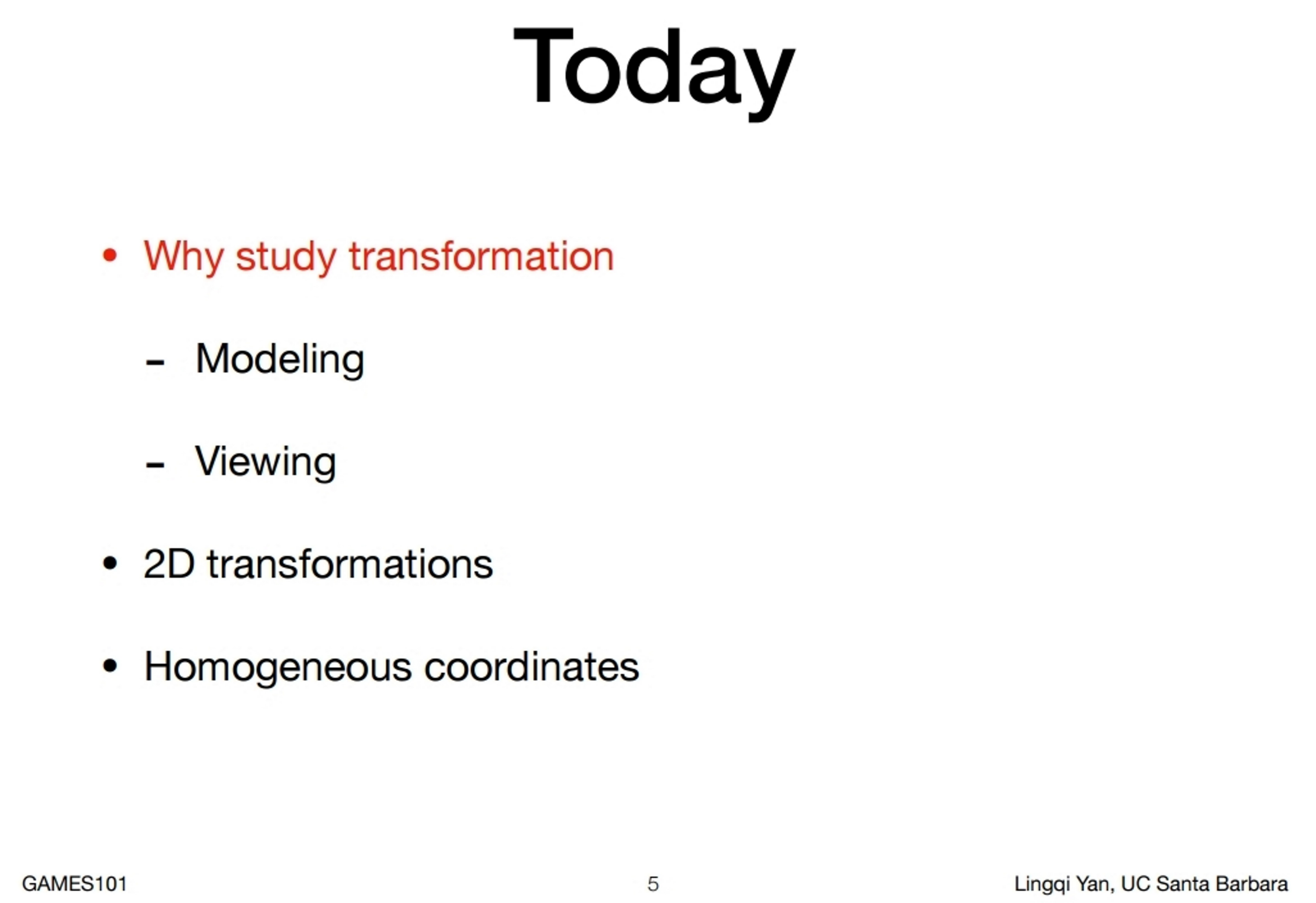
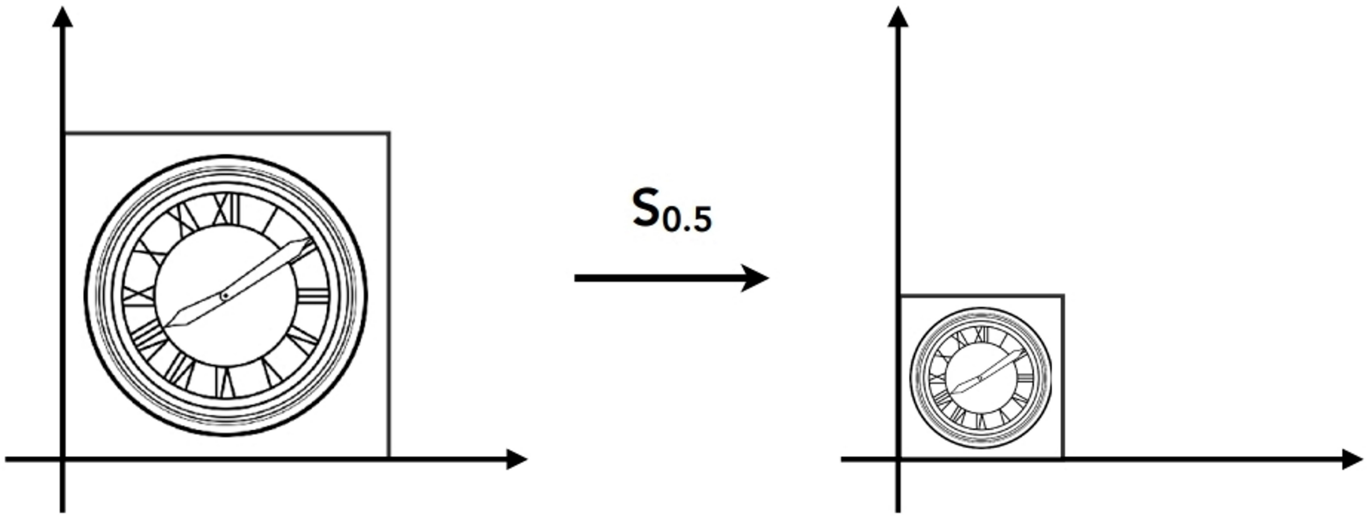
Lecture 03 Transformation

