Advanced Algorithms
Homework 1: Preliminaries, Divide and Conquer
CS 611, Spring 2020

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1 Problem 2: Asymptotic Growth Rates

1.1 (a) Rank by Order of Growth

Since there are 30 functions, we won't compare each one to each other; but will show why a function is adjacent to its neighbors in the queue. Therefore, if we can prove why two functions in the queue are adjacent, we can prove that the whole "order of growth queue" is in correct order.

1. 2^{2^n} is exponential of exponential, let's compare with the fastest growing factorial $(n+1)!$. See Figure 1

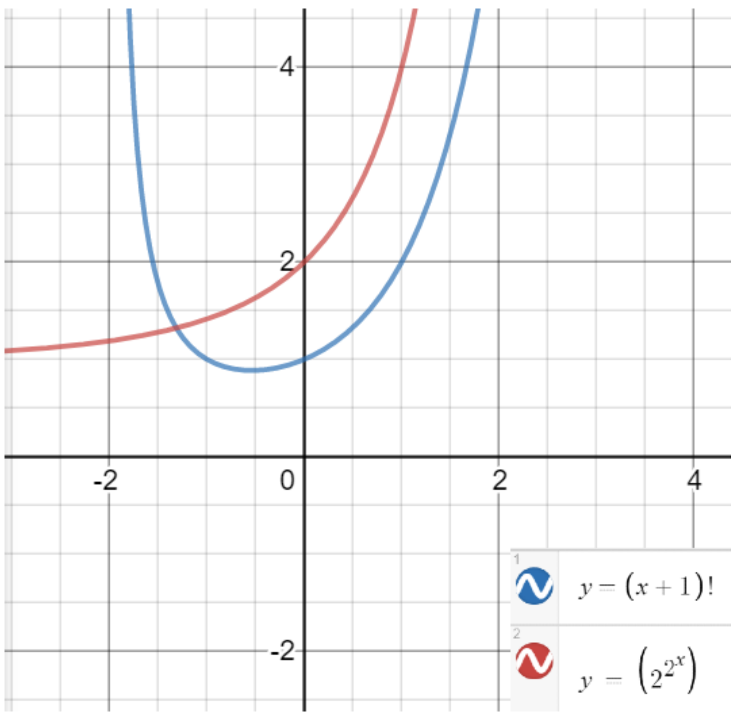

Figure 1: 2^{2^n} and $(n+1)!$ growth rate compared

We can see that at a certain x on X-axis, 2^{2^n} will grow faster than $(n+1)!$ for all $n > -1.314$

2. Obviously, $2^{2^{n+1}}$ grows faster than 2^{2^n} as for a certain point on X-axis, $n+1$ is greater than n

3. By same logic, $(n+1)!$ grows faster than $n!$

4. After factorials, exponentials will have fast rate of growth. There's a function $n \cdot 2^n$ is polynomial exponential, it grows at faster rate than e^n after $n \geq 1.669$. Note that we have not put the last factorial function $(\lg n)!$ right after $n!$ because there are exponential functions who grows faster than factorial of a log.

5. After $n \cdot 2^n$, there comes exponential function in fast to slower rate are: e^n, 2^n and $\left(\frac{3}{2}\right)^n$

6. Now since all the faster exponentials have been placed, we can put the log factorial function $(\lg n)!$ in the list.

7. Let's focus on remaining factorials and polynomials now:

8. Exponential will only grow at faster rate than polynomials if the exponent is greater than 1. In some functions given to us, the exponents are logarithmic, that means we can describe them in polynomial format, and compare their rates with other obvious polynomial functions.

9. Let's put $n^{\lg(\lg n)}$ after $(\lg n)!$, and compare it with n^3. We can see that 3 is a constant power but $\lg(\lg n)$ is not. Once, $\lg(\lg(n))$ value crosses $x = 3$, $n^{\lg(\lg n)}$ will grow faster than n^3.

10. Also, $n^{\lg(\lg n)}$ is equivalent to $(\lg(n))^{\lg(n)}$ by the logarithmic rule $a^{\lg_b c} = c^{\lg_b a}$. Thus they are in same class.

11. Then comes n^3, n^2 in that order. As naturally, $3 > 2$. There's a function which is equivalent to n^2.
$4^{\lg n} = 2^{2 \cdot \lg n} = 2^{\lg n^2} = n^{2 \cdot \lg 2^2} = n^2$. Thus they fall in same class.

12. Next polynomial function will be $n \lg n$ which grows faster than log as the exponent of 2; for n > 2.949. See Figure 2

13. $\lg(n!)$ is in the same class as $n \lg n$ because $n \lg n = \lg(n^n)$. And $n! = O(n^n)$ because $n! = n \cdot (n-1) \cdot (n-2) \ldots 1$

14. Now the polynomial cases are over (except one, which will be simplified to a constant), we should again try to put all the exponentials in place.

15. $2^{\lg n} = n^{\lg 2} = n^1 = n$ is also polynomial and both falls into same class.