2020年2月24日 10:29

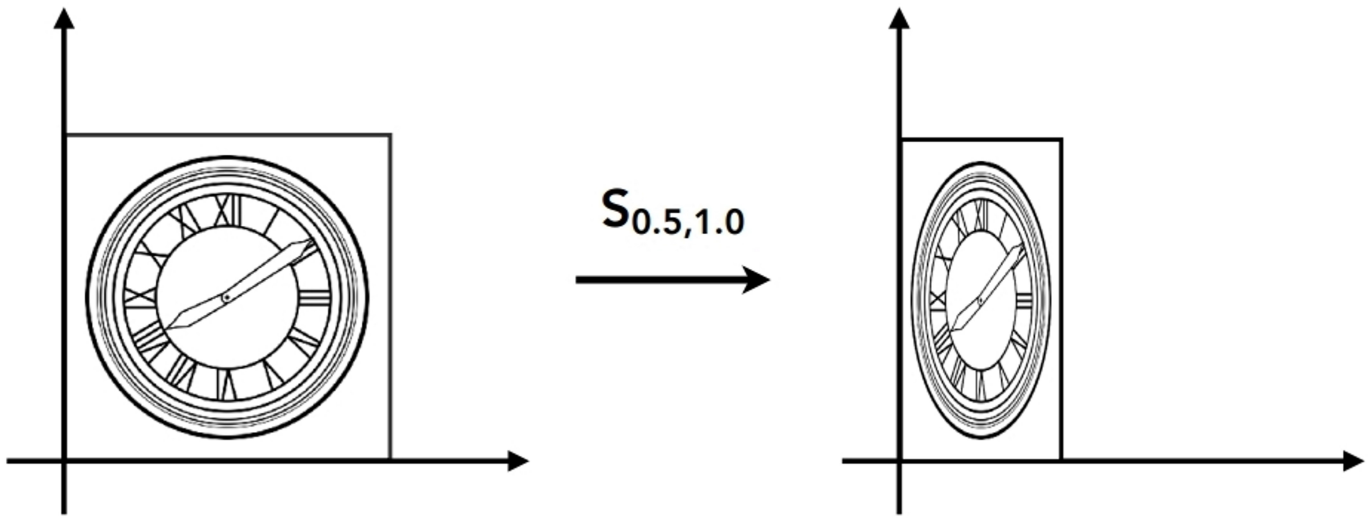


Scale Matrix



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

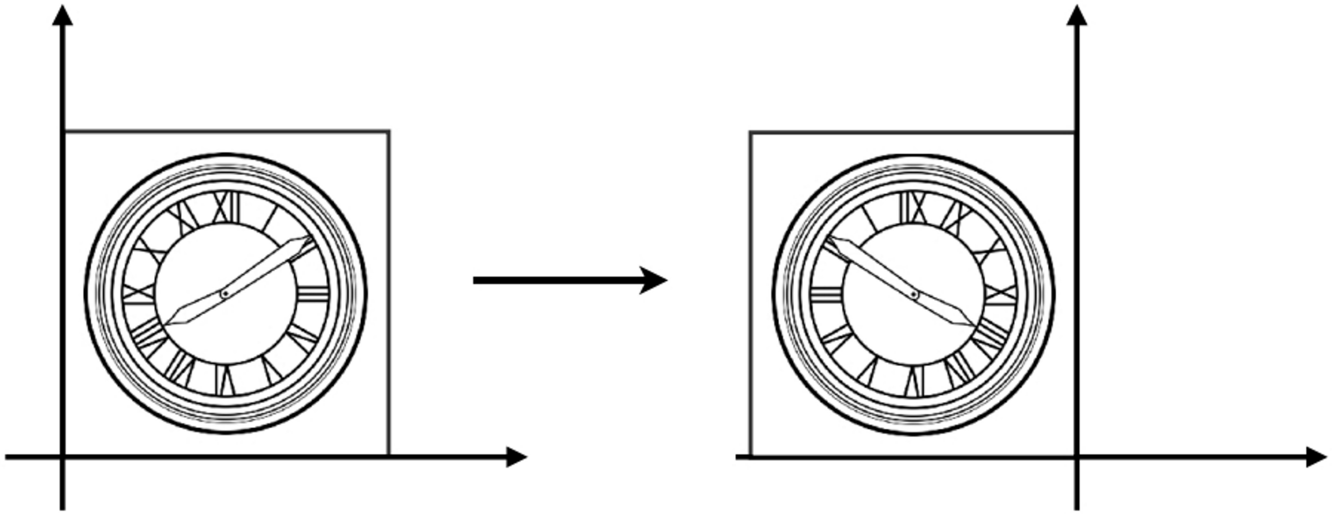