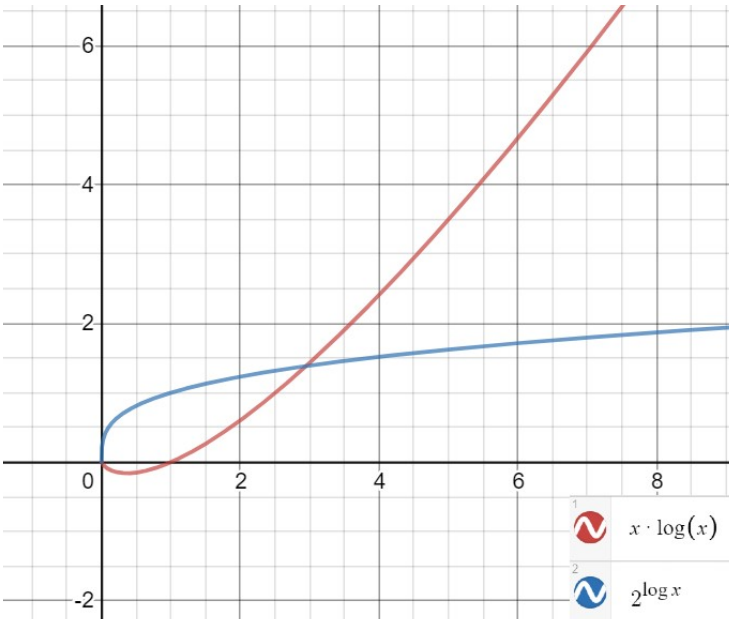

Figure 2: $n \lg n$ and $2^{\lg n}$ growth rate compared

16. Then comes an exponential with smaller base: $(\sqrt{2})^{\lg n}$.

17. Last exponential with log as a power is $2^{\sqrt{2 \lg n}}$. We put that in the list.

18. Now all that remains is logarithmic and iterative logarithmic functions. First is of course a log function with power greater than 1: $\lg^2(n)$

19. Next logarithmic function will be $\ln(n)$.

20. There are two log functions remaining. $\sqrt{\lg(n)}$ grows faster than $\ln(\ln(n))$ as the latter is a natural log of a natural log, which grows slower compared to square root of a log with base 2.

21. Next are iterative logarithms, because they grow very slowly:

$$\lg^* 2 = 1$$
$$\lg^* 4 = 2$$
$$\lg^* 16 = 3$$
$$\lg^* 65536 = 4$$
$$\lg^*(2^{65536}) = 5$$

22. Because of the reasons discussed above, an exponential function grows fastest in tis category: $2^{\lg^*(n)}$
23. Then comes iterative logarithm of log before logarithm of iterative log, because if iterative log is deeper in the nested brackets; the slower the function grows as the values in the bracket doesn't change much.
24. $\therefore \lg^*(\lg(n))$ and then $\lg(\lg^*(n))$.
25. Only constants remain now. Let's see how $n^{\frac{1}{\lg(n)}}$ is a constant.

$$n^{\frac{1}{\lg(n)}} = n^{\lg_2(n)^{-1}} = 2^{\lg_n(n)^{-1}} = 2^{1^{-1}} = 2^1 = 2$$

26. The function "1" is in same class as "2".

Finally, we put this in a tabular form to show it concisely.

$2^{2^{n+1}}$
2^{2^n}
$(n+1)!$
$(n)!$
$n \cdot 2^n$
e^n
2^n
$\left(\frac{3}{2}\right)^n$
$(\lg n)!$
$n^{\lg(\lg n)}$ and $\lg(n)^{\lg(n)}$
n^3
n^2 and $4^{\lg n}$
$n \lg n$ and $\lg(n!)$
$2^{\lg n}$ and n
$(\sqrt{2})^{\lg n}$
$2^{\sqrt{2 \lg n}}$
$\lg^2(n)$
$\ln(n)$
$\sqrt{\lg(n)}$
$\ln(\ln(n))$
$2^{\lg^*(n)}$
$\lg^*(n)$ and $\lg^*(\lg(n))$
$\lg(\lg *(n))$
1 and $n^{\frac{1}{\lg(n)}}$

1.2 (b) A nonnegative function $f(n)$, which is neither $O(g_i(n))$ nor $\Omega(g_i(n))$

Suppose our nonnegative function is:

$$f(n) = \begin{cases} 2^{2^{n+2}} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

For even n, there is

$$\lim_{n\to\infty} \frac{f(2n)}{g_i(2n)} \geq \lim_{n\to\infty} \frac{f(2n)}{g_1(2n)} = \lim_{n\to\infty} 2^{2^{(g_1(2n)-1)+2}} = \lim_{n\to\infty} 2^{2^{g_1(2n)+1}} = \boxed{\infty}$$

And for odd n, there is

$$\lim_{n\to\infty} \frac{f(2n+1)}{g_i(2n+1)} \leq \lim_{n\to\infty} \frac{f(2n+1)}{1} = \boxed{0}$$

We can now say that for even n, $f(n)$ is not $O(g_i(n))$ for any n. And for odd n, $f(n)$ is not $\Omega(g_i(n))$ for any n.