Scale (Non-Uniform)



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

缩放变换，用缩放矩阵表示

Reflection Matrix



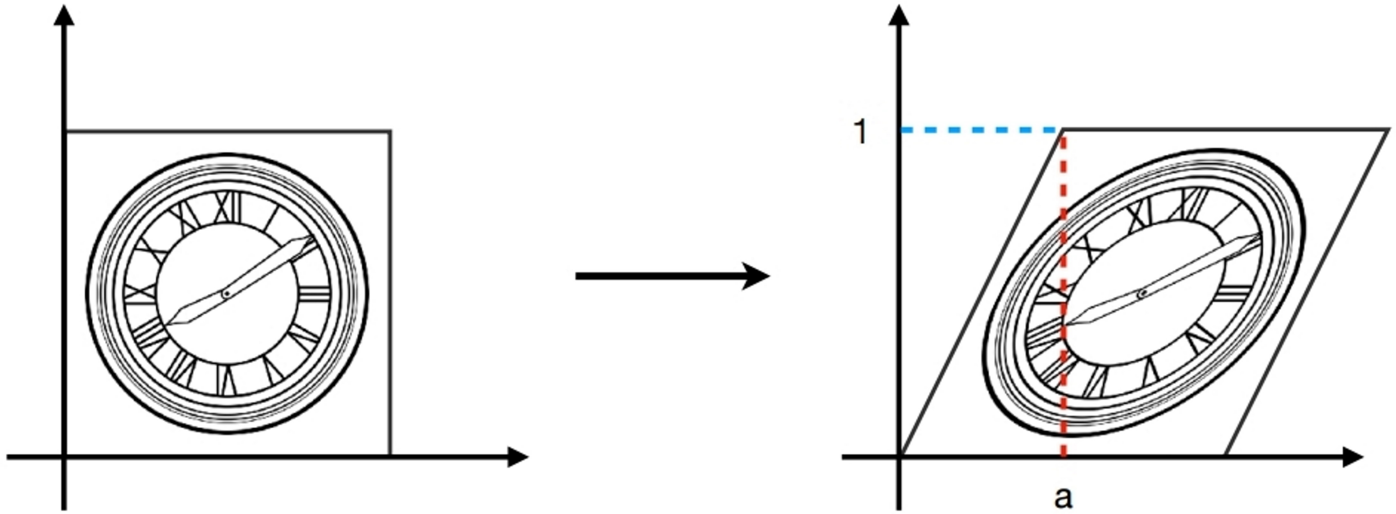
Horizontal reflection:

$$x' = -x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

反射（镜像）矩阵

Shear Matrix



Hints:

Horizontal shift is 0 at y=0

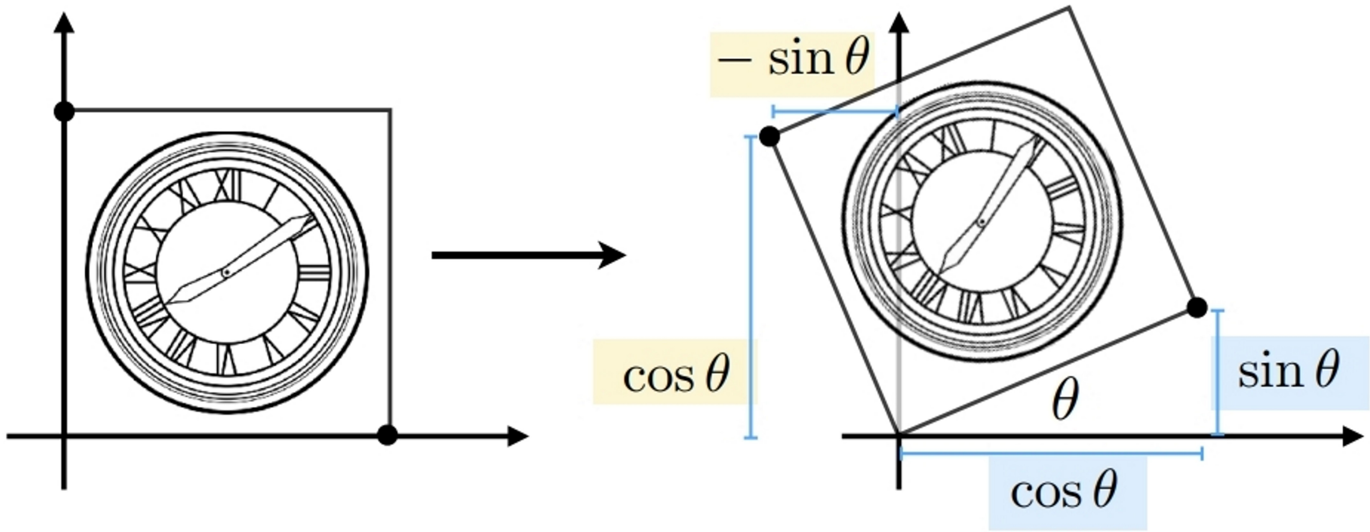
Horizontal shift is a at y=1

Vertical shift is always 0

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

切变矩阵

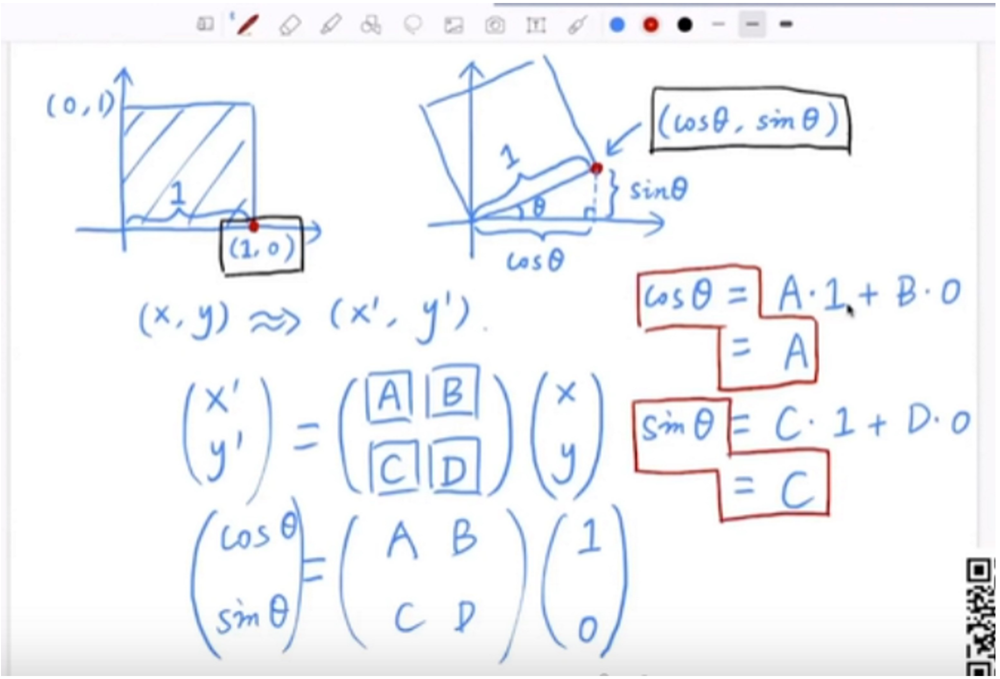
Rotation Matrix



$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

旋转矩阵

旋转的默认设定：（1）默认绕原点旋转；（2）默认逆时针旋转



旋转矩阵的推导（以特殊点(1,0)为例子）

Linear Transforms = Matrices

(of the same dimension)

$$x' = a\,x + b\,y$$
$$y' = c\,x + d\,y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{M}\,\mathbf{x}$$

线性变换：可以用一个矩阵表示的变换

齐次坐标

为什么引入齐次坐标?

Why Homogeneous Coordinates

- Translation cannot be represented in matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

(So, translation is NOT linear transform!)

- But we don't want translation to be a special case
- Is there a unified way to represent all transformations? (and what's the cost?)

因为平移变换不能直接用一个矩阵表示，必须加一个向量；但其他线性变换可以

人们发现，引入齐次坐标可以解决问题，让平移也能只用一个矩阵表示

Solution: Homogenous Coordinates

Add a third coordinate (*w-coordinate*)

- 2D point = $(x, y, 1)^T$
- 2D vector = $(x, y, 0)^T$

Matrix representation of translations

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

What if you translate a vector?

增加一个齐次项，规定点的齐次项是1，向量的齐次项是0

Homogenous Coordinates

Valid operation if w-coordinate of result is 1 or 0

- vector + vector = vector
- point – point = vector
- point + vector = point
- point + point = ??

In homogeneous coordinates,

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \text{ is the 2D point } \begin{pmatrix} x/w \\ y/w \\ 1 \end{pmatrix}, \quad w \neq 0$$

向量的齐次项是0，点的齐次项是1

向量 + 向量，结果齐次项是0，还是向量

点 - 点，得到的是一个向量，齐次项也变成0

点 + 向量，表示一个点的移动，结果齐次项是1，还是点。没毛病！

点 + 点是什么呢？齐次项变成2。将所有项除以2，齐次项又变为1
所以点 + 点结果实际上是两个点的中点。

Affine Transformations

Affine map = linear map + translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Using homogenous coordinates:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

仿射变换：线性变换加上一次平移
注意：先进行线性变换，再平移

2D Transformations

Scale

$$\mathbf{S}(s_x, s_y) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation

$$\mathbf{R}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Translation

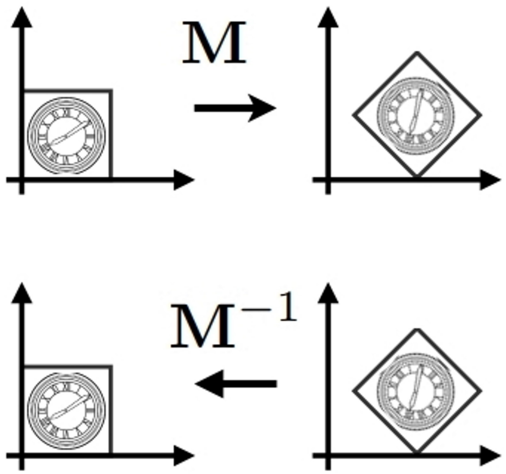
$$\mathbf{T}(t_x, t_y) = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

2D变换矩阵（缩放，旋转，平移变换）

Inverse Transform

$$\mathbf{M}^{-1}$$

$\mathbf{M}^{-1}$ is the inverse of transform $\mathbf{M}$ in both a matrix and geometric sense



逆变换

逆变换的矩阵就是原变换的逆矩阵

Transform Ordering Matters!

Matrix multiplication is <u>not</u> commutative

$$R_{45} \cdot T_{(1,0)} \neq T_{(1,0)} \cdot R_{45}$$

Note that matrices are applied right to left:

$$T_{(1,0)} \cdot R_{45} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

变换矩阵的顺序会影响结果（这也就是为什么矩阵运算不满足交换律）

Composing Transforms

Sequence of affine transforms A_1, A_2, A_3, ...

- Compose by matrix multiplication
  - Very important for performance!

$$A_n(\ldots A_2(A_1(\mathbf{x}))) = \underbrace{\mathbf{A}_n \cdots \mathbf{A}_2 \cdot \mathbf{A}_1}_{} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

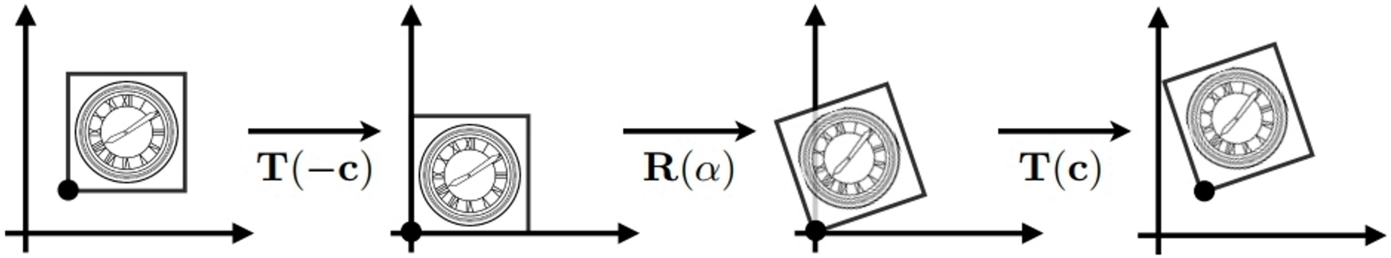
Pre-multiply *n* matrices to obtain a single matrix representing combined transform

合成：连续多个变换可以乘起来（矩阵运算满足结合律）

Decomposing Complex Transforms

How to rotate around a given point c?

1. Translate center to origin
2. Rotate
3. Translate back



Matrix representation?

$$\mathbf{T}(\mathbf{c}) \cdot \mathbf{R}(\alpha) \cdot \mathbf{T}(-\mathbf{c})$$

分解：变换可以分解，注意先后顺序是从右到左

3D Transformations

Use homogeneous coordinates again:

- **3D point = $(x, y, z, 1)^T$**
- **3D vector = $(x, y, z, 0)^T$**

In general, (x, y, z, w) ($w \neq 0$) is the 3D point:

$$(x/w, y/w, z/w)$$

三维空间中变换

3D Transformations

Use 4×4 matrices for affine transformations

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

What's the order?

Linear Transform first or Translation first?

三维空间中的变换也可以用齐次坐标描述

三维空间中的齐次变换，最后一行和二维变换类似，是0 0 0 1，平移还是在矩阵最后一列